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ESSAYS ON MECHANISM DESIGN WITHOUT TRANSFERS

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Abstract

This dissertation consists of three chapters.

In Chapter 1, we compare two widely used allocation methods for school choice, the Deferred Acceptance (DA) mechanism and the Boston mechanism, in terms of welfare. We consider a symmetric incomplete information setting in which students have independently drawn valuations for schools and all schools have an identical ranking of the students. Our main result is that when each school has one available seat and the number of schools and students is large, every type of every student has a higher interim utility under the Boston mechanism than under the DA mechanism. Although this strong result is not true when the number of schools is small, even in this case, the Boston mechanism is ex-ante welfare superior to DA under weak conditions on the distribution of valuations.

In Chapter 2, we consider the problem of allocating $n \geq 2$ indivisible distinct objects (possibly with multiple copies of each) to $m \geq 2$ agents without monetary transfers. We assume that agents have private preferences and consider mechanisms that depend only on agents' reported ordinal preferences. Full efficiency cannot be achieved in this environment, and so we look for a welfare maximizing, incentive compatible mechanism. We show that when agents' rankings over objects are independent of other agents' rankings and each possible ranking is equally likely, the so-called Ranking mechanism (first-order) stochastically dominates any other anonymous, neutral and incentive compatible ordinal mechanism. In particular, when agents' preferences over random allocations are responsive, every type of every agent has a higher interim welfare under the Ranking mechanism.

In Chapter 3, we consider the problem of allocating multiple objects to agents via an auction by using "points" as in the "Course Bidding System" that is used by several business schools. Each agent has a fixed amount of divisible points which can only be used for bidding and have no monetary value. Agents simultaneously bid for the objects, and each object is given to the agent who bids highest for that object. This game is equivalent to the classical "Colonel Blotto" game. We consider this game under incomplete information when agents have private values for the objects. For a class of value distributions, we solve for a Bayes-Nash equilibrium of this game. Furthermore, for all the value distributions for which we can solve for equilibrium in closed form, we show that every type of agent has a higher interim payoff under this allocation method than any other incentive compatible allocation method that depends only on ordinal preferences.

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Chapter 1

Welfare Comparison of School Choice Mechanisms under Incomplete Information

1.1 Introduction

It is impossible or impractical to use prices in many real-life allocation problems. An important example is that of assigning students to schools, the so-called school choice problem. In this chapter, we study two widely used mechanisms for assigning students to schools: the well-known Deferred Acceptance (DA) mechanism and the Boston mechanism. We compare these mechanisms in terms of welfare in an incomplete information setting. Calculations of welfare in such situations are known to be challenging, but we show that explicit formulae can be developed in our setting.

We consider a symmetric setting in which there is *incomplete* information both about student preferences and school preferences/priorities over students. Each student's type is a vector of cardinal values he or she derives from attending the different schools. These are drawn from distributions which are exchangeable (symmetric across schools) and independent of other students' types. In contrast, all the schools have identical priorities over students, say, because they all rank students according to some test score.[1] In this setting, when each school has one available seat, we show (Theorem 1, below):

> *When the number of schools and students is large, the Boston mechanism is welfare superior to the Deferred Acceptance mechanism.*

The Boston mechanism is shown to be superior in the following strong sense: every

[1] We elaborate more on this assumption later in this section.

type of every student has a higher interim utility under the Boston mechanism than under the DA mechanism. We also show that although this strong result does not hold when the number of schools is small, when there are 3 schools, regardless of the number of seats they have, the Boston mechanism is ex-ante welfare superior to the DA mechanism for a large class of value distributions.

Earlier work compared these two mechanisms mostly in terms of their incentive properties. The DA mechanism has been favored on these grounds since truthful revelation by students is a weakly dominant strategy under the DA. The Boston mechanism, on the other hand, is known to be vulnerable to strategic manipulation. That is, students may gain by misreporting their preferences under the Boston mechanism. In fact, after the pioneering work of Abdulkadiroğlu and Sönmez (2003), many school districts abandoned the Boston mechanism in favor of the DA because of this. However, strategy-proofness comes with a welfare cost. Here we quantify this cost in our setting.

The welfare comparison of the two mechanisms considered in here has, until very recently, concentrated only on the ex-post view of the problem. That is, student preferences and school priorities are assumed to be commonly known. Ergin and Sönmez (2006) show that the DA mechanism is superior to the Boston mechanism under complete information. They show via an example that when students have private information about their preferences, this result does not hold and some students may prefer the Boston mechanism. Again, in a complete information environment, Chen and Sönmez (2006) provide experimental evidence that the DA mechanism is better in terms of welfare. A couple of recent papers have considered these mechanisms under incomplete information. Abdulkadiroğlu, Che and Yasuda (2011) (henceforth, ACY) compare these two mechanisms in terms of welfare under incomplete information in a very special case. They assume that students' ordinal rankings are *identical*,[2] and schools have no priorities. They show that the Boston mechanism interim dominates the DA mechanism, that is, each student's interim payoff is weakly higher under the Boston mechanism in this special case. Troyan (2012), allowing for general priority structures for schools but still assuming that students' ordinal preferences are identical, shows that the Boston mechanism is ex-ante welfare superior to the DA mechanism. In another work, Featherstone and

[2]That is, each student's ranking over schools is the same. For example, school 1 is the first choice of *every* student, school 2 is the second choice of *every* student and so on.

Niederle (2008), experimentally show that the Boston mechanism may dominate the DA mechanism in terms of ex-ante welfare. When there is a continuum of students, Miralles (2009) shows that the DA mechanism performs poorly when all students have the same ordinal rankings.

In this study, we step away from the restrictive assumption that students have the same ordinal rankings over schools. Instead, we consider a polar opposite case: students' preferences are independently distributed and each ordinal ranking over schools is ex-ante equally likely. Under this assumption, the problem becomes significantly different from the one in which students have identical ordinal rankings over schools. For example, when all students have the same ranking and schools have no priorities over students, as in ACY, the DA mechanism becomes a trivial random assignment. More precisely, students are randomly assigned to schools in that case. This is a crucial simplification for welfare comparison in ACY. However, when students' ordinal rankings may differ from one another, as is the case in ours, the equivalence of DA to random assignment no longer holds, and analysis of the aforementioned mechanisms is more involved. Nevertheless, our symmetric setting allows us to do explicit welfare calculations which otherwise seem intractable. Although the symmetric setting in which each ranking is ex-ante equally likely is another extreme case, we believe that it is still important to understand how these two mechanisms perform in terms of welfare when students' preferences may differ and to the best our knowledge ours is the first attempt in this direction.

We now elaborate more on the assumption that schools have *identical* strict preferences/priorities over students. In many countries—for instance, China, Iran, Japan and Turkey—students take centralized tests and the scores on these are the sole determinant of school priorities. In some other systems, schools do not have a preference over students per se, and a central lottery is used to assign priorities to students. For example, in many U.S. school districts, schools give priority to students who live in the same neighborhood and/or have a sibling in the same school as opposed to those who live far away and/or do not have a sibling in that school. Lotteries determine the exact priority within each category.

The manner in which schools' preferences are determined has implications about what students know when they submit their preferences. In the test score interpretation, it is reasonable to assume that each student knows how he or she is ranked by the schools when submitting preferences. In the lottery interpretation, however, a student does not know his or her priority when submitting preferences. Our main result, however, is not sensitive to the two specifications—the result that the Boston mechanism yields a higher *interim* welfare than the DA mechanism holds in *both* cases.

The rest of this chapter is organized as follows. First, we formally describe the above-mentioned student assignment mechanisms, the Boston mechanism and the DA mechanism. In Section 2.2, we present the model and present the results for both large and small number of school cases. Then, we conclude. The proofs that are missing in the main body can be found in the appendix.

1.2 The School Choice Problem

The general *school choice problem* is one where a number of students are to be assigned to capacity constrained schools.[3] Each student has a *strict* preference ordering over schools and each school has a *strict* priority ordering, possibly determined by a lottery, over students.

Formally, we have a set of students $\mathcal{I} = \{i_1, ..., i_m\}$, a set of schools $\mathcal{S} = \{s_1, ..., s_n\}$ where school $s \in \mathcal{S}$ has $q_s \geq 1$ available seats. Each student i has a strict preference profile $P_i = (P_i(1), ..., P_i(n))$ and each school s has a strict priority list $\pi_s = (\pi_s(1), ..., \pi_s(m))$ where for all j, k such that $1 \leq j < k \leq n$, student i prefers school $P_i(j)$ strictly more than school $P_i(k)$ and similarly, for each j, k such that $1 \leq j < k \leq m$, student $\pi_s(j)$ has a higher priority than student $\pi_s(k)$ at school s.

A matching is a function $\mu : \mathcal{I} \to \mathcal{S}$ such that $\#\{i \in \mathcal{I} : \mu(i) = s\} \leq q_s$ for all s, where $\mu(i) = s$ means that student i is assigned to school s.

A **school choice mechanism** is a procedure that constructs a matching for each school choice problem. We will implicitly consider school choice mechanisms to be direct mechanisms that ask students to report their preference profiles and implement the corresponding procedure given these reports and school priority lists.

We now describe two commonly used school choice mechanisms.

[3] The total number of seats in schools is not less than the number of students given the fact that every student is entitled to receive education in public schools.

1.2.1 The Boston Mechanism

Given students' *strict* ordinal rankings over schools and schools' *strict* priorities over students, students are assigned to schools as follows:

Step 1: Each student applies to his *first* choice. Each school accepts students one at a time following the priority order over students until there is no seat available or there is no other student applying to that school and rejects the remaining students, if any. The number of available seats in each school is reduced by the number of students accepted by that school.

In general,

Step k: Students who are rejected in Step $(k-1)$ apply to their k^{th} choice. Seats of each school, if available, are assigned to those students one at a time following the priority order over students until there is no seat available at that school or there is no other student applying to that school and the remaining students, if any, are rejected. The number of available seats in each school is reduced by the number of students accepted by that school.

The algorithm stops when all seats of each school are filled or there are no unassigned students.

1.2.2 The Deferred Acceptance (DA) Mechanism

Again, given students' *strict* ordinal rankings over schools and *strict* priorities of schools over students, students are assigned to schools as follows:

Step 1: Each student applies to his most preferred school. Each school *tentatively* accepts students one at a time following the priority order over students until there is no seat available or there is no other student applying to that school and rejects the remaining students, if any.

In general,

Step k: Students who are rejected in Step $(k-1)$ apply to their next preferred school. Each school considers these students together *with* the students who were *tentatively* accepted in earlier steps and accepts students one at a time following the priority order over students until there is no seat available or there is no other student applying to that school and rejects the remaining students, if any.

The algorithm stops when there is no rejected student at some step. Students are assigned to the *final* school that they were tentatively accepted.

The main difference between these two procedures is that under the Boston mechanism each acceptance is final and immediate.[4]A student who ranks a school highly and gets accepted at earlier steps no longer faces any the competition from later applicants. On the other hand, each acceptance under the DA mechanism is tentative and the final matching is settled only after the last step. Hence, a student who ranks a school highly and is tentatively accepted still faces the competition with later applicants.

We are now ready to present our model in which we will consider a "symmetric" school choice problem.

1.3 Preliminaries

1.3.1 The Environment

Assume that there are $n \geq 2$ schools, $\mathcal{S} = \{s_1, ..., s_n\}$, each of which has $q \geq 1$ available seats and $m = nq$ students, $\mathcal{I} = \{i_1, ..., i_m\}$.[5] Each student i has a valuation v_j^i for school j. Student i's type (valuation vector) $\mathbf{v}^i = (v_1^i, ..., v_n^i)$ is independently drawn from

$$\boldsymbol{V}^i = \left\{\mathbf{v}^i = \left(v_1^i, ..., v_n^i\right) \in \mathbb{R}_+^n : v_j^i \neq v_k^i \text{ whenever } j \neq k\right\}$$

with an exchangeable distribution function F^i, that is, F^i is invariant under permutations of its arguments so that

$$F^i\left(v_1^i, ..., v_n^i\right) = F^i\left(z_1^i, ..., z_n^i\right)$$

whenever $z_1^i, ..., z_n^i$ is a permutation of $v_1^i, ..., v_n^i$. These assumptions imply that each student's ordinal ranking over schools is strict and that each strict ordinal ranking is

[4]Thus, the Boston mechanism may be justly called the "Immediate Acceptance" mechanism.

[5]In accordance with the fact that each student is entitled to get education in public schools, we assume that $m \leq nq$. For convenience, we assume that $m = nq$. All the results go through for $m < nq$ unless otherwise noted.

equiprobable.[6]

Schools' priorities over students are identical and drawn uniformly from the set of all strict ordinal rankings over students.[7] We assume that each student prefers getting into some school rather than remaining unassigned.

We consider the games induced by the Boston and the Deferred Acceptance mechanisms as follows: Each student learns his type and submits his strict ordinal ranking which may or may not be students' true ordinal ranking. Each student only knows the distribution of other students' preferences and school priorities when he submits his preferences. Then, based on students' submitted rankings and school priorities, the corresponding mechanism is implemented.

1.3.2 Incentive Compatibility

We, first, investigate the incentives of students. First, as is well known,[8] the Deferred Acceptance mechanism is strategy-proof, that is, each student at least weakly prefers to report his true ordinal ranking whatever the reports of other students are. On the other hand, in general, truthful reporting may not be an equilibrium under the Boston mechanism as is already been pointed out in the literature under complete information case. We now present an explicit example—with incomplete information—to illustrate how a student can become better off by manipulating his preference list.

Example 1. *There are* 3 *students* $\{i_1, i_2, i_3\}$ *and* 3 *schools* $\{s_1, s_2, s_3\}$ *each with one available seat. Assume that each student's type is independently drawn from a distribution such that*

$$\begin{aligned}\Pr(\mathbf{v} = (1, 0.8, 0)) &= p = \frac{3}{4}\\ \Pr(\mathbf{v} = (0.8, 1, 0)) &= 1 - p = \frac{1}{4}\end{aligned}$$

Assume that i_2 *and* i_3 *report truthfully and let's check whether it is a best response for* i_1 *to report truthfully.*

[6] For example, assume that $n = 2$. Then, there are two possible strict ordering over schools: (s_1, s_2) and (s_2, s_1). This assumption tells us student i's preference list is (s_1, s_2) with probability $\frac{1}{2}$ and (s_2, s_1) again with probability $\frac{1}{2}$.

[7] As discussed in the Introduction, we could also interpret this differently. Assume that schools have no priorities and a single random draw over the set of all possible rankings determines the identical priorities of schools.

[8] See Dubins and Freedman (1981) or Roth (1982), for example.

Consider i_1 *with type* $(1, 0.8, 0)$. *If he reports truthfully, that is if he reports his ranking as* (s_1, s_2, s_3), *his expected payoff is*

$$p^2\left(\frac{1}{3}(1)+\frac{1}{3}(0.8)+\frac{1}{3}(0)\right)+2p(1-p)\left(\frac{1}{2}(1)+\frac{1}{2}(0)\right)+(1-p)^2(1)=\frac{47}{80}$$

If he reports (s_2, s_1, s_3), *his expected payoff is*

$$p^2(0.8)+2p(1-p)\left(\frac{1}{2}(0.8)+\frac{1}{2}(0)\right)+(1-p)^2\left(\frac{1}{3}(1)+\frac{1}{3}(0.8)+\frac{1}{3}(0)\right)=\frac{51}{80}$$

Hence, student i_1 *with type* $(1, 0.8, 0)$ *can gain from reporting untruthfully. Thus, each student reporting truthfully is not an equilibrium.*

However, under symmetry assumptions, truthful reporting is an equilibrium under the Boston mechanism:

Proposition 1. *Assume that each school has a quota of* $q \geq 1$ *and each strict ordinal ranking over schools is equally likely for each student. Furthermore, assume that all schools have an identical ranking of the students. Then, truth-telling is an equilibrium under the Boston mechanism.*

Proof. See Appendix A.1. □

To get an idea to see why truth-telling becomes an equilibrium under the symmetry assumptions we impose, consider the above example. Student i_1 with type $(1, 0.8, 0)$ knows that other students' ordinal preferences over schools are *highly likely* to be $s_1 \succ s_2 \succ s_3$ which is also his true ranking. Hence, student i_1 with type $(1, 0.8, 0)$ can become better off by avoiding a highly likely tie. However, under symmetry assumptions, this kind of situation does not arise.

Note also that this result does not depend on whether students know their rankings when they report their preferences but it just requires that schools are ex-ante symmetric, that is, each school has an identical ranking of students.

1.3.3 Welfare Criteria

Given these results regarding incentives under these mechanisms, we will concentrate on truth-telling equilibrium for both the Boston and the DA mechanisms and compare them in terms of welfare. Two notions of welfare criteria we will be using are "Interim Dominance" and "Ex-ante welfare" criteria as described below.[9]

We will say that mechanism A *interim dominates* another mechanism B if the interim utility[10] of any type of student is weakly higher under mechanism A than under mechanism B when each student reports truthfully under each mechanism.

We will say that mechanism A is *ex-ante welfare superior* to mechanism B if

$$EU^A \geq EU^B$$

where

$$EU^X = E\left[u^X(\mathbf{V})\right]$$

, where $\mathbf{V}$ represents the random variable for value vector of a student and for any realization $\mathbf{v}$, $u^X(\mathbf{v})$ is the *interim utility* of type $\mathbf{v}$ when each student reports truthfully under mechanism $X \in \{A, B\}$ and $E[\cdot]$ is the expectation operator.

If mechanism A interim dominates mechanism B, then trivially A is also ex-ante welfare superior to B, but not the other way around.

1.4 Main Results

We first investigate the case when the number of schools and students are large.

1.4.1 Large Number of Schools

Assume that the number of schools is n and $q = 1$ and the number of students is n.[11] We will be interested in the case when n is large. We will be denoting $(P_k^n)^B$ as the interim probability of a student getting into his k^{th} choice under the Boston mechanism when there are n schools and n students and similarly we will denote $(P_k^n)^{DA}$ as the interim probability of a student getting into his k^{th} choice under the DA mechanism when there are n schools and n students.

Our main result is as follows:

[9]Note that here schools are merely objects to be consumed. Hence, when we refer to welfare, we mean welfare of students.

[10]By interim utility, we mean expected utility calculated when a student only knows his or her own type.

[11]Although $q = 1$ may not be a realistic assumption in school choice context, when $q > 1$, computation becomes intractable for a general n. Furthermore, later, in the small number school case, we show that q does not affect welfare comparison conclusions.

Theorem 1. *For sufficiently large n, the Boston mechanism interim dominates the Deferred Acceptance mechanism.*

Proof. **(Sketch)** To prove the theorem, we first establish an asymptotic stochastic dominance relation. Precisely, we show that for each $K \geq 1$, we have that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{K}(P_k^n)^B\right) > \lim_{n\to\infty}\left(\sum_{k=1}^{K}(P_k^n)^{DA}\right)$$

To prove this, we establish another interesting result. We show that, in the limit, as $n \to \infty$, the interim probability of getting into first choice is strictly higher[12] and the interim probability of getting into k^{th} choice for all $k \geq 2$ is strictly lower under the Boston mechanism compare to the DA. Although we relegate the formal proof to Appendix A.2, we give a brief sketch here. We first compute $(P_k^n)^{DA}$ for each k and n. To do so, we use the equivalence of the DA mechanism and another commonly known mechanism, Random Serial Dictatorship (RSD) when schools have identical priorities. By establishing a recursive relation, we calculate P_k^n for RSD for each n and $k \leq n$. (Lemma 3) Secondly, although it seems intractable to compute $(P_k^n)^B$ for each n and k, for each k we compute the limit probabilities as $n \to \infty$ and establish the above-mentioned result (Lemma 5). Given the limit inequality, we finally show that for sufficiently large n, any type of student has a higher interim utility under the Boston mechanism. □

Observe that the main advantage of the Boston mechanism over the DA mechanism is that the probability of a student getting into his top choice is higher under the Boston compare to the DA. It is easy to see this even without doing any computation. Note that under both mechanisms, each student applies to his first choice in the first step. If a student, say i_1, is accepted by a school, say s_1, he is placed to that school for sure under the Boston mechanism. However, under the DA mechanism, it may be the case that some other student, say, i_2, who is rejected in Step 1 by another school may apply to s_1 in later stages and i_2 may be chosen over i_1 and hence i_1 may end up not getting into s_1. Although it is not easy to see the relation between these mechanisms in getting into k^{th} choice in general for $k \geq 2$, we show that for large

[12] In fact, the probability of getting into first choice is strictly higher under the Boston mechanism for any $n \geq 3$.

enough n, actually the probability of getting into k^{th} choice is higher under the DA mechanism.

This result at hand, a natural question to ask is how much the welfare gain under the Boston mechanism is. We, below, show via an example that the gains can be quite large.

Example 2. *Assume that* $\mathbf{v} = (v_k)_{k=1}^n = \left(1, \frac{1}{n}, \frac{1}{2n}, ..., \frac{1}{(n-1)n}\right) \in \boldsymbol{V}$. *Now, consider this type of student. His expected payoff under the Boston mechanism is* $U_n^B(\mathbf{v}) = \sum_{k=1}^n (P_k^n)^B v_k$ *and similarly under the DA mechanism is* $U_n^{DA}(\mathbf{v}) = \sum_{k=1}^n (P_k^n)^{DA} v_k$. *Now, by calculations in A.2.*

$$\lim_{n\to\infty} U_n^B(\mathbf{v}) = \lim_{n\to\infty} (P_1^n)^B = \left(1 - \frac{1}{e}\right) \approx 0.632$$

and

$$\lim_{n\to\infty} U_n^{DA}(\mathbf{v}) = \lim_{n\to\infty} (P_1^n)^{DA} = 0.5$$

Note that the welfare gain under the Boston mechanism is more than 25% in this example.

1.4.2 Small Number of Schools

Second, we consider the case when the number of schools is small. We consider the cases when $n = 2$ and $n = 3$ and we assume that each school has $q \geq 1$ available seats and compare the two mechanisms.

When $n = 2$, then Boston and DA induce the same probability distribution over deterministic matchings and hence are outcome equivalent.

Proposition 2. *Assume that there are* 2 *schools each with* $q \geq 1$ *available seats and* $m = 2q$ *students. Then, the Boston mechanism and the DA mechanism are equivalent.*

Proof. See Appendix A.3. □

The first interesting case is when $n = 3$. We show via an example that when $n = 3$, the Boston mechanism and the DA mechanism are not equivalent and there is no interim dominance relation between the Boston mechanism and the DA mechanism.

Example 3. *Assume that there are* 3 *students and* 3 *schools each with one available seat. Consider a student, say* i_1*, and without loss of generality (wlog) say his ranking is* (s_1, s_2, s_3)*. We want to compute the probability of getting into school* s_i *under these mechanisms. Let* $(P_k^n)^X$ *denote the probability of getting into* k^{th} *ranked school (for* i_1*,* k^{th} *ranked school is school* s_k*) when the mechanism is* X *and when there are* n *schools.*

Firstly, consider the Boston mechanism. Note that there are 2 *more students other than* i_1*. Let* t *be the number of students who rank* s_1 *as their first choice, and* l *be the number of students who rank* s_2 *as their second choice from the other* 2 *students. Hence,* $2 - t - l$ *students rank* s_3 *as their first choice. Now,*

$$\begin{aligned} \left(P_1^3\right)^B &= \sum_{t=0}^{2} \left[\binom{2}{t}\left(\frac{1}{3}\right)^t \left(\frac{2}{3}\right)^{2-t}\right] \frac{1}{t+1} \\ &= 1 - \left(\frac{2}{3}\right)^3 = \frac{19}{27} \end{aligned}$$

where the term in the bracket is the probability that exactly t *students rank* s_1 *as their first choice and* $\frac{1}{t+1}$ *is the probability, when there are* t *other students who rank* s_1 *as their first choice, that* i_1 *will be ranked over other* t *students and hence assigned to* s_1*.*

To calculate P_2^B*, first, consider the case when* $t = 1$*.*[13] *Then,* $l \in \{0, 1\}$*. First, consider the case that* $l = 0$*. In this case, there is one student whose first choice is* s_3*. Hence, if* i_1 *is eliminated in step* 1 *which happens with probability* $\frac{1}{2}$*,* i_1 *will get into* s_2*. Second, consider the case that* $l = 1$*. In that case,* i_1 *can not get into* s_2 *since some other student got into* s_2 *in step* 1*.*

Now, consider $t = 2$*. If* i_1 *is eliminated in Step* 1 *which happens with probability* $\frac{2}{3}$*, there is one more student who is eliminated. If the other student's second choice is* s_3*,* i_1 *will be assigned to* s_2 *in step* 2*. If the other student's second choice is* s_2*,* i_1 *will be assigned to* s_2 *with probability* $\frac{1}{2}$*. Hence, combining these cases, we have*

$$\left(P_2^3\right)^B = \binom{2}{1}\left(\frac{1}{3}\right)^2\left(\frac{1}{2}\right) + \binom{2}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)\left(\frac{1}{2} + \frac{1}{2} \times \frac{1}{2}\right) = \frac{1}{6}$$

and

$$\left(P_3^3\right)^B = 1 - P_1^B - P_2^B = \frac{7}{54}$$

[13] Note that for i_1 to be assigned to s_2, it must be that he is not assigned to s_1 in Step 1. Therefore, t should be at least 1.

Hence,

$$\left(P^3\right)^B = \left(\left(P_k^3\right)^B\right)_{k=1}^3 = \left(\frac{19}{27}, \frac{1}{6}, \frac{7}{54}\right)$$

Furthermore,

$$\left(P^3\right)^{DA} = \left(\frac{2}{3}, \frac{2}{9}, \frac{1}{9}\right)$$

by Lemma 3 in Appendix A.2. Hence,

$$\left(P^3\right)^B - \left(P^3\right)^{DA} = \frac{1}{18}(2, -3, 1)$$

Note that there can not be an interim dominance relation between these two mechanisms given these probabilities. That is, it may be the case that some types prefer the Boston mechanism but others prefer the DA mechanism. The following example illustrates this.

Example 4. *Assume that* $\mathbf{v} = \left(1, \frac{9}{10}, \frac{1}{10}\right)$, $\mathbf{v}' = \left(1, \frac{1}{2}, \frac{1}{10}\right) \in \boldsymbol{V}$. *Now, expected utility difference between the Boston mechanism and the DA mechanism for type* $\mathbf{v}$ *is*

$$\frac{1}{18}\left((2, -3, 1) \cdot \left(1, \frac{9}{10}, \frac{1}{10}\right)\right) = -\frac{1}{30}$$

and expected utility difference between the Boston mechanism and the DA mechanism for type $\mathbf{v}'$ *is*

$$\frac{1}{18}\left((2, -3, 1) \cdot \left(1, \frac{1}{2}, \frac{1}{10}\right)\right) = \frac{1}{30}$$

Hence, type $\mathbf{v}$ *prefers the DA over the Boston and type* $\mathbf{v}'$ *prefers the Boston over the DA.*

Given that there is no interim dominance relation, we will compare these two mechanisms in terms of ex-ante welfare. Before doing that, we establish the following result for a general $q \geq 1$ which will enable us to make the ex-ante welfare comparison.

Proposition 3. *Assume that there are* 3 *schools each with a quota of* $q \geq 1$ *and* $m = 3q$ *students.*[14] *Then,*

$$P_1^B(q) - P_1^{DA}(q) = 2\left(P_3^B(q) - P_3^{DA}(q)\right)$$

[14]Same result holds for $2q < m < 3q$. When $m \leq 2q$, Boston and DA are equivalent which can be similarly shown as in the case of $n = 2$.

,and

$$P_1^B(q) - P_1^{DA}(q) = -\frac{2}{3}\left(P_2^B(q) - P_2^{DA}(q)\right)$$

where $P_k^X(q)$ denotes the probability of a student getting into his k^{th} choice when the mechanism is X and the quota is q.[15] *Furthermore,*

$$P_1^B(q) - P_1^{DA}(q) > 0$$

Thus,

$$P^B(q) - P^{DA}(q) = \left(P_k^B(q) - P_k^{DA}(q)\right)_{k=1}^3 = c_q(2, -3, 1)$$

for some $c_q > 0$.

Proof. See Appendix A.4. □

Hence, when $n = 3$, for any given $q \geq 1$, there is no interim dominance relationship between the Boston mechanism and the DA mechanism.

To make an ex-ante welfare comparison between these two mechanisms, we consider a special i.i.d. case and assume that each v_j^i is independently drawn from a continuous distribution function $F^i(.)$ with $F^i(0) = 0$.[16] Therefore, by Proposition 3, ex-ante welfare comparison between the two mechanisms, regardless of q, only depends on the sign of

$$2E[W_1] - 3E[W_2] + E[W_3]$$

,where $E[W_k]$ denotes the expectation of the random variable for the k^{th} order statistic for three i.i.d. random variables X_1, X_2, X_3 with distribution function $F^i(.)$. That is,

$$\begin{aligned} W_1 &= \max\{X_1, X_2, X_3\} \\ W_2 &= \sec\max\{X_1, X_2, X_3\} \\ W_3 &= \min\{X_1, X_2, X_3\} \end{aligned}$$

where $\sec\max\{X_1, X_2, X_3\}$ denotes the second highest of X_i s.

[15] We here make an abuse of notation and drop $n = 3$ from $(P_k^n)^B$ and $(P_k^n)^{DA}$.

[16] Note that this assumption does not satisfy the initial assumptions on the type space since ties can happen here. We assume that if there is a tie, a random draw breaks the tie. Note that ties would occur with zero probability theorem when distribution functions are continuous and will not affect the computation.

Let $w_{F^i} = 2E[W_1] - 3E[W_2] + E[W_3]$. We, now, verify that for a large class of distribution functions, we have that $w_{F^i} > 0$ which implies that the Boston mechanism is ex-ante welfare superior to the Deferred Acceptance mechanism when student i's values for school j is independently drawn from F^i.

Proposition 4. *Assume that each v_j^i is independently drawn from*
(i) $F^i(.)$ such that F^i has a decreasing density f^i, (F^i is concave), or,
(ii) $F^i(.)$ such that F^i has a log-convex density f^i, or,
(iii) $F^i(.)$ such that $1 - F^i$ is log-convex, or, F^i has a density f^i such that failure rate, $\frac{f^i(x)}{1-F^i(x)}$, is decreasing.
Then, $w_{F^i} > 0$.

Proof. See Appendix A.5. □

Note that for many other distribution functions that does not satisfy any of these sufficient conditions, we still have $w_F > 0$. For example, consider $F(x) = x^c$, $x \in [0,1]$. When $c > 1$, F does not satisfy any of the sufficient conditions listed. However, for each $c > 1$, we have $w_F > 0$.[17] Thus, these conditions are not necessary conditions and $w_F > 0$ for a larger class of distributions than stated here.

Some examples of distribution functions that satisfy the sufficient conditions listed in the above Proposition are as follows.

For example, consider Beta distribution, where the density of Beta distribution is

$$f_{a,b}(x) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, 0 \le x \le 1$$

where $a, b > 0$ and $B(.,.)$ is the Beta function. When $a, b < 1$, Beta distribution has a log-convex density. When $2 - b < a < 1$, then Beta distribution has a decreasing density.

Other examples of distributions that have log-convex density are widely used distributions such as Pareto (with cdf $F(x) = 1 - \frac{a}{x^\beta}$ (with support (a, ∞))) and Gamma (with density $f(x) = \frac{x^{m-1}\theta^m e^{-x\theta}}{\Gamma(m)}$, $x \ge 0$ $\theta, m > 0$) when $0 < m < 1$. Examples of concave distribution functions include well known Exponential (given by cdf $F(x) = 1 - e^{-\lambda x}$ with support $(0, \infty)$, $\lambda > 0$), Pareto distributions.

Weibull distribution (which has cdf $F(x) = 1 - e^{-x^c}$) satisfies (iii) when $1 > c > 0$.

[17] See Appendix A.5 for the computation.

Hence, when $n = 3$, for any $q \geq 1$, the Boston mechanism is ex-ante welfare superior to the DA mechanism for many widely used distributions. That is, as long as $n = 3$,when the number of students is small or large, same conclusion holds.

1.5 Conclusion

We have studied two widely used student assignment mechanisms, namely the Boston mechanism and the Deferred Acceptance mechanism under incomplete information. It is more realistic to assume that students have private information about their preferences. However, the analysis of the school choice problem under incomplete information has been restricted to a very special case in which every student has the same ordinal ranking over schools. We move away from this restrictive case and consider a symmetric environment in which students' ordinal preferences may differ from each other.

Possibly different ordinal preferences of students make the problem completely different and more involved than the one in which students' ordinal preferences are identical. Despite the difficulties this assumption introduces, we present explicit welfare calculations and compare these mechanisms in terms of welfare. We show that the Boston mechanism outperforms the DA mechanism in terms of welfare in this setting. More precisely, the Boston mechanism interim dominates the DA mechanism when the number of schools and students is large and when the number of schools is small, the Boston mechanism is ex-ante welfare superior to the DA mechanism for many widely used value distributions.

Chapter 2

Allocation without Transfers: A Welfare Maximizing Mechanism

2.1 Introduction

In many real-life allocation problems, it is not legal, fair or practical to use monetary transfers. Allocating office spaces to workers, tasks to employees, course seats to students are notable examples of such situations, among many others. In addition, allocations in such problems usually depend only on agents' ordinal preferences rather than their cardinal/monetary values.[1] In this chapter, we study the problem of allocating indivisible goods to individuals without the use of money, and consider mechanisms which depend only on the (reported) ordinal preferences of agents.

We consider a general problem in which $q \geq 1$ identical copies of $n \geq 2$ indivisible distinct object types are to be allocated to $m \geq 2$ agents who demand multiple objects. In our setting, each agent has a strict ranking over objects and this ranking is privately known. We assume that each agent's ranking over objects is independent of others' rankings and each possible ranking is equally likely. Furthermore, each agent's preference over random allocations is responsive—in the (first-order) stochastic dominance sense— with respect to his ranking over different objects.[2] More precisely, between two random allocations of objects, an agent prefers the one that stochastically dominates the other. One such preference structure is when different objects affect agent's utility in an additive fashion, that is, when preferences over bundles of objects admit additively separable utility representations. In this environment, we seek the

[1]As Kojima and Manea (2010) state, eliciting cardinal valuations may be difficult in many instances.

[2]This preference structure has also been considered by Pycia (2011).

"best" allocation method when transfers cannot be used.

Even in the absence of transfers, there can be several ways of allocating. We begin by describing the workings of a simple allocation method, the Ranking mechanism. For each object, agents who report the highest ranking for that object, up to the number of available copies, receive one copy. When there are more agents who has the highest ranking for an object than the number of available copies, these are distributed randomly among these agents.[3] As an example, suppose $n = 3$ objects, $\{o_1, o_2, o_3\}$, are to be allocated to $m = 3$ agents, $\{i_1, i_2, i_3\}$, when there is only one copy of each object. Assume that agents' reported preferences are as follows. Agent i_1 ranks objects as $o_1 \succ o_2 \succ o_3$, agent i_2 ranks as $o_1 \succ o_3 \succ o_2$ and agent i_3 ranks as $o_3 \succ o_2 \succ o_1$. Consider object o_1. Both i_1 and i_2 rank it as their first choice and i_3 ranks it as his third choice. Hence, object o_1 is given to i_1 and i_2 with probability $\frac{1}{2}$ each. There is no one who ranks object o_2 as his first choice but both i_1 and i_3 rank it as their second choice. Thus, i_1 and i_3 get object o_2 each with probability $\frac{1}{2}$. Finally, i_3 is the only agent who ranks object o_3 as his first choice, so i_3 gets object o_3.[4] Random Serial Dictatorship and Probabilistic Serial mechanisms, which will be explained later in this section, are some of the other popular and well-studied allocation methods.

Assuming that each agent's rankings over objects are independent of other agents' rankings and each possible ranking over objects is equally likely, our main result (Theorem 1, below) is:

> *The Ranking mechanism (first-order) stochastically dominates any other anonymous, neutral and incentive compatible ordinal mechanism.*[5]

This result implies that when agents' preferences for the objects are ex-ante symmetric and diverse, the Ranking mechanism is anonymous, neutral and incentive compatible and moreover, it is unambiguously better than any other such mechanism in terms of interim welfare of agents. Formally, the sum of interim probabilities of getting first $K \leq n$ choices under the Ranking mechanism is higher than that under any

[3]See Section 2.2.2 for a more detailed and formal explanation of the Ranking mechanism.

[4]To illustrate the case when there are multiple copies of each object, consider the same example but assume that there are $q = 2$ copies of each object. Then, agents i_1 and i_2 get a copy of object o_1, agents i_1 and i_3 get a copy of object o_2 and finally agents i_3 and i_2 get a copy of object o_3.

[5]A mechanism is *anonymous* if the allocation does not depend on the identity of agents; *neutral* if the allocation does not depend on the label of the objects and *incentive compatible* if truth-telling is a Bayes-Nash equilibrium.

other anonymous, neutral and incentive compatible ordinal mechanism. This implies that when agents' preferences are responsive, in particular, each agent has additively separable utility functions, for any set of utilities that are consistent with the ordinal rankings, the interim payoff of every agent is higher under the Ranking mechanism. That is, regardless of the preference intensities over objects, this result strongly favors the Ranking mechanism among other anonymous, neutral and incentive compatible ordinal mechanism mechanisms.

We next consider the case when we drop the assumption that each possible ranking over objects is equally likely. We first note that truth-telling may not be an equilibrium under the Ranking mechanism in this case. We next establish via an example that once this assumption is dropped, the strong welfare superiority of the Ranking mechanism does not hold anymore. In that example, truth-telling is still an equilibrium under the Ranking mechanism; however, another mechanism, namely the Random Serial Dictatorship (RSD) mechanism, which is described later in this section, yields a higher interim payoff than the Ranking mechanism for some type of agent. In addition to this observation, we investigate a special subclass in which agents' rankings over objects are common[6] and show that the Ranking mechanism is welfare superior to other incentive compatible mechanisms in the same strong sense—every type of every agent has a higher interim payoff under the Ranking mechanism.

Ordinal mechanisms have received great attention and a vast literature has emerged due to their practical applications. Many new mechanisms have been proposed, and most of them have been extensively used in practice. For example, as discussed in Chapter 1, for assigning students to schools, the so-called Boston mechanism and the well-known Deferred Acceptance (DA) mechanism (Gale and Shapley (1962)) are currently the most widely-used methods in many school districts in the US. Another ordinal mechanism, the Random Serial Dictatorship (RSD) mechanism (sometimes referred as the Random Priority (RP) mechanism) is a very popular mechanism that has been employed in many allocation problems. Under RSD, an order over agents is randomly determined and by following this order, the first agent is assigned his most preferred object, the second agent is assigned his most preferred object among those remaining, and so on. RSD has desirable features that make it a promising mechanism. It has straightforward rules and has been considered as a fair mechanism.

[6] Their preference intensities, or cardinal values, for the objects may differ, though.

Furthermore, it has desired incentive properties. When the number of agents is equal to the number of objects, truth-telling is a dominant strategy under RSD. However, as Bogomolnaia and Moulin (2001) illustrate, RSD may entail severe welfare loss. More precisely, they present an example with a set of preference profiles of agents in which another random allocation is unambiguously better than RSD for each agent.[7]

Given the inefficiency of this popular mechanism, Bogomolnaia and Moulin (2001) introduce a mechanism called the Probabilistic Serial (PS) mechanism. PS is based on a "probability eating" algorithm. When there is only one copy of each object type, it works as follows. Each object is considered to be a perfectly divisible good of probability shares. Agents start eating the probability share of their (reported) favorite object at equal speeds. Once an object is eaten up, that is, probability share of 1 is exhausted, the agents who were eating that object move to their next most preferred object among the available ones. The algorithm stops when all objects are "eaten up." We say that an agent gets an object with probability $p \in [0, 1]$ where p is the probability share of that object he has eaten. Bogomolnaia and Moulin (2001) show that PS is ordinally efficient, that is, there can not be any other mechanism such that for each preference profile, the random assignment of that mechanism stochastically dominates that of PS for each agent.[8] Of course, this result makes PS a promising mechanism. Note that although Bogomolnaia and Moulin (2001) consider PS when each agent demands only one object, the algorithm trivially extends to the case when agents may demand multiple objects as in our case. Although there are many other ordinal mechanisms that have been studied and received attention, PS and RSD appear to be the most "promising" mechanisms. We will explicitly compare the Ranking mechanism to PS and RSD in an example in Section 2.3.1.

Earlier work has mainly considered these mechanisms from an ex-post view, that is, under complete information when agents' preferences are commonly known. However, in most situations it is more realistic to assume that agents have private information regarding their preferences. To our surprise, there is little known about ordinal mechanisms such as their incentive properties and welfare properties when there is incomplete information. As noted in Chapter 1, there is a recent line of research that

[7]Manea (2009) shows that instances of such inefficiency of RSD are prevalent for large allocation problems. In particular, he shows that as the number of objects increase, the fraction of preference profiles for which this kind of inefficiency does not arise vanishes.

[8]This is not in contradiction with our result since Bogomolnaia and Moulin's result is an ex-post result but ours is interim. Furthermore, in their setting an agent can get only one object. We make an explicit comparison via an example in Section 2.3.1.

studies the school choice problem under incomplete information. Abdulkadiroglu, Che and Yasuda (2011), Troyan (2012), Miralles (2009) and Featherstone and Niederle (2008) are some of the examples. The analysis under incomplete information not only differs significantly from the one under complete information but also seems to suggest different conclusions.[9]

The analysis of ordinal mechanisms under incomplete information has been sporadic at best. In addition to the aforementioned papers in the school choice literature, Pesendorfer (2000) presents a formal model of cartel behavior in the school milk market where each agent has private information regarding its costs for the contracts (analogous to "object types" in our setting). He studies the optimal cartel both with and without transfers. His setting is analogous to a setting in which there is only one copy of each object type and these objects will be allocated to agents who have private information about their preferences. He shows that as the number of contracts goes to infinity, ex-ante expected payoff under the Ranking mechanism converges to the maximum ex-ante expected payoff. In another related work, McAfee (1992) considers the problem of assigning $n \geq 2$ objects to 2 agents–the problem of dissolving a partnership– both with and without transfers under incomplete information. In particular, for allocating objects without transfers, McAfee introduces the so-called Alternating Selection Mechanism (ASM) where agents choose objects one at a time in an alternating scheme. He provides sufficient conditions for the incentive compatibility of this mechanism and also investigates its welfare properties.

The remainder of this chapter is organized as follows. First, we formally describe the environment and present the preliminaries. Next, we consider a simple example in which $n = 3$ objects will be allocated to $m = 3$ agents and explicitly compare the Ranking mechanism to two "promising" mechanisms: RSD and PS. In Section 2.3, we state our main result and provide the proof for the case when there are $m = 2$ agents and relegate the more notationally involved proof for the case when there are $m > 2$ agents to Appendix B. Then, in Section 2.4, we consider the case when rankings over objects may not be equally likely. Finally, we conclude.

[9] For example, earlier results in the school choice literature favored the DA mechanism over the Boston mechanism in terms of the welfare of students under complete information.(See Ergin and Sönmez (2006), Chen and Sönmez (2006)) Aforementioned papers, however, suggest that the Boston mechanism may be welfare superior to the DA mechanism under incomplete information .

2.2 Preliminaries

2.2.1 Environment

Assume that there are $m \geq 2$ agents, $\mathcal{I} = \{i_1, ..., i_m\}$, $n \geq 2$ distinct object types, $\mathcal{O} = \{o_1, ..., o_n\}$ and there are $q \geq 1$ identical copies of each distinct object (type). Each agent i has a strict preference $\succ^i$ over $\mathcal{O}$ and this is privately known by agent i. We assume that each agent's preference ranking over objects is independent of other agents' rankings and each possible ranking is equally likely. Each agent demands only one copy of each object.

We denote that $o_j \succeq^i o_k$ if $o_j \succ^i o_k$ or $j = k$. We assume that each agent's preference over the random allocations is responsive—in the first order stochastic-dominance sense— with respect to $\succ^i$. Formally, let $p = \left(p^i_j\right)_{j=1}^n$ and $q = \left(q^i_j\right)_{j=1}^n$ be two random allocations, where p^i_j (and, similarly q^i_j) is the probability that agent i gets object o_j. Then, agent i (weakly) prefers random allocation p to q, $p \succeq^i q$, if for any $j \in \{1, ..., n\}$

$$\sum_{o_k \succeq_i o_j} p^i_k \geq \sum_{o_k \succeq_i o_j} q^i_k$$

Note that additively separable preferences over bundles of objects have this property. More precisely, if agents' preferences over the bundles of objects are additively separable, that is, preferences can be represented by utility functions $u^i : \mathcal{O} \longrightarrow \mathbb{R}$ such that utility of receiving a bundle of objects $O \subset \mathcal{O}$ is $\sum_{o \in O} u^i(o)$, agents derive a higher utility from random allocation p.[10] We will be focusing on additively separable preferences in the rest of the chapter but the main result holds for responsive preferences, in general.

After each agent privately learns his preference ranking, he reports a ranking over objects (which may or may not be his true preferences) and a predetermined allocation rule determines the assignments.

We start with preliminary definitions and lemmas to state and prove our main result.

[10] See Appendix B for a proof of this statement.

2.2.2 Ordinal Mechanisms

We define an *ordinal mechanism* as an allocation rule that assigns objects to agents given agents' strict ordinal rankings over objects. Formally, let us denote agent i's (reported) ordinal ranking over objects by $R^i = (R^i(1), R^i(2), ..., R^i(n))$ where $R^i(k)$ is the object that agent i ranks as his k^{th} choice. Let $R = (R^i)_{i=1,...,m}$ denote a set of preference profiles.

An ordinal mechanism φ is an allocation rule $\varphi(R) = (\varphi^i(R))_i$, where $\varphi^i(R) = \left(\varphi^i_j(R)\right)_{j=1,...,n}$ such that for each $j \in \{1, ..., n\}$

$$\sum_{i=1}^{m} \varphi^i_j(R) = q$$

and $\varphi^i_j(R) \in [0,1]$ for all i and j, where $\varphi^i_j(R)$ denotes the probability that agent i gets a copy of object o_j when the reported ordinal preferences are R.

Let us also denote $\varphi_j(R) = \left(\varphi^i_j(R)\right)_i$ as the probabilities that each agent gets object o_j.

Given any R and any permutation $\pi : \mathcal{I} \to \mathcal{I}$ over agents, let $R^\pi = \left(R^{\pi(1)}, ..., R^{\pi(m)}\right)$ and $(\varphi_j(R))^\pi = \left(\varphi^{\pi(1)}_j(R), ..., \varphi^{\pi(n)}_j(R)\right)$.

We will say that an ordinal mechanism is *anonymous* if for any permutation π over agents and any R

$$\varphi_j(R^\pi) = (\varphi_j(R))^\pi$$

for all j. That is, an ordinal mechanism is anonymous if the allocation does not depend on the identity of the agents.

Given any R and any permutation $\sigma : \mathcal{O} \to \mathcal{O}$ over objects, let, similarly, R^σ be the preference rankings in which for each agent rankings of objects j and $\sigma(j)$ are exchanged and $(\varphi^i(R))^\sigma$ be the allocation rule of agent i obtained from $\varphi^i(R)$ by exchanging objects j and $\sigma(j)$.

We will say that an ordinal mechanism is *neutral* if for any permutation σ over objects and any R

$$\varphi^i(R^\sigma) = \left(\varphi^i(R)\right)^\sigma$$

for all i. That is, an ordinal mechanism is neutral if the allocation does not depend on the label of the objects.

Some examples of ordinal mechanisms that satisfy anonymity and neutrality are:

The Ranking Mechanism:[11] A copy of an object is given to the agents, up to the number of available copies, who report the highest preference for that object and if there is a tie, that is, if the number of agents who rank the object highest is more than the number of available copies, the objects are allocated according to a random draw among those agents.

Formally:

- If there is some $k \in \{1, ..., n\}$ such that $j \geq q$ agents rank an object as their k^{th} choices and there is no other agent who reports a higher preference ranking for the same object, then each one of these j agents get a copy of that object with probability $\frac{q}{j}$.
- If there is some $k \in \{2, ..., n\}$ such that when there are $j \leq q$ agents who rank the object as one of $\{1, ..., (k-1)\} - th$ choice and $i \geq (q-j)$ agents who rank the object as a k^{th} choice, then each of the j agents will get a copy of each object and the remaining $(q-j)$ copies will be allocated to i agents who rank the object as their k^{th} choices, randomly if necessary.

Note that the Ranking mechanism is anonymous and neutral when each object has q identical copies.

The Random Serial Dictatorship (RSD) Mechanism: Assume that there are n agents and n objects with one copy of each. First, each possible ordering over agents is drawn with equal probability and then, by following the determined order, the first agent is assigned his most preferred object, the second agent is assigned his most preferred object among those remaining, and so on.[12] Note that RSD is anonymous and neutral.

[11] Pesendorfer (2000) defines the Ranking mechanism for the case when each object (type) has only one copy. We extend it to the case with multiple copies of objects.

[12] RSD has usually been studied under the settings when the number of agents is equal to the number of agents so that each agent chooses only once. Although there may be different versions to extend RSD to the case in which the number of objects is possibly more than the number of agents, a natural way seems to repeat the order over agents. That is, first randomly draw an order over agents. Each agent is assigned his most preferred object among those available, and once every agent is assigned an object, following the same order each agent is assigned another object.

The Alternating Selection Mechanism (ASM) (McAfee, 1992): Assume that there are $m = 2$ agents and $n \geq 2$ objects. ASM is the natural extension of RSD in which two agents will pick an object in an alternating scheme. This mechanism is anonymous, neutral.

The Probabilistic Serial Mechanism (PS) (Bogomolnaia and Moulin, 2001): Each object is considered to be a perfectly divisible good of probability shares under PS. Consider the case when there is one copy of each object. Each agent starts eating the probability share of his (reported) favorite object at equal speeds. Once an object is eaten up, that is probability share of 1 is exhausted, the agents who were eating that object move to their next most preferred objects among the available ones. The algorithm stops when all objects are eaten up. We say that an agent gets an object with probability $p \in [0, 1]$ where p is the probability share of that object he has eaten.[13] Trivially, PS is anonymous and neutral.

2.2.3 Incentive Compatibility

An ordinal mechanism is *incentive compatible* if agents cannot gain by misreporting their true preferences given that other agents report their true preferences. In other words, an ordinal mechanism is incentive compatible if truthful reporting is a Bayes-Nash equilibrium of the induced game.

We look for the welfare maximizing ordinal mechanism among incentive compatible mechanisms. Note that each agent is ex-ante symmetric in their ordinal preferences and each object is ex-ante symmetric for each agent. Hence, it will be natural to restrict ourselves to neutral and anonymous mechanisms.

We first note that under a neutral ordinal mechanism, when each possible ranking over objects is equally likely, the interim probability of obtaining an object only depends on the ranking of that object when other agents report their true rankings. To see this, without loss of generality (w.l.o.g.), consider agent 1 and consider two preference rankings

$$R^1_* = \left(R^1_*(1), ..., R^1_*(k-1), o_k, R^1_*(k+1), ..., R^1_*(n)\right)$$

[13]In general, if there are q identical copies, it is "probability share of q" instead of 1. See Kojima and Manea (2010) for a formal definition of PS when there are multiple copies of objects.

and

$$R^1_{**} = \left(R^1_{**}(1), ..., R^1_{**}(k-1), o_k, R^1_{**}(k+1), ..., R^1_{**}(n)\right)$$

Note that although rankings for the other objects may be different in these preference rankings, k^{th} choice is object o_k in both. We claim that the interim probability of obtaining object o_k is the same for both rankings given that other agents report their true rankings. Let, for any R^1,$P^\varphi_k(R^1)$ be the interim probability of getting object o_k for agent 1 when his preference ranking is R^1 given that other agents report their true rankings. Note that

$$P^\varphi_k\left(R^1\right) = \sum_{R^{-1}=(R^2,...,R^m)} \varphi^1_k\left(R^1, R^{-1}\right) \Pr\left(R^{-1}\right)$$

where summation is over all the possible preference rankings of other agents.

Now, consider the permutation σ over objects such that $\sigma\left(R^1_*(j)\right) = R^1_{**}(j)$ for all j. Note that, due to neutrality, since $\sigma(o_k) = o_k$, for any R^{-1}, we have that

$$\varphi^1_k\left(R^1_*, R^{-1}\right) = \varphi^1_k\left(R^1_{**}, \left(R^{-1}\right)^\sigma\right) \tag{2.1}$$

where $\left(R^{-1}\right)^\sigma$ is obtained from R^{-1} by relabeling objects according to permutation σ.

Now,

$$\begin{aligned}
P^\varphi_k\left(R^1_{**}\right) &= \sum_{R^{-1}=(R^2,...,R^m)} \varphi^1_k\left(R^1_{**}, R^{-1}\right) \Pr\left(R^{-1}\right) \\
&= \sum_{R^{-1}=(R^2,...,R^m)} \varphi^1_k\left(R^1_{**}, \left(R^{-1}\right)^\sigma\right) \Pr\left(\left(R^{-1}\right)^\sigma\right) \\
&= \sum_{R^{-1}=(R^2,...,R^m)} \varphi^1_k\left(R^1_*, R^{-1}\right) \Pr\left(R^{-1}\right) \\
&= P^\varphi_k\left(R^1_*\right)
\end{aligned}$$

where third equality is due to (2.1) and the fact that $\Pr\left(\left(R^{-1}\right)^\sigma\right) = \Pr\left(R^{-1}\right)$ since each ranking is equally likely.

Hence, we will characterize an anonymous, neutral mechanism φ by $\left(P^\varphi_k\right)^n_{k=1}$, where P^φ_k denotes the *interim* probability that an agent receives his (reported) k^{th} ranked choice when all the remaining agents report their true rankings under the mechanism φ.

Given this observation, it is easy to see that incentive compatibility of an anonymous and neutral mechanism φ is equivalent to P_k^{φ} to be (weakly) decreasing in k:

Remark 1. *An anonymous and neutral mechanism φ is incentive compatible iff $P_k^{\varphi} \geq P_j^{\varphi}$ whenever $k < j$.*

We will now verify that when each possible ranking is equally likely, truth-telling is an equilibrium under the Ranking mechanism.[14]

Lemma 1. *Assume that there are $m \geq 2$ agents and $n \geq 2$ distinct objects with $q \geq 1$ identical copies of each object. Assume further that each agent's preference ranking over objects is independent of other agents' rankings and each possible ranking is equally likely. Then, reporting true ordinal rankings is an equilibrium under the Ranking mechanism.*

Proof. Note that when there are equal number of copies of each object, the Ranking mechanism is anonymous and neutral. Then, as discussed earlier in the section, the interim probability of obtaining an object only depends on the rank of that object. W.l.o.g. consider agent i_1 and assume that $R^1 = (o_1, ..., o_n)$. That is, his preference is such that $o_1 \succ o_2 \succ ... \succ o_n$. Let $P_k^{rank}(R^1)$ be the interim probability of obtaining o_k for agent i_1 under the Ranking mechanism given that other agents report their true preferences. Now,

$$P_k^{rank}\left(R^1\right) = \sum_{R^{-1}=(R^2,...,R^m)} \mu_k^1\left(R^1, R^{-1}\right) \Pr\left(R^{-1}\right)$$

, where $\mu_k^1(R^1, R^{-1})$ is the probability that agent i_1 obtains object o_k when other agents report their preferences truthfully under Ranking mechanism. Now, w.l.o.g. suppose that agent i_1 reports that he has a higher preference for object o_2 than object o_1 although he prefers object o_1. Note that at any state[15] in which agent i_1 obtains object o_1 with probability 1 when he reports o_1 below o_2, he would obtain o_1 again with probability 1 if he reported his true preference. Furthermore, when he obtains object o_1 with probability $1 > p \geq 0$ when he misreported, he would obtain o_1 at least with probability p: If there is a tie with other agents when he is misreporting, he

[14] In general, reporting true rankings may not be an equilibrium under the Ranking mechanism. See Section 2.4.1 for an example.

[15] "State" refers to a set of preferences of other agents.

would instead obtain object o_1 with probability 1 and if he is not getting the object by misreporting, he may get the object with a positive probability if he tells the truth. Therefore, the interim probability of obtaining an object (weakly) increases by reporting a higher preference for that object. Thus, by Remark 1, the Ranking mechanism is incentive compatible. □

Indeed, the interim probability of obtaining the k^{th} choice under the Ranking mechanism may be explicitly computed. Note that when $q \geq m$, each agent receives one copy of each object. Assume that $1 \leq q \leq (m-1)$ Consider an agent, w.l.o.g. say agent i_1, and assume w.l.o.g. that his ranking over objects is $o_1 \succ o_2 \succ ... \succ o_n$. We want to compute the interim probability that agent i_1 obtains object o_k, which is his k^{th} choice. Now, letting j denote the number of other agents who rank object o_1 as their first choices, we have that

$$\begin{aligned} P_1^{rank} &= \sum_{j=0}^{q-1} \left[\binom{m-1}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{m-1-j} \right] \\ &\quad + \sum_{j=q}^{m-1} \left[\binom{m-1}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{m-1-j} \right] \left(\frac{q}{i+1}\right) \end{aligned}$$

since when $j \leq (q-1)$, agent i_1 gets a copy of object o_1 with probability 1 and if $j \geq q$ he gets object o_1 with probability $\frac{q}{i+1}$. The term in the squared brackets is the probability that exactly j agents (other than agent i_1) rank object o_1 as a first choice and the remaining $(m-1-j)$ agents rank object o_1 at a lower rank, that is as one of $\{2, 3, ..., n\} - th$ choice.

For $k \geq 2$, let i be the number of agents who report a higher preference for object o_k, that is who rank o_k as one of $\{1, ..., (k-1)\} - th$ choice and j be the number of agents (other than agent i_1) who rank object o_k as their k^{th} choice. Note first that agent i_1 can not get object o_k if $i \geq q$. If $i + j \leq (q-1)$, agent i_1 gets object o_k for sure. If $i + j \geq q$, then agents who rank object o_k as one of their $\{1, ..., (k-1)\} - th$ choice will get a copy of object o_k and the remaining $(q-i)$ copies of object o_k will be assigned randomly to agents who rank object o_k as their k^{th} choices. Thus, since there are $(j+1)$ agents in total who rank object o_k as their k^{th} choices, each of these agents will get a copy of object o_k with probability $\frac{q-1}{j+1}$. Thus, the interim probability of obtaining his k^{th} choice, $k \geq 2$, for agent i_1 under the Ranking mechanism is

$$\begin{aligned} P_k^{rank} &= \sum_{i=0}^{q-1}\sum_{j=1}^{q-i-1}\left[\binom{m-1}{i}\binom{m-1-i}{j}\left(\frac{k-1}{n}\right)^i\left(\frac{1}{n}\right)^j\left(\frac{n-k}{n}\right)^{m-1-i-j}\right] \\ &\quad +\sum_{i=0}^{q-1}\sum_{j=(q-i)}^{m-i-1}\left[\binom{m-1}{i}\binom{m-1-i}{j}\left(\frac{k-1}{n}\right)^i\left(\frac{1}{n}\right)^j\left(\frac{n-k}{n}\right)^{m-1-i-j}\right]\left(\frac{q-i}{j+1}\right) \end{aligned}$$

Although no closed-form expression is readily available in general, for the case of $q = 1$,

$$\begin{aligned} P_k^{rank} &= \sum_{i=0}^{m-1}\binom{m-1}{i}\left(\frac{1}{n}\right)^i\left(\frac{n-k}{n}\right)^{m-1-i}\frac{1}{i+1} \\ &= \sum_{i=0}^{m-1}\frac{(m-1)!}{i!\,(m-1-i)!}\left(\frac{1}{n}\right)^i\left(\frac{n-k}{n}\right)^{m-1-i}\frac{1}{i+1} \\ &= \sum_{i=0}^{m-1}\frac{(m)!}{(i+1)!\,(m-1-i)!}\frac{n}{m}\left(\frac{1}{n}\right)^{i+1}\left(\frac{n-k}{n}\right)^{m-1-i} \\ &= \frac{n}{m}\sum_{i=0}^{m-1}\binom{m}{i+1}\left(\frac{1}{n}\right)^{i+1}\left(\frac{n-k}{n}\right)^{m-1-i} \\ &= \frac{n}{m}\sum_{i=1}^{m}\binom{m}{i}\left(\frac{1}{n}\right)^{i}\left(\frac{n-k}{n}\right)^{m-i} \\ &= \frac{n}{m}\left[\left(\frac{n-k+1}{n}\right)^m-\left(\frac{n-k}{n}\right)^m\right] \\ &= \frac{1}{mn^{m-1}}\left[(n-k+1)^m-(n-k)^m\right] \end{aligned}$$

That is,

$$P_k^{rank} = \frac{(n-k+1)^m-(n-k)^m}{mn^{m-1}} \tag{2.2}$$

It can be easily checked that P_k^{rank} is decreasing in k in this case as we discussed earlier without explicit calculations.

We have established that the Ranking mechanism is incentive compatible under our assumptions. Some other incentive compatible mechanisms in this environment include (but not limited to) the ordinal mechanisms we mentioned before. Namely, the Random Serial Dictatorship (RSD) mechanism, the Alternating Selection Mechanism (ASM) and the Probabilistic Serial mechanism are incentive compatible when preference rankings over objects are equally likely. It is well-known that the Random Serial Dictatorship is even strategy-proof, that is, truth-telling is a weakly dominant strategy for each agent when the number of agents and objects are equal. For the ASM, it is easy to show that distributional assumptions we imposed satisfy the sufficient conditions for incentive compatibility of the ASM that are presented by McAfee (1992). The arguments for the incentive compatibility of the Probabilistic Serial mechanism are almost identical to the arguments for the Ranking mechanism. Since ranking an object at a higher spot can only increase the probability of obtaining that object for each possible preference of other agents and when each ranking over objects is equally likely, the interim probability of obtaining an object only depends on the reported rank of that object, truth-telling is an equilibrium. Furthermore, Kojima and Manea (2010) show that the Probabilistic Serial mechanism is even strategy-proof when the number of copies of each object is sufficiently large.

2.3 The Welfare Maximizing Ordinal Mechanism

Note that since each agent demands only one copy of each object, when $q \geq m$, it is best to give one copy of each object to each agent. Thus, we will assume that $1 \leq q \leq (m-1)$ to avoid this triviality. We begin with the following definition.

Definition 1. *A mechanism φ (first-order) stochastically dominates another mechanism φ' iff for all $K \in \{1, ..., n\}$,*

$$\sum_{k=1}^{K} P_k^{\varphi} \geq \sum_{k=1}^{K} P_k^{\varphi'}$$

where P_k^X is the interim probability of obtaining the k^{th} choice under mechanism $X \in \{\varphi, \varphi'\}$.

We next proceed with an example in which we are going to make explicit calculations to compare the Ranking mechanism to RSD and PS.

2.3.1 An Example: 3 objects and 3 agents

Let us consider a simple example in which $n = 3$ objects, $\{o_1, o_2, o_3\}$, are to be allocated to $m = 3$ agents, $\{i_1, i_2, i_3\}$, when there is only one copy of each object. We assume that each agent's preference ranking is independent of other agents' rankings and each possible ranking is equally likely. W.l.o.g., let us do our calculations from agent i_1's perspective and assume that his ordinal preference list over objects is $o_1 \succ o_2 \succ o_3$. That is, object o_k is agent i_1's k^{th} choice. Note that for the remaining agents, there are 6 possible rankings over objects for each agent.

Let

$$\begin{aligned} R_1 &:= (o_1 \succ o_2 \succ o_3) \\ R_2 &:= (o_1 \succ o_3 \succ o_2) \\ R_3 &:= (o_2 \succ o_1 \succ o_3) \\ R_4 &:= (o_2 \succ o_3 \succ o_1) \\ R_5 &:= (o_3 \succ o_1 \succ o_2) \\ R_6 &:= (o_3 \succ o_2 \succ o_1) \end{aligned}$$

Although we can get interim probabilities for the Ranking mechanism simply by using the formula (2.2) derived earlier in Section 2.2.3, let us do the calculations for each preference profile of agents i_2 and i_3 to give a better understanding of the workings of the Ranking mechanism.

For each case, the table below shows the probabilities that agent i_1 receives each object under the Ranking mechanism where columns are agent i_2's preferences and rows are agent i_3's preferences. More precisely, (p_1, p_2, p_3) in the row R_j, column R_k position denotes the probabilities that agent i_1 obtains object o_1, object o_2 and object o_3 when agent i_2 and agent i_3 report rankings R_j and R_k, respectively.

	R_1	R_2	R_3	R_4	R_5	R_6
R_1	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{3}, \frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, 0, \frac{1}{3}\right)$	$\left(\frac{1}{2}, 0, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{3}, 0\right)$
R_2	$\left(\frac{1}{3}, \frac{1}{2}, 0\right)$	$\left(\frac{1}{3}, 1, 0\right)$	$\left(\frac{1}{2}, 0, 0\right)$	$\left(\frac{1}{2}, 0, 0\right)$	$\left(\frac{1}{2}, 1, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$
R_3	$\left(\frac{1}{2}, 0, \frac{1}{3}\right)$	$\left(\frac{1}{2}, 0, 0\right)$	$\left(1, 0, \frac{1}{3}\right)$	$(1, 0, 0)$	$(1, 0, 0)$	$(1, 0, 0)$
R_4	$\left(\frac{1}{2}, 0, 0\right)$	$\left(\frac{1}{2}, 0, 0\right)$	$(1, 0, 0)$	$(1, 0, 0)$	$(1, 0, 0)$	$(1, 0, 0)$
R_5	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, 1, 0\right)$	$(1, 0, 0)$	$(1, 0, 0)$	$(1, 1, 0)$	$\left(1, \frac{1}{2}, 0\right)$
R_6	$\left(\frac{1}{2}, \frac{1}{3}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$(1, 0, 0)$	$(1, 0, 0)$	$\left(1, \frac{1}{2}, 0\right)$	$\left(1, \frac{1}{3}, 0\right)$

Since each ordinal ranking is equally likely for each agent, the interim probabilities for agent i_1 are just the averages of these numbers which are given by

$$P^{rank} = \left(\frac{19}{27}, \frac{7}{27}, \frac{1}{27}\right)$$

Second, we do the computations for RSD. Again, consider agent i_1. Recall that under RSD an order over agents will be randomly drawn and following the order each agent will choose an object from the available set of objects. Now, agent i_1 will be selected to be the first agent to pick with probability $\frac{1}{3}$ and hence will pick object o_1. With probability $\frac{1}{3}$, he will be the second agent to pick. Now, if the agent who picked first has picked object o_2 or o_3 which happens with probability $\frac{2}{3}$, agent i_1 will pick o_1. If the agent who picked first has picked object o_1 which happens with probability $\frac{1}{3}$, agent i_1 will pick o_2. Finally, with probability $\frac{1}{3}$, i_1 will be the last one to pick. Note that at the final step, each object will remain unpicked with equal probability, $\frac{1}{3}$. Thus, the interim probabilities for agent i_1 are

$$P_1^{RSD} = \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{3} = \frac{2}{3}$$

$$P_2^{RSD} = \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{3} = \frac{2}{9}$$

$$P_3^{RSD} = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

Hence,

$$P^{RSD} = \left(\frac{2}{3}, \frac{2}{9}, \frac{1}{9}\right)$$

Finally, we do the computations for PS.

	R_1	R_2	R_3	R_4	R_5	R_6
R_1	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$	$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$	$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$
R_2	$\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right)$	$\left(\frac{1}{3}, \frac{2}{3}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$
R_3	$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$	$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{2}{3}, 0, \frac{1}{3}\right)$	$\left(\frac{3}{4}, 0, \frac{1}{4}\right)$	$(1, 0, 0)$	$(1, 0, 0)$
R_4	$\left(\frac{1}{2}, \frac{1}{6}, \frac{1}{3}\right)$	$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$	$\left(\frac{3}{4}, 0, \frac{1}{4}\right)$	$(1, 0, 0)$	$(1, 0, 0)$	$(1, 0, 0)$
R_5	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$(1, 0, 0)$	$(1, 0, 0)$	$\left(\frac{2}{3}, \frac{1}{3}, 0\right)$	$\left(\frac{3}{4}, \frac{1}{4}, 0\right)$
R_6	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$(1, 0, 0)$	$(1, 0, 0)$	$\left(\frac{3}{4}, \frac{1}{4}, 0\right)$	$(1, 0, 0)$

To illustrate these computations we explicitly consider one case. The remaining cases are computed similarly.

Assume that agents i_2 and i_3 report R_2 and R_3, respectively. Then, agents i_1 and i_2 start to eat object o_1 and agent i_3 eats object o_2. After agents i_1 and i_2 both eat $\frac{1}{2}$ of object o_1, o_1 is exhausted. Then, agent i_1 starts to eat object o_2 and agent i_2

starts to eat object o_3. Note that $\frac{1}{2}$ of object o_2 was already eaten by agent i_3. Then, agents i_1 and i_3 eat the remaining $\frac{1}{2}$ of object o_2. Hence, agent i_1 eats $\frac{1}{4}$ of object o_2. Then, after eating up object o_2, agents i_1 and i_3 start eating object o_3. Since $\frac{1}{4}$ of object o_3 has already been eaten up by agent i_2, remaining $\frac{3}{4}$ is split between all agents and hence agent i_1 ends up $\frac{1}{4}$ of the total share of object o_3. Thus, agent i_1's shares are $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$.

Given these, we have that

$$P^{PS} = \left(\frac{71}{108}, \frac{51}{216}, \frac{23}{216}\right)$$

Note that

$$P^{rank} - P^{RSD} = \frac{1}{27}(1, 1, -2)$$

and

$$P^{rank} - P^{PS} = \frac{5}{216}(2, 1, -3)$$

Hence, the Ranking mechanism stochastically dominates both RSD and PS.

2.3.2 Main Result

After presenting a simple example, we now state and prove our main result that shows that the conclusion of this example holds more generally. We are going to present the proof here for the case when there are $m = 2$ agents. The proof for the case when $m > 2$ is similar but notationally more involved and is relegated to Appendix B.

Theorem 2. *The Ranking mechanism stochastically dominates any other anonymous, neutral and incentive compatible ordinal mechanism.*

Proof. $(\boldsymbol{m = 2})$ First, note that when $m = 2$, it must be that $q = 1$ since $q \leq (m - 1)$. Assume that φ is an anonymous, neutral and incentive compatible ordinal mechanism. Let us denote the set of agents by $\{1, 2\}$ and the set of objects by $\{o_1, o_2, ..., o_n\}$. Consider agent 1 and w.l.o.g. assume that his ranking is $R^1 = (o_1, o_2, ..., o_n)$. That is, object o_k is his k^{th} choice. Let μ stand for the Ranking mechanism. Remembering that P_k^{φ} is the interim probability of obtaining k^{th} choice given that other agents report truthfully, we have that

$$P_k^{\varphi} = \sum_{R^2} \varphi_k^1\left(R^1, R^2\right) \Pr\left(R^2\right)$$

Note that since each possible ranking over objects is equally likely, $\Pr(R^2) = \frac{1}{n!}$ for any R^2 since there are $(n!)$ possible orderings. Now, by definition of the Ranking mechanism we have that

$$\mu_1^1\left(R^1, R^2\right) = \begin{cases} 1 & \text{if} \quad o_1 \neq R^2(1) \\ \frac{1}{2} & \text{if} \quad o_1 = R^2(1) \end{cases}$$

since agent 1 will get object o_1 with probability 1 if object o_1 is not agent 2's first choice and if object o_1 is agent 2's first choice, then agent 1 will get object o_1 with probability $\frac{1}{2}$.

If $\varphi_1^1(R^1, R^2) \leq \mu_1^1(R^1, R^2)$ for all R^2, we have trivially that $P_1^\varphi \leq P_1^\mu$. Suppose that there is some R_*^2 such that $\varphi_1^1(R^1, R_*^2) > \mu_1^1(R^1, R_*^2)$. First, since $\varphi_1^1(R^1, R^2) \leq 1$ when $o_1 \neq R^2(1)$, it must be that $o_1 = R_*^2(1)$. Hence, $\varphi_1^1(R^1, R_*^2) = p > \frac{1}{2}$. That is, agent 1 gets object o_1 with probability $p > \frac{1}{2}$ when reported preferences of agent 1 and agent 2 are R^1 and R_*^2, respectively which in turn means that agent 2 receives object o_1 with probability $1 - p < \frac{1}{2}$. That is, $\varphi_1^2(R^1, R_*^2) = 1 - p < \frac{1}{2}$.

Now, if $R^1 = R_*^2$, due to anonymity, it must be that $\varphi_1^1(R^1, R_*^2) = \varphi_1^2(R^1, R_*^2) = \frac{1}{2}$, which would be a contradiction. Hence, it must be that $R^1 \neq R_*^2$. Note that by anonymity, $\varphi_1^1(R_*^2, R^1) = 1 - p$. Consider the permutation σ over objects such that $\sigma(R_*^2(k)) = R^1(k)$ for all $k \in \{1, ..., n\}$. Then, by neutrality, we have that $\varphi_1^1\left(R^1, (R^1)^\sigma\right) = 1 - p$ since $\sigma(o_1) = o_1$.

Now, if $(R^1)^\sigma = R_*^2$, then we would get a contradiction since we would have $\varphi_1^1(R^1, R_*^2) = p > \frac{1}{2} > 1 - p = \varphi_1^1\left(R^1, (R^1)^\sigma\right)$. Therefore, it must be that $(R^1)^\sigma \neq R_*^2$. Note also that R_*^2 and $(R^1)^\sigma$ are equally likely states. Hence, for an increase in probability of getting first choice at some state, we have found an equally likely state with the same amount of decrease in the probability of obtaining first choice. Note further that if there is another state, say $R_{**}^2 \neq R_*^2$, such that $\varphi_1^1(R^1, R_{**}^2) > \frac{1}{2}$ when $o_1 = R_{**}^2(1)$, we could find another state $(R^1)^{\sigma'}$ as we did for R_*^2 such that any increase in state R_{**}^2 comes with exactly a same decrease in another equally likely state. Note also that when $R_*^2 \neq R_{**}^2$, $(R^1)^{\sigma'} \neq (R^1)^\sigma$.

To sum up verbally, when agent 1 is gaining in probability of obtaining his first choice, object o_1, at some state under mechanism φ compare to the Ranking mechanism, it must mean that agent 2 is at a loss which is equal to the gain of agent 1. And, by anonymity and neutrality, when agent 1 is in agent 2 s position, it must be that agent 1 is at a loss in probability of obtaining his first choice. Since every state is equally likely, agent 1's interim probability of getting object o_1 can not be higher under mechanism φ. Thus, we have that $P_1^{\mu} \geq P_1^{\varphi}$.

If $\varphi_2^1(R^1, R^2) \leq \mu_2^1(R^1, R^2)$ for all R^2, we have trivially that $P_2^{\varphi} \leq P_2^{\mu}$ and hence $P_1^{\varphi} + P_2^{\varphi} \leq P_1^{\mu} + P_2^{\mu}$. Suppose that there exists some R_*^2 such that $\varphi_2^1(R^1, R_*^2) > \mu_2^1(R^1, R_*^2)$. Now, for any R^2,

$$\mu_2^1\left(R^1, R^2\right) = \begin{cases} 1 & \text{if} \quad o_2 = R^2(k) \text{ for some } k > 2 \\ \frac{1}{2} & \text{if} \quad o_2 = R^2(2) \\ 0 & \text{if} \quad o_2 = R^2(1) \end{cases}$$

since if agent 2 ranks object o_2 as his first choice, agent 1 will not get object o_2; if object o_2 is agent 2's second choice, like agent 1, then agent 1 will get object o_2 with probability $\frac{1}{2}$ and finally if agent 2 ranks object o_2 as his k^{th} choice for $k > 2$, then agent 1 will get object o_1 for sure.

Suppose first that $o_2 = R_*^2(2)$, thus, $\varphi_2^1(R^1, R_*^2) = p > \frac{1}{2}$. Then, agent 2 receives object o_2 with probability $1 - p < \frac{1}{2}$, that is, $\varphi_2^2(R^1, R_*^2) = 1 - p$. Then, by identical arguments as we did above, any increase in probability of agent 1's obtaining object o_2 in some state comes with exactly a same decrease in another equally likely state.

Second, suppose that $o_2 = R_*^2(1)$ and $\varphi_2^1(R^1, R_*^2) = p > 0$. Note that in this situation agent 2 gets object o_2 with probability $1 - p$. Formally, we have that $\varphi_2^2(R^1, R_*^2) = 1 - p$. Then, by anonymity, we have that $\varphi_2^1(R_*^2, R^1) = 1 - p$.

Now, consider a permutation σ over objects such that $\sigma(R_*^2(k)) = R^1(k)$ for all $k \in \{1, ..., n\}$. Then, by neutrality, $\varphi_1^1\left(R^1, (R^1)^{\sigma}\right) = 1 - p$ since $\sigma(o_2) = o_1$. That is, there exists a state, call it $R_{**}^2 = (R^1)^{\sigma}$, such that agent i_1 gets object o_1 with probability $1 - p$. Note that $\varphi_2^1(R^1, R_*^2) = p$, $\varphi_1^1(R^1, R_{**}^2) = 1 - p$ and $\mu_2^1(R^1, R_*^2) = 0$, $\mu_1^1(R^1, R_{**}^2) = 1$. Furthermore, R_*^2 and R_{**}^2 are equally likely states and hence an increase in obtaining the second choice comes with exactly same amount of decrease in obtaining the first choice compare to the Ranking mechanism.

Rephrasing what we did, when agent 1 is gaining in probability of obtaining his second choice at some state compare to the Ranking mechanism, it must mean that agent 2 is losing in probability of obtaining his first choice where his loss is equal to the gain of agent 1. And, by anonymity and neutrality, when agent 1 is in agent 2 s position, it must be that agent 1 is at a loss in probability of obtaining his first choice. Since every state is equally likely, the sum of interim probabilities of obtaining first two top choices can not exceed under mechanism φ compare to the Ranking mechanism.

Thus, $P_1^{\mu} + P_2^{\mu} \geq P_1^{\varphi} + P_2^{\varphi}$.

If we move to comparison for third choice, exact similar arguments[16] will yield the same conclusion. Continuing in this manner, we have that for any $K \in \{1, ..., n\}$

$$\sum_{k=1}^{K} P_k^{\mu} \geq \sum_{k=1}^{K} P_k^{\varphi}$$

which is the desired result.

We refer the reader to Appendix B for the proof of this result when there are $m > 2$ agents and $1 \leq q \leq (m-1)$ copies of each object type. □

Then, by Lemma 9 in Appendix B, we have the following result.

Corollary 1. *When agents' preferences over the bundles of objects are additively separable, each agent has a higher interim payoff under the Ranking mechanism than any other anonymous, neutral and incentive compatible mechanism for any set of utility functions.*

That is, regardless of the preference intensities, each agent has a higher interim payoff under the Ranking mechanism.

We, now, compare the Ranking mechanism to ASM for $n = 12, q = 1$ and $m = 2$ and by calculating the interim probabilities for the Ranking mechanism by using the general formula given by expression (2.2) which was derived earlier. McAfee (1992) also explicitly computes the interim probabilities for ASM when $n = 12, q = 1$.

Example 5. *When* $n = 12$

$$P^{ASM} = \left(\frac{23}{24}, \frac{115}{132}, \frac{69}{88}, \frac{23}{33}, \frac{161}{264}, \frac{23}{44}, \frac{3691}{8448}, \frac{2999}{8448}, \frac{3153}{11\,264}, \frac{1205}{5632}, \frac{3565}{22\,528}, \frac{231}{2048}\right)$$

$$P^{rank} = \left(\frac{23}{24}, \frac{21}{24}, \frac{19}{24}, \frac{17}{24}, \frac{15}{24}, \frac{13}{24}, \frac{11}{24}, \frac{9}{24}, \frac{7}{24}, \frac{5}{24}, \frac{3}{24}, \frac{1}{24}\right)$$

[16] The case of $\mu_3^1\left(R^1, R_*^2\right) > \frac{1}{2}$ when $o_3 = R^2(3)$ is identical to the arguments that we did for $\mu_2^1\left(R^1, R_*^2\right) > \frac{1}{2}$ and in the case $\mu_3^1\left(R^1, R_*^2\right) > 0$ when $o_3 = R^2(1)$ or $o_3 = R^2(2)$, any increase in obtaining third choice will result in same amount of decrease in obtaining first *or* second top choice. Hence, the sum of probabilities of the first three choices under φ will not exceed the one under the Ranking mechanism.

where $P^X = \left(P_k^X\right)_{k=1}^{k=n}$ *denotes the interim probability of obtaining* k^{th} *choice under mechanism* X.
Note that

$$P_1^{rank} = P_1^{ASM}$$

$$P_k^{rank} > P_k^{ASM} \text{ when } 1 < k < 10$$

and

$$P_k^{rank} < P_k^{ASM} \text{ when } 12 \geq k \geq 10$$

and hence the Ranking mechanism trivially stochastically dominates ASM, as it should by the above result. Therefore, every type has a higher interim payoff under the Ranking mechanism.

2.4 Discussions

2.4.1 Asymmetric Preferences

We have assumed that agents' preferences are such that each possible ordinal ranking is equally likely. We now consider the case when this is not the case.

First, we note that in general reporting true preferences may not be an equilibrium in this case. To illustrate this, consider the following example. Although there is complete information regarding preferences of agents in this particular example, it is easy to construct a similar example with incomplete information.

Example 6. *Consider the problem of assigning* $n = 3$ *objects,* $\{o_1, o_2, o_3\}$*, with only one copy of each, to* $m = 3$ *agents,* $\{i_1, i_2, i_3\}$*. Assume that each agent ranks objects as* $o_1 \succ o_2 \succ o_3$*. Assume that agents have additively separable preferences and* $u^1(o_1) = 1, u^1(o_2) = 0.95$ *and* $u^1(o_3) = 0$*. Assume that agents* i_2 *and* i_3 *report true rankings. Now, if agent* i_1 *reports truthfully, his payoff will be*

$$\frac{1}{3}(1 + 0.95 + 0) = 0.65$$

Instead, if agent i_1 *reports his ranking over objects as* $o_2 \succ o_1 \succ o_3$*, his payoff will be*

$$0.95 + \frac{1}{3}(0) = 0.95$$

Hence, truthful reporting is not an equilibrium.

Second, we ask the question whether the Ranking mechanism stochastically dominates other incentive compatible mechanisms when rankings over objects are not equally likely.

The following example shows that the answer to this question is "Not necessarily".

Assume again that there are $n = 3$ objects, $\{o_1, o_2, o_3\}$, $q = 1$ and $m = 3$ agents, $\{i_1, i_2, i_3\}$. Assume that each agent's ranking over objects is independently drawn from the following distribution:

$$\Pr\{o_1 \succ o_2 \succ o_3\} = p$$

and

$$\Pr\{o_2 \succ o_1 \succ o_3\} = 1 - p$$

with $p = \frac{4}{5}$.

Furthermore, assume that agent i_1 has additively separable preferences and that he derives utility of 1 from his first choice, a utility of 0.5 from his second choice and a utility of 0 from his third choice.

We first verify that truthful reporting is an equilibrium under the Ranking mechanism. Assume that agents i_2 and i_3 report truthfully and consider agent i_1 with preference $o_1 \succ o_2 \succ o_3$. If he reports truthfully, that is if he reports his ranking over objects as $o_1 \succ o_2 \succ o_3$, his expected payoff is

$$p^2\left(\frac{1}{3}(1 + 0.5 + 0)\right) + 2p(1-p)\left(\frac{1}{2}(1) + \frac{1}{3}(0)\right) + (1-p)^2\left(1 + \frac{1}{3}(0)\right) = 0.52$$

If he reports his ranking as $o_2 \succ o_1 \succ o_3$, his expected payoff is

$$p^2\left(0.5 + \frac{1}{3}(0)\right) + 2p(1-p)\left(\frac{1}{2}(0.5) + \frac{1}{3}(0)\right) + (1-p)^2\left(\frac{1}{3}(1 + 0.5 + 0)\right) = 0.42$$

Note further that he can not gain by reporting object o_3 as his first or second choice. Hence, truthful reporting is a best response to other agents' reporting truthfully.

Now, consider agent i_1 with preference $o_2 \succ o_1 \succ o_3$. Again, note that he can not gain by reporting object o_3 as his first or second choice. If he reports his true ranking, that is if he reports $o_2 \succ o_1 \succ o_3$, his expected payoff is

$$p^2\left(1 + \frac{1}{3}(0)\right) + 2p(1-p)\left(\frac{1}{2}(1) + \frac{1}{3}(0)\right) + (1-p)^2\left(\frac{1}{3}(1 + 0.5 + 0)\right) = 0.82$$

and if he reports $o_1 \succ o_2 \succ o_3$, his expected payoff is

$$p^2\left(\frac{1}{3}(1+0.5+0)\right)+2p(1-p)\left(\frac{1}{2}(0.5)+\frac{1}{3}(0)\right)+(1-p)^2\left(0.5+\frac{1}{3}(0)\right)=0.42$$

Thus, for type $o_2 \succ o_1 \succ o_3$ truthful reporting is a best response to other agents' strategy of reporting truthfully.

Hence, truthful reporting is an equilibrium.

Now, consider the RSD. Recall that RSD is strategy-proof. That is, reporting truthfully is a (weakly) dominant strategy. Consider agent i_1 type $o_1 \succ o_2 \succ o_3$. His expected payoff under RSD is

$$\frac{1}{3}(1)+\frac{1}{3}(p(0.5)+(1-p)(1))+\frac{1}{3}(0)=\frac{8}{15}\simeq 0.533\,33$$

The details for this computation are as follows. Agent i_1 will be chosen as the first agent to pick an object with probability $\frac{1}{3}$ and will choose object o_1 and get a payoff of 1. With probability $\frac{1}{3}$, he will be the second agent to pick an object. Note that the agent who is chosen to be the first one to pick is of type $o_1 \succ o_2 \succ o_3$ with probability p and in that case agent i_1 will pick object o_2 since object o_1 was picked by the first agent. Furthermore, the agent who is chosen to be the first one to pick is of type $o_2 \succ o_1 \succ o_3$ with probability $1-p$. In that case, first agent will pick object o_2 and hence agent i_1 will pick object o_1. Finally, agent i_1 will be chosen as the third agent to pick an object. In this case, objects o_1 and o_2 will be picked up by the first two agents and agent i_1 will end up getting object o_3.

Note that under the Ranking mechanism the interim payoff of this type was 0.52. Hence, the interim payoff is higher under RSD for this type.

2.4.2 Identical Ordinal Ranking

We have established that the result that the Ranking mechanism stochastically dominates any other anonymous, neutral and incentive compatible mechanism may not hold when rankings over objects are not equally likely. In this section, we consider a very specific but an interesting case in which rankings are not equally likely but ordinal ranking over objects is common across agents. That is, without loss, each agent ranks object o_1 as his first choice, object o_2 as his second choice and so on. We, furthermore, assume that agents preferences over bundles of objects are additively separable.

This special case might be relevant in some circumstances. For example, in some situations it may be reasonable to assume that each agent's utility from an object consists of a common value component (for example, "quality" of an object) plus a private value component (for example, "personal taste" for the object). Formally, agent i's value for object o_j, $v_j^i = q_j + \epsilon_j^i$ where q_j is the quality of the object which is commonly known and ϵ_j^i is personal taste of agent i for object o_j and is privately known by agent i. In a specific case, we assume that these common value and private value components are such that ordinal rankings of agents are identical.[17]

In a school choice problem, Abdulkadiroglu, Che and Yasuda (ACY, hereafter) (2011) discuss that the assumption that students' ordinal preferences are identical may be considered as a good proxy because it might be expected that students' preferences over schools are highly correlated.

For this special case in which agents' ordinal rankings are common, we establish the following result.

Proposition 5. *At any symmetric equilibrium of the Ranking mechanism, each agent's interim payoff is weakly higher than any other anonymous and incentive compatible ordinal mechanism, regardless of the utilities.*

Proof. The proof is similar to the proof of Theorem 1 of ACY (2011). Let $\mathcal{U}$ be type space, the set of possible utilities, where $\mathbf{u} = (u_1, ..., u_n) \in \mathcal{U}$ is the utilities derived from each object. We assume that the type space is finite for convenience. Let Π be the set of all strict preference orderings over the set of objects and $\Delta(\Pi)$ be the set of probability distributions over Π. A strategy is a mapping $\beta : \mathcal{U} \to \Delta(\Pi)$. Let β^* be a symmetric equilibrium under the Ranking mechanism. Let $P_j(\beta)$ denote the probability of obtaining object o_j when an agent follows strategy β while the remaining agents follow β^* under the Ranking mechanism. Now, for each j, it must be that

$$\sum_{\mathbf{u}\in\mathcal{U}} P_j(\beta^*(\mathbf{u})) f(\mathbf{u}) = \frac{q}{m}$$

where $f(\mathbf{u})$ is the probability of $\mathbf{u}$.

Note that for any anonymous and incentive compatible ordinal mechanism, for each j, it must be that

$$\overline{P}_j = \frac{q}{m}$$

[17] Agents' utilities for objects may differ from each other.

since agents' preference over objects are the same.

Consider a type $\tilde{\mathbf{u}} \in \mathcal{V}$. Assume that type $\tilde{\mathbf{u}}$ follows the following strategy under the Ranking mechanism: $\widetilde{\beta} := \sum_{\mathbf{v}\in\mathcal{V}} \beta^* (\mathbf{u}) f (\mathbf{u})$. That is, type $\tilde{\mathbf{u}}$ plays $\beta^* (\mathbf{v})$ with probability $f (\mathbf{u})$. Then, for type $\tilde{\mathbf{u}}$,

$$P_j \left(\widetilde{\beta}\right) = \sum_{\mathbf{v}\in\mathcal{V}} P_j \left(\beta^* (\mathbf{u})\right) f (\mathbf{u}) = \frac{q}{m} = \overline{P}_j$$

Since β^* is an equilibrium, it must be that

$$\sum_j \widetilde{u}_j P_j \left(\beta^* (\tilde{\mathbf{u}})\right) \geq \sum_j \widetilde{u}_j P_j \left(\widetilde{\beta}\right) = \sum_j \widetilde{u}_j \overline{P}_j$$

Hence, we have the result. □

The strategy-proof mechanisms are sometimes deemed desirable and strongly favored due to the strong incentive advantage: Agents cannot gain from misreporting their preferences regardless of other agents' reports. For example, a strategy-proof mechanism, the Deferred Acceptance mechanism, replaced the so-called Boston mechanism to allocate students to public schools in many U.S. school districts. This result implies that the Ranking mechanism is welfare superior to other incentive compatible ordinal mechanisms and hence to strategy-proof mechanisms in a very strong sense.

2.5 Conclusion

We have studied ordinal mechanisms when $n \geq 2$ objects (with possibly multiple copies of each) to be allocated to $m \geq 2$ agents when there is incomplete information. When agents have private information about their preferences and these preferences are ex-ante symmetric, we establish a very strong result: The Ranking mechanism stochastically dominates any other anonymous, neutral and incentive compatible mechanism in terms of interim welfare, which in particular implies that when agents' preferences over bundles of objects are additively separable, regardless of the utilities, every agent has a higher interim payoff under the Ranking mechanism.

As we discussed, although we verified that a similar result holds when agents' ordinal preferences are identical, this result may not hold in general in the "asymmetric" case when each ranking is not equally likely. The next natural question is which mechanisms are "good" in this case. Furthermore, we do not consider the case when there are complementarities. For example, in the course allocation problem, a student may prefer the bundle "MATH100 course and ECON100 course" to "ECON100 and ECON200 courses" although he likes ECON200 more than MATH100. In addition, an agent may care what other agents' allocations are. For example, in an office assignment problem, an agent may be better off having a "nicer" neighbor. These possible extensions do not readily follow from the analysis presented here and require possibly different techniques.

Chapter 3

Allocation with Points: Colonel Blotto Game under Incomplete Information

3.1 Introduction

To allocate course seats to students, many business and law schools have adopted a system, "Course Bidding System", that is based on an auction where "fake money" is used for bidding.[1] Students are given a positive bid endowment (so-called points), and they bid for the courses with these points. Students with highest bids for a course, up to the number of available seats, are assigned a seat in that course. Inspired by this system, we consider the problem of allocating goods to individuals via "fake money".

In particular, we assume that there are $n \geq 2$ distinct objects to be allocated to $m = 2$ agents. Each agent is given divisible points in the amount of $B > 0$ which can only be used for bidding. We consider a standard Bayesian setting in which each agent has private information regarding his values for the objects. Each agent simultaneously submits bids for each object such that sum of bids can not exceed B. Then, each object is given to the agent who submits the highest bid.[2] We, furthermore, assume that preferences of agents over bundles of objects are additively separable. That is, the payoff of an agent from a bundle is just the sum of his values for the objects included in that bundle.

This game is equivalent to the classical "Colonel Blotto" game, which was intro-

[1] Northwestern Kellogg, MIT Sloan, Wharton, Yale School of Management, Columbia Business School and University of Michigan Business School are among many others that employ variants of such an auction to allocate course seats to students. Further details on this system can be found in Sönmez and Üinver (2010).

[2] Ties are resolved by a fair coin toss.

duced by Borel (1921). In the original version of this game, there are two colonels and each has a unit of military resources. The colonels are going to fight in several battlefields.[3] Each colonel simultaneously chooses how to distribute his resources across the battlefields. At each battlefield, the colonel who has higher resources wins and each colonel's payoff is equal to the number of battlefields at which he has won the fight. That is, each battlefield is equally valued by each colonel. Note also that colonels only care whether they win the fight at a battlefield. That is, gone resources are sunk costs that do not affect colonels' payoffs.

Although the Colonel Blotto game has been first considered in such a military situation, it has several applications in a variety of political, social and other competitive situations in addition to the course bidding system as mentioned. Political campaigns constitute one example in which a similar situation arises. Political parties should decide how to allocate their resources to attract voters, and they only care whether they win the election. Another application might be trials in which lawyers have to decide how much resource (time, effort) to allocate into different lines of defense to be able to "win" against other lawyer. Several other situations such as lobbying, competing in sports can be considered as applications of this game.

We consider the Colonel Blotto game when each agent's values for objects are privately known and agents' values may be different from other agents' values. More precisely, each agent's private information is an n-dimensional vector that consists of values for each object. It is well-understood that it is in general very hard, if not impossible, to get explicit solutions in such problems. However, we get explicit solutions for an equilibrium of this game and furthermore make explicit welfare analysis for a class of value distributions.

First, we look for a Bayes-Nash equilibrium of this game. Although, as discussed, it is in general hard to obtain closed-form expressions for equilibria when there is multi-dimensional incomplete information, we were able to do so in some cases for the game at hand. We first consider the case when there are $n = 2$ objects. In that case, agents have a (weakly) dominant strategy to bid all their points in the object that they value more. Note that this strategy does not depend on any information regarding intensity of the preferences. It just depends on the ordinal ranking over objects. Next, we consider the case when $n > 2$. In this case, the equilibrium strategies depend on the intensity of preferences, namely on cardinal preferences. We obtain simple

[3] In the original version, there are 3 battlefields.

closed-form expressions for an equilibrium for a class of value distributions in this case.

Second aspect of our results will be on efficiency of this game. Auction with points[4] is one way to allocate "objects" to "agents" without monetary transfers, as in allocating course seats to students. Therefore, it is important to understand how this mechanism compares to the other mechanisms in terms of welfare of agents. In many real-life applications, we observe that mechanisms that only use ordinal preferences of agents over agents are predominantly used when transfers can not be used. As discussed in Chapter 1, in assigning students to public schools, students report their *ordinal* rankings over schools and a predetermined algorithm determines the assignments given students' ordinal preferences. The so-called Boston mechanism and the well-known Deferred Acceptance (DA) mechanism are currently the most widely-used mechanisms in many school districts in the US.[5] As introduced in Chapter 2, another ordinal mechanism, the Random Serial Dictatorship (RSD) mechanism in which an order over agents is randomly determined and following the order each agent chooses the object he wants among the available objects, is a very popular mechanism that has been used extensively in many allocation problems. Dominant use of ordinal mechanisms is mainly explained by the ease of implementing these mechanisms. Their rules are easy to apprehend for the agents. We compare the Blotto mechanism to incentive compatible ordinal mechanisms for the cases we could obtain analytic expressions for equilibrium. We show that each type of agent has a higher interim payoff under the Blotto mechanism than any other ordinal mechanism. To show this, we compare the Blotto mechanism to the welfare maximizing ordinal mechanism, the so-called Ranking mechanism.[6] Allocating via an auction with points, i.e. by letting agents play the Colonel Blotto game by endowing them with fake money, is a mechanism that is easy to implement in many real-life situations, which is already used in practice as in the aforementioned Course Bidding System. Therefore, welfare superiority of this mechanism over the widely-used ordinal mechanisms may have significant policy implications.

[4]We will refer to this mechanism as the "Blotto mechanism" from now on.

[5]Although school choice problem is different from the problem at hand since in school choice problem there are multiple copies of objects (school seats) and each agent (student) can only obtain one object and also it is a two-sided problem, we present these examples to emphasize the extensive use of ordinal mechanisms in practice.

[6]In Chapter 2, we show that each type of agent has a higher interim payoff under the Ranking mechanism than any other incentive compatible mechanism.

Related Literature. As mentioned, the original Colonel Blotto game dates back to 1921. The solution to this game is given by Borel and Ville (1938) when there are 3 battlefields, and later by Gross and Wagner (1953) for more than 3 battlefields, and valuations are identical both across battlefields and agents. Roberson (2006) analyzes the game allowing for asymmetric budgets. Hortalla-Vallve and Llorente-Saguer (2012) and Thomas (2013) consider the problem when colonels have asymmetric and heterogeneous battlefield valuations. Importantly, all these papers consider the problem under complete information: Each colonel knows the battleship valuations of the other colonel and also know the budget (total resources) of the other colonel. Adamo and Matros (2009) consider the problem when agents have private budget still assuming that battleship valuations are symmetric across agents and commonly known. Kovenock and Roberson (2011) consider a similar environment as ours in which agents have private information about their valuations and solve for the equilibrium for one special value distribution when there are 3 objects.

This study is also related to the literature on auctions with budget-constrained bidders. Such models are similar to the Colonel Blotto game in that bidders have scarce resources, but they differ in that bidders' resource is essentially "money" for which bidders care. Although the majority of the work in this literature has been under complete information similar to the Colonel Blotto game[7], there are a couple of papers that investigate the problem under incomplete information. Che and Gale (1996) consider standard single unit first-price and all-pay auctions when agents have private information about only their budgets and the value of the object is common across agents and is publicly known. Che and Gale (1998) study single unit first-price and second-price auctions when agents have private information regarding their values and also their budgets. Fang and Parreiras (2002, 2003) study equilibrium bidding in a two-bidder, second-price auction with private budget constraints allowing for affiliated and interdependent valuations.

The remainder of this chapter proceeds as follows. First, we introduce the formal model and define the game. In Section 3.3, we solve for equilibrium first for $n = 2$ and then we obtain an equilibrium for a class of value distributions when $n \geq 3$. In Section 3.4, we compare the welfare of agents under the Blotto mechanism to ordinal mechanisms. Finally, we conclude.

[7] See Szentes and Rosenthal (2002, 2003), Benoit&Krishna (2001)

3.2 Model

Assume that there are $m = 2$ agents and $n \geq 2$ objects. Each agent i derives a value $v_j^i \geq 0$ from obtaining object j. Assume that $\mathbf{v}^i = (v_1^i, ..., v_n^i)$ is independently drawn from a distribution G_i over $[\underline{v}^i, \overline{v}^i]^n$. Payoff of an agent is the sum of his values of objects he gets.

Each agent has divisible bid endowment (points) in the amount of $B > 0$ and this is commonly known. Agents, after privately observing their valuations, simultaneously choose how much to bid for each object where the sum of bids is B.[8] That is, agent i chooses a bid vector $(b_1^i, ..., b_n^i) \in [0, B]^n$ such that

$$\sum_{j=1}^{n} b_j^i = B$$

For each object, whoever has a higher bid gets the object and if there is a tie, the agent who gets the object is determined by a fair coin toss. Monetary transfers are not allowed and only points can be used for bidding.

3.3 Equilibrium

We want to solve for Bayesian Nash equilibrium of this game. We are going to denote a pure strategy as $\beta^i : [\underline{v}^i, \overline{v}^i]^n \to [0, B]^n$ such that $\beta^i = \left(\beta_j^i\right)_{j=1}^n$ and for all $\mathbf{v}^i = (v_1^i, ..., v_n^i)$, $\beta_j^i (\mathbf{v}^i) \in [0, B]$ and

$$\sum_{j=1}^{n} \beta_j^i \left(\mathbf{v}^i\right) = B$$

Firstly, we consider the case when $n = 2$.

Proposition 6. *Assume that there are $n = 2$ objects to be allocated and agent i's values are independently drawn from distribution $G_i (.,.)$. Then, the strategy*

$$\beta (v_1, v_2) = \begin{cases} (B, 0) & \textit{if} \quad v_1 > v_2 \\ \left(\frac{B}{2}, \frac{B}{2}\right) & \textit{if} \quad v_1 = v_2 \\ (0, B) & \textit{if} \quad v_1 < v_2 \end{cases}$$

[8] Since points have no value other than bidding, agents will always use all of their points.

is a weakly dominant strategy for each agent.

Proof. Consider agent 1 with type (v_1, v_2). Assume that $v_1 > v_2$. Consider a strategy of agent 2 and let B_1^2 denote the random variable of agent 2's bids on o_1. Now, by bidding $(B, 0)$, the expected payoff of agent 1 is

$$\Pr\left(B_1^2 = B\right)\left(\frac{v_1 + v_2}{2}\right) + \Pr\left(B_1^2 < B\right) v_1 \tag{3.1}$$

since if agent 2 bids B on first object this means that bids of agents for both objects are the same, therefore, each agent will get each object with probability $\frac{1}{2}$ and when if agent 2 bids less than B on o_1 which means that he bids a positive amount on o_2, agent 1 will get o_1 and agent 2 will get o_2.
Similarly, by bidding $(b, 1-b)$, $0 < b < 1$, his expected payoff is

$$\begin{aligned} & \Pr\left(B_1^2 > b\right) v_2 + \Pr\left(B_1^2 = b\right)\left(\frac{v_1 + v_2}{2}\right) + \Pr\left(B_1^2 < b\right) v_1 \qquad (3.2) \\ = \; & \Pr\left(B_1^2 = B\right) v_2 + \Pr\left(B > B_1^2 > b\right) v_2 + \Pr\left(B_1^2 = b\right)\left(\frac{v_1 + v_2}{2}\right) + \Pr\left(B_1^2 < b\right) v_1 \end{aligned}$$

By bidding $(0, B)$, his expected payoff

$$\Pr\left(B_1^2 > 0\right) v_2 + \Pr\left(B_1^2 = 0\right)\left(\frac{v_1 + v_2}{2}\right) \tag{3.3}$$

Note that (3.1) $\geq$ (3.2) $\geq$ (3.3). Thus, when $v_1 > v_2$, it is a (weakly) dominant strategy to bid $(B, 0)$. By symmetry for $v_1 < v_2$, it is a dominant strategy to bid $(0, B)$. For $v_1 = v_2$, agent is indifferent among bids. Hence, we have the result. □

Note that when there are $n = 2$ objects, the equilibrium of this game only depends on the ordinal preferences. More precisely, an individual bids all his points on the good he prefers more. However, when there are $n > 2$ objects, this is not true anymore and equilibrium strategies not only depend on the ranking but also on the preference intensities for each good. First, we solve for equilibrium of this game for a certain class of distributions when $n = 3$.

Proposition 7. *Assume that agent* 1 *and* 2 *s values are independently drawn from continuous distributions* $G_1(v_1, v_2, v_3)$ *and* $G_2(w_1, w_2, w_3)$*, respectively, such that densities are of the following form:*

$$g_1(v_1, v_2, v_3) \;=\; \widetilde{g}_1\left(\left(v_1^2 + v_2^2 + v_3^2\right)\right)$$

$$g_2(w_1, w_2, w_3) \;=\; \widetilde{g}_2\left(\left(w_1^2 + w_2^2 + w_3^2\right)\right)$$

where $\widetilde{g}_1$ *and* $\widetilde{g}_2$ *are measurable functions on* $\mathbb{R}_+$ *such that*[9]

$$\int_0^\infty \widetilde{g}_i(x)\, x^{\frac{1}{2}} dx = \frac{4}{\pi}$$

Then, the following is a symmetric equilibrium

$$\beta(v_1, v_2, v_3) = \left(\frac{v_1^2}{v_1^2 + v_2^2 + v_3^2} B, \frac{v_2^2}{v_1^2 + v_2^2 + v_3^2} B, \frac{v_3^2}{v_1^2 + v_2^2 + v_3^2} B\right)$$

Proof. See Appendix C. □

For any measurable function $\widetilde{g}_i$ that satisfies the given requirement, g_i becomes a density. We also note that we do not require that supports of distributions to be identical. As long as distribution functions satisfy the given conditions, we have the result.

Some examples of densities that satisfy the given condition are as follows.

Example 7. *Let* $\widetilde{g}(x) = \frac{8}{\pi\sqrt{\pi}} \exp(-(x))$ *when* $x \geq 0$. *Note that*

$$\int_0^\infty \frac{8}{\pi\sqrt{\pi}} \exp(-(x))\, x^{\frac{1}{2}} dx = \frac{4}{\pi}$$

Hence, for $g(v_1, v_2, v_3) = \frac{8}{\pi\sqrt{\pi}} \exp(-(v_1^2 + v_2^2 + v_3^2))$, $v_1, v_2, v_3 \geq 0$ *we have the result. Note that this is the density if* v_1, v_2, v_3 *comes independently from Generalized Gamma Distribution with parameters* $(a, d, p) = (1, 1, 2)$ *where general density of this distribution is given by* $\left(\frac{p}{a^d} \frac{x^{d-1} e^{-\left(\frac{x}{a}\right)^p}}{\Gamma\left(\frac{d}{p}\right)}\right)$, $x \geq 0$. *That is, each* v_i *is independently from a distribution with density*

$$f(x) = \frac{2}{\sqrt{\pi}} \exp(-x)\,,\; x \geq 0$$

Example 8. *Let* $\widetilde{g}(x) = \frac{6}{\pi}$ *if* $0 \leq x \leq 1$ *and* 0 *otherwise. Then,*

$$g(v_1, v_2, v_3) = \frac{6}{\pi} \text{ if } v_1^2 + v_2^2 + v_3^2 \leq 1$$

[9] this is to make sure that $g_i(v_1, v_2, v_3)$ is a density function.

That is, if (v_1, v_2, v_3) is uniformly distributed inside the unit sphere.

Example 9. *Let $\widetilde{g}(x) = \frac{6}{7\pi}$ if $1 < x \leq 4$ and 0 otherwise. Then,*

$$g(v_1, v_2, v_3) = \frac{6}{7\pi} \text{ if } 1 < v_1^2 + v_2^2 + v_3^2 \leq 4$$

That is, if (v_1, v_2, v_3) is uniformly distributed inside sphere with radius 2 but outside the unit sphere.

Similarly, we next solve for a symmetric equilibrium of the Colonel Blotto game for a class of distribution functions for $n > 3$. The proofs are relegated to Appendix C.

Proposition 8. *Assume that there are $n > 3$ objects. Assume that players' values are independently drawn from continuous distributions $G_1(v_1, ..., v_n)$ and $G_2(w_1, ..., w_n)$ such that densities are of the following form:*

$$g_1(v_1, v_2, v_3) = [v_1...v_n]^{\frac{3-n}{n-2}} \widetilde{g_1}\left(\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}\right)\right)$$
$$g_2(w_1, w_2, w_3) = [v_1...v_n]^{\frac{3-n}{n-2}} \widetilde{g_2}\left(\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}\right)\right)$$

where $\widetilde{g}_1$ and $\widetilde{g}_2$ are measurable functions on $\mathbb{R}_+$ such that[10]

$$\int_0^\infty \widetilde{g}_i(x)\, x^{\frac{1}{n-1}} dx = \Gamma\left(\frac{n}{n-1}\right)\left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right)^n \left(\frac{n-1}{n-2}\right)^n$$

Then, the following is a symmetric equilibrium

$$\beta(v_1, v_2..., v_n) = \left(\frac{v_1^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}}B, \frac{v_2^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}}B, ..., \frac{v_n^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}}B\right)$$

Proof. See Appendix C. □

One distribution example that satisfy the sufficient conditions listed in the Proposition is as follows. Several other examples can be easily constructed.

Example 10. *Assume that each v_i is independently drawn from Generalized Gamma Distribution with parameters $(a, d, p) = \left(1, \frac{1}{n-2}, \frac{n-1}{n-2}\right)$ where general density of this*

[10] Again, this makes sure that g_i s are density

distribution is given by $\left(\frac{p}{a^d}\frac{x^{d-1}e^{-\left(\frac{x}{a}\right)^p}}{\Gamma\left(\frac{d}{p}\right)}\right)$, $x \geq 0$. *That is, each* v_i *is independently drawn from a distribution function with density*

$$f(x) = \left(\frac{n-1}{n-2}\right)\left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right) x^{\frac{3-n}{n-2}} \exp\left(-x^{\frac{n-1}{n-2}}\right), x \geq 0$$

Then, the joint density will be

$$g(v_1,...,v_n) = \left(\frac{n-1}{n-2}\right)^n \left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right)^n v_1^{\frac{3-n}{n-2}}...v_n^{\frac{3-n}{n-2}} \exp\left(-\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}\right)\right)$$

,$v_1,...,v_n \geq 0$. *That is,* $\widetilde{g}(x) = \left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right)^n \left(\frac{n-1}{n-2}\right)^n \exp(-(x))$ *when* $x \geq 0$. *Note that*

$$\int_0^\infty \exp(-(x))\, x^{\frac{1}{n-1}} dx = \Gamma\left(\frac{n}{n-1}\right)\left(\frac{1}{\Gamma\left(\frac{1}{n-1}\right)}\right)^n \left(\frac{n-1}{n-2}\right)^n$$

3.4 Efficiency of the Blotto mechanism

As we have argued, variants of the Colonel Blotto game have been used in practice for allocating objects to agents as in the course allocation in business schools. Therefore, it is important to understand the efficiency of this mechanism. To do so, we are going to compare this mechanism to ordinal mechanisms. Ordinal mechanisms are widely-used in may real-life allocation problems. Agents report their ordinal rankings over objects and based on these reported preferences an allocation is determined.

To make the comparison, we are going to consider the so-called Ranking mechanism which was also defined in Chapter 2. The Ranking mechanism works as follows. Each object is assigned to the agent who ranks that object at a lower spot (that is, who has a higher preference ranking) and if some agents rank the object at the same place, object is given to each agent with equal probability. For example, assume that $n = 3, m = 2$ and first agent's strict ordinal ranking is given by $o_1 \succ o_2 \succ o_3$ and agent 2's strict ordinal ranking is $o_3 \succ o_2 \succ o_1$. Then, o_1 is given to agent $1, o_3$ is given to agent 2 and o_2 is given to each agent with probability $\frac{1}{2}$.

By the main result of Chapter 2, we know that the Ranking mechanism has strong welfare superiority to other ordinal mechanisms.

Proposition 9. *When each agent draws his valuation vector from an exchangeable distribution function, then every type of agent has a higher interim payoff under the Ranking mechanism compare to any other anonymous, neutral and incentive compatible ordinal mechanism.*

Now, this result at hand, we are going to compare the Blotto mechanism with the Ranking mechanism. Before that we note that when each agent draws his valuation vector from an exchangeable distribution function[11], the interim probability of an agent to receive his k^{th} choice under the Ranking mechanism is given by

$$P_k^{rank} = \frac{2n - 2k + 1}{2}$$

To see this, note that an agent, say agent 1, will obtain his k^{th} choice object with probability 1 if the other agent ranks this object as his j^{th} choice for $j > k$ which happens with probability $\frac{n-k}{n}$ and will obtain his k^{th} choice object with probability $\frac{1}{2}$ if the other agent ranks this object also as his k^{th} choice which happens with probability $\frac{1}{n}$. If the other agent ranks this object as his j^{th} choice for $j < k$, agent 1 can not obtain his k^{th} choice. Hence,

$$P_k^{rank} = 1 \times \frac{n-k}{n} + \frac{1}{2} \times \frac{1}{n} = \frac{2n - 2k + 1}{2n}$$

First, it is easy to see that when $n = 2$, the outcome of the Colonel Blotto game is equivalent to the Ranking mechanism. Therefore, we start with the case when $n = 3$.

Now, consider an agent with type (v_1, v_2, v_3). Then, his interim payoff under the Blotto mechanism is

$$\Pi^{Blotto}(v_1, v_2, v_3) = \sum_{i=1}^{3} \Pr\left(\frac{v_i^2}{v_1^2 + v_2^2 + v_3^2} B \geq \frac{W_i^2}{W_1^2 + W_2^2 + W_3^2} B\right) v_i$$

Then, by the Lemma (10) in Appendix C, we have that

$$\begin{aligned} & \Pi^{Blotto}(v_1, v_2, v_3) \\ = & \sum_{i=1}^{3} \sqrt{\frac{v_i^2}{v_1^2 + v_2^2 + v_3^2}} v_i \end{aligned}$$

[11] That is, when valuations are drawn from a distribution that is invariant under permutations of its arguments.

$$
\begin{aligned}
&= \sum_{i=1}^{3} \frac{v_i^2}{\sqrt{v_1^2+v_2^2+v_3^2}} \\
&= \sqrt{v_1^2+v_2^2+v_3^2}
\end{aligned}
$$

and his interim payoff under Ranking mechanism is

$$
\begin{aligned}
& \Pi^{rank}(v_1, v_2, v_3) \\
= & \frac{1}{6}(5v_1 + 3v_2 + v_3)
\end{aligned}
$$

since when $n = 3$,

$$P_k^{rank} = \frac{7-2k}{6}$$

for each $k \in \{1, 2, 3\}$.

We want to investigate the sign of $\sqrt{v_1^2+v_2^2+v_3^2} - \frac{1}{6}(5v_1 + 3v_2 + v_3)$, or, equivalently sign of $(v_1^2+v_2^2+v_3^2) - \frac{(5v_1+3v_2+v_3)^2}{36}$.

$$
\begin{aligned}
& v_1^2+v_2^2+v_3^2 - \frac{(5v_1+3v_2+v_3)^2}{36} \\
= & \frac{1}{36}\left(11v_1^2 + 27v_2^2 + 35v_3^2 - 30v_1v_2 - 10v_1v_3 - 6v_2v_3\right) \\
= & \frac{1}{36}\left((v_1 - 5v_3)^2 + (v_2 - 3v_3)^2 + (3v_1 - 5v_2)^2 + \left(v_1^2+v_2^2+v_3^2\right)\right) \\
\geq & \ 0
\end{aligned}
$$

Therefore, we have the following result.

Proposition 10. *Under distributional assumptions of Proposition* 7 *, any type of any agent has a higher interim payoff under the Blotto mechanism compare to the Ranking mechanism when* $n = 3$.

This result combined with the fact that the Ranking mechanism is superior to other anonymous, neutral and incentive compatible mechanisms in term of interim payoff of each type, we deduce that the Blotto mechanism is welfare superior to other ordinal mechanisms in a very strong sense.

Same result holds for the case when $n > 3$.

Proposition 11. *Under the distributional assumptions of Proposition* 8*, any type of any agent has a higher interim payoff under the Blotto mechanism compare to the Ranking mechanism when* $n > 3$*.*

Proof. See Appendix C. □

Hence, the Blotto mechanism has a welfare superiority over ordinal mechanisms in a very strong sense—every type has a higher interim payoff under the Blotto mechanism. Therefore, "fake market" environment seems promising to improve upon widely used ordinal mechanisms in terms of welfare.

3.5 Conclusion

We have considered the classical Colonel Blotto game under incomplete information. Although it is in general hard to come up with closed form expressions when there is multi-dimensional incomplete information, we were able to solve and obtain simple expressions for the equilibrium of this game. In addition, we have established a strong welfare superiority of this allocation mechanism to other ordinal mechanisms which are dominantly used in many real-life applications. This observation implies that creating "fake market" in situations when prices/transfers cannot be used seems beneficial for welfare over the widely used ordinal allocation methods.

Of course, it would be desirable to solve for equilibrium of this game in general and make further welfare analysis. Although it involves difficulties as in many problems with multi-dimensional incomplete information, we hope to be able to make additional progress in future work.

Appendix A

Proofs of Chapter 1

A.1 Proof of Proposition 1 (Truth-telling under the Boston mechanism)

Assume that each school has a quota of $q \geq 1$ and each strict ordinal ranking over schools is equally likely for each student. Furthermore, assume that all schools have an identical ranking of the students. We claim that truth-telling is an equilibrium under the Boston mechanism.[1]

Let the set of students be $\mathcal{I} = \{1, ..., m\}$, and the set of schools be $\mathcal{S} = \{s_1, ..., s_n\}$ where school $s \in \mathcal{S}$ has $q \geq 1$ available seats. Each student i has a strict preference profile $P_i = (P_i(1), ..., P_i(n))$ and each school s has a strict priority list $\pi_s = (\pi_s(1), ..., \pi_s(m))$ where for all j, k such that $1 \leq j < k \leq n$, student i prefers school $P_i(j)$ strictly more than school $P_i(k)$ and similarly, for each j, k such that $1 \leq j < k \leq m$, student $\pi_s(j)$ has a higher priority than $\pi_s(k)$ at school s. Let $P = (P_i)_{i=1,...,m}$ and let π be the schools' identical priority ranking over the students.

A school choice mechanism φ is an allocation rule $\varphi(P;\pi) = (\varphi_i(P;\pi))_i$ where $\varphi_i(P;R) = \left(\varphi_i^j(P;\pi)\right)_{j=1,...,n}$ such that for each $j \in \{1, ..., n\}$

$$\sum_{i=1}^{m} \varphi_i^j(P;\pi) = q$$

and $\varphi_i^j(P;\pi) \in [0,1]$ for all i and j where $\varphi_i^j(P;\pi)$ denotes the probability that student i is assigned to school s_j when the reported ordinal preferences of students are P and the school priorities are given by π.

[1] Featherstone and Niederle (2008) provide a similar result.

We first present the following definition.

Definition 2. *A school choice mechanism is neutral iff for any P and π and for any permutation $\sigma : \mathcal{S} \to \mathcal{S}$ over schools we have that*

$$\varphi_i (P^\sigma; \pi) = (\varphi_i (P; \pi))^\sigma$$

where P^σ is the preference rankings of students obtained from P according to permutation σ and $(\varphi_i (P; \pi))^\sigma$ is the allocation rule of student i again obtained from $\varphi_i (P; \pi)$ according to permutation σ, that is, by changing the name of school s_j to $\sigma (s_j)$ for each j.

That is, a school choice mechanism is neutral iff the assignments do not depend on the label of schools. If we relabel the schools, assignments change accordingly. Trivially, we have the following observation.

Lemma 2. *Assume that each school has $q \geq 1$ available seats and that all schools have an identical ranking of the students, the Boston mechanism is neutral.*

We next note that due to neutrality and the exchangeability assumption, given that other students report their true preferences, the interim probability of getting into some school only depends on the ranking for that school: Without loss of generality (W.l.o.g.), consider student 1 and consider two preference rankings

$$P_1^* = (P_1^* (1), ..., P_1^* (k-1), s_k, P_i^* (k+1), ..., P_i^* (n))$$

and

$$P_1^{**} = (P_1^{**} (1), ..., P_1^{**} (k-1), s_k, P_i^{**} (k+1), ..., P_i^{**} (n))$$

Note that although rankings for the other schools may be different in these preference rankings, k^{th} choice is school s_k in both. We claim that the interim probability of getting into school s_k is the same for both rankings given that other agents report their true rankings. Let $p_k^\varphi (P_1)$ be the interim probability of getting into school s_k for student 1 when his preference ranking is P_1 given that other agents report their true rankings. Note that

$$p_k^\varphi (P_1) = \sum_\pi \left(\sum_{P_{-1}=(P_2,...,P_m)} \varphi_1^k (P_1, P_{-1}; \pi) \Pr (P_{-1}) \right) \Pr (\pi)$$

Now, consider the permutation σ over schools such that $\sigma\left(P_1^*\left(j\right)\right) = P_1^{**}\left(j\right)$ for all j. Note that, due to neutrality, since $\sigma\left(s_k\right) = s_k$, for any P_{-1} and π, we have that

$$\varphi_1^k\left(P_1^*, P_{-1}; \pi\right) = \varphi_1^k\left(P_{**}^1, \left(P^{-1}\right)^\sigma; \pi\right) \tag{A.1}$$

where $\left(P^{-1}\right)^\sigma$ is obtained from P^{-1} by relabeling objects according to permutation σ.

Now,

$$\begin{aligned}
p_k^\varphi\left(P_1^{**}\right) &= \sum_\pi \left(\sum_{P_{-1}=(P_2,\ldots,P_m)} \varphi_1^k\left(P_1^{**}, P_{-1}; \pi\right) \Pr\left(P_{-1}\right) \right) \Pr\left(\pi\right) \\
&= \sum_\pi \left(\sum_{P_{-1}=(P_2,\ldots,P_m)} \varphi_1^k\left(P_1^{**}, \left(P^{-1}\right)^\sigma; \pi\right) \Pr\left(\left(P^{-1}\right)^\sigma\right) \right) \Pr\left(\pi\right) \\
&= \sum_\pi \left(\sum_{P_{-1}=(P_2,\ldots,P_m)} \varphi_1^k\left(P_1^{*}, P_{-1}; \pi\right) \Pr\left(P^{-1}\right) \right) \Pr\left(\pi\right) \\
&= p_k^\varphi\left(P_1^*\right)
\end{aligned}$$

where third equality is due to $(A.1)$ and the fact that $\Pr\left(\left(P^{-1}\right)^\sigma\right) = \Pr\left(P^{-1}\right)$ since each ranking over the schools is equally likely and independent of others rankings.

Hence, we will characterize a neutral school choice mechanism φ by $\left(p_k^\varphi\right)_{k=1}^n$, where p_k^φ denotes the *interim* probability that a student gets into his (reported) k^{th} ranked choice when all the remaining students report their true rankings under the mechanism φ.

Thus, truth-telling is an equilibrium under a neutral school choice mechanism iff p_k^φ is (weakly) decreasing in k.

We now claim that the interim probability that student 1 gets into his (reported) k^{th} ranked choice is decreasing in k when all the remaining students report their true rankings under the Boston mechanism. But, this is almost immediate. Consider any state (any preference report of other students and any priority list of schools) in which student 1 is assigned to some school. Increasing the rank of that school will also guarantee a spot at that school for student 1. Furthermore, there may exist states such that student 1 is not assigned to a school but by increasing the rank for that school student 1 may get into that school at that state. Hence, given the observation that the interim probability of getting into some school only depends on the rank for that school, the interim probability will always (weakly) increase once a school is given a higher ranking.

Hence, truth-telling is an equilibrium under the Boston mechanism.

A.2 Large Economy: Proof of Theorem 1

Before moving on to the proof, we consider another well known mechanism, **Serial Dictatorship,** which will be useful for the computation. Given a priority ranking list over students, let $\pi(i)$ denote the student in the i^{th} rank in the priority list.

Step 1: Student $\pi(1)$ chooses the school he wants to get into and he is permanently placed into that school.

Step 2: Student $\pi(2)$ chooses the school he wants to get into among the available ones and he is permanently placed into that school.

In general,

Step k : Student $\pi(k)$ chooses the school he wants to get into among the available ones and he is permanently placed into that school.

Remark 2. *When schools have identical priorities, then for any preference ranking of students, the outcome of the DA mechanism is equivalent to the outcome of Serial Dictatorship where the order for dictatorship is determined by the school priorities.*

This is a well-known equivalence[2] and given this equivalence, it will be easier to calculate the probabilities for Serial Dictatorship (SD) rather than the DA. We define **Random Serial Dictatorship** as the mechanism in which students are ordered randomly and then based on this order serial dictatorship is implemented. Hence, the Deferred Acceptance mechanism when school priorities are identical across schools, is equivalent to Random Serial Dictatorship.

Below, we give an explicit formula for the interim probabilities of getting into each school, that is the probabilities when a student knows his preferences but not the preferences of other students nor the priorities of schools, under the Random Serial Dictatorship mechanism, and hence under the DA mechanism.

Lemma 3. *When there are n schools with $q = 1$ and n students,*

$$(P_k^n)^{DA} = \frac{(n+1)}{k(k+1)n}$$

[2] See Balinski and Sonmez (1999), for example.

for each $k \in \{1, ..., n\}$ where $(P_k^n)^{DA}$ is the interim probability of getting into k^{th} ranked school when there are n schools and n students under DA.

Proof. We use recursive formulation and use RSD instead of the DA mechanism. We will simply use P_k^n instead of $(P_k^n)^{DA}$, where there is no confusion. Below, we will compute P_k^n for each $k \geq 1$ and $n \geq k$.

Note that there is a recursive relation between two successive stages of the RSD. After one step, some student is placed into some school and since each school has a quota of 1, no student will be able to get into that school in later stages. Hence, those student and the school is out of the problem in the next step and we are in the isomorphic problem with one less student and one less school. By using this observation, we will relate P_k^n to $P_j^{n-1}, j = 1, 2, ..., (n-1)$.

Consider a student, say i_1, and wlog, say his preference profile is such that his k^{th} choice is school k for all $k \geq 1$. Consider the probability of his getting into school 1 when there are n students and n schools. Now,

$$P_1^n = \frac{1}{n} + \frac{n-1}{n}\left(\frac{n-1}{n}P_1^{n-1}\right)$$

since with probability $\frac{1}{n}$, i_1 will be chosen in the first place and will get his first choice, school 1 and with probability $\frac{n-1}{n}$ some other student will be chosen in the first place. If that student chooses i_1 s top choice, school 1, which happens with probability $\frac{1}{n}$, i_1 can not get into school 1. And, with probability $\frac{n-1}{n}$, that student will choose a school different from school 1. Therefore, since one school, namely the school other student chose, and one student will be removed, we have $(n-1)$ students and schools and in that case the probability of i_1 s getting into school 1 is P_1^{n-1}.

It is easy to verify that if the claim is true for $(n-1)$, then claim is true for n:

$$\begin{aligned} P_1^n &= \frac{1}{n} + \frac{n-1}{n}\left(\frac{n-1}{n}P_1^{n-1}\right) \\ &= \frac{1}{n} + \frac{n-1}{n}\left(\frac{n-1}{n}\frac{n}{2(n-1)}\right) \\ &= \frac{1}{n} + \frac{n-1}{2n} = \frac{n+1}{2n} \end{aligned}$$

Note also that $P_1^1 = 1$. That is, claim is true for $n = 1$ and hence by above observation, by induction, we have the result.

Similarly, consider the probability of his getting his $k^{th}, n \geq k > 1$, choice when there are n students and n schools.

With probability $\frac{1}{n}$, i_1 will be chosen in the first place and will get his first choice and hence he won't get his k^{th} choice and with probability $\frac{n-1}{n}$ some other student will be chosen in the first place. If that student chooses one of i_1 s top $(k-1)^{th}$ choice, school $1, .., (k-1)$, which happens with probability $\frac{k-1}{n}$, school i becomes i_1 s $(k-1)^{th}$ choice among the remaining $n-1$ schools. And, with probability $\frac{1}{n}$, that student will choose school k. With probability $\frac{n-k}{n}$, that student chooses some school different from one of i_1 s top k choices. In that case school i remains to be his k^{th} choice among the remaining $(n-1)$ schools. Hence,

$$P_k^n = \frac{n-1}{n}\left[\frac{k-1}{n}P_{k-1}^{n-1} + \frac{n-k}{n}P_k^{n-1}\right]$$

It is easy to verify that for any $k \geq 2$, if the claim is true for $(n-1)$, it is true for n :

$$\begin{aligned}
P_k^n &= \frac{n-1}{n}\left[\frac{k-1}{n}P_{k-1}^{n-1} + \frac{n-k}{n}P_k^{n-1}\right] \\
&= \frac{n-1}{n}\left[\frac{k-1}{n}\frac{n}{k(k-1)(n-1)} + \frac{n-k}{n}\frac{n}{k(k+1)(n-1)}\right] \\
&= \frac{n-1}{n}\left[\frac{1}{k(n-1)} + \frac{n-k}{k(k+1)(n-1)}\right] \\
&= \frac{n-1}{n}\left[\frac{n+1}{k(k+1)(n-1)}\right] \\
&= \frac{(n+1)}{k(k+1)n}
\end{aligned}$$

Note that $P_2^2 = 1 - P_1^2 = \frac{1}{4}$ and hence by induction we have that $P_2^n = \frac{n+1}{6n}$. Given this, $P_3^3 = 1 - P_1^3 - P_2^3 = 1 - \frac{2}{3} - \frac{2}{9} = \frac{1}{9}$ and again by induction, we have that $P_3^n = \frac{n+1}{12n}$. Continuing in this manner, for a general $k \geq 2$, we have

$$\begin{aligned}
P_k^k &= 1 - \sum_{j=1}^{k-1} P_j^k \\
&= 1 - \sum_{j=1}^{k-1} \frac{k+1}{j(j+1)k} \\
&= 1 - \frac{k+1}{k}\sum_{j=1}^{k-1}\left(\frac{1}{j} - \frac{1}{j+1}\right)
\end{aligned}$$

$$= \quad 1 - \frac{k+1}{k}\left(\frac{k-1}{k}\right) = \frac{1}{k^2}$$

and hence by induction we have that $P_k^n = \frac{(n+1)}{k(k+1)n}$. By the equivalence of RSD and the DA mechanism,

$$\left(P_k^n\right)^{DA} = \frac{(n+1)}{k\,(k+1)\,n}$$

□

Although we were able to explicitly calculate these probabilities for the DA, unfortunately, it is not easy to calculate the probabilities for each n and k under the Boston mechanism. One tractable case is when $k = 1$. That is, the probability of getting into top choice. Assume that there n schools and n students with $q = 1$. Now, consider a student, say i_1, and let's compute his probability of getting into his first choice, wlog say his first choice is school 1. There are $(n-1)$ remaining students. Now, let the number of students whose first choice is school 1 be $j \in \{0, ..., n-1\}$. Note also that the probability of student 1 getting into school 1 is $\frac{1}{j+1}$ when there are j other students whose first choice is 1. Hence,

$$\begin{aligned}
P_1^n &= \sum_{j=0}^{n-1} \binom{n-1}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{n-1-j} \left(\frac{1}{j+1}\right) \\
&= \sum_{j=0}^{n-1} \binom{n}{j+1} \left(\frac{1}{n}\right)^{j+1} \left(\frac{n-1}{n}\right)^{n-1-j} \\
&= \sum_{j=1}^{n} \binom{n}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{n-j} \\
&= \left[\sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{n}\right)^j \left(\frac{n-1}{n}\right)^{n-j}\right] - \left(\frac{n-1}{n}\right)^n \\
&= 1 - \left(\frac{n-1}{n}\right)^n
\end{aligned}$$

As a future reference note that $P_1^n \to 1 - \frac{1}{e}$ as $n \to \infty$. Unfortunately, for $k > 1$, computation becomes intractable. However, by computing average number of seats filled at each step, we will be able to compute the limit probabilities as $n \to \infty$. Before moving on, we make couple of observations regarding the computation.

Note that under the Boston mechanism if a student, say student 1, is placed into some school in step k, that implies that student 1 is placed into his k^{th} choice.

Furthermore, the number of empty schools[3]after the allocation has been made at Step k is equivalent to the number of students that will be allocated in later steps.

Note also that under the Boston mechanism, a student competes only with students who applied to the same school at the exact same step. Furthermore, all the students applying to the same school at the same step are alike because all these students are eliminated in the earlier steps (eliminated $(k-1)$ times when we are at step k which means they rank below $(k-1)$ students in the school priority lists) and will only compete with each other. Thus, the probability of a student to be chosen at a step when there are, say z, other students applying to the same school is $\frac{q^*}{z+1}$ when $z \geq q^*$ and 1 if $z < q^*$ where q^* is the number of available seats unoccupied in earlier steps.

These observations will be key in the computation below.

Note that the probability of k "successes" out of N independent "trial"s when the probability of success is p is given by the Binomial Distribution

$$b\left(k; N, p\right) = \binom{N}{k} p^k \left(1-p\right)^{N-k}$$

In particular,

$$b\left(0; N, p\right) = \left(1-p\right)^N$$

Assume that there are n students and n schools with $q = 1$. We consider the Boston mechanism.

Step 1: Now, for any school, the probability that no student applies is

$$b\left(0; n, \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^n = \left(\frac{n-1}{n}\right)^n$$

Now,

$$r_1^n = \left(\frac{n-1}{n}\right)^n \to \frac{1}{e}$$

as $n \to \infty$. Call

$$q_1 = \frac{1}{e}$$

Hence, for any school, the probability that at least one student applies and hence a student gets placed is $1 - r_1^n$. Therefore, in average there are nr_1^n schools which have available seats in Step 1. In other words, $n\left(1 - r_1^n\right)$ students gets into their top

[3]By empty school, we mean schools to which no student is assigned.

choice. Hence, by symmetry,

$$P_1^B \to 1 - \frac{1}{e}$$

as $n \to \infty$.

Step 2: Now, in average there are nr_1^n students unassigned in Step 1 and hence nr_1^n students to assign in Step 2. For any school, the probability that no student applies in step 2

$$b\left(0; nr_1^n, \frac{1}{n-1}\right) = \left(\frac{n-2}{n-1}\right)^{nr_1^n}$$

since there are nr_1^n students and each student can apply to $(n-1)$ schools in Step 2. Note that nr_1^n may not be an integer. Hence, we are making an abuse of notation here but we are interested in the limit probabilities.

Let

$$r_2^n = \left(\frac{n-2}{n-1}\right)^{nr_1^n}$$

and

$$q_2 = \lim_{n\to\infty}\left(\frac{n-2}{n-1}\right)^{nr_1^n} = \exp\left(-q_1\right)$$

Thus, there remains (in average) $nr_1^n r_2^n$ schools which have available seats. In other words, $nr_1^n\left(1 - r_2^n\right)$ new students are placed in step 2. Hence, as $n \to \infty$

$$P_2^B \to q_1\left(1 - q_2\right)$$

In general,

Step k: There are in average $nr_1^n...r_{k-1}^n$ students to place. For any school, the probability that no student applies is

$$b\left(0; nr_1^n...r_{k-1}^n, \frac{1}{n-(k-1)}\right) = \left(\frac{n-k}{n-k+1}\right)^{nr_1^n...r_{k-1}^n}$$

and $n\left(\prod_{s=1}^{k-1} r_s^n\right)\left(1 - r_k^n\right)$ seats filled at step k where

$$r_k^n = \left(\frac{n-k}{n-k+1}\right)^{nr_1^n...r_{k-1}^n}$$

Note that

$$q_k = \lim_{n\to\infty}\left(\frac{n-k}{n-k+1}\right)^{nr_1^n...r_{k-1}^n} = \exp\left(-q_1...q_{k-1}\right)$$

hence, as $n \to \infty$,

$$P_k^B \to \left(\prod_{s=1}^{k-1} q_s\right)(1-q_k)$$

Given this computation, we will below prove that

$$\lim_{n\to\infty}\sum_{k=1}^{K} P_k^B > \lim_{n\to\infty}\sum_{k=1}^{K} P_k^{DA}$$

for all $K \geq 1$.

Let's consider the limiting probabilities of $(P_k^n)^B$ and $(P_k^n)^{DA}$ for each k as $n \to \infty$. We will make an abuse of notation and call the limiting probabilities as P_k^B and P_k^{DA} when there is no confusion. First, by Lemma 3 we have that for all $k \geq 1$

$$P_k^{DA} = \frac{1}{k(k+1)}$$

Note also that

$$P_k^B = \left(\prod_{s=1}^{k-1} q_s\right)(1-q_k)$$

where

$$q_1 = \frac{1}{e}$$

and

$$q_k = \exp\left(-\prod_{s=1}^{k-1} q_s\right)$$

for all $k \geq 2$.

Before moving on, we make two observations on the sequence $\{q_k\}_{k=1}^{\infty}$.

Lemma 4. *For each $k \geq 1$, $q_k \in (0,1)$ and q_k is strictly increasing in k.*

Proof. We first prove the former result and the proof is by induction. Note that $q_1 = \frac{1}{e} \in (0,1)$. Assume that $q_k \in (0,1)$ for all $k \in \{1, ..., (i-1)\}$. Then, $q_i \in (0,1)$ since $q_i = \exp\left(-\prod_{k=1}^{i-1} q_k\right)$ and $\left(\prod_{k=1}^{i-1} q_k\right) \in (0,1)$. Hence, $q_k \in (0,1)$ for all $k \geq 1$.

Secondly, we want to show that q_k is strictly increasing in k. To do so, we show that $q_k > q_{k-1}$ for all $k \geq 2$. Now, note that

$$\begin{aligned} q_{k+1} &= \exp\left(-\prod_{i=1}^{k} q_i\right) \\ &= \left(\exp\left(-\prod_{i=1}^{k-1} q_i\right)\right)^{q_k} \\ &= q_k^{q_k} \end{aligned}$$

Hence, defining

$$t\left(x\right) = x^x$$

we have that

$$q_{k+1} = t\left(q_k\right)$$

Now, for all $x \in (0, 1)$

$$t'\left(x\right) = x^x\left(1 + \ln x\right)$$

Hence, t is strictly increasing over $\left(\frac{1}{e}, 1\right)$. Noting that $q_2 = \exp\left(-\frac{1}{e}\right) > \frac{1}{e}$, we have the desired result. □

Now,

$$\begin{aligned} P_{k+1}^B &= \left(\prod_{s=1}^{k} q_s\right)\left(1 - q_{k+1}\right) \\ &= q_k\left(1 - q_{k+1}\right)\left(\prod_{s=1}^{k-1} q_s\right) \\ &= q_k\left(1 - q_{k+1}\right)\left(\prod_{s=1}^{k-1} q_s\right)\frac{1 - q_k}{1 - q_k} \\ &= q_k\frac{1 - q_{k+1}}{1 - q_k}\left(\left(\prod_{s=1}^{k-1} q_s\right)\left(1 - q_k\right)\right) \end{aligned}$$

, that is,

$$P_{k+1}^B = q_k\frac{1 - q_{k+1}}{1 - q_k}P_k^B$$

We, noting this observation, present and prove the following lemma:

Lemma 5.

$$\lim_{n\to\infty} P_1^B > \lim_{n\to\infty} P_1^{DA}$$

and

$$\lim_{n\to\infty} P_k^B < \lim_{n\to\infty} P_k^{DA}$$

for all $k \geq 2$.

Proof. Firstly, note that since $e = \sum_{k=0}^{\infty} \frac{1}{n!} \geq \sum_{k=0}^{2} \frac{1}{n!} = 1 + 1 + \frac{1}{2} = \frac{5}{2} > 2$, we have that

$$\lim_{n\to\infty} P_1^B = 1 - \frac{1}{e} > \frac{1}{2} = \lim_{n\to\infty} P_1^{DA}$$

Furthermore, we will show that

$$\lim_{n\to\infty} P_k^B < \lim_{n\to\infty} P_k^{DA}$$

for all $k \geq 2$.

Now, from Lemma 4 and its proof we know that

$$q_{k+1} = q_k^{q_k}$$

and that $q_k \in (0,1)$ for all k and that q_k is increasing in k. Furthermore, note that $q_k \to 1$ as $k \to \infty$.

Note that since

$$P_{k+1}^B = q_k \frac{1 - q_{k+1}}{1 - q_k} P_k^B$$

we have that

$$P_{k+1}^B = f(q_k) P_k^B$$

where

$$f(x) = \frac{x(1 - x^x)}{1 - x}$$

Now, we claim that $f'(x) > 0$ for all $x \in (0,1)$. Firstly, note that

$$f(x) = g(x) h(x)$$

where

$$g(x) = x$$

and

$$h(x) = \frac{1 - x^x}{1 - x}$$

Furthermore, $g(x), h(x) > 0$ for all $x \in (0,1)$ and $g'(x) > 0$ for all $x \in (0,1)$. Hence, if it is true that $h'(x) \geq 0$ for all $x \in (0,1)$, we will be done. Now,

$$h'(x) = \frac{1}{(1-x)^2} k(x)$$

where

$$k(x) = 1 - x^x - x^x (1 + \ln x)(1 - x)$$

Now,

$$\begin{aligned} k'(x) &= -x^x (1 + \ln x) - x^x (1 + \ln x)^2 (1 - x) - x^x \left(\frac{1-x}{x} - (1 + \ln x) \right) \\ &= -x^x (1 + \ln x)^2 (1 - x) - x^{x-1} (1 - x) \\ &\leq 0 \end{aligned}$$

for all $x \in (0,1)$. Note that $k(1) = 0$. Hence, $k(x) \geq 0$ for all $x \in (0,1)$. Thus, $h'(x) \geq 0$ for all $x \in (0,1)$, which is the desired result.

Hence, $q_k \frac{1-q_{k+1}}{1-q_k}$ is increasing in k since q_k is also increasing in k.

Second, we claim that $f''(x) > 0$ for all $x \in \left(\frac{1}{e}, 1\right)$. Note that

$$\begin{aligned} f'(x) &= \frac{\partial}{\partial x} \left(\frac{x(1-x^x)}{1-x} \right) \\ &= \left(\frac{1-x^x}{1-x} \right) + \left(\frac{x(1-x^x)}{(1-x)^2} \right) - \left(\frac{x^{x+1}(1+\ln x)}{1-x} \right) \end{aligned}$$

Hence,

$$\begin{aligned} & f''(x) \\ = {} & \frac{\partial}{\partial x} \left[\left(\frac{1-x^x}{1-x} \right) + \left(\frac{x(1-x^x)}{(1-x)^2} \right) - \left(\frac{x^{x+1}(1+\ln x)}{1-x} \right) \right] \\ = {} & \left(-\frac{x^x(1+\ln x)}{(1-x)} + \frac{1-x^x}{(1-x)^2} \right) + \left(\frac{1-x^x}{(1-x)^2} + x \left(-\frac{x^x(1+\ln x)}{(1-x)^2} + \frac{2(1-x^x)}{(1-x)^3} \right) \right) \\ & + \left(-\frac{x^x(x + x\ln x + 1)(1+\ln x)}{1-x} - x^{x+1} \frac{\frac{1-x}{x} + (1+\ln x)}{(1-x)^2} \right) \\ = {} & -\frac{x^x}{1-x} + \frac{2(1-x^x)}{(1-x)^2} + \frac{2x(1-x^x)}{(1-x)^3} - \frac{x^x(1+\ln x)}{1-x} \end{aligned}$$

$$
\begin{aligned}
&\quad + \left(-\frac{2x^{x+1}\left(1+\ln x\right)}{\left(1-x\right)^2} - \frac{x^{x+1}\left(1+\ln x\right)\left(\frac{1+x}{x}+\ln x\right)}{\left(1-x\right)} \right) \\
&= \frac{1}{\left(1-x\right)^3} m\left(x\right)
\end{aligned}
$$

where

$$
\begin{aligned}
& m\left(x\right) \\
=\ & -x^{x+1}\ln^2 x + 2x^{x+2}\ln^2 x - x^{x+3}\ln^2 x - 2x^x \ln x + 3x^{x+1} + x^{x+2} - x^{x+3} \\
& +4x^{x+2}\ln x - 2x^{x+3}\ln x - 5x^x + 2
\end{aligned}
$$

Now,

$$
\begin{aligned}
& m'\left(x\right) \\
=\ & -x^{x-1}\left(1-x\right)^2 \\
& * \left(x^2\ln^3 x + 3x^2\ln^2 x + 3x^2\ln x + x^2 + 3x\ln^2 x + 9x\ln x + 6x + 2 \right) \\
=\ & -x^{x-1}\left(1-x\right)^2 \left(x^2\left(1+3\ln x+3\ln^2 x+\ln^3 x\right) + 3x\left(\ln^2 x + 3\ln x + 2\right) + 2 \right) \\
=\ & -x^{x-1}\left(1-x\right)^2 \left(x^2\left(1+\ln x\right)^3 + 3x\left(2+\ln x\right)\left(1+\ln x\right) + 2 \right) \\
<\ & 0
\end{aligned}
$$

when $x \in \left(\frac{1}{e}, 1\right)$ since $2+\ln x > 1+\ln x > 0$ when $x > \frac{1}{e}$. Furthermore,

$$m\left(1\right) = 0$$

Hence,

$$f''\left(x\right) > 0$$

for all $x \in \left(\frac{1}{e}, 1\right)$. To sum up, f is strictly increasing and strictly convex over $\left(\frac{1}{e}, 1\right)$.

Let

$$v\left(x\right) = \frac{x}{x+2}$$

Note that v is strictly increasing and strictly concave for all $x \in \left(0, \infty\right)$.

Now, we claim that if $P_k^B < \frac{1}{k(k+1)}$, then $P_{k+1}^B < \frac{1}{(k+1)(k+2)}$. To do so, we will show that for all $k \geq 1$

$$q_k \frac{1-q_{k+1}}{1-q_k} < \frac{k}{k+2}$$

Suppose that $f(q_k) \geq v(k)$ for some k. That means $f(q_{k'}) > v(k')$ for all $k' > k$ since f is strictly convex and g is strictly concave and both are strictly increasing on $\left(\frac{1}{e}, 1\right)$. However,

$$\lim_{k \to \infty} f(q_k) = \lim_{x \to 1} f(x) = 1$$

and

$$\lim_{k \to \infty} v(k) = 1$$

, a contradiction. Thus, if $P_k^B < \frac{1}{k(k+1)}$, then $P_{k+1}^B < \frac{1}{(k+1)(k+2)}$

Now, $q_1 = \frac{1}{e}$ and hence $P_1^B = 1 - \frac{1}{e}$ and

$$q_1 \frac{1 - q_2}{1 - q_1} = f(q_1) = f\left(\frac{1}{e}\right) = \frac{1}{e}\left(\frac{1 - \left(\frac{1}{e}\right)^{\frac{1}{e}}}{1 - \frac{1}{e}}\right)$$

Hence,

$$P_2^B = \tfrac{1}{e}\left(\frac{1 - \left(\frac{1}{e}\right)^{\frac{1}{e}}}{1 - \frac{1}{e}}\right)\left(1 - \tfrac{1}{e}\right) = \tfrac{1}{e}\left(1 - \left(\tfrac{1}{e}\right)^{\frac{1}{e}}\right)$$

Now, we claim that $\frac{1}{e}\left(1 - \left(\frac{1}{e}\right)^{\frac{1}{e}}\right) < \frac{1}{6}$. To see this, note that by the mean value theorem, we have that

$$\ln(6) = \ln(6 - e) + e\left(\frac{1}{x^*}\right)$$

for some $x^* \in (6 - e, 6)$. Hence,

$$e(\ln(6) - \ln(6 - e)) = e^2 \frac{1}{x^*}$$

Now, since

$$e^2 = \sum_{k=0}^{\infty} \frac{2^n}{n!} \geq \sum_{k=0}^{3} \frac{2^n}{n!} = 1 + 2 + \frac{4}{2} + \frac{8}{6} = \frac{19}{3} > 6$$

we have that $e^2 \frac{1}{x^*} > \frac{e^2}{6} > 1$.

Hence,

$$e(\ln(6) - \ln(6 - e)) > 1$$

or

$$\left(\ln\left(\tfrac{6-e}{6}\right)\right) < -\tfrac{1}{e}$$

Hence, we have that

$$\tfrac{6-e}{6} < e^{-\frac{1}{e}} \iff 1 - \tfrac{e}{6} < \left(\tfrac{1}{e}\right)^{\frac{1}{e}} \iff \tfrac{1}{e}\left(1 - \left(\tfrac{1}{e}\right)^{\frac{1}{e}}\right) < \tfrac{1}{6}$$

Thus, we have that $P_2^B < \frac{1}{2*3} = \frac{1}{6}$. Then,

$$P_k^B < \frac{1}{k\,(k+1)}$$

for all $k \geq 2$. □

Note that Lemma 5 implies the desired result

$$\lim_{n\to\infty} \sum_{k=1}^{K} (P_k^n)^B > \lim_{n\to\infty} \sum_{k=1}^{K} (P_k^n)^{DA}$$

for all $K \geq 1$ due to the following lemma:[4]

Lemma 6. *Assume that there is $k^* \geq 1$ such that for all $k \leq k^*$,*

$$\lim_{n\to\infty} P_k^B > \lim_{n\to\infty} P_k^{DA}$$

and for all $k > k^*$

$$\lim_{n\to\infty} P_k^B < \lim_{n\to\infty} P_k^{DA}$$

Then,

$$\lim_{n\to\infty} \sum_{k=1}^{K} P_k^B > \lim_{n\to\infty} \sum_{k=1}^{K} P_k^{DA}$$

for all $K \geq 1$.

Proof. Now, the claim trivially holds for all $K \leq k^*$. For $K > k^*$ note that[5]

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{K} P_k^B &= \left(\lim_{n\to\infty} \sum_{k=1}^{\infty} P_k^B\right) - \left(\lim_{n\to\infty} \sum_{k=K+1}^{\infty} P_k^B\right) \\ &= 1 - \left(\lim_{n\to\infty} \sum_{k=K+1}^{\infty} P_k^B\right)\end{aligned}$$

[4]We will drop n from the notation below and write $(P_k)^B$ instead.

[5]For convenience, denote $(P_k^n)^B = 0$ when $k > n$.

and

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{K} P_k^{DA} &= \left(\lim_{n\to\infty}\sum_{k=1}^{\infty} P_k^{DA}\right) - \left(\lim_{n\to\infty}\sum_{k=K+1}^{\infty} P_k^{DA}\right)\\ &= 1 - \left(\lim_{n\to\infty}\sum_{k=K+1}^{\infty} P_k^{DA}\right)\end{aligned}$$

Note also that

$$\lim_{n\to\infty}\sum_{k=K+1}^{\infty} P_k^{B} < \lim_{n\to\infty}\sum_{k=K+1}^{\infty} P_k^{DA}$$

since $K > k^*$ and for all $k > k^*$ we have $\lim_{n\to\infty} P_k^B < \lim_{n\to\infty} P_k^{DA}$. Hence, we have the result. □

Now, we are ready to prove the theorem:

Theorem *For sufficiently large n, the Boston mechanism interim dominates the Deferred Acceptance mechanism.*

Proof. Consider some $k^* > 1$. Note that by Lemmas 6 and 5,for sufficiently large n, we have that

$$\sum_{k=1}^{K} (P_k^n)^B > \sum_{k=1}^{K} (P_k^n)^{DA}$$

for all $K \leq k^*$. Take any $\mathbf{v} = (v_k^n)_{k=1}^n \in \boldsymbol{V}$. We want to show that

$$\sum_{k=1}^{n} (P_k^n)^B v_k^n \geq \sum_{k=1}^{n} (P_k^n)^{DA} v_k^n$$

Now,

$$\begin{aligned}&\sum_{k=1}^{n} (P_k^n)^B v_k^n\\ =\;& (P_1^n)^B (v_1^n - v_2^n) + \left((P_1^n)^B + (P_2^n)^B\right)(v_2^n - v_3^n) + ... +\\ & \left((P_1^n)^B + ... + \left(P_{n-1}^n\right)^B\right)\left(v_{n-1}^n - v_n^n\right) + \left((P_1^n)^B + ... + (P_n^n)^B\right) v_n^n\\ =\;& v_n^n + \sum_{k=1}^{n-1}\left((P_1^n)^B + ... + (P_k^n)^B\right)\left(v_k^n - v_{k+1}^n\right)\end{aligned}$$

Similarly

$$\begin{aligned} & \sum_{k=1}^{n} \left(P_k^n\right)^{DA} v_k^n \\ = \quad & v_n^n + \sum_{k=1}^{n-1} \left(\left(P_1^n\right)^{DA} + ... + \left(P_k^n\right)^{DA}\right) \left(v_k^n - v_{k+1}^n\right) \end{aligned}$$

Thus,

$$\begin{aligned} & \sum_{k=1}^{n} \left(\left(P_k^n\right)^B - \left(P_k^n\right)^{DA}\right) v_k^n \\ = \quad & \sum_{k=1}^{n-1} \left[\left(\left(P_1^n\right)^B + ... + \left(P_k^n\right)^B\right) - \left(\left(P_1^n\right)^{DA} + ... + \left(P_k^n\right)^{DA}\right)\right] \left(v_k^n - v_{k+1}^n\right) \\ = \quad & \sum_{k=1}^{k^*} \left[\left(\left(P_1^n\right)^B + ... + \left(P_k^n\right)^B\right) - \left(\left(P_1^n\right)^{DA} + ... + \left(P_k^n\right)^{DA}\right)\right] \left(v_k^n - v_{k+1}^n\right) \\ & + \sum_{k=k^*+1}^{n} \left[\left(\left(P_1^n\right)^B + ... + \left(P_k^n\right)^B\right) - \left(\left(P_1^n\right)^{DA} + ... + \left(P_k^n\right)^{DA}\right)\right] \left(v_k^n - v_{k+1}^n\right) \\ > \quad & \left(\left(P_1^n\right)^B - \left(P_1^n\right)^{DA}\right) \left(v_1^n - v_2^n\right) - \left(1 - \left(\left(P_1^n\right)^B + ... + \left(P_{k^*+1}^n\right)^B\right)\right) v_{k+1}^* \end{aligned}$$

By earlier computations, we know that

$$\left(P_1^n\right)^B - \left(P_1^n\right)^{DA} = 1 - \left(\frac{n-1}{n}\right)^n - \frac{n+1}{2n}$$

Furthermore, note that

$$\begin{aligned} & \left(1 - \left(\left(P_1^n\right)^B + ... + \left(P_{k^*+1}^n\right)^B\right)\right) \\ \leq \quad & \left(1 - \left(\left(P_1^n\right)^B + ... + \left(P_{k^*}^n\right)^B\right)\right) \\ \leq \quad & \left(1 - \left(\left(P_1^n\right)^{DA} + ... + \left(P_{k^*}^n\right)^{DA}\right)\right) \end{aligned}$$

and

$$\sum_{k=1}^{k^*} \left(P_1^k\right)^{DA} = \frac{n+1}{n} \sum_{k=1}^{k^*} \frac{1}{k\left(k+1\right)} = \frac{n+1}{n} \left(1 - \frac{1}{k^*+1}\right)$$

Hence,

$$\left(\left(P_1^n\right)^B - \left(P_1^n\right)^{DA}\right) \left(v_1^n - v_2^n\right) - \left(1 - \left(\left(P_1^n\right)^B + ... + \left(P_{k^*+1}^n\right)^B\right)\right) v_{k^*+1}^n$$

$$
\begin{aligned}
&\geq \left(1-\left(\frac{n-1}{n}\right)^n-\frac{n+1}{2n}\right)\left(v_1^n-v_2^n\right)-\left(1-\frac{n+1}{n}\left(1-\frac{1}{k^*+1}\right)\right)v_{k^*+1}^n \\
&> 0
\end{aligned}
$$

for large enough k^* and n since $v_1^n - v_2^n > 0$, $\left(1-\left(\frac{n-1}{n}\right)^n-\frac{n+1}{2n}\right)$ is an increasing sequence in n that converges to $\frac{1}{e}-\frac{1}{2}>0$ and $1-\frac{n+1}{n}\left(1-\frac{1}{k^*+1}\right)$ converges to 0. □

A.3 Two-school Case: Proof of Proposition 2

Consider a student, say i_1, with ranking $1-2$ over schools. Let j be the number of students (except i_1) whose top choice is school 1. Hence, there are $2q-j-1$ students whose top choice is school 2.

Proof. Firstly, consider the case that $0 \leq j \leq (q-1)$. Under the Boston mechanism, i_1 gets a seat in school 1 for sure. Under the DA, in Step 1, no one is rejected by school 1 since there are at most q students applying to school 1. Now, there are $2q-j-1 \in [q, 2q-1]$ students who applied to school 2 is Step 1. Hence, q of them will be accepted and $q-j-1$ of them will be rejected by school 2. Those who are rejected by school 2 in Step 1 will apply to school 1 in Step 2. Hence, there are $(j+1)+(q-j-1)=q$ students applying to school 1 in Step 2. Note that no student is rejected by any school in Step 2 and hence algorithm stops. Therefore, i_1 gets into school 1 under the DA.

Secondly, consider the case $q \leq j \leq (2q-1)$. Now, under both mechanisms, i_1 is accepted by school 1 with probability $\frac{q}{j+1}$. If accepted in Step 1, i_1 will get a seat in school 1 under both DA and Boston. If rejected in Step 1, i_1 will end up getting into school 2 under both mechanisms.

Hence, the probability of entering into school 1 and school 2 for i_1 is the same for both mechanisms. □

A.4 Three-School Case: Proof of Proposition 3

Assume that there are 3 schools each with a quota of $q \geq 1$ and $3q$ students. We consider a student, say Student 1 ($S1$) ,who has the strict ordinal ranking of $1-2-3$ over schools. That is, his first choice is school 1, his second choice is school 2 and his third choice is school 3. Let $P^B(q) = \left(P_k^B(q)\right)$ denote the probability of $S1$

getting into each school under the Boston mechanism and P^{DA} similarly under the DA mechanism.[6]

Claim 1. *$P_1^B(q) - P_1^{DA}(q) = 2\left(P_3^B(q) - P_3^{DA}(q)\right)$ for each $q \geq 1$.*

Note that this result will imply that

$$P_1^B(q) - P_1^{DA}(q) = -\frac{2}{3}\left(P_2^B(q) - P_2^{DA}(q)\right)$$

since

$$\sum_{k=1}^{3} P_k^B(q) = \sum_{k=1}^{3} P_k^{DA}(q) = 1$$

Let m be the number of students (other than $S1$) who rank school 1 as first choice, and j be the number of students who rank school 2 as first choice and $k \equiv 3q-m-j-1$ be number of students who rank school 3 as first choice.

Before moving on, recall from earlier discussions that under the Boston mechanism the probability of a student to be accepted at a step when there are, say z, other students applying to the same school is $\frac{q^*}{z+1}$ when $z \geq q^*$ and 1 if $z < q^*$ where q^* is the number of available seats unoccupied in earlier steps.

However, this is not the case under DA. A student is competing not only with students applying to the same school at the same step but also with students who were tentatively accepted by that school in earlier steps. Note that these students are not alike because some students are eliminated and the number of students they competed with earlier may differ from each other. Furthermore, some students eliminated some other students which give additional information regarding the rank of these students. Hence, computing the probability of getting chosen is complicated in this case. For the computation below, we will use the following functions.

- $prob1(a, b, c, d)$: Probability of a student to be accepted by a school, say s, in Step 2 under DA when the quota is a, and there were $b \leq (a-1)$ other students whose first rank is s and d additional students apply to s in Step 2 who were eliminated by some school, say s', in the first step when there were in total c students applying to s'.

- $probc21(a, b, c, d)$: Probability of a student to be accepted by a school, say s, in Step 2 under DA when the quota is a, and he was eliminated by a school,

[6] Again we are suppressing n in the original notation P_k^n.

say s', in the first step together with d other students when there were in total $c+1$ students applying to s' and there were $b \leq a$ other students who applied to s in the first step.

- $prob2\,(a,b,c,d)$: Probability of a student to be accepted by a school, say s, in Step 2 under DA when the quota is a, and there were $b \geq a$ other students whose first rank is s and that student was accepted by s in the first step and d additional students apply to s in Step 2 who were eliminated by some other school, say s', in the first step when there were in total c students applying to s'.

- $probc22\,(a,b,c,d)$: Probability of a student to be accepted by a school, say s, in Step 2 under DA when the quota is a, and he was eliminated by a school, say s', in the first step together with other d students when there were $c+1$ students applying to s' and there were $b \geq a$ other students who applied to s in the first step.[7]

Although we will not need to compute these functions, as an illustration, let us consider the following simple example.

Example 11. *Assume that there are* 3 *students and* 3 *schools each with a quota of* 1*. Let the preference profiles be given as follows:*

$$\begin{aligned} i_1 &: (s_2, s_1, s_3) \\ i_2 &: (s_1, s_2, s_3) \\ i_3 &: (s_2, s_3, s_1) \end{aligned}$$

Let us compute the probabilities for student i_1 under both the Boston mechanism and the DA mechanism. In the first step, i_1 is chosen with probability $\frac{1}{2}$ under both Boston and DA since i_3 also applies to s_2. If i_1 is chosen then he gets into s_2 for sure under both mechanisms since under DA eliminated student i_3 will apply to s_3 and will be

[7]Note that $prob1$ and $prob2$ are the same except that one if for $b \leq (a-1)$ and one is for $b \geq a$. Theoretically, they could have been combined in one function which would be perfectly okay but conceptually there is a difference. When $b \leq (a-1)$, no one is eliminated in Step 1 and we gain no information but when $b > a$, there a people will be accepted and remaining $b-a$ people will be eliminated. This gives us additional information. We believe that it is a good idea to keep these functions separate to keep track of things. Similarly, for $probc1$ and $probc2$ functions.

accepted. If i_1 is eliminated in Step 1, then he will apply to s_1 in step 2. He will be accepted with probability 0 under Boston since i_2 was accepted in step 1 and hence there is not any available seat. Under DA, he will be accepted with probability $\frac{1}{3}$ since we do have the information that he was eliminated in Step 1 which implies that he is ranked below i_3 in the school priority list implying that the school priority list is (i_3, i_2, i_1) or (i_3, i_1, i_2) or (i_2, i_3, i_3) each with equal probability.

To save on space, we will define and use the following $f(q, m, j)$ function which is the probability that there are m students (other than $S1$) who rank school 1 as first choice, and j students who rank school 2 as first choice and $k \equiv 3q - m - j - 1$ students who rank school 3 as first choice.

Notation 1. $f(q, m, j) = \binom{3q-1}{m}\binom{3q-1-m}{j}\left(\frac{1}{3}\right)^{3q-1}$

A.4.1 $m \leq (q-1)$

In this case, under the Boston (B) mechanism, $S1$ will be assigned school 1 for sure. For DA, we consider the subcases:

A.4.1.1 $j \leq q-1$

In this case, $k \geq (q+1)$ and therefore $(k-q) = 2q - m - j - 1 \geq 1$ students will be eliminated from school 3 in the first step of DA. Assume that k' among those $k-q$ students apply to school 1 in the next step which happens with probability $\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{k'}\left(\frac{1}{2}\right)^{2q-j-m-1-k'}$.

Now, if $m + k' \leq (q-1)$, or $k' \leq q - 1 - m$ no one is eliminated in second step from school 1 again. In this case $S1$ will be placed into school 1 for sure.

Note that when $k' \geq (q-m)$, $prob1(q, m, k, k')$ is the probability of $S1$ getting accepted by school 1 is step 2. Note that if $S1$ is accepted, then he will be assigned to school 1. Hence, in this case, when $m, j \leq (q-1)$, the difference in probability of getting into school 1 between B and DA[8]is

$$\sum_{m=0}^{q-1}\sum_{j=0}^{q-1} f(q, m, j)\left(\sum_{k'=q-m}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-1}(1 - prob1(q, m, k, k'))\right) \tag{A.2}$$

Note that if $S1$ is not chosen, he will be assigned to school 2 under DA.

[8](probability under Boston)-(probability under DA)

A.4.1.2 $q \leq j \leq (2q - m - 1)$

In this case $j, k \geq q$ and $S1$ is assigned to school 1 for sure under DA since there can be at most q students applying to school 1 even in later steps.

A.4.1.3 $(2q - m) \leq j \leq (3q - m - 1)$

Then, $k \leq (q - 1)$ in this case. Similarly to above case, there will be $(j - q)$ students will be eliminated from 2. Assume that s of them applies to school 1 in second step.

If $s + m \leq (q - 1)$, or $s \leq q - 1 - m$, $S1$ will be assigned to school 1 for sure. If $s \geq (q - m)$, the probability of $S1$ getting chosen will be $prob1\,(q, m, j, s)$ by symmetry and if he is accepted, he is assigned to school 1 for sure. Hence, in this case the difference in probability of getting into 1 between B and DA is

$$\sum_{m=0}^{q-1} \sum_{j=2q-m}^{3q-m-1} f\,(q, m, j) \left(\sum_{s=q-m}^{j-q} \binom{j-q}{s} \left(\frac{1}{2}\right)^{j-q} (1 - prob1\,(q, m, j, s)) \right) \tag{A.3}$$

Note that $(A.2) = (A.3)$ by symmetry.

Note that $S1$ will end up going to school 3 if he is not chosen in step 2. Hence, in this case the difference in probability of getting into 3 between B and DA is

$$-\sum_{m=0}^{q-1} \sum_{j=2q-m}^{3q-m-1} f\,(q, m, j) \left(\sum_{s=q-m}^{j-q} \binom{j-q}{s} \left(\frac{1}{2}\right)^{j-q} (1 - prob1\,(q, m, j, s)) \right)$$

A.4.2 $q \leq m \leq (2q - 1)$

In this case, under both B and DA, $S1$ will be accepted in step 1 with probability $\frac{q}{m+1}$. Note that under B, $S1$ will be assigned to school 1 for sure if he is accepted in step 1. For DA, we again consider the subcases.

A.4.2.1 $0 \leq j \leq 2q - m - 1$

In this case, $j \leq (q - 1)$ and $k \geq q$.

Assume that $S1$ is accepted in first step. Again, $k - q$ students will be eliminated from school 3 and assume that k' among $k - q$ students eliminated apply to school 1 in the next step. Now, $prob2\,(q, m, k, k')$ is the probability of $S1$ getting chosen in this case in DA. Hence, the difference in probability of getting into school 1 between

B and DA is

$$\sum_{m=q}^{2q-1}\sum_{j=0}^{2q-m-1} f\left(q,m,j\right)\frac{q}{m+1}c_1 \tag{A.4}$$

where

$$c_1 = \left(\sum_{k'=0}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-1}\left(1-prob2\left(q,m,k,k'\right)\right)\right)$$

In this case if $S1$ is rejected in second step, he will be assigned to school 2 for sure under DA.

Secondly, assume that $S1$ is rejected in first step. Then, $S1$ will be assigned to school 2 for sure under both B and DA.

A.4.2.2 $(2q-m) \leq j \leq q$

Now, in this case $k \leq (q-1)$.

Now, if $S1$ is accepted in the first step, $S1$ will be assigned to school 1 under DA.

If $S1$ is eliminated in first step: There are $(m-q)$ additional students eliminated from 1 in the first step. Assume that m' of them applies to school 2 in second step.

If $m'+j \leq (q-1)$, $S1$ will be assigned to school 2 under both B and DA.

If $m'+j \geq q$, under B, $S1$ will be assigned to school 2 w/p $\frac{q-j}{m'+1}$ and will go to school 3 w/p 1$-$ $\frac{q-j}{m'+1}$. Under DA, $S1$ will be assigned to 2 w/p $probc21\left(q,m,j,m'\right)$ and will be assigned to 3 w/p 1$-$ $probc21\left(q,m,j,m'\right)$. Hence, in this case, the difference in probability of getting into 1 between B and DA is 0 and the difference in probability of getting into 3 between B and DA is

$$\sum_{m=q}^{2q-1}\sum_{j=2q-m}^{q} f\left(q,m,j\right)\frac{m+1-q}{m+1}c_2 \tag{A.5}$$

where

$$c_2 = \left(\sum_{m'=q-j}^{m-q}\binom{m-q}{m'}\left(\frac{1}{2}\right)^{m-q}\left(probc21\left(q,m,j,m'\right)-\frac{q-j}{m'+1}\right)\right)$$

A.4.2.3 $(q+1) \leq j \leq (3q-m-1)$

In this case $k \leq (q-2)$ and $j \geq (q+1)$.

Assume that $S1$ is accepted in first step. Then, he will be assigned to school 1 under B. For DA, again, $j-q$ students will be rejected by school 2 and assume that s among $j-q$ students eliminated goes to 1 in the next step. Now, $prob2\left(q,m,j,s\right)$ is the probability of $S1$ getting accepted in this case under DA. Hence, the difference in probability of getting into 1 between B and DA is

$$\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f\left(q,m,j\right)\frac{q}{m+1}\left(\sum_{s=0}^{j-q}\binom{j-q}{s}\left(\frac{1}{2}\right)^{j-q}\left(1-prob2\left(q,m,j,s\right)\right)\right) \tag{A.6}$$

Again, by symmetry $(A.4)=(A.6)$.

In this case if $S1$ is not chosen in second step, he will be assigned to school 3 for sure in DA. Hence, the difference in probability of getting into school 1 between B and DA is

$$-\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f\left(q,m,j\right)\frac{q}{m+1}\left(\sum_{s=0}^{j-q}\binom{j-q}{s}\left(\frac{1}{2}\right)^{j-q}\left(1-prob2\left(q,m,j,s\right)\right)\right) \tag{A.7}$$

Secondly, assume that $S1$ is not accepted in first step. Then, $S1$ will be assigned to school 3 for sure under B. Under DA, he will be assigned to school 2 w/p $probc22\left(q,j,m,m'\right)$ and to school 3 w/p $1-probc22\left(q,j,m,m'\right)$ where m' is the number of students who are eliminated from school 1 in the first step and goes to 2 in second step (other than $S1$). Hence, the difference in probability of getting into school 3 between B and DA is

$$\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f\left(q,m,j\right)\frac{m+1-q}{m+1}\left(\sum_{m'=0}^{m-q}\binom{m-q}{m'}\left(\frac{1}{2}\right)^{m-q}\left(probc22\left(q,j,m,m'\right)\right)\right) \tag{A.8}$$

A.4.3 $2q\leq m\leq(3q-1)$

Then, $j,k<q$. If $S1$ is chosen in first step, $S1$ will be assigned to school 1 under both B and DA.

If $S1$ is not chosen in first step: Let m' be the number of students who are eliminated from 1 in the first step and applies to 2 in second step (other than $S1$).

If $m'+j\leq(q-1)$, $S1$ will be assigned to school 2 under both B and DA.

If $m'+j\geq q$, under Boston, $S1$ will be assigned to school 2 w/p $\frac{q-j}{m'+1}$ and will be assigned to school 3 w/p $1-\frac{q-j}{m'+1}$. Under DA, $S1$ will be assigned to school 2 w/p $probc21\left(q,m,j,m'\right)$ and will be assigned to school 3 w/p 1− $probc21\left(q,m,j,m'\right)$. Hence, in this case, the difference in probability of getting into 1 between B and DA is 0 and the difference in probability of getting into 3 between B and DA is

$$\sum_{m=2q}^{3q-1}\sum_{j=0}^{3q-1-m} f\left(q,m,j\right)\frac{m+1-q}{m+1}c_3 \tag{A.9}$$

where

$$c_3=\left(\sum_{m'=q-j}^{m-q}\binom{m-q}{m'}\left(\frac{1}{2}\right)^{m-q}\left(probc21\left(q,m,j,m'\right)-\frac{q-j}{m'+1}\right)\right)$$

Hence, combining all the cases, we have that

$$\begin{aligned}&P_1^B\left(q\right)-P_1^{DA}\left(q\right)\\=\;&2\left[\sum_{m=0}^{q-1}\sum_{j=0}^{q-1} f\left(q,m,j\right)c_4+\sum_{m=q}^{2q-1}\sum_{j=0}^{2q-m-1} f\left(q,m,j\right)\frac{q}{m+1}c_5\right]\end{aligned}$$

where

$$\gamma\left(q,m,k,k'\right)=1-prob2\left(q,m,k,k'\right)$$

,

$$c_4=\sum_{k'=q-m}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-1}\left(1-prob1\left(q,m,k,k'\right)\right)$$

and

$$c_5=\sum_{k'=q-m}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-1}\left(1-prob1\left(q,m,k,k'\right)\right)$$

Note also that

$$\begin{aligned}&P_3^B\left(q\right)-P_3^{DA}\left(q\right)\\=\;&\\&-\left[\sum_{m=0}^{q-1}\sum_{j=2q-m}^{3q-m-1} f\left(q,m,j\right)\sum_{s=q-m}^{2q-m-j-1}\binom{j-q}{s}\left(\frac{1}{2}\right)^{j-q}\left(1-prob1\left(q,m,j,s\right)\right)\right]\\&+\sum_{m=q}^{2q-1}\sum_{j=2q-m}^{q} f\left(q,m,j\right)\frac{m+1-q}{m+1}c_6-\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f\left(q,m,j\right)\frac{q}{m+1}c_7\end{aligned}$$

$$+\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f(q,m,j)\frac{m+1-q}{m+1}c_8+\sum_{m=2q}^{3q-1}\sum_{j=0}^{3q-1-m} f(q,m,j)\frac{m+1-q}{m+1}c_9$$

where

$$c_6=\sum_{m'=q-j}^{m-q}\binom{m-q}{m'}\left(\frac{1}{2}\right)^{m-q}\left(probc21(q,m,j,m')-\frac{q-j}{m'+1}\right)$$

$$c_7=\sum_{s=0}^{j-q}\binom{j-q}{s}\left(\frac{1}{2}\right)^{j-q}(1-prob2(q,m,j,s))$$

$$c_8=\sum_{m'=0}^{m-q}\binom{m-q}{m'}\left(\frac{1}{2}\right)^{m-q}(probc22(q,m,j,m'))$$

and

$$c_9=\sum_{m'=q-j}^{m-q}\binom{m-q}{m'}\left(\frac{1}{2}\right)^{m-q}\left(probc21(q,m,j,m')-\frac{q-j}{m'+1}\right)$$

Note first that $P_1^B(q)-P_1^{DA}(q)>0.$

Two additional observations on probability functions defined:

Lemma 7. *When $m'+j\geq(q-1)$,*

$$probc21(q,m,j,m')=\frac{q-j(prob1(q,j-1,m+1,m'+1))}{m'+1}$$

Proof. Consider a situation in which $j\leq q$ students apply to school s in Step 1 and $m'+1$ other students who were eliminated from some other school when there were $m+1$ students applying to that school, say s', in step 1 applies to s in Step 2. Note that when $m'+1+j\geq q$, the probability of a student who applied to s in first step to be accepted in Step 2 is $prob1(q,j-1,m+1,m'+1)$ and the probability of a student who applied to s' in first step to be accepted by s in Step 2 is $probc21(q,m,j,m')$. Hence, it must be that $(m'+1)\,probc21(q,m,j,m')+j(prob1(q,j-1,m+1,m'+1))=q$ which implies the desired result. □

Lemma 8. *When $j\geq(q+1)$, $m\geq q$,*

$$probc22(q,m,j,m')=\frac{q(1-prob2(q,j-1,m+1,m'+1))}{m'+1}$$

Proof. Similar to the above lemma, we must have that $(m'+1)\,probc22(q,m,j,m')+q(prob2(q,j-1,m+1,m'+1))=q$ □

Hence, $P_3^B(q) - P_3^{DA}(q)$ becomes

$$
\begin{aligned}
& P_3^B(q) - P_3^{DA}(q) \\
= & \\
& -\left[\sum_{m=0}^{q-1}\sum_{j=2q-m}^{3q-m-1} f(q,m,j) \sum_{s=q-m}^{2q-m-j-1} \binom{j-q}{s} \left(\frac{1}{2}\right)^{j-q} (1 - prob1(q,m,j,s))\right] \\
& +\left[\sum_{m=q}^{2q-1}\sum_{j=2q-m}^{q} f(q,m,j)\, g(m,q) \sum_{m'=q-j}^{m-q} \binom{m-q}{m'} \left(\frac{1}{2}\right)^{m-q} z(m,j,m',q)\right] \\
& -\left[\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f(q,m,j)\, h(m,q) \sum_{s=0}^{j-q} \binom{j-q}{s} \left(\frac{1}{2}\right)^{j-q} (1 - prob2(q,m,j,s))\right] \\
& +\left[\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f(q,m,j)\, g(m,q)\, c_{10}\right] \\
& +\left[\sum_{m=2q}^{3q-1}\sum_{j=1}^{3q-1-m} f(q,m,j)\, g(m,q) \sum_{m'=q-j}^{m-q} \binom{m-q}{m'} \left(\frac{1}{2}\right)^{m-q} z(m,j,m',q)\right]
\end{aligned}
$$

where

$$g(m,q) = \frac{m+1-q}{m+1},\ h(m,q) = \frac{q}{m+1}$$

$$z(m,j,m',q) = \left(\frac{q - j\,(prob1(q,j-1,m+1,m'+1))}{m'+1} - \frac{q-j}{m'+1}\right)$$

$$c_{10} = \sum_{m'=0}^{m-q} \binom{m-q}{m'} \left(\frac{1}{2}\right)^{m-q} \left(\frac{q\,(1 - prob2(q,j-1,m+1,m'+1))}{m'+1}\right)$$

Noting that

$$
\begin{aligned}
& \left[\sum_{m=0}^{q-1}\sum_{j=2q-m}^{3q-m-1} f(q,m,j) \sum_{s=q-m}^{2q-m-j-1} \binom{j-q}{s} \left(\frac{1}{2}\right)^{j-q} (1 - prob1(q,m,j,s))\right] \\
= & \sum_{m=0}^{q-1}\sum_{j=0}^{q-1} f(q,m,j)\, c_{11}
\end{aligned}
$$

where

$$c_{11} = \left(\sum_{k'=q-m}^{2q-m-j-1} \binom{2q-j-m-1}{k'} \left(\frac{1}{2}\right)^{2q-j-m-1} (1 - prob1(q,m,k,k'))\right)$$

and

$$\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f(q,m,j)\,h(m,q)\sum_{s=0}^{j-q}\binom{j-q}{s}\left(\frac{1}{2}\right)^{j-q}(1-prob2(q,m,j,s))$$
$$=\sum_{m=q}^{2q-1}\sum_{j=0}^{2q-m-1} f(q,m,j)\,\frac{q}{m+1}c_{12}$$

where

$$c_{12}=\left(\sum_{k'=0}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-1}(1-prob2(q,m,k,k'))\right)$$

,to show that $P_1^B(q)-P_1^{DA}(q)=2\left(P_3^B(q)-P_3^{DA}(q)\right)$ it suffices to show the following equality:

$$2\left[\sum_{m=0}^{q-1}\sum_{j=0}^{q-1} f(q,m,j)\,c_{13}+\sum_{m=q}^{2q-1}\sum_{j=0}^{2q-m-1} f(q,m,j)\,\frac{q}{m+1}c_{14}\right]$$
$$=$$
$$\left[\sum_{m=q}^{2q-1}\sum_{j=2q-m}^{q} f(q,m,j)\,g(m,q)\sum_{m'=q-j}^{m-q}\binom{m-q}{m'}\left(\frac{1}{2}\right)^{m-q} z(m,j,m',q)\right]+$$
$$\left[\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f(q,m,j)\,g(m,q)\,c_{15}\right]+$$
$$\left[\sum_{m=2q}^{3q-1}\sum_{j=1}^{3q-1-m} f(q,m,j)\,g(m,q)\sum_{m'=q-j}^{m-q}\binom{m-q}{m'}\left(\frac{1}{2}\right)^{m-q} z(m,j,m',q)\right]$$

again $g(m,q)=\frac{m+1-q}{m+1}$ and $z(m,j,m',q)=\left(\frac{q-j(prob1(q,j-1,m+1,m'+1))}{m'+1}-\frac{q-j}{m'+1}\right)$ and

$$c_{13}=\left(\sum_{k'=q-m}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-1}(1-prob1(q,m,k,k'))\right)$$

$$c_{14}=\left(\sum_{k'=0}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-1}(1-prob2(q,m,k,k'))\right)$$

$$c_{15}=\sum_{m'=0}^{m-q}\binom{m-q}{m'}\left(\frac{1}{2}\right)^{m-q}\left(\frac{q\,(1-prob2(q,j-1,m+1,m'+1))}{m'+1}\right)$$

Calling the first sum as S_1 and the second as S_2 and by using the notation

$$\alpha\left(j,q,m'\right)=\left(j\left(1-prob1\left(q,j-1,m+1,m'+1\right)\right)\right)$$

and

$$\beta\left(j,q,m'\right)=\left(1-prob2\left(q,j-1,m+1,m'+1\right)\right)$$

, S_2 becomes

$$\sum_{m=q}^{2q-1}\sum_{j=2q-m}^{q-1} f\left(q,m,j\right)\frac{m+1-q}{m+1}\left(\sum_{m'=q-j}^{m-q}\binom{m-q}{m'}\frac{1}{m'+1}\left(\frac{1}{2}\right)^{m-q}\alpha\left(j,q,m'\right)\right)$$
$$+\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f\left(q,m,j\right)\frac{m+1-q}{m+1}\left(\sum_{m'=0}^{m-q}\binom{m-q}{m'}\frac{q}{m'+1}\left(\frac{1}{2}\right)^{m-q}\beta\left(j,q,m'\right)\right)$$
$$+\sum_{m=2q}^{3q-1}\sum_{j=0}^{3q-1-m} f\left(q,m,j\right)\frac{m+1-q}{m+1}\left(\sum_{m'=q-j}^{m-q}\binom{m-q}{m'}\frac{1}{m'+1}\left(\frac{1}{2}\right)^{m-q}\alpha\left(j,q,m'\right)\right)$$

which is equal to

$$\sum_{m=q}^{2q-1}\sum_{j=2q-m}^{q} f\left(q,m,j\right)\frac{1}{m+1}\sum_{m'=q-j}^{m-q}\binom{m+1-q}{m'+1}\left(\frac{1}{2}\right)^{m-q}\alpha\left(j,q,m'\right)$$
$$+\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f\left(q,m,j\right)\frac{q}{m+1}\sum_{m'=0}^{m-q}\binom{m+1-q}{m'+1}\left(\frac{1}{2}\right)^{m-q}\beta\left(j,q,m'\right)$$
$$+\sum_{m=2q}^{3q-1}\sum_{j=1}^{3q-1-m} f\left(q,m,j\right)\frac{1}{m+1}\sum_{m'=q-j}^{m-q}\binom{m+1-q}{m'+1}\left(\frac{1}{2}\right)^{m-q}\alpha\left(j,q,m'\right)$$

Claim 2.

$$2\sum_{m=q}^{2q-1}\sum_{j=0}^{2q-m-1} f\left(q,m,j\right)\frac{q}{m+1}c_{16}=\sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f\left(q,m,j\right)\frac{q}{m+1}c_{17}$$

where

$$c_{16}=\left(\sum_{k'=0}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-1}\left(1-prob2\left(q,m,k,k'\right)\right)\right)$$

$$c_{17}=\left(\sum_{m'=0}^{m-q}\binom{m+1-q}{m'+1}\left(\frac{1}{2}\right)^{m-q}\left(1-prob2\left(q,j-1,m+1,m'+1\right)\right)\right)$$

Proof. We want to show

$$\sum_{m=q}^{2q-1}\sum_{j=0}^{2q-m-1} f\left(q,m,j\right)\frac{q}{m+1}c_{18} = \sum_{m=q}^{2q-1}\sum_{j=q+1}^{3q-m-1} f\left(q,m,j\right)\frac{q}{m+1}c_{17}$$

where

$$c_{18} = \sum_{k'=0}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-2}\left(1-prob2\left(q,m,k,k'\right)\right)$$

Note that $prob2\left(q,m,k,0\right)=1$ and if $m=2q-1$, then $j=0$ and $2q-m-j-1=0$. Furthermore, if $j=2q-m-1$, then $2q-m-j-1=0$. Note also that $3q-m-1=q<(q+1)$ when $m=2q-1$. Hence, in effect, we want to show

$$\sum_{m=q}^{2q-2}\sum_{j=0}^{2q-m-2} f\left(q,m,j\right)\frac{1}{m+1}c_{19} = \sum_{m=q}^{2q-2}\sum_{j=q+1}^{3q-m-1} f\left(q,m,j\right)\frac{1}{m+1}c_{17}$$

where

$$c_{19} = \sum_{k'=1}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-2}\left(1-prob2\left(q,m,k,k'\right)\right)$$

Pick some (m,j) such that $m\in[q,2q-2]$ and $j\in[0,2q-m-2]$. Now, then consider $m^*=3q-m-j-2$ and $j^*=m+1$. Note that $m^*\in[q,2q-2]$ and $j^*\in[q+1,3q-m^*-1]$.

Recall that $f\left(q,m,j\right)=\binom{3q-1}{m}\binom{3q-1-m}{j}\left(\frac{1}{3}\right)^{3q-1}$ and we claim that

$$\binom{3q-1}{m}\binom{3q-1-m}{j}\frac{1}{m+1}c_{20}$$

$$=\binom{3q-1}{m^*}\binom{3q-1-m^*}{j^*}\frac{1}{m^*+1}c_{21}$$

where $l\left(q,j^*,m^*,m'\right)=\left(1-prob2\left(q,j^*-1,m^*+1,m'+1\right)\right)$,

$$c_{20} = \sum_{k'=1}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-2}\left(1-prob2\left(q,m,k,k'\right)\right)$$

and

$$c_{21} = \left(\sum_{m'=0}^{m^*-q} \binom{m^*+1-q}{m'+1} \left(\frac{1}{2}\right)^{m^*-q} \left(l\left(q, j^*, m^*, m'\right)\right)\right)$$

Easy to see that

$$\begin{aligned}
&\left(\sum_{k'=1}^{2q-m-j-1} \binom{2q-j-m-1}{k'} \left(\frac{1}{2}\right)^{2q-j-m-2} \left(1 - prob2\left(q, m, k, k'\right)\right)\right) \\
= &\left(\sum_{m'=0}^{m^*-q} \binom{m^*+1-q}{m'+1} \left(\frac{1}{2}\right)^{m^*-q} \left(l\left(q, j^*, m^*, m'\right)\right)\right)
\end{aligned}$$

We also claim that

$$\binom{3q-1}{m}\binom{3q-1-m}{j}\frac{1}{m+1} = \binom{3q-1}{m^*}\binom{3q-1-m^*}{j^*}\frac{1}{m^*+1}$$

Now,

$$\begin{aligned}
&\binom{3q-1}{m^*}\binom{3q-1-m^*}{j^*}\frac{1}{m^*+1} \\
= &\frac{(3q-1)!}{(3q-1-m^*)!m^*!}\frac{(3q-1-m^*)!}{(3q-1-m^*-j^*)!j^*!}\frac{1}{m^*+1} \\
= &\frac{(3q-1)!}{(3q-m-j-2)!}\frac{1}{j!\,(m+1)!}\frac{1}{3q-m-j-1} \\
= &\frac{(3q-1)!}{(3q-m-j-1)!}\frac{1}{j!\,(m+1)!}
\end{aligned}$$

and

$$\begin{aligned}
&\binom{3q-1}{m}\binom{3q-1-m}{j}\frac{1}{m+1} \\
= &\frac{(3q-1)!}{(3q-1-m)!m!}\frac{(3q-1-m)!}{(3q-1-m-j)!j!}\frac{1}{m+1} \\
= &\frac{(3q-1)!}{(m+1)!}\frac{1}{(3q-1-m-j)!j!}
\end{aligned}$$

Hence, we have proven the claim. Note that there is a one-to-one correspondence between (m, j) and (m^*, j^*) and hence we have the desired result. □

Claim 3.

$$2\sum_{m=0}^{q-1}\sum_{j=0}^{q-1} f(q,m,j)\left(\sum_{k'=q-m}^{2q-m-j-1}\binom{2q-j-m-1}{k'}\left(\frac{1}{2}\right)^{2q-j-m-1}(1-prob1(q,m,k,k'))\right)$$

$$\begin{aligned}= & \sum_{m=q}^{2q-1}\sum_{j=2q-m}^{q} f(q,m,j)\frac{1}{m+1}\left(\sum_{m'=q-j}^{m-q}\binom{m+1-q}{m'+1}\left(\frac{1}{2}\right)^{m-q}(\alpha(j,q,m'))\right)+\\ & \sum_{m=2q}^{3q-1}\sum_{j=1}^{3q-1-m} f(q,m,j)\frac{1}{m+1}\left(\sum_{m'=q-j}^{m-q}\binom{m+1-q}{m'+1}\left(\frac{1}{2}\right)^{m-q}(\alpha(j,q,m'))\right)\end{aligned}$$

where again $\alpha(j,q,m') = (j(1-prob1(q,j-1,m+1,m'+1)))$

Proof. Pick some (m,j) such that $m \in [0,q-1]$ and $j \in [0,q-1]$. Now, then consider $m^* = 3q-m-j-2$ and $j^* = m+1$. Then, by identical arguments as above claim, we have the result. □

Hence, we showed that $S_1 = S_2$ which was what we wanted to show.

A.5 Proof of Proposition 4

Proof. (i),(ii) and (iii) follow from Theorem $1.A.22$, Theorem $1.C.43$ and Theorem $1.B.32$ of Shaked and Shanthikumar (1994), respectively, which show that, under the stated assumptions, $(W_1 - W_2)$ first order stochastically dominates $(W_2 - W_3)$. Hence,

$$E[W_1] - E[W_2] \geq E[W_2] - E[W_3]$$

Thus, we have

$$2(E[W_1] - E[W_2]) \geq E[W_2] - E[W_3]$$

implying that

$$2E[W_1] - 3E[W_2] - E[W_3] \geq 0$$

□

Finally, we show that for $F(x) = x^k, 0 \leq x \leq 1$, $w_F > 0$. Assume that $F(x) = x^k$, $k > 0$. Now,

$$E[W_1] = \int_0^1 3xf(x)F^2(x)\,dx$$

$$E\left[W_2\right] = \int_0^1 6xf\left(x\right)\left(1 - F\left(x\right)\right)F\left(x\right)dx$$

$$E\left[W_3\right] = \int_0^1 x\left(3f\left(x\right)\left(1 - F\left(x\right)\right)^2\right)dx$$

$$\begin{aligned}
& 2E\left[W_1\right] - 3E\left[W_2\right] + E\left[W_3\right] \\
= \;& 2\left(\int_0^1 3xkx^{k-1}x^{2k}dx\right) - 3\left(-6k\int_0^1 x^{2k}\left(x^k - 1\right)\,dx\right) + 3k\int_0^1 x^k\left(x^k - 1\right)^2\,dx \\
= \;& 2\left(\frac{3k}{3k+1}\right) - 3\left(\frac{6k}{2k+1} - \frac{6k}{3k+1}\right) + \left(\frac{3k}{3k+1} - \frac{6k}{2k+1} + \frac{3k}{k+1}\right) \\
= \;& 6\frac{k}{6k^3 + 11k^2 + 6k + 1} > 0
\end{aligned}$$

Appendix B

Proofs of Chapter 2

B.1 Implication of Stochastic Dominance

We, here, show that when agents' preferences over bundles of objects are additively separable, then stochastic dominance implies that regardless of the preference intensities, stochastically dominating mechanism yields a higher interim payoff for each agent.

Lemma 9. *Assume that φ and φ' are anonymous, neutral and incentive compatible mechanisms and each agent's preference over the bundles of objects is additively separable. If φ stochastically dominates φ', then the interim payoff of any type of any agent is (weakly) higher under φ for any set of utility functions.*

Proof. Assume that agent i's preference over bundles of object is additively separable. That is, agent i's preference over bundles of objects can be represented by a utility function $u^i : \mathcal{O} \longrightarrow \mathbb{R}$ such that utility of receiving a bundle of objects $O \subset \mathcal{O}$ is $\sum_{o \in O} u^i(o)$ .Now, assume that for all $K \in \{1, ..., n\}$, we have that

$$\sum_{k=1}^{K} P_k^{\varphi} \geq \sum_{k=1}^{K} P_k^{\varphi'}$$

Consider agent i and w.l.o.g. assume that $o_j \succ_i o_k$ when $j < k$. Note that since both mechanisms are incentive compatible, submitted k^{th} choice is indeed his true k^{th} choice. That is, object o_k is agent i's k^{th} choice for any k. Now, denoting $u^i(o_k)$ as u_k^i, we have that

$$\Pi^{\varphi}\left(u^i\right) = \sum_{k=1}^{n} P_k^{\varphi} u_k^i$$

and

$$\Pi^{\varphi'}\left(u^i\right)=\sum_{k=1}^{n}P_k^{\varphi'}u_k^i$$

Now, notice that for any anonymous, neutral and incentive compatible mechanism φ, it must be that

$$m\left[\sum_{k=1}^{n}P_k^{\varphi}\right]=nq \tag{B.1}$$

Thus,

$$\begin{aligned}
&\sum_{k=1}^{n}P_k^{\varphi}u_k^i\\
=\;&P_1^{\varphi}\left(u_1^i-u_2^i\right)+\left(P_1^{\varphi}+P_2^{\varphi}\right)\left(u_2^i-u_3^i\right)+...+\left(P_1^{\varphi}+...+P_{n-1}^{\varphi}\right)\left(u_{n-1}^i-u_n^i\right)\\
&+\left(P_1^{\varphi}+...+P_n^{\varphi}\right)u_n^i\\
=\;&P_1^{\varphi}\left(u_1^i-u_2^i\right)+\left(P_1^{\varphi}+P_2^{\varphi}\right)\left(u_2^i-u_3^i\right)+...+\frac{nq}{m}u_n^i
\end{aligned}$$

by Abel's summation by parts and by equation $(B.1)$.

Similarly,

$$\begin{aligned}
&\sum_{k=1}^{n}P_k^{\varphi'}u_k^i\\
=\;&P_1^{\varphi'}\left(u_1^i-u_2^i\right)+\left(P_1^{\varphi'}+P_2^{\varphi'}\right)\left(u_2^i-u_3^i\right)+...+\left(P_1^{\varphi'}+...+P_n^{\varphi'}\right)u_n^i\\
=\;&P_1^{\varphi'}\left(u_1^i-u_2^i\right)+\left(P_1^{\varphi'}+P_2^{\varphi'}\right)\left(u_2^i-u_3^i\right)+...+\frac{nq}{m}u_n^i
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{k=1}^{n}P_k^{\varphi}u_k^i-\sum_{k=1}^{n}P_k^{\varphi'}u_k^i\\
=\;&\sum_{k=1}^{n-1}\left(\left(P_1^{\varphi}+...+P_k^{\varphi}\right)-\left(P_1^{\varphi'}+...+P_k^{\varphi'}\right)\right)u_k^i\\
\geq\;&0
\end{aligned}$$

where the last inequality is due to the fact that φ stochastically dominates φ'. □

B.2 Proof of Theorem 2

We prove our main result for the case when the number of agents is $m > 2$ and $q \in \{1, ..., (m-1)\}$.

Let the set of agents be $\{1, 2, ..., m\}$ and the set of objects be $\{o_1, o_2, ..., o_n\}$. Assume that φ is an anonymous, neutral and incentive compatible ordinal mechanism. Consider agent 1 and w.l.o.g. assume that his ranking is $R^1 = (o_1, o_2, ..., o_n)$. Let μ stand for the Ranking mechanism.

Note that

$$P_k^{\varphi} = \sum_{R^{-1}} \varphi_k^1 \left(R^1, R^{-1}\right) \Pr\left(R^{-1}\right)$$

denotes the interim probability of agent 1's getting object o_k, where $R^{-1} = (R^2, ..., R^m)$ denotes preference rankings of agents $\{2, ...m\}$ and $\Pr(R^{-1})$ is the probability of the state R^{-1}. Now, $\Pr(R^{-1}) = \frac{1}{(n!)^{m-1}}$ for all R^{-1}.

By definition of the Ranking mechanism, letting $\mathcal{J} = \{i \in \{2, ..., m\} \ : o_1 = R^i(1)\}$ and $J = |\mathcal{J}|$,

$$\mu_1^1\left(R^1, R^{-1}\right) = \begin{cases} 1 & \text{if} \quad J \leq (q-1) \\ \frac{q}{J+1} & \text{if} \quad J \geq q \end{cases}$$

That is, if the number of agents (other than agent 1) who rank object o_1 as a first choice is less than $(q-1)$, agent 1 gets one copy of object o_1 with probability 1 and if there are at least q other agents whose first choice is also object o_1, then each of these agents, and also agent 1, will get a copy of object o_1 with probability $\frac{q}{J+1}$.

Now, if $\varphi_1^1(R^1, R^{-1}) \leq \mu_1^1(R^1, R^{-1})$ for all R^{-1}, we have trivially that $P_1^{\mu} \geq P_1^{\varphi}$. Suppose that there is some R_*^{-1} such that $\varphi_1^1(R^1, R_*^{-1}) > \mu_1^1(R^1, R_*^{-1})$. First, since $\varphi_1^1(R^1, R_*^{-1}) \leq 1$ when $J \leq (q-1)$, it must be that $J \geq q$ where J is the number of agents (other than agent 1) whose first choice is object o_1 in R_*^{-1}.

Thus, $\varphi_1^1(R^1, R_*^{-1}) = p_1 > \frac{q}{J+1}$ when $J \geq q$. Let $\mathcal{J} = \{i \in \{2, ..., m\} \ : o_1 = R_*^i(1)\}$. W.l.o.g, assume that $\mathcal{J} = \{2, 3, ..., J, (J+1)\}$. Let

$$\varphi_1^i\left(R^1, R_*^{-1}\right) = p_i$$

for some $p_i \in [0,1]$ for all $i \in \mathcal{J}$. Then, by anonymity, for each $i \in \mathcal{J}$, we have that

$$\varphi_1^1\left(R_*^i, R_*^2, ., , R_*^{i-1}, R^1, R_*^{i+1}, ...R_*^m\right) = p_i$$

since we replace the preference of agent 1 with that of agent i.

Now, since $(p_1 + p_2 + ... + p_{j+1}) \leq q$ and $p_1 > \frac{q}{J+1}$, there must exist some set of agents $\mathcal{J}' \subset \mathcal{J}$ such that $p_i < \frac{q}{J+1}$ for all $i \in \mathcal{J}'$ and

$$\sum_{i \in \mathcal{J}'} \left(\frac{q}{J+1} - p_i \right) \geq \left(p_1 - \frac{q}{J+1} \right) \tag{B.2}$$

For each $i \in \mathcal{J}'$, consider the permutation σ^i over objects such that $\sigma^i \left(R_*^i \left(k \right) \right) = R^1 \left(k \right)$ for each k. That is, consider the permutation over objects that converts the ranking R_*^i to R^1. Note that for each $i \in \mathcal{J}'$, we have that $\sigma^i \left(R_*^i \left(1 \right) \right) = R^1 \left(1 \right)$ since each agent i's and also agent 1's first choice is object o_1 under R_*^i and R^1, respectively.

Then, by neutrality, we have that

$$\varphi_1^1 \left(R^1, \left(R_*^2 \right)^{\sigma^i}, .,, \left(R_*^{i-1} \right)^{\sigma^i}, \left(R^1 \right)^{\sigma^i}, \left(R_*^{i+1} \right)^{\sigma^i}, ... \left(R_*^m \right)^{\sigma^i} \right) = p_i$$

Note that each of these states[1] is equally likely and the sum of probabilities of getting object o_1 at these states together with the state R_*^1 under mechanism φ is

$$p_1 + \sum_{i \in \mathcal{J}'} p_i \tag{B.3}$$

but under μ , it is

$$\frac{q}{J+1} + \sum_{i \in \mathcal{J}'} \left(\frac{q}{J+1} \right) \tag{B.4}$$

Now, by $(B.2)$, we have that $(B.4) \geq (B.3)$.

To sum up in words, when the probability of getting object o_1 is higher under φ for agent 1 at some state, this means that there are some other agents who are at a loss which is at least as much as the gain of agent 1 in their probability of obtaining object o_1, which is also their first choice. Therefore, due to anonymity and neutrality, when agent 1 is in the position of other agents, agent 1 must be at a loss for obtaining object o_1 at those states and the loss at those states is at least as much as the gain at the original state. Hence, there cannot be any gain in interim probability of getting o_1 under φ compare to μ. Hence, $P_1^\mu \geq P_1^\varphi$.

If $P_2^\varphi \left(R^{-1} \right) \leq P_2^\mu \left(R^{-1} \right)$ for all R^{-1}, we have trivially that $P_2^\varphi \leq P_2^\mu$ and hence $P_1^\varphi + P_2^\varphi \leq P_1^\mu + P_2^\mu$. Suppose that there exists some R_*^{-1} such that $P_2^\varphi \left(R_*^{-1} \right) >$

[1] We refer to the states obtained by permutations σ^i for $i \in \mathcal{J}'$.

$P_2^{\mu}(R_*^{-1})$. Now, the allocation rule for the Ranking mechanism is given by

$$P_2^{rank}\left(R^{-1}\right) = \begin{cases} 1 & \text{if} \quad J + K \leq (q-1) \\ \frac{q-K}{J+1} & \text{if} \quad K \leq (q-1), \ J+K \geq q \\ 0 & \text{if} \quad K \geq q \end{cases}$$

where

$$\mathcal{K} = \left\{ i \in \{2, ..., m\} \ : o_2 = R^i(1) \right\} \text{ and } |\mathcal{K}| = K$$

and

$$\mathcal{J} = \left\{ i \in \{2, ..., m\} \ : o_2 = R^i(2) \right\} \text{ and } |\mathcal{J}| = J$$

That is, letting K to be the number of agents who rank object o_2 as a first choice and J to be the number of agents who rank object o_2 as a second choice, if there are at most $(q-1)$ other agents who rank object o_2 as a first or second choice, then agent 1 will get a copy of object o_2 with probability 1. If there are at most $(q-1)$ agents who rank object o_2 as their first choices and there are at least q agents (other than agent 1) who rank object o_2 as a first or second choice than agents who rank object o_2 as their first choices will get a copy of object o_2 for sure and the remaining copies will be randomly allocated to agents who rank object o_2 as a second choice. Hence, agent 1 will receive a copy with probability $\frac{q-K}{J+1}$. Finally, if there are at least q agents who rank object o_2 as their first choices, then no agent who rank object o_2 at a lower spot can get a copy of object o_2. Thus, since agent 1 ranks o_2 as his second choice, he cannot get a copy.

First, suppose that $\varphi_1^1(R^1, R_*^{-1}) = p_1 > \frac{q-K}{J+1}$ when $K \leq (q-1)$, $J+K \geq q$. Then, exactly by identical arguments for the first choice case above, there must exist other states such that agent 1 is now at a loss compare to the Ranking mechanism which is at least the gain he obtained at this state.

Second, suppose that $K \geq q$ and $\varphi_2^1(R^1, R_*^{-1}) = p > 0$. W.l.o.g., assume that $\mathcal{K} = \{2, 3, ..., K, (K+1)\}$. Let $\varphi_2^i(R^1, R_*^{-1}) = p_i$ for each $i \in \mathcal{K}$. That is, agent i obtains object o_2 when the reported preferences are (R^1, R_*^{-1}). Since $p + p_2 + ... + p_{k+1} \leq q$ and $p > 0$, there must exist a set of agents $\mathcal{K}' \subset \mathcal{K}$ such that for all $i \in \mathcal{K}'$, $p_i < \frac{q}{K}$ and

$$\sum_{i \in \mathcal{K}'} \left(\frac{q}{K} - p_i \right) \geq p \tag{B.5}$$

Now, by anonymity, we have that $\varphi_2^1(R_*^i, R_*^2, ..R_*^{i-1}, R^1, R_*^{i+1}, ...R_*^m) = p_i$ for each

$i \in \mathcal{K}'$. That is, if agent 1 and agent i reported preferences are interchanged than agent 1's new allocation must be agent i's original allocation and hence agent 1 will get a copy of object o_2 with probability p_i.

For each $i \in \mathcal{K}'$, consider the permutation σ^i over objects such that $\sigma^i\left(R^i_*\left(s\right)\right) = R^1\left(s\right)$ for each $s \in \{1, ..., n\}$. Then, since $\sigma^i\left(o_2\right) = o_1$ for each $i \in \mathcal{K}'$, by neutrality, we have that

$$\varphi^1_1\left(R^1, \left(R^2\right)^{\sigma^i}, .., \left(R^{i-1}\right)^{\sigma^i}, \left(R^1\right)^{\sigma^i}, \left(R^{i+1}\right)^{\sigma^i}, ... \left(R^m\right)^{\sigma^i}\right) = p_i$$

for each $i \in \mathcal{K}'$. In words, when we change the preferences such that object o_2 is now object o_1, the new allocation for object o_1 must be the original allocation for object o_2.

Now, $\mu^1_2\left(R^1, R^{-1}_*\right) = 0$ since there are at least q agents whose first choice is object o_2 under R^{-1}_* and object o_2 is agent 1's second choice. Furthermore,

$$\mu^1_1\left(R^1, \left(R^2\right)^{\sigma^i}, .., \left(R^{i-1}\right)^{\sigma^i}, \left(R^1\right)^{\sigma^i}, \left(R^{i+1}\right)^{\sigma^i}, ... \left(R^m\right)^{\sigma^i}\right) = \frac{q}{K}$$

for each $i \in \mathcal{K}'$ since there are in total K agents who rank object o_1 as a first choice.

Note that gain of agent 1 in probability of obtaining o_2 at state R^{-1}_* is p compare to the Ranking mechanism. However, the loss of agent 1 in probability of obtaining object o_1 at other states (as constructed above) is

$$\sum_{i \in \mathcal{K}'}\left(\frac{q}{K} - p_i\right)$$

compare to the Ranking mechanism. Thus, due to $(B.5)$, an increase in probability of obtaining the second choice under mechanism φ comes with a decrease (at least the same amount of increase in probability of getting second choice) in obtaining the first choice. In words, when the probability of getting object o_2 is higher under mechanism φ compare to the Ranking mechanism for agent 1 at some state, this means that there are some other agents who are at a loss which is at least as much as the gain of agent 1 in their probability of obtaining object o_1, their first choice. Therefore, due to anonymity and neutrality, when agent 1 is in the position of other agents, agent 1 must be at a loss for obtaining his first choice at those states where the loss is at least as much as the gain at the other state. Hence, sum of the interim probabilities of obtaining first two top choices can not exceed under mechanism φ

compare to the Ranking mechanism, $P_1^\mu + P_2^\mu \geq P_1^\varphi + P_2^\varphi$.

If we move to comparison for third choice, exact similar arguments[2] will yield the same conclusion. Continuing in this manner, we have that for any $K \in \{1, ..., n\}$

$$\sum_{k=1}^{K} P_k^{rank} \geq \sum_{k=1}^{K} P_k^\varphi$$

, which is what we wanted to show.

[2]In this case, any increase in obtaining third choice will result in same amount of decrease in obtaining first *or* second top choice. Hence, the sum of probabilities of obtaining first three choices under φ will not exceed the one under the Ranking mechanism.

Appendix C

Proofs of Chapter 3

C.1 Proof of Proposition 7

Lemma 10. *Define a random variable* $U \equiv \frac{V_1^2}{V_1^2+V_2^2+V_3^2}$, *where* (V_1, V_2, V_3) *is distributed according to a continuous distribution function that has density of the form* $g(v_1, v_2, v_3) \equiv \widetilde{g}((v_1^2 + v_2^2 + v_3^2))$ *for some measurable function* $\widetilde{g}$ *on* $\mathbb{R}_+$ *such that*

$$\int_0^\infty \widetilde{g}(x)\, x^{\frac{1}{2}} dx = \frac{4}{\pi}$$

Then, U *is distributed according to* $F(u) = u^{\frac{1}{2}}$ *with support* $[0, 1]$.

Proof. Let $V = \frac{V_2^2}{V_1^2+V_2^2+V_3^2}$ and $W = V_1^2 + V_2^2 + V_3^2$.

Now,

$$\begin{aligned} V_1^2 &= UW \\ V_2^2 &= VW \\ V_3^2 &= (1-U-V)\,W \end{aligned}$$

, or,

$$\begin{aligned} V_1 &= U^{\frac{1}{2}} W^{\frac{1}{2}} \\ V_2 &= V^{\frac{1}{2}} W^{\frac{1}{2}} \\ V_3 &= (1-U-V)^{\frac{1}{2}}\, W^{\frac{1}{2}} \end{aligned}$$

Then,

$$h(u,v,w) = g\left(u^{\frac{1}{2}}w^{\frac{1}{2}}, v^{\frac{1}{2}}w^{\frac{1}{2}}, (1-u-v)^{\frac{1}{2}}w^{\frac{1}{2}}\right)\det(J)$$

, where

$$J = \begin{bmatrix} \frac{1}{2}u^{-\frac{1}{2}}w^{\frac{1}{2}} & 0 & \frac{1}{2}u^{\frac{1}{2}}w^{-\frac{1}{2}} \\ 0 & \frac{1}{2}v^{-\frac{1}{2}}w^{\frac{1}{2}} & \frac{1}{2}v^{\frac{1}{2}}w^{-\frac{1}{2}} \\ -\frac{1}{2}(1-u-v)^{-\frac{1}{2}}w^{\frac{1}{2}} & -\frac{1}{2}(1-u-v)^{-\frac{1}{2}}w^{\frac{1}{2}} & \frac{1}{2}(1-u-v)^{\frac{1}{2}}w^{-\frac{1}{2}} \end{bmatrix}$$

Hence,

$$\begin{aligned} & h(u,v,w) \\ = \; & g\left(u^{\frac{1}{2}}w^{\frac{1}{2}}, v^{\frac{1}{2}}w^{\frac{1}{2}}, (1-u-v)^{\frac{1}{2}}w^{\frac{1}{2}}\right)\left(\frac{1}{8}\frac{w^{\frac{1}{2}}}{u^{\frac{1}{2}}v^{\frac{1}{2}}(1-u-v)^{\frac{1}{2}}}\right) \\ = \; & \widetilde{g}(w)\left(\frac{1}{8}\frac{w^{\frac{1}{2}}}{u^{\frac{1}{2}}v^{\frac{1}{2}}(1-u-v)^{\frac{1}{2}}}\right) \end{aligned}$$

$, 0 < u, v < 1,\ u+v < 1, w > 0$
Hence,

$$h(u,v,w) = \frac{1}{8}\widetilde{g}(w)\,w^{\frac{1}{2}}\left(\frac{1}{u^{\frac{1}{2}}v^{\frac{1}{2}}(1-u-v)^{\frac{1}{2}}}\right)$$

$, 0 < u, v < 1,\ u+v < 1, w > 0$. Now,

$$h(u,v) = \left(\frac{1}{8}\int_0^\infty \widetilde{g}(w)\,w^{\frac{1}{2}}dw\right)\left(\frac{1}{u^{\frac{1}{2}}v^{\frac{1}{2}}(1-u-v)^{\frac{1}{2}}}\right)$$

$0 < u, v < 1,\ u+v < 1$. Hence,

$$h(u,v) = \frac{1}{2\pi}\left(\frac{1}{u^{\frac{1}{2}}v^{\frac{1}{2}}(1-u-v)^{\frac{1}{2}}}\right)$$

,$0 < u, v < 1,\ u+v < 1$. Then,

$$\begin{aligned} h(u) &= \int_0^{1-u}\frac{1}{2\pi}\left(\frac{1}{u^{\frac{1}{2}}v^{\frac{1}{2}}(1-u-v)^{\frac{1}{2}}}\right)dv \\ &= \frac{1}{2}u^{-\frac{1}{2}} \end{aligned}$$

when $0 < u < 1$. Hence,

$$H(u) = u^{\frac{1}{2}}, \ 0 \le u \le 1$$

which proves the lemma. $\square$

Proof. **(of Proposition)** Assume that Player 2 follows the given strategy. Assume that Player 1 has values (v_1, v_2, v_3). Then, he solves

$$\max_{0 \le b_1, b_2 \le B} \Pr\left(b_1 \ge \frac{w_1^2}{w_1^2 + w_2^2 + w_3^2}\right) v_1 + \Pr\left(b_2 \ge \frac{w_2^2}{w_1^2 + w_2^2 + w_3^2}\right) v_2 + \\ + \Pr\left(B - b_1 - b_2 \ge \frac{w_3^2}{w_1^2 + w_2^2 + w_3^2}\right) v_3$$

Then, by above Lemma, the problem becomes

$$\max_{0 \le b_1, b_2 \le B} b_1^{\frac{1}{2}} v_1 + b_2^{\frac{1}{2}} v_2 + (B - b_1 - b_2)^{\frac{1}{2}} v_3$$

FOC:

$$\begin{aligned} b_1^{-\frac{1}{2}} v_1 &= (B - b_1 - b_2)^{-\frac{1}{2}} v_3 \\ b_2^{-\frac{1}{2}} v_2 &= (B - b_1 - b_2)^{-\frac{1}{2}} v_3 \end{aligned}$$

Hence,

$$b_2 = b_1 \left(\frac{v_2^2}{v_1^2}\right)$$

and

$$B - b_1 - b_2 = b_1 \left(\frac{v_3^2}{v_1^2}\right)$$

Thus,

$$\begin{aligned} B &= b_1 + b_2 + (B - b_1 - b_2) \\ &= b_1 \left(1 + \left(\frac{v_2^2}{v_1^2}\right) + \left(\frac{v_3^2}{v_1^2}\right)\right) \\ &= b_1 \left(\frac{v_1^2 + v_2^2 + v_3^2}{v_1^2}\right) \end{aligned}$$

Hence,

$$b_1 = B\frac{v_1^2}{v_1^2 + v_2^2 + v_3^2}$$

and

$$b_2 = B\frac{v_2^2}{v_1^2 + v_2^2 + v_3^2}$$

Hence, $\beta(.)$ is a best response to itself , giving the desired result. □

C.2 Proof of Proposition 8

Lemma 11. *Define a random variable* $U_1 \equiv \frac{V_1^{\frac{n-1}{n-2}}}{V_1^{\frac{n-1}{n-2}}+...+V_n^{\frac{n-1}{n-2}}}$, *where* $(V_1, ..., V_n)$ *is distributed according to a continuous distribution function that has density of the form* $g(v_1, ..., v_n) \equiv [v_1...v_n]^{\frac{3-n}{n-2}}\,\widetilde{g}\left(\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}\right)\right)$ *for some measurable function* $\widetilde{g}$ *on* $\mathbb{R}_+$ *such that*

$$\int_0^\infty \widetilde{g}(x)\, x^{\frac{1}{n-1}}dx = \left(\frac{\Gamma\left(\frac{n}{n-1}\right)}{\Gamma^n\left(\frac{1}{n-1}\right)}\right)\left(\frac{n-1}{n-2}\right)^n$$

Then, U_1 *is distributed according to* $F(u) = u^{\frac{1}{n-1}}$ *with support* $[0, 1]$

Proof. Let for any $i \in \{2, ..., n-1\}$, define $U_i \equiv \frac{V_i^{\frac{n-1}{n-2}}}{V_1^{\frac{n-1}{n-2}}+...+V_n^{\frac{n-1}{n-2}}}$ and let $W = V_1^{\frac{n-1}{n-2}} + ... + V_n^{\frac{n-1}{n-2}}$.Now,

$$\begin{aligned}
V_1^{\frac{n-1}{n-2}} &= U_1 W \\
&\;\;... \\
V_{n-1}^{\frac{n-1}{n-2}} &= U_{n-1} W \\
V_n^{\frac{n-1}{n-2}} &= (1 - U_1 - ... - U_{n-1})\, W
\end{aligned}$$

, or,

$$\begin{aligned}
V_1 &= U_1^{\frac{n-2}{n-1}} W^{\frac{n-2}{n-1}} \\
&\;\;... \\
V_{n-1} &= U_{n-1}^{\frac{n-2}{n-1}} W^{\frac{n-2}{n-1}} \\
V_n &= (1 - U_1 - ... - U_{n-1})^{\frac{n-2}{n-1}}\, W^{\frac{n-2}{n-1}}
\end{aligned}$$

Then,

$$\begin{aligned} & h\left(u_1,...,u_{n-1},w\right) \\ = \;& g\left(u_1^{\frac{n-2}{n-1}}w^{\frac{n-2}{n-1}},...,u_{n-1}^{\frac{n-2}{n-1}}w^{\frac{n-2}{n-1}},\left(1-u_1-...-u_{n-1}\right)^{\frac{n-2}{n-1}}w^{\frac{n-2}{n-1}}\right)\det\left(J\right) \end{aligned}$$

Now,[1]

$$\det\left(J\right)=\left(\frac{n-2}{n-1}\right)^n\frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}}...u_{n-1}^{\frac{1}{n-1}}\left(1-u_1-...-u_{n-1}\right)^{\frac{1}{n-1}}}$$

Hence,

$$\begin{aligned} & h\left(u_1,...,u_{n-1},w\right) \\ = \;& g\left(u_1^{\frac{n-2}{n-1}}w^{\frac{n-2}{n-1}},...,u_{n-1}^{\frac{n-2}{n-1}}w^{\frac{n-2}{n-1}},\left(1-u_1-...-u_{n-1}\right)^{\frac{n-2}{n-1}}w^{\frac{n-2}{n-1}}\right)* \\ & \left(\left(\frac{n-2}{n-1}\right)^n\frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}}...u_{n-1}^{\frac{1}{n-1}}\left(1-u_1-...-u_{n-1}\right)^{\frac{1}{n-1}}}\right) \\ = \;& \left[u_1...u_{n-1}\left(1-u_1-...-u_{n-1}\right)\right]^{\frac{3-n}{n-1}}\left(w^{n\frac{3-n}{n-1}}\right)\widetilde{g}\left(w\right)* \\ & \left(\left(\frac{n-2}{n-1}\right)^n\frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}}...u_{n-1}^{\frac{1}{n-1}}\left(1-u_1-...-u_{n-1}\right)^{\frac{1}{n-1}}}\right) \\ = \;& \left(\left(\frac{n-2}{n-1}\right)^n\right)\left[u_1...u_{n-1}\left(1-u_1-...-u_{n-1}\right)\right]^{\frac{2-n}{n-1}}\widetilde{g}\left(w\right)w^{\frac{1}{n-1}} \end{aligned}$$

Then,

$$\begin{aligned} h\left(u_1,...,u_{n-1}\right) &= \left(\int_0^\infty \widetilde{g}\left(w\right)w^{\frac{1}{n-1}}dw\right)\left(\left(\frac{n-2}{n-1}\right)^n\right)\left[u_1...u_{n-1}\left(1-u_1-...-u_{n-1}\right)\right]^{\frac{2-n}{n-1}} \\ &= \left(\frac{\Gamma\left(\frac{n}{n-1}\right)}{\Gamma^n\left(\frac{1}{n-1}\right)}\right)\left[u_1...u_{n-1}\left(1-u_1-...-u_{n-1}\right)\right]^{\frac{2-n}{n-1}} \end{aligned}$$

$0<u_1...,u_{n-1}<1$, $u_1+...+u_{n-1}<1$. But, this is a Dirichlet Distribution with parameters $\left(\frac{1}{n-1},\frac{1}{n-1},...,\frac{1}{n-1}\right)$ and hence

$$H\left(u\right)=u^{\frac{1}{n-1}},\; 0\leq u\leq 1$$

[1] See Lemma for a proof of this

which proves the lemma. □

Proof. **(of the Proposition)** Assume that Player 2 follows the given strategy. Assume that Player 1 has values $(v_1, ..., v_n)$. Then, he solves

$$\max_{\substack{0 \le b_1,...b_{n-1} \le B \\ b_1 + ... + b_{n-1} \le B}} \Pr(b_1 \ge X_1) v_1 + ... + \Pr(B - b_1 - ... - b_{n-1} \ge X_n) v_n$$

where for all $i \in \{1, ..., n\}$

$$X_i = \frac{w_i^{\frac{n-1}{n-2}}}{w_1^{\frac{n-1}{n-2}} + w_2^{\frac{n-1}{n-2}} + ... + w_n^{\frac{n-1}{n-2}}}$$

Then, by above Lemma, the problem becomes

$$\max_{\substack{0 \le b_1,...b_{n-1} \le B \\ b_1 + ... + b_{n-1} \le B}} b_1^{\frac{1}{n-1}} v_1 + ... + (B - b_1 - ... - b_{n-1})^{\frac{1}{n-1}} v_n$$

FOC:

$$\begin{aligned} b_1^{\frac{2-n}{n-1}} v_1 &= (B - b_1 - ... - b_{n-1})^{\frac{2-n}{n-1}} v_n \\ &\ldots \\ b_{n-1}^{\frac{2-n}{n-1}} v_{n-1} &= (B - b_1 - ... - b_{n-1})^{\frac{2-n}{n-1}} v_n \end{aligned}$$

Hence,

$$b_i = b_1 \left(\frac{v_i^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}}} \right)$$

for all $i \in \{2, .., n-1\}$ Thus,

$$\begin{aligned} B &= b_1 \left(1 + \left(\frac{v_2}{v_1} \right)^{\frac{n-1}{n-2}} + ... + \left(\frac{v_n}{v_1} \right)^{\frac{n-1}{n-2}} \right) \\ &= b_1 \left(\frac{v_1^{\frac{n-1}{n-2}} + v_2^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}}} \right) \end{aligned}$$

Hence,

$$b_1 = B \frac{v_1^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + v_2^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}}$$

and similarly others. □

Lemma 12 (Determinant of Jacobian)**.**

$$\det(J) = \left(\frac{n-2}{n-1}\right)^n \frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}} ... u_{n-1}^{\frac{1}{n-1}} \left(1 - u_1 - ... - u_{n-1}\right)^{\frac{1}{n-1}}}$$

Proof. Now,

$$J = \begin{bmatrix} \frac{n-2}{n-1} u_1^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & 0 & 0 & . & 0 & \frac{n-2}{n-1} u_1^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \\ 0 & \frac{n-2}{n-1} u_2^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & 0 & . & 0 & \frac{n-2}{n-1} u_2^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & \frac{n-2}{n-1} u_{n-1}^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & \frac{n-2}{n-1} u_{n-1}^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \\ -\frac{n-2}{n-1} u_n^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & -\frac{n-2}{n-1} u_n^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & . & . & -\frac{n-2}{n-1} u_n^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} & \frac{n-2}{n-1} u_n^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \end{bmatrix}$$

, where $u_n = 1 - u_1 - ... - u_{n-1}$. Denoting $J = (a_{i,j})_{i,j=1}^n$

$$\det(J) = \sum_{i=1}^{n} (-1)^{n+i} a_{n,i} M_{n,i}$$

where $M_{n,i}$ is the determinant of the $(n-1) \times (n-1)$ matrix obtained from J by deleting n-th row and $i-th$ column. Now, for all $1 \leq i \leq n-1$

$$M_{n,i} = (-1)^{n+i-1} \left(\frac{n-2}{n-1}\right)^{n-1} \left(u_i^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1}\right) \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} u_j\right)^{\frac{n-2}{n-1}-1} w^{\frac{(n-2)^2}{n-1}}$$

and

$$M_{n,n} = \left(\frac{n-2}{n-1}\right)^{n-1} \left(\prod_{j=1}^{n-1} u_j\right)^{\frac{n-2}{n-1}-1} w^{n-2}$$

Hence,

$$
\begin{aligned}
& \det(J) \\
= & \sum_{i=1}^{n} (-1)^{n+i} a_{n,i} M_{n,i} \\
= & \sum_{i=1}^{n-1} (-1)^{n+i} \left[-\left(\frac{n-2}{n-1} (1-u_1-...-u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}} \right) \right] * \\
& \left[(-1)^{n+i-1} \left(\frac{n-2}{n-1} \right)^{n-1} u_i^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} u_j \right)^{\frac{n-2}{n-1}-1} w^{\frac{(n-2)^2}{n-1}} \right] + \\
& + \left[\left(\frac{n-2}{n-1} \right) (1-u_1-...-u_{n-1})^{\frac{n-2}{n-1}} w^{\frac{n-2}{n-1}-1} \right] \left[\left(\frac{n-2}{n-1} \right)^{n-1} \left(\prod_{j=1}^{n-1} u_j \right)^{\frac{n-2}{n-1}-1} w^{n-2} \right] \\
= & \left[\left(\frac{n-2}{n-1} \right)^{n} (1-u_1-...-u_{n-1})^{\frac{n-2}{n-1}-1} (u_1...u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}-1} \right] * \\
& \left[w^{\frac{n-2}{n-1}+\frac{(n-2)^2}{n-1}} (u_1+...+u_{n-1}) + w^{n-2} (1-u_1-...-u_{n-1}) \right] \\
= & \left[\left(\frac{n-2}{n-1} \right)^{n} (1-u_1-...-u_{n-1})^{\frac{n-2}{n-1}-1} (u_1...u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}-1} \right] * \\
& \left[w^{(n-2)} (u_1+...+u_{n-1}) + w^{n-2} (1-u_1-...-u_{n-1}) \right] \\
= & \left[\left(\frac{n-2}{n-1} \right)^{n} (1-u_1-...-u_{n-1})^{\frac{n-2}{n-1}-1} (u_1...u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}-1} \right] \left[w^{(n-2)} \right] \\
= & \left(\frac{n-2}{n-1} \right)^{n} (1-u_1-...-u_{n-1})^{\frac{n-2}{n-1}-1} (u_1...u_{n-1})^{\frac{n-2}{n-1}-1} w^{\frac{n-2}{n-1}-1+(n-2)} \\
= & \left(\frac{n-2}{n-1} \right)^{n} \frac{w^{(n-2)-\frac{1}{n-1}}}{u_1^{\frac{1}{n-1}}...u_{n-1}^{\frac{1}{n-1}} (1-u_1-...-u_{n-1})^{\frac{1}{n-1}}}
\end{aligned}
$$

□

C.3 Proof of Proposition 11

Proof. Now, consider a bidder with type $(v_1, ..., v_n)$. Then, his expected payoff

$$
\Pi^{Blotto}(v_1, ..., v_n) = \sum_{i=1}^{n} \Pr \left(\frac{v_1^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}} \geq \frac{W_i^{\frac{n-1}{n-2}}}{W_1^{\frac{n-1}{n-2}} + ... + W_n^{\frac{n-1}{n-2}}} \right) v_i
$$

Then, by the lemma below, we have that

$$\begin{aligned}
&\Pi^{Blotto}(v_1, .., v_n) \\
= &\sum_{i=1}^{n} \left(\frac{v_i^{\frac{n-1}{n-2}}}{v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}}} \right)^{\frac{1}{n-1}} v_i \\
= &\sum_{i=1}^{n} \frac{v_i^{\frac{n-1}{n-2}}}{\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}} \right)^{\frac{1}{n-1}}} \\
= &\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}} \right)^{\frac{n-2}{n-1}}
\end{aligned}$$

and

$$\begin{aligned}
\Pi^{rank}(v_1, ..., v_n) &= \sum_{i=1}^{n} P_i v_i \\
&= \sum_{i=1}^{n} v_i \left(\frac{1}{2n} + 1 - \frac{i}{n} \right) \\
&= \left(\frac{1}{2n} + 1 \right) \left(\sum_{i=1}^{n} v_i \right) - \frac{1}{n} \left(\sum_{i=1}^{n} i v_i \right) \\
&= \frac{1}{2n} \left((2n+1) \left(\sum_{i=1}^{n} v_i \right) - 2 \left(\sum_{i=1}^{n} i v_i \right) \right)
\end{aligned}$$

Now,

$$\begin{aligned}
&d(v_1, .., v_n) \\
= &\Pi^{Blotto}(v_1, .., v_n) - \Pi^{rank}(v_1, ..., v_n) \\
= &\left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}} \right)^{\frac{n-2}{n-1}} - \frac{1}{2n} \left((2n+1) \left(\sum_{i=1}^{n} v_i \right) - 2 \left(\sum_{i=1}^{n} i v_i \right) \right)
\end{aligned}$$

We claim that $d(v_1, .., v_n) \geq 0$ for all $\mathbf{v} = (v_1, .., v_n) \in [\underline{v}, \overline{v}]^n$ for any $\overline{v} > \underline{v} \geq 0$ and $v_1 \geq v_2 \geq ... \geq v_n$. To do this we will show that

$$0 \leq \left[\min_{v_1 \geq v_2 \geq ... \geq v_n \geq \underline{v}} \left(v_1^{\frac{n-1}{n-2}} + ... + v_n^{\frac{n-1}{n-2}} \right)^{\frac{n-2}{n-1}} - \frac{1}{2n} \left((2n+1) \left(\sum_{i=1}^{n} v_i \right) - 2 \left(\sum_{i=1}^{n} i v_i \right) \right) \right]$$

Note that

$$
\begin{aligned}
& \left[\min_{\overline{v}\geq v_1\geq v_2\geq ...\geq v_n\geq \underline{v}}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}-\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n}v_i\right)-2\left(\sum_{i=1}^{n}iv_i\right)\right)\right] \\
\geq & \left[\min_{v_1\geq v_2\geq ...\geq v_n\geq 0}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}-\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n}v_i\right)-2\left(\sum_{i=1}^{n}iv_i\right)\right)\right] \\
\geq & \left[\min_{v_1, v_2, ... v_n\geq 0}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{n-2}{n-1}}-\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n}v_i\right)-2\left(\sum_{i=1}^{n}iv_i\right)\right)\right]
\end{aligned}
$$

Actually, it is easy to see that last two problems are equivalent, but weak inequality is sufficient for our purpose. Now, to show the desired result we will concentrate on the last minimization problem. First, observe that

$$
\begin{aligned}
& \frac{\partial}{\partial v_i}d\left(v_1,..,v_n\right) \\
= & v_i^{\frac{1}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{1}{n-1}}-\frac{2n+1}{2n}+\frac{i}{n} \\
= & \frac{v_i^{\frac{1}{n-2}}}{\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{1}{n-1}}}-\frac{2n+1}{2n}+\frac{i}{n}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial v_i \partial v_j}d\left(v_1,..,v_n\right) &= -\frac{1}{n-2}v_j^{\frac{1}{n-2}}v_i^{\frac{1}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{n}{n-1}} \\
&\leq 0
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\partial^2}{\partial v_i^2}d\left(v_1,..,v_n\right) \\
= & \frac{1}{n-2}v_i^{\frac{3-n}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{1}{n-1}}-\frac{1}{n-2}v_i^{\frac{2}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{n}{n-1}} \\
= & \frac{1}{n-2}v_i^{\frac{3-n}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{n}{n-1}}\left[\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)-v_i^{\frac{n-1}{n-2}}\right] \\
\geq & 0
\end{aligned}
$$

For a given $v_{-i} \neq \mathbf{0}$, $\frac{\partial}{\partial v_i}\left(d\left(v_1,..,v_n\right)\right) = \frac{v_i^{\frac{1}{n-2}}}{\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{1}{n-1}}} - \frac{2n+1}{2n} + \frac{i}{n} < 0$ for $v_i = 0$ since $i \leq n$, then the minimizing v_i for a given $v_{-i} \neq \mathbf{0}$ is such that it solves

$$\frac{v_i^{\frac{1}{n-2}}}{\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{\frac{1}{n-1}}} - \frac{2n+1}{2n} + \frac{i}{n} = 0$$

Furthermore, for $v_{-i} = \mathbf{0}$, the minimizing $v_i = 0$ since for $v_{-i} = \mathbf{0}$, minimization problem becomes

$$\min_{v_i \geq 0} v_i - \frac{1}{2n}\left[(2n+1) - 2i\right] v_i$$

or

$$\min_{v_i \geq 0} v_i \left(1 - \frac{2n+1-2i}{2n}\right)$$

$$\min_{v_i \geq 0} v_i \left(\frac{2i-1}{2n}\right)$$

Since $\left(\frac{2i-1}{2n}\right) > 0$, minimizing $v_i = 0$.

Given these suppose that minimizing $\mathbf{v} \neq \mathbf{0}$. That is, there is some j such that $v_j \neq 0$. But then, by above observation $v_i \neq 0$ for all $i \neq j$.

Now, then each i satisfies

$$v_i^{\frac{1}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{1}{n-1}} - \frac{2n+1}{2n} + \frac{i}{n} = 0$$

or

$$v_i^{\frac{1}{n-2}}\left(v_1^{\frac{n-1}{n-2}}+...+v_n^{\frac{n-1}{n-2}}\right)^{-\frac{1}{n-1}} = \frac{2n+1}{2n} - \frac{i}{n}$$

$$\frac{v_i^{\frac{1}{n-2}}}{v_j^{\frac{1}{n-2}}} = \frac{\frac{2n+1}{2n} - \frac{i}{n}}{\frac{2n+1}{2n} - \frac{j}{n}}$$

$$\frac{v_i}{v_j} = \left(\frac{\frac{2n+1}{2n} - \frac{i}{n}}{\frac{2n+1}{2n} - \frac{j}{n}}\right)^{n-2}$$

Hence,

$$v_i = v_1\left(\frac{\frac{2n+1}{2n} - \frac{i}{n}}{\frac{2n+1}{2n} - \frac{1}{n}}\right)^{n-2} \quad ,i > 1$$

or

$$v_i = v_1 \left(\frac{2n+1-2i}{2n-1}\right)^{n-2} \quad ,i > 1$$

Then, the objective function $d(.,..,.)$ becomes:

$$\begin{aligned}
& \left(\sum_{i=1}^{n} v_1^{\frac{n-1}{n-2}} \left(\frac{2n-2i+1}{2n-1}\right)^{n-1}\right)^{\frac{n-2}{n-1}} \\
& -\frac{1}{2n}\left((2n+1)\left(\sum_{i=1}^{n} v_1 \left(\frac{2n-2i+1}{2n-1}\right)^{n-2}\right) - 2\left(\sum_{i=1}^{n} v_1 i \left(\frac{2n-2i+1}{2n-1}\right)^{n-2}\right)\right) \\
= & \frac{v_1}{(2n-1)^{n-2}}\left[\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{n-2}{n-1}} - \frac{1}{2n}\left(\sum_{i=1}^{n}(2n+1-2i)(2n-2i+1)^{n-2}\right)\right] \\
= & \frac{v_1}{(2n-1)^{n-2}}\left[\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{n-2}{n-1}} - \frac{1}{2n}\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)\right] \\
= & \frac{v_1}{(2n-1)^{n-2}(2n)}\left[(2n)\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{n-2}{n-1}} - \left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)\right] \\
= & \left[\frac{v_1}{(2n-1)^{n-2}(2n)}\left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{n-2}{n-1}}\right]\left[2n - \left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{1}{n-1}}\right]
\end{aligned}$$

Note that the term in the first bracket is positive. □

Claim 4. $\left[2n - \left(\sum_{i=1}^{n}(2n-2i+1)^{n-1}\right)^{\frac{1}{n-1}}\right] > 0$ *for* $n \geq 3$

Proof. We want to show

$$(2n)^{n-1} > \sum_{i=1}^{n}(2n-2i+1)^{n-1}$$

or

$$n^{n-1} > \sum_{i=1}^{n}\left(\frac{2n-2i+1}{2}\right)^{n-1} = \sum_{i=1}^{n}\left(n-i+\frac{1}{2}\right)^{n-1}$$

Now, for each $i \in \{1,...,n\}$

$$\left(n-i+\frac{1}{2}\right)^{n-1} < \int_{n-i}^{n-i+1} t^{n-1}dt$$

since

$$
\begin{aligned}
\int_{n-i}^{n-i+1} t^{n-1} dt &= \int_{0}^{\frac{1}{2}} \left[(n-i+u)^{n-1} + (n-i+1-u)^{n-1} \right] du \\
&> \int_{0}^{\frac{1}{2}} 2 \left(n-i+\frac{1}{2} \right)^{n-1} du \\
&= \left(n-i+\frac{1}{2} \right)^{n-1}
\end{aligned}
$$

where strict inequality is due to $f(t) = t^{n-1}$ being a strictly convex function. Thus,

$$
\begin{aligned}
\sum_{i=1}^{n} \left(n-i+\frac{1}{2} \right)^{n-1} &< \sum_{i=1}^{n} \int_{n-i}^{n-i+1} t^{n-1} dt \\
&= \int_{0}^{n} t^{n-1} dt \\
&= n^{n-1}
\end{aligned}
$$

which is what we wanted to show. Thus, the value objective function is positive when $v_1 > 0$ but note that the value of objective function is 0 when $\mathbf{v} = \mathbf{0}$. Hence, the minimum is achieved at $v_i = v_j = 0$ for all i, j giving that for all $\mathbf{v}$

$$
d(v_1, .., v_n) \geq 0
$$

□

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