A PLANE WAVE SUPERPOSITION METHOD:
MODELING ACOUSTIC FIELDS INSIDE CAVITIES

A Thesis in
Acoustics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Master of Science

May 2014
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ABSTRACT

The Plane Wave Superposition Method (PWSM) presented here simply and efficiently calculates the pressure and velocity fields inside an arbitrarily shaped cavity with mixed boundary conditions and internal acoustic sources. In this boundary value problem, a superposition of N plane wave represents the acoustic pressure field, and the boundary conditions at N locations approximates the continuous boundary conditions. The solution’s sensitivity and accuracy is evaluated using the condition number and the maximum error in the boundary conditions.

The PWSM’s theory is illustrated using the following six examples: a one-dimensional tube, a circle, a square, a trapezoid, a sphere, and a cube where the pressure is specified on the boundaries. The tube example demonstrates that the PWSM and an analytic method give the same solution in one-dimensional problems. In all of the two and three dimensional examples, the continuous boundary conditions are well approximated with a finite number of points over a wide range of frequencies. The examples also illustrate that the condition number is proportional to the non-dimensional wavenumber and the number of points per wavelength used to approximate the boundary conditions. Further, the approximation’s accuracy is inversely proportional to the number of points per wavelength that are used to approximate the boundary conditions.
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ACKNOWLEDGEMENTS

First, I thank the U.S. Army Engineer Research and Development Center Construction Engineering Research Laboratory (ERDC-CERL), Champaign, IL for providing the financial support that allowed this research to occur. In addition, thank you to all of my family and friends that provided encouragement throughout this process.

Thank you to my thesis committee for providing helpful feedback on my thesis. Thank you to Mike Grissom who has overseen much of the work that I have completed at KCF Technologies in the past two years and has substantially contributed to my professional development. Lastly, I thank my advisor Gary Koopmann for providing me with many useful suggestions and wonderful insights.
Chapter 1

Introduction

Modeling acoustic variables such as pressure and velocity in enclosed spaces is a fundamental task in acoustics and has many applications. For example, an acoustic model of a car’s cabin reveals how the cabin’s geometry, wall materials, and speaker placement changes the sound in the cabin. This information can be used to optimize the space for its intended purpose. For instance, if a car manufacturer wants the sound system to provide uniform sound quality throughout the cabin, then the car manufacturer can adjust the speaker placement, the cabin’s geometry and layout, and the materials of the seats, dashboard, and walls to achieve the desired effect. Determining the change in the sound field that results from these alterations can be much more cost effective and practical in a model rather than making them in a real car.

Since modeling acoustic fields inside cavities is fundamentally and practically important, there are many different modeling methods available. For example, analytic methods (Kinsler et al., 2000; Koopman & Fahnline, 1997; Kuttruff, 2007), the finite element method (FEM; Astley, 2007; Fahy & Gardonio, 2007), the boundary element method (BEM; Herrin et al., 2007; Seybert & Cheng, 1987), and the Equivalent Source Method (Johnson et al., 1998) are all used to accomplish this task. Compared with the other methods, the analytic methods are relatively simple and computationally efficient, but they can only be used when the geometry fits the chosen coordinate system well (e.g. a rectangle in Cartesian coordinates or a circle in polar coordinates). The FEM is a powerful technique that can handle complex cases with nonlinearities and nonhomogeneous media (Cook et al., 2002). However, the FEM is a complicated procedure that requires expensive, specialized software, and extensive training to use. In addition, since the
entire cavity must be meshed and many integrals must be evaluated to calculate the coefficients in the stiffness and mass matrices (Cook et al., 2002), the FEM introduces unnecessary complexity and computational expense to linear problems with a homogenous medium. However, in many cases the computational expense of computing the integrals can be greatly reduced by analytically computing them beforehand. The BEM introduces unnecessary complexity because it has singularities when the pressure and velocity are evaluated on the boundary (Koopmann et al., 1989). The Equivalent Source Method can be simplified by summing plane waves instead of monopoles, which eliminates determining the sources’ locations.

The Plane Wave Superposition Method (PWSM) applies the simple procedure for analytically solving boundary value problems to arbitrarily shaped cavities. A boundary value problem has the following two steps: (1) find a solution to the governing differential equation and (2) enforce the boundary conditions. The PWSM uses a sum of plane waves as its solution to the wave equation and enforces the boundary conditions at a finite number of points. Enforcing the boundary conditions at a finite number of points instead of continuously enables the procedure to be applied to arbitrarily shaped cavities.

The PWSM has many advantages compared to the other methods. First, the PWSM is simple because it requires neither specialized software nor extensive training. Moreover, the PWSM is transparent and easy to check because every equation is physically meaningful. In addition, the PWSM yields continuous expressions for the pressure and velocity inside the cavity, which is equivalent to an analytic method in one-dimensional cases. Second, the PWSM is versatile because the cavity may have any geometry, any of the standard acoustic boundary conditions may be specified, internal acoustic sources may be included, and the accuracy of the solution may be improved by increasing the number of specified points. Third, the PWSM is computationally efficient because the PWSM produces a smaller system of equations, does not introduce any artificial singularities, and does not require any integration.
This thesis introduces the PWSM by explaining the theory that supports the PWSM. First, the PWSM is derived in one dimension; then, it is derived in two and three dimensions. The last theory section explains how internal acoustic sources can be incorporated into the PWSM. In addition, this thesis begins an evaluation of the PWSM by examining the condition number and the maximum error in the boundary conditions in several cases as a function of the number of points per wavelength and the normalized wavenumber. In the cases that are considered, the pressure and velocity fields can be modeled with both an acceptable level of accuracy and a relatively small number of specified points.
Chapter 2

Theory

This chapter elucidates the theory that supports the Plane Wave Superposition Method (PWSM). First, the PWSM is derived in one-dimension to emphasize that the PWSM and analytic solutions are identical in the one-dimensional case. Next, the PWSM is extended to two or more dimensions to demonstrate how to approximate the boundary conditions. The last section describes the PWSM with internal acoustic sources. The examples in this chapter give the theory a concrete context and are used throughout this thesis.

2.1 The PWSM in One-Dimension

Consider a one-dimensional problem with a homogeneous fluid that has a sound speed \( c \) and density \( \rho \). The interior pressure satisfies the one-dimensional wave equation

\[
\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0
\]

It is convenient to work in the frequency domain, so the pressure is \( p(x,t) = \hat{p}(x)e^{j\omega t} \) and the velocity is \( v(x,t) = \hat{v}(x)e^{j\omega t} \) where \( \omega \) is the angular frequency. Substitution of this pressure expression into the wave equation yields the one-dimensional Helmholtz equation
\[
\frac{d^2\hat{p}}{dx^2} + k^2 \hat{p} = 0
\]

where \(k\) is the wavenumber and \(k^2 = (\omega/c)^2\). A plane wave, \(\hat{p}(x) = Ae^{-jkx}\), is a solution to the Helmholtz equation. Since the Helmholtz equation is linear and a plane wave is a solution to the Helmholtz equation, then a sum of two plane waves

\[
\hat{p}(x) = A_1e^{-jkx} + A_2e^{jkx}
\]

(2.1)
is also a solution to the Helmholtz equation. The first plane wave is traveling in the positive \(x\)-direction and the second plane wave is traveling in the negative \(x\)-direction.

Two plane waves are summed because there are two boundary conditions, i.e. one boundary condition at each end. In addition, this pressure expression is a general solution to the one-dimensional Helmholtz equation when \(\omega \neq 0\) \(\text{rad/s}\) because the Helmholtz equation is a second order differential equation and the pressure expression contains two independent solutions to the Helmholtz equation. Thus, in one-dimensional problems the PWSM and analytic solutions are equivalent.

Next, the plane waves' amplitudes are found by enforcing the boundary conditions at both ends. The PWSM accepts pressure, velocity, and specific impedance boundary conditions. To apply the velocity and specific impedance boundary conditions, first the velocity must be found. Substituting the pressure expression into the one-dimensional, linear Euler equation,

\[
\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial x}
\]

produces the velocity

\[
\hat{v}(x) = \frac{1}{\rho c} (A_1e^{-jkx} - A_2e^{jkx})
\]

(2.2)
The specific impedance is the ratio of pressure to velocity, i.e. \( z = \frac{p}{v} \), but it is more useful to rearrange this equation to give \( p - zv = 0 \).

Therefore, if a pressure, velocity, or specific impedance is specified at \( x_s \), then the corresponding equation below is used:

\[
\hat{p}(x_s) = A_1 e^{-jkx_s} + A_2 e^{jkx_s} \\
\rho c \hat{v}(x_s) = A_1 e^{-jkx_s} - A_2 e^{jkx_s}
\]

\[0 = A_1 \left(1 - \frac{z(x_s)}{\rho c}\right) e^{-jkx_s} + A_2 \left(1 + \frac{z(x_s)}{\rho c}\right) e^{jkx_s}\]

Applying the boundary conditions creates a system of two linear equations with two unknowns plane waves’ amplitudes (\( A_1 \) and \( A_2 \)), which yields the plane waves’ amplitudes. Substituting the plane waves’ amplitudes into equations (2.1) and (2.2) yields the pressure and velocity inside the one-dimensional cavity.

**Example 2.1.1 Two Specified Pressures**

Consider a hollow, cylindrical tube with a length \( L = 1 \) m and a diameter that is much less than a wavelength, and let \( c = 343 \) m/s, \( \rho = 1.21 \) kg/m\(^3\), \( \hat{p}(0) = 0 \) Pa, and \( \hat{p}(L) = 1 \) Pa. Use \( f = 100 \) Hz, an off resonance frequency, and use \( f = 171.5 \) Hz, the fundamental resonance frequency. Compare the pressure at each end the prescribed pressures and plot the real part of the pressure.

**Solution:** Applying equation (2.2) at the boundaries generates

\[0 = A_1 + A_2\]

\[1 = A_1 e^{-jkL} + A_2 e^{jkL}\]
which when rewritten as a matrix equation,

\[
\begin{bmatrix}
    1 & 1 \\
    e^{-jkL} & e^{jkL}
\end{bmatrix}
\begin{bmatrix}
    A_1 \\
    A_2
\end{bmatrix} = \begin{bmatrix} 0 \\
    1 \end{bmatrix}
\]

is solved for the plane wave amplitudes,

\[
\begin{bmatrix}
    A_1 \\
    A_2
\end{bmatrix} = \begin{bmatrix}
    1 & 1 \\
    e^{-jkL} & e^{jkL}
\end{bmatrix}^{-1} \begin{bmatrix} 0 \\
    1 \end{bmatrix}
\]

If \( f = 100 \text{ Hz} \), then \( k = 2\pi f / c = 1.83 \text{ m}^{-1} \) and the plane waves’ amplitudes are \( A_1 = 0.518j \), and \( A_2 = -0.518j \). Substituting the plane waves’ amplitudes into equation (2.1) and evaluating at the ends produces \( \hat{p}(0) = 0 \text{ Pa} \), and \( \hat{p}(L) = 1 \text{ Pa} \), which means that the boundary conditions have been satisfied to within MATLAB’s working precision. A plot of the pressure is below.

![Figure 2-1. Pressure versus position where \( \hat{p}(0) = 0 \text{ Pa}, \hat{p}(L) = 1 \text{ Pa}, \text{ and } f = 100 \text{ Hz} \).](image)

If \( f = 171.5 \text{ Hz} \), then \( k = 3.14 \text{ m}^{-1} \) and the plane wave amplitudes are \( A_1 = 0.5 - \)
$1.55 \times 10^{15} j$, and $A_2 = -0.5 + 1.55 \times 10^{15} j$. Dissimilar to the last case, the imaginary parts of the plane waves’ amplitudes are very large because the system is at resonance, so the amplitudes are diverging. Substituting the plane waves’ amplitudes into equation (2.1) and evaluating at the ends produces $\hat{p}(0) = -5.55 \times 10^{-17} \text{ Pa}$, and $\hat{p}(L) = 1 \text{ Pa}$.

A plot of the pressure is below. Notice that the scale of the y-axis is very large; the values are multiplied by $10^{15}$.

![Figure 2-2. Pressure versus position where $\hat{p}(0) = 0 \text{ Pa}$, $\hat{p}(L) = 1 \text{ Pa}$, and $f = 171.5 \text{ Hz}$.

In both cases, the boundary conditions are satisfied to the working precision of MATLAB. In addition, the discretization error is on the order of the working precision of MATLAB because in the one-dimensional case, the PWSM and analytic solutions are equivalent. In contrast to FEM, the computational effort remains the same for any frequency or tube length.
Again, the $y$-scale is very different between these two plots: $1 \text{ Pa}$ versus $3 \times 10^{15}$. If the plane waves’ amplitudes were calculated with infinite precision, then the resonance condition would have an infinite amplitude. However, MATLAB uses finite precision, which gives the finite amplitude that is shown. This problem is discussed more in chapter 3.

### 2.2 The PWSM in Two or More Dimensions

Next, the PWSM is extended to more than one dimension. As in the one-dimensional case, a plane wave, $\hat{\rho}(x) = Ae^{-jkx}$, is a solution to the Helmholtz equation. Now, $x$ and $k$ are vectors with the appropriate number of elements, and the exponential contains a dot product of the two vectors. Physically, $\vec{k} = k\vec{k}$ where $k = \omega/c$ and $\vec{k}$ is a unit vector that points in the direction that the plane wave is traveling. $k$ is the wavenumber, which is the same for all of the plane waves, and $\vec{k}$ will be called the plane waves’ orientations, which is different for every plane wave. Again, since the Helmholtz equation is linear and one plane wave is a solution to the Helmholtz equation, then a sum of $N$ plane waves

$$ \hat{\rho}(x) = \sum_{m=1}^{N} A_m e^{-jk_m x} \quad (2.6) $$

is a solution to the Helmholtz equation.

In the one-dimensional case, $N = 2$ because there were only two points on the boundary; however, in the multi-dimensional case there are infinitely many points on the boundary. Instead of continuously enforcing the boundary conditions at every point, $N$
sample points are used to approximate the boundary conditions. Expressing the pressure as a sum of N plane waves produces a square coefficient matrix. Another option is to let the number of specified points on the boundary be greater than the number of plane waves. Then, the method of least-squares yields the plane waves’ amplitudes. However, since the number of plane waves is arbitrary, this approach has not been found to be useful in the PWSM.

The plane waves’ amplitudes, $A_m$, are found by enforcing the boundary conditions at a finite number of points. Again, the plane wave superposition method accepts pressure, normal velocity, and specific impedance boundary conditions, so an expression for the normal velocity is needed. Figure 2-3 illustrates the setup.

![Figure 2-3](image_url)

Figure 2-3. Schematic showing the general setup of a two-dimensional problem. The pressure, normal velocity, or the specific impedance is specified at each of the dots.

Substitution of equation (2.6) into the Euler equation yields the following expression for
the velocity

\[ \hat{v}(x) = \frac{1}{\rho c} \sum_{m=1}^{N} A_m \vec{k}_m e^{-j k_m \cdot x} \]  

(2.7)

Thus, the normal velocity is \( \hat{v}_n(x_s) = \hat{v}(x_s) \cdot \vec{n}_s \) where \( \vec{n}_s \) is a unit vector that is normal to the boundary at \( x_s \). The specific impedance is \( z(x) = \hat{p}(x)/\hat{v}_n(x) \), which can be rearranged to give \( 0 = p(x) - z(x) v_n(x) \).

Therefore, if the pressure, normal velocity, or specific impedance is known at \( x_s \), the higher dimensional versions of equations (2.3), (2.4), and (2.5) are

\[ \hat{p}(x_s) = \sum_{m=1}^{N} A_m e^{-j k_m \cdot x_s} \]  

(2.8)

\[ \rho c \hat{v}_n(x_s) = \vec{n}_s \cdot \sum_{m=1}^{N} A_m \vec{k}_m e^{-j k_m \cdot x_s} \]  

(2.9)

\[ 0 = \sum_{m=1}^{N} A_m \left[ 1 - \frac{z(x_s)}{\rho c} \vec{k}_m \cdot \vec{n}_s \right] e^{-j k_m \cdot x_s} \]  

(2.10)

If the pressure, normal velocity, or specific impedance is known at \( N \) locations, then these three equations can be used to create a system of \( N \) equations with \( N \) unknown plane waves’ amplitudes.

However, the system of equations cannot be solved because the \( \vec{k}_m \)'s are not specified. In the one-dimensional case, plane waves can only travel in two directions: to the left and to the right. In the multi-dimensional case, plane waves can travel in infinitely many different directions. Nevertheless, since the system is underdetermined, the \( \vec{k}_m \)'s can be chosen. Some choices are better than others, but optimizing the plane waves’ orientations is beyond the scope of this thesis. For this thesis, let the plane
waves' orientations be the specified position vectors that are normalized to length one. For example, let the specified points be $x_s = \{1,1; -1,1; -1,-1; 1,-1\}$. Since $|x_s| = \{\sqrt{2}; \sqrt{2}; \sqrt{2}; \sqrt{2}\}$, then $\vec{k}_m = x_s / \sqrt{2}$.

Since the $\vec{k}_m$s are now specified, equations (2.8-2.10) and the boundary conditions together produce a system of N linear equation with N unknowns, which can be solved for the plane waves' amplitudes. Then, inserting the plane waves' amplitudes into equations (2.6) and (2.7) generates the pressure and velocity everywhere inside the cavity.

**Example 2.2.1 Circle with Pressure Specified on the Boundary**

Consider a hollow, cylinder with a radius $a = 1$ m and a height that is much less than a wavelength, and let $c = 343$ m/s, $\rho = 1.21$ kg/m$^3$, $|\hat{p}(0,\theta)| < \infty$, $\hat{p}(a,\theta) = 1$ Pa, and $f = 100$ Hz, an off resonance frequency. Compute the maximum relative error at the specified points and plot the real part of the pressure.

**Solution:** This problem is a two-dimensional problem. For a first approximation, the boundary conditions are approximated by five points $(x_s = \{1,0; 0.31,0.95; -0.81,0.59; -0.81,-0.59; 0.31,-0.95\})$, which are shown in Figure 2-4 as black dots. Next, let the plane waves' orientations be $\vec{k}_m = x_s$, which are shown in Figure 2-4 as blue arrows. The position vectors do not need to be normalized because all of the vectors already have a length of one. Applying equation (2.8) at the five locations produces five equations with five unknown plane waves' amplitudes. Solving for the plane waves' amplitudes, substituting them into equation (2.6), and
evaluating the resulting equation at $x_s$ shows that the largest relative error at $x_s$ is $5.0 \times 10^{-16}$. Figure 2-4 illustrates that the pressure field inside of a circle can be approximated using plane waves, which means that curved shapes can be modeled with the PWSM.

Figure 2-4. Pressure versus position for a circle where $\hat{p}(a, \theta) = 1$ Pa and $f = 100$ Hz. The black dots give the specified pressure locations, and the blue arrows give the plane waves’ orientations.

The pressure field in Figure 2-4 is very good for only using five points. Since finite elements commonly uses linear or quadratic variation between the points, the finite element method would not produce a pressure field with this level of accuracy with only five points. In addition to satisfying the pressure condition at the five points, the plot indicates that the approximation of the boundary conditions is doing well on the entire boundary. This observation is discussed more in chapter 4.
Example 2.2.2 Square with Pressure Specified on the Boundary

Consider a hollow, rectangular prism with side lengths $L = 1$ m and $W = 1$ m and a height that is much less than a wavelength, and let $c = 343$ m/s, $\rho = 1.21$ kg/m$^3$, $\hat{p}(x,0) = \hat{p}(L,y) = \hat{p}(x,W) = \hat{p}(0,y) = 1$ Pa, and $f = 100$ Hz, an off resonance frequency. Compute the maximum relative error at the specified points and plot the real part of the pressure.

Solution: This problem is a two-dimensional problem. For a first approximation, the boundary conditions are approximated by eight points: the four corners and the midpoint on each side, which are shown below in Figure 2-5 by the black dots. Next, let the plane waves' orientations be the normalized position vectors. In this case, the corner position vectors need to be multiplied by the square root of two and the side position vectors need to be multiplied by two. The plane waves' orientations are shown as blue arrows in Figure 2-5 below. However, the plane waves' orientations have been shorten to make the plot neater. Applying equation (2.8) at the eight locations produces eight equations with eight unknown plane waves' amplitudes. Solving for the plane waves' amplitudes, substituting them into equation (2.6), and evaluating the resulting equation at $x_s$ shows
that the largest relative error at $x_s$ is $2.0 \times 10^{-15}$. Below is a plot of the pressure.

Figure 2-5. Pressure versus position for a square where $\hat{p}(x, 0) = \hat{p}(L, y) = \hat{p}(x, W) = \hat{p}(0, y) = 1$ Pa and $f = 100$ Hz. The black dots give the specified pressure locations, and the blue arrows give the plane waves’ orientations.

Figure 2-5 illustrates that the pressure field inside of a square can be approximated using plane waves, which indicates that the PWSM can model corners. Again, the pressure field looks very good for only using eight points because the entire boundary appears to be converged.

**Example 2.2.3 Trapezoid with Pressure Specified on the Boundary**

Consider the trapezoid that is shown in Figure 2-6 where the right side is $3/4^{th}$ the length of the left side, and let $c = 343$ m/s, $\rho = 1.21$ kg/m$^3$, $\hat{p} = 1$ Pa on the boundary, and $f = 100$ Hz, an off resonance frequency. Compute the maximum relative error at the specified points and plot the real part of the pressure.

**Solution:** For a first approximation, the boundary conditions are approximated by eleven
points: the four corners and two points on each side except for the shortest side, which only has one point. The points’ locations are shown as black dots in Figure 2-6. Next, let the plane waves’ orientations be given by the normalized position vectors, which are shown as blue arrows in Figure 2-6 below. To make the figure neater, the length of the unit vector orientations have been shorten. Applying equation (2.8) at the eleven locations produces eleven equations with eleven unknown plane waves’ amplitudes. Solving for the plane waves’ amplitudes, substituting them into equation (2.6), and evaluating the resulting equation at $x_s$ shows that the largest relative error at $x_s$ is $3.4 \times 10^{-15}$. Below is a plot of the pressure.

![Figure 2-6. Pressure versus position for a trapezoid where $\rho = 1$ Pa on the boundary and $f = 100$ Hz. The black dots give the specified pressure locations, and the blue arrows give the plane waves’ orientations.](image)

Figure 2-6 illustrates that the pressure field inside of a trapezoid can be approximated using plane waves, which demonstrates that the PWSM can model non-separable geometries. Non-separable geometries cannot be modeled with analytic methods. Again,
the pressure field looks very good for only using eleven points because the entire boundary appears to be converged.

**Example 2.2.4 Sphere with Pressure Specified on the Boundary**

Consider a hollow sphere with a radius $a = 1$ m, and let $c = 343$ m/s, $\rho = 1.21$ kg/m$^3$, $\hat{p}(a) = 1$ Pa, and $f = 100$ Hz, an off resonance frequency. Compute the maximum relative error at the specified points and plot the real part of the pressure.

**Solution:** This problem is a three-dimensional problem. For a first approximation, the boundary conditions are approximated by twelve points that are evenly distributed on the sphere. Their locations were found using the Coulomb force method (Peake et al., 2014) and are shown in Figure 2-7 as black dots. Physically, the Coulomb force method treats each of the points as a positive charge on a metal sphere and marches forward in time until the solution converges.

Next, let the plane waves’ orientations be the position vectors of the specified points. In this case, $\vec{k}_m = \vec{x}_s$ because $a = 1$ m. Applying equation (2.8) at the twelve locations produces twelve equations with twelve unknown plane waves’ amplitudes. Solving for the plane waves’ amplitudes, substituting them into equation (2.6), and evaluating the resulting equation at $\vec{x}_s$ shows that the largest relative error at $\vec{x}_s$ is
6.9 \times 10^{-16}. Below is a plot of the pressure.

![Figure 2-7. Pressure versus position for a sphere where $\hat{p}(\alpha) = 1$ Pa and $f = 100$ Hz. The black dots give the specified pressure locations.](image)

Figure 2-7 illustrates that the pressure field inside of a sphere can be approximated using plane waves, which demonstrates that the PWSM can be used in three dimensions and can model curved shapes. Again, the pressure field looks very good for only using twelve points because the entire boundary appears to be converged.

**Example 2.2.5 Cube with Pressure Specified on the Boundary**

Consider a hollow, cube with side lengths $W = 1$ m, and let $c = 343$ m/s, $\rho = 1.21$ kg/m$^3$, $\hat{p} = 1$ Pa on the boundary, and $f = 100$ Hz, an off resonance frequency. Compute the maximum relative error at the specified points and plot the real part of the pressure.
Solution: This problem is a three-dimensional problem. For a first approximation, the boundary conditions are approximated by twenty-six points: the eight corner points, twelve mid-side points, and six mid-face points, which are shown as black dots in Figure 2-8. Next, let the plane waves’ orientations be the normalized position vectors of the specified pressures. Applying equation (2.8) at the twenty-six locations produces twenty-six equations with twenty-six unknown plane waves’ amplitudes. Solving for the plane waves’ amplitudes, substituting them into equation (2.6), and evaluating the resulting equation at \( x_s \) shows that the largest relative error at \( x_s \) is \( 3.3 \times 10^{-14} \). Figure 2-8 illustrates that the pressure field inside of a cube can be approximated using plane waves, which demonstrates that the PWSM can be used in three dimensions and can model corners. Again, the pressure field looks very good for only using twenty-six points because the entire boundary appears to be converged.

Figure 2-8. Pressure versus position for a cube where \( \bar{p} = 1 \text{ Pa} \) on the boundary and \( f = 100 \text{ Hz} \). The black dots give the specified pressure locations.
2.3 The PWSM with an Internal Source

Using the plane wave superposition method, an acoustic source (e.g., a monopole or dipole), can be added inside the cavity with little change to the overall procedure. Since the plane wave method assumes linear acoustics, then superposition holds and the pressure inside the cavity is the sum of the pressure from the source in a free-field ($\hat{p}_{src}$), which is the particular solution of the inhomogeneous wave equation for the given source, and from the plane waves,

$$\hat{p}(x) = \hat{p}_{src}(x) + \sum_{m=1}^{N} A_m e^{-j k_m \cdot x}$$  \hspace{1cm} (2.11)

The source will also have some velocity contribution ($\hat{v}_{src}$), which can be summed with the velocity from the plane waves to give the total velocity

$$\hat{v}(x) = \hat{v}_{src}(x) + \frac{1}{\rho c} \sum_{m=1}^{N} A_m \vec{k}_m e^{-j k_m \cdot x}$$  \hspace{1cm} (2.12)

The above pressure and velocity expressions together produce the more general versions of equations (2.8-2.10) that can include an internal source term:

$$\hat{p}(x_s) - \hat{p}_{src}(x_s) = \sum_{m=1}^{N} A_m e^{-j k_m \cdot x_s}$$  \hspace{1cm} (2.13)

$$\rho c[\hat{v}_n(x_s) - \hat{v}_{src,n}(x_s)] = \vec{n}_s \cdot \sum_{m=1}^{N} A_m \vec{k}_m e^{-j k_m \cdot x_s}$$  \hspace{1cm} (2.14)

$$z(x_s) \hat{v}_{src,n}(x_s) - \hat{p}_{src}(x_s) = \sum_{m=1}^{N} A_m \left[ 1 - \frac{z(x_s)}{\rho c} \frac{1}{\vec{k}_m \cdot \vec{n}_s} \right] e^{-j k_m \cdot x_s}$$  \hspace{1cm} (2.15)
where $\hat{v}_{src,n}(x_s) = \hat{v}_{src}(x_s) \cdot \mathbf{n}_s$. Evaluating equations (2.13 – 2.15) at the known boundary conditions produces a system of $N$ linear equations with $N$ unknown plane waves’ amplitudes, which can be solved for the plane waves’ amplitudes. Substituting the plane waves’ amplitudes into equations (2.11) and (2.12) generates the pressure and velocity inside the cavity.

Since all of the source terms appear on the left side of the equations, then the matrix on the right side only needs to be inverted once for any source term, which is beneficial because inverting the matrix is the most computationally expensive part of the PWSM.
Chapter 3

Numerical Error

Chapters 1 and 2 introduced and explained the PWSM’s theory, and chapter 3 and 4 discuss some of the potential errors when using the PWSM. Specifically, chapters 3 evaluates sources of error that are inherent to solving a system of equations. The PWSM solves a system of N linear equations for N unknowns, which can be represented as a matrix equation of the following form: \( Ax = b \) where \( A \) is an \( N \times N \) matrix and where \( x \) and \( b \) are \( N \times 1 \) column vectors. Numerically solving for \( x \) results in numerical error, i.e. rounding error and manipulation error (Cook et al., 2002). Rounding error is the difference between an infinitely precise number and its finitely precise approximation and is necessary to store the elements of \( A \) and \( b \) in a computer. Manipulation error includes any error that results from manipulating the rounded numbers. Often the numerical error is negligible, but cases when the numerical error is known to be large are discussed in this chapter.

3.1 Ill-Conditioned Coefficient Matrices

The numerical error is large when the \( A \) matrix is ill-conditioned, i.e., close to singular. To be concrete, reconsider example 2.1.1, which is a one-dimensional problem
with the pressure specified at both ends. Away from resonance at \( f = 100 \) Hz, the coefficient matrix is
\[
\begin{bmatrix}
1 & 1 \\
e^{-jkL} & e^{jkL}
\end{bmatrix} = \begin{bmatrix}
-0.25808 & 1 \\
-0.96612j & -0.25808 + 0.96612i
\end{bmatrix},
\]
which has a determinant of \( 1.9322j \). In this case, the matrix is not close to singular, so the plane waves’ amplitudes can be found with negligible numerical error. However, at resonance \( (f = 171.5 \) Hz), calculated analytically \( kL = \pi \), so the coefficient matrix is
\[
\begin{bmatrix}
1 \\
-1 \\
-1
\end{bmatrix},
\]
which is singular, the determinant is zero, and the plane waves’ amplitudes diverge. However, calculated numerically in MATLAB the coefficient matrix is
\[
\begin{bmatrix}
1 \\
-1 + 3.2162 \times 10^{-16}j \\
-1 - 3.2162 \times 10^{-16}j
\end{bmatrix}
\]
and the determinant is \( -1.0344 \times 10^{-31} - 6.4325 \times 10^{-16}j \). The numerical error is small. However, this error causes the plane waves’ amplitudes to remain finite instead of diverging. The condition number, which is described in the next section, warns when this problem may occur.

### 3.2 The Condition Number

The condition number is a useful measure of the conditioning of a matrix and is defined as \( C(A) = \mu_{\text{max}}/\mu_{\text{min}} \) where \( \mu_{\text{max}} \) and \( \mu_{\text{min}} \) are the largest and smallest eigenvalues of \( A \) (Cook et al., 2002; Watkins, 2010). The condition number is useful because it is used to estimate the largest number of digits of accuracy that may be lost in solving the matrix equation \( Ax = b \). Specifically, \( d_{\text{orig}} - \log_{10}(C) \leq d_{\text{acc}} \leq d_{\text{orig}} \).
where $d_{\text{orig}}$ is the original number of digits of precision and $d_{\text{acc}}$ is digits of accuracy in the computed values (Cook et al., 2002). Thus, if the condition number is small, then the number of digits of accuracy is approximately equal to the original number of digits of precision. However, if the condition number is large, then the number of digits of accuracy may (but not necessarily) be much less than the original number of digits of precision.

### 3.3 The Condition Number in One Dimension

Again return to example 2.1.1. On the one hand, away from resonance when $f = 100$ Hz, the condition number is 1.30, so the matrix is well-conditioned and $14.8 \leq d_{\text{acc}} \leq 15$ (MATLAB uses 15 digits of precision). On the other hand, at resonance when $f = 171.5$ Hz, the condition number is $6.2 \times 10^{15}$, so the matrix is relatively ill-conditioned. In addition, since MATLAB uses approximately 15 digits of precision, then $0 \leq d_{\text{acc}} \leq 15$, which means that the values of the plane waves’ amplitudes could be meaningless. In that case, more information needs to be used to determine the credibility (or the lack thereof) of the models outputs. In this case, the model is not credible because the large pressure amplitude breaks the PWSM’s assumption of linear acoustics.

Nevertheless, a more general theory of when the condition number becomes large is needed. The determinant of the coefficient matrix is zero when $0 = e^{jkl} - e^{-jkl} = 2j \sin(kL)$. Thus, the determinant is zero when $kL = n\pi$ where $n$ is any integer, which matches the resonance frequencies when homogeneous boundary conditions are
specified. This result is plausible because the specified pressures are not used to calculate the condition number. Figure 3-1 shows a plot of the condition number versus $kL$.

![Figure 3-1](image)

Figure 3-1. $\log_{10}$ of the condition number versus normalized wavenumber for a one-dimensional case where the pressure is specified at both ends.

The condition number is large only at the system’s resonances and at zero hertz. Assuming that $\omega > 0$, the condition number becomes large because the matrix is singular when $kL = n\pi$. Physically, when there is a non-homogeneous boundary condition and $kL = n\pi$, both boundary conditions cannot be satisfied. When both boundary conditions are homogeneous and $kL = n\pi$, both boundary conditions are satisfied without specifying the plane wave amplitudes, so the plane wave amplitudes cannot be determined. Assuming that $\omega = 0$, the pressure becomes independent of time, i.e.
\[ p(x, t) = \hat{p}(x)e^{i\omega t} = \hat{p}(x), \]
so the wave equation becomes \( d^2p/dx^2 = 0 \). The solution to this differential equation is a line, and specifically in this case with \( \omega = 0 \) the solution is \( p(x, t) = x/L \). The condition number increases as \( \omega \to 0 \) because the solution cannot be represented by the assumed form. This case demonstrates that there are no problems away from resonance, but suggests that there could be problems near resonance frequencies in higher dimensional problems.

### 3.4 The Condition Number in Two and Three Dimensions

Consider the circle from example 2.2.1. The two dimensional case is more complicated than the one dimensional case because the continuous boundary conditions need to be approximated with a finite number of points. Thus, in addition to plotting the condition number versus normalized wavenumber, the condition number is also plotted versus the number of points per wavelength, which corresponds to the quality of the approximation. To make Figure 3-2, the condition number was found for thousands of matrices for a range of frequencies and numbers of points per wavelength.

Figure 3-2 plots the condition number versus the wavelength divided by the minimum distance between two specified points \( (\lambda/\delta_{\text{min}}) \) and the wavenumber times the circle’s radius \( (k\alpha) \). Figure 3-2 shows a clear trend that as both \( \lambda/\delta_{\text{min}} \) and \( k\alpha \) increase, the condition number increases, which suggests a relationship of the following form: \( C \propto (\lambda/\delta_{\text{min}})^\alpha(k\alpha)^\beta \) where \( \alpha > \beta > 0 \). We have tried to determine \( \alpha \) and \( \beta \) in this case using a least squares fit of a portion of the data in Figure 3-2, but the values vary greatly
depending on what portion of the data is used. For comparison, the condition number in this case for a finite element model is $C \propto (h_{max}/h_{min})N_{els}$ where $h_{max}$ and $h_{min}$ are the maximum and minimum distance between nodes in the mesh and $N_{els}$ is the number of elements in the finite element model (Cook et al., 2002).

Figure 3-2. $\log_{10}$ of the condition number versus the normalized wavenumber and the number of points per wavelength for a circle with the pressure specified at the edge.

Thus, in finite elements analysis the larger the number of elements and the smaller the minimum distance between nodes, the larger the condition number is. This aspect is similar between the FEM and PWSM. However, Figure 3-2 suggests a much higher dependence on these parameters than a linear one in FEM. Nevertheless, if $\lambda/\delta_{min}$ is sufficiently small, then there is a large range of frequencies where the plane waves’
amplitudes can be found with a large number of digits of precision (e.g. 6 digits of precision).

Looking at the bottom of the plot, there are vertical strips of blue in the dark blue. These vertical strips occur when $ka$ is a zero of the cylindrical Bessel function ($j = 2.40, 3.83, 5.14, 5.52, 6.38 \ldots$). The corresponding frequencies are the resonance frequencies of the circle. Recall that in the one-dimensional case, the condition number becomes very large at resonance. However, Figure 3-2 indicates that in higher dimensional examples, the condition number only increases a relatively small amount. The difference between the one-dimensional case and the higher-dimensional cases stems from the boundary conditions being enforce exactly in the one-dimensional case and only approximated in the higher-dimensional cases.

Next, to show that our conclusions have some generality, we will do the same analysis for the other examples given in chapter 2. The results for the square in example 2.2.2 are in Figure 3-3 below where L is the length of a diagonal of the square. While some of the finer structure is different than the circle, the general trend is that as both $\lambda/\delta_{min}$ and $ka$ increase, the condition number increases. Thus, if $\lambda/\delta_{min}$ is sufficiently small, then there is a large range of frequencies where the plane waves’ amplitudes can be found with a large number of digits of precision.
Figure 3-3. $\log_{10}$ of the condition number versus the normalized wavenumber and the number of points per wavelength for a square with the pressure specified at the edge.

In contrast to the circle, the square has big sweeping bands where the condition number plot is not continuous on the edges. For example, there is a big band that sweeps from (6,10) down and the right to (13,4.5). These discontinuities are an artifact of how the plot is generated. Within each of the bands, the number of points that are used to approximate the boundary conditions remains constant. Since Figure 3-3 is for a square, each side needs an additional point at the same time, so four points are added each time a point is needed, which creates the discontinuity. In addition, the vertical strips at $kL = 2\pi, \sqrt{10}\pi, 4\pi$ correspond to the (1,1); (2,1); and (2,2) resonance frequencies of the square.
Next, the results for the trapezoid in example 2.2.3 are in Figure 3-4 below where $L$ is the length of a longest diagonal of the trapezoid.

![Figure 3-4](image)

Figure 3-4. Log$_{10}$ of the condition number versus the normalized wavenumber and the number of points per wavelength for a trapezoid with the pressure specified at the edge.

While some of the finer structure is different from the previous examples, the general trend that as both $\lambda/\delta_{min}$ and $k\alpha$ increase, the condition number increases. Thus, if $\lambda/\delta_{min}$ is sufficiently small, then there is a large range of frequencies where the plane waves’ amplitudes can be found with a large number of digits of precision.

Again, the bands appear in the plot for the trapezoid; however, the lines between the bands are blurred. For each single line in Figure 3-3, there are two lines in Figure 3-4. The line split because not all sides of the trapezoid are the same length, so points are
added to the sides of different lengths for a different number of points per wavelength. In
addition, the effects of the trapezoid’s resonances are not noticeable at this scale.

Next, the results for the sphere in example 2.2.4 are in Figure 3-5 below where \( a \) is the radius of the sphere. This plot has a much smaller y-axis range than the other plots,
which also makes the range for the condition number much smaller. The condition
number is acceptable for all of the points that are plotted in Figure 3-5.

![Figure 3-5. Log\(_{10}\) of the condition number versus the normalized wavenumber and the number of
points per wavelength for a sphere with the pressure specified on the boundary.](image)

While some of the finer structure is different from the previous examples, the general
trend that as both \( \lambda/\delta_{min} \) and \( ka \) increase, the condition number increases. Thus, if
If \( \lambda / \delta_{\text{min}} \) is sufficiently small, then there is a large range of frequencies where the plane waves’ amplitudes can be found with a large number of digits of precision.

The finer structure for the sphere is similar to that of the circle because both cases allow points to be added one at a time. Figure 3-5 also shows the effects of the resonance frequencies well because the range of the condition number is much smaller than in the other plots. In this case, the vertical strips are at the zeroes of the spherical Bessel function, which correspond to the resonance frequencies of the sphere.

Next, the results for the cube in example 2.2.5 are in Figure 3-6 below where \( L \) is the length of a longest diagonal of the cube.

![Figure 3-6. Log10 of the condition number versus the normalized wavenumber and the number of points per wavelength for a cube with the pressure specified on the boundary.](image)
While some of the finer structure is different from the previous examples, the general trend that as both $\lambda/\delta_{\text{min}}$ and $ka$ increase, the condition number increases. Thus, if $\lambda/\delta_{\text{min}}$ is sufficiently small, then there is a large range of frequencies where the plane waves’ amplitudes can be found with a large number of digits of precision.

The plot for the cube has the same sweeping bands that the plot for the square had because, like the square, many points are added at the same time. For the bands in Figure 3-6, $N = 1, 2, 4, 8, 26, 56, 98, 152, 218, 296$. In addition, the resonances of the cube do not make a noticeable difference at this scale. It remains to be shown that if $\lambda/\delta_{\text{min}}$ is small enough to know the plane wave amplitudes with many digits of precision, then it is also possible to accurately approximate the boundary conditions, which is the subject of the next chapter.
Chapter 4

Convergence

This chapter discusses how well the continuous boundary conditions are approximated by the specified points. In one-dimensional problems, the boundary conditions are exact, so there is not an error associated with approximating the boundary conditions and the PWSM gives the same result as an analytic method. However, in two or more dimensions, the continuous boundary conditions are only enforced at a finite number of points. A general theory of how well the boundary conditions are approximated has not been found; instead, the five two/three dimensional cases from chapter 2 will be explored to learn about the quality of the approximation in each case.

Before these cases can be considered, a definition of convergence needs to be established. The calculated boundary values are sampled with a large number (e.g. hundreds) of points, and the relative error in the boundary condition at each point is calculated. If the maximum relative error is less than 10%, then the model is converged. 10% is a good starting point because it gives one or two significant figures in the answer. If a more accurate solution is desired, then a smaller relative error would be required.

Since only non-zero pressures are specified on the boundary in the cases under consideration, convergence will be measured by sampling the pressure on the boundary and then finding the maximum relative error between the true pressure ($\hat{p}$) and the approximated pressure ($\hat{p}^*$), which mathematically is written as
Thus, if $E_{\text{max}} < 0.1$, then the model is converged.

### 4.1 Convergence for a Unit Circle

Consider the circle from example 2.2.1 where the pressure (1 Pa) is specified on the boundary. Figure 4-1 shows this example for $N = 2, 3, 4, 5$.

![Figure 4-1](image.png)

Figure 4-1. Pressure for a circle with $N = 2, 3, 4, 5$. As $N$ increases the approximation becomes better.
These plots qualitatively demonstrate that as the number of points increases, the maximum error in the boundary conditions generally decreases. Furthermore, the calculated maximum error in the boundary conditions for each plot quantitatively demonstrates that the maximum error generally decreases as the number of specified points increases.

Consider Figure 4-1 by starting with the $N = 2$ case. In this case, the pressure field looks like a standing wave, which satisfies the boundary conditions at the two specified points but very poorly approximates the boundary conditions away from those two points. For example, at the top of the circle the pressure is approximately -3.9 Pa instead of the specified 1 Pa, which is why the maximum relative error in that case is approximately 4.9. The approximation for the $N = 3$ case is much better than the approximation for the $N = 2$ case. Specifically, it is about 5 times better. Nonetheless, the pressure field is not converged yet because the plot shows a triangle shape in the pressure, which is not appropriate for the geometry and is only an artifact of only using three plane waves. The improvement continues when a fourth point is added. Specifically, the $N = 4$ approximation is about 4 times better than the $N = 3$ approximation, but the pressure still has a square shape to it. However, the $N = 5$ case is different because there are enough plane waves that the pressure does not look like a pentagon. Instead, the pressure looks circular, as it should.

So far, only one frequency (i.e. 100 Hz) has been considered while changing the number of specified points/plane waves. Similar to the condition number, the maximum relative error changes with respect to the number of specified points per wavelength.
(λ/δ_{min}) and non-dimensional wavenumber (ka). Ideally, there are values of λ/δ_{min} for a wide range of ka values where both the condition number and the maximum relative error are acceptable.

Figure 4-2 plots $E_{max}$ for the circle as a function of λ/δ_{min} and ka.

![Log10 of the maximum error versus the normalized wavenumber and the number of points per wavelength for a circle with the pressure specified at the edge.](image)

In this case, the boundaries converge quite rapidly. The vast majority of plotted points shows that the boundaries are all within 10% of the true value. In addition, all of the points that are dark blue are converged to the same accuracy as the points that were specified. However, there are certain frequencies where the error is larger than the general trend. These frequencies of increased error occur at the resonance frequencies of the circle. The zeroes of the zeroth order cylindrical Bessel function of the first kind are
\( j_{0,n} = \{2.40, 5.52, 8.65, \ldots\} \), which correspond very well with the vertical strips. Lastly, examination of Figure 4-2 with Figure 3-2 illustrates that there are certain numbers of points per wavelength at many different frequencies where the condition number is reasonable and the approximation of the boundary conditions is good.

### 4.2 Convergence for a Unit Square

Consider the square from example 2.2.2 where the pressure (1 Pa) is specified on the boundary. Again, plotting the pressure for 1, 2, 4, and 8 specified points/plane waves illustrates that the PWSM’s solution converges as the number of specified points increases. Consider Figure 4-3 by starting with the \( N = 1 \) case. In this case, the pressure field looks like a traveling wave that is traveling to the top right corner, which satisfies the pressure at the top right corner but is a very poor representation of the pressure field in a square. The poor quality of this approximation is reflected in the large maximum error value. For the \( N = 2 \) case, the pressure field looks like a standing wave between lower left corner and the upper right corner, which satisfies the boundary conditions at the two specified points but very poorly approximates the boundary conditions away from those two points. For example, at \((-0.5,0.5)\) the pressure is approximately 4 Pa instead of the specified 1 Pa. The approximation for the \( N = 4 \) case is qualitatively and quantitatively much better than the approximation for the \( N = 2 \) case. Specifically, the \( N = 4 \) approximation is an order of magnitude better than the \( N = 2 \) approximation. However, the \( N = 4 \) approximation still is not converged because while all the corners are approximated well, the sides are not. The improvement continues going to the \( N = \)
case, which is converged. Comparing the visual results with the calculated maximum error shows that the convergence criteria accurately describes this case.

Figure 4-3. Pressure for a square with N = 1, 2, 4, and 8. As N increases the approximation becomes better.

Figure 4-4 plots $E_{\text{max}}$ for the square as a function of $\lambda/\delta_{\text{min}}$ and $kL$. In this case, the maximum error decreases as the number of points increases, but not nearly as rapidly as for the circle. An explanation for why this case converges more slowly has not been found. However, the rate of convergence for the circle appears to the special and the rate
of convergence for the square appears to be more common. However, most plotted points still show that the pressure on the boundaries are all within 10% of the true value.

Figure 4-4. $\log_{10}(E_{\text{max}})$ of the maximum error versus the normalized wavenumber and the number of points per wavelength for a square with the pressure specified at the edge.

In addition, there is much more structure in Figure 4-4 than in Figure 4-2. The large bands that sweep from the upper left to the lower right are the same bands that occurred in the condition number plot for the square (Figure 3-3). Recall that inside the bands the number of specified points is constant. Lastly, examination of Figure 4-4 with Figure 3-3 illustrates that there is a certain number of points per wavelength at many different frequencies where the condition number is reasonable and the approximation of the boundary conditions is good.
4.3 Convergence for a Trapezoid

Consider the trapezoid from example 2.2.3 where the pressure (1 Pa) is specified on the boundary. Again, plotting the pressure for 2, 4, 7, and 11 specified points/planes waves illustrates that the PWSM’s solution converges as the number of specified points increases.

Figure 4-5. Pressure for a trapezoid with \( N = 2, 4, 7, \) and 11. As \( N \) increases the approximation becomes better.
Consider Figure 4-5 by starting with the $N = 2$ case. Similar to the square, the pressure field looks like a standing wave between the lower left corner and the upper right corner, which satisfies the boundary conditions at the two specified points but very poorly approximates the boundary conditions away from those two points. For example, at $(-0.5,0.5)$ the pressure is approximately 4 Pa instead of the specified 1 Pa. The approximation for the $N = 4$ case is qualitatively and quantitatively much better than the approximation for the $N = 2$ case. Specifically, the $N = 4$ approximation is an order of magnitude better than the $N = 2$ approximation. Now, all the corners are approximated well, but the sides are not. There is not much improvement when the number of specified points increases to $N = 7$ because the right side does not have a specified point yet. However, when $N = 11$ all of the sides have at least one point and the pressure on the boundary is converged. Comparing the visual results with the calculated maximum error shows that the convergence criteria accurately describes this case.

Figure 4-6 plots $E_{\text{max}}$ for the trapezoid as a function of $\lambda/\delta_{\text{min}}$ and $kL$. In this case, the maximum error decreases as the number of points increases. Most plotted points show that the pressure on the boundaries are all within 10% of their true value. Again, the large bands that sweep from the upper left to the lower right are the where the number of points are constant and are the same bands that occurred in the condition number plot for the trapezoid (Figure 3-4). The poor convergence at $kL = 6.8$ is due to the fundamental resonance of the trapezoid. It is not known why this resonance has a much larger impact on the maximum error than the resonances for the other shapes. Optimizing the plane waves’ orientations, which was not done in this case, should
mitigate or possibly eliminate the problem.

Figure 4-6. $\log_{10}(E_{\text{max}})$ of the maximum error versus the normalized wavenumber and the number of points per wavelength for a trapezoid with the pressure specified at the edge.

Lastly, examination of Figure 4-6 with Figure 3-4 illustrates that there is a certain number of points per wavelength at many different frequencies where the condition number is reasonable and the approximation of the boundary conditions is good.

4.4 Convergence for a Unit Sphere

Consider the sphere from example 2.2.4 where the pressure (1 Pa) is specified on the boundary. Again, plotting the pressure for 2, 4, 6, and 8, specified points/plane waves illustrates that the PWSM’s solution converges as the number of specified points
Figure 4-7. Pressure for a sphere with $N = 2, 4, 6,$ and $8$. As $N$ increases the approximation becomes better.

Consider Figure 4-7 by starting with the $N = 2$ case. Similar to the square, the pressure field looks like a standing wave between the bottom and the top of the sphere, which satisfies the boundary conditions at the two specified points but very poorly approximates the boundary conditions away from those two points. For example, at $(0,0)$ the pressure is approximately $-4$ Pa instead of the specified $1$ Pa. The approximation for
the $N = 4$ case is qualitatively and quantitatively much better than the approximation for the $N = 2$ case. Specifically, the $N = 4$ approximation is an order of magnitude better than the $N = 2$ approximation. The improvement continues as the number of points increases. Comparing the visual results with the calculated maximum error shows that the convergence criteria accurately describes this case.

Figure 4-8 plots $E_{\text{max}}$ for the sphere as a function of $\lambda/\delta_{\text{min}}$ and $kL$.

![Figure 4-8. Log$_{10}$ of the maximum error versus the normalized wave number and the number of points per wavelength for a sphere with the pressure specified on the boundary.](image)

In this case, the maximum error decreases as the number of points increases. Many plotted points show that the pressure on the boundaries are all within 10% of their true value, which shows that the PWSM can have converged solutions with as little as 2-3 points per wavelength. In the FEM, the general rule of thumb is to use 6-10 nodes per wavelength. On the conservative end, Astley suggests using 10 nodes per wavelength.
(Astley, 2007). Since for the sphere points can be added one at a time, Figure 4-8 lacks the large sweeping bands that were present in the plots for the square and the trapezoid. Also, the resonance frequencies appear in the plot as an increase in the maximum error at \( k\alpha = 3.14, 6.28 \), which are zeroes of the zeroth order spherical Bessel function. However, the additional resonance frequencies at \( k\alpha = 4.49, 5.76, 6.99 \), which did affect the condition number in Figure 3-5, do not appear in Figure 4-8 because those resonances are not excited by the symmetrical boundary condition. In contrast to the trapezoid, using 4-5 points per wavelength at the resonance frequencies should be sufficient to obtain a converged solution. Lastly, examination of Figure 4-8 with Figure 3-5 also illustrates that there is a certain number of points per wavelength at many different frequencies where the condition number is reasonable and the approximation of the boundary conditions is good.

### 4.5 Convergence for a Unit Cube

Consider the cube from example 2.2.5 where the pressure (1 Pa) is specified on the boundary. Again, plotting the pressure for 2, 4, 8, and 26 specified points/plane waves illustrates that the PWSM’s solution converges as the number of specified points increases. Consider Figure 4-9 by starting with the \( N = 2 \) case. In this case, the pressure field looks like a standing wave between \((-0.5,-0.5,-0.5)\) and \((0.5,0.5,0.5)\), which satisfies the boundary conditions at the two specified points but very poorly approximates the boundary conditions away from those two points. For example, at some places on the
boundary, the pressure is less than -60 Pa instead of the specified 1 Pa.

The approximation for the $N = 4$ case is qualitatively and quantitatively much better than the approximation for the $N = 2$ case. Specifically, the $N = 4$ approximation is 2 orders of magnitude better than the $N = 2$ approximation. Now, all the corners are approximated well, but the sides and faces are not. The $N = 8$ is not much better than the $N = 4$ approximation. The pressure on the sides and faces still has larger errors. The
$N = 26$ approximation is much better than any of the previous approximations and is the only approximation that satisfies the convergence criterion.

Figure 4-10 plots $E_{\text{max}}$ for the cube as a function of $\lambda/\delta_{\text{min}}$ and $kL$.

![Figure 4-10](image)

Figure 4-10. Log$_{10}$ of the maximum error versus the normalized wavenumber and the number of points per wavelength for a cube with the pressure specified on the boundary.

In this case, many plotted points show that the boundaries are all within 10% of the true value. In addition, similar to the square, the cube shows the large sweeping bands where the number of specified points is constant. This example illustrates that there is a minimum number of points that is needed to define a geometry. The first two sweeping bands have large areas that are not converged because they correspond to the $N = 4$ and $N = 8$ approximations. The next band, the first band that is completely converged in this plot, is the $N = 26$ approximation, which has one point on each corner, side, and face.
Lastly, examination Figure 4-10 with Figure 3-6 illustrates that there is a certain number of points per wavelength at many different frequencies where the condition number is reasonable and the approximation of the boundary conditions is good.

4.6 Contrasting the PWSM and the FEM

The PWSM produces a smaller system of equations than the FEM for three reasons. First, the PWSM only needs specified points on the boundaries of the space whereas the FEM needs nodes throughout the space. For a cube with N points per wavelength along each edge, the PWSM would need $6N^2$ points whereas the FEM would need $N^3$ points. This is the same reasoning for why boundary element models are smaller than finite element models. Second, the PWSM converges with fewer points per wavelength than the FEM. In the two examples that we have considered in this chapter, many of the frequencies have converged solutions with just 3 points per wavelength. In addition, for the circle at the high frequencies, where it is most important, just over two points per wavelength could be used. Thus, in many cases the number of points per wavelength can be reduced by 50% or more. Third, in contrast to the FEM, the PWSM does not have any discretization error, so the number of points per wavelength does not need to be increased in order to have a gradient of the pressure (e.g. velocity) converge.

However, while the PWSM is faster than the FEM in some cases because of the smaller system of equations, the PWSM is not always faster because the PWSM matrix is harder to solve than the FEM matrix. The PWSM matrix is full, complex, and non-symmetric, and the FEM is sparse and possibly symmetric. In general, for matrices of the
same size a sparse matrix is faster to solve than a full matrix and a symmetric matrix is faster to solve than a non-symmetric matrix. Thus, the method that will be faster will depend on the problem.
Chapter 5

Conclusion

The Plane Wave Superposition Method (PWSM) is a numerical algorithm to find continuous approximation of the pressure and velocity fields inside an arbitrarily shaped cavity with mixed boundary conditions and internal acoustic sources. Compared with other numerical methods, the PWSM has the following advantages:

1. Simple
   - The PWSM generates continuous approximations of the pressure and velocity fields
   - Neither specialized software nor extensive training is required; instead, writing a PWSM program or incorporating it into another program is easy
   - For one-dimensional problems, the PSWM’s and analytic method’s solutions are equivalent

2. Versatile
   - The cavity may be any geometry
   - Any standard acoustic boundary condition (pressure, normal velocity, and specific impedance) may be specified
   - Internal acoustic sources may be included
   - The approximation may be improved by increasing the number of specified points

3. Computationally efficient
- The PWSM solves a smaller system of equations
- The PWSM does not introduce any singularities
- Integration over elements is not required

Therefore, the PWSM has many advantages compared to other methods.

Nevertheless, there are at least two areas where the PWSM can be improved. The most important computational area is the optimization of the plane waves’ orientations. Throughout this thesis, the plane wave orientations were the normalized position vectors because of its simplicity and repeatability. However, since the plane waves’ orientations affect the condition number and the maximum error, then the plane waves’ orientations affect the computational speed and accuracy of the PWSM. Thus, determining the optimal plane waves’ orientations for an arbitrarily shaped cavity could be greatly beneficial.

From the user’s perspective, the most important area of improvement is the user interface. Currently, it is relatively difficult to input the geometry and boundary conditions and to access the outputs. These inputs could be defined in a program (e.g. Solidworks) and then analyzed using the PWSM. The outputs could then be displayed in an easy to use graphical user interface. Standard FEM software is already capable of a better user experience, so the same techniques would just need to be applied to the PWSM.

In addition, the PWSM can be extended to solve exterior problems by using the PWSM to find the pressures and velocities on the surface of the vibrating object. Then, substituting those quantities into the Kirchhoff-Helmholtz integral equation yields the pressures and velocities off of the surface. The advantage of this method over the current
BEM is that it would avoid the singularities in the self-terms of the BEM without increasing computational expense. This indirect method must be implemented because the current formulation of the PWSM does not satisfy the Sommerfeld radiation condition (Pierce, 1994), which is required in exterior problems. A potentially more elegant method would not need to use the Kirchhoff-Helmholtz integral equation, but would instead find a way to satisfy the Sommerfeld radiation condition within the PWSM. However, this would require moving away from plane waves and towards some other source. A superposition of simple acoustic sources has already been accomplished with the Superposition Method (Koopmann, 1989).

Lastly, a vision of the Plane Wave Superposition Method is to digitize acoustic space. To digitize an acoustic space is to represent the acoustic fields of that space by a finite set of points where the pressure, velocity, or impedance is known. Currently, the PWSM can accomplish this task in a few cases, but in order to digitize any acoustic space, the method needs to be standardized and improved as suggested above. Digitizing acoustic space would make storage compact and computationally efficient.
Appendix A
Example MATLAB Code

This appendix contains MATLAB code for one- and two-dimensional problems.

A.1 One-Dimensional MATLAB Code

%Matthew Kamrath 01/07/2014
%Analytic Solutions for 1D Cases
clear
clc
format short

%Inputs
%frequency, Hz
f = 100;

%sound speed, m/s
v = 343;

%medium density, kg/m^3
rho = 1.21;

%length of tube, m
L = 1;

%left boundary condition
p1 = [0];
v1 = [];
z1 = [];

%right boundary condition
p2 = [1];
v2 = [];

z2 = [];

%Code/Calculations
%----------------------------------------
%angular frequency, rad/s
w = 2*pi*f;

%wavenumber, m^-1
k = w/c;

if ~isempty(p1)
    if ~isempty(p2)
        M = [1 1;
             exp(-1i*k*L) exp(1i*k*L)];
        C = M[p1;p2];
    elseif ~isempty(v2)
        M = [1 1;
             exp(-1i*k*L) -exp(1i*k*L)];
        C = M[p1;v2*c*rho];
    else
        M = [1 -1;
             (1-z2/(rho*c))*exp(-1i*k*L)
             (1+z2/(rho*c))*exp(1i*k*L)];
        C = M[p1;0];
    end
elseif ~isempty(v1)
    if ~isempty(p2)
        M = [1 -1;
             exp(-1i*k*L) exp(1i*k*L)];
        C = M[rho*c*v1;p2];
    elseif ~isempty(v2)
        M = [1 -1;
             exp(-1i*k*L) -exp(1i*k*L)];
        C = M[rho*c*v1;v2*c*rho];
    else
        M = [1 -1;
             (1-z2/(rho*c))*exp(-1i*k*L)
             (1+z2/(rho*c))*exp(1i*k*L)];
        C = M[rho*c*v1;0];
    end
else
    if ~isempty(p2)
        M = [1-z1/(rho*c) 1+z1/(rho*c);
             exp(-1i*k*L) exp(1i*k*L)];
        C = M[0;p2];
    else
        ...
elseif ~isempty(v2)
    M = [1-z1/(rho*c) 1+z1/(rho*c);
        exp(-1i*k*L) -exp(1i*k*L)];
    C = M\[0;v2*c*rho];
else
    M = [1-z1/(rho*c) 1+z1/(rho*c);
        (1-z2/(rho*c))*exp(-1i*k*L)
        (1+z2/(rho*c))*exp(1i*k*L)];
    C = M\[0;0];
end
end

x = 0:L/100:L;
p = C(1)*exp(-1i*k*x)+C(2)*exp(1i*k*x);
v = 1/(rho*c)*(C(1)*exp(-1i*k*x)-C(2)*exp(1i*k*x));

figure(1)
clf
plot(x,real(p),'linewidth',3)
set(gca,'fontsize',12)
xlabel('x (m)')
ylabel('Re[p] (Pa)')
ylim([0 1.1])
grid

disp('Condition Number = ')
disp(cond(M))
disp(['p1 p2';
      'v1 v2';
      'z1 z2'])
disp([p(1) p(end);
      v(1) v(end);
      p(1)/v(1) p(end)/v(end)])

A.2 Two-Dimensional MATLAB Code

%Matthew Kamrath 01/14/2014
%2D Plane Wave Method
clear
clc
format shortg

%INPUTS
%Frequency, Hz
f = 100;

%Medium's sound speed, m/s
c = 343;

%Medium's density, kg/m^3
rho = 1.21;

%Specified pressures, Pa (column vector)
N_cir = 5;
ps = ones(N_cir,1);

%Specified pressures' locations, m (#ps x 2)
r_cir = 1;
phi_cir = (0:N_cir-1).*2*pi/N_cir;
psx = r_cir.*cos(phi_cir);
psy = r_cir*sin(phi_cir);
r_ps = [psx psy];

%Specified velocities, m/s (column vector)
vs = [];

%Specified velocities' locations, m (#vs x 2)
r_vs = [];

%Unit Normal vectors at the velocities' locations, unitless
n_vs = [];

%Specified impedances, m/s (column vector)
zs = [];

%Specified impedances' locations, m (#vs x 2)
r_zs = [];

%Unit Normal vectors at the impedances' locations, unitless
n_zs = [];

xmin = -r_cir;
xmax = r_cir;
dx = xmax/40;

ymin = -r_cir;
ymax = r_cir;
dy = ymax/40;

%CODE/CALCULATIONS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Number of specified pressures
N_p = size(ps,1);

%Number of specified velocities
N_v = size(vs,1);

%Number of specified impedances
N_z = size(zs,1);

%Total number of specified points
N_tot = N_p+N_v+N_z;

%wavenumber, m^-1
k = 2*pi*f/c;

%b in the matrix equation Ax = b
b = [ps; rho*c*vs; zeros(size(zs))];

%angle of each of the plane waves
phi = (2*pi*(0:N_tot-1)/N_tot).';

%plane waves' direction unit vector
k_hats = [cos(phi),sin(phi)];

%Plane waves' direction vector
k_vecs = k.*k_hats;

%A in the matrix equation Ax = b
T = zeros(N_tot,N_tot);
for d1 = 1:N_p
    T(d1,:) = exp(-1i.*dot(k_vecs,ones(N_tot,1)*r_ps(d1,:),2));
end
for d1 = 1:N_v
    T(d1+N_p,:) = dot(ones(N_tot,1)*n_vs(d1,:),k_hats,2).*...
exp(-
1i.*dot(k_vecs,ones(N_tot,1)*r_vs(d1,:),2));
end
for d1 = 1:N_z
    if abs(zs(d1))<=rho*c
        T(d1+N_p+N_v,:)=(1-zs(d1)/rho./c.*...
dot(ones(N_tot,1)*n_zs(d1,:),k_hats,2)).*...
    exp(-
1i.*dot(k_vecs,ones(N_tot,1)*r_zs(d1,:),2));
    else
        T(d1+N_p+N_v,:)=(dot(ones(N_tot,1)*zhats(d1,:),k_hats,2)/rho./c-...
        1./zs(d1)).*exp(-
1i.*dot(kvecs,ones(N_tot,1)*r_zs(d1,:),2));
    end
end

%plane wave amplitudes, Pa
A = T\b;

%calculate the error
error1=[mean(abs(T*A-b)) max(abs(T*A-b))];
disp('error1 = ')
disp(error1)

%displaying the condition number
condnum=cond(T);
disp('Condition Number = ')
disp(condnum)

%Making the grid of points
X = xmin:dx:xmax;
Y = ymin:dy:ymax;
[x,y] = meshgrid(X,Y);

%Calculating the pressure at each mesh point
pmmesh = zeros(length(Y),length(X),N_tot);
for d1 = 1:length(Y)
    for d2 = 1:length(X)
        pmmesh(d1,d2,:) = A.*exp(-
1i.*dot(k_vecs,ones(N_tot,1)*[X(d2),Y(d1)],2));
    end
end
pmesh = sum(pmmesh,3);
% Plotting the pressure
figure(1)
clf
contourf(x,y,real(pmesh),250,'linestyle','none')
xlabel('x (m)',['FontSize',14,'FontWeight','Bold'])
ylabel('y (m)',['FontSize',14,'FontWeight','Bold'])
colormap jet
h = colorbar;
set(get(h,'title'),'string','Re[p (Pa))','FontSize',12,'FontWeight','Bold');
axis square
hold on
line(r_cir*cos(0:2*pi/100:2*pi),r_cir*sin(0:2*pi/100:2*pi),
'color','k','linewidth',2)
hold off

A.3 Three-Dimensional MATLAB Code

%Matthew Kamrath 03/01/2014
%3D Plane Wave Method
clear
clc
format shortg

% INPUTS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% % Frequency, Hz
f = 100;
%f = 297; % fundamental resonance
%f = 343; % lambda = 1 m, the side length of the cube

% Number of points/wavelength
N_pts = 7;
%N_pts = 2.15;

% Medium's sound speed, m/s
c = 343;

% Medium's density, kg/m^3
rho = 1.21;

% distance between points
delta = c/f/N_pts;

xmin = -0.5;
xmax = 0.5;
xhalf = (xmax+xmin)/2;
dx = (xmax-xmin)/50;

ymin = -0.5;
ymax = 0.5;
yhalf = (ymax+ymin)/2;
dy = (ymax-ymin)/50;

zmin = -0.5;
zmax = 0.5;
zhalf = (zmax+zmin)/2;
dz = (zmax-zmin)/50;

L = sqrt((xmax-xmin)^2+(ymax-ymin)^2+(zmax-zmin)^2);

f_fund = c/2*sqrt((1/(xmax-xmin))^2+(1/(ymax-ymin))^2+(1/(zmax-zmin))^2);

if xmax-xmin > 2*delta
    x_pat = (xmin+delta:delta:xmax-delta).';
    x_pat = x_pat+((xmax-xmin)-delta*(length(x_pat)+1))./2;
else
    x_pat = [];
end

if ymax-ymin > 2*delta;
    y_pat = (ymin+delta:delta:ymax-delta).';
    y_pat = y_pat+((ymax-ymin)-delta*(length(y_pat)+1))./2;
else
    y_pat = [];
end

if zmax-zmin > 2*delta;
    z_pat = (zmin+delta:delta:zmax-delta).';
    z_pat = z_pat+((zmax-zmin)-delta*(length(z_pat)+1))./2;
else
    z_pat = [];
end

Lxp = length(x_pat);
Lyp = length(y_pat);
Lzp = length(z_pat);

if Lxp > 0

corners = [xmin;xmax;xmin;xmax;xmin;xmax;xmin;xmax];
ycorners = [ymin;ymin;ymax;ymax;ymin;ymin;ymax;ymax];
zcorners = [zmin;zmin;zmin;zmin;zmax;zmax;zmax;zmax];

xedge = [x_pat;x_pat;xmin*ones(Lyp,1);xmax*ones(Lyp,1);...
         xmin*ones(Lzp,1);xmax*ones(Lzp,1);xmin*ones(Lzp,1);xmax*ones(Lzp,1);...
         x_pat;x_pat;xmin*ones(Lyp,1);xmax*ones(Lyp,1)];

yedge = [ymin*ones(Lxp,1);ymax*ones(Lxp,1);y_pat;y_pat;...
         ymin*ones(Lzp,1);ymin*ones(Lzp,1);ymax*ones(Lzp,1);ymax*ones(Lzp,1);...
         ymin*ones(Lxp,1);ymin*ones(Lxp,1);ymax*ones(Lxp,1);y_pat;y_pat];

zedge = [zmin*ones(Lxp,1);zmin*ones(Lxp,1);zmin*ones(Lyp,1);zmin*ones(Lyp,1);...
         z_min;z_min;z_min;z_min;zmax;zmax;zmax;zmax];

xface_z0 = x_pat*ones(1,Lyp); xface_z0 = reshape(xface_z0,Lxp*Lyp,1);
yface_z0 = ones(Lxp,1)*y_pat.'; yface_z0 = reshape(yface_z0,Lxp*Lyp,1);
zface_z0 = zmin*ones(Lxp*Lyp,1);

xface_z1 = x_pat*ones(1,Lyp); xface_z1 = reshape(xface_z1,Lxp*Lyp,1);
yface_z1 = ones(Lxp,1)*y_pat.'; yface_z1 = reshape(yface_z1,Lxp*Lyp,1);
zface_z1 = zmax*ones(Lxp*Lyp,1);

xface_y0 = x_pat*ones(1,Lzp); xface_y0 = reshape(xface_y0,Lxp*Lzp,1);
yface_y0 = ymin*ones(Lxp*Lzp,1);
zface_y0 = ones(Lxp,1)*z_pat.'; zface_y0 = reshape(zface_y0,Lxp*Lzp,1);
xface_y1 = x_pat*ones(1,Lzp); xface_y1 = reshape(xface_y1,Lxp*Lzp,1); yface_y1 = ymax*ones(Lxp*Lzp,1); zface_y1 = ones(Lxp,1)*z_pat.'; zface_y1 = reshape(zface_y1,Lxp*Lzp,1);

xface_x0 = xmin*ones(Lyp*Lzp,1); yface_x0 = y_pat*ones(1,Lzp); yface_x0 = reshape(yface_x0,Lyp*Lzp,1); zface_x0 = ones(Lyp,1)*z_pat.'; zface_x0 = reshape(zface_x0,Lyp*Lzp,1);

xface_x1 = xmax*ones(Lyp*Lzp,1); yface_x1 = y_pat*ones(1,Lzp); yface_x1 = reshape(yface_x1,Lyp*Lzp,1); zface_x1 = ones(Lyp,1)*z_pat.'; zface_x1 = reshape(zface_x1,Lyp*Lzp,1);

xface = [xface_z0;xface_z1;xface_y0;xface_y1;xface_x0;xface_x1];
yface = [yface_z0;yface_z1;yface_y0;yface_y1;yface_x0;yface_x1];
zface = [zface_z0;zface_z1;zface_y0;zface_y1;zface_x0;zface_x1];

r_ps = [xcorners,ycorners,zcorners;
       xedge,yedge,zedge;
       xface,yface,zface];

elseif delta < xmax-xmin
xcorners = [xmin;xmax;xmin;xmax;xmax;xmin;xmin;xmax];
ycorners = [ymin;ymin;ymax;ymax;ymin;ymin;ymax;ymax];
zcorners = [zmin;zmin;zmax;zmax;zmin;zmin;zmax;zmax];
r_ps = [xcorners,ycorners,zcorners];

elseif delta < sqrt((xmax-xmin)^2+(ymax-ymin)^2)
xcorners = [xmin;xmax;xmax;xmin];
ycorners = [ymin;ymin;ymax;ymin;ymax;ymin;ymax];
zcorners = [zmin;zmin;zmax;zmax;zmin;zmin;zmax;zmax];
r_ps = [xcorners,ycorners,zcorners];

else
xcorners = [xmin;xmax];
ycorners = [ymin;ymax];
zcorners = [zmin;zmax];
r_ps = [xcorners,ycorners,zcorners];

end

%Specified pressures, Pa (column vector)
ps = ones(size(r_ps,1),1);

%Specified velocities' locations, m (#vs x 2)
r_vs = [];

%Specified velocities, m/s (column vector)
vs = [];

%Unit Normal vectors at the velocities' locations, unitless
n_vs = [];

%Specified impedances' locations, m (#vs x 2)
r_zs = [];

%Specified impedances, m/s (column vector)
zs = [];

%Unit Normal vectors at the impedances' locations, unitless
n_zs = [];

%CODE/CALCULATIONS
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%Number of specified pressures
N_p = size(ps,1);

%Number of specified velocities
N_v = size(vs,1);

%Number of specified impedances
N_z = size(zs,1);

%Total number of specified points
N_tot = N_p+N_v+N_z;

%wavenumber, m^-1
k = 2*pi*f/c;
%b in the matrix equation Ax = b
b = [ps; rho*c*vs; zeros(size(zs))];
disp('N = ')
disp(N_tot)

%plane waves' direction unit vector
%k_hats = xlsread(['k_hats_',num2str(N_tot)]);
k_hats = zeros(size(r_ps));
for d1 = 1:size(r_ps,1)
    k_hats(d1,:) = r_ps(d1,:)./sqrt(sum(r_ps(d1,:).^2));
end

%Plane waves' direction vector
k_vecs = k.*k_hats;

%A in the matrix equation Ax = b
T = zeros(N_tot,N_tot);
for d1 = 1:N_p
    T(d1,:) = exp(-1i.*dot(k_vecs,ones(N_tot,1)*r_ps(d1,:),2));
end
for d1 = 1:N_v
    T(d1+N_p,:) = dot(ones(N_tot,1)*n_vs(d1,:),k_hats).*...
    exp(-1i.*dot(k_vecs,ones(N_tot,1)*r_vs(d1,:),2));
end
for d1 = 1:N_z
    if abs(zs(d1))<=rho*c
        T(d1+N_p+N_v,:)=(1-zs(d1)./rho./c.*...
        dot(ones(N_tot,1)*n_zs(d1,:),k_hats,2)).*...
        exp(-1i.*dot(k_vecs,ones(N_tot,1)*r_zs(d1,:),2));
    else
        T(d1+N_p+N_v,:)=(dot(ones(N_tot,1)*zhat preca(d1,:),k_hats,2)./rho./c-...
        1./zs(d1)).*exp(-1i.*dot(kvecs,ones(N_tot,1)*r_zs(d1,:),2));
    end
end

%plane wave amplitudes, Pa
A = T\b;
%A = pinv(T)*b
% calculate the error
error1 = [mean(abs(T*A-b)) max(abs(T*A-b))];
disp('error1 = ')
disp(error1)

% displaying the condition number
condnum = cond(T);
disp('Condition Number = ')
disp(condnum)

% Making the grid of points
X = xmin:dx:xmax;
Y = ymin:dy:ymax;
Z = zmin:dz:zmax;

[x,y,z] = meshgrid(X,Y,Z);

% Calculating the pressure at each mesh point
pmmesh = zeros(length(Y),length(X),length(Z),N_tot);
for d1 = 1:length(Y)
    for d2 = 1:length(X)
        for d3 = 1:length(Z)
            pmmesh(d1,d2,d3,:) = A.*exp(-1i.*
                dot(k_vecs,ones(N_tot,1)*[X(d2),Y(d1),Z(d3)],2));
        end
    end
end

clear d1 d2 d3

pmesh = sum(pmmesh,4);

E_max = 0;
for d1 = 1:length(X)*length(Y)*length(Z)
    if x(d1)==xmin || x(d1)==xmax || y(d1)==ymin ||
    y(d1)==ymax || ... 
        z(d1)==zmin || z(d1)==zmax
        E_max = max(E_max,abs(1-pmesh(d1))/1);
    end
end
disp('E_max = ')
disp(E_max)

% edges of the box
xx = [xmin*ones(1,4),xmax*ones(1,6),xmin*ones(1,3),xmax*ones(1,2),xmin*ones(1,1)];
yy = [ymin, ymax*ones(1,6), ymin*ones(1,5), ymax*ones(1,2), ymin*ones(1,2)];
zz = [zmin, zmin, zmax, zmin, zmin, zmax, zmin, zmin, zmax, zmin, zmax*ones(1,5)];

%setting the slices
xslice1 = xhalf;
yslice1 = yhalf;
zslice1 = zhalf;
xslice2 = [xmin, xmax];
yslice2 = [ymin, ymax];
zslice2 = [zmin, zmax];
xslices = [xslice1, xslice2];
yslices = [yslice1, yslice2];
zslices = [zslice1, zslice2];

%Plotting the pressure
figure(3)
clf
slice(x,y,z,real(pmesh),xslice1,yslice1,zslice1)
set(gca,'FontSize',20,'FontWeight','Bold')
title('')
xlabel('x (m)')
ylabel('y (m)')
zlabel('z (m)')
shading interp
axis tight
colormap jet
h = colorbar;
set(get(h,'title'),'string','Re[p (Pa)', 'FontSize',20,'FontWeight','Bold');
set(h,'FontSize',20,'FontWeight','Bold')
axis equal

figure(4)
clf
slice(x,y,z,real(pmesh),xslice2,yslice2,zslice2)
hold on
line(xx,yy,zz,'color','k')
hold off
set(gca,'FontSize',20,'FontWeight','Bold')
title('')
xlabel('x (m)')
ylabel('y (m)')
zlabel('z (m)')
shading interp
axis tight
colormap jet
h = colorbar;
set(get(h, 'title'), 'string', 'Re[\rho] (Pa)', 'FontSize', 20, 'FontWeight', 'Bold');
set(h, 'FontSize', 20, 'FontWeight', 'Bold')
axis equal

p = 1.01;

[x, y, z] = meshgrid([xmin, xmax], [ymin, ymax], [zmin, zmax]);

c1 = 0.85*ones(2,2,2);
c1(2,2,1) = 1;
c1(2,2,2) = 0;

figure(5)
clf
scatter3(p*r_ps(:,1),p*r_ps(:,2),p*r_ps(:,3),150,'f','k')
hold on
%line(xx,yy,zz,'color','k')
slice(x,y,z,c1,xslice2,yslice2,zslice2)
hold off
set(gca, 'FontSize', 20, 'FontWeight', 'Bold')
xlabel('x (m)')
ylabel('y (m)')
zlabel('z (m)')
axis equal
colormap gray
axis tight
References


