FRACTIONAL QUANTUM HALL STATES IN CONTINUUM AND
LATTICE SYSTEMS

A Dissertation in
Physics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2014
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Abstract

In this dissertation, I will present theoretical studies on several aspects of quantum Hall states in both continuum and lattice systems.

In the continuum case, one can understand the quantum Hall states starting from the Landau levels of charged particles moving in a magnetic field. If an integral number of Landau levels are completely filled with non-interacting fermions due to the Pauli principle, we can get an integer quantum Hall state. When there are strong interactions between fermions or bosons, fractional quantum Hall states can be realized. These states can not be understood using perturbation theory but requires emergent concepts for their interpretations. In the composite fermion theory, fractional quantum Hall states of physical particles are mapped to integer quantum Hall states of emergent particles known as composite fermions. It is demonstrated that the edge excitations of the fermionic fractional quantum Hall states at filling factor 2/3 and a variety of quantum Hall states for two-component bosons can be explained using the composite fermion theory.

When the quantum Hall problem is formulated on lattices, the topological aspects of the quantum Hall states can be revealed. A topological invariant called the Chern number may be defined for the Bloch bands of lattice systems. An integer quantum Hall state occurs in a fermionic system when the total Chern number of the occupied bands of the system is non-zero. Fractional quantum Hall states can appear for interacting particles in a nearly flat Bloch band with Chern number 1. It is shown that such states are adiabatically connected to those in continuum. Lattice models which possess bands with Chern numbers larger than 1 are constructed and exotic fractional topological phases are identified in such bands. The similarities and differences between these lattice models and conventional multi-layer quantum Hall systems are examined in detail.
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List of Symbols

$h$ the Planck constant
$h$ the reduced Planck constant
$e$ absolute value of the electron charge
$c$ the speed of light
$A$ electromagnetic vector potential
$B$ magnetic field for the underlying particles
$B^*$ magnetic field for the composite fermions
$\nu$ filling factor of the underlying particles
$\nu^*$ filling factor of the composite fermions
$\ell_B$ magnetic length
$z$ complex coordinate of two-dimensional space
$2Q$ magnetic flux through a sphere
$N_\phi$ magnetic flux through a torus
$\chi_n$ wave function of $n$ completely filled Landau levels
$\mathcal{P}_{\text{LLL}}$ lowest Landau level projection operator
$C$ Chern number of Bloch bands
$H_2$ many body Hamiltonian consists of two-body interactions
$H_3$ many body Hamiltonian consists of three-body interactions
Acknowledgments

The Ph.D. years in graduate school would be very difficult without the guidance, friendship, and help from many people. I feel very fortunate to be able to get into the fascinating world of condensed matter physics under the guidance of my advisor Jainendra Jain. His insight and patience are of fundamental importance in helping me to become an qualified scientist.

I am grateful to Kai Sun for teaching me many useful things in our fruitful collaborations. I have enjoyed many discussions with Chaoxing Liu through which my understanding of physics improved substantially. I am indebted to Nicolas Regnault for sharing the wonderful DiagHam package, which played a vital role in my research. I learned a lot about numerical methods from Arkadiusz Wójs for which I am always grateful. I would like to thank Vijay Shenoy for his great help on many things when I visited the Indian Institute of Science.

I would like to thank Xinxin Cai for her help on making some figures. I also thank Alex Archer, Ajit Balram, Ashley DaSilva, Jimmy Hutasoit, Hsiu-Chuan Hsu, Sangzi Liang, Xin Liu, Yufei Shen, G. J. Sreejith, Qingze Wang, Wei Xu, Gang Yang, Ruixing Zhang, and Yuhe Zhang.

The staff members of the physics department, especially Carol Deering, have helped me on many aspects in the past few years. The high-performance computing resources for the numerical works in this thesis are provided by the Research Computing and Cyberinfrastructure. Their staff members have always been very helpful when I have problems about the computer clusters.
Dedication

*to those who love me and those who I love*
Chapter 1

Introduction

Physics is about the principles that govern objects ranging from elementary particles to the whole universe. It is often the case that one first studies the properties of the individual components of a system and then proceeds to understand the whole system. For condensed matter systems, we have $\sim 10^{23}$ nuclei and electrons that are accurately described by non-relativistic quantum mechanics. When the interactions between the constituents of a system are weak enough, we can explain the properties of the system starting from non-interacting particles and including interaction effects perturbatively. One very useful framework following this line of thought is the Landau Fermi liquid theory which successfully describes the metallic behavior of many condensed matter systems. The essential idea of this theory is adiabaticity. We can start from a non-interacting fermionic system and gradually turn on the interactions. If the ground state of the interacting system is adiabatically connected to that of the free system, the interacting system at low temperatures can be described by a set of quasiparticles. These quasiparticles behave like ordinary fermions in the sense that they have well-defined spins and charges, but they are only weakly interacting and can be treated using perturbation theory [117]. Another example is the Bose-Einstein condensation. In a three-dimensional non-interacting bosonic system, the fraction of bosons in the zero momentum eigenstate becomes macroscopic for temperatures below a certain threshold. The weak interactions between bosons can be approximately described using the Gross-Pitaevskii equation [157]. The properties of strongly interacting systems are much more difficult to understand as they cannot be studied using
perturbation theory. One very famous example is the high temperature superconductivity in cuprates [18], which was discovered in 1986 but its origin remains controversial even now. The studies of cuprates have stimulated the developments of many theoretical tools and experimental techniques such as dynamical mean field theory [63] and angle-resolved photoemission spectroscopy [43].

The focus of this thesis is the quantum Hall effect, which occurs in two-dimensional electron gas placed in magnetic field and is signified by exponentially small longitudinal conductance and quantized Hall conductance. The quantized values of the Hall conductance are integers and some special fractions in appropriate units. While those with integral quantized conductances can be understood in a non-interacting picture, the ones with fractional quantized conductances only occur when there are strong interactions between the electrons. On the experimental side, the remarkable progresses such as modulation doping in semiconductors [50] and synthesis of graphene [61] are of fundamental importance in creating platforms for quantum Hall physics. The desire of having a better understanding of the quantum Hall states has also been a driving force for improving the experimental methods. On the theoretical side, many new ideas and techniques have been developed to understand the quantum Hall states. Large scale numerical exact diagonalizations of microscopic Hamiltonians on computers have played a crucial role in the theoretical investigations of quantum Hall effect. The success of this approach relies heavily on the improvement of memory and CPU of compute systems, but it is fundamentally limited by the exponential growth of the Hilbert space dimension. An intuitive and powerful physical understanding of many quantum Hall states is given by the composite fermion theory, which maps a strongly interacting system of electrons to a weakly interacting system of emergent ingredients named as composite fermions. This emergence is intrinsically non-perturbative and cannot be captured by any perturbative calculations. The quantum Hall states have also motivated the studies of topological phenomena in condensed matter systems. The concept of topological invariant was introduced to understand the quantization of the Hall conductance in quantum Hall states and recently generalized to describe other systems such as topological insulators and topological superconductors [22]. The low-energy and long-wavelength effective descriptions of quantum Hall states have been proposed to be topological quantum field theories and the concept of
topological order has been discussed extensively [214, 215].

1.1 Classical and Quantum Hall Effect

We first describe the classical Hall effect discovered in 1879 [75]. Although the classical Hall effect appear in three-dimensional systems, we shall consider two-dimensional systems here as the quantum Hall effect to be described below can only be found in two dimensions. As illustrated schematically in Fig. 1.1, the sample used here is a thin film and placed in a perpendicular magnetic field. When there are mobile particles with negative charges, they experience the Lorentz force and get diverted to accumulate on one side of the system. The other side of the system then have net positive charges. The charges on the two sides generate an electric potential which results in another force whose direction is opposite to the Lorentz force. When the system is in equilibrium, the electric force and the magnetic force cancel each other. To analyze the transport properties quantitatively, we study the conductivity tensor

\[
\sigma = \begin{pmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{yx} & \sigma_{yy}
\end{pmatrix}
\] (1.1)

which relates the current density and the electric field via

\[
J_\mu = \sigma_{\mu\nu} E_\nu
\] (1.2)

Let us assume that the charge carriers all have the same charge \(e\) and the density of charge carriers is \(\rho\). The current density along the \(x\) direction is

\[
J_x = e\rho v
\] (1.3)

where \(v\) is the velocity of the charge carriers, and there is no current along the \(y\) direction. The electric field \(E_y\) along the \(y\) direction in equilibrium is determined by the condition \(eE_y = evB\) which ensures that the charged particles experience zero net force. These considerations lead to the conclusion that \(\sigma_{xy} = e\rho/B\). One can get the conductance tensor \(G_{\mu\nu}\) from the conductivity tensor. In particular,
Figure 1.1. Schematic experimental setup for measuring longitudinal and Hall conductance. $I$ is the current through the sample, $V_L$ is the longitudinal voltage, and $V_H$ is the Hall voltage. The longitudinal conductance is $G_L = I/V_L$ and the Hall conductance is $G_H = I/V_H$.

The conductance $G_{xy}$ of a two-dimensional system is given by

$$G_{xy} = \sigma_{xy}$$

(1.4)

For a system with rotational invariance, $G_{xx}$ and $G_{yy}$ have the same role, they are called longitudinal conductance which we denote as $G_L$. The quantity $G_{xy}$ is the Hall conductance usually written as $G_H$.

In the discussion of the classical Hall effect, we have assumed that the system under consideration is two-dimensional. This is because the quantum Hall effect that we are going to study below only occurs in two-dimensional systems. As schematically shown in Fig. 1.2, two-dimensional electron gases can be realized in semiconductor quantum wells and heterostructures (typically made of GaAs and AlGaAs). There is a strong confinement potential along one direction so the electrons are effectively confined to a two-dimensional plane at low temperatures.

It was shown above using classical physics that the Hall conductance $G_H$ of a system is inversely proportional to the magnetic field. This view dramatically
changed when von Klitzing, Dorda, and Pepper [100] discovered that there are plateaus in the plot of Hall conductance versus magnetic field on which $G_H$ takes quantized values while the longitudinal conductance $G_L$ is exponentially small. This phenomenon is called integer quantum Hall (IQH) effect, because the Hall conductance on the plateaus can be written as

$$G_H = f \frac{e^2}{h}$$  \hspace{1cm} (1.5)

with $f$ being an integer for a given plateau (note that $e^2/h = 3.874 \times 10^{-5} S$ has the unit of conductance). Shortly after this discovery, Tsui, Strömer, and Gossard [199] found a quantized plateau with Hall conductance $(1/3)e^2/h$, i.e. $f = 1/3$. With improving sample quality and measurement techniques, many other plateaus for which the Hall conductances are fractional multiples of $e^2/h$ have been observed and they are collectively referred to as fractional quantum Hall (FQH) states.

It is clear that the appearance of quantized values of Hall conductance can only be explained using quantum mechanics. To begin with, we consider the quantum mechanical motion of one particle in two dimensions in a perpendicular magnetic field. This problem was solved by Landau [108] in 1932, so the energy levels are called Landau levels. We mention the main results here and defer the mathematical details to Sec. 1.3. The single-particle kinetic energy have the form $\hbar \Omega (m + 1/2)$ where $\Omega$ is called the cyclotron frequency and $m$ is an integer labeling the Landau levels. There are multiple degenerate eigenstates within each Landau level which all have the same kinetic energy. The number of states in one Landau level is determined by the area of the sample (the small correction due to the

![Figure 1.2. Schematics of semiconductor quantum well and semiconductor heterostructure.](image-url)
boundary of the sample can be neglected for a large enough sample). We define the filling factor $\nu$ as the ratio between the number of particles and the number of available states. Because of the Pauli principle, two electrons cannot occupy the same state and the number of electrons that one Landau level can accommodate is finite. When the lowest $N$ Landau levels are completely filled with electrons and the higher Landau levels are empty, the Hall conductance of the system is $Ne^2/h$. If we compress the system to reduce its area, the number of states in each Landau level decreases so some electrons are pushed into the unoccupied Landau levels at higher energies. This means that a finite amount of energy is needed to compress the system, i.e. the system is incompressible. The presence of plateaus on which the Hall conductance takes quantized values can be explained as due to disorder in the system [88]. The values of $G_H$ at the plateaus are very precise, which has motivated the topological interpretation of IQH states [195] that will also be discussed in Sec. 1.3.

If the interactions between electrons are neglected, one can only get IQH states. Indeed, for a Landau level that is partially filled with non-interacting electrons, there are many possible configurations with the same kinetic energy so the system is not incompressible. To understand the FQH states, we need to include the Coulomb interactions between the electrons in our study. This problem then becomes very difficult because there is no small parameter in the problem on which one can build well-controlled perturbative calculations. We shall tackle the strongly interacting problem with numerical methods and the details are presented Sec. 1.4. The numerical results that one can obtain suggest that there are gaps in the energy spectra of interacting electrons in a partially filled lowest Landau level or second Landau level at some filling factors. However, these calculations do not provide a physical picture for the FQH effect. A microscopic understanding based on the composite fermion theory can be used to explain the FQH states in the lowest Landau level. In the composite fermion theory, many FQH states are mapped to simple IQH states of emergent particles known as the composite fermions. The fractions at which robust FQH states appear correspond to IQH states of composite fermions. The ground states and the excitations of the FQH states of electrons are modeled as the ground states and excitations of the IQH states of composite fermions. The details of this theory are given below in Sec. 1.5.
We will briefly discuss in Sec. 1.6 the more complicated and less well understood FQH states in the second Landau level, for which a unified framework has not yet been established.

In addition to the semiconductor quantum wells and heterostructures, quantum Hall states have also been observed in other systems such as graphene. As an atomically thin layer of carbon atoms, graphene is an ideal host of two-dimensional electron gas. One special feature of graphene is that the electrons have a valley degree of freedom (as well as the usual spin degree of freedom), which leads to a rich variety of quantum Hall states \[26, 52, 56, 145, 146, 232\]. Furthermore, multiple layers of graphene can be stacked in different ways that give rise to distinct electronic structures. For example, Bernal stacked bilayer graphene and ABC stacked trilayer graphene have additional two-fold and three-fold orbital degeneracies respectively \[15\]. Quantum Hall states have been observed in these systems \[14, 79, 97, 103, 105, 127, 231\] and new states may be realized in the future.

The quantum Hall states that we have discussed so far are realized using electrons in condensed matter systems. One may wonder whether the quantum Hall physics can also be studied in bosonic systems. It is unfortunate that nature does not provide two-dimensional systems of charged bosons that can be coupled to a magnetic field, but recent advances in atomic and optical physics might eventually create systems which can simulate quantum Hall physics using bosons. Atomic gases can be cooled to the quantum degenerate regime routinely nowadays and strong interactions between them can be induced \[25, 34\]. One challenge is that the atoms are charge neutral and do not couple to the magnetic field as electrons do, so it is necessary to create artificial gauge potentials for them. The conceptually simplest way is to rotate the system because the Coriolis force in the rotating frame has the same functional form as a magnetic field in the laboratory frame \[40, 203\]. Another way that has been actively pursued in recent years is to couple atoms to spatially modulated laser beams in both continuum and optical lattices \[42, 67\]. The idea of creating artificial gauge potential has also been studied in photonic systems \[28\] and bosonic quantum Hall states may also be realized here. We note that bosonic quantum Hall states can occur only in strongly interacting systems.
1.2 Non-Technical Summary of Other Chapters

Before getting into the theoretical details about the quantum Hall effect, we give a brief overview of the topics that will be studied in the subsequent chapters.

A real experimental system occupies a finite spatial region to which the particles are confined. The Landau levels are bended at the boundary due to the confinement potential. For the IQH states, gapless excitations can be created in the vicinity of the boundary. The particles experience an electric field due to the confinement potential. The edge excitations are unidirectional (which defines the forward-moving direction) as determined by the magnetic field and the electric field. For the FQH states, the existence of gapless edge excitations is not very intuitive but has also been theoretically predicted and experimentally tested [30]. It was proposed that the edge excitations of FQH states provide a window to probe the universal properties of FQH states [214, 215]. The edge excitations are unidirectional for some FQH states but a more interesting possibility is the presence of edge excitations that propagate in both directions. In Chapter 2, we will study the spin-unpolarized and spin-polarized $\nu = 2/3$ FQH states which were predicted to have forward-moving charged edge excitations and backward-moving neutral edge excitations. We test the predictions of topological field theory from a microscopic perspective using exact diagonalization and composite fermion theory.

In the simplest cases, the particles forming quantum Hall states have no internal degrees of freedom. On the experimental side, the spin degree of freedom of electrons in condensed matter systems is frozen for high enough magnetic fields. It comes into play in moderate or small magnetic fields [51, 106, 154, 220]. Furthermore, one can make multiple layers of two-dimensional electron gases or use the valley degree of freedom in the electronic band structures to study multi-component FQH states. On the theoretical side, FQH states of one-component fermions and bosons have been found to share many similarities but also exhibit important differences [31, 39, 160, 161]. This motivates us to ask what are the possible quantum Hall states of bosons that have internal degrees of freedom. In Chapter 3, we will study quantum Hall states of two-component bosons. A variety of quantum Hall states that are in parallel with two-component fermionic systems can be identified.

In Chapters 4 and 5, we turn to quantum Hall states in lattice systems. A
topological invariant $C$ known as the Chern number can be defined to characterize the Bloch bands in lattice systems [195]. If the Chern number of a band is non-zero, the system will exhibit IQH effect when the band is completely filled with charged fermions. When a Bloch band with $C = 1$ is nearly flat and partially filled with fermions or bosons, the interactions between the particles can produce FQH states at the filling factors that one encounters in continuum Landau levels [163, 175, 206, 207, 222]. We demonstrate in Chapter 3 that these FQH states in topological flat bands with $C = 1$ are adiabatically connected to those in continuum Landau levels. To search for states in lattice systems that may not have counterparts in the continuum Landau levels, we proceed to consider topological flat bands with $C > 1$ in Chapter 5. A class of models with Bloch bands that can have arbitrary Chern numbers are constructed and our numerical calculations suggest that exotic fractional topological phases can be realized in them. We also examine the similarities and differences between topological flat bands with $C > 1$ and conventional quantum Hall multi-layers.

Appendix A provides a pedagogical example of calculating entanglement spectrum, which could be useful for understanding some results presented in Chapters 3 and 4.

### 1.3 One Body Problem

#### 1.3.1 Landau Levels in Continuum

Let us denote the unit vectors along the three spatial directions as $\hat{e}_x$, $\hat{e}_y$, and $\hat{e}_z$. The particles under consideration are taken to be positively charged, confined to the $xy$ plane, and the magnetic field has strength $B$ along the positive $z$ direction. The solution we present below can also be used to describe particles with negative charges if one reverses the direction of the magnetic field. The single-particle Hamiltonian is

$$H_0 = \frac{1}{2m} \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right)^2$$  \hspace{1cm} (1.6)
where \( A \) is the electromagnetic vector potential producing the magnetic field \( B \hat{e}_z \) via \( \nabla \times A = B \hat{e}_z \). One may choose a specific gauge for the vector potential \( A \) to solve for the eigenvalues and eigenstates, but we will first try to understand the physics without fixing the gauge. To begin with, we define the canonical momentum operators as \( \pi_\mu = p_\mu - eA_\mu/c \) which satisfy the commutation relation

\[
[\pi_\mu, \pi_\nu] = i\epsilon_{\mu\nu}\frac{\hbar}{\ell_B^2}
\]  

with \( \ell_B = \sqrt{\hbar c/eB} \) being the magnetic length. With this commutation relation, a pair of ladder operators can be defined as

\[
a = \frac{\ell_B}{\sqrt{2\hbar}} (\pi_x + i\pi_y) \quad a^\dagger = \frac{\ell_B}{\sqrt{2\hbar}} (\pi_x - i\pi_y)
\]

which satisfy \([a, a^\dagger] = 1\). The single-particle Hamiltonian is expressed using these ladder operators as

\[
H_0 = \hbar \Omega \left(a^\dagger a + \frac{1}{2}\right)
\]

with \( \Omega = eB/mc \). If a state \( |\psi_m\rangle \) is an eigenstate of \( a^\dagger a \) with eigenvalue \( m \), then it is also an eigenstate of \( H_0 \) with eigenvalue \( \hbar \Omega (m + 1/2) \). The eigenstates are not completely specified by the operators \( a \) and \( a^\dagger \). To see this, we define another set of ladder operators \( b \) and \( b^\dagger \) as

\[
b = \frac{1}{\sqrt{2\ell_B}} (R_x + iR_y) \quad b^\dagger = \frac{1}{\sqrt{2\ell_B}} (R_x - iR_y)
\]

where the guiding center coordinate \( R_\mu = r_\mu + \ell_B^2 \epsilon_{\mu\alpha}\pi_\alpha/\hbar \). It can be easily shown that \( a \) and \( a^\dagger \) commute with \( b \) and \( b^\dagger \), so the eigenstates which have the same eigenvalue \( m \) can be further distinguished by their eigenvalues with respect to \( b^\dagger b \). A generic eigenstate can be written as

\[
|\phi_{mn}\rangle = \frac{(a^\dagger)^m(b^\dagger)^n}{\sqrt{m!n!}} |\phi_{00}\rangle
\]
where $|\psi_{00}\rangle$ is the vacuum state that is simultaneously annihilated by $a$ and $b$ as $a|\psi_{00}\rangle = b|\psi_{00}\rangle = 0$.

In practice, four cases corresponding to different choices of gauges and boundary conditions are commonly used, where the two-dimensional surface has the shape of disk, cylinder, sphere, and torus as shown in Fig. 1.3.

### 1.3.1.1 disk

For Landau levels on disk, we use the symmetric gauge with \( A = (-By/2, Bx/2, 0) \). The Hamiltonian is rotationally invariant which means that angular momentum is a good quantum number. The single-particle wave functions are

$$
\psi_{n\alpha}(x, y) = \frac{1}{\sqrt{2\pi 2^{n+\alpha+n}|\alpha|\alpha!}} \left( \frac{\bar{z}}{2} - 2 \frac{\partial}{\partial \bar{z}} \right)^n \left( \frac{z}{2} - 2 \frac{\partial}{\partial z} \right)^\alpha \exp \left( -\frac{|z|^2}{4} \right) (1.12)
$$

where $z = (x + iy)/\ell_B$ is the rescaled complex coordinate in two dimensions, $n$ is the Landau level index, and $\alpha$ labels the states within a Landau level. The angular
momentum operator is given by

$$\hat{L}_z = \hbar \left( \frac{\partial}{\partial z} - \bar{z} \frac{\partial}{\partial \bar{z}} \right)$$ (1.13)

so the angular momentum eigenvalue of $D_{n\alpha}(x, y)$ is $\alpha - n$.

### 1.3.1.2 cylinder

For Landau levels on cylinder, we use the Landau gauge $\mathbf{A} = (0, Bx, 0)$. The Hamiltonian does not depend on $y$ and we impose periodic boundary condition along the $y$ direction. The single-particle wave functions are

$$\psi_{n\alpha}(x, y) = \frac{1}{(\sqrt{\pi} 2n! L_2 \ell_B)^{1/2}} \exp \left\{ -\frac{1}{2} \left[ \frac{x}{\ell_B} - \frac{2\pi \alpha \ell_B}{L_2} \right]^2 \right\} \times \exp \left\{ \frac{i 2\pi \alpha y}{L_2} \right\} H_n \left[ \frac{x}{\ell_B} - \frac{2\pi \alpha \ell_B}{L_2} \right]$$ (1.14)

where $L_2$ is the length of the $y$ direction, $n$ is the Landau level index, and $\alpha$ labels the states within a Landau level. To satisfy the periodic boundary condition $\psi_{n\alpha}(x, y + L_2) = \psi_{n\alpha}(x, y)$, we need to choose $\alpha$ to be integers.

The disk and cylinder both have edges, so they are not ideal for studying the bulk physics of quantum Hall states. To this end, the compact sphere and torus geometries are more widely used in numerical studies.

### 1.3.1.3 sphere

To solve for the Landau levels on sphere, we consider a system of charged particles moving on a two-dimensional sphere of radius $R$ embedded in three-dimensional space. The magnetic field perpendicular to the sphere is along the radial direction which can be generated by a hypothetical magnetic monopole at the center of the sphere. Because of the Dirac quantization condition, the strength of the magnetic monopole is $2Q\hbar c/e$ with $2Q$ being an integer. This system is described by the Hamiltonian

$$H_0 = \frac{1}{2mR^2} \left| \mathbf{R} \times (\mathbf{p} - e\mathbf{A}) \right|^2$$ (1.15)
that is slightly different from the previous one. The single-particle wave functions \([219]\)

\[
\psi^{Q \alpha}_{n \alpha}(\theta, \phi) = \left(\frac{2l + 1}{4\pi} \frac{(l - \alpha)!}{(l + Q)!} \frac{(l + \alpha)!}{(l + Q)!}\right)^{1/2} u^{Q + \alpha} v^{Q - \alpha} \\
\times \sum_{s=0}^{l-\alpha} (-1)^{l-\alpha+s} \binom{l-Q}{s} \binom{l+Q}{l-\alpha-s} (u^* u)^s (v^* v)^{l-Q-s} \tag{1.16}
\]

are called monopole harmonics, where \(n\) is the Landau level index, \(l = |Q| + n\) is the angular momentum, \(\alpha = -l, -l + 1, \ldots, l-1, l\) is the \(z\) component of the angular momentum, \(\theta\) and \(\phi\) are the azimuthal and radial angles, and \(u = \cos(\theta/2)e^{i\phi/2}\), \(v = \sin(\theta/2)e^{-i\phi/2}\) are the spinor coordinates.

1.3.1.4 torus

We imposed periodic boundary condition along the \(y\) direction to get Landau levels on a cylinder. If periodic boundary condition is also imposed along the \(x\) direction, the two-dimensional surface becomes a torus which we take to be spanned by the two vectors \(\mathbf{L}_1 = L_1 \hat{e}_x\) and \(\mathbf{L}_2 = L_2 \hat{e}_y\) with \(\hat{e}_v = \hat{e}_x \sin \theta + \hat{e}_y \cos \theta\). The reciprocal lattice vectors are given by

\[
\mathbf{G}_1 = \frac{2\pi}{L_1 \sin \theta} \hat{e}_x \quad \mathbf{G}_2 = \frac{2\pi}{L_2} (\hat{e}_y - \hat{e}_x \cot \theta) \tag{1.17}
\]

We define the guiding center momentum operator as \(K_\mu = \pi_\mu + \hbar \epsilon_{\mu\alpha} r_\alpha / \ell_B^2\) which satisfy the following commutation relations

\[
[r_\mu, K_\nu] = i\hbar \delta_{\mu\nu} \quad [\pi_\mu, K_\nu] = 0 \quad [K_\mu, K_\nu] = i \frac{\hbar^2}{\ell_B^2} \epsilon_{\mu\nu} \tag{1.18}
\]

These relations tell us that \(K\) commutes with the Hamiltonian and can be used to define the translation operator with displacement \(a\) as

\[
O(a) = \exp \left\{i \frac{a \cdot K}{\hbar} \right\} \tag{1.19}
\]
Recalling that $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$ if $[A,[A,B]] = [B,[A,B]] = 0$, we get the commutation relation

$$[O(a), O(b)] = -2i \sin \left\{ \frac{\hat{e}_z \cdot (a \times b)}{2\ell_B^2} \right\} O(a + b) \tag{1.20}$$

The periodic boundary conditions require that the wave functions are invariant under translations of $L_1$ and $L_2$. This means that the two translation operators $O(L_1)$ and $O(L_2)$ have to commute with each other

$$[O(L_1), O(L_2)] = -2i \sin \left\{ \frac{\hat{e}_z \cdot (L_1 \times L_2)}{2\ell_B^2} \right\} O(L_1 + L_2) = 0 \tag{1.21}$$

This condition can be satisfied if $\hat{e}_z \cdot (L_1 \times L_2) = 2\pi \ell_B^2 N_\phi$ where $N_\phi$ is an integer describing the number of magnetic flux through the torus. The single-particle wave functions are

$$\psi_{n\alpha}^{N_\phi}(x,y) = \frac{1}{(\sqrt{\pi} 2^n n! L_2 \ell_B)^{1/2}} \sum_k \exp \left\{ -\frac{1}{2} \left[ \frac{x}{\ell_B} - \frac{2\pi \ell_B}{L_2} (\alpha + kN_\phi) \right]^2 \right\} \times \exp \left\{ \frac{i 2\pi y}{L_2} (\alpha + kN_\phi) \right\} \exp \left\{ -i \frac{2\pi \ell_B^2}{L_2^2} \cot \theta (\alpha + kN_\phi)^2 \right\} \times H_n \left[ \frac{x}{\ell_B} - \frac{2\pi \ell_B}{L_2} (\alpha + kN_\phi) \right] \tag{1.22}$$

where $n$ is the Landau level index and $\alpha \in [0, 1, \cdots, N_\phi - 1]$ labels the states within a Landau level [20].

As can be seen from the explicit solutions, the wave functions are localized Gaussian wave packets with a characteristic size $\ell_B$. This helps us to calculate the number of states in a given area. For a sample of size $A$ with particle density $\rho$, the number of states is $A/2\pi \ell_B^2$ and the number of electrons is $\rho A$. The ratio between these two quantities defines the filling factor $\nu = 2\pi \rho \ell_B^2 = \rho hc/eB$.

### 1.3.2 Bloch Bands on Lattice

Although the experimental systems are crystals, it is usually sufficient to neglect lattice effects when one tries to understand the quantum Hall effect. This is because the magnetic length $\ell_B$, which characterize the typical size of the wave functions,
is much larger than the lattice constant. Nevertheless, one may study the problem of particles moving in the presence of both a uniform magnetic field and a periodic potential. This model has been studied extensively and is now commonly referred to as the Harper-Hofstadter model [8, 78, 81, 155, 204]. Thouless et. al. used this model and demonstrated that the Hall conductance is the first Chern number times the conductance quantum $e^2/h$ [195]. This work tells us that integer quantum Hall effect results if a band with non-zero Chern number is filled with electrons. In the Harper-Hofstadter model, the non-zero Chern number is due to the presence of the uniform external magnetic field. This was shown to be unnecessary by Haldane who constructed a two band model on honeycomb lattice in which the two bands have Chern numbers $\pm 1$ [74]. The particles in this model experience a staggered magnetic field whose average strength is zero.

We describe electrons in a periodic potential using tight binding Hamiltonians. For the general case, there are multiple orbitals which are denoted as $|i\alpha\rangle$ in each unit cell. The creation operator corresponding to the state $|i\alpha\rangle$ is denoted as $C_{i\alpha}^\dagger$. The index $i$ labels the unit cell at position $r_i$ and $\alpha$ is the index for the orbital at displacement $d_\alpha$ relative to $r_i$. A tight binding Hamiltonian

$$H_0 = \sum_{ij} \sum_{\alpha\beta} C_{i\alpha}^\dagger \mathcal{H}_{ij}^{\alpha\beta} C_{j\beta}$$

(1.23)

can be transformed to momentum space as

$$H_0 = \sum_{\mathbf{k}} \sum_{\alpha\beta} C_{k\alpha}^\dagger \mathcal{H}_{\mathbf{k}}^{\alpha\beta}(\mathbf{k}) C_{\mathbf{k}\beta}$$

(1.24)

using the Fourier transforms

$$C_{i\alpha}^\dagger = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot (r_i + d_\alpha)} C_{\mathbf{k}\alpha}^\dagger \quad C_{j\beta} = \frac{1}{\sqrt{N}} \sum_{\mathbf{k}} e^{-i\mathbf{k} \cdot (r_j + d_\beta)} C_{\mathbf{k}\beta}$$

$$\mathcal{H}_{ij}^{\alpha\beta} = \mathcal{H}_{\mathbf{k}}^{\alpha\beta}(\mathbf{r}_i + d_\alpha - \mathbf{r}_j - d_\beta) = \frac{1}{N} \sum_{\mathbf{k}} \mathcal{H}_{\mathbf{k}}^{\alpha\beta}(\mathbf{k}) e^{-i\mathbf{k} \cdot (r_i + d_\alpha - r_j - d_\beta)}$$

The matrices $\mathcal{H}_{\mathbf{k}}^{\alpha\beta}(\mathbf{k})$ at different $\mathbf{k}$ decouple and can be diagonalized separately

$$\sum_{\beta} \mathcal{H}_{\mathbf{k}}^{\alpha\beta}(\mathbf{k}) u_{\beta}^\alpha(\mathbf{k}) = E_{\mathbf{k}\alpha} u_{\alpha}^\alpha(\mathbf{k})$$

(1.25)
where \( u^n_\alpha(k) \) are the eigenvectors. We can define normal mode operators \( \gamma_{kn} \) via
\[
C_{kn} = \sum_n u^n_\alpha(k) \gamma_{kn}
\]
to transform the Hamiltonian into a diagonal form
\[
H_0 = \sum_k \sum_{nm} \sum_{\alpha\beta} \gamma_k^\dagger u^n_\alpha(k) h^{\alpha\beta}(k) u^m_\beta(k) \gamma_{km} = \sum_k \sum_n E_{kn} \gamma_k^\dagger \gamma_{kn} \tag{1.26}
\]
There exist gaps in the energy spectra of insulators and only the occupied bands are important at low temperatures. When the dynamics of the system is projected to the \( B \) occupied bands, the adiabatic evolution of the Bloch states \( |u^n(k)\rangle \) in the Brillouin zone can be described using the Berry vector potential
\[
\mathcal{A}^m_{\mu}(k) = i \langle u^m(k)|\partial_\mu|u^n(k)\rangle \tag{1.27}
\]
We note that \( \mathcal{A}_\mu(k) \) is a \( B \times B \) matrix and \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) may not commute with each other. One can define the Berry curvature \( \mathcal{F}(k) \) as in non-Abelian gauge theory
\[
\mathcal{F}_{\mu\nu}(k) = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + i [\mathcal{A}_\mu, \mathcal{A}_\nu] \tag{1.28}
\]
Using linear response theory [22], we can prove that the Hall conductance of the system is given by the total Chern number of the occupied bands
\[
C = \frac{1}{2\pi} \sum_{m=1}^B \int_{BZ} d^2k \ Tr [\mathcal{F}^{mn}(k)] \tag{1.29}
\]
in units of \( e^2/h \).

For the Hofstadter model describing particles in both a uniform magnetic field and a periodic potential, the real space tight-binding Hamiltonian is
\[
H = \sum_{ij} e^{i\theta_{ij}} C_i^\dagger C_j,
\]
where \( \theta_{ij} \) is the phase associated with the hopping from site \( j \) to \( i \) as given by the Peierls substitution [155]. If the magnetic flux per plaquette is \( 2\pi/q \) with \( q \) being an integer, translational symmetry is preserved on the scale of magnetic unit cells that each contains \( q \) plaquettes. As a specific example, we choose \( q = 4 \) in the model shown in Fig. 1.4 which results in the following momentum
Figure 1.4. A Hofstadter model on square lattice with $\pi/2$ flux per plaquette with one unit cell enclosed by the blue rectangle for illustration. The indices of orbitals in a magnetic unit cell are shown in parentheses and the numbers on the bonds indicate the phases of the complex hopping amplitudes along the $y$ direction in units of $\pi$.

The space Hamiltonian

$$\mathcal{H}(k) = \begin{pmatrix} 2 \cos(k_y) & e^{ik_x/4} & 0 & e^{-ik_x/4} \\ e^{-ik_x/4} & 2 \cos(k_y + \pi/2) & e^{ik_x/4} & 0 \\ 0 & e^{-ik_x/4} & 2 \cos(k_y + \pi) & e^{ik_x/4} \\ e^{ik_x/4} & 0 & e^{-ik_x/4} & 2 \cos(k_y + 3\pi/2) \end{pmatrix} \quad (1.30)$$

and the lowest band has Chern number 1.

### 1.4 Many Body Problem

To make the FQH states theoretically tractable, we need to make some assumptions to simplify the problem. As we have seen above, the Landau levels are separated by cyclotron gaps, so we do not need to consider very high Landau levels at low temperatures. The particles under investigation may have internal degrees of free-
dom and associated energy scales, e.g. the electrons have spins which couple to the magnetic field through Zeeman terms. If all the other energy scales are much larger than the interaction strength, we can limit ourselves to the partially occupied Landau level and neglect all the other Landau levels above or below it. When there are multiple nearly degenerate Landau levels with distinct internal indices (which may be spin, valley, and/or layer), we shall include all of them.

In physical systems, the realistic interactions are usually pairwise as described by potentials like $V(r_1 - r_2)$. One may also use more exotic interaction potentials such as $V(r_1, r_2, \cdots, r_k)$ that simultaneously depends on the coordinates of multiple particles. In numerical calculations, we construct the Fock states in the second quantized form and express the many body Hamiltonian in the Fock state basis. The low-lying eigenstates of a Hamiltonian can be obtained using the Lanczos algorithm. The non-interacting wave function corresponding to a Fock state is a Slater determinant for fermions and a symmetric monomial for bosons. As an example, we consider the single-particle states on disk in the lowest Landau level for which the wave functions are powers of $z$ (the Gaussian factors can be neglected as they are ubiquitous). The Slater determinant with $N$ fermions filling the angular momentum states $0, 1, \cdots, N - 1$ in the lowest Landau level is

$$
\begin{pmatrix}
z_1^0 & \cdots & z_N^0 \\
z_1^1 & \cdots & z_N^1 \\
\vdots & \cdots & \vdots \\
z_1^{N-1} & \cdots & z_N^{N-1}
\end{pmatrix}
= \prod_{i<j}(z_i - z_j) \tag{1.31}
$$

A general interacting many body state in the lowest Landau level on disk is a polynomial whose power is determined by the total angular momentum. This fact is very useful when one constructs trial wave functions for the FQH states.

### 1.4.1 Interactions in Landau Levels

In the previous section, we presented the wave functions for Landau levels in different geometries. They are all labeled by a Landau level index $n$ and another index $\alpha$ which distinguishes the states within a Landau level. In the second quantized
notation, a many body Hamiltonian consists of two-body interactions is given by

\[ H_2 = \frac{1}{2} \sum_{n_1a_1} \sum_{n_2a_2} \sum_{n_3a_3} \sum_{n_4a_4} V_{n_1n_2n_3n_4} C_{n_1a_1}^\dagger C_{n_2a_2} C_{n_3a_3} C_{n_4a_4} \] (1.32)

where \( C_{na}/C_{na}^\dagger \) is the annihilation/creation operator corresponding to the state \( \phi_{na} \) and the two-body interaction matrix element \( V_{n_1n_2n_3n_4} \) is

\[ \int d^2r_1 \int d^2r_2 [\psi_{n_1a_1}(r_1)]^\dagger [\psi_{n_2a_2}(r_2)]^\dagger V(r_1 - r_2) \psi_{n_4a_4}(r_2) \psi_{n_3a_3}(r_1) \] (1.33)

For some purposes, we may need to study three-body interactions which can be represented in the second quantized form in a similar way

\[ H_3 = \frac{1}{6} \sum_{n_1a_1} \sum_{n_2a_2} \sum_{n_3a_3} \sum_{n_4a_4} \sum_{n_5a_5} \sum_{n_6a_6} V_{n_1n_2n_3n_4n_5n_6} C_{n_1a_1}^\dagger C_{n_2a_2}^\dagger C_{n_3a_3}^\dagger C_{n_4a_4} C_{n_5a_5} C_{n_6a_6} \] (1.34)

1.4.1.1 disk

For particles on disk with Coulomb interaction, the two-body matrix element for Coulomb interaction is

\[ \int d^2r_1 \int d^2r_2 [\psi_{n_1a_1}(r_1)]^\dagger [\psi_{n_2a_2}(r_2)]^\dagger \frac{1}{|r_1 - r_2|} \psi_{n_4a_4}(r_2) \psi_{n_3a_3}(r_1) \] (1.35)

and we shall only use the ones in the lowest Landau level (\( i.e. n_1 = n_2 = n_3 = n_4 = 0 \)). There are several different versions for the matrix elements. A very convenient one is given by Tsipen [198]

\[ V_{a_1a_2a_3a_4}^{0000} = \sqrt{\frac{\alpha_1!\alpha_4!}{\alpha_2!\alpha_3!}} \frac{\Gamma(\alpha_1 + \alpha_2 + 3/2)}{\pi^{2\alpha_1+\alpha_2+2}} \left[ A_{\alpha_3\alpha_2}^\alpha B_{\alpha_2\alpha_3}^\alpha + A_{\alpha_2\alpha_3}^\alpha B_{\alpha_3\alpha_2}^\alpha \right] \] (1.36)

where

\[ A_{\beta\gamma}^\alpha = \sum_{i=0}^{\beta} \binom{\beta}{i} \frac{\Gamma(i + 1/2)\Gamma(\alpha + i + 1/2)}{(\alpha + i)!\Gamma(\alpha + \gamma + i + 3/2)} \] (1.37)

\[ B_{\beta\gamma}^\alpha = \sum_{i=0}^{\beta} \binom{\beta}{i} \frac{\Gamma(i + 1/2)\Gamma(\alpha + i + 1/2)}{(\alpha + i)!\Gamma(\alpha + \gamma + i + 3/2)} (\alpha + 2i + 1/2) \] (1.38)
1.4.1.2 cylinder

We will not use the cylinder geometry for numerical calculations so the results are omitted here.

1.4.1.3 sphere

To evaluate the two-body matrix elements on sphere, we need to use the following identities of the monopole harmonics

\[
\left[\psi_{n,\alpha}^Q\right]^* = (-1)^{Q-\alpha} \psi_{n,-\alpha}^Q
\]

\[
\psi_{n_1,\alpha_1}^Q \psi_{n_2,\alpha_2}^Q = (-1)^{Q-\alpha} \sum_n F(Q, n, \alpha; Q_1, n_1, \alpha_1; Q_2, n_2, \alpha_2) \phi_{n,\alpha}^Q
\]  

(1.39)

\[
\int d\theta d\phi \left[\psi_{n_1,\alpha_1}^Q \psi_{n_2,\alpha_2}^Q \psi_{n,-\alpha}^Q\right] = F(Q, n, \alpha; Q_1, n_1, \alpha_1; Q_2, n_2, \alpha_2)
\]  

(1.40)

where \( Q = Q_1 + Q_2 \) and \( \alpha = \alpha_1 + \alpha_2 \). The coefficient \( F(Q, n, \alpha; Q_1, n_1, \alpha_1; Q_2, n_2, \alpha_2) \) is given by

\[
(-1)^{l_1+l_2+l} \left[\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}\right]^{1/2} J_{l_1, l_2, l}^{n_1, n_2, n} J_{Q_1, Q_2, -Q}^{l_1, l_2, l}
\]  

(1.41)

where \( l_1 = |Q_1| + n_1 \), \( l_2 = |Q_2| + n_2 \), \( l = |Q| + n \), and

\[
J_{n_1, n_2, n}^{l_1, l_2, l} = \frac{(-1)^{l_1-l_2-\alpha}}{\sqrt{2l+1}} \langle l_1, \alpha_1; l_2, \alpha_2 | l, -\alpha \rangle
\]  

(1.42)

with \( \langle l_1, \alpha_1; l_2, \alpha_2 | l, \alpha \rangle \) being the Clebsch-Gordon coefficient. For the Coulomb interaction, we can decompose it as

\[
\frac{1}{|r_1 - r_2|} = \frac{4\pi}{R} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{\alpha=-n}^{n} \left[ S^0_{n,\alpha}(\theta_1, \phi_1) \right]^* S^0_{n,\alpha}(\theta_2, \phi_2)
\]  

(1.43)

in the integral

\[
\int d\theta_1 d\phi_1 \int d\theta_2 d\phi_2 \left[ \psi_{n_1,\alpha_1}^Q(\theta_1, \phi_1) \right]^* \left[ \psi_{n_2,\alpha_2}^Q(\theta_2, \phi_2) \right]^* \times \frac{1}{|r_1 - r_2|} \psi_{n_4,\alpha_4}^Q(\theta_2, \phi_2) \psi_{n_3,\alpha_3}^Q(\theta_1, \phi_1)
\]  

(1.44)
and obtain
\[
\frac{4\pi}{R} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{\alpha=-n}^{n} \int d\theta_1 d\phi_1 (-1)^{Q-\alpha_1-\alpha} \psi_{n,-\alpha}^0 (\theta_1, \phi_1) \psi_{n_3,\alpha_3}^Q (\theta_1, \phi_1) \psi_{n_1,-\alpha_1}^Q (\theta_1, \phi_1) \\
\times \int d\theta_2 d\phi_2 (-1)^{Q-\alpha_2} \psi_{n,\alpha}^0 (\theta_2, \phi_2) \psi_{n_4,\alpha_4}^Q (\theta_2, \phi_2) \psi_{n_2,-\alpha_2}^Q (\theta_2, \phi_2) \\
= \frac{4\pi}{R} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sum_{\alpha=-n}^{n} \delta_{\alpha,\alpha_1+\alpha_3} \delta_{\alpha,\alpha_2+\alpha_4} (-1)^{2Q-\alpha_1-\alpha_2} \\
\times F(Q, n_1, \alpha_1; 0, n, -\alpha; Q, n_3, \alpha_3) F(Q, n_2, \alpha_2; 0, n, \alpha; Q, n_4, \alpha_4) \tag{1.45}
\]

1.4.1.4 torus

The periodic structure of the torus allows us to define
\[V(r_1 - r_2) = \frac{1}{L_1 L_2 \sin \theta} \sum_q V(q) e^{i\mathbf{q}(r_1 - r_2)} \tag{1.46}\]
\[
\rho_q = \int d^2 r_1 \sum_{n_{1\alpha_1}} \sum_{n_{3\alpha_3}} \left[ \psi_{n_{1\alpha_1}}^{N\phi} (\mathbf{r}_1) \right]^* e^{i\mathbf{q} \cdot \mathbf{r}_1} \psi_{n_{3\alpha_3}}^{N\phi} (\mathbf{r}_1) C_{n_{1\alpha_1}}^\dagger C_{n_{3\alpha_3}} \tag{1.47}\]
\[
\rho_{-q} = \int d^2 r_2 \sum_{n_{2\alpha_2}} \sum_{n_{4\alpha_4}} \left[ \psi_{n_{2\alpha_2}}^{N\phi} (\mathbf{r}_2) \right]^* e^{-i\mathbf{q} \cdot \mathbf{r}_2} \psi_{n_{4\alpha_4}}^{N\phi} (\mathbf{r}_2) C_{n_{2\alpha_2}}^\dagger C_{n_{4\alpha_4}} \tag{1.48}\]

with \(q = q_1 \mathbf{G}_1 + q_2 \mathbf{G}_2\). The Hamiltonian can be expressed as
\[H = \frac{1}{2L_1 L_2 \sin \theta} \sum_q V(q) : \rho_q \rho_{-q} : \tag{1.49}\]

where : \(\cdots\) : enforces normal ordering of the creation and annihilation operators.

Because of the periodicity of the wave functions, the projected density operator \(\rho_q\) can be evaluated in the region \(x \in [0, L_1 \sin \theta]\) and \(y \in [0, L_2]\)

\[
\int d^2 r_1 \sum_{s\alpha} \sum_{t\beta} \left[ \psi_{s\alpha}^{N\phi}(\mathbf{r}_1) \right]^* e^{i\mathbf{q} \cdot \mathbf{r}_1} \psi_{t\beta}^{N\phi}(\mathbf{r}_1) \\
= \sum_{s\alpha} \sum_{t\beta} \frac{1}{\sqrt{2^2 + t^2 \sin^2 \theta}} \exp \left\{ -\frac{1}{4} \mathbf{q}^2 \ell_B^2 \right\} \exp \left\{ i \frac{\pi q_1}{N\phi} (\alpha + \beta) \right\} \sum_k \min[s,t] 2^k k! \binom{s}{k} \binom{t}{k} \times \left\{ i \left[ \frac{2\pi \ell_B}{L_1 \sin \theta} q_1 - \frac{2\pi \ell_B \cot \theta}{L_2} q_2 - \frac{2\pi \ell_B}{L_2} q_2 \right] \right\}^{s-k} \]

\[\times \left\{ i \left[ \frac{2\pi \ell_B}{L_1 \sin \theta} q_1 - \frac{2\pi \ell_B \cot \theta}{L_2} q_2 \right] + \frac{2\pi \ell_B}{L_2} q_2 \right\}^{t-k} \]
\[= \sum_{\alpha} \sum_{t\beta} \exp \left\{ -\frac{1}{4} q^2 B \right\} \exp \left\{ i \frac{\pi q_1}{N_\phi} (\alpha + \beta) \right\} F_{st}(q_1, q_2) \tilde{\delta}_{\alpha, \beta + q_y} \] (1.50)

where \( \tilde{\delta}_{\alpha, \beta + q_y} \) is a periodic Kronecker symbol \([\text{it is non-zero only if } \alpha \mod N_\phi = (\beta + q_y) \mod N_\phi] \) and we have defined \( F_{st}(q_1, q_2) \) to simplify the notation. The Hamiltonian is

\[ H = \sum_{q_1, q_2} V(q) \rho(q) \rho(-q) = \sum_{q_1, q_2} \sum_{n_1, \alpha_1} \sum_{n_2, \alpha_2} \sum_{n_3, \alpha_3} \sum_{n_4, \alpha_4} C_{n_1}^{\dagger} C_{n_2}^{\dagger} C_{n_3} C_{n_4} \] \( \hat{N}_{i\alpha} \hat{N}_{j\beta} : \]
\[\times V(q) \exp \left\{ -\frac{1}{2} q^2 B \right\} F_{n_1 n_3}(q_1, q_2) F_{n_2 n_4}(-q_1, -q_2) \]
\[\times \exp \left\{ i \pi q_1 \right\} \frac{1}{N_\phi} (\alpha_1 + \alpha_3 - \alpha_2 - \alpha_4) \tilde{\delta}_{\alpha_1, \alpha_3 + q_2} \tilde{\delta}_{\alpha_2, \alpha_4 - q_2} \] (1.51)

### 1.4.2 Interactions in Lattice Models

The many body interactions on lattice have very similar forms as those presented above when expressed using second quantized operators. A general two-body Hamiltonian can be written as

\[ H_2 = \sum_{i\alpha} \sum_{j\beta} V_{i\alpha}^{ij} : \hat{N}_{i\alpha} \hat{N}_{j\beta} : \] (1.52)

where \( i \) and \( j \) label the unit cells, \( \alpha \) and \( \beta \) label the orbitals with in a unit cell, \( \hat{N}_{i\alpha} = C_{i\alpha}^{\dagger} C_{i\alpha} \) is the particle density operator, and \( : \cdots : \) enforces normal ordering of the creation and annihilation operators. In most cases, the coefficients \( V_{\alpha\beta}^{ij} \) that we will use only couples a few nearby lattice sites. For three-body interactions, the Hamiltonian can be written as

\[ H_3 = \sum_{i\alpha} \sum_{j\beta} \sum_{k\gamma} V_{i\alpha\beta}^{ijk} : \hat{N}_{i\alpha} \hat{N}_{j\beta} \hat{N}_{k\gamma} : \] (1.53)

using the same symbols as defined above.
1.5 Composite Fermion Theory

The composite fermion theory was originally proposed to explain the FQH states of electrons [83], but it can be generalized to bosonic systems as well. The central postulate of this theory is that the particles in a system form bound states with vortices to minimize the strong repulsive interactions between the particles. The bound states are always fermionic so they are called composite fermions. An even (odd) number of vortices are bound to fermions (bosons) to create composite fermions. The composite fermions move in an effective magnetic field whose strength is determined by the actual magnetic field and the vortices bound to the underlying particles. The effective magnetic field may be parallel or antiparallel to the actual magnetic field. As a first order approximation, we take the composite fermions to be non-interacting (or weakly interacting) so their states can be determined easily (e.g. IQH states).

Without doing any calculation, the composite fermion theory can already explain many features of the FQH effect. In particular, it predicts the existence of FQH states at some particular filling factors. Let us denote the magnetic field as $B$ (a positive number), the charge of a particle as $e$, the particle density as $\rho$, and the number of vortices attached to a particle as $p$, then the effective magnetic field for the composite fermions is

$$B^* = B - p\frac{\rho hc}{e}$$

which may be a positive or negative number. The density of the composite fermions is also $\rho$, so one can define the filling factor for the composite fermions as $\nu^* = \rho hc/e|B^*|$ and express the filling factor of the underlying particles as

$$\nu = \frac{\rho hc}{eB} = \frac{\nu^*}{p\nu^* \pm 1}.$$  

where the $+$ ($-$) sign corresponds to positive (negative) $B^*$. The composite fermions form their own Landau-like levels (i.e. composite fermion Landau levels) in this effective magnetic field $B^*$. If the composite fermion filling factor $\nu^*$ is an integer, they can form a gapped IQH state which gives rise to a FQH state of
the underlying particles. The mapping between the composite fermions and the underlying particles not only holds for the ground states but also the low-lying excitations. For a given IQH ground state of the composite fermions \( i.e. \) a FQH ground state of the underlying particles), one can add a composite fermion to the lowest empty composite fermion Landau level or remove a composite fermion from the highest full composite fermion Landau level. These excited states of composite fermions correspond to quasiparticle or quasihole excitations of the FQH state of the underlying particles. The composite fermions can also form their own FQH states due to interactions with each other, which will be mapped to new FQH states of the underlying particles [140, 141]. For GaAs based systems, the FQH states in the lowest Landau level all fit into this theory [88, 89]. The case with \( B^* = 0 \), where there are no composite fermion Landau levels, is also very interesting. Here the composite fermions may simply form a Fermi sea (as non-interacting ordinary fermions would do) and the corresponding state of the underlying particles is then a composite fermion Fermi sea [77, 165, 166]. A more exotic state occurs when there are weak attractive interactions between the composite fermions, which then form a Bardeen-Cooper-Schriffer superconducting state [69, 137, 171].

The mathematical description of composite fermions is most conveniently formulated on disk and sphere due to the properties of the wave functions that have been mentioned earlier. The concrete trial wave functions for the strongly interacting fermionic or bosonic FQH states have been confirmed by extensive numerical studies [88]. Let us first consider the case in which particles with no internal degree of freedom are confined to the lowest Landau level. The vortex attachment is facilitated by the Jastrow factor \( \prod_{i<j}(z_i - z_j) \) where the subscripts label the particles. The IQH states with \( n \) filled Landau levels is denoted as \( \chi_n \) when the effective magnetic field for composite fermions is parallel to the real magnetic field. On the other hand, one should use the complex conjugate \([\chi_n]^*\) if the effective magnetic field for composite fermions is antiparallel to the real magnetic field. The wave functions for the strongly interacting fermionic and bosonic FQH states are

\[
\Psi_F^F(\{z\}) = \chi_n(\{z\}) \prod_{i<j}(z_i - z_j)^p
\]

(1.56)

\[
\Psi_B^F(\{z\}) = [\chi_n(\{z\})]^* \prod_{i<j}(z_i - z_j)^p
\]

(1.57)
\[ \Psi_P^B(\{ z \}) = \chi_n(\{ z \}) \prod_{i<j} (z_i - z_j)^p \]  
(1.58)

\[ \Psi_R^B(\{ z \}) = [\chi_n(\{ z \})]^* \prod_{i<j} (z_i - z_j)^p \]  
(1.59)

where the superscripts \( P \) and \( R \) refer to parallel and reverse flux attachment, the subscripts \( F \) and \( B \) refer to fermions and bosons, and \( p \) is an even (odd) integer for fermions (bosons). These wave functions can be easily generalized to multi-component cases in which the extra degrees of freedom could be spin, valley, and/or layer. To model such systems, we assign extra degrees of freedom to the composite fermions and write the factor \( \chi_n \) as \( \chi_{n_1,n_2,\ldots} \) where the number of subscripts is the number of different components.

### 1.6 The Second Landau Level

The FQH states in the lowest Landau level can be understood extremely well using the composite fermion theory. The fractional quantum Hall states have also been observed in the second Landau level they are not well understood. The well-established states in the lowest Landau level all have odd denominators, but in the second Landau level an incompressible state appears at filling factor \( \nu = 5/2 \) [217]. There is evidence for FQH states at \( \nu = 3/8 \) and \( 3/10 \) in the lowest Landau level [149], whose origins are likely related to the nature of the \( 5/2 \) state [140].

The leading candidate for the \( 5/2 \) state is the Pfaffian state [137]

\[ \Psi(\{ z \}) = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i<j} (z_i - z_j)^2 \]  
(1.60)

The symbol Pf denotes the Pfaffian of an antisymmetric matrix defined as

\[ \text{Pf} \left( \frac{1}{z_i - z_j} \right) = \sum_{P} \text{sgn} P \prod_{i=1}^{2N} \frac{1}{z_{P(2i-1)} - z_{P(2i)}} \]  
(1.61)

where \( 2N \) is the total number of particles, the sum is over all permutations of \( 2N \) objects, and \( \text{sgn} P \) is the signature of the permutation \( P \). The particle-hole conjugate of the Pfaffian state is another possible candidate [114,115]. As for the composite fermion wave functions presented above, the \( \prod_{i<j}(z_i - z_j)^2 \) factor im-
plements vortex attachment and transforms the electrons into composite fermions. The Pf factor describes a chiral $p$ wave pairing of composite fermions. There are also many other FQH states in the second Landau level but a unified framework for them has not been established.
Chapter 2

Edge Excitations of Fractional Quantum Hall States at $\nu = 2/3$

This chapter is a reproduction of the published paper Phys. Rev. B 86, 115127 by the present author in collaboration with G. J. Sreejith and J. K. Jain. It has been modified slightly to fit into this thesis. In this chapter, we study the edge of spin-unpolarized or spin-polarized $\nu = 2/3$ fractional quantum Hall states. These states have been predicted by the effective theory to support a backward-moving neutral mode in addition to a forward-moving charge mode. We study this issue from a microscopic perspective where these states are identified with an effective filling factor of 2 of composite fermions, but with an effective magnetic field that is antiparallel to the external field. A simple counting from the composite fermion description suggests that there might be two backward-moving edge modes, but explicit calculations show that one of these is projected out of the low-energy sector, while the remaining mode provides a good microscopic account of the actual counterpropagating edge mode. The forward-moving modes are identified as “Schur modes”, obtained by multiplying the ground-state wave function by the symmetric Schur polynomials. The edge of the $2/3$ spin unpolarized state provides a particularly striking realization of spin-charge separation in one-dimensional Tomonaga-Luttinger liquids, with the spin and charge modes moving in opposite directions.
2.1 Introduction

Two-dimensional electron systems have been the platform for many interesting phenomena. In particular, the integer and fractional quantum Hall effects \([100,199]\) occur when a two-dimensional electron system (2DES) is placed in a magnetic field. Integer quantum Hall (IQH) states occur when an integer number of Landau levels are completely filled with electrons. For a partially filled Landau level (LL), interactions between electrons can produce incompressible states at certain fillings and lead to fractional quantum Hall (FQH) states. These are characterized by the formation of composite fermions \([83]\), where a composite fermion (CF) is the bound state of an electron and an even number of vortices. A strongly interacting state of electrons in a magnetic field \(B\) is described by a weakly interacting state of composite fermions in an effective magnetic field \(B^*\), whose direction can be either parallel or antiparallel to \(B\). The composite fermions form Landau-like levels (called \(\Lambda\) levels) in the field \(B^*\), in analogy to the LLs of non-interacting electrons. The FQH states of electrons are described as IQH states of composite fermions, which correspond to situations where composite fermions occupy an integer number of \(\Lambda\) levels. This results in FQH effect at the prominently observed fractions

\[
\nu = \frac{n}{2pn \pm 1} \tag{2.1}
\]

where \(+\) (\(-\)) indicate that the direction of the effective magnetic field is parallel (antiparallel) to the real magnetic field.

Since FQH states occur in the presence of a large magnetic field, one might at first expect that the spin degree of freedom is frozen. However, in the most widely studied GaAs system, the \(g\) factor is very small, and unpolarized or partially polarized quantum Hall states have been found to occur. The CF theory predicts the possible spin polarizations at various fractions in terms of composite fermions filling both up and down spin \(\Lambda\) levels (ALs) \([154,220]\). The spin polarization is determined by a competition between the CF cyclotron energy and the Zeeman energy \(E_Z = g\mu_B B\); at very small Zeeman energies the state with smallest spin polarization is obtained, and transitions into larger spin polarizations occur as the Zeeman energy is increased. This physics has been found to be in good qualitative...
and semi-quantitative agreement with experiments [51, 106]. In particular, both the 2/5 and the 2/3 FQH states map into filling factor 2 of composite fermions (with effective magnetic field antiparallel to the applied field for 2/3); the spin unpolarized state maps into the state in which $0^{\uparrow}$ and $0^{\downarrow}$ ALs are occupied, and the fully polarized state is described as the one in which $0^{\uparrow}$ and $1^{\uparrow}$ ALs are occupied.

Our concern in this paper is with the physics of the edge excitations of the FQH states. The FQH states are gapped in the bulk but there are gapless excitations residing at the boundary [212], which provide a realization of a nontrivial one-dimensional Tomonaga-Luttinger liquid [30, 214, 215]. Several theoretical approaches have been used to study the edge states, especially the Chern-Simons theory [123, 214, 215]. In general, the FQH state at $n/(2pn + 1)$ has $n$ edge modes, one corresponding to each AL. A surprising prediction of the edge theory has been the presence of backward moving neutral modes for the FQH states at $n/(2pn - 1)$ for which the effective magnetic field for composite fermions is antiparallel to the real magnetic field [58, 96, 214, 215]. Evidence of such counter-propagating edge modes has been seen in a recent experiment [24].

We will consider the FQH state at 2/3, which is the simplest state where backward-moving modes are theoretically predicted. The bulk physics of both the spin-unpolarized and spin-polarized FQH states at 2/3 is closely related to that of the spin-unpolarized and spin-polarized IQH states at 2, as has been demonstrated by Wu, Dev and Jain [220]. In this paper, we study the edge states at $\nu = 2/3$ using a combination of the parton method, exact diagonalization, and the microscopic CF theory to test predictions of the effective field theory, and also to gain further insight into the physics of the backward-moving edge modes.

The presence of upstream modes can be motivated in different ways. For fully spin polarized states, the presence of such modes appears naturally for the FQH states at

$$\nu = 1 - \frac{n}{2pn + 1}$$  \hspace{1cm} (2.2)

which are particle-hole symmetric to the principal states. Consider the fully spin-polarized 2/3 FQH state for example, which can be viewed as the 1/3 state of holes in the background of one filled Landau level. In this picture the 2/3 state is surrounded by a $\nu = 1$ state at the boundary, which in turn is surrounded
by vacuum [82,93]. The edge between 1 and vacuum supports a forward-moving mode, whereas the edge between 2/3 and 1 supports a backward-moving mode. The physics suggested by the CF theory has similarity to the picture described above. At filling factor 2 we have two edges, one separating 2 and 1, and the other between 1 and 0 (i.e. vacuum). After antiparallel flux attachment, filling factors 2, 1 and 0 turn into 2/3, 1, and 0, thus again producing a $\nu = 1$ region separating the 2/3 state and the vacuum. One can therefore expect a counter-propagating mode by the reasoning given above.

In constrast, an analogous picture is not available for the spin singlet 2/3 state. This state cannot be viewed as the hole partner of any principal state, because particle-hole symmetry in the presence of spin relates $\nu$ to $2 - \nu$. The picture for the edge of the spin-unpolarized 2/3 state is qualitatively different. The edge of the spin-unpolarized state at 2 goes directly from 2 to 0, implying a 2/3-0 edge for the spin-unpolarized 2/3 state. Because of the absence of $\nu = 1$ at the boundary, it is not obvious why there should be an upstream edge mode.

Nonetheless, an effective $K$ matrix description of Wen [214,215] indicates a backward-moving edge mode for both spin-unpolarized and spin-polarized states. The $K$ matrix can be obtained in the CF basis straightforwardly by noting that the $2 \times 2$ $K$ matrix for filling factor 2 is $K_{jk} = -\delta_{jk}$ for antiparallel field. Composite fermionization of electrons by attachment of two vortices amounts to adding 2 to each element of $K$, giving

$$K_{2/3} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

for both the spin-unpolarized and spin-polarized 2/3 states. This matrix has one positive and one negative eigenvalue, implying one downstream and one upstream edge mode. As explained in Ref. [134], this structure of $K_{2/3}$ possesses a hidden $SU(2)$ symmetry, with an $SU(2)$ algebra generated by the neutral modes. On the contrary, the $K$ matrix of the 2/5 state in the CF basis is given by

$$K_{2/5} = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix},$$

obtained by adding 2 to each element of $\delta_{jk}$; both eigenvalues of this matrix are...
positive, hence no backward-moving modes (neglecting edge reconstruction).

Yet another way to intuitively understand why there should be backward-moving modes in the \( n/(2n - 1) \) FQH state is to note that composite fermions experience a negative effective field. Because switching the direction of the magnetic field reverses the direction of the \( \mathbf{E} \times \mathbf{B} \) drift, we can expect composite fermions at the edge moving in the backward direction. More explicitly, consider the wave function in Eq. (2.9) below, written in the symmetric gauge. The edge modes at \( n/(2n - 1) \) derive from the edge modes of \( \Phi_n \). The state \( \Phi_n \) has \( n \) independent edge modes, and the energy of a single excitation at each edge increases with its angular momentum (relative to the ground state). However, because of the complex conjugation of \( \Phi_n \), an increase in the angular momentum in \( \Phi_n \) translates into a decrease in the angular momentum at \( n/(2n - 1) \), thus producing a mode moving in the opposite direction. While this seems to give a rather nice picture for the origin of backward-moving modes, it suggests that there are as many backward-moving modes at \( n/(2n - 1) \) as there are forward moving modes at \( n \), which is inconsistent with the effective \( K \) matrix description that produces a single backward-moving edge mode.

Our aim in this work is to gain a microscopic understanding of the edge excitations of the negative flux CF states by considering the example of 2/3, and to bring consistency between the different approaches. We show, by an explicit construction of the wave functions, that only the neutral combination of the upstream edge modes survives at low energies, which is also in good agreement with exact diagonalization studies. In addition, we identify the forward moving modes with the so called “Schur” modes, which are obtained by multiplying the ground state wave function by the symmetric Schur polynomials; these are analogous to the edge modes at filling factor 1. We study both polarized and unpolarized states at 2/3, and also show results of the \( \nu = 2/5 \) state for comparison since it is also described as \( \nu = 2 \) filled ALs but with parallel flux attachment.

The edge states of \( \nu = 2/3 \) have also been studied by exact diagonalization. Johnson and MacDonald [93] and Hu et. al. [82] model the spin-polarized 2/3 state as 1/3 of holes inside a \( \nu = 1 \) droplet. Moore and Haldane [138] have studied the spin-unpolarized 2/3 state by exact diagonalization to demonstrate the presence of a backward-moving mode. The validity of the CF theory for electron droplets in
the disk geometry, which can contain complex edges, has been studied extensively in a number of previous articles [86].

We note that the edge of the spin-unpolarized $2/3$ state provides a striking realization of spin charge separation, with pure spin and pure charge modes moving in opposite directions, which should in principle be observable. These modes have been labeled “spinons' and “chargeons' by Balatsky and Stone [9].

This chapter is organized as follows. We introduce the CF wave functions and present the parton construction based on them in Sec. 3.1. Our model and numerical methods are briefly explained in Sec. 3.2. The energy spectra of spin-unpolarized $\nu = 2$ and $2/5$ states are studied in Sec. 3.3. The energy spectra of spin-unpolarized and polarized $\nu = 2/3$ state are analyzed in Sec. 3.4 and 3.5, respectively. The conclusions of our study are summarized in Sec. 3.6.

### 2.2 Composite Fermion and Parton Construction

In order to study the edge properties of a FQH state, we choose the disk geometry where the $n^{th}$ LL single particle states in the symmetric gauge are given by

$$\eta_{n,m}(z) = \frac{(-1)^n}{\sqrt{2\pi}} \sqrt{\frac{n!}{2^m(m+n)!}} e^{-|z|^2/4} z^m L_n^m \left( \frac{|z|^2}{2} \right), \quad (2.5)$$

where $L_n^m(x)$ is the associated Laguerre polynomial, $n$ and $m$ denote the LL index and angular momentum index respectively, $z = x - iy$ is the complex representation of electron coordinates, and all lengths are measured in units of the magnetic length $\ell_B$. The lowest Landau level (LLL) states ($n = 0$)

$$\eta_{0,m}(z) = \frac{z^m e^{-|z|^2/4}}{\sqrt{2\pi 2^m m!}} \quad (2.6)$$

are of special importance. The wave function of the completely filled LLL is

$$\Phi_1(\{z_i\}) = \prod_{i<j} (z_i - z_j). \quad (2.7)$$

where we have omitted the ubiquitous Gaussian factor for notational ease and also the normalization coefficient.
In the CF theory, the system of strongly correlated electrons at filling factor given by Eq. (2.1) is mapped into a system of weakly interacting composite fermions at filling factor $\nu^* = n$. The wave function of this state is constructed as [83]

$$
\Psi_{gs}^{\frac{n}{2n+1}} = \mathcal{P}_{LLL} [\Phi_1 \Phi_1 \Phi_n]
$$

or

$$
\Psi_{gs}^{\frac{n}{2n-1}} = \mathcal{P}_{LLL} [\Phi_1 \Phi_1 \Phi_n^*]
$$

where $\Phi_n$ is a wave function at filling factor $\nu^* = n$. The first (second) of the above equation refers to situation when the effective magnetic field is parallel (antiparallel) to the external magnetic field. The spin degree of freedom is incorporated by assigning spins to the composite fermions [154, 220]. To form an incompressible state, the composite fermions of each spin species independently occupy an integer number $n_\uparrow$ and $n_\downarrow$ of $\Lambda$ levels, with $n = n_\uparrow + n_\downarrow$. $\Phi_n$ in the above equations is replaced by $\Phi_{n\uparrow,n\downarrow}$, where $n_\uparrow$ spin-up $\Lambda$s and $n_\downarrow$ spin-down $\Lambda$s are filled:

$$
\Psi_{gs}^{\frac{n}{2n+1}} = \mathcal{P}_{LLL} [\Phi_1 \Phi_1 \Phi_{n\uparrow,n\downarrow}]
$$

$$
\Psi_{gs}^{\frac{n}{2n-1}} = \mathcal{P}_{LLL} [\Phi_1 \Phi_1 \Phi_{n\uparrow}^*,n_\downarrow]
$$

In particular, for the spin-polarized $\nu = 2/3$ and $2/5$ ground states considered below, the composite fermions are polarized and occupy the lowest two spin-up $\Lambda$s, whereas for the spin-unpolarized $\nu = 2/3$ and $2/5$ ground states, both spin-up and spin-down composite fermions occupy one $\Lambda$.

One can expect that the above wave functions also give a correspondence between the edge states at $\nu = n/(2n\pm1)$ and $\nu^* = n$. In the disk geometry, the total angular momentum plays the role of the momentum. At $\nu^* = n$ there are $n$ edge modes, one corresponding to each LL, moving in the forward direction. They produce $n$ forward moving CF edge modes at $n/(2n + 1)$, one for each $\Lambda$, which is consistent with the description of the edge by other methods. However, for antiparallel flux attachment, they produce $n$ backward-moving modes, because of the negative effective magnetic field for composite fermions, as indicated by the complex conjugation of $\Phi_n$ in Eq. (2.9). This simple view is in disagreement with the edge behavior from other methods.

A more systematic description of the edge was developed by Wen [213] using
Figure 2.1. Schematics of the parton construction for the edge modes at 2/5 and 2/3. When spin is included, the symbol $\Phi_2$ is to be replaced by $\Phi_{2,0}$ for the spin polarized states and by $\Phi_{1,1}$ for the spin unpolarized states. The dispersions of various edge modes of the partons, and also of the edge modes of 2/5 and 2/3 after projection into the physical space. The two colors represent charge (blue) and neutral (green) modes. $E$ and $k$ are the energy and wave vector.

the parton model of the FQHE [84]. In this model one imagines breaking each electron into an odd number of fictitious fermions (called partons) and describing an incompressible state as one in which each parton occupies an IQH state (which could be in either parallel or antiparallel magnetic field). For the states of Eqs. (2.8) and (2.9), we have three partons, individually occupying states with filling factor 1, 1, and $n$ [the last being in negative magnetic field for Eq. (2.9)]. Constraints on the charge and filling factors of various partons can be derived straightforwardly [84]. If the partons are treated as independent, we have $n$ edge modes of the partons at filling $n$ (one from each LL), and one edge mode for each parton at filling one. However, the partons are obviously not independent degrees of freedom and must be identified in the calculation. Wen showed [213] that imposing a constraint that annihilates all relative density oscillations produces an edge description that is consistent with the effective field theory description. This method can be straightforwardly applied to $n/(2n - 1)$ states.

We first briefly review the hydrodynamic approach for the edge physics [214, 215] of quantum Hall systems, which will be used below in the parton construction. Consider a Hall droplet with filling factor $\nu$. The electric field generated by the
confining potential generates a current with speed \( v = E/B \) along the edge,

\[
j = \nu \frac{e^2}{h} \vec{B} \times \vec{E}
\]

(2.11)

where \( \vec{B} \) is the direction of the magnetic field. The edge wave can be described by

the one-dimensional density \( \rho(x) = n h(x) \) where \( h(x) \) is the displacement of the

edge, \( x \) is the coordinate along the edge, and \( n \) is the bulk electron density. The

propagation of edge waves is described by

\[
\partial_t \rho \pm v \partial_x \rho = 0
\]

(2.12)

where the + (−) sign applies when the magnetic field in the positive (negative) \( z \)

direction.

The Hamiltonian of the edge wave is given by

\[
H = \int dx \frac{1}{2} e h \rho E = \int dx \, \pi \nu \rho^2
\]

(2.13)

In momentum space the wave equation and the Hamiltonian can be rewritten as

\[
\dot{\rho}(k) = \pm i v k \rho(k) \quad H = 2 \pi \nu \sum_{k>0} \rho(k) \rho(-k)
\]

(2.14)

where \( \rho(k) = \int dx \frac{1}{\sqrt{L}} e^{i k x} \rho(x) \), and \( L \) is the length of the edge. Comparing with the standard Hamiltonian equation, \( \rho_k |_{k>0} \) may be identified as the "coordinates" and their corresponding canonical "momenta" are \( p(k) = \pm i 2 \pi \rho(-k) / \nu k \). This theory

is quantized by imposing the canonical commutation relation between \( \rho(k) \) and \( p(k) \), \([\rho(k), p(k')] = i \delta_{k k'}\), which leads to the so-called \( U(1) \) Kac-Moody algebra

\[
[\rho(k), \rho(k')] = \pm \frac{\nu}{2 \pi} k \delta_{k+k'} \quad k, k' = \text{integer} \times \frac{2 \pi}{L}
\]

(2.15)

In what follows, there will be several partons occupying one LL each; in that case, their density operators \( \rho_\lambda \)'s, with \( \lambda \) labeling different partons, obey the commutation relation

\[
[\rho_\lambda(k), \rho_\mu(k')] = \pm \frac{k}{2 \pi} \delta_{\lambda \mu} \delta_{k+k'}
\]

(2.16)
2.2.1 spin-polarized states

For fully spin-polarized states, the ground state wave functions of 2/5 and 2/3 are given by Eq. (2.8) and (2.9), where \( \Phi_n \) is \( \Phi_2 \), the wave function of two filled spin-up LLLs. There are three types of partons which carry charges \( 2e/5, 2e/5 \) and \( e/5 \) for the 2/5 state, and \( 2e/3, 2e/3 \) and \( -e/3 \) for the 2/3 state [84]. Following Wen [213], we introduce density operators \( \rho_1, \rho_2, \rho_3 \) and \( \rho_4 \) where \( \rho_{1,2} \) describe the edges of the two \( \Phi_1 \) state and \( \rho_{3,4} \) describe the edges of the two filled LLLs in \( \Phi_2 \) or \( \Phi_2^* \). The commutators of \( \rho_1 \) and \( \rho_2 \) are given by Eq. (2.16) with positive sign, and those of \( \rho_3 \) and \( \rho_4 \) with positive (negative) sign for the 2/5 (2/3) state. To get a physical state of electrons from a state of partons, one must project away the unphysical degrees of freedom introduced through the fictitious partons. For this purpose, we use the fact [213] that the density fluctuations associated with \( \tilde{\rho}_C = C_1 \rho_1 + C_2 \rho_2 + C_3 (\rho_3 + \rho_4) \) are unphysical for any \( C_\alpha \) satisfying \( \sum_{\alpha=1}^3 C_\alpha = 0 \). Therefore, a physical operator must commute with \( \tilde{\rho}_C \)

\[
[\hat{O}, \tilde{\rho}_C] = 0 \quad (2.17)
\]

Before projecting to the physical Hilbert space, the edge excitations contain four branches described by the \( \rho_\mu \)’s. One can check that the edge density operators

\[
\begin{align*}
  j_0 &= \sqrt{\frac{2}{5}} \left[ \rho_1 + \rho_2 + \frac{1}{2} (\rho_3 + \rho_4) \right] \quad j_1 = \sqrt{\frac{1}{2}} (\rho_3 - \rho_4) \\
  j_0 &= \sqrt{\frac{2}{3}} \left[ \rho_1 + \rho_2 - \frac{1}{2} (\rho_3 + \rho_4) \right] \quad j_1 = \sqrt{\frac{1}{2}} (\rho_3 - \rho_4)
\end{align*}
\]

(2.18)

for the 2/5 state and

\[
\begin{align*}
  j_0 &= \sqrt{\frac{2}{3}} \left[ \rho_1 + \rho_2 - \frac{1}{2} (\rho_3 + \rho_4) \right] \quad j_1 = \sqrt{\frac{1}{2}} (\rho_3 - \rho_4) \\
  j_0 &= \sqrt{\frac{2}{5}} \left[ \rho_1 + \rho_2 + \frac{1}{2} (\rho_3 + \rho_4) \right] \quad j_1 = \sqrt{\frac{1}{2}} (\rho_3 - \rho_4)
\end{align*}
\]

(2.19)

for the 2/3 state commute with \( \tilde{\rho}_C \). The physical edge excitations thus have two branches described by \( j_0 \) and \( j_1 \). This conclusion is consistent with the predictions of Chern-Simons theory [214, 215] and numerical calculations [186]. We note that only \( j_0 \) couples to the external electric potential through \( j_0 A_0 \). The commutation
relations between the $j_\mu$'s are

$$[j_\mu(k), j_\lambda(k')] = \pm \frac{k}{2\pi} \delta_{\mu\lambda} \delta_{k+k'}$$  \hspace{1cm} (2.20)

for $\mu, \lambda = 0$ and 1. The sign $\pm$ is only for the $j_1$ operator of the spin-polarized $2/3$ state, which describes a neutral backward-moving edge mode. The parton construction is schematically shown in Fig. 2.1 for $2/3$ and $2/5$.

### 2.2.2 spin-unpolarized states

The CF ground state wave functions for spin-unpolarized $\nu = 2/5$ and $2/3$ states are given by Eq. (2.10) with $\Phi_{n_1, n_1}$ being $\Phi_{1,1}$, the wave function of two filled LLs with spin-up and one spin-down. The parton construction for these states is analogous, again with four density operators defined as above. There are three types of partons for the unpolarized $\nu = 2/5$ and $2/3$ states, which carry charges $2e/5, 2e/5$ and $e/5$ and $2e/3, 2e/3$ and $-e/3$ in these two cases. Four operators $\rho_1, \rho_2, \rho_3$ and $\rho_4$ are introduced to describe the edges of the parton states, where $\rho_{1,2}$ describe the edges of the two $\Phi_1$ state and $\rho_{3,4}$ describe the edges of $\Phi_{1,1}$ or $\Phi_{1,1}^*$, where $\rho_3$ and $\rho_4$ denote the density operators for spin-up and spin-down electrons of the $\nu = 2$ spin-unpolarized state. Following the same arguments as those used in the spin-polarized case, these operators satisfy the $U(1)$ Kac-Moody algebra Eq. (2.16). It is convenient to combine these operators in a form that reflects the symmetry of the system under rotation in the spin space. Following Moore and Haldane [138], we introduce operators $\rho_s = (\rho_3 - \rho_4)/\sqrt{2}$ and $\rho_c = (\rho_3 + \rho_4)/\sqrt{2}$ which commute with $S^2$ and $S_z$. They describe the spin and charge edge modes and their commutators are given by Eq. (2.16). The density fluctuation operator is $\tilde{\rho}_C = C_1 \rho_1 + C_2 \rho_2 + \sqrt{2} C_3 \rho_c$. The commutators of $\rho_{1,2}$ are given by Eq. (2.16) with positive sign. The commutators of $\rho_{c,s}$ are given by Eq. (2.16) with positive (negative) sign for the $2/5$ ($2/3$) state. A physical operator must commute with $\tilde{\rho}_C$ as shown in Eq. (2.17). The edge density operators

$$j_c = \sqrt{\frac{2}{5}} \left( \rho_1 + \rho_2 + \frac{\rho_c}{\sqrt{2}} \right) \hspace{1cm} j_s = \rho_s$$  \hspace{1cm} (2.21)
for the 2/5 state and

\[ j_c = \sqrt{\frac{2}{3}} \left( \rho_1 + \rho_2 - \frac{\rho_c}{\sqrt{2}} \right) \quad j_s = \rho_s \]  \hspace{1cm} (2.22)

for the 2/3 state are physical operators. We note that only \( j_c \) couples to the external electric potential through \( j_c A_0 \). The commutation relations between \( j_\mu \)'s are given by Eq. (2.20) where the sign is \(-\) only for the \( j_\mu \) operator of the spin-unpolarized 2/3 state, which describes a neutral spin mode moving in the backward direction. The parton constructions for spin-unpolarized states are also schematically shown in Fig. 2.1.

### 2.3 Model and Methods

To test these ideas we have performed extensive numerical studies in various systems. In this section, we briefly explain our model and methods.

#### 2.3.1 exact diagonalization

A semi-realistic confinement potential \[211\] in the disk geometry can be modeled by a uniformly distributed positive charge background on a disk separated from the electron disk by a distance \( d \). The Hamiltonian of such a system is

\[
H = E_K + V_{ee} + V_{eb} + V_{bb} + E_Z = \sum_j \frac{1}{2m_b} \left( \mathbf{p}_j + \frac{e}{c} \mathbf{A}_j \right)^2 \\
+ \sum_{j<k} \frac{e^2}{\epsilon |\mathbf{r}_j - \mathbf{r}_k|} - \rho_0 \sum_j \int_{\Omega_N} d^2r \frac{e^2}{\epsilon \sqrt{\mathbf{r}_j - \mathbf{r}}^2 + d^2} \\
+ \rho_0^2 \int_{\Omega_N} \int_{\Omega_N} d^2r d^2r' \frac{e^2}{\epsilon |\mathbf{r} - \mathbf{r}'|} + g\mu_B BS_z 
\]  \hspace{1cm} (2.23)

Here \( m_b \) is the band mass of the electrons, \( \mathbf{p}_j \) and \( \mathbf{r}_j \) are the momentum and position operators of the \( j \)th electron, respectively. The quantity \( \mathbf{A}_j \) is the vector potential of the magnetic field at \( \mathbf{r}_j \), \( \rho_0 = v/2\pi\ell_B^2 \) is the positive charge density spread over the background disk of radius \( R_N \), and \( \epsilon \) is the dielectric constant of the system. \( S_z \) is the total spin in the \( z \)-direction. The \( V_{bb} \) term is a constant and
does not affect the result, so we will drop it in what follows. It has been found that changing the distance $d$ can cause edge reconstruction [211], but the universal properties of edge states should not depend sensitively on the detailed nature of the confinement potential, so we also use a parabolic confinement potential given by $U(r) = \alpha r^2$ to simplify some calculations. Confining to the LLL and neglecting LL mixing, the Hamiltonian in the second quantized representation is given by

$$H = \frac{1}{2} \sum_{r,s,t,u} \langle r,s|V_{ee}|t,u\rangle a_r^\dagger a_s^\dagger a_t a_u + \sum_m \langle m|V_{eb}|m\rangle a_m^\dagger a_m$$

(2.24)

The two body electron-electron interaction coefficients and electron-background interaction coefficients are

$$\langle r,s|V_{ee}|t,u\rangle = \int d^2r_1 d^2r_2 \eta^*_r(r_1)\eta^*_s(r_2) \frac{e^2}{\epsilon r_{12}} \eta_t(r_1)\eta_u(r_2)$$

(2.25)

$$\langle m|V_{eb}|m\rangle = -\rho_0 \int d^2r_1 \int \Omega d^2r_2 \frac{|\eta_m(r_1)|^2}{\sqrt{r_{12}^2 + d^2}}$$

(2.26)

The structure of edge excitations of the $\nu = 2/3$ spin-unpolarized state is not very apparent for the pure Coulomb interaction, and following Ref. [138] we consider a screened Coulomb interaction for which the edge excitations can be more readily identified. The two body matrix elements for screened Coulomb potential can be obtained by inserting a factor $\exp(-r^2/4(\kappa \ell_B)^2)$ in Eq. (2.25), where $\kappa \ell_B$ is the screening length. The effect of parabolic confinement potential is very simple: it introduce an additive term $\beta M \left[ \beta = \hbar(\sqrt{(eB/m_0 c)^2 + 8\alpha/m - eB/m_0 c})/2 \right]$ to the total energy of a state in the absence of confinement, where $M$ is the total angular momentum of the state. We shall shift the zero of the energy by changing this term to $\beta(M - M_0)$ where $M_0$ is the angular momentum of the ground state. The value of the confinement strength $\beta$ is chosen such that the state at $M_0$ becomes the ground state.

The Zeeman term $g\mu_B BS_z$ commutes with other terms in the Hamiltonian and thus can be considered separately. When this term is set to zero, the Hamiltonian commutes with the orbital and spin angular momentum operators, so the total orbital and spin angular momentum $M$ and $S^2$ are both good quantum numbers. We shall diagonalize in subspaces with fixed $M$ and $S_z = 0$. The spin eigenvalue
of a state can be calculated using
\[ \hat{S}^2 = \hat{S}_- \hat{S}_+ + \hat{S}_z^2 + \hat{S}_z \] (2.27)

The effect of the Zeeman term can be incorporated straightforwardly at the end of the calculation.

### 2.3.2 lowest Landau level projection

In general, the wave function of composite fermions occupying \( n \) \( \Lambda \) levels, prior to LLL projection, is written as
\[
\Psi(\{\{z\}\}) = \text{Det} \begin{pmatrix}
\phi_1(z_1) & \cdots & \phi_1(z_N) \\
\phi_2(z_1) & \cdots & \phi_2(z_N) \\
\vdots & \cdots & \vdots \\
\phi_N(z_1) & \cdots & \phi_N(z_N)
\end{pmatrix} \prod_{i<j}^N (z_i - z_j)^2
\] (2.28)

where \( \phi_i \)'s are single particle states of the lowest \( n \) LLs. This must be projected into the lowest LL to determine the low energy behavior.

There are two ways of performing the LLL projection. In the first one, first used by Dev and Jain [47,48], the LLL projection is achieved by replacing the anti-holomorphic coordinates \( \bar{z} \) in the Slater determinant with \( 2\partial/\partial z \). The evaluation of the projected wave function essentially amounts to expanding the unprojected wave function fully and then projecting each term into the LLL. The projection cannot be evaluated for a large number of electrons, because it requires keeping track of all basis states (Slater determinants) whose number grows exponentially with \( N \).

One can simplify the book-keeping process using the Jack polynomial formalism. It has been shown that Jastrow factor belongs to a special class of polynomials, i.e. the Jack polynomials [19]. The Schur function Eq. (2.41) that will be used later is also a Jack polynomial. The properties of Jack polynomials are briefly reviewed below for completeness. We can label a non-interacting \( N \)-particle state using a partition \( \lambda = [\lambda_1, \cdots, \lambda_N] \) which contains all the occupied states or a occupation configuration \( n(\lambda) = n_m(\lambda), \ m = 0,1,2,\cdots \) where \( n_m(\lambda) \) gives the number of
particles in the angular momentum state \( m \). The wave functions corresponding to these non-interacting states are symmetric monomials for bosons and Slater determinants for fermions. An interacting many-body state is a superposition of many non-interacting basis states indexed by \( \lambda \)'s with coefficients \( c_{\lambda} \). For a non-interacting Fock state in which the orbitals \( m_1 \) and \( m_2 \) are occupied (\( m_1 < m_2 - 1 \)), we define the elementary squeezing operation as the procedure which shifts the two particles as \( n_{m_1} \rightarrow n_{m_1} - 1 \), \( n_{m_2} \rightarrow n_{m_2} + 1 \). A partition \( \lambda \) is said to dominate \( \mu \) (denoted as \( \lambda > \mu \)) if \( \mu \) can be generated by squeezing \( \lambda \). A bosonic Jack polynomial can be expanded using symmetric monomials as

\[
J_\lambda^\alpha = \sum_{\kappa \leq \lambda} c_{\lambda\kappa}(\alpha) M_\kappa,
\]

where the summation over \( \kappa \) runs over all the partitions that are dominated by or equal to the root partition \( \lambda \) and \( M_\kappa \) is the symmetric monomial labeled by \( \kappa \) [19].

To do the LLL projection, we need to expand the the Jastrow factor of \( N \) particles, \( \prod_{i<j}^N (z_i - z_j)^2 \), whose root partition is \([2N, 2N-2, \cdots, 0]\) and \( \alpha = -2 \). There is a recursive relation [194] for the expansion coefficients \( c_{\lambda\kappa}(\alpha) \)

\[
c_{\lambda\kappa}(\alpha) = \frac{2/\alpha}{\rho_\lambda(\alpha) - \rho_\kappa(\alpha)} \sum_{\kappa < \mu \leq \lambda} [((l_i + t) - (l_j - t)) c_{\mu\kappa}(\alpha)],
\]

where the sum is over all the partitions \( \mu \) that strictly dominate \( \kappa \) but are dominated by or equal to \( \lambda \) and the function \( \rho \) is defined as

\[
\rho_\lambda(\alpha) = \sum_i \lambda_i \left[ \lambda_i - 1 - \frac{2}{\alpha} (i - 1) \right]
\]

Once the expansion is obtained, one can act the derivative on each symmetric monomial and sort the results to Slater determinants, i.e. Fock states of fermions. This method simplifies the LLL projection but the computational time still grows exponentially, so it is typically not possible to perform the calculations for systems with more than 10 particles.

The second method is the Jain-Kamilla projection [87]. In this method we
abrupt the Jastrow factor into the Slater determinant to write

\[
\Psi(\{z\}) = \text{Det} \begin{pmatrix}
\phi_1(z_1)J_1 & \cdots & \phi_1(z_N)J_N \\
\phi_2(z_1)J_1 & \cdots & \phi_2(z_N)J_N \\
\vdots & \cdots & \vdots \\
\phi_N(z_1)J_1 & \cdots & \phi_N(z_N)J_N
\end{pmatrix}
\]  

(2.32)

where \( J_i = \prod_k' (z_i - z_k) \) and the summation runs over all indices \( k \neq i \). Instead of applying \( P_{\text{LLL}} \) to the whole expression, one apply it to each matrix element individually and then evaluate the determinant. This method does not require decomposition of the wave function in the Slater determinant basis, and thus can be applied to very large systems for both parallel and antiparallel flux attachments [45, 87, 135].

The two methods for projection do not produce identical wave functions. However, explicit calculations have shown that they are very close for fully spin-polarized states. The spin-unpolarized states are somewhat more sensitive to which projection is used, and we have found that the states obtained from the Dev-Jain projection are closer to the exact Coulomb states. In our calculations below, the Jain-Kamilla projection has been used for the spin-polarized states, and Dev-Jain projection for the spin-unpolarized states.

\[\Delta M\] 
<table>
<thead>
<tr>
<th>( S_z = 0 )</th>
<th>( S_z = \pm 1 )</th>
<th>( S_z = \pm 2 )</th>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>1</td>
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<tr>
<td>3</td>
<td>10</td>
<td>5</td>
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<tr>
<td>4</td>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>36</td>
<td>20</td>
</tr>
</tbody>
</table>

Table 2.1. Number of all edge modes for various values of \( \Delta M \) and \( S_z \).

2.4 Spin-Unpolarized \( \nu = 2 \) and \( \nu = 2/5 \) State

Both the spin-unpolarized \( 2/3 \) and \( 2/5 \) states are closely related to the spin-unpolarized \( \nu = 2 \) state. The edge modes at \( \nu = 2 \) consist of one pure charge branch and one pure spin branch. Their counting can be obtained straightfor-
Figure 2.2. Pure spin excitations at $\nu = 2$ and $2/5$. Energy spectra of the edge excitations of the unpolarized $\nu = 2$ (panel (a)) and $2/5$ (panel (b)) states with 6 particles. The ground states have $M = 6$ and $M = 36$, respectively. The pure spin modes are enclosed by green boxes, and the black dots in the lower panel show the energies of CF wave functions. The full spectrum at $\nu = 2$ consists of pure spin, pure charge (Fig. 2.3), and mixed excitations. At $\nu = 2/5$, the spectrum also includes excitations in the interior of the system where composite fermions are excited across $\Lambda_L$s. In this and the subsequent figures, eigenstates with different spin quantum number are shown in different colors, with the color coding indicated on the figures, and also horizontally shifted for clarity.

Figure 2.3. Pure charge excitations at $\nu = 2$ and $2/5$. Energy spectra of the unpolarized $\nu = 2$ (panel (a)) and $2/5$ (panel (b)) states with 6 particles. The ground states have $M = 6$ and $M = 36$, respectively. The pure charge excitations of $\nu = 2$ are enclosed by grey boxes. The pure charge excitations at $\nu = 2/5$ are harder to identify for small systems because they lie in the continuum of the bulk excitations.
Table 2.2. Number of pure spin edge modes for various values of $\Delta M$ and $S_z$.

<table>
<thead>
<tr>
<th>$\Delta M$</th>
<th>$S_z = 0$</th>
<th>$S_z = \pm 1$</th>
<th>$S_z = \pm 2$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>3</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>5</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2.3. Number of pure charge edge modes for several $\Delta M$. They all have $S_z = 0$.

<table>
<thead>
<tr>
<th>$\Delta M$</th>
<th>$S_z = 0$</th>
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<tr>
<td>1</td>
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<td>7</td>
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wardly [138]; the number of edge excitations in the subspaces with fixed $S_z$ values are shown in Table 2.1 for some values of $\Delta M$. Note that a $S_z = \pm A$ state appears only if $\Delta M \geq A^2$. Among these states, some are pure spin states, some are pure charge states, and some mixed. Tables 2.2 and 2.3 show the number of pure spin and pure charge modes. Since the states form $SU(2)$ multiplets, the number of state in the $S_z = A(A + 1)$ sector can be obtained by subtracting the number of state in the $S_z = A + 1$ sector from that of the $S_z = A$ sector; some instances are summarized in Table 2.4. We expect identical counting for 2/5.

Fig. 2.2 shows the spectra for $\nu = 2$ and $\nu = 2/5$ states with the Zeeman energy set to zero. The coefficient $\beta$ due to the parabolic confinement potential is chosen to be 0.6 and 0.06 for the $\nu = 2$ and 2/5 state, respectively, so as to make the

<table>
<thead>
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<td>1</td>
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<tr>
<td>2</td>
<td>0,1</td>
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<td>3</td>
<td>0,1,1</td>
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<tr>
<td>4</td>
<td>0,0,1,1,2</td>
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<td>5</td>
<td>0,0,1,1,2,2</td>
</tr>
</tbody>
</table>

Table 2.4. Number of pure spin edge modes for given $\Delta M$ and $S$. 
compact state the ground state. The energy eigenstates are also $S^2$ eigenstates and different colors are used to represent the different $S^2$ eigenvalues. We also shift the energy levels in the horizontal direction for clarity. The ground states are marked by green arrows. We see in both spectra low energy states (enclosed by green boxes), which we identify as pure spin edge states; these are well separated from other states. This counting matches that of the pure spin mode as shown in Table 2.4. We introduce the following operators to describe the edge modes of the $\nu = 2$ spin-unpolarized state (the superscripts “c” and “s” refer to “pure charge” and “pure spin” respectively):

$$C^{\dagger,c}_m = \sum_n a^{\dagger}_{\uparrow m+n} a^{\uparrow n} + a^{\dagger}_{\downarrow m+n} a^{\downarrow n}$$ (2.33)

$$C^{\dagger,s}_{m,0} = \sum_n a^{\dagger}_{\uparrow m+n} a^{\uparrow n} - a^{\dagger}_{\downarrow m+n} a^{\downarrow n}$$ (2.34)

$$C^{\dagger,s}_{m,1} = \sum_n a^{\dagger}_{\uparrow m+n} a^{\downarrow n}$$ (2.35)

$$C^{\dagger,s}_{m,-1} = \sum_n a^{\dagger}_{\downarrow m+n} a^{\uparrow n}$$ (2.36)

where $a_{\sigma m}$ is the annihilation operator for an electron with spin $\sigma$ and angular momentum $m$. They have the following commutation relations with the spin operators $S^-$ and $S^+$:

$$[S^+, C^{\dagger,s}_{m,0}] = C^{\dagger,s}_{m,1}$$ (2.37)

$$[S^-, C^{\dagger,s}_{m,1}] = C^{\dagger,s}_{m,0}$$ (2.38)

$$[S^-, C^{\dagger,s}_{m,0}] = C^{\dagger,s}_{m,-1}$$ (2.39)

and all other commutators vanish. The states obtained by acting these operators on the ground state are in general not eigenstates of the Coulomb Hamiltonian. However, the two appear to be adiabatically connected. We can construct a model Hamiltonian $H^c = \sum_m C^{\dagger,c}_m C^{c}_m$ for which the pure spin excitations appear as zero modes. These zero mode states can be obtained by acting $C^{\dagger,s}_{m,0}$ operators on the ground state, and are adiabatically connected to the spin edge modes in the $\nu = 2$ spectrum with Coulomb interaction.

Due to the strongly interacting nature of the state at $\nu = 2/5$, it is not possible to construct similar operators explicitly, but the trial wave functions for the the
ground state and also excitations of the unpolarized $\nu = 2/5$ state can be obtained by composite fermionizing the corresponding states at $\nu = 2$. The edge excitations of unpolarized $2/5$ state can be obtained via

$$\Psi_{2/5}^{\Delta M} = \mathcal{P}_{\text{LLL}} \left[ \Phi_1^2 \Phi_{1,1}^{\Delta M} \right]$$

where $\Phi_{1,1}^{\Delta M}$ is an edge state of the unpolarized $\nu = 2$ state with angular momentum $\Delta M$ relative to its ground state. The pure spin modes of the unpolarized $\nu = 2/5$ state can be obtained from the $\nu = 2$ pure spin modes in this way, and we show the energies of such CF state using black dots in panel (b) of Fig. 2.2. The lowest LL projection has been performed by the method in Ref. [87].

How about the pure charge modes moving in the forward direction? We construct these modes by multiplying the ground state wave functions with Schur functions, which are symmetric polynomials defined as

$$S_{\lambda^B}(\{z\}) = \frac{\text{Det}_{\lambda^F}(\{z\})}{\prod_{i<j} (z_i - z_j)}$$

where $\lambda^B = [\lambda_1, \lambda_2, \cdots, \lambda_N]$ is a bosonic partition and $\text{Det}_{\lambda^F}(\{z\})$ is a Slater determinant with fermionic index $\lambda^F = [\lambda_1 + N - 1, \lambda_2 + N - 2, \cdots, \lambda_N]$. Multiplication by this function increases the angular momentum by $\Delta M = \sum_i \lambda_i^B$. The number of independent Schur functions at $\Delta M$ is equal to the number of partitions of integer $\Delta M$, which is consistent with the pure charge mode counting in Table 2.3. In Fig. 2.3, we show the comparison of Schur modes with exact states for unpolarized $\nu = 2$ and $2/5$ states. At $\nu = 2$ the Schur modes and exact states match very well. While all states shown in the $\nu = 2$ spectrum are edge excitations (as excitations to higher LLs are suppressed), the exact $\nu = 2/5$ spectrum in Fig. 2.3 also contains bulk excitations. The pure charge modes are not clearly separated from the bulk states (in contrast to the pure spin modes discussed above), indicating a larger velocity for the pure charge mode. This is a finite size effect, however, and we expect that for large $N$ a well defined edge branch will appear.
Figure 2.4. Energy spectrum of the edge excitations of the unpolarized $\nu = 2/3$ state. The ground state, marked by a green arrow, occurs at total angular momentum $M = 44$. In panel (a), backward moving pure spin modes are enclosed by green boxes and forward moving pure charge modes are enclosed by grey boxes. In panel (b), the dots show the energoes of the Schur states, and the nearby numbers show their overlaps with the exact states. For comparison, the pure spin and pure charge edge excitations from exact diagonalization spectrum of panel (a) are also shown in panel (b).

Figure 2.5. Density profiles of excited states identified as edge and bulk excitations of unpolarized $\nu = 2/3$ state for $N = 8$ at $\Delta M = 1$ ($M = 45$). For comparison, the density profile of the ground state at $M = 44$ is also shown.

2.5 Spin-Unpolarized $\nu = 2/3$ State

We next come to the edge excitations of $2/3$, where we expect counter propagating modes. In panel (a) of Fig. 2.4, we show the spectra of spin-unpolarized $\nu = 2/3$ with 8 particles for negative and positive $\Delta M$. The coefficient $\beta$ is 0.015 here. The ground state is marked by the green arrow. As explained before, we use a screened
Coulomb interaction with screening length $\kappa l = l$ here. For negative $\Delta M$, some states, marked by the green boxes in panel (a) of Fig. 2.4, are well separated from others. The counting of these states suggests that they are pure spin modes. It is not possible to construct explicit operators that would create the edge excitations of the unpolarized $\nu = 2/3$ state, but trial wave functions can be obtained in CF theory using

$$\Psi_{2/3}^{-\Delta M} = \mathcal{P}_{\text{LLL}} \left[ \Phi_1^{2} (\Phi_{1,1}^{\Delta M})^* \right]$$

(2.42)

We note both the forward moving edge modes of the unpolarized $\nu = 2$ state are converted to backward moving modes of the $\nu = 2/3$ state according to this transformation; because of the complex conjugation the factor $(\Phi_{1,1}^{\Delta M})^*$ contributes a negative angular momentum. Thus it appears that there would be two backward moving branches. However, we find that some CF states are annihilated by the LLL projection (evaluated by the method of Ref. [47,48], and among the surviving ones, many are pushed to higher energies. The counting of the remaining low energy states matches with that predicted by the bosonized theory of the edge within the parton description. The black dots and nearby numbers in panel (b) of Fig. 2.4 show the energies of the CF trial states and their overlaps with the exact states enclosed by green boxes in panel (a). These results demonstrate that the CF theory captures the qualitative behavior, and while the agreement is quantitatively not as good as it is for the $n/(2n+1)$ states, it clearly gives a semiquantitative account of the backward moving edge. The CF states with high energies (typically higher than the range of Fig. 2.4) are not shown.

For positive $\Delta M$, it is theoretically predicted that there are pure charge ($i.e.$ $S = 0$) excitations. In numerical calculations, we cannot identify any states well separated from the others at positive $\Delta M$’s and the states with lowest energies do not have $S = 0$. This is to be expected, however, as the charge modes have higher velocity and therefore rapidly merge into the bulk excitations. By analogy to the discussion of the pure charge modes of the unpolarized $\nu = 2$ and $2/5$ states, we expect that the pure charge modes of the $2/3$ state are also Schur modes. In panel (a) of Fig. 2.4, the states enclosed by grey boxes are identified as pure charge modes. The red dots and nearby numbers in panel (b) show the energies of the Schur modes and their overlaps with these exact states. We have also calculated the energy spectrum of 6 particles. The energy differences between these Schur states...
Figure 2.6. Energy spectrum of the edge excitations of the spin polarized $\nu = 2/3$ state. The ground state occurs at total angular momentum $M = 69$. In panel (a), the backward moving neutral modes are enclosed by green boxes and the forward moving charge modes by grey boxes. The two insets show the CF configurations at $M = 69$ and $73$. In panel (b), the dots and nearby numbers show the energies of CF and Schur states and their overlap with exact states (i.e. boxed states in panel (a)). The relatively poor comparison for $M = 71$ is attributed to the fact that the low energy modes here can also be viewed as the backward moving modes emanating from the ground state at $M = 73$, and the two description compete; this will not be an issue for larger systems.

and the lowest energy state decreases as the system size increases, which shows that the Schur modes will become the lowest energy states in the thermodynamic limit. To further support our identification of the edge and bulk excitations, we plot the density profiles of some states at $\Delta M = 1$ in Fig. 2.5; the state shown in panel (a) only exhibits density variations in the vicinity of the edge, whereas those in panel (b) deviate also in the bulk. Similar behavior is confirmed at $\Delta M = 2$.

2.6 Spin-Polarized $\nu = 2/3$ State

CF wave functions of the edge excitations of the spin polarized $\nu = 2/3$ state are constructed from the edge excitations of the spin-polarized $\nu = 2$ state via the mapping

$$\Psi_{2/3}^{-\Delta M} = \mathcal{P}_{\text{LLL}} \left[ \Phi_1^2(\Phi_2^{\Delta M})^* \right]$$

(2.43)

where $(-) \Delta M$ is the angular momentum measured relative to their respective ground states. More details about the construction of such wave functions can be found in Ref. [90]. We denote the state with $N_1$ composite fermions in the lowest and $N_2$ composite fermions in the second $\Lambda$L by $[N_1, N_2]$. Note that in
order to have two independent CF edge branches, we should choose the number of composite fermions in the second ALs to be sufficiently smaller than that in the lowest $\Lambda$ level, so as to eliminate transitions of composite fermions from the $2^{nd}$ $\Lambda$ to the $1^{st}$ $\Lambda$ (which would happen only at a large $\Delta M$). This also corresponds to the experimental situation where for a typical confinement potential we expect that the lowest $\Lambda$ level would extend farther than the second. (Note that the spin polarized $2/3$ is different from the spin unpolarized $2/3$ in this respect.)

We show the energy spectrum of spin-polarized $2/3$ state with 10 particles in panel (a) of Fig. 2.6. The coefficient $\beta$ is chosen to be 0.075 here. The ground state $[7,3]$ occurs at angular momentum $M = 69$; it is shown by the left inset of Fig. 2.6. The backward moving edge states are enclosed by green boxes. By construction, all the forward-moving $\nu^* = 2$ edge states are transformed to backward-moving $\nu = 2/3$ edge states, and one may expect two backward moving modes. We find that, similarly to the spin-unpolarized $\nu = 2/3$ state, some of the CF states are projected out and some are pushed to high energies (typically outside the range of Fig. 2.6, leaving only a few at low energies, which match very well with exact states. A comparison with the exact states is shown in panel (b) of Fig. 2.6, where black dots show the energies of the CF states and the numbers are the overlaps.

It is again natural to associate the forward moving mode with positive $\Delta M$ with Schur excitations. We again encounter the finite size difficulty of the absence of a clear gap in some cases. The red dots and nearby numbers in panel (b) of Fig. 2.6 show the energies of Schur modes and their overlaps with exact states. As we increase the angular momentum from 69 to 73, we get another CF configuration $[6,4]$, which can also serve as a finite size representation of the $2/3$ state. That sets a finite size limitation on the angular momentum $\Delta M$ one can study in the forward or backward direction. For example, the state at angular momenta smaller than 73 may also be viewed as backward moving edge states emanating from $M = 73$. This complicates the counting of the edge modes for finite size systems, and is also the likely cause of the mismatch between the exact spectra and the expectation from the effective theory.
2.7 Conclusion

We have studied the edge states of spin unpolarized and polarized $\nu = 2/3$ states within the framework of the CF theory, both using parton construction and the microscopic CF wave functions. The parton construction of composite fermions produces one forward moving charge mode and one backward moving neutral mode for both spin unpolarized and polarized $2/3$ states, which agrees with the predictions of Chern-Simons effective field theory. Backward moving modes also appear naturally from the observation that at $2/3$ composite fermions experience negative magnetic field, and thus move at the edge in a direction opposite to that of electrons. A naive counting would suggest two backward moving edge modes, one from each $\Lambda$ level, but we have shown, by explicit construction of the CF wave functions, that one mode is projected out of the low energy sector and the remaining excitations are good approximations of the exact states for both spin unpolarized and spin polarized $2/3$ state. The forward moving modes are Schur modes; they are harder to identify in the exact spectra of small systems because, due to the larger velocity of these modes, they quickly enter into the continuum of bulk excitations across $\Lambda$ levels. Nonetheless, a careful examination of the density profiles has allowed us to identify the forward moving edge states and to compare them with Schur modes. We have thus shown that the description from the microscopic approach is consistent with the Chern-Simons or the parton approach, albeit only after a nontrivial reduction of the edge excitations upon projection into the low energy space. Annihilation of mean field CF states upon projection has been found in previous numerical studies in other contexts as well [47,48,221], but no understanding exists of the general mathematical structure underlying such annihilations.

Before closing, we note that a number of effects have been left out in our study. While we have only focused on state counting in this paper, the effective description in terms of bosons also makes predictions for the spectral function, i.e., matrix elements relating ground to excited states through the electron creation operator [94,95,148,209,210,235], which we have not investigated. We have also not considered subtle questions regarding the antisymmetry of the electron operator in the projected edge state space [236], or the role of AL mixing. Similarly, the possibility of edge reconstruction [29,92,148,211] has not been incorporated into...
our calculations, which, if it occurs, will fundamentally alter the nature of the edge. The effects of finite thickness, LL mixing and disorder have also been neglected.
Chapter 3

Quantum Hall States of Two-Component Bosons

This chapter is a reproduction of the published paper Phys. Rev. B 87, 245123 by the present author in collaboration with J. K. Jain. It has been modified slightly to fit into this thesis. In this chapter, we investigate the feasibility of many candidate quantum Hall states for two-component bosons in the lowest Landau level. We identify interactions for which spin-singlet incompressible states occur at filling factors $\nu = \frac{2}{3}, \frac{4}{5},$ and $\frac{4}{3}$, and partially spin-polarized states at filling factors $\frac{3}{4}$ and $\frac{3}{2}$, where “spin” serves as a generic label for the two components. We study ground states, excitations, edge states, and entanglement spectrum for systems with up to 16 bosons and construct explicit trial wave functions to clarify the underlying physics. The composite fermion theory very accurately describes the ground states as well as excitations at $\nu = \frac{2}{3}, \frac{4}{5},$ and $\frac{3}{4}$, although it is less satisfactory for the $\nu = 3/2$ state. For $\nu = \frac{4}{3}$ a “non-Abelian spin-singlet” state, which is the exact ground state of a three-body contact interaction, has been proposed to occur even for a two-body contact interaction; our trial wave functions are very accurate for the excitations of the three-body interaction, but they do not describe the excitations of the two-body interaction very well. Instead, we find that the $\nu = \frac{4}{3}$ state is more likely to be a spin-singlet state of reverse-flux-attached composite fermions at filling factor $\nu^* = 4$. We also consider incompressible states at integral filling factors $\nu = 1$ and $2$. The incompressible state at $\nu = 1$ is shown to be well described by the parton-based Jain spin-singlet wave function, and the
incompressible state at \( \nu = 2 \) as the spin-singlet state of reverse-flux-attached composite fermions at \( \nu^* = 2 \), which provides an example of the bosonic integer topological phase

### 3.1 Introduction

The study of two-component fractional quantum Hall (FQH) effect has revealed a tremendous amount of new physics. The earlier studies were performed on GaAs systems \([51, 54, 70, 71, 104, 106, 107, 131, 181, 182, 196, 229, 230]\), where the Landé g-factor is small and therefore both components of spin can be active at relatively small magnetic fields. More recently, two-component FQH effect has been studied in systems where valleys play the role of spin, as in AlAs quantum wells \([64, 147]\) and H-terminated Si(111) surface \([102]\); here the Zeeman energy is large enough to freeze the spin degree of freedom for typical experimental parameters. In graphene, the two components could be either spins or valleys, depending on parameters \([49, 56]\). Experiments have shown that in general, FQH states with several spin/valley polarization can occur at a given filling factor, and transitions between them can be caused by tuning the Zeeman/valley splitting. These level crossing transitions are understood in terms a competition between the composite fermion (CF) cyclotron energy and the Zeeman/valley splitting. A quantitative understanding of this physics has been achieved through the multi-component wave functions proposed by Halperin \([76]\), and more generally through the theory of spinful composite fermions \([44, 88, 154, 220]\).

Given a rich diversity of strongly correlated states of fermions involving the spin physics, it is natural to ask what new physics can be learned from the study of two-component Bose gases, such as those made up of two hyperfine spin states of the same atoms, in the FQH regime. Neutral bosons can in principle be driven into the FQH regime by rapid rotation \([40, 203]\). Strongly correlation among particles is achieved as the number of vortices \( N_V \) in a rotating Bose-Einstein condensate (BEC) becomes comparable with the number of atoms \( N \) as quantified by the filling factor \( \nu = N/N_V \). For simplicity, we will refer to the two components as spins, but the results apply to any two-component bosons for which the interaction is (approximately) independent of the component index. There has been much recent
study of bosonic quantum Hall states [13, 27, 31, 35, 37–39, 101, 130, 150, 160, 161, 173, 180, 202, 216]. It has been shown that the vortex lattice that forms at large $\nu$ melts and that a series of FQH states appear at various filling factors, which include, for appropriately chosen interactions, Laughlin [111], Jain [83], Moore-Read [137] and Read-Rezayi [159] states. While FQH effect in cold atom systems has not yet been observed in a convincing manner, substantial progress in that direction has been reported [62]. Other ingenious methods to simulate the effect of magnetic field have also been explored [41, 118, 156]. We consider below two-component bosons in the FQH regime. Aside from the experimental interest, a motivation for studying this problem is the possibility of realization of new structures that are not available in electronic FQH effect. In particular, we will see that some FQH states require a tuning of the interaction, which is more easily accomplished in ultracold atomic systems.

The theoretical study of FQH effect has relied on the notion of formation of emergent quasiparticles, description in terms of which provides a tangible way to understand the physical properties of an inherently hard quantum $N$-body problem. The physics of emergent quasiparticles is captured by appropriate wave functions, which, in turn, represent topological phases. To fully classify all topological phases is a formidable task, but progress has been made in the context of topological insulators and superconductors [98, 172]. However, one can take specific examples and ask if they occur for models with realistic interactions. We consider in this article several bosonic spin-singlet and spin-partially-polarized states, and ask for what kinds of interaction they would be realized. Some of these support excitations with Abelian braid statistics, whereas some with non-Abelian braid statistics.

This chapter is organized as follows. In Sec. 4.1 we introduce all of the trial wave functions that we study in the subsequent sections. Sec. 4.2 describes our model, and our methods for evaluating the wave functions, exact diagonalization and entanglement spectra. Sec. 4.3 presents the results. Sec. 4.4 summarizes the conclusions of our study.
3.2 Trial Wave Functions

In this section, we shall use the symmetric gauge on disk geometry where the lowest Landau level (LLL) wave functions are particular simple as given by

\[ \phi_m(z) = \frac{z^m \exp(-|z|^2/4)}{\sqrt{2\pi^2m!}} \]  

(3.1)

where \( z = x + iy \) is the complex coordinate of particles on the disk. The ubiquitous exponential factor will be omitted in the rest of this paper. A bosonic Fock state is represented using symmetric monomials and a many-body state is the superposition of all monomials with appropriate quantum numbers. We use the convention that the coordinates \( \{ z^\uparrow \} \) and \( \{ z^\downarrow \} \) denote, respectively, spin-up and spin-down particles, whereas \( \{ z \} \) denote all particles.

The general wave function of two-component bosons (with \( N^\uparrow \) spin-up and \( N^\downarrow \) spin-down bosons) at filling factor \( \nu \) has the form

\[ \chi_\nu = S\left[ \Psi_\nu(\{ z \})u_1 \cdots u_{N^\uparrow}d_1 \cdots d_{N^\downarrow} \right] \]  

(3.2)

where \( \Psi_\nu(\{ z \}) \) is the spatial part, \( u \) and \( d \) refer to the two components, and \( S \) denotes symmerization. It is sufficient to consider \( \Psi_\nu(\{ z \}) \) provided it satisfies appropriate symmetries. An acceptable wave function with spin \( S = S_z \) must satisfy Fock’s cyclic condition, which means that the state \( \chi_\nu \) is annihilated by an attempt to antisymmetrize a spin-down particle with respect to the spin-up particles. This condition is satisfied for the wave functions considered below.

(i) The Halperin 221 state at \( \nu = 2/3 \) state is given by

\[ \Psi_{221}(\{ z \}) = \prod_{i<j}(z_i^\uparrow - z_j^\uparrow)^2(z_i^\downarrow - z_j^\downarrow)^2 \prod_{i,j}(z_i^\uparrow - z_j^\downarrow) \]  

(3.3)

This form of multi-component wave functions were introduced by Halperin for electronic FQH states [76]. The Halperin 221 wave function vanishes as the third power of distance between particles when two particles are brought together, regardless of their spin configuration. It is the exact ground state for the contact interaction \( \sum_{i<j} \delta(z_i - z_j) \).
(ii) The Jain’s CF (JCF) states at $\nu = n/(n\pm1)$ are given by

$$
\Psi^{[n_\uparrow,n_\downarrow]}_{\frac{\nu-1}{n+1}}(\{z\}) = \mathcal{P}_{\text{LLL}} \left[ \Phi_{n_\uparrow}(\{z^\uparrow\})\Phi_{n_\downarrow}(\{z^\downarrow\})J(\{z\}) \right]
$$

(3.4)

$$
\Psi^{[-n_\uparrow,-n_\downarrow]}_{\frac{\nu+1}{n-1}}(\{z\}) = \mathcal{P}_{\text{LLL}} \left[ \Phi_{-n_\uparrow}(\{z^\uparrow\})\Phi_{-n_\downarrow}(\{z^\downarrow\})J(\{z\}) \right]
$$

(3.5)

where $\Phi_{-n_\uparrow} \equiv \Phi^{*}_{n_\uparrow}$, $\Phi_{-n_\downarrow} \equiv \Phi^{*}_{n_\downarrow}$ and $J(\{z\}) = \prod_{i<j}(z_i - z_j)$ is the Jastrow factor for all particles; $\Phi_{n_\uparrow}$ and $\Phi_{n_\downarrow}$ are two Slater determinants for the spin-up and spin-down particles at fillings $n_\uparrow$ and $n_\downarrow$, respectively, and their complex conjugates $\Phi_{-n_\uparrow}$ and $\Phi_{-n_\downarrow}$ represent filled LL states in opposite magnetic field; $n = n_\uparrow + n_\downarrow$; and the symbol $\mathcal{P}_{\text{LLL}}$ represents the LLL projection operator. The spin polarization is given by

$$
P = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow}
$$

(3.6)

Those with $n_\uparrow = n_\downarrow$ are spin-singlet, while those with $n_\uparrow \neq n_\downarrow$ (i.e. odd $n$) are spin-partially-polarized (or spin-polarized).

These wave functions are closely related to those studied previously for electronic FQH effect [154,220], where they represent the physics of electrons capturing two vortices to turn into composite fermions, which then form integer quantum Hall (IQH) states. In the present case, the bosons capture one vortex each to form composite fermions, which experience a reduced effective magnetic field $B^* = B - \rho hc/e$ ($B$ is the external field and $\rho$ is the density) and condense into IQH states (with filling factor denoted as $\nu^*$) to produce incompressibility. An intuitive reason for why bosons convert into composite fermions is because this builds good correlations that keep the particles away from one another and thus reduce the interaction energy. For $n_\downarrow = 0$ these wave functions reduce to fully spin polarized bosons which have been considered previously [31,160]. The wave functions in Eq. (3.4) and Eq. (3.5) are interpreted as the states in which composite fermions fill $n_\uparrow$ spin-up and $n_\downarrow$ spin-down $\Lambda$ levels ($\Lambda$Ls), where $\Lambda$Ls are Landau-like levels of composite fermions.

A noteworthy aspect of the analogy to IQH effect is that it goes beyond the ground state and also allows construction of wave functions for the excitations of the $\nu = n/(n\pm1)$ state in terms of the known excitations of the IQH states. In
fact, the CF theory implies a one-to-one correspondence between the excitations at $\nu^* = n$ and those at $\nu$, because an IQH wave function with a given spin and angular momentum quantum numbers produces, through Eqs. (3.4) or (3.5), a wave function at $\nu$ with the same quantum numbers. In particular, neutral and charged excitations of the IQH state at $\nu^* = n$ produce neutral and charged excitations of the state at $\nu$. In what follows, the JCF wave function $\Psi^{[\pm n_+, \pm n_\downarrow]}_{n/(n\pm 1)}$ will collectively represent wave functions for the ground state as well as neutral and charged excitations.

We study below $\Psi^{[1,1]}_{2/3}$, $\Psi^{[2,2]}_{4/5}$, $\Psi^{[-2,-2]}_{4/3}$, $\Psi^{[2,1]}_{3/4}$, $\Psi^{[-2,-1]}_{3/2}$ and $\Psi^{[-1,-1]}_{2}$, including ground state and excitations. We note that for $n_\uparrow = n_\downarrow = 1$ the ground state wave function is given by

$$\Psi^{[1,1]}_{2/3, G.S.}(\{z\}) = \mathcal{P}_{LLL} \left[ \Phi_1(\{z^\uparrow\}) \Phi_1(\{z^\downarrow\}) J(\{z\}) \right] = \prod_{i<j} (z^\uparrow_i - z^\downarrow_j) \prod_{ij} (z_i - z_j)$$  

(no LLL projection is required in this case) which is identical to the Halperin 221 wave function. In other words, the Halperin-221 state is interpreted as the $\nu^* = 2$ spin-singlet state of composite fermions. This interpretation also allows a construction of the excitations of the $2/3$ state by correspondence with the excitations of the $\nu^* = 2$ spin-singlet IQH state $\Phi_1(\{z^\uparrow\}) \Phi_1(\{z^\downarrow\})$.

(iii) The simplest non-Abelian spin-singlet (NASS) state [5, 6] at filling factor $\nu = 2k/3$ can be written as a symmetrized product of $k$ copies of the Halperin 221 state

$$\Psi^{NASS}_{2k/3}(\{z\}) = \mathcal{S}_{\uparrow\downarrow} \left[ \Psi^{221}_{2/3}(\{z^\alpha\}) \Psi^{221}_{2/3}(\{z^\beta\}) \cdots \Psi^{221}_{2/3}(\{z^k\}) \right]$$  

where the particles are divided into $k$ groups with coordinates $\{z^\alpha\}, \{z^\beta\}, \cdots, \{z^k\}$ and $\mathcal{S}_{\uparrow\downarrow}$ denotes the separate symmetrization of the spin-up and spin-down particles. It may be viewed as a spin-singlet generalization of Read-Rezayi $Z_k$ states [159] whose excitations obey non-Abelian braiding statistics [5, 6]. It is the exact zero energy ground state of a model $(k + 1)$-body contact interaction. It has recently been suggested [59,68] that the 4/3 NASS state may be realized even for the 2-body contact interaction.

We will also study excitations of this state. The quasiholes, excitations, obtained
by adding flux quanta, also have zero energy for the \((k + 1)\)-body interaction, and can be explicitly constructed [7, 55]. The neutral excitations and the quasiparticles of the \((k + 1)\)-body Hamiltonian are nontrivial and do not have zero energy. To construct trial wave functions for them, we generalize Eq. (3.8) to

\[
\Psi_{NASS}^{\frac{1}{4}} \{\{z\}\} = S^{\uparrow\downarrow} \left[ \Psi^{[1,1]}_{\frac{1}{4}} \{\{z^\alpha\}\} \Psi^{[1,1]}_{\frac{1}{4}} \{\{z^\beta\}\} \cdots \Psi^{[1,1]}_{\frac{1}{4}} \{\{z^k\}\} \right] \tag{3.9}
\]

This reproduces the wave function of Eq. (3.8) when all factors \(\Psi^{[1,1]}_{\frac{1}{2}/3}\) are chosen as the ground states (i.e. the Halperin 221 state), but also produces excitations by appropriate choice of excited states on the right hand side. For example, the lowest energy neutral excitations corresponds to a CF exciton in a single factor \(\Psi^{[1,1]}_{\frac{1}{2}/3}\). This approach for constructing excitations follows a “multipartite CF” representation investigated recently to study the excitations of the Moore-Read state [167, 184, 185] and the Read-Rezayi \(Z_3\) state [187]. The NASS state can also be generalized to produce other candidate incompressible states by replacing the Halperin 221 state with \(\Psi_{\frac{n}{n+1}}^{[\pm n_1, \pm n_2]}\).

(iv) Moran et. al. [139] recently studied the Jain spin-singlet (JSS) wave function for fermions, which they argued contains topological \(d\)-wave pairing structure. We consider here its bosonic analog at \(\nu = 1\)

\[
\Psi_{JSS}^{1} \{\{z\}\} = \mathcal{P}_{\text{LLL}} \left[ \Phi_2(\{z\}) \prod_{i<j} (z_i^\dagger - z_j^\dagger)(z_i^\dagger - z_j^\dagger) \right] \tag{3.10}
\]

where \(\Phi_2\) is the wave function of two filled Landau levels. This does not belong to the \(\Psi_{\frac{n}{n+1}}^{[\pm n_1, \pm n_2]}\) states considered above, but follows from the parton construction of FQH states [84]. In this construction, each boson is viewed as the bound states of two fictitious species of fermions (partons), one of which carries spin while the other is spinless. The spinful fermions occupy the spin-singlet state at \(\nu = 2\) whereas the spinless ones occupy the fully spin-polarized state at \(\nu = 2\). The fermionic version of this state (obtained by multiplication by another full Jastrow factor) describes a spin-singlet incompressible state at \(\nu = 1/2\); it was introduced in Ref. [84] and considered as a possible candidate for the spin-singlet \(5/2\) FQH state [16, 17], but was abandoned when it was realized that the Coulomb \(5/2\) state is fully spin-polarized.
We will see below in Sec. 4.3 that this state is realizable for a 2-body interaction. This result is of interest because $\Psi^{JSS}_1$ is the simplest “parton” state that goes beyond the CF interpretation (all states of composite fermions admit a parton construction but the converse is not true). The excitations of this state are more complicated. One may naively expect that the low-lying energy levels can be obtained by creating excitations in either $\Phi_2$ or $\prod_{i<j}(z_i^\uparrow - z_j^\uparrow)(z_i^\downarrow - z_j^\downarrow)$ in Eq. (3.10). However, it turns out that neither of them gives a very accurate description of the excitations.

Many of the above wave functions involve $\Phi_n$, the Slater determinant wave function of $n$ filled LLs, on the right hand side. While $\Phi_n$ is uniquely defined for a compact geometry, where the number of single-particle states in each Landau level is finite, that is not the case in the disk geometry. For example, in the disk geometry $\Phi_2$ can be defined with $N_1$ particles in the lowest Landau level and $N_2$ particles in the second Landau with the constraints that $N = N_1 + N_2$. Different possible choices of $N_1, N_2$ complicates the analysis of the edge excitations of the states involving $\Phi_2$, as has been found to be the case for spin-polarized fermions at 2/5 [168, 187].

3.3 Models and Methods

We consider a bosonic system with two internal states in a rapidly rotating harmonic trap. These neutral particles experience forces in the rotating reference frame which mathematically has the same description as charged particles moving in a uniform magnetic field. We specialize to the case where single-particle cyclotron energy is much larger than the many-body gap, so the bosons can be treated as in the lowest Landau level only and effects due to Landau levels mixing are neglected. The number of particles, the number of spin-up particles and the number of spin-down particles are denoted using $N$, $N_\uparrow$ and $N_\downarrow$, respectively.

3.3.1 spherical and disk geometry

We will use the spherical geometry [72] for most of our calculations. The flux enclosed by the sphere is denoted as $2Q$, which is related to the numbers of particles
\( N \) and the filling factor \( \nu \) via \( 2Q = N/\nu - S_h \). The quantity \( S_h \) is called the “shift.” Sometimes there is an ambiguity when two states at different fillings “alias,” i.e., occur at the same flux. In such cases, it is important to study several values of \( N \) to draw unambiguous information. The compact spherical geometry is very convenient for studying the bulk properties of a FQH state, due to absence of edges. For studying the structure of edge excitations, there are two ways of proceeding. One can study either the states in the disk geometry, or the entanglement spectrum in the spherical geometry [116].

The single-particle eigenstates on a sphere are the so-called monopole harmonics [228]

\[
Y_{Qlm} = \frac{(2l + 1)}{4\pi} \frac{(l - m)!}{(l + m)!} \frac{(l + Q)!}{(l - Q)!} (-1)^{l-m} u^{Q+m} v^{Q-m} \\
\times \sum_{s=0}^{l-m} (-1)^s \binom{l - Q}{s} \binom{l + Q}{l - m - s} (u^* u)^s (v^* v)^{l-Q-s} \tag{3.11}
\]

where \( l = Q + n \) (\( n \) is the Landau level index) is the angular momentum, \( m \) is the \( z \) component of angular momentum, and \( \theta \) and \( \phi \) are the azimuthal and radial angles. The spinor coordinates \( u = \cos(\theta/2)e^{i\phi/2} \), \( v = \sin(\theta/2)e^{-i\phi/2} \).

### 3.3.2 lowest Landau level projection

When Eq. (3.4) and Eq. (3.5) are constructed on a sphere, the flux \( 2Q^* \) experienced by composite fermions, that is, the flux of the IQH states \( \Phi_{\pm n_i}(\{z^1\})\Phi_{\pm n_i}(\{z^1\}) \), is related to the actual flux by \( 2Q = 2Q^* + (N - 1) \). Once the IQH states are constructed using the above single-particle wave functions, we multiply them by the Jastrow factor \( J \) and then project the products to the LLL. An efficient Jain-Kamilla method [87] has been developed that applies to states of the form \( \mathcal{P}_{\text{LLL}} J^{2p}\Phi_n \), where the projected wave function can be constructed for rather large \( N \) without the need for expanding it in basis functions. This method requires \textit{even} exponent of \( J \) for technical reasons. In Ref. [31] this method was applied to spinless bosons, by writing \( \mathcal{P}_{\text{LLL}} J\Phi_n \) as \( J^{-1}\mathcal{P}_{\text{LLL}} J^{2}\Phi_n \). Unfortunately, this method does not work for spin-singlet states, because \( J^{-1}\mathcal{P}_{\text{LLL}} J^{2}\Phi_{n_1,n_1} \) is a singular, non-normalizable wave function, as \( \mathcal{P}_{\text{LLL}} J^{2}\Phi_{n_1,n_1} \) does not vanish when two particles
with opposite spins coincide. Therefore, we must evaluate the LLL projection by using its expansion in terms of the symmetric monomials for the spin-singlet states [48, 220]. The following identity of monopole harmonics discovered by Wu and Yang [219] are useful in the LLL projection

\[
Y_{Q_1l_1m_1}Y_{Q_2l_2m_2} = (-1)^{m_3-Q_3} \sum_{l_3} S(\{Q_i, l_i, m_i\}) Y_{Q_3l_3m_3}
\] (3.12)

where we have defined the following quantities

\[
S(\{Q_i, l_i, m_i\}) = (-1)^{l_1+l_2+l_3} \frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi}^{1/2}
\times F_{l_1l_2l_3}^{l_1l_2l_3} F_{Q_1Q_2-Q_3}
\]

\[
F_{l_1l_2l_3}^{l_1l_2l_3} = \frac{(-1)^{l_1-l_2-m_3}}{\sqrt{2l_3+1}} \langle l_1, m_1; l_2, m_2|l_3, -m_3 \rangle
\] (3.13)

\[
Q_3 = Q_1 + Q_2, \quad m_3 = m_1 + m_2 \quad \text{and} \quad \langle l_1, m_1; l_2, m_2|l_3, m_3 \rangle \quad \text{is the Clebsch-Gordon coefficient.}
\]

The computational time to perform the LLL projection grows factorially with the number of particles, since one must consider all possible permutations of the indices. As a result, \( N = 14 \) or 16 is the maximum number of particles that we can study in a reasonable amount of time.

### 3.3.3 exact diagonalization

Interaction between particles can be parametrized by the Haldane pseudopotential in the 2-body case and their generalizations in the 3-body case [44, 179]. We study Hamiltonians containing 2-body and 3-body interactions, denoted as \( H_2 \) and \( H_3 \), respectively:

\[
H_2 = \sum_\alpha \sum_{ij} c_{ij} [P_{ij}(\alpha, 1) + P_{ij}(\alpha, 0)]
\] (3.15)

\[
H_3 = \sum_{ijk} [P_{ijk}(0, 3/2) + P_{ijk}(0, 1/2)]
\] (3.16)

where \( P_{ij}(L, S) \) projects out a pair of particles \( i, j \) with relative angular momentum \( L \) and total spin \( S \), and \( P_{ijk}(L, S) \) projects out a triple of particles \( i, j, k \) with
relative angular momentum $L$ and total spin $S$. The natural interaction for bosons is the contact interaction, which corresponds, in units of $c_0$, to

$$H_2^{\text{con}} = \sum_{ij} [P_{ij}(0, 1) + P_{ij}(0, 0)]$$

(3.17)

This will be the interaction used unless otherwise stated. Non-zero values for $c_1$ and $c_2$ in $H_2$, and the 3-body Hamiltonian $H_3$ will also be used sometimes, to stabilize certain interesting states. Since the interaction is rotationally invariant and spin-independent, the energy eigenstates are also eigenstates of orbital angular momentum $\hat{L}^2$ [with eigenvalue $L(L + 1)$] and spin angular momentum $\hat{S}^2$ [with eigenvalue $S(S + 1)$]. In the figures shown below, the energy levels are labeled by their angular momentum and spin quantum numbers $L$ and $S$ and are also shifted horizontally according to their $S$ values for clarity.

To study edge excitations, we use the disk geometry. The Hamiltonian can also be represented using 2-body Haldane pseudopotentials

$$\tilde{H}_2 = \sum_\alpha \sum_{ij} \tilde{c}_\alpha [P_{ij}(\alpha, 1) + P_{ij}(\alpha, 0)] + \omega_c(\hat{L}_z - L_0)$$

(3.18)

where $\hat{L}_z$ is the $z$-component angular momentum operator and the term $\omega_c(\hat{L}_z - L_0)$ is due to a parabolic confinement potential whose strength is controlled by the parameter $\omega_c$. We choose the coefficients $\tilde{c}_\alpha$ to have the same values as their counterparts in the spherical geometry Hamiltonian and tune the coefficient $\omega_c$ to make sure that the state at angular momentum $L_0$ has the lowest energy, where the counting of edge excitations starts.

### 3.3.4 entanglement spectrum

In addition to comparing the wave functions with exact eigenstates obtained in finite systems, we also study the entanglement spectrum [116] in some cases, because it can provide additional insight into the physics of the FQH states. In particular, it has been found that the entanglement spectrum contains information about the edge excitations; specifically, entanglement spectrum can reproduce the counting of the edge states (which provides a method of study edge excitations in the spheri-
cal geometry). To obtain the entanglement spectrum for an incompressible ground state $|\Psi\rangle$, one divides the Hilbert space into two parts labeled as $A$ and $B$ and then decomposes the ground state as

$$|\Psi\rangle = \sum_{\alpha\beta} C_{\alpha\beta} |\Psi^A_{\alpha}\rangle \otimes |\Psi^B_{\beta}\rangle = \sum_i e^{-\xi_i/2} |\Psi^A_i\rangle \otimes |\Psi^B_i\rangle$$

(3.19)

where $|\Psi^A_{\alpha}\rangle$ and $|\Psi^B_{\beta}\rangle$ are two sets of basis states for $A$ and $B$, respectively. The second step is achieved through a singular value decomposition (SVD) of the matrix $C_{\alpha\beta}$, which also changes the basis states to $|\Psi^A_i\rangle$ and $|\Psi^B_i\rangle$. A plot of the “eigenvalues” $\xi_i$ versus the conserved quantum numbers in region $A$ comprises the entanglement spectrum. We shall calculate the “real space entanglement spectrum” [53, 168, 189] (RSES), where the cut is made along the equator and the southern hemisphere is chosen as $A$, with $N^A_\uparrow$ ($N^A_\downarrow$) spin-up (spin-down) particles. Due to the choice of cut, the levels in the RSES can be labeled by the $z$ component of the total angular momentum $L^A_z$ and the total spin quantum number $S^A$ of the particles in $A$. To compare the edge excitations with the RSES, we will calculate energy spectra on disk geometry when the edge counting cannot be predicted exactly. For example, the counting of edge excitations of the NASS state can be predicted in several ways and does not require exact diagonalization, but the counting of the edge excitations of JCF state and JSS state are more complicated.

3.4 Results and Discussions

3.4.1 $\nu = 2/3$ and $4/5$ states

(i) The Halperin 221 state is the unique exact zero energy state of the hard-core interaction $H_{2}^{\text{con}}$ at flux $2Q = 3N/2 - 2$. The quasihole states, obtained by adding flux, are also exact zero energy states of $H_{2}^{\text{con}}$, whose counting can be predicted in several ways and the wave functions are also known exactly [7, 55].

Exact solutions are not known for the neutral excitations and the quasiparticles, which do not have zero energy with respect to $H_{2}^{\text{con}}$. For these we use the trial wave functions $\Psi_{2/3}^{[1,1]} = \mathcal{P}_{\text{LLL}}[\Phi_1({\{z^\uparrow}\}})\Phi_1({\{z^\downarrow}\}})J({\{z\}})]$. The lowest energy neutral excitations correspond to a particle-hole excitation in one of the $\Phi_1$ factors. When
Figure 3.1. Energy spectra (lines) of the $\nu = 2/3$ state for the 2-body contact Hamiltonian $H^\text{con}_2$. The lines are colored according to their spin quantum numbers and are also shifted in the horizontal direction for clarity. The same conventions are used in all other figures. The crosses represent the energies of the wave functions $\Psi^{[1,1]}$ for the ground and excited states. The panels correspond to (a) $N^\uparrow = 4$, $N^\downarrow = 4$ and $2Q = 10$; (b) $N^\uparrow = 5$, $N^\downarrow = 5$ and $2Q = 13$; (c) $N^\uparrow = 5$, $N^\downarrow = 5$ and $2Q = 12$. The inset in (a) shows the color scheme for all panels. Panels (a) and (b) correspond to incompressible states where the uniform ground state has $L = 0$ and $S = 0$, and the excitations are neutral particle-hole pairs of composite fermions. Panel (c) corresponds to a system containing two quasiparticles; the low energy band contains all possible states of these quasiparticles.
Figure 3.2. Energy spectra of the $\nu = 4/5$ state for the Hamiltonian $H_2^{con}$. The crosses represent the energies of the wave functions $\Psi^{[2,2]}_{4/5}$. (a) $N_\uparrow = 4$, $N_\downarrow = 4$ and $2Q = 7$; (b) $N_\uparrow = 6$, $N_\downarrow = 6$ and $2Q = 12$; (c) $N_\uparrow = 5$, $N_\downarrow = 5$ and $2Q = 10$. The inset in (a) shows the color scheme for all panels. Panels (a) and (b) correspond to incompressible states where the uniform ground state has $L = 0$ and $S = 0$, and the excitations are neutral particle-hole pairs of composite fermions. Panel (c) corresponds to a system containing two quasiholes; the low energy band contains all possible states of these quasiholes.
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Table 3.1. Overlaps between the trial states $\Psi_{2/3}^{[1,1]}$ and $\Psi_{4/5}^{[2,2]}$ and corresponding exact eigenstates shown in Figs. 3.1 and 3.2. $L$ is the orbital angular momentum, $S$ is the spin quantum number, and “-” means that there is no trial state in that $(L, S)$ sector. The total number of linearly independent $(L, S)$ multiplets is given below each overlap. The same conventions are used in all other tables.
the flux is reduced by one unit, each $\Phi_1$ factor on the right hand side contains one particle in the second LL. We construct $L$ and $S$ eigenstates by taking appropriate linear combinations. Fig. 3.1 gives the energies (shown by crosses) of the trial wave functions of the neutral excitations in panels (a) and (b) and of quasiparticle excitations in panel (c). The overlaps between the trial wave functions and the exact eigenstates are shown in Table 3.1. These comparisons show that the CF theory provides an excellent description of the excitations of the 2/3 spin-singlet state.

(ii) The incompressible $\Psi_{4/5}^{[2,2]}$ state occurs at $2Q = 5N/4 - 3$. We find that the system at this flux value is incompressible for up to 12 particles as shown in panels (a) and (b) of Fig. 3.2. We have explicitly constructed the wave function $\Psi_{4/5}^{[2,2]} = \mathcal{P}_{LLL}[\Phi_2(\{z_i^\uparrow\})\Phi_2(\{z_i^\downarrow\})J(\{z\})]$ for the ground states and excitations. Their energies are shown by crosses in Fig. 3.2, and their overlaps with the corresponding exact states are shown in Table 3.1, which have excellent agreement. We note in passing that another candidate at $\nu = 4/5$ is a spin-singlet Gaffnian state [46], but it is likely to describe a gapless or critical state rather than an incompressible state since it is given by the conformal blocks of a non-unitary conformal field theory.

For the 2/3 state, the edge energy spectrum is trivial and it has been found that the counting of levels in RSES matches predictions [7, 55]. In contrast, the edge spectrum of the 4/5 state is expected to be complicated, containing several branches, because composite fermions occupy two $\Lambda$ levels. The studies of fermionic 2/5 state tell us that such structures can only be seen for a rather large number of particles [168, 186]. The systems studied here are too small to bring out the edge physics.

### 3.4.2 $\nu = 3/4$ and 3/2 states

The 3/4 state $\Psi_{3/4}^{[1,2]}$ occurs at $2Q = 4N/3 - 8/3$ and the 3/2 state $\Psi_{3/2}^{[1,-2]}$ occurs at $2Q = 2N/3 + 2/3$. These are spin-partially-polarized states. They are both derived from the spin-partially-polarized IQH state at $\nu^* = 3$, one with parallel flux attachment and the other with reverse flux attachment. Fig. 3.3 shows the energy spectra for the contact interaction $H_2^{\text{con}}$ at these two filling factors and their comparison with the trial wave functions for the ground state as well as neutral
Table 3.2. Overlaps between the trial states $\Psi_{3/4}^{[1,2]}$ and $\Psi_{3/2}^{[-1,-2]}$ and corresponding exact eigenstates shown in Fig. 3.3. The stars mark $(L, S)$ quantum numbers where the CF theory produces two independent states; the overlaps in these cases are defined as $\sqrt{\sum_{ij} \langle \Psi_i^E | \Psi_j^T \rangle^2}$ where the summation is over the lowest two exact states $|\Psi_i^E\rangle$ and trial states $|\Psi_j^T\rangle$ in the same $(L, S)$ sector. The total number of linearly independent $(L, S)$ multiplets is given below each overlap.

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Figure 3.3. (a) Energy spectrum of the $\nu = 3/4$ state for the Hamiltonian $H_2^{\text{con}}$ with $N_\uparrow = 3$, $N_\downarrow = 8$ and $2Q = 12$. The crosses represent the energies of the wave functions $\Psi_{3/4}^{[1,2]}$. (b) Energy spectrum of the $\nu = 3/2$ state for the Hamiltonian $H_2^{\text{con}}$ with $N_\uparrow = 4$, $N_\downarrow = 10$ and $2Q = 10$. The crosses represent the energies of the states $\Psi_{3/2}^{[-1,-2]}$. The insets show the color schemes for the panels.

excitations. Table 3.2 gives the overlaps of the trial states and exact states shown in Fig. 3.3. In some orbital and spin angular momentum sectors, there are two trial states and we define the overlap as $\sqrt{\sum_{ij} \left[ \langle \Psi_i^E | \Psi_j^T \rangle \right]^2}$ where the summation is over the lowest two exact states $|\Psi_i^E\rangle$ and trial states $|\Psi_j^T\rangle$. These results show that the actual $3/4$ state is very well described by the CF theory, whereas this theory is less accurate for $3/2$.

Note that the number of particles in each spin component is fixed (because the Hamiltonian $H_2^{\text{con}}$ conserves the $z$-component of spin), so only states with total spin $S \geq |N_\uparrow - N_\downarrow|/2$ may occur. (Should we allow the spins to flip, these spin-partially-polarized state will not be ground states.) It is interesting to note that the low energy part of the spectrum contains states with $S = |N_\uparrow - N_\downarrow|/2$, with the states with higher values of $S$ appearing at much higher energies. This feature
is nicely explained by the CF theory as follows. The 3/4 and 3/2 states map into [1, 2] and [−1, −2] of composite fermions, and the lowest energy excitations (without changing $S_z$) contain a single CF exciton either in the spin-up sector or in spin-down sector. The resulting states satisfy the Fock condition (all occupied states in the spin-up sector are definitely occupied in the spin-down sector, and therefore the wave function is annihilated upon further antisymmetrization), and thus represent states with $S = |S_z| = |N_\uparrow - N_\downarrow|/2$. To produce a state with $S > |N_\uparrow - N_\downarrow|/2$ one must consider CF configurations containing at least two CF excitons, which are expected to lie at higher energies.

3.4.3 $\nu = 4/3$ state

The filling factor $4/3$ has been considered [59,68] because it may provide a realization of the simplest NASS state $\Psi_{4/3}^{\text{NASS}}$. At the same time, the CF theory provides another candidate $\Psi_{4/3}^{[-2,-2]}$ here. It is therefore of interest to ask what kinds of interaction would favor these states. The states $\Psi_{4/3}^{[-2,-2]}$ and $\Psi_{4/3}^{\text{NASS}}$ occur at different shifts with $2Q = 3N/4 + 1$ and $2Q = 3N/4 - 2$, respectively, on the spherical geometry.

Let us first consider the 2-body interaction. For the contact interaction $H_2^{\text{con}}$, the spectrum for 12 particles at $2Q = 3N/4 + 1$ is shown in Fig. 3.4(a), and for $2Q = 3N/4 - 2$ in Fig. 3.6(a). The overlaps of trial states and exact eigenstates are shown in Table 3.3 and 3.4, respectively. Given that the JCF ground state has a higher overlap (0.985) than the NASS ground state (0.918) in spite of a larger Hilbert space (646 independent $L = S = 0$ multiplets as opposed to 79 for NASS), these comparisons suggest that the states $\Psi_{4/3}^{[-2,-2]}$ is favored for the contact interaction. Comparison is also shown for excitations.

To test the stability of the JCF and NASS states at filling factor $\nu = 4/3$, we further test their performances when changing the coefficients $c_\alpha$ for $\alpha = 1, 2$ in the Hamiltonian $H_2$. The results are shown in Tables 3.5. Both states remain good approximations for small values of $c_1$ and $c_2$, but are destroyed at large enough values for these parameters. We should emphasize that these numbers are not to be compared directly since the two states occur at different shift, and the dimensions of the subspaces with fixed $L$ and $S$ quantum numbers are different.
Table 3.3. Overlaps between the trial states $\Psi^{[-2,-2]}_{4/3}$ and corresponding exact eigenstates shown in Fig. 3.4. $L$ is the orbital angular momentum, $S$ is the spin quantum number, and “$-$” means that there is no trial state in that $(L, S)$ sector. The total number of linearly independent $(L, S)$ multiplets is given below each overlap.
Table 3.4. Overlaps between the NASS trial states $\Psi_{4/3}^{\text{NASS}}$ (with excitations created within a spinful bipartite CF representation) and corresponding exact eigenstates shown in Figs. 3.5 and 3.6. $L$ is the orbital angular momentum, $S$ is the spin quantum number, and “−” means that there is no trial state in that $(L, S)$ sector. The total number of linearly independent $(L, S)$ multiplets is given below each overlap.
Table 3.5. Comparing the JCF and NASS trial states at $4/3$ ($\Psi_{4/3}^{[-2,-2]}$ and $\Psi_{4/3}^{\text{NASS}}$, respectively) with the exact ground states at the corresponding flux ($2Q$) values as a function of pseudopotential parameters. The calculations are for $N\uparrow = 6$ and $N\downarrow = 6$ with respect to $c_1$ (columns) and $c_2$ (rows); we set $c_0 = 1$. The upper number in each block gives the overlap of $\Psi_{4/3}^{[-2,-2]}$ with the corresponding exact ground state. The lower number in each block gives the overlap of $\Psi_{4/3}^{\text{NASS}}$ with the corresponding exact ground state.
As mentioned previously, the NASS is the exact ground state for the 3-body contact interaction $H_3$. The energy spectra corresponding to the NASS shift are shown in Fig. 3.6 for this 3-body interaction. From the energy comparisons shown in this figure, and the overlaps shown in Table 3.4, the excitations are very well described by the trial wave functions which create CF excitations in individual
Figure 3.5. Energy spectra of the $\nu = 4/3$ state for the Hamiltonian $H_2^{\text{con}}$. The crosses represent the energies of the wave functions $\Psi_{\text{NASS}}^{4/3}$ obtained from the spinful bipartite CF theory. (a) $N_\uparrow = 6$, $N_\downarrow = 6$ and $2Q = 7$; (b) $N_\uparrow = 7$, $N_\downarrow = 7$ and $2Q = 8$; (c) $N_\uparrow = 8$, $N_\downarrow = 8$ and $2Q = 10$. The inset of panel (a) shows the color scheme for all panels.

factors of Eq. (3.9).

We have also compared the RSES of the exact NASS state and the 2-body ground state in Fig. 3.7. The RSES are similar, as would be expected from the reasonably high overlaps. We also show the energy spectrum in the disk geometry, which, however, does not has very similar structure as the RSES. In fact, the energy spectrum in Fig. 3.7(c) is better understood as reverse-flux-attached CF state, as
Figure 3.6. Energy spectra of the $\nu = 4/3$ state for the 3-body Hamiltonian $H_3$. The crosses represent the energies of the wave functions $\Psi_{\text{NASS}}^{4/3}$ obtained from the spinful bipartite CF theory. (a) $N_\uparrow = 6, N_\downarrow = 6$ and $2Q = 7$; (b) $N_\uparrow = 7, N_\downarrow = 7$ and $2Q = 8$. The inset of panel (a) shows the color scheme for both panels.

We note that any spectrum in Figs. 3.4 and 3.5 can be interpreted in two different ways. For example, the $N_\uparrow = 6, N_\downarrow = 6$ and $2Q = 7$ state in Fig. 3.4(a) can be thought of as excitations of $\Psi_{4/3}^{[-2,-2]}$, but here the NASS gives a satisfactory account of the exact spectrum. On the other hand, for Figs. 3.4(b) and 3.5(b), both interpretations work comparably well (although they predict different numbers of states), as seen from the overlaps in Tables 3.3 and 3.4.

Taking all of these results into account, while our studies do not rule out the NASS state, they suggest that the 4/3 ground state for the contact interaction is

described below in Sec. 3.4.4. We have not studied the RSES for $\Psi_{4/3}^{[-2,-2]}$ or the corresponding exact ground state. Since composite fermions occupy two $\Lambda$ levels in both spin sectors in the 4/3 state, we do not expect the RSES to give very useful information using the system sizes that are accessible to exact diagonalization or for which $\Psi_{4/3}^{[-2,-2]}$ can be explicitly generated.
Figure 3.7. RSES and edge excitations of the $\nu = 4/3$ NASS state. (a) $N_{\uparrow} = 8$, $N_{\downarrow} = 8$, $N_{\uparrow}^A = 4$, and $N_{\downarrow}^A = 4$, using the exact NASS state; (b) $N_{\uparrow} = 8$, $N_{\downarrow} = 8$, $N_{\uparrow}^A = 4$, and $N_{\downarrow}^A = 4$, using the ground state of the 2-body Hamiltonian $H_2$; (c) $N_{\uparrow} = 4$ and $N_{\downarrow} = 4$, energy spectrum on disk geometry of the Hamiltonian $\tilde{H}_2$ with confinement potential parameter $\omega_c = 0.4$. The inset of panel (a) shows the color scheme for all panels. The arrow in panel (c) indicates the ground state and the arrows in panel (a) and panel (b) show the corresponding levels in the RSES.
likely to be $\Psi_{4/3}^{-2,-2}$ with Abelian excitations.

It would be useful to compare these two candidate states in the torus geometry where they compete directly. Recently, composite fermion wave functions have been successfully constructed in the torus geometry [80] for spin-polarized state at filling factors $2/3$ (for 10 bosons) and $2/5$ (for 6 fermions). Generalizing this method to spinful cases could be very interesting, although the numerical implementation of such schemes is expected to be very difficult.

### 3.4.4 $\nu = 2$ state

We consider the state $\Psi_{2}^{-1,-1} = P_{\text{LLL}}[\Phi_{-1}(\{z^{\uparrow}\})\Phi_{-1}(\{z^{\downarrow}\})J(\{z\})]$, obtained from the $\nu^{*} = 2$ spin-singlet state with reverse flux attachment. This state is analogous to the spin-singlet 2/3 state of fermions [220]. It has attracted special interest recently as an example of symmetry protected bosonic integer topological states [32, 126, 174], which refer to states with no topological order (i.e., Abelian or non-Abelian fractional excitations) but are still topologically non-trivial.

As an initial test, we find that the ground state of the 2-body contact interaction $H_{2}^{\text{con}}$ at the $2Q$ values corresponding to $\Psi_{2}^{-1,-1}$ indeed has $L = 0$ and $S = 0$ for up to 18 particles. Fig. 3.8 shows the energy spectra. [For 18 particles the dimension of the Fock space is very large (with 58,130,756 states in the $L_{z} = S_{z} = 0$ sector), and producing eigenstates by the Lanczos method is computationally time consuming; we have only obtained the lowest few eigenstates to confirm that the ground state has $L = S = 0$ and is separated from the excitations by a reasonable gap.] The overlaps of trial states and exact eigenstates for $N_{\uparrow} = N_{\downarrow} = 6$ and $N_{\uparrow} = N_{\downarrow} = 7$ are shown in Table 3.6. For 12 (14) particles, the exact ground state has overlap 0.943 (0.888) with $\Psi_{2}^{-1,-1}$; for 16 particles we are not able to generate the trial state as explained in Sec. 4.2 B. We also study the stability of the state under addition of longer-range interaction. Table 3.7 shows the evolution of overlaps between $\Psi_{2}^{-1,-1}$ and exact ground state for a range of values of $c_1, c_2$ (with $c_0 = 1$), demonstrating that $\Psi_{2}^{-1,-1}$ remains a good description of the ground state for a wide range of parameters. We should point out that the trial wave functions for excitations are not as accurate as the ground states as one can see from Fig. 3.8 and Table 3.6.

We also study the RSES and the edge spectrum. The CF theory implies a
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Table 3.6. Overlaps between the trial states $\Psi_2^{[-1,-1]}$ and corresponding exact eigenstates shown in Fig. 3.8. $L$ is the orbital angular momentum, $S$ is the spin quantum number, and “−” means that there is no trial state in that $(L, S)$ sector. The total number of linearly independent $(L, S)$ multiplets is given below each overlap.
Table 3.7. Comparison of $\Psi_2^{[-1,-1]}$ at $\nu = 2$ and $\Psi_1^{\text{JSS}}$ at $\nu = 1$ for $N_\up = N_\down = 7$ with the corresponding exact ground states as a function of pseudopotential parameters. We set $c_0 = 1$ and vary $c_1$ (columns) and $c_2$ (rows). The upper number in each block gives the overlap of $\Psi_2^{[-1,-1]}$ with the corresponding exact ground state and the lower number of $\Psi_1^{\text{JSS}}$ with the corresponding exact ground state.

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</table>
Figure 3.8. Energy spectra of the $\nu = 2$ ground states for the 2-body Hamiltonian $H_2^{\text{con}}$. The cross represents the energy of the wave functions $\Psi_{[-1,-1]}$. (a) $N_\uparrow = 6$, $N_\downarrow = 6$ and $2Q = 6$; (b) $N_\uparrow = 7$, $N_\downarrow = 7$ and $2Q = 7$; (c) $N_\uparrow = 8$, $N_\downarrow = 8$ and $2Q = 8$. The inset of panel (a) shows the color scheme for all panels.

behavior similar to that of the $\nu = 2/3$ spin-singlet fermionic state, which has been studied in Refs. [138] and [224]. In particular, one expects a backward-moving mode that carries spin but no charge, and a forward-moving mode that carries charge but no spin [138,224]. We show in Fig. 3.9 the RSES of the ground state of a bosonic system at $\nu = 2$ with $N_\uparrow = N_\downarrow = 8$ particles and the edge excitations of
a system with $N_{\uparrow} = N_{\downarrow} = 4$ particles on a disk. (We add a parabolic confinement potential of an appropriate strength in the disk geometry, to ensure that the ground state has the angular momentum given by $\Psi_2^{-1,-1}$. ) We see a strong similarity between the RSES and the spectrum of edge excitations on the disk. In particular, the RSES nicely captures the backward-moving mode marked by the arrows in Fig. 3.9. The counting of states for the backward-moving modes is also consistent with that found for the $\nu = 2/3$ spin- unpolarized fermionic states [138, 224]. The forward-moving mode is not clearly identifiable in both the RSES and the disk edge spectrum, as was also the case for spin-singlet 2/3 state [138, 224]; this can be understood by noting that the velocity of this mode is sufficiently large that it rapidly merges into the continuum for the small systems accessible to our study.
Figure 3.10. Energy spectra of the $\nu = 1$ ground states for the 2-body Hamiltonian $H_2$ with $c_0 = 1$, $c_2 = 0.3$ and all other $c_\alpha = 0$ for $\alpha \neq 0, 2$. The crosses represent the energies of the wave functions $\Psi_{1^{\text{SS}}}$. (a) $N_\uparrow = 6$, $N_\downarrow = 6$ and $2Q = 9$; (b) $N_\uparrow = 7$, $N_\downarrow = 7$ and $2Q = 11$. The inset of panel (a) shows the color scheme for both panels.

The incompressibility at $\nu = 2$ for bosons occurs because of interactions between them, and is therefore closer to the FQH (rather than the IQH) of fermions. One may ask what is the charge of the excitations. Identifying an isolated CF particle or CF hole in one of the factors of $\Phi_1$, it is straightforward to see that the charge excess associated with it is equal to a unit charge.

3.4.5 $\nu = 1$ state

We now explore the validity of $\Psi_{1^{\text{SS}}}^{1\text{SS}}(\{z\})$. This state is an excellent description of the ground state at filling factor $\nu = 1$ with $2Q = N - 4$ if some amount of $c_2$ interaction is turned on, as shown Fig. 3.10. The evolution of overlap between trial states and exact ground states with the coefficients $c_1$ and $c_2$ of the Hamiltonian $H_2$ is shown in Table 3.7.

It is natural to construct wave functions for the excitations of $\Psi_{1^{\text{SS}}}^{1\text{SS}}(\{z\}) = \ldots$
Figure 3.11. RSES and edge excitations of the $\nu = 1$ state. (a) RSES for $\Psi_{1}^{\mathrm{ISS}}$ for $N_{\uparrow} = 7$, $N_{\downarrow} = 7$, $N_{\uparrow}^{A} = 3$, and $N_{\downarrow}^{A} = 3$. (b) RSES for the ground state of the 2-body Hamiltonian $H_{2}$ for $N_{\uparrow} = 7$, $N_{\downarrow} = 7$, $N_{\uparrow}^{A} = 3$, and $N_{\downarrow}^{A} = 3$; the parameters of the Hamiltonian are $c_{0} = 1$, $c_{2} = 0.3$ and all other $c_{\alpha} = 0$ for $\alpha \neq 0, 2$. (c) Energy spectrum on disk for $N_{\uparrow} = 3$ and $N_{\downarrow} = 3$ for the Hamiltonian $\tilde{H}_{2}$ with $\tilde{c}_{2} = 0.3$ and the confinement potential parameter $\omega_{c} = 0.4$. The inset of panel (a) shows the color scheme for all panels. The arrows in (c) indicate the states obtained with four different choices for $\Phi_{2}$ in the wave function Eq. (3.10), which are $[3,3]$, $[4,2]$, $[5,1]$ and $[6,0]$ (from left to right). The arrows in (a) and (b) show the corresponding levels in the RSES, which nicely match the starting points of various edge branches.
\[ \mathcal{P}_{\text{LLL}}[\Phi_2(\{z\}) \prod_{i<j}(z_i^\dagger - z_j^\dagger) \prod_{i<j}(z_i^\dagger - z_j^\dagger)] \] by analogy to excitations of either the factor \( \Phi_2 \) or one of the two Jastrow factors on the right hand side. We have constructed such wave functions for the excited states, but neither of them gives very accurate description of the excitations.

We also study the RSES at \( \nu = 1 \). Fig. 3.11 shows the RSES of the JSS wave function and the exact ground state wave function for a certain choice of parameters \((c_0 = 1, c_2 = 0.3 \text{ and all other } c \text{'s are set to zero})\) for 14 particles. The two have similar low-lying levels. For many trial wave functions that are exact zero energy solutions of certain simple pseudopotential Hamiltonians, such as the Laughlin or Moore-Read wave functions, the entanglement spectrum contains only universal levels, i.e., all levels represent edge excitations. That, however, is not true in general. For electronic systems, the RSES of the exact Coulomb eigenstates at \( 1/3 \) or \( 5/2 \) contain “non-universal” levels, as is also true of the either the exact states at \( n/(2n + 1) \) or the JCF wave functions for those states. The trial state \( \Psi_1^{\text{JSS}}(\{z\}) \) also has many non-universal levels as it is not the exact zero energy state of a simple pseudopotential Hamiltonian and its construction requires LLL projection. The RSES of the exact state contains even more “non-universal” levels.

We also show the edge excitation spectrum on disk geometry in Fig. 3.11, and some similarities between the RSES and edge spectrum can be seen even for such a small system. A noteworthy feature is that there are several branches of edge excitations, and the starting points of these branches (indicated by arrows in Fig. 3.11) match nicely in both the RSES and the edge spectrum. [Note that the minimum value of angular momentum in (a) and (b) is \(-33\) while the minimum value in (c) is 0, so the positions of the arrows in (a) and (b) match exactly with those in (c) if the angular momentum values in (a) and (b) are relabeled by adding \(33\).] The starting points of edge excitations can be simply predicted using parton method: they correspond to different choices for the number of particles \([N_1, N_2]\) in the two \( \Lambda \) levels in the \( \Phi_2 \) part of Eq. (3.10), given in the figure caption. (While the starting points of the edge branches are identifiable, they quickly spread and merge into the non-universal part, making an identification of the edge states difficult.) The existence of multiple branches in the edge excitation spectrum and the RSES have been observed before for spin-polarized fermionic \( 2/5 \) state [168, 186], which is also due to the appearance of a \( \Phi_2 \) factor in the trial wave functions. In short,
the RSES and edge studies provide support to the identification of the exact state with \( \Psi_1^{\text{JSS}}(\{z\}) \), and, in particular, bring out features that can be understood by analogy to two filled ALs of composite fermions.

### 3.5 Conclusion

We have carried out an extensive study of quantum Hall effect for two-component bosons, studying a number of candidate states at fractional as well as integral fillings. Here is a summary of our findings:

(i) We have shown that for 2/3 and 4/5, the wave functions \( \Psi_{2/3}^{[1,1]} \) and \( \Psi_{4/5}^{[2,2]} \) provide an accurate representation of the spin singlet states of the contact interaction, for the ground state as well as excitations.

(ii) We have also considered partially polarized states at 3/4 and 3/2. For the former the state \( \Psi_{3/4}^{[2,1]} \) provides an accurate description of the ground state and excitations. For 3/2, \( \Psi_{3/2}^{[2,-2]} \) is not accurate.

(iii) For \( \nu = 4/3 \) we consider two candidates, \( \Psi_{4/3}^{[-2,-2]} \) and \( \Psi_{4/3}^{\text{NASS}} \), which have Abelian and non-Abelian excitations, respectively. Previous works [59, 68] suggested that the NASS state is realized at this filling factor. We find, from a direct comparison with the exact solution, that \( \Psi_{4/3}^{[-2,-2]} \) is more likely for the 2-body contact interaction.

(iv) For \( \nu = 4/3 \), the NASS state has been known to be the exact ground state for a 3-body interaction. We find that the exact excited states of this 3-body interaction correspond to CF excitations in the individual factors, confirming a spinful bipartite CF structure for this state.

(v) For \( \nu = 2 \), \( \Psi_2^{[-1,-1]} \) provides an accurate representation of the exact ground state for in certain parameter range of the 2-body Hamiltonian. The trial wave functions for the excitations are less accurate. The RSES and the disk energy spectrum provide a consistent description of the edge structure, both nicely displaying a backward-moving edge mode (which is similar to that found previously for the fermionic 2/3 spin-singlet state [138, 224]).

(vi) For \( \nu = 1 \), the JSS state accurately represents the ground state for a 2-body interaction that contains terms beyond contact interaction. The RSES and edge excitation studies provide further confirmation of the validity of \( \Psi_1^{\text{JSS}} \), and,
in particular, demonstrate the existence of several edge branches, which are fully consistent with the expectation from an underlying two-filled AL state. Our trial wave functions are not very accurate for the excitations.

(vii) We note a systematic effect as a function of the filling factor: the agreement between the CF and the exact spectra becomes worse with increasing filling factor. There are two possible reasons for that. One, the JCF wave functions of the states with large fillings, e.g. $\nu = 3/2, 4/3$ and 2, namely $\Psi_{3/2}^{[-2,-1]}$, $\Psi_{4/3}^{[-2,-2]}$ and $\Psi_{2}^{[-1,-1]}$, all require reverse flux attachment. We have found that for spinful particles, the wave functions involving reverse flux attachment are less accurate than those with parallel flux attachment. Two, from general arguments one expects that bosons at very high fillings are not in the FQH regime, because vortex lattice or other weakly interacting states may be preferred energetically.
Chapter 4

Adiabatic Continuity between Hofstadter and Chern Insulator States

This chapter is a reproduction of the published paper Phys. Rev. B 86, 165129 by the present author in collaboration with J. K. Jain and Kai Sun. It has been modified slightly to fit into this thesis. In this chapter, we show that the topologically nontrivial bands of Chern insulators are adiabatic cousins of the Landau bands of Hofstadter lattices. We demonstrate adiabatic connection also between several familiar fractional quantum Hall states on Hofstadter lattices and the fractional Chern insulator states in partially filled Chern bands, which implies that they are in fact different manifestations of the same phase. This adiabatic path provides a way of generating many more fractional Chern insulator states and helps clarify that nonuniformity in the distribution of the Berry curvature is responsible for weakening or altogether destroying fractional topological states.

4.1 Introduction

The phenomena of integer and fractional quantum Hall (FQH) effects have motivated remarkable developments. Of particular significance in this context is the topological interpretation of these effects. Thouless, Kohmoto, Nightingale and den Nijs [195] considered electrons in a periodic lattice exposed to a magnetic field,
and showed that the Hall conductance of a filled band is related to the first Chern number $C$. Specifically, the Bloch wave function for a magnetic unit cell has the form

$$|\Psi^m_n(k)\rangle = e^{ikr}|u^m_n(k)\rangle$$  \hspace{1cm} (4.1)$$

where $k$ is the wave vector in the first Brillouin zone, and $n$ is the band index. One defines the non-Abelian Berry connection as

$$A^{mn}_\mu(k) = i\langle u^m(k)|\partial_\mu|u^n(k)\rangle$$  \hspace{1cm} (4.2)$$

where $\partial_\mu$ is the shorthand notation for $\partial/\partial k_\mu$. The Berry curvature $F$ is then defined as

$$F_{\mu\nu}(k) = \partial_\mu A_{\nu} - \partial_\nu A_{\mu} + i [A_{\mu}, A_{\nu}]$$  \hspace{1cm} (4.3)$$

Thouless et al. showed that the Hall conductance is given, in units of $e^2/h$, by the Berry curvature integrated over the Brillouin zone:

$$\sigma_H = \frac{1}{4\pi} \int_{BZ} d^2k \epsilon_{\mu\nu} \text{Tr} [F_{\mu\nu}(k)] = C$$  \hspace{1cm} (4.4)$$

where the trace is over the occupied bands. The Chern number is a topological index provided the sum is over filled bands and the Fermi level lies in a gap. If there is only one occupied band, as we will assume below in this paper, the above expressions simplify to

$$A_\mu(k) = i\langle u(k)|\partial_\mu|u(k)\rangle \quad F(k) = \partial_1 A_2 - \partial_2 A_1 \quad C = \frac{1}{2\pi} \int_{BZ} d^2k F(k)$$  \hspace{1cm} (4.5)$$

This gives a topological interpretation of the quantized Hall conductance and also clarified that the essential property of a Landau level that distinguishes it from an ordinary band is its nonzero Chern number. Subsequently, Haldane [74] showed that a uniform magnetic field is not required to produce bands with nonzero Chern numbers. For this purpose he constructed an explicit model of an electron hopping on the honeycomb (graphene) lattice, with complex hopping matrix elements; this model has no net magnetic field (although it has staggered magnetic field) but produces bands with nonzero Chern numbers in certain regions of the parameter space. This system has integrally quantized Hall conductance in the absence of
a uniform magnetic field. Systems with bands of nontrivial topology (nonzero Chern number) in the absence of a uniform magnetic field are now called “Chern insulators,” to distinguish them from the Landau levels that occur in the presence of a uniform magnetic field. A number of other models have been proposed for Chern insulators, some of which have nearly flat bands [143, 192, 193].

The next natural question is whether Chern bands can also support FQH-like states; that is incompressible states in a partially filled Chern band with a fractionally quantized Hall conductance. Such states will obviously require interactions, and have been dubbed fractional Chern insulator (FCI) states. Exact numerical diagonalizations have demonstrated FCI states at filling factors \( \nu = 1/3 \) [163, 175, 222], 1/2 [20], 2/5 and 3/7 [122] for fermions, and 1/2 [206], 1 [207], and 2/3 [122] for bosons. These states require specific forms of interaction, and sometimes fine tuning of parameters. Trial wave functions for FCIs have been proposed [125, 158, 223]. Flat-band models with \( C > 1 \) have also been constructed and produce strongly-interacting topological states [120, 197, 205, 208, 228].

While the FCI states appear similar to the FQH states found in the lowest Landau level, no direct connection between them has yet been established. It remains unclear why some fractions occur while others do not, and what is the role of lattice symmetry and the type and range of the interaction in establishing various FCI states. It also remains unclear to what extent the extensive physics of the FQH effect and composite fermions is possible in Chern insulators. Progress in this direction has been made by Murthy and Shankar [142], who exploit the modified algebra of the density operator projected into the lowest Chern band to motivate composite fermion physics.

We address below this issue by demonstrating an adiabatic continuity between the ordinary quantum Hall states in a Landau level and the corresponding states in a Chern insulator. Because the latter are defined on a lattice, we work with a lattice model of electrons in a uniform magnetic field. This problem of Bloch electrons in a magnetic field was studied in a number of papers, including those by Peierls [155], Harper [78] Wannier [204], Azbel [8], and Hofstadter [81], with the last article presenting the band structure in a pictorially appealing form that is now known as the Hofstadter butterfly. For appropriately chosen flux per plaquette, the low-lying Bloch bands of this system are essentially Landau levels; they approximate
Landau levels of the continuum very accurately for a sufficiently fine lattice. We call them “Hofstadter bands,” and the filled band states “Hofstadter insulators.” For a given Hofstadter lattice, not all FQH states of the continuum will occur, and which ones survive is an interesting problem in its own right, but will not be addressed in this article (some work along these lines can be found in the literature [136,183]). However, we can certainly construct a Hofstadter lattice that approximates the continuum arbitrarily closely, by taking the flux per plaquette to be sufficiently small, and thus it produces all of the quantum Hall states seen in continuum. (Strictly speaking, the electrons in GaAs quantum wells are not in a continuum but feel the periodic potential of the lattice.) We will study a possible adiabatic connection between the quantum Hall states on a Hofstadter lattice and the corresponding states in a Chern insulator.

An intuitive understanding for why an adiabatic connection between a Chern and a Hofstadter lattice may exist can be gained by noting that a Hofstadter insulator in a uniform magnetic field can be transformed into a Chern insulator in zero net magnetic field by a simple gauge transformation. For a Hofstadter lattice, the total magnetic field passing through each magnetic unit cell is $2q\pi$ ($q$ is an integer). Here and below, one flux quantum is defined as $\phi_0 = 2\pi\hbar c/e = 2\pi$ in units with $\hbar = c = e = 1$. Let us now insert a $-2q\pi$ flux at an arbitrary point in each magnetic unit cell to produce a new problem, called Hofstadter’ (“Hofstadter prime”). The insertion of the $-2q\pi$ flux in a tight-binding model, however, is simply a gauge choice that has no physical consequence, and hence leaves all properties of the system unchanged: the energy bands of the Hofstadter’ lattice are identical to those of the original Hofstadter lattice, and the eigenfunctions of the two are related by a gauge transformation. In particular, the bands of the new lattice continue to have nonzero Chern numbers. At the same time, if we treat the (enlarged) magnetic unit cell as our unit cell, then the total magnetic field through it is zero. The Hofstadter’ lattice is thus a Chern insulator. (In fact, this Chern insulator has flat bands and uniform Berry curvature.) Every Hofstadter insulator thus has a corresponding Chern insulator with identical properties. This implies that all of the physics of FQH effect and composite fermions is, in principle, possible for Chern insulators, provided that we allow Chern insulators with a sufficiently complex unit cell.
For simple Chern lattices, not all FQH states occur. In what follows, we consider certain previously introduced Chern insulator models, construct for each a Hofstadter’ lattice whose magnetic unit cell coincides with the unit cells of the Chern insulator, and show, using exact numerical methods, that the familiar FQH states at filling factors 1/3 and 1/2 of the Hofstadter’ model adiabatically evolve into the corresponding FCI states in the presence of appropriate repulsive interactions. (Because Hofstadter’ lattice is trivially related to the Hofstadter lattice, we will dispense with the prime below.) We show that not only does the ground state evolve in this manner, but so do the quasiholes and the entanglement spectra, lending further credence to such an adiabatic relationship. This demonstrates that the origin of these states is governed by the same underlying physics. Furthermore, this adiabatic connection also enables us to investigate the role of the Berry curvature distribution in the momentum space. We find that nonuniformity in the distribution of the Berry curvature weakens, and can even destroy, FQH states. Our results show that such nonuniformities effectively translate into an enhancement of the residual interaction between composite fermions, and as a result can eliminate states of the sequence $p/(2p \pm 1)$ with relatively small gaps (all these fractions would occur for noninteracting composite fermions). Nonetheless, several FQH states are surprisingly robust to nonuniformities of the Berry curvature.

The paper is organized as follows. We present two single-particle tight-binding models with topologically nontrivial lowest bands in Sec. 5.1. The FQH states on these lattice models with appropriately chosen interactions are studied in Sec. 5.2. We conclude with a discussion of the implications of our results in Sec. 5.3. Since the posting of the first version of this work as arXiv:1207.4439v1, some new results [110, 121, 169] have appeared, which are also discussed in Sec. 5.3.

4.2 Lattice Models and Integer Quantum Hall Effect

We consider two popular models for Chern insulators: the checkerboard and the kagome lattices. In either case, our goal is to write a more general model that extrapolates between a Hofstadter lattice and a Chern insulator lattice. For this
purpose we add many more lattice sites to the Chern insulator lattice to create a Hofstadter lattice, and arrange the flux per plaquette so that the Hofstadter lattice has the same magnetic unit cell as the Chern insulator being considered, and also has a net zero magnetic field passing through the magnetic unit cell. With this arrangement both the lattices have the same symmetries (although they have different numbers of bands, because they have different numbers of lattice sites in a unit cell) and it is sensible to envision an adiabatic evolution from one to the other. We first study the single-particle band structures to demonstrate an adiabatic connection between a Landau level and a Chern band for the two lattice models mentioned above.

4.2.1 square-checkerboard hybrid lattice

The checkerboard lattice model was considered in Refs. [192] and [175]. It supports a topologically nontrivial band in the presence of nearest-neighbor, next-nearest-neighbor, and next-next-nearest-neighbor hopping terms, with the nearest neighbor hopping carrying a nonzero phase. By appropriate choice of parameters, the lowest band can be made very nearly flat, which is an important consideration for the discussion of the FCI states, which require that the interaction energy dominate the kinetic energy. The checkerboard lattice is shown in Fig. 4.1(a) by the encircled dots, with its two sublattices marked by blue and red rectangles. The checkerboard Hamiltonian \( H_{cb} \) is given by

\[
H_{cb} = -t_1 \sum_{\langle ij \rangle} e^{i\phi_{ij}} c_i^\dagger c_j - t_2 \sum_{\langle\langle ij \rangle \rangle} s_{ij} c_i^\dagger c_j - t_3 \sum_{\langle\langle\langle ij \rangle \rangle \rangle} c_i^\dagger c_j + \text{h.c.}
\]  

(4.6)

where \( t_1 = 1 \), \( t_2 = 1 - \sqrt{2}/2 \), \( t_3 = (\sqrt{2} - 1)/2 \), \( \phi_{ij} \) is the phase acquired during hopping between nearest neighbors \( \langle ij \rangle \), \( s_{ij} \) is positive (negative) one for next-nearest-neighbor hoppings between sites encircled by blue (red) rectangles, and \( \langle\langle\langle ij \rangle \rangle \rangle \) denotes next-next-nearest neighbors.

As shown in Fig. 4.1(a), we embed the checkerboard lattice inside a square Hofstadter lattice (all black dots) with only nearest neighbor hopping. It is assumed that each square of this lattice has a magnetic flux \( 2\pi/n \) passing through it (\( n = 16 \) for Fig. 4.1(a)), with the exception of one plaquette (indicated with
Figure 4.1. Lattice model and band structure for the square-checkerboard lattice model. Panel (a) shows the lattice structure. The black dots shows the lattice sites in a Hofstadter lattice with flux $2\pi/16$ through each plaquette. The dashed black square marks a magnetic unit cell containing 16 sites. The sites marked by the dashed rectangles form a checkerboard lattice where the two different orientations of the rectangles represent two sublattices. The Hamiltonian $H_{\text{sq-cb}}(R)$ in Eq. (4.7) interpolates between the Hofstadter and Chern insulator limits as $R$ is varied from 0 to 1. Panels (b-d) show the band structures at three values of $R$ (0, 0.5 and 1, respectively) along the contour $\Gamma \rightarrow M \rightarrow X \rightarrow \Gamma$ in the momentum space. In panel (b), flat Landau levels carry Chern number $C = 1$ while the two non-flat bands at the middle have a total $C = -14$. In panel (d), the top and bottom bands of the checkerboard model have nontrivial Chern numbers $C = \mp 1$.

a star in Fig. 4.2) which has an additional $2\pi$ flux passing through it in the opposite direction (so the net flux is $-(15/16)2\pi$ through this plaquette), so as to make the total flux through the magnetic unit cell equal to zero. (In other words, the lattice is what we had called Hofstadter'). The hopping matrix elements are complex, with our choice of phases shown in Fig. 4.2. The phases are chosen to obey periodic boundary conditions, but that does not fix them uniquely; we further impose the convention that the phase coming up to a site is the same as the phase going out of it toward right, as shown in the figure. It is straightforward to verify
Figure 4.2. Phases of the square lattice Hofstadter model. The numbers and arrows indicate the hopping phases along the bonds in units of $\pi$, and the star marks the plaquette where a $-2\pi$ flux is inserted.

Figure 4.3. Square-checkerboard hybrid lattice (a) The single-particle band gap (top curve) and bandwidth (lower curve) as a function of $R$. The flatness ratio (band gap over bandwidth) is shown as the inset. (b) The Berry curvature at different $k$-points with $R = 0.0$, 0.5 and 1.0 (dotted, dashed, and solid lines, respectively).
that the phases correspond to a flux of $2\pi/16$ through each square, except for the starred one which has an additional flux of $-2\pi$ through it. The Hamiltonian of this Hofstadter model on square lattice is denoted by $H_{sq}$.

We note that we could have taken a finer lattice with a larger number ($n$) of squares per unit cell; Landau levels are recovered in the weak lattice limit of $n \gg 1$. For our purposes the current choice with 16 squares in a magnetic unit cell will suffice, as seen below in the explicit numerical calculations. It is also noteworthy that if we only had the Hofstadter lattice (no checkerboard lattice), then we could have chosen a linear magnetic unit cell with 16 sites along a single line; this magnetic unit cell arises naturally in the Landau gauge, and was the choice made by Hofstadter. However, our objective of matching the Hofstadter problem to the checkerboard problem forces us to choose identical unit cells for both of them. We are not able to explicitly write a real space gauge potential that would produce the desired phases for the hopping matrix elements, but an appropriate gauge choice is made directly at the level of the phases of the hopping matrix elements.

Having defined the checkerboard and the Hofstadter Hamiltonians separately, we now define an interpolating Hamiltonian

$$H_{sq-cb}(R) = (1-R)H_{sq} + RH_{cb},$$  \hspace{1cm} (4.7)

which evolves continuously from Hofstadter to checkerboard as $R$ increases from 0 to 1. This interpolation scheme selects one specific path in the parameter space connecting the Hofstadter and the checkerboard Hamiltonians. We shall see that this path will suffice for demonstrating adiabatic continuity for many situations. By Fourier transformation, the above Hamiltonian can be converted into its momentum-space form $H_{sq-cb} = \sum_k \sum_{\alpha\beta} c_k^\dagger c_k H_{0\alpha\beta}^{sq-cb}(k)$. We list below the nonzero components of $H_{0\alpha\beta}^{sq-cb}$ (dropping the subscript “sq-cb” for simplicity)

$${H^0}^1 = \exp(ik_x/4)$$  \hspace{1cm} \(H^0)^0 = \exp(-i(k_x/4 + 3\pi/80))

$${H^0}^4 = \exp(i(k_x/4 + 3\pi/80))$$  \hspace{1cm} \(H^0)^{12} = \exp(-i k_y/4)

$${H^1}^{12} = \exp(i(k_x/4 + \pi/16))$$  \hspace{1cm} \(H^1)^{15} = \exp(i(k_y/4 + \pi/5))

$${H^1}^{13} = \exp(-i(k_y/4 + \pi/16))$$  \hspace{1cm} \(H^2)^{23} = \exp(i(k_x/4 + 3\pi/40))$$
\[ H^{26} = \exp(i(k_x/4 + 37\pi/80)) \quad H^{2,14} = \exp(-i(k_y/4 + 3\pi/40)) \]
\[ H^{37} = \exp(i(k_y/4 + 39\pi/40)) \quad H^{3,15} = \exp(-i(k_y/4 + 3\pi/80)) \]
\[ H^{45} = \exp(i(k_x/4 + 3\pi/80)) \quad H^{47} = \exp(-i(k_x/4 + 39\pi/40)) \]
\[ H^{48} = \exp(i(k_y/4 + 3\pi/40)) \quad H^{56} = \exp(i(k_x/4 + \pi/5)) \]
\[ H^{59} = \exp(i(k_y/4 + 19\pi/80)) \quad H^{67} = \exp(i(k_x/4 + 37\pi/80)) \]
\[ H^{6,10} = \exp(i(k_y/4 + 2\pi/5)) \quad H^{7,11} = \exp(i(k_y/4 + 37\pi/80)) \]
\[ H^{89} = \exp(i(k_x/4 + 3\pi/40)) \quad H^{8,11} = \exp(-i(k_x/4 + 37\pi/80)) \]
\[ H^{8,12} = \exp(i(k_y/4 + \pi/16)) \quad H^{9,10} = \exp(i(k_x/4 + 19\pi/80)) \]
\[ H^{9,13} = \exp(i(k_y/4 + 7\pi/40)) \quad H^{10,11} = \exp(i(k_x/4 + 2\pi/5)) \]
\[ H^{10,14} = \exp(i(k_x/4 + 19\pi/80)) \quad H^{11,15} = \exp(i(k_y/4 + \pi/5)) \]
\[ H^{12,13} = \exp(i(k_x/4 + \pi/16)) \quad H^{12,15} = \exp(-i(k_x/4 + \pi/5)) \]
\[ H^{13,14} = \exp(i(k_x/4 + 7\pi/40)) \quad H^{14,15} = \exp(i(k_x/4 + 19\pi/80)) \]
\[ H^{44} = -2t_2 \left[ \cos(k_x + 7\pi/40) + \cos(k_y - 47\pi/40) \right] \\
+ 2t_3 \left[ \cos(k_x - k_y + 47\pi/20) + \cos(k_x + k_y) \right] \\
H^{4,14} = -t_1 \left[ \exp(i(k_x/2 + k_y/2 + 3\pi/10)) + \exp(i(k_x/2 - k_y/2 + 79\pi/40)) \right] \\
+ t_1 \left[ i(-k_x/2 + k_y/2 - 3\pi/8) + \exp(i(-k_x/2 - k_y/2 + 3\pi/10)) \right] \\
H^{14,14} = -2t_2 \left[ \cos(k_x + 47\pi/40) + \cos(k_y - 7\pi/40) \right] \\
+ 2t_3 \left[ \cos(k_x - k_y + 47\pi/20) + \cos(k_x + k_y) \right] \quad (4.8) \]

The actual values of \( H^{44} \), \( H^{4,14} \) and \( H^{14,14} \) have another multiplicative factor \( R \), whereas the remaining elements should be multiplied by \( 1 - R \). The other elements can be obtained by complex conjugation. Diagonalization of this \( 16 \times 16 \) Hamiltonian produces the eigenstates and eigenvalues for the 16 bands as a function of the two-dimensional wave vector \( \mathbf{k} \) over the entire Brillouin zone.

We have considered a large number of values of \( R \) to reach our conclusions noted below, but, for brevity, we will show results only for \( R = 0 \), \( R = 1 \) and an intermediate value \( R = 0.5 \). Fig. 4.1 (b), (c) and (d) show the band structure at these \( R \) values. Explicit calculation shows that the lowest band always remains gapped as a function of \( R \), which implies that the lowest "Landau level" of the Hofstadter model adiabatically evolves into the lowest band of the Chern insulator,
carrying along its Chern number. In Fig. 4.3 (a), we show the band gap and the band width of as functions of $R$. The Chern insulator band can thus be considered a renormalized Landau level. The integer quantum Hall states in the Chern insulator thus are adiabatically connected to their counterparts in Landau level systems. Even though the flatness ratio (band gap divided by band width) remains large, the Berry curvature changes drastically as shown in Fig. 4.3 (b).

One may note that as $R$ approaches 1, 14 of the 16 bands of the Hofstadter lattice become degenerate at zero energy, reflecting the fact that 14 of the lattice sites in each unit cell essentially drop out of the problem, being completely disconnected from other sites. The band structure at $R = 1$ thus contains two dispersive bands and 14 degenerate bands at $E = 0$. This drastic rearrangement of higher bands underscores the nontriviality of the adiabatic evolution of the lowest band.

### 4.2.2 triangular-kagome hybrid lattice

The kagome lattice was introduced in Ref. [193], and has been found to be a excellent platform of FCI states [122, 222]. In Fig. 4.4, the kagome lattice is indicated with the encircled dots, with its three sublattices marked by blue, red and green circles. In the original proposal [193], with complex hopping terms, nearly flat lowest band with Chern number 1 can be obtained by tuning parameters. The Hamiltonian of this model is

$$H_{ka} = -t_1 \sum_{\langle ij \rangle} c_i^\dagger c_j - t_2 \sum_{\langle\langle ij \rangle\rangle} c_i^\dagger c_j + \text{h.c.}$$

where $t_1 = 1.0 + 0.28i$, $t_2 = -0.3 - 0.2i$, $\langle ij \rangle$ denotes nearest neighbors and $\langle\langle ij \rangle\rangle$ next-nearest neighbors and $t_1$ and $t_2$ are complex hopping coefficients. We also embed the kagome lattice inside a Hofstadter lattice, which is chosen to be a triangular lattice with sixteen lattice sites in each magnetic unit cell, as shown in Fig. 4.4. There is $\pi/16$ magnetic flux passing through each triangle as shown in Fig. 4.5 except the one indicated with a star, where an additional $2\pi$ flux passes through it in the opposite direction. As in the square-checkerboard lattice case, we define the gauge through an explicit choice of the phases as shown in Fig. 4.5. The triangular Hofstadter lattice Hamiltonian $H_{\text{tri}}$ with nearest-neighbor...
Figure 4.4. Lattice model and band structure for the triangular-kagome lattice model. Panel (a) shows the lattice structure. The black dots show the lattice sites in a triangular Hofstadter lattice with flux $\pi/16$ in each triangle. The dashed lines mark a magnetic unit cell, which contains 16 sites. The sites marked by the dashed circles form a kagome lattice where the three different colors represent the three sublattices. Panels (b-d) shows the band structures at different values of $R$ (0.0, 0.5, and 1, respectively) along the contour $\Gamma \to K \to M \to \Gamma$ in momentum space. Hopping carrying these phases give almost flat lowest band and nearly constant Berry curvature. An interpolating Hamiltonian between the triangular and kagome limits is defined as

$$H_{\text{tri-ka}}(R) = (1 - R)H_{\text{tri}} + RH_{\text{ka}}$$

(4.10)

The momentum-space Hamiltonian is given by $H_{\text{tri-ka}} = \sum_k \sum_{\alpha, \beta} \hat{c}_k^\dagger \mathcal{H}_{\alpha\beta}^{\text{tri-ka}}(k) \hat{c}_k$. The nonzero components of $\mathcal{H}_{\alpha\beta}^{\text{tri-ka}}$ are (dropping the subscript “tri-ka” for simplicity)

$$\mathcal{H}^{01} = \exp(ik_x/4) \quad \mathcal{H}^{03} = \exp(-ik_x/4)$$

$$\mathcal{H}^{04} = \exp(ik_y/4) \quad \mathcal{H}^{07} = \exp(i(-k_x/4 + k_y/4 - 25\pi/16))$$
Figure 4.5. Phases of triangular lattice Hofstadter model. With the exception of the outermost hopping bonds, the phase factor associated with a bond, in units of $\pi$, is indicated by the arrow on it and the number either below or to the left of it, and the star marks the triangle where a $-2\pi$ flux is inserted.

Figure 4.6. Triangular-kagome hybrid lattice (a) The single-particle band gap (top curve) and bandwidth (lower curve) as a function of $R$. The flatness ratio (band gap over bandwidth) is shown in the inset. (b) The Berry curvature at different $k$-points with $R = 0.0, 0.5$ and $1.0$ (dotted, dashed and solid lines respectively).

\[
\mathcal{H}^{0,12} = \exp(ik_y/4) \quad \mathcal{H}^{0,13} = \exp(i(k_x/4 - k_y/4 - \pi/16)) \\
\mathcal{H}^{12} = \exp(ik_x/4) \quad \mathcal{H}^{14} = \exp(i(-k_x/4 + k_y/4 + \pi/16)) \\
\mathcal{H}^{15} = \exp(i(k_y/4 + \pi/8)) \quad \mathcal{H}^{1,13} = \exp(-i(k_y/4 + \pi/8)) \\
\mathcal{H}^{1,14} = \exp(i(k_x/4 - k_y/4 - 3\pi/16)) \quad \mathcal{H}^{23} = \exp(i(k_x/4) \\
\mathcal{H}^{25} = \exp(i(-k_x/4 + k_y/4 + 3\pi/16)) \quad \mathcal{H}^{26} = \exp(i(k_y/4 + \pi/4)) \\
\mathcal{H}^{2,14} = \exp(-i(k_y/4 + \pi/4)) \quad \mathcal{H}^{2,15} = \exp(i(k_x/4 - k_y/4 - 5\pi/16))
\]
whereas the remaining elements should be multiplied by 1. The actual values of $H_R$ bands does not close as we change $R$ show the band structures at $R = 0$. The model, the Berry curvature changes significantly even though the energy dispersion $R$ the band gap and band width as functions of $R$. We see that, similarly to the square-checkerboard lattice model, the Berry curvature changes significantly even though the energy dispersion remains quite flat at all $R$.

\[
\begin{align*}
\mathcal{H}^{36} &= \exp(i(-k_x/4 + k_y/4 + 5\pi/16)) \quad \mathcal{H}^{37} = \exp(i(k_y/4 + 3\pi/8)) \\
\mathcal{H}^{3,12} &= \exp(i(k_x/4 - k_y/4 + \pi/16)) \quad \mathcal{H}^{3,15} = \exp(-i(k_y/4 + 3\pi/8)) \\
\mathcal{H}^{45} &= \exp(ik_x/4) \quad \mathcal{H}^{47} = \exp(-i(k_x/4 + 3\pi/2)) \\
\mathcal{H}^{48} &= \exp(ik_y/4) \quad \mathcal{H}^{4,11} = \exp(i(-k_x/4 + k_y/4 - 17\pi/16)) \\
\mathcal{H}^{56} &= \exp(ik_x/4) \quad \mathcal{H}^{58} = \exp(i(-k_x/4 + k_y/4 + \pi/16)) \\
\mathcal{H}^{59} &= \exp(i(k_y/4 + \pi/8)) \quad \mathcal{H}^{67} = \exp(ik_x/4) \\
\mathcal{H}^{69} &= \exp(i(-k_x/4 + k_y/4 + 3\pi/16)) \quad \mathcal{H}^{6,10} = \exp(i(k_y/4 + \pi/4)) \\
\mathcal{H}^{7,10} &= \exp(i(-k_x/4 + k_y/4 + 5\pi/16)) \quad \mathcal{H}^{7,11} = \exp(i(k_y/4 + 3\pi/8)) \\
\mathcal{H}^{89} &= \exp(ik_x/4) \quad \mathcal{H}^{8,11} = \exp(-i(k_x/4 + \pi)) \\
\mathcal{H}^{8,12} &= \exp(ik_y/4) \quad \mathcal{H}^{8,15} = \exp(i(-k_x/4 + k_y/4 - 9\pi/16)) \\
\mathcal{H}^{9,10} &= \exp(ik_x/4) \quad \mathcal{H}^{9,12} = \exp(i(-k_x/4 + k_y/4 + \pi/16)) \\
\mathcal{H}^{9,13} &= \exp(i(k_y/4 + \pi/8)) \quad \mathcal{H}^{10,11} = \exp(ik_x/4) \\
\mathcal{H}^{10,13} &= \exp(i(-k_x/4 + k_y/4 + 3\pi/16)) \quad \mathcal{H}^{10,14} = \exp(i(k_y/4 + \pi/4)) \\
\mathcal{H}^{11,14} &= \exp(i(-k_x/4 + k_y/4 + 5\pi/16)) \quad \mathcal{H}^{11,15} = \exp(i(k_y/4 + 3\pi/8)) \\
\mathcal{H}^{12,13} &= \exp(ik_x/4) \quad \mathcal{H}^{12,15} = \exp(-i(k_y/4 + \pi/2)) \\
\mathcal{H}^{13,14} &= \exp(ik_x/4) \quad \mathcal{H}^{14,15} = \exp(ik_x/4), \\
\mathcal{H}^{02} &= -2 \left[t_1 \cos(k_x/2) + t_2 \cos(-k_x/2 + k_y)\right] \\
\mathcal{H}^{28} &= -2 \left[t_1 \cos(-k_x/2 + k_y/2) + t_2 \cos(k_x/2 + k_y/2)\right] \\
\mathcal{H}^{80} &= -2 \left[t_1 \cos(k_y/2) + t_2 \cos(k_x - k_y/2)\right]
\end{align*}
\]

(4.11)
4.3 Fractional Quantum Hall Effect

Having shown that the $C = 1$ bands of the Chern insulators are adiabatically connected to the Landau bands of Hofstadter lattices, we proceed to FQH states in these systems. As mentioned above, a Hofstadter lattice with a sufficiently small flux per plaquette simulates the continuum, and thus displays all the standard FQH states. Given that a Hofstadter insulator can trivially be converted into a Chern insulator, it follows, as a matter of principle, that all the FQH physics is also possible for Chern insulators, provided one chooses a sufficiently detailed unit cell. In this section, we will ask if the FCI states in the checkerboard and the kagome lattices can be understood as adiabatic evolutions of the corresponding states in Hofstadter lattices. Our exact diagonalization results below show that to be the case for the $1/3$ Laughlin state [111] and the $1/2$ Moore-Read state [137] in both models. However, the evidence for the $2/5$ Jain state is inconclusive. We stress that the FCI states at all these fractions had been established previously; our aim here is to show that they are adiabatic evolutions of the familiar FQH states at these fractions, thus establishing that the two are essentially the same.

4.3.1 model

Our calculations are performed on lattices with periodic boundary conditions in the $x$ and $y$ directions with lengths $L_x$ and $L_y$. The number of particles and the number of magnetic unit cells in the $x$- and $y$-directions are denoted by $N$, $N_x$ and $N_y$. The adiabatic continuity between the lowest Landau level of the Hofstadter band and the Chern band clarifies that the filling factor is to be defined as $\nu = N/(N_xN_y)$. Following Ref. [163], we project out all high-energy bands to reduce the size of the Hilbert space, which is analogous to the lowest Landau level approximation routinely made in studies of the FQH states. We assume that the electrons are fully spin polarized. We also set the width of the lowest band to zero to eliminate the effect of the band curvature, which is a good approximation when the interaction energy is large compared to the bandwidth of the lowest band but small compared to the gap separating it to the first excited band.

A short range interaction is known to produce FQH states at $n/(2n \pm 1)$ in
a continuum Landau level, and, in particular, the 1/3 Laughlin state is the exact ground state of a short range interaction potential. For the FCI states at 1/3 and 2/5, we use a two-body interaction Hamiltonian

\[ H_2 = \sum_{i,j} U_{ij} \hat{n}_i \hat{n}_j \] (4.12)

Here \( \hat{n}_i = c_i^\dagger c_i \) is the particle number operator on site \( i \). In order to make sure that the interaction has a nontrivial effect, we must make it sufficiently long ranged that it survives in the checkerboard limit. We will arrange it so that it becomes a nearest neighbor interaction in that limit. Specifically, we choose \( U_{ij} = 0.5/r_{ij}^2 \) if the distance \( r_{ij} \) between the sites \( i \) and \( j \) is smaller than or equal to a cutoff distance \( r_c \), and \( U_{ij} = 0 \) otherwise. For the square-checkerboard and triangular-kagome models, \( r_c \)'s are chosen to be \( \sqrt{2}/2 \) and \( 1/2 \), respectively. All distances here and below are quoted in units of the lattice constant of the magnetic unit cell. We have also considered a truncated \( 1/r \) interaction and found that the results are consistent with our conclusions below; for simplicity, we will only show results for the truncated \( 1/r^2 \) interaction.

For the FCI states at 1/2 we use a three-body interaction Hamiltonian

\[ H_3 = \sum_{i,j,k} V_{ijk} \hat{n}_i \hat{n}_j \hat{n}_k \] (4.13)

where we choose \( V_{ijk} \) to be \( 0.5/(r_{ij} r_{jk} r_{ki})^2 \) if the distances \( r_{ij}, r_{jk}, r_{ki} \) satisfy the cutoff conditions given below and 0 otherwise, such that they become the nearest neighbor three body interaction in the checkerboard and kagome models. For the square-checkerboard lattice model, the condition is that \( r_{ij,jk,ki} \leq 1 \) and \( r_{ij} r_{jk} r_{ki} \leq 1/2 \). For the triangular-kagome lattice model, the condition is that \( r_{ij,jk,ki} \leq 1/8 \) and \( r_{ij} r_{jk} r_{ki} \leq 1/2 \).

### 4.3.2 exact diagonalization

We first calculate the eigenstates \( u^\alpha(k) \) of the single-particle Hamiltonians \( H_{sq-cb}^{\alpha} \) or \( H_{tri-ka}^{\alpha} \). Then the Hamiltonians \( H_{sq-cb} \) or \( H_{tri-ka} \) are transformed to diagonal form by defining \( c_{k\alpha} = \sum_n u_{\alpha}^n(k) \gamma_{kn} \), where \( \gamma_{kn} \) is the creation operators for a
particle of momentum \( k \) in the \( n \)-th band. A many-body Slater basis state in the lowest band is given by \( \gamma_{k_1,0}^\dagger \gamma_{k_2,0}^\dagger \ldots \gamma_{k_N,0}^\dagger |0\rangle \), which has total momentum \( k_1 + k_2 + \ldots + k_N \). Since the Hamiltonians \( H_2 \) and \( H_3 \) commute with the translation operators in the \( x \) and \( y \) directions, they are block diagonal in the many-body basis. We decompose the Hilbert space into different sectors indexed by momentum quantum numbers \((K_x, K_y)\), which are the sum of the momentum all \( N \) particles modulo \((N_x, N_y)\) in units of \((2\pi/L_x, 2\pi/L_y)\). To calculate the many-body matrix elements of \( H_2 \) and \( H_3 \), we transform the Hamiltonians to momentum space in which they are expressed using \( c_{k \alpha}^\dagger \) and \( c_{k \alpha} \). The many-body Slater basis are defined only using the operators \( \gamma_{k,0}^\dagger \), so the operator \( c_{k \alpha} \) is replaced by \( u_{\alpha}(k) \gamma_{k,0} \) when acting on these basis states.

### 4.3.3 ground states

Fig. 4.7 and Fig. 4.8 shows the energy spectra at 1/3 \((N = 8, N_x = 4, \text{and } N_y = 6)\) and 1/2 \((N = 10, N_x = 4 \text{ and } N_y = 5)\) fillings at \( R = 0.0, 0.5 \text{ and } 1.0 \). For 1/3 filling, we observe 3 quasidegenerate states at \((K_x, K_y) = (0, 0), (0, 2) \text{ and } (0, 4)\), while 6 quasidegenerate states are found at 1/2 filling: one for \((K_x, K_y) = (0, 0)\) or \((2, 0)\) and two for \((1, 0)\) or \((3, 0)\). The gap does not close as \( R \) is increased from 0 to 1, as shown in Fig. 4.19, thus establishing an adiabatic continuity.

In Fig. 4.9 and Fig. 4.10, we show the energy spectra of \( H_2 \) at filling factor 2/5 with \( N = 8, N_x = 4, N_y = 5 \text{ and } N = 10, N_x = 5, N_y = 5 \), respectively. We find that the 2/5 states only show adiabatic continuity for the square-checkerboard model with \( N = 8 \). For the triangular-kagome model with \( N = 8 \) and for both models with \( N = 10 \), however, the gap closes during the evolution. One may attribute the gap closing for the \( N = 10 \) systems in the square-checkerboard model to a combination of the small gap and the fact that all “ground states” occur at the same momenta and therefore are susceptible to significant mixing in finite systems; a study of larger systems will be necessary to clarify the fate of the 2/5 state in the square-checkerboard model.

The earlier work on FCI states at 1/3, 1/2 and 2/5 in the checkerboard and kagome lattices used the degeneracy and momenta of the quasidegenerate ground states as criteria for identifying them with FQH-like states; these quantities are the
Figure 4.7. Energy spectra at filling $1/3$ ($N = 8$, $N_x = 4$, $N_y = 6$) for the square-checkerboard (left panels) and triangular-kagome (right panels) models at $R = 0.0$, 0.5 and 1.0 (top to bottom). There are 3 quasidegenerate states at $(K_x, K_y) = (0, 0), (0, 2)$ and $(0, 4)$.

same as the known degeneracy and momenta of the FQH ground states in the torus geometry for the same aspect ratio, and can be determined using root partitions and certain folding rules given by Bernevig and Regnault [20]. The folding rule relates the degeneracy $N_{\text{FQH}}(K_x, K_y)$ of low energy FQH states in the $(K_x, K_y)$ momentum sector and the approximate degeneracy $N_{\text{FCI}}(K_x, K_y)$ for the FCI case.
**Figure 4.8.** Energy spectra at filling $1/2$ ($N = 10$, $N_x = 4$, $N_y = 5$) for the square-checkerboard (left panels) and triangular-kagome (right panels) models at $R = 0.0$, 0.5 and 1.0 (top to bottom). There are 6 quasidegenerate states: one each at $(K_x, K_y) = (0, 0)$ or $(2, 0)$, and two for $(1, 0)$ or $(3, 0)$.

via the following equation

$$N_{\text{FCI}}(K_x, K_y) = \sum_{K_x',K_y'=0}^{N-1} \delta_{K'_x \text{mod } N_x,0} \delta_{K'_y \text{mod } N_y,0} \frac{N_{x0} N_{y0}}{N_0} N_{\text{FQH}}(K'_x, K'_y)$$

(4.14)

where $N_{x0} = \text{GCD}(N,N_x)$, $N_{y0} = \text{GCD}(N,N_y)$ and $N_0 = \text{GCD}(N,N_x N_y)$ (GCD denotes the greatest common divisor). For the 1/3 Laughlin state and the 1/2 Moore-Read state, the degeneracy of ground states and quasihole states (discussed
Figure 4.9. Energy spectra at filling $2/5$ ($N = 8, N_x = 4, N_y = 5$) for the square-checkerboard (left panels) and triangular-kagome (right panels) models at $R = 0.0, 0.5$ and $1.0$. There are 5 quasi-degenerate states in each panel at $(K_x, K_y) = (0, 0), (0, 1), (0, 2), (0, 3)$ and $(0, 4)$.

below) $\mathcal{N}_{\text{FQH}}(K_x, K_y)$ can be obtained using a generalized Pauli principle [19] and the many-body translational symmetry [73]. For general composite fermion states, the usage of the generalized Pauli principle is limited, but we can still directly compare the energy spectrum of a FQH system on torus and its counterpart in a FCI to check the validity of Eq. (4.14). For our considerations, however, an a priori knowledge of the counting is not necessary, as we directly establish adiabatic continuity with the reference FQH state in the Hofstadter limit. This becomes important when the generalized Pauli principle does not apply, e.g. for the quasiparticle spectra (below).

Further proof that the state has a fractional Hall conductance (or a fractional
Figure 4.10. Energy spectra at filling $2/5$ ($N = 10$, $N_x = 5$, $N_y = 5$) for the square-checkerboard (left panels) and triangular-kagome (right panels) models at $R = 0.0$, 0.5, 0.6, 0.8 and 1.0 (top to bottom). The 5 quasidegenerate states all appear in the $(K_x, K_y) = (0, 0)$ sector and interaction causes splittings.
Figure 4.11. Evolution of the 1/3 ground states for square-checkerboard (left panels) and triangular-kagome (right panels) models shown in Fig. 4.7 upon flux insertion in the \( y \)-direction. The quasidegenerate ground states are separated from the excited state at each point. Note that at \( R = 0.0 \), the states are perfectly degenerate at each flux value and there is no obvious spectral flow.

Chern number) can be demonstrated by looking at the evolution of the quasidegenerate ground states upon flux insertion along the \( x \) or \( y \) direction. The effect of inserting a flux \( \Phi \) in either of the two directions is implemented by letting the single-particle momenta \( k_{x,y} \rightarrow k_{x,y} + \Phi \). For non-FQH states, a state will come back to itself after one flux insertion, whereas a FQH state returns to the original state only after insertion of several flux quanta. Fig. 4.11 show that at 1/3, one of the quasidegenerate ground state evolves into a second state after one flux quantum, and into a third after two flux quanta, before returning to the original state. This demonstrates a Hall conductance of 1/3. Note that there is no level crossing.
Figure 4.12. Evolution of the 1/2 ground states for square-checkerboard (left panels) and triangular-kagome (right panels) models shown in Fig. 4.8 upon flux insertion in the $x$-direction. The quasidegenerate ground states are separated from the excited state at each point. Note that at $R = 0.0$, the states are perfectly degenerate at each flux value and there is no obvious spectral flow.

with higher energy states. Similarly, in Fig. 4.12, the ground states only evolve back to themselves, without crossing higher-energy levels, after inserting two flux quanta, which reveals the 1/2 Hall conductance.

4.3.4 quasiholes and quasiparticles

A FQH state is characterized not only by its ground state but also by the nature of its quasiholes and quasiparticles, in particular the number of quasidegenerate states when one or several quasiholes or quasiparticles are created. For FQH states, the composite fermion (CF) theory has been shown (in the spherical geometry) to
Figure 4.13. Quasihole spectra at 1/3 filling ($N = 7, N_x = 4, N_y = 6$) for the square-checkerboard (left panels) and triangular-kagome (right panels) models at $R = 0.0, 0.5$ and 1.0 (top to bottom). There are 12 states in the low-energy manifold (below the blue lines) in each momentum sector.

give a complete account of states containing quasiparticles or quasiholes for the fractions of the form $n/(2pn \pm 1)$, such as the number of quasi-degenerate states, their quantum numbers (orbital angular momenta for the spherical geometry), and their wave functions; [88] this demonstrates that the quasiparticles are composite fermions in a nearly empty $\Lambda$ level and quasiholes are missing composite fermions from an almost full $\Lambda$ level. Unfortunately, formulation of the CF theory in the torus geometry is not yet available, but we can take the solution in the Hofstadter limit as our definition of the quasihole or quasiparticle spectrum (provided a low energy band can be clearly identified). For quasiholes, the number of states in each
**Figure 4.14.** Quasihole spectra at filling $1/2$ ($N = 10$, $N_x = 3$, $N_y = 7$) for the square-checkerboard (left panels) and triangular-kagome (right panels) models at $R = 0.0$, 0.5 and 1.0 (top to bottom). There are 6 states in the low-energy manifold (below the blue lines) in each momentum sector.

Momentum sector can also be obtained using the generalized Pauli principle and the folding rules [20].

In Fig. 4.13 and Fig. 4.14, the quasihole spectra with $N = 7$, $N_x = 4$ and $N_y = 6$ and $N = 10$, $N_x = 3$ and $N_y = 7$ are presented, which correspond to a $\nu = 1/3$ and $1/2$ states with three and two quasiholes, respectively. The principal observation is that the gap between the low energy quasihole manifold and the higher energy states does not close as $R$ is increased from 0 to 1, as shown by the circles and asterisks in Fig. 4.19.

In Fig. 4.15, we show energy spectra of $H_2$ with $N = 9$, $N_x = 4$ and $N_y = 6$ which correspond to the $1/3$ state with 3 quasiparticles (i.e., three composite
Figure 4.15. Quasiparticle spectra at filling $1/3$ ($N = 9, N_x = 4, N_y = 6$) for the square-checkerboard (left panels) and triangular-kagome (right panels) models at $R = 0.0$, $0.5$ and $1$ (top to bottom). The number of states below the blue lines obey the FQH to FCI mapping in Eq. (4.14).

fermions in the second $\Lambda$ level). Here, it is not clear, even at $R = 0$, how to identify the quasiparticle band. For this purpose, we show in Fig. 4.16 the energy spectra for the corresponding FQH state (9 electrons on a torus interacting via the Coulomb interaction in the presence of 24 flux quanta) on a torus. This system has a well defined quasiparticle band, which allows us to also identify the quasiparticle bands at $R = 0$ in the current problem as well, as marked by the blue lines. This band is seen to evolve continuously, without gap closing, in the square-checkerboard model, but not in the triangular-kagome limit. The gaps in the quasiparticle spectra are not as clear as those in the quasihole cases; as a confirmation of our assignment of the quasiparticle bands, we also studied flux
insertion in these systems and found that the states marked under the blue lines do not mix with higher-energy states above the lines.

The fact that the quasiparticle band is not very well defined is already an indication that the $2/5$ state will be either weak or absent in the checkerboard lattice and absent in the kagome lattice. The large bandwidth of the quasiparticle band implies substantial residual interactions between composite fermions in the second $\Lambda$ level, which can weaken or destroy the two-filled-$\Lambda$-level $2/5$ state. The $n/(2pn \pm 1)$ states, which are the prominent FQH states in the lowest Landau level of the continuum, are even more unlikely to occur in Chern bands for $n \geq 3$.

### 4.3.5 particle entanglement spectra

The entanglement spectrum [116] has been used to probe the topological properties of many FQH states [153, 162, 188]. For the torus geometry used here, the particle entanglement spectrum (PES) [188] has proven particularly useful. Given $d$ (quasi-)degenerate ground states $\{|\psi_i\rangle\}$, the density matrix is defined as $\rho = d^{-1} \sum_{i=1}^{d} |\psi_i\rangle\langle\psi_i|$. We make a cut in the particle space by dividing the $N$ particles into two groups $A$ and $B$ with $N_A$ and $N_B$ particles. The reduced density matrix $\rho_A = \text{Tr}_B \rho$ is obtained by tracing out the particles in $B$. The translational symmetries along the $x$ and $y$ directions are preserved in this process,
so we can plot the eigenvalues $\exp(-\xi)$ ($\xi$ is usually called the entanglement energy) of $\rho_A$ versus the momenta of their corresponding eigenstates. As previously found [20, 163, 222], the numbers of low-lying levels in the PES are determined by the numbers of quasihole states that $N_A$ particles can form on an $N_x \times N_y$ lattice. There are also levels at higher entanglement energies separated from the low-lying universal ones by “entanglement gaps.” It is further demonstrated that the PES can differentiate FQH states from charge density wave states which occur in the thin torus limit [21].

The PES are presented in Fig. 4.17 and Fig. 4.18 for $1/3$ and $1/2$ fillings. We trace out 5 and 6 particles in these two cases, respectively. For the $1/3$ PES, the
low energy band (below the blue lines) consists of 46 states at each momentum in the $K_y = 0$, 3 momentum sectors and 45 states in other sectors. For the 1/2 PES, the low energy band has 200, 196, 201 and 196 states in the $K_x = 0$, 1, 2 and 3 momentum sectors, respectively. These numbers agree with theoretical predictions [20] and the entanglement gap does not close for any value of $R$ between 0 and 1.
Figure 4.19. The plus signs, circles, squares and asterisks show the gaps in the spectra of the $1/3$ ground states, $1/3$ quasihole states, $1/2$ ground states, and $1/2$ quasihole states at several $R$ for the square-checkerboard model (upper panel) and triangular-kagome model (lower panel). The continuous line shows the deviation of the Berry curvature $\delta F$ (normalized by the average value $\bar{F}$) as a function of $R$.

4.3.6 role of Berry curvature

We now ask what weakens or destroys FQH states as the Hofstadter lattices evolve to Chern insulators. It is clear that the nonflatness of the bands is not relevant here, both because the lowest bands are quite flat over the entire evolution as shown in panels (a) of Fig. 4.3 and Fig. 4.6, and because we have set the bands to be strictly flat by hand. We believe that the relevant quantity in this respect is the nonuniformity (in momentum space) of the Berry curvature. The adiabatic path between a FQH insulator and a FCI offers a natural way to explore the role of the distribution of the Berry curvature in momentum space.

Panels (b) of Fig. 4.3 and Fig. 4.6 show the distribution of the Berry curvature ($F$) as a function of $R$ along certain lines in the Brillouin zone for our models. In both cases, the Berry curvature of the lowest band is flat at $R = 0$, as expected for a Landau level. However, in the the square-checkerboard model, the value of $F$ reduces near the $\Gamma$ and $M$ points, and has a peak at the $X$ point as $R$ increases. In the triangular-kagome model, a peak of $F$ emerges at the $K$ point while its value near the $\Gamma$ point goes to zero. Although the integrated Berry curvature in the whole Brillouin zone remains constant ($2\pi$ times the Chern number), its fluctuations become nearly as large as its mean value as $R$ approaches unity.

This change in the distribution of the Berry curvature has a direct correlation
with the many body properties. As a quantitative measure of the deviation of the Berry curvature from its average value \( \bar{F} \), we define the standard deviation of Berry curvature as

\[
\delta F = \sqrt{\frac{1}{A} \int_{BZ} d^2 k (F(k) - \bar{F})^2}
\]

with

\[
\bar{F} = \frac{1}{A} \int_{BZ} d^2 k F(k)
\]

where \( A \) is the area of the Brillouin zone. We now argue that this quantity determines the robustness of the FQH states. We see from Fig. 4.19 that the spectral gaps for the ground and quasihole states at different fillings follow the same trend, changing rapidly for \( R < 0.7 \) and then saturating at \( R \sim 0.7 \). These many-body gaps have a strong (anti)correlation with \( \delta F \), indicating its important role in the FCI states. While the topological properties remain intact over a wide range of \( \delta F \), an increase in \( \delta F \) reduces the size of the gap and thus the robustness of the FCI states. Essentially, the variations in \( F \) enhance the residual interactions between composite fermions, as we see from the increase in the bandwidths of the quasihole and quasiparticle states, which causes a weakening of the FCI states. However, interactions between composite fermions open up the possibility of new emergent structure.

4.4 Conclusion

A remark on the form of the interaction is in order. By construction, the interaction becomes a nearest-neighbor interaction in the checkerboard and the kagome limits, independent of what short range decay is assumed. However, the robustness of the states in the Hofstadter limit depends on the form of the interaction. As mentioned previously, we find that using an exponent of 1 rather than 2 in \( H_2 \) does not change our results qualitatively for the 1/3 state, in the sense that adiabatic continuity can still be established and the gap decreases as the deviation of Berry curvature \( \delta F \) increases. On the contrary, if we choose the exponent in \( H_3 \) to be 1 rather
than 2 then the 1/2 FQH state in the Hofstadter lattice (i.e. \( R = 0 \)) is much weaker, with small spectral gaps and relatively large ground state splittings; in this case, the gap in the checkerboard limit is actually larger than that in the Hofstadter limit, so our conclusion that the gap decreases with increasing \( \delta F \) does not hold. This is physically understandable. The Moore-Read Pfaffian state is the exact zero-energy state of a short-range three-body interaction in the lowest Landau level, but adding longer-range components weakens, and even eliminates, this state [218]. A smaller exponent in \( H_3 \) means a longer-range interaction in the Hofstadter limit, which renders the 1/2 FQH states weaker.

The equivalence of the FQH and FCI states has been studied from the perspective of density algebra, [65, 151] based on the observation that the commutators of momentum-space density operators have the same form in the long-wavelength (small-momentum) limit and for flat Berry curvature. This suggests the same low energy physics for the two problems. However, all momenta are relevant to the FCI states since the bands are exactly flat and the low-energy excitations may have large momenta. Very recently, Roy [169] demonstrated that the density algebra in a lattice model has the same form as that of the Landau level in continuum for all momenta if the Fubini-Study metric satisfies a certain condition. The Hofstadter models that we construct have nearly flat Berry curvature and thus the folding rule is almost exact. Our work shows that the low energy physics can evolve adiabatically as the Berry curvature changes, and thus provides justification for the assumption of flat Berry curvature in the aforementioned works.

We use the particle entanglement spectrum as a probe of the quasihole physics of our models. Ref. [121] obtains the orbital entanglement spectra [109, 116] for the kagome model at both 1/3 and 1/2 fillings, which reveal the edge modes of these states and provide further support for the adiabatic continuity between the FQH states in continuum and FCI states on lattice.

We do not find conclusive evidence for adiabatic continuity for the 2/5 state in either model, which is due to absence of the 2/5 FCI states. Recent papers have proposed that 2/5 states can be obtained in the checkerboard model [110] and the kagome model [122] by either using tilted samples or fine tuning of parameters. We believe that these states are also adiabatically connected to the 2/5 state in the Hofstadter lattice, but we have not confirmed this. Similarly, if the quasiparticle
spectra of the Chern insulator models can be obtained from the FQH quasiparticle spectra on torus via the folding rule at $R = 1$ after fine tuning of parameters, we do expect adiabatic continuity between the quasiparticle spectra at $R = 0$ and $R = 1$.

In conclusion, we have shown, by studying the ground states, quasihole and quasiparticle states, and their particle entanglement spectra, that the integer and fractional states in the Hofstadter and Chern insulators are adiabatically connected. Our study reveals that the nonuniform distribution of the Berry curvature reduces the gap and increases the interaction strength between quasiparticles. In addition, our work shows how Chern insulators with arbitrarily uniform Berry curvature can be constructed by allowing more complex lattices, which should produce many other FCI states. Time-reversal-invariant fractional topological insulators can be constructed from the $p/(2p + 1)$ states by introducing a spin, and are expected to be topologically stable for odd $p$ [57].
Chapter 5

Fractional Topological Phases in Color-Entangled Hofstadter Models

This chapter is a reproduction of the preprint arXiv:1309.1698 by the present author in collaboration with J. K. Jain and Kai Sun. It has been modified slightly to fit into this thesis. In this chapter, we construct generalized Hofstadter models that possess “color-entangled” flat bands and study interacting many body states in such bands. For a system with periodic boundary conditions and appropriate interactions, there exist gapped states at certain filling factors for which the ground state degeneracy depends on the number of unit cells along one particular direction. This puzzling observation can be understood intuitively by mapping our model to a single-layer or a multi-layer system for a given lattice configuration. We discuss the relation between these results and the previously proposed “topological nematic states”, in which lattice dislocations have non-Abelian braiding statistics. Our study also provides a systematic way of stabilizing various fractional topological states in $C > 1$ flat bands and provides some hints on how to realize such states in experiments.
5.1 Introduction

The topological structure of two-dimensional space plays an fundamental role in understanding the quantum Hall effect [100, 199]. It was proved by Thouless et al. [195] that, for a system of non-interacting electrons, the Hall conductance is proportional to the Chern number $C$ defined as the integral of Berry curvature over the Brillouin zone (BZ) [178]. This clarifies the topological origin of the integer quantum Hall effect because the Landau levels generated in an uniform magnetic field all have $C = 1$. Haldane demonstrated subsequently that an uniform external magnetic field is not necessary by showing that a two-band model on honeycomb lattice with suitable parameters can have $C = \pm 1$ bands [74]; such systems are now termed “Chern insulators” [143,192,193]. When interactions between particles in a partially filled Landau level are taken into account, fractional quantum Hall (FQH) states can appear at certain filling factors. For a sufficiently flat band with a nonzero Chern number, fractional topological states may also be realized for suitable interactions [20,23,41,65,112,113,121,122,142,151,152,158,163,170,175,201,206,207,222,223,225]. Many states in $C = 1$ flat bands are shown to be adiabatically connected to those in Landau levels [121,170,225], which provides a simple way of characterizing their properties.

In constrast to a Landau level which has $C = 1$, a topological flat band can have an arbitrary Chern number. This motivates one to ask what is the nature of the fractional topological phases in $C > 1$ flat bands [120,191,197,208,228] and whether it is possible to realize some states that may not have analogs in conventional Landau levels. Ref. [226] demonstrates that the incompressible ground states in a $C > 1$ flat band at filling factor $\nu = 1/(C + 1)$ [$\nu = 1/(2C + 1)$] for bosons (fermions) can be interpreted as Halperin states [76] with special flux insertions (i.e. boundary conditions) in some cases, but the nature of other states remains unclear. Ref. [10] proposed that some bilayer FQH states would have special properties if they are realized in $C = 2$ systems and dubbed them as “topological nematic states”, but numerical evidence for such states has not been found. In this Letter, we construct generalized Hofstadter models and demonstrate that there are interacting systems whose ground state degeneracy (GSD) depends on the number of unit cells along one direction. It is found that our models can be mapped
either to a single-layer or to a multi-layer quantum Hall system depending on the lattice configuration, which provides a simple physical picture that helps us to understand the puzzling properties of \( C > 1 \) flat bands. The change of GSD is one signature of the topological nematic states \([10]\), but there are other subtle issues, \textit{e.g.} qualitative differences between bosons and fermions and the nature of the symmetry reduction, which we explain using our models.

### 5.2 Color-Entangled Hofstadter Models

We construct flat bands with arbitrary Chern numbers by generalizing the Hofstadter model \([8, 78, 81, 155, 204]\) using the generic scheme of Ref. \([228]\). As shown in Fig 5.1 (a), the Hofstadter model describes particles in both a uniform magnetic field and a periodic potential. The tight-binding Hamiltonian for the model on a square lattice is

\[
H = \sum_{ij} t_{ij} e^{i\theta_{ij}} \hat{a}_i^\dagger \hat{a}_j + \text{H.c.},
\]

where \( \theta_{ij} \) is the phase associated with the hopping from site \( j \) to \( i \) \([155]\), \( \hat{a}_i^\dagger \) is the creation operator on site \( i \) and H.c. means Hermitian conjugate. If the magnetic flux per plaquette is \( \frac{2\pi}{n_\phi} \) with \( n_\phi \) being an integer, translational symmetry is preserved on the scale of magnetic unit cell which contains \( n_\phi \) plaquettes. The momentum space Hamiltonian is

\[
H(k) = \Psi^\dagger(k) \mathcal{H} \Psi(k),
\]

where \( \Psi^\dagger(k) = [\hat{a}_0^\dagger(k), \hat{a}_1^\dagger(k), \ldots, \hat{a}_{n_\phi-1}^\dagger(k)] \) and the subscript of \( \hat{a}_i^\dagger \) marks different sites within a magnetic unit cell. \( \mathcal{H}(k) \) is a \( n_\phi \times n_\phi \) matrix whose non-zero matrix elements are \( \mathcal{H}_{nm}(k) = 2 \cos(k_y + 2m\pi/n_\phi) \) and

\[
\mathcal{H}_{nm}(k) = \mathcal{H}_{nm}^*(k) = \exp(ik_x/n_\phi)
\]

for \( n = (m+1) \mod n_\phi \). The \( n_\phi \to \infty \) limit recovers the continuous limit in which Landau levels arise. One important advantage of starting from the Hofstadter model is that the Berry curvature of the lowest band can be made uniform over the entire BZ. This is desirable because a nonuniformity in the Berry curvature usually tends to weaken or even destroy incompressible states \([225]\).

We next stack two identical Hofstadter lattices together in two different ways. In Fig. 5.1 (b), the same orbitals in different layers are aligned together, which results in a conventional bilayer quantum Hall system. In Fig. 5.1 (c), the \( m \)-th orbital in one layer is aligned with the \( (m + n_\phi/2) \)-th orbital in the other layer.

The latter stacking pattern reduces the size of the magnetic unit cell by half \([228]\), so the model shown in Fig 5.1 (c) possess a single lowest band with \( C = 2 \) instead
Figure 5.1. In panel (a), we give an example of Hofstadter lattice with $n_{\phi} = 4$ that are used in panels (b) and (c). The indices of orbitals in a magnetic unit cell are shown in parentheses and the numbers on the bonds indicate the phases of the complex hopping amplitudes along the $y$ direction in units of $\pi$. In panel (b), a bilayer Hofstadter model is obtained by stacking the two layers together. In panel (c), the two Hofstadter layers are shifted relative to each other and then stacked together. The method used in panel (c) gives a color-entangled Hofstadter model in which the size of the magnetic unit cell is reduced by a factor of two and the lowest band has $C = 2$. There are two orbitals on each lattice site (colored in red and blue) for both models and their indices are given in parentheses.
of having two degenerate $C = 1$ bands. It should be emphasized that these two systems are equivalent insofar as the behavior of the bulk is concerned, since they correspond to two different gauge choices. However, as shown below, they behave differently when periodic boundary conditions (PBCs) are imposed because PBCs are not invariant under a change of gauge. In general, one can get a band with an arbitrary Chern number $C$ by stacking $C$ layers of Hofstadter lattices and aligning the orbitals labeled by $m$, $m + n\phi/C$, $\cdots$, and $m + (C - 1)n\phi/C$ in different layers (where $m \in [0, 1, \cdots, n\phi/(C - 1)]$). The momentum space Hamiltonian is very similar to the single-layer Hofstadter model, except that the off-diagonal term $\mathcal{H}_{mn}(k)$ is replaced by $\exp(ik_x C/n\phi)$.

The generalized Hofstadter model we construct here is closely related to the “color-entangled Bloch basis” [226] for Landau levels. This suggests that our models, as well as those constructed in Ref. [228], can be referred to as “color-entangled” topological flat band models. To understand the connection, we briefly explain the color-entangled Bloch basis following Ref. [226]. Let us consider a torus defined by vectors $L_1 = L_1\hat{e}_x$ and $L_2 = L_2\hat{e}_y$ with $\hat{e}_x = \sin \theta \hat{e}_x + \cos \theta \hat{e}_y$, where $\hat{e}_x$ and $\hat{e}_y$ are the unit vectors along the $x$ and $y$ directions. The magnetic field through the torus is $B$ along the $z$ direction and the electromagnetic vector potential is $\mathbf{A}(\mathbf{r}) = Bx\hat{e}_y$. One can define the magnetic translation operator as $T(a) = e^{-iK \cdot a}$ ($K = -i\hbar \nabla - e\mathbf{A} + e\mathbf{B} \times \mathbf{r}$), then the periodic boundary condition is implemented as $T(L_\alpha) = 1$. This requires that the number of magnetic flux through the torus must be an integer $N\phi = L_1 L_2 \sin \theta/(2\pi \ell_B^2)$. For these given conditions, the lowest Landau level wave functions are

$$\langle \mathbf{r}|j\rangle = \frac{1}{(\sqrt{\pi}L_2\ell_B)^{1/2}} \sum_n \exp \left[ -\frac{x^2}{2\ell_B^2} + 2\pi(j + nN\phi) \frac{x + iy}{L_2} \right]$$

$$\exp \left[ -i\frac{\pi L_1 e^{-i\theta}}{N\phi L_2} (j + nN\phi)^2 \right]$$

(5.1)

with index $j \in [0, 1, \cdots, N\phi - 1]$. For a pair of integers $N_x$ and $N_y$ satisfying $N\phi = N_x N_y$, we can construct the Bloch basis states as

$$|k_x, k_y\rangle = \frac{1}{\sqrt{N_x}} \sum_{m=0}^{N_x-1} e^{i2\pi mk_x/N_x} |j = mN_y + k_y\rangle$$

(5.2)
with indices \( k_x \in [0, 1, \cdots, N_x - 1] \) and \( k_y \in [0, 1, \cdots, N_y - 1] \). One can show that these states are eigenstates of translation operators \( T(L_1/N_x) \) and \( T(L_2/N_y) \), so they are well-defined Bloch states.

When the Landau levels have internal degrees of freedom (which we call color), we introduce two operators \( P \) and \( Q \) that act on a color eigenstate \( |s\rangle \)
\[
P|s\rangle = |s + 1 \text{ (mod } C\rangle) \quad Q|s\rangle = e^{i2\pi s/C}|s\rangle
\]

In short words, the operator \( P \) flips the color index and the operator \( Q \) induces a color-dependent phase. We then define two commuting operators \( \tilde{T}_x = T(L_1/N_x)P \) and \( \tilde{T}_y = T(L_2/N_y)Q \), which entangle translation in real space and rotation in the internal color space. The basis states can be chosen as
\[
\langle r, s|k_x, k_y\rangle = \frac{1}{(\sqrt{\pi N_x L_2 \ell_B})^{1/2}} \sum_n \exp \left[-\frac{x^2}{2\ell_B^2} + 2\pi \left( k_y + nN_y + \frac{s}{C}N_y\right) \frac{x + iy}{L_2} \right] 
\]
\[
\times \exp \left[i\frac{2\pi k_x}{N_x}(nC + s) - i\frac{\pi L_1 e^{-i\theta}}{N_\phi L_2} \left( k_y + nN_y + \frac{s}{C}N_y\right)^2\right]
\]

with \( N_\phi = N_x N_y/C \) and \( s \in [0, 1, \cdots, C - 1] \). These states are eigenstates of the color-entangled translation operator which satisfy \( \tilde{T}_\alpha|k_x, k_y\rangle = \exp(-i2\pi k_\alpha/N_\alpha)|k_x, k_y\rangle \).

If one adopts the color-entangled boundary conditions \( \tilde{T}_\alpha^{N_\alpha} = 1 \), the \( k_\alpha\)'s are constrained to be integers and the momentum values can be chosen as \( k_x \in [0, 1, \cdots, N_x - 1] \) and \( k_y \in [0, 1, \cdots, N_y - 1] \).

We use the four band \( C = 2 \) model shown in Fig. 5.1 as an example to demonstrate that the Bloch basis discussed above is closely related to the generalized Hofstadter models. The matrix \( \mathcal{H}(k) \) for this system is
\[
\begin{pmatrix}
2 \cos(k_y) & e^{ik_x/2} & 0 & e^{-ik_x/2} \\
-e^{-ik_x/2} & 2 \cos(k_y + \pi/2) & e^{ik_x/2} & 0 \\
0 & e^{-ik_x/2} & 2 \cos(k_y + \pi) & e^{ik_x/2} \\
e^{ik_x/2} & 0 & e^{-ik_x/2} & 2 \cos(k_y + 3\pi/2)
\end{pmatrix}
\]
The single-particle Hamiltonian can be rewritten as

\[ H(k) = \Psi^{\dagger}_{02}(k)e^{ik_y} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi} \end{pmatrix} \Psi_{02}(k) + \Psi^{\dagger}_{13}(k)e^{ik_y/2} \begin{pmatrix} e^{i\pi/2} & 0 \\ 0 & e^{i\pi} \end{pmatrix} \Psi_{13}(k) + \Psi^{\dagger}_{02}(k)e^{ik_x/2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Psi_{13}(k) + \Psi^{\dagger}_{13}(k)e^{ik_x/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi_{02}(k) + \text{H. c.} \]  

(5.6)

where \( \Psi^{\dagger}_{02} = (a^{\dagger}_0, a^{\dagger}_2) \) and \( \Psi^{\dagger}_{13} = (a^{\dagger}_1, a^{\dagger}_3) \). On the right hand side of Eq. (5.6), the first two terms describe hopping in the \( y \) direction, where the second component always has an additional \( \pi \) phase relative to the first; the third term originates from hopping in the \( x \) direction within a unit cell; the fourth term comes from hopping in the \( x \) direction across a boundary separating adjacent unit cells. The color index is flipped when the particle hops across the boundary, as the hopping matrix connecting \( \Psi^{\dagger}_{13} \) and \( \Psi_{02} \) is off-diagonal. These effects are the same as the color-entangled boundary conditions discussed above.

### 5.3 Interacting Many Body Systems

The differences between a bilayer Hofstadter model and a \( C = 2 \) color-entangled Hofstadter model become transparent when one studies interacting many body systems. We consider \( N \) particles on a periodic lattice with \( N_x \) and \( N_y \) magnetic unit cells along the \( x \) and \( y \) directions. The total number of plaquettes (to be distinguished with the numbers of magnetic unit cells) along the \( x \) and \( y \) directions are denoted as \( L_x \) and \( L_y \), respectively. The magnetic unit cell is always chosen to contain only one plaquette in the \( y \) direction so we have \( L_y = N_y \). It was proposed in Ref. [10] that the following bilayer quantum Hall wave functions

\[ \Psi_B(\{z^1\}, \{z^2\}) = \Phi_{p+1}(\{z^1\})\Phi_{p+1}(\{z^2\}) \]  

(5.7)
\[ \Psi_F(\{z^1\}, \{z^2\}) = \prod_{i<j} (z^1_i - z^1_j)^3 \prod_{i<j} (z^2_i - z^2_j)^3 \prod_{i,j} (z^1_i - z^2_j) \]  

(5.8)

are topological nematic states in \( C = 2 \) bands, where \( z = x + iy \) is the complex coordinate and its superscript indicates the layer it resides in. Eq. (5.7) describes bosonic systems with two decoupled layers and Eq. (5.8) is the Halperin 331 state [76] for fermions. The value of \( p \) is chosen to be 1 or 2 and the associated wave functions are the Laughlin 1/2 state \( \Phi_{1/2}(\{z^a\}) \) and the Jain 2/3 state \( \Phi_{2/3}(\{z^a\}) \) (they are the bosonic analogs of the Laughlin 1/3 state [111] and the Jain 2/5 state [83] for fermions). When the states represented by Eq. (5.7) and Eq. (5.8) are realized on a torus with PBCs, the GSDs are \((p + 1)^2\) and 8 respectively [73, 133].

### 5.3.1 numerical results

For certain color-independent Hamiltonians and the \( C \)-color Bloch basis, zero energy ground state occur at filling factor \( \nu = 1/(C + 1) \) [\( \nu = 1/(C + 1) \)] for bosons (fermions) [226]. These states correspond to color-dependent flux inserted version of the Halperin states [76,226] when \( N_x \) is a multiple of \( C \) (i.e. in the cases where the Bloch basis can be mapped to a multi-layer system). We have confirmed that these FQH states also appear in our color-entangled Hofstadter models with Chern number \( C \) by using the Hamiltonian

\[
\tilde{H}_B = \sum_i \sum_{\sigma} U_{\sigma\sigma} : \hat{n}_i(\sigma) \hat{n}_i(\sigma) : + \sum_i \sum_{\sigma \neq \tau} : \hat{n}_i(\sigma) \hat{n}_i(\tau) :
\]

(5.9)

for bosons and the Hamiltonian

\[
\tilde{H}_F = \sum_{\langle ij \rangle} \sum_{\sigma} : n_i(\sigma) n_j(\sigma) : + \sum_i \sum_{\sigma \neq \tau} : n_i(\sigma) n_i(\tau) :
\]

\[
+ \sum_{\langle ij \rangle} \sum_{\sigma \neq \tau} : n_i(\sigma) n_j(\tau) :
\]

(5.10)

for fermions. The energy spectra as well as particle entanglement spectra match the results obtained using color-entangled Bloch basis.

The wave functions Eq. (5.7) and Eq. (5.8) can be realized in a bilayer Landau level system in continuum when intra-layer interaction is stronger than inter-layer
interaction. This motivates us to study the Hamiltonian

\[ H_B = \sum_i \sum_{\sigma} U_{\sigma\sigma}^B : \hat{n}_i(\sigma)\hat{n}_i(\sigma) : + \sum_i \sum_{\sigma \neq \tau} V_{\sigma\tau}^B : \hat{n}_i(\sigma)\hat{n}_i(\tau) : \quad (5.11) \]

for bosons and the Hamiltonian

\[ H_F = \sum_{\langle ij \rangle} \sum_{\sigma} U_{\sigma\sigma}^F : \hat{n}_i(\sigma)\hat{n}_j(\sigma) : + \sum_i \sum_{\sigma \neq \tau} V_{\sigma\tau}^F : \hat{n}_i(\sigma)\hat{n}_i(\tau) : \quad (5.12) \]

for fermions, where : \cdots : enforces normal ordering, \( \hat{n}_i(\sigma) \) is the number operator for particle of color \( \sigma \) on site \( i \), and \( \langle ij \rangle \) denotes nearest neighbors. The parameters are chosen as \( U_{\sigma\sigma}^B = 1, V_{\sigma\tau}^B = 0.03, U_{\sigma\sigma}^F = 1 \) and \( V_{\sigma\tau}^F = 0.5 \). In other words, we use intra-color onsite interactions for bosonic systems (with a small perturbation given by the \( V_{\sigma\tau}^B \) terms to lift the degeneracies between some energy levels, which increases the efficiency of convergence in exact diagonalization) and both intra-color NN and inter-color onsite interactions for fermionic systems. The eigenstates of these Hamiltonians are labeled by their momenta \( K_x \) and \( K_y \) along the two directions. The many-body Hamiltonians are projected into the partially occupied lowest band(s) [163] and the filling factor is defined as \( \nu = N/(MN_xN_y) \), where \( M \) is the number of bands that are kept in the projection (i.e., \( M = 2 \) for the bilayer Hofstadter model and \( M = 1 \) for the \( C = 2 \) color-entangled Hofstadter model). This means that Eq. (5.7) and Eq. (5.8) have filling factors \( \frac{p}{p+1} \) and \( \frac{1}{4} \) respectively.

The number of plaquettes in the \( x \) direction for given \( N_x \) and \( N_y \) values is chosen to ensure that this system is close to isotropic (i.e. has aspect ratio close to 1). As the size of the unit cell increases, the wave function of a particle spreads over a larger area and the interaction between two particles becomes weaker. To compare systems with different \( n_\phi \), we normalize the energy scale using the total energy of two particles in a system with \( N_x = 1 \) and \( N_y = 1 \) (\( N_x = 1 \) and \( N_y = 2 \) for bosons (fermions).

The GSDs of Eq. (5.7) and Eq. (5.8) given above were derived using the continuum wave functions, but we have found that they are still valid for bosonic systems with Hamiltonian \( H_B \) and fermionic systems with Hamiltonian \( H_F \) in the bilayer Hofstadter model. The results in the \( C = 2 \) color-entangled Hofstadter
model are very different as presented in Fig. 5.2 and Fig. 5.3. The special feature of the $C = 2$ systems is that the GSD depends on $N_x$ and it only agrees with the result in the bilayer Hofstadter model when $N_x$ is even. For bosonic systems, the GSD at $1/2$ is $2$ if $N_x$ is odd and $4$ if $N_x$ is even; the GSD at $2/3$ is $3$ if $N_x$ is odd and $9$ if $N_x$ is even. The gaps of the bosonic states survive in the presence of small inter-color onsite interaction $V_{B}^{B}$ but disappear if $V_{B}^{B}$ becomes comparable to $U_{B}^{B}$. The phase boundary can not be determined precisely because a realliable finite-size scaling is difficult here. For fermionic systems, the GSD is $4$ for $N_x = 1$ and $8$ for $N_x = 2$. The quasi-degenerate ground states have a more pronounced splitting than the bosonic cases. The gaps become less clear for larger $N_x$ and there is no well-defined set of quasi-degenerate ground states when $N = 8$, $N_x = 4$, $N_y = 8$, and $L_x = 16$. 

**Figure 5.2.** Energy spectra of bosons on the $C = 2$ model at filling factors $1/2$ [(a) and (b)] and $2/3$ [(c) and (d)]. The system parameters are given in square brackets as $[N, N_x, N_y, L_x]$. The numbers above some energy levels indicate degeneracies that may not be resolved by inspection.
5.3.2 boundary condition, topology, and symmetry

The key to understanding the physics of a $C = 2$ band is that it may have two fundamentally distinct topologies determined by the parity of $N_x$. As illustrated in Fig. 5.4, this originates from the twisted hoppings along the $x$ direction at the boundary between two magnetic unit cells (i.e. the color index of a particle is flipped). For odd $N_x$ [Fig. 5.4 (a)], it can be unfolded to produce a single Hofstadter layer with $C = 1$ by tracking the black lines which represent hopping terms along the $x$ direction. For even $N_x$ [Fig. 5.4 (b)], it contains two decoupled Hofstadter layers each having $C = 1$. This mapping is sufficient to explain why the GSD change in the bosonic systems: a single-layer $p/(p + 1)$ state with GSD $p + 1$ is realized for odd $N_x$, while two decoupled $p/(p + 1)$ states with GSD $(p + 1)^2$ appear when $N_x$ is even. The fermionic case is more complicated, but it was argued that the GSD of the Halperin 331 state in a $C = 2$ band is 8 when $N_x$ is even and 4 if $N_x$ is odd [10].

The fact that the unfolding of the model depends only on the parity of $N_x$ but not of $N_y$ signifies a reduction of rotational symmetry. However, in our systems the $C_4$ symmetry is not broken spontaneously, as in previously studied nematic states [58,99], but results from the model Hamiltonian itself through boundary conditions. To gain insight into this issue, we note that the simple square lattice Hofstadter model has four-fold rotational symmetry $C_4$ (up to gauge transformations) even though a magnetic unit cell usually has only two-fold rotational sym-
Figure 5.4. This figure shows a slice of the $C = 2$ model constructed in Fig. 5.1 but the two orbitals are plotted separately for clarity. In panels (a) and (b), the unit cells are labeled by Roman numbers and the black lines represent the hopping terms along the $x$ direction. When $N_x$ is odd in (a) [even in (b)], this model maps into a single-layer (bilayer) system. The hopping terms along the $y$ direction do not change this mapping. Panel (c) shows certain interaction terms: 1. intra-color onsite term; 2. inter-color onsite term; 3. intra-color NN term within one unit cell; 4. intra-color NN term across the boundary of a unit cell; 5. inter-color NN term within one unit cell; 6. inter-color NN term across the boundary of a unit cell.

metry $C_2$. This conclusion is valid when the system contains an integral number of magnetic unit cells, which is also satisfied automatically for a Hofstadter multilayer. In contrast, since the unit cell of the color-entangled $C = 2$ Hofstadter model is half as large as the original magnetic unit cell, the $C_4$ symmetry of the parent Hofstadter model is inherited only when $N_x$ is even, but is reduced to $C_2$ symmetry for odd $N_x$.

Although the GSDs of bosonic systems confirm the theoretical predictions, the fermionic 331 state seems less stable. This puzzle is resolved when we analyze the 2-body interaction terms shown in Fig. 5.4 (c), which also have different effects depending on the parity of $N_x$. For both even and odd $N_x$, the term (1) in Fig. 5.4 (c) is still an onsite term and both (3) and (6) turn out to be intra-layer nearest neighbor (NN) terms. If $N_x$ is even, (2), (4) and (5) become, respectively, an inter-layer onsite term, an inter-layer NN term, and an inter-layer NN term. On the other hand, when $N_x$ is odd, (2), (4) and (5) all result in interactions, in the single unfolded layer, that extend over a range comparable to the system size. In a fermionic system with even $N_x$, the intra-color NN terms across boundaries...
between unit cells turn into inter-layer NN terms when the model is mapped to a bilayer system, which is expected to weaken or destroy the 331 state. If one carefully design the Hamiltonian to make sure that it contains no inter-layer NN terms after mapping into a bilayer system, then one can get a 331 state with a clear gap. One can also choose the Hamiltonian such that the Halperin 330 state is realized. It is unclear whether this is related to the pseudopotential Hamiltonian proposed in Ref. [113].

Based on our previous analysis, the $C$-color Hofstadter model or Bloch basis can be mapped to a single layer if $N_x$ is not a multiple of $C$, but the nature of the gapped states here is unclear. As shown in Fig. 5.2, some local interaction terms in Eq. (5.9) and Eq. (5.10) induce special long-range correlations when the system is unfolded to a single layer, but their exact forms in the continuum are not known without analytical calculations. To test this interpretation more explicitly, we have tested many different Hamiltonians for particles in a one-component Landau level on torus and found that some choices of system-dependent unnatural long-range interactions (in addition to short-range ones) indeed produce gapped ground states at filling factor $1/3$ ($1/5$) for bosons (fermions).

### 5.3.3 a square lattice $C = 2$ model

Our considerations can be generalized to the square lattice $C = 2$ model [228], which can be obtained by stacking two checkerboard lattices together and shift them relative to each other along the $a_x$ direction defined in Fig. 5.5. The checkerboard lattice model [192] contains two orbitals per unit cell and there are NN, next NN and second next NN hopping terms. The NN hopping terms connect the two types of orbitals and this brings out certain additional subtleties. For the usual choice of the primitive translation vectors $a_x$ and $a_y$ (Fig. 5.5), the hoppings along both these directions are associated with changes of color index. A system is mapped to a single layer when one of $N_x$ and $N_y$ is odd, and two decoupled layers when both are even. Insight into the physics of this model is given by choosing instead $a_x$ and $\tilde{a}_y$ to define the unit cell. In this case, the color index of a particle does not change during hopping along the $\tilde{a}_y$ direction, and a system may be mapped to a single layer or two layers depending only on the parity of $N_x$. The
Figure 5.5. (color online) Square lattice two-orbital model with Chern number \( C = 2 \). The red and blue colors on each site represent the two orbitals.

The momentum space single-particle Hamiltonian in this case is

\[
\mathcal{H}_S = 2t_3 \left[ \cos(2k_x) + \cos(2k_y - 2k_x) \right] I + \sqrt{2}t_1 \left[ \cos(k_x) + \cos(k_y - k_x) \right] \sigma_x \\
- \sqrt{2}t_1 \left[ \cos(k_x) - \cos(k_y - k_x) \right] \sigma_y - 4t_2 \sin(k_x) \sin(k_y - k_x) \sigma_z
\]

(5.13)

where \( t_1 = 1 \), \( t_2 = 1/(2 + \sqrt{2}) \) and \( t_3 = 1/(2\sqrt{2} + 2) \). \( I \) is the identity matrix and \( \sigma_{x,y,z} \) are the Pauli matrices. This Hamiltonian is different from the one given in Ref. [228], because we are using different lattice translation vectors. We use 2-body onsite interaction given by \( H_S = \sum_i : n_i(A)n_i(A) + n_i(B)n_i(B) + 0.06n_i(A)n_i(B) : \) (the small interaction between \( A \) and \( B \) is used to split some degeneracies and increase the speed of exact diagonalization). The energy spectra of bosons on this model are shown in Fig. 5.6. We see that the GSD is 2 when \( N_x = 5 \) and 4 when \( N_x = 4 \), which can be understood along the same lines as for the color-entangled Hofstadter models.

How relevant is the above analysis using PBCs for real physical systems with open boundaries? The topology of a lattice is determined by \( N_x \) because the hoppings along the \( x \) direction at each boundary between two magnetic unit cells flip the color indices of particles. It was proposed that edge dislocations have a similar effect [10] and they have projective non-Abelian braiding statistics [11]: there are multiple degenerate states given a fixed configuration of dislocations; an exchange of two dislocations results in a unitary evolution in this degenerate space; two such exchanges may not commute with each other; the overall phases of the braiding operations are undetermined. These exotic properties may be
Figure 5.6. Ground state energy spectra of bosons on the square lattice $C = 2$ model at filling factor $\nu = 1/2$. (a) $N = 10, N_x = 5, N_y = 4$; (b) $N = 10, N_x = 4, N_y = 5$.

demonstrated in tunneling and interferometric measurements [12]. The physics of defects in various topological phases have also been studied [33, 36, 119, 200].

5.4 Experimental Realization

The standard Hofstadter model has recently been realized for $^{87}$Rb [3, 132]. It is possible that a practical method to realize the color-entangled Hofstadter models with $C > 1$ can be designed along similar lines. While a model that can be realized easily using current experimental techniques has not been found, our progress on this problem is summarized below.

An essential ingredient in the experimental realization of the simple Hofstadter model [3, 132] is laster-assisted tunneling which generates hopping terms with appropriate phases. To find an experimental protocol for the color-entangled Hofstadter model, we note that a particle may acquire a phase as well as rotate in the internal color space when it hops between two lattice sites. This is a special form of spin-orbit coupling which can in principle be generated if the particles are coupled to suitable non-Abelian gauge fields. By transforming the momentum space Hamiltonian back to real space

$$
\Psi^\dagger(mn)e^{iA_{mn}^x} \Psi(m + 1, n) + \Psi^\dagger(mn)e^{iA_{mn}^y} \Psi(m, n + 1) + \text{H.c.}
$$

one can see what are the necessary gauge fields $A_{x,y}^{mn}$.
We still use the four band \( C = 2 \) model shown in Fig. 5.1 as an example. It is easy to check that one would obtain very unnatural gauge fields \( A_{x,y}^{mn} \) when using the original Hamiltonian Eq. (5.6), so we want to change it to another form which gives more realistic gauge fields \( A_{x,y}^{mn} \). Using gauge transformations \( \Psi_{02} \rightarrow U_0 \Psi_{02} \) and \( \Psi_{13} \rightarrow U_1 \Psi_{13} \), where \( U_0 = \exp[i\phi(1-\sigma_x)] \) and \( U_1 = \exp[i\theta(1-\sigma_x)] \) induce rotations in the internal color space along the \( x \) axis, we can change the Hamiltonian to

\[
\Psi_{02}^\dagger(k) e^{ik_y} \begin{pmatrix} \cos(2\phi) & -i \sin(2\phi) \\ i \sin(2\phi) & -\cos(2\phi) \end{pmatrix} \Psi_{02}(k) \\
+ \Psi_{13}^\dagger(k) e^{i(k_y+\pi/2)} \begin{pmatrix} \cos(2\theta) & -i \sin(2\theta) \\ i \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \Psi_{13}(k) \\
+ \Psi_{02}^\dagger(k) e^{ik_x/2} e^{i(\theta-\phi)} \begin{pmatrix} \cos(\theta - \phi) & -i \sin(\theta - \phi) \\ -i \sin(\theta - \phi) & \cos(\theta - \phi) \end{pmatrix} \Psi_{13}(k) \\
+ \Psi_{13}^\dagger(k) e^{ik_x/2} e^{i(\pi/2 - \theta + \phi)} \begin{pmatrix} \cos(\pi/2 - \theta + \phi) & -i \sin(\pi/2 - \theta + \phi) \\ -i \sin(\pi/2 - \theta + \phi) & \cos(\pi/2 - \theta + \phi) \end{pmatrix} \Psi_{02}(k) \\
+ \text{H.c.} \tag{5.15}
\]

By choosing \( \phi = 0 \) and \( \theta = \pi/4 \), we get

\[
\Psi_{02}^\dagger(k) e^{ik_y} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_{02}(k) + \Psi_{13}^\dagger(k) e^{i(k_y+\pi/2)} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Psi_{13}(k) \\
+ \Psi_{02}^\dagger(k) e^{i(k_x+\pi/2)} \begin{pmatrix} \cos(\pi/4) & -i \sin(\pi/4) \\ -i \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \Psi_{13}(k) \\
+ \Psi_{13}^\dagger(k) e^{i(k_x+\pi/2)} \begin{pmatrix} \cos(\pi/4) & -i \sin(\pi/4) \\ -i \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \Psi_{02}(k) + \text{H.c.} \tag{5.16}
\]

which means that

\[
A_x^{mn} = e^{i\pi/4} \begin{pmatrix} \cos(\pi/4) & -i \sin(\pi/4) \\ -i \sin(\pi/4) & -\cos(\pi/4) \end{pmatrix} \\
A_y^{mn} = e^{im\pi/2} \begin{pmatrix} \cos(m\pi/2) & -i \sin(m\pi/2) \\ i \sin(m\pi/2) & -\cos(m\pi/2) \end{pmatrix} \tag{5.17}
\]

If this method were to be realized in experiments, we need to facilitate rotations
in the internal color space when a particle hops in both the \( x \) and \( y \) directions. It can be simplified if we start from a slightly different Hamiltonian

\[
\begin{align*}
\Psi_{02}^\dagger(k) e^{i k_y} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi_{02}(k) + \Psi_{13}^\dagger(k) e^{i (k_y + \pi/2)} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Psi_{13}(k) \\
+ \Psi_{02}^\dagger(k) e^{i k_x/2} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Psi_{13}(k) + \Psi_{13}^\dagger(k) e^{i k_x/2} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_{02}(k) \\
+ & \text{H.c.}
\end{align*}
\]

This Hamiltonian is equivalent to the color-entangled Bloch basis if the \( P \) and \( Q \) operators are exchanged in the construction. Using gauge transformations \( U_0 = \exp(-i \phi \sigma_y) \) and \( U_1 = \exp(-i \theta \sigma_y) \), this Hamiltonian is changed to

\[
\begin{align*}
\Psi_{02}^\dagger(k) e^{i k_y} & \begin{pmatrix} 0 & e^{i \phi} \\ e^{-i \phi} & 0 \end{pmatrix} \Psi_{02}(k) + \Psi_{13}^\dagger(k) e^{i (k_y + \pi/2)} & \begin{pmatrix} 0 & e^{i \theta} \\ e^{-i \theta} & 0 \end{pmatrix} \Psi_{13}(k) \\
+ \Psi_{02}^\dagger(k) e^{i k_x/2} & \begin{pmatrix} 1 & 0 \\ 0 & e^{i (\theta - \phi)} \end{pmatrix} \Psi_{13}(k) + \Psi_{13}^\dagger(k) e^{i k_x/2} & \begin{pmatrix} 1 & 0 \\ 0 & e^{i (\pi - \theta + \phi)} \end{pmatrix} \Psi_{02}(k) \\
+ & \text{H.c.}
\end{align*}
\]

By choosing \( \phi = 0 \) and \( \theta = \pi/2 \), we have

\[
A_x^{mn} = \begin{pmatrix} 0 & e^{im\pi} \\ 1 & 0 \end{pmatrix}, \quad A_y^{mn} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}
\]

and a particle only rotates in the internal color space when it hops along the \( x \) direction.

The tunability of interaction in ultracold atomic systems [25, 34] would be essential for stabilizing the various states discussed above since the intra-color interaction should be weaker than the inter-color interaction. The topological structure and edge states of a bosonic system in optical lattice can be probed to extract the Chern number [1, 66, 234]. One may also devise some methods to directly measure the braiding statistics following the proposals presented in other contexts [2, 91, 233].
5.5 Conclusion

In conclusion, we have constructed color-entangled Hofstadter models with arbitrary Chern numbers and demonstrated the existence of fractional topological phases in such systems by extensive exact diagonalization studies. The models we use help to clarify many aspects of topological flat bands with $C > 1$ in a physically intuitive manner.
Appendix

Entanglement Spectrum

Entanglement is a very fundamental and peculiar property of quantum mechanics. It has been widely investigated in the context of quantum information and quantum computation for its usefulness [144]. The concepts developed there has been brought into condensed matter physics in recent years and applied in many situations [4]. To characterize entanglement in a quantum condensed matter system, one can divide a system into multiple subsystems. One commonly used choice is a bipartition with two subsystems called $A$ and $B$, then the reduced density matrix of part $A$ for a given state $|\Psi\rangle$ is obtained by tracing out subsystem $B$ as $ho_A = \text{Tr}_B |\Psi\rangle \langle \Psi|$. The von Neumann entropy defined as $S = -\text{Tr}(\rho_A \log \rho_A)$ can quantify the entanglement present in the system. One may be able to get more information from the entanglement spectrum [116] defined as the negative logarithm $\xi$ of the eigenvalues $\exp(-\xi)$ of $\rho_A$. For two given sets of basis states $|\Phi^A_i\rangle$ and $|\Phi^B_j\rangle$ of subsystems $A$ and $B$, one can write the state of the whole system as

$$|\Psi\rangle = \sum_{ij} C_{ij} |\Phi^A_i\rangle \otimes |\Phi^B_j\rangle$$ (1)

and then $\rho_A$ can be written as $CC^\dagger$. We can perform a singular value decomposition of the matrix $C$ to obtain

$$|\Psi\rangle = \sum_{ij} C_{ij} |\Phi^A_i\rangle \otimes |\Phi^B_j\rangle = \sum_i e^{-\xi_i/2} |\Psi^A_i\rangle \otimes |\Phi^B_i\rangle$$ (2)
In the last equation we have written the singular values of $C$ as $\exp(-\xi_i/2)$. It is then easy to show that the eigenvalues of $\rho_A$ are given by $\exp(-\xi_i)$.

To study the FQH states, the cut of Hilbert space is usually made in the orbitals $[116]$, particles $[188]$, or real space $[53,168,190]$. Let us use the Laughlin $1/2$ state $[111]$ with 4 particles on sphere as an example to illustrate how to calculate the entanglement spectrum. The flux enclosed by the sphere is $2Q = 6$ and we use $|m_0, m_1, m_2, m_3\rangle$ to denote the state in which the $z$ components of the angular momenta of the 4 bosons are $m_{0,1,2,3}$. The Laughlin $1/2$ state with 4 particles can be expanded as

\[
|\Psi_{1/2}\rangle = c_0|3, 1, -1, -3\rangle + c_1|3, 1, -2, -2\rangle + c_2|3, 0, 0, -3\rangle + c_3|3, 0, -1, -2\rangle \\
+ c_4|3, -1, -1, -1\rangle + c_5|2, 2, -1, -3\rangle + c_6|2, 2, -2, -2\rangle + c_7|2, 1, 0, -3\rangle \\
+ c_8|2, 1, -1, -2\rangle + c_9|2, 0, 0, -2\rangle + c_{10}|2, 0, -1, -1\rangle + c_{11}|1, 1, 1, -3\rangle \\
+ c_{12}|1, 1, 0, -2\rangle + c_{13}|1, 1, -1, -1\rangle + c_{14}|1, 0, 0, -1\rangle + c_{15}|0, 0, 0, 0\rangle
\]

where the coefficients are $c_0 = 0.384841$, $c_1 = -0.351310$, $c_2 = -0.408185$, $c_3 = 0.272124$, $c_4 = -0.243395$, $c_5 = -0.351310$, $c_6 = 0.320701$, $c_7 = 0.272124$, $c_8 = -0.128280$, $c_9 = -0.136062$, $c_{10} = 0.099365$, $c_{11} = -0.243395$, $c_{12} = 0.099365$, $c_{13} = 0.051312$, $c_{14} = -0.081637$, and $c_{15} = 0.070700$. The simplest cut is to divide the orbitals into two parts. We choose the subsystem $A$ to be the orbitals with angular momenta $0, 1, 2$ and the remaining orbitals form subsystem $B$. The particle number $N_A$ and total angular momentum $L_A^z$ of the subsystem $A$ are conserved quantities, so the reduced density matrix $\rho_A$ can be decomposed to block diagonal form. For the block with $N_A = 2$ and $L_A^z = -4$, we have two basis states $|-1, -3\rangle$ and $|2, -2\rangle$ for subsystem $A$ and two basis states $|3, 1\rangle$ and $|2, 2\rangle$ for subsystem $B$. This block of the $C$ matrix is

\[
\begin{pmatrix}
0.384841 & -0.351310 \\
-0.351310 & 0.320701
\end{pmatrix}
\]

The other blocks of the $C$ matrix can be built in a similar way.
Bibliography


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