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ROBUST OPTIMIZATION MODELS FOR INVENTORY
CONTROL UNDER DEMAND AND LEAD TIME UNCERTAINTY

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Abstract

Calculating optimal inventory ordering policies for systems under supply and demand uncertainty is essential in supply chain management in order to reduce the costs of holding inventory and backlogging orders while maintaining customer satisfaction. This thesis develops a general methodology based on robust optimization for an inventory control problem subject to uncertain lead time and uncertain demand.

First, we focus on an inventory control problem where lead times are uncertain and demand is fixed. We present a robust optimization model of this problem where the uncertainty is in the recourse matrix and is of column-wise nature. A variation of Benders’ decomposition is proposed to compute robust optimal (i.e., best worst-case) policy parameters under several policies (static, basestock, and affinely adjustable). Insight is provided into the computation and performance of these policies.

Then, we extend this approach to handle both demand and lead time uncertainty. In order to produce solutions which are not overly conservative, we investigate the performance of several types of uncertainty sets: budget uncertainty sets, central limit theorem-based uncertainty sets, and statistical hypothesis test-based uncertainty sets. Our numerical results indicate that our robust approach performs well compared to a sample average approximation approach in terms of stability measured by average cost, variance, and maximum cost.

Finally, we develop models for inventory control under demand uncertainty where limited distributional information is available. In this section the support, mean, and an upper bound for the second moment is assumed to be known. We develop a solution method to solve a deterministic reformulation of the minimax expectation problem, and prove that it converges to the optimal solution under mild conditions.
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Chapter 1  Introduction

1.1 Problem Statement

Calculating optimal inventory ordering policies for systems under supply and demand uncertainty is essential in supply chain management in order to reduce the costs of holding inventory and backlogging orders while maintaining customer satisfaction. Lead times may be uncertain for several reasons, including variable set-up or processing time at the supplier or transportation delays. Formulating the problem under lead time uncertainty is difficult because of a phenomenon called “order crossover”, which is the arrival of orders in a sequence different than the sequence they were placed (He et al. 1998). Order crossover distorts the lead time distribution which complicates the analysis.

Many results in literature rely on the assumption that order crossovers do not occur and that orders are independent and identically distributed (i.i.d) but the i.i.d. assumption is contradictory and this assumption will not always hold (Hayya et al. (2008)). In fact, crossovers are likely to occur more frequently in the future for several reasons. Riezebos (2006) examines the changes in modern supply chain management and concludes that some of the reasons for the increasing frequency of order crossovers are “a reduction of the time between issuing an order, more suppliers, more frequent ordering, larger distances between supplier and firm, more supply options with different lead time consequences, and dependable but more variable total lead times.”

Many policies have been proposed for inventory problems under stochastic demand and constant lead time. For example, the basestock policy (also called
the “order-up-to” policy), which involves placing an order for \( S - x \) units when the inventory position \( x \) falls below \( S \), was initially proven to be optimal for serial supply chains when demand follows a known distribution by Clark and Scarf (1960) and it has been shown to be optimal for more general systems since then (the reader is referred to Zipkin (2000) where many extensions can be found). Due to its simplicity, the basestock policy has been widely adopted in industry.

Stochastic dynamic programming (SDP) is often used to compute the optimal policy parameters. However, SDP is intractable for large problems as it suffers from “the curse of dimensionality” (Zipkin 2000). Another downside of SDP is that for many real world problems the true probability distribution may not be known.

An alternate approach to handle uncertainty is using uncertainty sets in a robust optimization (RO) framework. The objective of a robust optimization problem is to find the best worst-case solution under a predefined uncertainty set. RO has been studied in supply chain problems showing promising computational results (e.g., see Ben-Tal et al. 2005; Bertsimas and Thiele 2006; Bienstock and Ozbay 2008).

In this thesis a general methodology is proposed based on robust optimization for an inventory control problem subject to uncertain lead times as well as uncertain demand. Prior research has examined this problem using stochastic dynamic programming under the assumption of no order crossovers and full distributional knowledge of lead times. Important characteristics of the proposed approach is that it does not assume distributional knowledge, it makes no assumption regarding order crossovers, and it is tractable in a practical sense.

In Chapter 2, we consider a distribution-free robust optimization model for the inventory of a retailer under uncertain lead time and fixed demand. Chapter 3 is an extension in which we consider uncertain lead time and uncertain demand concurrently, and we develop richer uncertainty sets for lead time uncertainty modeling in particular. In Chapter 4 we consider a slightly different inventory problem where demand is uncertain, however limited distributional information is available including the mean, support, and an upper bound on the variance. We develop a solution method to solve the deterministic reformulation of the minimax expectation problem.
1.2 Contributions

In particular, the contributions of this thesis include the following:

In Chapter 2:

- We develop an approach using robust optimization to handle lead time uncertainty in an inventory control problem. In order to highlight the effect of lead time uncertainty, we consider an inventory problem where lead time is uncertain and demand is fixed (in a similar vein to Hayya and Harrison (2010)). This approach is extended to other lead time uncertainty models in the proceeding chapter.

- We model lead time uncertainty using a column-wise and binary uncertainty set and we make no assumption about the specific lead time distribution or about order crossovers.

- We propose a Benders’ approach to solve inventory control problems under three policies (static, basestock, and the affinely adjustable robust counterpart (AARC)) where the uncertainty is column-wise and lies in the recourse matrix. The non-convex subproblems are reformulated as mixed-integer programs for each policy. Especially for the basestock policy, our computational results are promising.

- We extend the notion of affine decision rules to robust optimization problems under column-wise uncertainty and show that these problems can be solved using the Benders’ approach. The AARC is a challenging problem because the uncertainty in the recourse matrix creates nonconvex bilinear terms and may be intractable in general, however we eliminate the bilinear terms by reformulating the Benders’ subproblems as a mixed-integer program.

- We provide insight into the solution quality of these policies and compare the solution to well-known heuristics from literature. We also examine the value of information for the AARC policy.

In Chapter 3:
• We extend the model from the preceding chapter to develop an approach using robust optimization to concurrently handle lead time uncertainty and demand uncertainty in an inventory control problem.

• We present a new budget-type lead time uncertainty set, as well as a CLT-type uncertainty set and a so-called data-driven, statistical hypothesis test-based uncertainty set to handle lead time and demand uncertainty, respectively.

• The numerical results indicate that our approach compares well to Sample Average Approximation (SAA).

In Chapter 4:

• We propose a distributionally robust optimization approach to handle demand uncertainty when partial distribution information is known, such as the support, mean, and variance. The main tool used for this modeling approach is infinite-dimensional duality to reformulate the min-max expectation problem as a deterministic minimization problem.

• For the demand uncertainty model, we develop a Benders’ decomposition algorithm to solve the problem and we compare the results to the models from the previous chapters. We prove the convergence of this algorithm.

There are two categories of literature related to this work which are discussed in the remainder of this chapter. The first category of literature is work examining robust optimization and its methodologies and applications, which has been studied for the past fifteen years. The second category of literature is work examining inventory control problems under order lead time uncertainty.

1.3 Robust Optimization Literature

Robust optimization (RO) is a methodology used to solve optimization problems that involve uncertainty in the parameter values without considering probability distributions. Typically, the objective of robust optimization is to find the best worst-case solution over an uncertainty set specified by the modeler. The set-based robust optimization is significantly different than the scenario-based robust optimization developed in Mulvey et al. (1995). In scenario-based robust optimization a finite set
of scenarios are considered and the solution may violate the constraints involving these scenarios; a penalty function is included in the objective to account for the violations. For the remainder of the thesis, when we write RO we are referring to set-based robust optimization.

The downside of the RO approach is that the solutions are usually more conservative than solutions generated using complementary methods such as stochastic optimization, although the conservatism can be controlled somewhat by the design of the uncertainty sets (Ben-Tal et al. 2009). The main advantage of robust optimization over alternative methods (e.g., stochastic dynamic programming) is that the approach is computationally tractable (i.e., polynomially solvable) for many cases. For example, a robust linear program with an ellipsoidal uncertainty set can be reformulated as a second-order cone problem which is tractably solvable using interior-point methods (Ben-Tal and Nemirovski 1999), and a robust linear program with polyhedral uncertainty can be reformulated as a (slightly larger) linear program (Bertsimas and Sim 2004). However, these reformulations are possible because of an approach using duality to handle subproblems involving uncertain parameters in the constraints of the optimization problems. Therefore, uncertainty sets which are row-wise dependent are considered. We refer the reader to Gabrel et al. (2013), a recent survey paper on applications of robust optimization, which describes this reformulation approach. When uncertainty is dependent within the columns (as is the case for the lead time models presented in this thesis), this reformulation tactic does not work (see section Differences with Bertsimas and Sim’s Approach in Minoux (2009)).

Widely cited as the first paper considering RO, Soyster (1973) considers a model using column-wise uncertainty sets for linear programming problems where the uncertainty sets are ellipsoids. The author shows that if each column of the constraint matrix belongs to a convex set, then solving the problem amounts to solving a linear program with the matrix coefficients equal to their worst-case value. While this is beautifully simple, it is also an extremely conservative modeling approach. The formulation by Soyster was largely dismissed for many years because of its extreme conservatism. Ben-Tal and Nemirovski (1999) formulate a robust optimization model for a linear program where the uncertainty is row-wise and ellipsoidal. They show that their model can be reformulated as a conic quadratic program. Bertsimas and Sim (2004) formulate an alternative robust optimization
model using a row-wise and polyhedral uncertainty set in which a budget parameter is used to control the conservatism of the model. The main advantage of this method is that the robust counterpart of a linear program under budget uncertainty remains a (slightly larger) linear program, while the robust counterpart of a linear programming under the Ben-Tal and Nemirovski uncertainty set becomes a conic quadratic program. Bandi and Berstimas (2012) develop a new approach to constructing uncertainty sets using conclusions from probability. The approach incorporates distributional information which can be estimated with historical data and the central limit theorem is assumed to hold which is used to construct the uncertainty sets. Rather than using a priori reasoning, Bertsimas et al. (2013) develop a data-driven approach to construct uncertainty sets based on statistical hypothesis tests.

The models discussed above are static models where all decisions are “here and now" decisions that are made before uncertainties are realized. Ben-Tal et al. (2004) present a dynamic approach called adjustable RO (ARO) approach where “wait and see” decisions are made after uncertainty realizations occur. This approach can produce less conservative solutions that the static approach but it may be computationally intractable. The authors formulate an affinely adjustable RO (AARO) approach (this is also called the linear decision rule approach in the literature) by restricting the decisions to be affine function of uncertain data which is shown to be tractable for certain cases.

While enjoying greater tractability, the linear decision rule approach may produce suboptimal solutions except in special cases. See and Sim (2010) study a multi-period inventory control problem with ambiguous demand and propose a truncated linear decision rule which is piecewise linear with respect to demand realizations which outperforms the linear decision rule approach. Bertsimas and Georghiou (2013) derive the structure and propose a methodology to compute optimal decision rules for continuous and binary variables which are piecewise linear and piecewise constant, respectively. An alternative approach to handle “wait and see” decisions is proposed by Bertsimas and Caramanis (2010). In their approach, the uncertainty set is partitioned into a finite number of subsets and each subset has its own decision variable. This approach can be used to model problems which involve continuous adjustable decision variables as well as integer adjustable decision variables.
With the exception of Soyster (1973), the papers previously discussed have considered row-wise uncertainty sets. However, column-based uncertainty sets are still important since they can represent uncertainty in processes (Soyster and Murphy 2013). Since it is straightforward that the dual formulation corresponding to a robust problem with row-wise uncertainty has column-based uncertainty, there has been a stream of research investigating this relationship. Minoux (2009) considers robust linear programming with right-hand side uncertainty, a special case of column-wise uncertainty. The main results are that the dual of the robust model is not equivalent to the robust version of the dual and that the formulation of a two-stage approach to produce less conservative solutions for column-wise robust problems than the Soyster model. Beck and Ben-Tal (2009) show that the dual of the robust counterpart is the same as the optimistic counterpart of the dual, where the optimistic counterpart is defined as a solution to a problem that satisfies the constraints for at least one realization in the uncertainty set. Minoux (2011) further investigates the two-stage approach to robust linear programming under right-hand side uncertainty which is shown to be NP-hard for the general case and the author shows several polynomially solvable special cases.

A robust optimization problem can be solved using either a direct reformulation approach or what is called an adversarial approach. First we describe the various solution approaches utilizing a direct reformulation. Soyster (1973) shows that the robust counterpart of a linear program under column-wise, ellipsoidal uncertainty is a linear program. Ben-Tal and Nemirovski (1999) show that the robust counterpart of a linear program under row-wise, ellipsoidal uncertainty is a conic quadratic program. Bertsimas and Sim (2004) show that the robust counterpart under budget uncertainty (polyhedral) remains a linear program. For multistage robust optimization, the AARO approach can be reformulated as a linear program (or conic quadratic program) when there is fixed recourse and the uncertainty set is polyhedral (or ellipsoidal) (Ben-Tal et al. 2004).

A cutting plane approach is proposed in Bienstock and Ozbay (2008) which involves solving the robust problem over a finite set of uncertainty scenarios. The worst-case uncertainty scenario for that problem is then identified by solving an adversarial problem and added to the set and then the problem is re-solved using the updated set of uncertainty scenarios. This process continues in an iterative fashion until the robust solution is obtained.
1.3.1 RO: Supply Chain Applications Under Demand Uncertainty

A large proportion of the literature on RO applied to supply chain problems deals with demand uncertainty. For example, Ben-Tal et al. (2005) solves the two-echelon, multi-period retailer-supplier flexible commitment contract problem under demand uncertainty using the AARO approach. Also using the AARO approach, Ang et al. (2012) examine the storage assignment problem in unit-load warehouses under demand uncertainty. Zhang (2011) shows that two-stage minimax regret robust uncapacitated lot-sizing problems are polynomially solvable when demand uncertainty is characterized using interval uncertainty sets.

Many supply chain applications of robust optimization in the literature involve static RO modeling approaches, and we mention only a few here. For example, Bertsimas and Thiele (2006) use a static RO with budget uncertainty sets to solve an inventory problem under demand uncertainty and demonstrate its computational tractability compared to the dynamic programming approach. Bienstock and Ozbay (2008) use Benders’ decomposition to compute robust basestock levels under two demand uncertainty models (risk budget demand and bursty demand). Rikun (2011) extends this by considering a multi-echelon system with more cost structure as well as a polyhedral uncertainty set motivated by the central limit theorem. Wei et al. (2011) formulate a robust production planning problem with uncertain demands and returns using a static RO model with budget uncertainty sets. José Alem and Morabito (2012) examines a production planning problem for a furniture company with uncertain costs and demands using a static RO model with budget uncertainty sets. Aouam and Brahimi (2013) formulate a static RO model for a production planning problem with order acceptance decisions under demand uncertainty using the budget uncertainty set approach. Carlsson et al. (2014) study the distribution and inventory planning problem for a large pulp producer using a static RO model with budget uncertainty sets.

1.3.2 RO: Supply Chain Applications Under Supply Uncertainty

In the above papers involving demand uncertainty, the supply-side is assumed to be deterministic and order lead times are assumed to be either zero or fixed. There have been far fewer papers on robust optimization with supply uncertainty,
and they mainly deal with yield uncertainty and raw material supply uncertainty. Bohle et al. (2010) develop a static RO model using the budget uncertainty set approach for wine grape harvesting scheduling where productivity of the manual harvesting method (i.e., yield) is uncertain. Alvarez and Vera (2011) formulate a static RO model for a sawmill planning problem with yield uncertainty using the budget uncertainty set approach. Similarly, Varas et al. (2014) formulate a static RO model for a sawmill planning problem under demand and raw material supply uncertainty and use a budget uncertainty sets. Movahed and Zhang (2013) present a scenario-based robust approach to compute \((s, S)\) parameters for an inventory problem under demand and lead time uncertainty, and we emphasize that the modeling approach (scenario-based RO) is based on Mulvey et al. (1995), which is a different methodology than the set-based RO approach considered in this thesis.

A multi-period RO approach for these problems involving supply uncertainty is not a trivial extension. Ben-Tal et al. (2005) mention that the treatment of random yield in an AARO model is difficult because the model would no longer have fixed recourse since the coefficients for the adjustable variables are the uncertain parameters. For models without fixed recourse, AARO may be intractable (Ben-Tal et al. 2004). However, in this thesis we consider a multi-period inventory problem under lead time uncertainty (which lies in the recourse matrix) where the uncertainty set is binary and we develop a solution method based on Benders’ decomposition to solve it.

In the next section we describe the row-wise and column-wise robust optimization models in more detail.

1.3.3 Row-wise Uncertainty Sets

The linear RO problem under row-wise uncertainty is formulated as

\[
\begin{align*}
\min_x & \quad c^T x \\
\text{subject to} & \quad a_i^T x \leq b_i \quad \forall a_i \in U_i, \ i = 1, \ldots, m
\end{align*}
\] (1.1)
where \( U_i \) is a convex set and \( a_i \) is a row vector. This problem has infinitely many constraints. However, each row has an associated subproblem of the form

\[
\max_{a_i \in U_i} a_i^T x \leq b
\]  

(1.2)

For each row \( i \), the entries \( a_{ij}, j = 1, \ldots, n \) may vary within the uncertainty set \( U_i \). \( U_i \) may involve an budget of uncertainty parameter, \( \Gamma_i \), which controls the number of uncertain terms that may vary from their nominal values. The seminal work of Bertsimas and Sim (2004) describes this type of uncertainty set for the first time which is reviewed in more detail next.

The values of \( a_{ij} \) are modeled as symmetric, independent, and bounded random variables \( \tilde{a}_{ij} \) with the center at \( a_{ij} \). In this model, for each row \( i \), the entries \( a_{ij}, j = 1, \ldots, n \) takes values in \([a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]\). The scaled deviation from the nominal value is denoted as \( z_{ij} = (\tilde{a}_{ij} - a_{ij})/\hat{a}_{ij} \) which can take values in \([-1, 1]\). Then the following constraint bounds the total scaled deviation for each row \( i \):

\[
\sum_j |z_{ij}| \leq \Gamma_i, \forall i.
\]

When \( \Gamma_i = 0, \forall i \), the model represents the nominal, or deterministic, case; when the budget parameter is large, the model allows all parameters to reach their worst-case values (which is equivalent to the column-wise approach in the next section). The parameter \( \Gamma_i \) can be set by the decision maker according to their level of risk aversion, allowing the decision maker to control the level of conservatism of the robust solutions.

An advantage of this uncertainty set is that the robust counterpart of a linear program can be reformulated as a linear program (see Bertsimas and Sim (2004) for details). A disadvantage of this method is that it is unclear what value \( \Gamma_i \) should be. One approach is to show the full spectrum of setting the budget from \( \Gamma_i = 0 \) (which is equivalent to the nominal formulation) to \( \Gamma_i = J_i \), where \( J_i \) is the maximum number of parameters in row \( i \) that may vary using sensitivity analysis (Bertsimas and Thiele 2006, José Alem and Morabito 2012). This allows the decision maker to examine the robustness tradeoff, or “the price of robustness” (Bertsimas and Sim 2004). Another approach is to set the upper and lower bounds of the \( a_{ij} \) parameters according to a \((1 - \alpha)\%\) confidence interval and then set the budget parameter to a low value if the confidence interval is very wide since it is unlikely that many of the \( a_{ij} \)'s will reach their bounds, or set the budget parameter high if the confidence interval is narrower (Denton et al. 2010).
1.3.4 Column-wise Uncertainty Sets

The linear RO problem under column-wise uncertainty is formulated as

\[
\begin{align*}
\min_x c^T x \\
\text{subject to} \\
Ax &\leq b \quad \forall A_j \in K_j, \ j = 1, \ldots, n
\end{align*}
\]  

(1.3)

where the column vectors \( A_j \) for each column \( j \) belong to some convex uncertainty set \( K_j \). Soyster (1973) shows that the solution to this problem can easily be found by solving the following LP

\[
\begin{align*}
\min_x c^T x \\
\text{subject to} \\
Ax &\leq b
\end{align*}
\]  

(1.4)

where the elements of \( A \) are defined as \( a_{ij} = \max_{a_{ij} \in K_j} \{a_{ij}\} \forall i, j \)

The drawback of this approach is that it is much more conservative than the row-wise modeling approach. In order to obtain less conservative solutions than Soyster’s model, Minoux (2009) investigates a special case of robust column-wise uncertainty using a two-stage approach where the uncertainty is on the right-hand-side vector. The problem to solve is

\[
\begin{align*}
\min_x c^T x \\
\text{subject to} \\
Ax + By &\leq b \quad \forall b \in B \\
x &\geq 0 \\
y &\geq 0
\end{align*}
\]  

(1.5)

where \( x \) are first-stage “here and now” variables, \( y \) are second-stage “wait and see” variables, and \( b \in B \) is a compact uncertainty set. In this model, robustness means that a solution \( x \) is feasible for any realization \( b \in B \) by using the second-stage variable \( y \). Explicitly, the robust feasible set is \( X = \{ x : x \geq 0 \text{ and } \forall b \in B, \exists y \geq 0 : By \leq b - Ax \} \). The drawback of using this approach is that, in general, these problems are NP-hard (for example, when the uncertainty set
is polyhedral), although special cases are shown to be polynomially solvable in Minoux (2011).

1.4 Literature on Inventory Control Under Lead Time Uncertainty

Because of the analytical complexity caused by order crossovers, many studies on inventory problems under lead time uncertainty assume that orders do not cross. In early work, Kaplan (1970) examines the finite horizon, periodic review inventory problem under stochastic demand and lead times and uses dynamic programming to prove that the \((s, S)\) policy is optimal assuming that order crossovers cannot occur. Ehrhardt (1984) extended these results to an infinite horizon model and propose a heuristic to compute the approximately optimal policy.

Liberatore (1979) examines a continuous review inventory model under stochastic lead time and deterministic demand and generalizes the economic order quantity (EOQ) model assuming that demands are non-interchangeable, thus a single-cycle analysis is appropriate. Bagchi et al. (1986) studies the impact of lead time variability on stockouts at U.S. Air Force using a model that does not allow order crossovers and justifies the approach by assuming that there is only one vendor and only a single order outstanding at any time. Zipkin (1986) shows that the no order crossing assumption is valid under the assumption that the supply process is exogenous and sequential. That is, the supply system is independent of orders and demands and operates sequentially. Using the framework of an inventory model assuming an exogenous and sequential supply system, Song (1994) examines the impact of stochastic lead times on basestock levels and optimal cost. Research that has focused on continuous review \((Q, r)\) models under constant demand and stochastic lead time includes Bookbinder and Cakanyildirim (1999), where the authors assume stock is always above the reorder level after order arrival, so order crossovers cannot occur. Ayanso et al. (2006) examine a continuous review \((Q, r)\) model where crossovers are not allowed and develop a Monte Carlo simulation method to determine inventory policy parameters for internet retailers where the lead time and demand processes are both stochastic.

Most literature on inventory models which allow order crossovers is more recent.
Riezebos (2006) argues that crossovers are likely to occur more frequently in the future. One reason for this is due to dynamic lead time fluctuations caused by “contract changes, expediting policies, dual-sourcing policies from different geographical areas, transportation mode changes, etc” (Riezebos and Gaalman 2009). Another reason is that manufacturers have reduced order sizes and increased order frequency for materials from distant sources (Wensing and Kuhn 2014). Another factor that makes crossover more frequent is the increasing use of global suppliers, in particular as buyers are increasingly contracting multiple global suppliers simultaneously (e.g., for sourcing commodities) (Srinivasan et al. 2011).

Robinson et al. (2001) show that order policies should be based on the shortfall distribution (i.e., number of outstanding orders) instead of the lead time demand distribution in order to significantly reduce inventory cost, even if the probability of order crossover is small. Bradley and Robinson (2005) examine a periodic review inventory model and develop an upper bound for the variance of the number of outstanding orders, and this bound is further tightened in Robinson and Bradley (2008). The authors use this bound to approximate the shortfall distribution and develop several heuristic basestock approximation policies. Their policies outperform the commonly used method of approximating the lead time demand distribution with a normal distribution. Hayya et al. (2008) show that the shortfall distribution has the same mean but lower variance than the original lead time distribution, which motivates lower safety stocks than those computed when crossovers are ignored. Hayya et al. (2009) examine a continuous review \((Q,r)\) model where demand and lead time are stochastic (allowing crossovers) and develop regression equations for the optimal cost and \((Q,r)\) policy parameters using the distribution of effective lead time demand, or the time between placing the first order and receiving the first delivery.

In order to highlight the effects of lead time uncertainty, in Chapter 2 we focus on inventory problems under deterministic demand, uncertain lead times, and backorders. The reason to focus on uncertain lead time where demand is deterministic is to isolate the effect of lead time uncertainty. When both demand and lead time are uncertain, it may be difficult to separate the effects since the total demand during the lead time will be a compound of both distributions. The problem where lead time is uncertain and demand is deterministic is also of practical interest. Applications where constant demand rate may be common include the
stocking of raw materials as inputs to a production process where the production facility is at full capacity. When working at full capacity, it can be assumed that the consumption rate of raw materials is known. The lead time of these materials will involve uncertainty due to internal and external activities.

There have been several papers on inventory models under constant demand, stochastic lead times, and backorders. Liberatore (1979) considers the continuous-review EOQ model under deterministic demand and stochastic lead time. Although no closed-form analytical solution is presented, the optimal order quantity and time between placing orders can be computed by solving a system of equations. Sphicas and Nasri (1984) obtains a closed-form expression for the EOQ with backorders when the range of the lead time distribution is finite. Nasri et al. (1990) consider an extension of this model where setup costs can be reduced by investments in technology. The authors present a logarithmic investment function and derive closed-form solutions for optimal lot size, setup cost, and total cost. Similarly, Paknejad et al. (1992) look at the synergistic effect of mechanisms for setup cost reduction as well as lead time variability reduction. Kim et al. (2004) use a regression approximation of the continuous review, reorder point, order quantity \((s, Q)\) model where the lead time follows the Erlang distribution. Kouvelis and Tang (2012) examine optimal expediting policies for inventory systems with stochastic lead times. They assume a constant demand rate, and their model adopts the non-interchangeable demand assumption which allows a single-cycle analysis to be conducted. A sequence of papers have made the same assumption to study the use of a backup supplier under lead time uncertainty Kouvelis and Li (2008), and to examine several contingency strategies to deal with lead time uncertainty Kouvelis and Li (2012).

An assumption made in all the papers mentioned in the previous paragraph is that orders may not cross. In another work in the constant demand and uncertain lead time setting, He et al. (1998) use a heuristic multi-cycle analysis for the continuous review \((Q, r)\) policy and show that single-cycle analyses overestimate the system cost and order quantity under high lead time variability. Recently, Hayya and Harrison (2010) examine several continuous review \((Q, r)\) inventory models, where crossovers are allowed, and develop response surfaces for the cost, order quantity, and safety factor in terms of the problem parameters.

Srinivasan et al. (2011) show that the optimal policy for periodic review inventory
models with stochastic demand and stochastic lead times depends on the inventory level as well as the size and age of outstanding orders. The authors also demonstrate that calculating the optimal policy parameters using stochastic dynamic programming is computationally intractable, and they propose a simulation-based approach to calculate approximate parameters. In the next chapter we propose an alternative approach, based on robust optimization, to calculate basestock policy parameters for a periodic review inventory model where lead time is uncertain and demand is fixed.
In this chapter, a general methodology is proposed based on robust optimization for an inventory control problem subject to uncertain lead times. Prior research has examined this problem using stochastic dynamic programming under the assumption of no order crossovers and full distributional knowledge of lead times.

Riezebos (2006) argues that crossovers are likely to occur more frequently in the future. One reason for this is due to dynamic lead time fluctuations caused by “contract changes, expediting policies, dual-sourcing policies from different geographical areas, transportation mode changes, etc” (Riezebos and Gaalman 2009). Another reason is that manufacturers have reduced order sizes and increased order frequency for materials from distant sources (Wensing and Kuhn 2014). Another factor that makes crossover more frequent is the increasing use of global suppliers, in particular as buyers are increasingly contracting multiple global suppliers simultaneously (e.g., for sourcing commodities) (Srinivasan et al. 2011).

Recently, heuristic approximations have been developed in literature for systems that allow order crossovers. The approach proposed in this thesis does not assume distributional knowledge, makes no assumption regarding order crossovers, and is tractable in a practical sense. We show that this problem is an application of robust optimization under column-wise uncertainty where the uncertainty is in the recourse matrix. A variation of Benders’ decomposition is proposed to compute optimal robust (i.e., best worst-case) policy parameters under several policies (static,
basestock, and affinely adjustable). We show it is less conservative than the column-wise robust approaches in literature. The notion of affine decision rules is extended to robust optimization problems under column-wise uncertainty in the recourse matrix and we show that these problems can be solved using the proposed approach. Finally, we provide insight into these policies through computational experiments where sensitivity analysis is performed on the lead time and cost parameters and we benchmark the solution to commonly used heuristics in literature. Our results show that affinely adjustable policies may perform marginally better than static policies, however having information about order arrivals for several future periods can produce large improvements.

2.1 The Nominal Inventory Problem

We consider an inventory problem (with periodic review) for a single facility over a finite time horizon of $T$ time periods. The planner places an order, $u_i$, at the beginning of the time period $i$ which incurs a variable cost $c$. Then demand, $d_i$, occurs during the period. At the end of each period costs are incurred for holding positive inventory (holding cost, $h$), or for negative inventory (backorder cost, $b$). The initial inventory level is $x_0$.

2.1.1 Static Policy Model

The nominal problem, with no uncertainty, and lead times of zero, is the following:

$$\min_{u \geq 0} \sum_{t=1}^{T} \left( cu_t + \max \left\{ h \left( x_0 + \sum_{i=1}^{t} (u_i - d_i) \right), -b \left( x_0 + \sum_{i=1}^{t} (u_i - d_i) \right) \right\} \right)$$

(2.1)
This is equivalent to the following epigraph formulation:

\[
\min_{u, y} \sum_{i=1}^{T} (cu_i + y_i)
\]

subject to

\[y_k \geq hx_{k+1} \quad k = 1, \ldots, T.\]
\[y_k \geq -bx_{k+1} \quad k = 1, \ldots, T.\]
\[x_{k+1} = x_0 + \sum_{i=1}^{k} (u_i - d_i) \quad k = 1, \ldots, T.\]
\[u_i \geq 0 \quad i = 1, \ldots, T.\]

The third constraint in Formulation (2.2) describes the evolution of inventory position, \(x_k\). The ordering policy that is found using Formulation (2.2) places no restrictions on \(u_i\) besides nonnegativity. We call this the static policy because the ordering decisions for the entire time horizon are determined at time zero. As this is the nominal model without uncertainty, it is clear that the optimal decision is to place an order in each time period for the demand that occurs. We present this model, as well as the next model, in order to provide a starting point for when we add uncertainty to the model. We note that the static policy can be implemented dynamically by resolving the problem at each time epoch when new information arises in a receding horizon fashion.

2.1.2 Basestock Policy Model

Next, we restrict the ordering policy to basestock type, where orders are of the form

\[
u_i = \begin{cases} 
\sigma - x_i & x_i < \sigma \\
0 & \text{otherwise}
\end{cases}
\]

where \(\sigma\) is the order-up-to level. We use the standard big-M method from mixed-integer programming to formulate the problem (Bosch and Trick 2005).

**Proposition 2.1.1.** The following mixed-integer linear programming (MILP) formulation yields a basestock policy:
\[
\min_{u, x, y, z, \sigma} \sum_{i=1}^{T} (c u_i + y_i)
\]

subject to
\[
\begin{align*}
    y_k & \geq hx_{k+1} & k = 1, \ldots, T. \\
    y_k & \geq -bx_{k+1} & k = 1, \ldots, T. \\
    u_k & \leq Mz_k & k = 1, \ldots, T. \\
    \sigma - x_k & \leq u_k \leq \sigma - x_k + M(1 - z_k) & k = 1, \ldots, T. \\
    x_{k+1} & = x_0 + \sum_{i=1}^{k} (u_i - d_i) & k = 1, \ldots, T. \\
    u_k & \geq 0 & k = 1, \ldots, T. \\
    z_k & \in \{0, 1\} & k = 1, \ldots, T. \\
    \sigma & \geq 0
\end{align*}
\] (2.4)

where \(M\) is a large constant.

**Proof.** Notice that when \(z_i\) equals zero, \(u_i\) is forced to zero. Otherwise, when \(z_i\) equals one, then \(u_i = \sigma - x_i\). This is exactly the relationship we want from Equation (2.3):

- **Case 1:** \(x_i > \sigma\) ("Order Nothing" case)
  If \(z_i\) is set to one, the third constraint is relaxed, and the fourth constraint is infeasible since \(u_i = \sigma - x_i < 0\). Thus, \(z_i\) must be set to zero; then, the fourth constraint is relaxed and the third constraint becomes \(u_i \leq 0\), but since \(u_i\) is nonnegative, \(u_i\) is set to zero.

- **Case 2:** \(x_i < \sigma\) ("Order" case)
  If \(z_i\) is set to zero, then the third constraint forces \(u_i\) to zero, and the fourth constraint forces \(u_i \geq \sigma - x_i > 0\) which is infeasible. Thus, \(z_i\) is set to one to relax the third constraint, and the fourth constraint yields \(u_i = \sigma - x_i\).

Formulations (2.2) and (2.4) assume zero lead times for all orders. As noted in Bertsimas and Thiele (2006), any constant lead time \(L\) can be considered by
adjusting the evolution of inventory level to

\[ x_{t+1} = x_0 + \sum_{i=1}^{t-L} u_i - \sum_{i=1}^{t} d_i \]  

(2.5)

However, this only addresses deterministic lead time. Next we discuss our approach to model uncertainty.

### 2.2 Lead Time Uncertainty Model and Robust Formulations

#### 2.2.1 Characterization of Lead Time Uncertainty

It is assumed that all orders arrive in full (i.e., there are no split orders). In order to generalize stock evolution to allow different lead times for orders we define a lead time parameter \( \delta_{ik} \) as the following:

**Definition 2.2.1.**

\[
\delta_{ik} = \begin{cases} 
0 & \text{LT}(i) + i > k \\
1 & \text{otherwise}
\end{cases}
\]

(2.6)

where \( \text{LT}(i) \) represents the lead time (measured in integer number of periods) of the order in period \( i \), where \( i = 1, \ldots, T \) and \( k = i, \ldots, T \). If \( \delta_{ik} = 1 \), the order placed in period \( i \) has arrived by period \( k \). If \( \delta_{ik} = 0 \), the order placed in period \( i \) has not yet arrived by period \( k \).

Then it is clear that the inventory level at each time period, \( x_t \), evolves according to

\[ x_{t+1} = x_0 + \sum_{i=1}^{t} \left( \delta_{it} u_i - d_i \right) \]

(2.7)

This description of lead time can be useful for mathematical programming formulations when the lead time of each order is known a priori. However, in order to use a robust optimization approach to mitigate lead time uncertainty, we must construct an uncertainty set to which \( \delta_{ik} \) may belong. With that in mind, we
present a discrete uncertainty set for lead time of orders in period \( i \):

\[
\Omega_i = \{\delta_i \in \{0, 1\}^{T-i} : \delta_{ik} \leq \delta_{i,k+1} \quad k = i, \ldots, T-1, \quad \delta_{i,i+LT_{max}} = 1\} \tag{2.8}
\]

where \( \delta_i \) := (\delta_{ii}, \ldots, \delta_{iT}) \). In this uncertainty set, \( LT_{max} \) is the maximum (integer) periods of lead time for an order. We define \( \Omega = \prod_i \Omega_i \) to be the discrete lead time uncertainty set for the problem. Values of \( \delta_{ik} \) should clearly be non-decreasing in \( k \) since it represents the time of order arrivals. The last inequality in the uncertainty set ensures lead times are at most \( LT_{max} \) periods. The lead time uncertainty set includes column-wise dependencies for the delivery times of each order. These uncertainty sets do not include dependencies between orders. Therefore we make no assumption about order crossings.

### 2.2.2 Robust Static and Basestock Models

The robust inventory problem under lead time uncertainty involves computing the best worst-case solution over the lead time uncertainty set. Starting from Formulation (2.1), the explicit formulation of the robust inventory problem under lead time uncertainty using a static policy (RIPS) is the following:

\[
\min_{u \geq 0} \left\{ \max_{\delta \in \Omega} \sum_{t=1}^{T} \left( cu_t + \max \{ h(x_0 + \sum_{i=1}^{t} (\delta_{it}u_i - d_i)), -b(x_0 + \sum_{i=1}^{t} (\delta_{it}u_i - d_i)) \} \right) \right\} \tag{2.9}
\]

where \( \delta := \{\delta_{ij} : i = 1, \ldots, T, \ j = i, \ldots, T\} \). This model is not equivalent to the robust counterpart of the epigraph formulation (Formulation (2.2)). Theorem 2.2.2 describes the relationship between these models.

**Proposition 2.2.2.** Formulation (2.9) can produce less conservative solutions than robust counterpart of the epigraph formulation (Formulation (2.2)).

**Proof.** Denote \( \tilde{x}^t = x_0 + \sum_{i=1}^{t} (\delta_{it}u_i - d_i) \). The following is the robust counterpart of
the epigraph formulation (Formulation (2.2)):

\[
\begin{align*}
\min_{u,y} & \sum_{i=1}^{T} (cu_i + y_i) \\
\text{s.t.} & \quad y_k \geq h\tilde{x}^k \quad \forall \delta_{ik} \in \Omega \quad k = 1, \ldots, T. \\
& \quad y_k \geq -b\tilde{x}^k \quad \forall \delta_{ik} \in \Omega \quad k = 1, \ldots, T. \\
& \quad u_i \geq 0 \quad i = 1, \ldots, T. \\
\end{align*}
\]

Starting with Formulation (2.9),

\[
\begin{align*}
\min_{u \geq 0} & \max_{\delta \in \Omega} \sum_{t=1}^{T} \left( cu_t + \max \left\{ h\tilde{x}^t, -b\tilde{x}^t \right\} \right) \\
\leq & \min_{u \geq 0} \left\{ \sum_{t=1}^{T} \left( \max \left\{ cu_t + \max \left\{ h\tilde{x}^t, -b\tilde{x}^t \right\} \right\} \right\} \\
= & \min_{u \geq 0} \left\{ \sum_{t=1}^{T} \left( cu_t + \max_{\delta \in \Omega} \max \left\{ h\tilde{x}^t, -b\tilde{x}^t \right\} \right) \right\} \\
= & \min_{u \geq 0} \sum_{t=1}^{T} (cu_t + y_t) \\
\text{s.t} & \quad y_i \geq \max_{\delta \in \Omega} \left\{ h\tilde{x}^i \right\} \quad i = 1, \ldots, T. \\
& \quad y_i \geq \max_{\delta \in \Omega} \left\{ -b\tilde{x}^i \right\} \quad i = 1, \ldots, T. \\
\end{align*}
\]

In the first line we are taking the worst case (maximum cost) uncertainty realization over the sum, and this is less than or equal to the second line where we are taking the worst case (maximum cost) uncertainty realization for each term in the sum (This relationship was first discussed in literature by Gorissen and Den Hertog (2013)). The first equality holds since \( cu_i \) does not involve \( \delta \), the second equality holds by switching the order of max functions, and the third equality holds by the standard LP reformulation of minimax problems. Formulation (2.11) is clearly equivalent to the Formulation (2.10).

Intuitively this result makes sense since Formulation (2.10) involves finding the
worst-case scenario for each constraint and Formulation (2.9) involves finding a single worst-case scenario for the set of constraints.

Formulation (2.10) is a column-wise uncertain robust problem which is ultra-conservative and can be solved easily using the approach described in Soyster (1973). A common approach in literature on robust optimization for supply chain applications involves solving the robust counterpart of epigraph formulations (e.g., Bertsimas and Thiele 2006, and Ben-Tal et al. 2005).

Solving Formulation (2.9) computes the true worst-case cost which is less conservative, and this is our approach in this chapter. We also solve a true worst-case inventory problem where the ordering decisions follow a basestock policy. Using the Big-M method, we formulate the robust basestock policy (RIPB) formulation as the following:

\[
\min_{u \geq 0} \left\{ \max_{\delta \in \Omega} \sum_{t=1}^{T} \left( cu_t + \max \left\{ h \left( x_0 + \sum_{i=1}^{t} (\delta t u_i - d_i) \right), -b \left( x_0 + \sum_{i=1}^{t} (\delta t u_i - d_i) \right) \right\} \right\} \right.
\]
\[
\text{s.t.} \quad u_i \leq Mz_i \quad i = 1, \ldots, T.
\]
\[
\sigma - x_i \leq u_i \leq \sigma - x_i + M(1 - z_i) \quad i = 1, \ldots, T.
\]
\[
\pi_{k+1} = x_0 + \sum_{i=1}^{k} u_i - \sum_{i=1}^{k} d_i \quad k = 1, \ldots, T.
\]
\[
z_i \in \{0, 1\} \quad i = 1, \ldots, T.
\]

where inventory position (the sum of current inventory and inventory on order minus backorders) evolves according to \( \pi_t \).

### 2.2.3 Affinely Adjustable Robust Counterpart Model

In this section, we extend the model to allow decision variables \( u_i \) at each time period \( i \) to be a function of uncertainty realizations in other time periods. Such “adjustable” decisions for RO problems were first introduced in Ben-Tal et al. (2004), and the solutions to such problems can be significantly less conservative than solutions using the static RO approach. They show that the problem can be computationally intractable if the recourse matrix is uncertain, as is the case for our problem. Introducing “adjustable” order decisions at each period \( t \) involves
substituting $u_t(\delta_{jk})$, where $(j, k) \in \mathcal{I}^t$ into Formulation (2.2) for each $u_t$. We call $\mathcal{I}^t$ the information index set which may include all uncertain parameters of index prior to time $t$ or a subset of these uncertain parameters. For example, if we restrict the decision variables $u_t$ to be affine functions of lead time uncertainty realizations at period $t-1$, we define the information index set $\mathcal{I}^t_L = \{(j, k) : t - LT_{max} \leq j \leq t - 1, k = t - 1\}$ Figure 3.4 depicts the order dependence under information index set $\mathcal{I}^t_L$.

![Figure 3.4](image)

**Figure 2.1.** Depiction of the affine dependence of orders ($u_i$) on lead time uncertainty coefficients ($\delta_{ik}$) in the AARC formulation

The mathematical formula is as follows:

$$u_t = u_t^0 + \sum_{(j,k) \in \mathcal{I}^t} \delta_{j,k} u_j^t \quad t = 1, \ldots, T \tag{2.13}$$

where the variables are the $u_i^t$’s. The $y_t$ variables, which represent the inventory/backorder costs incurred each period, are also reformulated as affine functions of previous uncertainty realizations.

$$y_t = y_t^0 + \sum_{(j,k) \in \mathcal{I}^t} \delta_{j,k} y_j^t \quad t = 1, \ldots, T \tag{2.14}$$

The inventory state variables, $x_t$, are also functions of previous uncertainty realizations:

$$x_t = x_t^0 + \sum_{(j,k) \in \mathcal{I}^t} \delta_{j,k} x_j^t \quad t = 1, \ldots, T \tag{2.15}$$

The AARC inventory policy (AARIP) formulation can now be stated as the following:
\[
\min \left\{ \max_{u \geq 0} \sum_{t=1}^{T} (c_{t}u_{t} + \max\{h_{t}x_{t+1} - bx_{t+1}\}) \right\} \\
\text{s.t. } x_{k} = x_{k}^{0} + \sum_{(i,j) \in \mathcal{I}^{k}} \delta_{i,j}x_{k}^{i,j} \quad k = 1, \ldots, T. \\
u_{k} = u_{k}^{0} + \sum_{(i,j) \in \mathcal{I}^{k}} \delta_{i,j}u_{k}^{i,j} \quad k = 1, \ldots, T. \\
x_{k+1}^{0} + \sum_{(i,j) \in \mathcal{I}^{k+1}} \delta_{i,j}x_{k+1}^{i,j} = x_{k}^{0} + \sum_{(i,j) \in \mathcal{I}^{k}} \delta_{i,j}x_{k}^{i,j} + \delta_{k}(u_{k}^{0} + \sum_{(i,j) \in \mathcal{I}^{k}} \delta_{i,j}u_{k}^{i,j}) \\
+ \sum_{i=1}^{k-1} \left( (\delta_{i,k} - \delta_{i,k-1}) (u_{i}^{0} + \sum_{(j,k) \in \mathcal{I}^{i}} \delta_{j,k}u_{i}^{j,k}) \right) - d_{k} \quad k = 1, \ldots, T.
\]

The right-hand side of the inventory balance equation involves four parts: (1) the inventory level at the beginning of period \(k\); (2) arrival of an order with zero lead time at period \(k\); (3) arrival of orders with positive lead time at period \(k\); and (4) demand that occurs at period \(k\).

### 2.3 Benders’ Decomposition Solution Approach

We use a cutting plane approach to solve RIPS, RIPB, and AARIP and we observe that the approach can produce less conservative solutions than existing methods for column-wise problems. Related to our approach is the work by Bienstock and Ozbay (2008), where the authors develop a cutting plane approach to solve an inventory problem under demand uncertainty for several demand uncertainty models. Rikun (2011) extends this approach to the more general polyhedral demand uncertainty with promising computational results.

The algorithm involves maintaining a working list containing lead time uncertainty scenarios (denoted as \(\tilde{\Omega}\)). The Decision Maker Problem (DM) involves minimizing the maximum cost over the scenarios in \(\tilde{\Omega}\). Using the order vector from DM, the Adversary Problem (AP) chooses the worst-case lead time scenario over the uncertainty set which maximizes the cost. The lead time scenario returned by AP is then added to the list \(\tilde{\Omega}\). DM is then re-solved using the updated \(\tilde{\Omega}\) cuts and the algorithm iterates until the solutions to DM and AP converge. The Benders’
decomposition algorithm is described in Algorithm 1.

**Algorithm 1 Benders’ Approach**

1. **Initialization Step:**
   
   - Set $\tilde{\Omega} = \emptyset$.
   - Choose initial scenario $\Omega^0 = \{\delta_{ij}^0 : i = 1 \ldots, T, \; j = i, \ldots, T\}$, and add $\Omega^0$ to $\tilde{\Omega}$.
   - Set $i = 0$, $L = 0$, and $U = \infty$.

2. **Decision Maker Problem:**
   
   - Set $L =$ Objective value from DM formulation
   - Set $\bar{u} =$ Solution from DM formulation

3. **Adversarial Problem:**
   
   - Set $U =$ min\{U, Objective value from AP Formulation\}
   - Set $\bar{\delta} =$ Solution from AP Formulation with order vector $\bar{u}$, and set $\Psi^{i+1} = \bar{\delta}$

4. **Terminate if** $U - L < \epsilon$.
   
   Otherwise, add $\Omega^{i+1} \in \tilde{\Omega}$, set $i = i + 1$, and return to Step 2.

The DM and AP formulations for each of the three policies (RIPS, RIPB, and AARIP) are discussed next.

The DM formulation for the RIPS is:

$$\min Z$$

s.t. 

$$Z \geq \sum_{i=1}^{T} (cu_i + y_{i,\omega}) \quad \omega = 0, \ldots, |\tilde{\Omega}|$$

$$y_{i,\omega} \geq h \left( x_0 + \sum_{i=1}^{t} (\delta_{ii}^u u_i - d_i) \right) \quad i = 1, \ldots, T, \; \omega = 0, \ldots, |\tilde{\Omega}| \quad (2.17)$$

$$y_{i,\omega} \geq -b \left( x_0 + \sum_{i=1}^{t} (\delta_{ii}^w u_i - d_i) \right) \quad i = 1, \ldots, T, \; \omega = 0, \ldots, |\tilde{\Omega}|$$

$$u_i \geq 0 \quad i = 1, \ldots, T.$$

The DM formulation for the RIPB policy is Formulation (2.17) with the constraints from Formulation (2.12) appended to it. The DM formulation for AARIP is the following:

$$\min Z$$

s.t. 

$$\quad (2.18)$$

26
\[ Z \geq \sum_{i=1}^{T} (cu_{i,\omega} + y_{i,\omega}) \quad \omega = 0, \ldots, |\tilde{\Omega}| \]

\[ y_{i,\omega} \geq h \left( x_0 + \sum_{i=1}^{t} (\delta_{i,t}^\omega u_{i,\omega} - d_i) \right) \quad i = 1, \ldots, T, \ \omega = 0, \ldots, |\tilde{\Omega}| \]

\[ y_{i,\omega} \geq -b \left( x_0 + \sum_{i=1}^{t} (\delta_{i,t}^\omega u_{i,\omega} - d_i) \right) \quad i = 1, \ldots, T, \ \omega = 0, \ldots, |\tilde{\Omega}| \]

\[ u_{i,\omega} \geq 0 \quad i = 1, \ldots, T, \ \omega = 0, \ldots, |\tilde{\Omega}| \]

\[ u_{k,\omega} = u_{k}^0 + \sum_{(i,j) \in \mathcal{F}^k} \delta_{i,j}^\omega u_{k}^{i,j} \quad k = 1, \ldots, T, \ \omega = 0, \ldots, |\tilde{\Omega}| \]

\[ x_{k+1} = \sum_{(i,j) \in \mathcal{F}^{k+1}} \delta_{i,j}^\omega x_{k+1}^{i,j} = x_k + \sum_{(i,j) \in \mathcal{F}^k} \delta_{i,j}^\omega x_k^{i,j} + \sum_{k} \left( u_k^0 + \sum_{(i,j) \in \mathcal{F}^k} \delta_{i,j}^\omega u_{k}^{i,j} \right) \]

\[ + \sum_{i=1}^{k-1} \left( \delta_{i,k}^\omega - \delta_{i,k-1}^\omega \right) \left( u_i^0 + \sum_{(j,k) \in \mathcal{F}^i} \delta_{j,k}^\omega u_{i}^{j,k} \right) - d_k, \quad k = 1, \ldots, T, \ \omega = 0, \ldots, |\tilde{\Omega}| \]

The AP formulation for both RIPS and RIPB is the following MILP (adapted from Bienstock and Ozbay (2008), section 2.1.2) shown in Formulation (2.20), which returns the worst-case lead time scenario for the order vector \( \bar{n} \).
\[
\max_{\delta, I, B, p} \sum_{i=1}^{T} (c\bar{u}_i + I_i + B_i)
\]

s.t. 
\[
I_k \geq h \left( x_0 + \sum_{i=1}^{k} (\delta_{ik} \bar{u}_i - d_i) \right) \quad k = 1, \ldots, T
\]
\[
I_k \leq h \left( x_0 + \sum_{i=1}^{k} (\delta_{ik} \bar{u}_i - d_i) \right) + M(1 - p_k) \quad k = 1, \ldots, T
\]
\[
I_k \leq M(p_k) \quad k = 1, \ldots, T
\]
\[
B_k \geq -b \left( x_0 + \sum_{i=1}^{k} (\delta_{ik} \bar{u}_i - d_i) \right) \quad k = 1, \ldots, T
\]
\[
B_k \leq -b \left( x_0 + \sum_{i=1}^{k} (\delta_{ik} \bar{u}_i - d_i) \right) + M(p_k) \quad k = 1, \ldots, T
\]
\[
B_k \leq M(1 - p_k) \quad k = 1, \ldots, T
\]
\[
\delta_{ik} \in \Omega \quad i = 1, \ldots, T \quad k \geq i
\]
\[
p_k \in \{0, 1\}
\]

where \(M\) is a large constant.

The AP formulation for AARIP, where \(u_i\)'s are fixed, and the decision variables are \(\delta_{ik}\)'s, is a difficult mixed-integer nonlinear program (MINLP). This is because the uncertainty is in the coefficients for the adjustable variables which results in bilinear terms. However, since the uncertainty set \(\Omega\) is binary, the Adversarial Step can be reformulated as a MILP using three additional constraints and one additional continuous variable for each bilinear term. That is, terms of the type \(\delta_i \delta_j\) where \(\delta_i, \delta_j \in \{0, 1\}\) can be replaced by a variable \(\omega_{ij} \in \mathbb{R}_+\) along with the constraints

\[
\omega_{ij} \leq \delta_i
\]
\[
\omega_{ij} \leq \delta_j
\]
\[
\omega_{ij} \geq 1 - (1 - \delta_i) - (1 - \delta_j)
\]

Using this reformulation technique, the AP formulation for AARIP (using informa-
tion index set $\mathcal{I}_L$) is shown as follows:

$$\max_{\delta,L,B,p} \sum_{i=1}^{T} \left( c \left( u_i^0 + \sum_{i'=i-\text{LT}_{\text{max}}}^{i-1} \delta_{i',i-1} u_i' \right) + I_i + B_i \right)$$

s.t.

$$I_k \geq h \left( x_0 + \sum_{i=1}^{k} \left( \delta_{ik} u_i^0 + \sum_{j=i-\text{LT}_{\text{max}}}^{i-1} (w_{i,j,k,i-1} u_i' - d_i) \right) \right) \forall k$$

$$I_k \leq h \left( x_0 + \sum_{i=1}^{k} \left( \delta_{ik} u_i^0 + \sum_{j=i-\text{LT}_{\text{max}}}^{i-1} (w_{i,j,k,i-1} u_i' - d_i) \right) \right) + M(1-p_k) \forall k$$

$$I_k \leq M(p_k) \forall k$$

$$B_k \geq -b \left( x_0 + \sum_{i=1}^{k} \left( \delta_{ik} u_i^0 + \sum_{j=i-\text{LT}_{\text{max}}}^{i-1} (w_{i,j,k,i-1} u_i' - d_i) \right) \right) \forall k$$

$$B_k \leq -b \left( x_0 + \sum_{i=1}^{k} \left( \delta_{ik} u_i^0 + \sum_{j=i-\text{LT}_{\text{max}}}^{i-1} (w_{i,j,k,i-1} u_i' - d_i) \right) \right) + M(p_k) \forall k$$

$$B_k \leq M(1-p_k) \forall k$$

$$\delta_{ik} \in \Omega \quad i = 1, \ldots, T, \quad \forall k \geq i$$

$$p_k \in \{0,1\} \quad \forall k$$

$$w_{i,j,k,i-1} \leq \delta_{ik} \quad \forall i, j, k$$

$$w_{i,j,k,i-1} \leq \delta_{j,i-1} \quad \forall i, j, k$$

$$w_{i,j,k,i-1} \geq \delta_{ik} + \delta_{j,i-1} - 1 \quad \forall i, j, k$$

(2.22)

where $M$ is a large constant.

### 2.3.1 Example of Benders’ Applied to Static Policy

In this section we use an instance of RIPS, Formulation (2.10), to illustrate Proposition 2.2.2 (i.e., that the Benders’ method computes a minimax solution which is less conservative than the column-wise model from Soyster (1973) applied to an epigraph robust formulation). The data for the instance is as follows: There are four time periods ($T = 4$), holding cost $h$ is 5, backorder cost $b$ is 2, variable cost $c$ is 1, and demand is $(10, 8, 5, 5)$. Maximum lead time for an order is 2 time periods. The problem converges to an objective value of 90.8 in four iterations as shown in
Table 2.1, along with the lead time uncertainty realizations $\Psi^i$ added to $\tilde{\Psi}$ at each iteration $i$.

Table 2.1. Objective values for Decision Maker Step and Adversarial Step and uncertainty realizations for Adversarial Step

<table>
<thead>
<tr>
<th>Iteration</th>
<th>DM Objective</th>
<th>Adv Objective</th>
<th>$\delta_{11}$</th>
<th>$\delta_{12}$</th>
<th>$\delta_{22}$</th>
<th>$\delta_{23}$</th>
<th>$\delta_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>28</td>
<td>110</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>89.67</td>
<td>95.33</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>90.43</td>
<td>95.03</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>90.8</td>
<td>90.8</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The optimal ordering decisions are $u_1^* = 19.6$, $u_2^* = 0$, $u_3^* = 3.4$, and $u_4^* = 5$. Clearly there does not exist a lead time realization that would cost more than 90.8 when using the optimal ordering decisions. When $\Psi^4$ and $u^*$ is used as input to DM Problem (2.17), we find the optimal inventory/holding costs at each period are $y_1^* = 20$, $y_2^* = 36$, $y_3^* = 6.8$, and $y_4^* = 0$.

However, this solution is not feasible to the epigraph robust formulation, Formulation (2.10). Consider the realization $\hat{\delta}_{11} = 1$. Then the constraint for $y_1^*$ in Formulation (2.10) would state: $y_1^* = 20 \geq 5(19.6 - 10)$, or $20 \geq 48$ which does not hold. The worst-case solution for Formulation (2.10) involves ordering decisions $u_1^* = 14$, $u_2^* = 0$, $u_3^* = 0$, and $u_4^* = 14$, with inventory/holding costs at each period are $y_1^* = 20$, $y_2^* = 36$, $y_3^* = 18$, and $y_4^* = 0$. This solution incurs a cost of 102, which is a 12% higher cost than the RIPS solution.

### 2.4 Results

Several computational experiments were conducted. In the first part, we examine the practical tractability and scalability of our approach for the basestock and static policies. In the second part, we examine the solution quality of both policies. The robust basestock policy is benchmarked against several approximation methods. The computational performance and solution quality of the AARC model is then examined.
2.4.1 Computational Performance: Static and Basestock Models

In this section, the computational performance of the Benders’ decomposition algorithm for calculating static and basestock inventory policies is examined. The termination criterion used for the Benders’ algorithm was $\epsilon = 0.01$. For the first experiment, we considered four lead time scenarios: Maximum lead time of 1, 2, 3, and 4 periods, with minimum lead time equal to zero for all scenarios. For the static problem we tested time horizons of 10, 20, and 40 time periods. For the basestock problem, due to better computational performance, we considered time horizons of 10, 20, 40, and 100 time periods. Parameters were set as follows: Demand, $d$, was sampled from a discrete uniform distribution from 7 to 13; variable cost, $c$, was sampled from a discrete uniform distribution from 1 to 5; holding cost, $h$, was sampled from a discrete uniform distribution from 5 to 10; and backorder cost, $b$, was sampled from a discrete uniform distribution from 10 to 20. Fifty instances of the static and basestock inventory problems were solved, and the computation times and iteration counts are shown in Tables (2.2) and (2.3). It is evident that basestock problems are solved much faster than the static problems.

<table>
<thead>
<tr>
<th>Table 2.2. Computational results for solving RIPS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T = 10$</td>
</tr>
<tr>
<td>LT = 1</td>
</tr>
<tr>
<td>LT = 2</td>
</tr>
<tr>
<td>LT = 3</td>
</tr>
<tr>
<td>LT = 4</td>
</tr>
<tr>
<td>$T = 20$</td>
</tr>
<tr>
<td>LT = 1</td>
</tr>
<tr>
<td>LT = 2</td>
</tr>
<tr>
<td>LT = 3</td>
</tr>
<tr>
<td>LT = 4</td>
</tr>
<tr>
<td>$T = 40$</td>
</tr>
<tr>
<td>LT = 1</td>
</tr>
<tr>
<td>LT = 2</td>
</tr>
<tr>
<td>LT = 3</td>
</tr>
<tr>
<td>LT = 4</td>
</tr>
</tbody>
</table>
Table 2.3. Computational results for solving RIPB

<table>
<thead>
<tr>
<th></th>
<th>Avg Time</th>
<th>Min Time</th>
<th>Max Time</th>
<th>Avg Iter</th>
<th>Min Iter</th>
<th>Max Iter</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>T = 10</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LT = 1</td>
<td>0.45</td>
<td>0.33</td>
<td>1.71</td>
<td>4.66</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>LT = 2</td>
<td>0.37</td>
<td>0.23</td>
<td>1.15</td>
<td>3.98</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>LT = 3</td>
<td>0.26</td>
<td>0.20</td>
<td>1.28</td>
<td>3.60</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>LT = 4</td>
<td>0.24</td>
<td>0.19</td>
<td>0.62</td>
<td>3.56</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td><strong>T = 20</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LT = 1</td>
<td>0.74</td>
<td>0.58</td>
<td>0.82</td>
<td>4.80</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>LT = 2</td>
<td>0.59</td>
<td>0.43</td>
<td>0.73</td>
<td>4.18</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>LT = 3</td>
<td>0.49</td>
<td>0.36</td>
<td>0.64</td>
<td>3.96</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>LT = 4</td>
<td>0.45</td>
<td>0.35</td>
<td>0.60</td>
<td>3.70</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td><strong>T = 40</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LT = 1</td>
<td>1.45</td>
<td>1.15</td>
<td>2.24</td>
<td>4.84</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>LT = 2</td>
<td>1.20</td>
<td>1.03</td>
<td>1.40</td>
<td>4.44</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>LT = 3</td>
<td>1.07</td>
<td>0.75</td>
<td>1.30</td>
<td>4.18</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>LT = 4</td>
<td>1.00</td>
<td>0.71</td>
<td>1.37</td>
<td>3.92</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td><strong>T = 100</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LT = 1</td>
<td>4.28</td>
<td>3.47</td>
<td>5.23</td>
<td>4.90</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>LT = 2</td>
<td>3.66</td>
<td>3.30</td>
<td>4.55</td>
<td>4.16</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>LT = 3</td>
<td>3.75</td>
<td>3.18</td>
<td>4.98</td>
<td>4.28</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>LT = 4</td>
<td>4.14</td>
<td>3.36</td>
<td>5.85</td>
<td>4.56</td>
<td>4</td>
<td>5</td>
</tr>
</tbody>
</table>

2.4.2 Solution Quality: Static and Basestock Models

We compare the performance of the static and basestock policies by using a 20-period example problem and comparing the results. Demand is generated randomly between 5 and 20, and we set $c = 1$, $h = 2$, and $b = 5$. We obtain the robust policies for five lead time scenarios (maximum lead time from one to five periods) using the Benders’ decomposition algorithm. Then, we simulate 100 realizations of order lead times for the time horizon and calculate costs. Table (2.4) shows descriptive statistics regarding the simulated costs and shows that static policies perform significantly better than basestock, particularly for small lead times. However, the outperformance diminishes as lead time uncertainty increases, as the difference between average cost for RIPB is 27% higher than RIPS for lead time of one period but only 3% higher for lead times of five periods.

Next, we examine the quality of the solution of the robust basestock model by benchmarking the solution to several heuristic approximations. First, we consider a common heuristic for setting the basestock level, which we call LTD-Heuristic, which is derived under the assumption that lead time demand is normally distributed. Second, we consider an improved approximation that takes into account order crossover called BR05-Heuristic (Bradley and Robinson 2005). It produces solutions
Table 2.4. Simulation results for RIPS and RIPB. The first four rows correspond to the respective descriptive statistic for the sample. “Worst Case” is the objective function value from the Benders’ decomposition algorithm.

<table>
<thead>
<tr>
<th></th>
<th>LT1 RIPS</th>
<th>LT1 RIPB</th>
<th>LT2 RIPS</th>
<th>LT2 RIPB</th>
<th>LT3 RIPS</th>
<th>LT3 RIPB</th>
<th>LT4 RIPS</th>
<th>LT4 RIPB</th>
<th>LT5 RIPS</th>
<th>LT5 RIPB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Avg Cost</td>
<td>582.0</td>
<td>740.8</td>
<td>708.9</td>
<td>854.1</td>
<td>964.9</td>
<td>980.6</td>
<td>1071.8</td>
<td>1127.6</td>
<td>1211.9</td>
<td>1247.4</td>
</tr>
<tr>
<td>Min Cost</td>
<td>545.2</td>
<td>624.3</td>
<td>551.2</td>
<td>684.6</td>
<td>729.9</td>
<td>724.9</td>
<td>760.9</td>
<td>847.9</td>
<td>871.7</td>
<td>765.9</td>
</tr>
<tr>
<td>Max Cost</td>
<td>621.9</td>
<td>858.9</td>
<td>859.3</td>
<td>1078.9</td>
<td>1285.1</td>
<td>1332.2</td>
<td>1504.3</td>
<td>1595.7</td>
<td>1818.3</td>
<td>1984.1</td>
</tr>
<tr>
<td>Worst Case</td>
<td>621.9</td>
<td>900.4</td>
<td>961.5</td>
<td>1210.9</td>
<td>1370.3</td>
<td>1634.2</td>
<td>1785.0</td>
<td>2121.7</td>
<td>2150.3</td>
<td>2640.5</td>
</tr>
</tbody>
</table>

with lower inventory than LTD-Heuristic because the variance of shortfall is smaller than the variance of lead times.

With LTD-Heuristic, the basestock level is set to $S = \mu_{LTD} + \sigma_{LTD} \Phi^{-1}(b/(b + h))$, where $\mu_{LTD}$ and $\sigma_{LTD}$ are the mean and standard deviation of lead time demand and $\Phi()$ is the standard normal distribution. They are calculated as follows:

$$\mu_{LTD} = (\mu_L + 1)\mu_D$$

$$\sigma_{LTD}^2 = (\mu_L + 1)\sigma_D^2 + \mu_D^2\sigma_L^2$$

BR05-Heuristic calculates basestock levels using an approximation of the shortfall distribution in place of the lead time demand distribution, where $S = \mu_{SF} + \hat{\sigma}_{SF} \Phi^{-1}(b/(b + h))$, where $\mu_{SF} = \mu_{LTD}$, and the approximation to the variance of the shortfall is calculated with the following equations (see Bradley and Robinson (2005) for details):

$$\hat{\sigma}_{SF}^2 = (\mu_N + 1)\sigma_D^2 + \mu_D^2\sigma_N^2$$

$$\hat{\sigma}_N^2 = \min \{\sigma_L^2, \mu_L, \sigma_L/\sqrt{3}\}$$

As an illustrative example, consider an instance where demand is fixed at a constant rate (10 units) and lead time is assumed to belong to a certain distribution - in particular, a discrete uniform distribution (between zero and three periods). Then, $\mu_D = 10$, $\sigma_D^2 = 0$, $\mu_L = 1.5$, and $\sigma_L^2 = 1.25$. Set $h = 5$ and $b = 10$ (call this Scenario A). The LTD-Heuristic for the basestock level yields $S = 29.81$, and the BR05-Heuristic produces $S = 28.46$. Using our robust optimization approach and a time horizon of ten periods, the basestock level is set to 30, which is slightly more conservative than the heuristic levels.

Changing the costs to $h = 1$ and $b = 19$ (Call this Scenario B) gives the LTD-
Heuristic basestock level of \( S = 43.38 \), while the BR05-Heuristic gives \( S = 38.21 \). The robust optimization approach, again with a time horizon of ten periods chooses a basestock level of \( S = 38.5 \), which is close to the BR05-Heuristic solution. Using a time horizon of 30 periods \( (T = 30) \), the computed robust levels remained the same \( (S = 30 \) for Scenario A, \( S = 38.5 \) for Scenario B). In Table (2.5), the performance of the heuristics and the robust basestock levels is examined under Scenario A and B for the ten period problem and the thirty period problem. We simulate 100 realizations of lead time (chosen randomly from a uniform distribution between 0 and 3) for a ten period problem (and a thirty period problem) and examine descriptive statistics regarding the costs (sample average, sample minimum, sample maximum, and sample standard deviation). The robust basestock approach performs well in comparison to the heuristics in this experiment.

Table 2.5. Descriptive statistics for heuristic basestock and robust basestock policies for a 10 period problem and a 30 period problem

<table>
<thead>
<tr>
<th>Scenario A (T=10)</th>
<th>Scenario B (T=10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robust</td>
<td>LTD-Heur</td>
</tr>
<tr>
<td>Basestock (S)</td>
<td>30</td>
</tr>
<tr>
<td>Avg Cost</td>
<td>652.0</td>
</tr>
<tr>
<td>Min Cost</td>
<td>350</td>
</tr>
<tr>
<td>Max Cost</td>
<td>1100</td>
</tr>
<tr>
<td>Std Dev</td>
<td>171.9</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Scenario A (T=30)</th>
<th>Scenario B (T=30)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Robust</td>
<td>LTD-Heur</td>
</tr>
<tr>
<td>Avg Cost</td>
<td>1200.5</td>
</tr>
<tr>
<td>Min Cost</td>
<td>720.0</td>
</tr>
<tr>
<td>Max Cost</td>
<td>1920.0</td>
</tr>
<tr>
<td>Std Dev</td>
<td>217.3</td>
</tr>
</tbody>
</table>

Figures (2.2) and (2.3) show the average total cost over 100 realizations of uncertain lead times when using different basestock levels (robust basestock level and comparison heuristics are labeled). The robust basestock levels are near the minimum values for this simulation. It is observed that the basestock levels computed using the robust optimization approach can produce good results despite its conservatism (i.e., high basestock values in Scenario A), and may compute a policy close to the BR-Heuristic value (as in Scenario B).
Figure 2.2. Costs of Scenario A (a 30 period problem) with various basestock levels

Figure 2.3. Costs of Scenario B (a 30 period problem) with various basestock levels
2.4.3 Computational Results: AARIP Model

We show the solution time and iteration count when solving 100 randomly generated instances using the static policy (RIPS) as well as the affinely adjustable policy (AARIP) using several different information index sets in Table (2.6). Orders at each time period are affine functions of all uncertain parameters for information index set “Full”. Orders at each time period \( t \) are affine functions of only uncertain parameters from period 1 to \( t - 1 \) in information index set “Last”. Information index sets “One Ahead” and “Two Ahead” append one and two time periods, respectively, to the information index set “Last”. Conceptually, these models consider that the decision maker can anticipate if orders will be arriving the next day or two days out. The choice of the information index set impacts the size of the model and the computational performance.

Table 2.6. Average computation time (and average number of iterations) for RIPS and AARIP for a ten-period problem

<table>
<thead>
<tr>
<th>LT</th>
<th>Static</th>
<th>Last</th>
<th>1 Ahead</th>
<th>2 Ahead</th>
<th>Full</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.6 (10.9)</td>
<td>78.9 (52.2)</td>
<td>39.7 (27.7)</td>
<td>34.2 (24.5)</td>
<td>95.0 (57.7)</td>
</tr>
<tr>
<td>2</td>
<td>1.2 (14.3)</td>
<td>306.6 (102.3)</td>
<td>356.1 (131.5)</td>
<td>618.9 (184.2)</td>
<td>531.9 (135.9)</td>
</tr>
<tr>
<td>3</td>
<td>1.3 (13.6)</td>
<td>477.3 (140.9)</td>
<td>1122.2 (147)</td>
<td>2234.4 (216.4)</td>
<td>2274.3 (202.2)</td>
</tr>
</tbody>
</table>

2.4.4 Solution Quality: AARIP Model

A small example is used to demonstrate the performance of the AARIP model and compare the solution to the basestock and static models. In this example, there are five time periods. The cost parameters are \( c = 8 \), \( h = 50 \), and \( b = 23 \). Demand is 20 units at each period. The maximum lead time of each order is one period. Hence, the only uncertain terms are \( \delta_{tt} \) for \( t = 1, 2, 3, 4, 5 \). In order to determine the best possible AARIP solution, we use an information index set \( \mathcal{I}_{ALL} = \{(j, k) : (1, 1), (2, 2), (3, 3), (4, 4), (5, 5) \} \) for each \( t \). That is, the order at each time period \( t \) is an affine function of all uncertainties in the model. Since the order at each \( t \) is an affine function of uncertainties at periods \( s = 1, \ldots, T \), this can be considered an omniscient model that will produce the best AARIP solution. Table (2.7) compares the solution for the three policies. The AARIP worst-case
Cost is 20.1% lower than RIPS and 33% lower than RIPB.

The solutions for RIPS and AARIP under different information index sets for several generated sets of data (shown in Table (2.8)) are shown in Table (2.9). Datasets S1 and S2 are ten period problems using the same demand but different cost parameters. Dataset S3 is a dataset with constant demand and high inventory holding cost. An interesting observation of Table (2.9) is that an affinely adjustable policy with only information from previous periods available performs marginally better than the static policy (zero to 6.9% improvement), while an affinely adjustable policy that has information from previous periods as well as the current period can achieve higher improvements over the static policy (up to 25.3% improvement).

**Table 2.7.** Solutions, worst case costs, and average costs for AARIP, RIPS, and RIPB models for a small example. WC Cost is the worst-case objective value, and Avg Cost is the average cost calculated using simulation (n=100)

<table>
<thead>
<tr>
<th>t</th>
<th>d</th>
<th>$u^{RIPS}$</th>
<th>$u^{RIPB}$</th>
<th>$u^{AARIP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>29.2</td>
<td>26.3</td>
<td>40 - 20δ_{11}</td>
</tr>
<tr>
<td>2</td>
<td>20</td>
<td>15.77</td>
<td>20</td>
<td>20 + 10δ_{11} - 20δ_{22}</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>21.95</td>
<td>20</td>
<td>20 + 10δ_{11} + 20δ_{22} - 20δ_{33}</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>13.09</td>
<td>20</td>
<td>20 + 20δ_{33} - 20δ_{44}</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>20</td>
<td>20</td>
<td>20δ_{44}</td>
</tr>
<tr>
<td>WC Cost</td>
<td>2155.1</td>
<td>2570.7</td>
<td>1720.0</td>
<td></td>
</tr>
<tr>
<td>Avg Cost</td>
<td>1995.6</td>
<td>2498.1</td>
<td>1529.1</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2.8.** Several datasets

<table>
<thead>
<tr>
<th>Dataset</th>
<th>T</th>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$d_3$</th>
<th>$d_4$</th>
<th>$d_5$</th>
<th>$d_6$</th>
<th>$d_7$</th>
<th>$d_8$</th>
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<th>$d_{10}$</th>
<th>h</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>10</td>
<td>19</td>
<td>20</td>
<td>11</td>
<td>18</td>
<td>10</td>
<td>5</td>
<td>17</td>
<td>17</td>
<td>10</td>
<td>15</td>
<td>2</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>S2</td>
<td>10</td>
<td>19</td>
<td>20</td>
<td>11</td>
<td>18</td>
<td>10</td>
<td>5</td>
<td>17</td>
<td>17</td>
<td>10</td>
<td>15</td>
<td>15</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>S3</td>
<td>6</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
<td>10</td>
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<td>10</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>20</td>
</tr>
</tbody>
</table>

37
<table>
<thead>
<tr>
<th>Dataset S1</th>
<th>LT&lt;sub&gt;max&lt;/sub&gt;</th>
<th>RIPS</th>
<th>AARIP(Last)</th>
<th>AARIP(1 Ahead)</th>
<th>AARIP(2 Ahead)</th>
<th>AARIP(Full)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>365.79</td>
<td>365.79(0%)</td>
<td>365.79(0%)</td>
<td>333.00(8.9%)</td>
<td>333.00(8.9%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>636.57</td>
<td>628.78(1.2%)</td>
<td>624.92(1.8%)</td>
<td>614.56(3.4%)</td>
<td>610.00(4.1%)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>910.57</td>
<td>887.33(2.6%)</td>
<td>875.45(3.9%)</td>
<td>860.60(5.5%)</td>
<td>854.00(6.2%)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dataset S2</th>
<th>LT&lt;sub&gt;max&lt;/sub&gt;</th>
<th>RIPS</th>
<th>AARIP(Last)</th>
<th>AARIP(1 Ahead)</th>
<th>AARIP(2 Ahead)</th>
<th>AARIP(Full)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>548.62</td>
<td>548.62(0%)</td>
<td>410.00(25.3%)</td>
<td>410.00(25.3%)</td>
<td>410.00(25.3%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>926.51</td>
<td>877.50(5.3%)</td>
<td>762.00(17.8%)</td>
<td>762.00(17.8%)</td>
<td>762.00(17.8%)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1265.37</td>
<td>1177.01(6.9%)</td>
<td>1049.44(17.1%)</td>
<td>977.16(22.8%)</td>
<td>977.16(22.8%)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Dataset S3</th>
<th>LT&lt;sub&gt;max&lt;/sub&gt;</th>
<th>RIPS</th>
<th>AARIP(Last)</th>
<th>AARIP(1 Ahead)</th>
<th>AARIP(2 Ahead)</th>
<th>AARIP(Full)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>232.22</td>
<td>232.22(0%)</td>
<td>180(22.5%)</td>
<td>180(22.5%)</td>
<td>180(22.5%)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>372</td>
<td>360.42(3.1%)</td>
<td>300(19.4%)</td>
<td>300(19.4%)</td>
<td>300(19.4%)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>480.00</td>
<td>462.75(3.6%)</td>
<td>393.91(17.9%)</td>
<td>362.86(24.4%)</td>
<td>340(29.2%)</td>
<td></td>
</tr>
</tbody>
</table>

### 2.5 Concluding Remarks

In this chapter a general methodology is proposed based on robust optimization for an inventory control problem subject to uncertain lead times and fixed demand. By parameterizing the lead time at each period we are able to construct a column-wise binary uncertainty set. To the best of our knowledge, this is the first paper to propose a robust optimization-based method to mitigate lead time uncertainty. The column-wise robust optimization problem is solved under three policies: static, affinely adjustable and basestock. A summary of insights is as follows:

- In our computational experiments the static policy outperforms the basestock policy, however the outperformance diminishes as the lead time variability increases.

- A simulation study shows that the robust basestock policy performs well compared to several heuristics and the robust basestock level may be greater...
than or less than the heuristic parameters depending on the instance.

- Our results show that affinely adjustable policies may perform marginally better than static policies, however having information about order arrivals for several future periods can produce large improvements.
Chapter 3  
Robust Inventory Control Under Uncertain Lead Times And Uncertain Demand

In this chapter we propose several robust optimization-based models for an inventory control problem under uncertain demands and uncertain lead times. We consider static and basestock policies. First, three uncertainty sets for robust optimization problems are compared: the budget uncertainty set, a set based on the central-limit theorem (CLT), and a set based on statistical hypothesis tests (SHT). We illustrate the SHT approach for solving a newsvendor problem. We then develop several robust optimization models for a periodic review, finite horizon inventory control problem subject to uncertain demands and uncertain lead times. We solve them using the epigraph reformulation method and we extend the Benders’ approach from Chapter 2 to handle both demand and lead time uncertainty. Our numerical results demonstrate that the epigraph reformulation approach is overly conservative even when costs are stationary. Several lead time uncertainty sets are proposed, and the results are compared to the sample average approximation (SAA) method. The subproblems of the Benders’ method for the budget and CLT-based models turn out to be mixed-integer linear programs (MILP) since the uncertainty set is polyhedral.
3.1 Introduction

In supply chain, future demand may be difficult to predict and lead times may be uncertain because of reasons such as variable processing times at the supplier or transportation delays. There is a rich body of literature dealing with inventory control under demand uncertainty, and there is a much smaller body of literature concerning inventory control under lead time uncertainty.

RO has been studied in supply chain problems showing promising computational results for problems under demand uncertainty (e.g., see Ben-Tal et al. 2005, Bertsimas and Thiele 2006, Bienstock and Ozbay 2008). In these papers robust optimization is cast as a distribution-free approach that does not take into account information about probability distributions or available data on past parameter realizations. However, a new approach to RO is expressed in Bertsimas et al. (2013) where statistical hypothesis tests are used to leverage information from sample data to build uncertainty sets, and an approach in Bandi and Bertsimas (2012) proposes a method to build uncertainty sets using the conclusions of probability theory (i.e., the central limit theorem). While a sound theory has been developed for these two approaches, they have not been applied to problems such as inventory control to date. In this chapter, these approaches are applied to inventory control under demand and lead time uncertainty.

Over the past decade there have been many papers that have applied RO to problems in inventory control, as discussed in Chapter 1. Bertsimas and Thiele (2006) develop an RO approach for inventory control where demand is uncertain using a budget uncertainty set first derived in Bertsimas and Sim (2004). The approach involves applying RO to a reformulation of the inventory problem. Recently, Gorissen and Den Hertog (2013) discuss the conservatism of such a formulation. Despite the conservatism many papers take this approach in applying RO to problems in operations management (e.g., Ben-Tal et al. 2005, José Alem and Morabito 2012, Wei et al. 2011, Aouam and Brahim 2013). Bienstock and Ozbay (2008) alleviate this conservatism for an inventory problem under demand uncertainty where RO is applied to the true problem and solve it using a Benders’ approach.

In the above papers involving demand uncertainty, the supply-side is assumed to be deterministic and order lead times are assumed to be either zero or fixed. There have been far fewer papers on robust optimization on supply uncertainty,
and they mainly deal with yield uncertainty and raw material supply uncertainty. Yield uncertainty is considered from a robust optimization perspective in Bohle et al. (2010) and Alvarez and Vera (2011). Similarly, Varas et al. (2014) formulate a static RO model for a sawmill planning problem under demand and raw material supply uncertainty and use a budget uncertainty sets.

In this chapter we apply these approaches to an inventory control problem under uncertain demand and uncertain lead times. The contributions of this chapter are as follows:

- We propose a data-driven RO model for an inventory control problem under uncertain demand and lead time. While there are several results in literature for robust inventory control under uncertain demand, this is the first that jointly considers uncertain demand and uncertain lead time from an RO perspective.

- This study is the first to use a Benders’ approach to handle robust optimization problems under CLT and SHT uncertainty sets. This approach has been shown to produce less conservative solutions than the alternative method in literature of robustifying the epigraph formulation of inventory control problems (Gorissen and Den Hertog 2013).

- Using a computational study for an inventory control problem we compare the performance of several uncertainty set modeling approaches (budget, CLT, and SHT). This is the first attempt to compare the performance of these sets.

In Section 3.2 we describe the RO methodology including several modeling approaches for uncertainty sets. We present the robust inventory problem in Sections 3.3 and 3.4. The solution approach is described in Section 3.5. In Section 3.6, we extend the approach to involve more general uncertainty sets. Computational results follow in Section 3.7. Concluding remarks are made in Section 3.8.
3.2 Robust Optimization

In this section we describe the RO methodology and several uncertainty sets that are used in this chapter. Consider the following nominal linear program:

\[
\begin{align*}
\min_{x} & \quad c^\top x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]  

(3.1)

where the decision variable is \( x \in \mathbb{R}^n \), and parameters are \( A \in \mathbb{R}^{m \times n} \), \( c \in \mathbb{R}^n \), and \( b \in \mathbb{R}^m \). The robust version of Formulation (3.1) is the following:

\[
\begin{align*}
\min_{x} \max_{(A,b,c) \in U} & \quad c^\top x \\
\text{subject to} & \quad Ax \leq b
\end{align*}
\]  

(3.2)

where \( U \) is an uncertainty set. The solution to Formulation (3.2) remains feasible for any realization of data uncertainty within \( U \), and it achieves the best worst-case objective value. This is a semi-infinite optimization problem because there are infinitely many constraints, however for many uncertainty sets the problem can be reformulated to be tractably solvable (these reformulations are often called “robust counterparts” in the literature). Next, several uncertainty sets are described.

3.2.1 Budget uncertainty sets

Budget uncertainty sets were first introduced in Bertsimas and Sim (2004). Without loss of generality, consider a problem where the only uncertain parameters are the elements of \( A \) where \( a_{ij} \in [\underline{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij} + \hat{a}_{ij}] \). Define the scaled deviation from the nominal value as \( z_{ij} = (a_{ij} - \bar{a}_{ij}) / \hat{a}_{ij} \), so that \( |z_{ij}| \leq 1 \). The cumulative deviation from the nominal value for each row \( i \) is bounded by a budget parameter, that is, \( \sum_{j=1}^{n} z_{ij} \leq \Gamma_i \). The budget uncertainty set is the following:

\[
U_i^b = \{(a_{i1}, \ldots, a_{in}) : |z_{ij}| \leq 1 \ \forall j, \sum_{j=1}^{n} z_{ij} \leq \Gamma_i\}
\]  

(3.3)
An advantage of this uncertainty set is that the robust counterpart of a linear program can be reformulated as a linear program (see Bertsimas and Sim (2004) for details). A disadvantage of this method is that it is unclear what value $\Gamma_i$ should be. One approach is to show the full spectrum of setting the budget from $\Gamma_i = 0$ (which is equivalent to the nominal formulation) to $\Gamma_i = J_i$, where $J_i$ is the maximum number of parameters in row $i$ that may vary using sensitivity analysis (Bertsimas and Thiele 2006, José Alem and Morabito 2012). This allows the decision maker to examine the robustness tradeoff, or “the price of robustness' (Bertsimas and Sim 2004). Another approach is to set the upper and lower bounds of the $a_{ij}$ parameters according to a $(1 - \alpha)\%$ confidence interval and then set the budget parameter to a low value if the confidence interval is very wide since it is unlikely that many of the $a_{ij}$'s will reach their bounds, or set the budget parameter high if the confidence interval is narrower (Denton et al. 2010).

As a remark, this approach does not require distributional information or past data, and if these are available it is not straightforward how to use such information.

### 3.2.2 Central limit theorem-based uncertainty sets

The CLT-based uncertainty set is introduced in Bandi and Berstimas (2012) where the authors replace the axioms of probability theory and the concept of random variables with uncertainty sets derived from conclusions of probability, in particular the Central Limit Theorem. Consider i.i.d. random variables $X_i$, $i = 1, \ldots, n$ with mean $\mu$ and variance $\sigma^2$, and define the sum $Y_n = \sum_{i=1}^{n} X_i$. The Central Limit Theorem states that, as $n \to \infty$, $(Y_n - n\mu)/\sigma \sqrt{n}$ is asymptotically distributed as a standard normal variable. The CLT-based uncertainty set is based on the $z$-test where the input to this $z$-test are the uncertainty realizations. The CLT-based uncertainty set is the following:

$$U^{CLT} = \{(X_1, \ldots, X_n) : |\sum_{i=1}^{n} X_i - n\mu| \leq \Gamma \sigma \sqrt{n}\} \quad (3.4)$$

Using standard normal tables we can find probabilities, e.g.,

$$\mathbb{P}(|(Y_n - n\mu)/\sigma \sqrt{n}| \leq 2) \approx 0.95$$
so the $\Gamma$ parameter can be specified by the modeler to satisfy some asymptotic probability guarantee (e.g., in this example $\Gamma = 2$, satisfies a 95% guarantee). This property is the main advantage of this uncertainty set. Also, like the budget uncertainty set, the CLT-based uncertainty set is polyhedral which allows the robust counterpart of a linear program to be formulated as a linear program. The disadvantage of this method is that the modeler must assume the data belongs to a specific probability distribution, and furthermore, the approach does not provide a finite-sample probability guarantee.

Also note that the budget uncertainty set is directly applied to the $A$ matrix of the uncertain problem while the CLT-based uncertainty set is applied to parameters which behave like a sequence of i.i.d. random variables. While this approach can easily be applied to problems such as the inventory control problem in this paper where there is a sequence of demands and lead times, it will not be applicable in general, as is the budget approach.

### 3.2.3 Statistical hypothesis test-based uncertainty sets

SHT-based uncertainty sets are introduced in Bertsimas et al. (2013) and involve using sample data and statistical hypothesis tests to design the uncertainty sets to construct $U$ such that the constraint

$$a^\top x^* \leq b, \quad \forall a \in U$$

(3.5)

is guaranteed to hold with probability $1 - \epsilon$. Here, we only present the uncertainty set based on the $\chi^2$ test. It is assumed a priori that the true distribution of the data, $P^*$, has known finite support on $\{\hat{a}_0, \ldots, \hat{a}_{n-1}\}$. The null hypothesis is $p_i = P_0(\hat{a} = \hat{a}_i)$ for $i = 0, \ldots, n-1$. The empirical distribution is given by $\hat{p}_i$ which is calculated using sample data. Consider the following set:

$$P_{\chi^2} = \{p : e^\top p = 1, \ p_i \geq 0 \ i = 0, \ldots, n-1, \sum_{i=0}^{n-1} \frac{(p_i - \hat{p}_i)^2}{p_i} \leq \frac{1}{N} \chi^2_{n-1, 1-\delta}\}$$

(3.6)

where $\chi^2_{n-1, 1-\delta}$ is the critical value of the chi-squared distribution with $n-1$ degrees of freedom at level $\delta$. This set represents all values of $p$ that satisfy the null
hypothesis of the chi-squared goodness of fit test with probability $1 - \delta$.

Bertsimas et al. (2013) show that this set is equivalent to the following set which only contains linear and second-order cone constraints.

$$
P_{\chi^2} = \{ p : \exists t \in \mathbb{R}^n \ e^T p = 1, p_i \geq 0 \ i = 0, \ldots, n - 1, \\
\sum_{i=0}^{n-1} t_i \leq \frac{1}{2N} \chi^2_{n-1,1-\delta}, \\
\| \left( \frac{2(p_i - \hat{p}_i)}{p_i - 2t_i} \right) \|_2 \leq p_i + 2t_i, \ i = 0, \ldots, n - 1 \}$$

Finally, the set

$$
U_{\chi^2} = \{ a \in \mathbb{R}^d : a = \sum_{j=0}^{n-1} q_j \hat{a}_j, q \leq \frac{1}{\epsilon} p, \ p \in P_{\chi^2}, \\
e^T q = 1, q_i \geq 0 \ i = 0, \ldots, n - 1 \}
$$

implies that Constraint (3.5) is satisfied with probability of at least $1 - \epsilon$ (see details in Bertsimas et al. (2013)). The set $U_{\chi^2}$ can be reformulated as a second-order cone program so this set is computationally tractable. The parameters $N, \delta,$ and $\epsilon$ can be used as levers to control the conservatism of the uncertainty set: the first two adjust the values of $p$ allowed in the uncertainty set corresponding to the $\chi^2$ goodness of fit test, and the latter controls the allowed constraint violation probability.

SHT-based uncertainty sets are based on empirical data, as opposed to CLT-based uncertainty sets which are based on a specific hypothesis test (in particular the $z$-test) where the input to this $z$-test are the uncertainty realizations. The main advantage of using SHT-based uncertainty sets is that they are based on empirical data, which is not the case for the other two types of uncertainty sets.

Table 3.1 summarizes the key differences between the uncertainty sets described above.
Table 3.1. Several key differences between uncertainty sets

<table>
<thead>
<tr>
<th>Parameters controlling conservatism</th>
<th>Budget</th>
<th>CLT</th>
<th>SHT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum of uncertain parameters in the constraint required</td>
<td>✓</td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Data can be directly used as input</td>
<td></td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Distributional information required</td>
<td></td>
<td>✓</td>
<td></td>
</tr>
<tr>
<td>Structure of uncertainty set</td>
<td>Polyhedral</td>
<td>Polyhedral</td>
<td>Second-order cones</td>
</tr>
</tbody>
</table>

3.3 Robust Newsvendor Problem

In this section, a newsvendor problem is used to illustrate the implementation of SHT-based uncertainty sets for robust inventory control. In the (single stage) newsvendor problem discussed here, the planner first orders $u$ units of product (which incurs a variable cost $c$) at the beginning of the day and then the demand $d$ is realized. Either a holding cost $h$ is incurred for each unit of product in inventory that is not satisfied by the demand or a backorder cost $b$ is incurred for each unit of inventory that is unable to be satisfied by the order.

There are a number of results in literature on the newsvendor problem where the demand distribution is unknown, as it is one of the most fundamental problems in operations management (we refer the reader to Khouja (1999) for a comprehensive review of the literature). There are several proofs in literature on the optimal ordering quantity to maximize expected profit under the worst distribution when the mean and variance of demand are known (Scarf et al. 1958, Gallego and Moon 1993). Several researchers have approached this problem using a distributionally robust optimization approach with a minimax regret modeling formulation (Perakis and Roels 2008, Yue et al. 2006, Zhu et al. 2013). In this section, we implement a data-driven robust optimization approach using SHT-based uncertainty sets.

We assume demand belongs to a SHT-based uncertainty set, $d \in U^{x^2}$. A linear
programming formulation is the following:

\[
\begin{align*}
\text{minimize } & cu + y \\
\text{subject to } & \\
y & \geq h(u - d), \quad \forall d \in U^x \\
y & \geq b(d - u), \quad \forall d \in U^x \\
u & \geq 0
\end{align*}
\] (3.9)

Note that, because the uncertainty set is the intersection of linear and second-order cone constraints, the above formulation has infinitely many constraints. Next we use duality to reformulate this into an LP with finitely many constraints (Bertsimas and Sim 2004).

**Proposition 3.3.1.** The following second-order cone program (SOCP) is the robust counterpart of the Newsvendor Problem (3.9):

\[
\begin{align*}
\text{min} & \quad cu + y \\
\text{subject to } & \\
y & \geq h(u - d), \quad \forall d \in U^x \\
y & \geq b(d - u), \quad \forall d \in U^x \\
u & \geq 0 \\
-w_k^p - w_k^q - \frac{1}{\epsilon}w_{kj}^p - 2y_{1kj} - y_{2kj} - z_{kj} = 0, & k = 1, 2, \quad j = 0, \ldots, n - 1 \quad \text{(3.10)} \\
w_k^1 + 2y_{2kj} - 2z_{kj} = 0, & k = 1, 2, \quad j = 0, \ldots, n - 1 \\
a_j - v_1^q - w_{ij} + w^p q_{1j} = 0, & j = 0, \ldots, n - 1 \\
-a_j - v_2^q - w_{2j}^p + w^p q_{2j} = 0, & j = 0, \ldots, n - 1 \\
\left\| \begin{bmatrix} y_{1kj}^1 \\ y_{1kj}^2 \end{bmatrix} \right\|_2 \leq z_{kj}, & k = 1, 2, \quad j = 0, \ldots, n - 1
\end{align*}
\]

**Proof.** In order for the first set of constraints in Model (3.9) to be feasible for all
demand in the uncertainty set $U^2$, we must solve the following subproblem:

$$
\begin{align*}
\max_{d \in U^2} \{ hu - hd \} \\
= hu + \max_{d \in U^2} \{-hd\} \\
= hu - h \left( \min_{d \in U^2} \{ d \} \right)
\end{align*}
$$

(3.11)

Writing out this minimization problem, we have

$$
\begin{align*}
\min \sum_{j=0}^{n-1} q_j a_j \\
\text{subject to} \\
\sum_{j=0}^{n-1} p_j &= 1 \\
\sum_{j=0}^{n-1} q_j &= 1 \\
-p_j &\leq 0, \quad j = 0, \ldots, n-1 \\
-q_j &\leq 0, \quad j = 0, \ldots, n-1 \\
p_j - \frac{1}{\epsilon} q_j &\leq 0, \quad j = 0, \ldots, n-1 \\
\sum_{j=0}^{n-1} t_j &\leq \frac{1}{2N} \lambda_{n-1,1-\delta}^2 \\
\| \begin{pmatrix} 2(p_j - \hat{p}_j) \\ p_j - 2t_j \end{pmatrix} \|_2 &\leq p_j + 2t_j, \quad j = 0, \ldots, n-1
\end{align*}
$$

(3.12)

We define dual variables $v_1^p$, $v_1^q$ for the equality constraints, $w_1^p \in \mathbb{R}_+^n$ for the $p$ inequalities, $w_1^q \in \mathbb{R}_+^n$ for the $q$ inequalities, $w_1^{pq} \in \mathbb{R}_+^n$ for the inequalities containing both $p$ and $q$, $w_1^t \in \mathbb{R}_+$ for the $t$ inequality, and $(y_{ij}^1, y_{ij}^2, z_{ij}) \in \mathbb{R}^3$ for each second-order cone constraint. It is easy to show using the Lagrangian that the following is the dual:
\[
\max v_1^p + v_1^q - \frac{1}{2N} \chi^2 w_1^t + \sum_j 2\hat{p}_j y_{1j}^1
\]

subject to

\[
- v_1^p - w_{1j}^p - \frac{1}{\epsilon} w_{1j}^{pq} - 2y_{1j}^1 - y_{1j}^2 - z_{1j} = 0, \quad j = 0, \ldots, n - 1
\]

\[
w_1^t + 2y_{1j}^2 - 2z_{1j} = 0, \quad j = 0, \ldots, n - 1
\]

\[
a_j - v_1^q - w_{1j}^q + w_{1j}^{pq} = 0, \quad j = 0, \ldots, n - 1
\]

\[
\|\begin{pmatrix} y_{1j}^1 \\ y_{1j}^2 \end{pmatrix}\|_2 \leq z_{1j}, \quad j = 0, \ldots, n - 1
\]

\[
w_{1j}^p, w_{1j}^q, w_{1j}^{pq}, w_1^t \geq 0, \quad j = 0, \ldots, n - 1
\]

Similarly for the second set of constraints, we must solve the following subproblem:

\[
\max_{d \in U^x_2} \{bd - bu\}
\]

\[
= -bu + b(\max_{d \in U^x_2} \{d\})
\]

\[
= -bu - b(\min_{d \in U^x_2} \{-d\}) \tag{3.14}
\]

In the robust counterpart formulation, there are two sets of dual variables: one for the subproblem from the holding constraint (index \(k = 1\)), and the other for the subproblem for the backorder constraint (index \(k = 2\)).

This SOCP can be solved in polynomial time using commercial solvers such as MOSEK or CPLEX.

### 3.3.1 Illustrational example

In this section a small illustrational example is presented to demonstrate how SHT-uncertainty sets can control conservatism of solutions for the newsvendor problem. Consider an instance where \(c = 1\), \(h = 5\), \(b = 10\). We assume demand belongs to the set \(\{0, 1, 2, 3, 4\}\). Using a sample of 500 demands, we compute \(\hat{p}\) (Figure 3.1).
Using $\alpha = 0.05$, the chi-squared value is equal to 9.488. Recall that the constraints are satisfied with probability $1 - \epsilon$. In Figure 3.2 the bounds on the demand uncertainty set are shown for the full range of $\epsilon$. For very high probability guarantees (97% and higher), the uncertainty set of this instance is $d \in [0, 4]$, however the uncertainty set shrinks for $\epsilon \geq 0.04$. When $\epsilon = 1$ the uncertainty set is $d \in [2.023, 2.176]$. This set can be made even smaller by setting $\alpha$ to a lower level since this acts as another lever to control the size of the uncertainty set, which is a control on the level of conservatism of the solution. Figure 3.3 shows the worst-case cost and order decisions for the full range of $\epsilon$. 

**Figure 3.1.** Empirical probability distribution of demand
As a remark, the other uncertainty sets described thus far provide no benefit for the newsvendor problem. Budget uncertainty sets, which are used to control the worst-case realization of sums of uncertain parameters, can only adjust conservatism in a simple way using upper and lower bounds for the demand parameter. This simple case is equivalent to box uncertainty and it is because this problem has only
a single uncertain parameter in each constraint. CLT-based uncertainty sets cannot be used to control conversatism in this newsvendor problem for the same reason.

### 3.4 Robust Multi-Period Inventory Model

The problem setup is as follows. We consider a periodic review inventory control problem for a single facility over a finite time horizon of \( T \) time periods. The planner places an order, \( u_i \), at the beginning of the time period \( i \) which incurs a variable cost \( c \). Then demand, \( d_i \), occurs during the period. Then order \( u_i \) arrives if the lead time is zero; if the lead time is \( LT(i) \), then \( u_i \) arrives in \( LT(i) \) periods. At the end of each period costs are incurred for holding positive inventory (holding cost, \( h \)), or for negative inventory (backorder cost, \( b \)). The initial inventory level is \( x_0 \).

The lead time parameter \( \delta_{ik} \) is defined as follows:

**Definition 3.4.1.**

\[
\delta_{ik} = \begin{cases} 
0 & LT(i) > k - i \\
1 & \text{otherwise}
\end{cases}
\]  

(3.15)

where \( LT(i) \) represents the lead time (measured in integer number of periods) of the order in period \( i \), where \( i = 1, \ldots, T \) and \( k \leq i \). If \( \delta_{ik} = 1 \), the order placed in period \( i \) has arrived by period \( k \). If \( \delta_{ik} = 0 \), the order placed in period \( i \) has not yet arrived by period \( k \).

Assuming fixed demand and fixed lead time, the inventory level at the end of period \( t \) is \( x_{t+1} \), and the inventory balance constraints are the following:

\[
x_{t+1} = x_0 + \sum_{i=1}^{t} (\delta_{it} u_i - d_i)
\]

(3.16)

The inventory problem is to minimize the total cost \( Z \) which can be expressed as follows:

\[
Z = \min_{u \geq 0} \left\{ \sum_{t=1}^{T} \left( cu_t + \max\{hx_{t+1}, -bx_{t+1}\} \right) \right\}
\]

(3.17)

Problem (3.17) is nonlinear because of the max function in the objective. It is equivalent to the following linear program (called the epigraph formulation):
The solution of this problem when demand is known and lead times are zero is trivial where \( u_t \) is set to \( d_t \) for each time period \( t \). However, when lead times and demand are uncertain the solution is unclear. In this chapter we model the uncertainty by allowing the demand and lead time of each order to belong to several uncertainty sets and solving a min-max problem.

### 3.4.1 Demand uncertainty sets

We model demand using the budget uncertainty set as follows:

\[
\mathcal{D}^b_t = \{ \mathbf{d} \in \mathbb{R}^t, \mathbf{z} \in \mathbb{R}^t : d_i = \bar{d}_i + z_i \tilde{d}_i, \ i = 1, \ldots, t; \\
-1 \leq z_i \leq 1, \ i = 1, \ldots, t; \\
\sum_{i=1}^{t} |z_i| \leq \Gamma_t, \} \tag{3.19}
\]

The budget parameter \( \Gamma_t \) is set to \( \alpha t \) where \( \alpha \in [0, 1] \). When \( \alpha = 0 \) the uncertainty set is reduced to \( d_t = \bar{d}_t \forall i \). When \( \alpha = 1 \) the demand uncertainty set is most conservative, allowing all demands to reach their worst-case values simultaneously.

### 3.4.2 Lead time uncertainty sets

For our first model of lead time uncertainty of the order placed at time period \( t \), we use the following uncertainty set, similar to the set introduced in Chapter 2:
\[ \mathcal{L}_t = \{ \delta_t \in \mathbb{R}^{T-t} : 0 \leq \delta_{ti} \leq 1, \ i = t, \ldots, T; \]
\[ \delta_{ti} \leq \delta_{t,i+1}, \ i = 1, \ldots, T - t - 1; \]
\[ \delta_{t,t+LT_{\text{max}}} = 1 \} \]

where \( \delta_t := (\delta_{tt}, \ldots, \delta_{tT}) \). The uncertainty set allows the possible realizations of uncertain order lead times at time period \( t \) to be between zero and \( LT_{\text{max}} \). A major structural difference between this uncertainty set and the demand budget uncertainty set is that \( \mathcal{L}_t \) is a column-wise set, while the demand uncertainty set is row-wise. As another remark about the set \( \mathcal{L}_t \), note that this uncertainty set relaxes the binary structure of the \( \delta_{ij} \) terms (as originally defined in Definition (3.15)). (The difference between this set and the uncertainty set from Chapter 2 is that the \( \delta_{ij} \) parameters are continuous in this set). The implication of this relaxation is that partial orders may be delivered. For example, \( \delta_{11} = 0.5 \) means that half of the order that was placed in period 1 is delivered with zero lead time. In many supply chains partial deliveries may occur if a supplier cannot fulfill the entire order on time and ships a portion early or if a supplier has longer lead times and the customer uses an alternative supplier to satisfy immediate needs. In order to use standard robust optimization techniques involving duality to derive a robust counterpart it is required that the uncertainty sets are convex. Therefore, we introduce the lead time parameter relaxation into the uncertainty set instead of imposing binary values.

### 3.4.3 Robust Counterpart of Inventory Control Problem Under Uncertain Demand and Lead Time

The robust problem involves solving the epigraph formulation where the constraints must hold for any realization of uncertainty:
\[ Z = \min_{u, y} \left\{ \sum_{t=1}^{T} \left( cu_t + y_t \right) \right\} \]

subject to

\[ y_t \geq h(x_0 + \sum_{i=1}^{t} (\delta_{it} u_i - d_i)), \quad t = 1, \ldots, T; \quad \forall d \in \mathcal{D}_t, \quad \forall \delta_i \in \mathcal{L}_i, \quad i = 1, \ldots, t \]

\[ y_t \geq -b(x_0 + \sum_{i=1}^{t} (\delta_{it} u_i - d_i)), \quad t = 1, \ldots, T; \quad \forall d \in \mathcal{D}_t, \quad \forall \delta_i \in \mathcal{L}_i, \quad i = 1, \ldots, t \]

\[ u_t \geq 0, \quad t = 1, \ldots, T \]

(3.21)

We now show that the inventory problem under demand uncertainty and lead time uncertainty can also be formulated as a linear program.

**Proposition 3.4.2.** The robust counterpart of Model (3.21) is the following linear program:

\[ Z = \min_{u, y} \left\{ \sum_{t=1}^{T} \left( cu_t + y_t \right) \right\} \]

subject to

\[ y_k \geq h\left(x_0 - \sum_{i=1}^{k} d_i + \sum_{i=1}^{k} s^H_{ik} - \sum_{i=1}^{k \leq LT_{\max}} v^H_{ik} + \Gamma_k q_k + \sum_{i=1}^{k} r_{ik} \right), \quad k = 1, \ldots, T \]

\[ y_k \geq -b\left(x_0 - \sum_{i=1}^{k} d_i - \sum_{i=1}^{k} s^B_{ik} + \sum_{i=1}^{k \leq LT_{\max}} v^B_{ik} - \Gamma_k q_k - \sum_{i=1}^{k} r_{ik} \right), \quad k = 1, \ldots, T \]

\[ q_k + r_{ik} \geq \bar{d}_i, \quad i = 1, \ldots, k, \quad k = 1, \ldots, T \]

\[ s^H_{ik} - v^H_{ik} \geq u_i, \quad i = 1, \ldots, k, \quad k - i \geq LT_{\max}, \quad k = 1, \ldots, T \]

\[ s^H_{ik} \geq u_i, \quad i = 1, \ldots, k, \quad k - i < LT_{\max}, \quad k = 1, \ldots, T \]

\[ s^B_{ik} - v^B_{ik} \geq -u_i, \quad i = 1, \ldots, k, \quad k - i \geq LT_{\max}, \quad k = 1, \ldots, T \]

\[ s^B_{ik} \geq -u_i, \quad i = 1, \ldots, k, \quad k - i < LT_{\max}, \quad k = 1, \ldots, T \]

\[ q_k, u_k \geq 0, \quad k = 1, \ldots, T \]

(3.22)
Proof. In order for the holding constraint in Model (3.21) at time period \( k \) to be feasible for any demand or lead time realization within the respective uncertainty set, it must satisfy

\[
y_k \geq \max_{\delta \in \mathcal{D}_k, \delta \in \mathcal{L}} h(x_0 + \sum_{i=1}^{k} (\delta_{ik} u_i - d_i))
\]

\[
= hx_0 + h(\max_{\delta \in \mathcal{L}} \sum_{i=1}^{k} (\delta_{ik} u_i) + \max_{d \in \mathcal{D}_k} \sum_{i=1}^{k} d_i)
\]

\[
= h(x_0 - \sum_{i=1}^{k} \bar{d}_i) + h(\max_{\delta \in \mathcal{L}} \sum_{i=1}^{k} (\delta_{ik} u_i) + \max_{d \in \mathcal{D}_k} \sum_{i=1}^{k} (-z_i \bar{d}_i))
\]

(3.23)

Bertsimas and Thiele (2006) show that the robust inventory problem under demand uncertainty (using budget uncertainty sets) can be formulated as a linear program by applying principles of duality to the demand subproblem. The demand subproblem for this holding constraint \( k \) is the following linear program (dual variables in parentheses):

\[
\max \sum_{i=1}^{k} z_i \bar{d}_i
\]

subject to

\[
z_i \leq 1, \quad i = 1, \ldots, k \quad (r_{ik})
\]

\[
\sum_{i=1}^{k} z_i \leq \Gamma_k \quad (q_k)
\]

\[
z_i \geq 0, \quad i = 1, \ldots, k
\]

Its dual is the following:

\[
\min_{q, r} \Gamma_k q_k + \sum_{i=1}^{k} r_{ik}
\]

subject to

\[
q_k + r_{ik} \geq \bar{d}_i, \quad i = 1, \ldots, k
\]

\[
q_k \geq 0, \quad r_{ik} \geq 0, \quad i = 1, \ldots, k;
\]

(3.25)

Similarly, the lead time subproblem for holding constraint \( k \) is the following linear program (dual variables in parentheses).
\[
\max_{\delta} \delta \sum_{i=1}^{k} (\delta_{ik} u_i)
\]

subject to
\[
\delta_{ik} \leq 1, \quad i = 1, \ldots, k \\
-\delta_{ik} \leq -1, \quad \forall (i, k) \text{ s.t. } k - i \geq LT_{\max} \\
\delta_{ik} \geq 0, \quad i = 1, \ldots, k
\]

(3.26)

Formulation (3.26) involves the lead time parameters \( \delta_{ik}, \ldots, \delta_{kk} \). Thus the constraints of type \( \delta_{ti} \leq \delta_{t,i+1} \) from the lead time uncertainty set (3.20) are not involved in this subproblem. The dual of this problem is

\[
\min_{s, v} \left( \sum_{i=1}^{k} (s_{ik}) - \sum_{i \leq k - LT_{\max}} (v_{ik}) \right)
\]

subject to
\[
s_{ik} - v_{ik} \geq u_i, \quad i = 1, \ldots, k, \quad k - i \geq LT_{\max} \\
s_{ik} \geq u_i, \quad i = 1, \ldots, k, \quad k - i < LT_{\max} \\
s_{ik}^H, v_{ik}^H \geq 0, \quad i = 1, \ldots, k
\]

(3.27)

The lead time subproblem for backorder constraint \( k \) is similar. We substitute the duals for these subproblems back into the original problem and obtain the robust counterpart.

\[ \square \]

### 3.5 Solution Methods

Two approaches are used to solve this problem in this paper. First, the epigraph reformulation (ER) approach involves solving Model (3.22) directly using an optimization solver such as CPLEX or GUROBI which can be solved quickly since it is a linear programming problem. However, it may produce solutions that are too conservative. To find less conservative solutions we solve the “true min-max problem” using the Adversarial approach. The “true min-max problem” is the following:
\[
\min \left\{ \max_{u \geq 0, \delta \in \Omega, d \in D} \sum_{t=1}^{T} \left( c u_t + \max \{ h \tilde{x}^t, -b \tilde{x}^t \} \right) \right\}
\]  
(3.28)

where \( \delta := \{ \delta_{ij} : i = 1, \ldots, T, j = i, \ldots, T \} \) and \( \tilde{x}^t = x_0 + \sum_{i=1}^{t} (\delta_{it} u_i - d_i) \)

This "true min-max problem" is solved under two ordering policies: static, and basestock, where an order-up-to level is calculated and an ordering scheme following Equation (2.3) is followed. Model (3.28) bears a close resemblance to Model (2.9) (called the static policy) from Chapter 2; however, in Chapter 2 the demand was assumed to be fixed, and here the demand is uncertain. The Adversarial approach involves maintaining a finite list of demand scenarios and minimizing the maximum cost over these scenarios in the Decision Maker Subproblem (DS). Worst-case demand scenarios are generated in the Adversarial subproblem (AS).

Algorithm 2 is a high level description of the approach. For the problem discussed in this chapter, the DS is formulated as a linear program for the static policy and a mixed integer program for the basestock policy, and the AS is formulated as a mixed-integer linear program for both static and basestock policies.

### Algorithm 2 Adversarial Approach

1. **Initialization Step:**
   Set \( \tilde{\Omega} = \emptyset \).
   Choose initial lead time and demand scenario \( \Omega^0 \) and add it to \( \tilde{\Omega} \).
   Set \( i = 0, L = 0, \) and \( U = \infty \).

2. **Decision Maker Subproblem:**
   Set \( L = \) Objective value from DM formulation
   For static policy problem: Set \( \bar{u} = \) Solution from DM formulation
   For basestock policy problem: Set \( \bar{\sigma} = \) solution from DM formulation

3. **Adversarial Subproblem:**
   Set \( U = \min \{ U, \) Objective value from AP Formulation \} \)
   Set \( (\bar{\delta}, \bar{d}) = \) Solution from AP Formulation with order vector \( \bar{u}, \) and set \( \Omega^{i+1} = (\bar{\delta}, \bar{d}) \)

4. **Terminate if** \( U - L < \epsilon \)
   Otherwise, add \( \Omega^{i+1} \) to \( \Omega \), set \( i = i + 1 \), and return to Step 2.

### 3.5.1 Benders’ Subproblems for Static Policy

For the static policy, the explicit formulations of the Decision Maker Subproblem and Adversarial Problem are the following. The Decision Maker Subproblem for
the static policy is the following linear program:

\[
\min Z \\
\text{s.t.} \quad Z \geq \sum_{i=1}^{T} (cu_i + y_{i,\omega}) \quad \omega = 0, \ldots, |\tilde{\Omega}|
\]

\[
y_{i,\omega} \geq h \left( x_0 + \sum_{i=1}^{t} (\delta_{it}^\omega u_i - d_i^\omega) \right) \quad i = 1, \ldots, T, \ \omega = 0, \ldots, |\tilde{\Omega}| \quad (3.29)
\]

\[
y_{i,\omega} \geq -b \left( x_0 + \sum_{i=1}^{t} (\delta_{it}^\omega u_i - d_i^\omega) \right) \quad i = 1, \ldots, T, \ \omega = 0, \ldots, |\tilde{\Omega}|
\]

\[
u_i \geq 0 \quad i = 1, \ldots, T.
\]

The Adversarial Subproblem for the static policy is the following mixed-integer linear program:

\[
\max_{\delta, I, B, p, d} \sum_{i=1}^{T} (c\tilde{u}_i + I_i + B_i)
\]

\[
\text{s.t.} \quad I_k \geq h \left( x_0 + \sum_{i=1}^{k} (\delta_{ik} \tilde{u}_i - d_i) \right) \quad k = 1, \ldots, T
\]

\[
I_k \leq h \left( x_0 + \sum_{i=1}^{k} (\delta_{ik} \tilde{u}_i - d_i) \right) + M(1-p_k) \quad k = 1, \ldots, T
\]

\[
I_k \leq M(p_k) \quad k = 1, \ldots, T
\]

\[
B_k \geq -b \left( x_0 + \sum_{i=1}^{k} (\delta_{ik} \tilde{u}_i - d_i) \right) \quad k = 1, \ldots, T \quad (3.30)
\]

\[
B_k \leq -b \left( x_0 + \sum_{i=1}^{k} (\delta_{ik} \tilde{u}_i - d_i) \right) + M(p_k) \quad k = 1, \ldots, T
\]

\[
B_k \leq M(1-p_k) \quad k = 1, \ldots, T
\]

\[
\delta_{ik} \in \Omega \quad i = 1, \ldots, T \quad k \geq i
\]

\[
d_i \in \mathcal{D}_i^b \quad i = 1, \ldots, T
\]

\[
p_k \in \{0,1\}
\]

where \(M\) is a large constant.
### 3.5.2 Benders’ Subproblems for Basestock Policy

For the basestock policy, the explicit formulations of the Decision Maker Subproblem and Adversarial Problem are the following. The Decision Maker Subproblem for the basestock policy is the following mixed-integer linear program:

\[
\begin{align*}
\min & \quad Z \\
\text{s.t.} & \quad Z \geq \sum_{i=1}^{T} (cu_i^\omega + y_{i,\omega}) \quad \omega = 0, \ldots, |\tilde{\Omega}| \\
& \quad y_{i,\omega} \geq h \left( x_0 + \sum_{i=1}^{t} (\delta_{it}^u u_i^\omega - d_i^\omega) \right) \quad i = 1, \ldots, T, \omega = 0, \ldots, |\tilde{\Omega}| \\
& \quad y_{i,\omega} \geq -b \left( x_0 + \sum_{i=1}^{t} (\delta_{it}^u u_i^\omega - d_i^\omega) \right) \quad i = 1, \ldots, T, \omega = 0, \ldots, |\tilde{\Omega}| \\
& \quad u_i^\omega \geq 0 \quad i = 1, \ldots, T, \omega = 0, \ldots, |\tilde{\Omega}| \\
& \quad u_i^\omega \leq M z_{i,\omega} \quad i = 1, \ldots, T, \omega = 0, \ldots, |\tilde{\Omega}| \\
& \quad \sigma - \overline{x}_{i+1}^\omega \leq u_i^\omega \leq \sigma - \overline{x}_{i+1}^\omega + M (1 - z_{i,\omega}) \quad i = 1, \ldots, T, \omega = 0, \ldots, |\tilde{\Omega}| \\
& \quad \overline{x}_{k+1} = x_0 + \sum_{i=1}^{k} u_i^\omega - \sum_{i=1}^{k} d_i^\omega \quad k = 1, \ldots, T, \omega = 0, \ldots, |\tilde{\Omega}| \\
& \quad z_{i,\omega} \in \{0, 1\} \quad i = 1, \ldots, T, \omega = 0, \ldots, |\tilde{\Omega}| 
\end{align*}
\]

In Formulation (3.31), where both lead time and demand is uncertain, the orders at each period are indexed by the uncertainty realization \(\omega\). This was not the case in the fixed-demand case from Chapter 2. This is because the basestock policy (defined by Equation (2.3)) is a dynamic policy based on inventory position which is a function of demand.

The Adversarial Subproblem for the basestock policy is the following mixed-integer linear program:
\[
\max_{\delta, I, B, p, d} \sum_{i=1}^{T} (c u_i + I_i + B_i)
\]

s.t. \( u_t \leq \bar{\sigma} - \sum_{i<t} (u_i - d_i) + M (1 - \xi_t), \quad t = 1, \ldots, T \)
\( u_t \geq \bar{\sigma} - \sum_{i<t} (u_i - d_i), \quad t = 1, \ldots, T \)
\( u_t \leq M(\xi_t), \quad t = 1, \ldots, T \)
\( x_{k+1} = x_0 + \sum_{i=1}^{k} (\tilde{u}_{ik} - d_i), \quad k = 1, \ldots, T \)
\( \tilde{u}_{tk} \leq u_t, \quad t = 1, \ldots, T, \quad k = t, \ldots, t + LT_{\max} - 1 \)
\( \tilde{u}_{tk} \geq u_t - M(1 - \delta_{tk}), \quad t = 1, \ldots, T, \quad k = t, \ldots, t + LT_{\max} - 1 \)
\( \tilde{u}_{tk} = u_t, \quad t = 1, \ldots, T, \quad k = t + LT_{\max}, \ldots, T \)
\( I_k \geq h(x_{k+1}) \quad k = 1, \ldots, T \)
\( I_k \leq h(x_{k+1}) + M(1 - p_k) \quad k = 1, \ldots, T \)
\( I_k \leq M(p_k) \quad k = 1, \ldots, T \)
\( B_k \geq -b(x_{k+1}) \quad k = 1, \ldots, T \)
\( B_k \leq -b(x_{k+1}) + M(p_k) \quad k = 1, \ldots, T \)
\( B_k \leq M(1 - p_k) \quad k = 1, \ldots, T \)
\( \delta_{ik} \in \Omega \quad i = 1, \ldots, T, \quad k \geq i \)
\( d_i \in D_i^b \quad i = 1, \ldots, T \)
\( u_k, \tilde{u}_{ik}, I_k, B_k, x_k \geq 0, \quad i = 1, \ldots, T, \quad k = 1, \ldots, T \)
\( p_k \in \{0, 1\}, \quad k = 1, \ldots, T \)
\( \xi_k \in \{0, 1\}, \quad k = 1, \ldots, T \)

where \( M \) is a large constant. The first, second, and third constraints are Big M constraints to force the orders to follow a basestock policy. Recall that inventory level follows Equation (3.16), which involves \( \delta_{tk} u_t \) terms. As long as \( \delta_{tk} \) is binary, the inventory level \( x_t \) at each period \( t \) can be expressed by the fourth through seventh constraints which determine the inventory level, \( x_t \) at each period \( t \), where
the lead time parameter $\delta_{tk}$ acts as an indicator variable. The next two sets of constraints determine the inventory holding cost or backorder cost that is incurred at each time period $t$. The next constraints enforce that the lead time parameters must belong to the lead time uncertainty set, the demand parameters must belong to the lead time uncertainty set, and the variables $p_k$ and $\xi_k$ are binary.

### 3.5.3 On Over conservatism of the Epigraph Reformulation Approach

First, we use a small problem instance (using a time horizon $T = 10$) to illustrate the differences between solutions obtained with the ER approach and the Adversarial approach. The problem instance is described as follows: Demand uncertainty is characterized using the budget demand uncertainty set $\mathcal{D}^b$ where the average demand $d_t = 15$ and $\tilde{d}_t = 5$ at each time period $t$. The costs are $h = 5$, $b = 10$, and $c = 1$.

First consider the case where lead time is fixed at zero and the demand uncertainty budget is at the most conservative level, $\Gamma_t = t$. Table (3.2) shows the ER solution and costs for each period in the second and third columns. The Adversarial solution is shown in the fourth column. The two demand scenarios that were generated in the Adversarial algorithm are shown in the next two columns. The cost of the Adversarial solution for the two demand scenarios is shown in the next two columns. In the final column we show the ER cost when the Adversarial solution is used as the solution for the ER approach.

The total cost for the ER is only 2.2% higher than the total cost for the Adversarial solution, which is not surprising. Bertsimas and Thiele (2006) show that the ER approach works well for an inventory problem under demand uncertainty, when costs are stationary as they are in this example. However, the solutions themselves are quite different. In the ER approach, a constant amount is ordered per period, while in the Adversarial approach a higher amount is ordered in earlier periods tapering off in the last few periods.

Table (3.2) also illustrates how the ER approach can produce overly conservative solutions. The worst case cost for each time period of the ER using the solution from the Adversarial approach is shown in the last column. Notice that the worst-case for period seven corresponds to Demand Scenario 2 where high inventory is carried at
Table 3.2. Comparing ER and Adversarial solution approaches for an instance with demand uncertainty and fixed lead time of zero

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<td>50</td>
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<td>1957.5</td>
<td>3120.00</td>
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<td></td>
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</tr>
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</table>

each period, while the worst-case for period eight corresponds to Demand Scenario 1 where demand is high from periods one through eight, thus requiring backorders in period eight. Obviously these worst-case costs of period seven and eight cannot occur simultaneously. However, the ER approach allows such worst-case costs to occur since the robust optimization methodology is applied to the epigraph formulation of the inventory problem which effectively decouples the worst-case subproblems for each time period.

Figure (3.4) shows the percentage increase in the worst-case cost of the ER approach over the Adversarial approach for the same example for the full range of \( \alpha \) when lead time uncertainty is also present. We characterize lead time uncertainty using the lead time uncertainty set \( \mathcal{L}_t \) for each period \( t \) and vary the parameter \( LT_{max} \) between zero and three. When costs and demands are stationary the ER solution may overestimate the worst-case solution by close to 25% when lead times are between zero and three periods. These numerical results demonstrate that the direct reformulation approach is overly conservative even when costs are stationary. The overconservatism of the ER is increased when average demand is time varying, as shown in Figure (3.5) which is for an example with average demand \( \bar{d}_t \) generated from a discrete uniform distribution, \( U(5, 25) \).
Figure 3.4. The percentage of increase of ER solution over Adversarial solution for an instance with stationary demand.
3.6 Extension to More General Uncertainty Sets

3.6.1 Lead Time Uncertainty Sets: A Budget Approach

The uncertainty sets described so far for the multi-period inventory problem include a budget-type demand uncertainty set and a lead time uncertainty set. The benefit of budget uncertainty sets is the ability to control the conservatism by varying the degree of uncertainty from the nominal case (without uncertainty) to the worst-case (with full uncertainty). The lead time uncertainty set $\mathcal{L}_t$ for the order lead time at period $t$ does not involve budget-like constraints.

It is not straightforward how to develop a budget-type lead time uncertainty set when using the ER approach. Budget constraints involve requiring a sum of deviations to be less than or equal to some budget parameter. In lead time uncertainty set $\mathcal{L}_t$, lead times can deviate from zero to $LT_{max}$. Let us denote $X_t$
as the lead time of an order placed in period \( t \). We remark that \( X_t \) is equal to \( \sum_{j=0}^{LT_{\text{max}}} (1 - \delta_{t,t+j}) \). However, Formulation (3.26), the subproblem for time period \( k \), involves only lead time parameters \( \delta_{ik}, \ldots, \delta_{kk} \) which represents whether orders at periods \( 1, \ldots, k \) have arrived at period \( k \). Within the holding or backorder constraint for time \( t \), there is not information available pertaining to the actual arrival period of previous orders. Therefore a ‘budget’ on a sum of deviations for \((X_1, \ldots, X_k)\) is not directly implementable for the ER approach.

However, next we present a budget-type lead time uncertainty set which can be used in the Adversarial approach since the Adversarial approach considers all uncertainties simultaneously. We assume that the nominal lead time value is the integer value \( \lfloor \frac{LT_{\text{max}}}{2} \rfloor \). For the nominal case (without uncertainty), per Definition (3.4.1), all values of \( \delta_{it} \) are set to zero for \( i < \lfloor \frac{LT_{\text{max}}}{2} \rfloor + t \), and set to one for \( i \geq \lfloor \frac{LT_{\text{max}}}{2} \rfloor + t \). The budget lead time uncertainty set is the following:

\[
\mathcal{L}^b = \{ \delta_t \in \mathcal{L}_t \mid t = 1, \ldots, T: \\
\sum_{t=1}^{\tau} \sum_{j=t}^{t+\lfloor \frac{LT_{\text{max}}}{2} \rfloor-1} \delta_{tj} \leq \Gamma_a^\tau, \quad \tau = 1, \ldots, T \\
\sum_{t=1}^{\tau} \sum_{j=t+\lfloor \frac{LT_{\text{max}}}{2} \rfloor}^{t+LT_{\text{max}}-1} (1 - \delta_{tj}) \leq \Gamma_b^\tau, \quad \tau = 1, \ldots, T \}
\]

where \( \delta_t := (\delta_{ht}, \ldots, \delta_{hT}) \) and \( \Gamma_a^\tau, \Gamma_b^\tau \) are budget parameters. We set \( \Gamma_a^\tau \) to \( \beta \tau \lfloor \frac{LT_{\text{max}}}{2} \rfloor \) and \( \Gamma_b^\tau \) to \( \beta \tau (LT_{\text{max}} - \lfloor \frac{LT_{\text{max}}}{2} \rfloor) \) where \( 0 \leq \beta \leq 1 \). When \( \beta = 0 \), all lead time parameters are forced to their nominal values. When \( \beta = 1 \), then the uncertainty set allows all order lead times to reach their maximum values. The major value of budget uncertainty sets is to allow the decision maker to make tradeoffs between worst-case cost and robustness.

### 3.6.2 Uncertainty Sets: A Central Limit Theorem Based Approach

So far, we have not used distributional information in constructing the lead time uncertainty sets. In practice, there is often historical data that can be used to estimate the distribution of orders. While stochastic optimization methods aim to use the assumed probability distribution directly, we follow the ideas set forth in
Bandi and Berstimas (2012) to use conclusions of probability theory, specifically the central limit theorem (CLT), to model the lead time uncertainty set.

The central limit theorem states that if \( X_i, i = 1, \ldots, n \) are i.i.d. random variables with mean \( \mu \) and variance \( \sigma^2 \), for a sufficiently large \( n \), the random variable \( Z = \sum_{i=1}^{n} X_i \) is distributed as a standard normal. We can use standard normal tables to compute \( \Pr(|Z| \leq \Gamma) \) for a given \( \Gamma \) parameter (For example, \( \Pr(|Z| \leq 2) = 0.95 \)). Then the CLT-uncertainty set is defined as

\[
U_\epsilon = \{(X_1, \ldots, X_n) | -\Gamma \sigma \sqrt{n} \leq \sum_{i=1}^{n} X_i - n\mu \leq \Gamma \sigma \sqrt{n}\} \tag{3.34}
\]

First, we examine the case where the assumed distribution of lead times is discrete uniform. The mean and variance of the discrete uniform distribution are \( \mu = \frac{a+b}{2} \) and \( \sigma^2 = \frac{(b-a+1)^2-1}{12} \) respectively.

**Proposition 3.6.1.** The following is a CLT-based uncertainty set for random variables \( Z_i \sim U[0, LT_{max}] \), where \( 1 \leq i \leq T \):

\[
U_{LT}^{\epsilon} = \{(Z_1, \ldots, Z_T) | T\hat{\mu} - \hat{\sigma} \sqrt{T} \Gamma_{LT}^{\epsilon} \leq \sum_{i=1}^{T} Z_i \leq T\hat{\mu} + \hat{\sigma} \sqrt{T} \Gamma_{LT}^{\epsilon}\} \tag{3.35}
\]

where \( Z_i \) is the lead time of order \( i \), \( \hat{\mu} = LT_{max}/2 \), \( \hat{\sigma} = \sqrt{\frac{LT_{max}^2+2LT_{max}}{12}} \), and \( \Gamma_{LT}^{\epsilon} \) is chosen empirically.

**Proof.** Using the lead time parameter of our model, we define

\[
Z_i = \frac{\min \{T, j + LT_{max} - 1\}}{\sum_{j=i}^{\min \{T, j + LT_{max} - 1\}}} (1 - \delta_{ij}) \tag{3.36}
\]

where the min function in the upper bound of the summation is due to imposing the requirement that all orders are received by the end of the time horizon. The relationship between \( Z_i \) and \( \delta_{ik} \) is illustrated in Table 3.3 for lead time parameters \( \delta_{ij} \). There is a one-to-one correspondance of \( Z_i \) to the values of \( \delta_{ik} \) parameters under binary \( \delta_{ij} \)'s. \( \square \)
Table 3.3. $Z_i$, defined in Equation (3.36) for various realizations of $\delta_{ij}$

<table>
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<th>$\delta_{i,i}$</th>
<th>$\delta_{i,i+1}$</th>
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<th>$\delta_{i,i+LT_{max}-2}$</th>
<th>$\delta_{i,i+LT_{max}-1}$</th>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>$LT_{max}$</td>
</tr>
</tbody>
</table>

Using the relationship in Equation (3.36), we rewrite Equation (3.35) as

$$U_{LT} = \{ \delta_{ij} | i = 1, \ldots, T, j = i, \ldots, T \} \leq \sum_{j=i}^{\min\{T,j+LT_{max}-1\}} (1 - \delta_{ij}) \geq T \hat{\mu} - \hat{\sigma} \sqrt{T} \Gamma_{\epsilon}$$

$$U_{LT} = \{ \delta_{ij} | i = 1, \ldots, T, j = i, \ldots, T \} \leq \sum_{j=i}^{\min\{T,j+LT_{max}-1\}} (1 - \delta_{ij}) \leq T \hat{\mu} + \hat{\sigma} \sqrt{T} \Gamma_{\epsilon}$$

Thus, we define the CLT-based lead time uncertainty set as $\Omega_{CLT} = \Omega \cap U_{LT}^{LT}$.

Similarly, the following is a demand uncertainty set based on the CLT.

$$U_{d} = \{ (d_1, \ldots, d_T) | T \hat{\mu} - \hat{\sigma} \sqrt{T} \Gamma_{\epsilon} \leq \sum_{i=1}^{T} d_i \leq T \hat{\mu} + \hat{\sigma} \sqrt{T} \Gamma_{\epsilon} \}$$

where $d_i$ is the demand at time $t$, $\hat{\mu}$ and $\hat{\sigma}$ the the mean and standard deviation of demand, respectively, and $\Gamma_{\epsilon}^d$ is chosen empirically.

### 3.7 Computational Results

#### 3.7.1 Results for Budget Uncertainty

We continue examining the illustrative example from earlier ($T = 10$, demand uncertainty is characterized using the budget demand uncertainty set $D_b$ where the average demand $\bar{d}_t = 15$ and $\bar{d}_t = 5$ at each time period $t$. The costs are $h = 5$, $b = 10$, and $c_1 = 1$). The surface plots in Figures (3.6), (3.7), and (3.8) show how the worst-case cost changes for instances where the maximum order lead time can be from zero to one, two, or three periods as the budget for demand ($\alpha$) and the budget for lead times ($\beta$) are varied.
Figure 3.6. Surface plots where budget for demand and lead time uncertainty sets are varied
Figure 3.7. Surface plots where budget for demand and lead time uncertainty sets are varied.
However, it is important to note that the worst-case costs determined by the optimization model shown in Figures (3.6), (3.7), and (3.8) could be significantly higher if the uncertainty budgets underestimate the realized uncertainty levels. We illustrate this relationship between the chosen lead time budget set and the performance realized by the model by using a Monte Carlo simulation with 100 realizations of uncertainties for the full time horizon. Table (3.4) summarizes the input data used in this first computational study.

In the Appendix we provide charts that show the average, standard deviation, and maximum total cost, as well as the average holding and backorder cost for this simulation. These surface charts show these costs as a function of budget uncertainty level for both demand ($\alpha$) and lead time ($\beta$), where $\alpha$ and $\beta$ range from zero to one in increments of 0.1. The purpose of the simulation is to gain insight into the sensitivity of the budget parameters to different distributions. We now discuss the results from this simulation.

Figure (A.1) shows that the budget setting producing the lowest average cost when demand is uniform and lead time is uniform is ($\alpha = 0.1$, $\beta = 0.3$) with an
Table 3.4. Input data

<table>
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<th>Parameter</th>
<th>Range</th>
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<tr>
<td>Demand per period</td>
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<tr>
<td>Maximum lead time</td>
<td>Three periods</td>
</tr>
<tr>
<td>Lead time distribution</td>
<td>Uniform, Two point**, Triangular***</td>
</tr>
<tr>
<td>(h, p, c) cost values</td>
<td>(5,10,1)</td>
</tr>
</tbody>
</table>

\(*\alpha = 5, \beta = 1, A = 10, B = 20\)

\(**\) for Two point distribution, \(f_0 = .25, f_3 = .75\)

\(**\) for Triangular distribution, \(f_0 = .1, f_1 = .4, f_2 = .4, f_3 = .1\)

average cost of 1178. The highest average cost is when \((\alpha = \beta = 1)\) with an average cost of 1933. The budget setting corresponding to the model from Chapter 2 is the model where demand is assumed at the nominal level \((\alpha = 0)\) and lead time uncertainty is most conservative \((\beta = 1)\) which has a cost of 1266 which is on the low end of the spectrum for this simulation.

Under uniform demand and the two-point lead time distribution, Figure (A.2) shows that the nominal budget setting \((\alpha = \beta = 0)\) has a lower average cost than most settings, and low budget parameters produce the lowest average cost, with \((\alpha = 0.3, \beta = 0)\) producing the lowest average cost of 990.

Under uniform demand and triangular lead time distribution, Figure (A.3) shows that the lowest average cost of 1118 is attained at \((\alpha = 0.6, \beta = 0.5)\). Interestingly, the solution \((\alpha = 0, \beta = 0.5)\) also works well with average cost 1171, which involves mitigating lead time uncertainty and setting the demand budget to zero (the nominal case). The nominal budget setting performs quite poorly (average cost 1555) compared to moderate settings. The nominal budget setting also performed poorly under uniform demand and triangular lead time when maximum cost is used as a metric, as shown in Figure (A.6).

Figure (A.4) shows that under uniformly distributed lead times and demand, the maximum cost that was highest in the simulation is for \((\alpha = \beta = 0)\) with a cost of 3950. The lowest maximum cost for this distribution is under \((\alpha = 0.6, \beta = 1)\) with a cost of 2222. This demonstrates the robustness tradeoff that must be made, since a low setting of \(\alpha\) and \(\beta\) produces high maximum cost but lower average cost.

Figure (A.5) shows the maximum cost for the case of uniformly distributed demands and two-point lead times, and the resulting surface chart is quite jagged,
lacking easily identifiable trends. This is mainly due to one realization of uncertainties in the simulation which consisted of 7 orders with a lead time of 3 periods, which produced the maximum cost for many of the \((\alpha, \beta)\) combinations.

Another set of surface charts in the Appendix correspond to the simulation where demand follows a Beta distribution and lead time follows three different distributions (Figures (A.16 - A.30)). Under uniform and triangular lead time distributions, the surface charts indicate that budget uncertainty set perform better (measured by average cost, maximum cost, and standard deviation) when budget levels are set to higher values.

Figures (A.7), (A.8), and (A.9) show the standard deviation of total cost for the simulation. For each distribution the variability of cost is generally decreasing in \(\alpha\) and \(\beta\).

Figures (A.10), (A.11), and (A.12) show the average holding costs for the simulation, and Figures ((A.13), (A.14), and (A.15) show the average backorder costs. In general, an increase in \(\alpha\) and \(\beta\) parameters results in increased holding costs, and the inverse (and weaker) relationship is observed for backorder cost.

We now summarize the results from the simulation. When demand and lead times are both simulated as uniform distributions, the simulation suggests that low budget settings can produce good average costs but there is a risk of attaining a large cost. This risk can be mitigated using the budget parameters. Under uniform demand and two-point lead times, the nominal settings were observed to perform well on an average cost basis and there was no clear improvement that can be observed by setting the budget parameters to even moderate values. For uniform demand and triangular lead times, moderate to high budget settings can produce improved average cost with a low risk of high costs.

### 3.7.2 Robust Optimization vs. Sampling Based Stochastic Programming

Sample Average Approximation (SAA) is a stochastic programming method to minimize the expected value of a stochastic program by generating random samples and solving a deterministic program to optimize the sample average objective value. We compare our robust optimization-based approach to the SAA approach. The following is the SAA formulation for the static policy.
minimize $\frac{1}{N} \left( \sum_{i=1}^{T} (cu_i + y_{i,n}) \right) \quad n = 1, \ldots, N$

\[ y_{i,n} \geq h \left( x_0 + \sum_{i=1}^{t} (\delta_{it} u_i - d_{it}^n) \right) \quad i = 1, \ldots, T, \quad n = 1, \ldots, N \]

\[ y_{i,n} \geq -b \left( x_0 + \sum_{i=1}^{t} (\delta_{it} u_i - d_{it}^n) \right) \quad i = 1, \ldots, T, \quad n = 1, \ldots, N \]

\[ u_i \geq 0 \quad i = 1, \ldots, T. \]

(3.39)

where each sample in set $\mathcal{N} = \{\xi_1, \ldots, \xi_N\}$ is a realization of lead time and demand uncertainty for the full time horizon. We implement SAA in the following way. For our comparison, we draw 500 samples of demands and lead times for the full time horizon where the demands belong to a uniform distribution between 20 and 30, and the lead times belong to a discrete uniform distribution between 0 and 3. Then we take the ordering decisions from the SAA problem, and run a Monte Carlo simulation using 100 new samples. Tables (3.5), (3.6), and (3.7) shows some sample statistics from this Monte Carlo simulation.

Since the robust optimization problem is solved for the full range of $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$ in increments of 0.1, we have $11 \times 11 = 121$ solutions to each problem. In the following tables, we report the lowest, median, and highest values to have a comparison to the SAA problem (The full range of solutions for these RO problems are shown in the surface charts in Appendix).

Table 3.5. Average Cost from Monte Carlo simulation (100 replications) for Robust Optimization Problem and SAA Problem where demands and lead times are drawn from uniform distributions

<table>
<thead>
<tr>
<th>Lead Time Distribution</th>
<th>Uniform</th>
<th>2-point</th>
<th>Triangular</th>
</tr>
</thead>
<tbody>
<tr>
<td>Demand Distr</td>
<td>Uni</td>
<td>Beta</td>
<td>Uni</td>
</tr>
<tr>
<td>SAA</td>
<td>1142</td>
<td>2258</td>
<td>992</td>
</tr>
<tr>
<td>Robust(lowest)</td>
<td>1178</td>
<td><strong>1842</strong></td>
<td><strong>990</strong></td>
</tr>
<tr>
<td>Robust(median)</td>
<td>1373</td>
<td><strong>2101</strong></td>
<td>1412</td>
</tr>
<tr>
<td>Robust(highest)</td>
<td>1933</td>
<td>2942</td>
<td>2068</td>
</tr>
</tbody>
</table>
Table 3.6. Maximum Cost from Monte Carlo simulation (100 replications) for Robust Optimization Problem and SAA Problem where demands and lead times are drawn from uniform distributions

<table>
<thead>
<tr>
<th>Lead Time Distribution</th>
<th>Demand Distr</th>
<th>Uniform</th>
<th>2-point</th>
<th>Triangular</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Uni</td>
<td>Beta</td>
<td>Uni</td>
</tr>
<tr>
<td>SAA</td>
<td></td>
<td>3269</td>
<td>3441</td>
<td>1734</td>
</tr>
<tr>
<td>Robust(lowest)</td>
<td></td>
<td>1178</td>
<td>2715</td>
<td>1844</td>
</tr>
<tr>
<td>Robust(median)</td>
<td></td>
<td>1373</td>
<td>3147</td>
<td>2220</td>
</tr>
<tr>
<td>Robust(highest)</td>
<td></td>
<td>1933</td>
<td>4205</td>
<td>2603</td>
</tr>
</tbody>
</table>

Table 3.7. Standard Deviation of Cost from Monte Carlo simulation (100 replications) for Robust Optimization Problem and SAA Problem where demands and lead times are drawn from uniform distributions

<table>
<thead>
<tr>
<th>Lead Time Distribution</th>
<th>Demand Distr</th>
<th>Uniform</th>
<th>2-point</th>
<th>Triangular</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Uni</td>
<td>Beta</td>
<td>Uni</td>
</tr>
<tr>
<td>SAA</td>
<td></td>
<td>364.2</td>
<td>550.8</td>
<td>248.8</td>
</tr>
<tr>
<td>Robust(lowest)</td>
<td></td>
<td>215.8</td>
<td>243.1</td>
<td>183.7</td>
</tr>
<tr>
<td>Robust(median)</td>
<td></td>
<td>296.0</td>
<td>437.4</td>
<td>257.6</td>
</tr>
<tr>
<td>Robust(highest)</td>
<td></td>
<td>532.2</td>
<td>618.1</td>
<td>395.3</td>
</tr>
</tbody>
</table>

Appendix, show that the RO solution performs well compared to SAA for several ranges of the budget parameters even though SAA explicitly minimizes average cost while the RO objective is to minimize the maximum cost. SAA dominates RO in terms of average cost under the uniform lead time, uniform demand scenario - which makes sense, since these are the distributions from which SAA was sampled. Tables (3.6) and (3.7) indicate that RO provides more stable and robust solutions compared to SAA in terms of standard deviation and worst case solution.

Table (3.8) shows two specific robust optimization solutions compared to SAA. SAA is generated using uniform demand and uniform lead time distributions. RO(\(\alpha = \beta = 0.3\)) is a RO model with low uncertainty budget, while RO(\(\alpha = \beta = 0.9\)) is a RO model where a high uncertainty budget is chosen. In the top section of the table, the out-of-sample distribution is the same as the in-sample SAA distribution. We see that RO (\(\alpha = \beta = 0.3\)) performs similarly to SAA with slightly increased average cost, and decreased maximum cost and better stability. The average cost of RO (\(\alpha = \beta = 0.9\)) is 54.6% higher than SAA, but the maximum
cost is much lower, and the stability is higher.

In the second set of results, the out-of-sample distribution is Beta for demand, and same as in-sample for lead time. Both RO solutions outperform SAA in all three performance measures. In particular, the stability for RO \((\alpha = \beta = 0.9)\) is better.

In the third set of results, the out-of-sample distribution is same as in-sample for demand, and Two-Point for lead time. RO \((alpha=beta=0.9)\) has better stability than SAA, and lower maximum cost, with the tradeoff of increased average cost (20\% higher). RO \((\alpha = \beta = 0.3)\) has higher average cost and maximum cost than SAA. This may be due to the under-estimate for budget.

Finally, when the out-of-sample distribution is Beta for demand, and Two-Point for lead time, both RO solutions have good performance compared to SAA in this case.

### Table 3.8. Comparison of two RO solutions to SAA

<table>
<thead>
<tr>
<th>Out-Of Sample Dist.</th>
<th>Measure</th>
<th>SAA</th>
<th>RO ((\alpha = \beta = 0.3))</th>
<th>RO ((\alpha = \beta = 0.9))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d \sim \text{Uniform}, ) (LT \sim \text{Uniform})</td>
<td>Avg Cost</td>
<td>1142</td>
<td>1238 (8.4%)</td>
<td>1765 (54.6%)</td>
</tr>
<tr>
<td></td>
<td>Max Cost</td>
<td>3269</td>
<td>2762 (-15.5%)</td>
<td>2463 (-24.7%)</td>
</tr>
<tr>
<td></td>
<td>Std. Dev</td>
<td>364.2</td>
<td>326.7 (-10.3%)</td>
<td>257.9 (-26.7%)</td>
</tr>
<tr>
<td>(d \sim \text{Beta}, ) (LT \sim \text{Uniform})</td>
<td>Avg Cost</td>
<td>2258</td>
<td>2103 (-6.9%)</td>
<td>2035 (-9.9%)</td>
</tr>
<tr>
<td></td>
<td>Max Cost</td>
<td>3341</td>
<td>3172 (-5.1%)</td>
<td>2821 (-15.6%)</td>
</tr>
<tr>
<td></td>
<td>Std. Dev</td>
<td>550.8</td>
<td>512.6 (-6.9%)</td>
<td>257.0 (-53.3%)</td>
</tr>
<tr>
<td>(d \sim \text{Uniform}, ) (LT \sim \text{Two – Point})</td>
<td>Avg Cost</td>
<td>1420</td>
<td>1572 (10.7%)</td>
<td>1704 (20.0%)</td>
</tr>
<tr>
<td></td>
<td>Max Cost</td>
<td>2963</td>
<td>3575 (20.6%)</td>
<td>2407 (-18.8%)</td>
</tr>
<tr>
<td></td>
<td>Std. Dev</td>
<td>469.5</td>
<td>441.8 (-5.9%)</td>
<td>246.5 (-52.5%)</td>
</tr>
<tr>
<td>(d \sim \text{Beta}, ) (LT \sim \text{Two – Point})</td>
<td>Avg Cost</td>
<td>2896</td>
<td>2908 (0.4%)</td>
<td>2259 (-22.0%)</td>
</tr>
<tr>
<td></td>
<td>Max Cost</td>
<td>4203</td>
<td>4103 (-2.3%)</td>
<td>2922 (-30.5%)</td>
</tr>
<tr>
<td></td>
<td>Std. Dev</td>
<td>627.4</td>
<td>640.8 (2.1%)</td>
<td>350.8 (-44.1%)</td>
</tr>
</tbody>
</table>

### 3.7.3 Results for the Central Limit Theorem Based Uncertainty Set

In this section we investigate, through computational experiments, the benefits of adding to our model uncertainty sets based on the central limit theorem for
demand, $U^d_\epsilon$, and lead times, $U^{LT}_\epsilon$ which were previously defined. Specifically, we investigate the reduction of conservatism in the solution that may be attributed to this uncertainty set.

In order for the central limit theorem to hold, so that the partial sums are approximately normally distributed, a well-known rule of thumb is the requirement of a sample size of 30. In this section we examine only instances where $T \geq 30$ in order to satisfy this requirement. Furthermore, we focus on basestock policy models in this section for two reasons: First, the basestock models tend to be solved much faster than static models (see Chapter 2 for computational results). Second, the scalar decision variable (the order-up-to level) that is determined by the basestock policy optimization may be an effective, and simple, comparison tool between instances.

Table (3.9) shows the effect of tightening CLT uncertainty set parameters on the worst-case cost for robust basestock models where full budget uncertainty is allowed (i.e., $\alpha = \beta = 1$). Certain extreme realizations cannot occur when the CLT constraint is binding, such as “all orders arriving with zero lead time”, or “all orders arriving with maximum lead time.”

For the case of $h = b = 10$ (top table), the difference in worst-case cost between $\Gamma = 0$ and full CLT uncertainty ($\Gamma = 100$) ranges from 6.3% to 16.4%. Note that there is no clear trend for basestock levels when $\Gamma$ is changed. One reason for this may be that an equal value for holding and backorder costs allows worst-case values to have either high inventory or high backorders.

For the case of $h = 10, b = 50$ (middle table), the trend for basestock levels is more clear. As the solution becomes less conservative (i.e., worst-case cost decreases) the basestock level increases. The same trend holds for the bottom table where $h = 10, b = 100$ (bottom table). The difference in worst-case cost between $\Gamma = 0$ and full CLT uncertainty ($\Gamma = 100$) for these two instances ranges from about 3-13%.

In the above discussion of CLT results the computational results show an improvement in best worst-case cost for problems where full budget uncertainty is used. The reduction in worst-case cost decreases when a less conservative budget is used. For instance, Table (3.10) shows the worst-case cost for a problem where $\alpha = \beta = 0.5$. 

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Table 3.9. Objective values (WC is the best worst-case solution) and corresponding basestock levels (BS) for the solutions. Gamma=100 corresponds to relaxing the CLT uncertainty set so it is non-binding. (Note: $\Gamma$ refers to the parameter setting for both $\Gamma_{LT}$ and $\Gamma_d$)

<table>
<thead>
<tr>
<th>Gamma</th>
<th>$LT_{max} = 2$</th>
<th>$LT_{max} = 4$</th>
<th>$LT_{max} = 6$</th>
<th>$LT_{max} = 8$</th>
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<tbody>
<tr>
<td></td>
<td>WC</td>
<td>BS</td>
<td>WC</td>
<td>BS</td>
</tr>
<tr>
<td>100</td>
<td>9060</td>
<td>36</td>
<td>15250</td>
<td>57</td>
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<td>4</td>
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<td>3</td>
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<td>2</td>
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<td>28</td>
<td>14981</td>
<td>43</td>
</tr>
<tr>
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<td>8918</td>
<td>20</td>
<td>14449</td>
<td>26</td>
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<tr>
<td>0</td>
<td>8480</td>
<td>20</td>
<td>13782</td>
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<table>
<thead>
<tr>
<th>Gamma</th>
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<th>$LT_{max} = 4$</th>
<th>$LT_{max} = 6$</th>
<th>$LT_{max} = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>WC</td>
<td>BS</td>
<td>WC</td>
<td>BS</td>
</tr>
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<td>15893</td>
<td>52</td>
<td>30877</td>
<td>86</td>
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<td>4</td>
<td>15403</td>
<td>53</td>
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<td>87</td>
</tr>
<tr>
<td>3</td>
<td>15017</td>
<td>57</td>
<td>30127</td>
<td>93</td>
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<td>57</td>
<td>29407</td>
<td>93</td>
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<td>14177</td>
<td>57</td>
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<td>27967</td>
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<table>
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<th>Gamma</th>
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<th>$LT_{max} = 4$</th>
<th>$LT_{max} = 6$</th>
<th>$LT_{max} = 8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>WC</td>
<td>BS</td>
<td>WC</td>
<td>BS</td>
</tr>
<tr>
<td>100</td>
<td>19901</td>
<td>56</td>
<td>42610</td>
<td>92</td>
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<td>4</td>
<td>18966</td>
<td>58</td>
<td>41644</td>
<td>96</td>
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<td>58</td>
<td>40924</td>
<td>96</td>
</tr>
<tr>
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<td>58</td>
<td>40204</td>
<td>96</td>
</tr>
<tr>
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<td>58</td>
<td>39484</td>
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<tr>
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<td>17185</td>
<td>58</td>
<td>38764</td>
<td>96</td>
</tr>
</tbody>
</table>

3.8 Concluding Remarks

In this paper, several robust optimization models for inventory control under uncertain demand and uncertain lead time are proposed. We describe state-of-the-art modeling approaches for robust optimization, which includes: budget
Table 3.10. Objective values (WC is the best worst-case solution) and corresponding basestock levels (BS) for the solutions. Gamma=100 corresponds to relaxing the CLT uncertainty set so it is non-binding. (Note: \( \Gamma \) refers to the parameter setting for both \( \Gamma_{LT} \) and \( \Gamma_d \))

<table>
<thead>
<tr>
<th>Gamma</th>
<th>( LT_{max} = 2 ) WC</th>
<th>BS</th>
<th>( LT_{max} = 4 ) WC</th>
<th>BS</th>
<th>( LT_{max} = 6 ) WC</th>
<th>BS</th>
<th>( LT_{max} = 8 ) WC</th>
<th>BS</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>13004</td>
<td>53</td>
<td>24577</td>
<td>87</td>
<td>34726</td>
<td>119</td>
<td>49016</td>
<td>148</td>
</tr>
<tr>
<td>4</td>
<td>12972</td>
<td>53</td>
<td>24391</td>
<td>88</td>
<td>34603</td>
<td>120</td>
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<td>147</td>
</tr>
<tr>
<td>3</td>
<td>12964</td>
<td>53</td>
<td>24342</td>
<td>89</td>
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<td>120</td>
<td>48203</td>
<td>146</td>
</tr>
<tr>
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<td>123</td>
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</tr>
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<td>90</td>
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<td>89</td>
<td>33932</td>
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<td>47190</td>
<td>148</td>
</tr>
</tbody>
</table>

uncertainty sets, CLT-based uncertainty sets, and a SHT-based uncertainty set. For the single-stage newsvendor problem, we present the robust counterpart in the form of a second-order cone program and show how it can be used to control solution conservatism.

Our numerical results indicate that standard robust optimization methods using epigraph reformulations are overly conservative for the multi-period inventory control problem under demand and lead time uncertainty due to their inherent overestimation of total cost even when costs and demands are stationary. This is due to the decoupling of the robust optimization subproblems that are solved. We present a new budget-type lead time uncertainty set, a CLT-type uncertainty set, and a so-called data-driven, statistical hypothesis test-based uncertainty set to handle demand and lead time uncertainty.

We compare our demand and lead time budget RO approach to the sampling-based stochastic programming SAA method. The RO solution performs well compared to SAA for several ranges of the budget parameters even though SAA explicitly minimizes average cost while the RO objective is to minimize the maximum cost. We do not argue that RO is “better” in general, however our computational results indicate that RO can provide more stable and robust solutions compared to SAA in terms of standard deviation and worst case solutions, especially when the realized distribution is different than the sampled distribution.
Chapter 4  
An Approach For Inventory Control Under Demand Uncertainty with Limited Distributional Information

4.1 Introduction and Literature Review

A common approach for decision-making under uncertainty is stochastic programming. In problems related to operations management, such as inventory control and production planning, this can involve minimizing the expected cost or maximizing expected profit over a multiperiod time horizon. It is often assumed that the probability distributions of the underlying random variables are known exactly, however this assumption may not hold for many real world applications. In this chapter we develop several distributionally robust models for inventory control under demand and lead time uncertainty. We demonstrate why this is a difficult problem even for demand uncertainty, and develop a Benders’ approach to solve the models. By distributionally robust, we mean a minimax stochastic optimization problem where the objective function minimizes the maximum expected value over a family of probability distributions that are parameterized by their support and first few moments.

There is a long history of using probability distribution information for inventory control problems in the literature. Scarf et al. (1958) first proposed the distribu-
tionally robust stochastic program, in an application paper where he developed a closed-form solution for the optimal order quantity of the single period newsvendor problem when only the mean and variance of demand are known. There are several studies for the single period newsvendor problem where partial information about demand is known (Yue et al. 2006, Zhu et al. 2013). By using a regret criterion (that is, minimizing the maximum opportunity cost) Perakis and Roels (2008) formulate closed-form expressions for the minimax regret order quantity.

Much of the research on distributionally robust stochastic problems is on deriving tractable approaches which is often achieved by reformulating these problems into conic programs. For example, Bertsimas et al. (2010) develop a semidefinite optimization model for minimax two-stage stochastic linear programs. Delage and Ye (2010) show that some minimax stochastic optimization problems are polynomial time solveable by using an ellipsoidal algorithm. Recently, Ang et al. (2014) use the linear decision rule approach for a two-stage stochastic linear program with partial distributional information and show that it can be reformulated as a second-order cone program. Gao et al. (2014) examine the same problem and develop a deterministic second-order cone approach by assuming knowledge of the extreme points of the dual polyhedron. Mak et al. (2013) develop a distributionally robust approach to appointment scheduling and sequencing and show that the model can be reformulated as a conic program because of the structural properties of their model.

See and Sim (2010) propose a robust optimization approach to approximate a $T$ stage stochastic optimization model for a multiperiod inventory control problem under demand uncertainty where partial distributional information is known (mean, support, and a notion called directional deviations). An important tool used in their study is an approximation of $\mathbb{E}(\cdot)^+$, which is an upper bound first proposed in Chen and Sim (2009). This upper bound, which takes the form of a second order cone program, replaces the max function in the holding/backorder cost function. Our model is similar to See and Sim (2010) in that we consider a $T$ stage stochastic program. Our approach, which involves LP duality in infinite-dimensional spaces, does not require us to use an approximation for the $\mathbb{E}(\cdot)^+$ terms. Because we solve the model using the exact holding/backorder cost function our model turns out to be theoretically intractable, however a solution approach is developed in this chapter.
4.2 Demand Uncertainty Models

For our model, we consider a periodic review inventory problem in discrete time for a single facility over a finite time horizon of $T$ time periods. The planner places an order, $u_i$, at the beginning of the time period $i$ which incurs a variable cost $c$. The order arrives immediately (i.e., lead time is zero). Then demand, $d_i$, occurs during the period. At the end of each period costs are incurred for holding positive inventory (holding cost, $h$), or for negative inventory (backorder cost, $b$). The initial inventory level is $x_0$. For our model, we minimize the total cost $f$ which is a function of the decision variable, orders $u_t$ placed at time $t$, and the parameters include the demands $d_t$, per-unit variable cost $c$, inventory holding cost $h$, and backorder cost $b$.

$$
\min_u f(u, d) = \sum_{t=1}^{T} f_t(u, d) \tag{4.1}
$$

$$
f_t(u, d) = cu_t + \max \{ h \sum_{i=1}^{t} (u_i - d_i), -b \sum_{i=1}^{t} (u_i - d_i) \}, \quad t = 1, \ldots, T
$$

In previous chapters we have focused on robust optimization models to mitigate uncertainty for this problem by developing uncertainty sets and then solving a min-max problem. In this chapter, we use an alternative approach that assumes that partial distributional information is known. Then, we solve the corresponding min-max expectation problem.

Demand is denoted by $d_t$, $t = 1, \ldots, T$ and it is a random variable and it has some joint distribution. We know the support of $d_t$ and it is denoted by $D_t$. Let $\mathbb{D} = D_1 \times \ldots \times D_T$. The $q^{th}$ moment of $d_t$ is denoted as $M_{tq}$. In this paper, we examine a model where we assume known support, first moment, an an upper bound for the second moment for $d_t$, where $t = 1, \ldots, T$. The exact joint demand
distribution $F$ is unknown however we know that

\[
\int_{\mathcal{D}} dF(d) = 1 \]
\[
\int_{\mathcal{D}} d_i dF(d) = M_{i1}, \quad 1 \leq i \leq T
\]
\[
\int_{\mathcal{D}} d_i^2 dF(d) \leq M_{i2}, \quad 1 \leq i \leq T
\]
\[
dF(d) \geq 0, \quad \forall d \in \mathcal{D}
\]

and the family of distributions that satisfy (4.2) is denoted as $\mathcal{F}(\mathcal{D}, Q)$.

The $T$ stage distributionally robust optimization model is the following:

\[
\min_u \max_{F \in \mathcal{F}(\mathcal{D}, Q)} \sum_{t=1}^{T} \mathbb{E}[f_t(u, d)]
\]  

(4.3)

The inner maximization problem can be explicitly written as

\[
\max_{F \in \mathcal{F}(\mathcal{D}, Q)} \sum_{t=1}^{T} \int_{D_1} \int_{D_2} \ldots \int_{D_T} f_t(u, d) dF(d_1) dF(d_2) \ldots dF(d_T)
\]  

(4.4)

Since the sum of an expectation is equal to the expectation of a sum, we have

\[
\max_{F \in \mathcal{F}(\mathcal{D}, Q)} \int_{D_1} \int_{D_2} \ldots \int_{D_T} \sum_{t=1}^{T} f_t(u, d) dF(d_1) dF(d_2) \ldots dF(d_T)
\]  

(4.5)

This is the same as

\[
\max_{F \in \mathcal{F}(\mathcal{D}, Q)} \int_{D_1} \int_{D_2} \ldots \int_{D_T} f(u, d) dF(d_1) dF(d_2) \ldots dF(d_T)
\]  

(4.6)

The following one stage model is an equivalent form of Problem (4.3).

\[
\min_u \max_{F \in \mathcal{F}(\mathcal{D}, Q)} \mathbb{E}[f(u, d)]
\]  

(4.7)

Let’s focus on the inner maximization problem of Problem (4.7),

\[
\max_{F \in \mathcal{F}(\mathcal{D}, Q)} \mathbb{E}[f(u, d)]
\]  

(4.8)

Using duality (Bertsimas and Popescu 2005), for fixed $u$, the dual of Prob-
The constraints in Problem (4.9) can be rewritten as

\[
\theta \geq \max_{d \in \mathcal{D}} \{ f(u, d) - \sum_{i=1}^{T} \sum_{q \in Q} p_{iq} d_i^q \} \tag{4.10}
\]

Substituting the definition of \( f(u, d) \) into the constraint yields

\[
\theta \geq \max_{d \in \mathcal{D}} \left\{ \left( \sum_{i=1}^{T} (cu_i + \max \left\{ h \sum_{k=1}^{i} (u_k - d_k), -b \sum_{k=1}^{i} (u_k - d_k) \right\} - \sum_{q \in Q} p_{iq} d_i^q ) \right) \right\} \tag{4.11}
\]

Fixing \( \theta \) and \( \rho_{iq} \) and maximizing the right-hand side over \( d \) involves solving a nonconvex program due to the presence of the piecewise convex holding/backorder cost function. In the next section, we develop an approach for solving the following equivalent reformulation of Problem (4.3):

\[
\min_{\theta, \rho} \left( \theta + \sum_{i=1}^{T} \sum_{q \in Q} p_{iq} d_i^q \right)
\]

\[
\text{s.t. } \theta \geq \max_{d \in \mathcal{D}} \left\{ \sum_{i=1}^{T} (cu_i + \max \left\{ h \sum_{k=1}^{i} (u_k - d_k), -b \sum_{k=1}^{i} (u_k - d_k) \right\} - \sum_{q \in Q} p_{iq} d_i^q ) \right\}
\]

\[
\rho_{i2} \geq 0, \quad \forall i \tag{4.12}
\]

### 4.2.1 Benders’ Solution Approach

One approach to solve this problem is to use Benders’ decomposition. The first step in developing this approach is to identify the Decision Maker Problem (DM) and Adversarial Problem (AP). The AP corresponds to the inner maximization
problem in Problem (4.7), and the DM corresponds to the outer minimization in Problem (4.7). In our approach, for the Decision Maker Problem (DM) we solve a relaxed formulation of Problem (4.12) where we solve the problem over a finite set of demand realizations, denoted \( \tilde{D} \). Then we solve a subproblem to generate a demand realization that renders the DM infeasible, and add the scenario to \( \tilde{D} \), and we iterate until no such demand realization makes the DM infeasible. We now describe the formulations used in this algorithm.

4.2.1.1 Decision Maker Problem

The Decision Maker (DM) Problem is the following:

\[
\begin{align*}
\min_{\theta, \rho, u} & \quad \theta + \sum_{i=1}^{T} \sum_{q \in Q} M_{iq} \rho_{iq} \\
\text{s.t.} & \quad \theta \geq \sum_{i=1}^{T} \left( cu_i + \max \left\{ h \sum_{k=1}^{i} (u_k - d_k), -b \sum_{k=1}^{i} (u_k - d_k) \right\} - \sum_{q \in Q} \rho_{iq} d_i^q \right), \quad \forall d \in \tilde{D} \\
\rho_{i,2} & \geq 0, \quad i = 1, \ldots, T
\end{align*}
\]

This has an equivalent linear programming reformulation, which is the following:

\[
\begin{align*}
\min_{\theta, \rho, u} & \quad \theta + \sum_{i=1}^{T} \sum_{q \in Q} M_{iq} \rho_{iq} \\
\text{s.t.} & \quad \theta \geq \sum_{i=1}^{T} \left( cu_i + y_{i,r} - \sum_{q \in Q} \rho_{iq} (d_i^q) \right), \quad r = 1, \ldots, |\tilde{D}| \\
y_{i,r} & \geq h \sum_{k=1}^{i} (u_k - d_k), \quad i = 1, \ldots, T, r = 1, \ldots, |\tilde{D}| \quad (4.14) \\
y_{i,r} & \geq -b \sum_{k=1}^{i} (u_k - d_k), \quad i = 1, \ldots, T, r = 1, \ldots, |\tilde{D}| \\
\rho_{i,2} & \geq 0, \quad i = 1, \ldots, T
\end{align*}
\]
4.2.1.2 Adversarial Problem

Since the DM is a relaxed version of Problem (4.12) with a finite set of demand scenarios, the AP involves generating new demand scenarios that render DM infeasible. Essentially it solves the nonconvex maximization on the RHS of Constraint (4.12) by formulating it as a mixed-integer program. Given $u$, $\rho$, and $\theta$ from the DM, we solve

$$\max \sum_{i=1}^{T} cu_i + I_i + B_i - \sum_{q \in Q} \rho_{iqd_i^q}$$

s.t. $h \sum_{i=1}^{k} (u_i - d_i) \leq I_k \leq h \sum_{i=1}^{k} (u_i - d_i) + Mp_k, \quad k = 1, \ldots, T$

$$-b \sum_{i=1}^{k} (u_i - d_i) \leq B_k \leq -b \sum_{i=1}^{k} (u_i - d_i) + M(1 - p_k), \quad k = 1, \ldots, T$$

$$\mu_i - d_i \leq d_i \leq \mu_i + \bar{d}_i, \quad i = 1, \ldots, T$$

$$p_i \in \{0, 1\}, \quad i = 1, \ldots, T$$

(4.15)

This is a mixed integer quadratic program (MIQP). The objective function is concave in $d_i$ since the $\rho_{iq}$ parameter is always positive. We note that the concavity in $d_i$ is due to having information on the upper bound of the second moment, instead of exact information. Had we assumed exact knowledge, the dual variable $\rho_{iq}$ from Problem (4.9) would be unrestricted, and this subproblem would be much harder to solve. It is necessary that no other demand realization can be added that would make the constraint infeasible (i.e., Equation (4.10) holds).

Algorithm 3 describes the pseudocode for this Benders’ algorithm.
Algorithm 3 Benders’ approach to Solving Problem (4.7)

1. Initialization Step:
Set $r = 1$, $L = 0$, and $U = \infty$
Set $\tilde{D} = \emptyset$.
Choose initial demand scenario $\tilde{d}^r = (\mu_1, \ldots, \mu_T)$, and add $\tilde{d}^r$ to $\tilde{D}$.

2. Decision Maker Problem:
Solve Problem (4.14).
Set $L = $ Objective value from Problem (4.14)
Set $(\bar{u}, \bar{\theta}, \bar{\rho}) = $ Solution from Problem (4.14)

3. Adversarial Problem:
Solve Problem (4.15).
Set $\tilde{d}^{r+1} = d$
Set $U = $ Objective value from Problem (4.15)
If $\bar{\theta} - U \geq 0$, go to Step 4.
Otherwise add $\tilde{d}^{r+1}$ to $\tilde{D}$, set $r = r + 1$, and return to Step 2.

4. Terminate if $U - L < \epsilon$
Otherwise, return to Step 2.

Remark about Initialization Step: The initial demand scenario chosen must satisfy the inequality $(\tilde{d}^r_i)^2 < M_{i2}$ for all $i$. Consider the DM Problem (4.14) where all variables are fixed with the exception of $\rho_{i2}$ and $\theta$. Since this is a minimization problem, it is clear by inspecting the constraints that a unit increase of $\rho_{i2}$ will allow $\theta$ to decrease by $(\tilde{d}^r_i)^2$ units. In the objective function the coefficient of $\rho_{i2}$ is $M_{i2}$ and the coefficient of $\theta$ is 1, the objective value will be unbounded when $(\tilde{d}^r_i)^2 > M_{i2}$. After the first iteration, the value of $(\tilde{d}^{r+1}_i)^2$ in future iterations can exceed $M_{i2}$ without issue since the initializing cut will remain as a lower bound on $\theta$.

We now prove that Algorithm 3 converges to the optimal solution.

Theorem 4.2.1. Denote $\mathbb{T}$ as the index set of iterations and $X$ as the feasible region of the DM problem. Define $z^*$ as the optimal objective value of Problem (4.12), $x_\tau = (u_\tau, \rho_\tau)$ as the decision variables for the DM Problem at iteration $\tau \in \mathbb{T}$, and $z_\tau$ is the objective value. Denote $d_\tau$ as the solution that maximizes adversarial problem at iteration $\tau \in \mathbb{T}$ given the decision maker’s solution $x_\tau$. Under the conditions (i) that $\|(x_\tau, d_\tau)\| \leq M$ for all $\tau \in \mathbb{T}$, and (ii) $z^*$ is bounded from
above, then the algorithm converges to the optimal solution.

Proof. First, we denote the objective function of Problem (4.12) as

\[
\phi(x) = \max_{d \in D} \left( \Delta_1(x, d) + \Delta_2(x) \right)
\]

where

\[
\Delta_1(x, d) = f(x, d) - \sum_{i=1}^{T} \sum_{q \in Q} \rho_{iq} d_i
\]

\[
\Delta_2(x) = \sum_{i=1}^{T} \sum_{q \in Q} M_{iq} \rho_{iq}
\]

By definition, \( z^* = \min_{x \in X} \phi(x) \).

We know

\[
z_\tau \leq z_{\tau+1} \leq \ldots \leq z^*
\]

since in each iteration we add a cut. Therefore by the monotone convergence theorem, \( \{z_\tau\}_T \) is a convergent sequence with the mild assumption that \( z^* \) is bounded from above. We define \( z_\infty \) as the limit point of this sequence, i.e.,

\[
z_\tau \xrightarrow{\tau \to \infty} z_\infty
\]

Second, note that each DM problem generates a feasible solution \( x_{\tau} \), i.e., \( x_{\tau} \in X \), the following holds:

\[
z^* \leq \phi(x_{\tau}), \quad \forall \tau
\]

Third, notice that by definitions of \( x_{\tau+1}, d_{\tau+1} \) and \( d_{\tau} \),

\[
\Delta_1(x_{\tau+1}, d_{\tau+1}) + \Delta_2(x_{\tau+1}) = \max_{d \in D} \Delta_1(x_{\tau+1}, d) + \Delta_2(x_{\tau+1}) = \phi(x_{\tau+1})
\]

\[
\geq \min_{x \in X} \max_{d \in D} \Delta_1(x, d) + \Delta_2(x, d)
\]

\[
\geq \min_{x \in X} \max_{0 \leq i \leq \tau} \Delta_1(x_{\tau+1}, d_i) + \Delta_2(x_{\tau+1}) = z_{\tau+1}
\]

\[
\geq \Delta_1(x_{\tau+1}, d_{\tau}) + \Delta_2(x_{\tau+1})
\]

From the assumption that \( \|(x_{\tau}, d_{\tau})\| \leq M \), there exists a convergent subsequence
\{x_t, d_t\}_T$, where $T \subseteq \mathbb{T}$. We define $(x_\infty, d_\infty)$ as the limit point of this subsequence, i.e., $(x_t, d_t) \xrightarrow{t \to \infty} (x_\infty, d_\infty)$. By noticing the continuity of the function $\Delta_1(x, d) + \Delta_2(x)$ in $(x, d)$ and that of $\phi(\cdot)$, we have that

\[
\lim_{t \to \infty} [\Delta_1(x_{t+1}, d_{t+1}) + \Delta_2(x_{t+1})] = \lim_{t \to \infty} [\Delta_1(x_{t+1}, d_t) + \Delta_2(x_{t+1})] = \Delta_1(x_\infty, d_\infty) + \Delta_2(x_\infty) = \phi(x_\infty).
\]

(4.26)

Invoking (4.21)-(4.24)-(4.25), we have, when $t \to \infty$, the following inequality holds:

\[
\phi(x_\infty) = \Delta_1(x_\infty, d_\infty) + \Delta_2(x_\infty) \geq z_\infty \geq \Delta_1(x_\infty, d_\infty) + \Delta_2(x_\infty)
\]

(4.29)

\[
\Rightarrow z_\infty = \phi(x_\infty) = \Delta_1(x_\infty, d_\infty) + \Delta_2(x_\infty)
\]

(4.30)

Therefore, by the continuity of $\phi(\cdot)$ and invoking (4.20), we have

\[
z_\infty = \phi(x_\infty) \geq z^*
\]

(4.31)

Further invoking (4.18), we have

\[
z^* \geq z_\infty \geq z^* \Rightarrow \phi(x_\infty) = z_\infty = z^*.
\]

(4.32)

which immediately provides the result.

\[\square\]

### 4.3 Computational Results

In this section, we present numerical results for our model. The objective of this section is to illustrate the method. Figure (4.1) shows the the trajectory of the DM solution (blue line) and the AP solution (red line) for a 5 period problem, with convergence to the optimal solution after 250 iterations. Figure (4.2) illustrates the sensitivity of the model to the assumed upper bound on the second moment: For this problem where demand is between zero and 10 units per period with a mean of 5 units per period, the optimal decision is to order 10 units per period when the model assumes an upper bound on the second moment corresponding to a standard deviation of four units. When the upper bound is lower, the optimal
decision is less conservative.

Figure 4.1. The convergence of the Benders’ algorithm is shown for a typical problem instance
Figure 4.2. The optimal ordering decisions determined by the algorithm for a 5 time period instance assuming demand is between 0 and 10, the mean is five, c=1, h=3, b=10, and the upper bound for the standard deviation varies.

The outcomes illustrated in Figure (4.2) indicate that there is a threshold on the second moment above which the decision to trivially order the upper bound of demand is optimal. Tables (4.1),(4.2),(4.3), and (4.4) show the solutions for a five-period problem where demand is supported on [0,40] with mean 20 under varied second moment information. The difference between these tables is the critical ratio used, or $\frac{b}{h+b}$. For these four tables, they are: 1/2, 1/3, 10/11, and 20/21.
Table 4.1. Costs are $c = 1$, $h = 10$, $b = 10$. "Iter" is the number of iterations Algorithm 3 takes to solve the problem, "WC Cost" is the optimal objective value, $u_i$ is the order amount in period $i$, and $\sigma$ is the upper bound for the standard deviation assumed in the model.

<table>
<thead>
<tr>
<th>$\sigma \leq$</th>
<th>Iter</th>
<th>WC Cost</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>254</td>
<td>247.1</td>
<td>20.1</td>
<td>19.8</td>
<td>21.3</td>
<td>19.6</td>
<td>15.9</td>
</tr>
<tr>
<td>3</td>
<td>230</td>
<td>541.4</td>
<td>20.9</td>
<td>19.0</td>
<td>26.5</td>
<td>13.4</td>
<td>10.9</td>
</tr>
<tr>
<td>5</td>
<td>207</td>
<td>835.6</td>
<td>22.1</td>
<td>23.1</td>
<td>14.6</td>
<td>24.9</td>
<td>0.0</td>
</tr>
<tr>
<td>7</td>
<td>227</td>
<td>1129.9</td>
<td>20.0</td>
<td>29.8</td>
<td>18.7</td>
<td>10.28</td>
<td>0.0</td>
</tr>
<tr>
<td>9</td>
<td>211</td>
<td>1426.3</td>
<td>27.112</td>
<td>15.0</td>
<td>33.3</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>11</td>
<td>205</td>
<td>1723.2</td>
<td>21.923</td>
<td>34.1</td>
<td>16.2</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>15</td>
<td>174</td>
<td>2319.3</td>
<td>28.8</td>
<td>30.9</td>
<td>8.8</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

Table 4.2. Costs are $c = 1$, $h = 5$, $b = 10$. "Iter" is the number of iterations Algorithm 3 takes to solve the problem, "WC Cost" is the optimal objective value, $u_i$ is the order amount in period $i$, and $\sigma$ is the upper bound for the standard deviation assumed in the model.

<table>
<thead>
<tr>
<th>$\sigma \leq$</th>
<th>Iter</th>
<th>WC Cost</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>250</td>
<td>204.8</td>
<td>20.0</td>
<td>21.7</td>
<td>19.0</td>
<td>23.1</td>
<td>14.7</td>
</tr>
<tr>
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<td>264</td>
<td>414.3</td>
<td>20.2</td>
<td>22.4</td>
<td>25.4</td>
<td>19.5</td>
<td>8.0</td>
</tr>
<tr>
<td>5</td>
<td>234</td>
<td>623.9</td>
<td>24.4</td>
<td>22.4</td>
<td>26.0</td>
<td>13.6</td>
<td>6.2</td>
</tr>
<tr>
<td>7</td>
<td>233</td>
<td>833.4</td>
<td>21.3</td>
<td>27.1</td>
<td>35.0</td>
<td>6.3</td>
<td>0.0</td>
</tr>
<tr>
<td>9</td>
<td>218</td>
<td>1044.2</td>
<td>24.9</td>
<td>26.5</td>
<td>37.1</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>11</td>
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<td>1256.2</td>
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<td>25.9</td>
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<td>0.0</td>
</tr>
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<td>1668.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
Table 4.3. Costs are $c = 1$, $h = 5$, $b = 50$. "Iter" is the number of iterations Algorithm 3 takes to solve the problem, "WC Cost" is the optimal objective value, $u_i$ is the order amount in period $i$, and $\sigma$ is the upper bound for the standard deviation assumed in the model.

<table>
<thead>
<tr>
<th>$\sigma \leq$</th>
<th>Iter</th>
<th>WC Cost</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>99</td>
<td>141.9</td>
<td>20.3</td>
<td>19.7</td>
<td>19.9</td>
<td>20.4</td>
<td>19.5</td>
</tr>
<tr>
<td>3</td>
<td>130</td>
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<td>20</td>
<td>22.2</td>
<td>22.3</td>
<td>18.7</td>
<td>16.6</td>
</tr>
<tr>
<td>5</td>
<td>109</td>
<td>770.0</td>
<td>27</td>
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<td>7.9</td>
<td>29.9</td>
<td>10.0</td>
</tr>
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<td>28.4</td>
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<td>6.7</td>
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<td>0</td>
</tr>
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<td>15</td>
<td>8</td>
<td>1700</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
<td>40</td>
</tr>
</tbody>
</table>

Table 4.4. Costs are $c = 1$, $h = 5$, $b = 100$. "Iter" is the number of iterations Algorithm 3 takes to solve the problem, "WC Cost" is the optimal objective value, $u_i$ is the order amount in period $i$, and $\sigma$ is the upper bound for the standard deviation assumed in the model.

<table>
<thead>
<tr>
<th>$\sigma \leq$</th>
<th>Iter</th>
<th>WC Cost</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>118</td>
<td>173.7</td>
<td>20.1</td>
<td>20.3</td>
<td>19.6</td>
<td>20.6</td>
<td>19.4</td>
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<td>596.6</td>
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<td>28.6</td>
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<tr>
<td>7</td>
<td>20</td>
<td>1700.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
</tr>
<tr>
<td>9</td>
<td>20</td>
<td>1700.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
</tr>
<tr>
<td>11</td>
<td>18</td>
<td>1700.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
<td>40.0</td>
</tr>
</tbody>
</table>

Each iteration of this algorithm takes about half a second. So for the problems solved here, the computational time for the algorithm ranges from four seconds to two minutes. Table (4.1) shows the case where $h = b = 10$; even for fairly high variability, the ordering decisions do not reach the bounds of the demand. Since holding cost is equal to backorder cost, it is understandable that the optimal decision would be something in between. However, the problem in Table (4.2)
reaches the threshold at $\sigma = 15$, the problem in Table (4.3) at $\sigma = 9$, and the problem in Table (4.4) at $\sigma = 7$.

Table (4.5) shows the ratio of worst-case SAA objective value to DRO objective value for various instances with different costs, and different upper bounds for the second moment. The parameters used in this instance are: $T = 10$, $c = 1$, $d_t \in [0, 40]$, $\mu_t = 20$. SAA is computed using 500 samples of $d_t \sim N(\mu_t, \sigma^2)$. The worst-case expected cost of SAA is computed using Benders’ decomposition with the solution from SAA fixed. The results show that the DRO method produces better results in the worst-case for instances characterized by high backorder cost, especially when the second moment is larger. However, when backorder cost and holding cost are of the same order (i.e., in the second and third columns of Table (4.5)), the DRO approach performs similarly to SAA.

Table 4.5. Comparison of worst-case SAA solution to DRO solution

<table>
<thead>
<tr>
<th>Second Moment</th>
<th>h=10,b=10</th>
<th>h=5,b=10</th>
<th>h=5,b=50</th>
<th>h=5,b=100</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma \leq 1$</td>
<td>1.05%</td>
<td>1.43%</td>
<td>-0.27%</td>
<td>13.18%</td>
</tr>
<tr>
<td>$\sigma \leq 3$</td>
<td>1.42%</td>
<td>2.04%</td>
<td>9.70%</td>
<td>23.72%</td>
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Chapter 5  |
Conclusions and Future Research Directions

This thesis presents a general robust optimization-based methodology for modeling demand uncertainty and, in particular, lead time uncertainty using robust optimization. To the best of our knowledge, this is the first study of inventory control under lead time uncertainty from a robust optimization perspective. First, we develop several models (static, basestock, and affinely adjustable) for an inventory control problem under uncertain lead time where demand is fixed (deterministic). In our computational experiments we find that the static policy outperforms the basestock policy, however the outperformance diminishes as the lead time variability increases. A simulation study also indicates that the robust basestock policy performs well compared to several heuristics from literature. We then show that the affinely adjustable policy may perform marginally better than static policies, however, having information about order arrivals for several future periods can produce large improvements in some instances (up to 26%).

Next, we describe a robust optimization approach for an inventory problem where both demand and lead time are uncertain. We show that the standard robust optimization approach (i.e., epigraph reformulation approach) is overly conservative compared to our Benders’ approach for multi-period inventory problems under demand and lead time uncertainty. This over-conservatism is due to the decoupling of the robust optimization subproblems that are solved. We develop several uncertainty sets for lead times based on the budget approach, and the central limit theorem. The former requires no distributional information and the budget parameters can be controlled by the decision maker, while the latter requires the

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assumption that the uncertainties follow a specified distribution in order for the central-limit theorem to hold. This approach is compared to the sampling-based stochastic programming Sample Average Approximation (SAA) method. The RO solution performs well compared to SAA for multiple ranges of budget parameters even though SAA explicitly minimizes average cost while RO has a minimax objective. Our computational results indicate that our method can provide more stable and robust solutions compared to SAA in terms of standard deviation and worst case solution, especially when the realized distribution is different than the sampled distribution.

Finally we consider an approach to solve an inventory control problem under demand uncertainty where there is limited distributional information. In this problem the objective function is a minimization of the maximum expected value over a family of distributions that are parameterized by the first and second moment and the support. We adopt an inventory model with an exact holding/backorder cost function. The main tool used is infinite-dimensional duality to reformulate the min-max expectation problem into a deterministic optimization problem. Even after this reformulation the problem is intractable. We devise an algorithm to solve it, and prove its convergence.

Using robust optimization methodology in this thesis allows us to forego the computational intractabilities that accompany probabilistic analysis of stochastic systems and instead approach it using mathematical optimization tools. We developed several algorithms to solve nonconvex robust optimization problems under columnwise and rowwise uncertainty in the context of an inventory control problem.

One of the limitations of this research is that it was based on simulated data, and to the best of our knowledge, there is not a standard test bed of instances to compare against other benchmark methods. Future work will aim to obtain real world data to test this approach as well as the development of a test bed of instances. The focus in this thesis is limited to the inventory at a retailer; there are many ways to extend this work, for example the extension to supply chain networks (multi-echelon systems). In terms of modeling lead time using robust optimization, it would be interesting to consider alternative adaptive policies beyond the affine and basestock policies discussed in this thesis, in particular adaptive policies that explicitly consider the number and age of outstanding orders in the
ordering decisions.
Appendix | Figures from Chapter 3

1 Simulation Results for Budget Uncertainty

Figure A.1. Simulated average cost (uniform demand, uniform lead times)
Figure A.2. Simulated average cost (uniform demand, two-point lead times)

Figure A.3. Simulated average cost (uniform demand, triangular lead times)
Figure A.4. Simulated maximum cost (uniform demand, uniform lead time)

Figure A.5. Simulated maximum cost (uniform demand, two-point lead time)
Figure A.6. Simulated maximum cost (uniform demand, triangular lead time)

Figure A.7. Simulated standard deviation (uniform demand, uniform lead time)
Figure A.8. Simulated standard deviation (uniform demand, two-point lead time)

Figure A.9. Simulated standard deviation (uniform demand, triangular lead time)
Figure A.10. Simulated holding cost (uniform demand, uniform lead time)

Figure A.11. Simulated holding cost (uniform demand, two-point lead time)
Figure A.12. Simulated holding cost (uniform demand, triangular lead time)

Figure A.13. Simulated backorder cost (uniform demand, uniform lead time)
Figure A.14. Simulated backorder cost (uniform demand, two-point lead time)

Figure A.15. Simulated backorder cost (uniform demand, triangular lead time)
Figure A.16. Simulated average cost (Beta demand, uniform lead time)

Figure A.17. Simulated average cost (Beta demand, two-point lead time)
Figure A.18. Simulated average cost (Beta demand, triangular lead time)

Figure A.19. Simulated maximum cost (Beta demand, uniform lead time)
Figure A.20. Simulated maximum cost (Beta demand, two-point lead time)

Figure A.21. Simulated maximum cost (Beta demand, triangular lead time)
Figure A.22. Simulated standard deviation (Beta demand, uniform lead time)

Figure A.23. Simulated standard deviation (Beta demand, two-point lead time)
Figure A.24. Simulated standard deviation (Beta demand, triangular lead time)

Figure A.25. Simulated holding cost (Beta demand, uniform lead time)
Figure A.26. Simulated holding cost (Beta demand, two-point lead time)

Figure A.27. Simulated holding cost (Beta demand, triangular lead time)
Figure A.28. Simulated backorder cost (Beta demand, uniform lead time)

Figure A.29. Simulated backorder cost (Beta demand, two-point lead time)
Figure A.30. Simulated backorder cost (Beta demand, triangular lead time)


Vita

Andreas Thorsen

Andreas Thorsen is a Ph.D. candidate in the Harold and Inge Marcus Department of Industrial and Manufacturing Engineering at The Pennsylvania State University and will earn his degree in Industrial Engineering and Operations Research in August 2014. He received his Bachelor of Science in Industrial and Manufacturing Engineering from The Pennsylvania State University in 2005. After earning his Bachelor’s degree, Andreas worked for The Sherwin-Williams Company in Columbus, Ohio for three years in several management and engineering roles before enrolling in graduate school. During graduate school he spent two summers working at RAND Corporation as a Summer Associate. He worked as a teaching assistant for several courses before being awarded the Academic Computing Fellowship. Shortly after receiving his Ph.D. degree, Andreas Thorsen will join the Jake Jabs College of Business & Entrepreneurship at Montana State University as Assistant Professor of Management. Andreas Thorsen’s research area includes supply chain optimization, decision-making under uncertainty, and network design. He is a member of Institute of Industrial Engineers (IIE) and Institute for Operations Research and the Management Sciences (INFORMS).