PRODUCTION DATA ANALYSIS OF NATURALLY FRACTURED RESERVOIRS:
A DENSITY-BASED APPROACH

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ABSTRACT

Significant amounts of oil and gas are trapped in naturally fractured reservoirs, a phenomenon which has attracted growing attention as the shale boom has evolved. The dual-porosity concept has been commonly used in modeling these naturally fractured reservoirs. In this model, the fluid flows through the fracture system in the reservoir, while matrix blocks are segregated by the fractures and act as the fluids sources for them. This model was originally developed for liquid in naturally fractured systems and therefore inadequate for capturing pressure-dependent effects in gas systems.

This study presents a rigorous derivation of a gas interporosity flow equation that accounts for the effects of pressure-sensitive properties. A numerical simulator using the gas interporosity flow equation is built and demonstrates a significant difference in system response from that of a simulator implementing a liquid-form interporosity flow equation. For this reason, rigorous modeling of interporosity flow is considered essential to decline curve analysis for naturally fractured gas reservoirs. State-of-the-art approaches to decline curve analysis have typically used pseudo-functions, yet these approaches remain limited in utility as demonstrated in many previous comparisons between analytical results and production data that revealed discrepancy. In this study, we show the gas interporosity flow equation eliminates the discrepancy at the decline stage and enables rigorous decline curve analysis for production at constant bottomhole pressure. We investigate the applicability of a density-based approach for decline curve analysis for production at constant bottomhole pressure in dual-porosity gas systems. This approach relates gas production profiles to their liquid counterparts by decoupling pressure-dependent effects from pressure depletion. This study further demonstrates the process of rigorous derivation for density-based decline curve analysis in dual-porosity gas systems. The interporosity flow equation for gas is used, and a deliverability equation for dual-porosity systems is rigorously derived in the process.

In light of density-based approach for production at constant bottomhole pressure in dual-porosity gas systems, a density-based, rescaled exponential model for variable pressure drawdown/variable rate production was developed for dual-porosity gas systems. We also explore straight-line analysis for convenient prediction of OGIP and production rate at variable pressure drawdown/rate production. This density-based model was tested in a variety of scenarios to showcase its validity.

Furthermore, based on Warren and Root’s model, a density-based exponential model for variable pressure drawdown/rate in dual-porosity liquid systems is proposed and verified. Then, a straight-line analysis is proposed to enable explicit OOIP prediction and convenient future production calculation. Aside from these, we develop a double-exponential decline model under constant BHP for liquid which is not only applicable to both decline stages but also convenient to implement.
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NOMENCLATURE

**Roman**

\( a_m \) radius of spherical matrix, ft

\( A \) area, \( \text{ft}^2 \)

\( b_{D_PSS} \) pseudo-steady component, dimensionless

\( B \) dimensionless intrinsic dependency of viscosity-compressibility with density, dimensionless

\( \bar{B} \) average dimensionless intrinsic dependency of viscosity-compressibility with density, dimensionless

\( B_o \) oil formation volume factor, RB/STB

\( c_1 \) matrix compressibility, \( 1/\text{psi} \)

\( c_2 \) fracture compressibility, \( 1/\text{psi} \)

\( c_f \) addition of fracture compressibility and liquid compressibility, \( 1/\text{psi} \)

\( c_{gf} \) gas compressibility in fracture, \( 1/\text{psi} \)

\( c_{gi} \) initial gas compressibility in fracture, \( 1/\text{psi} \)

\( c_{gm} \) gas compressibility in matrix, \( 1/\text{psi} \)

\( c_l \) liquid compressibility, \( 1/\text{psi} \)

\( c_m \) addition of matrix compressibility and liquid compressibility, \( 1/\text{psi} \)

\( C \) constant, dimensionless

\( C_A \) Dietz’s reservoir shape factor, dimensionless

\( D^e_{di} \) initial decline coefficient for density-based dual-porosity model, \( 1/\text{day} \)

\( D^f_{di} \) initial decline coefficient for density-based model, \( 1/\text{day} \)

\( G_p \) cumulative gas production, Mscf/D

\( h \) thickness, ft

\( k_f \) fracture permeability, md

\( k_m \) matrix permeability, md

\( K \) rescaling component for dual-porosity system, dimensionless

\( m \) \( \frac{D_f(b-1)}{q_{gi}} \)

\( m_{in} \) mass flow rate into the control volume, \( \frac{\text{lb}}{\text{cf} \cdot \text{s}} \)

\( m_{out} \) mass flow rate out of the control volume, \( \frac{\text{lb}}{\text{cf} \cdot \text{s}} \)

\( \bar{m}(p_f) \) average pseudo-pressure in fracture, \( \frac{\text{psia}}{\text{cp}} \)
\( \bar{m}(p_m) \) average pseudo-pressure in matrix, \( \text{psia}^2 \text{cp} \)

\( m(p_f) \) pseudo-pressure in fracture, \( \text{psia}^2 \text{cp} \)

\( \bar{m}(p_f) \) volume-averaged pseudo-pressure in fracture, \( \text{psia}^2 \text{cp} \)

\( m(p_i) \) initial pseudo-pressure in matrix, \( \text{psia}^2 \text{cp} \)

\( m(p_m) \) pseudo-pressure in matrix, \( \text{psia}^2 \text{cp} \)

\( \bar{m}(p_m) \) volume-averaged pseudo-pressure in matrix, \( \text{psia}^2 \text{cp} \)

\( m(p_{wf}) \) pseudo-pressure corresponding to flowing bottom-hole pressure, \( \text{psia}^2 \text{cp} \)

\( MW \) molecular weight of gas, lbm/lbmol

\( n_r \) number of gridblocks, dimensionless

\( N \) \[
\frac{2\theta \rho_{sc} T_s k_f h}{50294 \phi_m V_{res} \rho_{sc} T (\log_r e - \frac{1}{4} + \frac{s}{4}) \mu g_i c_i}
\]

\( O \) \[
\frac{k_f h}{25147 \rho_{sc} (\log_r e - \frac{1}{4} + \frac{s}{4}) \mu g_i c_i}
\]

\( OGIP \) original gas in place, Mscf

\( p \) pressure, psia

\( \bar{p} \) average pressure, psia

\( p_{Df} \) dimensionless fracture pressure, dimensionless

\( p_{Dm} \) dimensionless matrix pressure, dimensionless

\( p_f \) fracture pressure, psia

\( \bar{p}_f \) average fracture pressure, psia

\( p_i \) initial pressure, psia

\( \bar{p}_m \) average matrix pressure, psia

\( p_m \) matrix pressure, psia

\( p_{sc} \) pressure at standard conditions, psia

\( p_{wf} \) wellbore pressure, psia

\( q_D \) dimensionless flow rate definition 1, dimensionless

\( \bar{q}_D \) dimensionless flow rate in Laplace space, dimensionless

\( q_D^{gas} \) dimensionless gas flow rate, dimensionless

\( q_D^{liq} \) dimensionless liquid flow rate, dimensionless

\( q_f \) net gas production from fracture system to well, c[\text{f/s}]
\( q_{fsc} \)  net flow rate out of the fracture system at standard condition, scf/s

\( q_{gsc} \)  gas flow rate at standard conditions, Mcf/D

\( q_{adi} \)  initial decline rate for density-based dual-porosity model, Mcf/D

\( q_{adi}^e \)  initial decline rate for density-based dual-porosity model under full potential drawdown, Mcf/D

\( q_m \)  gas production from matrix to fracture system, cf/s

\( q_{msc} \)  total flow rate out of the matrix system at standard condition, scf/s

\( q_{sc} \)  flow rate at standard conditions, Mcf/D

\( r \)  radius, ft

\( r_D \)  dimensionless radius, dimensionless

\( r_e \)  external radius, ft

\( r_{eD} \)  dimensionless radius at reservoir radius, dimensionless

\( r_w \)  wellbore radius, ft

\( r_p \)  wellbore-to-initial density ratio, dimensionless

\( r_p' \)  modified wellbore-to-initial density ratio, dimensionless

\( R \)  molar gas constant, 10.73 \( \text{psia-ft}^3\text{lbmol}^\circ\text{R} \)

\( s \)  skin factor or Laplace variable, dimensionless.

\( S_{wi} \)  connate water saturation, dimensionless

\( SG \)  specific gravity, dimensionless

\( t \)  time, days

\( t_a \)  normalized pseudo-time, days

\( t_D \)  dimensionless time, dimensionless

\( T \)  temperature, \( ^\circ\text{R} \)

\( T_{sc} \)  temperature at standard conditions, \( ^\circ\text{R} \)

\( u \)  \( \rho_m r \cdot \text{lb-f}t \)

\( u_{mr} \)  fluid flow rate in matrix, ft/s

\( u_r \)  fluid flow rate in porous media, ft/s

\( v \)  volume, Mcf

\( V_m \)  volume of gas in matrix system, Mcf

\( V_{res} \)  reservoir volume, Mcf

\( W_a \)  rate of mass accumulation in the control volume, lb/s

\( \chi \)  \( \frac{\lambda_{wf}B_D t}{(1+K)} \)
\[ x = \frac{q^{g_i}}{OGI P}, \text{ 1/day} \]

\[ y = \left( \frac{1+K}{\lambda_{wf}q_{gsc}} \right)^{\frac{B}{1+B}} \]

\[ Y = q^{e} r_p, \text{ Mscf/D} \]

\[ Z \] compressibility factor, dimensionless

\[ Z_f \] compressibility factor of fluid in fracture system, dimensionless

\[ Z_i \] compressibility factor at initial reservoir conditions, dimensionless

\[ Z_m \] compressibility factor of fluid in matrix system, dimensionless

**Greek**

\[ \alpha \] shape factor, 1/ft\(^2\)

\[ \beta \] time-averaged \( \lambda \), dimensionless

\[ \bar{\beta} \] time-averaged \( \bar{\lambda} \), dimensionless

\[ \bar{\beta}_d \] time-averaged \( \bar{\lambda}_d \), dimensionless

\[ \beta_m \] time-averaged \( \lambda_m \), dimensionless

\[ \beta_{m}^{*} \] time-averaged \( \lambda_{m}^{*} \), dimensionless

\[ \gamma \] Euler’s constant, 0.5772156649

\[ \Delta t \] time increment, s

\[ \theta \] \( \frac{RT}{MW} \) or angle of control volume, rad

\[ \lambda \] viscosity-compressibility dimensionless ratio, dimensionless

\[ \bar{\lambda} \] space-averaged viscosity-compressibility ratio for dual-porosity system, dimensionless

\[ \bar{\lambda}_d \] modified space-averaged viscosity-compressibility ratio for dual-porosity system, dimensionless

\[ \lambda_f^{*} \] viscosity-compressibility ratio for fracture fluid, dimensionless

\[ \bar{\lambda}_m \] average viscosity-compressibility dimensionless ratio between average matrix pressure and flowing bottom-hole pressure, dimensionless

\[ \lambda_m^{*} \] viscosity-compressibility ratio for matrix fluid, dimensionless

\[ \lambda_{m_f}^{*} \] average viscosity-compressibility dimensionless ratio between average matrix pressure and fracture pressure, dimensionless

\[ \lambda_{wf} \] average viscosity-compressibility dimensionless ratio between initial reservoir pressure and flowing bottom-hole pressure, dimensionless

\[ \mu \] liquid viscosity, cp

\[ \mu_{gf} \] fracture gas viscosity, cp
\( \bar{\mu}_{gf} \)  
average fracture gas viscosity, cp

\( \mu_{gi} \)  
initial gas viscosity, cp

\( \mu_{gm} \)  
fracture gas viscosity, cp

\( \bar{\mu}_{gm} \)  
fracture gas viscosity, cp

\( \mu_{gw} \)  
average fluid viscosity-compressibility between initial reservoir condition and bottom-hole condition, cp/psi

\( \xi \)  
Interporosity flow coefficient, dimensionless

\( \bar{\rho} \)  
average density of reservoir fluid, lb/cf

\( \bar{\rho}_{Df} \)  
dimensionless density in Laplace space, dimensionless

\( \rho_f \)  
fracture fluid density, lb/cf

\( \bar{\rho}_f \)  
Average fracture fluid density, lb/cf

\( \rho_m \)  
matrix fluid density, lb/cf

\( \bar{\rho}_m \)  
average matrix fluid density, lb/cf

\( \rho_{sc} \)  
density of fluid at standard condition, lb/scf

\( \rho_{wf} \)  
density at wellbore condition, lb/cf

\( \phi_m \)  
matrix porosity, dimensionless

\( \phi_f \)  
fracture porosity, dimensionless

\( \omega \)  
storativity ratio, dimensionless

\[
\bar{\omega} = \frac{\phi_f (\bar{\rho}_g + c_2)}{\phi_m (1-S_{wf}) (\bar{\rho}_g + c_1) + \phi_f (\bar{\rho}_g + c_2)}
\]

Subscript

1  
matrix

2  
fracture

avg  
average

f  
fracture

in  
flow into the element

m  
matrix

out  
flow out of the element

sc  
standard condition

source  
source term

Superscript

av  
average
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INTRODUCTION

Naturally fractured reservoirs traps significant amount of oil and gas. The naturally fractured reservoirs are characterized by matrix and fractures that are continuous in many cases. Exploration on the naturally fractured reservoirs has been practiced for decades, and the interest in them from industry has been growing since the shale boom started. The behavior and performance of the naturally fractured systems have been extensively studied. (Barenblatt et al., 1960; Warren and Root, 1963; Kazemi, 1969; Kazemi et al., 1969; Crawford et al. 1976; De Swaan, 1976; Boulton and Streltsova, 1977; Mavor and Cinco-Ley, 1979; Da Prat et al., 1981; Serra, 1981; Moench, 1984; Barker, 1985; Chen, 1989; Gatens et al. 1989; Zimmerman et al., 1993; Lim and Aziz, 1995; Spivey and Semmelbeck, 1995; Sobbi and Badakhshan, 1996; Rodriguez-Roman and Camacho, 2005; Gerami et al., 2007; Ranjbar and Hassanzadeh, 2011; Sureshjani et al., 2012) However, forecasting the performance and original hydrocarbon in naturally fractured reservoirs remains a major challenge. In this study, we advocate the dual-porosity model and the implementation of a pseudo-pressure-based interporosity flow equation to modeling the gas flow between the matrix and fracture properly. The density-based decline curve analysis is investigated and rigorously derived. This study also reveals the rescaling relationship between the liquid analytical solutions and the gas numerical responses of dual-porosity systems. Pseudo functions successfully linearize the governing equations for dual-porosity gas systems, provided that the pseudo-pressure-based interporosity flow equation is applied to model the fluid transfer between the matrix and the fracture. Moreover, we derive a double-exponential model for the convenient computation of the liquid production behavior at constant bottomhole pressure. What’s more, a density-based decline curve analysis method is proposed for naturally fractured liquid system producing at variable pressure drawdown/variable rate.

Chapter 1 presents the rigorous derivation of the proposed interporosity flow equation, which captures the effects of the pressure-dependent properties. An in-house simulator, DPS 2, is then built based on the dual-porosity model with the proposed interporosity flow equation. We then investigate the applicability of rescaling approach, density-based decline curve analysis and pseudo-function-based approach for constant-bottomhole-pressure decline curve analysis.

Chapter 2 rigorously derives the density-based decline curve analysis and the rescaling approach for the production at constant bottomhole pressure in the dual-porosity systems. The derivation is based on a diffusivity equation and the proposed pseudo-function-based interporosity flow equation. An exponential decline for dual-porosity liquid system is derived at the second decline stage using Warren and Root model (1963). Chapter 2 also discovers the relationship between average fracture pseudo-pressure and average matrix pseudo-pressure. Moreover, a gas rate equation relating production rate and average matrix pseudo-pressure is derived from the diffusivity equation.
Chapter 3 proposes a density-based rescaled exponential model for variable pressure drawdown/variable rate production in dual-porosity gas systems. The density-based rescaled exponential model corroborates the decline curve analysis approach. The straight-line analysis is then implemented at variable pressure drawdown/variable rate production. We also propose a density-based exponential model for variable pressure drawdown/variable rate in dual-porosity liquid systems. Following the procedure of derivation for its gas counterpart, we then developed a decline curve analysis approach to facilitate convenient decline curve analysis. Moreover, Chapter 3 proposes a double-exponential model for the convenient computation of the liquid production behavior at constant bottomhole pressure.
Chapter 1
Decline Curve Analysis with Pseudo-Pressure-Based Interporosity Flow Equation for Naturally Fractured Gas Reservoirs

1.1 Chapter Summary

Naturally fracture reservoirs reserve great amounts of hydrocarbon, which is of significant economic interest. Dual-porosity model is commonly used to modeling naturally fractured reservoirs. Such models assumes uniform matrix blocks that are segregated by continuous fracture network. However, Previous dual-porosity models have been inadequate to fully capture pressure-dependent effects in gas systems.

This study derives a gas interporosity flow equation rigorously, which accounts for effects of pressure-sensitive properties. Furthermore, we develops an in-house simulator that applies the gas interporosity flow equation. The simulator’s simulation results demonstrate significant difference than that from CMG-IMEX, which assumes a liquid-form interporosity flow equation. We advocate the derived interporosity flow equation for gas since it captures the viscosity-compressibility change from the matrix to the fracture.

State-of-the-art decline curve analysis has not proven accurate as many comparisons between analytical results and production data demonstrated discrepancy in the previous works. In this study, we showed that the gas interporosity flow equation eliminated the discrepancy at decline stage and enable rigorous decline curve analysis. The applicability of density-based approach in dual-porosity gas systems is also investigated. This approach reveals the equality of gas production profile and the rescaled liquid solution using depletion-driven parameter, \( \lambda \) and \( \beta \). Application of this approach demonstrated that, at the late decline stage, gas production profile shifted from its liquid counterpart is identical to gas numerical responses with gas interporosity flow equation in effects. This study demonstrates the applicability of the density-based method and the pseudo-functions in decline curve analysis for dual-porosity systems.

1.2 Introduction

Naturally fractured reservoirs are widely distributed around the world. Considerable amount of natural gas reservoirs, both conventional and unconventional, are naturally fractured. As a result of shale boom, naturally fractured reservoirs are supplying increasing amount of oil and gas to the U.S. market. The natural fractures result from various reasons such as tectonic movement, lithostatic pressure
changes, thermal stress, and high fluid pressure. The fractures are either connected or discrete. Good interconnectivity between fractures yields fracture network dividing matrix into individual blocks, which is found in many reservoirs. Fluid flow in fractures is treated as Darcy flow. The fractures have large flow capacity but small storage capacity. On the contrary, matrix is characterized by small flow capacity but large storage capacity. In such a system, flow throughout the reservoir occurs in fracture system, and matrix blocks act as source of fluids.

Barenblatt et al. (1960) first proposed dual-porosity model for liquid flow in naturally fractured reservoirs. Warren and Root (1963) applied Barenblatt’s et al. (1960) ideas into well testing with pseudo-steady state interporosity flow equation presented as follows:

$$\frac{\partial (\phi_m \rho_m)}{\partial t} = \rho_m \frac{\alpha k_m}{\mu} (p_f - p_m)$$

Equation 1-1

where $\phi_m$ is matrix porosity, $\rho_m$ is fluid density in matrix, $\alpha$ is shape factor, $k_m$ is matrix permeability, $\mu$ is liquid viscosity, $p_m$ is matrix fluid pressure, and $p_f$ is fracture fluid pressure.

$\alpha$ is a constant in Warren and Root model but differs with the matrix blocks’ shape. Zimmerman et al. (1993) demonstrated the rigorous derivation of Equation 1-1 and shape factor for slab-like matrix block. Lim and Aziz (1995) used the same approach to generate shape factors for different matrix shapes.

Equation 1-1 assumes liquid, namely constant viscosity and constant compressibility, in its development. With drastic pressure change as fluids flow from matrix to fracture, it is thereupon inadequate for modeling interporosity gas flow in naturally fractured gas system. Rigorous interporosity flow equation for gas needs to be in place for reliable production data analysis in such systems. Though Equation 1-1 fails to incorporate viscosity-compressibility change of gas, it is still applied. For example, CMG-IMEX utilizes the equation for liquid with viscosity and compressibility evaluated at matrix pressure when modeling dual-porosity gas systems.

Azom and Javadpour (2012) used a modified pseudo-pressure approach and obtained an adequate matrix-fracture shape factor for interporosity gas flow. They presented a two-dimensional implicit dual-continuum reservoir simulator for naturally fractured reservoirs with single-phase compositional setting. However, implementing the model requires numerical simulation. Sureshjani et al. (2012) derived explicit rate-time solution of single-phase interporosity gas flow assuming quasi-steady state flow for dual-porosity system. In the derivation, they approximated pseudo-time to time when integrating outflow from matrix block and, moreover, $\frac{\mu_c}{\mu} \approx \frac{P_i}{Z_i}$ is assumed. Ranjbar and Hassanzadeh (2011) developed semi-analytical solutions for nonlinear diffusion equation in gas bearing reservoir before back-calculating matrix-fracture shape factor with the developed solution. However,
the solutions contain two unknown parameters determined by matching data generated by numerical simulator for corresponding matrix and fluid type.

Incorporation of the aforementioned interporosity equation for gas in decline curve analysis hasn’t been investigated thoroughly. State-of-the-art methodologies of decline curve analysis for naturally fracture gas reservoirs has been using the liquid-form interporosity flow equation for development or validation purposes. Spivey and Semmelbeck (1995) combined transient radial model, adjusted pressure, and desorption term together and developed a production-prediction method for shale gas and dewatered coal seams producing at constant bottom-hole pressure. Adjusted pseudo-time and adjusted pseudo-pressure were used instead of real time and real pressure in the analytical solution for Warren and Root model. This approach produces error less than 10% when $\xi r_{ed} \geq 1$ with a slab-like dual-porosity model. This study didn’t specify the details on the interporosity flow equation implemented in the simulator. In addition, the direct substitution of pseudo-pressure and pseudo-time into the liquid analytical solution is not supported by the governing equations.

Gerami et al. (2007) applied pseudo-time and pseudo-pressure to dual-porosity reservoirs, and, without derivation, they proposed a pseudo-pressure-based interporosity flow equation for gas. However, the verification doesn’t adopt a simulator with the gas interporosity flow equation. The error increases with production when he compared the results obtained semi-analytically with those from CMG-IMEX.

In this study, a pseudo-steady state interporosity flow equation for single-phase gas is rigorously derived. Application of the new model is found to enable pseudo-functions-based decline curve analysis in dual-porosity gas systems.

For the case of single-porosity systems, Ye and Ayala (2012; 2013), and Ayala and Ye (2012; 2013) proposed a density-based approach for decline curve analysis. With depletion-driven dimensionless variables $\lambda$ and $\beta$, Ye and Ayala (2012) was able to rescale dimensionless gas rate solution under constant bottom-hole pressure from their liquid counterparts, which thereupon facilitates the decline curve analysis based on density. Zhang and Ayala (2014a) provided rigorous derivation for the density-based approach and improved the methods for analyzing data at variable pressure drawdown/rate at decline stage (Ayala and Zhang, 2013; Zhang and Ayala, 2014b). In our study, the applicability of the density-based approach is investigated and a match is found between density-based prediction and gas numerical responses with the application of gas interporosity flow equation.
1.3 Pseudo-Steady State Interporosity Flow Equation for Gas

The interporosity flow equation in Barenblatt et al. (1960) and Warren and Root (1963) is proposed for pseudo-steady state liquid flow from matrix blocks to fracture system. Starting from physical basis, Zimmerman et al. (1993) derived this interporosity flow equation for liquid with spherical matrix shape. The development procedure assumes the quasi-steady-state approximation, which treats fracture pressure on the outer boundary, $p_f$, as constant throughout the derivation. However, the interporosity flow equation for liquid can be proven inadequate for gas flow. Since gas compressibility and viscosity are pressure-dependent, the gas flow out of the matrix gridlock experiences change in pressure-dependent properties and demonstrates different behavior from liquid flow. This difference could be drastic with large contrast between fracture pressure and matrix pressure.

In this study, we develop an interporosity equation for gas with quasi-steady-state assumption. Spherical matrix block is assumed throughout the derivation, as illustrated in Figure 1-1.

\[ -\nabla \cdot (\rho_m u_m) = \frac{\partial (\phi_m \rho_m)}{\partial t} \]

Equation 1-2

where $u_m$ is Darcy velocity of fluid flow.

Substituting Darcy’s law into Equation 1-2 gives:

\[ \nabla \cdot (\rho_m \frac{k_m}{\mu_{gm}} \nabla p_m) = \frac{\partial (\phi_m \rho_m)}{\partial t} \]
Assuming incompressible matrix rock, multiplying both sides by $\theta$, adding term $\mu_g c_g / (\mu_g c_g)$ on the RHS and substituting $dm(p_m) = 2\theta dp_m / (\mu_g c_g)$ gives:

\[
\nabla \cdot (k_m \nabla m(p_m)) = \phi_m \mu_g c_g \frac{\partial m(p_m)}{\partial t}
\]

Taking $k_m$ out of the divergence term, dividing both sides by $\phi_m \mu_g c_g$ and substituting $\lambda^*_m = \mu_{gi} c_{gi} / (\mu_g c_g)$ into Equation 1-4 gives:

\[
\frac{\partial m(p_m)}{\lambda_m^* \partial t} = \frac{k_m}{\phi_m \mu_{gi} c_{gi}} \nabla^2 m(p_m)
\]

Denote $\tilde{\lambda}_m^*$ as the $\lambda_m^*$ evaluated average pressure in the matrix block and substituting $\beta^*_m = \int \tilde{\lambda}_m^* dt / t$ into Equation 1-5 gives:

\[
\frac{\partial m(p_m)}{\partial (\beta^*_m t)} = \frac{k_m}{\phi_m \mu_{gi} c_{gi}} \nabla^2 m(p_m)
\]

where $\beta^*_m t$ is equivalent to normalized pseudo-time. For gas reservoirs, the average reservoir pressure is utilized to evaluate pseudo-time, which has been proven to work well during boundary dominated flow. Expanding Equation 1-6 in spherical coordinates and taking $u(r, t) = m(p_m) r$ gives:

\[
\frac{\partial u}{\partial (\beta^*_m t)} = \frac{k_m}{\phi_m \mu_{gi} c_{gi}} \frac{\partial^2 u}{\partial r^2}
\]

We take spherical matrix shape with radius $a_m$. With the fracture surrounding the matrix, the pressure at matrix surface is the same as fracture pressure. The boundary conditions are written as follows:

\[
\begin{align*}
\quad u(0, \beta^*_m t) &= 0 \\
\quad u(a_m, \beta^*_m t) &= a_m m(p_f) \\
\quad u(r, 0) &= rm(p_i)
\end{align*}
\]

Solving Equations 1-8 to 1-10 for $m(p_m)$ distribution and calculating the average pseudo-pressure gives (Crank, 1975):
\[
\frac{m_{avg}(p_m) - m(p_i)}{m(p_f) - m(p_i)} = 1 - \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{\pi^2 k_m n^2 \beta_m^* t}{\mu_{gi} c_{gi} \phi_m a_m^2} \right)
\]

Equation 1-11

where \(m_{avg}(p_m)\) is average pseudo-pressure throughout the matrix block. Long-term approximation truncates to the first term of the infinite series, giving:

\[
\frac{m_{avg}(p_m) - m(p_i)}{m(p_f) - m(p_i)} = 1 - \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m \beta_m^* t}{\mu_{gi} c_{gi} \phi_m a_m^2} \right)
\]

Equation 1-12

Lim and Aziz (1995) validated the long-term approximation in their derivation for liquid system. This approximation is accurate for \(\pi^2 k_m \beta_m^* t/\mu_{gi} c_{gi} \phi_m a_m^2 > 0.5\) as shown in Figure 8 of their work. For a wide variety of cases, the approximation is valid at decline stage. Moving terms in Equation 1-12 gives:

\[
\frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m \beta_m^* t}{\mu_{gi} c_{gi} \phi_m a_m^2} \right) = \frac{m(p_f) - m_{avg}(p_m)}{m(p_f) - m(p_i)}
\]

Equation 1-13

Taking derivative of Equation 1-13 with respect to \(\beta_m^* t\) gives:

\[
\frac{1}{m(p_f) - m(p_i)} \frac{d}{d(\beta_m^* t)} \left( m_{avg}(p_m) \right) = \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m \beta_m^* t}{\mu_{gi} c_{gi} \phi_m a_m^2} \right) \frac{\pi^2 k_m}{\mu_{gi} c_{gi} \phi_m a_m^2}
\]

Equation 1-14

Writing \(d\left( m_{avg}(p_m) \right)/d(\beta_m^* t)\) in Equation 1-14 as \(\frac{d(m_{avg}(p_m))}{dt} \frac{dt}{d(\beta_m^* t)}\) and substituting \(d(\beta_m^* t) = \bar{\lambda}_m^* dt\) gives into the resulting equation gives:

\[
\frac{1}{m(p_f) - m(p_i)} \frac{d}{dt} \left( m_{avg}(p_m) \right) = \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m \beta_m^* t}{\mu_{gi} c_{gi} \phi_m a_m^2} \right) \frac{\pi^2 k_m}{\mu_{gi} c_{gi} \phi_m a_m^2} \bar{\lambda}_m^*
\]

Equation 1-15

Substituting Equation 1-13 into Equation 1-15 gives:

\[
\frac{d(m_{avg}(p_m))}{\bar{\lambda}_m^* dt} = \frac{\pi^2 k_m}{\mu_{gi} c_{gi} \phi_m a_m^2} \left( m(p_f) - m_{avg}(p_m) \right)
\]

Equation 1-16

\(\frac{\pi^2}{a_m^2}\) is a constant known as shape factor, \(\alpha\), that changes with the geometry of matrix. Moreover, replacing average pseudo-pressure in matrix volume, \(m_{avg}(p_m)\), with point-specific matrix pseudo-pressure and substituting \(\bar{\lambda}_m^*\) with \(\lambda_m^*\) since matrix is point-specific as represented by the governing equations gives:
\[ \phi_m \frac{d(m(p_m))}{\lambda_m^* dt} = \frac{2\mu_m}{\mu_{gi} c_{gi}} (m(p_f) - m(p_m)) \]

Equation 1-17

Application of definition of \( \lambda_m^* \) and \( d(m(p_m)) = 2\theta d\rho_m/\mu_g m c_{gm} \) into Equation 1-17 gives:

\[ \phi_m \frac{d\rho_m}{dt} = \frac{2\mu_m}{\mu_{gi} c_{gi}} (m(p_f) - m(p_m)) \]

Equation 1-18

This interporosity flow equation is rigorously derived for gas incorporating viscosity-compressibility effect. An important characteristic of this model is the same shape factor to that in Lim and Aziz (1995) for Warren and Root model. For slab-like matrix, shape factor is \( \pi^2/4L^2 \) where \( L \) denotes fracture half spacing. Equation 1-18 is in the same form as interporosity flow equation written by Gerami \textit{et al.} (2007) without derivation\footnote{Per personal communication with Dr. Pooladi-Darvish where he indicated that they wrote it in analogy with the liquid formulation.} if we consider incompressible matrix and fracture and no connate water. Sureshjani \textit{et al.} (2012) proposed the same interporosity flow equation in a different form utilizing two approximations in a different derivation:

\[ \frac{\mu_{gi} c_{gi}}{\mu_g c_g} \approx \frac{p/Z}{(p/Z)_i} \]

Equation 1-19

\[ t \approx t_a \]

Equation 1-20

In this study, we prove that the pseudo-steady state interporosity flow equation is valid without invoking the approximations. Sureshjani \textit{et al.} (2012) built a fine grid single-porosity numerical simulator to model flow between matrix and fracture with slab shape. Both matrix and fracture are represented by fine grid blocks. The shape factor is back-calculated and compared against \( \pi^2/4L^2 \) demonstrating great match at decline stage. The results of the comparison validate the accuracy of the pseudo-steady state interporosity flow equation for gas. The back calculation is rewritten as follows:

\[ \alpha = \frac{2\theta \phi_m}{k_m (m(p_f) - m(p_m))} \frac{d\rho_m}{dt} \]

Equation 1-21

Substituting \( dm(p_m) = 2\theta d\rho_m/\mu_g m c_{gm} \) into Equation 1-18 and canceling \( 2\theta \) gives:

\[ \phi_m \frac{d\rho_m}{dt} = \alpha k_m \left( \int_0^{p_f} \frac{1}{\mu_{gf} c_{gf}} dp_f - \int_0^{p_m} \frac{1}{\mu_g m c_{gm}} dp_m \right) \]

Equation 1-22
With constant viscosity and compressibility, Equation 1-22 could collapse to interporosity flow equation in Warren and Root model, which is developed for liquid. Replacing $\mu_{gf} c_{gf}$ and $\mu_{gm} c_{gm}$ with constant $\mu c$ in Equation 1-22 gives:

$$
\phi_m \frac{d \rho_m}{dt} = \frac{\alpha k_m}{\mu c} (\rho_f - \rho_m)
$$

Equation 1-23

Liquid systems have close $\rho_f$ and $\rho_m$ due to small compressibility. Thus, by substituting $p_f - p_m = \ln(\rho_f/\rho_m)/c_t$ and $\ln(\rho_f/\rho_m) \approx (\rho_f - \rho_m)/\rho_m$ into Equation 1-22 gives interporosity flow equation in Warren and Root model:

$$
\frac{\phi_m}{\rho_m} \frac{1}{\rho_m} \frac{d \rho_m}{dt} = \frac{\alpha k_m}{\mu} (p_f - p_m)
$$

Equation 1-24

The biggest obstacle for using Warren and Root model in gas Scenarios is the difference between viscosity and compressibility in fracture system and matrix system for gas. The derived interporosity flow equation for gas incorporated effects of pressure-dependent properties by invoking pseudo-functions. The development embraces pseudo-steady state interporosity flow and long-term approximation of series that requires $\pi^2 k_m \beta^2_m t / \mu_{gi} c_{gi} \phi_m a^2_m > 0.5$ for spherical matrix block. When applied to infinite acting period in matrix, this model could also be a reasonable approximation.

1.4 Effects of Interporosity Flow Equation on Production Behavior

We investigated the difference in production behavior brought by different interporosity flow models on production from the comparisons later. A simulator with the proposed interporosity flow equation is in need for validation of the developed tools in the previous section but no such simulator is available. CMG-IMEX is using the liquid-form interporosity flow equation as in Warren and Root model, in which the viscosity and compressibility are evaluated at matrix gas pressure. The model can be written as:

$$
\frac{\partial (\phi_m \rho_m)}{\partial t} = \rho_m \frac{\alpha k_m}{\mu_{gm}} (p_f - p_m)
$$

Equation 1-25

An in-house dual-porosity reservoir simulator, Dual-Porosity Simulator 2 (DPS 2), is established for modeling dual-porosity gas reservoir with the pseudo-steady state interporosity flow equation for gas. The buildup procedure is adjusted from that in Abou-Kassem et al. (2006) that is provided for single-porosity system. A circular reservoir is considered with a well fully penetrated with no skin at the center. The reservoir is homogeneous and isotropic. Logarithmic discretization is taken owing to its
radial nature. Equation discretization is implicit, and simple-iteration method (SIM) acts as pressure advancing algorithm. Viscosity is calculated with method by Lee et al. (1966). The Abou-Kassem et al. (1990) is used for determining compressibility, and compressibility factor calculation follows Dranchuk and Abou-Kassem (1975).

We illustrate the significant impact of different interporosity flow model by comparing the gas numerical responses from DPS 2 and that from CMG-IMEX as they assume gas interporosity flow equation and liquid-form interporosity flow equation respectively. An imaginary case is established as described in Table 1-1 and Table 1-2. Specific gravity of natural gas, \( r_g \), is 0.55. Matrix porosity and fracture porosity are taken as 0.15 and 0.01 respectively. Permeability in matrix and fracture are changed to 0.005 md and 50 md respectively to guarantee evident dual-porosity behavior. Shape factor is assumed \( 9.98959 \times 10^{-05} \) \( 1/\text{ft}^2 \). A summary of relevant properties are provided in Table 1-1. Three Scenarios with different reservoir sizes are used for generating production data and are presented in Table 1-2.

**Table 1-1. Reservoir and Fluid Properties**

<table>
<thead>
<tr>
<th>Properties</th>
<th>Units</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrix permeability, ( k_m )</td>
<td>md</td>
<td>0.005</td>
</tr>
<tr>
<td>Fracture permeability, ( k_f )</td>
<td>md</td>
<td>50</td>
</tr>
<tr>
<td>Matrix porosity, ( \phi_m )</td>
<td></td>
<td>0.15</td>
</tr>
<tr>
<td>Fracture porosity, ( \phi_f )</td>
<td></td>
<td>0.01</td>
</tr>
<tr>
<td>Pay zone thickness, ( h )</td>
<td>ft</td>
<td>300</td>
</tr>
<tr>
<td>Gas specific gravity, SG (air=1)</td>
<td></td>
<td>0.55</td>
</tr>
<tr>
<td>Wellbore radius, ( r_w )</td>
<td>ft</td>
<td>0.25</td>
</tr>
<tr>
<td>Initial pressure, ( p_i )</td>
<td>psia</td>
<td>5000</td>
</tr>
<tr>
<td>Initial temperature, ( T )</td>
<td>°F</td>
<td>200</td>
</tr>
<tr>
<td>Specified wellbore flowing pressure, ( p_{wf} )</td>
<td>psia</td>
<td>100</td>
</tr>
<tr>
<td>Shape factor, ( \alpha )</td>
<td>( 1/\text{ft}^2 )</td>
<td>( 9.98959 \times 10^{-05} )</td>
</tr>
<tr>
<td>Storativity ratio, ( \omega )</td>
<td></td>
<td>0.0625</td>
</tr>
<tr>
<td>Interporosity flow coefficient, ( \xi )</td>
<td></td>
<td>( 6.2435 \times 10^{-10} )</td>
</tr>
</tbody>
</table>
Table 1-2. Reservoir Size and OGIP for Three Scenarios

<table>
<thead>
<tr>
<th>Scenario 1:</th>
<th>Units</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir outer radius, ( r_e )</td>
<td>ft</td>
<td>175</td>
</tr>
<tr>
<td>Dimensionless outer radius, ( r_{eD} )</td>
<td></td>
<td>700</td>
</tr>
<tr>
<td>Drainage area</td>
<td>acres</td>
<td>2.21</td>
</tr>
<tr>
<td>OGIP</td>
<td>Bscf</td>
<td>1.199</td>
</tr>
<tr>
<td>Scenario 2:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reservoir outer radius, ( r_e )</td>
<td>ft</td>
<td>350</td>
</tr>
<tr>
<td>Dimensionless outer radius, ( r_{eD} )</td>
<td></td>
<td>1400</td>
</tr>
<tr>
<td>Drainage area</td>
<td>acres</td>
<td>8.84</td>
</tr>
<tr>
<td>OGIP</td>
<td>Bscf</td>
<td>4.796</td>
</tr>
<tr>
<td>Scenario 3:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reservoir outer radius, ( r_e )</td>
<td>ft</td>
<td>700</td>
</tr>
<tr>
<td>Dimensionless outer radius, ( r_{eD} )</td>
<td></td>
<td>2800</td>
</tr>
<tr>
<td>Drainage area</td>
<td>acres</td>
<td>35.34</td>
</tr>
<tr>
<td>OGIP</td>
<td>Bscf</td>
<td>19.184</td>
</tr>
</tbody>
</table>

Figure 1-2 shows, for instance, Scenario 1’s gas numerical responses in terms of \( q_{sc} \) vs. \( t \) from DPS 2 under the consideration in Table 1-1 and Table 1-2. Comparatively, Figure 1-2 presents \( q_{sc} \) vs. \( t \) in Scenario 1 from CMG which assumes liquid-form interporosity flow equation rather than the gas interporosity flow equation. Similar to liquid system’s behavior, the flow rate shows a rapid decline in dual-porosity systems at first, and then enters a second decline. As is pointed out by Moench (1984), in the first decline stage, the production is primarily from fracture storage, and matrix storage doesn’t contribute primarily to production until the end of this stage. Figure 1-2 clearly demonstrates this characteristic as we observe no difference between production rates of two systems with different interporosity flow models.

Figures 1-3 to 1-5 plot gas numerical responses in terms of \( q_{sc} \) vs. \( t \) from DPS 2 and CMG in Scenarios 1, 2 and 3 as described in Table 1-1 and Table 1-2 respectively. They shows, more evidently, the difference in \( q_{sc} \) vs. \( t \) with different interporosity flow equations by looking into the second decline stages in the three Scenarios. Blue curve represents results from CMG-IMEX, while dashed red curve corresponds to production data from DPS 2.

The comparisons shown in Figures 1-3 to 1-5 demonstrated DPS 2, which employs gas interporosity flow equation and CMG-IMEX, which uses liquid-form interporosity flow equation, show very
different production behaviors in the same Scenarios. The major difference between our gas interporosity flow equation and liquid-form equation in Warren and Root’s model is that we have considered the viscosity-compressibility change as gas flow through matrix to fracture. Behaviors of the two systems are similar when interporosity flow is not significant in the contribution to production. With the depletion of the fracture system, interporosity flow becomes dominant in contribution to gas production rate and consideration of viscosity-compressibility change from matrix to fracture in interporosity flow equation brings a major change. The relative difference between production rates from the two systems begins at the end of the first decline stage with production rate from CMG-IMEX larger than the other. The difference increases and then decreases before the two production rates meet each other early in the second decline stage. Late in the second decline stage, production rate from DPS 2 surpasses the other, and the relative difference increases along the time line. Notably, the gas production rates from DPS 2 could be nearly twice those from CMG-IMEX throughout the late stage. The interporosity flow equation in CMG-IMEX assumes constant viscosity and compressibility for fluids in its derivation, which misrepresents gas flow from matrix blocks to fracture. The gas interporosity flow equation is derived rigorously from physical basis and provide a better representation of the gas flow between matrix and fracture systems. Using the liquid-form interporosity flow equation in Warren and Root model can lead to significant error and distort the production behavior. Thus, using liquid-form interporosity flow equation for dual-porosity gas systems is not recommended.

Figure 1-2. Comparison of Production Behaviors between CMG-IMEX and DPS 2 for Scenario 1
Figure 1-3. Comparison of Production Behaviors between CMG-IMEX and DPS 2 at the Second Decline Stage for Scenario 1

Figure 1-4. Comparison of Production Behaviors between CMG-IMEX and DPS 2 at the Second Decline Stage for Scenario 2
**Figure 1-5.** Comparison of Production Behaviors between CMG-IMEX and DPS 2 at the Second Decline Stage for Scenario 3

### 1.5 Observations on the Applicability of Pseudo-Functions

Spivey and Semmelbeck’s (1995) and Gerami et al.’s (2007) studied the applicability of pseudo-pressure and pseudo-time in dual-porosity system but results show deviations, extreme in certain cases, between proposed methods and gas numerical responses. Spivey and Semmelbeck (1995) used the liquid-form interporosity flow equation, and Gerami et al. (2007) proposed the gas interporosity flow equation but didn’t implement it in the numerical model. We reexamine the applicability of pseudo-pressure and pseudo-time with DPS 2 which invokes the gas interporosity flow equation. Three scenarios as described by Table 1 and Table 2 are used for validation purpose. Figure 1-6 shows the gas numerical response in terms of $q_D = q_D(r_D, t_D)$ for three scenarios as described by Table 1-1 and Table 1-2. Figure 1-6 also presents the comparisons between such numerical gas responses and the gas response with Gerami et al.’s (2007) method, both in dimensionless term. $\bar{\omega}$ is a constant in this case and no iteration for analytical solution is needed.
Figure 1-6. Dimensionless Gas Rate from Simulation vs. Dimensionless Gas Rate Generated with Gerami et al.’s (2007) Approach.

Figure 1-6 demonstrates that pseudo-time and pseudo-pressure could effectively linearize the governing equations of dual-porosity gas systems, especially in the second decline stage during the production life. Therefore, applying pseudo-time and pseudo-pressure, the liquid analytical solution to dual-porosity system could accurately describe the behavior of its gas counterpart at second decline stage. Particularly, this linearization requires gas interporosity flow equation, Equation 17, to be employed for modeling gas flow between matrix and fracture. Taking into account comparison between responses of DPS 2 and CMG-IMEX reveals that deviation between gas analytical responses and gas numerical responses from CMG-IMEX in long-term-is because of assuming liquid-form interporosity equation in simulation. The pseudo-time evaluated at average reservoir pressure successfully linearizes the governing equations provided that the proposed interporosity gas flow equation is applied to model fluid transfer between matrix and fracture. Provided incompressible matrix and fracture, production data analysis with the concept of pseudo-functions could predict long-term gas production rate and OGIP, as illustrated in Gerami et al. (2007).
1.6 Observation on Rescaling Approach

Ye and Ayala (2012; 2013), and Ayala and Ye (2012; 2013) proposed a density-based approach to analyze unsteady state flow of single-porosity gas reservoirs. Using depletion-driven dimensionless variables $\lambda$ and $\beta$, they successfully decouple pressure-dependent effects from pressure depletion. Ye and Ayala (2012) showed that dimensionless gas rate solution under constant bottom-hole pressure can be rescaled from their liquid counterparts with depletion-driven dimensionless variables $\bar{\lambda}$ and $\bar{\beta}$. Zhang and Ayala (2014a) provided rigorous derivation for the rescaling approach. The relation is written by Zhang and Ayala (2014a) as follows:

$$q^\text{gas}_D(t_D) = \bar{\lambda} \cdot q^\text{liq}_D(\bar{\beta} t_D)$$

**Equation 1-26**

where $q^\text{gas}_D$ is dimensionless gas flow rate, $q^\text{liq}_D$ is the liquid counterpart, $\bar{\lambda}$ and $\bar{\beta}$ are depletion-driven dimensionless variables defined as follows:

$$\bar{\lambda} = \frac{\mu_{gi} c_{gi}}{2 \theta (\bar{\rho} - \rho_{wf})} \frac{\overline{m}(p)}{m(p_{wf})}$$

**Equation 1-27**

where $\bar{\rho}$ is average reservoir gas density, $\rho_{wf}$ is gas density at bottom-hole condition, $\overline{m}(p)$ is average pseudo-pressure of reservoir fluids, $m(p_{wf})$ is pseudo-pressure of gas at bottom-hole condition, $\mu_{gi}$ and $c_{gi}$ are initial gas viscosity and initial gas compressibility respectively, $\theta = RT/MW$, $T$ is temperature, $MW$ is molecular weight.

$$\bar{\beta} = \int_0^t \bar{\lambda} \frac{dt}{t}$$

**Equation 1-28**

$\bar{\rho}$ could be obtained from material balance equation assuming tank model for the reservoir, $\overline{m}(p)$ is evaluated at pressure corresponding to $\bar{\rho}$. It was demonstrated that $\bar{\lambda}$ and $\bar{\beta}$ are able to capture the effects of pressure-sensitive properties on single-porosity systems’ behaviors.

When it comes to dual-porosity reservoir, two systems-fracture system and matrix system overlap with each other at the same place, and two systems are communicated through interporosity flow described by interporosity flow equation. The behavior of production rate at constant bottom-hole pressure in dual-porosity systems is different from that in single-porosity systems. At the end of the first decline stage, fluids originally in the fracture system are relatively depleted compared to the matrix system. Therefore, with decreasing fracture fluid pressure, interporosity flow develops and becomes dominant in the second decline stage, namely, flow out of matrix into fracture has dominant contribution in gas
production. The matrix blocks is thereupon treated as the only storage sites at the second decline stage and is the representative pressure for evaluating pressure-dependent effects. It is then a reasonable guess that $\bar{\lambda}$ and $\bar{\beta}$ could capture the response of dual-porosity system in the second decline stage since the pressure-sensitive effects are controlled by matrix pressure only. In this perspective, the $\bar{\lambda}$ and $\bar{\beta}$ for dual-porosity systems is written into $\bar{\lambda}_m$ and $\bar{\beta}_m$ since they are evaluated at average matrix pressure, which is complicated to achieve in practices. Considering the fact that matrix fluids account for vast majority of reservoir fluids, the fracture pressure’s influence on the average reservoir pressure would be negligible. Thus, simple material balance equation for single-porosity system would predict average pressure for, from which $\bar{\lambda}_m$ and $\bar{\beta}_m$ is then calculated.

The proposed rescaling approach is validated in a variety of scenarios which exhibit strong dual-porosity behaviors. First we test the rescaling approach against three scenarios described by Table 1-1 and Table 1-2. DPS 2 is used to generate production data. The rate-time production data are transformed into dimensionless form and compared against rescaled dimensionless gas production rate from liquid analytical solution.

The dimensionless flow rate produced at constant bottom-hole pressure in a bounded circular reservoir in Laplace space is given as follows (Da Prat, 1981):

$$q_D = \frac{\sqrt{s f(s)}(l_1(\sqrt{s f(s)}r_{eD})K_1(\sqrt{s f(s)}) - K_1(\sqrt{s f(s)}r_{eD})l_1(\sqrt{s f(s)}))}{s(l_0(\sqrt{s f(s)}K_1(\sqrt{s f(s)}r_{eD}) + K_0(\sqrt{s f(s)})l_1(\sqrt{s f(s)}r_{eD}))}$$

**Equation 1-29**

By numerical inversion such as Stehfest algorithm (Stehfest, 1970) from Laplace space to real space, Dimensionless liquid flow rate is obtained. Similar to its counterpart in single-porosity system (Ye and Ayala, 2012), the definition of $q_D^{gas}$ is:

$$q_D^{gas} = \frac{\rho_{sc} \mu_i c_{gi} q_{gsc}}{2\pi k_f h (\rho_i - \rho_{wf})}$$

**Equation 1-30**

where $\rho_{sc}$ is gas density at standard condition and $k_f$ is fracture permeability.

Figure 1-7 presents the comparison between rescaled production rate and dimensionless production rate with data generated by DPS 2. Dashed curve and dotted curve are dimensionless liquid rate and dimensionless gas rate rescaled from it respectively while the solid curve represents production data from the DPS 2. Figure 1-7 presents the well-known constant-pressure liquid solutions of dual-porosity system in terms of $q_D = q_D(r_D, t_D)$ with Equation 1-29 for the three different reservoir sizes under consideration in Table 1-1 and Table 1-2. Figure 1-7 also presents the comparisons between such liquid analytical responses and the gas numerical responses under the same consideration as described in Table 1-1 and Table 1-2, both expressed in dimensionless terms. Comparatively, Figure 1-8 shows
rescaled $q_{D}^{gas}$ and gas numerical responses DPS 2 for the three scenarios under consideration in Table 1-1 and Table 1-2, both expressed in dimensionless forms. Dashed curve and dotted curve are dimensionless dual-porosity liquid rate and rescaled dimensionless gas rate respectively while the solid curve represents production data from DPS 2.

Figure 1-7 and Figure 1-8 demonstrate that liquid analytical solution for dual-porosity reservoir could be used to predict the long-term behavior of corresponding dual-porosity gas reservoir accurately by rescaling from dual-porosity liquid solution using $\lambda_m$ and $\beta_m$. What’s more, it is observed that $\lambda$ and $\beta$ rescaling is able to capture the gas production behavior at the very beginning.

Great match between rescaled $q_{D}^{gas}$ and $q_{D}^{gas}$ from DPS 2 leads us to the conclusion that depletion-driven dimensionless variables $\lambda$ and $\beta$ could successfully decouple pressure-dependent effects from pressure depletion for dual-porosity system at second decline stage. The liquid analytical solution for dual-porosity system as described by Equation 1-29 could be readily used to accurately predict the corresponding natural gas reservoir’s analytical responses by transforming the liquid traces based on the depletion-driven dimensionless parameters $\lambda_m$ and $\beta_m$. This method circumvents the complexity involved in Gerami et al’s (2007) method. This implies the applicability of production data analysis developed previously by Zhang and Ayala (2014b) based on density.

![Graph](image-url)

**Figure 1-7.** Density-Based Approximation vs. Liquid Analytical Solution for Scenarios 1, 2, and 3
1.7 Observation on Density-Based Straight-Line Analysis

Zhang and Ayala (2014a, 2014b) and Ayala and Zhang (2013) have shown that single-porosity gas reservoirs can be analyzed with rescaled straight-line analysis for prediction of OGIP. The governing equation for the straight-line analysis is:

\[
\frac{\bar{\lambda}}{q_{gsc}} \frac{r_p}{G_p} = \frac{1}{OGIP} \frac{\bar{\lambda}}{q_{gsc}} + \frac{1}{q_{gi}^e}
\]

Equation 1-31

where \(r_p = 1 - \frac{\rho_{wf}}{\rho_i}\), \(G_p\) is cumulative gas production, \(q_{gi}^e\) is a constant defined as follows:

\[
q_{gi}^e = \frac{2\pi \rho_i kh}{b_{D, PSS} \phi_{gi} \mu_{gi} \epsilon_{gi}}
\]

Equation 1-32

The method originates from a gas rate equation essential for rigorous proof of \(\lambda\) and \(\beta\) rescaling. Therefore, it is reasonable guess that dual-porosity systems satisfy a similar equation at the second decline stage as well, in which \(\tilde{\lambda}_m\) replaces \(\tilde{\lambda}\).
Same steps for OGIP prediction to that for single-porosity system utilizing Equation 1-31 are taken except for replacing $\bar{\lambda}$ by $\bar{\lambda}_m$. Steps are detailed in Zhang and Ayala (2014b). Data of Scenarios 1 to 3 is used for validating the OGIP prediction methods for dual-porosity reservoirs. With derived OGIP by plotting $G_p/q_{gsc}$ vs. $r_p/q_{gsc}$, we plot $\bar{\lambda}_m G_p/q_{gsc}$ vs. $\bar{\lambda}_m r_p/q_{gsc}$ and obtain the best-fit straight line through the points in late-decline stage, which reveals the gradient $1/\text{OGIP}$ hence OGIP. If the difference is large, we could replot $\bar{\lambda}_m G_p/q_{gsc}$ vs. $\bar{\lambda}_m r_p/q_{gsc}$ with new OGIP. The resulting plot after 4 rounds of OGIP prediction is shown in Figures 1-9 to 1-11 corresponding to six scenarios. For Scenario 1, the fitted straight lines for the late decline stage yields slopes of $8.2214 \times 10^{-7}$ Mscf$^{-1}$, which corresponds to a OGIP estimation of 1.2162 Bscf with a relative error of 1.430% with respect to actual $G_i$. The slope of straight line for Scenarios 2 and 3 are $2.0567 \times 10^{-7}$ Mscf$^{-1}$ and $5.1415 \times 10^{-8}$ Mscf$^{-1}$ respectively, yielding OGIP estimations of 4.8621 Bscf and 19.458 Bscf with relative error of 1.380% and 1.384% respectively. The intercepts for Figures 1-9 1-10 and 1-11 are $1.4101 \times 10^{-6} \left(\frac{\text{Mscf}}{D}\right)^{-1}$, $4.96 \times 10^{-7} \left(\frac{\text{Mscf}}{D}\right)^{-1}$ and $2.7862 \times 10^{-7} \left(\frac{\text{Mscf}}{D}\right)^{-1}$ respectively.

![Figure 1-9. $\bar{\lambda}_m G_p/q_{gsc}$ vs. $\bar{\lambda}_m r_p/q_{gsc}$ Straight-Line Analysis for Scenario 1](image-url)
Figure 1-10. $\frac{\bar{\lambda}_m G_p}{q_{gsc}}$ vs. $\frac{\bar{\lambda}_m r_p}{q_{gsc}}$ Straight-Line Analysis for Scenario 2

Figure 1-11. $\frac{\bar{\lambda}_m G_p}{q_{gsc}}$ vs. $\frac{\bar{\lambda}_m r_p}{q_{gsc}}$ Straight-Line Analysis for Scenario 3
A notable feature of Figures 1-11 to 1-13 is how readily the production data at late stage fall along a straight line. Also we observe negligible deviation between derived OGIP and actual OGIP in three scenarios after 4 rounds of iterations. It implies that the gas production from dual-porosity system at the second decline stage follows:

$$\bar{\lambda}_m \frac{r_p}{q_{gsc}} = \frac{1}{OGIP} \bar{\lambda}_m \frac{G_p}{q_{gsc}} + \frac{1}{q_{gi}^e}$$

Equation 1-33

where $q_{g_i}^e$ is constant. $q_{g_i}$ is not used because theoretical $q_{g_i}$ are different from the inverse of intercepts which could be due to fitting error. Significantly, this correlation could be readily used for predicting OGIP of dual-porosity systems without any calculation of pseudo-pressure or pseudo-time variables. Moreover, with OGIP predicted and $q_{g_i}^e$ known from the intercept, production rate at the second decline stage is then predicted based on Equation 1-33 following the procedure:

1. Knowing $\bar{\lambda}_m G_p/q_{gsc}$ at last time step, increase $\bar{\lambda}_m G_p/q_{gsc}$ by a small value and calculate corresponding $\bar{\lambda}_m r_p/q_{gsc}$.
2. Calculate $r_p/G_p$ with $\bar{\lambda}_m r_p/q_{gsc}/(\bar{\lambda}_m G_p/q_{gsc})$ and then calculate $G_p$ knowing $r_p \cdot \frac{\bar{p}}{\bar{Z}}$ can be obtained with material balance equation:

$$\frac{\bar{p}}{\bar{Z}} = \frac{p_i}{Z_i} (1 - \frac{G_p}{OGIP})$$

Equation 1-34

3. Calculate pressure corresponding to $\bar{p}/\bar{Z}$ at each time point by interpolating $\bar{p}/\bar{Z}$ in the generated $\bar{p}/\bar{Z}$ vs. $\bar{p}$ table. Then we calculate $\bar{\lambda}_m$ and then $q_{gsc}$ knowing $\bar{\lambda}_m r_p/q_{gsc}$ and $r_p$.
4. Calculate time using

$$(t)^i = (t)^{i-1} + 2 \frac{(G_p)^i - (G_p)^{i-1}}{(q_{gsc})^i + (q_{gsc})^{i-1}}$$

Equation 1-35

where $i$ denotes time level.
5. Repeat steps 1 to 4.
1.8 Concluding Remarks

Starting from physical basis, this study derived a pseudo-steady state interporosity flow equation for single-phase gas. This model incorporates viscosity-compressibility change as fluids flow from matrix to fracture compared to its counterpart in Warren and Root model that holds liquid assumptions. We showed that the liquid-form interporosity flow model (Warren and Root, 1963) is a special case of the derived interporosity flow equation for gas when liquid assumptions hold true. Comparisons between production behaviors from two simulators with the gas interporosity flow equation and liquid-form one respectively demonstrate large difference at the second decline stage, which points out the necessity to use the gas interporosity flow equation for naturally fractured gas reservoirs as it encompasses pressure-dependent effects of gas. The derived interporosity flow equation for gas is a reasonable approximation even when matrix block is at infinite acting stage.

The application of pseudo-pressure and pseudo-time for decline curve analysis with the simulator invoking the gas interporosity flow equation is tested to be a success. Applying the pseudo-functions to the liquid analytical solution captures the behavior of its gas counterpart at the second decline stage. Important to realize, this linearization is contingent on gas interporosity flow equation. A reason why, in long term, the verification for constant pressure production demonstrated deviation in Gerami et al. (2007) is because of using data generated by CMG-IMEX, which applies liquid-form interporosity flow equation.

Investigation on density-based approach found it applicable to dual-porosity systems, provided that gas interporosity flow equation is implemented. Shifting the liquid traces with the depletion-driven dimensionless parameters $\tilde{\lambda}_m$ and $\tilde{\beta}_m$, the liquid analytical solution could rigorously predict the responses of natural gas reservoirs at the second decline stage. Notably, $\tilde{\lambda}_m$ and $\tilde{\beta}_m$, two depletion-driven parameters, are simply dependent on average pressure and bottomhole pressure and do not require pseudo-time, making this method easier to implement than the pseudo-functions-based approach. Furthermore, Density-based decline curve analysis, which was derived for single-porosity system, accurately predicts OGIP of dual-porosity systems. Our results presented success in extending density-based method and pseudo-functions-based linearization to dual-porosity gas systems. The reason is not completely clear yet. Further investigation is required to draw more rigorous conclusion. Nevertheless, our studies highlight the role of gas interporosity flow equation. Finally, the success on pseudo-function linearization and density-based decline approach could have important implications in decline curve analysis in naturally fractured gas reservoirs.
Chapter 2

Constant-Bottomhole-Pressure Decline Curve Analysis of Dual-Porosity Gas Systems Using a Density-Based Approach

2.1 Chapter Summary

The development of naturally fractured gas reservoirs often requires the deployment of rigorous techniques for production data analysis incorporating dual-porosity gas behavior. In dual-porosity gas systems, fluids in the two overlapping continua may be found at different pressures throughout the system, thereby leading to markedly different viscosity and compressibility values in the matrix and fractures, respectively. As a result, it is difficult to linearize and analytically solve the corresponding flow equations. Recent studies have shown that, with the application of a pseudo-pressure-based interporosity flow equation, a density-based approach may be able to accurately predict the gas flow rate and estimate the amount of original gas in place (OGIP) for these systems. The methodology can also accurately predict the gas production rate by transforming its liquid counterpart response via a decoupling of the pressure-dependent effects using dimensionless depletion-driven parameters.

This study further rigorously derives the density-based decline curve analysis and rescaling relationship in dual-porosity gas systems using the diffusivity equation and the interporosity flow equation. The derivation yields a deliverability equation for dual-porosity systems. Results show that the density-based approach is able to successfully capture the dual-porosity behavior of gas. It is also able to estimate OGIP and predict production performance.
2.2 Introduction

Naturally fractured reservoirs are heterogeneous in nature. Many of them comprise discrete volumes of matrix rock separated by fractures. The fractures disrupt the matrix blocks and form a continuous network. Fractures are highly permeable and occupy a small percentage of total reservoir volume. The matrix, on the other hand, is of low permeability and has a large storage capacity. Because of high fracture permeability, most of the fluid flow within a reservoir occurs in the fracture network, while the majority of the fluid is stored in the reservoir’s matrix. Figure 2-1 from Barenblatt et al. (1960) illustrates this structure.

![Figure 2-1: Schematic Representation of Fissured Rock (Barenblatt et al., 1960)](image)

Studies of naturally fractured reservoirs have been conducted for decades. This research has proven increasingly important because of the shift from conventional reservoirs to unconventional reservoirs. Barenblatt et al. (1960) proposed a dual-porosity model for liquid, using an interporosity flow equation and a diffusivity equation for the fracture system. The fracture system is treated as a continuous porous medium, while the matrix system is treated as a source of hydrocarbon that provides flow into the fracture system. Both systems are homogenized and communicate with each other. Warren and Root (1963) used Barenblatt’s et al. (1960) ideas for well test analysis of oil reservoirs. They assumed a systematic array of identical, rectangular parallelepiped representing matrix blocks throughout an anisotropic reservoir. Warren and Root (1963) analytically solved their governing equations for constant-rate production, examined pressure-buildup performance, and obtained asymptotic solutions. Techniques for analyzing drawdown and build-up data were then proposed. They pointed out that the storativity ratio and the interporosity flow coefficient are controlling parameters in dual-porosity behavior. Of the two controlling parameters, the storativity ratio, $\omega$, is a measure of the fluid capacitance of the fracture system, and the interporosity flow coefficient, $\xi$, represents the degree of
heterogeneity in the dual-porosity system. The equations for describing the dual-porosity reservoir for homogeneous and isotropic reservoirs following Warren and Root’s model in dimensionless form are written as follows:

\[
\frac{\partial^2 p_{fD}}{\partial r_D^2} + \frac{1}{r_D} \frac{\partial p_{fD}}{\partial r_D} = (1 - \omega) \frac{\partial p_{mD}}{\partial t_D} + \omega \frac{\partial p_{fD}}{\partial t_D}
\]

**Equation 2-1**

\[
(1 - \omega) \frac{\partial p_{mD}}{\partial t_D} = \xi (p_{fD} - p_{mD})
\]

**Equation 2-2**

where \( \omega = \frac{\phi_f c_f}{\phi_f c_f + \phi_m c_m} ; \quad \xi = \alpha \frac{k_m}{k_f} r_w^2 ; \quad p_{fD} = \frac{k_f h (p_i - p_f)}{q B_o \mu} \), dimensionless fracture pressure; \( p_{Dm} = \frac{k_f h (p_i - p_m)}{q B_o \mu} \), dimensionless matrix pressure; \( r_D = \frac{r}{r_w} \), dimensionless radius; \( t_D = \frac{k_f t}{(\phi_f c_f + \phi_m c_m) \mu r_w^2} \), dimensionless time; \( c_f \) is fracture compressibility plus liquid compressibility; \( c_m \) is matrix compressibility plus liquid compressibility; \( k_f \) is fracture permeability; \( k_m \) is matrix permeability; \( \alpha \) is the shape factor; \( \phi_m \) is matrix porosity; and \( \phi_f \) is fracture porosity. Note that \( \xi \) is used instead of \( \lambda \) to avoid confusion with \( \lambda \) in the \( \lambda \) and \( \beta \) rescaling approach. Equation 2-2 is the pseudo-steady state interporosity flow equation in dimensionless form, as used in the Warren and Root model proposed for liquid. Zimmerman et al. (1993) showed the derivation of the dimensional form of Equation 2-2 assuming quasi-steady state flow. Crawford et al. (1976) analyzed pressure buildup data from known naturally fractured reservoirs and found that the Warren and Root model can successfully describe the behavior of these reservoirs. Da Prat et al. (1981) derived a constant bottom-hole solution in a closed, circular dual-porosity system using the Warren and Root model. Moench (1984) proposed a transient interporosity flow model incorporating the skin effect at the interface between matrix and fracture. The skin may result from underground water circulating in groundwater and geothermal reservoirs. Moench found that the pseudo-steady state interporosity flow applies even in early stages when the skin effect is strong enough.

On the gas side, efforts have been made to linearize these governing equations using the concepts of pseudo-pressure and pseudo-time. However, for gas dual-porosity systems, fluids in the two overlapping continua may be found at different pressures at any given location, leading to markedly different gas properties. Resulting flow equations thus become difficult to linearize and solve analytically. As a result, traditional production decline analysis with pseudo-functions has yet to be proven fully successful.

Spivey and Semmelbeck (1995) used pseudo-time and pseudo-pressure to forecast the long-term gas production in shale gas and dewatered coal seams. Pseudo-time accounts for desorption effects with
modified compressibility. This may predict the long-term gas production for a wide range of parameters. However, it is not applicable when there is a small $\xi r^2_{eD}$ or drawdown. Moreover, the applicability of this approach has not been rigorously proven. Gerami et al. (2007) used pseudo-functions to study dual-porosity systems only. Pseudo-time is defined so as to incorporate water saturations and compressibilities. They presented a new interporosity flow equation for gas without derivation. The equation they presented is similar to the equation for liquid but with a changing storativity ratio, $\bar{\omega}$, given as a function of pressure, which implies both nonlinearity and an unsolvable nature. To tackle this problem, Gerami et al. (2007) first solved the equations assuming a constant $\bar{\omega}$. The authors assumed that the same analytical result may be used for gas if $\bar{\omega}$ is updated at each time step. However, this error increases with the CMG-IMEX results as production goes on.

Recently Ye and Ayala (2012; 2013) and Ayala and Ye (2012; 2013) have proposed a $\lambda$ and $\beta$ density-based approach to analyzing unsteady state flow under constant bottom-hole pressure. Zhang and Ayala (2014a) rigorously derived $\lambda$ and $\beta$ and extended them to the variable pressure drawdown/variable rate systems in later work (Ayala and Zhang, 2013; Zhang and Ayala, 2014a; Zhang and Ayala, 2014b). This density-based approach mostly circumvents the concepts of pseudo-functions and enables straightforward OGIP prediction and gas well performance forecasting. In this study, we extend the density-based approach to dual-porosity systems and show that the behavior of constant bottom-hole pressure dual-porosity systems is able to be captured by this extended approach at late stages. Moreover, we utilize a new interporosity flow equation for single-phase gas for decline-curve purposes. Based on this equation, a gas deliverability equation for dual-porosity system is rigorously derived.

### 2.3 Interporosity Flow Equation

The interporosity flow equation employed in Barenblatt et al. (1960) and Warren and Root (1963) has been proposed for a pseudo-steady state liquid flow from the matrix blocks to the fracture system. Starting from a physical basis, Zimmerman et al. (1993) derived this interporosity flow equation for liquid with spherical matrix blocks. The development procedure assumed the “quasi-steady state” approximation, which treats fracture pressure on the outer boundary, $p_f$, as constant throughout the derivation.

Owing to the pressure-dependent nature of compressibility and viscosity for gas, the gas flow out of the matrix gridlock behaves differently than the liquid flow. Thus, it is not adequate to use the interporosity flow equation for liquid. Instead, a gas interporosity flow equation is necessary to provide a better description of the gas flow between the matrix blocks and the fracture system. Following
Zimmerman’s et al. (1993) approach and incorporating the compressibility-viscosity effects over time, we are able to symbolically derive an interporosity flow equation for gas from a physical basis as follows:

\[ \phi_m \frac{d \rho_m}{dt} = \frac{\alpha k_m}{2\theta} (m(p_f) - m(p_m)) \]

**Equation 2-3**

where \( \rho_m \) is matrix fluid density, \( \theta = RT/MW \), and \( m(p_f) \) and \( m(p_m) \) are fracture pseudo-pressure and matrix pseudo-pressure, respectively. Appendix B shows the development of Equation 2-3 in detail. In the development process, an incompressible matrix was assumed for simplicity. We consider no desorption of slippage effects and focus instead on dual-porosity gas behavior. Equation 2-3 is in the same form as the interporosity flow equation written by Gerami et al. (2007) without derivation in the case of an incompressible matrix and fractures and no connate water. Equation 2-1 takes into account the compressibility-viscosity effects for gas flow out of the matrix blocks. By substituting the definition of pseudo-pressure, we can rewrite Equation 2-1 as:

\[ \phi_m \frac{d \rho_m}{dt} = \alpha k_m \left( \int_0^{p_f} \frac{1}{\mu_{gf} c_{gf}} d\rho_f - \int_0^{p_m} \frac{1}{\mu_{gm} c_{gm}} d\rho_m \right) \]

**Equation 2-4**

In the case of liquid, Equation 2-4 collapses to

\[ \phi_m \frac{d \rho_m}{dt} = \frac{\alpha k_m}{\mu c} (\rho_f - \rho_m) \]

**Equation 2-5**

Equation 2-5 is a different form of liquid-form interporosity flow equation in Warren and Root model. With assumptions in Warren and Root model, Equation 2-5 is rigorously derived in Appendix A. Substituting \( p_f - p_m = \frac{1}{c_l} \ln\left( \frac{\rho_f}{\rho_m} \right) \) and \( \ln\left( \frac{\rho_f}{\rho_m} \right) \approx \frac{\rho_f}{\rho_m} - 1 \) into Equation 2-5 gives:

\[ \phi_m \frac{1}{\rho_m} \frac{\partial \rho_m}{\partial t} = \frac{\alpha k_m}{\mu} (p_f - p_m) \]

**Equation 2-6**

Equation 2-6 demonstrates that the gas interporosity flow equation collapses to the liquid interporosity flow equation in the Warren and Root model. Owing to different levels of pressure in the fracture system and the matrix system in the presence of gas, the viscosity-compressibility terms are different for the gas system, unlike in the liquid system. The interporosity flow equation written for liquid in the Warren and Root model is unable to capture the viscosity-compressibility effects and hence unable to accurately capture the behavior of gas.
2.4 Deliverability Equation for Dual-Porosity System

The deliverability test plays an important role in determining the production capacity of a given reservoir. The deliverability equation for conventional single-porosity reservoirs has been well established. With proper manipulation of the interporosity flow equation, we are able to derive the interporosity flow equation for a dual-porosity system, which is then able to be utilized for the deliverability test.

Given a circular reservoir, the diffusivity equation in the fracture system can be written as:

$$-\frac{1}{r} \frac{\partial}{\partial r} (r \rho_f u_r) = \frac{\partial (\phi_f \rho_f)}{\partial t} + \frac{\partial (\phi_m \rho_m)}{\partial t}$$

Equation 2-7

Assuming Darcy flow in the fracture system, substituting Darcy’s law and

$$d \left( m(p_f) \right) = 2 \theta \frac{1}{\mu_f e_B} d \rho_f$$

into Equation 2-7 gives:

$$\frac{MW k_f}{2RT} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = \frac{\partial (\phi_f \rho_f)}{\partial t} + \frac{\partial (\phi_m \rho_m)}{\partial t}$$

Equation 2-8

Multiply both sides in Equation 2-8 by reservoir volume, $V_{res}$, to yield:

$$V_{res} \frac{MW k_f}{2RT} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = V_{res} \frac{\partial (\phi_f \rho_f + \phi_m \rho_m)}{\partial t}$$

Equation 2-9

For boundary-dominated flow, we propose a valid approximation. In other words, we propose that density change rates with respect to time are similar throughout the reservoir, and that they thus are able to be represented by the average density change rate in the system:

$$V_{res} \frac{\partial (\phi_f \rho_f + \phi_m \rho_m)}{\partial t} \approx V_{res} \frac{d(\phi_f \bar{\rho}_f + \phi_m \bar{\rho}_m)}{dt} = -q_{gsc} \rho_{sc}$$

Equation 2-10

where $\bar{\rho}_f$ and $\bar{\rho}_m$ represent average fracture fluid density and average matrix fluid density, respectively.

Substituting Equation 2-10 into Equation 2-9 gives:

$$V_{res} \frac{MW k_f}{2RT} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = -q_{gsc} \rho_{sc}$$

Equation 2-11

Substituting $V_{res} = \pi r_e^2 h$, $\rho_{sc} = p_{sc} MW / RT_{sc}$ into Equation 2-11 gives:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = -\frac{2p_{sc} T}{\pi r_e^2 h k_f T_{sc}} q_{gsc}$$
Equation 2-12

Equation 2-12 yields the gas rate equation for a dual-porosity reservoir using the techniques provided in Dake (1978). Integrating both sides of Equation 2-12 with respect to radius twice and eliminating constants after integration with respect to the closed boundary condition gives:

\[ m(p_f) - m(p_{wf}) = \left( \frac{p_{sc}q_{gsc}}{\pi h k_f T_{sc}} \right) \left[ \ln \left( \frac{r}{r_w} \right) - \frac{r^2}{2r_e^2} \right] \]

Equation 2-13

Following Dake (1978), we calculate volume-averaged \( m(p_f) \) from Equation 2-13 and extend the resulting equation to a reservoir with an arbitrary reservoir shape following Dake (1978). Writing the result in customary units gives:

\[ q_{gsc} = \frac{T_{sc} k_f h}{50294 p_{sc} T_{b_{D, PSS}}} \left( \bar{m}(p_f) - m(p_{wf}) \right) \]

Equation 2-14

where \( b_{D, PSS} \) is the pseudo-steady state component and \( \bar{m}(p_f) \) is the volume-averaged fracture pseudo-pressure. Equation 2-14 correlates the gas production rate with the average fracture pseudo-pressure. However, it is hard to quantify the average fracture pseudo-pressure from production data or reservoir parameters collected from monitors. For this reason, a technique transferring the average fracture pseudo-pressure to predictable average reservoir pseudo-pressure is developed in Appendix D. Applying the technique yields correlation among average fracture pseudo-pressure, average matrix pseudo-pressure, and bottom-hole pseudo-pressure.

Equation D-15 gives:

\[ \bar{m}(p_f) = \frac{1}{1 + K} \bar{m}(p_m) + \frac{K}{1 + K} m(p_{wf}) \]

Equation 2-15

where \( K = \frac{0.0062841 k_f h}{ak_m n_{res} b_{D, PSS}} \). Then we can write Equation 2-14 in terms of \( \bar{m}(p_m) \) instead of \( \bar{m}(p_f) \) by substituting Equation 2-15 into it:

\[ q_{gsc} = \frac{T_{sc} k_f h}{50294 p_{sc} T b_{D, PSS}(1 + K) \left( \bar{m}(p_m) - m(p_{wf}) \right) \left( \bar{m}(p_m) - m(p_{wf}) \right)} \]

Equation 2-16

In the long-term, the average matrix pressure \( \bar{p}_m \) corresponding to \( \bar{m}(p_m) \) is able to be approximated using the average pressure \( \bar{p} \) from the material balance equation that treats the reservoir as a tank for conventional reservoirs. This approximation is based on the physical fact that the majority of production in a naturally fractured reservoir originates in the matrix as fluids are depleted in the fracture system. This results in a greater interporosity flow. Replacing \( \bar{m}(p_m) \) with \( \bar{m}(p) \) gives:
\[ q_{gsc} = \frac{T_{sc} k_f h}{50294 \phi_m V_{res} P_{sc} T b_{D,PSS}(1 + K)} (\bar{m}(p) - m(p_{wf})) \]

Equation 2-17

where \( \bar{m}(p) \) is pseudo-pressure corresponding to average pressure from the material balance equation with the tank model.

### 2.5 \( \lambda \) and \( \beta \) Rescaling for Dual-Porosity Systems

Ye and Ayala (2012; 2013), and Ayala and Ye (2012; 2013) proposed a density-based approach to analyzing the unsteady state flow of single-porosity gas reservoirs. Using depletion-driven dimensionless variables \( \lambda \) and \( \beta \), they successfully decoupled pressure-dependent effects from depletion. Ye and Ayala (2012) were able to show that the dimensionless gas rate solution under constant flowing bottom-hole pressure can be rescaled from its liquid counterpart using the depletion-driven dimensionless variables \( \lambda \) and \( \beta \). Zhang and Ayala (2014a) subsequently rigorously derived \( \lambda \) and \( \beta \). The rescaling relationship and \( \lambda \) and \( \beta \) as defined by Zhang and Ayala (2014a) are as follows:

\[ q_D^{gas}(t_D) = \bar{\lambda} \cdot q_D^{liq-s}(\beta t_D) \]

Equation 2-18

\[ \bar{\lambda} = \frac{\mu g_i c_{gi}}{2\theta(\bar{\rho} - \rho_{wf})} \frac{m(p) - m(p_{wf})}{\bar{m}(p) - m(p_{wf})} \]

Equation 2-19

\[ \bar{\beta} = \frac{\int_0^t \lambda \, dt}{t} \]

Equation 2-20

where \( q_D^{gas} \) is the predicted analytical gas flow rate response at constant bottom-hole pressure in a single-porosity system with a closed boundary, and \( q_D^{liq-s} \) is the analytical response of the liquid single-porosity system with a closed boundary at constant bottom-hole pressure. For a single-porosity system, \( \lambda \) and \( \beta \) have been proven to successfully capture the effects of pressure-dependent properties on the system response. With Equation 2-16, \( \lambda \) and \( \beta \) rescaling is found to be an intrinsic characteristic of dual-porosity systems.

Substituting Equation D-4 into Equation 2-16 and rearranging the terms assuming incompressible matrix gives:

\[ \frac{d(\bar{\rho}_m)}{dt} = -\frac{\rho_{sc} T_{sc} k_f h}{50294 \phi_m V_{res} P_{sc} T b_{D,PSS}(1 + K)} (\bar{m}(p_m) - m(p_{wf})) \]
Define \( \bar{\lambda}_m \) as follows:

\[
\bar{\lambda}_m = \frac{\mu_{gi} c_{gi}}{2 \theta (\bar{\rho}_m - \rho_{wf})}/m(\bar{\rho}_m) - m(p_{wf})
\]

Equation 2-22

For long-term behavior, the fracture pressure’s fraction in the average reservoir pressure is minimal considering that the matrix stores the majority of the remaining gas. \( \bar{\lambda}_m \) is equivalent to \( \bar{\lambda} \) and evaluated at average pressure, \( \bar{p} \), as obtained from the material balance equation for conventional reservoirs.

Deriving the representation of \( m(\bar{\rho}_m) - m(p_{wf}) \) in terms of \( \bar{\lambda}_m \) from Equation 2-22 gives:

\[
m(\bar{\rho}_m) - m(p_{wf}) = \frac{2 \theta (\bar{\rho}_m - \rho_{wf}) \bar{\lambda}_m}{\mu_{gi} c_{gi}}
\]

Equation 2-23

Substituting Equation 2-23 into Equation 2-21 and rearranging the terms gives:

\[
d\frac{\bar{\rho}_m}{\bar{\rho}_m - \rho_{wf}} = \frac{\theta \rho_{sc} T_{sc} k h}{25147 \phi_m V_{res} p_{sc} T b_{D, PSS} \mu_{gi} c_{gi}} (1 + K) \bar{\lambda}_m(\bar{\rho}_m - \rho_{wf}) dt
\]

Equation 2-24

Take the notation:

\[
N = \frac{\theta \rho_{sc} T_{sc} k_f h}{25147 \phi_m V_{res} p_{sc} T b_{D, PSS} \mu_{gi} c_{gi}}
\]

Equation 2-25

Substituting the notation into Equation 2-24 and multiplying both sides by \( dt/(\bar{\rho}_m - \rho_{wf}) \) gives:

\[
\frac{d(\bar{\rho}_m - \rho_{wf})}{(\bar{\rho}_m - \rho_{wf})} = -\frac{N}{1 + K} \bar{\lambda}_m dt
\]

Equation 2-26

where \((-\rho_{wf})\) is added in the differential term because of constant bottom-hole pressure. Integrating Equation 2-26 and substituting \( \bar{\lambda}_m = \frac{d(\bar{\rho}_m t)}{dt} \) gives:

\[
\int_{\bar{\rho}_m}^{\bar{\rho}_0} d\left(\frac{\bar{\rho}_m - \rho_{wf}}{\bar{\rho}_0 - \rho_{wf}}\right) = \frac{-N}{1 + K} \int_{0}^{t} d\bar{\rho}_m t
\]

Equation 2-27

Finishing the integration in Equation 2-27 gives:

\[
\ln \left(\frac{\bar{\rho}_m - \rho_{wf}}{\bar{\rho}_0 - \rho_{wf}}\right) = \frac{-N}{1 + K} \bar{\rho}_m t
\]

Equation 2-28
Taking the exponentials of both sides and transforming the resulting equations gives:

\[
(\bar{\rho}_m - \rho_{wf}) = (\rho_i - \rho_{wf}) \exp \left( \frac{-N}{1 + K} \bar{\rho}_m t \right)
\]

Equation 2-29

Substituting Equation 2-23 and Equation 2-29 into Equation 2-16 gives:

\[
q_{sc} = -\frac{2\theta T_{sc} k_f h}{50294 p_{sc} T b_{D,PSS} \mu_{gi} c_{gi}} \frac{\bar{\lambda}_m}{1 + K} (\rho_i - \rho_{wf}) \exp \left( \frac{-N}{1 + K} \bar{\rho}_m t \right)
\]

Equation 2-30

Introduce the definitions of \( q_{D}^{gas} \) and \( t_D \) as follows:

\[
q_{D}^{gas} = \frac{158021 \rho_{sc} \mu_{gi} c_{gi} q_{gsc}}{2\pi k_f h (\rho_i - \rho_{wf})}
\]

Equation 2-31

\[
t_D = \frac{0.0063283 k_f t}{\mu_{gi} c_{gi} (\phi_m + \phi_f) r_w^2}
\]

Equation 2-32

Substituting Equation 2-25, Equation 2-31, and Equation 2-32 into Equation 2-30 in sequence gives:

\[
q_{D}^{gas} = \frac{1}{b_{D,PSS}} \cdot \frac{\lambda_m}{1 + K} \exp \left( - \frac{1}{b_{D,PSS}} \frac{2\pi r_w^2}{A} \frac{1}{1 + K} \phi_m + \phi_f \frac{\phi_m + \phi_f}{\phi_m} t_D \right)
\]

Equation 2-33

Equation 2-33 is derived similarly to those of Zhang and Ayala (2014a). The exponential solution for a single-porosity liquid reservoir with fracture permeability, \( k_f \), and matrix porosity, \( \phi_m \), in dimensionless form is taken from Fetkovich (1980):

\[
q_{D}^{liq} = \frac{1}{b_{D,PSS}} \exp \left( - \frac{1}{b_{D,PSS}} \frac{2\pi r_w^2}{A} \frac{\phi_m + \phi_f}{\phi_m} t_D \right)
\]

Equation 2-34

where the definition of \( t_D \) is modified to be \( \frac{0.0063283 k_f t}{\mu c(\phi_m + \phi_f) r_w^2} \).

A comparison between Equation 2-33 and Equation 2-34 produces the \( \lambda \) and \( \beta \) rescaling approach from a single-porosity liquid solution to dual-porosity gas behavior in the long-term, is shown as:

\[
q_{D}^{gas} = \bar{\lambda}_d t^{-1} q_{D}^{liq} \left( \bar{\beta}_d t_D \right)
\]

Equation 2-35

where \( \bar{\lambda}_d = \frac{\bar{\lambda}_m}{1 + K} ; \bar{\lambda}_m = \frac{\mu_{gi} c_{gi}}{2\theta (\bar{\rho}_m - \rho_{wf})} \); \( \bar{\beta}_d = \frac{\bar{\beta}_m}{1 + K} ; \bar{\beta}_m = \frac{\int_0^t \bar{\rho}_m \, dt}{t} \).
Moreover, with \( \tilde{\lambda}_m \) and \( \tilde{\beta}_m \), rescaling from a dual-porosity liquid solution is found to yield a match between the behavior of the liquid solution and its gas counterpart. To explain the match, we derive the liquid rate equation for a well producing in a closed-boundary dual-porosity reservoir. By following the derivation process of its gas counterpart, as explained in Equations 7-14, we are able to obtain the liquid rate equation:

\[
q_{sc} = \frac{k_f \pi h}{79014 \mu c_l \rho_{sc} b_{D,\text{PSS}}} (\tilde{\rho}_f - \rho_{wf})
\]

Equation 2-36

where \( q_{sc} \) is the liquid flow rate, \( \tilde{\rho}_f \) is the average fracture liquid density, and \( c_l \) is liquid compressibility. Equation 2-5 can be used to derive the average interporosity flow equation for liquid by integrating Equation 2-5 with respect to volume and dividing the reservoir volume on both sides. The resultant equation is written as follows:

\[
\frac{\partial (\phi_m \tilde{\rho}_m)}{\partial t} = 0.0063288 \frac{a_k m}{\mu c_l} (\tilde{\rho}_f - \tilde{\rho}_m)
\]

Equation 2-37

We subsequently develop an exponential liquid rate equation for a dual-porosity system in a manner similar to its gas counterpart’s derivation, giving:

\[
q_D^{liq} = \frac{1}{1 + K b_{D,\text{PSS}}} \frac{1}{2 \pi n_w^2 A} \frac{1}{1 + K} \frac{\phi_m + \phi_f}{(\phi_m + \phi_f) t_D}
\]

Equation 2-38

where \( q_D^{liq-d} = \frac{158021 \rho_{sc} \mu c_l \pi}{2 \pi k_f h (\rho_i - \rho_{wf})} \) and \( t_D = \frac{0.0063283 k_f t}{\mu c_l (\phi_m + \phi_f) r_w^2} \).

Comparing Equation 2-33 and Equation 2-38 gives:

\[
q_D^{gas} = \tilde{\lambda}_m q_D^{liq-d} (\tilde{\beta}_m t_D)
\]

Equation 2-39

where \( q_D^{gas} \) is the predicted analytical gas flow rate response at constant bottom-hole pressure in a dual-porosity system with a closed boundary, and \( q_D^{liq-s} \) is the analytical response of the liquid dual-porosity system with a closed boundary at constant bottom-hole pressure. The representation of \( q_D^{liq-s} \) in Laplace space is provided in the literature (Raghavan, 1993). Da Prat et al. (1981) presented \( q_D^{liq-d} \) in Laplace space. By numerically inverting \( q_D^{liq-s} \) and \( q_D^{liq-d} \) to real space by means of the Stehfest algorithm, we are able to obtain the exact values of \( q_D^{liq-s} \) and \( q_D^{liq-d} \) for rescaling purposes.
2.6 Case Study

Two proposed $\lambda$ and $\beta$ rescaling approaches are validated against three numerically generated reservoir scenarios that exhibit strong dual-porosity behaviors. Because of the lack of simulators with the rigorously derived interporosity flow equation for gas, we developed an in-house dual-porosity reservoir simulator, DPS 2. An isotropic circular reservoir is described by its radial system. The well is at the center and fully penetrated with no skin. The buildup procedure follows that for a single-porosity system as described in Abou-Kassem et al. (2006). The simulator adopts implicit discretization and the simple-iteration method (SIM). The method proposed by Lee et al. (1966) is used for the viscosity calculation. The compressibility calculation follows the method proposed in Abou-Kassem et al. (1990), and Dranchuk and Abou-Kassem’s (1975) method is used for the compressibility factor calculation. Gas with $r_g=0.55$ is assumed in the case. Table 2-1 provides a summary of the reservoir and fluid properties. Table 2-2 presents three scenarios with different reservoir sizes. The rate-time production data are transformed into dimensionless form by means of their definitions and are compared against rescaled dimensionless gas production rates from analytical liquid flow rates.

Table 2-1: Reservoir and fluid properties

<table>
<thead>
<tr>
<th>Properties</th>
<th>Units</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrix permeability, $k_m$</td>
<td>md</td>
<td>0.005</td>
</tr>
<tr>
<td>Fracture permeability, $k_f$</td>
<td>md</td>
<td>50</td>
</tr>
<tr>
<td>Matrix porosity, $\phi_m$</td>
<td></td>
<td>0.15</td>
</tr>
<tr>
<td>Fracture porosity, $\phi_f$</td>
<td></td>
<td>0.01</td>
</tr>
<tr>
<td>Pay zone thickness, h</td>
<td>ft</td>
<td>300</td>
</tr>
<tr>
<td>Gas specific gravity, SG (air=1)</td>
<td></td>
<td>0.55</td>
</tr>
<tr>
<td>Wellbore radius, $r_w$</td>
<td>ft</td>
<td>0.25</td>
</tr>
<tr>
<td>Initial pressure, $p_i$</td>
<td>psia</td>
<td>5000</td>
</tr>
<tr>
<td>Initial temperature, $T$</td>
<td>°F</td>
<td>200</td>
</tr>
<tr>
<td>Specified wellbore flowing pressure, $p_{wf}$</td>
<td>psia</td>
<td>100</td>
</tr>
<tr>
<td>Shape factor, $\alpha$</td>
<td>$1/ft^2$</td>
<td>$9.98959 \times 10^{-5}$</td>
</tr>
<tr>
<td>Storativity ratio, $\omega$</td>
<td></td>
<td>0.0625</td>
</tr>
<tr>
<td>Interporosity flow coefficient, $\xi$</td>
<td></td>
<td>$6.2435 \times 10^{-10}$</td>
</tr>
</tbody>
</table>
Table 2-2: Reservoir Size and OGIP for Three Scenarios

<table>
<thead>
<tr>
<th>Scenario 1:</th>
<th>Units</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reservoir outer radius, ( r_e )</td>
<td>ft</td>
<td>1750</td>
</tr>
<tr>
<td>Dimensionless outer radius, ( r_{eD} )</td>
<td></td>
<td>7000</td>
</tr>
<tr>
<td>Drainage area</td>
<td>acres</td>
<td>221</td>
</tr>
<tr>
<td>OGIP</td>
<td>Bscf</td>
<td>119.9</td>
</tr>
<tr>
<td>Scenario 2:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reservoir outer radius, ( r_e )</td>
<td>ft</td>
<td>3500</td>
</tr>
<tr>
<td>Dimensionless outer radius, ( r_{eD} )</td>
<td></td>
<td>14000</td>
</tr>
<tr>
<td>Drainage area</td>
<td>acres</td>
<td>884</td>
</tr>
<tr>
<td>OGIP</td>
<td>Bscf</td>
<td>479.6</td>
</tr>
<tr>
<td>Scenario 3:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Reservoir outer radius, ( r_e )</td>
<td>ft</td>
<td>7000</td>
</tr>
<tr>
<td>Dimensionless outer radius, ( r_{eD} )</td>
<td></td>
<td>28000</td>
</tr>
<tr>
<td>Drainage area</td>
<td>acres</td>
<td>3534</td>
</tr>
<tr>
<td>OGIP</td>
<td>Bscf</td>
<td>1918.4</td>
</tr>
</tbody>
</table>

Figure 2-2 presents the rescaling from a single-porosity liquid solution to a gas analytical solution by means of Equation 2-35, and Figure 2-3 shows the comparison between a gas analytical solution and production data from DPS 2. The dashed curve and the square curve represent the dimensionless liquid rate and the dimensionless gas rate rescaled from it, respectively, while the solid curve represents production data from DPS 2. Figure 2-2 and Figure 2-3 demonstrate that the liquid analytical solution for a single-porosity reservoir is able to be rescaled to accurately predict the long-term behavior of the corresponding dual-porosity gas reservoir. We observe two declines from the numerically generated data for every scenario in Figure 2-3. The first decline denotes the shift from the production mainly contributed by fracture fluids to the production contributed by matrix fluids as depletion in the fracture system develops. The derivation of the rescaling approach originates from assumptions about the boundary-dominated flow and \( \bar{\rho}_m = \bar{\rho} \) that apply in the second decline stage. Systems with smaller fracture storage capacities, larger reservoir sizes, or stronger interporosity flows satisfy the approximation better than systems with the opposite properties. The more the system satisfies the assumptions, the better the match is.
**Figure 2-2:** Proposed Density Approximation vs. Single-Porosity Liquid Analytical Solution

**Figure 2-3:** Proposed Density Approximation from Single-Porosity Liquid Solution vs. Numerically Generated Profile
Note that the rescaling is from the single-porosity liquid solution for the gas rate. Now we apply the rescaling approach from the dual-porosity liquid rate solution using Equation 2-39. Figure 2-4 presents the rescaling approach from the dual-porosity liquid solution, and Figure 2-5 shows the comparison against results from DPS 2. The dashed curve and square curve represent the dimensionless dual-porosity liquid rate and rescaled dimensionless gas rate, respectively, while the solid curve represents production data from DPS 2. Figure 2-4 and Figure 2-5 demonstrate that the liquid analytical solution for a dual-porosity reservoir may be used to accurately predict the long-term behavior of a corresponding dual-porosity gas reservoir using the $\lambda$ and $\beta$ rescaling approach from the dual-porosity liquid solution. The deviation in Figure 2-5 originates from the assumptions $\bar{\rho}_m = \bar{\rho}$, which are not satisfied at the transition stage. It is also observed that $\lambda$ and $\beta$ rescaling is able to capture gas production behavior at the very beginning.

**Figure 2-4:** Proposed Density Approximation vs. Dual-Porosity Liquid Analytical Solution
**Figure 2-5**: Proposed Density Approximation from Dual-Porosity Liquid Solution vs. Numerically Generated Profile

### 2.7 Original Gas in Place Calculations

Following Zhang and Ayala (2014a), we develop a rate-transient analysis for variable pressure drawdown/variable rate production in a dual-porosity gas system. Substituting Equation 2-23 into Equation 2-16 and adding $\rho_i/\rho_l$ into the RHS gives:

$$q_{gsc} = \frac{\rho_i T_{sc} k_f h}{50294 p_{sc} T b_{D,PS} \left(1 + K \frac{2\theta \left(\frac{\bar{\rho}_m}{\rho_l} - \frac{\rho_{wf}}{\rho_l}\right)}{\mu_{gi} c_{gi}}\right)}$$  \hspace{1cm} \text{Equation 2-40}

Consider two correlations:

$$\frac{\bar{\rho}_m}{\rho_l} = 1 - \frac{G_p}{OGIP}$$  \hspace{1cm} \text{Equation 2-41}

$$r_p = 1 - \frac{\rho_{wf}}{\rho_l}$$  \hspace{1cm} \text{Equation 2-42}

Substituting Equation 2-41 and Equation 2-42 into Equation 2-40 gives:
\[
q_{gsc} = \frac{2\rho_lk_fh}{50294\rho_{sc}b_{D, PSS}\mu_{gi}c_{gi}} \frac{1}{1 + K} \left( r_p - \frac{G_p}{OGIP} \right) \tilde{\lambda}_m
\]

Equation 2-43

Denote:

\[
q_{gdi}^e = \frac{2\rho_lk_fh}{50294\rho_{sc}b_{D, PSS}\mu_{gi}c_{gi}(1 + K)}
\]

Equation 2-44

With Equation 2-44, Equation 2-43 develops into:

\[
\tilde{\lambda}_m \frac{r_p}{q_{gsc}} = \frac{1}{OGIP} \tilde{\lambda}_m \frac{G_p}{q_{gsc}} + \frac{1}{q_{gdi}^e}
\]

Equation 2-45

Equation 2-45 could be used for OGIP prediction following the steps described below:

- First of all, we plot \( \frac{r_p}{q_{gsc}} \) data in a Cartesian plot, obtain the best-fit straight line through the points at the late-decline stage, and obtain the gradient \( \frac{1}{OGIP} \) to calculate OGIP.

- For the second step, using the OGIP derived from last step, we plot \( \tilde{\lambda}_m \frac{G_p}{q_{gsc}} \) vs. \( \tilde{\lambda}_m \frac{r_p}{q_{gsc}} \) and obtain the best-fit straight line through the points at the late-decline stage. We also obtain the gradient \( \frac{1}{OGIP} \) to calculate OGIP.

- Thirdly, we calculate the difference between the two OGIP values in the last two iterations. If the difference is large, we repeat steps two and three; otherwise, we use the OGIP as calculated.

Equation 2-45 may be utilized for predicting future production by solving its discretized form and discretized material balance equation together after obtaining the OGIP.

2.8 Case Study Revisited

Production data from the three scenarios is utilized to validate the proposed rate-transient analysis for predicting OGIP in dual-porosity reservoirs. The resulting plots after four rounds of OGIP prediction are shown in Figure 2-6, Figure 2-7, and Figure 2-8, corresponding to the three aforementioned scenarios.
Figure 2-6: $\frac{\overline{\lambda_m G_p}}{q_{gsc}}$ vs. $\frac{\overline{\lambda_m r_p}}{q_{gsc}}$ Straight-Line Analysis for Scenario 1

Figure 2-7: $\frac{\overline{\lambda_m G_p}}{q_{gsc}}$ vs. $\frac{\overline{\lambda_m r_p}}{q_{gsc}}$ Straight-Line Analysis for Scenario 2
Negligible differences between the derived OGIP and the actual OGIP are observed. For Scenario 1, the slopes for the fitted straight lines are $8.2317 \times 10^{-9} \text{ Mscf}^{-1}$, yielding a $G_i$ estimation of 121.5 Bscf with a relative error of 1.32% with respect to the actual OGIP. The OGIP estimations for Scenarios 2 and 3 are 485 Bscf and 1903.6 Bscf with relative errors of 1.113% and -0.75%, respectively. The straight lines in Figures 2-6, 2-7, and 2-8 have intercepts of $1.4101 \times 10^{-6} \left( \frac{\text{Mscf}}{D} \right)^{-1}$, $4.96 \times 10^{-7} \left( \frac{\text{Mscf}}{D} \right)^{-1}$, and $2.7862 \times 10^{-7} \left( \frac{\text{Mscf}}{D} \right)^{-1}$, respectively.

2.9 Concluding Remarks

This study demonstrates that the long-term gas production performance in dual-porosity reservoirs can be successfully captured using a density-based approach. An interporosity flow equation for gas is rigorously derived incorporating viscosity-compressibility effects that the interporosity flow equation for liquid in Warren and Root’s model does not fully capture. A new gas deliverability equation was proposed for dual-porosity systems using rigorous derivation. Assumptions, the development of diffusivity equations, and proof of the $\lambda$ and $\beta$ rescaling approach were all presented.
An in-house simulator for dual-porosity gas systems using the newly derived interporosity flow equation for gas was used to validate the proposed $\lambda$ and $\beta$ rescaling approach for dual-porosity reservoirs. Successful matches between the dimensionless gas flow rate and the rescaled dimensionless flow rate with $\lambda$ and $\beta$ were obtained, showing that the $\lambda$ and $\beta$ rescaling approach successfully captures well performance under constant bottom-hold pressure production.

Moreover, a rate-transient analysis method was proposed using a density-based approach to predict OGIP and gas production rates. The successful implementation of the method proves the validity of the density-based approach.
Chapter 3

Production Data Analysis of Variable Pressure Drawdown/Variable Rate Dual-Porosity System: A Density-Based Approach

3.1 Chapter Summary

Naturally fractured reservoirs exist widely around the world. A common method to representing naturally fractured reservoirs is dual-porosity model. However, creating production data analysis models for dual-porosity systems has proven difficult due to intrinsic characteristics of dual-porosity systems and their unique rate-transient profile. Past studies on dual-porosity systems assume constant bottom-hole pressure or constant rate production, both of which impose significant constraint on production data analysis.

Recently, a density-based approach for analyzing production data for single-porosity systems has been proven to successfully describe gas well behavior under BDF. Using depletion-driven parameters, $\lambda$ and $\beta$, this state-of-the-art approach circumvents pseudo-functions. It has since been extended to variable pressure drawdown/variable rate gas systems and proven successful.

In this study, a rigorously derived interporosity flow equation for gas is advocated. This equation is able to capture the pressure-dependent effects of interporosity flow between the matrix and the fracture. Using the interporosity flow equation for gas, this study then derives a density-based, rescaled exponential model for variable pressure drawdown/variable rate production in dual-porosity gas systems. The straight-line analysis for OGIP prediction and production rate forecasting is developed. The density-based model’s validity is demonstrated by testing against a variety of scenarios.

Moreover, based on Warren and Root’s model, a density-based model for dual-porosity liquid systems is developed and fully verified. This approach is applicable at variable pressure drawdown/variable rate scenarios. The corresponding straight-line analysis enables explicit OOIP prediction and convenient production forecasting. What’s more, we develop a double-exponential decline model as a convenient substitute for the analytical solution of production rate at constant bottomhole pressure in radial dual-porosity liquid systems.
3.2 Introduction

Production data analysis on naturally fractured reservoirs has been conducted for decades. The associated reserve estimates and production forecasts have key implications in oil and gas development. Naturally fractured reservoirs can be described by dual-porosity models. Barenblatt et al. (1960) proposed the first dual-porosity model to describe liquid naturally fractured reservoirs by idealizing the fracture and matrix systems. In this model, the fracture system is treated as continuous porous media connecting discrete matrix blocks. Barenblatt et al. (1960) developed an equation describing flow between matrix and fracture, or interporosity flow. Warren and Root (1963) applied Barenblatt’s et al.’s (1960) idea to oil reservoirs and proposed a pseudo-steady state interporosity flow equation for liquid and derived the analytical solution for constant-rate production. Zimmerman et al. (1993) rigorously derived the interporosity flow equation in Warren and Root’s model, assuming quasi-steady state flow. Crawford et al. (1976) observed that pressure buildup data from known naturally fractured reservoirs could be described by Warren and Root’s model. Warren and Root model is known to produce error in early stages due to the pseudo-steady state assumption built into interporosity flow (Kazemi, 1969; Najurieta, 1980). Mavor and Cinco-Ley (1979) extended Warren and Root’s model by taking wellbore storage and skin effect into consideration and solved the constant bottom-hole production problem in an infinite dual-porosity system.


Applying pseudo-functions is a common technique to approaching gas systems. However, past studies have found this technique unable to accurately match dual-porosity system behavior. Watson et al. (1990) provided a model with history matching and model selection from analyzing production data. They evaluated reservoir properties through the procedure and conducted subsequent production forecasts. Serra’s (1981) analytical solution substituted with pseudo-pressure and pseudo-time is integrated in the model. The pseudo-pressure and pseudo-time is calculated from average reservoir pressure.
Spivey and Semmelbeck (1995) applied pseudo-functions in predicting long-term production rate in shale gas and dewatered coal seams. However, error brought by this approach is larger than 10 percent at small $\xi r_{eD}^2$ or small drawdown scenarios. Gerami et al. (2007) used a new interporosity flow equation without proof to linearize the system for pseudo-function applications. The storativity ratio in his equations is a function of reservoir pressure. In Gerami et al.’s equation, an analytical solution is derived with constant $\bar{\omega}$ first. Then, the problem is solved numerically by updating $\bar{\omega}$ in the analytical solution at each time step. However, growing error has been observed when comparing analytical results against results from CMG-IMEX.

Ye and Ayala (2012, 2013) and Ayala and Ye (2012, 2013) proposed a density-based approach for analyzing gas-well performance. The rescaled exponential model was demonstrated to be convenient and successful in describing decline in BDF. In this model, the use of depletion-driven parameters, $\lambda$ and $\beta$, enables gas production data analysis without pseudo-functions in BDF. This $\lambda$ and $\beta$ rescaling approach was then rigorously proven based on fundamental physical principles by Zhang and Ayala (2014a). Moreover, the density-based, rescaled exponential model and production data analysis was extended to variable pressure drawdown/variable rate systems by Zhang and Ayala (2014a, 2014b) and Ayala and Zhang (2013).

In the present study, a density-based, rescaled exponential model for variable pressure drawdown/variable rate production in dual-porosity gas systems is proposed as well as straight-line analysis for OGIP prediction and production rate forecast. The density-based model is tested in a variety of scenarios in order to showcase its validity. The development procedure is similar to that in Ayala and Zhang (2013).

We developed a density-based exponential model for variable pressure drawdown/variable rate in dual porosity liquid systems based on the Warren and Root model. On the basis of this exponential model, we introduced the straight-line analysis for dual-porosity liquid systems. Moreover, we developed a double exponential model for constant bottom-hole pressure production with the aid of Mathematica 9, which is proven a more convenient tool than analytical solution in Laplace space (Da Prat, 1981).

### 3.3 Rescaled Exponential Model for Variable Rate/Pressure-Drawdown in Dual-Porosity Gas System

The deliverability equation for dual-porosity gas systems was derived by Zhang (2014) as follows:

$$q_{gsc} = \frac{T_{sc}k_f h}{50294p_{sc}T_{bD, PSS}(1 + K)}(\bar{m}(p_m) - m(p_{wf}))$$

Equation 3-1
Equation 3-1 adopts an interporosity flow equation for gas flow. The development procedure for the gas interporosity flow equation is presented in Zhang (2014). The modified Carter’s viscosity-compressibility ratio (Carter, 1985) for the matrix system is defined as:

$$\bar{\lambda}_m = \frac{\mu_{gi} c_{gi}}{2\theta (\bar{\rho}_m - \rho_{wf})}$$

Equation 3-2

where $\theta = \frac{RT}{MW}$. Equation 3-2 can be recast into:

$$m(\bar{p}_m) - m(p_{wf}) = \frac{2\theta (\bar{\rho}_m - \rho_{wf}) \bar{\lambda}_m}{\mu_{gi} c_{gi}}$$

Equation 3-3

Substituting Equation 3-3 into Equation 3-1 gives:

$$q_{gsc} = \frac{k_f h (\bar{\rho}_m - \rho_{wf}) \bar{\lambda}_m}{25147 \rho_{sc} b_{D, PSS} (1 + K) \mu_{gi} c_{gi}}$$

Equation 3-4

The tank material balance model can represent gas production rate in terms of average reservoir density change with time during BDF:

$$q_{gsc} = -\frac{V_{res}}{\rho_{sc}} \frac{d(\bar{\rho}_m \bar{\rho}_m)}{dt}$$

Equation 3-5

Substituting Equation 3-5 into Equation 3-4 gives:

$$\frac{1}{\bar{\lambda}_m} \frac{1}{(\bar{\rho}_m - \rho_{wf})} \frac{d(\bar{\rho}_m)}{dt} = \frac{k_f}{25147 A \phi_m b_{D, PSS} \mu_{gi} c_{gi} (1 + K)} = -D_{di}^e$$

Equation 3-6

where $D_{di}^e$ is the decline constant for the dual porosity decline model. Defining $\bar{\beta}_m$ as the time-averaged evolution of $\bar{\lambda}_m$ gives:

$$\bar{\lambda}_m = \frac{d(\bar{\beta}_m t)}{dt}$$

Equation 3-7

Substituting Equation 3-7 into Equation 3-6 gives:

$$\frac{1}{(\bar{\rho}_m - \rho_{wf})} \frac{d(\bar{\rho}_m)}{dt} = -D_{di}^e$$

Equation 3-8
Multiplying both sides of Equation 3-8 by \( d(\bar{\rho}_m t) \) and integrating on both sides between time \( t_j \) and \( t \) gives:

\[
\int_{\bar{\rho}_m (t)}^{\bar{\rho}_m (t + \Delta t)} \frac{1}{\bar{\rho}_m - \rho_{wf}} d(\bar{\rho}_m) = - \int_{\bar{\rho}_m (t_j)}^{\bar{\rho}_m (t)} D_{di}^e d(\bar{\rho}_m t)
\]

Equation 3-9

where \( \bar{\rho}_{mj} \) is average density of matrix fluids at time \( t_j \), and \( \bar{\rho}_m \) is \( \bar{\rho}_m \) at time \( t_j \). With constant bottom-hole pressure in this period, we conducted integration on both sides of Equation 3-9, and labeling \( \rho_{wf} \) in the interval \( j \) as \( \rho_{wf,j} \) gives:

\[
\left( \bar{\rho}_m - \rho_{wf,j} \right) = \left( \bar{\rho}_{mj} - \rho_{wf,j} \right) \exp\left[ -D_{di}^e \left( \bar{\rho}_m t - \bar{\rho}_{mj} t_j \right) \right]
\]

Equation 3-10

Substituting Equation 3-10 in Equation 3-4 gives:

\[
q_{gsc} = \frac{k_f h}{25147 \rho_{sc} b_{P, PSS} \mu_{gi} c_{gi} (1 + K)} \lambda_m \left( \bar{\rho}_{mj} - \rho_{wf,j} \right) \exp\left[ -D_{di}^e \left( \bar{\rho}_m t - \bar{\rho}_{mj} t_j \right) \right]
\]

Equation 3-11

We know material balance equation at time \( t_j \):

\[
\bar{\rho}_{mj} \approx 1 - \frac{G_{pj}}{OGIP}
\]

Equation 3-12

Also, we define drawdown ratio \( r_{\bar{\rho}, j} \) at time \( t_j \) and \( q_{gdi}^e \) as follows:

\[
r_{\bar{\rho}, j} = \frac{\rho_i - \rho_{wf,j}}{\rho_i}
\]

Equation 3-13

\[
q_{gdi}^e = \frac{k_f h \rho_i}{25147 \rho_{sc} b_{P, PSS} \mu_{gi} c_{gi} (1 + K)}
\]

Equation 3-14

Following Zhang (2013), we represent \( \bar{\rho}_{mj} - \rho_{wf,j} \) as follows by combining Equation 3-6 and Equations 3-12 to 3-14:

\[
\bar{\rho}_{mj} - \rho_{wf,j} = \rho_i \left( 1 - \frac{G_{pj}}{OGIP} \right) - \rho_{wf,j} = \rho_i \left( r_{\bar{\rho}, j} - \frac{D_{di}^e}{q_{gdi}^e} G_{pj} \right) = \rho_i R_{\bar{\rho}}
\]

Equation 3-15

where \( R_{\bar{\rho}} = r_{\bar{\rho}, j} - \frac{D_{di}^e}{q_{gdi}^e} G_{pj} \), and \( R_{\bar{\rho}} \) is a constant between time \( t_j \) and time \( t \).

Substituting Equations 3-14 and 3-15 into Equation 3-11 gives:
\[ q_{gsc} = q_{gd'i} \tilde{\lambda}_m R_p \exp \left[ -D_{di}^e \left( \tilde{\beta}_m t - \tilde{\beta}_{m_j} t_j \right) \right] \]

Equation 3-16

Equation 3-16 may be integrated to obtain the associated equation predicting cumulative production as follows:

\[ G_p - G_{p_j} = \frac{q_{gd'i}}{D_{di}^e} R_p \left[ 1 - \exp \left( -D_{di}^e \left( \tilde{\beta}_m t - \tilde{\beta}_{m_j} t_j \right) \right) \right] \]

Equation 3-17

Substituting Equation 3-16 into Equation 3-17 gives:

\[ G_p - G_{p_j} = \frac{q_{gd'i}}{D_{di}^e} R_p \left( 1 - \frac{q_{gsc}}{q_{gd'i} \tilde{\lambda}_m R_p} \right) \]

Equation 3-18

Substituting Equation 3-15 into Equation 3-18 and combining terms gives:

\[ q_{gsc} = \tilde{\lambda}_m q_{gd'i} \left( r_p - D_{di}^e G_p \right) \]

Equation 3-19

We now have the production rate forecasting model for dual-porosity systems based on density that does not employ pseudo-time concepts. The model applies to variable rate/variable pressure drawdown systems. For variable rate systems, Equation 3-19 is written in terms of \( r_p \) due to specified \( q_{gsc} \):

\[ r_{p_j} = \frac{1}{\tilde{\lambda}_m} \frac{q_{gsc}}{q_{gd'i}} + \frac{D_{di}^e}{q_{gd'i}} G_p \]

Equation 3-20

Substituting \( \frac{D_{di}^e}{q_{gd'i}} = OGIP \) into Equation 3-19 and moving terms gives:

\[ \tilde{\lambda}_m \frac{r_{p_j}}{q_{gsc}} = \frac{1}{OGIP} \tilde{\lambda}_m \frac{G_p}{q_{gsc}} + \frac{1}{q_{gd'i}} \]

Equation 3-21

Equation 3-21 is convenient for OGIP prediction. This is done in an iterative approach. By plotting \( \tilde{\lambda}_m \frac{r_{p_j}}{q_{gsc}} \) vs. \( \tilde{\lambda}_m \frac{G_p}{q_{gsc}} \) on a Cartesian plot and fitting a straight line in the second decline stage, OGIP is determined as the inverse of the straight-line gradient. The derived OGIP generates \( \tilde{\lambda}_m \) at each time step for the same operations at the next iteration level. The \( \tilde{\lambda}_m \) in the first iteration level is assumed 1 or assigned an estimated value. The guessed OGIP, apparently, should exceed the last \( G_p \).
3.4 Case Studies

3.4.1 Case Study A: Dual-Porosity Gas System Producing at Variable $p_{wf}$

Due to adopting a more appropriate interporosity flow equation than the Warren and Root model used in dual-porosity gas systems, we established an in-house dual-porosity simulator, DPS 2, to simulate naturally fractured reservoirs. The configurations are described in Chapter 2. Gas specific gravity, $r_g$, is 0.55. Matrix porosity is 0.15 while fracture porosity is taken as 0.01. Matrix permeability is taken as 0.005 md, while fracture permeability is taken as 50 md. Shape factor is $9.98959E-05 \, 1/\text{ft}^2$. Table 3-1 list relevant reservoir and fluid properties.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Units</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Matrix permeability, $k_m$</td>
<td>md</td>
<td>0.005</td>
</tr>
<tr>
<td>Fracture permeability, $k_f$</td>
<td>md</td>
<td>50</td>
</tr>
<tr>
<td>Matrix porosity, $\phi_m$</td>
<td></td>
<td>0.15</td>
</tr>
<tr>
<td>Fracture porosity, $\phi_f$</td>
<td></td>
<td>0.01</td>
</tr>
<tr>
<td>Pay zone thickness, $h$</td>
<td>ft</td>
<td>300</td>
</tr>
<tr>
<td>Gas specific gravity, SG (air=1)</td>
<td></td>
<td>0.55</td>
</tr>
<tr>
<td>Wellbore radius, $r_w$</td>
<td>ft</td>
<td>0.25</td>
</tr>
<tr>
<td>Initial pressure, $p_i$</td>
<td>psia</td>
<td>5000</td>
</tr>
<tr>
<td>Initial temperature, $T$</td>
<td>°F</td>
<td>200</td>
</tr>
<tr>
<td>Specified wellbore flowing pressure, $p_{wf}$</td>
<td>psia</td>
<td>100</td>
</tr>
<tr>
<td>Shape factor, $\alpha$</td>
<td>1/ft$^2$</td>
<td>$9.98959 \times 10^{-5}$</td>
</tr>
<tr>
<td>Storativity ratio, $\omega$</td>
<td></td>
<td>0.0625</td>
</tr>
<tr>
<td>Interporosity flow coefficient, $\xi$</td>
<td></td>
<td>$6.2435 \times 10^{-10}$</td>
</tr>
<tr>
<td>Reservoir outer radius, $r_e$</td>
<td>ft</td>
<td>1750</td>
</tr>
<tr>
<td>Dimensionless outer radius, $r_{eD}$</td>
<td></td>
<td>7000</td>
</tr>
<tr>
<td>Drainage area</td>
<td>acres</td>
<td>221</td>
</tr>
<tr>
<td>OGIP</td>
<td>Bscf</td>
<td>119.9</td>
</tr>
</tbody>
</table>
The original-gas-in-place is 119.9 Bscf. A variable pressure drawdown production is run on DPS 2 with reservoir and fluids settings shown in Table 3-1. The bottom-hole pressure schedule is shown in Table 3-2.

**Table 3-2: Bottom-Hole Pressure Schedule**

<table>
<thead>
<tr>
<th>( t, ) days</th>
<th>( p_{wf}, ) psia</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-300</td>
<td>4000</td>
</tr>
<tr>
<td>300-700</td>
<td>2000</td>
</tr>
<tr>
<td>700-1100</td>
<td>1000</td>
</tr>
<tr>
<td>1100-2000</td>
<td>500</td>
</tr>
<tr>
<td>2000-14244</td>
<td>300</td>
</tr>
</tbody>
</table>

Figure 3-1 plots numerically generated gas rate response against the gas rate predicted by modified exponential and density-based models as described in Equations 3-16 and 3-19 respectively for dual-porosity systems. Gas rates generated from the two methods are observed to match each other in the second decline stage, in which production is mainly supported by matrix fluids and the reservoir flow is boundary-dominated. Mismatch between gas rates lies in the assumption of boundary-dominated flow and negligible fracture storage capacity in exponential and density models.

It is important to note that the analytical models may predict deviated gas rate at the end of the second depletion stage due to increasing error of approximating matrix average pressure as average reservoir pressure in the analytical models. The error tends to be eliminated with decreasing storage capacity of fracture system.
With the applicability of exponential and density-based decline models validated, we utilize Equation 3-21, which is a rewritten form of the previous models for OGIP prediction. The detailed procedure is illustrated in Zhang and Ayala (2014b). First, \( \frac{r_{p_l}}{q_{gsc}} \) vs. \( \frac{q_{sc}}{q_{gsc}} \) is plotted, a fitted straight line in the second decline stage yielding OGIP by inversing its gradient. With the derived OGIP, \( \lambda_m \) is calculated and the same step is done iteratively. The straight-line trend is easy to identify, even at the first trial, because the large \( q_{gdi} \) yields negligible intercept. The resulting plot in Figure 3-2, with slope \( 8.2669 \times 10^{-9} \) Mscf\(^{-1}\), yields OGIP prediction of 120.96 Bscf. The intercept or \( \frac{1}{q_{gdi}} \) is fitted as \( 1.3633 \times 10^{-6} \) (Mscf/D)\(^{-1}\). The derived OGIP with a density-based approach is significantly close to the actual OGIP, 119.9 Bscf, proving that a density-based approach is a convenient tool for OGIP prediction at variable drawdown cases in dual-porosity gas systems.
Figure 3-2 $\frac{\bar{\lambda}_m G_p}{q_{gsc}}$ vs. $\frac{\bar{\lambda}_m r_p}{q_{gsc}}$ Straight-Line Analysis for Variable Drawdown Dual-Porosity Gas System (Case Study A)

### 3.4.2 Case Study B: Dual-porosity Gas System Producing at Constant $q_{gsc}$

We use the identical dual-porosity gas reservoir for validation of the density-based model at constant $q_{gsc}$. The only difference between Case Study B and Case Study A lies in the production inner boundary condition. The reservoir in Case Study B produces at a constant rate ($q_{gsc} = 5000 \text{ Mscf/D}$). Production data is generated with DPS 2.

As explained in Zhang (2013) for single-porosity cases, the density-based approach developed for variable drawdown still applies at constant or variable $q_{gsc}$ case if we consider the $t$ in the exponential term $\exp[-D^e_i \left( \bar{\beta}_m t - \bar{\beta}_m t_j \right)]$ is extremely close to $t_j$. Thus, Equation 2-20 still hold true.

Substituting $\frac{D^e_i q_{gdi}}{q_{gdi}} = OGIP$ into Equation 2-20 gives:
\[ r_{\rho j} = \frac{1}{\bar{\lambda_m}} \frac{q_{gsc}}{q_{gdi}} + \frac{G_p}{OGIP} \]

Equation 3-22

Equation 3-22 is able to predict density drawdown ratio, hence \( p_{wf} \), with the knowledge of gas properties, production, OGIP and \( q_{gdi}^g \). Figure 3-3 compares \( r_{\rho} \) prediction with Equation 22 for Case Study B against actual \( r_{\rho} \) from reservoir simulation. The analytical derived \( r_{\rho} \) is observed to match simulated \( r_{\rho} \) in a gradual manner in the second decline stage. Deviation due to average reservoir pressure approximation in the density-based model is observed at the very end. This observation proves the applicability of a density-based approach for constant-rate production in a dual-porosity gas system.

![Figure 3-3 Density Drawdown Ratio Comparison between Density-Based Model and Numerical Simulation for Constant \( q_{gsc} \) Dual-Porosity System (Case Study B)](image)

We then plot \( \bar{\lambda}_m \frac{r_{\rho j}}{q_{gsc}} \) vs. \( \bar{\lambda}_m \frac{G_p}{q_{gsc}} \) following the procedure illustrated in Case Study A. The resulting plot in Figure 3-4 is with slope \( 8.2813 \times 10^{-9} \) Mcf\(^{-1}\). By inversing the gradient, we generate an OGIP prediction of 120.75 Bscf. The intercept, namely \( \frac{1}{q_{gdi}} \), is fitted as \( 1.0469 \times 10^{-6} \) (Mscf/D)\(^{-1}\). The derived OGIP with a density-based approach closely matches the actual OGIP, 119.9 Bscf, which
proves that a density-based approach is applicable for OGIP prediction at constant $q_{gsc}$ cases in dual-porosity gas systems.

![Graph showing straight-line analysis for constant $q_{gsc}$ dual-porosity system (Case Study B).]

**Figure 3-4** $\lambda_{m, G_p}/q_{gsc}$ vs. $\lambda_{m, r_p}/q_{gsc}$ Straight-Line Analysis for Constant $q_{gsc}$ Dual-Porosity System (Case Study B)

### 3.4.3 Case Study C: Dual-porosity Gas System Producing at Variable $q_{gsc}$

We apply a density-based model to a variable rate scenario detailed in Table 3-3. The reservoir and fluids properties remain the same as those in Table 3-1. Production data is generated by DPS 2. $r_p$ is analytically calculated at each time step by applying production data into Equation 3-22.
Table 3-3 \( q_{gsc} \) Schedule for Case Study C

<table>
<thead>
<tr>
<th>( t, ) days</th>
<th>( q_{gsc}, ) Mcf/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-2000</td>
<td>8000</td>
</tr>
<tr>
<td>2000-4017</td>
<td>6000</td>
</tr>
<tr>
<td>4017-6025</td>
<td>7000</td>
</tr>
<tr>
<td>6025-8004</td>
<td>5000</td>
</tr>
<tr>
<td>8004-10000</td>
<td>6000</td>
</tr>
<tr>
<td>10000-14244</td>
<td>4000</td>
</tr>
</tbody>
</table>

Figure 3-5 Density Drawdown Ratio Comparison between a Density-Based Model and Numerical Simulation for Variable \( q_{gsc} \) Dual-Porosity System (Case Study C)

Figure 3-5 plots \( r_{p} \) calculated using \( p_{wf} \) from the simulator against the analytically derived \( r_{p} \). Excellent match in the second decline stage between them is observed. Contrary to the behavior of
most single-porosity systems, the magnitude of $p_{wf}$ variation upon flow rate change is not significant in dual-porosity systems. Because of relatively large fracture permeability and its limited storage capacity, pressure across the fracture system approaches $p_{wf}$ closely at the second decline stage in this case. With large deliverability of the fracture system, a small change in $p_{wf}$ could be sufficient for a relatively large $q_{gsc}$ jump. Moreover, it requires a much more significant change in $q_{gsc}$ to have an obvious $p_{wf}$ jump than that in a single-porosity system with similar reservoir and fluids configurations. The small discontinuities in Figure 3-5 denote $p_{wf}$ variation due to changed $q_{gsc}$.

We then utilize $\frac{\lambda_m G_p}{q_{gsc}}$ vs. $\frac{\lambda_m r_p}{q_{gsc}}$ straight-line analysis to predict OGIP following the procedure in Case Study A. Sudden change in flow rate generates space shown along the line. The resulting straight line has a slope of $8.2617 \times 10^{-9}$ Mscf$^{-1}$. OGIP prediction is hence 121.04 Bscf, which is accurate compared to actual OGIP, 119.9 Bscf. $\frac{1}{q_{gdi}}$ is fitted as $1.1307 \times 10^{-6}$ (Mscf/D)$^{-1}$. The slight difference between predicted OGIP and actual OGIP proves the validity of a density-based approach at constant $q_{gsc}$ cases in dual-porosity gas systems.

![Graph](image)

**Figure 3-6** $\frac{\lambda_m G_p}{q_{gsc}}$ vs. $\frac{\lambda_m r_p}{q_{gsc}}$ Straight-Line Analysis for Variable $q_{gsc}$ Dual-Porosity System (Case Study C)
3.5 Exponential Model for Variable Rate/Pressure-Drawdown in a Dual-Porosity Liquid System

First of all, we derive the deliverability equation from a diffusivity equation and an interporosity flow equation for liquid production. For a circular dual porosity reservoir in a radial system, the diffusivity equation for a liquid system is written symbolically as follows:

\[-\frac{1}{r}\frac{\partial}{\partial r}(r \rho_f u_r) - \frac{\partial (\phi_m \rho_m)}{\partial t} = \frac{\partial (\phi_f \rho_f)}{\partial t}\]

Equation 3-23

where \(u_r\) is fluid flow rate in the fracture system. Substituting Darcy’s law and \(c_l = \frac{1}{\rho_m \partial p_m}\) into Equation 3-23 and moving terms gives:

\[-\frac{1}{r}\frac{\partial}{\partial r}(r \frac{k_f}{\mu c_l} \frac{\partial \rho_f}{\partial r}) - \frac{\partial (\phi_m \rho_m)}{\partial t} = \frac{\partial (\phi_f \rho_f)}{\partial t}\]

Equation 3-24

where \(\mu\) is liquid viscosity, \(c_l\) is liquid compressibility. Moving terms, combining two accumulation terms, and adding \(V_{res}/V_{res}\) on the RHS gives:

\[-\frac{1}{r}\frac{\partial}{\partial r}(r \frac{\partial \rho_f}{\partial r}) = \frac{\mu c_l V_{res}}{V_{res} k_f} \frac{\partial (\phi_f \rho_f)}{\partial t}\]

Equation 3-25

where \(V_{res}\) is reservoir volume. For boundary-dominated flow, we propose a valid approximation that the changing rates of fluid density with time are close across the reservoir so that they could be represented by an average density change rate. This approximation is represented as follows:

\[V_{res} \frac{\partial (\phi_f \rho_f + \phi_m \rho_m)}{\partial t} \approx V_{res} \frac{\partial (\bar{\rho}_f \bar{\rho}_f + \bar{\rho}_m \bar{\rho}_m)}{\partial t} = -q_{osc} \rho_{sc}\]

Equation 3-26

where \(\bar{\rho}_f\) and \(\bar{\rho}_m\) represent average fracture fluids density and average matrix fluid density respectively. Substituting Equation 3-26 into Equation 3-25 gives:

\[-\frac{1}{r}\frac{\partial}{\partial r}(r \frac{\partial \rho_f}{\partial r}) = -\frac{\mu c_l}{V_{res} k_f} q_{osc} \rho_{sc}\]

Equation 3-27

Equation 3-27 could yield a gas rate equation for dual-porosity reservoirs with techniques provided in Dake (1978). By integrating both sides of Equation 3-27 with respect to radius twice with closed boundary conditions and calculating volume-averaged \(\rho_f\), we have in customary units:
\[ q_{sc} = \frac{0.01266\pi k_f h}{\mu c_l \rho_{sc} b_D P_{SS}} (\bar{\rho}_f - \rho_{wf}) \]

\textbf{Equation 3-28}

In Appendix B, we derive an interporosity flow equation for liquid from a physical basis in customary units:

\[ \phi_m \frac{d(\rho_m)}{dt} = 0.0063288 \frac{ak_m}{\mu c_l} (\rho_f - \rho_m) \]

\textbf{Equation 3-29}

This equation is equivalent to Warren and Root’s model. To prove this equivalency, we write interporosity flow equation in Warren and Root’s model first:

\[ \phi_m \frac{c_m}{c_l} \frac{1}{\rho_m} \frac{\partial \rho_m}{\partial t} = 0.0063288 \frac{ak_m}{\mu} \frac{1}{\rho_m} \ln \left( \frac{\rho_f}{\rho_m} \right) \]

\textbf{Equation 3-30}

For slightly compressible fluids, we have

\[ p_f - p_m = \frac{1}{c_l} \ln \left( \frac{\rho_f}{\rho_m} \right) \]

derived from the compressibility definition. Substituting the relation into Equation 3-30 gives:

\[ \phi_m \frac{c_m}{c_l} \frac{\partial \rho_m}{\partial t} = 0.0063288 \frac{ak_m}{\mu} \frac{1}{\rho_m} \ln \left( \frac{\rho_f}{\rho_m} \right) \]

\textbf{Equation 3-31}

For \( \frac{\rho_f}{\rho_m} \) close to 1, the following approximation holds:

\[ \ln \left( \frac{\rho_f}{\rho_m} \right) \approx \frac{\rho_f}{\rho_m} - 1 = \frac{\rho_f - \rho_m}{\rho_m} \]

\textbf{Equation 3-32}

Substituting Equation 3-32 into Equation 3-31 gives Equation 3-29, hence proving the interchangeability of the density-based interporosity flow equation and that in Warren and Root’s model. Integrating Equation 3-29 with respect to volume and dividing reservoir volume on both sides of the results gives:

\[ \phi_m c_m \frac{\partial \bar{\rho}_m}{\partial t} = 0.0063288 \frac{ak_m}{\mu} (\bar{\rho}_f - \bar{\rho}_m) \]

\textbf{Equation 3-33}

where \( \bar{\rho}_m \) and \( \bar{\rho}_f \) are volume-averaged matrix density and fracture density respectively. Substituting Equation 3-5 into Equation 3-28 and writing the result in customary units gives:

\[ V_{res} \phi_m \frac{d(\bar{\rho}_m)}{dt} = \frac{0.012656 k_f \pi h}{1000 \mu c_l b_D P_{SS}} (\bar{\rho}_f - \rho_{wf}) \]

\textbf{Equation 3-34}

Substituting Equation 3-33 into Equation 3-34 and moving terms gives:
\[
\tilde{\rho}_f = \frac{1}{1 + K} \tilde{\rho}_m + \frac{K}{1 + K} \rho_{wf}
\]

where:
\[
K = \frac{0.0062841k_fh}{\alpha k_m V_{res} b_{D,PS}}
\]

Substituting Equation 3-35 into Equation 3-28 and moving terms gives
\[
\frac{1}{(\tilde{\rho}_m - \rho_{wf})} \frac{d(\tilde{\rho}_m)}{dt} = -\frac{0.01266\pi k_fh}{1000V_{res} \phi_m \mu c_l b_{D,PS} (1 + K)} = -D_{di}^e
\]

Multiplying both sides of Equation 3-37 by \(dt\) and integrating on both sides of the result between two arbitrary time \(t_j\) and \(t\) gives:
\[
\int_{\tilde{\rho}_m}^{\tilde{\rho}_m_j} \frac{1}{(\tilde{\rho}_m - \rho_{wf})} d(\tilde{\rho}_m) = \int_{t_j}^{t} -D_{di}^e d(t)
\]

Provided bottom-hole pressure stays constant in this period, integrating on both sides of Equation 38 and labeling \(\rho_{wf}^j\) in the interval \(j\) as \(\rho_{wf,j}\) gives:
\[
(\tilde{\rho}_m - \rho_{wf,j}) = (\tilde{\rho}_m_j - \rho_{wf,j}) \exp[-D_{di}^e(t - t_j)]
\]

Substituting Equation 3-35 and Equation 3-39 into Equation 3-28 gives:
\[
q_{osc} = \frac{0.0022548\pi k_fh}{\mu c_l \rho_{sc} b_{D,PS} (1 + K)} (\tilde{\rho}_m_j - \rho_{wf,j}) \exp[-D_{di}^e(t - t_j)]
\]

We know material balance equation at time \(t_j\):
\[
\frac{\tilde{\rho}_m_j}{\rho_i} \approx \frac{\tilde{\rho}}{\rho_i} = 1 - \frac{N_{p_j}}{\text{OOIP}}
\]

Drawdown ratio \(r_{pj}\) at time \(t_j\) and \(q_{odi}^e\) are defined as follows:
\[
r_{pj} = \frac{\rho_i - \rho_{wf,j}}{\rho_i}
\]
\[
q_{odi}^e = \frac{0.0022548\pi k_fh\rho_i}{\mu c_l \rho_{sc} b_{D,PS} (1 + K)}
\]
With Equation 3-42 and Equation 3-43, we can represent \( \bar{\rho}_{mj} - \rho_{wf_j} \) as follows:

\[
\bar{\rho}_{mj} - \rho_{wf_j} = \rho_i \left( 1 - \frac{N_{pj}}{OOIP} \right) - \rho_{wf_j} = \rho_i \left( \frac{r_p}{q_{odi}^e} \right) - \frac{D_{di}^e N_{pj}}{q_{odi}^e} = \rho_i R_p
\]

Equation 3-43

where \( R_p = r_p j - \frac{D_{di}^e}{q_{odi}^e} N_{pj} \), and \( R_p \) is a constant between time \( t_j \) and time \( t \).

Substituting Equation 3-43 and Equation 3-44 into Equation 3-40 gives:

\[
q_{osc} = q_{odi}^e R_p \exp \left( -D_{di}^e (t - t_j) \right)
\]

Equation 3-44

For a well subject to a constant drawdown scenario, Equation 3-45 collapses to

\[
q_{osc} = q_{odi}^e r_p \exp(-D_{di}^e t)
\]

Equation 3-45

Equation 3-45 may be integrated to obtain the associated equation predicting cumulative production as follows:

\[
N_p - N_{pj} = q_{odi}^e R_p \left\{ 1 - \exp \left[ -D_{di}^e (t - t_j) \right] \right\}
\]

Equation 3-46

Substituting Equation 3-45 into Equation 3-47 gives:

\[
N_p - N_{pj} = \frac{q_{odi}^e}{D_{di}^e} R_p \left( 1 - \frac{q_{osc}}{q_{odi}^e R_p} \right)
\]

Equation 3-48

Substituting \( R_p = r_p j - \frac{D_{di}^e}{q_{odi}^e} N_{pj} \) into Equation 3-48 and combining terms gives:

\[
q_{osc} = q_{odi}^e \left( r_p j - \frac{D_{di}^e}{q_{odi}^e} N_p \right)
\]

Equation 3-49

By substituting \( \frac{D_{di}^e}{q_{odi}^e} = \frac{1}{OOIP} \), Equation 3-49 could be rewritten into:

\[
r_p j = \frac{q_{osc}}{q_{odi}^e} + \frac{N_p}{OOIP}
\]

Equation 3-50

Equation 3-50 can also be written as follows:

\[
\frac{r_p j}{q_{osc}} = \frac{1}{OOIP} \frac{N_p}{q_{osc}} + \frac{1}{q_{odi}^e}
\]

Equation 3-51
The density-based analytical model is applicable to variable $q_{osc}$ cases if we consider that $t_j$ infinitely approaches $t$ in the development. Equation 3-51 could be used for OOIP prediction by plotting $\frac{r_p}{q_{osc}}$ vs. $\frac{N_p}{q_{osc}}$ data in a Cartesian plot and fitting a straight line through the points at a late stage. The gradient of the straight line, namely $\frac{1}{OOIP}$, yields OOIP. Equation 3-51 can also be utilized for rate-transient analysis by solving its discretized form as well as a discretized material balance equation after obtaining the OOIP.

The density-based, exponential model and associated rate-transient analysis methodology are developed for liquid dual-porosity systems and are applicable at variable rate/pressure-drawdown conditions at the second decline stage.

### 3.6 Liquid Double-Exponential Decline Model

A special case of variable pressure-drawdown/rate production is constant bottom-hole pressure production. In a circular reservoir with a centered well, accurate liquid rate prediction for a dual-porosity system is solved from governing equations and boundary conditions. The dimensionless flow rate produced at constant bottom-hole pressure in a bounded circular reservoir in Laplace space (Da Prat, 1981) is given as below:

$$\tilde{q}_D = \frac{K_1(\sqrt{s})I_1(r_{ed}\sqrt{s}) - K_1(r_{ed}\sqrt{s})I_1(\sqrt{s})}{\sqrt{s}[K_1(r_{ed}\sqrt{s})I_0(\sqrt{s}) + I_1(r_{ed}\sqrt{s})K_0(\sqrt{s})]}$$

**Equation 3-52**

Obtaining the exact value requires transforming liquid rate from Laplace space to real space. A common method for this inversion is the Stehfest algorithm (Stehfest, 1970), which is computationally intense, along with other techniques. Appendix F develops a double-exponential decline model for accurately predicting circular liquid reservoir behavior in the decline stage:

$$q_D = \frac{(C + D)e^{Gt} + (C - D)e^{Ht}}{C : F}$$

**Equation 3-53**

where:

$$C = \sqrt{-\frac{8\xi(-\frac{3}{4} + \log(r_{ed}))}{r_{ed}^2(1 - \omega)\omega} + \frac{2\xi[-\frac{3}{4} + \frac{2(1 - \omega)^2}{\xi r_{ed}^2} + \log(r_{ed})]}{r_{ed}^2(1 - \omega)\omega}^2}$$

**Equation 3-54**
\[ D = \frac{2}{r_e \delta} + \frac{4(1 - \omega)}{r_e \delta \omega} \left\{ \frac{3}{4} + \frac{2(1 - \omega)^2 + \log(r_e \delta)}{r_e \delta^2 (1 - \omega) \omega} \right\} \]

Equation 3-55

\[ E = -\frac{1}{r_e \delta^2 (-\frac{3}{4} + \log(r_e \delta))} \left\{ \frac{3}{4} + \frac{2(1 - \omega)^2 + \log(r_e \delta)}{r_e \delta^2 (1 - \omega) \omega (-\frac{3}{4} + \log(r_e \delta))} \right\} \]

Equation 3-56

\[ F = 2(-\frac{3}{4} + \log(r_e \delta)) \]

Equation 3-57

\[ G = E - \frac{C}{F} \]

Equation 3-58

\[ H = E + \frac{C}{F} \]

Equation 3-59

In the next section we apply the double-exponential decline model, density-based decline model, and rigorous analytical model to a constant bottom-hole pressure production scenario for validation purposes.

3.7 Case Studies Revisited

3.7.1 Case Study D: Dual-Porosity Liquid System Producing at Constant \( p_{wf} \)

The reservoir described in Table 3-1 is considered for validation except for fluid properties. In this case study, liquid with constant compressibility and viscosity replaces gas in the reservoir. We generate dimensionless production data semi-analytically by inverting the analytical solution in Laplace space (Da Prat, 1981) with the Stehfest algorithm (Stehfest, 1970). This data is then compared against results from the density-based model and double-exponential model. Figure 3-7 demonstrates the comparisons.
Figure 3-7 Double-Exponential Model and Density-Based Model against Analytical Solution for Liquid Production at Constant $p_{wf}$ in a Dual-Porosity System

Figure 3-7 showcases that the double-exponential model is accurate throughout the life cycle except for very early stages. This analytical model in real space is better than an analytical solution in Laplace space (Da Prat, 1981) in terms of computational time. No inversion is demanded in the double-exponential model, while accurate prediction is achieved. It is proven a convenient tool for production rate prediction at constant $p_{wf}$ in liquid dual-porosity systems.

Liquid rates generated from a density-based model and analytical solution are observed to match each other in the second decline stage. This observation is in accordance with the assumption that production is mainly supported by matrix fluids in the density-based model. Original fracture fluids contribute to the majority of production at the first decline stage. Reservoir flow, both fracture flow and interporosity flow, should be boundary-dominated to satisfy a density-based model. Both a double-exponential model and a density-based model are fully verified for constant $p_{wf}$ condition in dual-porosity liquid systems.
3.7.2 Case Study E: Dual-Porosity Liquid System Producing at Variable $p_{wf}$

We intend to validate a density-based model developed for liquid dual-porosity systems at variable $p_{wf}$. A successful match would inferentially validate the applicability of a density-based approach at constant $q_{sc}$ condition. Liquid is assumed in the same reservoir instead of gas. The matrix and fracture are assigned $1 \times 10^{-14}$ psi$^{-1}$ in terms of compressibility or near incompressibility. Stock tank oil density is 50 lb/cf. Oil compressibility and viscosity are $1 \times 10^{-6}$ psi$^{-1}$ and 1 cp respectively with zero pressure dependency. Table 3-4 describes the liquid properties.

Table 3-4: Liquid Properties for Case Study E

<table>
<thead>
<tr>
<th>Properties</th>
<th>Units</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compressibility, $c$</td>
<td>psi$^{-1}$</td>
<td>$1 \times 10^{-6}$</td>
</tr>
<tr>
<td>Viscosity, $\mu$</td>
<td>cp</td>
<td>1</td>
</tr>
<tr>
<td>Matrix compressibility, $c_m$</td>
<td>psi$^{-1}$</td>
<td>$1 \times 10^{-14}$</td>
</tr>
<tr>
<td>Fracture compressibility, $c_f$</td>
<td>psi$^{-1}$</td>
<td>$1 \times 10^{-14}$</td>
</tr>
<tr>
<td>Stock tank oil density, $\rho_{sc}$</td>
<td>lb/cf</td>
<td>50</td>
</tr>
</tbody>
</table>

Bottom-hole pressure schedule described by Table 3-5 is input in CMG-IMEX along with reservoir and fluid properties.

Table 3-5: Bottom-Hole Pressure Schedule for Case Study E

<table>
<thead>
<tr>
<th>t, days</th>
<th>$p_{wf}$, psia</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-100</td>
<td>4000</td>
</tr>
<tr>
<td>100-200</td>
<td>2000</td>
</tr>
<tr>
<td>200-300</td>
<td>1000</td>
</tr>
<tr>
<td>300-400</td>
<td>500</td>
</tr>
<tr>
<td>400-500</td>
<td>300</td>
</tr>
</tbody>
</table>

Figure 3-8 plots the numerically generated gas rate response from CMG-IMEX against the liquid rate predicted by modified exponential and density-based models for dual-porosity systems. Two different liquid rates are observed to match each other in the second decline stage. This complies with the
density-based model assumption that the majority of production is supported by matrix fluids. Boundary dominated flow in both fracture and matrix are assumed as well, which contribute to the mismatch at the early stage. Similar to the density-based models for gas, the density-based models for liquid may predict distorted liquid rate as matrix pressure closely approaches fracture pressure. Decreasing pressure difference decreases interporosity flow and removes the dominance of matrix fluids in production, which dissatisfies the assumption of the density-based model. As fracture porosity decreases, the error gradually diminishes.

Figure 3-8 Comparison of Gas Rate Forecast of a Dual-Porosity System between Proposed Analytical Model and Numerical Simulation for Variable Drawdown System (Case Study E)

The above study validated the applicability of exponential and density-based decline models for liquid dual-porosity systems. Therefore, OOIP prediction could be accomplished using Equation 3-51. Unlike the gas case, this procedure is not iterative for liquid due to the absence of $\overline{\lambda}_m$, $N_P$ vs. $r_P$ in Cartesian plot is fitted with a straight line in the second decline stage. We obtain OOIP by inversing the gradient of the fitted straight line. The resulting plot in Figure 3-9 is with slope $1.2060 \times 10^{-8}$ STB$^{-1}$, which yields an OOIP prediction of 82.916 MMSTB. The intercept for the fitted straight line, namely $\frac{1}{q_{odi}}$, is $6.5092E-07$ (STB/D)$^{-1}$. The derived OOIP and the actual OOIP, 82.664 MMSTB, demonstrate a close
match. Therefore, we conclude that the density-based approach is applicable to liquid production at variable drawdown cases in dual-porosity systems.

![Graph](image)

**Figure 3-9** $\frac{N_p}{q_{osc}}$ vs. $\frac{r_p}{q_{osc}}$ - Straight-Line Analysis for Variable Drawdown Dual-Porosity Liquid System (Case Study E)

### 3.7.3 Case Study F: Dual-Porosity Liquid System Producing at Variable $q_{osc}$

We apply the density-based model to a variable rate scenario detailed in Table 3-6. The reservoir and fluids properties remain the same as that in Table 3-1. Production data is generated by DPS 2. $r_p$ is analytically calculated at each time step by applying production data into Equation 3-49.
Table 3-6: $q_{osc}$ Schedule for Case Study F

<table>
<thead>
<tr>
<th>$t$, days</th>
<th>$q_{osc}$, STB/D</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-100</td>
<td>600</td>
</tr>
<tr>
<td>100-200</td>
<td>800</td>
</tr>
<tr>
<td>200-300</td>
<td>500</td>
</tr>
<tr>
<td>300-400</td>
<td>700</td>
</tr>
<tr>
<td>400-700</td>
<td>400</td>
</tr>
</tbody>
</table>

**Figure 3-10**: Density Drawdown Ratio Comparison between Density-Based Model and Numerical Simulation of Liquid Dual-Porosity System Producing at Variable $q_{osc}$ (Case Study F)

Figure 3-10 plots $p_{wf}$-derived $r_p$ from the CMG-IMEX against the $r_p$ from a density based model, demonstrating near match in the second decline stage. The mismatch at the early stage lies with the assumption that matrix fluid flow into the fracture supports the production from the well, and fracture fluid flow is boundary-dominated. With depleting fracture fluid, the dual-porosity system behaves
closer to our analytical model and demonstrates near match in the second decline stage. The small discontinuities in Figure 3-10 denote altered $p_{wf}$ due to $q_{gsc}$ change. The analytical model at variable $q_{osc}$ is less likely to produce distorted liquid with large enough $q_{osc}$ because considerable interporosity flow is in place to support the liquid production for variable rate scenarios.

With the applicability of a density-based model validated, we take $\frac{N_p}{q_{osc}}$ vs. $\frac{r_p}{q_{osc}}$ straight-line analysis for OGIP prediction. The straight line trend is captured in Figure 3-11. Observable discontinuities stem from sudden changes of $q_{osc}$. The slope of the fitted straight line is $1.2116 \times 10^{-8}$ STB$^{-1}$. Hence the OOIP prediction is 82.534 MMSTB, demonstrating near match against actual OOIP, 82.534 MMSTB. $\frac{1}{q_{odi}}$ is fitted to be $6.1777 \times 10^{-7}$ (STB/D)$^{-1}$. Thus, we conclude that the density-based approach is applicable to variable $q_{sc}$ cases in liquid dual-porosity systems.

![Figure 3-11](image)

**Figure 3-11** $\frac{N_p}{q_{osc}}$ vs. $\frac{r_p}{q_{osc}}$ Straight-Line Analysis for Variable $q_{osc}$ Dual-Porosity System (Case Study F)
3.8 Concluding Remarks

This study developed a density-based, rescaled exponential model for single-porosity gas systems and proves its applicability for dual-porosity gas systems in the second decline stage. The validation demonstrates a good match between simulation results and the proposed model for a variety of scenarios including variable pressure-drawdown, constant rate, and variable rate conditions. Straight-line analysis for OGIP prediction and production forecast in dual-porosity gas systems has been developed and validated for the same scenarios. Both the rescaled exponential model and the straight-line analysis demonstrated capability to capture pressure change, continuous or discrete, at the bottom hole in the second decline stage. The model adopted a pseudo-steady state gas interporosity flow equation, which was rigorously developed for compressible fluids. The density-based model and associated straight-line analysis served as a convenient tool for production data analysis in naturally fractured gas reservoirs. This study also derived a density-based, exponential model for dual-porosity liquid systems, which applies to variable pressure-drawdown/variable rate conditions and, inferentially, constant rate/drawdown scenarios in the second decline stage. By comparing simulation results from CMG-IMEX and those from the density-based model, the validity of the density-based, exponential model was verified. Straight-line analysis based on the analytical model was developed and verified for variable pressure-drawdown/variable rate scenarios. It provided an easy tool for OOIP prediction and oil rate forecast in a dual-porosity system with no iteration required. Moreover, we proposed the double-exponential model for constant bottom-hole pressure production in dual-porosity systems. This model produced an accurate production rate in the decline stage without numerical inversion of an analytical solution in Laplace Space (Da Prat, 1981).
Conclusions and Recommendations

This study demonstrates that the gas production performance at the second decline stage in dual-porosity gas systems can be successfully captured using density-based decline curve analysis. This approach enables an accurate prediction of production rate and OGIP by circumventing nonlinearity in the governing equations of dual-porosity gas systems. The density-based decline curve analysis applies to not only constant pressure drawdown/rate cases but also variable pressure drawdown/variable rate cases. We advocate a pseudo-pressure-based interporosity flow equation for gas for modeling gas transfer between the matrix and fracture because this equation account for pressure-dependent effects of gas. We also provide a rigorous derivation for the equation. An in-house simulator for naturally fractured gas systems is built based on dual-porosity model and the proposed interporosity flow equation. The production results of the in-house simulator demonstrate apparent deviation from those of CMG-IMEX which employs a liquid-form interporosity flow equation, highlighting the effects of pressure-dependent effects of gas. Using depletion-driven parameters λ and β, this study shows that the gas reservoir production behavior could be rescaled from its liquid counterparts. Successful matches between dimensionless gas flow rate and rescaled dimensionless flow rate are obtained under constant bottom-hole pressure. We also obtain a new deliverability equation for dual-porosity gas system. The rigorous representation exponential decline for liquid at the second decline stage is presented.

The study then derived a convenient method for decline curve analysis for variable pressure drawdown/rate inner boundary conditions in dual-porosity systems, of which constant pressure drawdown/rate are special cases. This model features explicit calculation of OOIP and rigorous oil rate prediction. Moreover, we derive a double-exponential model for convenient calculation of liquid analytical solution (Da Prat, 1981) without numerical inversion.

The implementation of pseudo-functions successfully linearizes the governing equations in dual-porosity gas system using production data of the in-house simulator, implying the applicability of pseudo-function-based decline curve analysis. More work is to be done to fully understand the role pseudo-functions could play in decline curve analysis.

Appendix E presents the type curve for wide-open production in dual-porosity gas system, which provides an alternative to OGIP prediction. This appendix also introduces a different technique for decline analysis and validates it. An approximation to viscosity-compressibility ratio between fracture pressure and matrix pressure for methane-dominated gas production is proposed in Appendix G. This approximation significantly reduces the amount of work required for calculating λ.

Some cues for further study include considerations of compressibility effects in the development of density-based model, integration of desorption and slippage into the rescaling approach and decline
curve analysis method. Compressibility effects play an important role in reservoir with compaction effects. The compressibility effects may be incorporated by the modification of $\lambda$ and $\beta$. More work is required to fully capture the compressibility effects with density-based approach. Sorption and desorption are dominant mechanism for storing and releasing gas in large amounts of naturally fractured reservoirs, and slippage effects frequently characterizes gas flow as Darcy flow is inadequate to describe the gas flow behavior in many cases. More work is required to account for desorption and slippage in the density-based model for dual-porosity systems.
References


Appendix A  Density Diffusivity Equation Development and Liquid Solution in Density

This appendix presents symbolic derivation for density diffusivity equation and the liquid solution of density. Assuming radial reservoir, no connate water, then, for a control volume, we have:

\[
\frac{-\rho_f u_r (r + \Delta r) \theta h}{\Delta t} + \frac{\{\rho_f u_r - \Delta(\rho_f u_r)\} r \theta h}{\Delta t} - \frac{(r \Delta r \phi_f \rho_f)_{t+\Delta t} - (r \Delta r \phi_f \rho_f)_t}{\Delta t} = \frac{(\phi_f \rho_f)_{t+\Delta t} - (\phi_f \rho_f)_t}{\Delta t}
\]

Equation A-1

The mass flow rate into the control volume of fracture during time period \(\Delta t\) is as follows:

\[
m_{\text{in}} = -\rho_f u_r (r + \Delta r) \theta h
\]

Equation A-2

where \(u_r\) is flow velocity, \(m\) is mass flow.

The mass flow rate out of the control volume of fracture during time period \(\Delta t\) is as follows:

\[
m_{\text{out}} = -[\rho_f u_r - \Delta(\rho_f u_r)] r \theta h
\]

Equation A-3

The mass flow rate from matrix to fracture in the control volume during time period \(\Delta t\) is as follows:

\[
m_{\text{source}} = \frac{(r \Delta r \phi_f \rho_f)_{t+\Delta t} - (r \Delta r \phi_f \rho_f)_t}{\Delta t}
\]

Equation A-4

Rate of mass accumulation in the control volume may be written as follows:

\[
W_a = \frac{(r \Delta r \phi_f \rho_f)_{t+\Delta t} - (r \Delta r \phi_f \rho_f)_t}{\Delta t}
\]

Equation A-5

Now we can express conservation equation as:

\[
-\rho_f u_r (r + \Delta r) \theta h - \{\rho_f u_r - \Delta(\rho_f u_r)\} r \theta h - \frac{(r \Delta r \phi_f \rho_f)_{t+\Delta t} - (r \Delta r \phi_f \rho_f)_t}{\Delta t} = \frac{(\phi_f \rho_f)_{t+\Delta t} - (\phi_f \rho_f)_t}{\Delta t}
\]

Equation A-6

Dividing Equation A-6 by the bulk volume of the control volume, \(hr\Delta r\) gives:

\[
-\frac{\rho_f u_r (r + \Delta r)}{r \Delta r} + \frac{\rho_f u_r - \Delta(\rho_f u_r)}{\Delta r} - \frac{(\phi_m \rho_m)_{t+\Delta t} - (\phi_m \rho_m)_t}{\Delta t} = \frac{(\phi_f \rho_f)_{t+\Delta t} - (\phi_f \rho_f)_t}{\Delta t}
\]
Then we simplify the left side of Equation A-7 by factoring out \( \frac{1}{r \Delta r} \) for the first two terms, yielding:

\[
\frac{1}{r \Delta r} \left[ -\rho_f u_r (r + \Delta r) + \rho_f u_r r - \Delta (\rho_f u_r) r \right] - \frac{(\phi_m \rho_m)_{t+\Delta t} - (\phi_m \rho_m)_t}{\Delta t} = \frac{(\phi_f \rho_f)_{t+\Delta t} - (\phi_f \rho_f)_t}{\Delta t}
\]

Equation A-8

Canceling out terms in Equation A-8 gives:

\[
\frac{1}{r \Delta r} \left[ -\rho_f u_r \Delta r - \Delta (\rho_f u_r) r \right] - \frac{(\phi_m \rho_m)_{t+\Delta t} - (\phi_m \rho_m)_t}{\Delta t} = \frac{(\phi_f \rho_f)_{t+\Delta t} - (\phi_f \rho_f)_t}{\Delta t}
\]

Equation A-9

Canceling out and combining terms in Equation A-9 gives:

\[-\frac{1}{r} \left[ \rho_f u_r + \frac{\Delta (\rho_f u_r)}{\Delta r} r \right] - \frac{\Delta (\phi_m \rho_m)}{\Delta t} = \frac{\Delta (\phi_f \rho_f)}{\Delta t}
\]

Equation A-10

Taking limits of Equation A-10 gives:

\[-\frac{1}{r} \left[ \rho_f u_r + \frac{\partial (\rho_f u_r)}{\partial r} r \right] - \frac{\partial (\phi_m \rho_m)}{\partial t} = \frac{\partial (\phi_f \rho_f)}{\partial t}
\]

Equation A-11

Equation A-11 can be further written as follows:

\[-\frac{1}{r} \left[ \frac{\partial}{\partial r} (r \rho_f u_r) \right] - \frac{\partial (\phi_m \rho_m)}{\partial t} = \frac{\partial (\phi_f \rho_f)}{\partial t}
\]

Equation A-12

**Liquid Diffusivity Equation**

Substituting Darcy’s law into Equation A-12 gives:

\[-\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( -r \rho_f \frac{k_f \rho_f}{\mu} \frac{\partial p_f}{\partial r} \right) \right] - \frac{\partial (\phi_m \rho_m)}{\partial t} = \frac{\partial (\phi_f \rho_f)}{\partial t}
\]

Equation A-13

Canceling out minus sign in Equation A-13 gives:

\[\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \rho_f \frac{k_f \rho_f}{\mu} \frac{\partial p_f}{\partial r} \right) \right] - \frac{\partial (\rho_m \phi_m)}{\partial t} = \frac{\partial (\rho_f \phi_f)}{\partial t}
\]

Equation A-14

Expanding terms in Equation A-14 gives:

\[
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \rho_f \frac{k_f \rho_f}{\mu} \frac{\partial \rho_f}{\partial r} \right) \right] - \frac{\partial (\rho_m \phi_m)}{\partial t} \frac{\partial \rho_m}{\partial \rho_f} \frac{\partial \rho_m}{\partial \rho_f} = \frac{\partial (\rho_f \phi_f)}{\partial \rho_f} \frac{\partial \rho_f}{\partial \rho_f} \frac{\partial \rho_f}{\partial \rho_f}
\]
Expanding $\frac{\partial (\rho_m \phi_m)}{\partial p_m}$ and $\frac{\partial (\rho_f \phi_f)}{\partial p_f}$ in Equation A-15 gives:

$$
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \rho_f \frac{k_f \partial p_f}{\mu} \frac{\partial p_f}{\partial r} \right) \right] - \left( \rho_m \frac{\partial \phi_m}{\partial p_m} + \phi_m \frac{\partial \rho_m}{\partial p_m} \right) \frac{\partial p_m}{\partial t} \frac{\partial \rho_m}{\partial t} = \left( \frac{\partial \phi_f}{\partial p_f} + \frac{\partial \rho_f}{\partial p_f} \right) \frac{\partial \rho_f}{\partial t}
$$

Equation A-16

Taking $\frac{\partial (\rho_f \phi_f)}{\partial p_f}$ and $\frac{\partial p_f}{\partial p_f}$ into adjacent brackets respectively in Equation A-16 gives:

$$
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \phi_f \frac{1}{\mu} \frac{\partial p_f}{\partial r} \right) \right] - \phi_f \left( \frac{1}{\phi_f \frac{\partial \phi_f}{\partial p_f}} \frac{\partial \rho_f}{\partial \rho_f} + \frac{\partial \rho_f}{\partial \rho_f} \right) \frac{\partial \rho_f}{\partial t} = \phi_f \left( \frac{1}{\phi_f \frac{\partial \phi_f}{\partial p_f}} \frac{\partial \rho_f}{\partial \rho_f} \right) \frac{\partial \rho_f}{\partial t}
$$

Equation A-17

Substituting $c_l = \frac{1}{\phi_f \frac{\partial \rho_f}{\partial p_f}} = \frac{1}{\rho_m \frac{\partial \rho_m}{\partial p_m}}$ for liquid, $c_1 = \frac{1}{\phi_m \frac{\partial \phi_m}{\partial p_m}}$ and $c_2 = \frac{1}{\phi_f \frac{\partial \phi_f}{\partial p_f}}$ into Equation A-17 gives:

$$
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{k_f \partial p_f}{\mu c_l} \frac{\partial \rho_f}{\partial r} \right) \right] - \phi_m \left( 1 + \frac{c_1}{c_l} \right) \frac{\partial \rho_m}{\partial t} = \phi_f \left( 1 + \frac{c_2}{c_l} \right) \frac{\partial \rho_f}{\partial t}
$$

Equation A-18

Considering constant $\mu$ and constant $c_l$ by liquid assumption, we have:

$$
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial p_f}{\partial r} \right) \right] - \frac{\mu}{k_f} \phi_m (c_l + c_l) \frac{\partial \rho_m}{\partial t} = \frac{\mu}{k_f} \phi_f (c_l + c_l) \frac{\partial \rho_f}{\partial t}
$$

Equation A-19

Making $c_l + c_1 = c_m, c_l + c_2 = c_f$, we have:

$$
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial p_f}{\partial r} \right) \right] - \frac{\mu}{k_f} \phi_m c_m \frac{\partial \rho_m}{\partial t} = \frac{\mu}{k_f} \phi_f c_f \frac{\partial \rho_f}{\partial t}
$$

Equation A-20

If fracture and matrix are incompressible, the diffusivity equation could be written as follows:

$$
\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial p_f}{\partial r} \right) \right] - \frac{\mu}{k_f} \phi_m c_l \frac{\partial \rho_m}{\partial t} = \frac{\mu}{k_f} \phi_f c_l \frac{\partial \rho_f}{\partial t}
$$

Equation A-21
Gas Diffusivity Equation

Substituting Darcy’s law into Equation A-12 for gas gives:

$$\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \rho_f k_f \frac{\partial p_f}{\partial r} \right) \right] - \frac{\partial (\rho_m \phi_m)}{\partial t} = \frac{\partial (\rho_f \phi_f)}{\partial t}$$

Equation A-22

Expanding $\frac{\partial (\rho_m \phi_m)}{\partial t}$ and $\frac{\partial (\rho_f \phi_f)}{\partial t}$ in Equation A-22 gives:

$$\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \rho_f \phi_f \frac{\partial p_f}{\partial r} \right) \right] - \frac{\partial (\rho_m \phi_m)}{\partial t} = \frac{\partial (\rho_f \phi_f)}{\partial t}$$

Equation A-23

Expanding $\frac{\partial (\rho_m \phi_m)}{\partial p_m}$ and $\frac{\partial (\rho_f \phi_f)}{\partial p_f}$ in Equation A-23 gives:

$$\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \rho_f \phi_f \frac{\partial p_f}{\partial r} \right) \right] - \frac{\partial (\rho_m \phi_m)}{\partial t} = \frac{\partial (\rho_f \phi_f)}{\partial t}$$

Equation A-24

Taking $\frac{\partial (\rho_f \phi_f)}{\partial p_f}$ and $\frac{\partial p_f}{\partial \rho_f}$ into adjacent brackets respectively in Equation A-24 gives:

$$\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \rho_f \phi_f \frac{\partial p_f}{\partial r} \right) \right] - \frac{\partial (\rho_m \phi_m)}{\partial t} = \frac{\partial (\rho_f \phi_f)}{\partial t}$$

Equation A-25

Substituting $c_{gf} = \frac{1}{\rho_f} \frac{\partial p_f}{\partial \rho_f}$, $c_{gm} = \frac{1}{\rho_m} \frac{\partial p_m}{\partial \rho_m}$, $c_1 = \frac{1}{\phi_m} \frac{\partial \phi_m}{\partial \rho_m}$ and $c_2 = \frac{1}{\phi_f} \frac{\partial \phi_f}{\partial \rho_f}$ into Equation A-17 gives:

$$\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{k_f}{\mu_{gf} c_{gf}} \frac{\partial \rho_f}{\partial r} \right) \right] - \phi_m \left( 1 + \frac{c_1}{c_{gm}} \right) \frac{\partial \rho_m}{\partial t} = \phi_f \left( 1 + \frac{c_2}{c_{gf}} \right) \frac{\partial \rho_f}{\partial t}$$

Equation A-26

in which $c_1 = \frac{1}{\phi_m} \frac{\partial \phi_m}{\partial \rho_m}$ and $c_2 = \frac{1}{\phi_f} \frac{\partial \phi_f}{\partial \rho_f}$

Liquid Solution in Density

We know from Appendix B that interporosity flow equation for liquid in density is as follows:
\[
\phi_m c_m \frac{\partial \rho_m}{\partial t} = \frac{\alpha k_m}{\mu} (\rho_f - \rho_m)
\]

Equation A-27

Writing Equation A-20 in dimensionless form gives:

\[
\frac{1}{r_D} \left[ \frac{\partial}{\partial r_D} \left(r_D \frac{\partial \rho_D f}{\partial r_D} \right) \right] = (1 - \omega) \frac{\partial \rho_D m}{\partial t_D} + \omega \frac{\partial \rho_D f}{\partial t_D}
\]

Equation A-28

\[
(1 - \omega) \frac{\partial \rho_D m}{\partial t_D} = \xi (\rho_D f - \rho_D m)
\]

Equation A-29

Where \( t_D = \frac{k_f t}{\mu_i c_i (\phi_m + \phi_f) r_w} \), \( \rho_D f = \frac{\rho_i - \rho_f}{\rho_i} \), and \( \rho_D m = \frac{\rho_i - \rho_m}{\rho_i} \)

The boundary condition in dimensionless form is:

\[
\rho_D f(r_D, 0) = 0
\]

Equation A-30

\[
(\rho_D f)_{r_D=1} - s \left( \frac{\partial \rho_D f}{\partial r_D} \right)_{r_D=1} = 1
\]

Equation A-31

\[
\lim_{r_D \to r_e} \rho_D f(r_D, t_D) = 0
\]

Equation A-32

Applying Laplace transform to the Equation 28 to 32, the dimensionless density and dimensionless flow rate in Laplace space may be solved and written as follows:

\[
\tilde{\rho}_D
\]

\[
= \frac{K_i(\sqrt{sf(s)} r_e) l_0(\sqrt{sf(s)} r_0) + l_1(\sqrt{sf(s)} r_e) K_0(\sqrt{sf(s)} r_0)}{s(l_0(\sqrt{sf(s)} K_1(\sqrt{sf(s)} r_e) + K_0(\sqrt{sf(s)} l_1(\sqrt{sf(s)} r_e) + \lim_{s \to 0} \sqrt{sf(s)} (K_1(\sqrt{sf(s)} l_1(\sqrt{sf(s)} r_e) - l_1(\sqrt{sf(s)} K_1(\sqrt{sf(s)} r_e))))}
\]

Equation A-33

\[
\tilde{q}_D
\]

\[
= \frac{\sqrt{sf(s)} l_1(\sqrt{sf(s)} r_e) K_1(\sqrt{sf(s)} r_0) - K_0(\sqrt{sf(s)} r_e) l_1(\sqrt{sf(s)} r_0)}{s(l_0(\sqrt{sf(s)} K_1(\sqrt{sf(s)} r_e) + K_0(\sqrt{sf(s)} l_1(\sqrt{sf(s)} r_e) + \lim_{s \to 0} \sqrt{sf(s)} (K_1(\sqrt{sf(s)} l_1(\sqrt{sf(s)} r_e) - l_1(\sqrt{sf(s)} K_1(\sqrt{sf(s)} r_e))))}
\]

Equation A-34

\[
f(s) = \frac{\omega (1 - \omega) s + \xi}{(1 - \omega) s + \xi}
\]

Equation A-35
Appendix B  Interporosity Flow Equation for Liquid and Gas

This appendix derived symbolically interporosity flow equation for liquid and gas in different forms. Zimmerman (1993) demonstrated derivation for interporosity flow equation in Warren and Root form. Lim and Aziz (1995) derived shape factor for different geometries of matrix block. The interporosity flow equations in density or pseudo-pressure are developed in a similar approach in this appendix for spherical matrix block.

Interporosity Flow Equation in Warren and Root Model

This derivation follows similar approach as Zimmerman (1993) and Lim and Aziz (1995). Diffusivity equation in density for gas in matrix:

\[-\nabla \cdot (\rho_m u_{mr}) = \frac{\partial (\phi_m \rho_m)}{\partial t}\]

Equation B-1

Substituting Darcy’s law into Equation B-1 gives:

\[\nabla \cdot (\rho_m \frac{k_m}{\mu} \nabla p_m) = \frac{\partial (\phi_m \rho_m)}{\partial t}\]

Equation B-2

Factoring out \(\rho_m \frac{k_m}{\mu}\) of divergence and rearranging terms gives:

\[\nabla \cdot (\nabla p_m) = \frac{\phi_m \mu}{k_m} \frac{1}{\rho_m} \frac{\partial (\rho_m)}{\partial t}\]

Equation B-3

Substituting \(c_1 = \frac{1}{\rho_m} \frac{\partial \rho_m}{\partial t}\) into Equation B-3 gives:

\[\nabla^2 p_m = \frac{\phi_m \mu c_i}{k_m} \frac{\partial (p_m)}{\partial t}\]

Equation B-4

Writing Equation B-4 into:

\[\frac{k_m}{\phi_m \mu c_i} \nabla^2 p_m = \frac{\partial (p_m)}{\partial t}\]

Equation B-5

For a spherical system, Equation B-5 is written as follows:

\[\frac{\partial p_m}{\partial t} = \frac{k_m}{\mu c_i \phi_m} \left(\frac{\partial^2 p_m}{\partial r^2} + \frac{2}{r} \frac{\partial p_m}{\partial r}\right)\]
Following Crank (1975), we take \( u(r, t) = p_m r \) and obtain:

\[
\frac{\partial u}{\partial t} = -\frac{k_m}{\mu c_l \phi_m} \frac{\partial^2 u}{\partial r^2}
\]

We have the boundary condition:

\[
u(0, t) = 0
\]

\[
u(a_m, t) = a_m p_f
\]

\[
u(r, 0) = r p_i
\]

where quasi-steady assumption has been taken for \( u(a_m, t) = a_m p_f \).

Then we have the average pressure expressed as follows:

\[
\frac{p_m^{\text{av}} - p_i}{p_f - p_i} = 1 - \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{\pi^2 k_m n^2 t}{\mu c_l \phi_m a_m^2}\right)
\]

Equation B-11

We then take the long-term approximation of Equation B-11, yielding the following equation:

\[
\frac{p_m^{\text{av}} - p_i}{p_f - p_i} = 1 - \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right)
\]

Equation B-12

Equation B-12 is rearranged to be as follows:

\[
\frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right) = 1 - \frac{p_m^{\text{av}} - p_i}{p_f - p_i} = \frac{p_f - p_m^{\text{av}}}{p_f - p_i}
\]

Equation B-13

Differentiating Equation B-13 with respect to \( t \) gives the following equation:

\[
\frac{d}{dt} \left( \frac{p_m^{\text{av}} - p_i}{p_f - p_i} \right) = \frac{d}{dt} \left( 1 - \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right) \right)
\]

Equation B-14

Simplifying \( \frac{d}{dt} \left( \frac{p_m^{\text{av}} - p_i}{p_f - p_i} \right) \) in Equation B-14 gives:

\[
\frac{1}{p_f - p_i} \frac{d(p_m^{\text{av}})}{dt} = \frac{d}{dt} \left( -\frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right) \right)
\]
Substituting \( \frac{d}{dt} \left[ \frac{-\frac{\pi^2 k_m t}{\mu c i \phi_m a_m^2}}{\rho_f - \rho_i} \right] \) in Equation B-15 gives:

\[
\frac{1}{\rho_f - \rho_i} \frac{d(p_{mav})}{dt} = \frac{6}{\pi^2} \exp \left( -\frac{\pi^2 k_m t}{\mu c_i \phi_m a_m^2} \right) \frac{\pi^2 k_m}{\mu c_i \phi_m a_m^2} (\rho_f - \rho_i)
\]

Equation B-16

Calculating \( \frac{d}{dt} \left[ \frac{-\frac{\pi^2 k_m t}{\mu c_i \phi_m a_m^2}}{\mu c_i \phi_m a_m^2} \right] \) in Equation B-16 gives:

\[
\frac{1}{\rho_f - \rho_i} \frac{d(p_{mav})}{dt} = \frac{6}{\pi^2} \exp \left( -\frac{\pi^2 k_m t}{\mu c_i \phi_m a_m^2} \right) \frac{\pi^2 k_m}{\mu c_i \phi_m a_m^2} \frac{\rho_f - \rho_i}{(\rho_f - \rho_i) 
\]

Equation B-17

Multiplying both sides by \( (\rho_f - \rho_i) \) in Equation B-17 gives:

\[
\frac{d(p_{mav})}{dt} = \frac{6}{\pi^2} \exp \left( -\frac{\pi^2 k_m t}{\mu c_i \phi_m a_m^2} \right) \frac{\pi^2 k_m}{\mu c_i \phi_m a_m^2} (\rho_f - \rho_i)
\]

Equation B-18

Substituting Equation B-13 into Equation B-18 gives:

\[
\frac{d(p_{mav})}{dt} = \frac{\rho_f - p_{mav}}{\rho_f - \rho_i} \frac{\pi^2 k_m}{\mu c_i \phi_m a_m^2} (\rho_f - \rho_i)
\]

Equation B-19

Canceling out terms in Equation B-20 gives:

\[
\phi_m \frac{d(p_{mav})}{dt} = \frac{\pi^2 k_m}{\mu c_i \phi_m a_m^2} (\rho_f - p_{mav})
\]

Equation B-20

Equation B-21 can be written as:

\[
\phi_m \frac{\partial p_{mav}}{\partial t} = \frac{k_m}{\mu c_i} (\rho_f - p_{mav})
\]

Equation B-22

where \( \alpha \) is shape factor which can be derived for different shapes. And \( \alpha = \frac{\pi^2}{a_m^2} \) in the case of spherical matrix with a radius \( a_m \). The shape factors for other shapes of matrix blocks may be easily derived.

The shape factors are constants.

Considering each point in a dual-porosity model corresponds to a different matrix gridblock, we use \( p_m \) to represent matrix block pressure at a certain point in dual-porosity model, yielding:

\[
\phi_m \frac{\partial p_m}{\partial t} = \frac{k_m}{\mu c_i} (\rho_f - p_m)
\]

where \( \alpha \) is shape factor which can be derived for different shapes. And \( \alpha = \frac{\pi^2}{a_m^2} \) in the case of spherical matrix with a radius \( a_m \). The shape factors for other shapes of matrix blocks may be easily derived. The shape factors are constants.
Interporosity Flow Equation for Liquid in Density

Diffusivity equation in density for gas in matrix is written as follows:

\[-\nabla \cdot (\rho_m u_m) = \frac{\partial (\phi_m \rho_m)}{\partial t}\]

Equation B-22

Substituting Darcy’s law into Equation B-23 gives:

\[\nabla \cdot (\rho_m \frac{k_m}{\mu} \nabla p_m) = \frac{\partial (\phi_m \rho_m)}{\partial t}\]

Equation B-23

Assuming incompressible matrix, Equation B-24 turns into:

\[\nabla \cdot \left( \frac{k_m}{\mu c_l} \nabla \rho_m \right) = \phi_m \frac{\partial \rho_m}{\partial t}\]

Equation B-24

Factoring out \(\frac{k_m}{\mu c_l}\) from the divergence in Equation B-25 gives:

\[\frac{k_m}{\mu c_l} \nabla \cdot (\nabla \rho_m) = \phi_m \frac{\partial \rho_m}{\partial t}\]

Equation B-25

Combining divergence operator and gradient operator in Equation B-26 gives:

\[\frac{k_m}{\mu c_l \phi_m} \nabla^2 \rho_m = \frac{\partial \rho_m}{\partial t}\]

Equation B-26

For a spherical system, Equation B-27 is written as follows:

\[\frac{\partial \rho_m}{\partial t} = \frac{k_m}{\mu c_l \phi_m} \left( \frac{\partial^2 \rho_m}{\partial r^2} + \frac{2 \partial \rho_m}{r \partial r} \right)\]

Equation B-27

Following Crank (1975), We take \(u(r, t) = \rho_m r\) to obtain the following equation:

\[\frac{\partial u}{\partial t} = \frac{k_m}{\mu c_l \phi_m} \frac{\partial^2 u}{\partial r^2}\]

Equation B-28

Also we have boundary condition as follows:

\[u(0, t) = 0\]

Equation B-29
\[ u(a_m, t) = a_m \rho_f \]
\[ u(r, 0) = r \rho_i \]

where quasi-steady assumption is taken for \( u(a_m, t) = a_m \rho_f \).

Then we have the average pressure expressed as follows:

\[ \frac{\rho_{av}}{\rho_i} = 1 - \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{\pi^2 k_m n^2 t}{\mu c_l \phi_m a_m^2}\right) \]

\[ \text{Equation B-33} \]

Taking the long-term approximation of Equation B-33 yields the equation below:

\[ \frac{\rho_{av} - \rho_i}{\rho_i} = 1 - \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right) \]

\[ \text{Equation B-34} \]

Equation B-34 is rearranged to be as follows:

\[ \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right) = 1 - \frac{\rho_{av} - \rho_i}{\rho_i} = \frac{\rho_{av} - \rho_i}{\rho_i} \]

\[ \text{Equation B-35} \]

Differentiating Equation B-35 with respect to \( t \) gives:

\[ \frac{d}{dt}\left(\frac{\rho_{av}}{\rho_i} - \rho_i\right) = \frac{d}{dt}\left(1 - \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right)\right) \]

\[ \text{Equation B-36} \]

Simplifying \[ \frac{d\left(\frac{\rho_{av} - \rho_i}{\rho_i - \rho_i}\right)}{dt} \] gives the following equation:

\[ \frac{1}{\rho_i} \frac{d(\rho_{av})}{dt} = \frac{d}{dt}\left(1 - \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right)\right) \]

\[ \text{Equation B-37} \]

Substituting \[ \frac{d\left(\frac{\pi^2 (\pi^2 k_m t)}{\mu c_l \phi_m a_m^2}\right)}{dt} = -\frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right) \]

in Equation B-37 gives the following equation:

\[ \frac{1}{\rho_i} \frac{d(\rho_{av})}{dt} = -\frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m t}{\mu c_l \phi_m a_m^2}\right) \]

\[ \text{Equation B-38} \]
Calculating \( \frac{d\left(-\frac{n^2kmt}{\mu c_i\phi_m a_m^2}\right)}{dt} \) in Equation B-38 gives the following equation:

\[
\frac{1}{\rho_f - \rho_i} \frac{d(\rho_m^{av})}{dt} = \frac{6}{\pi^2} \exp\left(-\frac{\pi^2k_m t}{\mu c_i\phi_m a_m^2}\right) \frac{\pi^2k_m}{\mu c_i\phi_m a_m^2}
\]

\text{Equation B-39}

Multiplying both sides by \((\rho_f - \rho_i)\) in Equation B-39 gives the following equation:

\[
\frac{d(\rho_m^{av})}{dt} = \frac{6}{\pi^2} \exp\left(-\frac{\pi^2k_m t}{\mu c_i\phi_m a_m^2}\right) \frac{\pi^2k_m}{\mu c_i\phi_m a_m^2} (\rho_f - \rho_i)
\]

\text{Equation B-40}

Substituting Equation B-35 into Equation B-40 gives the following equation:

\[
\frac{d(\rho_m^{av})}{dt} = \frac{\rho_f - \rho_m^{av}}{\rho_f - \rho_i} \frac{\pi^2k_m}{\mu c_i\phi_m a_m^2} (\rho_f - \rho_i)
\]

\text{Equation B-41}

Canceling out terms in Equation B-41 gives the following equation:

\[
\phi_m \frac{d(\rho_m^{av})}{dt} = \frac{\pi^2k_m}{\mu c_i a_m^2} (\rho_f - \rho_m^{av})
\]

\text{Equation B-42}

Equation B-42 can be written as follows:

\[
\phi_m \frac{\partial\rho_m^{av}}{\partial t} = \frac{k_m}{\mu c_i} (\rho_f - \rho_m^{av})
\]

\text{Equation B-43}

where \( \alpha \) is shape factor which can be derived for different shapes. \( \alpha = \frac{\pi^2}{a_m^2} \) is the shape factor for spherical matrix block with a radius \( a_m \). The shape factor for other shapes of matrix blocks can also be derived.

**Interporosity Flow Equation for Gas**

a) Derivation of Gas Interporosity Flow Equation with Density

Diffusivity equation in density for gas in matrix:

\[
-\nabla \cdot (\rho_m u_{mr}) = \frac{\partial(\phi_m \rho_m)}{\partial t}
\]

\text{Equation B-44}

Substituting Darcy’s law into Equation B-44 gives:
\[
\n\nabla \cdot (\rho_m \frac{k_m}{\mu_{gm}} \nabla p_m) = \frac{\partial (\phi_m \rho_m)}{\partial t}
\]

Equation B-45

Assuming incompressible matrix, we have from Equation B-45:

\[
\nabla \cdot (\frac{k_m}{\mu_{gm} c_{gm}} \nabla \rho_m) = \phi_m \frac{\partial \rho_m}{\partial t}
\]

Equation B-46

Expanding the divergence term in Equation B-46 gives:

\[
\nabla \left( \frac{k_m}{\mu_{gm} c_{gm}} \right) \nabla \rho_m + \frac{k_m}{\mu_{gm} c_{gm}} \nabla \cdot (\nabla \rho_m) = \phi_m \frac{\partial \rho_m}{\partial t}
\]

Equation B-47

Ye and Ayala (2012) showed intrinsic relationship between \(\lambda_f\) and density:

\[
\nabla \lambda_m^* = B \frac{\rho_m \nabla \rho_m}{\rho_m}
\]

Equation B-48

where

\[
\lambda_m^* = \frac{\mu_i c_i}{\mu_{gm} c_{gm}}
\]

Equation B-48a

Substituting B-48a into B-47 gives the following equation:

\[
\frac{k_m}{\mu_{gi} c_{gi}} \nabla (\lambda_m^*) \nabla \rho_m + \frac{k_m}{\mu_{gm} c_{gm}} \nabla \cdot (\nabla \rho_m) = \phi_m \frac{\partial \rho_m}{\partial t}
\]

Equation B-49

Substituting B-48 into B-49 gives the equation as follows:

\[
\frac{k_m}{\mu_{gi} c_{gi}} B \frac{\nabla \rho_m}{\rho_m} \nabla \rho_m + \frac{k_m}{\mu_{gm} c_{gm}} \nabla \cdot (\nabla \rho_m) = \phi_m \frac{\partial \rho_m}{\partial t}
\]

Equation B-50

Equation B-50 can be written as follows:

\[
\frac{k_m}{\mu_{gi} c_{gi}} \frac{B (\nabla \rho_m)^2}{\rho_m} + \frac{k_m}{\mu_{gm} c_{gm}} \nabla \cdot (\nabla \rho_m) = \phi_m \frac{\partial \rho_m}{\partial t}
\]

Equation B-51

Take approximation that term with \((\nabla \rho_m)^2\) is negligible, Equation B-51 is written into:

\[
\frac{k_m}{\mu_{gm} c_{gm}} \nabla \cdot (\nabla \rho_m) = \phi_m \frac{\partial \rho_m}{\partial t}
\]

Equation B-52
Equation B-52 is then written into:

$$\frac{k_m}{\mu_{gm} c_{gm}} \nabla^2 \rho_m = \phi_m \frac{\partial \rho_m}{\partial t}$$

Equation B-53

Assume spherical matrix with radius $a_m$, we then solve the diffusivity equation with boundary conditions.

Expanding Equation B-53 to spherical coordinates gives:

$$\frac{\partial \rho_m}{\partial t} = \frac{k_m}{\mu_{gm} c_{gm} \phi_m} \nabla^2 \rho_m = \frac{k_m}{\mu_{gm} c_{gm} \phi_m} \left( \frac{\partial^2 \rho_m}{\partial r^2} + \frac{2 \partial \rho_m}{\partial r} \right)$$

Equation B-54

Boundary conditions are as follows:

$$\rho_m(x_m, t = 0) = \rho_i$$

Equation B-55

$$\rho_m(|x| = a_m, t > 0) = \rho_f$$

Equation B-56

Taking $u(r, t) = \rho_m r$, we have the following equation from Equation B-30:

$$\frac{\partial u}{\partial t} = \frac{k_m}{\mu_{gm} c_{gm} \phi_m} \frac{\partial^2 u}{\partial r^2}$$

Equation B-57

Or:

$$\frac{\partial u}{\lambda_m^* \partial t} = \frac{k_m}{\mu_{gi} c_{gi} \phi_m} \frac{\partial^2 u}{\partial r^2}$$

Equation B-58

Defining the following relationship:

$$\beta_m^* = \frac{\int_0^t \lambda_m^* dt}{t}$$

Equation B-59

where $\beta_m^* t$ is normalized pseudo-time in terms of $\lambda_m^*$, which is $\lambda_m^*$ corresponding to average pressure in the matrix block. For gas reservoirs, the average reservoir pressure is utilized to evaluate pseudo-time, which works effectively at boundary-dominated-flow stage.

Equation B-58 is written into:

$$\frac{\partial u}{\partial (\beta_m^* t)} = \frac{k_m}{\mu_{gi} c_{gi} \phi_m} \frac{\partial^2 u}{\partial r^2}$$

Equation B-60
Note that this $\beta_m^*$ is a spaced-averaged property.

Also we have boundary condition as follows:

$$u(0, \beta_m^* t) = 0$$  \hspace{1cm} \text{Equation B-61}

$$u(a_m, \beta_m^* t) = a_m \rho_f$$  \hspace{1cm} \text{Equation B-62}

$$u(r, 0) = r \rho_i$$  \hspace{1cm} \text{Equation B-63}

Solving the ODE gives:

$$\frac{\rho_m^{av} - \rho_i}{\rho_f - \rho_i} = 1 - \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp(-\frac{\pi^2 k_m n^2 \beta_m^* t}{\mu g_i c g_i \phi_m a_m^2})$$  \hspace{1cm} \text{Equation B-64}

Truncating polynomial on the RHS of Equation B-64 to the first term and taking differential with respect to time in Equation B-64 gives:

$$\frac{1}{\rho_f - \rho_i} \frac{d(\rho_m^{av})}{dt} = \frac{6}{\pi^2} \exp(-\frac{\pi^2 k_m \beta_m^* t}{\mu g_i c g_i \phi_m a_m^2}) \frac{\pi^2 k_m}{\mu g_i c g_i \phi_m a_m^2} \lambda_m^*$$  \hspace{1cm} \text{Equation B-65}

Truncating the first term in the polynomial in Equation B-64 and moving terms gives:

$$\frac{6}{\pi^2} \exp(-\frac{\pi^2 k_m \beta_m^* t}{\mu g_i c g_i \phi_m a_m^2}) = \frac{\rho_f - \rho_m^{av}}{\rho_f - \rho_i}$$  \hspace{1cm} \text{Equation B-66}

Substituting Equation B-66 into Equation B-65 gives:

$$\frac{1}{\rho_f - \rho_i} \frac{d(\rho_m^{av})}{dt} = \frac{\rho_f - \rho_m^{av}}{\rho_f - \rho_i} \frac{\pi^2 k_m}{\mu g_i c g_i \phi_m a_m^2} \lambda_m^*$$  \hspace{1cm} \text{Equation B-67}

Canceling out terms in Equation B-67 gives:

$$\frac{d(\rho_m^{av})}{\lambda_m dt} = \frac{\pi^2 k_m}{\mu g_i c g_i \phi_m a_m^2} (\rho_f - \rho_m^{av})$$  \hspace{1cm} \text{Equation B-68}

Since $\frac{\pi^2}{a_m^2}$ is the shape factor, we can write the above equation in a more general form as follows:

$$\frac{d(\rho_m^{av})}{\lambda_m dt} = \frac{a k_m}{\mu g_i c g_i \phi_m} (\rho_f - \rho_m^{av})$$  \hspace{1cm} \text{Equation B-69}
where \( \alpha \) is the shape factor, and it can be derived for different geometry of matrix.

Since matrix is treated as points when solving the equations, replacing average pseudo-pressure in matrix volume, \( m_{\text{avg}}(p_m) \), and \( \lambda^*_m \) with point-specific matrix pseudo-pressure and point-specific \( \lambda^*_m \) respectively gives:

\[
\frac{d(p_m)}{\lambda^*_m dt} = \frac{\alpha k_m}{\mu_g c_g \phi_m} (P_f - p_m)
\]

Equation B-70

b) Derivation of Gas Interporosity Flow Equation in Pseudo-Pressure

Diffusivity equation in density for gas in matrix:

\[
-\frac{1}{r} \frac{\partial}{\partial r} (r \rho_m u_m) = \frac{\partial (\phi_m \rho_m)}{\partial t}
\]

Equation B-71

Substituting Darcy’s law into Equation B-71 gives:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{k_m}{\mu_g c_g} \frac{\partial p_m}{\partial r} \right) = \frac{\partial (\phi_m \rho_m)}{\partial t}
\]

Equation B-72

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{k_m}{\mu_g c_g} \frac{\partial \rho_m}{\partial r} \right) = \phi_m \frac{\partial \rho_m}{\partial t}
\]

Equation B-73

Multiplying both sides of Equation B-73 by \( \theta \) and adding term \( \mu_g c_g \frac{1}{\mu_g c_g} \) at the RHS gives:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r k_m \theta \frac{1}{\mu_g c_g} \frac{\partial \rho_m}{\partial r} \right) = \phi_m \mu_g c_g \theta \frac{1}{\mu_g c_g} \frac{\partial \rho_m}{\partial t}
\]

Equation B-74

where \( \theta = RT/MW \)

Remember definition for pseudo-pressure in matrix:

\[
dm(p_m) = 2\theta \frac{1}{\mu_g c_g} d\rho_m
\]

Equation B-75

Substituting Equation B-75 into Equation B-74 gives:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r k_m \frac{\partial m(p_m)}{\partial r} \right) = \phi_m \mu_g c_g \frac{\partial m(p_m)}{\partial t}
\]

Equation B-76
Dividing both sides of Equation B-76 by $k_m$ gives:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial m(p_m)}{\partial r} \right) = \frac{\phi_m \mu c g_m}{k_m} \frac{\partial m(p_m)}{\partial t}$$

Equation B-77

Writing Equation B-77 in terms of operators gives:

$$\nabla^2 m(p_m) = \frac{\phi_m \mu c g_m}{k_m} \frac{\partial m(p_m)}{\partial t}$$

Equation B-78

Dividing both sides of Equation B-78 by $\frac{\phi_m \mu c g_m}{k_m}$ gives:

$$\frac{\partial m(p_m)}{\partial t} = \frac{k_m}{\phi_m \mu c g_i} \nabla^2 m(p_m)$$

Equation B-79

Take spherical matrix shape with radius $a_m$. The fracture is around the matrix. The pressure at matrix surface is equal to fracture pressure.

Substituting definition of $\lambda_m^*$ into Equation B-79 gives:

$$\frac{\partial m(p_m)}{\partial t} = \frac{k_m}{\phi_m \mu c g_i} \lambda_m^* \nabla^2 m(p_m)$$

Equation B-80

Dividing both sides of Equation B-80 by $\lambda_m^*$ gives:

$$\frac{\partial m(p_m)}{\lambda_m^* \partial t} = \frac{k_m}{\phi_m \mu c g_i} \nabla^2 m(p_m)$$

Equation B-81

Substituting Equation B-59 into Equation B-81 gives:

$$\frac{\partial m(p_m)}{\partial (\beta_m^* t)} = \frac{k_m}{\phi_m \mu c g_i} \nabla^2 m(p_m)$$

Equation B-82

Expanding Equation B-82 in spherical coordinates gives:

$$\frac{\partial m(p_m)}{\partial (\beta_m^* t)} = \frac{k_m}{\mu c g_i \phi_m} \nabla^2 m(p_m) = \frac{k_m}{\mu c g_i \phi_m} \left( \frac{\partial^2 m(p_m)}{\partial r^2} + \frac{2 m(p_m)}{r} \frac{\partial m(p_m)}{\partial r} \right)$$

Equation B-83

Taking $u(r, t) = m(p_m) r$, we have from Equation B-83:

$$\frac{\partial u}{\partial (\beta_m^* t)} = \frac{k_m}{\mu c g_i \phi_m} \frac{\partial^2 u}{\partial r^2}$$

Equation B-84
Take spherical matrix shape with radius $a_m$. The fracture is around the matrix. And the pressure at matrix surface is equal to fracture pressure.

Also we have boundary condition:

$$u(0, \beta_m^* t) = 0$$  \hspace{1cm} \text{Equation B-85}$$

$$u(a_m, \beta_m^* t) = a_m m(p_m)$$  \hspace{1cm} \text{Equation B-86}$$

$$u(r, 0) = rm(p_m)$$  \hspace{1cm} \text{Equation B-87}$$

Solving Equation B-84 to B-87 gives:

$$\frac{m_{avg}(p_m) - m(p_m)}{m(p_f) - m(p_i)} = 1 - \frac{6}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp\left(-\frac{\pi^2 k_m n^2 \beta_m^* t}{\mu_g c_g \phi_m a_m^2}\right)$$  \hspace{1cm} \text{Equation B-88}$$

Long-term approximation truncates to the first term of the polynomial, giving:

$$\frac{m_{avg}(p_m) - m_m(p_i)}{m(p_f) - m(p_i)} = 1 - \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m \beta_m^* t}{\mu_g c_g \phi_m a_m^2}\right)$$  \hspace{1cm} \text{Equation B-89}$$

Moving terms in Equation B-89 gives:

$$\frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m \beta_m^* t}{\mu_g c_g \phi_m a_m^2}\right) = \frac{m(p_f) - m_{avg}(p_m)}{m(p_f) - m(p_i)}$$  \hspace{1cm} \text{Equation B-90}$$

Take derivative of Equation B-90 with respect to $\beta_m^* t$ gives:

$$\frac{1}{m(p_f) - m(p_i)} \frac{d}{d(\beta_m^* t)} \left(\frac{m_{avg}(p_m)}{\beta_m^* t}\right) = \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m \beta_m^* t}{\mu_g c_g \phi_m a_m^2}\right) \frac{\pi^2 k_m}{\mu_g c_g \phi_m a_m^2}$$  \hspace{1cm} \text{Equation B-91}$$

Writing $\frac{d(m_{avg}(p_m))}{d(\beta_m^* t)}$ in Equation B-91 as $\frac{d(m_{avg}(p_m))}{dt} \frac{dt}{d(\beta_m^* t)}$ gives:

$$\frac{1}{m(p_f) - m(p_i)} \frac{d}{dt} \left(\frac{m_{avg}(p_m)}{\beta_m^* t}\right) = \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m \beta_m^* t}{\mu_g c_g \phi_m a_m^2}\right) \frac{\pi^2 k_m}{\mu_g c_g \phi_m a_m^2}$$  \hspace{1cm} \text{Equation B-92}$$

Substituting Equation B-59 into Equation B-92 gives:

$$\frac{1}{m(p_f) - m(p_i)} \frac{d}{dt} \left(\frac{m_{avg}(p_m)}{\beta_m^* t}\right) = \frac{6}{\pi^2} \exp\left(-\frac{\pi^2 k_m \beta_m^* t}{\mu_g c_g \phi_m a_m^2}\right) \frac{\pi^2 k_m}{\mu_g c_g \phi_m a_m^2} \chi_m$$
Substituting Equation B-90 into Equation B-93 gives:

\[
\frac{1}{m(p_f) - m(p_i)} \frac{d(m_{avg}(p_m))}{dt} = \frac{m(p_f) - m_{avg}(p_m)}{m(p_f) - m(p_i)} \frac{\pi^2 k_m}{\mu_{gi} c_{gi} \phi_m a_m^2} \lambda_m^*
\]

Canceling terms in Equation B-94 gives:

\[
\frac{d(m_{avg}(p_m))}{\lambda_m^* dt} = \frac{\pi^2 k_m}{\mu_{gi} c_{gi} \phi_m a_m^2} (m(p_f) - m_{avg}(p_m))
\]

\[\frac{\pi^2}{a_m^2}\] is a the shape factor that changes with the geometry of matrix. replacing it with \(\alpha\):

\[
\phi_m \frac{d(m_{avg}(p_m))}{\lambda_m^* dt} = \frac{\alpha k_m}{\mu_{gi} c_{gi}} (m(p_f) - m_{avg}(p_m))
\]

We replace average pseudo-pressure in matrix volume, \(m_{avg}(p_m)\), with point-specific matrix pseudo-pressure and \(\lambda_m^*\), with point-specific viscosity-compressibility ratio \(\lambda_m^*\), since matrix is treated as points in dual-porosity system. You could understand this as each point representing a matrix block in a dual-porosity model.

\[
\phi_m \frac{d(m(p_m))}{\lambda_m^* dt} = \frac{\alpha k_m}{\mu_{gm} c_{gm}} (m(p_f) - m(p_m))
\]

Application of definition of \(\lambda_m^*\) into Equation B-97 gives:

\[
\phi_m \frac{d(m(p_m))}{dt} = \frac{\alpha k_m}{\mu_{gm} c_{gm}} (m(p_f) - m(p_m))
\]

Which is in the similar form to Gerami et al.'s (2007) equation if matrix compressibility, fracture compressibility, and connate water saturation are negligible.

Recalling:

\[
d m(p_m) = 2\theta \frac{1}{\mu_{gm} c_{gm}} d \rho_m
\]

Substituting Equation B-99 into Equation B-98 gives:

\[
\phi_m \frac{2\theta \frac{1}{\mu_{gm} c_{gm}} d \rho_m}{dt} = \frac{\alpha k_m}{\mu_{gm} c_{gm}} (m(p_f) - m(p_m))
\]
There is more than one way to adapt the interporosity flow equation in pseudo-pressure. Define:

\[
\lambda_{mf}^* = \frac{\mu_{gi}c_{gi}}{2\theta(p_m - p_f)} \frac{m(p_m) - m(p_f)}{m(p_m) - m(p_f)}
\]

Equation B-102

Then we have:

\[
m(p_f) - m(p_m) = \frac{2\theta(p_f - p_m)\lambda_{mf}^*}{\mu_{gi}c_{gi}}
\]

Equation B-103

Interporosity flow equation for pseudo-pressure can be written as:

\[
\phi_m \frac{d(m(p_m))}{dt} = \frac{ak_m}{\mu_{gm}c_{gm}} \frac{2\theta(p_f - p_m)\lambda_{mf}^*}{\mu_{gi}c_{gi}}
\]

Equation B-104

Recall:

\[
dm(p_m) = 2\theta \frac{1}{\mu_{gm}c_{gm}} d\rho_m
\]

Equation B-105

Substituting Equation B-105 into Equation B-104 gives:

\[
\frac{2\theta}{\mu_{gm}c_{gm}} \frac{1}{d\rho_m} \frac{d\rho_m}{dt} = \frac{ak_m}{\mu_{gm}c_{gm}} \frac{2\theta(p_f - p_m)\lambda_{mf}^*}{\mu_{gi}c_{gi}}
\]

Equation B-106

Canceling out terms in Equation B-106 gives:

\[
\phi_m \frac{d\rho_m}{\lambda_{mf}^* dt} = \frac{ak_m}{\mu_{gi}c_{gi}} (\rho_f - \rho_m)
\]

Equation B-107

A relation for calculating \(\lambda_{mf}^*\) is proposed and discussed in Appendix G. A good approximation for methane dominated gas mixture is proposed as follows:

\[
\lambda_{mf}^* = \left( \frac{\sqrt{\lambda_m^*} + \sqrt{\lambda_f^*}}{4} \right)^2
\]
Interporosity Equation in CMG-IMEX

Warren and Root (1963) propose a pseudo-steady state equation for liquid:

$$\phi_m \frac{1}{\rho_m} \frac{\partial \rho_m}{\partial t} = \frac{\alpha k_m}{\mu} (p_f - p_m)$$

Equation B-109

And CMG-IMEX modifies it for application into gas by changing $\mu$ that was a constant for liquid into $\mu_{gm}$ that viscosity of gas in matrix:

$$\phi_m \frac{1}{\rho_m} \frac{\partial \rho_m}{\partial t} = \frac{\alpha k_m}{\mu_{gm}} (p_f - p_m)$$

Equation B-110
Appendix C  Gas Rate Equation in Dual-Porosity Systems

Start from diffusivity equation in pseudo-pressure, for a radial dual porosity reservoir:

$$-\frac{1}{r} \frac{\partial}{\partial r} (r \rho_f u_r) - \frac{\partial (\phi_m \rho_m)}{\partial t} = \frac{\partial (\phi_f \rho_f)}{\partial t}$$

Equation C-1

Substituting Darcy’s law into Equation C-1 gives:

$$-\frac{1}{r} \frac{\partial}{\partial r} (r \rho_f (-\frac{k_f}{\mu_{gf}} \frac{dp_f}{dr})) - \frac{\partial (\phi_m \rho_m)}{\partial t} = \frac{\partial (\phi_f \rho_f)}{\partial t}$$

Equation C-2

Expanding $\frac{\partial (\phi_m \rho_m)}{\partial t}$ and $\frac{\partial (\phi_f \rho_f)}{\partial t}$ of Equation C-2 and substituting compressibility definitions into the resulting equation gives:

$$1 \frac{\partial}{\partial r} \left( r \rho_f \frac{k_f}{\mu_{gf}} \frac{dp_f}{dr} \right) = \rho_m \phi_m (c_1 + c_{gm}) \frac{\partial p_m}{\partial t} + \rho_f \phi_f (c_2 + c_{gf}) \frac{\partial p_f}{\partial t}$$

Equation C-3

Replacing $\rho_f$ by $p_f MW / Z_f RT$ in Equation C-3 gives:

$$\frac{MW k_f}{2RT} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{2p_f}{\mu_{gf} Z_f} \frac{dp_f}{dr} \right) = \rho_m \phi_m (c_1 + c_{gm}) \frac{\partial p_m}{\partial t} + \rho_f \phi_f (c_2 + c_{gf}) \frac{\partial p_f}{\partial t}$$

Equation C-4

Substituting definition of pseudo-pressure into Equation C-4 gives:

$$\frac{MW k_f}{2RT} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = \frac{\mu_{gm} MW}{2RT} \phi_m (c_1 + c_{gm}) \frac{2p_m}{\mu_{gm} Z_m} \frac{\partial p_m}{\partial t} + \frac{\mu_{gf} MW}{2RT} \phi_f (c_2 + c_{gf}) \frac{2p_m}{\mu_{gm} Z_m} \frac{\partial p_f}{\partial t}$$

Equation C-5

Adding $\mu_{gm}/\mu_{gf}$ on the RHS or Equation C-5 and moving the terms gives:

$$\frac{MW k_f}{2RT} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = \frac{\mu_{gm} MW}{2RT} \phi_m (c_1 + c_{gm}) \frac{2p_m}{\mu_{gm} Z_m} \frac{\partial p_m}{\partial t} + \frac{\mu_{gf} MW}{2RT} \phi_f (c_2 + c_{gf}) \frac{2p_m}{\mu_{gm} Z_m} \frac{\partial p_f}{\partial t}$$

Equation C-6

Substituting definition of pseudo-pressure into Equation C-6 gives:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = \frac{\mu_{gm}}{k_f} \phi_m (c_1 + c_{gm}) \frac{dm(p_m)}{dt} + \frac{\mu_{gf}}{k_f} \phi_f (c_2 + c_{gf}) \frac{dm(p_f)}{dt}$$

Equation C-7
Assuming incompressible matrix and fracture and taking properties as volume averaged ones on the RHS of Equation C-7 gives:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial m(p_f)}{\partial r} \right) = \mu_{gm} \phi_m \xi_{gm} \frac{\partial \bar{m}(p_m)}{\partial t} + \frac{\mu_{gf}}{k_f} \phi_f \xi_{gf} \frac{\partial \bar{m}(p_f)}{\partial t}
\]

Equation C-8

The following relationship holds:

\[
\bar{c}_{gm} V_m \frac{\partial \bar{p}_m}{\partial t} = - \frac{1}{V_m} \frac{\partial V_m}{\partial \bar{p}_m} V_m \frac{\partial \bar{p}_m}{\partial t} = - \frac{\partial V_m}{\partial \bar{t}} = - q_m
\]

Equation C-9

where:

\( V_m = \) matrix gas volume in reservoir condition

\( \bar{c}_{gm} = \) average gas compressibility in matrix system

\( q_m = \) gas production from matrix to fracture system; positive sign denotes production

Dividing both sides in Equation C-9 by \( \bar{c}_{gm} V_m \) gives:

\[
\frac{\partial \bar{p}_m}{\partial t} = - \frac{q_m}{\bar{c}_{gm} V_m}
\]

Equation C-10

For a radial system, we have:

\[
\frac{\partial \bar{p}_m}{\partial t} = - \frac{q_m}{\bar{c}_{gm} \pi r_e^2 h \phi_m}
\]

Equation C-11

Expanding \( \frac{\partial \bar{m}(p_m)}{\partial t} \) and substituting Equation C-11 gives:

\[
\frac{\partial \bar{m}(p_m)}{\partial t} = \frac{2 \bar{p}_m}{\mu_{gm} Z_m} \frac{\partial \bar{p}_m}{\partial t} = - \frac{2 \bar{p}_m}{\mu_{gm} Z_m \bar{c}_{gm} \pi r_e^2 h \phi_m} \frac{q_m}{2}
\]

Equation C-12

Similarly, we have for fracture system the following relationship:

\[
\frac{\partial \bar{m}(p_f)}{\partial t} = - \frac{2 \bar{p}_f}{\mu_{gf} Z_f \phi_f} \frac{q_f}{\bar{c}_f \pi r_e^2 h \phi_f}
\]

Equation C-13

\( q_f \) and \( q_m \) at standard condition satisfy the relation:

\[
q_{fsc} = q_{gsc} - q_{msc}
\]

Equation C-14

where:

\( q_{gsc} \) = Total flow rate out of the reservoir at standard condition
\( q_{fsc} \) = Net flow rate out of the fracture system at standard condition
\( q_{msc} \) = Total flow rate out of the matrix system at standard condition

Substituting C-12 and C-13 into C-8 gives:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = -\frac{2\bar{p}_m q_m}{Z_m \pi r_e^2 h k_f} + \frac{2\bar{p}_f q_f}{Z_f \pi r_e^2 h k_f}
\]

Equation C-15

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = -\left( \bar{p}_m \frac{2}{Z_m} \pi r_e^2 h k_f \right) + \left( \bar{p}_f \frac{2}{Z_f} \pi r_e^2 h k_f \right)
\]

Equation C-16

Equation of State gives the following relationship:

\[
\frac{\bar{p}_m q_m}{Z_m} = \frac{p_{sc} q_{msc} T}{T_{sc}}
\]

Equation C-17

\[
\frac{\bar{p}_f q_f}{Z_f} = \frac{p_{sc} q_{fsc} T}{T_{sc}}
\]

Equation C-18

Substituting Equation C-17 and Equation C-18 into Equation C-16 gives:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = -\left( \frac{p_{sc} q_{msc} T}{T_{sc}} \frac{2}{\pi r_e^2 h k_f} + \frac{2p_{sc} q_{fsc} T}{T_{sc}} \frac{2}{\pi r_e^2 h k_f} \right)
\]

Equation C-19

Combining terms in Equation C-19 gives:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = -\left( \frac{2p_{sc} T}{\pi r_e^2 h k_f T_{sc}} q_{msc} + \frac{2p_{sc} T}{\pi r_e^2 h k_f T_{sc}} q_{fsc} \right)
\]

Equation C-20

Taking \( \frac{2p_{sc} T}{\pi r_e^2 h k_f T_{sc}} \) out of Equation C-20 gives:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = -\left( \frac{2p_{sc} T}{\pi r_e^2 h k_f T_{sc}} \right) (q_{msc} + q_{fsc})
\]

Equation C-21

Substituting Equation C-14 into Equation C-21 gives:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{dm(p_f)}{dr} \right) = -\frac{2p_{sc} T}{\pi r_e^2 h k_f T_{sc}} q_{gsc}
\]

Equation C-22

Combining terms in Equation C-22 gives:
\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dm(p_f)}{dr} \right) = -\frac{p_{sc} q_{gsc} T}{T_{sc}} \frac{2}{\pi r_e^2 h k_f}
\]

Equation C-23

Multiplying both sides of Equation C-23 by rdr gives:

\[
d \left( r \frac{dm(p_f)}{dr} \right) = -\frac{p_{sc} q_{gsc} T}{T_{sc}} \frac{1}{\pi r_e^2 h k_f} dr^2
\]

Equation C-24

Integrating both sides of Equation C-24 from \( r_w \) to \( r \) gives:

\[
\int d \left( r \frac{dm(p_f)}{dr} \right) = \int_{r_w}^{r} -\frac{p_{sc} q_{gsc} T}{T_{sc}} \frac{1}{\pi r_e^2 h k_f} dr^2
\]

Equation C-25

The integration gives:

\[
r \frac{dm(p_f)}{dr} = -\frac{p_{sc} q_{gsc} T}{\pi h k_f T_{sc}} \left( r^2 - r_w^2 \right) + C
\]

Equation C-26

Dividing both sides in Equation C-26 by \( r \) gives:

\[
\frac{dm(p_f)}{dr} = \left( \frac{p_{sc} q_{gsc} T}{\pi h k_f T_{sc}} \right) \left( \frac{r_w^2}{r_e^2} - \frac{r}{r_e^2} \right) + \frac{C}{r}
\]

Equation C-27

Making \( \left( \frac{dm(p_f)}{dr} \right)_{r=r_e} = 0 \) in Equation C-27 gives:

\[
\left( \frac{dm(p_f)}{dr} \right)_{r=r_e} = \left( \frac{p_{sc} q_{gsc} T}{\pi h k_f T_{sc}} \right) \left( \frac{r_w^2}{r_e^2} - \frac{1}{r_e} \right) + \frac{C}{r_e}
\]

Equation C-28

\( \frac{r_w^2}{r_e^2} \) is negligible and \( C \) should be equal to \( \frac{p_{sc} q_{gsc} T}{\pi h k_f T_{sc}} \), hence Equation C-28 produces:

\[
\frac{dm(p_f)}{dr} = \left( \frac{p_{sc} q_{gsc} T}{\pi h k_f T_{sc}} \right) \left( \frac{r_w^2}{r_e^2} - \frac{r}{r_e^2} + \frac{1}{r} \right)
\]

Equation C-29

Multiplying both sides by \( dr \) and integrating resulting equation gives:

\[
\int_{p_{wf}}^{p_f} dm(p_f) = \int_{r_w}^{r_e} \left( \frac{p_{sc} q_{gsc} T}{\pi h k_f T_{sc}} \right) \left( \frac{r_w^2}{r_e^2} - \frac{r}{r_e^2} + \frac{1}{r} \right) dr
\]

Equation C-30

Equation C-30 gives:
\[
\int_{p_{wf}}^{p_f} dm(p_f) = \int_{r_w}^{r} \left( \frac{p_{sc}q_{gsc}T}{\pi h k_f T_{sc}} \right) d\left( \frac{r_w^2}{r_e^2} \right) \ln\left( \frac{r}{r_w} \right) - \frac{1}{2r_e^2} r^2
\]

Equation C-31

Finishing the integration gives:

\[
\begin{align*}
m(p_f) - m(p_{wf}) &= \left( \frac{p_{sc}q_{gsc}T}{\pi h k_f T_{sc}} \right) \ln\left( \frac{r}{r_w} \right) - \frac{1}{2r_e^2} (r^2 - r_w^2) \\
&\quad - \frac{1}{2r_e^2} (r^2 - r_w^2)
\end{align*}
\]

Equation C-32

Expanding Equation C-32 gives:

\[
\begin{align*}
m(p_f) - m(p_{wf}) &= \left( \frac{p_{sc}q_{gsc}T}{\pi h k_f T_{sc}} \right) \ln\left( \frac{r}{r_w} \right) - \frac{r^2}{2r_e^2} + \frac{r_w^2}{2r_e^2}
\end{align*}
\]

Equation C-33

Neglecting \( \frac{r_w^2}{r_e^2} \) in Equation C-33 gives:

\[
m(p_f) - m(p_{wf}) = \left( \frac{p_{sc}q_{gsc}T}{\pi h k_f T_{sc}} \right) \ln\left( \frac{r}{r_w} \right) - \frac{r^2}{2r_e^2}
\]

Equation C-34

Incorporate skin factor in Equation C-34, yielding the following equation:

\[
m(p_f) - m(p_{wf}) = \left( \frac{p_{sc}q_{gsc}T}{\pi h k_f T_{sc}} \right) \ln\left( \frac{r}{r_w} \right) - \frac{r^2}{2r_e^2} + s
\]

Equation C-35

Volume-averaged pseudo-pressure in fracture system is calculated as follows:

\[
\bar{m}(p_f) = \frac{\int_{r_w}^{r} m(p_f) dv}{\int_{r_w}^{r} dv}
\]

Equation C-36

Equation C-36 is rewritten into:

\[
\bar{m}(p_f) = \frac{\int_{r_w}^{r} m(p_f) 2\pi r h \phi dr}{\pi (r_e^2 - r_w^2) h \phi}
\]

Equation C-37

Equation C-37 is simplified as:

\[
\bar{m}(p_f) = \frac{2}{r_e^2 - r_w^2} \int_{r_w}^{r} m(p_f) r dr
\]

Equation C-38

Substituting C-35 into Equation C-38 gives:
\[ \bar{m}(p_f) = \frac{2}{r_e^2} \int_{r_w}^{r_e} \left( \frac{p_{sc}q_{gsc}T}{\pi h k_f T_{sc}} \right) (\ln \left( \frac{r}{r_w} \right) - \frac{1}{2} + s) + m(p_{wf}) \, dr \]

\text{Equation C-39}

Factoring out common term of the integral gives:
\[ \bar{m}(p_f) = \frac{2}{r_e^2} \left( \frac{p_{sc}q_{gsc}T}{\pi h k_f T_{sc}} \right) \int_{r_w}^{r_e} (\ln \left( \frac{r}{r_w} \right) - \frac{r^2}{2r_e^2} + s) r \, dr \]

\text{Equation C-40}

Equation C-40 after integration is written as:
\[ \bar{m}(p_f) - m(p_{wf}) = \frac{p_{sc}q_{gsc}T}{\pi h k_f T_{sc}} \left( \ln \frac{r_e}{r_w} - \frac{3}{4} + s \right) \]

\text{Equation C-41}

Dividing both sides by \((\bar{m}(p_f) - m(p_{wf}))/q_{sc}\) gives:
\[ q_{gsc} = \frac{\pi h k_f T_{sc}}{p_{sc}T (\log \frac{r_e}{r_w} - \frac{3}{4} + s)} (\bar{m}(p_f) - m(p_{wf})) \]

\text{Equation C-42}

where \(\bar{m}(p_f)\) is the space-averaged pseudo-pressure in the fracture system, and \(m(p_{wf})\) is the pseudo-pressure at bottomhole pressure.

This equation is derived in a similar approach as Dake (1978). Writing Equation C-38 in the field units for different shapes of reservoirs following Dake (1978) yields:
\[ q_{gsc} = \frac{T_{sc}k_f h}{50294 p_{sc} T b_{D,PSS}} (\bar{m}(p_f) - m(p_{wf})) \]

\text{Equation C-43}

where:
\[ b_{D,PSS} = \frac{1}{2} \ln \left( \frac{4}{e \gamma} \frac{A}{C^2 k_f r_w^2} \right) \]

\text{Equation C-43a}

\( h \) = formation thickness, ft
\( k \) = permeability, md
\( p_{sc} \) = standard pressure, psia
\( q_{gsc} \) = gas flow rate, Mscf/D
\( r_w' \) = effective wellbore radius, ft
\( T \) = reservoir temperature, °R
\( T_{sc} \) = standard temperature, °R
\( \gamma \) = Euler’s constant, 0.5772156649
Appendix D  Rescaling Approach for Dual-Porosity Systems

From Appendix C, we know:

\[ q_{gsc} = \frac{T_{sc} k_f h}{50294 p_{sc} T_b D_{D, PSS}} (\bar{m}(p_f) - m(p_{wf})) \]  
Equation D-1

where:

\[ b_{D, PSS} = \frac{1}{2} \ln \left( \frac{4}{e^\gamma} \frac{A}{\epsilon \sigma \rho_w r_w^2} \right) \]
Equation D-1a

\( h \) = formation thickness, ft
\( k \) = permeability, md
\( p_{sc} \) = standard pressure, psia
\( q_{gsc} \) = gas flow rate, Mscf/D
\( r_w' \) = effective wellbore radius, ft
\( T \) = reservoir temperature, °R
\( T_{sc} \) = standard temperature, °R
\( \gamma \) = Euler's constant, 0.5772156649

Take a valid assumption for dual-porosity reservoir at late stage:

\[ \frac{d\bar{\rho}_m}{dt} = \frac{d\bar{\rho}}{dt} \]
Equation D-2

Then we have material balance equation at the second decline stage in differential form as:

\[ q_{sc} \rho_{sc} = -V_{res} \frac{d(\bar{\phi}_m \bar{\rho}_m)}{dt} \]
Equation D-3

Dividing both sides by \( \rho_{sc} \) gives:

\[ q_{sc} = -\frac{V_{res}}{\rho_{sc}} \frac{d(\bar{\phi}_m \bar{\rho}_m)}{dt} \]
Equation D-4

Substituting Equation D-3 into Equation D-1 gives:

\[ \frac{V_{res}}{\rho_{sc}} \frac{d(\bar{\phi}_m \bar{\rho}_m)}{dt} = -\frac{T_{sc} k_f h}{50294 p_{sc} T_b D_{D, PSS}} (\bar{m}(p_f) - m(p_{wf})) \]
Equation D-5
Assuming boundary-dominated flow, Appendix C derives gas rate equation for dual-porosity reservoir as described by Equation C-43, which is presented below:

\[ q_{gsc} = \frac{T_{sc}k_fh}{50294\rho_{sc}T_{bd,PSS}} (\bar{m}(p_f) - m(p_{wf})) \]

Equation D-6

For the convenience of transferring average pseudo-pressure in fracture to a more measurable variable, we derive average interporosity flow equation for gas at boundary-dominated-flow stage. Integrating with respect to volume and dividing reservoir volume on both sides of Equation B-101 in customary units gives:

\[ \int \phi_m \frac{d\rho_m}{dt} dv = \frac{ak_m}{158.0253 \times 2\theta} \left( \int \frac{m(p_f)dv}{V_{res}} - \int \frac{m(p_m)dv}{V_{res}} \right) \]

Equation D-7

Moving \( V_{res} \) inside the integral and \( dt \) outside the integral in Equation D-7 gives:

\[ \frac{d}{dt} \int \frac{\phi_m \rho_m}{V_{res}} dv = \frac{ak_m}{158.0253 \times 2\theta} \left( \int \frac{m(p_f)dv}{V_{res}} - \int \frac{m(p_m)dv}{V_{res}} \right) \]

Equation D-8

Equation D-8 is written into:

\[ \frac{d(\bar{\phi}_m \bar{\rho}_m)}{dt} = \frac{ak_m}{158.0253 \times 2\theta} (\bar{m}(p_f) - \bar{m}(p_m)) \]

Equation D-9

where \( \bar{\rho}_m, \bar{m}(p_f), \) and \( \bar{m}(p_m) \) are volume-averaged density, fracture pseudo-pressure and matrix pseudo-pressure respectively.

Combining Equation D-9 with Equation D-5, we have:

\[ \frac{V_{res}}{\rho_{sc}} \int \frac{\phi_m}{V_{res}} dv = \frac{ak_m}{158.0253 \times 2\theta} (\bar{m}(p_f) - \bar{m}(p_m)) = - \frac{T_{sc}k_fh}{50294\rho_{sc}T_{bd,PSS}} (\bar{m}(p_f) - m(p_{wf})) \]

Equation D-10

Moving terms in Equation D-10 yields:

\[ (\bar{m}(p_f) - \bar{m}(p_m)) = - \frac{0.006284062k_fh}{ak_mV_{res}b_{D,PSS}} (\bar{m}(p_f) - m(p_{wf})) \]

Equation D-11

Take the notation as follows:

\[ K = \frac{0.006284062k_fh}{ak_mV_{res}b_{D,PSS}} \]

Equation D-12
With Equation D-12, Equation D-11 is transformed into:

\[
\bar{m}(p_f) - \bar{m}(p_m) = -K(\bar{m}(p_f) - m(p_{wf}))
\]

Equation D-13

Then we have representation of \(\bar{m}(p_m)\) in \(\bar{m}(p_f)\):

\[
\bar{m}(p_m) = (1 + K)\bar{m}(p_f) - Km(p_{wf})
\]

Equation D-14

From Equation D-14 we obtain the representation of \(\bar{m}(p_f)\) in \(\bar{m}(p_m)\) and \(m(p_{wf})\):

\[
\bar{m}(p_f) = \frac{1}{1 + K} \bar{m}(p_m) + \frac{K}{1 + K} m(p_{wf})
\]

Equation D-15

Then we can write Equation D-5 in \(\bar{m}(p_m)\) instead of \(\bar{m}(p_f)\) by substituting Equation D-13 into it:

\[
\frac{V_{res}}{\rho_{sc}} \frac{d(\bar{m} \bar{p}_m)}{dt} = -\frac{T_{sc} k_f h}{50294 \rho_{sc} p_{sc} T D_{D, PSS}} \left( \frac{1}{1 + K} \bar{m}(p_m) + \frac{K}{1 + K} m(p_{wf}) - m(p_{wf}) \right)
\]

Equation D-16

Combining \(\frac{K}{1 + K} m(p_{wf}) - m(p_{wf})\) gives:

\[
\frac{V_{res}}{\rho_{sc}} \frac{d(\bar{m} \bar{p}_m)}{dt} = -\frac{T_{sc} k_f h}{50294 \rho_{sc} p_{sc} T D_{D, PSS}} \left( \frac{1}{1 + K} \bar{m}(p_m) - \frac{1}{1 + K} m(p_{wf}) \right)
\]

Equation D-17

Given incompressible matrix and fracture and constant porosity, \(\phi_m\), throughout the reservoir, we take \(\phi_m\) out of the differential term, yielding:

\[
\frac{d(\bar{p}_m)}{dt} = -\frac{\rho_{sc} T_{sc} k h}{50294 \phi_m V_{res} p_{sc} T D_{D, PSS}(1 + K)} (\bar{m}(p_m) - m(p_{wf}))
\]

Equation D-18

Define \(\lambda_m\) as follows for transferring from pseudo-pressure into density:

\[
\lambda_m = \frac{\mu_{gi} c_{gi}}{2\theta (\bar{p}_m - \rho_{wf})} m(\bar{p}_m) - m(p_{wf})
\]

Equation D-19

\[
m(\bar{p}_m) - m(p_{wf}) = \frac{2\theta (\bar{p}_m - \rho_{wf}) \lambda_m}{\mu_{gi} c_{gi}}
\]

Equation D-20

Substituting Equation D-20 into Equation D-18, we have:

\[
\frac{d(\bar{p}_m)}{dt} = -\frac{\rho_{sc} T_{sc} k h}{50294 \phi_m V_{res} p_{sc} T D_{D, PSS}(1 + K)} \frac{2\theta (\bar{p}_m - \rho_{wf}) \lambda_m}{\mu_{gi} c_{gi}}
\]

Equation D-21
Moving terms in Equation D-21 gives:

\[
\frac{d(\bar{\rho}_m)}{dt} = -\frac{\theta \rho_{sc} T_{sc} kh}{25147 \phi_m V_{res} p_{sc} T_{bD_PSS} \mu_{gi} c_{gi}} \lambda_m (\bar{\rho}_m - \rho_{wf})
\]

Equation D-22

Take the notation as follows:

\[
N = \frac{\theta \rho_{sc} T_{sc} k_f h}{25147 \phi_m V_{res} p_{sc} T_{bD_PSS} \mu_{gi} c_{gi}}
\]

Equation D-23

Then Equation D-21 may be written as:

\[
\frac{d(\bar{\rho}_m)}{dt} = -\frac{N}{(1 + K)} \lambda_m (\bar{\rho}_m - \rho_{wf})
\]

Equation D-24

Dividing both sides by \((\bar{\rho}_m - \rho_{wf})\) in Equation D-24 gives:

\[
\frac{d(\bar{\rho}_m - \rho_{wf})}{(\rho_m - \rho_{wf})} = -\frac{N}{1 + K} \lambda_m dt
\]

Equation D-25

Integrating Equation D-25 from time zero to a time point, \(t\), gives:

\[
\int_{\rho_i}^{\bar{\rho}_m} \frac{d(\bar{\rho}_m - \rho_{wf})}{(\rho_m - \rho_{wf})} = \int_{0}^{t} \frac{-N}{1 + K} \lambda_m dt
\]

Equation D-26

Taking \(-\frac{N}{1+K}\) out of the integral since it is a constant and substituting \(\lambda_m = \frac{d(\beta_m t)}{dt}\) into the resultant equation gives:

\[
\int_{\rho_i}^{\bar{\rho}_m} d\ln(\bar{\rho}_m - \rho_{wf}) = \frac{-N}{1 + K} \int_{0}^{t} d\beta_m t
\]

Equation D-27

Equation D-27 is finally written into:

\[
\ln \left( \frac{\bar{\rho}_m - \rho_{wf}}{\rho_i - \rho_{wf}} \right) = \frac{-N}{1 + K} \beta_m t
\]

Equation D-28

Taking exponential of both sides and transforming the resultant equations gives:

\[
(\bar{\rho}_m - \rho_{wf}) = (\rho_i - \rho_{wf}) \exp \left( \frac{-N}{1 + K} \beta_m t \right)
\]

Equation D-29

Equation D-22 can be rewritten into:
\[
\frac{V_{\text{res}} \phi_m d(\bar{\rho}_m)}{\rho_{sc} \Delta t} = -\frac{\theta T_{sc} k_f h}{25147 p_{sc} T b_{D,PSS} \mu_{gi} c_{gi} (1 + K)} \lambda_m (\bar{\rho}_m - \rho_{wf})
\]

Equation D-30

Or:

\[
q_{sc} = -\frac{\theta T_{sc} k_f h}{25147 p_{sc} T b_{D,PSS} \mu_{gi} c_{gi} (1 + K)} \lambda_m (\bar{\rho}_m - \rho_{wf})
\]

Equation D-31

Substituting Equation D-29 into Equation D-31, we have:

\[
q_{sc} = -\frac{\theta T_{sc} k_f h}{25147 p_{sc} T b_{D,PSS} \mu_{gi} c_{gi} (1 + K)} \lambda_m (\rho_i - \rho_{wf}) \exp\left(-\frac{N}{1 + K \beta_m t}\right)
\]

Equation D-32

Or:

\[
q_{sc} = O \cdot \frac{\lambda_m}{1 + K} (\rho_i - \rho_{wf}) \exp\left(-\frac{N}{1 + K \beta_m t}\right)
\]

Equation D-33

where:

\[
K = \frac{0.006284062 k_f h}{\alpha k_m V_{\text{res}} b_{D,PSS}}
\]

Equation D-33a

\[
N = \frac{k_f h}{25147 \phi_m V_{\text{res}} b_{D,PSS} \mu_{gi} c_{gi}}
\]

Equation D-33b

\[
O = \frac{k_f h}{25147 \rho_{sc} b_{D,PSS} \mu_{gi} c_{gi}}
\]

Equation D-33c

Moving terms in Equation D-33 gives:

\[
\frac{\rho_{sc} \mu_{gi} c_{gi} q_{gsc}}{k_f h (\rho_i - \rho_{wf})} = \frac{1}{25147 b_{D,PSS}} \cdot \frac{\lambda_m}{1 + K} \exp\left(-\frac{N}{1 + K \beta_m t}\right)
\]

Equation D-34

Add a term in Equation D-34 as follows:

\[
\frac{158021 \rho_{sc} \mu_{gi} c_{gi} q_{gsc}}{2\pi k_f h (\rho_i - \rho_{wf})} = \frac{1}{25147 b_{D,PSS}} \cdot \frac{\lambda_m}{1 + K} \exp\left(-\frac{N}{1 + K \beta_m t}\right)
\]

Equation D-35

Definition of \(q_{D}^{gas}\) is as follows:
\[ q_D^{gas} = \frac{158021 \mu_{sc} \mu_{gi} c_{gi} q_{gsc}}{2\pi k_f h (\rho_i - \rho_{wf})} \]  

Equation D-36

Substituting Equation D-36 into Equation D-35 gives:

\[ q_D^{gas} = \frac{1}{b_{D,PSS}} \cdot \frac{\lambda_m}{1 + K} \cdot \exp(-\frac{N}{1 + K} \beta_m t) \]  

Equation D-37

Substituting definition of \( N \) into Equation D-37 gives:

\[ q_D^{gas} = \frac{1}{b_{D,PSS}} \cdot \frac{\lambda_m}{1 + K} \cdot \exp(-\frac{1}{1 + K} \cdot \frac{25147 V_{res} b_{D,PSS} \mu_{gi} c_{gi} \phi_m}{k_f t} \beta_m) \]  

Equation D-38

Rearranging terms in Equation D-38 gives:

\[ q_D^{gas} = \frac{1}{b_{D,PSS}} \cdot \frac{\lambda_m}{1 + K} \cdot \exp(-\frac{1}{1 + K} \cdot \frac{25147 V_{res} b_{D,PSS} \mu_{gi} c_{gi} \phi_m}{k_f t} \beta_m) \]  

Equation D-39

Definition of \( t_D \) is:

\[ t_D = \frac{0.0063283 k_f t}{\mu_{gi} c_{gi} (\phi_m + \phi_f) r_w^2} \]  

Equation D-40

Substituting Equation D-40 into Equation D-39 gives:

\[ q_D^{gas} = \frac{1}{b_{D,PSS}} \cdot \frac{\lambda_m}{1 + K} \cdot \exp(-\frac{1}{1 + K} \cdot \frac{25147 V_{res} b_{D,PSS} \mu_{gi} c_{gi} \phi_m}{k_f t} \beta_m) \]  

Equation D-41

Substituting \( V_{res} [Mcf] = \frac{V_{res} [cf]}{1000} \) and \( V_{res} [cf] = h [ft] A [ft^2] \) into Equation D-41 gives:

\[ q_D^{gas} = \frac{1}{b_{D,PSS}} \cdot \frac{\lambda_m}{1 + K} \cdot \exp(-\frac{1}{b_{D,PSS}} \cdot \frac{2\pi r_w^2}{A} \cdot \frac{1}{1 + K} \cdot \frac{\phi_m + \phi_f}{\phi_m} \cdot t_D) \]  

Equation D-42

Exponential solution for a single-porosity liquid reservoir with fracture permeability, \( k_f \), and matrix porosity, \( \phi_m \), in dimensionless form is:

\[ q_D^{liq} = \frac{1}{b_{D,PSS}} \cdot \frac{\lambda_m}{1 + K} \cdot \exp(-\frac{1}{b_{D,PSS}} \cdot \frac{2\pi r_w^2}{A} \cdot \frac{\phi_m + \phi_f}{\phi_m} \cdot t_D) \]  

Equation D-43

Comparison between Equation D-42 and Equation D-43 produces \( \lambda \) and \( \beta \) rescaling approach for dual-porosity reservoirs in the second decline stage.
\[ q_D^\text{gas} = \bar{\lambda}_D q_D^{\text{liq-s}} (\bar{\beta}_D t_D) \]

**Equation D-44**

where:

\[ \bar{\lambda}_D = \frac{\lambda_m}{1 + K} \]

**Equation D-44a**

\[ \bar{\beta}_D = \frac{\beta_m}{1 + K} \]

**Equation D-44b**

Note that \( q_D^{\text{liq-s}} \) is dimensionless liquid flow rate in a single-porosity reservoir with the same size, permeability, \( k_f \), and porosity, \( \phi_m \), while \( q_D^\text{gas} \) is dimensionless gas flow rate in dual-porosity reservoirs.

Moreover, rescaling from a dual-porosity liquid solution with \( \bar{\lambda}_m \) and \( \bar{\beta}_m \) yields a match between the rescaled liquid solution and its gas counterpart. The match is proved in following steps. First of all, following the derivation process of its gas counterpart, as explained in Equations 7-14, we derive the liquid rate equation for a vertical well producing in a closed-boundary dual-porosity reservoir as follows:

\[ q_{sc} = \frac{k_f \pi h}{79014 \mu c_l \rho_{sc} b_{D,PSS}} (\bar{\rho}_f - \rho_{wf}) \]

**Equation D-45**

where \( q_{sc} \) is the liquid flow rate, \( \bar{\rho}_f \) is the average fracture liquid density, and \( c_l \) is liquid compressibility. Integrating Equation 2-5 with respect to volume and dividing both sides by reservoir volume gives:

\[ \frac{\partial (\phi_m \bar{\rho}_m)}{\partial t} = 0.0063288 \frac{\alpha k_m}{\mu c_l} (\bar{\rho}_f - \rho_m) \]

**Equation D-46**

Following steps for deriving its gas counterpart, we subsequently develop an exponential liquid rate equation for dual-porosity systems presented as follows:

\[ q_D^{\text{liq}} = \frac{1}{1 + K \frac{1}{b_{D,PSS}}} \exp\left(-\frac{1}{b_{D,PSS}} \frac{2\pi r_w^2}{A} \frac{1}{1 + K} \frac{\phi_m + \phi_f}{\phi_m + \phi_f} t_D \right) \]

**Equation D-47**

where \( q_D^{\text{liq-d}} = \frac{158021 \rho_{sc} \mu c_l \varrho}{2\pi k_f h (\rho_i - \rho_{wf})} \) and \( t_D = \frac{0.0063283 k_f t}{\mu_g c_g (\phi_m + \phi_f) r_w^2} \).

Comparing Equation D-42 and Equation D-47 gives:

\[ q_D^\text{gas} = \bar{\lambda}_m q_D^{\text{liq-d}} (\bar{\beta}_m t_D) \]
where $q_D^{gas}$ is the predicted analytical gas flow rate response at constant bottom-hole pressure in a dual-porosity system with a closed boundary, and $q_D^{liq-s}$ is the analytical response of the liquid dual-porosity system with a closed boundary at constant bottom-hole pressure. Literature such as Raghavan (1993) provided the representation of $q_D^{liq-s}$ in Laplace space. Da Prat et al. (1981) presented $q_D^{liq-d}$ in Laplace space. Inverting $q_D^{liq-s}$ and $q_D^{liq-d}$ to real space using the Stehfest algorithm yields the exact values of $q_D^{liq-s}$ and $q_D^{liq-d}$ for rescaling purposes.
Appendix E  Rate-Transient Analysis and Type Curve in Dual-Porosity Systems

Rate Transient Analysis 1

Appendix C derives gas rate equation in dual-porosity reservoirs:

\[ q_{gsc} = \frac{T_{sc}kh}{50294p_{sc}T b_{D,PSS}} (m(\bar{p}_f) - m(p_{wf})) \]

**Equation E-1**

where \( b_{D,PSS} \) is \( \frac{1}{2} \ln \left( \frac{A}{e^{\gamma} c_A r_w^2} \right) \), \( A \) is area in ft\(^2\), \( r_w \) represent wellbore radius in ft, and \( \gamma \) is Euler’s constant, 0.5772156649.

Knowing from Appendix D that:

\[ m(\bar{p}_f) = \frac{1}{1+K} m(\bar{p}_m) + \frac{K}{1+K} m(p_{wf}) \]

**Equation E-2**

Substitution of Equation E-2 into Equation E-1 gives:

\[ q_{gsc} = \frac{T_{sc}kh}{50294p_{sc}T b_{D,PSS}} \left( \frac{1}{1+K} m(\bar{p}_m) - \frac{1}{1+K} m(p_{wf}) \right) \]

**Equation E-3**

Taking \( \frac{1}{1+K} \) out of the parentheses gives:

\[ q_{gsc} = \frac{T_{sc}kh}{50294p_{sc}T b_{D,PSS}} \frac{1}{1+K} (m(\bar{p}_m) - m(p_{wf})) \]

**Equation E-4**

Define:

\[ \bar{\lambda}_m = \frac{\mu_{gi} c_{gi}}{2\theta (\bar{p}_m - p_{wf}) \bar{m}(p_m) - m(p_{wf})} \]

**Equation E-5**

Rearranging terms in Equation E-5 to represent \( m(\bar{p}_m) - m(p_{wf}) \) gives:

\[ m(\bar{p}_m) - m(p_{wf}) = \frac{2\theta (\bar{p}_m - p_{wf}) \bar{\lambda}_m}{\mu_{gi} c_{gi}} \]

**Equation E-6**

Substituting Equation E-6 into Equation E-4 gives:

\[ q_{gsc} = \frac{T_{sc}kh}{50294p_{sc}T b_{D,PSS}} \frac{1}{1+K} \frac{2\theta (\bar{p}_m - p_{wf}) \bar{\lambda}_m}{\mu_{gi} c_{gi}} \]
Adding $\rho_i \frac{1}{\rho_i}$ and arranging terms gives:

$$q_{gsc} = \frac{\rho_i T_{sc} k h}{50294 p_{sc} T_{b, PSS}} \frac{1}{1 + K} \frac{2\theta \left( \frac{\rho_m}{\rho_i} - \frac{\rho_{wf}}{\rho_i} \right)}{\mu_{gi} c_{gi}} \bar{\lambda}_m$$

Equation E-7

Assuming negligible fracture porosity and boundary-dominated flow, we have:

$$\frac{\rho_m}{\rho_i} = 1 - \frac{G_p}{OGIP}$$

Equation E-8

Substituting Equation E-9 into Equation E-8 gives:

$$q_{gsc} = \frac{\rho_i T_{sc} k h}{50294 p_{sc} T_{b, PSS}} \frac{1}{1 + K} \frac{2\theta \left( 1 - \frac{G_p}{OGIP} - \frac{\rho_{wf}}{\rho_i} \right)}{\mu_{gi} c_{gi}} \bar{\lambda}_m$$

Equation E-9

Take a notation as follows:

$$r_p = 1 - \frac{\rho_{wf}}{\rho_i}$$

Equation E-10

Substituting Equation E-11 into Equation E-10 gives:

$$q_{gsc} = \frac{\rho_i T_{sc} k h}{50294 p_{sc} T_{b, PSS}} \frac{1}{1 + K} \frac{2\theta \left( r_p - \frac{G_p}{OGIP} \right)}{\mu_{gi} c_{gi}} \bar{\lambda}_m$$

Equation E-11

Substituting $\theta = \frac{RT}{MW}$ into Equation E-12 and canceling out terms gives:

$$q_{gsc} = \frac{\rho_i k_f h}{25147 \rho_{sc} b_{D, PSS} \mu_{gi} c_{gi}} \frac{1}{1 + K} \left( r_p - \frac{G_p}{OGIP} \right) \bar{\lambda}_m$$

Equation E-12

Denote:

$$q_{gdi}^e = \frac{\rho_i k_f h}{25147 \rho_{sc} b_{D, PSS} \mu_{gi} c_{gi}(1 + K)}$$

Equation E-13

Substitution of Equation E-14 into Equation E-13 gives:

$$q_{gsc} = q_{gdi}^e \left( r_p - \frac{G_p}{OGIP} \right) \bar{\lambda}_m$$

Equation E-14

Equation E-15
Dividing both sides by \(q_{gdi}^e\) gives:
\[
\frac{q_{gsc}}{q_{gdi}^e} = \frac{1}{\bar{\lambda}_m \rho_p - \bar{\lambda}_m \frac{G_p}{OGIP}}
\]
Equation E-16

Dividing both sides by \(q_{gsc}\) gives:
\[
\frac{1}{q_{gdi}^e} = \frac{1}{\bar{\lambda}_m \frac{\bar{\rho}_p}{q_{gsc}} - \frac{1}{OGIP} \frac{\bar{\lambda}_m}{q_{gsc}} \frac{G_p}{q_{gsc}}}
\]
Equation E-17

Equation E-17 is written as follows:
\[
\bar{\lambda}_m \frac{\bar{\rho}_p}{q_{gsc}} = \frac{1}{OGIP} \frac{\bar{\lambda}_m}{q_{gsc}} \frac{G_p}{q_{gsc}} + \frac{1}{q_{gdi}^e}
\]
Equation E-18

An iterative procedure could then be implemented to predict OGIP rigorously.

**Rate Transient Analysis 2**

Appendix D derives gas rate equation in exponential form as follows:
\[
q_{sc} = -\frac{\theta T_{sc} k_f h}{25147 p_{sc} T_{bD,PSS} \mu_{g} c_{gi}} \bar{\lambda}_m \left( \rho_i - \rho_{wf} \right) \exp\left( \frac{-N}{1 + K} \bar{\beta}_m t \right)
\]
Equation E-19

Dividing both sides by \(\bar{\lambda}_m\) gives:
\[
\frac{q_{sc}}{\bar{\lambda}_m} = -\frac{\theta T_{sc} k_f h}{25147 p_{sc} T_{bD,PSS} \mu_{g} c_{gi}} \left( \rho_i - \rho_{wf} \right) \exp\left( \frac{-N}{1 + K} \bar{\beta}_m t \right)
\]
Equation E-20

Taking logarithm on both sides yields:
\[
\ln\left( \frac{q_{sc}}{\bar{\lambda}_m} \right) = \ln\left[ -\frac{\theta T_{sc} k_f h}{25147 p_{sc} T_{bD,PSS} \mu_{g} c_{gi}} \left( \rho_i - \rho_{wf} \right) \exp\left( \frac{-N}{1 + K} \bar{\beta}_m t \right) \right]
\]
Equation E-21

Taking \(\exp\left( \frac{-N}{1 + K} \bar{\beta}_m t \right)\) out of the logarithm gives:
\[
\ln\left( \frac{q_{sc}}{\bar{\lambda}_m} \right) = \ln\left[ -\frac{\theta T_{sc} k_f h}{25147 p_{sc} T_{bD,PSS} \mu_{g} c_{gi}} \left( \rho_i - \rho_{wf} \right) \right] + \frac{-N}{1 + K} \bar{\beta}_m t
\]
Equation E-22

Substituting \(\theta = \frac{RT}{MW}\) into Equation E-22 and canceling out terms gives:
\[
\ln\left(\frac{q_{sc}}{\lambda_m}\right) = \ln\left[\frac{k_fh}{25147\rho_{sc} b_{D,PSS}\mu_{gi}c_{gi}} \left(\frac{\rho_i - \rho_{wf}}{1 + K}\right)\right] + \frac{-N}{1 + K}\bar{\beta}_m t
\]

Equation E-23

Substituting definition of N into Equation E-23 gives:
\[
\ln\left(\frac{q_{sc}}{\lambda_m}\right) = \ln\left[\frac{k_fh}{25147\rho_{sc} b_{D,PSS}\mu_{gi}c_{gi}} \left(\frac{\rho_i - \rho_{wf}}{1 + K}\right)\right] + \frac{-1}{1 + K}\frac{k_fh}{25147\phi_m V_{res} b_{D,PSS}\mu_{gi}c_{gi}} \bar{\beta}_m t
\]

Equation E-24

Adding \(\frac{\rho_{sc}}{\rho_{sc}}\) in the last term in Equation E-24 gives:
\[
\ln\left(\frac{q_{sc}}{\lambda_m}\right) = \ln\left[\frac{k_fh}{25147\rho_{sc} b_{D,PSS}\mu_{gi}c_{gi}} \left(\frac{\rho_i - \rho_{wf}}{1 + K}\right)\right] + \frac{-1}{1 + K}\frac{k_fh\rho_i}{\rho_{sc}} \frac{1}{25147\phi_m V_{res} b_{D,PSS}\mu_{gi}c_{gi}} \bar{\beta}_m t
\]

Equation E-25

Substituting \(\frac{(\phi_m + \phi_f)V_{res}\rho_i}{\rho_{sc}} = OGIP\) into Equation E-25 gives:
\[
\ln\left(\frac{q_{sc}}{\lambda_m}\right) = \ln\left[\frac{k_fh}{25147\rho_{sc} b_{D,PSS}\mu_{gi}c_{gi}} \left(\frac{\rho_i - \rho_{wf}}{1 + K}\right)\right] - \frac{(\phi_m + \phi_f)k_fh\rho_i}{25147\phi_m OGIP \rho_{sc} b_{D,PSS}\mu_{gi}c_{gi}} \frac{1}{1 + K}\bar{\beta}_m t
\]

Equation E-26

Substituting definition of \(r_p\) into Equation E-26 gives:
\[
\ln\left(\frac{q_{sc}}{\lambda_m}\right) = \ln\left[\frac{k_fh\rho_i}{25147\rho_{sc} b_{D,PSS}\mu_{gi}c_{gi}} \frac{r_p}{1 + K}\right] - \frac{1}{OGIP} \frac{(\phi_m + \phi_f)k_fh\rho_i}{25147\phi_m \rho_{sc} b_{D,PSS}\mu_{gi}c_{gi}} \frac{1}{1 + K}\bar{\beta}_m t
\]

Equation E-27

Equation E-27 can be written as follows:
\[
\ln\left(\frac{q_{sc}}{\lambda_m}\right) = \ln(Y) - X \cdot \bar{\beta}_m t
\]

Equation E-28

where:
\[
X = \frac{\phi_m + \phi_f}{\phi_m} \frac{q_{gd}^e}{OGIP}
\]

Equation E-28a

\[
Y = q_{gd}^e r_p
\]

Equation E-28b

\[
q_{gd}^e = \frac{\rho_i k_fh}{25147\rho_{sc} b_{D,PSS}\mu_{gi}c_{gi}(1 + K)}
\]

Equation E-28c

Plotting \(\beta_m t\) vs. \(\ln\left(\frac{q_{sc}}{\lambda_m}\right)\) and obtaining X and Y may produce OGIP as follows:
**OGIP**  
\[ \frac{\phi_m + \phi_f}{\phi_m} \frac{Y}{r_p X} \approx \frac{Y}{r_p X} \]  
\text{Equation E-29}

An iterative procedure could then be implemented to predict OGIP rigorously, which is detailed in section “Implementation of Rate-Transient Analysis 2”. Implementations as demonstrated by Figures E-1 to E-6 prove the validity of this rate-transient analysis method.

**Type Curve for Wide-Open Flow Condition**

Appendix D derives gas rate at the second decline stage equation in exponential form:

\[ q_{gsc} = \frac{k_f h \rho_i}{25147 \rho_{sc} b_{D,PSS} \mu_{gi} c_{gi}} \frac{r_p}{1 + K} \tilde{\lambda}_m \exp\left(- \frac{1}{OGIP} \frac{k_f h \rho_i}{25147 \rho_{sc} b_{D,PSS} \mu_{gi} c_{gi}} \frac{1}{1 + K} \tilde{\beta}_m \phi_m + \phi_f t \right) \]  
\text{Equation E-34}

We also write Equation E-34 as follows:

\[ q_{gsc} = q_{gdi} \tilde{\lambda}_m \exp(-D_{di}^e \tilde{\beta}_m t) \]  
\text{Equation E-35}

where:

\[ q_{gi}^e = \frac{\rho_i k_f h}{25147 \rho_{sc} b_{D,PSS} \mu_{gi} c_{gi} (1 + K)} \]  
\text{Equation E-35a}

\[ q_{gdi} = r_p q_{gi}^e = \frac{\rho_i k_f h r_p}{25147 \rho_{sc} b_{D,PSS} \mu_{gi} c_{gi} (1 + K)} \]  
\text{Equation E-35b}

\[ D_{di}^e = \frac{1}{OGIP} \frac{k_f h \rho_i}{25147 \rho_{sc} b_{D,PSS} \mu_{gi} c_{gi} (1 + K)} = \frac{1}{OGIP} \frac{q_{gi}^e}{q_{gdi}} \]  
\text{Equation E-35c}

Assuming negligible error in gas production due to using the second-decline-stage gas rate equation in early stage, which holds in a number of naturally fractured reservoirs because early stage is short compared to the second decline stage. Cumulative gas production is calculated by taking integral of Equation E-35, as shown below:

\[ G_p = \int_0^t q_{gsc} dt = \int_0^t q_{gdi} \tilde{\lambda}_m \exp(-D_{di}^e \tilde{\beta}_m t) dt \]  
\text{Equation E-36}

Taking \( q_{gdi} \) out of the integral gives:
\[ G_p = q_g di \int_0^t \exp(-D_{\text{di}}^e \frac{1}{1 + K} \ddot{\beta}_m t) \ddot{\lambda}_m dt \]

Equation E-37

Rearranging Equation E-37 so that the differential term is \( d \left( -D_{\text{di}}^e \frac{1}{1 + K} \ddot{\beta}_m t \right) \) gives:

\[ G_p = -\frac{q_g dt}{D_{\text{di}}^e} \int_0^t \exp(-D_{\text{di}}^e \frac{1}{1 + K} \ddot{\beta}_m t) d \left( -D_{\text{di}}^e \frac{1}{1 + K} \ddot{\beta}_m t \right) \]

Equation E-38

Equation E-38 gives:

\[ G_p = -r_p OGLP \left[ \exp \left( -D_{\text{di}}^e \frac{1}{1 + K} \ddot{\beta}_m t \right) - 1 \right] \]

Equation E-39

Ye and Ayala (2012) proposed \( \lambda = \left( 1 - \frac{G_p}{OGLP} \right)^{\frac{B}{\beta}} \). Per Equation 27 in Zhang and Ayala (2014a), this relationship is transformed as follows:

\[ \frac{\ddot{\lambda}_m}{\lambda_{wf}} = \left( 1 - \frac{G_p}{OGLP} \right)^{\frac{B}{\beta}} \]

Equation E-40

Applying Equation E-40 into Equation E-39 gives:

\[ 1 - \frac{1}{r_p} \left[ 1 - \left( \frac{\ddot{\lambda}_m}{\lambda_{wf}} \right)^{\frac{1}{\frac{B}{\beta}}} \right] = \exp \left( -D_{\text{di}}^e \frac{1}{1 + K} \ddot{\beta}_m t \right) \]

Equation E-41

For maximum rates and wide-open production, \( r_p = 1 \):

\[ \left( \frac{\ddot{\lambda}_m}{\lambda_{wf}} \right)^{\frac{1}{\frac{B}{\beta}}} = \exp \left( -D_{\text{di}}^e \ddot{\beta}_m t \right) \]

Equation E-42

\[ \ddot{\lambda}_m = \lambda_{wf} \exp^{\frac{B}{\beta}} \left( -D_{\text{di}}^e \ddot{\beta}_m t \right) \]

Equation E-43

Substituting \( \ddot{\lambda}_m = \frac{d(\ddot{\beta}_m t)}{dt} \) into Equation E-43 yields:

\[ \frac{d(\ddot{\beta}_m t)}{dt} = \lambda_{wf} \exp^{\frac{B}{\beta}} \left( -D_{\text{di}}^e \frac{1}{1 + K} \ddot{\beta}_m t \right) \]

Equation E-44

Dividing both sides by \( \frac{\lambda_{wf} \exp^{\frac{B}{\beta}} \left( -D_{\text{di}}^e \frac{1}{1 + K} \ddot{\beta}_m t \right)}{dt} \) and taking integral gives:
\[
\frac{1}{\lambda_{wf}} \int_0^t \frac{d(\bar{\beta}_m t)}{\exp(-D^e_{di} \bar{\beta}_m t)} = \int_0^t dt
\]

Equation E-45

Transferring differential term into \( d(\bar{B}^e t \bar{\beta}_m t) \) gives:

\[
\frac{1}{\lambda_{wf} D^e_{di}} \int_0^t \frac{d(\bar{B}^e D^e_{di} \bar{\beta}_m t)}{\exp(-\bar{B}^e D^e_{di} \bar{\beta}_m t)} = \int_0^t dt
\]

Equation E-46

Equation E-46 is rewritten into:

\[
\frac{1}{\lambda_{wf} D^e_{di}} \int_0^t \exp(\bar{B}^e D^e_{di} \bar{\beta}_m t) d(\bar{B}^e D^e_{di} \bar{\beta}_m t) = \int_0^t dt
\]

Equation E-47

Equation E-47 gives:

\[
\frac{1}{\lambda_{wf} D^e_{di}} \frac{1}{\bar{B}} \left[ \exp(\bar{B}^e D^e_{di} \bar{\beta}_m t) - 1 \right] = t
\]

Equation E-48

Multiplying both sides by \( \lambda_{wf} D^e_{di} \bar{B} \) and taking logarithm of both sides gives:

\[
\bar{B}^e D^e_{di} \bar{\beta}_m t = \ln(\lambda_{wf} \bar{B}^e D^e_{di} t + 1)
\]

Equation E-49

The analytical solution of \( \bar{\beta}_m \) is obtained by dividing both sides by \( \bar{B}^e D^e_{di} t \):

\[
\bar{\beta}_m = \frac{\ln(\lambda_{wf} \bar{B}^e D^e_{di} t + 1)}{\bar{B}^e D^e_{di} t}
\]

Equation E-50

Thus, \( \bar{\lambda}_m \) is explicitly calculated by substituting Equation E-50 into Equation E-43:

\[
\bar{\lambda}_m = \frac{\left[ \frac{\bar{B}^e D^e_{di} t}{\lambda_{wf} (1 + K)} + 1 \right]^{-1}}{\lambda_{wf}}
\]

Equation E-51

Substituting Equation E-50 and Equation E-51 into Equation E-35 gives:

\[
q_{gsc} = q_{gdi} \lambda_{wf} \left( \lambda_{wf} \bar{B}^e D^e_{di} t + 1 \right)^{-1} \exp \left[ \frac{1}{\beta} \ln(\lambda_{wf} \bar{B}^e D^e_{di} t + 1) \right]
\]

Equation E-55

Calculating the exponential term in Equation E-55 gives:

\[
q_{gsc} = q_{gdi} \lambda_{wf} \left( \lambda_{wf} \bar{B}^e D^e_{di} t + 1 \right)^{-1} (\lambda_{wf} \bar{B}^e D^e_{di} t + 1)^{-\frac{1}{\beta}}
\]

Equation E-56
Combining the two term with exponent in Equation E-56 gives:

\[ q_{gsc} = q_{gd} \lambda_w f (\lambda_w f \bar{B} D_{di}^e t + 1)^{-\frac{1}{B-1}} \]  

Equation E-57

Rewriting Equation E-57 as follows:

\[ q_{gsc} = q_{gd} \lambda_w f \frac{1}{(\lambda_w f \bar{B} D_{di}^e t + 1)^{\frac{1}{B+1}}} \]  

Equation E-58

Putting exponent \( \frac{B}{1+B} \) on both sides gives:

\[ \left( \frac{1}{q_{gd} \lambda_w f} q_{gsc} \right)^{\frac{\bar{B}}{1+B}} = \frac{1}{\lambda_w f \bar{B} D_{di}^e t + 1} \]  

Equation E-59

Equation E-59 reveals universal type curve in wide-open decline:

\[ y = \frac{1}{x+1} \]  

Equation E-60

where:

\[ y = \left( \frac{1 + K}{\lambda_w f q_{gd}} q_{gsc} \right)^{\frac{\bar{B}}{1+B}} \]  

Equation E-60a

\[ x = \frac{\lambda_w f \bar{B} D_{di}^e t}{1 + K} \]  

Equation E-60b

Plotting \( t \) vs. \( \frac{B}{1+B} \) and \( x \) vs. \( y \) on the same plot and matching them against each other, we could have

\[ \left( \frac{1}{\lambda_w f q_{gd} \lambda_w f \bar{B} D_{di}^e} \right)^{\frac{\bar{B}}{1+B}} \text{ and } \lambda_w f \bar{B} D_{di}^e \]:

\[ OGIP = \frac{\bar{B}}{r_p \frac{1}{\lambda_w f q_{gd}} \cdot \lambda_w f \bar{B} D_{di}^e} \]  

Equation E-61

Implementation of Rate-Transient Analysis 2

Equation E-29 could be used for OGIP prediction following the steps:

1. Calculate \( G_p \) at the last time point and estimate OGIP by \( 2G_p \).
2. With derived OGIP from last step, plotting \( \bar{\beta}_m t \) vs. \( \ln \left( \frac{q_{gsc}}{\bar{\lambda}_m} \right) \) and obtain the best fitted straight line through the points in late-decline stage and obtaining X and Y to calculate OGIP with the Equation E-29.

3. Calculate the difference between two OGIPs in the last two iterations, if the difference is larger than criterion, repeat steps 2 and 3, otherwise use the OGIP calculated.

Knowing OGIP, associated rate-transient analysis could be conducted by solving Equation E-62, Equation E-63, E-64 and E-65 for \( q_{gsc}^i \) at each time step:

\[
\ln \left( \frac{q_{gsc}^i}{\bar{\lambda}_m^i} \right) = \ln(Y) - X \cdot \bar{\beta}_m^i t^i
\]

\text{Equation E-62}

\[
G_p^i = G_p^{i-1} + \frac{q_{gsc}^i + q_{gsc}^{i-1}}{2}(t^i - t^{i-1})
\]

\text{Equation E-63}

\[
\frac{\bar{\rho}_m^i}{\rho_i} = 1 - \frac{G_p^i}{OGIP}
\]

\text{Equation E-64}

\[
\bar{\lambda}_m^i = \frac{\mu_{gi} c_{gi}}{2\theta (\bar{\rho}_m^i - \rho_{wf})} \frac{m^i (p_m - p_{wt})}{m^i (p_m - p_{wt})}
\]

\text{Equation E-65}

where \( i \) denotes the number of time steps.

Same reservoir and fluid properties as Chapter 2 is utilized for validating rate-transient analysis and OGIP prediction method 2. Calculate \( G_p \) at the last time point and estimate OGIP by \( 2G_p \). OGIP corresponding to Scenarios 1, 2 and 3 are 119.9 Bscf, 479.6 Bscf and 1918.4 Bscf respectively.

With derived OGIP, we plot \( \bar{\beta}_m t \) vs. \( \ln \left( \frac{q_{gsc}}{\bar{\lambda}_m} \right) \) before obtaining a fitted straight line through the points in late-decline stage and obtaining X and Y and hence OGIP. The resulting plots are presented as Figures E-1, E-2, and E-3 corresponding to three scenarios. The trend of plotted data points is a curve when wrong OGIP is applied. For accuracy, the straight lines are calculated by applying linear regression to data points at the second decline stage.
Figure E-1 $\bar{\beta}_m t$ vs. $\ln\left(\frac{q_{scx}}{\lambda_m}\right)$ Straight-Line Analysis for Scenario 1

Figure E-2 $\bar{\beta}_m t$ vs. $\ln\left(\frac{q_{scx}}{\lambda_m}\right)$ Straight-Line Analysis for Scenario 2
Figure E-3 $\bar{\beta}_m t$ vs. $\ln \left( \frac{q_{sc}}{\lambda_m} \right)$ Straight-Line Analysis for Scenario 3

The resulting OGIP are calculated with the estimated OGIPs. The exact OGIPs, and prediction errors are listed in Table E-1.

Table E-1 Estimated OGIP vs. Exact OGIP from $\bar{\beta}_m t$ vs. $\ln \left( \frac{q_{sc}}{\lambda_m} \right)$ Straight-Line Analysis

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$X, 1$/day</th>
<th>$\ln(Y)$</th>
<th>Estimated OGIP, Bscf</th>
<th>Exact OGIP, Bscf</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>0.000607951</td>
<td>10.72544751</td>
<td>80.3</td>
<td>119.9</td>
<td>33.07%</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>0.000557872</td>
<td>12.17042367</td>
<td>370</td>
<td>479.6</td>
<td>22.93%</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>0.000431405</td>
<td>13.48272883</td>
<td>1740</td>
<td>1918.4</td>
<td>9.41%</td>
</tr>
</tbody>
</table>

If estimated OGIP is smaller than $G_p$ from production data, use $1.01G_p$ as new OGIP. If the difference between new OGIP and guessed OGIP is larger than the criterion, repeat the steps with the new OGIP. The resulting plot after 2 rounds of OGIP prediction is shown in Figures E-4, E-5, and E-6 corresponding to three scenarios.
Figure E- 4 $\bar{\beta}_m t$ vs. $\ln\left(\frac{q_{gsc}}{\lambda_m}\right)$ Straight-Line Analysis for Scenario 1 at Run 3

Figure E- 5 $\bar{\beta}_m t$ vs. $\ln\left(\frac{q_{gsc}}{\lambda_m}\right)$ Straight-Line Analysis for Scenario 2 at Run 3
Figure E-6 $\bar{\beta}_m t$ vs. $\ln \left( \frac{q_{gsc}}{\lambda_m} \right)$ Straight-Line Analysis for Scenario 3 at Run 3

The resulting OGIP are calculated with the estimated OGIPs. The exact OGIPs, and prediction errors are listed in Table E-2.

Table E-2 Estimated OGIP vs. Exact OGIP from $\bar{\beta}_m t$ vs. $\ln \left( \frac{q_{gsc}}{\lambda_m} \right)$ Straight-Line Analysis at Run 3

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$X$, 1/day</th>
<th>$\ln(Y)$</th>
<th>Estimated OGIP, Bscf</th>
<th>Exact OGIP, Bscf</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scenario 1</td>
<td>0.006034278</td>
<td>13.46544684</td>
<td>118.3</td>
<td>119.919</td>
<td>1.36%</td>
</tr>
<tr>
<td>Scenario 2</td>
<td>0.004266943</td>
<td>14.5009273</td>
<td>470.6</td>
<td>479.6</td>
<td>1.89%</td>
</tr>
<tr>
<td>Scenario 3</td>
<td>0.001927918</td>
<td>15.07823697</td>
<td>1855.1</td>
<td>1918.4</td>
<td>3.30%</td>
</tr>
</tbody>
</table>

In this case, $G_p$ are close to OGIP and 1.01 $G_p$ are larger than OGIP. Derived OGIP from straight-line analysis tend to be slightly smaller than $G_p$ every time OGIP is set 1.01 $G_p$. The estimated OGIP tend to oscillate around the true OGIP and we take average value of the oscillation on both sides.
Appendix F  A Double-Exponential Model for Dual-Porosity Liquid Systems

A double exponential solution for boundary dominated flow in dual-porosity reservoir is found accurate:

$$q_D = \frac{E e^{Et} + F e^{Ht}}{l}$$

Equation F-1

$$E = \frac{2}{r_{eD}^2} + \frac{4(1 - \omega)}{r_{eD}^2 \omega} \left[ \frac{\xi \left[ -\frac{3}{4} + \frac{2(1 - \omega)^2}{\xi r_{eD}^2} + \log(r_{eD}) \right]}{(1 - \omega) \omega} \right]$$

$$+ \sqrt{\frac{-8 \xi \left[ -\frac{3}{4} + \log(r_{eD}) \right]}{r_{eD}^2 (1 - \omega) \omega} + \frac{2}{r_{eD}^2} + \frac{\xi \left[ -\frac{3}{4} + \frac{2(1 - \omega)^2}{\xi r_{eD}^2} + \log(r_{eD}) \right]}{(1 - \omega) \omega}}^2$$

Equation F-2

$$F = -\frac{2}{r_{eD}^2} - \frac{4(1 - \omega)}{r_{eD}^2 \omega} \left[ \frac{\xi \left[ -\frac{3}{4} + \frac{2(1 - \omega)^2}{\xi r_{eD}^2} + \log(r_{eD}) \right]}{(1 - \omega) \omega} \right]$$

$$+ \sqrt{\frac{-8 \xi \left[ -\frac{3}{4} + \log(r_{eD}) \right]}{r_{eD}^2 (1 - \omega) \omega} + \frac{2}{r_{eD}^2} + \frac{\xi \left[ -\frac{3}{4} + \frac{2(1 - \omega)^2}{\xi r_{eD}^2} + \log(r_{eD}) \right]}{(1 - \omega) \omega}}^2$$

Equation F-3

$$G = -\frac{1}{r_{eD}^2 \left( -\frac{3}{4} + \log(r_{eD}) \right)} \left\{ \frac{\xi \left[ -\frac{3}{4} + \frac{2(1 - \omega)^2}{\xi r_{eD}^2} + \log(r_{eD}) \right]}{2(1 - \omega) \omega \left( -\frac{3}{4} + \log(r_{eD}) \right)} \right\}$$

$$- \sqrt{\frac{-8 \xi \left[ -\frac{3}{4} + \log(r_{eD}) \right]}{r_{eD}^2 (1 - \omega) \omega} + \frac{2}{r_{eD}^2} + \frac{\xi \left[ -\frac{3}{4} + \frac{2(1 - \omega)^2}{\xi r_{eD}^2} + \log(r_{eD}) \right]}{(1 - \omega) \omega}}^2$$

Equation F-4
\[ H = - \frac{1}{r_e D^2 \left( -\frac{3}{4} + \log(r_e D) \right)} - \frac{\xi \left[ -\frac{3}{4} + \frac{2(1 - \omega)^2}{\xi r_e D^2} + \log(r_e D) \right]}{2(1 - \omega) \omega \left( -\frac{3}{4} + \log(r_e D) \right)} \]

\[ + \sqrt{-\frac{8 \xi \left( -\frac{3}{4} + \log(r_e D) \right)}{r_e D^2 (1 - \omega) \omega} + \frac{2 \frac{3}{4} + \frac{2(1 - \omega)^2 + \log(r_e D)}{\xi r_e D^2}}{(1 - \omega) \omega} \]^{-2} \]

\[ 2 \left( -\frac{3}{4} + \log(r_e D) \right) \]

**Equation F-5**

\[ I = 2 \left( -\frac{3}{4} + \log(r_e D) \right) \sqrt{-\frac{8 \xi \left( -\frac{3}{4} + \log(r_e D) \right)}{r_e D^2 (1 - \omega) \omega} + \frac{2 \frac{3}{4} + \frac{2(1 - \omega)^2 + \log(r_e D)}{\xi r_e D^2}}{(1 - \omega) \omega} \]^{-3} \]

**Equation F-6**

This equation is proven to be rigorous by comparing it against numerical inversion of \( q_D \) in Laplace Space.

**Derivation Procedure**

Laplace space solution in dual-porosity system under closed boundary and constant \( p_{wf} \):

\[ \tilde{q}_D = \frac{\sqrt{s f(s)} l_1(\sqrt{s f(s)} r_e D) K_1(\sqrt{s f(s)}) - K_1(\sqrt{s f(s)} r_e D) l_1(\sqrt{s f(s)})}{s l_0(\sqrt{s f(s)}) K_1(\sqrt{s f(s)} r_e D) + K_0(\sqrt{s f(s)}) l_1(\sqrt{s f(s)} r_e D)} \]

**Equation F-7**

Laplace space solution in dual-porosity system under closed boundary and constant \( q \) is written as follows:

\[ \tilde{p}_{WD} = \frac{K_1(\sqrt{s f(s)} r_e D) l_0(\sqrt{s f(s)}) + l_1(\sqrt{s f(s)} r_e D) K_0(\sqrt{s f(s)})}{s \sqrt{s f(s)} [l_1(\sqrt{s f(s)} r_e D) K_1(\sqrt{s f(s)}) - l_1(\sqrt{s f(s)}) K_1(\sqrt{s f(s)} r_e D)]} \]

**Equation F-8**

Equation F-8 is written as follows:

\[ \frac{1}{\tilde{p}_{WD}} = \frac{s \sqrt{s f(s)} [l_1(\sqrt{s f(s)} r_e D) K_1(\sqrt{s f(s)}) - l_1(\sqrt{s f(s)}) K_1(\sqrt{s f(s)} r_e D)]}{K_1(\sqrt{s f(s)} r_e D) l_0(\sqrt{s f(s)}) + l_1(\sqrt{s f(s)} r_e D) K_0(\sqrt{s f(s)})} \]

**Equation F-9**

Equation F-7 and F-8 satisfy:

\[ \tilde{q}_D = \frac{1}{s^2 \tilde{p}_{WD}} \]
Warren and Root (1963) gives late-time approximation for \( p_{wD} \) as follows:

\[
P_{wD}(t_D, 1) \approx \frac{2}{r_{eD}^2 - 1} \left( \frac{1}{4} + t_D + \frac{(1 - \omega)^2}{\xi} \left\{ 1 - \exp \left[ -\frac{\xi t_D}{\omega(1 - \omega)} \right] \right\} \right) - \frac{3r_{eD}^4 - 4r_{eD}^4 \ln r_{eD} - 2r_{eD}^2 - 1}{4(r_{eD}^2 - 1)^2}
\]

Equation F-10

Given large \( r_{eD} \), Equation F-11 can be written in a simplified form as follows:

\[
P_{df}(t_D, 1) \approx \frac{2}{r_{eD}^2} \left( \frac{1}{4} + t_D + \frac{(1 - \omega)^2}{\xi} \left\{ 1 - \exp \left[ -\frac{\xi t_D}{\omega(1 - \omega)} \right] \right\} \right) - \frac{3r_{eD}^4 - 4r_{eD}^4 \ln r_{eD} - 2r_{eD}^2 - 1}{4r_{eD}^4}
\]

Equation F-11

Equation F-12 may be written as follows:

\[
P_{df}(t_D, 1) \approx \frac{2}{r_{eD}^2} \left( \frac{1}{4} + t_D + \frac{(1 - \omega)^2}{\xi} \left\{ 1 - \exp \left[ -\frac{\xi t_D}{\omega(1 - \omega)} \right] \right\} \right) - \frac{3r_{eD}^4 - 4r_{eD}^4 \ln r_{eD} - 2r_{eD}^2 - 1}{4r_{eD}^4} - \frac{1}{4r_{eD}^4}
\]

Equation F-12

Simplifying last four terms on the RHS in Equation F-13 gives:

\[
P_{df}(t_D, 1) \approx \frac{2}{r_{eD}^2} \left( \frac{1}{4} + t_D + \frac{(1 - \omega)^2}{\xi} \left\{ 1 - \exp \left[ -\frac{\xi t_D}{\omega(1 - \omega)} \right] \right\} \right) - \frac{3}{4} + \ln r_{eD} + \frac{1}{2r_{eD}^2} + \frac{1}{4r_{eD}^4}
\]

Equation F-13

Rearranging terms in Equation F-14 gives:

\[
P_{df}(t_D, 1) \approx \frac{1}{4r_{eD}^2} + \frac{2}{r_{eD}^2} t_D + \frac{2}{r_{eD}^2} (1 - \omega)^2 \left\{ 1 - \exp \left[ -\frac{\xi t_D}{\omega(1 - \omega)} \right] \right\} - \frac{3}{4} + \ln r_{eD}
\]

Equation F-14

Combining terms in Equation F-15 gives:

\[
P_{df}(t_D, 1) \approx -\frac{2(1 - \omega)^2}{\xi^2 r_{eD}^2} \exp \left[ -\frac{\xi t_D}{\omega(1 - \omega)} \right] + \frac{2}{r_{eD}^2} t_D + \frac{2(1 - \omega)^2}{\xi^2 r_{eD}^2} - \frac{3}{4} + \ln r_{eD}
\]

Equation F-15

Substituting Equation F-16 into the Equation F-10 gives:

\[
\bar{q}_D = \frac{1}{s^2 p_{wd}} = \frac{1}{s^2 L \left( -\frac{2(1 - \omega)^2}{\xi^2 r_{eD}^2} \exp \left[ -\frac{\xi t_D}{\omega(1 - \omega)} \right] + \frac{2}{r_{eD}^2} t_D + \frac{2(1 - \omega)^2}{\xi^2 r_{eD}^2} - \frac{3}{4} + \ln r_{eD} \right)}
\]

Equation F-16

Combining terms in Equation F-17 gives:
\[ \tilde{q}_D = \frac{1}{s^2(-\frac{2(1-\omega)^2}{\xi r_{ed}^2} - \frac{1}{s} + \frac{2}{\omega(1-\omega)} + \frac{2(1-\omega)^2 - \frac{3}{4} + \ln r_{ed}}{s})} \]

Equation F-18

Bring \( s^2 \) into the bracket in Equation F-18 gives:

\[ \tilde{q}_D = \frac{1}{-\frac{2(1-\omega)^2}{\xi r_{ed}^2} - \frac{s^2}{s} + \frac{2(1-\omega)^2 - \frac{3}{4} + \ln r_{ed}}{s} + \frac{2}{r_{ed}^2}} \]

Equation F-19

Equation F-19 is then inversed analytically with Mathematica 9, yielding:

\[ q_D = \frac{E e^{Gt} + F e^{Ht}}{l} \]

Equation F-20

\[ E = \frac{2}{r_{ed}^2} \frac{4(1-\omega)}{r_{ed}^2 \omega} - \frac{\xi[-\frac{3}{4} + \frac{2(1-\omega)^2}{\xi r_{ed}^2} + \ln(r_{ed})]}{(1-\omega)\omega} \]

\[ + \frac{8\xi(-\frac{3}{4} + \ln(r_{ed}))}{r_{ed}^2(1-\omega)\omega} + \left\{ \frac{2}{r_{ed}^2} + \frac{\xi[-\frac{3}{4} + \frac{2(1-\omega)^2}{\xi r_{ed}^2} + \ln(r_{ed})]}{(1-\omega)\omega} \right\}^2 \]

Equation F-21

\[ F = -\frac{2}{r_{ed}^2} \frac{4(1-\omega)}{r_{ed}^2 \omega} + \frac{\xi[-\frac{3}{4} + \frac{2(1-\omega)^2}{\xi r_{ed}^2} + \ln(r_{ed})]}{(1-\omega)\omega} \]

\[ + \frac{8\xi(-\frac{3}{4} + \ln(r_{ed}))}{r_{ed}^2(1-\omega)\omega} + \left\{ \frac{2}{r_{ed}^2} + \frac{\xi[-\frac{3}{4} + \frac{2(1-\omega)^2}{\xi r_{ed}^2} + \ln(r_{ed})]}{(1-\omega)\omega} \right\}^2 \]

Equation F-22

\[ G = -\frac{1}{r_{ed}^2(-\frac{3}{4} + \ln(r_{ed}))(2(1-\omega)\omega(-\frac{3}{4} + \ln(r_{ed}))} \]

\[ - \frac{8\xi(-\frac{3}{4} + \ln(r_{ed}))}{r_{ed}^2(1-\omega)\omega} + \left\{ \frac{2}{r_{ed}^2} + \frac{\xi[-\frac{3}{4} + \frac{2(1-\omega)^2}{\xi r_{ed}^2} + \ln(r_{ed})]}{(1-\omega)\omega} \right\}^2 \]

Equation F-22
\[ H = -\frac{1}{r_e D^2 \left( -\frac{3}{4} + \log(r_e D) \right)} \left[ \frac{\xi \left[ -\frac{3}{4} + \frac{2(1-\omega)^2}{\xi r_e D^2} + \log(r_e D) \right]}{2(1-\omega) \omega \left( -\frac{3}{4} + \log(r_e D) \right)} \right] \]

Equation F-23

\[ I = 2\left( -\frac{3}{4} + \log(r_e D) \right) \sqrt{\frac{8\xi \left( -\frac{3}{4} + \log(r_e D) \right)}{r_e D^2 (1-\omega) \omega} + \left[ \frac{2}{r_e D^2} + \frac{\xi \left[ -\frac{3}{4} + \frac{2(1-\omega)^2}{\xi r_e D^2} + \log(r_e D) \right]}{(1-\omega) \omega} \right]^2} \]

Equation F-24
Appendix G Calculation of $\lambda_{mf}^*$

Appendix B shows the derivation of gas interporosity flow equation in pseudo-pressure:

$$\phi_c \frac{d(m(p_m))}{dt} = \frac{\alpha k_m}{\mu g_m c_g m} (m(p_f) - m(p_m))$$

Equation G-1

Define $\lambda_{mf}^*$ as below:

$$\lambda_{mf}^* = \frac{\mu g_i c_{gi}}{2\theta (\rho_m - \rho_f)} \frac{m(\rho_m) - m(\rho_f)}{\mu g_i c_{gi}}$$

Equation G-2

The interporosity flow equation can be written as:

$$\phi_c \frac{d\rho_m}{\lambda_{mf}^* dt} = \frac{\alpha k_m}{\mu g_i c_{gi}} (\rho_f - \rho_m)$$

Equation G-3

An approximation is found to produce a good estimate of $\lambda_{mf}^*$ without pseudo-pressure calculations for methane-dominated gas, namely gas with small SG:

$$\lambda_{mf}^* = \left( \sqrt{\lambda_m^*} + \sqrt{\lambda_f^*} \right)^2$$

Equation G-4