DYNAMIC AND DIFFERENTIAL GAMES IN ELECTRIC POWER MARKETS

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by
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Abstract

Over the last two decades, the electricity industry has shifted from regulation of monopolistic and centralized utilities towards deregulation and promoted competition. With increased competition in electric power markets, system operators are recognizing their pivotal role in ensuring the efficient operation of the electric grid and the maximization of social welfare.

In this research, we introduce a model of dynamic spatial network equilibrium among consumers, system operators and electricity generators as the solution of a dynamic Stackelberg game. In that game, generators form an oligopoly and act as Cournot-Nash competitors who non-cooperatively maximize their own profits. The market monitor attempts to increase social welfare by intelligently employing equilibrium congestion pricing anticipating the actions of generators. The market monitor influences the generators by charging network access fees that influence power flows towards a perfectly competitive scenario. Our approach anticipates uncompetitive behavior and minimizes the impacts upon society. The resulting game is modeled as a Mathematical Program with Equilibrium Constraints (MPEC). We present an illustrative example as well as a stylized 15-node network of the
Western European electric grid.

We also introduce two unique differential oligopolistic games with excess demand and sticky price dynamics for feedback Nash equilibria applied to electric power markets. The resulting differential Nash games can be reformulated into a standard linear quadratic differential game framework. The linear quadratic differential games can then be solved using optimal feedback control by subsequently solving a system of coupled algebraic feedback Nash Riccati equations. We introduce a computable algorithm to solve the feedback Nash equilibria of the differential games. The solution of a 28 market, 96 player differential game is presented using the proposed numerical algorithm.
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Chapter 1

Introduction

With increased competition in electric power markets, system operators are recognizing their pivotal role in ensuring the efficient operation of the electric grid and the maximization of net social welfare. For now, we define social welfare as a measure of the aggregate utility of a set of economic decisions upon a society. This need primarily stems from the fact that over the last two decades, the electricity industry has seen shifted from regulation of monopolistic and centralized utilities towards deregulation and promoted competition. These efforts were made by governments hoping a renaissance of competition among firms would lead to increased societal benefits such as lower prices, increased innovation and reduced barriers to entry (Nanduri and Das, 2009; Momoh, 2009). Therefore, the need for more advanced decision support models has arisen for both private and public parties. In this research, we provide a framework of model electricity power markets as both dynamic and differential games. Dynamic optimization and differential games provides increased model fidelity as well a robust framework to study inter temporal links that may go unseen otherwise.

We model the dynamic spatial network equilibrium among consumers, the mar-
ket monitor and electricity generators as the solution of a differential Stackelberg game. In the game, generators form an oligopoly and act as Cournot-Nash competitors who noncooperatively maximize their own profits. The market monitor acts as the Stackelberg leader and maximizes total economic surplus by deciding the access charges generators pay to transmit electricity. The differential game combining the leaders and the followers behaviors is expressed as a mathematical program with equilibrium constraints (MPEC). The market monitor is an independent entity charged with task of minimizing uncompetitive behavior in the market.

The advantage of this proposed mechanism is that the market monitor has new found intelligence instead of simply reacting to what dominant firms have already decided. The market monitor derives their strategic insights by employing a new market mechanism of equilibrium congestion pricing. The market monitor sets the charges of transmitting power not only to efficiently clear the transmission market (i.e. allocate transmission capacity with the least amount of congestion) but also to increase social welfare. Interestingly, our research has shown that it possible to increase social welfare with equilibrium congestion pricing when compared to the Cournot-Nash model. Thus, social welfare can be increased by allowing the market monitor to employ equilibrium congestion pricing.

1.1 Electricity Market Design

A key feature that distinguishes game-theoretic models of electricity markets is how the treatment of bid and transactions of power. Bid-based systems, typically referred to as POOLCO models, represent pool based systems where firms bid a supply curve to the central operator of the grid Ventosa et al. (2005). The supply
curve is typically submitted as quantity increments and corresponding prices. The central operator collects all bids from firms and then decides how best to operate the grid. The specific decisions depend on the planning horizon in which the operator is interested. For example, a day ahead planning model may be referred to as a unit commitment model (UC) where the central operator will inform each firm which of the generation assets will need to be operational the next day. The central planner may not yet know how much power will be required from each firm but allows firms to plan their own operations with greater certainty. An economic dispatch problem (EDP) is typically run either an hour or five minutes ahead of when the power will be consumed. In this scenario, the central operator dispatches the generator to produce certain amounts of power into the grid Wood and Wollenberg (1996). These models require a fixed demand for the planning-horizon of interest.

There are several pricing mechanisms to determine what price each generator receives. We refer the readers to (Meier, 2006; Momoh, 2009; Wood and Wollenberg, 1996) for a more detailed overview of the pricing mechanisms and the general operation of electricity markets. In contrast to pool based systems, bilateral transaction markets feature generators directly engaging in exchanging electricity and money. The central operator is therefore primarily concerned with the security and reliability of the market. Hybrid systems also exist in which pool based systems also allow bilateral transactions between agents.

Both the UC and EDP are tactical planning tools. The central planner in conjunction with the transmission grid operator must also, typically in parallel, solve an optimal power flow (OPF) model. OPF models usually resemble UC and EDP model closely with the addition of transmission, security and reliability constraints Momoh (2009). These constraints ensure the the resultant flows from the UC and
EDP model are feasible for the actual transmission network. One common area of infusibility is the overloading of the thermal capacity of a transmission line. Such an infeasibility, if allowed to transpire, would cause potential loss of power to customers as well as reduced reliability of the transmission network far beyond the local area surrounding the transmission line Meier (2006). In this dissertation, we provide models for both pool based and bilateral transaction markets with emphasis on the latter.

The market monitor, or often referred to as market monitoring and mitigating group, is an independent entity charged with minimizing uncompetitive behavior in electricity markets and ensuring the market power among participants does not endanger other grid participants (Güler and Gross, 2005). Market monitors have a plethora of tools at their disposable but a majority of them are ex-ante tools. A more detailed discussion about the role that market monitors play in electricity market may be found in (Rahimi and Sheffrin, 2003; Güler and Gross, 2005).

We use game theoretic models to facilitate the computation of interactions between market participants resulting in an equilibrium. The models serve as a decision support tool rather than as a replacement, to the economic dispatch, unit commitment or other scheduling models. There exists some overlap in the phenomena both types of models try to represent; however game theoretic models are proxies for real world models so that market designs and other theoretical exercises may be performed with increased fidelity. Game theoretic models typically do not intend to model all portions of the system models. Rather they focus on mimicking certain aspects of the model to explore various interactions and outcomes that result from competition and interaction of generators, consumers, operators and regulators. Game theoretic models do not perfectly represent the actual models that central planners use to operate the grid. However, they do provide insight
that allows the exploration of decision support and the potential for new market
design and operational policies.

Our work is primarily focused on the equilibrium of market participants. Equilibrium
models are especially suited for decision support since the resulting model
output is derived from the interaction of market participants rather than
specifically assuming behavior of other participants. Ventosa et al. (2005) makes the
distinction between equilibrium models and single firm optimization problems as
well as simulation models. Single firm optimization models assume a market par-
ticipant maximizes their own objection function, typically profit or social welfare,
within a known competing market. Market quantities, such as prices and quanti-
ties, are usually derived from functions given the single firm’s decision variables.
Simulation models represent market phenomena that may be too complex to model
in traditional optimization or equilibrium models. Simulation models are descrip-
tive models and offer insight how phenomena of interest may behave. In contrast,
prescriptive models such as optimization and equilibrium models, offer the modeler
information on how to make a decision.

We also focus on modeling imperfectly competitive markets as they offer the
biggest challenge for market participants and regulators in modeling decision and
behavior. One such area is that of oligopolistic competition where firms have the
ability to influence the price of electricity by their actions.

1.2 Welfare Economics

Game-theoretic models where the public are modeled as agents require a mecha-
nism to incorporate their preferences. The objective of maximizing the benefits
received by the public is not a trivial exercise. An analyst must consider the
Within the context of multiple-decision maker models, our work is concerned what Cohon (1978), pg 214 describes as “techniques for the aggregation of multiple preference orderings into a single ordering.” Creating such a function is crucial to provide computable solutions that are beneficial to the public and stakeholders. Welfare economics provides a branch of literature of creating methods to create a social ordering function. Cohon (1978) claims “aggregation rests on the definition of public interest in a democratic society as the combination of each individual’s interest.” The fundamental question to solve is how to select a set of actions that have an effect on the well being of various people? In this section, we provide a brief overview of how each branches of welfare economists have contributed to solving our decision maker problem. We follow the style presented in Cohon (1978).

We present the mathematical statement originally published in Samuleson (1947). The total amount of commodity $j$ that is consumed $i$ is defined as $q_{ij}$. The aggregate quantity of commodity is denoted by $Q_j$. The total amount of each commodity must be less than or equal to the aggregate availability of the commodity. The above constraint is enforced for all $n$ commodities. The constraint is written as

$$\sum_i q_{ij} - Q_j \leq 0, \quad j = 1, 2, \ldots n$$  \hspace{1cm} (1.1)

Each individual is endowed resources that may be supplied to the production process. We define $v_{ik}^*$ as the initial endowment of resource $k$ for individual $i$. $v_{ik}$ is the amount of resource $k$ that is supplied by individual $i$. A total of $m$ resources in total and the aggregate quantity of resource $k$ is $V_k$. An individual’s supply may not exceed their initial endowment described by
\[ v_{ik} \leq v^*_ik, \quad i = 1, 2, \ldots, p \quad \text{and} \quad k - 1, 2, \ldots, r \] (1.2)

The generation expression of how resources may be transformed into commodities for consumption is

\[ T(Q_1, Q_2, \ldots Q_n : v_1, v_2, \ldots v_p) = 0 \] (1.3)

Lastly, we require non negativity on all variables.

\[ q_{ij}, v_{ik}, Q_j \geq 0, \quad \forall i, j, k \] (1.4)

We seek to maximize social welfare. The term social welfare distinguishes one feasible social state from another. The concept of utility is closely related as social welfare is merely the aggregation of individual’s utility functions. The utility \( U_i \) on an individual is obtained from the commodities it consumes, hence \( U_i(q_i) \). Mathematically we can define a general social welfare function as

\[ \text{maximize} \quad U = [U_1(q_1), U_2(q_2), \ldots, U_p(q_p)] \] (1.5)

where \( U \) is a \( p \)-dimensional vector of utilities. One key assumption of the aggregated utility function is that individual’s utility is purely a function of quantity. The primary challenge is solving the above math program is how to structure \( U \). Some further assumptions are required on how to make value judgements between individual’s utility functions.
1.3 Preliminary Mathematics

1.3.1 Differential Nash Game

The concept of a differential Nash game is at the core of the type of model we set forth in the dissertation. We present the concise and abstract differential Nash game published in Friesz (2010).

Suppose there are \( N \) agents, each of which chooses a feasible strategy vector \( u^i \) from the strategy set \( \Omega_i \) which is independent of the other players’ strategies. Furthermore, every agent \( i \in [1, N] \) has a cost (disutility) functional \( J_i(u) : \Omega \rightarrow \mathbb{R}^1 \) that depends on all agents’ strategies where

\[
\Omega = \prod_{i=1}^{N} \Omega_i \tag{1.6}
\]

\[
u = (u^i : i = 1, ..., N) \tag{1.7}
\]

Every agent \( i \in [1, N] \) seeks to solve the problem

\[
\min J_i(u^i, u^{-i}) = K_i\left[ x^i(t_f), t_f \right] + \int_{0}^{t_f} \Theta_i(x^i, u^i, x^{-i}, u^{-i}, t) dt \tag{1.8}
\]

subject to
\[
\frac{dx^i}{dt} = f^i(x^i, u^i, t) \quad (1.9)
\]
\[
x^i(t_0) = x^i_0 \quad (1.10)
\]
\[
\Psi[x^i(t_f), t_f] = 0 \quad (1.11)
\]
\[
u^i \in \Omega_i, \quad (1.12)
\]

for each fixed yet arbitrary non-own control tuple

\[
u^{-i} = (u_j : j \neq i) \quad (1.13)
\]

where \(x^i_0\) is a vector of initial values of \(x^i\), the state tuple of the \(i^{th}\) agent and

\[
x^{-i} = (x_j : j \neq i) \quad (1.14)
\]
is the corresponding non-own tuple.

### 1.3.2 Complementarity

The principle of complementarity is a powerful tool in solving differential and dynamics games in electricity markets and several other domains Gabriel and Leuthold (2010). Complementarity allows for the efficient computation of certain types of optimization problems that model game-theoretic models. A prerequisite in understanding complementarity is the concept of orthogonality. The symbol \(\perp\) signifies orthogonality of two vectors. For example, consider the vectors \(A\) and \(B\) such that \(A = (a_1, a_2, \ldots, a_i)\) and \(B = (b_1, b_2, \ldots, b_i)\) with the same cardinality (i.e.
The orthogonality of $A$ and $B$, $0 \leq A \perp B \geq 0$ suggests:

\[ a_i \cdot b_i = 0 \quad \forall i, \]
\[ a_i \geq 0 \]
\[ b_i \geq 0 \]

Orthogonality describes a relationship of vectors where their product is zero but both have to be strictly positive. This property is especially convenient because we often see this relationship among constraints in optimization problems. It is seen when relating constraints with their associated dual variables. This occurs since a dual variable can only be greater than zero when a constraint is non-binding. A complementarity problem is a specific type of optimization that uses the principle of orthogonality. The properties of the underlying vectors describe the type of complementarity problem is constitutes. In this dissertation, we are concerned with mixed complementarity problems defined below

\[ y \text{ solves } MCP(h(\cdot, \cdot), B) \]

(1.15)

The variables $x$ refer to the upper level variables while the variables $y$ are associated with the lower level as described by the MCP above consisting of both the function $h(x, \cdot)$ and bounds $B$. The constraints $g$ can be a function of both types of variables. The variables $y$ are a solution to the MCP. A point $y$ with $a_l \leq y_l \leq b_l$ solves (1.15) if, for each $l$, at least one of the following holds
\[ h_t(x, y) = 0 \] \hspace{1cm} (1.16)
\[ h_t(x, y) \geq 0, y_t = a_t \] \hspace{1cm} (1.17)
\[ h_t(x, y) \leq 0, y_t = b_t \] \hspace{1cm} (1.18)

1.4 About This Dissertation

1.4.1 Motivation

Proposing a new market mechanism adds another potential decision support model in the arsenal that market monitors and other central planning agencies have in mitigating imperfect competition and market power in the new world of deregulated electricity markets. The proposed dynamic Stackelberg model can be used to analyze the potential benefits of incorporating new computational tools and models to the existing process of market monitoring. The optimal feedback control models are motivated by the curiosity of investigating the impact that the currently acceptable assumption of open-loop equilibrium models have upon the quality of solutions. Closed-loop models are typically regarded as a more realistic approximation of how agents behave in the real-world but extremely little work has been published in the electricity market literature. Worgin et al. (2012) provides one the few examples in which feedback equilibrium is achieved for electricity markets. As much as this paper pushes the envelope of the feedback literature in electricity markets, the paper presents analytical solutions of a single time period example. Our goal is to provide researchers and practitioners with a framework of computable models that can be both applied to electricity markets and closed-
loop equilibrium information structures while still have practical relevance to both communities.

### 1.4.2 Contributions

A key contribution to our work is the proposed market design where the market monitor serves as an intelligent agent representing both the operation of the power grid and societal contributor. The traditional role of the market monitor is that of penalizing generator’s behavior that it has deemed *uncompetitive*. This has been replaced with a proposal of anticipating both generators’ and consumers’ behaviors and reactions to intelligently use equilibrium congestion pricing to increase social welfare. We advance the state of the dynamic and differential game-theoretic models of electricity markets and networks. The dynamic Stackelberg model considers a new market design mechanism that also includes the following numerous realistic and computable features: oligopolistic competition, inter-temporal constraints, dynamic production constraints, time-varying demand, transmission constrained network and multi generator assets.

We present two models of oligopolistic competition solved using optimal feedback control. The models of competition employ excess demand and sticky price dynamics respectively. A large scale numerical algorithm is implemented to provide optimal feedback control of a 12 player, 7 markets differential Nash game. To the best of our knowledge, this is research presents the first large scale numerical results for feedback equilibrium of a differential electricity market game.
1.4.3 Organization

We organize the document as follows: Chapter 2 provides a brief literature review of welfare economics and industrial organization, dynamic and differential games, electricity markets and different type of dynamics relevant to electricity markets. Chapter 3 provides an overview of foundational non dynamic electricity market games. Chapter 4 presents a dynamic oligopolistic Cournot-Nash game. Chapter 5 sets forth a new paradigm in modeling how a proposed market monitor or central planners utilizes equilibrium congestion pricing to increase social welfare in oligopolistic competition as a dynamic Stackelberg game. Chapter 6 sets fourth a dynamic closed-loop optimal control of an electric power model as a standard linear quadratic optimal control problem. Chapter 7 provides numerical examples to all models presented in the dissertation with an emphasis placed on the new work presented in chapter 5 and 6. We end the document with conclusions and future work.
Literature Review

Our work is in the domain of computable game theory and equilibria with the specific application of electricity markets. Game theory allows us to model decision making under competition. The computability of our state-space game allows us to apply our framework to large problems that would not have been able to be accomplished with normal form games. Specifically, our framework lies within the area of differential games where the state of the game evolves with time according to a differential equation. Our application area of electricity markets poses unique challenges, primarily stemming from a physical transmission network connecting agents and properties of electricity generation, consumption and transmission. The economic conditions of electricity markets further serve as a connection between the electric grid and the computable framework we put forth. The subsequent sections divulge further details of the literature that exists in each of the above areas and fields of study.
2.1 Differential and Dynamic Games

Game theory can be originally attributed to the works of Von Neumann and Morgenstern (1944). However, the computable game theory framework that we build upon in this research is largely a result of Isaacs (1999). We have chosen to accept the widely-held definition that a dynamic game requires that the game evolves over time (Dockner et al., 2000). We utilize the normal form of games, or referred to as state-space games, where variables representing states describe the behavior of the game at any point in time (Mehlmann, 1988). Our differential game is a subset of dynamic games where we utilize ordinary differential equations to describe the evolution over time of the state of the game. We limit our analysis to noncooperative differential games where decision makers, or agents, do not enter in agreements with other agents. Our work also employs deterministic functions where stochasticity is not used but simple scenario-based perturbation can be employed to mimic levels of degrees of uncertainty. Furthermore, we analyze our dynamic Stackelberg game in open-loop equilibrium while our linear quadratic electricity game utilizes feedback equilibrium.

2.2 Electricity Markets

Electricity market models typically describe the generation, sales and flows of electricity on a transmission network. The literature varies with the information structure of the games, agents and markets as well as time horizon, degree of uncertainty and the objective or goal of the analysis. Ventosa et al. (2005) classified electricity market models into three distinct categories. First is the optimization problem for one firm, in which the firm assumes either an exogenous price or has
a known demand function. Second, the simulation models are descriptive models that attempt to describe market interaction typically via discrete-event simulation or agent based simulation. Lastly, market equilibrium firms consider all firms and are prescriptive in nature, and typically utilizing a Cournot or supply function equilibrium viewpoint. Cournot competition is a structure in which firms compete by deciding their quantities. Supply function equilibrium models the offer curves that generation firms submit to the system operator. In this research, we focus on the market equilibrium considering all firms, assuming Cournot competition. A more detailed survey of Cournot and supply function equilibrium models can be found in Hobbs (2001) and Day et al. (2002), respectively. Wu et al. (1996) provides a readable summary and interpretation of folk theorems that have been developed in the electricity market literature with respect to transmission access. Specifically, the authors provide details of “Nodal prices, congestion revenues, transmission capacity rights and compensation of transmission access.” Key formulations of the economic dispatch, optimal power flow models are provided for spot and bilateral transaction markets.

A common goal of system planners is to mitigate market power, typically done by incentivizing or penalizing firms that earn excess profit above a certain threshold, or are excessively depended upon for the successful operation of the grid. This concern is valid when the market exudes imperfect competition. Our research assumes imperfect competition. Specifically, we focus on oligopolies where a few firms dominate the market and can directly influence the price of electricity. Nanduri and Das (2009) and Blumsack et al. (2002) provides a brief survey of market mitigation and imperfect competition literature.

Rivier et al. (2001) presents a hydrothermal coordination assuming Cournot equilibrium and utilizes a Mixed Complementarity Problem (MCP) framework.
Jing-Yuan and Smeers (1999) used Variational Inequalities (VI) to analyze congestion in a spatial network assuming Cournot equilibrium. Mookherjee et al. Mookherjee et al. (2010) utilized a MCP to analyze a Cournot-Nash equilibrium between generation firms competing as oligopolies, simultaneously with a transmission clearing system operator.

Our research goes one step beyond imperfect competition and utilizes a Stackelberg framework. A Stackelberg game is a bilevel game, often represented as a Mathematical Program with Equilibrium Constraints (MPEC), where a leader has the advantage of deciding their variables before the followers decide their own variables; Ventosa et al. (2002); Murphy and Smeers (2005), Gabriel and Leuthold (2010) and Hobbs (2001) all provide unique contributions to the MPEC problem. For a more comprehensive review of literature of equilibrium in electricity markets see Ventosa et al. (2005); Nanduri and Das (2009); Daxhalet and Smeers (2001) and Hobbs and Helman (2004).

Hobbs et al. (2000) set forth a framework to model imperfect competition (electricity prices rise above marginal cost) of electricity markets via an MPEC approach. The model and procedure calculates oligopolistic price equilibria assuming ”supply function equilibrium” in which generators decide on their bid curves (acceptable price vs quantity generated for each firm) on the belief that rival generators will not change their own bid curves. Specifically, the model is relevant when the economy consists of several dominant firms. The dominant firm acts as the leader (upper level) of the MPEC. They choose their bid curves first and in anticipation of rival generator’s bids and the ISO’s actions. The ISO, modeled as the follower, solves a single commodity spatial price equilibrium problem. The ISO decides quantity of power as to maximize social welfare given the upper level bid curves. Quantity is an input to bid curves and price equilibrium emerges. Thus,
the dominant firms can strategically set their bid curves to maximize profit while anticipating the actions of the ISO and non dominant firms.

2.3 Feedback Optimal Control

Feedback control has its roots in antiquity with invention of a water clock by the Greek Ktesibious about 270 B.C. (Lewis, 1992). The clock uses a self regulated flow to produce a constant flow of water to a lower tank. A steady flow of water accumulates with known elapsed time by measuring the level of the bottom tank. Kwakernaak and Sivan (1972) provides a highly regarded text detailing the modern advances of linear feedback control theory dating back to the late 1950s. The authors emphasize the state space approach for linear feedback control problems. Modern advances refer to the inability of classical approaches to provide a solution to multi-input and multi-output systems as well as time-varying systems. Starr and Ho (1969) are recognized as key originators of nonzero-sum differential games with a specific focus on linear quadratic differential games. The authors built upon the foundation created by Isaacs (1999) with a generalization of zero-sum game theory. The authors also advanced the solution properties of the feedback finite-planning horizon linear quadratic differential game by providing a sufficiency condition for existence.

Lukes (1971) demonstrated that if a sufficiently small enough planning-horizon is chosen, the linear quadratic differential game has a unique linear feedback Nash equilibrium for every initial state. Also, Lukes provided a crucial method for solving feedback problems. He presented a computation scheme for the equilibrium solution by solving a system of feedback Nash Riccati differential equations. Papavassilopoulos and Olsder (1984) showed the case in which an infinite planning-
horizon game may have multiple, unique solutions with no equilibria although the comparable finite planning-horizon game has a unique feedback Nash solution. In the publication, the authors also provide a solution procedure for solving the feedback equilibrium related to algebraic Riccati equations (ARE) associated with the infinite-planning horizon game. Cruz and Chen (1971) and Gzgüber and Perkins (1977) provide practical insight in solving the feedback Riccati differential equations associated with the finite planning-horizon game. Li and Gajic (1995) provides an iterative scheme to calculate the feedback equilibrium of AREs associated with the infinite planning-horizon game. The algorithm relies on calculating an initial solution by successively relaxing the coupled nature of the system of AREs. Lyapunov iterations are then performed until the solution matrices have converged. The success of iterative algorithms to solve Riccati equations are generally dependent on good initial solutions and convergence properties. A more detailed review of linear quadratic differential games and optimal feedback control may be found in Engwerda (2005).

An obvious gap in the literature lies in providing computable models of feedback equilibria to game-theoretic models that can be applied to industries and research areas of interest. This dissertation focuses on electric power models, but the aforementioned statements apply to a broad set of topics. Another gap in the literature is the need for more ubiquitous results for a broader class of problem outside the realm of games with a linear quadratic structure. A small portion of that gap may be filled with an increased awareness and theory of reformulating non-standard game into standard games well established in the literature. Furthermore, publications in feedback optimal control have fallen behind on the size of large scale computations relative to math programming and optimization in general. Lastly, few articles exist in applying optimal control as well as feedback
equilibrium solutions applied to electricity markets. We strongly feel that a need exists to explore the assumptions that govern the information structure of game-theoretic models used to study electric power markets. Some areas that can reduce the barriers to implementing optimal feedback control in electric power markets include a broader class of models, techniques and computation tractability. In this dissertation, we provide remediation to all three barriers and advance the literature in applying optimal feedback control of electric power markets. It is our hope that this research inspires a broader participation between the rigorous literature of optimal feedback control and the diverse area of electric power markets.
Chapter 3

Non-Dynamic Electricity Market Games

In this chapter, we provide an introductory overview of non-dynamic electricity market games from the literature. These basic models provide a theoretical foundation for more advanced models covered in this dissertation leading up to our Stackelberg game presented in subsequent chapters.

3.1 Short Run and Perfectly Competitive Market

We present an introductory model, also presented in Chapter 11 of Gabriel et al. (2013). Generators $f$ and consumers are spatially located on a transmission network represented by nodes $i$. Generators are able to sell power to consumers located at different markets or node by paying to transmit power over the grid. The resulting equilibrium often produces results where prices vary from one location to another. We assume the market is perfectly competitive. The transmission
network is modeled as a linearized direct current (DC) network. We present a simple assumption that the transmission system operator (TSO) directs the flow of power from the generators to the consumers. It is assumed that the TSO is an arbitrage entity such that it buys power from markets where the supply is high and sells it to markets where supply is low. The short run time horizon of this model simply analyzes generations and sales while ignoring investment. The model for a single time period is provided below.

Each generator is assumed to be a price-take and therefore cannot influence the price of electricity with its decisions. The generator’s optimization problem is given by

\[
\begin{align*}
\text{Maximize} & \quad \sum_i p_i x_i - \sum_i C_{fi}(x_{fi}) \\
\text{subject to} & \quad x_{fi} - K_{fi} \leq 0 \quad \forall i \in I_f \\
& \quad x_{fi} \geq 0 
\end{align*}
\]

where \(I_f\) is the set of nodes that firm \(f\) has generation equipment. The generator \(f\) decides the value of \(x_{fi}\), representing the amount of electricity to be produced at node \(i\) in Mega-Watt (MW) units. \(C_{fi}(x_{fi})\) represents the generation cost at node \(i\). \(K_{fi}\) denotes the capacity limit and \(p_i\) represents the price of electricity at node \(i\). The dual variable \(\mu_{fi}\) is shown for computational purposes. The generator’s model ignores unit commitment or other inter temporal constraints that would typically be necessary for real world operation of the grid. As mentioned
previously, the consumer’s behavior is modeled as an inverse demand function denoted by

\[ q_i = f_{d,i}(p_i) \]  

(3.4)

where \( q_i \) is the quantity of electricity demand at each node \( i \).

The TSO is assumed to be a nonprofit entity that has an incentive to efficiently operate the grid. Specifically their role is to maximize the benefits that or the market receives from the transmission grid assets it controls. The TSO’s optimization problem is provided below.

\[
\text{Maximize}_{a_i} \quad \sum_i p_i a_i
\]  

(3.5)

subject to

\[
\sum_i a_i = 0 \quad (\gamma)
\]  

(3.6)

\[
\sum_i PTDF_{ik} a_i \leq T_k \quad (\lambda_k), \quad \forall k
\]  

(3.7)

where \( a_i \) denotes the sales of power to consumers at node \( i \) that did not originate at node \( i \). \( PTDF_{ik} \) are the power transfer distribution factors representing the proportion of power that flows on line \( k \) given a unit of power injected at the hub node. The use of these factors approximates the actual flows governed by nonlinear alternating current (AC) laws by assuming a linearized direct current (DC) network. \( T_k \) is the upper bound of electricity flow for each line \( k \) (Meier, 2006).

A market clearing condition connects each generator’s problem, the TSO’s op-
timization problem and the consumer’s behavior is required to achieve equilibrium. Here, market clearing is defined as when supply is matched with demand at each node $i$ and is given by

$$q_i - a_i - \sum_{f \in F_i} x_{fi} = 0 \quad (p_i \text{ unrestricted}), \quad \forall i$$ (3.8)

where $F_i$ is the set of firms with generation equipment at node $i$.

The solution of finding the equilibrium between all three types of agents can be modeled as a complementarity problem by taking advantage of the first order optimality conditions of both the generator’s and TSO’s optimization problems in conjunction with the dual variables with associated constraints. The resulting complementarity problem is given by

Find $\{x_{fi}, \mu_{fi}, q_i, a_i, \gamma, \lambda_k, p_i\}$ satisfying

$$0 \leq x_{fi} \perp p_i - \frac{dC_{fi}}{dx_{fi}} - \mu_{fi} \leq 0, \quad \forall f, i \in G_f$$ (3.9)

$$0 \leq \mu_i \perp x_{fi} - K_{fi} \leq 0, \quad \forall f, i \in G_f$$ (3.10)

$$q_i = f_{d,i}(p_i) \quad (q_i \text{ unrestricted}), \quad \forall i$$ (3.11)

$$\sum_i a_i = 0 \quad (\gamma \text{ unrestricted})$$ (3.12)

$$p_i - \gamma - \sum_k PTDF_{ik} \lambda_k = 0 \quad (a_i \text{ unrestricted}), \quad \forall i$$ (3.13)

$$0 \leq \lambda_k \perp \sum_k PTDF_{ik} a_i - T_k \leq 0, \quad \forall k$$ (3.14)

$$q_i - a_i - \sum_f x_{fi} = 0 \quad (p_i \text{ unrestricted}), \forall i$$ (3.15)
3.2 Cournot Model of Oligopolistic Competition

In this section, we present a natural extension to the short run, perfectly competitive equilibrium model with an oligopolistic Cournot model (Gabriel et al., 2013). A key feature of oligopolistic competition is that firms possess the ability to influence prices by their actions. Any form of competition differing from perfect competition is often referred to as imperfect competition. Typically, by exerting market power, firms in imperfect competition can raise prices above the marginal cost of the commodity.

In this model, it is assumed that generators have the ability to sell both to consumers, often referred to as bilateral contracts, as well as central buyers, such as the TSO. In Cournot competition firms simultaneously decide on their output while believing that their actions will not influence those of their competitors. The generators play a quantity game in which they decide on the quantity of sales at each market node \( i \). It is also assumed that generators pay the TSO for transmission services by a per unit value of \( w_i \). This fee is the cost to transmit power from an arbitrary hub node to node \( i \). Thus a generator’s profit function includes terms of revenue, generation cost and transmission. Generators do not believe that altering their output will influence the fee \( w_i \). The model also adopts the previously mentioned system of the TSO acting as an arbitrage entity. We assume the same linearized DC network and PTDF mentioned in the previous section.

Consumers are again represented by an inverse demand curve below.

\[
p_i = f_{d, i}^{-1} (a_i + \sum_f s_{fi}) \tag{3.16}
\]

where \( p_i \) is the price of electricity. The function considers the quantity of electricity
supplied to the market by both the amount $a_i$ supplied by the TSO arbitrage actions and the sum of sales $s_{fi}$.

The generator model is given by

$$\text{Maximize } \sum_i \left[ f_{d,i}^{-1}(a_i + \sum_f s_{fi}) - w_i \right] s_{fi} - \left[ \sum_{i \in I_f} C_{fi}(x_{fi}) - w_i x_{fi} \right]$$

(3.17)

subject to

$$x_{fi} - K_{fi} \leq 0 \ (\mu_{fi}) \ \forall i \in I_f$$

(3.19)

$$\sum_i s_{fi} - \sum_{i \in I_f} x_{fi} = 0 \ (\phi_f)$$

(3.20)

$$x_{fi} \geq 0 \ \forall i \in I_f$$

(3.21)

Similar to the perfectly competitive model, the TSO facilitates bilateral transactions with the variable $y_i$ and arbitrage transactions are denoted by $a_i$. The TSO transacts power such that the difference of price between two nodes is equal to the difference of the cost of wheeling electricity between them. The model is given by

$$\text{Maximize}_{a_i,y_i} \ \sum_i p_i a_i + \sum_i w_i y_i$$

(3.22)

subject to
\[ \sum_i a_i = 0 \quad (\gamma) \quad (3.23) \]

\[ \sum_i PTDF_{ik}a_i \leq T_k \quad (\lambda_k), \quad \forall k \quad (3.24) \]

The market clearing equilibrium condition requires that the amount of electricity that is transmitted \(y_i\) is balanced with the net sales for each node \(i\), as stated below.

\[ y_i - \sum_f (x_{fi} - s_{fi}) = 0 \quad (w_i), \quad \forall i \quad (3.25) \]

Concatenating and applying the same complementarity process as seen in the perfectly competitive model results in the following complete model:
Find \( \{x_f, s_{fi}, \mu_{fi}, \phi_f, q_i, y_i, a_i, \gamma, \lambda_k, w_i, p_i\} \) satisfying

\[
0 \leq x_{fi} \perp - \left( \frac{dC_{fi}}{dx_{fi}} - w_i + \mu_{fi} \right) + \phi_f \leq 0, \quad \forall f, i \in I_f \tag{3.26}
\]

\[
0 \leq s_{fi} \perp \left( p_i + s_{fi} \frac{dp_i}{ds_{fi}} \right) - (w_i + \phi_f) \leq 0, \quad \forall f, i \in I_f \tag{3.27}
\]

\[
0 \leq \mu_i \perp x_{fi} - K_{fi} \leq 0, \quad \forall f, i \in I_f \tag{3.28}
\]

\[
\sum_i s_{fi} - \sum_{i \in I_f} x_{fi} = 0 \quad (\phi_f \text{ unrestricted}) \tag{3.29}
\]

\[
p_i - \gamma - \sum_k PTDF_{ik} \lambda_k = 0 \quad (a_i \text{ unrestricted}), \quad \forall i \tag{3.30}
\]

\[
w_i - \sum_k PTDF_{ik} \lambda_k = 0 \quad (y_i \text{ unrestricted}), \quad \forall i \tag{3.31}
\]

\[
\sum_i a_i = 0 \quad (\gamma \text{ unrestricted}) \tag{3.32}
\]

\[
0 \leq \lambda_k \perp \sum_k PTDF_{ik} a_i - T_k \leq 0, \quad \forall k \tag{3.33}
\]

\[
y_i - \sum_f (x_{fi} - s_{fi}) = 0 \quad (w_i \text{ unrestricted}), \quad \forall i \tag{3.34}
\]

\[
p_i - f_{d,i}^{-1}(a_i + \sum_f s_{fi}) = 0 \quad (p_i \text{ unrestricted}), \quad \forall i \tag{3.35}
\]

where

\[
p_i + \frac{dp_i}{ds_{fi}} = f_{d,i}^{-1}(a_i + \sum_f s_{fi}) + \frac{df_{d,i}^{-1}(a_i + \sum_f s_{fi})}{ds_{fi}} \tag{3.36}
\]

is the marginal revenue term commonly found in Cournot oligopoly models.
3.3 Conjectured Supply Function of Oligopolistic Competition in Power Networks

Day et al. (2002) present a unique view on how generation firms anticipate their competitor’s sales in equilibrium under the assumption of oligopolistic competition. Each generator makes a conjecture on how a competing firm will react to their own changes in sales. Day et al. argue that the conjectured supply function approach is more general than the Cournot equilibrium and can handle circumstances in which the demand elasticity is zero. In this section, we present the POOLCO model derived by Day et al. (2002). POOLCO is a pool based market mechanism where in each firms bids a supply function for each of their generation assets. The TSO, or independent system operator (ISO), then selects or dispatches an amount of electricity that each generator may produce and inject into the transmission network, depending on the time horizon considered.

We begin with introductory notation and definitions. The anticipated sales $s_{-fi}$ of its competing firms is defined as

$$ s_{-fi} = \sum_{g \neq f} s_{gi} \quad (3.37) $$

while the sales $s_{fi}$ is defined as

$$ s_{fi} = \sum_{h} g_{ifh}, \quad \forall i, f \quad (3.38) $$

where $s_{fi}$ is the sales of firm $f$ to node $i$. The total sales to consumers is also defined as
\[ q_i = s_{fi} + s_{-fi} + a_i \]  \hspace{1cm} (3.39)

where \( a_i \) is the net sales of the arbitrage agents.

The generator equilibrium model is given by

\[
\max \Pi_f = \sum_i \sum_h \left[ p_{fi}(s^*_{-fi}, p^*_i, \sum_n g_{fin}, a_{fi}) - C_{fih} \right] g_{fih} \hspace{1cm} (3.40)
\]

subject to

\[
p_{fi}(s^*_{-fi}, p^*_i, \sum_n g_{fin}, a_{fi}) - p_{f, hub}(s^*_{-fi}, p^*_i, \sum_n g_{fin}, a_{fi}) - w^*_i = 0 \hspace{1cm} \forall i \neq hub \hspace{1cm} (3.41)
\]

\[
\sum_i a_{fi} = 0 \hspace{1cm} (3.42)
\]

\[
g_{fih} \leq G_{ifh} (\mu_{ifh}) \hspace{1cm} \forall i, h \hspace{1cm} (3.43)
\]

\[
g_{fih} \geq 0 \hspace{1cm} \forall i, h \hspace{1cm} (3.44)
\]

where \( p_{fi} \) is the price firm \( f \) anticipates at node \( i \). In equilibrium, \( p_{fi} \) are equal \( \forall f \) at every node \( i \). The quantity \( p^*_i \) represents the equilibrium value of price at every node. Each firm produces electricity \( g_{fih} \) for every node \( i \) that it poses \( h \) generation facilities.

Each firm anticipates how the TSO or (ISO) will modify the bids. The ISO maintains a locational marginal pricing (LMP) relationship such that the price at every node \( i \) is equivalent to the price to transfer electricity to the hub node plus the cost of transmission. Mathematically, the LMP relationship is stated as
\[ p_{hub}^* + w_i^* = p_i^* \quad \forall i \neq \text{hub} \quad (3.45) \]

The market clearing conditions for the POOLCO market are defined as

\[ y_i = a_i \quad \forall i \quad (3.46) \]

\[ p_{fi}(s_{-fi}^*, p_i^*, \sum_n g_{fin}, a_i) = p_i^* \quad \forall i, f \quad (3.47) \]

\[ s_{-fi}^* = \sum_{m \neq f} \sum_h g_{fh} \quad \forall i, f \quad (3.48) \]

\[ a_i = a_{fi} \quad \forall i, f \quad (3.49) \]

The transmission feasibility conditions are

\[ w_i^* - \sum_k PTDF_{ik} \lambda_k = 0 \quad \forall i \quad (3.50) \]

\[ 0 \leq \lambda_k \perp \left[ \sum_i PTDF_{ik} y_i - T_k \right] \leq 0 \quad (3.51) \]

where \( PTDF_{ik} \) is the proportion of power that flows to transmission line \( k \) given an injection of power at the hub node and withdrawal at node \( i \). \( T_k \) is the capacity of transmission line \( k \). \( w_i^* \) is the equilibrium value of the cost to transmit electricity to node \( i \).

The POOLCO equilibrium model is formulated by combining the Karush Kuhn Tucker (KKT) conditions with the market clearing conditions and the transmission feasibility conditions. The resulting model is a mixed complementarity problem (MCP).
Dynamic Oligopolistic Competition
in a Electric Power Network

Below we present the Cournot-Nash equilibrium as published in Mookherjee et al. (2010) in which generators, under oligopolistic competition, sell electric power over a spatially distributed electricity network. The Independent System Operator (ISO) acts as a central nonprofit entity that oversees the operation of the electricity, specifically over the transmission lines connecting the nodes, each representing an electricity market. In the model framework presented, the ISO controls the transmission line capacity by utilizing congestion pricing such that generators are cost prohibited from transmitting electricity beyond the thermal limits of the transmission lines. These fees are referred to wheeling fees and represent the price per unit of electricity transmitted between locations in the network. The competing generators receive revenue from selling electricity by allocating a quantity of electricity that is generated and sold at each node of the network. The resulting game is modeled such that each generator has a unique, discrete math program. The equilibrium problem is molded as a system of coupled math programs represented
by a nonlinear complementarity problem (NCP).

A key feature of the model is that a dynamic approach is taken. As opposed to static models, the dynamic model represents constraints and equilibrium evolving over time. Furthermore, the physical limit of ramping rate of the electrical generation equipment is considered as a critical constraint. The model incorporates the common assumption of a linearized DC load flow network utilizing power transmission distribution factors (PTDFS) representing the approximate proportion of power routed to each transmission line given an injection of electricity at any node.

### 4.1 Notation

A summary of the notation used in Mookherjee et al. (2010) is provided in this section.

<table>
<thead>
<tr>
<th>Sets</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A}$</td>
<td>set of transmission lines (arcs) in the network</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>set of generating firms</td>
</tr>
<tr>
<td>$\mathcal{M}$</td>
<td>set of nodes at which there are markets for power</td>
</tr>
<tr>
<td>$\mathcal{N}$</td>
<td>set of nodes in the network</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Variables</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{i,t}^f$</td>
<td>allocation of output to consumption by firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$ in period $t$</td>
</tr>
<tr>
<td>$q_{i,t}^f$</td>
<td>generation in MW by firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$ in period $t$</td>
</tr>
<tr>
<td>$r_{i,t}^f$</td>
<td>ramping rate of unit owned by firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$</td>
</tr>
<tr>
<td>$V_{i,t}^f$</td>
<td>generation cost for firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$ in period $t$</td>
</tr>
<tr>
<td>$w_{i,t}$</td>
<td>wheeling fee for transmission of power into or out of market $i \in \mathcal{M}$ in period $t$</td>
</tr>
<tr>
<td>$\pi_{i,t}$</td>
<td>inverse demand function for market $i \in \mathcal{M}$ in period $t$</td>
</tr>
<tr>
<td>$V_{i,t}^f$</td>
<td>generation cost function for firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$</td>
</tr>
<tr>
<td>$R_{i,t}^f$</td>
<td>ramping cost for firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$ in period $t$</td>
</tr>
<tr>
<td>Parameters</td>
<td>Description</td>
</tr>
<tr>
<td>------------</td>
<td>-------------</td>
</tr>
<tr>
<td>$b_{1,i}^f$</td>
<td>Intercept of first section of two piece generation cost for firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$</td>
</tr>
<tr>
<td>$b_{2,i}^f$</td>
<td>Intercept of second section of two piece generation cost for firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$</td>
</tr>
<tr>
<td>$m_{1,i}^f$</td>
<td>Slope of first section of the two piece generation cost for firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$</td>
</tr>
<tr>
<td>$m_{2,i}^f$</td>
<td>Slope of second section of the two piece generation cost for firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of time periods</td>
</tr>
<tr>
<td>$PTDF_{i,a}$</td>
<td>Power transmission distribution factors giving MW flow through arc $a \in \mathcal{A}$ due to a unit</td>
</tr>
<tr>
<td>$q_{i,\text{max},t}^f$</td>
<td>Upper bound on generation for firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$ in period $t$</td>
</tr>
<tr>
<td>$r_{i,\text{min}}^f$</td>
<td>Minimum ramp rate of generator owned by firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$</td>
</tr>
<tr>
<td>$r_{i,\text{max}}^f$</td>
<td>Maximum ramp rate of generator owned by firm $f \in \mathcal{F}$ at market $i \in \mathcal{M}$</td>
</tr>
<tr>
<td>$T_{a,t}$</td>
<td>Transmission capacity of arc $a \in \mathcal{A}$ in period $t$</td>
</tr>
<tr>
<td>$a_{i,t}$</td>
<td>Fixed term of inverse demand function at node $i \in \mathcal{M}$ in period $t$</td>
</tr>
<tr>
<td>$b_{i,t}$</td>
<td>Linear coefficient of inverse demand function at node $i \in \mathcal{M}$ in period $t$</td>
</tr>
</tbody>
</table>

The following vector concatenations are used, when applicable, to simplify the notation.

\[ q^f : q_{i,t}^f \text{ for all } i \in \mathcal{M}, \; t = 0...N \]
\[ c^f : c_{i,t}^f \text{ for all } i \in \mathcal{M}, \; t = 0...N \]
\[ r^f : r_{i,t}^f \text{ for all } i \in \mathcal{M}, \; t = 1...N \]
\[ w : w_{i,t} \text{ for all } i \in \mathcal{M}, \; t = 0...N \]
4.2 Generator’s Extremal Problem

In this section, we present the optimization problem that each generator solves. Each firm maximizes profit subject to its production constraints. Each generator must consider the actions of all other generators. Each generator’s optimization problem is

\[
\max J_1 \left( c^f, q^f; r^f; c_{-f}, w \right) = \sum_{t=0}^{N} \sum_{i \in M} \left\{ \pi_{i,t} \left( \sum_{g \in F} c_{i,t}^g \right) \cdot c_{i,t}^f - V_{i,t}^f \left( q_{i,t}^f \right) \right. \\
\left. - w_{i,t} \cdot \left( c_{i,t}^f - q_{i,t}^f \right) \right\} 
\]

subject to

\[
\sum_{i \in M} q_{i,t}^f = \sum_{i \in M} c_{i,t}^f \quad \text{for all } t = 0 \ldots N \quad (4.2)
\]

\[
\pi_{i,t} = a_{i,t} - b_{i,t} \sum_{g \in F} c_{i,t}^g \quad \text{for all } i \in M, \ t = 0 \ldots N \quad (4.3)
\]

\[
V_{i,t}^f = \max \left( m_{1,i}^f q_{i,t}^f + b_{1,i}^f, \ m_{2,i}^f q_{i,t}^f + b_{2,i}^f \right) \quad \text{for all } i \in M, \ t = 0 \ldots N \quad (4.4)
\]

\[
r_{i,t}^f = q_{i,t-1}^f - q_{i,t}^f \quad \text{for all } i \in M, \ t = 1 \ldots N \quad (4.5)
\]

\[
0 \leq q_{i,t}^f \leq q_{i,t}^{\text{max}} \quad \text{for all } i \in M, \ t = 0 \ldots N \quad (4.6)
\]

\[
r_{i,\text{min}}^f \leq r_{i,t}^f \leq r_{i,\text{max}}^f \quad \text{for all } i \in M, \ t = 1 \ldots N \quad (4.7)
\]

The objection function of profit maximization includes the terms of revenue, generation cost and wheeling fees paid to the ISO. The generators sell directly to consumers of electricity and thus a bilateral market is modeled. The wheeling fee represents the per unit cost of transmitting electricity from the hub node to the node of interest in the network. The wheeling fee may be positive or negative.
The ISO determines the wheeling fee as a result of the interactions between the generators and the thermal limits of the transmission lines.

The generators assume the wheeling fees are exogenous parameters in their own optimization problem. The generators are in equilibrium with each other by playing a Cournot, or quantity, game with each other. Specifically, each generator decides how much quantity to provide to each market and the resulting actions determines the market prices and subsequent flow and profit terms. The coupling arises in the inverse demand function \( \pi \). Given the formulation for the generators’ problems, we will now show the problem the ISO solves.

### 4.3 ISO’s Problem

The ISO sets the wheeling fees \( w \) to ensure generators do not exceed the thermal limits of the transmission line network. The ISO accomplishes this task by solving the following math program

\[
\begin{align*}
\max J_2(t) &= \sum_{i \in M} y_{i,t} \cdot w_{i,t} \\
\text{subject to} & \sum_{i \in M} PDF_{i,a} \cdot y_{i,t} \leq T_{a,t} \text{ for all } a \in \mathcal{A}
\end{align*}
\]

where \( \mathcal{A} \) is the arc set of the electric power network, \( T_{a,t} \) is the transmission capacity on arc \( a \in \mathcal{A} \) at time \( t \) where as \( PDF_{i,a} \) is the power transmission distribution factors (PTDF) that describe how much MW flow occurs through arc \( a \) as a result of a unit MW injection at an arbitrary hub node and a unit withdrawal at node \( i \).
4.4 Complete NCP Formulation

Concatenating both the generators’ and the ISO’s problems, a nonlinear complementarity problem is formulated for both the generator’s equilibrium problem and the ISO by employing the necessary conditions derived from the Karush-Kuhn-Tucker (KKT) conditions. Concatenating both NCPS results in the following complete NCP formulation

\[
\begin{bmatrix}
2b_{i,t}c^f_{i,t} - a_{i,t} + b_{i,t} \sum_{g \in \mathcal{F}, g \neq f} c^g_{i,t} - w_{i,t} - \zeta^+_t + \zeta^-_t \\
\zeta^+_t - \zeta^-_t + m^f_{1,i} \gamma_{i,t} + m^f_{2,i} \eta_{i,t} + \sigma^f_{i,t} - \mu^f_{i,t} + \theta^f_{i,t} \\
1 - \gamma^f_{i,t} - \eta^f_{i,t} \\
- \sum_{i \in \mathcal{M}} q^f_{i,t} + \sum_{i \in \mathcal{M}} c^f_{i,t} \\
- \sum_{i \in \mathcal{M}} q^f_{i,t} - \sum_{i \in \mathcal{M}} c^f_{i,t} \\
V^f_{i,t} - m^f_{1,i} (q^f_{i,t}) - b^f_{1,i} \\
V^f_{i,t} - m^f_{2,i} (q^f_{i,t}) - b^f_{2,i} \\
- q^f_{i,t} + q^f_{i,\max} \\
q^f_{i,t} - q^f_{i,t-1} - r^f_{i,\min} \\
- q^f_{i,t} + q^f_{i,t-1} + r^f_{i,\max} \\
T_a - \sum_{i \in \mathcal{M}} PDF_{i,a} \cdot \sum_{f \in \mathcal{F}} (s^f_{i,t} - q^f_{j,t})
\end{bmatrix}
= G(y) \perp y =
\begin{bmatrix}
c^f_{i,t} \\
q^f_{i,t} \\
V^f_{i,t} \\
\zeta^+_t \\
\zeta^-_t \\
\gamma^f_{i,t} \\
\eta^f_{i,t} \\
\phi^f_{i,t} \\
\delta^f_{i,t} \\
\mu^f_{i,t} \\
\sigma^f_{i,t} \\
\theta^f_{i,t} \\
\alpha^f_{a,t}
\end{bmatrix}
\tag{4.10}
\]

The resulting NCP can be efficiently and optimally solved by commercially available software such as the PATH solver in GAMS. We refer the reader to Mookherjee et al. (2010) for more details regarding the presented dynamic game.
In this chapter, we set forth a proposed market mechanism that serves as an additional tool for market monitors to ensure uncompetitive behavior is minimized. The proposed framework may serve as a starting point for a new class of models that aids market monitors to anticipate and prevent uncompetitive behavior ex-ante any electricity is transmitted. This is in contrast to most of the tools available to market monitors that look ex-post to the alleged uncompetitive behavior.

Specifically, we presented a model in which the market monitor employs equilibrium congestion pricing of a bilateral transaction market in which generators and consumers are in equilibrium.

We consider an electric power grid operating with consumers, a market monitor and generators forming an oligopoly. Specifically, each generator possesses the power of influencing the price of electricity with their own production and sales plan. Our consumers are generalized to include retail consumers, utilities and Load Serving Entities (LSE). We assume the market monitor utilizes equilibrium
congestion pricing for the purposes of clearing the transmission market to prevent
generators from exceeding the physical limitations of the network. Furthermore,
we assume the market monitor has the power to use equilibrium congestion pric-
ing to represent society’s interests as a whole; with the objective of maximizing
economic surplus or commonly referred to as social welfare. Each generator max-
imizes individual profit in a Cournot-Nash game with other generators given the
access charges the market monitor has set forth. Our unique Stackelberg approach
models the market monitor as a single leader, with the generators acting as follow-
ers. The leader has complete anticipatory knowledge of the generators’ equilibrium
problem and decides access charges such that the generators produce a production
and sales schedule that is optimal from a societal perspective. Furthermore, we
model the interaction of the game’s agents with the use of dynamics. The dynamic
approach allows us to represent a higher fidelity model, which advances the level of
market design tools available for the system operators to analyze the competitive
implications of oligopolies in electric power markets.

We start with presenting a general dynamic Stackelberg game of a electric
power oligopoly. The market monitor is represented as a leader maximizing social
welfare. The lower level consists of a Cournot-Nash equilibrium among genera-
tion firms. The chapter begins with preliminary notation and assumptions while
Section 5.2 presents a continuous time formulation of dynamic Stackelberg game.
Section 5.3 contains our discrete formulations of the Stackelberg game as well as
the Mathematical Program with Equilibrium Constraints (MPCC) into which we
reformulate our original game. We conclude the chapter with the proposed future
work that this dissertation aims to accomplish.
5.1 Notation and Assumptions

The price of electricity ($/MWH) of each node where electricity is consumed is a known function of sales and continuous time since we assume an oligopoly market structure. $\pi(t)$ is defined as

$$\pi(t) \in L^2[t_0, t_f]$$

where $L^2[t_0, t_f]$ is the space of square-integrable functions. Moreover, we stipulate that the price is a square-integrable function of time. We further assume that every firm is an oligopoly and that no firms are price-takers. An oligopoly is an economic market structure in which firms can influence the market price through their own sales and generations while price-takers do not have any influence on price and sell at a price determined by the market. Our model is general enough to include price-takers but are omitted at this point to convey a homogenous market structure. Furthermore, we assume our decision structure is deterministic and open loop. “Open loop” signifies that firms simultaneously determine the decision variable for all time periods within the planning horizon. We further assume perfect initial information and a finite time interval $[t_0, t_f] \subseteq \mathbb{R}_+^1$, where $t_0 \in \mathbb{R}_+^1$ is the fixed initial time, $t_f \in \mathbb{R}_+^f$ is the fixed terminal time and $t_f > t_0$.

The ramping rate $r(t)$ describes each generators’ instantaneous rate of change of output $q(t)$ with respect to time. Each generator’s output rate is $q(t)$ with associated generation cost $V(q(t))$ with producing $q(t)$ units of electricity (MW). The rate of power sales is denoted as $c(t)$ while $w(t)$ represents the access charges that the market monitor charges to transmit power from the hub node to the node.
of interest. The controls

\( r(\cdot) \in L^2[t_0, t_f] \)
\( c(\cdot) \in L^2[t_0, t_f] \)
\( w(\cdot) \in L^2[t_0, t_f] \)

determine the generators’ dynamics \( q(t) \)

\[
\frac{dq_i^f(t)}{dt} = r_i^f(t) \quad (5.1)
\]

which may also be expressed as the operator

\[
q(r(t) : L^2[t_0, t_f] \times L^2[t_0, t_f] \rightarrow H^1[t_0, t_f]
\]

where \( H^1[t_0, t_f] \) is Sobolev space for the real interval \( [t_0, t_f] \in \mathbb{R}_+^1 \). We impose the following upper bounds on generation and ramping respectively

\[
q_{\text{max}} \in \mathbb{R}_{++}^f \\
r_{\text{max}} \in \mathbb{R}_{++}^f \\
r_{\text{min}} \in \mathbb{R}_{++}^f
\]

Our model focuses on spatial equilibrium since any electric grid of interest spans a network. We denote each node of the network using the index \( i \) that belongs to the set \( N \) consisting of every node of the network. We also define the set of nodes \( M \) in which there are markets for power since it is possible to sell and generate
power at different nodes simultaneously. Furthermore, we assume a linearized DC power flow as an approximation to the real world AC flow electric grid as published in Scheppe et al. (1988). This common model approximation allows us to easily represent Kirchoff’s laws with a parameter $PTDF_{i,a}$ representing the proportion of power that flows on each transmission line of the network when power is transmitted to node $i$. Furthermore, firms behave non-cooperatively (i.e. no collusion).

A summary of sets, variables and parameters is shown below in continuous time:

---

**Sets**

$F$ Set of generating firms  
$N$ Set of nodes in the power network  
$M$ Set of nodes at which there are markets for power  
$A$ Set of transmission lines (arcs) in the network  
$T$ Set of time periods in planning horizon  

**Variables**

$q_i^f(t)$ Generation in MW by firm $f \in F$ at node $i \in N$  
$c_i^f(t)$ Sales (consumption) in MW by firm $f \in F$ at market $i \in M$  
$w_i(t)$ Access Charge ($$/MW) for market $i \in M$  
$r_i(t)$ Ramping rate for firm $f \in F$ at node $i \in N$  
$\pi_i(c,t)$ Inverse demand function($$/MW) at market $i \in M$  
$V_i^f(q_t,t)$ Generation cost function for firm $f \in F$ at node $i \in N$  

---

**Parameters**
Transmission capacity of arc $a$

$q_{i,\text{max}}^f$ Upper bound of generation $f \in F$ at node $i \in N$

$PTDF_{i,a}$ Describes how much MW occurs through transmission line (“arc”) $a$ as a result of a unit MW injection at the hub node and a withdrawal at node $i$.

$r_{i,\text{min}}^f$ Minimum ramping for firm $f \in F$ at node $i \in N$

$r_{i,\text{max}}^f$ Maximum ramping for firm $f \in F$ at node $i \in N$

We distinguish $\pi_i(c,t)$ and $V_i^f(q,t)$ as explicit functions that have both the arguments $c_i^f(t)$ and $q_i^f(t)$ respectively and the time $t$ as a parameter. The variables and parameters presented above will be used in the subsequent section in discrete time by substituting $t$ as a subscript for a continuous function.

5.2 Dynamic Stackelberg Game

We assume the market monitor uses its influence to maximize social welfare or economic surplus. In this work we assume that the only means of reaching their objective is by way of enforcing access charges paid by the generators per MW of electricity transmitted on the network.

We define access charges as tariffs per units of power transmitted from the hub node to the node of interest. The hub node is single location within the network that all electricity sales are assumed to pass through in order calculate the cost of transmission. For example, if 1 MW is sold from a generator at location A to a consumer at location B, for transmission pricing purposes only, the transaction is divided into two transactions: a 1 MW transfer from location A to the hub node, and a 1 MW transfer from the hub node to location B. The hub node is arbitrarily chosen and the charge can be simply thought as the price to transmit one MW of
electricity to the desired node of sales or consumption. We define the access charge to transmit to a node may be either positive or negative.

Every feasible set of unique access charges influences each generator’s profit and thus, the Cournot-Nash equilibrium of the generators as a whole. The generators take the access charges as exogenous variables and play a Cournot-Nash game with other generators to maximize their individual profit. The consumers are represented by an inverse demand function at each node, making the game a complete market. Standard congestion pricing formulations set the price of transmitting electricity equal to the difference of marginal pricing between nodes. Locational marginal pricing (LMP) is one of such pricing schemes (Wood and Wollenberg, 1996). Thus, the price of electricity is equal at every node when combined with the transmission cost for each node. We propose a new congestion pricing scheme where the access charge is the market monitor’s decision variable. Our formulation gives the responsibility of the market monitor to set the access charges so that: 1) Generators do not transmit electricity beyond the capacity of each transmission line; and 2) The electricity flows between generators and consumers maximize net economic surplus.

5.2.1 Market Monitor’s Upper Level Problem

Social welfare, or interchangeably economic surplus, represents the benefits that all agents in a market receive from the economic participation of purchasing and selling goods. Gabriel et al. (2013) states that "It is the standard measure of market efficiency.” It is the sum of consumer, producer and market monitor surplus. Each surplus is defined as the monetary gain experienced from purchasing/selling a good for less/more than what they are willing to pay/sell for the good. The consumer
surplus can be thought of the gain each consumer receives from willing to pay for a quantity of electricity less what they actually paid for it. It can also be thought as the psychological or perceived benefit derived from consuming a good beyond the opportunity cost to purchase it. Producer surplus is completely synonymous with its profit in producing and selling electricity. Note that in addition to the generation cost of producing electricity, each generator must also pay an access charge to transmit the electricity it produces. The market monitor surplus must also be considered since it is an active player in the economy of interest. The market monitor surplus is derived from the network assess charge revenue it collects from the generators. Thus the gain of the market monitor surplus is 100% at the loss of the generator surplus. This access charge is specifically included in the calculation of producer’s surplus as seen later within this section. We directly stipulate for a given time period that the total market monitor surplus must be non negative. A negative market monitor surplus would indicate that market monitor would subsidize the users of the network and thus require outside funding. In practice, it may be desired for the market monitor to act as a non-profit entity serving the needs of the economy as a whole and thus the revenue would be returned to the users of the network via a financial mechanism.

We present a general form of social welfare that can be evaluated at equilibrium values of power consumption, generation and access charges. The market monitor’s objective function is to maximize social welfare $SW(c^*, q^*)$ for every node where a total of $c^*$ units of power were sold, $q^*$ units of power were generated and $w^*$ dollars of access charges summed for every firm, node and time period in the planning horizon. We, for the time being, drop the subscripts for nodes, firms and time for the sake of clear exposition. $SW(c^*, q^*)$ can be generally defined as the summation of all surpluses associated with each agent in the market economy. Our
problem of interest consists of consumers, generators (also referred to as producers) and the market monitor. We define consumer surplus $CS(c^*)$ as

$$CS(c^*) = \int_0^{c^*} \pi(x)dx - [\pi(c^*) \cdot c^*]$$  \hspace{1cm} (5.2)$$

where $c^*$ is the equilibrium value of sales, $\pi$ is the inverse demand function and $\pi(c^*)$ is the equilibrium price that the consumers and producers pay and receive respectively. The first term denotes integrating every consumer’s benefit derived from consuming electricity from the the first unit of electricity up to $c^*$ units. Note that consumers do not receive any benefits beyond $c^*$ units simply because we define $c^*$ as the equilibrium units that are sold. The second term refers to the cost, price multiplied by quantity of sales, that the consumers paid to the generators for their consumption.

Producer surplus $PS(c^*, q^*, w^*)$ is defined as the profit of the electricity generation industry given by revenue less costs. We define producer surplus as

$$PS(c^*, q^*, w^*) = \pi(c^*) \cdot c^* - V(q^*) - w^* \cdot q^*$$  \hspace{1cm} (5.3)$$

where $V(q^*)$ is the total cost to generate $q^*$. In our current case of dropped node subscripts, $c^* := q^*$ since we have not explicitly defined social welfare for the network. The first term of (5.4) represents the revenue received. The second and third terms denote the generation cost and network access charges respectively associated with $q^*$ and $w^*$.

The market monitor surplus $MMS(c^*, q^*, w^*)$ is the revenue the market monitor receives from placing access charges on the network. We define market monitor surplus as
\[ MMS(c^*, q^*, w^*) = w^* \cdot q^* \quad (5.4) \]

We can now state that social welfare is a summation of all surplus as seen in equations (5.5)-(5.7).

\[
SW(c^*, q^*, w^*) = CS(c^*) + PS(c^*, q^*, w^*) + MMS(c^*, q^*, w^*)
\]
\[
= \left\{ \int_0^{c^*} \pi(x)dx - \left[ \pi(c^*) \cdot c^* \right] + \pi(c^*) \cdot c^* - V(q^*) - w^* \cdot q^* \right\}
\]
\[ + w^* \cdot q^* \quad (5.6) \]
\[ SW(c^*, q^*) = \int_0^{c^*} \pi(x)dx - V(q^*) \quad (5.7) \]

Simplification of equation (5.6) results in the dropping of the revenue terms of electricity and access charges as they are net neutral in the calculation of social welfare. Equation (5.7) shows that social welfare is not directly influenced by the access charges. Instead, the access charges \( w^* \) influence \( c^* \) and \( q^* \) which turn are used to calculate social welfare.

We now present a more formal notation and elaboration of social welfare maximization as it relates to our problem, with the inclusion of the subscripts for nodes, firms and time. The market monitor determines access charges \( w_{i,t} \) such that social welfare is maximized as seen in equation (5.8). The access charges represent the cost to transmit power from the hub node to node \( i \) for time period \( t \). These fees are set to clear the transmission market to ensure that generators do not send power on a transmission line beyond its physical limitations. We sum social welfare across all time periods within the planning horizon, as well as nodes...
where power is consumed or generated. The summation of $V$ across $f$ represents the total cost of generation for all firms $f$. With the planning horizon assumed to be approximately one day, net present value (NPV) is not accounted for in this stage. Our modeling approach is general enough to include longer time horizons and NPV calculations.

$$\max_{w_i(t)} Z(c^f, q^f) = \int_{t_0}^{t_f} \sum_{i \in N} \left\{ \int_0^{\sum_{g \in F} c^g_i(t)} \pi_i(x, t) \, dx - \sum_{f \in F} V^f_i(q^f_i(t)) \right\} dt \quad (5.8)$$

Note that the market monitor’s objective function does not specifically contain the decision variable $w_i(t)$. The market monitor uses $w_i(t)$ to influence the equilibrium quantity of $c$ and $q$ determined in the lower level. Specific details of the lower level are presented in section 5.2.2. The clearing of the transmission markets is modeled in Equations (5.9) and (5.10). The quantity $\sum_{f \in F} (c^f_i - q^f_i)$ is the net power flow from the hub node to node $i$. The parameter $PTDF_{i,a}$ is multiplied with the net power flow to determine what proportion of the power flows on arc $a$. The summation of all power flows across nodes $i$ results in the total net power flow on arc $a$. $T_a$ simply bounds the operating capacity of the transmission line. Electricity is modeled as either a positive and negative quantity representing the direction of travel along a transmission line. Therefore, we must account for both directions not exceeding transmission capacity as indicated by Equations (5.9) and (5.10). These two equations represent the upper level constraints of our Stackelberg game.
\[
\sum_{i \in N} PTDF_{i,a} \cdot \left[ \sum_{f \in F} (c_f^i(t) - q_f^i(t)) \right] \leq T_a \quad \forall a \in A \tag{5.9}
\]

\[
\sum_{i \in N} PTDF_{i,a} \cdot \left[ \sum_{f \in F} (c_f^i(t) - q_f^i(t)) \right] \geq -T_a \quad \forall a \in A \tag{5.10}
\]

The market monitor must also be concerned that the surplus be strictly positive.
A negative surplus would indicate a subsidy provided by the market monitor to generators and consumers. Equation (5.11) states the constraint on the market monitor surplus in the upper level problem.

\[
\sum \sum w_i(t)[c_f^i(t) - q_f^i(t)] \geq 0 \tag{5.11}
\]

### 5.2.2 Generator’s Lower Level Problem

Each generating firm acts as a follower to the market monitor leader and plays a Cournot-Nash game with all other firms given access charges \( w_{i,t} \) set by the market monitor leader in the upper level. “Cournot” refers to the fact that each firm competes with other firms by determining their “quantity.” A “Nash” game indicates that the solution defines an equilibrium such that no firm has an incentive to deviate from their strategy.

Each firm maximizes their individual profit function \( J_f \) consisting of production costs and access charges subtracted from revenue to transmit their net balance of sales and production at each node. The firms determine \( c \) and production \( q \) while in equilibrium with other generation firms. Other firms’ sales are denoted by \( c^{-f} \) where \( c^{-f} : c^g \ \forall g \neq f \). The resulting game is represented in Equations (5.12) through (5.16).
\[
\text{Max} \quad J_f(c^f, q^f; c^{-f}, w) = \int_{t_0}^{t_f} \left\{ \sum_{i \in M} \pi_i \left( \sum_{g \in F} c_i^g(t) \right) \cdot c_i^f(t) - V_i^f(q_i^f(t)) \right. \\
- w_i(t)[c_i^f(t) - q_i^f(t)] \left\} \, dt \quad (5.12)
\]

subject to

\[
\sum_{i \in N} q_i^f(t) = \sum_{i \in M} c_i^f(t) \quad \forall i \in M \quad (5.13)
\]

\[
q_i^f(t) \leq q_{i,\text{max}}^f \quad \forall i \in M \quad (5.14)
\]

\[
\frac{dq_i^f(t)}{dt} = r_i^f(t) \quad (5.15)
\]

\[
r_{i,\text{min}}^f \leq r_i^f(t) \leq r_{i,\text{max}}^f \quad \forall i \in M \quad (5.16)
\]

The oligopoly market structure that we assume has a direct consequence on the equilibrium of the game since every decision to sell power affects the market price of power and thus, all other firm’s profit functions. This is contrast to perfectly competitive markets in which firms have no influence on price and thus would be considered price takers. A game theoretic approach is utilized, taking advantage of the oligopoly market structure. Each agent in the economy has a direct or indirect influence on the decisions and outcomes made and experienced by other agents. Equation (5.13) ensures that all the power that a firm generates in the boundary of the network is sold due to the assumption that electricity cannot be economically stored in a meaningful capacity. Equation (5.14) bounds each firm’s production at each node. The differential equation described in Equation (5.15) defines the ramping rate \( r_i^f(t) \) each generator \( i \) experiences at each of their facilities.
as a function of time $t$. Equation (5.17) imposes a lower and upper bound on $r_i^f(t)$. These constrained dynamics add a level of modeling sophistication that ensures our Stackelberg game follows real world limitations that generators face. Our model is general enough to allow different firms and facility locations to have different ramping bounds. These ramping limitations correspond to electricity generation technology that prevents sudden deviations in the generation plan. In practice, generators use several forecasts at multiple time scales to make both investment and operating decisions concerning generation equipment and deployment of resources. However, as the planning horizon shortens to the period of about one day, physical limitations are imposed on each generator on how quickly they can deviate from the operational plan. These limits on the agility of electricity generation in the short term are referred to as ramping rate bounds.

5.2.3 Complete MPEC Formulation

We present the concatenated bi-level game consisting of both the upper and lower level in equations (5.17) through (5.25).

$$\max_{w_i(t)} Z(c^f, q^f) = \int_{t_0}^{t_f} \sum_{i \in N} \left\{ \int_0^{\sum_{g \in F} r_{i}^{g}(t)} \pi_i(x, t) \, dx - \sum_{f \in F} V_{i}^{f}(q_{i}^{f}(t)) \right\} \, dt \quad (5.17)$$
subject to

$$\sum_{i \in N} PTDF_{i,a} \cdot \left[ \sum_{f \in F} (c^f_i(t) - q^f_i(t)) \right] \leq T_a \quad \forall a \in A$$

(5.18)

$$\sum_{i \in N} PTDF_{i,a} \cdot \left[ \sum_{f \in F} (c^f_i(t) - q^f_i(t)) \right] \geq -T_a \quad \forall a \in A$$

(5.19)

$$\sum_{i \in N} \sum_{F \in F} w_t(t)(c^f_i(t) - q^f_i(t)) \geq 0$$

(5.20)

where $w_t(t)^*$ is the minimizer of the objective function (5.17) and solves the following Cournot-Nash game:

$$\max_{J_f(c^f, q^f, c^{-f}, w)} \int_{t_0}^{t_f} \left\{ \sum_{i \in M} \pi_i \left( \sum_{g \in F} c^g_i(t) \right) \cdot c^f_i(t) - V_i^f(q^f_i(t)) \right. \left. - w_t(t)(c^f_i(t) - q^f_i(t)) \right\} dt$$

(5.21)

subject to

$$\sum_{i \in N} q^f_i(t) = \sum_{i \in M} c^f_i(t) \quad \forall i \in M$$

(5.22)

$$q^f_i(t) \leq q^f_{i,max} \quad \forall i \in M$$

(5.23)

$$\frac{dq^f_i(t)}{dt} = r^f_i(t)$$

(5.24)

$$r^f_{i,min} \leq r^f_i(t) \leq r^f_{i,max} \quad \forall i \in M$$

(5.25)

The resulting game has the unique feature of having hierarchy relating the market monitor’s optimization problem with $|F|$ number of constrained optimization problems. This unique game theoretic approach can be classified as an optimization problem constrained by other optimization problems (OPcOP). Furthermore, the optimization problems consisting of the lower level can also be classified as
a set of equilibrium constraints when viewed together with the upper level problem. This classification leads us to the structure of a mathematical program with equilibrium constraints (MPEC). The hierarchal structure is also in fact a Stackelberg game, as seen in microeconomic analysis, since we assume leader-follower relationship. Specifically, the leader’s game is modeled in the upper level and has the luxury of deciding its variables before the followers modeled in the lower level.

5.3 Discrete Time Formulations

We first present a general discrete time formulation stated in section 5.2. We then reformulate the game as a Mathematical Program with Complementarity Constraints (MPCC) in Section 5.3.2.

5.3.1 Discrete MPEC Formulation

In this subsection, we reformulate our continuos-time MPEC into a discrete-time MPEC. A simple subscript $t$ has been substituted for continuous time with $T$ equal to the total number of time periods in the planning horizon. We also rewrite the lower level constraints of the form “less than or equal to zero” in order to be conducive to further mathematical manipulations in later sections. The dual variables of each lower level constraint are presented in parentheses to the right of each constraint. These dual variables represents the marginal increase of the objective function per additional unit of the constraint. This concept is particularity powerful in conjunction with complementarity as it allows for efficient reformulation and subsequent computation of MPECs. More details clarifying the role of dual variables and complementarity are provided in Sections 5.3.2 and 5.3.3.
\[ \max_{w_{i,t}} Z(c^f, q^f) = \sum_{t \in T} \sum_{i \in M} \left\{ - \int_0^{\sum_g c_{i,t}^g} \{ \pi_{i,t}(x) \} \, dx - \sum_{f \in F} V_{i,t}^f(q_{i,t}) \right\} \] (5.26)

subject to the upper level constraints

\[ \sum_{i \in N} PTD F_{i,a} \cdot \left[ \sum_f (c_{i,t}^f - q_{i,t}^f) \right] \leq T_a \quad \forall a \in A, \forall t \in T \] (5.27)

\[ \sum_{i \in N} PTD F_{i,a} \cdot \left[ \sum_f (c_{i,t}^f - q_{i,t}^f) \right] \geq -T_a \quad \forall a \in A, \forall t \in T \] (5.28)

\[ \sum_{i \in N} \sum_{f \in F} w_{i,t} [c_{i,t}^f - q_{i,t}^f] \geq 0 \quad \forall t \in T \] (5.29)

where \( w_{i,t}^* \) is the minimizer of the objective function (5.17) and solves the following Cournot-Nash game:

\[ \min J_f(c^f, q^f; c^-f, w) = -\sum_t \sum_i \pi_{i,t} \cdot c_{i,t}^f - V_{i,t}^f - w_{i,t} [c_{i,t}^f - q_{i,t}^f] \] (5.30)
subject to the lower level equilibrium constraints

$$
\sum_{i \in N} q_{i,t}^f - \sum_{i \in M} c_{i,t}^f \leq 0 \quad \forall t \in T \quad (\zeta_t^+)^f
$$

(5.31)

$$
- \sum_{i \in N} q_{i,t}^f + \sum_{i \in M} c_{i,t}^f \leq 0 \quad \forall t \in T \quad (\zeta_t^-)^f
$$

(5.32)

$$
- c_{i,t}^f \leq 0 \quad \forall i \in M, \forall t \in T \quad (\phi_{i,t}^f)
$$

(5.33)

$$
- q_{i,t}^f \leq 0 \quad \forall i \in M, \forall t \in T \quad (\rho_{i,t}^f)
$$

(5.34)

$$
q_{i,t}^f - q_{i,t}^{f,\text{max}} \leq 0 \quad \forall i \in M, \forall t \in T \quad (\sigma_{i,t}^f)
$$

(5.35)

$$
\frac{q_{i,t}^f - q_{i,t-\Delta t}^f}{\Delta t} - r_{i,t}^{f,\text{max}} \leq 0 \quad \forall i \in M, \forall t = 1, ..., T \quad (\mu_{i,t}^f)
$$

(5.36)

$$
- \frac{q_{i,t}^f + q_{i,t-\Delta t}^f}{\Delta t} + r_{i,t}^{f,\text{min}} \leq 0 \quad \forall i \in M, \forall t = 1, ..., T \quad (\theta_{i,t}^f)
$$

(5.37)

where $\Delta t$ is a user-chosen time step parameter used to approximate the derivative $\frac{dq_i(t)}{dt}$. Equation (5.16) and (5.17) can be transformed into the set of discrete time constraint as seen in equations (5.36) and (5.37). The term $\frac{dq_i(t)}{dt}$ can be expressed as the quantity $q - q_{t-\Delta t}$.

### 5.3.2 Complementarity Conditions for Generating Firms

We can now transform the discrete math program described in equations (5.30) through (5.37) as a complementarity problem by formulating the necessary conditions for the generating firms’s game. Our lower level program has the convenient property of containing only linear constraints and thus Abadie’s constraint qualification holds. This property allows us to compose the Karush-Kuhn-Tucker (KKT) conditions. These necessary conditions are combined to form a Mixed Complementarity Problem (MCP) or more precisely, a Nonlinear Complementarity Problem (NCP). The KKT identities, with respect to $c$ and $q$, are found to be respectively
\[ 0 = -\pi_{i,t} \left( \sum_{g \in F} c_{i,t}^g \right) - c_{i,t}^f \cdot \pi_{i,t}^f \left( \sum_{g \in F} c_{i,t}^g \right) + w_{i,t} - \zeta_{i,t}^{-f} + \zeta_{i,t}^{+f} - \phi_{i,t}^f \quad (5.38) \]

\[ 0 = -w_{i,t} + \zeta_{i,t}^{+f} - \zeta_{i,t}^{-f} + \sigma_{i,t}^f - \mu_{i,t}^f + \theta_{i,t}^f - \rho_{i,t}^f \quad (5.39) \]

where \( \pi_{i,t}^f \left( \sum_{g \in F} c_{i,t}^g \right) \) denotes the derivative of \( \pi \) with respect to \( c_{i,t}^f \). The following complementarily slackness conditions accompany the KKT identities

\[ 0 \leq \left[ - \sum_{i \in N} q_{i,t}^f + \sum_{i \in M} c_{i,t}^f \right] \perp \zeta_{i,t}^{+f} \geq 0 \quad (5.40) \]

\[ 0 \leq \left[ \sum_{i \in N} q_{i,t}^f - \sum_{i \in M} c_{i,t}^f \right] \perp \zeta_{i,t}^{-f} \geq 0 \quad (5.41) \]

\[ 0 \leq c_{i,t}^f \perp \phi_{i,t}^f \geq 0 \quad (5.42) \]

\[ 0 \leq q_{i,t}^f \perp \rho_{i,t}^f \geq 0 \quad (5.43) \]

\[ 0 \leq -q_{i,t}^f + q_{i,\max}^f \perp \sigma_{i,t}^f \geq 0 \quad (5.44) \]

\[ 0 \leq \frac{q_{i,t}^f - q_{i,t-\Delta t}^f}{\Delta t} - r_{i,\min}^f \perp \mu_{i,t}^f \geq 0 \quad (5.45) \]

\[ 0 \leq \frac{-q_{i,t}^f + q_{i,t+\Delta t}^f}{\Delta t} + r_{i,\max}^f \perp \phi_{i,t}^f \geq 0 \quad (5.46) \]

The symbol \( \perp \) signifies orthogonality of two vectors. For example, consider the vectors \( A \) and \( B \) such that \( A = (a_1, a_2, ... a_i) \) and \( B = (b_1, b_2, ... b_i) \) with the same cardinality (i.e. \( |A| = |B| \)). The orthogonality of \( A \) and \( B \), \( 0 \leq A \perp B \geq 0 \)
suggests:

\[ a_i \cdot b_i = 0 \quad \forall i, \]
\[ a_i \geq 0 \]
\[ b_i \geq 0 \]

### 5.3.3 Discrete MPCC Formulation

A Mathematical Program with Complementarity Constraints (MPCC) is simply a math program in which complementarity constraints are supplemented. In general, the \( y \) variables of the upper level program solve a Mixed Complementarity Problem (MCP) representing the lower level.

\[
\min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} f(x, y) \quad (5.47)
\]

subject to

\[ g(x, y) \leq 0 \quad (5.48) \]

and

\[ y \text{ solves } MCP(h(\cdot, \cdot), B) \quad (5.49) \]

The variables \( x \) refers to the upper level variables while the variables \( y \) are associated with the lower level as described by the MCP above consisting of both
the function \( h(x, \cdot) \) and bounds \( B \). The constraints \( g \) can be a function of both types of variables. The variables \( y \) are a solution to the MCP. A point \( y \) with \( a_l \leq y_l \leq b_l \) solves (5.49) if, for each \( l \), at least one of the following holds

\[
    h_l(x, y) = 0 \quad (5.50)
    h_l(x, y) \geq 0, y_l = a_l \quad (5.51)
    h_l(x, y) \leq 0, y_l = b_l \quad (5.52)
\]

We now combine the math program representing the market monitor’s upper level program with the lower levels MCP to form our discrete MPCC presented in equations (5.53) through (5.65). The formulation of these equations is discussed in Chapter 7.

\[
    \text{Max}_{w_i,t} \quad Z(c^f, q^f) = \sum_{t \in T} \sum_{i \in M} \left\{ -\int_0^{\Sigma_g c^g_{i,t}} \left\{ \pi_{i,t}(x) \right\} dx - \sum_{f \in F} V_{i,t}^f(q_{i,t}) \right\} \quad (5.53)
\]
subject to

\[
\sum_{i \in N} PTDF_{i,a} \cdot \left[ \sum_f \left( c^f_{i,t} - q^f_{i,t} \right) \right] \leq T_a \quad \forall a \in A, \forall t \in T \quad (5.54)
\]

\[
\sum_{i \in N} PTDF_{i,a} \cdot \left[ \sum_f \left( c^f_{i,t} - q^f_{i,t} \right) \right] \geq -T_a \quad \forall a \in A, \forall t \in T \quad (5.55)
\]

\[
\sum_{i \in N} \sum_{F \in F} w^f_{i,t} \left[ c^f_{i,t} - q^f_{i,t} \right] \geq 0 \quad \forall t \in T \quad (5.56)
\]

\[
0 \leq c^f_{i,t} \perp \left[ -\pi^t_{i,t} \left( \sum_{g \in F} c^g_{i,t} \right) - c^f_{i,t} \cdot \pi'^t_{i,t} \left( \sum_{g \in F} c^g_{i,t} \right) + w^t_{i,t} - \zeta^+_t - \zeta^-_t \right] = \phi^f_{i,t} \geq 0 \quad (5.57)
\]

\[
0 \leq q^f_{i,t} \perp \left[ -w^t_{i,t} + \zeta^+_t - \zeta^-_t + \sigma^f_{i,t} - \mu^f_{i,t} + \theta^f_{i,t} \right] = \rho^f_{i,t} \geq 0 \quad (5.58)
\]

\[
0 \leq \left[ -\sum_{i \in N} q^f_{i,t} + \sum_{i \in M} c^f_{i,t} \right] \perp \zeta^+_t \geq 0 \quad (5.59)
\]

\[
0 \leq \left[ \sum_{i \in N} q^f_{i,t} - \sum_{i \in M} c^f_{i,t} \right] \perp \zeta^-_t \geq 0 \quad (5.60)
\]

\[
0 \leq c^f_{i,t} \perp \phi^f_{i,t} \geq 0 \quad (5.61)
\]

\[
0 \leq q^f_{i,t} \perp \rho^f_{i,t} \geq 0 \quad (5.62)
\]

\[
0 \leq -q^f_{i,t} + q^f_{i,max} \perp \sigma^f_{i,t} \geq 0 \quad (5.63)
\]

\[
0 \leq q^f_{i,t} - q^f_{i,t-1} - r^f_{i,min} \perp \mu^f_{i,t} \geq 0 \quad (5.64)
\]

\[
0 \leq -q^f_{i,t} + q^f_{i,t-1} + r^f_{i,max} \perp \theta^f_{i,t} \geq 0 \quad (5.65)
\]

Numerical experiments are presented in Chapter 7

**Future Work**

Our modeling framework is flexible so as to add a host of modeling extensions as well as analysis tools that enhance the capability of market monitor maximizing
social welfare in the context they see fit. The tools also serve to help generating firms anticipate the market outcomes of their own investments and operations as well as anticipate the effects of hypothetical policy decisions of the market monitor or similar external regulatory body. We present some extensions that expand the repertoire of the modeling sophistication of Stackelberg competition and market design of electricity markets.

Our problem can be extended to consider multi-attribute distribution of economic surpluses. It is imagined the market monitor may want to place a higher priority of certain nodes receiving more consumer surplus compared to other nodes. This may reflect societies’ desire to funnel monetary gains to areas that have lacked investment in the past or play a strategic role in the country. The market monitor may also perhaps prioritize the allocation of producer surplus among generating firms, types of generation equipment or location of facilities. This could serve as an alternate method of modeling economic incentives and punishments that regulators could award/penalize generation firms.
This chapter sets forth two differential Nash games solved utilizing the tools of optimal control theory. We model the interactions of player or agents that have utility functions, often modeled as profit, related to each other. We assume that each player is noncooperative in that there is no collusion among players. We refer the readers to section 2.3 for a brief overview of the game theoretic framework of differential Nash games using optimal control theory.

Up to this point of the dissertation, we have only considered an information structure that inherently assumes that firms decide their variables simultaneously for all periods of time Dockner et al. (2000). This assumed information structure is defined as open-loop equilibrium. On the contrary, feedback equilibrium allows each player to observe the state of the game before deciding on their controls. Thus, feedback equilibrium is typically a more realistic type of game information structure since it does not require agents to decide their variables simultaneously. Specifically, feedback equilibrium, also referred to as Markov and closed-loop equi-
librium, allows the optimal control path to be a function of the state of the system. Instead of deciding a single solution, feedback results in a solution that can adjust as the state of the system shifts to other agents’ decisions. However, optimal feedback control is vastly more difficult to compute.

An underutilized yet robust body of literature on feedback optimal control provides efficient computation of linear quadratic optimal control problems. Works by Engwerda (2005); Dockner et al. (2000); Mehlmann (1988) provide readable expositions of techniques developed thus far. See Section 2.3 for a more thorough exposition of literature. The work presented in this chapter is largely based upon the succinct formulations provided by the aforementioned authors. The theoretical foundations presented follow the notation and style published in (Engwerda, 2000a,b, 2003; van den Broek et al., 2003; Weeren et al., 1999).

We also show that a wide range of optimal control problems have the potential of being reformulated into linear quadratic optimal control problems. We again assume that the game is deterministic and non-cooperative. We organize this chapter as follows: Sections 6.1 and 6.2 each present a unique economic application of oligopolistic competition in electricity markets. Section 6.3 details the necessary treatment and theory of linear quadratic optimal control problems. We present numerical results in Chapter 7.

6.1 Oligopolistic Competition with Excess Demand Price Dynamics

The oligopolistic firms described represent electricity generators competing in equilibrium for sales. The resulting game is modeled in an infinite time horizon rep-
resenting the steady state of competition among generators. Consumers are described by demand functions internally derived by the sales output \( s^f \) of each generator \( f \in F \). The competition of generators is modeled as a differential game where the dynamics describe the price of electricity evolving over time. The dynamics are assumed to follow an excess demand price mechanism. Each firm’s decision variables, also referred to as controls, have an influence on the price of electricity, state of the system, at each point in time. Furthermore, the game is dynamic characterized by the fact that all functions are continuous and time-varying. It is assumed that electricity represents a homogenous commodity.

### 6.1.1 Economic Model

We employ slightly different notation as seen in previous sections. Continuous time is denoted by the scalar \( t \in \mathbb{R}^1 \). The set of \( F \) describes firms or, interchangeably referred to as generators, while the subset \( f \) refers to a specific firm.

Each firm \( f \in F \) controls production rates, Megawatt (MW), \( s^f(t) \) which exactly corresponds to sales to the consumer. It is assumed that every unit of electricity sold is consumed. This assumption corresponds to a setting where demand of electricity in the long term is elastic. Furthermore, it is also assumed the storage of electricity is not economical in large or sufficient quantities. The price of electricity \( \pi(t) \) are state variables determined by the controls.

Each firm maximizes their individual profit function \( J_f \) consisting of revenue, less production costs, to meet the demands of electricity of consumers represented by the market modeled. For each firm \( f \in F \), the profit is defined as

\[
J_f(s^f; s^{-f}) = \int_0^\infty e^{-\rho t} \left[ \pi(t) \cdot s^f(t) - V^f(s^f, t) \right] dt,
\]

(6.1)
where $\rho \in \mathbb{R}^{1++}$ is a constant nominal rate of discount, $\pi$ is the price of electricity, $s^f$ is the sale of electricity of firm $f$ and $V^f(s^f,t)$ is the variable cost production for firm $f$. $J_f(s^f; s^{-f})$ is a functional wholly determined by the controls $s^f$ where non-own allocations to sales

$$s^{-f} = (s'^{f'} : f' \neq f)$$

are taken as exogenous information by firm $f$. We will elaborate further on the information structure of the game in subsequent sections of this chapter. The firm term in the objective function $J_f(s^f; s^{-f})$ in (6.28) is firm’s revenue while the second term signifies the firm’s variable production cost.

Firm $f$ solves an optimal control problem to determine their sales $s^f$ thereby also influencing the price $\pi$ of electricity of the market via dynamics described by

$$\frac{d\pi}{dt} = F(E(t))$$

(6.2)

where $E(t)$ represents excess demand of the market. Excess demand is generally defined as

$$E(t) = D(t) - S(t)$$

(6.3)

where $D(t)$ and $S(t)$ represent demand and supply of electricity respectively. Excess demand is a positive quantity when consumers indicate it is willing to consume more than what firms are offering. Conversely, excess demand is negative when firms offer more quantity into the market than what consumers are willing to consume. We assume that over the long term that excess demand converges to zero.
since in equilibrium firms will match their supply to the market’s demand. However, in the short term it is possible for the market to be in *disequilibrium* where excess demand can take a positive or negative value. We assume demand is solely depends on the price of electricity given by

$$ D(t) = F(\pi, t) \quad (6.4) $$

Our disregard of perfect competition in favor for oligopolistic competition further requires refinement of the concept of supply. Rather than utilizing a traditional form of an exogenous supply function, we simply employ the notion that the supply of electricity is equal to the summation of sales provided by firms. More formally, we define the supply of electricity as

$$ S(t) = \sum_{g \in F} s^g(t) \quad (6.5) $$

Thus we can restate excess demand as

$$ E(t) = F(\pi, \sum_{g \in F} s^g, t) \quad (6.6) $$

Further restating allows the price dynamics of $\pi$ to take on the form of

$$ \frac{d\pi}{dt} = F(\pi, \sum_{g \in F} s^g, t) \quad (6.7) $$

Note that expression (6.7) states a differential equation in which the rate of change of price is a function of the state itself as well as the controls of all firms. We now assume that the function $F$ governing $\frac{d\pi}{dt}$ takes on the form of
\[
\frac{d\pi}{dt} = k E(\pi, \sum_{g \in F} s^g, t) \tag{6.8}
\]

where \(k\) is a constant exogenous parameter. The expression above states the rate of change of the price of electricity with respect to time is proportional to the excess demand of the market. Substituting the individual demand and supply functions we have used in (6.4) and (6.5) into (6.8) yields

\[
\frac{d\pi}{dt} = k \left[ D(\pi, t) - \sum_{g \in F} s^g(t) \right] \tag{6.9}
\]

Combining the objective function (6.28) and the price dynamics (6.9) allows us to write the complete model as

\[
J_f(s^f; s^{-f}) = \int_0^\infty e^{-\rho t} \left[ \pi(t) \cdot s^f(t) - V^f(s^f, t) \right] dt, \tag{6.10}
\]

subject to the dynamical system

\[
\frac{d\pi}{dt} = k \left[ D(\pi, t) - \sum_{g \in F} s^g(t) \right], \quad \pi(0) = \pi_0 \tag{6.11}
\]

to be solved by every firm \(f \in F\) for every instant of time \(t \in [0, \infty]\). Note that we intentionally omit further constraints on the system and the controls themselves.

### 6.1.2 The Linear Quadratic Optimal Control Formulation

It can be shown that the Nash equilibria solution to (6.10) and (6.11) can be solved both in open and closed loop equilibrium by subsuming the optimal control problem can be formulated as a linear quadratic optimal control problem.

A linear quadratic optimal control problem is a specific class of optimal control
problems in which the highest degree of the criterion function is two, while the
dynamics are linear in both state and control. We remind the reader the general
form of an infinite time horizon linear quadratic optimal control below. We consider
the problem of finding a control function \( u(\cdot) = F(x, \cdot) \) where \( F \) is a time-invariant
matrix for each \( x_0 \in \mathbb{R}^n \) that minimizes the cost functional

\[
J(x_0, u) = \int_0^\infty \{ x^T(t)Qx(t) + u^T(t)Ru(t) \} dt, \quad (6.12)
\]

subject to the dynamical system

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \quad (6.13)
\]

where \( R \) is a positive definite matrix and \( Q, R \) and \( QT \) are symmetric matrices.

Engwerda (2005) described six extensions of the standard linear quadratic
framework that allows a wide array of problems to be reformulated as linear
quadratic optimal control problems. Those six classes of extensions or reformu-
lation strategies are: 1) the N-Player case; 2) discounting (discount future values
to describe the net present value of economic terms; 3) cost functions containing
cross products (state multiplied by control in the performance criterion); 4) affine
systems and cost functions; 5) Infinite horizon affine systems and discounted affine
cost functions; and 6) tracking systems (players prefer to track a pre specified state
trajectory using a pre specified ideal control path.

We define our sales variable \( s(\cdot) \) as the vector of variables that can be used
to control the system as equivalent to \( u(\cdot) \) in the system described in equations
(7.82) and 7.83. Similarly, we define our electricity price state \( \pi(t) \) as the state of
the system equivalent to \( x(t) \). Let us assume that Demand \( D(t) \) is linear in price
and that \( V^f \) is linear, or at most quadratic. The system described by (6.10) and
Demand $D(t)$ is linear and of the form

$$D(t) = a - b \cdot \pi(t) \quad (6.14)$$

where $a$ and $b$ are time-invariant exogenous parameters. We also assume electricity generation cost $V_f$ is quadratic and is of the form

$$V_f(t) = h \cdot [s_f(t)]^2 \quad (6.15)$$

For exposition purposes, we assume a duopolistic market structure thus implying two firms $f = \{1, 2\}$. Our non-standard linear quadratic optimal control problem now takes on the form of

$$J_f(s^f; s^{-f}) = \int_0^\infty e^{-\rho t} \left\{ \pi(t) \cdot s^f(t) - h \cdot [s^f(t)]^2 \right\} dt, \quad (6.16)$$

subject to the dynamical system

$$\frac{d\pi}{dt} = k \left[ a - b \cdot \pi(t) - c^1(t) - c^2(t) \right], \quad \pi(0) = \pi_0 \quad (6.17)$$

Notice that the system described by equations (6.16) and (6.17) is quadratic only in control and linear in state for the functional $J_f$ while linear in both state in control for the state dynamics. We will employ the reformulation strategy of discounting followed by cost functions functions containing cross products.

Our first step to reformulate our problem into the linear quadratic optimal control problem is to introduce the new variables following the methodology presented under extension two of Engwerda (2005) to handle the discount term $e^{-\rho t}$
\[ x_1 = e^{-\frac{1}{2} \rho t} \pi(t) \quad (6.18) \]
\[ x_2 = e^{-\frac{1}{2} \rho t} \quad (6.19) \]
\[ \tilde{s}^f = e^{-\frac{1}{2} \rho t} s^f(t), \quad f = 1, 2. \quad (6.20) \]

With these variables and \( x^T := [x_1(t), x_2(t)] \), the model described by equations (6.16) and (6.17) can be rewritten as the problem to find the solution to

\[
\begin{aligned}
\min_{c^{f(t)}} & = \int_0^\infty \begin{bmatrix} x_1(t) & x_2(t) & \tilde{s}^f(t) \end{bmatrix} \\
& \times \begin{bmatrix}
0 & 0 & -\frac{1}{2} \\
0 & 0 & 0 \\
-\frac{1}{2} & 0 & h
\end{bmatrix} \\
& \times \begin{bmatrix} x_1(t) \\
x_2(t) \\
\tilde{s}^f(t)
\end{bmatrix} dt \\
& \text{subject to the dynamics}
\end{aligned}
\]

\[ \dot{x}(t) = \begin{bmatrix}
-kb - \frac{1}{2} \rho & ak \\
0 & -\frac{1}{2} \rho
\end{bmatrix} x(t) + \begin{bmatrix} -k \\
0
\end{bmatrix} \tilde{s}^1(t) + \begin{bmatrix} -k \\
0
\end{bmatrix} \tilde{s}^2(t) \quad (6.22) \]

We now utilize the steps in extension three to manipulate the cross product \( pi(t) \cdot s^f(t) \). Consider the new control variables

\[ v_f(t) := \tilde{s}^f(t) + \frac{1}{h} \begin{bmatrix} -\frac{1}{2} & 0 \end{bmatrix} x(t) \quad (6.23) \]

Using these variables allows us to reformulate the problem described by (6.21) and (6.42) into
subject to the dynamics

\[
\dot{x}(t) = \begin{cases}
\begin{bmatrix}
-k & \frac{1}{2} \\
0 & -\frac{1}{2}
\end{bmatrix} - \frac{2}{\hbar} \begin{bmatrix}
-k \\
0
\end{bmatrix} - \frac{1}{2} \begin{bmatrix}
-k \\
0
\end{bmatrix} \begin{bmatrix}
-k & \frac{1}{2} \\
0 & -\frac{1}{2}
\end{bmatrix} x(t)
\end{cases}
\]

+ \begin{bmatrix}
-k \\
0
\end{bmatrix} \tilde{v}_1(t) + \begin{bmatrix}
-k \\
0
\end{bmatrix} \tilde{v}_2(t)
\]

Our model reduces to the game defined by

\[
J_f := \int_0^\infty \{x^T(t)Q_f x(t) + v_{tf}^T(t)R_f v_{tf}(t)\} dt,
\]

subject to the dynamical system

\[
\dot{x}(t) = Ax(t) + B_1 v_1(t) + B_2 v_2(t) \quad x^T(0) = [\pi_0 \ 0],
\]
where

\[
A = \begin{bmatrix}
-k(b + \frac{1}{h}) - \frac{1}{2}\rho & ak \\
0 & -\frac{1}{2}\rho
\end{bmatrix},
\]

\[
B_f = \begin{bmatrix}
-k \\
0
\end{bmatrix},
\]

\[
Q_f = \begin{bmatrix}
-\frac{1}{4h} & 0 \\
0 & 0
\end{bmatrix},
\]

\[
R_f = h
\]

This completes our reformulation into the standard linear quadratic optimal control framework. We present a numerical example in section 7.3.

### 6.2 Oligopolistic Competition with Sticky Prices

#### Dynamics

We study the model originally presented by Fershtman and Kamien (1987). The authors provide the infinite time horizon case to the duopolistic finite time horizon presented in Engwerda (2005). Other versions of duoplistic competition can be found in Engwerda (2005); Mehlmann (1988); Dockner et al. (2000).

#### 6.2.1 Economic Model

We will now consider the infinite time horizon case. Duopolistic competition is oligopolistic competition when there are only two firms selling a homogenous good. The term sticky prices refers to the phenomena that the price of the homogenous
good does not adjust instantaneously from the price stipulated by the demand function at the current level of joint output and current price of the good. Thus, the firms are able to capitalize on the lag in adjustment of the price. We frame the model to suit our electricity market context by stipulating the homogenous good represent a contract of electricity rather than electricity itself. The notion of an electricity contract is a promise to deliver a fixed quantity of electricity at time period $t$. We choose to model a contract because they are prevalent among electricity markets (Ventosa et al., 2005). Some firms choose to limit their exposure to the price of electricity in the short term by engaging in consumption or production contracts of electricity. In the short term, the price of the contract will vary from the instantaneous spot price of electricity, as calculated from the instantaneous demand function at current output levels. However, we assume that the price of contracts will, with some speed, converge to the instantaneous price since, with enough time, both parties have access to both demand and output data. We model our problem as a differential game where the dynamics describe the price of the electricity contract evolving over time.

Our notation and assumptions follow a similar thread as presented in the excess demand dynamics seen in section 6.1. Consumers are represented by instantaneous demand functions internally derived by the sales output $c_f$ of each generator $f \in F$. Each firm’s decision variables, also referred to as controls, have an influence on the contract price of electricity and state of the system at each point in time. Furthermore, the game is dynamic since since all functions are continuous and time-varying. It is assumed that electricity represents a homogenous commodity. We let continuous time be denoted by the scalar $t \in \mathbb{R}^1$. The set of $F$ described firms or also interchangeably referred to as generators while the subset $f$ refers to a specific firm. Each firm $f \in F$ controls production rates, Megawatt (MW), $c_f(t)$
which exactly correspond to sales to the consumer. It is assumed that every unit of electricity sold is consumed. This assumption corresponds to a setting where demand of electricity in the long term is elastic. Furthermore, it is also assumed the storage of electricity is not economical in large or sufficient quantities. The price of electricity $\pi(t)$ are state variables determined by the controls. We are interested in the long term equilibrium of the competition among the firms. We model the resulting game as an infinite time horizon optimal control problem.

Each firm maximizes their individual profit function $J_f$ consisting of revenue less production costs to meet the demands of electricity of consumers represented by the market modeled. For each firm $f \in F$, the profit is defined as

$$J_f(s^f; s^{-f}) = \int_0^\infty e^{-\rho t} [\pi(t) \cdot s^f(t) - V^f(c^f, t)] dt, \quad (6.28)$$

where $\rho \in \mathbb{R}^1_{++}$ is a constant nominal rate of discount, $\pi$ is the price of electricity, $s^f$ is the sale of electricity of firm $f$ and $V^f(s^f, t)$ is the variable cost production for firm $f$. $J_f(s^f; s^{-f})$ is a functional wholly determined by the controls $s^f$ where non-own allocations to sales

$$s^{-f} = \{s^{f'} : f' \neq f\}$$

are taken as exogenous information by firm $f$. We will elaborate further on the information structure of the game in subsequent sections of this chapter. The firm term in the objective function $J_f(s^f; s^{-f})$ in (6.28) is firm’s revenue while the second term signifies the firm’s variable production cost.

Firm $f$ solves an optimal control problem to determine their sales $c^f$ thereby
also influencing the contract price $\pi$ of electricity of the market via dynamics described by

$$\frac{d\pi}{dt} = s [P(t) - \pi(t)] \quad (6.29)$$

where $k \in (0, \infty)$ is the speed of adjustment parameter and $P(t)$ represents the inverse demand curve at time period $t$. The inverse demand function calculates the instantaneous spot price of electricity. We assume the instantaneous price of electricity only depends on the generation output of the firms and time. Specifically, we assume $P(t)$ is of the form

$$P(t) = F(s_f, t) \quad \forall f \in F \quad (6.30)$$

For example, a generator and consumer will agree on a price of electricity $\pi_0$ for the time period $t$. At time period $t+1$ the instantaneous price of the electricity increases by 10 units. The price of the contract of electricity in time period $t+1$ will only increase by $s \cdot 10$ units since there is a lag in the adjustment of the contract price. However, if the instantaneous price of electricity remains constant, the price of the electricity contract will converge to the $\pi + 10$ units level. The converse would be true if the price of instantaneous electricity were to drop. Our complete optimal control problem is given by equations (6.31) and (6.32).

$$J_f(s^f; s^{-f}) = \int_0^{\infty} e^{-\rho t} \left[ \pi(t) \cdot s^f(t) - V^f(s^f, t) \right] dt, \quad (6.31)$$

subject to the dynamics

$$\frac{d\pi}{dt} = k [P(s_f, t) - \pi(t)] \quad (6.32)$$
6.2.2 The Linear Quadratic Optimal Control Formulation

In this section, we show that the dynamic duopolistic optimal control model with sticky prices defined in (6.31) and (6.32) may be formulated as a linear optimal control problem through a similar style presented in section 6.1.2. We first assume the instantaneous price $P(t)$ is the linear function

$$P(s_f, t) = a - b \sum_{f \in F} s_f(t)$$  \hspace{1cm} (6.33)

Substituting equation (6.33) with $f = 1, 2$ leads to our duopolistic instantaneous price

$$P(s_f, t) = a - b [s_1(t) + s_2(t)]$$  \hspace{1cm} (6.34)

where $a$ and $b$ are exogenous parameters. We assume the electricity cost function $V^f$ is quadratic and is of the form

$$V^f(s_f, t) = g \cdot s_f + h \cdot s_f^2$$  \hspace{1cm} (6.35)

Our non-standard linear quadratic optimal control problem is now of the form

$$J_f(s_f; s_{-f}) = \int_0^\infty e^{-\rho t} \left[ \pi(t) \cdot s_f(t) - g \cdot s_f - h \cdot s_f^2 \right] dt,$$  \hspace{1cm} (6.36)

subject to the dynamics

$$\frac{d\pi}{dt} = k [a - b [s_1(t) + s_2(t)] - \pi(t)]; \quad \pi(0) = \pi_0$$  \hspace{1cm} (6.37)

The system described by equations (6.36) and (6.37) contains both linear and quadratic control terms and a linear state term in the functional $J_f$. The state
function $\pi$ is linear both in the state and control. We utilize the reformulation strategy of discounting followed by cost function containing affine and cross product terms.

We begin the reformulation of the model into the standard linear quadratic optimal control problem by introducing new variables as per the methodology of Extension two presented by Engwerda (2005) to reformulate the discount term $e^{-\rho t}$

\begin{align*}
x_1 &= e^{-\frac{1}{2}\rho t} \pi(t) \\
x_2 &= e^{-\frac{1}{2}\rho t} \\
\tilde{s}_f &= e^{-\frac{1}{2}\rho t} s_f(t), \quad f = 1, 2.
\end{align*}

With these variables and $x^T := [x_1(t), x_2(t)]$, the model described by equations (6.36) and (6.37) can be rewritten as the problem to find the solution to

\begin{equation}
-\min_{s_f(\cdot)} \int_0^\infty [x_1(t) \quad x_2(t) \quad \tilde{s}_f(t)] \begin{bmatrix} 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2}g \\ -\frac{1}{2} & \frac{1}{2}g & h \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \tilde{s}_f(t) \end{bmatrix} dt
\end{equation}

subject to the dynamics

\begin{equation}
\dot{x}(t) = \begin{bmatrix} -k - \frac{1}{2}\rho & ak \\ 0 & -\frac{1}{2}\rho \end{bmatrix} x(t) + \begin{bmatrix} -kb \\ 0 \end{bmatrix} \tilde{s}_1(t) + \begin{bmatrix} -kb \\ 0 \end{bmatrix} \tilde{s}_2(t)
\end{equation}

We now utilize the steps in Extension three to manipulate the cross product $\pi(t) \cdot s_f(t)$. Consider the new control variables
Using these variables, we reformulate the problem described by (6.21) and (6.42) into

\[- \min_{s(\cdot)} = \int_0^\infty x^T(t) \left( \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \frac{1}{h} \begin{bmatrix} -\frac{1}{2} & \frac{1}{2}g \\ \frac{1}{2}g & \frac{1}{2}g \end{bmatrix} \right) x(t) + h[v_f(t)]^2 \] (6.44)

subject to the dynamics

\[ \dot{x}(t) = \left\{ \begin{bmatrix} -k - \frac{1}{2}\rho k & \frac{1}{2}\rho s \\ 0 & -\frac{1}{2}\rho s \end{bmatrix} - \frac{2}{h} \begin{bmatrix} -s & \frac{1}{2}g \\ \frac{1}{2}g & \frac{1}{2}g \end{bmatrix} \right) x(t) \\
+ \begin{bmatrix} -s_b \\ 0 \end{bmatrix} \tilde{v}_1(t) + \begin{bmatrix} -k_b \\ 0 \end{bmatrix} \tilde{v}_2(t) \} \] (6.45)

Our model reduces to the game defined by

\[ J_f := \int_0^\infty \{ x^T(t)Q_f x(t) + v^T_f(t)R_f v_f(t) \} dt, \] (6.46)

subject to the dynamical system

\[ \dot{x}(t) = A x(t) + B_1 v_1(t) + B_2 v_2(t) \quad x^T(0) = [\pi_0 \ 0], \] (6.47)
where

\[
A = \begin{bmatrix}
-k(1 + \frac{1}{n}) - \frac{1}{2} \rho & s (a + \frac{2}{n}) \\
0 & -\frac{1}{2} \rho
\end{bmatrix}
\]

\[
B_f = \begin{bmatrix}
-k \\
0
\end{bmatrix}
\]

\[
Q_f = \begin{bmatrix}
-\frac{1}{4h} & \frac{1}{2} g \\
\frac{1}{2} g & -\frac{1}{2} g^2
\end{bmatrix}
\]

\[
R_f = h
\]

Thus completing our reformulation into the standard linear quadratic optimal control framework. We present a numerical example in section 7.4.

### 6.3 Linear Quadratic Optimal Control Theory

The linear-quadratic problem (LQP) enjoys much fame in the domain of optimal control problems due to the rich theory of computation and applicability to real world problems Engwerda (2000a). Specifically, the results that the Hamilton-Jacobi partial differential equation corresponding to the LQP can be solved quite efficiently. The formulation we will use in this chapter refers to
\[
f_{0}(x,u,t) = \frac{1}{2} x^T A(t) x + \frac{1}{2} u^T B(t) u
\]

\[
f(x,u,t) = F(t) x + G(t) u
\]

\[
K[x(t_f),t_f] = \frac{1}{2} [x(t_f)]^T S(t_f) x(t_f)
\]

\[U = V\], the entire vector space

so that the problem we face is

\[
\min J[x(t),u(t)] = K[x(t_f),t_f] + \int_{t_0}^{t_f} f_{0}[x(t),u(t),t] dt
\]

\[
= \frac{1}{2} (x^T S x)_{t=t_f} + \frac{1}{2} \int_{t_0}^{t_f} (x^T A x + u^T B u) dt \quad (6.48)
\]

subject to

\[
\frac{dx}{dt} = f(x(t),u(t),t) = Fx + Gu \quad (6.49)
\]

\[x(t_0) = x_0 \quad (6.50)
\]

where \( t_0 \in \mathbb{R}^+_+ \) is known and we have suppressed time dependencies of the matrices \( A, B, F, G, \) and \( S \). There is no terminal time constraint, and both the initial time \( (t_0) \) and the terminal time \( (t_f) \) are fixed. Likewise \( x_0 \) is fixed. Furthermore, we assume that the matrices \( A \) and \( S \) are positive semidefinite and the matrix \( B \) is positive definite. In this chapter, we will present several ways to solve linear quadratic optimal control problems. We first begin by stating the optimality conditions using the Hamilton-Jacobi equation followed by the separation
of variables technique. A well established methodology of using matrix Riccati differential equations (RDE) to solve feedback control is presented to solve the finite planning-horizon problem. Finally, we present the feedback equilibria solution techniques utilizing the steady-state matrix Riccati equations or referred to as Algebraic Riccati Equations (ARE), for the infinite planning horizon problem.

6.3.1 Linear Quadratic Optimality Conditions

We present the methodology used by Friesz (2010) to derive an equivalent two-point boundary-value formulation of the linear quadratic optimal control problem. The necessary conditions are also sufficient due to the assumptions of positive definiteness. Specifically,

\[ H(x, u, \lambda) = \frac{1}{2} (x^T Ax + u^T Bu) + \lambda^T (Fx + Gu) \]  \tag{6.52}

\[- \frac{d\lambda}{dt} = \frac{\partial H}{\partial x} = Ax + F^T \lambda \]  \tag{6.53}

\[ u = \arg \left\{ \frac{\partial H}{\partial u} = Bu + G^T \lambda = 0 \right\} \]  \tag{6.54}

\[ \Rightarrow u = -B^{-1}G^T \lambda \]  \tag{6.55}

Hence

\[
\begin{bmatrix}
\dot{x} \\
\dot{\lambda}
\end{bmatrix} =
\begin{bmatrix}
F & -GB^{-1}G^T \\
-A & -F^T
\end{bmatrix}
\begin{bmatrix}
x \\
\lambda
\end{bmatrix}
\]  \tag{6.56}
with

\[ x(t_0) = x_0 \]  
\[ \lambda(t_f) = \left[ \frac{\partial K}{\partial x} \right]_{t=t_f} = \left[ \frac{\partial}{\partial x} \left( x^T S x \right) \right]_{t=t_f} \]

Expressions (6.56), (6.57), and (6.58) constitute a linear two-point boundary-value problem.

However, a direct solution of the Hamilton-Jacobi partial differential equation is attempted:

\[ H^* \left( x, \frac{\partial J^*}{\partial x} \right) + \frac{\partial J^*}{\partial t} = 0 \]

(6.59)

where \( J^* \) is the optimal-value function and \( H^* \) is obtained by evaluating the Hamiltonian along its optimal trajectory using the control law and the identity

\[ \lambda^T = \frac{\partial J^*}{\partial x} \implies \lambda = \left( \frac{\partial J^*}{\partial x} \right)^T \]

(6.60)
also valid along the optimal trajectory. Therefore

\[ H^* \left( x, \frac{\partial J^*}{\partial x} \right) \equiv \min_{u \in U} \left[ H \left( x, u, \lambda \right) \right] \]

\[ \lambda = \left( \frac{\partial J^*}{\partial x} \right)^T \]

\[ \equiv [ H \left( x, u, \lambda \right)]_{u = -B^{-1}G^T \lambda, \lambda = \left( \frac{\partial J^*}{\partial x} \right)^T} \]

\[ = \left[ \frac{1}{2} \left( x^T Ax + u^T Bu \right) + \lambda^T \left( Fx + Gu \right) \right]_{u = -B^{-1}G^T \lambda, \lambda = \left( \frac{\partial J^*}{\partial x} \right)^T} \]

\[ = \frac{1}{2} x^T Ax + \frac{1}{2} \left[ B^{-1}G^T \left( \frac{\partial J^*}{\partial x} \right)^T \right]^T B \left[ B^{-1}G^T \left( \frac{\partial J^*}{\partial x} \right)^T \right] \]

\[ + \left( \frac{\partial J^*}{\partial x} \right)^T Fx + \left( \frac{\partial J^*}{\partial x} \right)^T G \left[ -B^{-1}G^T \left( \frac{\partial J^*}{\partial x} \right)^T \right] \]

\[ (6.61) \]

Note that

\[ \left[ B^{-1}G^T \left( \frac{\partial J^*}{\partial x} \right)^T \right]^T B \left[ B^{-1}G^T \left( \frac{\partial J^*}{\partial x} \right)^T \right] \]

\[ = \left[ \left( \frac{\partial J^*}{\partial x} \right)^T \right]^T \left( G^T \right)^T \left( B^{-1} \right)^T \left( BB^{-1} \right) \left[ G^T \left( \frac{\partial J^*}{\partial x} \right)^T \right] \]

\[ = \left( \frac{\partial J^*}{\partial x} \right)^T G \left( B^{-1} \right)^T G^T \left( \frac{\partial J^*}{\partial x} \right)^T = \left( \frac{\partial J^*}{\partial x} \right)^T GB^{-1}G^T \left( \frac{\partial J^*}{\partial x} \right)^T \]

\[ (6.62) \]

since \( GB^{-1}G^T \) is a symmetric matrix. Results (6.61) and (6.62) give

\[ H^* \left( x, \frac{\partial J^*}{\partial x} \right) = \frac{1}{2} x^T Ax + \left( \frac{\partial J^*}{\partial x} \right)^T Fx - \frac{1}{2} \left( \frac{\partial J^*}{\partial x} \right)^T GB^{-1}G^T \left( \frac{\partial J^*}{\partial x} \right)^T \]

\[ (6.63) \]
so that the Hamilton-Jacobi partial differential equation is

$$\frac{1}{2} x^T A x + \left( \frac{\partial J^*}{\partial x} \right) F x - \frac{1}{2} \left( \frac{\partial J^*}{\partial x} \right) G B^{-1} G^T \left( \frac{\partial J^*}{\partial x} \right)^T + \frac{\partial J^*}{\partial t} = 0$$  \hspace{1cm} (6.64)

with

$$J = \frac{1}{2} [x (t_f), t_f] S (t_f) x (t_f)$$  \hspace{1cm} (6.65)

as the boundary condition.

### 6.3.2 Separation of Variables for the LQP

Friesz (2010) presents a clear summary of how to employ a separation of variables to solve expression 6.64 subject to 6.65.

We start with the following transformation:

$$J^* = \frac{1}{2} x^T Z (t) x$$  \hspace{1cm} (6.66)

where $Z (t)$ is an unknown time-dependent symmetric matrix. Substituting (6.66) into (6.64) yields

$$\frac{1}{2} x^T \left[ \frac{dZ}{dt} + Z F + F^T Z - Z G B^{-1} G^T Z + A \right] x = 0$$  \hspace{1cm} (6.67)

Therefore

$$\frac{dZ}{dt} + Z F + F^T Z - Z G B^{-1} G^T Z + A = 0$$  \hspace{1cm} (6.68)

which is known as the **matrix Riccati equation** and is subject to the boundary condition

$$Z (t_f) = S (t_f)$$  \hspace{1cm} (6.69)
6.3.3 Feedback Equilibrium of the Finite Planning-Horizon Problem

We consider the minimization of

\[ J = \int_0^T \left\{ x^T(t)Qx(t) + u^T(t)Ru(t) \right\} dt + x^T(t)Q_Tx(T), \]  

(6.70)

subject to

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \]  

(6.71)

where \( R \) is a positive definite matrix and \( Q, R \) and \( Q_T \) are symmetric matrices.

We seek a fully dynamic solution to the above linear quadratic optimal control. We define fully dynamic as a solution that represents the optimal state and control before steady state is achieved. Steady state refers to when the time derivative of the state and control vanishes as time approaches infinity.

**Theorem 1.** The LQP defined by equations (6.70) and (6.71) has for every initial state \( x_0 \), a solution if and only if the Riccati differential equation (6.72) has a symmetric solution \( K(\cdot) \) on \([0, T]\).

\[
\dot{K}(t) = -A^T K(t) - K(t) A + K(t) S K(t) - Q, \quad K(T) = Q_T \]  

(6.72)

where \( S = BR^{-1}B^T \).

If the LQP has a solution, then it is unique and the optimal control in feedback form is

\[
u^*(t) = -R^{-1}B^T K(t)x(t),\]
whereas in open-loop form it is

\[ u^*(t) = -R^{-1}B^TK(t)\Phi(t, 0)x_0 \]

with \( \Phi \) is the solution of the transition equation

\[ \dot{\Phi}(t, 0) = (A - SK(t))\Phi(t, 0); \Phi(0, 0) = I, \]

Moreover, \( J(u^*) = x_0^TK(0)x_0. \)

**Proof.** Notice the equivalence of the following expression

\[
\int_0^T \frac{d}{dt} \{x^T(t)K(t)x(t)\} dt = x^T(T)K(T)x(T) - x^T(0)K(0)x(0). \tag{6.73}
\]

As a result, the cost function (6.70) can be rewritten as

\[
J = \int_0^T \{x^T(t)Qx(t) + u^T(t)Ru(t)\} dt + \int_0^T \frac{d}{dt} \{x^T(t)K(t)x(t)\} dt + x^T(0)K(0)x(0) \\
+ x^T(T) [Q_T - K(T)] x(T) \\
= \int_0^T \{Qx(t) + u^T(t)Ru(t) + \frac{d}{dt} \{x^T(t)K(t)x(t)\}\} dt + x^T(0)K(0)x(0) \\
+ x^T(T) [Q_T - K(T)] x(T). 
\]

Using equations (6.72) and (6.71), the integrand can be rewritten as the expression below. We purposely omit the dependence of time for notation purposes only.
\[ x^T Q x + u^T R u + \frac{d}{dt} \{ x^T K x \} = x^T Q x + u^T R u + x^T K x + x^T K \dot{x} \]

\[ = x^T Q x + u^T R u + (A x + B u)^T K x + x^T K (A x + B u) \]

\[ = x^T (Q + A^T K + K A + \dot{K}) x + u^T R u + u^T B^T K x + x^T K B u \]

\[ = x^T K S K x + u^T R u + u^T B^T K x + x^T K B u \]

\[ = (u + R^{-1} B^T K x)^T R (u + R^{-1} B^T K x). \]

Subsequently

\[ J = \int_0^T (u(t) + R^{-1} B^T K(t)x(t))^T R (u(t) + R^{-1} B^T K(t)x(t)) dt + x^T(0) K(0) x(0). \]

(6.74)

It is clear from expression (6.74) that \( J \geq x^T(0) K(0) x(0) \) for all \( u \) and that equality is achieved when \( u(\cdot) \) satisfies (6.107).

We now analyze whether the converse statement can be proven. Assume that \( u^*(\cdot) \) is the optimal control solution of the linear quadratic control problem; \( x^*(t) \) is the state trajectory; and \( J^*(0, x_0) \) is the associated minimum cost. A challenge in handling the converse statement of Theorem 1 is to show that if \( J^*(0, x_0) \) exists for an arbitrary initial state, the minimum \( J^*(t, x_0) \) exists in the optimization problem (6.75) below, for an arbitrary initial state \( x_0 \), and for an arbitrarily chosen \( t \in [0, T] \). We consider the following slightly different linear quadratic optimal control problem (6.75) in order to prove this statement. We show that the problem’s solution will be quadratic given a solution may be found. Specifically, consider the optimal control problem of finding the infimum of the quadratic cost function.
\[ J(t, x_0) = \int_t^T \{ x^T(t)Qx(t) + u^T(t)Ru(t) \} dt + x^T(T)Q_T x(T), \] (6.75)

subject to

\[ \dot{x}(t) = Ax(t) + Bu(t), \quad x(t) = x_0, \] (6.76)

where \( Q, Q_T \) and \( R \) satisfy the assumptions given in equations (6.70) and (6.71).

Then under the assumption that this infimum, denoted by \( J^\text{inf}(t, x_0) \), exists

\[ J^\text{inf}(t, x_0(t)) = x_0^T(t)P(t)x_0(t), \] (6.77)

where, without loss of generality, \( P(t) \) is assumed to be symmetric. The complete proof may be found in Molinari (1975).

\[ \square \]

**Definition 2.** The set of control actions \( u^*_i(t) = \gamma^*_i(t, x(t)) \) constitute a feedback Nash equilibrium solution if these strategies provide a Nash equilibrium for the truncated game \( \Gamma(t_1, x_{t_1}) \), for all \( t_1 \in [0, T] \), and \( x_{t_1} \in \mathbb{R}^n \) that are reachable from some initial state at time \( t = 0 \).

Since by definition the equilibrium actions are a function of the current state of the system, they can be interpreted as policy rules (Reinganum and Stokey 1985). They require no recommitment of the players, and hence are also applicable if players are not credible.

For notational simplicity we again confine ourselves to the two-player case from now on. We next introduce \( J^*_i(t, x) \) as the minimum cost to go for player \( i \) if the he evaluates his cost \( J \) at time \( t \) starting in the initial state \( x \) and knowing the
action of his opponent.

**Theorem 3.** Assume \( u_i^*(\cdot), i = 1, 2 \), provide a feedback Nash equilibrium and let \( x^*(t) \) be the corresponding closed-loop state trajectory. Let \( V_i(t, x) := J_i^*(t, x) \) and assume that both partial derivatives of \( V_i \) exist and, moreover, \( \frac{\partial V_i}{\partial x} \) is continuous and \( \frac{d}{dt}V_i(t, x(t)) \) exist. Then, for \( t_0 \leq t \leq T \), \( V_i : [0, T] \times \mathbb{R}^n \to \mathbb{R}, i = 1, 2 \), satisfy the partial differential equations:

\[
-\frac{\partial V_1(t, x_1)}{\partial t} = \min_{u_1 \in U_1} \left[ \frac{\partial V_1(t, x_1)}{\partial x_1} (Ax_1 + B_1u_1 + B_2u_2^*) + x_1^T Q_1 x_1 + u_1^T R_{11} u_1 + u_2^T R_{12} u_2 \right]
\]

\[
V_1(T, x_1) = x_1^T(T) Q_{1T} x_1(T) \tag{6.79}
\]

\[
-\frac{\partial V_2(t, x_2)}{\partial t} = \min_{u_2 \in U_2} \left[ \frac{\partial V_2(t, x_2)}{\partial x_2} (Ax_2 + B_1u_1 + B_2u_2) + x_2^T Q_2 x_2 + u_1^T R_{21} u_1 + u_2^T R_{22} u_2 \right]
\]

\[
V_2(T, x_2) = x_2^T(T) Q_{2T} x_2(T) \tag{6.81}
\]

where \( x_1 \) and \( x_2 \) are the solutions of the differential equations

\[
\dot{x}_1 = Ax_1 + B_1u_1 + B_2u_2^*, \quad x_1(0) = x(0) \tag{6.82}
\]

\[
\dot{x}_2 = Ax_2 + B_1u_1 + B_2u_2, \quad x_2(0) = x(0), \tag{6.83}
\]

respectively.

If there exists \( V_i, i = 1, 2 \) with the above mentioned properties such that they satisfy the set of partial differential equations (6.78)-(6.80) and...
\[ u_1^* = \arg\min_{u_1 \in U_1} \left[ \frac{\partial V_1(t, x_1)}{\partial x_1} (Ax_1 + B_1u_1 + B_2u_2^*) + x_1^T Q_1 x_1 + u_1^T R_{11} u_1 + u_2^T R_{12} u_2^* \right] \]

(6.84)

\[ u_2^* = \arg\min_{u_2 \in U_2} \left[ \frac{\partial V_2(t, x_2)}{\partial x_2} (Ax_2 + B_1u_1^* + B_2u_2) + x_2^T Q_2 x_2 + u_1^* R_{21} u_1^* + u_2^T R_{22} u_2^* \right] \]

(6.85)

then \( u_i^*, i = 1, 2 \), provide a feedback Nash equilibrium. This equilibrium is strongly time consistent and the minimum costs for player \( i \) are \( J_i^* = V_i(t_0, x_0) \).

Since the system we are considering is linear, it is often argued that the equilibrium actions should be a linear function of the state as well. This argument implies that we should consider either a refinement of the feedback Nash equilibrium concept of strategy spaces that only certain functions of the above-mentioned type. The first option amounts to considering only those feedback Nash equilibria which permit a linear feedback synthesis as being relevant. For the second option one has to consider the strategy spaces defined by

\[ \Gamma_i^{\text{lfb}} := \{ u_i(0, T) | u_i(t) = F_i(t)x(t) \} \]

(6.86)

where \( F_i(\cdot) \) is a piecewise continuous function, \( i = 1, 2 \) and consider Nash equilibrium actions \((u_1^*, u_2^*)\) within the strategy space \( \Gamma_1^{\text{lfb}} \times \Gamma_2^{\text{lfb}} \).

It turns out that both equilibrium concepts yield the same characterization of these equilibria for the linear quadratic differential game, which will be presented in Theorem 9. Therefore, we will define just one equilibrium concept here and leave the formulation and proof of the other concept as an exercise for the reader.

**Definition 4.** The set of controls actions \( u_i^*(t) = F_i^*(t)x(t) \) constitute a linear
feedback Nash equilibrium solution if both

\[ J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*) \] and \[ J_2(u_1^*, u_2^*) \leq J_1(u_1^*, u_2), \] \hspace{1cm} (6.87)

for all \( u_i \in \Gamma_i \).

**Theorem 5.** The two-player linear quadratic differential game (6.70) and (6.71) has, for every initial state, a linear feedback Nash equilibrium if and only if the following set of coupled Riccati differential equations has a set of of symmetric solutions \( K_1, K_2 \) on \([0, T]\)

\[
\dot{K}_1(t) = -(A - S_2K_2(t))^TK_1(t) - K_1(t)(A - S_2K_2(t)) + K_1(t)S_1K_1(t) - Q_1 - K_2(t)S_{21}K_2(t), \hspace{1cm} (6.88)
\]

\[ K_1(T) = Q_{1T} \hspace{1cm} (6.89) \]

\[
\dot{K}_2(t) = -(A - S_1K_1(t))^TK_2(t) - K_2(t)(A - S_1K_1(t)) + K_2(t)S_2K_2(t) - Q_2 - K_1(t)S_{12}K_1(t), \hspace{1cm} (6.90)
\]

\[ K_2(T) = Q_{2T} \hspace{1cm} (6.91) \]

Moreover, in that case there is a unique equilibrium. The equilibrium actions are

\[ u_i^*(t) = -R_i^{-1}B_i^TK_i(t)x(t), \quad i = 1, 2. \hspace{1cm} (6.92) \]

The cost incurred by player \( i \) is \( x_0^T K_i(0)x_0 \), \( i = 1, 2. \)

**Proof.** Assume \( u_i^*(t) = F_i^*(t)x(t), \quad t \in [0, T], \quad i = 1, 2. \) is a set of linear feedback equilibrium actions. Then, according to the definition of a linear feed-
back equilibrium, the following linear quadratic regulator problem has a solution $u^*_1(t) = F^*_1(t)x_1(t)$, for all $x_0$.

$$
\min \int_0^T \{x^T_1(s)(Q_1 + F^*_2(s)R_12F^*_2(s))x_1(s) + u^T_1(s)R_1u_1(s)\} ds + x^T_1Q_1 Tx_1(T),
$$

subject to the system

$$
\dot{x}_1(t) = (A + B_2F^*_2(t))x_1(t) + B_1u_1(t), \quad x_1(0) = x_0.
$$

According to Theorem 1 this regulator problem has a solution if and only if the Riccati differential equation

$$
\dot{K}_1(t) = -(A - B_2F^*_2(t))^TK_1(t) - K_1(t)(A - B_2F^*_2(t)) + K_1(t)S_1K_1(t) - (Q_1 + F^*_2(t)R_12F^*_2(t)), \quad K_1(T) = Q_1T
$$

has a symmetric solution $K_1(.)$ on $[0, T]$. Moreover, the solution for this optimization problem is unique, and is given by

$$
u^*_1(t) = -R_1^{-1}B^TK_1(t)x_1(t).
$$

So, we conclude that $F^*_1(t) = -R_1^{-1}B^TK_1(t)$. Similarly, it follows by definition that $u^*_2(t) = F^*_2(t)x_2(t)$ solves the problem.
\[
\min \int_0^T \{ x_2^T(s)(Q_2 + K_1(s)S_1K_1^*(s))x_2(s) + u_2^T(s)R_2u_2(s)\} ds + x_2^TQ_2x_2(T),
\]
subject to the system
\[
\dot{x}_2(t) = (A - S_1K_1(t))x_2(t) + B_2u_2(t), \quad x_2(0) = x_0.
\]

According to Theorem 1 this regulator problem has a solution if and only if the Riccati differential equation

\[
\dot{K}_2(t) = -(A - S_1K_1(t))^T K_2(t) - K_2(t)(A - S_1K_1(t)) + K_2(t)S_2K_2(t) - Q_2,
\]

\[
K_2(T) = Q_2T
\]

has a symmetric solution \(K_2(.)\) on \([0, T]\). Moreover, the solution for the optimization problem is unique, and is given by

\[
u_2^*(t) = -R_2^{-1}\ B_2^T K_2(t)x_2(t).
\]

Therefore, \(F_2^*(t)\) must coincide with \(-R_2^{-1}\ B_2^T K_2(t)\). Substituting this result into equation (6.96) then shows that it is necessary for the set of coupled Riccati equations (6.89) and (6.91) to have a set of symmetric solutions \((K_1(.), K_2(.))\) on \([0, T]\).

The converse statement, that if the set of coupled Riccati differential equations (6.89) and (6.91) has a symmetric solution the strategies
\begin{equation}
    u_i^*(t) = -R_i^{-1}B_i^T K_i(t)x(t).
\end{equation}

constitute an equilibrium solution, follows directly from Theorem 3 by considering \(V_i(t, x) = x^T(t)K_i(t)x(t)\). That there is only one equilibrium follows from the first part of the proof. Finally, by Theorem 3, the cost incurred by player \(i\) is \(x_0^T K_i(0)x_0\). The formulation requires that the matrix pair \((A, B_i)\) is controllable.

**Definition 6.** The dynamical system described or the pair \((A, B)\) is said to be controllable if, for any initial state \(x_0, t_1 > t_0\) and final state \(x_1\), there exists a piecewise continuous input \(u(\cdot)\) such that the solution \(K\) satisfies \(x(t_1) = x_1\). Otherwise, the system or the pair \((A, B)\) is said to be uncontrollable.

\[\Box\]

### 6.3.4 Feedback Equilibrium of the Infinite Planning-Horizon Problem

We present the game in which each player \(i\) considers the minimization of the functional

\begin{equation}
    J_i(x_0, u_1, u_2) = \lim_{t \to \infty} J_i(x_0, u_1, u_2, t)
\end{equation}

for player \(i\), \(i = 1, 2\), where

\begin{equation}
    J_i(x_0, u_1, u_2, T) = \int_0^T \{x^T(t)Q_i x(t) + u_i^T(t)R_{ii}u_i(t) + u_j^T(t)R_{ij}u_j(t)\} dt, \quad j \neq i
\end{equation}

subject to the dynamical system
\[ \dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t), \quad x(0) = x_0 \]  

(6.105)

where \( Q_i \) and \( R_{ij}, \ i, j = 2 \) are symmetric matrices and \( R_{ii}, \ i, j = 2 \) is positive definite.

We seek a steady state solution to the above linear quadratic optimal control as we are concerned with the optimal state and control for the infinite-planning horizon. Steady state refers to when the time derivative of the state and control vanishes as time approaches infinity.

**Theorem 7.** Assume that \((A, B)\) is stabilizable and \(u = Fx\), with \(F \in F\). The linear quadratic control problem (6.104) and (6.105) has a minimum \(\hat{F} \in F\). for \(J(F)\) for each \(x_0\) if and only if the algebraic Riccati equation (6.106) has a symmetric stabilizing solution \(K\).

\[
Q + A^T X + XA - XSX = 0 \quad (6.106)
\]

If the linear quadratic control problem has a solution, then the solution is uniquely given by \(\hat{F} = -R^{-1}B^TK\) and the optimal control in feedback form is

\[
u^*(t) = -R^{-1}B^TKx(t) \quad (6.107)
\]

Moreover, \(J(u^*) = x_0^T K x_0\)

**Proof.** Let \(K\) be the stabilizing solution of the algebraic Riccati equation (ARE). Then

\[
\int_0^\infty \frac{d}{dt} \{x^T(t)Kx(t)\} dt = 0 - x^T(0)Kx(0). \quad (6.108)
\]
As a result, a completion of squares again within the cost function results in

\[ J = \int_{0}^{\infty} \{ x^T Q x(t) + u^T(t) R u(t) \} dt + \int_{0}^{\infty} \{ x^T K x(t) \} dt + x^T(0) K x(0) \] (6.109)

\[ = \int_{0}^{\infty} \{ x^T Q x(t) + u^T(t) R u(t) + \frac{d}{dt} \{ x^T K x(t) \} \} dt + x^T(0) K x(0) \] (6.110)

An arbitrary \( F \in \mathcal{F} \), with \( u = F x, \dot{x} = (A + BF) x \). Using this and the ARE (6.106), the integrand can be rewritten as

\[ x^T Q x + u^T R u + \frac{d}{dt} \{ x^T K x \} = x^T Q x + x^T F^T R F x + x^T(A + BF) x + x^T K(A + BF) x \] (6.111)

\[ = x^T(Q + A^T K + KA)x + x^T F^T R F x + x^T F^T B^T K x + x^T K B F x \] (6.112)

\[ = x^T (Q + A^T K + KA)x + x^T F^T R F x + x^T F^T B^T K x + x^T K B F x \] (6.113)

\[ = x^T K S K x + x^T F^T R F x + x^T F^T B^T K x + x^T K B F x \] (6.114)

\[ = x^T (F + R^{-1} B^T K)^T R (F + R^{-1} B^T K) x. \] (6.115)

Hence

\[ J = \int_{0}^{\infty} \{ x^T(t)(F + R^{-1} B^T K)^T R (F + R^{-1} B^T K) x(t) \} dt + x^T(0) K x(0). \] (6.116)

We can see that expression (6.116) will always result in \( J \geq x^T(0) K x(0) \) for all \( F \) and that equality is achieved if \( F = -R^{-1} B^T K \).

\[ \square \]

**Definition 8.** \((F_1^*, F_2^*) \in \mathcal{F} \) is called a stationary linear feedback Nash equilibrium if the following inequalities hold
\[ J_1(x_0, F_1^*, F_2^*) \leq J_1(x_0, F_1, F_2^*) \text{ and } J_2(x_0, F_1^*, F_2^*) \leq J_2(x_0, F_1^*, F_2) \]

for each \( x_0 \) and for each state feedback matrix \( F_i, i = 1, 2 \) such that \( (F_1^*, F_2) \) and \( (F_1, F_2^*) \in F \)

Next, consider the set of coupled algebraic Riccati equations

\[
0 = (A - S_2 K_2)T K_1 - K_1 (A - S_2 K_2) + K_1 S_1 K_1 - Q_1 - K_2 S_2 K_2 \quad (6.117)
0 = (A - S_1 K_1)T K_2 - K_2 (A - S_1 K_1) + K_2 S_2 K_2 - Q_2 - K_1 S_1 K_1 \quad (6.118)
\]

Theorem 9 states that feedback Nash equilibria are completely characterized by symmetric stabilizing solutions of equations 6.117 and 6.118. That is, by symmetric solutions \((K_1, K_2)\) for which the closed-loop system matrix \( A - S_1 K_1 - S_2 K_2 \) is stable.

Consider the set of coupled algebraic Riccati equations:

\[
0 = - (A - S_2 K_2)^T K_1 - K_1 (A - S_2 K_2) + K_1 S_1 K_1 - Q_1 - K_2 S_2 K_2 \quad (6.119)
0 = - (A - S_1 K_1)^T K_2 - K_2 (A - S_1 K_1) + K_2 S_2 K_2 - Q_2 - K_1 S_1 K_1 \quad (6.120)
\]

Theorem 9 below states that feedback Nash equilibria are completely characterized by symmetric stabilizing solutions of equations (6.119) and (6.120). That is, by symmetric solutions \((K_1, K_2)\) for which the closed-loop system matrix \( A - S_1 K_1 - S_2 K_2 \) is stable.

**Theorem 9.** Let \((K_1, K_2)\) be a symmetric stabilizing solution of equations (6.119) and (6.120) and define \( F_i^* = -R_i^{-1} B_i^T K_i \) for \( i = 1, 2 \). Then \((F_1^*, F_2^*)\) is a feedback Nash equilibrium. Moreover, the cost incurred by player \( i \) by playing this equilibrium action is \( x_i^T K_i x_0 \), \( i = 1, 2 \). Conversely, if \((F_1^*, F_2^*)\) is a feedback Nash equilibrium, there exists a symmetric stabilizing solution \((K_1, K_2)\) of equations
(6.119) and (6.120) such that $F^*_i = -R^{-1}_{ii}B^T_i K_i$.

The ultimate take away is that $F^*_i = -R^{-1}_{ii}B^T_i K_i$ is a feedback Nash equilibrium where $(K_1, K_2)$ is the solution of the coupled algebraic Riccati equations.

**Proof.** Let $(K_1, K_2)$ be a stabilizing solution of equations (6.119) and (6.120) and define $F^*_2 := -R^{-1}_{22}B^T_2 K_2$. We then look at the minimization by player one of the cost function

$$J_1(x_0, u_1, F^*_2) = \int_0^\infty \left\{ x^T(Q_1 + F^*_2 R_{12} F^*_2) x + u_1^T R_{11} u_1 \right\} dt,$$

subject to the system $\dot{x}(t) = (A - S_2 K_2) x(t) + B_1 u_1(t), \ x(0) = x_0$.

We assume, according to equation (6.119), the equation

$$0 = -(A - S_2 K_2)^T X - X (A - S_2 K_2) + X S_1 X - Q_1 - K_2 S_2 K_2$$

(6.122)

has a stabilizing solution $X = K_1$. So according to Theorem 7, the above minimization problem has a solution. The solution is given by $u_1^* = -R^{-1}_{11} B^T_1 X$ and the associated minimum cost by $x_0^T X x_0$. Since $X = K_1$ it follows that $F_1 = F_1^*$ solves the minimization problem. Specifically,

$$J_1(x_0, F^*_1, F^*_2) \leq J_1(x_0, F_1, F^*_2)$$

(6.123)

Along a similar line of thought, we show that the minimization problem for player two is solved by $F^*_2$, which proves the first part of the claim.

Next, assume that $(F^*_1, F^*_2) \in F$ is a feedback Nash equilibrium. By definition we can state the following expressions.
\[ J_1(x_0, F_1^*, F_2^*) \leq J_1(x_0, F_1, F_2^*) \text{ and } J_2(x_0, F_1^*, F_2^*) \leq J_1(x_0, F_1^*, F_2^*) \] (6.124)

for all \( x_0 \) and for all state feedback matrices \( F_i, i = 1, 2 \), such that \( (F_1^*, F_2) \) and \( (F_1, F_2^*) \) \( \in \mathcal{F} \). Therefore, according to Theorem 7, there exist real symmetric matrices \( K_i, i = 1, 2 \), satisfying the set of equations

\[
0 = -(Q_1 + F_2^* R_{12} F_2^*) - (A + B_2 F_2^*)^T K_1 - K_1 (A + B_2 F_2^*) + K_1 S_1 K_1 \] (6.125)

\[
0 = -(Q_2 + F_1^* R_{21} F_1^*) - (A + B_1 F_1^*)^T K_2 - K_2 (A + B_1 F_1^*) + K_2 S_2 K_2 \] (6.126)

We require that \( A + B_2 F_2^* - S_1 K_1 \) and \( A + B_1 F_1^* - S_2 K_2 \) are stable.

Theorem 7 implies, that \( F_1^* = -R_{ii}^{-1} B_i^T K_i \) and \( J_i(x_0, F_i^*, F_2^*) = x_0^T K_i x_0 \). Substituting \( F_2^* \) in equation (6.125) by \( -R_{12}^{-1} B_2^T K_2 \) and \( F_1^* \) in equation (6.126) by \( -R_{11}^{-1} B_1^T K_1 \) shows that \( (K_1, K_2) \) is a solution to the the coupled system of AREs (6.119) and (6.120). Moreover, substituting \( F_1^* \) in \( A + B_1 F_1^* - S_2 K_2 \) by \( -R_{11}^{-1} B_1^T K_1 \) shows that matrix \( A - S_1 K_1 - S_2 K_2 \) is stable, which completes the proof.

**Definition 10.** The dynamical system \( \dot{x} = Ax \) is said to be stable if all the (possibly) complex eigenvalues of \( A \) are in the open left of the complex plane, i.e. the real part of every eigenvalue of \( A \) is strictly smaller than zero. A matrix \( A \) with such property is said to be stable.

**Definition 11.** The dynamical system \( \dot{x} = Ax + Bu \) or the pair \( (A, B) \) is said to be stabilizable if, for any initial state \( x_0 \), there exists a piecewise continuous input \( u(\cdot) \) such that the solution converges to zero.

From this definition it is clear that whenever a system is controllable it is also
stabilizable. This observation follows directly from the the well-known theorem which gives several characterizations for stabilizability of a system (Zhou, Doyle and Glover, 1996).

N-Player Matrix Case

We distinguish between the terms matrix and scalar. Matrix refers to a circumstance where the linear quadratic game has matrices as their input parameters while scalar represents the simplified game where the inputs to the linear quadratic game are scalar numbers.

We can without much effort extend the results of Theorem 9 for the N-player game defined by equations (6.119) and (6.120).

\[
0 = - \left( A - \sum_{j \neq i}^N S_j K_j \right)^T K_i - K_i \left( A - \sum_{j \neq i}^N S_j K_j \right) + K_i S_i K_i - Q_i - \sum_{j \neq i}^N K_j S_{ij} K_j \tag{6.127}
\]

The matrix \( A - \sum_{i}^N S_i K_i \) must be stable. The resulting solution \( K_i \) is a symmetric matrix for each player. See Appendix A on an algorithm to solve the above system of coupled ARE. The optimal state can be found by solving the following system of linear differential equations

\[
\dot{x} = \left[ A - \sum_{i}^N S_i K_i \right] x \quad \tag{6.128}
\]

where \( x \) is a column vector of size equal to the number of states \( V \) in the game. Thus, it is required that a system of \( N \) homogenous linear differential equations be solved to obtain the optimal state trajectory \( x(t) \). A similar generalization can be performed to obtain the optimal control trajectory of each player.
\[ u_i(t)^* = -R_{ii}^{-1}B_i^T K_i x(t)^* \]  

(6.129)

The optimal control trajectory is a linear combination of the term \(-R_{ii}^{-1}B_i^T\) and the column vector \(x(t)^*\).

**Two-Player Scalar Case**

We consider the scalar two-player game; players are not interested in the control actions pursued by the other player. In that case it is possible to derive analytic results. In particular we will show that in this game never more than three equilibria occur. Furthermore, a complete characterization of parameters that give rise to either zero, one, two or three equilibria will be given.

The object of study is the next game:

\[ J_i(x_0, u_1, u_2) = \int_0^\infty \left\{ q_i x^2(t) + r_i u_i^2 \right\} dt, \quad i = 1, 2 \]  

(6.130)

subject to the dynamical system

\[ \dot{x}(t) = a x(t) + b_1 u_1(t) + b_2 u_2(t), \quad x(0) = x_0 \]  

(6.131)

The algebraic Riccati equations that provide the key to finding the feedback Nash equilibria for this game (see Theorem 9) are obtained from equations (6.119) and (6.120) by substitution of \(R_{12} = R_{21} = 0, A = a, B_i = b_i, Q_i = q_i, R_{ii} = r_i\) and \(s_i = b_i^2/r_i, i = 1, 2\), into these equations. By Theorem 9 a pair of control actions \(f_i^* \doteq -\frac{b_i}{r_i} k_i, i = 1, 2\), then constitute a feedback Nash equilibrium if and only if
the following equations have a solution $x_i = k_i$, $i = 1, 2$,

$$s_1x_1^2 + 2s_2x_1x_2 - 2ax_1 - q_1 = 0 \quad (6.132)$$
$$s_2x_2^2 + 2s_1x_1x_2 - 2ax_2 - q_2 = 0 \quad (6.133)$$
$$a - s_1x_1 - s_2x_2 < 0 \quad (6.134)$$

Geometrically, the equations (6.132) and (6.133) represent two hyperbolas in the $(x_1, x_2)$ plane, whereas the inequality (6.134) divides this plane into a ‘stable’ and an ‘anti-stable’ region. So, all feedback Nash equilibria are obtained as the intersection points of both hyperbolas in the ‘stable’ region.

**Theorem 12.** 1. Assume that $(k_1, k_2)$ is a feedback Nash equilibrium strategy. Then the negative of the corresponding closed-loop system parameter $\lambda \doteq -a + \sum_{i=1}^{2} s_i k_i > 0$ is an eigenvalue of the matrix

$$M \doteq \begin{bmatrix} -a & s_1 & s_2 & 0 \\ -q_1 & a & 0 & -s_2 \\ -q_2 & 0 & a & -s_1 \\ 0 & \frac{1}{3}q_2 & \frac{1}{3}q_1 & \frac{1}{3}a \end{bmatrix}$$

Furthermore, $[1, k_1, k_2, k_1k_2]^T$ is a corresponding eigenvector and $\lambda^2 \geq \sigma_{\text{max}}$.

2. Assume that $[1, k_1, k_2, k_1k_2]^T$ is an eigenvector corresponding to a positive eigenvalue $\lambda$ of $M$, satisfying $\lambda^2 \geq \sigma_{\text{max}}$, and that the eigenspace corresponding to $\lambda$ has dimension one. Then, $(k_1, k_2)$ is a feedback Nash equilibrium.

**Theorem 13.** 1. Assume that $(k_1, k_2)$ is a feedback Nash equilibrium strategy. Then the negative of the corresponding closed-loop system parameter $\lambda \doteq -a + \sum_{i=1}^{2} s_i k_i > 0$ is an eigenvalue of the matrix
Furthermore, $[1, k_1, k_2, k_1 k_2]^T$ is a corresponding eigenvector and $\lambda^2 \geq \sigma_{\text{max}}$.

2. Assume that $[1, k_1, k_2, k_1 k_2]^T$ is an eigenvector corresponding to a positive eigenvalue $\lambda$ of $M$, satisfying $\lambda^2 \geq \sigma_{\text{max}}$, and that the eigenspace corresponding to $\lambda$ has dimension one. Then, $(k_1, k_2)$ is a feedback Nash equilibrium.

**Algorithm to solve the N player scalar case**

We present an algorithm to solve the N-player scalar linear feedback problem originally presented in Engwerda (2005). The algorithm below calculates all feedback Nash equilibria of the linear quadratic differential game (6.130) and (6.131).

**Step 1** Calculate matrix $M$

$$M \doteq \begin{bmatrix} -a & s_1 & s_2 & 0 \\ -q_1 & a & 0 & -s_2 \\ -q_2 & 0 & a & -s_1 \\ 0 & \frac{1}{3} q_2 & \frac{1}{3} q_1 & \frac{1}{3} a \end{bmatrix}$$

and $\sigma \doteq \max_i \frac{b_i^2 q_i}{r_i}$.

**Step 2** Calculate the eigenstructure $(\lambda_i, m_i)$, $i = 1, \ldots, k$, of $M$, where $\lambda_i$ are the eigenvalues and $m_i$ the corresponding algebraic multiplicities.

**Step 3** For $i = 1, \ldots, k$ repeat the following steps.
3.1. If (i) \( \lambda_i \in \mathbb{R} \), (ii) \( \lambda_i > 0 \) and (iii) \( \lambda_i^2 \geq \sigma \) then proceed with Step 3.2 of the algorithm. Otherwise, return to Step 3.

3.2. If \( m_i = 1 \) then carry out the following.

3.2.1 Calculate an eigenvector \( v \) corresponding with \( \lambda_i \) of \( M \). Denote the entries of \( v \) by \( \left[ v_0, v_1, v_2, \ldots, \right]^T \). Calculate \( k_j \doteq \frac{v_j}{v_0} \) and \( f_j \doteq -\frac{b_j k_j}{r_j} \). Then, \((f_1, \ldots, f_N)\) is a feedback Nash equilibrium and \( J_j = k_j x_0^2, \ j = 1, \ldots, N \). Return to Step 3.

If \( m_i > 1 \) then carry out the following

3.2.2. Calculate \( \sigma_i \doteq \frac{b_i^2 r_i}{r_i} \).

3.2.3. For all \( 2^N \) sequences \((t_1, \ldots, t_N)\), \( t_k \in \{-1, 1\} \),

(i) calculate

\[
y_j \doteq \lambda_i + t_j \sqrt{\lambda_i^2 - \sigma_j}, \quad j = 1, \ldots, N;
\]

(ii) if \( \lambda_i = -a + \sum_{j=1,\ldots,N} y_j \) then calculate \( k_j \doteq \frac{y_j r_j}{r_j^2} \) and \( f_j \doteq -\frac{b_j k_j}{r_j} \); then \((f_1, \ldots, f_N)\) is a feedback Nash equilibrium and \( J_j = k_j x_0^2, \ j = 1, \ldots, N \).

Step 4 End of the algorithm.

6.3.5 Fully Dynamic and Steady State Solutions

The Matrix Riccati differential equation (RDE) is used to solve finite planning-horizon linear quadratic problems while the algebraic Riccati equations (ARE) are used for the infinite planning horizon problem. The RDE provides fully dynamic solutions in the sense that it offers an accurate depiction of the state and control trajectory before reaching steady state. The ARE provides steady state solutions.
We define steady state as where the state only moves an infinitesimal amount as time increases. In mathematical terms, the derivative of the state approaches zero as time approaches infinity. For the infinite planning-horizon, it is a natural assumption for the derivative to vanish and thus the ARE is used. The two types of Riccati equations are closely related in that their only differences is that the derivative of $K$ with respect to time is assumed to vanish in the ARE while it remains a quantity to solve in the RDE. To further illustrate these points, we define the matrix Riccati differential equation is defined as

$$\frac{dK}{dt} = -ATK - KA + KSK - Q, \quad K(t_f) = Q_{t_f} \quad (6.135)$$

where

$K$ is the solution of the differential equation

$S = BB^T R^{-1}$

$A, B, R$ and $Q$ are symmetric matrices of the following linear quadratic game

minimize

$$J = \int_0^T \{ x^TQx(t) + u^TRu(t) \} dt + x^T(T)QTx(T) \quad (6.136)$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \quad (6.137)$$

We assume that the Riccati differential equation has an asymptotically stable solution $K(t)$ for when $T \to \infty$, thus subsequently assuming the infinite planning-horizon problem.

$$\lim_{t \to \infty} K(t) = K_\infty \quad (6.138)$$
Then the time derivative vanishes

$$\lim_{t \to \infty} \frac{dK(t)}{dt} = 0 \quad (6.139)$$

Substituting (6.139) into (6.135) yields

$$0 = -A^T K - KA + KSK - Q = 0 \quad (6.140)$$

Equation (6.140) is known as the Algebraic Riccati Equation, or also referred to as the steady-state Riccati Equation or non dynamic Riccati. It is relevant when $T$ is set to $\infty$ in the infinite-planning horizon problem. Hence the infinite-planning horizon does assume steady-state while the finite-planning horizon does not. In this chapter we presented two optimal control models reformulated into linear quadratic optimal problems. We then presented theoretical methods to solve for the optimal feedback control of the resulting differential games. We now continue to numerical examples of the models presented in the dissertation.
Chapter 7

Numerical Examples

7.1 Data Sources and Methodology

A key feature of any proposed model is how computable it is with near real-world data sets. In this chapter, we present an illustrative example as well as the stylized 15-node network of the Western European electric grid. All examples are shown to be computable with the commercially available NLPEC solver within GAMS. The illustrative example solved in less than one second while the Western European network solved in eight minutes. We chose $\Delta t = 1$ that resulted in 24 discrete time periods. All computations were performed on the Network-Enabled Optimization System (NEOS) server Czyzyk et al. (1998). NEOS is a free high performance computing platform that allows researchers to use a variety of software packages and optimization solvers.
7.2 Dynamic Stackelberg Model

7.2.1 Specific Formulation

We now present a specific formulation of equations (5.53) through (5.65) assuming specific function forms for the generating cost \( V \) and inverse demand \( \pi \).

We assume \( \pi \) follows a well studied linear inverse demand where

\[
\pi_{i,t} = a_{i,t} - b_{i,t} \cdot \sum_f c_{i,t}^f
\]  

(7.1)

The parameter \( a_{i,t} \) represents that amount of power (MW) that consumers at node \( i \) would use if prices were set to zero, and \( b_{i,t} \) signifies the slope of the inverse demand curve. We have previously defined \( V(q) \) as an explicit function. A realistic generation cost function typically exerts a quadratic or nonlinear property as a result of decreasing returns to scale (Varian, 2006). The model is general enough to handle higher order functions that would model to scale both increasing and decreasing returns. We approximate a quadratic cost curve assuming a piecewise linear function as described in (7.2).

\[
V_{i,t}^f = \max(m_{1,i}^f q_{i,t}^f + b_{1,i}^f, m_{2,i}^f q_{i,t}^f + b_{2,i}^f)
\]  

(7.2)

We can further express \( V_{i,t}^f \) as a variable subject to the following inequalities
This clever reformulation, including $V$ as a variable and additional inequalities, all us to remove any nonlinear effects that a quadratic $V$ function might have imposed. The two constraints represented by (7.3) and (7.4) form a feasible region bounded below. The term $V_{i,t}^f$ is always minimized by each generator and therefore is guaranteed to be on the lower boundary of the feasible region set. This property allows $V_{i,t}^f$ to be equal the piecewise linear segment, approximating a quadratic cost function. In our example, we approximated the quadratic cost function with two affine segments. However, the model is general enough to handle greater number of affine segments. A greater number of segments increases the quality of the approximation.

We substitute $V$ and $\pi$ into the upper level market monitor problems in Equations (5.53) through (5.55) along with the constraints (7.3) and (7.4). Equations (7.10) and (7.12) have been modified accordingly to handle the addition of $V$ as a variable.

$$\begin{align*}
V_{i,t}^f & \geq m_{1,i}^f q_{i,t}^f + b_{1,i}^f \\
V_{i,t}^f & \geq m_{2,i}^f q_{i,t}^f + b_{2,i}^f
\end{align*}$$

(7.3)

(7.4)

\[ \text{Max} \quad Z(c_i^f, q_i^f) = \sum_{t \in T} \sum_{i \in M} \left\{ -a_{i,t} \sum_f c_{i,t}^f + \frac{b_{i,t}^f}{2} \left[ \sum_f c_{i,t}^f \right]^2 + \sum_f V_{i,t}^f \right\} \quad (7.5) \]
subject to

\[
\sum_{i \in N} PTDF_{i,a} \cdot \left[ \sum_f (c^f_{i,t} - q^f_{i,t}) \right] \leq T_a \quad \forall a \in A, \forall t \in T \quad (7.6)
\]

\[
\sum_{i \in N} PTDF_{i,a} \cdot \left[ \sum_f (c^f_{i,t} - q^f_{i,t}) \right] \geq -T_a \quad \forall a \in A, \forall t \in T \quad (7.7)
\]

\[
\sum_{i \in N} \sum_{F \in F} w_{i,t} [c^f_{i,t} - q^f_{i,t}] \geq 0 \quad \forall t \in T \quad (7.8)
\]

\[
0 \leq c^f_{i,t} \perp \left[ 2b_{i,t}c^f_{i,t} - a_{i,t} + b_{i,t} \sum_{g \in F, g \neq f} c^g_{i,t} + w_{i,t} - \zeta^+_t + \zeta^-_t \right] = \phi^f_{i,t} \geq 0 \quad (7.9)
\]

\[
0 \leq q^f_{i,t} \perp \left[ -w_{i,t} + \zeta^+_t - \zeta^-_t + m^f_{1,i} \gamma^+_t + m^f_{2,i} \gamma^-_t + \sigma^f_{i,t} - \mu^f_{i,t} + \theta^f_{i,t} \right] = \rho^f_{i,t} \geq 0 \quad (7.10)
\]

\[
0 \leq V^f_{i,t} \perp \left[ 1 + \gamma^+_t - \gamma^-_t \right] = \delta^f_{i,t} \geq 0 \quad (7.11)
\]

\[
0 \leq \left[ - \sum_{i \in N} q^f_{i,t} + \sum_{i \in M} c^f_{i,t} \right] \perp \zeta^+_t \geq 0 \quad (7.12)
\]

\[
0 \leq \left[ \sum_{i \in N} q^f_{i,t} - \sum_{i \in M} c^f_{i,t} \right] \perp \zeta^-_t \geq 0 \quad (7.13)
\]

\[
0 \leq \left[ V^f_{i,t} - m^f_{1,i} q^f_{i,t} - b^f_{1,i} \right] \perp \gamma^+_t \geq 0 \quad (7.14)
\]

\[
0 \leq \left[ V^f_{i,t} - m^f_{2,i} q^f_{i,t} - b^f_{2,i} \right] \perp \gamma^-_t \geq 0 \quad (7.15)
\]

\[
0 \leq c^f_{i,t} \perp \phi^f_{i,t} \geq 0 \quad (7.16)
\]

\[
0 \leq V^f_{i,t} \perp \delta^f_{i,t} \geq 0 \quad (7.17)
\]

\[
0 \leq q^f_{i,t} \perp \rho^f_{i,t} \geq 0 \quad (7.18)
\]

\[
0 \leq -q^f_{i,t} + q^f_{i,max} \perp \sigma^f_{i,t} \geq 0 \quad (7.19)
\]

\[
0 \leq q^f_{i,t} - q^f_{i,t-1} - r^f_{i,min} \perp \mu^f_{i,t} \geq 0 \quad (7.20)
\]

\[
0 \leq -q^f_{i,t} + q^f_{i,t-1} + r^f_{i,max} \perp \theta^f_{i,t} \geq 0 \quad (7.21)
\]
7.2.2 Illustrative Example

Our first numerical example is a simple 3-node network based on the small network presented in Gabriel and Leuthold (2010). All three nodes have demand, but the primary load resides at node 3. There are two firms that act as followers, while the market monitor is modeled as the only leader. Firm 1 possess facilities at nodes 1 and 2. Firm 2 has a plant only at node 2. We present numerical results for two congestion scenarios; one with no congestion and the other in which the transmission line connecting node 2 to 3 is congested.

7.2.2.1 Formulation

We assumed a planning horizon of one time period. Ramping bounds $r_{i,\text{min}}$ and $r_{i,\text{max}}$ were not applicable since ramping only applies when more than one discrete time period is under consideration. A summary of the model topology and parameters of the network can be seen in Figure 7.1 and Tables 7.1 through 7.4 respectively.

![Figure 7.1. 3 node network](image)

The direction arrows in Figure 7.1 describe the sign of electricity flow. For example, a MW flowing from node 1 to node 2 would be +1 MW while a MW flowing from node 3 to node 1 would be -1 MW. The inverse demand functions
Table 7.1. Inverse Demand data $\pi_i$

<table>
<thead>
<tr>
<th>Node</th>
<th>$a_i$</th>
<th>$b_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 7.2. Plant Production and Capacity data $V$ and $q_{max}$

<table>
<thead>
<tr>
<th>Firm</th>
<th>Node</th>
<th>$m_1$</th>
<th>$b_1$</th>
<th>$m_2$</th>
<th>$b_2$</th>
<th>$q_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

described by table 7.1 assert that the primary load demand resides in node 3, a moderate amount in node 1 and small demand in node 2. This simple example is equivalent to node 3 residing in a high population area while the other nodes represent rural generation nodes. Table 7.2 describes the cost of electricity generation and capacity. For illustrative purposes, the piecewise production $V$ is transformed into a simple linear function. Firm 1 has a low-cost generation facility located at node 2 in addition to a moderately low-cost facility at node 2. Firm 2 has a high priced facility at node 2. Each facility has a capacity limit of 10 MWs. Table 7.3 describes the two congestion scenarios with the transmission line connection node 2 and 3 limited to 4 MW in the congestion phase.

We compare our Stackelberg game detailed in Chapter 5 with perfect competitive, Cournot-Nash equilibrium and our dynamic Stackelberg game. Perfect

Table 7.3. Transmission Arc Capacity $T_a$

<table>
<thead>
<tr>
<th>Arc</th>
<th>No Congestion $T_a$</th>
<th>Congestion $T_a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-2</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>2-3</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>1-3</td>
<td>10</td>
<td>10</td>
</tr>
</tbody>
</table>
Table 7.4. $PTDF_{i,a}$ data with Node 3 as Hub

<table>
<thead>
<tr>
<th>Node</th>
<th>Arc 1-2</th>
<th>Arc 2-3</th>
<th>Arc 1-3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1/3</td>
<td>-1/3</td>
<td>-2/3</td>
</tr>
<tr>
<td>2</td>
<td>1/3</td>
<td>-2/3</td>
<td>-1/3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

competition serves as an ideal comparison of what any central planner or market monitor hopes to achieve in a market. In a perfectly competitive market, producers are forced to generate electricity with a profit equal to zero. Thus, all economic surpluses reside with the consumer and market monitor. As a consequence, the price of electricity at each node is equal to lowest marginal cost to produce an additional unit of electricity. Furthermore, the concept of a Nash equilibrium between generators is dissolved. Generators must sell and generate electricity such that the consumer’s and market monitor’s surpluses are maximized. However, it is not practically obtainable or reasonable in real world electricity markets. The scenarios serves merely as a comparison with other market structures. It also serves as a pseudo-upper bound of the possible improvement of market efficiency.

Is noted that we do interchange perfectly competitive and perfect competition. Perfect competition is a specific market structure that we do not possess since we have limited buyers and sellers. Firms have an impact on price and we do not allow additional firms to enter the market within our planning time horizon.

The Cournot-Nash equilibrium for this specific scenario is a game originated in Mookherjee et al. (2010) in which the ISO determines the wheeling simultaneously with the Cournot-Nash game played by the generators. In Mookherjee and colleagues’ model, the ISO’s only concern is to efficiently allocate scarce transmission line capacity among the generators. The details of the formulation can been seen
in chapter. Our Stackelberg equilibrium describes the market monitor possess the ability to impose access charges on the network to incentivize generators to use transmission line capacity efficiently while simultaneously maximizing social welfare.

7.2.2.2 Discussion

Figure 7.2 demonstrates a comparison of all the models described above for consumer, producer and congestion surplus, social welfare for the no congestion scenario. We remind the reader that consumer’s surplus is defined at utility derived from the difference of the maximum price consumers are willing to pay and the actual price the pay for electricity. Producer’s surplus is simply the net profit producers receive from revenue less generation cost and access charges paid to the market monitor. Congestion surplus is defined as the total revenue the market monitor receives from imposing access charges to the generators.1

The goal of the market monitor is to achieve economic surpluses as close as possible to perfectly competitive scenario we described previously. This serves as a benchmark to compare all models. Analyzing the perfectly competitive for our illustrative example yielded the highest consumer surplus and social welfare; by definition, producer surplus was zero while consumer and congestion surpluses were maximized. The Cournot-Nash and Stackelberg equilibrium models produced identical economic surpluses. These results suggest that the market monitor did not utilize equilibrium congestion pricing to increase social welfare. We posit that the Stackelberg equilibrium model is nearly identical to the Cournot-Nash game when little or no congestion exists as socially optimal electricity flows are not encumbered to transmission line constraints. It is only in the presence of

1Note, we distinguish congestion rent from the term “No congestion” describing the scenario.
Figure 7.2. Model comparison under no congestion: Economic surplus and quantities

congestion that there exists an opportunity to incentivize generators to alter their
electricity flows.

An interesting phenomena is observed in comparing the perfectly competitive
solution with both equilibrium solutions. In the perfectly competitive scenario, the
market monitor forces generators to produce electricity although they produce zero
profit. From a generator’s perspective, producing a social welfare-optimal genera-
tion yields the same profit as not producing at all. In realistic market structures,
generators cannot be forced to generate electricity and thus have to be incentivized
by profit to generate power. A producer’s surplus of 20 units is required for the
generators to produce power at socially optimal levels. Any restriction placed on
the producer’s surplus would result in lower social welfare, assuming the realistic
market structure described by Cournot-Nash and Stackelberg equilibrium.

Figure 7.5 displays the wheeling/congestion fees, sales and production for nodes and firms for the no congestion scenario. The Stackelberg equilibrium demonstrated all equal wheeling fees. Our computational results indicate multiples with equal wheeling fees. This is a result of the directionality of the transmission lines we have defined. The Cournot-Nash equilibrium solution produced identical sales and generation to the Stackelberg equilibrium without the use of access charges/wheeling fees. We remind the reader that wheeling fees and access charges refers to fees per unit power to transmit electricity to a node for the Cournot-Nash and Stackelberg models respectively.

![Access Charges, Sales and Generation: No Congestion](image)

**Figure 7.3.** Model comparison under no congestion: Access charges, sales and generation
We now turn to the congested version of the network as demonstrated by Figure 7.4. The transmission line connecting nodes 2 and 3 was reduced from a capacity of 10 to 3 units. The remaining transmission lines were unchanged.

![Economic Surpluses and Quantities: Congestion](image)

**Figure 7.4.** Model comparison under congestion: Economic surpluses and quantities

The congestion reduced social welfare and consumer surplus of the perfect competition model by 8.5 units and 7.5 units, respectively. Interestingly, the congestion surplus increased from zero to 1 unit in the presence of congestion. The Stackelberg equilibrium did achieve a higher social welfare than the Cournot-Nash equilibrium as expected since its objective was directly stipulated to maximize social welfare. This provides evidence that our model has the potential to achieve higher social welfare compared to the Cournot-Nash equilibrium, given all else equal. This result is at the very heart of the contribution presented in this paper.
Social welfare was increased by employing equilibrium congestion pricing even for such a small network presented. Furthermore, the increase was achieved without directly coercing generators to produce electricity. Simply, the access charges incentivized generators to sell and produce electricity in a manner consistent with social welfare optimal electricity flows. It is also noteworthy to state that all production and demand characteristics were held constant between the Stackelberg equilibrium and Cournot-Nash equilibrium scenarios.

Our analysis has shown the existence of multiple optimal solutions in presence of congestion. Specifically, the access charges vary among solutions, suggesting the market monitor has a certain amount of latitude in determining the set of access charges to use within the network. It makes sense that congestion allows access charges to influence the flow of electricity since congestion creates price differentials between nodes.

Figure 7.5 shows the wheeling fees, sales and production for nodes and firms for the no congestion scenario. The first subscript each quantities refers to nodes and the second subscript indicates firms. It is clear that the Stackelberg equilibrium had higher sales and production than the Cournot-Nash since social welfare was improved. Production and sales were more diverse among the equilibrium models than the perfectly competitive.

### 7.2.3 Western European Electric Grid

We now present a stylized network based upon the Western European electric grid presented in Neuhoff et al. (2005). Analysis and results are presented similarly to the previous illustrative example.
7.2.3.1 Data Formulation

The network consists of 15 nodes, 28 arcs and 12 generating firms. Specifically, the set of firms is $F = \{1, ..., 12\}$, the set of nodes is $N = \{1, ..., 15\}$, the set of nodes in which there are markets is $M = \{4, 5, 6, 8, 9, 14, 15\}$ and the set of arcs is $A = \{1, ..., 28\}$. A total of 24 discrete time periods were considered in our planning horizon used to mimic each hour of a day.

All data is formulated in the style of Mookherjee et al. (2010), in which the authors state the inverse demand was created synthetically, based upon demand patterns obtained from California Independent System Operator (CAISO). The data successfully represents the daily load profiles of a typical node in California.
This double hump profile matches the peak profiles of consumers utilizing electricity in the middle of the day and again returning home from work. The ramping bounds were created synthetically. Additionally, the piecewise linear generation cost, PTDF and transmission line capacities originate the Energy research Centre of the Netherlands (ECN). Figure 7.6 provides a topology of our network of interest at time period 19.

![Figure 7.6. Transmission flows for dynamic Stackelberg game (t=19)](image)

### 7.2.3.2 Discussion

Figure 7.6 provides a clever manner of displaying the flow in MW of our optimal solution found for $t = 19$. Each node in gray represents markets while the color of the transmission arc corresponds to the scale of flow on the left. The figure shows a large difference between arcs as well as the phenomenon that a majority of the load resides in the upper portion of the network. The figure demonstrates the intricacies that real world electric networks exhibit. For example, two of the three
lines connected to node 2 have extremely large flows while, the third transmission line contains barely any flow.

![Figure 7.7. Generation at node 4 for all 24 hours](image)

Figure 7.7. Generation at node 4 for all 24 hours

Figure 7.8 shows the generation profile at node 4 for all 24 hours in the planning horizon. Firm 7 produces a nearly constant generation profile as it has a zero marginal cost while firms 2 and 6 have traditional upward sloping cost curves. The sudden drop in generation at time period 14 marks a point in which congestion fees rose and demand shifted to another node.

Figure 7.8 displays the power sales at node 15 across all time periods. The upper double humped curve corresponds to the total sales of the node. This profile mimics the traditional load profile experienced in California and many areas with similar daily consumption patterns. The interaction among firms is particularly interesting in the first time period, as significant market-share changes are experienced. This could be explained, in part, by the time periods required for the ramping dynamics to adjust themselves.
Figure 7.8. Sales at node 15 for all 24 hours

Figure 7.9. Economic surpluses and quantities for our dynamic Stackelberg game and the Cournot-Nash equilibrium model. All quantities were scaled to a value of 1 for display purposes, since large social welfare numbers are difficult to compare. All economic quantities were relatively similar with the slight exception of consumer and producer surplus. It appears that our dynamic Stack-
Elberg game shifted some economic benefits from producers to consumers without increasing social welfare by a large amount. The production quantities on the right hand axis did vary. The congestion surplus for both models was extremely small in comparison to other economic surpluses. This suggests that even for a large network, large access charges may not be needed for the market monitor to influence electricity flows and subsequently social welfare.

However, it is noted that a comparison between our Stackelberg model to the Cournot-Nash game is not completely relevant due to the large difference of each games’ information structure. Our dynamic Stackelberg game anticipates and calculates the lower level equilibrium in tandem. Hence, all agents execute their solutions simultaneously and in equilibrium. This differs from the Cournot-Nash game where the generation firms use the model as a decision support tool for what their generation and sales will be given their forecasts of access charges. Our dynamic Stackelberg game has the feature in which the optimal access charges are announced to all generators and then an equilibrium is found amongt the firms.

**Future Work**

Our model has the flexibility of analyzing a host of energy-related questions. In this section, we foresee two extensions of particular interest. First, we posit what the competitive market effects of future renewable energy facilities would be. Second, we explore how consumers and producers would be affected by a substantial increase in micogrids.

Renewable energy is of high interest to regulators, consumers and generators. A typical feature of renewable generation facilities is that they tend to have high initial capital costs and nearly zero marginal costs. Do markets become more competitive? Does the market monitor gain or loose power to influence social welfare? Do the profits of fossil-burning generators decrease? These are the type
of questions that may be solved with the framework provided.

Another area of exploration may involve renewable portfolio standards (RPS) and renewable energy certificates (RECS). The dynamic Stackelberg model may be able to analyze the equilibrium between conventional sources of power and renewable energy requirements given several versions of government incentives and requirements acting as a Stackelberg leader. The impact of Microgrids upon the competition of firms may yet be another extension. Microgrids are the addition of extremely small generation facilities to the network at the source of consumption; typically installed by consumers as renewable energy sources such as solar energy.

7.3 Oligopolistic Competition with Excess Demand Price Dynamics

In this section we present a small illustrative example that can be solved analytically for both the open loop and feedback equilibrium case. We then present a large scale example consisting of 12 players and 7 states.

7.3.1 Illustrative Example

In this section, we present a numerical example of linear quadratic game presented for both open and closed loop equilibrium as presented in Section 6.1

Our model is a linear quadratic optimal control problem defined by

$$ J_f := \int_0^\infty \{x^T(t)Q_f x(t) + v_f^T(t)R_f v_f(t)\}dt, $$

subject to the dynamical system
\[ \dot{x}(t) = A x(t) + B_1 v_1(t) + B_2 v_2(t) \quad x^T(0) = [\pi_0 \ 0], \quad (7.23) \]

where

\[
A = \begin{bmatrix} -k(b + \frac{1}{h}) - \frac{1}{2}\rho & ak \\ 0 & -\frac{1}{2}\rho \end{bmatrix}
\]

\[
B_f = \begin{bmatrix} -k \\ 0 \end{bmatrix}
\]

\[
Q_f = \begin{bmatrix} -\frac{1}{4h} & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
R_f = h
\]

For the numerical example we assume there are two firms \( f = 1, 2 \) competing in an electricity market. We make every attempt to make the results as parametrically general as possible. However, for the sake of exposition compactness, we initially chose values for \( b \) and \( h \). We assume, for the remainder of the numerical computations, that \( b = 1 \) and \( h = \frac{1}{2} \). Our chosen linear quadratic optimal control problem of interest now becomes equations (7.85) and (7.86) where
7.3.1.1 Open-Loop Equilibrium

The first step in computing the open-loop equilibrium is to calculate the eigenvalues of Matrix $M$. Matrix $M$ is defined as

$$M = \begin{bmatrix} A & -S_1 & -S_2 \\ -Q_1 & -AT & 0 \\ -Q_2 & 0 & -AT^T \end{bmatrix}$$ (7.24)

where, in matrix notation, $S_f$ is defined as

$$S_f = B_f B_f^T R_f^{-1}$$ (7.25)

simple substitution yields

\[
A = \begin{bmatrix} -3k - \frac{1}{2}\rho & ak \\ 0 & -\frac{1}{2}\rho \end{bmatrix}
\]

\[
B_f = \begin{bmatrix} -k \\ 0 \end{bmatrix}
\]

\[
Q_f = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}
\]

\[
R_f = \frac{1}{2}
\]
\[ S_f = \begin{bmatrix} 2k^2 & 0 \\ 0 & 0 \end{bmatrix} \quad f = 1, 2 \]  

(7.26)

The transpose of \( A \) is given as

\[ A^T = \begin{bmatrix} -3k - \frac{1}{2} \rho & 0 \\ a \kappa & -\frac{1}{2} \rho \end{bmatrix} \]  

(7.27)

The specific matrix \( M \) of interest is

\[
M = \begin{bmatrix} 
-3k - \frac{1}{2} \rho & ak & -2k^2 & 0 & -2k^2 & 0 \\
0 & -\frac{1}{2} \rho & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & 3k + \frac{1}{2} \rho & 0 & 0 & 0 \\
0 & 0 & -ak & \frac{1}{2} \rho & 0 & 0 \\
\frac{1}{2} & 0 & 0 & 0 & 3k + \frac{1}{2} \rho & 0 \\
0 & 0 & 0 & 0 & -ak & \frac{1}{2} \rho 
\end{bmatrix}  
\]  

(7.28)

The eigenvalues of \( M \) are \( \{ -\frac{1}{2} \rho, -\lambda_1, \frac{1}{2} \rho, \frac{1}{2} \rho, \frac{1}{2} \rho + 3k, \lambda_1 \} \) where

\[ \lambda_1 = \sqrt{17k^2 + 10k\rho + \rho^2} \]  

(7.29)

We find that matrix \( M \) has two stable and four unstable eigenvalues. We can conclude that the only candidate open-loop equilibrium is found by calculating the
eigenspaces of $M$ corresponding to the eigenvalues $-\frac{1}{2}\rho$ and $-\lambda_1$. The associated eigenvalues and eigenspaces are

\[
\mathcal{T}_1 = \text{Span}\{ T_1 \} \quad \text{where} \quad T_1 = \begin{bmatrix} \frac{-2a\rho(\rho + 3k)}{a}, -\frac{\rho(6\rho + 14k)}{a}, ak, \rho, ak \end{bmatrix}^T \tag{7.30}
\]

\[
\mathcal{T}_2 = \text{Span}\{ T_1 \} \quad \text{where} \quad T_2 = \begin{bmatrix} -2 \left( \frac{1}{2}\rho + 3k + \lambda_1 \right) \left( \frac{1}{2}\rho + \lambda_1 \right), 0, \frac{1}{2}\rho + \lambda_1, v, \frac{1}{2}\rho + \lambda_1, ak \end{bmatrix}^T \tag{7.31}
\]

We notice that the eigenspace corresponding to $\{-\frac{1}{2}\rho, -\lambda_1\}$ is a graph subspace.

The next step is to consider the Hamiltonian matrix $H$

\[
H = \begin{bmatrix} A & -S_i \\ -Q_i & -A^T \end{bmatrix} \tag{7.34}
\]

The matrix $H$ has two negative real eigenvalues $-\frac{1}{2}\rho$ and $- \left( \left( \frac{1}{2}\rho + 3k^2 \right)^2 - k^2 \right)$. It is verified that eigenspace corresponding to these eigenvalues is a graph subspace.

The algebraic Riccati equation

\[
0 = A^T K_f + K_f A - K_f S_f K_f + Q_f \tag{7.35}
\]

has a stabilizing solution $K_f$, $f = 1, 2$.

Since $(A, B_f)$, $f = 1, 2$ are stabilizable, the unique open-loop equilibrium actions are
\[ v_f = -2[-k\ 0]P_f x(t), \quad (7.36) \]

where

\[ P_f = \begin{bmatrix} \rho & \frac{1}{2}\rho + \lambda_1 \\ ak & ak \end{bmatrix} \cdot \begin{bmatrix} -\frac{2a\rho(\rho+3k)}{a} & -2 \left( \frac{1}{2}\rho + 3k + \lambda_1 \right) \left( \frac{1}{2}\rho + \lambda_1 \right) \\ -\frac{\rho(6\rho+14k)}{a} & 0 \end{bmatrix}^{-1} \quad (7.37) \]

\[ = \begin{bmatrix} f_0 & g_0 \\ h_0 & l_0 \end{bmatrix}, \quad f = 1, 2. \quad (7.38) \]

where

\[ f_0 = -\frac{2}{5k + \rho + \lambda_1} \quad (7.40) \]
\[ g_0 = -\frac{a}{3\rho + 4k} + \frac{2a(\rho + 4k)}{(3\rho + 4k)(5k + \rho + \lambda_1)} \quad (7.41) \]

Using the equilibrium actions

\[ u^*_f = (1 + kf_0)x_1(t) + (kg_0)x_2(t) \quad (7.42) \]

the resulting open-loop system is

\[ \dot{x}(t) = \begin{bmatrix} -\frac{1}{2}k - \frac{1}{2}\lambda_1 & \frac{1}{2}(k - \rho + \lambda_1) \frac{a(\rho+2k)}{3\rho+4k} \\ 0 & -\frac{1}{2}\rho \end{bmatrix} x(t) \quad (7.43) \]

We now turn our attention to reformulate the results into our original model.
parameters. Substitution for $x_1$ and $x_2$ in (7.43) yields the differential equation describing the price of electricity $\pi$

$$\dot{\pi}(t) = \frac{1}{2}(\rho - k - \lambda_1) \left[ \pi(t) - \frac{a(\rho + 2k)}{3\rho + 4k} \right]$$  \hspace{1cm} (7.44)

Solving the above differential equation for $\pi(t)$ results in

$$\pi(t) = \alpha e^{\frac{1}{2}(\rho - k - \lambda_1)} + \frac{a(\rho + 2k)}{3\rho + 4k},$$  \hspace{1cm} (7.45)

where

$$\alpha = \pi_0 - \frac{a(\rho + 2k)}{3\rho + 4k}.$$

(7.46)

We can compute that the price path converges to the constant level

$$\pi^* := \frac{a(\rho + 2k)}{3\rho + 4k}.$$

(7.47)

The equilibrium controls are

$$v_f^*(t) = (1 + kf_0)\pi(t) + kg_0, \quad f = 1, 2.$$

(7.48)

Let us assume the following specific model parameters to allow computation of the game.
Figure 7.10 plots the equilibrium price of electricity up to the 10th year. It is clear the price has approximately converged to the steady state around the 5th time period.

Figure 7.10. Price of electricity up to $t = 10$

Figure 7.11 shows the marginal cost of electricity generation and the price of
electricity on the same scale. This small symmetric two-player example results in a relatively large profit margin for each generator.

Figure 7.12 plots the optimal generation path of the generators. The firms initially respond to the high price by generating higher output of electricity since demand was greater than supply. However, they reduced their generation in response to falling prices in order to maximize their profits.
7.3.1.2 Feedback Equilibrium

The feedback equilibrium of our model consists of the information structure in which firms have the ability to observe the state of the system before committing to their control actions. It is potentially a more realistic information structure when compared to open-loop equilibrium since firms do need to commit to their controls simultaneously at the beginning of the planning-horizon.

According to Theorem 9 in Chapter 6, the linear quadratic optimal control problem has a feedback Nash equilibrium if and only if equations 6.117 and 6.118 have a set of stabilizing solutions. With

![Electricity Generation](image_url)

**Figure 7.12.** Electricity generation up to $t = 10$
\[ S_f := \begin{bmatrix} 2k^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad f = 1, 2; \quad S_{12} = S_{21} = 0 \] (7.54)

and

\[ K_1 := \begin{bmatrix} l_1 & l_2 \\ l_2 & l_3 \end{bmatrix} \] (7.55)

\[ K_2 := \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} \] (7.56)

Note that we distinguish between the parameter \( k \) and \( K_1 \) representing the solution of the algebraic Riccati equations (ARE) found in Theorem 9.

Substituting \( K_1 \) and \( K_2 \) into equations 6.117 and 6.118 results in the following six equations

\[-2l_1 \left( -\frac{1}{2} \rho - 3k - 2k^2k_1 \right) + 2k^2l_1^2 + \frac{1}{2} = 0 \] (7.57)

\[ 2k^2k_1^2 - 2k_1 \left( -\frac{1}{2} \rho - 3k - 2k^2l_1 \right) + \frac{1}{2} = 0 \] (7.58)

\[(\rho + 3k + 2k^2k_1 + 2k^2l_1)l_2 + 2k^2l_1k_2 - akl_1 = 0 \] (7.59)

\[ 2k^2k_1l_2 + (\rho + 3k + 2k^2k_1 + 2k^2l_1)k_2 - akk_1 = 0 \] (7.60)

\[ \frac{1}{\rho} \left( 2l_2 (ak - 2k^2k_2) - 2k^2l_2^2 \right) - l_3 = 0 \] (7.61)

\[ \frac{1}{\rho} \left( 2k_2 (ak - 2k^2l_2) - 2k^2k_2^2 \right) - k_3 = 0 \] (7.62)
The stability requirement, that matrix $A - S_1K_1 - S_2K_2$ must be stable, reduces to

$$\frac{1}{2} \rho + 3k + 2k^2(l_1 + k_1) > 0 \quad (7.63)$$

We can solve the system of equations (7.57)–(7.62) by cascading manner. We first solve for $(k_1, l_1)$ from equations 7.57 and 7.58. The terms $k_2$ and $l_2$ are then determined by the set of linear equations (7.59) and (7.60). Finally, equations (7.61) and (7.62) then result in the solution of $l_3$ and $k_3$.

The subtraction of equation (7.58) from equation (7.57) yields

$$(l_1 - k_1)(\rho + 6k + 2k^2(l_1 + k_1)) = 0 \quad (7.64)$$

If the term $\rho + 6k + 2k^2(l_1 + k_1) = 0$ in equation (7.64), we have a contradiction with the stability requirement (7.63). Therefore, we can conclude that $l_1 = k_1$. As a result, the equations (7.59) and (7.60) become symmetric in $(l_2, k_2)$. It is also verified that $l_2 = k_2$. Finally, we can deduce upon inspection of equations (7.61) and (7.62) that $l_3 = k_3$. Note that this statement is equivalent to $K_1 = K_2$.

The equations (7.57) and (7.59) can be rewritten as

$$6k^2k_1^2 + (\rho + 6k)k_1 + \frac{1}{2} = 0, \quad (7.65)$$

and
respectively.

From the stability condition (7.63) it next follows that the appropriate solution \( k_1 \) in equation (7.65) above is

\[
k_1 = \frac{-(6k + \rho) + \sqrt{(\rho + 6k)^2 - 12k^2}}{12k^2}
\] (7.67)

The equilibrium actions now follow from reformulating into our original model variables. Specifically,

\[
u_f(t) = (2kk_1 + 1)\pi(t) + 2kk_2, \quad f = 1, 2.
\] (7.68)

with \( k_1 \) given by equation (7.67) and \( k_2 \) by equation (7.66). The resulting dynamics of the equilibrium state path \( \pi(t) \) are

\[
\dot{\pi}(t) = (-4k^2k_1 - 3k)\pi(t) + k(a - 4kk_2)
\] (7.69)

Restating \( \pi(t) \) in non differential form yields

\[
\pi(t) = \alpha e^{(-4k^2k_1 - 3k)t} + \frac{a - 4kk_2}{4kk_1 + 3}
\] (7.70)
where \( \alpha = \pi_0 - \frac{a-4kk_2}{4kk_1+3} \)

In a similar fashion to the open-loop equilibrium, we calculate the feedback equilibrium price \( \pi(t) \) converges to the stationary value

\[
\bar{\pi} := \frac{a - 4kk_2}{4kk_1 + 3} \quad (7.71)
\]

Let us assume the following specific model parameters to allow computation of the game. They are purposely chosen to match the open-loop equilibrium case to facilitate comparison.

\[
b = 1 \quad (7.72)
\]

\[
k = .4 \quad (7.73)
\]

\[
\rho = .07 \quad (7.74)
\]

\[
a = 250 \quad (7.75)
\]

\[
\pi_0 = 130 \quad (7.76)
\]

Figure 7.13 plots the equilibrium price of electricity up to the 10th year for both the open-loop and feedback equilibrium models. The prices converged to $107.17 and $120.17 for the feedback and open-loop case, respectively. Furthermore, the feedback converged to the steady state faster than the open-loop case.

Figure 7.14 shows the marginal cost of electricity generation and the price of electricity on the same scale for both open-loop and feedback equilibrium. The marginal cost of the feedback is higher than in the open-loop case primarily because of the higher production output and lower prices realized. We can surmise that
the generators behave more competitively in the feedback case compared to the open-loop information structure.

Figure 7.15 plots the optimal generation path of the firm for the open-loop and feedback equilibrium models. The feedback case consistently has higher generation than the open-loop case.

In summary, the feedback equilibrium had higher generation and lower prices when compared to the open-loop equilibrium. The feedback information structure allows generation firms to observe the price of electricity before committing to generation. We postulate that the knowledge that other firms can react to rising and following prices forces each generator to behave more competitively since they know that their competitors can quickly react to changing electricity prices. Therefore, each firm is less willing to adopt a strategy that makes them susceptible
Figure 7.14. Marginal cost and price of electricity up to $t = 10$

Figure 7.15. Electricity generation up to $t = 10$
to less profits in the future.
7.3.2 Computation of Large Scale Problems

Up to this point, we have made assumptions to solve the excess demand price dynamics optimal control problem analytically and in closed form. This advantage has been as result of the small number of firms and states resulting in the ability to solve the coupled ARE (Algebraic Riccati Equation) described by equations (6.119) and (6.120) easily. Furthermore, the optimal state trajectory is a trivial solution of a linear ODE (Ordinary Differential Equation) as well solving for the optimal control trajectory. Increasing the size of the problem increases the difficulty of solving the ARE as well as obtaining an explicit solution for the optimal state and control functions.

In this subsection, we discuss the scalability, tractability and computational performance of our proposed numerical algorithm for large scale problems. We present and solve the excess demand price dynamics model for 96 unique players and 28 unique markets. Each player chooses electricity production to maximize profit defined as revenue from sales, less generation cost. We make similar assumptions already presented in this section in which transmission and generation constraints are ignored. We account for generation capacity by ensuring the function $V(s,t)$ is monotonically increasing. We also make the assumption that each firm sells the same quantity of electricity to each market.

We modify equation (6.28) to handle multiple firms and states as seen in equation (6.28)

$$J_f(s') = \int_0^\infty e^{-\rho t} \left[ \sum_{v \in W_f} \{ \pi_v(t) \cdot s'(t) \} - V^f(s', \gamma_f, t) \right] dt, \quad \forall f \in F$$

(7.77)
We define $W_f$ as the markets that firm $f$ sells electricity to and thus each firm receives revenue from selling to all markets in $W_f$. Each player’s control is defined as the quantity of electricity to be sold at each market $v \in W_f$. The second term is the production function of producing $s^f$ units with the additional parameter $\gamma_f$, as previously mentioned. We define $\gamma_f$ as the number of markets firm $f$ sells to. The parameter $\gamma_f$ is needed since a total of $\gamma_f \cdot s^f$ units will be generated by player $f$.

The system of states or electricity price dynamics of each market are

$$
\frac{d\pi_v}{dt} = k_v \left[ a_v - b_v \cdot \pi_v(t) - \sum_{g \in G_v} s^g(t) \right], \quad \pi_v(0) = \pi_{v0}, \quad \forall v \in V \quad (7.78)
$$

where $V$ is the set of all state or electricity markets and $G_v$ is the set of players that sell electricity to the market $v$. We make similar assumptions that were presented in the illustrative example to allows us to reformulate the model described by equations (7.80) and (7.81) into a linear quadratic optimal control problem. We assume we have 96 firms and 28 states. We assume $V^f$ is of the form

$$
V^f(t) = \gamma_f \cdot s^f(t) + h_f \left[ \gamma_f \cdot s^f(t) \right]^2, \quad \forall f \in F \quad (7.79)
$$

Substitution of equation (7.79) into equation (6.28) results in a complete description of the differential Nash game

$$
J_f(s^f) = \int_0^\infty e^{-\rho t} \left[ \sum_{v \in W_f} \{\pi_v(t) \cdot s^f(t)\} - \gamma_f \cdot s^f(t) - h_f \left[ \gamma_f \cdot s^f(t) \right]^2 \right] dt, \quad \forall f \in F \quad (7.80)
$$
subject to the dynamical system

\[
\frac{d\pi_v}{dt} = k_v \left[ a_v - b_v \cdot \pi_v(t) - \sum_{g \in G_v} s^g(t) \right], \quad \pi_v(0) = \pi_{v0}, \quad \forall v \in V \quad (7.81)
\]

We assume the following value of parameters for our specific numerical example. The data is mostly derived synthetically with the exception that the sets of markets and firms correspond to the ECN data used in the section 7.2.3.

We perform a similar reformulation as seen in section 6.1.2 by handling both crossproduct terms and discounting. Our model reduces to the game defined by

\[
J_f := \int_0^\infty \left\{ x^T(t)Q_f x(t) + v_f^T(t)R_f v_f(t) \right\} dt, \quad (7.82)
\]

subject to the the dynamical system

\[
\dot{x}(t) = Ax(t) + B_1 v_1(t) + B_2 v_2(t) \quad x^T(0) = [\pi_0 \quad 0], \quad (7.83)
\]

The computation of the feedback optimal control problem requires numerically solving the N-player generalized system of coupled Algebraic Riccati Equations (ARE) described by (6.127). We utilize the algorithm presented in Appendix A to solve the system of coupled ARE.

The resulting solution \( K \) of the ARE facilities the computation of the optimal control and state variables. For the simple 2-player and single state case, it was relatively straight-forward to obtain the optimal state and controls given the solution \( K \). However, the multiplayer and multi state required difficult computation of the optimal state and control trajectories since a system of homogenous linear differential equations has to be solved. We refer the reader to section 6.3.4 on how
the optimal states and controls can be computed.

Figure 7.16 shows the optimal price of electricity in (\$/MWhr) of market 7 across time. We remind the reader the game is modeled in infinite time. The figure displays up to $t = 50$ years. The price approximately converged to steady state value around $t = 25$ years. Figure 7.17 demonstrates the price of electricity at market 5.

**Figure 7.16.** Price of electricity at market 7

**Figure 7.17.** Electricity generation up to $t = 10$
Figures 7.18 through 7.20 plot the amount of electricity that each respective player sold to each market in which they operate.

**Figure 7.18.** Electricity sales of firm 1

**Figure 7.19.** Electricity generation up to $t = 10$

**Computational performance**

A key advantage of reformulating our optimal control as a linear quadratic game are the unique capabilities for computational performance, scalability and
tractability Engwerda (2005). We define computational performance of our proposed numerical algorithm both by how quickly the distance between iterations converged to a user specified $\epsilon$ and the computational time required to reach convergence. The scalability and tractability is defined in context with problems faced by practitioners in the real world. We show performance measures of the algorithm as the number of players and markets are increased.

The computational performance of the algorithm for the numerical experiment can be defined as maximum distance as the maximum norm between $K$ at each iteration for each firm. The distance between two consecutive iterates is defined as

$$D_{\text{max}} = \max_{1 \leq i \leq 96} \| K_{i}^{j+1} - K_{i}^{j} \|$$ (7.84)

Figure 7.21 displays the maximum distance at each iteration. The solution converged rapidly to a solution in less than 15 iterations. The computation time was less than one second on a Windows-based computer running Matlab 8.1. The

**Figure 7.20.** Electricity generation up to $t = 10$
machine used a 2.2 GHz processor with 8 GB of RAM with standard hardware and software configurations.

![Figure 7.21. Maximum distance at each iteration](image)

We note that our numerical algorithm attempts to start with a relatively good solution. However, our algorithm does not emphasize the quality of the solution. We specify a solution that solves a similar system of coupled ARE with a minor degree of coupling relaxed. However, the distance at initial iterates is often quite large in comparison to the $\epsilon$ value we chose.

Table 7.5 provides a summary of the computational time of network with various number of markets or nodes and number of players. Each record in the table denotes a unique network with specified number of unique markets and firms. We denotes each network as small, medium and large based upon our own understanding of the electric power market literature. We define computational time as the cpu time in seconds (s) or minutes (m) required to execute our numerical algorithm. Our algorithm is composed of an input data preprocessing phase followed by an iterative algorithm that solves the system of coupled AREs. The input
data preprocessing time is required to specify the input matrices for the iterative
algorithm. The algorithm portion is the time required to execute the routine to
produce the solution of the system of coupled ARE.

<table>
<thead>
<tr>
<th>Table 7.5. Computational time by problem type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input Data</td>
</tr>
<tr>
<td>Computation Time</td>
</tr>
<tr>
<td>Network</td>
</tr>
<tr>
<td>Small</td>
</tr>
<tr>
<td>Small</td>
</tr>
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<td>Large</td>
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<tr>
<td>Large</td>
</tr>
</tbody>
</table>

We remind the reader that we are using synthetic data based upon the ECN
network used in the previous dynamic Stackelberg computations. Hence, the the 7
market 12 firm small network may be used as foundational data set. Each network
test problem is scaled up by repeating various segments of the foundation input
data. Each scaled test problem usually involves doubling the number of markets
or firms from a previous network test problem. Table 7.5 contains test problems
in which the markets are held constant and the firms are increased as well as
when firms are held constant and the number of markets are increased. This
methodology allows us to investigate both the tractability and scalability of the
proposed numerical algorithm as the number of markets and firms are increased.

Figure 7.22 and 7.23 plot the computation time of the algorithm and input
data preprocessing respectively for each test problem. Our analysis shows that
for every test problem that the input data preprocessing takes significantly more
computational time than the execution of the algorithm. The 112 market 12 firm
Figure 7.22. Algorithm computation time for each test problem size

Figure 7.23. Input data preprocessing computation time for each test problem size
test problem took the longest computation time in terms of the algorithm but only equated to 14 seconds of algorithmic computation time in comparison to 40 minutes of input data preprocessing. The computation time of algorithm appears to be more heavily influenced by the number of markets than the number of firms. The time to process the input data is approximately equally influenced by the number of markets and firms. We conclude that our algorithm is computationally efficient for the following reasons:

- the small network problems provide useful insight to practitioners and can be solved in less than one second.
- large scale problems involving over a hundred markets or firms can be solved in an under an hour of total computation time.
- all test problems were able to reach convergence with ease without any numerical tricks or tweaks to parameters thus indicating tractability of the algorithm.
- the algorithm is scalable since it is able to handle increasing number of markets and firms without require significant computational resources.

A possible extension to the large scale model is to relax the assumption that each firm sells the same quantity of electricity to each market in which they operate in. In order to maintain the linear quadratic optimal control formulation, a set of dummy players may be created to represent each unique market to which each player sells electricity. For example, Player 1 sells to markets 1, 2, and 3. Player 1’s functional would be replaced by three dummy players deciding the optimal sales trajectory in each of the three respective markets. It is clear that the total number
of players’ variables in this proposed extension would grow more rapidly as the number of firms is increased.

### 7.4 Oligopolistic Competition with Sticky Price Dynamics

In this section, we present a numerical example of linear quadratic game presented for both open and closed loop equilibrium.

Our model is the game defined by

\[
J_f := \int_0^\infty \{x^T(t)Q_f x(t) + v_f^T(t)R_f v_f(t)\} dt,
\]

subject to the dynamical system

\[
\dot{x}(t) = A x(t) + B_1 v_1(t) + B_2 v_2(t) \quad x^T(0) = [\pi_0 \quad 0],
\]

where

\[
A = \begin{bmatrix} -s(1 + \frac{1}{h}) - \frac{1}{2}\rho & s \left( a + \frac{g}{h} \right) \\ 0 & -\frac{1}{2}\rho \end{bmatrix},
\]

\[
B_f = \begin{bmatrix} -s \\ 0 \end{bmatrix},
\]

\[
Q_f = \begin{bmatrix} -\frac{1}{4h} & \frac{1}{2} g \\ \frac{1}{2} g & -\frac{1}{2} g^2 \end{bmatrix},
\]

\[
R_f = h.
\]
For the numerical example we assume there are two firms \( f = 1, 2 \) competing in an electricity market. We make every attempt to make the results as parametrically general as possible. However, for the sake of exposition compactness, let \( h = \frac{1}{2} \). Our chosen linear quadratic optimal control problem of interest now becomes equations (7.85) and (7.86) where

\[
A = \begin{bmatrix}
-3s - \frac{1}{2}\rho & s(a + 2g) \\
0 & -\frac{1}{2}\rho
\end{bmatrix}
\]

\[
B_f = \begin{bmatrix}
-s \\
0
\end{bmatrix}
\]

\[
Q_f = \begin{bmatrix}
-\frac{1}{2} & \frac{1}{2}g \\
\frac{1}{2}g & -\frac{1}{2}g^2
\end{bmatrix}
\]

\[
R_f = \frac{1}{2}
\]

### 7.4.1 Open-Loop Equilibrium

The first step in computing the open-loop equilibrium is to calculate the eigenvalues of Matrix \( M \). Matrix \( M \) is defined as

\[
M = \begin{bmatrix}
A & -S_1 & -S_2 \\
-Q_1 & -A^T & 0 \\
-Q_2 & 0 & -A^T
\end{bmatrix}
\]  \hspace{1cm} (7.87)

where, in matrix notation, \( S_f \) is defined as
\[ S_f = B_f B_f^T R_f^{-1} \]  \hspace{1cm} (7.88)

simple substitution yields

\[
S_f = \begin{bmatrix} 2s^2 & 0 \\ 0 & 0 \end{bmatrix} \quad f = 1, 2
\]  \hspace{1cm} (7.89)

The transpose of \( A \) is given as

\[
A^T = \begin{bmatrix} -3s - \frac{1}{2} \rho & 0 \\ s(a + 2g) & -\frac{1}{2} \rho \end{bmatrix}
\]  \hspace{1cm} (7.90)

The specific matrix \( M \) of interest is

\[
M = \begin{bmatrix}
-3k - \frac{1}{2} \rho & s(a + 2g) & -2s^2 & 0 & -2s^2 & 0 \\
0 & -\frac{1}{2} \rho & 0 & 0 & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} g & 3s + \frac{1}{2} \rho & 0 & 0 & 0 \\
-\frac{1}{2} g & \frac{1}{2} g^2 & -s(a + 2g) & \frac{1}{2} \rho & 0 & 0 \\
\frac{1}{2} & -\frac{1}{2} g & 0 & 0 & 3s + \frac{1}{2} \rho & 0 \\
-\frac{1}{2} g & \frac{1}{2} g^2 & 0 & 0 & -s(a + 2g) & \frac{1}{2} \rho 
\end{bmatrix}
\]  \hspace{1cm} (7.91)

The eigenvalues of \( M \) are \( \{ -\frac{1}{2} \rho, -\lambda_1, \frac{1}{2} \rho, \frac{1}{2} \rho, \frac{1}{2} \rho + 3s, \lambda_1 \} \) where
\[ \lambda_1 = \sqrt{17s^2 + 10s\rho + \rho^2} \]  

We find that matrix \( M \) has two stable and four unstable eigenvalues. We can conclude that the only candidate open-loop equilibrium is found by calculating the eigenspaces of \( M \) corresponding to the eigenvalues \( -\frac{1}{2}\rho \) and \( -\lambda_1 \). The associated eigenvalues and eigenspaces are

\[
\mathcal{T}_1 = \text{Span}\{T_1\} \quad \text{where} \\
T_1 = \left[ \frac{2\rho(2gs - (a + 2g)(\rho + 3s))}{a-g}, -\frac{\rho(6\rho + 14s)}{a-g}, as - gs - gp, \rho, as - gs - gp \right]^T
\]  

\[
\mathcal{T}_2 = \text{Span}\{T_1\} \quad \text{where} \\
T_2 = \left[ -2\left(\frac{1}{2}\rho + 3s + \lambda_1\right)\left(\frac{1}{2}\rho + \lambda_1\right), 0, \frac{1}{2}\rho + \lambda_1, v, \frac{1}{2}\rho + \lambda_1, v \right]^T
\]

where

\[ v = as - g\left(\frac{1}{2}\rho + s + \lambda_1\right) \]

We notice that the eigenspace corresponding to \( \{-\frac{1}{2}\rho, -\lambda_1\} \) is a graph subspace.

The next step is to consider the Hamiltonian matrix \( H \)
$$H = \begin{bmatrix} A & -S_i \\ -Q_i & -A^T \end{bmatrix} \quad \text{(7.98)}$$

The matrix $H$ has two negative real eigenvalues $-\frac{1}{2}\rho$ and $-\left(\frac{1}{2}\rho + 3s^2\right)^2 - s^2)$. It is verified that the eigenspace corresponding to these eigenvalues is a graph subspace. The algebraic Riccati equation

$$0 = A^T K_f + K_f A - K_f S_f K_f + Q_f \quad \text{(7.99)}$$

has a stabilizing solution $K_f, \ f = 1, 2$.

Since $(A, B_f), \ f = 1, 2$ are stabilizable, the unique open-loop equilibrium actions are

$$v_f = -2[-s \ 0]P_f x(t), \quad \text{(7.100)}$$

where

$$P_f = \begin{bmatrix} \rho & \frac{1}{2}\rho + \lambda_1 \\ as - gs - g\rho & as - g\left(\frac{1}{2}\rho + s + \lambda_1\right) \\ \frac{2\rho(2gs-(a+2g)(\rho+3s))}{a-g} & -2\left(\frac{1}{2}\rho + 3s + \lambda_1\right) \left(\frac{1}{2}\rho + \lambda_1\right) \\ \frac{-\rho(6\rho+14s)}{a-g} & 0 \end{bmatrix}^{-1} \quad \text{(7.101)}$$

$$= \begin{bmatrix} f_0 & g_0 \\ h_0 & l_0 \end{bmatrix}, \ f = 1, 2. \quad \text{(7.102)}$$
where

\[ f_0 = -\frac{2}{5s + \rho + \lambda_1} \]  \hspace{1cm} (7.105)  

\[ g_0 = \frac{-(a - g)}{3\rho + 4s} + \frac{2(as + (a + 2g)(\rho + s))}{(3\rho + 4s)(5s + \rho + \lambda_1)} \]  \hspace{1cm} (7.106)

Using the equilibrium actions

\[ u_f^* = (1 + sf_0)x_1(t) + (sg_0)x_2(t) \]  \hspace{1cm} (7.107)

the resulting open-loop system is

\[ \dot{x}(t) = \begin{bmatrix} -\frac{1}{2}s - \frac{1}{2}\lambda_1 & \frac{1}{2}(s - \rho + \lambda_1) \frac{a(\rho + 2s)}{3\rho + 4s} \\ 0 & -\frac{1}{2}\rho \end{bmatrix} x(t) \]  \hspace{1cm} (7.108)

We now turn our attention to reformulate the results into our original model parameters. Substitution for \( x_1 \) and \( x_2 \) in (7.108) yields the differential equation describing the price of electricity \( \pi \)

\[ \dot{\pi}(t) = \frac{1}{2}(\rho - s - \lambda_1) \left[ \pi(t) - \frac{as + (a + 2g)(\rho + s)}{3\rho + 4s} \right] \]  \hspace{1cm} (7.109)

Solving the above differential equation for \( \pi(t) \) results in

\[ \pi(t) = \alpha e^{\frac{1}{2}(\rho - s - \lambda_1)} + \frac{as + (a + 2g)(\rho + s)}{3\rho + 4s}, \]  \hspace{1cm} (7.110)

where
\[ \alpha = \pi_0 - \frac{as + (a + 2g)(p + s)}{3\rho + 4s} \] (7.111)

We can compute that the price path converges to the constant level

\[ \pi^* := \frac{as + (a + 2g)(p + s)}{3\rho + 4s} \] (7.112)

The equilibrium controls are

\[ v^*_f(t) = (1 + sf_0)\pi(t) + kg_0, \quad f = 1, 2. \] (7.113)

We assume the following specific model parameters to facilitate computation of the game.

\[ b = 1 \] (7.114)
\[ k = .2 \] (7.115)
\[ \rho = .07 \] (7.116)
\[ a = 175 \] (7.117)
\[ g = 3 \] (7.118)
\[ \pi_0 = 75 \] (7.119)

Figure 7.24 plots the equilibrium price of electricity contract up to the 10th year. It is clear the price has approximately converged to the steady state around the 5th time period.

Figure 7.25 shows the marginal cost of electricity generation and the price of
electricity on the same scale. This small symmetric two player example results in a relatively large profit margin for each generator.

Figure 7.26 plots the optimal generation path of the generators. The firms initially respond to the high price since demand was greater than supply by generating a high value. However, they reduced their generation in response to falling prices in order to maximize their profits.

### 7.4.2 Feedback Equilibrium

In this subsection, we follow a similar but unique solution procedure seen in subsection 7.3.1.2. We remind the reader that the specific linear quadratic optimal control problem of interest can be seen in equations 7.85 and 7.86. The feedback equilibrium of our model differs from open-loop equilibrium in that in the feedback
Figure 7.25. Marginal cost and price of electricity contract up to \( t = 10 \)

Figure 7.26. Electricity generation up to \( t = 10 \)
case, firms do not have to commit their controls from the start of the game. Firms have the benefit of observing the state of the system continuously while determining their optimal control path. Again, the feedback equilibrium information structure is potentially more realistic when compared to open-loop equilibrium.

According to Theorem 9 in Chapter 6, the linear quadratic optimal control problem has a feedback Nash equilibrium if and only if equations 6.117 and 6.118 have a set of stabilizing solutions. With

$$ S_f := \begin{bmatrix} 2s^2 & 0 \\ 0 & 0 \end{bmatrix}, \quad f = 1, 2; \quad S_{12} = S_{21} = 0 \quad (7.120) $$

and

$$ K_1 := \begin{bmatrix} l_1 & l_2 \\ l_2 & l_3 \end{bmatrix} \quad (7.121) $$

$$ K_2 := \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} \quad (7.122) $$

Substituting $K_1$ and $K_2$ into equations 6.117 and 6.118 results in the following six equations.
\[-2l_1 \left( -\frac{1}{2} \rho - 3s - 2s^2 k_1 \right) + 2s^2 l_1^2 + \frac{1}{2} = 0 \]  \hspace{1cm} (7.123)

\[2s^2 k_1^2 - 2k_1 \left( -\frac{1}{2} \rho - 3s - 2s^2 l_1 \right) + \frac{1}{2} = 0 \]  \hspace{1cm} (7.124)

\[(\rho + 3s + 2s^2 k_1 + 2k^2 l_1)l_2 + 2s^2 l_1 k_2 = \frac{1}{2} g + (a + 2g)sl_1 \]  \hspace{1cm} (7.125)

\[2s^2 k_1 l_2 + (\rho + 3s + 2s^2 k_1 + 2s^2 l_1)k_2 = \frac{1}{2} g + (a + 2g)sk_1 \]  \hspace{1cm} (7.126)

\[\frac{1}{\rho} (2l_2 ((a + 2g)s - 2s^2 k_2) - 2s^2 l_2^2) - l_3 = 0 \]  \hspace{1cm} (7.127)

\[\frac{1}{\rho} (2k_2 ((a + 2g)s - 2s^2 l_2) - 2s^2 k_2^2) - k_3 = 0 \]  \hspace{1cm} (7.128)

The stability requirement, that matrix $A - S_1 K_1 - S_2 K_2$ must be stable, reduces to

\[\frac{1}{2} \rho + 3s + 2s^2 (l_1 + k_1) > 0 \]  \hspace{1cm} (7.129)

We can solve the system of equations (7.123)- (7.128) by cascading manner. We first solve for $(k_1, l_1)$ from equations 7.123 and 7.124. The terms $k_2$ and $l_2$ are then determined by the set of linear equations (7.125) and (7.126). Finally, equations (7.127) and (7.128) then result in the solution of $l_3$ and $k_3$.

The subtraction of equation (7.124) from equation (7.123) yields

\[(l_1 - k_1)(\rho + 6s + 2s^2 (l_1 + k_1)) = 0 \]  \hspace{1cm} (7.130)

If the term $\rho + 6s + 2s^2 (l_1 + k_1) = 0$ in equation (7.130), we have a contradiction
with the stability requirement (7.129). Therefore, we can conclude that \( l_1 = k_1 \). As a result, the equations (7.125) and (7.126) become symmetric in \((l_2, k_2)\). It is also verified that \( l_2 = k_2 \). Finally, we can deduce upon inspection of equations (7.127) and (7.128) that \( l_3 = k_3 \). Note that this statement is equivalent to \( K_1 = K_2 \).

The equations (7.123) and (7.125) can be rewritten as

\[
6s^2k_1^2 + (\rho + 6s)k_1 + \frac{1}{2} = 0, 
\] (7.131)

and

\[
k_2 = \frac{\frac{1}{2}g + (a + 2g)sk_1}{\rho + 3s + 6s^2k_1}, 
\] (7.132)

respectively.

From the stability condition (7.129) it next follows that the appropriate solution \( k_1 \) in equation (7.131) above is

\[
k_1 = \frac{-(6s + \rho) + \sqrt{(\rho + 6s)^2 - 12s^2}}{12s^2} 
\] (7.133)

The equilibrium actions now follow from reformulating into our original model variables. Specifically,

\[
u_f(t) = (2sk_1 + 1)\pi(t) + 2sk_2 - g, \quad f = 1, 2. 
\] (7.134)
with \( k_1 \) given by equation (7.133) and \( k_2 \) by equation (7.132). The resulting dynamics of the equilibrium state path \( \pi(t) \) are

\[
\dot{\pi}(t) = (-4s^2k_1 - 3s)\pi(t) + s(a - 4sk_2 + 2g) \tag{7.135}
\]

Restating \( \pi(t) \) in non differential form yields

\[
\pi(t) = \alpha e^{(-4s^2k_1 - 3s)t} + \frac{a - 4sk_2 + 2g}{4sk_1 + 3} \tag{7.136}
\]

where \( \alpha = \pi_0 - \frac{a - 4sk_2 + 2g}{4sk_1 + 3} \)

In a similar fashion to the open-loop equilibrium, we calculate the feedback equilibrium price \( \pi(t) \) converges to the stationary value

\[
\bar{\pi} := \frac{a - 4sk_2 + 2g}{4sk_1 + 3} \tag{7.137}
\]

Let us assume the following specific model parameters to allow computation of the game. They are purposely chosen to match the open-loop equilibrium case to facilitate comparison.
Figure 7.27 plots the equilibrium price of electricity up to the 10th year for both the open-loop and feedback equilibrium models. The prices converged to $75.54 and $83.03 for the feedback and open-loop case respectively.

\begin{align*}
b &= 1 & (7.138) \\
k &= .2 & (7.139) \\
\rho &= .07 & (7.140) \\
a &= 250 & (7.141) \\
\pi_0 &= 50 & (7.142) \\
g &= 3 & (7.143)
\end{align*}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7.27.png}
\caption{Price of electricity up to \( t = 10 \)}
\end{figure}
Figure 7.28. Marginal cost and price of electricity up to $t = 10$

Figure 7.28 shows the marginal cost of electricity generation and the price of electricity on the same scale for both open-loop and feedback equilibrium. The marginal cost of the feedback is lower than the open-loop.

Figure 7.29 plots the optimal generation path of the generators for the open-loop and feedback equilibrium models. The feedback case consistently has consistently greater generation than the open-loop case.
The feedback equilibrium has higher generation and lower prices when compared to the open-loop equilibrium. Similar to the excess price dynamics model, the feedback information structure allows generation firms to observe the price of electricity before committing to generation. We postulate that the knowledge that other firms can react to rising and following prices forces each generator to behave more competitively since they know that their competitors can quickly react to changing electricity prices. Therefore, each firm is less willing to adopt a strategy that makes them susceptible of less profits in the future.
Conclusions

In this research, we have presented continuous and discrete-time formulations of a unique dynamic Stackelberg game of an electric power oligopoly. Our single leader is represented by the market monitor using equilibrium congestion pricing to increase social welfare. Our model may be used as a decision support tool for the existing operations the market monitor conducts. The followers are generators that play a Cournot-Nash game with other generators to sell and transmit power over an electric network assuming an oligopoly market structure. The congestion pricing determined by the market monitor solves the Cournot-Nash equilibrium problem in the lower level. We described the evolution of ramping rates over time for each generator via a differential equation as well employed a multi-period time horizon.

We advance the state of the dynamic and differential game-theoretic models of electricity markets and networks. The dynamic Stackelberg model considers a new market design mechanism that also includes the following numerous realistic and computable features: oligopolistic competition, inter-temporal constraints, dynamic production constraints, time-varying demand, transmission constrained
network and multi-generator assets. We were able to compute the dynamic Stackelberg game by first reformulating it as a Mathematical Program with Complementarity Constraints (MPCC) and utilizing the commercially available NLPEC solver within GAMS. We solved a 15-node Western European Electric Network in approximately eight minutes. Our numerical experimentation concluded that equilibrium congestion pricing can increase social welfare.

We utilized optimal feedback control to solve several models of differential games applied to oligopolistic electricity markets. Optimal feedback control allows differential Nash games to be solved in closed-loop as opposed to the ubiquitous open-loop model. Closed-loop allows players of the game to react to the state of the system at every point versus simultaneously decide all of their actions for the planning-horizon. Computational tractability and a narrow class of models suitable to feedback control limit its prevalence in electricity market research. We advance the literature by bridging the gap between the optimal feedback control and electric power communities through presentation of several computable models relevant to electricity markets. We presented two models of oligopolistic competition solved using optimal feedback control. The models of competition employ excess demand and sticky price dynamics, respectively. A large scale numerical algorithm was implemented to provide optimal feedback control of a 28 market, 96 player differential Nash game.

Future research primarily lies in extending the dynamic Stackelberg game presented in Chapter 5 for a multi-attribute objective function and explore the impacts of renewable energy and micro-grids. One additional area of future research is to solve the feedback equilibria of the finite planning-horizon problem. This may be achieved by modifying the algorithm presented in Appendix A to include the differential Riccati equation. To the best of our knowledge, this is research presents
the first large scale numerical results for feedback equilibrium of a differential electricity market game.
Appendix A

System of Coupled Algebraic Riccati Equations Algorithm

The iterative algorithm presented solves the equilibria of the infinite horizon linear quadratic optimal control problem associated with a system of coupled algebraic Riccati equations. It is loosely based on the two player algorithm presented in Li and Gajic (1995).

**Step 1** : Determine the stabilizing solution $K_0^1$ of

$$A^TK_0^1 + K_0^1 A + Q_1 - K_0^1 S_1 K_0^1 = 0 \quad (A.1)$$

Next determine the stabilizing solution $K_0^2$ of the Riccati equation

$$(A - S_1 K_0^1)^T K_0^2 + K_0^2 (A - S_1 K_0^1) + Q_2 + K_0^1 S_1 K_0^1 - K_0^2 S_2 K_0^2 = 0. \quad (A.2)$$

Repeat until an initial solution is obtained for all $K_j$, where $j = 1, 2...N$
Step 2: Let $i := 0$. Repeat the next iterations until the matrices $K_j^i$, $j = 1, 2...N$ below have converged. Here $A_i^i := A - \sum_{l \neq j}^N S_l K_l$, and $K_j^{i+1}$, $j = 1, 2$, are the solutions of the

$$A_i^i K_1^{i+1} + K_1^{i+1} A_i^i = K_1^i S_1 K_1^i - Q_1 - \sum_{l \neq 1}^N K_l S_{1l} K_l \quad (A.3)$$

$$A_i^i K_2^{i+1} + K_2^{i+1} A_i^i = K_2^i S_2 K_2^i - Q_2 - \sum_{l \neq 2}^N K_l S_{2l} K_l \quad (A.4)$$

respectively, for $i = 0, 1, ...$

Step 3: End of the algorithm.

In the above algorithm, the hope is that the matrices $A_i^i$ will be stable for all $i$. In case the stability condition is violated at some iteration, this might indicate that the iteration at hand will have no appropriate solution.
Bibliography


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