GEOMETRIC QUANTIZATION, REDUCTION AND K-HOMOLOGY

A Dissertation in Mathematics
by
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Abstract

Let $G$ be a compact connected Lie group acting on a stable complex manifold $M$ that has an equivariant vector bundle $E$ on it. In addition, suppose that $\phi$ is an equivariant map from $M$ to the Lie algebra $\mathfrak{g}$. We define an equivalence relation on the triples $(M, E, \phi)$ such that the set of equivalence classes forms an abelian group. We prove that this group is isomorphic to a completion of character ring $R(G)$. This leads to a geometric proof of the quantization commutes with reduction conjecture in the non-compact setting.
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Chapter 1

Introduction

1.1 Geometric Quantization

The goal of quantization is, starting from a classical mechanical system which is formalized as a symplectic manifold \((M, \omega)\), to produce a quantum mechanical system formalized as a Hilbert space \(H\). Moreover, since the classical observables consist of smooth functions \(C^\infty(M)\), while the quantum observables are realized as skew-Hermitian operators on \(H\), the process of quantization should give correspondence

\[ Q : C^\infty(M) \rightarrow \text{End}(H), f \rightarrow Q_f \]

satisfying

\[ Q\{f,g\} = \frac{\sqrt{-1}}{2\pi} [Q_f, Q_g], \text{ and } Q_1 = \text{Id}, \tag{1.1.1} \]

for at least some family of functions on \(M\). Neither existence nor uniqueness of quantization can be easily resolved. See [Woo92] for further details.

We start by recalling some notations.

Definition 1.1.1. A symplectic manifold is a smooth manifold, \(M\), equipped with a closed nondegenerate differential 2-form, \(\omega\), called the symplectic form.

The simplest example of a symplectic manifold is the phase space \(\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n\) of elementary mechanics. Its symplectic form is given by

\[ \omega = \sum_k dq_k \wedge dp_k, \]

where \(q_1, \ldots, q_n, p_1, \ldots p_n\) are coordinates on the first and second copies of \(\mathbb{R}^n\).
Definition 1.1.2. Any smooth function on $M$ uniquely determines a vector field, $f \rightarrow V_f$, defined by
\[ \omega(V_f, \cdot) = -df. \]
In addition, we define the Poisson bracket by the formula:
\[ \{\cdot, \cdot\} : C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M) : \{f, g\} = V_f(g). \]
For instance, let $M$ be the phase space $\mathbb{R}^{2n}$. The Poisson bracket is given by
\[ \{f, g\} = \sum \left( \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} - \frac{\partial g}{\partial p_k} \frac{\partial f}{\partial q_k} \right). \]
We should point out that not every symplectic manifold can be quantized. Experience shows that we should at least require a pre-quantum condition (or integral condition).

Definition 1.1.3. We say that a symplectic manifold $(M, \omega)$ satisfies the pre-quantum condition if there exists a Hermitian line bundle $E$ with a Hermitian connection $\nabla^E$ such that
\[ \frac{\sqrt{-1}}{2\pi} (\nabla^E)^2 = \omega. \]
In this case, we call $(M, E, \omega)$ pre-quantum data and $E$ the pre-quantum line bundle.

With this pre-quantum line bundle, we can now define the pre-quantum Hilbert space by
\[ H_0 = L^2(M, E). \]

Definition 1.1.4. For any $f \in C^\infty(M)$, we define an operator $Q_f : H_0 \rightarrow H_0$:
\[ Q_f(s) = \frac{\sqrt{-1}}{2\pi} \nabla^E_{V_f} s + f \cdot s, \quad s \in H_0. \]
One can check that $Q_f$ satisfies all the conditions in (1.1.1) for all smooth. The map
\[ Q : f \rightarrow \frac{1}{\sqrt{-1}} Q_f \]
indeed gives a Lie algebra homeomorphism:
\[ Q : (C^\infty(M), \{\cdot, \cdot\}) \longrightarrow (\text{skew Hermitian operators}, [\cdot, \cdot]). \]
Unfortunately, this is not the correct solution to the quantization problem. From
the viewpoint of geometric quantization theory, we need to reduce $H_0$ to an appropriate size by equipping pre-quantum data $(M, E, \omega)$ with a polarization (e.g. a complex structure). However, we are not going to discuss polarization in this dissertation. Instead, we will study quantization from the perspective of $K$-theory and index theory.

1.2 Quantization as an Index

Let $E$ be a smooth, oriented, even-dimensional Euclidean vector bundle over a smooth manifold $M$. A spinor bundle for $E$ is a smooth $\mathbb{Z}_2$-graded Hermitian vector bundle

$$\mathcal{E} = \mathcal{E}_+ \oplus \mathcal{E}_-$$

over $M$ that is equipped with a grading operator $\gamma$ and a $\mathbb{R}$-linear map

$$c : E \to \text{End}(\mathcal{E}),$$

which associates to each $v \in E_m$ a skew-adjoint, odd-graded endomorphism such that

$$[c(v)]^2 = -\|v\|^2 \cdot I.$$  

Moreover, the map $c$ induces an isomorphism

$$c : \text{Cliff}(E) \cong \text{End}(\mathcal{E}).$$

See [LM89, BD82] for details.

**Definition 1.2.1.** Suppose that $M$ is an even dimensional, smooth manifold with a spinor bundle $\mathcal{E}$ for $TM$. Let $L$ be a Hermitian vector bundle over $M$. We can define a $\text{Spin}^c$-Dirac operator $D^L$ in the following way. First, we choose two Hermitian connections $\nabla^E, \nabla^L$ on $\mathcal{E}$ and $L$. Form the tensor product

$$\nabla = \nabla^E \otimes I + I \otimes \nabla^L.$$  

We define

$$D^L = \sum c(e_i) \nabla_{e_i} : L^2(M, \mathcal{E}_+ \otimes L) \to L^2(M, \mathcal{E}_- \otimes L),$$

where $e_1, e_2, \ldots, e_{\dim M}$ is an orthonormal basis of $TM$.

**Remark 1.2.2.** When $M$ is odd-dimensional, there are also parallel definitions of spinor bundles and $\text{Spin}^c$-Dirac operators. For details, see [LM89, BHS07].
In the case when $M$ is symplectic, we can construct the spinor bundle and associated Spin$^c$-Dirac operator explicitly. Let $J$ be an almost-complex structure on $TM$ such that the formula

$$g^{TM}(v, w) = \omega(v, Jw)$$

defines a Riemannian metric on $TM$. Let $TM_C$ be the complexification of tangent bundle $TM$ and extend $J$ to $TM_C$ by complex linearity. Then $J$ gives a canonical splitting:

$$TM_C = T^{0,1}M \oplus T^{1,0}M,$$

where

$$T^{(1,0)}M = \{ z \in TM_C | Jz = \sqrt{-1}z \},$$
$$T^{(0,1)}M = \{ z \in TM_C | Jz = -\sqrt{-1}z \}.$$

We decompose

$$\Lambda^*(T^*M) \otimes \mathbb{C} = \bigoplus_{i,j} \Lambda^{i,j}T^*M,$$

where

$$\Lambda^{i,j}T^*M = \Lambda^i(T^{(1,0)*}M) \otimes \Lambda^j(T^{(0,1)*}M).$$

Here $T^{(1,0)*}M$ and $T^{(0,1)*}M$ are the complex vector bundle duals of $T^{(1,0)}M$ and $T^{(0,1)}M$.

For any $X \in TM$, its complexification decompose to

$$X = X_1 + X_2 \in T^{(1,0)}M \oplus T^{(0,1)}M.$$

The formula

$$c(X) = \sqrt{2}X_1^* \wedge -\sqrt{2}X_2^*,$$

where $X_1^* \in T^{(1,0)*}M$ is the vector dual of $X_1$, defines the canonical Clifford action of $X$ on $\Lambda^{0,*}T^*M$. Hence, the bundle $\mathcal{E} = \Lambda^{0,\text{odd}}T^*M \oplus \Lambda^{0,\text{even}}T^*M$ is a spinor bundle associated to the almost-complex structure $J$.

**Definition 1.2.3.** Let $(M, E, \omega)$ be pre-quantum data. As in Definition 1.2.1, we can construct a Spin$^c$-Dirac operator $D^E$. When $M$ is compact, we define its geometric quantization

$$Q(M, E) = \text{Ind}(D^E) = \ker(D^E) - \text{coker}(D^E),$$

which is a finite-dimensional virtual vector space.
We can see that the dimension of $Q(M, E)$ is determined by the almost-complex structure $J$. We sometimes call $Q(M, E)$ the **almost-complex quantization**. An additional advantage over this approach is that $M$ is not required to possess a symplectic structure, or even almost-complex structure as we shall now see.

**Definition 1.2.4.** A **stable-complex structure** (or **weakly complex structure**) on a real vector bundle $V$ over $M$ is a complex vector bundle structure (compatible with the underlying real vector bundle structure) on some direct sum $V \oplus \mathbb{R}^k$, where $\mathbb{R}^k$ is the trivial rank $k$ vector bundle on $M$. Two stable-complex structures are equivalent if there is an isomorphism of complex vector bundles

$$V \oplus \mathbb{R}^{k_1} \oplus \mathbb{C}^{n_1} \cong V \oplus \mathbb{R}^{k_2} \oplus \mathbb{C}^{n_2}$$

for some $n_1$ and $n_2$. A **stable-complex manifold** is a smooth manifold $M$ with a stable-complex structure on its tangent bundle $TM$.

**Remark 1.2.5.** A stable-complex structure gives a specific choice of lifting from real $K$-theory to complex $K$-theory.

**Remark 1.2.6.** We define an equivariant stable complex structure by requiring the complex structure to be invariant and the isomorphisms to be equivariant. Here, the group acts on $TM$ by the natural lifting of its action on $M$. It is very important to point out that we require that $G$ act trivially on $\mathbb{C}^n$ but $\mathbb{R}^k$ does not have to be equipped with trivial $G$-action [GGK02, Appendix D].

Suppose that $M$ is a stable-complex manifold. In this case, we can define a spinor bundle and a Spin$^c$-Dirac operator associated to the stable-complex structure $J$ [GGK02]. Its index is defined to be the **stable-complex quantization**.

**Remark 1.2.7.** In the equivariant case in which a compact group $G$ acts on $M$ preserving stable-complex structure, and also acts on $E$ preserving $\nabla^E$, we can define an equivariant Spin$^c$-Dirac operator, whose index takes value in the character ring of $G$, that is

$$Q(M, E) \in R(G).$$

### 1.3 Quantization Commutes with Reduction

The laws describing the mechanical systems studied in physics usually display a high degree of symmetry, which correspond to the conserved quantities familiar in physics like energy and momentum. Symplectic reduction is a sort of quotient operation defined for group actions on symplectic manifolds. The goal is to factor out the symmetries so as to obtain a simpler system that is easier to analyze.
Definition 1.3.1. Let $G$ be a compact, connected Lie group. We say that $G$ acts on $M$ in a Hamiltonian fashion if there is a map $\mu : M \to g^*$ such that

$$d\mu_\xi = \iota_{\xi_M} \omega,$$

where $\xi \in g$ and $\xi_M$ is the induced infinitesimal vector field, and

$$\{\mu_X, \mu_Y\} = \mu_{[X,Y]}, \quad X, Y \in g.$$

We call $\mu$ the moment map associated to the $G$-action. Unless $g$ is semi-simple, the moment map is not uniquely defined. For if $\mu : M \to g^*$ satisfies the conditions above, so does $\mu + c$, where $c$ is in the annihilator of $[g, g] \subseteq g$.

Theorem 1.3.2 (Kostant). Let $(M, E, \omega)$ be pre-quantum data with Hamiltonian $G$-action. The map defined by the formula

$$\mu_\xi = \frac{\sqrt{-1}}{2\pi} (\nabla^E_{\xi_M} - L_\xi), \xi \in g,$$

where $L_\xi$ is the Lie derivative on $E$, satisfies all the conditions in Definition 1.3.1. We call it the Kostant’s moment map.

Proposition 1.3.3. If the origin in $g^*$ is a regular value of $\mu$, then $\mu^{-1}(0)$ is a $G$-invariant submanifold of $M$ on which $G$ acts locally freely.

Proof. [GS82]

If we further assume that $G$ acts on $\mu^{-1}(0)$ freely, then the orbit space

$$M_0 = \mu^{-1}(0)/G$$

is a smooth manifold (indeed a symplectic manifold). Let $\pi : \mu^{-1}(0) \to M_0$ be the projection, $\iota : \mu^{-1}(0) \hookrightarrow M$ the inclusion, and $E_0$ a line bundle over $M_0$ defined by

$$E_0 = \iota^*(E)/G.$$

Proposition 1.3.4. There exists a unique symplectic form $\omega_0$ on $M_0$ such that

$$\pi^* \omega_0 = \iota^* \omega.$$

In addition, we can choose a connection $\nabla_0$ on $E_0$, such that

$$\pi^* \nabla_0 = \iota^* \nabla^E, \quad \frac{\sqrt{-1}}{2\pi} (\nabla_0)^2 = \omega_0.$$
Proof. [GS82, Wei79]

**Definition 1.3.5.** Given any pre-quantum data \((M, E, \omega)\), we define the **reduced pre-quantum data** to be \((M_0, E_0, \omega_0)\).

**Remark 1.3.6.** One can extend the symplectic reduction to a general “integral” coadjoint orbit \(O_\gamma \subset g^*\) using a shifting trick. To be precise, let us first fix a maximal torus \(T\) and positive Weyl chamber \(t_+\) [Hum78]. Let \(\gamma \in t_+\) be a dominate weight and \(O_\gamma\) the coadjoint orbit through \(\gamma\). It is well-known that \(O_\gamma\) is automatically a symplectic manifold, with a natural pre-quantum line bundle denoted by \(L_\gamma\). The famous Borel-Weil theorem says that the space of all the holomorphic sections \(H_\gamma = \Gamma_{\text{hol}}(L_\gamma)\) is an irreducible representation of \(G\). Moreover, this actually gives an 1-1 correspondence between dominant weights and irreducible representations \(\hat{G}\) [Kir04].

Form the product \(M \times O_\gamma\), which is a symplectic manifold, with associated moment map defined by

\[
\mu_\gamma : M \times O_\gamma \to g^* : \mu_\gamma(m, z) = m + z.
\]

If we apply symplectic reduction to \(M \times O_\gamma\), then we will obtain reduced pre-quantum data \((M_\gamma, E_\gamma, \omega_\gamma)\), where

\[
M_\gamma = \mu_\gamma^{-1}(0)/G = \mu_\gamma^{-1}(O_\gamma)/G,
\]

and

\[
E_\gamma = (E|_{\mu_\gamma^{-1}(O_\gamma)})/G.
\]

**Remark 1.3.7.** If the \(G\)-action on \(\mu_\gamma^{-1}(O_\gamma)\) is only locally free, then \(M_\gamma\) is an orbifold which we will discuss later. The case when \(\gamma\) is a singular value was discussed in [MS99].

We have defined the reduction in classical mechanics. But there is also a process of reduction within quantum mechanics. Thus, we can ask, is it consistent with classical reduction? In other words, if we start with a classical system, reduce it in order to simplify it, then quantize it in order to carry out calculations appropriate to physics at a small scale, will we end up with the same calculations as if we quantized first, then reduced the quantum mechanical system? This is the quantization commutes with reduction problem.

Throughout this paper, we will assume that the \(G\)-action on \(M\) lifts to an action on the line bundle \(E\).

**Definition 1.3.8.** For any \(\chi \in R(G)\) and \(\gamma \in \hat{G}\), we denote by \([\chi]^\gamma \in \mathbb{Z}\) the \(\gamma\)-component of \(\chi\).
Theorem 1.3.9. Let $G$ be a compact, connected Lie group, and $(M,E,\omega)$ pre-quantum data with Hamiltonian $G$-action. If $M$ is compact and $0$ is a regular value of the moment map $\mu$, then we have that

$$[Q(M,E)]^0 = Q(M_0,E_0) \in \mathbb{Z}. \quad (1.3.1)$$

This conjecture was first stated and proved by Guillemin and Sternberg [GS82] for Kähler manifolds in 1982. Thereafter, a number of people (Vergne, Jeffery, Kirwan, etc) contributed to the conjecture. The general case was solved by Meinrenken, Meinrenken-Sjamaar in 1996 [Mei98, MS99] using symplectic cutting surgery [Ler95]. Some further interesting development have occurred subsequently. In [TZ98], Tian and Zhang gave a completely new analytic approach to the conjecture. A cobordism approach developed by Guillemin-Ginzburg-Karshon [GGK99], as well as Paradan’s proof [Par01] based on transversally elliptic operator provide different understanding of the $[Q, R] = 0$ conjecture. This dissertation was inspired by their works.

By the “shifting trick” described in Remark 1.3.6, Theorem 1.3.9 can be generalized to arbitrary irreducible representation of $G$.

Theorem 1.3.10. Under the same assumption, if $\gamma \in \hat{G}$ is a regular value of moment map $\mu$, then $(M_\gamma,E_\gamma)$ is pre-quantum data. Moreover,

$$[Q(M,E)]^\gamma = Q(M_\gamma,E_\gamma) \in \mathbb{Z}.$$

1.4 Noncompact Case

It is natural to try to generalize the $[Q, R] = 0$ theorem to the case when the symplectic manifold is non-compact. The non-compact quantization conjecture was first made by Vergne in her ICM2006 Plenary lecture, and solved by Ma and Zhang [MZ09]. Later, Paradan gave a different proof [Par11].

The basic problem in the non-compact $[Q, R]=0$ theorem is “how to quantize non-compact manifolds”. Even in the classical $[Q, R]=0$ theorem, it becomes important to consider quantizing an open manifold. To be more precise, given any compact pre-quantum data $(M,E,\mu)$, let us choose a regular value $c$ of the function $H = \|\mu\|^2 : M \to \mathbb{R}$. We cut $(M,E)$ into two pieces:

$$(U_0,E|_{U_0}) \text{ and } (U_+,E|_{U_+}),$$

where $U_0 = H^{-1}((0,c))$, $U_+ = H^{-1}((c, +\infty))$. If we can quantize the two open pieces in a reasonable way, one can see that only $Q(U_0,E|_{U_0})$ will contribute to the
multiplicity of trivial representation in $Q(M, E)$. As we take $c$ close to 0, this will lead to the $[Q, R] = 0$ theorem.

In addition, the non-compact $[Q, R] = 0$ theorem has some interesting applications in representation theory of reductive Lie group [Par12].

To formulate the $[Q, R] = 0$ theorem in the non-compact setting, we need some additional assumptions on the pre-quantum data $(M, E, \omega)$.

**Definition 1.4.1.** We say that the moment map $\mu : M \to \mathfrak{g}^*$ is proper, if the inverse image of a compact subset is compact.

Under the assumption of properness of $\mu$, the submanifold $\mu^{-1}(O_\gamma)$ is compact. Hence, when $\gamma \in \hat{G}$ is a regular value of moment map $\mu$, the reduced space

$$M_\gamma = \mu^{-1}(O_\gamma)/G$$

is compact, and

$$n_\gamma = Q(M_\gamma, E_\gamma) \in \mathbb{Z}$$

is finite.

When $M$ is non-compact, the left-hand side of (1.3.1) is not well-defined, because the index of Spin$^c$-Dirac operators for non-compact manifolds are no longer finite dimensional. To address this problem, Ma and Zhang introduced a formal geometric quantization for $(M, E, \mu)$, by means of the Atiyah-Patodi-Singer-type index [APS76] for Dirac-type operators on manifolds with boundary, denoted by $Q_{APS}(M, E)$ [MZ09].

**Definition 1.4.2.** For any compact, connected Lie group, we define $\hat{R}(G)$ the completion of character ring:

$$\hat{R}(G) = \text{Hom}_\mathbb{Z}(R(G), \mathbb{Z}).$$

Note that $\hat{R}(G)$ is a $R(G)$-module, but no longer a ring.

**Theorem 1.4.3** (Ma-Zhang). Let $G$ be a compact, connected Lie group, and $(M, E, \omega)$ pre-quantum data with proper moment map. There exists a well-defined formal geometric quantization

$$Q_{APS}(M, E) \in \hat{R}(G).$$

Moreover, if 0 is a regular value of $\mu$, then

$$[Q_{APS}(M, E)]^0 = Q(M_0, E_0) \in \mathbb{Z}.$$
**Remark 1.4.4.** In the case when $M$ is compact, the moment map $\mu$ is automatically proper. One can check that the formal geometric quantization $Q_{APS}(M, E)$ is the same as the usual geometric quantization $Q(M, E)$. Thus, the Guillemin-Sternberg conjecture is a special case of Ma-Zhang theorem.

**Theorem 1.4.5 (Ma-Zhang).** Under the same assumptions above, if $\gamma \in \hat{G}$ is a regular value of moment map $\mu$, then

$$[Q_{APS}(M, E)]^\gamma = Q(M_\gamma, E_\gamma) \in \mathbb{Z}. $$

**Remark 1.4.6.** When the manifold $M$ is non-compact, the shifting trick is highly non-trivial since the definition of $Q_{APS}$ depends on the moment map $\mu$. Hence, Theorem 1.4.5 is not a direct corollary of Theorem 1.4.3.

### 1.5 Main Results

In this paper, we examine the problem of quantizing non-compact pre-quantum data from a topological perspective. One of the significant advantage of the topological approach is avoiding defining the quantization directly, but constructing the inverse of the quantization map by organizing all the pre-quantum data as a group. This idea comes from the definition of geometric $K$-homology by Baum and Douglas [BD82].

As before, let $G$ be a compact, connected Lie group. Instead of pre-quantum data, we consider more general data $(M, E, \phi)$:

- $M$ is a stable complex $G$-orbifold, possibly non-compact.
- $E$ is a $G$-equivariant orbifold vector bundle over $M$, not necessarily a line bundle.
- $\phi$ is a $G$-equivariant map from $M$ to $\mathfrak{g} \cong \mathfrak{g}^*$ (throughout this paper, we identify $\mathfrak{g}$ and its dual by making a choice of invariant inner product on $\mathfrak{g}$).

In Section 3.1, we will give the precise definition of these $K$-cycles which contains most interesting examples (e.g pre-quantum data with proper moment map) and the equivalence relation between them. The equivalence classes of all $K$-cycles form an abelian group, denoted by $\hat{K}(G)$. The addition operation is given by disjoint union and the additive inverse of a $K$-cycle is obtained by reversing the stable complex structure.
For any $K$-cycle $(M, E, \phi)$, we denote by $n \cdot (M, E, \phi)$ the disjoint union of $n$ copies of $(M, E, \phi)$. Take $\Gamma = \sum_{\gamma \in G} n_\gamma V_\gamma \in \hat{R}(G)$ and define

$$O_\Gamma = \bigsqcup_{\gamma \in G} n_\gamma \cdot (O_{\gamma}, E_{\gamma}, \iota_{\gamma}),$$

where $O_{\gamma}$ is the coadjoint orbit through $\gamma \in \mathfrak{t}_+^*$; $E_{\gamma}$ is the natural line bundle \cite{Bot65, Kir04}; and $\iota_{\gamma}: O_{\gamma} \hookrightarrow \mathfrak{g}^*$ is the inclusion. The following theorems constitute the main results of this paper.

**Theorem 1.5.1.** The map $P_{\text{TOP}}: \hat{R}(G) \to \hat{K}(G)$ defined by the formula

$$P_{\text{TOP}}: \Gamma \mapsto O_\Gamma$$

gives an isomorphism of abelian groups and $R(G)$-modules.

**Corollary 1.5.2.** When $M$ is compact, the inverse map $Q_{\text{TOP}} = P_{\text{TOP}}^{-1}: \hat{K}(G) \to \hat{R}(G)$ is consistent with the classical geometric quantization for compact manifold, that is

$$Q_{\text{TOP}}(M, E, \phi) = Q(M, E) \in R(G).$$

**Corollary 1.5.3** (Shifting Trick). Let $(M, E, \phi_1)$ and $(N, F, \phi_2)$ be two pre-quantum data. When $\phi_1$ is proper and $N$ is compact, we have that

$$Q_{\text{TOP}}(M, E, \phi_1) \times Q_{\text{TOP}}(N, F, \phi_2) = Q_{\text{TOP}}(M \times N, E \boxtimes F, \hat{\phi}_1 + \hat{\phi}_2),$$

where $\hat{\phi}_1, \hat{\phi}_2$ are the pullbacks of $\phi_1$ and $\phi_2$ respectively.

**Theorem 1.5.4.** Let $(M, E, \omega)$ be pre-quantum data with proper moment map $\mu$. If $\gamma \in \hat{G}$ is a regular value of $\mu$, then

$$[Q_{\text{TOP}}(M, E, \mu)]^\gamma = Q(M_\gamma, E_\gamma) \in \mathbb{Z}.$$
Chapter 2

Preliminaries

2.1 The Index Theorem

There are close connections between index theory and $K$-theory. Let us start with the following definition.

**Definition 2.1.1.** Let $E$ be an Euclidean vector bundle over a compact manifold $M$, and $\mathcal{E}_E$ a spinor bundle for $E$. Let $\hat{\mathcal{E}}_E$ be the pullback of $\mathcal{E}_E$ to the total space $E$. The *Thom class* $\beta(E) \in K(E)$ is the complex given by

\[
\{E_v : \hat{\mathcal{E}}_E|_v \to \hat{\mathcal{E}}_E|_v\}_{v \in E},
\]

where $E_v = c(v) \cdot \gamma$ is the Clifford multiplication operator associated to $v \in E$ ($\gamma$ is the grading operator).

**Definition 2.1.2.** If $\mathcal{E}$ is a spinor bundle for an Euclidean vector bundle $E$, then we denote by $\bar{\mathcal{E}}$ the complex conjugate of the underlying Hermitian bundle.

**Definition 2.1.3.** Let $M$ is a smooth manifold, and $E$ an Euclidean vector bundle over $M$. Suppose that both $M$ and the total space $E$ have stable-complex structures, whose associated spinor bundles are denoted by $\mathcal{E}_M$ and $\hat{\mathcal{E}}_E$. We embed $\iota : M \hookrightarrow E$ as zero sections. We say that the embedding $\iota$ *preserves stable-complex structures* if there exists a spinor bundle $\mathcal{E}_E$ for $E$ such that

\[
\mathcal{E}_M \otimes \mathcal{E}_E \cong \hat{\mathcal{E}}_E|_M
\]

(2.1.1)

In this case, we define the *wrong-way map*

\[
\iota_* : K^0(M) \to K^0(E)
\]
by multiplication by $[\beta(E)] \in K^0(E)$.

Suppose that $f : M \hookrightarrow N$ is a smooth embedding between two stable-complex manifolds. Let $NF$ be the normal bundle of $M$ in $N$. Then $NF$ can be identified as a tubular neighborhood $U$ of $M$. We say that $f$ preserves stable-complex structures if the zero-section embedding $\iota : M \hookrightarrow NF$ preserves the stable-complex structures as above. Then we define the wrong-way map

$$f_* : K^0(M) \rightarrow K^0(N)$$

to be the composition of

$$K^0(M) \rightarrow K^0(NF) \cong K^0(U) \rightarrow K^0(N),$$

where the first map is $\iota_*$ defined above and the last one is the natural extension map.

For any continuous map between two stable-complex manifolds $f : M \rightarrow N$, we can define the wrong-way map

$$f_* : K^0(M) \rightarrow K^0(N)$$

in the following way. We embed $\iota : M \hookrightarrow V$ into some complex vector space such that the embedding

$$p : M \hookrightarrow V \times N : m \rightarrow (\iota(m), f(m))$$

preserves stable complex structures. The wrong-way map $f_*$ is defined to be the composition of $p_*$ with the Thom isomorphism $K^0(V \times N) \cong K^0(N)$.

**Proposition 2.1.4.** The wrong-way map $f_*$ depends only on the continuous map $f$ and the stable complex structures on $M$ and $N$. The wrong-way map is functorial and homotopy invariant.

**Theorem 2.1.5.** [AS68b] Suppose that $M$ is a closed, stable-complex manifold and $p : M \rightarrow \text{pt}$. Let $[E] \in K^0(M)$ We have that

$$Q(M, E) = \text{Ind}(D^E) = p_*([E]) \in K^0(\text{pt}) \cong \mathbb{Z}.$$

### 2.2 Localization

Let $G$ be a compact Lie group. See [Seg68] for the definition of equivariant $K$-theory.
Lemma 2.2.1. If the action of $G$ on a space $X$ is trivial, then the natural map

$$K_0(X) \otimes R(G) \rightarrow K_0^G(X)$$

is an isomorphism.

In this section, we are particularly interested in the case when $G = S^1$. We can identify $R(S^1)$ with the ring of Laurent polynomials in integer coefficients:

$$R(S^1) \cong \mathbb{Z}[t, t^{-1}].$$

Definition 2.2.2. We denote by $R_{\text{loc}}$ the ring obtained from $R(S^1)$ by inverting all the Laurent polynomials $(1 - t^k)$ with $k \neq 0$.

Definition 2.2.3. Any equivariant $K$-theory group $K^0_{S^1}(X)$ is a $R(S^1)$-module. We define

$$K^0_{S^1}(X)_{\text{loc}} = K^0_{S^1}(X) \otimes_{R(S^1)} R_{\text{loc}}.$$ 

Suppose that $M$ is a $S^1$-equivariant, stable-complex manifold and $F = M_{S^1}$ is the fixed point. Let $\iota : F \hookrightarrow M$ be the inclusion, with normal bundle $NF$.

Lemma 2.2.4. The submanifold $F$ is a stable-complex manifold and the embedding $\iota : F \hookrightarrow M$ preserves stable-complex structures.

Proof. Suppose that the stable-complex structure on $M$ is represented by an almost-complex structure on $TM \oplus \mathbb{R}^n$. As a $S^1$-equivariant complex vector bundle, we can split $(TM \oplus \mathbb{R}^n)|_F$ into two complex vector bundle:

$$(TM \oplus \mathbb{R}^n)|_F \cong E_1 \oplus E_2,$$

where $S^1$ acts trivially on $E_1$, and $S^1$ fixes no non-zero vector on $E_2$. Since $F$ is the fixed point, we have that

$$TM|_F \cong TF \oplus NF,$$

where $S^1$ acts trivially on $TF$, and $S^1$ fixes no non-zero vector on $NF$. Therefore,

$$E_1 \cong TF \oplus \mathbb{R}^{n_1}, E_2 \cong NF \oplus \mathbb{R}^{n-n_1}.$$

This actually gives stable-complex structures on $TF$ and $NF$. What left is to check condition 2.1.1, which is straightforward. \qed
If we decompose the $S^1$-equivariant vector bundle $E_2$ into weight bundles:

$$E_2 \cong \bigoplus L_\alpha \otimes C_\alpha,$$

we have that the Thom class

$$\beta(NF) = \prod (1 - t^\alpha [L_\alpha]) \in K_{S^1}(F).$$

**Lemma 2.2.5 ([Seg68]).** The element $\beta(NF)$ is invertible in the localized group $K^0_{S^1}(F)_{\text{loc}}$.

**Theorem 2.2.6 ([Seg68, AS68a]).** The wrong-way map

$$\iota_* : K_{S^1}(F) \to K_{S^1}(M)$$

induces an isomorphism after localization. That is,

$$\iota_* : K_{S^1}(F)_{\text{loc}} \cong K_{S^1}(M)_{\text{loc}},$$

whose inverse is given by $\beta^{-1}(NF) \cdot \iota^*$.

**Theorem 2.2.7** (Atiyah-Bott Fixed Point Theorem). *For every connected component $F$ of $M^{S^1}$, form the class

$$[E|_F] \cdot \beta^{-1}(NF) \in K^0_{S^1}(F)_{\text{loc}} \cong K^0(F) \otimes R_{\text{loc}}.$$*

Let $p_F : F \to \text{pt}$. Then

$$Q(M, E) = \sum_F p_{F,*} ([E|_F] \cdot \beta^{-1}(NF)) \in R(S^1)_{\text{loc}}, \quad (2.2.1)$$

where $F$ ranges over all the connected components of fixed points.

**Corollary 2.2.8** ([MS99]). *If $S^1$ acts trivially on $E|_F$, then its equivariant index $Q(M, E)$ concentrates at the trivial representation in $R(S^1)$.*

**Proof.** Suppose that the stable-complex structure on $NF$ is represented by an almost-complex structure on $NF \oplus \mathbb{R}^n$. Decompose the complex vector bundle $NF \oplus \mathbb{R}^n$ into weight bundles:

$$NF = \sum N_\alpha \otimes C_\alpha.$$
Assume for simplicity that each $N_\alpha$ is a trivial bundle. Then, we have that
\[ p_*(\lbrack E \rbrack) = \sum_F (\prod (1 - t^\alpha)^{-1}) \cdot p_{F,*}([E|_F]) \in R(S^1)_{\text{loc}}. \] (2.2.2)

The “rational function” is regular, and indeed vanishes, at all $t^k$ for $k < 0$.

On the other hand, if we replace all $t$ by $t - 1$, we will get that (2.2.2) vanishes at all $t^k$ for $k > 0$. Combining the two facts together, we prove the lemma.

**Example 2.2.9.** Let us consider $M = \mathbb{C}P^1$, on which $S^1$ acts by rotation. The fixed points of $S^1$-action on $\mathbb{C}P^1$ are given by the north and south pole. Suppose that $E$ is a line bundle over $\mathbb{C}P^1$ with fiber weights equal to 1 at north pole and -1 at south pole. The associated Spin$^c$-Dirac operator $D$ gives an equivariant index given by:
\[ \text{Index}(D^E) = \chi_+ + \chi_- = \frac{t^{-1}}{1-t} + \frac{t}{1-t^{-1}} = t^{-1} + 1 + t \in R(S^1), \] (2.2.3)
where $\chi_\pm$ are the contributions by poles.

### 2.3 Comparision with $[Q, R] = 0$

We now exploit the connection between the index theorem and $K$-theory to prove the $[Q, R] = 0$ conjecture for the $G = S^1$ case.

In Example 2.2.9, $(M, E)$ indeed gives pre-quantum data with a moment map $\mu : M \to \mathbb{R}$. In this case, $\mu^{-1}(0)$ is the equator and $M_0 = \mu^{-1}(0)/S^1 = \text{pt}$. From the viewpoint of Atiyah-Bott fixed point theorem, $M_0$ will not contribute to the index (only the south and north poles will) which makes the connection between the $[Q, R] = 0$ theorem and index theory unclear. Hence, we need to revise the fixed point theorem.

**Definition 2.3.1.** We denote by $R^+(S^1)$ the ring of positive series of $t$:
\[ R^+(S^1) = \{ \sum c_k \cdot t^k | c_k \in \mathbb{Z}, c_{-k} = 0 \text{ when } k \gg 0 \}; \]
and $R^-(S^1)$ the ring of negative series of $t$:
\[ R^-(S^1) = \{ \sum c_k \cdot t^k | c_k \in \mathbb{Z}, c_k = 0 \text{ when } k \gg 0 \}. \]

In addition, let $\xi$ be an element in $\hat{R}(S^1)$:
\[ \xi = (\cdots + t^{-1} + 1 + t + \ldots). \]
Lemma 2.3.2. The inclusion of rings

\[ R(S^1) \hookrightarrow R^+(S^1) \]
and

\[ R(S^1) \hookrightarrow R^+(S^1) \]

extend uniquely to inclusion of rings

\[ R(S^1)_{loc} \hookrightarrow R^+(S^1) : \chi \to [\chi]^+ \]
and

\[ R(S^1)_{loc} \hookrightarrow R^-(S^1) : \chi \to [\chi]^-. \]

Theorem 2.3.3 (Revised Fixed Point Theorem). Suppose that \( M \) is a stable complex, closed, \( S^1 \)-equivariant manifold with an equivariant vector bundle \( E \). Let \( \phi : M \to \mathbb{R} \) be an equivariant function on \( M \). When 0 is a regular value of \( \phi \) and \( S^1 \) acts on \( \phi^{-1}(0) \) freely, we have the following formula:

\[ Q(M, E) = \sum_{F \subset M^+} [\chi_F]^+ + \sum_{F \subset M^-} [\chi_F]^+ + Q(M_0, E_0) \cdot \xi \in \hat{R}(G), \]

where

\[ M_\pm = \{ m \in M | \pm \phi(m) > 0 \}, \]

and

\[ M_0 = \phi^{-1}(0)/S^1, \; E_0 = (E|_{\phi^{-1}(0)})/S^1. \]

Proof. By the assumption that 0 is a regular value of \( \phi \), one can choose a number \( \alpha \) which is very close to 0 such that any \( c \in (-2\alpha, 2\alpha) \), \( \phi^{-1}(c) \) is equivariantly diffeomorphic to \( \phi^{-1}(0) \). Following the idea of symplectic cutting [Ler95], let us consider

\[ Z = \{ (m, z) \in M \times \mathbb{C} : -R \leq \phi(m) \pm \frac{2\alpha}{3} z^2 - |z|^2 \leq \frac{\alpha^2}{9} \}, \]

where \( R \) is a positive number large enough so that \( |\phi(M)| < R \). It turns out that \( Z \) is a stable complex, smooth manifold on which the diagonal action of \( S^1 \) is free. Let \( \pi \) be the projection map

\[ \pi : Z \to W = Z/S^1. \]

The quotient space \( W \) is a compact manifold with boundary. In fact, \( W \) gives a
bordism between $M$ and $M_{\text{cut}}$, where $M_{\text{cut}}$ consists of three disjoint components:

$$
M_1 \cong \{ m \in M \mid \phi(m) > \alpha \} \sqcup \phi^{-1}(\alpha)/S^1;
$$

$$
M_2 \cong \{ m \in M \mid \phi(m) < -\alpha \} \sqcup \phi^{-1}(-\alpha)/S^1;
$$

$$
M_3 \cong \{ m \in M \mid -\frac{\alpha}{3} < \phi(m) < \frac{\alpha}{3} \} \sqcup \phi^{-1}(\pm\frac{\alpha}{3})/S^1.
$$

(2.3.1)

They are all $S^1$-equivariant, compact, smooth manifolds.

**Lemma 2.3.4.** The fixed point of $M_{\text{cut}}$ contains two parts (under the identification (2.3.1)):

1. 
   $$
   \mathcal{F}_+ = M_1^{S^1} \cap M_+ , \mathcal{F}_- = M_2^{S^1} \cap M_-,
   $$

   which are inherited from $M$, and $\mathcal{F}_+ \sqcup \mathcal{F}_- = M^{S^1}$;

2. 
   $$
   \mathcal{F}_0 = \phi^{-1}(\pm\alpha)/S^1 \sqcup \phi^{-1}(\pm\frac{\alpha}{3})/S^1,
   $$

   which are created during the cutting.

Moreover, for any component $F \in \mathcal{F}_0$, the normal bundle of $F \hookrightarrow M_{\text{cut}}$ is given by

$$
\pi^*(F) \times_{S^1} \mathbb{C}.
$$

**Proof.** [CdSKT00, Proposition 6.1]

Let $i: \partial Z \hookrightarrow Z$ be the inclusion. There exists a (non-canonical) isomorphism of vector bundles over $\partial Z$:

$$
i^*(TZ) \cong \pi^*(T\partial W) \oplus TS \oplus NZ,
$$

where $TS$ is the sub-bundle of $TZ$ consisting of vectors $\xi_Z$, which are tangent to $S^1$-orbits, and $NZ$ is the normal bundle of $\partial Z$ in $Z$. By assumption, the vectors $J\xi_Z$ are always transverse to $\partial Z$.

Suppose that the stable complex structure on $Z$ is represented by a complex structure $J$ on $TZ \oplus \mathbb{R}^k$. We can choose a positive frame on $i^*(TZ \oplus \mathbb{R}^k)$

$$
\eta_1, J\eta_1, \ldots, \eta_n, J\eta_n, \xi_Z, \pm J\xi_Z,
$$

such that

$$
\eta_1, J\eta_1, \ldots, \eta_n, J\eta_n
$$
gives a frame of $T\partial W \oplus \mathbb{R}^k$. We can see that different choices of sign of $J_\xi Z$ determines different complex structure on $T\partial W \oplus \mathbb{R}^k$, thus stable complex structures on $\partial W$. However, only one of them is the “right” choice which means that give the same stable complex structure on $M \subseteq \partial W$ as we start from. And the choice in turn endows $M_{\text{cut}} \subseteq \partial W$ with a stable complex structure.

For the vector bundle, we first pullback $E$ to $M \times \mathbb{C}$, and then restrict it to $Z$. Its quotient descends to a vector bundle $E_{\text{cut}}$ on $M_{\text{cut}}$.

To sum up, we obtain a bordism (in the sense of Definition 3.1.4) between $(M, E)$ and $(M_{\text{cut}}, E_{\text{cut}})$. If we apply fixed point theorem, we get that

$$\text{Ind}(D_M) = \sum_{F \in M_{S^1}^1} \chi_F.$$  

By bordism invariance of index map, we have that

$$\text{Ind}(D_M) = \sum \text{Ind}(D_{M_i}) = \sum_{F \in M_{\text{cut}}^{S^1}} [\chi_F]^+ \in R^+(S^1). \quad (2.3.2)$$

If we rearrange (2.3.2), we will obtain

$$\text{Ind}(D_M) = \sum_{F \in M_{1}^{S^1}} [\chi_F]^+ + \sum_{F \in M_{2}^{S^1}} [\chi_F]^+ + \sum_{F \in M_{3}^{S^1}} [\chi_F]^+$$

$$= \sum_{F \in M_{1}^{S^1}} [\chi_F]^+ + \sum_{F \in M_{2}^{S^1}} [\chi_F]^+ + \sum_{F \in M_{3}^{S^1}} [\chi_F]^+$$

$$+ \sum_{F \in M_{0}^{S^1}} [\chi_F]^+ + \sum_{F \in M_{2}^{S^1}} [\chi_F]^+ \in \hat{R}(S^1), \quad (2.3.3)$$

By lemma 2.3.4 and the fact that $F \cong M_0$ for all $F \in M_0$, one can verify that the sum of last three terms above equals to

$$Q(M_0, E_0) \cdot \xi \in \hat{R}(S^1).$$

This completes the proof.

$\square$

**Remark 2.3.5.** It is necessary to point out that the cutting space $M_{\text{cut}}$ does not have to be symplectic even if we start from a symplectic manifold $M$. Thus, to carry out the bordism argument, it is natural to leave symplectic manifolds to stable complex manifolds.

**Remark 2.3.6.** If we apply Theorem 2.3.3 to Example 2.2.9, we will obtain a new
formula different from (2.2.3),

\[ \text{Index}(D_{S^2}) = \chi_+ + \chi_0 + \chi_- \in \hat{R}(S^1), \]

where

\[ \chi_- = \left[ \frac{t^{-1}}{1-t} \right]^- = -t^2 - t^3 \cdots \in R^-(S^1); \]
\[ \chi_+ = \left[ \frac{t}{1-t^{-1}} \right]^+ = -t^2 - t^3 \cdots \in R^+(S^1); \tag{2.3.4} \]
\[ \chi_0 = \xi = \cdots + t^{-1} + 1 + t + \cdots \in \hat{R}(S^1). \]

**Theorem 2.3.7.** Suppose that \((M, E)\) is \(S^1\)-equivariant pre-quantum data and \(\phi\) is the associated moment map. For any component \(F \subset M_{S^1}^+\), \([\chi_F]^+\) does not contain any constant terms; and neither does \([\chi_F]^-\) for \(F \subset M_{S^1}^-\).

**Proof.** The proof is similar to corollary 2.2.8. \qed

Therefore, the \([Q, R] = 0\) conjecture is a direct corollary of the two theorems above.

## 2.4 Orbifolds

We use [Kaw79, MS99] as our primary reference for orbifolds. We now review some definitions and basic properties.

**Definition 2.4.1.** Let \(M\) be a locally compact topological space. An orbifold chart for \(M\) is a triple \((U, V, F)\):

- \(U\) is an open subset of \(M\);
- \(V\) is an open subset of an Euclidean space;
- \(F\) is a finite group;
- a homeomorphism \(U \cong V/F\).

**Definition 2.4.2.** Let \(\{(U_i, V_i, F_i)\}\) be a collection of orbifold charts such that \(\{U_i\}\) cover \(M\) and they are compatible in the following sense. For any \(m \in U_i \cap U_j\), there exists a chart \((U_m, V_m, F_m)\) such that \(m \in U_m \subseteq U_i \cap U_j\). There also exists an embedding \(\rho : V_m \hookrightarrow V_i\), together with an injective group homomorphism
$\phi : F_m \to F_i$ such that the diagram

$$
\begin{array}{ccc}
V_m & \xrightarrow{\rho} & V_i \\
\downarrow & & \downarrow \\
U_m & \leftarrow & U_i
\end{array}
$$

commutes and

$$f_i \cdot \rho(U_m) \cap \rho(U_m) \neq \emptyset \Rightarrow f_i \in \phi(F_m).$$

In this case, we say that $M$ is an orbifold and $\{(U_i, V_i, F_i)\}$ give the orbifold structure on $M$.

**Definition 2.4.3.** Suppose that $M$ is an orbifold. For any point $m \in M$, let $(U_m, V_m, F_m)$ be a chart such that the preimage of $m$ in $V_m$ are fixed by $F_m$. We call $F_m$ the isotropy group of $m$, which is unique up to isomorphism. For any finite group $F$, we define

$$M(F) = \{m \in M | F_m \sim F\}.$$ 

In fact, $M(F)$ are all smooth submanifolds and we thus get a decomposition of $M$:

$$M = \bigsqcup_F M(F).$$

Moreover, for any connected compact orbifold $M$, there exists a unique finite group $F$ such that $M(F)$ is an open, dense submanifold of $M$.

**Remark 2.4.4.** There is an alternative way of defining orbifold. Let $G$ be a compact Lie group acting locally freely on a manifold $P$. The orbit space $M = P/G$ is an orbifold. Any orbifold can be written in this form.

**Definition 2.4.5.** Let $M$ be an orbifold with charts $\{(U_i, V_i, F_i)\}$. By an orbifold vector bundle $E$ over $M$, we mean a family of $F_i$-equivariant vector bundles $\{E_i \to V_i\}$, together with suitable compatibility conditions. In addition, we say that $s : M \to E$ is a $C^\infty$-section if, for each orbifold chart $(U_i, V_i, F_i)$, $s|_{U_i} : U_i \to E_i/F_i$ is covered by a $F_i$-invariant $C^\infty$-section $\hat{s}_i : V_i \to E_i$ in a compatible way.

For example, the tangent bundle of $M$ is an orbifold vector bundle, with fiber at $m \in M$ equals to $T_m V_m/F_m$.

**Definition 2.4.6.** A stable complex structure $J$ on an orbifold $M$ is given by a family of $F_i$-invariant stable complex structure $J_i$ on $V_i$ satisfying compatibility conditions.
Continuing in this fashion, one can define symplectic structure, spin$^c$ structure, etc on orbifolds.

Let $\Gamma^\infty_\mathcal{V}(M, E)$ be the set of smooth sections on an orbifold vector bundle $E$. We can define a differential operator

$$D : \Gamma^\infty_\mathcal{V}(M, E) \to \Gamma^\infty_\mathcal{V}(M, E)$$

to be a family of $F_\mathcal{V}$-invariant differential operators

$$D_i : \Gamma^\infty(V_i, E_i) \to \Gamma^\infty(V_i, E_i),$$

which cover $D$, and are compatible with attaching maps. As in the manifold case, the $V$-index of $D$ is defined to be

$$\text{Index}(D) = \dim[\ker(D)] - \text{codim}[\ker(D)].$$

Furthermore, we still have various index theorems for orbifolds. For details, see [Kaw79, Mei98].

### 2.5 K-Homology

Various topics in index theory become more conceptual when viewed from the perspective of K-homology. This is a dual theory to $K$-theory, characterized by functorial pairings:

$$K_0(X) \otimes K^0(X \times Y) \xrightarrow{\text{slant product}} K^0(Y).$$

For the purpose of defining $K$-homology, we fix two locally compact spaces $X, Y$.

**Definition 2.5.1 ([Hig87]).** Let $T_Z$ be a family of $K(Z)$-module homomorphism in $Z$-variable

$$T_Z : K^*(X \times Z) \to K^*(Y \times Z).$$

We say that $T$ is a continuous transformation if the following diagram is commutative:

$$\begin{array}{ccc}
K(X \times Z) \otimes K(W) & \xrightarrow{T_Z \otimes Id} & K(Y \times Z) \otimes K(W) \\
\downarrow & & \downarrow \\
K(X \times Z \times W) & \xrightarrow{T_{Z \times W}} & K(Y \times Z \times W)
\end{array}$$
For example, for any map \( f : Y \to X \), the operation of pullback via \( f \) gives a continuous transformation.

**Definition 2.5.2.** We define \( KK(X, Y) \) the abelian group of continuous transformations from \( K^*(X) \) to \( K^*(Y) \).

**Remark 2.5.3.** In this paper, we shall be working with compact spaces \( X, Y \) that are Euclidean neighborhood retracts [Hig87]. In this case, the group defined above is the same as Kasparov’s \( KK \)-group [Kas75].

**Definition 2.5.4.** For any compact space \( X \) which is Euclidean neighborhood retract, its \( K \)-homology group \( K^*(X) \) is defined to be \( KK(X, pt) \).

The \( K \)-homology groups (and its equivariant counterparts) can be also defined using homotopy theory, geometry or analysis. Following a suggestion of Atiyah, Kasparov gave a functional-analytic definition of \( K \)-homology group based on analytic cycles \( (H, F) \) [Kas75, HR00], where

- \( H \) is a Hilbert space with an action of \( C(X) \), and
- \( F \) is a pseudolocal Fredholm operator on \( H \).

The equivalence relation is homotopy.

**Definition 2.5.5** ([Kas75, HR00]). The set of equivalence classes of analytic cycles \( (H, F) \) forms an abelian group \( K^\text{*ana}(X) \).

**Example 2.5.6.** Suppose that \( M \) is a closed, stable complex manifold. Let \( S \) be the spinor bundle associated to stable complex structure and let \( H = L^2(M, S) \), on which continuous function \( f \in C(M) \) acts by pointwise multiplication. We can construct a Spin\(^c\)-Dirac operator \( D \) acting on \( H \) [Roe98] and denote

\[
F = \text{Phase}(D) = \frac{D}{\sqrt{1 - D^2}}.
\]

Then \( (H, F) \) give an analytic cycle and thus an element \([M] \in K^\text{*ana}(M)\). We usually call it the *fundamental class* associated to the stable complex manifold \( M \).

Besides Kasparov’s approach, Baum and Douglas [BD82] introduced a geometric definition of \( K \)-homology (using manifolds, bordisms, and so on) in connection with work on the Riemann-Roch problem.

**Definition 2.5.7.** A geometric *\( K \)-chain* is given by a triple consisting of

- A compact, stable complex manifold \( M \);
• A complex vector bundle $E$ over $M$;
• A continuous map $f : M \to X$.

We say a triple is a $K$-cycle when $M$ has no boundary.

Next, we define equivalence relations between $K$-cycles. Let us start with bordism.

**Definition 2.5.8.** Suppose that $(W, L, f)$ is a $K$-chain, whose boundary is divided into two parts $M \sqcup N$. We obtain two $K$-cycles:

$$(M, L|_M, f|_M) \text{ and } (N, L|_N, f|_N).$$

And the stable complex structures of $M$ and $N$ are inherited from $W$. We say that the first $K$-cycle is **bordant** to (the opposite of) the second.

**Definition 2.5.9.** Suppose that $(M_1, E_1, f_1), (M_2, E_2, f_2)$ are two $K$-chains. We define their direct sum to be

$$(M_1, E_1, f_1) + (M_2, E_2, f_2) = (M_1 \sqcup M_2, E_1 \sqcup E_2, f_1 \sqcup f_2).$$

In a special case when $M_1 \cong M_2$, and $f_1 = f_2$, we define

$$(M_1, E_1, f_1) + (M_2, E_2, f_2) = (M_1, E_1 \oplus E_2, f_1).$$

We have one more operation on $K$-cycles to introduce, that is, **bundle modification**.

**Definition 2.5.10.** Suppose that $P$ is a principal bundle over $M$ whose structure group is a compact Lie group $H$. Let $N$ be a compact, even dimensional, $H$-equivariant, stable complex manifold. We define

$$\hat{M} = P \times_H N,$$

$\hat{E}$ the pullback of $E$, and $\hat{f}$ the composition of $f$ with the projection from $\hat{M}$ to $M$. We say that $(\hat{M}, \hat{E}, \hat{f})$ is a **bundle modification** of $(M, E, f)$, if the fundamental class $[N] \in K^H(N)$ descends to $[1] \in K^H(pt) \cong R(H)$.

Here is an example of bundle modification.

**Example 2.5.11.** If $\hat{M}$ is a bundle of projective spaces over $M$, then

$$(\hat{M}, \hat{E}, \hat{f}) \sim (M, E, f) \in K_*(X).$$
Definition 2.5.12. [BD82, BHS07] Denote by $K_{*,\text{geo}}(X)$ the set of equivalence classes of $K$-cycles, for the equivalence relations are generated by direct sum, bordism and bundle modification.

In [BHS07], Baum, Higson and Schick showed that we can associate to any geometric cycle $(M, E, f)$ an analytic cycle $(H, F)$. To be precise, let

$$H = L^2(M, S \otimes E)$$

and the Fredholm operator

$$F = \text{Phase}(D).$$

Here $S$ is the spinor bundle, $H$ is viewed as a representation space for $C(X)$ via $f$, and $D$ is the Spin$^c$-Dirac operator associated to stable complex structure. In another word,

$$(H, F) = f_*([M] \cap [E]).$$

Theorem 2.5.13. The map

$$Q : K_{*,\text{geo}}(X) \to K_{*,\text{ana}}(X)$$

defined above is an isomorphism.

The letter $Q$ stands for quantization. When $X = \text{pt}$, the map $Q(M, E) \in K_*(\text{pt}) \cong \mathbb{Z}$ is the usual stable complex quantization [CdSKT00, GGK02].

Remark 2.5.14. The geometric and analytic constructions of $K$-homology both generalized to equivariant case (when $G$ is compact), giving the groups $K^G_*(X)$ [Kas88, BOOSW10]. The equivariant $K$-homology groups are modules over the character ring $R(G)$.

Remark 2.5.15. For any $K$-chain $(M, E, f)$, if we allow $M$ to be orbifold, then we can still define the same $K$-homology group with slight modification.
Chapter 3

Generalized $K$-Cycles

3.1 Definition of $K$-Cycles

Definition 3.1.1. Given a $G$-equivariant map $\phi : M \mapsto g$, define a vector field $V^\phi$ by the formula:

$$V^\phi(m) := \frac{d}{dt} \bigg|_{t=0} \exp(-t\phi(m)) \cdot m, \ \forall m \in M.$$ 

Let $M^\phi$ be the vanishing subset in $M$:

$$M^\phi = \{m \in M | V^\phi(m) = 0\}.$$ 

We do not require $\phi$ to be a moment map. Rather, we relax the moment map condition in the following way.

Definition 3.1.2. We say that $\phi$ is compatible with $E$, if there exists a constant $K$ such that

$$\|\sqrt{-1} L_\xi + \langle \phi(m), \xi \rangle \cdot I_m \| \leq K \|\xi\|, \ \text{for all } m \in M^\phi,$$ 

(3.1.1)

where $I_m$ is the identity map from $E|_m$ to itself, $L_\xi$ is the Lie derivative on $E|_m$, and $\xi$ lies in the isotropy Lie algebra $g_m$.

Remark 3.1.3. When $E$ is an actual orbifold line bundle with a moment map $\mu$, (3.1.1) is equivalent to

$$|\langle \mu(m), \xi \rangle - \langle \phi(m), \xi \rangle| \leq K \cdot \|\xi\|.$$ 

(3.1.2)
Notice that (3.1.2) does not depend on the choice of moment map $\mu$.

**Definition 3.1.4.** We say that a triple $(M, E, \phi)$ in which $M$ may have boundary is a *K-chain* if $\phi$ is proper over $M^\phi$ and compatible with $E$. When $M$ has no boundary, we say that $(M, E, \phi)$ is a *K-cycle*. In particular, all pre-quantum data $(M, E, \mu)$ and compact triples are *K-cycles*.

The equivalence relations between *K*-chain in Chapter 2 can be generalized to the non-compact, orbifold cases without any trouble. The only relation need to be specified is bundle modification.

**Definition 3.1.5.** Suppose that $P$ is a principal bundle over $M$ whose structure group is the compact Lie group $H$. Let $\mathcal{N}$ be a compact, even dimensional, stable complex $H$-orbifold. The orbifold index([Kaw79]) of the associated Spin$^c$-Dirac operator gives an element $[D_N] \in R(H)$. And, if $[D_H] = [1]$, then

$$(M, E, \phi) \sim (\hat{M}, \hat{E}, \hat{\phi}),$$

where $\hat{M} = P \times_H \mathcal{N}$, $\hat{E}$ and $\hat{\phi}$ are the pullbacks of $E$ and $\phi$ respectively.

**Definition 3.1.6.** We denote by $\hat{K}(G)$ the set of equivalence classes of *K*-cycles.

### 3.2 Basic Properties of *K*-Cycles

In this section, we will discuss some basic properties of *K*-cycles.

**Definition 3.2.1.** For convenience, we define some special *K*-chain (*K*-cycle):

- We say that a *K*-chain or a *K*-cycle $(M, E, \phi)$ has *compact vanishing set* if $M^\phi \subseteq M$ is compact.

- We say that a *K*-cycle is *closed* if $M$ is compact and has no boundary.

- We say that a *K*-cycle is *discrete* if it has the following form:

$$\bigcup_{k=1}^{\infty} (N_k, E_k, \rho_k) \in \hat{K}(G),$$

where $N_k$ are closed orbifolds.

These lemmas follow immediately from the definition of *K*-cycles.
Lemma 3.2.2. When $M$ is compact, we have that
\[(M, E, \phi) \sim (M, E, 0).\]

Theorem 3.2.3 (Localization Theorem). Suppose $\{U_\alpha\}$ is a family of disjoint $G$-invariant open subsets such that $M^\phi \subseteq \bigsqcup U_\alpha$. We have
\[(M, E, \phi) \sim \bigsqcup_{\alpha} (U_\alpha, E|_{U_\alpha}, \phi|_{U_\alpha}).\]

Proof. Let $W = M \times [0, 1]$ and $\hat{E}$ be the pullback of $E$. Define a map
\[\hat{\phi} : W \rightarrow \mathfrak{g} : (m, t) \rightarrow \phi(m).\]
In addition, let $F = M \setminus (\bigsqcup U_\alpha)$ and $\hat{W} = W \setminus (F \times \{1\})$. We also denote by $\hat{E}$ and $\hat{\phi}$ their restrictions to $\hat{W}$. It is easy to verify that $(\hat{W}, \hat{E}, \hat{\phi})$ is a $K$-chain which gives desired bordism.

Remark 3.2.4. When $G$ is abelian, Ginzburg, Guillemin, and Karshon proved a similar theorem [GGK02, Chapter 4] in which $\phi$ was required to be an abstract moment map.

Corollary 3.2.5. Let $(M, E, \phi)$ be an arbitrary $K$-cycle. Suppose that $\Sigma \subset M$ is a smooth $G$-invariant hypersurface in $M$ and $\Sigma$ cuts $M$ into two oriented pieces: $M \setminus \Sigma = M_+ \sqcup M_-$. We obtain two $K$-cycles:
\[(M_+, E|_{M_+}, \phi|_{M_+})\] and \[(M_-, E|_{M_-}, \phi|_{M_-}).\]

When the vector field $V^\phi$ does not vanish over $\Sigma$, we have that
\[(M, E, \phi) \sim (M_+, E|_{M_+}, \phi|_{M_+}) + (M_-, E|_{M_-}, \phi|_{M_-}).\]

At last, let us point out that there is a natural embedding of geometric $K$-homology of a point into $\hat{K}(G)$:
\[B : K^G(\text{pt}) \rightarrow \hat{K}(G) : B(M, E) = (M, E, 0).\]

### 3.3 Vanishing Set of $K$-Cycles

The main goal of this section and the next one is to study $K$-cycles with compact vanishing set. These two sections are the heart of the paper.
For all $K$-cycles $(M, E, \phi)$, the vanishing set $M^\phi$ will play an important role in defining their quantization. In general, $M^\phi$ is a subset of $M$, which may be very complicated. However, the following theorem shows that $M^\phi$ can be separated into compact parts.

**Theorem 3.3.1.** Let $(M, E, \phi)$ be a $K$-cycle. There exists a covering of $M^\phi$ by disjoint $G$-invariant open subsets $\{U_\alpha\}$, such that each $F_\alpha = U_\alpha \cap M^\phi$ is compact.

**Remark 3.3.2.** When $(M, E, \phi)$ is pre-quantum data and $\phi$ is proper, Theorem 3.3.1 is trivial. In fact, let $H = \|\phi\|^2 : M \to \mathbb{R}$. We can find a series of regular values of $H$

$$c_1, c_2, \ldots, c_n, \ldots$$

such that $\lim_{n \to \infty} c_n = \infty$. Over $H^{-1}(c_i) \subseteq M$, the vector field $V^\phi$ does not vanish. Hence,

$$\{ U_i = H^{-1}((c_i, c_i+1)) \}_{i=1}^\infty$$

give the desired covering. In addition, Paradan discussed the case in which $\phi$ is just an arbitrary abstract moment map [Par01, Lemma 6.3].

In general, the proof of Theorem 3.3.1 is based on Lemma 3.3.3 and Lemma 3.3.4. For simplicity, we will assume that $E$ is a line bundle throughout the proof. However, one shall see that this assumption is not necessary.

To begin with, let $T$ be a maximal torus in $G$, $t$ be the Lie algebra of $T$, and $t_+$ be a chosen positive Weyl chamber. We observe that $m \in M^\phi$ if and only if $m \in M^\gamma \cap \phi^{-1}(\gamma)$, where $\gamma = \phi(m) \in g$. Thus,

$$M^\phi = \bigcup_{\gamma \in \mathfrak{g}} (M^\gamma \cap \phi^{-1}(\gamma)) = \bigcup_{\gamma \in t_+} G.(M^\gamma \cap \phi^{-1}(\gamma)). \quad (3.3.1)$$

Put an invariant connection on $E$ and denote by $\mu$ the associated moment map. Let $\{H_i\}_{i=1}^\infty$ be the set of stabilizers for the action of maximal torus $T$ on $M$. Since $H_i$ are subgroups of $T$, we can identify their Lie algebras $\mathfrak{h}_i$ as subspaces in $t$. Let

$$\mu_t : M \to t$$

be the moment map associated to the action of $T$ and $\mu_{\mathfrak{h}_i}$ be the composition of $\mu_t$ with the projection from $t$ to $\mathfrak{h}_i$. From (3.3.1), we have

$$M^\phi = \bigcup_{\mathfrak{h}_i \in \Gamma} G.(M^{H_i} \cap \phi^{-1}(\mathfrak{h}_i)), \quad (3.3.2)$$

where $\Gamma$ is the set of Lie algebras $\mathfrak{h}_i$ such that $\mathfrak{h}_i \cap t_+ \neq \{0\}$. 
Lemma 3.3.3. For every $h_i \in \Gamma$, the set
\[ G \cdot (M^{H_i} \cap \phi^{-1}(h_i)) \]
can be separated by a set of disjoint $G$-invariant open subsets $\{V^i_k\}_{k=1}^{\infty}$ in $(G \cdot M^{H_i})$. Moreover, the map $\phi$ is bounded on all $V^i_k$.

Proof. Notice that the map $\mu_{h_i}$ is locally constant on $M^{H_i}$. The set $\mu_{h_i}(M^{H_i}) \cap \mathfrak{t}_+$ has to be a set of countable many points in $h_i \cap \mathfrak{t}_+$. Hence, we can find a set of disjoint open subsets $\{V^i_k\}_{k=1}^{\infty}$ which covers $G \cdot (M^{H_i} \cap \phi^{-1}(h_i))$ and
\[ \mu_{h_i}(V^i_k) \cap \mathfrak{t}_+ = \gamma_k, \quad (3.3.3) \]
where $\gamma_k \in h_i \cap \mathfrak{t}_+$. In addition, we can choose $V^i_k$ small enough so that
\[ \phi(V^i_k) \cap \mathfrak{t}_+ \subseteq \{ x \in \mathfrak{t}_+ \mid \text{distance}(x, h_i) < \epsilon \}, \quad (3.3.4) \]
where $\epsilon$ is a small positive number. From (3.1.2), (3.3.3), and (3.3.4), we know that
\[ \phi(V^i_k) \cap \mathfrak{t}_+ \subseteq B(\gamma_k, K + \epsilon) = \{ x \in \mathfrak{t}_+ \mid \| x - \gamma_k \| < K + \epsilon \}. \]
It follows that $\phi$ is bounded on all $V^i_k$. \qed

Obviously, the union $\bigcup_{i,k} V^i_k$ covers the vanishing set $M^\phi$. We choose $\{U_\alpha\}$ to be a set of disjoint $G$-invariant neighborhoods of connected components of $\bigcup_{i,k} V^i_k$ in $M$. To prove Theorem 3.3.1, it remains to show that $F_\alpha = M^\phi \cap U_\alpha$ is compact.

Lemma 3.3.4. The set $F_\alpha = M^\phi \cap U_\alpha$ is compact for all $\alpha$.

Proof. Because $\phi$ is proper over $M^\phi$, it is equivalent to prove that $\phi$ is bounded on $U_\alpha$.

Suppose that $\phi$ is unbounded on $U_\alpha$. According to Lemma 3.3.3, there must exist an infinite chain in $\{V^i_k\}$,
\[ V_1, V_2, \ldots, V_m \ldots \quad (3.3.5) \]
such that
\[ V_i \cap V_{i+1} \neq \emptyset \quad \text{and} \quad V_i \subseteq U_\alpha. \]
As in (3.3.3), let us denote \( \gamma_i = \mu_{h_k}(V^i_i) \cap t_+ \). It is clear that

\[
\lim_{i \to \infty} \| \gamma_i \| = \infty.
\]

(3.3.6)

Without loss of generality, we can assume that for any constant \( T \), there exists \( N_T \geq T \) and a chain

\( \gamma_{N_T}, \ldots, \gamma_{N_T + \dim(T)} \),

such that they are two by two distinct.

For \( N_T \leq k \leq N_T + \dim(T) - 1 \), pick an arbitrary point \( m_k \in V_k \cap V_{k+1} \cap \phi^{-1}(t_+) \) and denote by \( H_{m_k} \) its isotropy group of \( T \)-action. Let \( \omega_k \) be the fiber weight of the \( E|_m \) for the \( H_{m_k} \)-action. Here, we can identify \( \omega_k \) with a point in \( t \). By the choice of \( \omega_k, \gamma_k \) and properties of moment map, we have the following orthogonal conditions:

\[
\omega_k - \gamma_k \perp \gamma_k \quad \text{and} \quad \omega_k - \gamma_{k+1} \perp \gamma_{k+1}.
\]

(3.3.7)

By (3.3.4), we have

\[
\text{distance}(\phi(m_k), h_k) < \epsilon.
\]

(3.3.8)

It follows from (3.1.2) that

\[
\| \omega_k - \phi(m_k) \| \leq K.
\]

(3.3.9)

From (3.3.7)-(3.3.9), we get

\[
\| \gamma_k - \omega_k \| \leq K + \epsilon.
\]

Similarly, we also have

\[
\| \gamma_{k+1} - \omega_k \| \leq K + \epsilon.
\]

Therefore, these points \( \{ \omega_k, \gamma_k \}_{k=N_T}^{N_T + \dim(T)} \) are within finite distance from each other. Moreover, they must lie in the integer lattice of \( t \) since they are all weights of \( T \)-action. By the orthogonal condition (3.3.7), there are only finitely many possibilities of \( \{ \omega_k, \gamma_k \}_{k=N_T}^{N_T + \dim(T)} \). This leads to a contradiction to (3.3.6).

\[ \square \]

Remark 3.3.5. In Theorem 3.3.1, the cover \( \{ U_\alpha \} \) has the property that \( \phi \) is uniform bounded. That is, there exists a constant \( R \) such that for all \( \alpha \),

\[
\| \phi(x) - \phi(y) \| \leq R, \text{ for } x, y \in U_\alpha \cap \phi^{-1}(t_+).
\]

Remark 3.3.6. Suppose that \( (M, E, \phi) \) is a \( K \)-cycle with compact vanishing set.
We can assume that $M = \text{int}(\overline{M})$ and denote $\partial M = \overline{M} \setminus M$. Given any small neighborhood $U$ of $\partial M$ in $\overline{M}$, we can identify
\[ U \cong \partial M \times [0,1). \] (3.3.10)

In fact, by rescaling, we can assume that $\phi(m)$ tends to infinity as $m$ tends to $\partial M$. Take $\mathcal{H} = \|\phi\|^2 : M \to \mathbb{R}$ and pick a regular value $c$. Let $M_c$ be a subset of $M$ defined by
\[ M_c = \{ m \in M | \mathcal{H}(m) < c \}. \]

When $c$ is large enough, we have $M^\phi \subseteq M_c$. By Theorem 3.2.3,
\[ (M, E, \phi) \sim (M_c, E|_{M_c}, \phi|_{M_c}), \]
where the second $K$-cycle satisfies (3.3.10). Unless stated otherwise, from here when we refer to the $K$-cycle with compact vanishing set, we always assume that it automatically satisfies (3.3.10).

**Definition 3.3.7.** Let $(M, E, \phi)$ be a $K$-cycle with compact vanishing set. We can extend the orbifold vector bundle $E$ and map $\phi$ to $\partial M$, denoted by $\partial E$ and $\partial \phi$ respectively. We define $\partial (M, E, \phi)$ to be the $K$-cycle $(\partial M, \partial E, \partial \phi)$.

**Remark 3.3.8.** Let $(M_1, E_1, \phi_1)$ and $(M_2, E_2, \phi_2)$ be two $K$-cycles with compact vanishing set. Suppose that there is a diffeomorphism $f : \partial M_1 \cong \partial M_2$. By Remark 3.3.6, the map $f$ also induces a diffeomorphism:
\[ \hat{f} : U_1 \cong U_2, \]
where $U_i$ are necks of $\partial M_i$. Notice that we can alway vary the map $\phi$ without changing $K$-cycle class. When the map $\hat{f}$ lifts to an isomorphism between the two orbifold vector bundles: $E_1|_{U_1}$ and $E_2|_{U_2}$, and preserves the stable complex structures, we say that
\[ \partial (M_1, E_1, \phi_1) \cong \partial (M_2, E_2, \phi_2). \]

In this case, we can obtain a compact $K$-cycle by gluing the two $K$-cycles using the map $\hat{f}$.
3.4  $K$-Cycles with Compact Vanishing Set

In this section, we will focus on $(M, E, \phi)$, a $K$-cycles with compact vanishing set. We want to show that $(M, E, \phi)$ is equivalent to a discrete $K$-cycle (see Definition 3.2.1). The general strategy is to build a “cap”, another $K$-cycle, so that we can compactify $(M, E, \phi)$ by gluing on the cap. The geometric construction of the cap is the main part of this section.

**Theorem 3.4.1.** Let $(M, E, \phi)$ be a $K$-cycle with compact vanishing set. There is a $K$-cycle with compact vanishing set $(W, L, \psi)$ such that\[ \partial (W, L, \psi) \cong \partial (M, E, \phi). \]
In addition, the $K$-cycle $(W, L, \psi)$ is equivalent to a discrete $K$-cycle.

**Corollary 3.4.2.** Every $K$-cycle with compact vanishing set is equivalent to a discrete $K$-cycle.

**Proof.** Given any $K$-cycle with compact vanishing set $(M, E, \phi)$, by Theorem 3.4.1 and Remark 3.3.8, we can obtain a compact $K$-cycle by gluing $(W, L, \psi)$ and $(M, E, \phi)$ together. The corollary follows directly from the second part of Theorem 3.4.1.

**Corollary 3.4.3.** Suppose that $\Sigma$ is a closed $G$-manifold and $\phi : \Sigma \to g$ is an equivariant map. If the vector field $V^\phi$ induced by $\phi$ is nowhere vanishing over $\Sigma$, then $\Sigma$ is a boundary.

3.4.1 Circle Case

Take $\Sigma = \partial M$. By assumption, we know that $S^1$ acts locally freely on $\Sigma$. Let $D^2$ be the open disk with standard $S^1$-action and $W$ be the associated disk bundle\[ W = \Sigma \times_{S^1} D^2, \]
which is indeed an orbifold. We define an orbifold vector bundle on $W$ by\[ L = \pi^* (\partial E)/S^1, \]
where $\pi$ is the projection from $\Sigma \times D^2$ to $\Sigma$, and $S^1$-action is the diagonal action.

The map $(\partial \phi) \circ \pi$ over $\Sigma \times D^2$ descends to a map over $W$ denoted by $\psi$. The stable complex structure on $W$ is inherited from $\Sigma$ and $D^2$. In particular, if
$E_{J_S}, E_{J_{D^2}}, E_{J_W}$ are the canonical line bundles associated to their stable complex structures respectively, then $E_{J_W} = [E_{J_S} \otimes E_{J_{D^2}}]/S^1$.

We can verify that $(W, L, \psi)$ constitutes a $K$-cycle with the property that

$$\partial(W, L, \psi) \cong \partial(M, E, \phi).$$

**Remark 3.4.4.** An alternative description of the construction above is symplectic cutting. From that viewpoint, it is easier to see that $(W, L, \psi)$ naturally inherits various structures from $M$ so that they can be glued together[GGK99, CdSKT00].

It remains to show that $(W, L, \psi)$ is equivalent to a discrete $K$-cycle, beginning with the following lemma.

**Lemma 3.4.5.** Consider the $K$-cycle

$$(D^2, C, f(z)),$$

where $C$ is the trivial line bundle on $D^2$ with trivial $S^1$-action, and $f(z)$ is a positive function on $D^2$. Then, we have that

$$(D^2, C, f) \sim \sum_{n=0}^{\infty} (S^2, F_n, f_n),$$

(3.4.1)

where $S^2$ is the sphere with standard $S^1$-action, $F_n$ are trivial line bundles on $S^2$ with $S^1$-weight equals to $n$, and $f_n$ are equivariant functions on $S^2$ which take the values $n$ and $(n+1)$ on the north and south poles.

**Proof.** We break every compact $K$-cycle $(S^2, F_n, f_n)$ into two pieces:

$$(S^2, F_n, f_n) \sim (S_+, F_n|_{S_+}, f_n|_{S_+}) + (S_-, F_n|_{S_-}, f_n|_{S_-}),$$

where $S_\pm$ are hemispheres. In particular, we have that

$$(S_+, F_0|_{S_+}, f_0) \cong (D^2, C, f).$$

Then we rearrange those $K$-cycles $(S_\pm, F_n|_{S_\pm}, f_n|_{S_\pm})$. Let $(S^2, F^{n+1})$ be the compact $K$-cycle obtained by gluing

$$(S_-, F_n|_{S_-}, f_n) \text{ and } (S_+, F_{n+1}|_{S_+}, f_{n+1}).$$

The line bundle $F^{n+1}$ on $S^2$ has fiber weights equal to $n$ at the south pole and $(n+1)$ at the north pole. By Atiyah-Bott fixed point theorem, we can calculate
that
\[ \text{Index}(S^2, F_n^{n+1}) = 0 \in R(S^1). \] (3.4.2)

Hence, according to [BHS07], the $K$-cycle $(S^2, F_n^{n+1})$ must be equivalent to an empty $K$-cycle through bordism and bundle modification. This completes the proof.

Remark 3.4.6. If $f(z)$ is a negative function on $D^2$, then we have a similar result:
\[ (D^2, \mathbb{C}, f) \sim -\sum_{n=-1}^{-\infty} (S^2, F_n, f_n). \] (3.4.3)

**Lemma 3.4.7.** Let $(M, E, f)$ be a $\mathbb{T}$-equivariant $K$-cycle and $\Sigma$ be an orbifold on which $\mathbb{T}$ acts locally freely. If $(M', E', f')$ is equivalent to $(M, E, f)$ then
\[ (\Sigma \times \mathbb{T} M, \hat{E}, \hat{f}) \sim (\Sigma \times \mathbb{T} M', \hat{E}', \hat{f}'), \]
where $\hat{E}, \hat{f}, \hat{E}', \hat{f}'$ are the pullbacks of $E, f, E', f'$ respectively.

**Proof.** In the case when $(M', E', f')$ is bordant to $(M, E, f)$, whose bordism is given by a $K$-chain $(W, L, \psi)$, we have that $(\Sigma \times \mathbb{T} W, \hat{L}, \hat{\psi})$ gives a bordism between
\[ (\Sigma \times \mathbb{T} M, \hat{E}, \hat{f}) \text{ and } (\Sigma \times \mathbb{T} M', \hat{E}', \hat{f}'). \]

Suppose now that $(M', E', f')$ is a bundle modification of $(M, E, f)$. Let us assume that $M' = Q \times_H N$, where $Q$ is a $\mathbb{T}$-equivariant principle $H$-bundle over $M$. We define $\hat{Q} = \Sigma \times \mathbb{T} Q$, which is a principle $H$-bundle over $(\Sigma \times \mathbb{T} M)$. We notice that
\[ \Sigma \times \mathbb{T} M' = \Sigma \times \mathbb{T} (Q \times_H N) \cong \hat{Q} \times_H N. \]
One can check that $(\Sigma \times \mathbb{T} M', \hat{E}', \hat{f}')$ is also a bundle modification of $(\Sigma \times \mathbb{T} M, \hat{E}, \hat{f})$ (respect to $\hat{Q}$).

Because equivalence relations between $K$-cycles are generated by bordism and bundle modification, this completes the proof.

**Proposition 3.4.8.** The $K$-cycle $(W, L, \psi)$ is equivalent to a discrete $K$-cycle.

**Proof.** Let $P = \Sigma \times_{S^1} S^2$, whose stable complex structure is inherited from that on $\Sigma$ and $S^2$. An orbifold vector bundle $L_n$ over $P$ is defined by
\[ L_n = [E \boxtimes F_n]/S^1, \]
where the $S^1$-action is the diagonal action. The function

$$\Psi : \Sigma \times S^2 \to \mathbb{R} : \Psi(m, x) = \phi(m) + f_n(x)$$

descends to a function on $P$, denoted by $\psi_n$. The triple $(P, L_n, \psi_n)$ forms a closed $K$-cycle. When $\psi$ is positive, we have that

$$\langle W, L, \psi \rangle \sim \sum_{n=0}^{\infty} \langle P, L_n, \psi_n \rangle; \quad (3.4.4)$$

and when $\psi$ is negative,

$$\langle W, L, \psi \rangle \sim -\sum_{n=-1}^{-\infty} \langle P, L_n, \psi_n \rangle. \quad (3.4.5)$$

3.4.2 Torus Case

As before, we denote $\Sigma = \partial M$, which is a compact orbifold together with a $T$-action.

**Definition 3.4.9.** Let $\{U_i\}_{i=1}^n$ be an open cover of $\Sigma$. We say that $\{U_i, S_i\}_{i=1}^n$ is a **good cover** if it satisfies all the conditions below:

- Every $U_i$ is $T$-invariant.
- Every $S_i$ is a circle subgroup of $T$, and $S_i$ acts locally freely on $U_i$.
- Let us define
  $$\mathcal{A} = \{ I \subseteq \{1, \ldots, n\} | U_I = \bigcap_{i \in I} U_i \neq \emptyset \}.$$

  For any $I \in \mathcal{A}$, $\{S_i\}_{i \in I}$ generate an $|I|$-dimensional subgroup $S_I$ and it acts locally freely on $U_I$.

When $\Sigma$ has a good cover, we want to construct a cap for $\Sigma$. If we naively build the caps locally, that is, defining $W_i = U_i \times_{S_i} D^2$ as in the circle case, then there is a problem that $\{W_i\}$ may not be glued together easily. In order to overcome this difficulty, we define the local caps in a more subtle way using the third condition in Definition 3.4.9.
Lemma 3.4.10. For any $I \in \mathcal{A}$, we can define an orbifold $W_I$ with the property that

$$(W_I \setminus \text{int}(W_I)) \cong U_I. \quad (3.4.6)$$

Moreover, for any $I, J \in \mathcal{A}$ and $I \subseteq J$, there exists a diffeomorphism $\Phi^I_J$ from an open subset $U_{I,J}$ of $W_I$ to an open subset $U_{J,I}$ of $W_J$:

$$\Phi^I_J : U_{I,J} \cong U_{J,I},$$

satisfying the coherence condition: for $I \subseteq J \subseteq K$

$$U_{I,K} = U_{I,J} \cap (\Phi^J_I)^{-1}(U_{J,K}) \text{ and } \Phi^I_J \circ \Phi^K_J = \Phi^K_I. \quad (3.4.7)$$

Proof. To begin with, we choose a partition of unity $\{\varphi_i\}_{i=1}^n$ subordinate to the open cover $\{U_i\}_{i=1}^n$. For any $I \in \mathcal{A}$, we define a map $\rho_I$ to be

$$\rho_I : U_I \times [1, \infty) \to \mathbb{R}^{|I|} : (\rho_I,i)_{i \in I} = (t \cdot \varphi_i(m))_{i \in I}. \quad (3.4.8)$$

We can assume $\rho_I$ to be proper.

Let us fix $n$ numbers $\{\alpha_1, \ldots, \alpha_n\}$ such that

- $\alpha_i > 1$ for all $i$;
- for any $I \in \mathcal{A}$, $\prod_{i \in I} \alpha_i$ is a regular value for $\rho_I$.

For any $I \in \mathcal{A}$, let us define

$$\tilde{I} = \bigcup_{K \in \mathcal{A}, J \subseteq K} (K \setminus I). \quad (3.4.9)$$

Let $V_I$ be an open subset in $U_I \times [1, \infty)$ defined by

$$V_I = \{(m, t) \in U_I \times [1, \infty) | t \cdot \varphi_i(m) < \alpha_i \text{ for all } i \in \tilde{I}\}.$$ 

We observe that $S_I$ acts locally freely on $V_I$, and $(V_I \setminus \text{int}(V_I)) \cong U_I$.

Let $\mathbb{C}$ be the complex plane with a standard circle action, and $\mathbb{C}^{|I|}$ be the product of $|I|$ copies of $\mathbb{C}$ with a standard $|I|$-dimensional torus action. Let $T_I$ be the diagonal torus action on $V_I \times \mathbb{C}^{|I|}$ and define a map $\chi_I$ to be:

$$\chi_I : V_I \times \mathbb{C}^{|I|} \to \mathbb{R}^{|I|} : \chi_I(m, t, z) = (\alpha_i - t \varphi_i(m) - |z_i|^2)_{i \in I}.$$ 

It is clear that 0 is a regular value for $\chi_I$ and $T_I$-action is locally free on $\chi_I^{-1}(0)$. 
Therefore,
\[ W_I = \chi_I^{-1}(0)/\mathbb{T}_I \] (3.4.10)
defines an orbifold satisfying (3.4.6).

For the second part, let \( V_I^J \) be an orbifold defined by
\[ V_I^J = \{ (m, t) \in U_J \times [1, \infty) | t \cdot \varphi_i(m) < \alpha_i, \text{ for all } i \in \mathcal{I} \}. \]
As the construction before, we can define a map \( \tilde{\chi}_J \) on \( V_I^J \times \mathbb{C}^{|\mathcal{I}|} \) and form
\[ U_I^J = \tilde{\chi}_J^{-1}(0)/\mathbb{T}_J. \]

For \( I \subsetneq J \), we have that \( U_I \subseteq U_J \) and \( \mathcal{J} \subseteq \mathcal{I} \). Hence, \( V_I^J \) is an open subset of both \( V_J \) and \( V_I \). As a result, \( U_I^J \) can be identified as an open suborbifold of both \( W_I \) and \( W_J \). We define the diffeomorphism \( \Phi_I^J \) to be the map between \( W_I \) and \( W_J \) factoring through \( U_I^J \). The verification of (3.4.7) is straightforward.

**Theorem 3.4.11.** Let \( \Sigma \) be a compact orbifold. If it has a good cover, then we can construct an orbifold \( W \) such that
\[ \partial W \cong \Sigma. \]

**Proof.** By gluing all the pieces \( \{W_I\}_{I \in \mathcal{A}} \) using \( \phi_I^J \), we obtain an orbifold \( \tilde{W} \). Then, \( W = \text{int}(\tilde{W}) \) gives a desired orbifold. \( \square \)

Now, let \((M, E, \phi)\) be a \( K \)-cycle with compact vanishing set and \( \Sigma = \partial M \). We will show that \( \Sigma \) has a good cover.

**Definition 3.4.12.** Let \( \mathfrak{h} \) be an isotropy Lie algebra of \( \mathbb{T} \)-action on \( \Sigma \) and
\[ \Sigma_\mathfrak{h} = \{ x \in \Sigma | t_x = \mathfrak{h} \}, \]
where \( t_x \) is the isotropy Lie algebra of \( x \). For any connected component \( F \) of \( \Sigma_\mathfrak{h} \), we denote by \( U_F \) a \( \mathbb{T} \)-invariant neighborhood of \( F \) in \( \Sigma \). We can choose \( U_F \) small enough so that
\begin{itemize}
  \item for all \( x \in U_F \), we have that \( t_x \subseteq \mathfrak{h} \);
  \item for any other component \( F' \) of \( \Sigma_\mathfrak{h} \), we have that \( U_F \cap U_{F'} = \emptyset \).
\end{itemize}
In this case, we say that \( U_F \) has *level* equals to \( \dim(\mathfrak{h}) \). As \( \mathfrak{h} \) ranges over all the possible isotropy Lie algebra, we obtain a covering of \( \Sigma \), denoted by \( \{U_i\} \).
Lemma 3.4.13. Every point \( x \in \Sigma \) can be covered by at most \((\dim(\mathbb{T}) - \dim(\mathbb{T}_x))\) open sets in \( \{U_i\} \).

**Proof.** The lemma follows from the fact that every point \( x \in \Sigma \) can only be covered by open set \( U_i \) with level no lower than \( \dim(\mathbb{T}_x) \). \(\square\)

**Proposition 3.4.14.** There exists a good cover on \( \Sigma \).

**Proof.** It is enough to associate every \( U_i \) with a compatible circle action. We will complete the proof by induction on the levels of \( \{U_i\} \).

If \( U_i \) has the highest level, then we can always find a circle action \( S_i \) which acts locally freely on \( U_i \), due to the fact that \( \phi \) induces a nowhere vanishing vector field on \( \Sigma \).

Suppose that we have already associated the open sets whose levels are greater than \( K \) with compatible circle actions.

Let \( U_k \) be an open set with level equals to \( K \). Suppose that \( I \) is a subset of \( \{1, \ldots, n\} \) such that

\[
U_k \cap (\bigcap_{i \in I} U_i) \neq \emptyset,
\]

and every \( \{U_i\}_{i \in I} \) has level greater than \( K \). From Lemma 3.4.13, we know that \( |I| \leq \dim(\mathbb{T}) - K - 1 \). Meanwhile, for any point \( x \in U_k \), the isotropy group \( \mathbb{T}_x \) has dimension no greater than \( K \). Hence, we can associate \( U_k \) with a circle action which is compatible with others. \(\square\)

In order to build a cap for the \( K \)-cycle \((M, E, \phi)\), we should also construct an orbifold vector bundle \( L \) and an equivariant map \( \psi \) on \( W \).

**Proposition 3.4.15.** For any fixed weight \( \gamma \), we can construct a \( K \)-cycle \((W, L, \psi)\) such that

\[
\partial(W, L, \psi) \cong \partial(M, E, \phi)
\]

and \( L|_x \) has a \( \mathbb{T} \)-weight equals to \( \gamma \) for any \( x \in W^T \).

**Proof.** We use the same notation as in Lemma 3.4.10. For every \( I \in \mathcal{A} \), \( S_I \) is a subgroup of \( \mathbb{T} \). Thus, by restriction, we obtain a \( S_I \)-weight \( \gamma_I \). Let \( F_{\gamma_I} \) be the trivial line bundle on \( C^{|I|} \), on which an \( |I| \)-dimensional torus acts with weight \( \gamma_I \).

Recall that \( \chi_I^{-1}(0) \) is a suborbifold of \( V_I \times C^I \) and the diagonal action \( \mathbb{T}_I \) is locally free. Thus,

\[
L_I = ((E \boxtimes F_{\gamma_I})|_{\chi_I^{-1}(0)})/\mathbb{T}_I \tag{3.4.11}
\]

defines an orbifold vector bundle on \( W_I = \chi_I^{-1}(0)/\mathbb{T}_I \).
For the equivariant map, let $\hat{\phi}$ be the pullback of $\phi$ to $V_I \times \mathbb{C}^{|I|}$. After restricting to $\chi_I^{-1}(0)$, $\hat{\phi}$ descends to a map on $W_I$.

Using the diffeomorphisms $\{\phi_J^I\}$ in Lemma 3.4.10, we can get an orbifold vector $L$ and an equivariant map $\psi$ by gluing. The triple $(W, L, \psi)$ constitute a $K$-cycle satisfying all the desired properties. \hfill \square

Next, we are going to show that $(W, L, \psi)$ is bordant to a discrete $K$-cycle.

Definition 3.4.16. Let $\mathbb{D}^n = D^2 \times \cdots \times D^2$ be the product of $n$ copies of open disks. A $n$-dimensional torus $T^n = S^1 \times \cdots \times S^1$ acts on $\mathbb{D}^n$ in such a way that the $i$-th factor of $S^1$ acts on the $i$-th Disk by rotation.

For every $I \in \mathcal{A}$, let us define $K_I = \rho_I^{-1}(\alpha_I) \cap V_I$, that is

$$K_I = \{(m, t) \in U_I \times [1, \infty) | t \cdot \varphi_i(m) = \alpha_i, i \in I \text{ and } t \cdot \varphi_j(m) < \alpha_j, j \in \bar{I}\}.$$ 

We notice that $K_I$ does not have to be closed unless $\bar{I} = \emptyset$. However, we have the following:

Lemma 3.4.17. For every $I \in \mathcal{A}$, we can obtain a closed orbifold by gluing

$$\left( \bigsqcup_{J \in \mathcal{A}, I \subset J} K_J \times T_I^J \mathbb{D}^{||J|-|I||} \right) \sqcup K_I,$$

where $T_I^J$ is a $(|J| - |I|)$-dimensional torus and it acts on $K_J$ through $S^I_J = \prod_{i \in J \cap \bar{I}} S_i$.

Proof. It is straightforward based on the fact that $\alpha_I$ are regular values for

$$\rho_I : U_I \times [1, \infty) \to \mathbb{R}^{|I|}.$$

\hfill \square

Lemma 3.4.18. The $K$-cycle $(W, L, \psi)$ is bordant to a $K$-cycle in the following form:

$$(W, L, \psi) \sim \sum (M_i \times T_i, \mathbb{D}^{||T_i||}, E_i, \phi_i) + \sum (N_j, F_j, \psi_j),$$

where $\{M_i, N_j\}$ are closed orbifolds.

Proof. If we identify $K_I/\mathbb{T}_I$ with a subset in $W_I$, then the set

$$Z_I = K_I \times T_I \mathbb{D}^{||I||}$$

(3.4.12)
can be identified as a neighborhood of $K_I/T_I$ in $W_I$. We define $Z$ to be the orbifold obtained by gluing $\{Z_I\}$ together. Since $V^\phi$ is nowhere vanishing on $\Sigma$, the vanishing set $W^\psi$ in $W$ must be contained in $Z$. Therefore,

$$(W, L, \psi) \sim (Z, L|_Z, \psi|_Z).$$

Without loss of generality, we can furthermore assume that

$$(Z, L|_Z, \psi|_Z) \sim \sum_I (Z_I, L|_{Z_I}, \psi|_{Z_I}). \quad (3.4.13)$$

Therefore, the Lemma follows from (3.4.12), (3.4.13), and Lemma 3.4.17.

We want to prove that the $K$-cycle $(W, L, \psi)$ is equivalent to a discrete $K$-cycle. Lemma 3.4.18 reduces the problem to a special case in which $W$ is in the form of $N \times_{\mathbb{T}} \mathbb{D}^k$, where $N$ is a closed orbifold. In this case, one can complete the proof by induction and Lemma 3.4.5.

### 3.4.3 Nonabelian Case

We assume now that $G$ is a compact connected Lie group, $\mathbb{T}$ is a maximal torus, and $t_+$ is a fixed positive Weyl chamber. For any $x \in t_+$, we denote by $G_x$ the isotropy group of adjoint action at $x$.

**Lemma 3.4.19.** Let $\Delta$ be a face of $t_+$. For any point in int$(\Delta)$, they have the same isotropy group, denoted by $G_\Delta$. Moreover, we have that $G_\Delta \subseteq G_{\Delta'}$ if and only if $\Delta$ is a sub-face of $\Delta'$.

Let us recall the symplectic cross-section theorem.

**Theorem 3.4.20 (Cross-section).** Let $(M, \omega)$ be a compact connected symplectic orbifold with a moment map $\mu : M \to g^*$ arising from an action of a compact Lie group $G$. For any face $\Delta$ in $t_+$, let $V_\Delta$ be a small neighborhood of int$(\Delta)$ in $t_+$ such that $G_x \subseteq G_\Delta$ for any $x \in V_\Delta$. If we denote $U_\Delta = G_\Delta \cdot V_\Delta$, then the cross section $R = \mu^{-1}(U_\Delta)$ is a $G_\Delta$-invariant symplectic sub-orbifold and

$$U = G \cdot R \cong G \times_{G_\Delta} R$$

is an open subset of $M$. Moreover, if $A_\Delta$ is the abelian part of $G_\Delta$, then the $A_\Delta$-action on $R$ extends in a unique way to an action on $U$ which commutes with the $G$-action.
Proof. See [GS90, LMTW98, Mei98].

In this paper, we are considering stable complex orbifolds instead of symplectic orbifolds. Hence, the symplectic cross-section theorem does not apply. However, the idea of building the cap in nonabelian case comes from the symplectic cross-section theorem and symplectic surgery by Meinrenken [Mei98].

Let $(M, E, \phi)$ be a $K$-cycle with compact vanishing set and $\Sigma = \partial M$. Suppose that $\Sigma$ is connected.

**Definition 3.4.21.** For each $m \in \Sigma$, let $\mathfrak{g}_m \subset \mathfrak{g}$ be the corresponding isotropy Lie algebra. It is clear that $\mathfrak{g}_{g_m} = \text{Ad}(g)(\mathfrak{g}_m)$. We call the set of subalgebras

$$\left( \mathfrak{g}_m \right) = \{ \text{Ad}(g)(\mathfrak{g}_m) \mid g \in G \}$$

the orbit type of $m$. **There are only finite many orbit types of $\Sigma$. Moreover, there is a unique orbit type $(\mathfrak{g}_0)$ such that the set**

$$\Sigma_{(\mathfrak{g}_0)} = \{ m \in \Sigma \mid (\mathfrak{g}_m) = (\mathfrak{g}_0) \}$$

**is a dense, open subset in $\Sigma$ [GS90, LMTW98, Mei98].**

Let $(\mathfrak{h})$ be an arbitrary orbit type of $\Sigma$. There exists a face $\Delta$ with maximum dimension such that $(\mathfrak{h})$ is subconjugated to $(\mathfrak{g}_\Delta)$. Suppose that $F$ is a connected component of $\Sigma_{(\mathfrak{h})}$ and $U$ is a small $G$-invariant neighborhood of $F$ in $\Sigma$ such that for any $m \in U$, $(\mathfrak{g}_m)$ is subconjugated to $(\mathfrak{g}_\Delta)$. In this case, as in the symplectic cross-section theorem, we can find a $G_\Delta$-invariant subset $R$ of $U$ such that

$$U = G \cdot R \cong G \times_{G_\Delta} R,$$

where the isomorphism map is given by

$$G \times_{G_\Delta} R \to G \cdot R, \quad [a, u] \to a \cdot u.$$

Moreover, the $A_\Delta$-action on $R$ extends to an action on $U$, which commutes with the $G$-action [Woo96, LMTW98, Mei98]. As $(\mathfrak{h})$ ranges over all the possible orbit types, we obtain a covering of $\Sigma$.

**Lemma 3.4.22.** There exists an open covering $\{U_i\}$ of $\Sigma$ with circle actions $\{S_i\}$ such that

- Every $U_i$ is $G$-invariant.
- The circle action $S_i$ acts locally freely on $U_i$ and commutes with $G$-action.
For all \( I \in \mathcal{A} \), \( \{S_i\}_{i \in I} \) generate an \(|I|\)-dimensional torus and it acts locally freely on \( U_I \).

**Proof.** The proof is similar to the torus case except the circle actions are induced from the locally defined \( A_\Delta \)-actions. \( \square \)

By a parallel argument as in the torus case, we can prove Theorem 3.4.1 in general.

### 3.5 General K-Cycles

In this section, we will study general \( K \)-cycles. The main goal is to prove the following theorem.

**Theorem 3.5.1.** Every \( K \)-cycle \((M, E, \phi)\) is equivalent to a discrete \( K \)-cycle:

\[
(M, E, \phi) \sim \bigcup_{k=1}^{\infty} (M_k, E_k, \phi_k).
\]

(3.5.1)

In addition, for any irreducible representation \( \gamma \in \hat{G} \), there exists a constant \( T_\gamma \) such that

\[
[\text{Index}(M_k, E_k)]^\gamma = 0, \quad \text{if} \quad k \geq T_\gamma.
\]

(3.5.2)

To begin with, by Theorem 3.2.3, we know that

\[
(M, E, \phi) \sim \bigcup_k (U_k, E|_{U_k}, \phi|_{U_k}),
\]

(3.5.3)

where the right-hand side consists of \( K \)-cycles with compact vanishing set. For each \((U_k, E|_{U_k}, \phi|_{U_k})\), we can build a cap \((W_k, L_k, \psi_k)\). Although the constructions of \( L_k \) are not unique, we can build \( L_k \) in a way such that \( \{(W_k, L_k, \psi_k)\} \) form a global \( K \)-cycle by putting all them together.

**Lemma 3.5.2.** We can build \((W_k, L_k, \psi_k)\) such that

\[
\bigcup_k (W_k, L_k, \psi_k)
\]

form a \( K \)-cycle.

**Proof.** Let us denote \( \Sigma_k = \partial U_k \). Every \( W_k \) is constructed in section 4. Since the maps \( \{\psi_k\} \) are induced from \( \phi|_{\Sigma_k} \), they satisfy the properness condition. The
main problem is how to construct the orbifold vector bundles \( \{L_k\} \) so that they are compatible with \( \{\psi_k\} \) globally. In another word, there exists a constant \( C \) such that for all \( k \),

\[
\| \frac{\sqrt{-1}}{2\pi} L_\xi + \langle \psi_k(m), \xi \rangle \| \leq C \cdot \| \xi \|, \quad m \in W_k^{\psi_k} \text{ and } \xi \in g_m. \tag{3.5.4}
\]

Due to Remark 3.3.5, there exists a constant \( R \) and a series of dominant weights \( \{\gamma_k\} \in t_+^+ \) such that

\[
\phi_k(\Sigma_k) \cap t_+ \subseteq B(\gamma_k, R) = \{ x \in t_+ \| x - \gamma_k \| \leq R \} \tag{3.5.5}
\]

and \( \lim_{k \to \infty} \| \gamma_k \| = \infty \). As in Proposition 3.4.15, we can construct \( L_k \) using the fixed weights \( \gamma_k \). One can check that these \( \{L_k\} \) satisfy condition (3.5.4).

We know from section 4 that every \( (W_k, L_k, \psi_k) \) is equivalent to a discrete \( K \)-cycle:

\[
(W_k, L_k, \psi_k) \sim \bigsqcup_i (N_k^i, F_k^i, \rho_k^i).
\]

It is natural to ask that if we put all them together, do they form a \( K \)-cycle?

**Lemma 3.5.3.** The infinite sum

\[
\bigsqcup_{k,i} (N_k^i, F_k^i, \rho_k^i)
\]

constitute a \( K \)-cycle.

**Proof.** Let \( N = \bigsqcup_{k,i} N_k^i \), \( F \) be the orbifold vector bundle such that \( F|_{N_k^i} = F_k^i \), and \( \rho \) be the map on \( N \) such that \( \rho|_{N_k^i} = \rho_k^i \).

The orbifold vector bundles \( F_k^i \) are induced from \( L_k \). By Lemma 3.5.2, they must satisfy the compatibleness condition. To check that \( (N, F, \rho) \) is a \( K \)-cycle, the key is to show that \( \rho \) is proper. To be more precise, for every \( R > 0 \), we want to show that there exists a constant \( T_R \) such that for all \( i, k \geq T_R \),

\[
\rho_k^i(N_k^i) \cap B_R = \emptyset,
\]

where \( B_R = \{ x \in t_+ \| x \| \leq R \} \).

Given any fixed \( k \), we know from section 4 that when \( i \) is large enough

\[
\rho_k^i(N_k^i) \cap B_R = \emptyset.
\]
On the other hand, there exists a constant $R' \gg R$ such that if $\psi_k(W_k) \cap B_{R'} = \emptyset$, then

$$\rho_k^i(N_k^i) \cap B_R = \emptyset$$

for all $i$. By (3.5.5), we have that $\psi_k(W_k) \cap B_{R'} = \emptyset$ as $k \to \infty$. This completes the proof.

The first part of theorem 3.5.1 follows from Lemma 3.5.2 and Lemma 3.5.3. Suppose that we have

$$(M, E, \phi) \sim \bigcup_{k=1}^{\infty} (M_k, E_k, \phi_k),$$

where $M_k$ are all closed orbifolds. Moreover, from the discussion before, there exists a constant $R$ and a series of weights $\{\gamma_k\} \in t_+$ such that

$$\phi_k(M_k) \cap t_+ \subseteq B(\gamma_k, R),$$

(3.5.6)

and $\lim_{k \to \infty} \|\gamma_k\| = \infty$.

**Lemma 3.5.4.** Let $(M_k, E_k, \phi_k)$ be a closed $K$-cycle satisfying (3.5.6). For any fixed irreducible representation $\gamma \in \hat{G}$, there exists a constant $T_\gamma$, such that when $\|\gamma_k\| \geq T_\gamma$,

$$[\text{Index}(M_k, E_k)]^\gamma = 0.$$

**Proof.** The lemma is an analog of Corollary 2.2.8. For simplicity, we will only prove the manifold case.

**Abelian Case:** When $\|\gamma_k\|$ is large enough, we know from (3.5.6) that there exists a cyclic unit vector $\xi \in \text{Lie}(\mathbb{T})$ such that

$$\langle \phi_k(x), \xi \rangle \gg 0,$$

for all $x \in M_k^T$.

By (3.1.1), we have that

$$\|\frac{\sqrt{-1}}{2\pi L_\xi}\| \gg 0,$$ on $E_k|_{M_k^T}$

which gives an estimation of weights of $E_k$. One can thus prove the lemma as in Corollary 2.2.8.

**General Case:** The idea of the proof in nonabelian case is due to N.Higson. Let us consider a complex $\lambda_{-1}$ on $M$:

$$\wedge^0(g/t)^* \xrightarrow{d} \wedge^1(g/t)^* \xrightarrow{d} \cdots \xrightarrow{d} \wedge^{\text{top}}(g/t)^*,$$
where the differentials \( d \) at \( m \in M \) is given by widget with \( (\phi_k(m))^* \). Notice that the support of \( \lambda_{-1} \) is contained in \( \phi_k^{-1}(t) \).

Let \( \phi_k^t \) be the composition of \( \phi_k \) with projection from \( g \) to \( t \). Suppose that \( W \) is the Weyl group. By the Weyl Character formula, to prove the lemma, it is equivalent to show that as a \( T \)-equivariant cycle, \( \text{Index}(M_k, E_k \otimes \lambda_{-1})^{W \cdot \gamma} = 0 \). Thus, we can apply the same argument as in abelian case to the \( T \)-equivariant \( K \)-cycle: \( (M_k, E_k \otimes \lambda_{-1}, \phi_k^t) \).

Therefore, Theorem 3.5.1 follows from Lemma 3.5.2-3.5.4. \( \square \)
Chapter 4

Geometric Quantization and K-Homology

4.1 Quantization Map

In this chapter, we will first give the definition of quantization map for all $K$-cycles:

$$Q_{TOP} : \{(M, E, \phi)\} \rightarrow \hat{R}(G).$$

Then, we will show that $Q_{TOP}$ induces an isomorphism from $\hat{K}(G)$ to $\hat{R}(G)$.

**Definition 4.1.1.** Let $\bigcup_{k=1}^{\infty} (M_k, E_k, \phi_k)$ be a discrete $K$-cycle satisfying (3.5.2), we define the quantization map $Q_{TOP}$ to be

$$Q_{TOP} \left( \bigcup_{k=1}^{\infty} (M_k, E_k, \phi_k) \right) = \sum_{k=1}^{\infty} \text{Index}(M_k, E_k) \in \hat{R}(G).$$

Suppose that $(M, E, \phi)$ is an arbitrary $K$-cycle. As in (3.5.1), it is equivalent to a discrete $K$-cycle:

$$(M, E, \phi) \sim \bigcup_{k=1}^{\infty} (M_k, E_k, \phi_k). \quad (4.1.1)$$

If we can prove that $Q_{TOP}$ is invariant under equivalence relation between $K$-cycles, then we get a well-defined quantization map for all $K$-cycles:

$$Q_{TOP}(M, E, \phi) = Q_{TOP} \left( \bigcup_{k=1}^{\infty} (M_k, E_k, \phi_k) \right).$$

**Remark 4.1.2.** For cases in which $(M, E, \phi)$ is a $K$-cycle with compact vanishing
set, Braverman [Bra02] introduced a quantization using transversally elliptic index. Moreover, he showed that the quantization was invariant under bordism (through manifolds with compact vanishing set).

**Lemma 4.1.3.** Suppose that \((W, L, \psi)\) is a \(K\)-chain. For any constant \(R > 0\), there exists a hypersurface \(\Sigma_R\) in \(W\) such that

- the vector field \(V^\psi\) is nowhere vanishing over \(\Sigma_R\);
- \(\Sigma_R\) subdivides \(W\) into two parts: a bounded part \(W_-\) and an unbounded part \(W_+\), with the property that
  \[
  \psi(W_+) \cap B_R = \emptyset.
  \]

**Proof.** Let \(\mathcal{H} = \|\psi\|^2 : W \to \mathbb{R}\). By Theorem 3.3.1, the vanishing set \(W^\psi\) can be covered by a set of disjoint open subsets \(\{U_\alpha\}\) such that \(F_\alpha = W^\psi \cap U_\alpha\) is compact. Further, by Remark 3.3.5, there exists a constant \(K\) such that for all \(\alpha\),

\[
\|\psi(x) - \psi(y)\| \leq K, \text{ for all } x, y \in U_\alpha \cap \psi^{-1}(t_+). \tag{4.1.2}
\]

By (4.1.2) and the fact that \(\psi\) is proper over the vanishing set, there are only finitely many \(\alpha\) such that

\[
F_\alpha \cap \mathcal{H}^{-1}(R + K) \neq \emptyset. \tag{4.1.3}
\]

Let us denote them by \(\{\alpha_1, \ldots, \alpha_n\}\). Define an equivariant non-negative function \(\rho : W \to \mathbb{R}\) such that

- \(\rho(x) = 0\) for all \(x \not\in \bigcup_{i=1}^n U_{\alpha_i}\);
- \(\rho(x) \gg 2(R + K)\) for \(x \in F_{\alpha_i}, i = 1, \ldots, n\).

Select a regular value \(c\) for \(\mathcal{H} + \rho\), which is very close to \(R + K\). The hypersurface \(\Sigma_R = (\mathcal{H} + \rho)^{-1}(c)\) satisfies all the conditions.

\(\square\)

**Proposition 4.1.4.** Let \((M, E, \phi)\) and \((M', E', \phi')\) be two \(K\)-cycles. If they are bordant, then

\[
Q_{\text{TOP}}(M, E, \phi) = Q_{\text{TOP}}(M', E', \phi') \in \hat{R}(G).
\]
Proof. It is enough to show that they have the same multiplicity of any fixed irreducible representation $\gamma$:

\[ [Q_{\text{TOP}}(M, E, \phi)]^\gamma = [Q_{\text{TOP}}(M', E', \phi')]^\gamma. \]

We can build bordisms between $K$-cycles and discrete $K$-cycles. Suppose that

\[(M, E, \phi) \sim \bigsqcup_{k=1}^{\infty} (M_k, E_k, \phi_k)\]

and

\[(M', E', \phi') \sim \bigsqcup_{k=1}^{\infty} (M'_k, E'_k, \phi'_k).\]

This induces a bordism between

\[\bigsqcup_k (M_k, E_k, \phi_k) \sim \bigsqcup_k (M'_k, E'_k, \phi'_k). \tag{4.1.4}\]

Let us assume that the $K$-chain $(W, L, \psi)$ realizes the bordism above.

By Lemma 4.1.3, we construct a hypersurface $\Sigma_R$ which cuts $(W, L, \psi)$ into two pieces: $(W_-, L|_{W_-}, \psi|_{W_-})$ and $(W_+, L|_{W_+}, \psi|_{W_+})$. We observe that $W_-$ is an orbifold with boundary consisting of

- the parts which do not intersect with $\Sigma_R$;
- the parts which do intersect with $\Sigma_R$.

Notice that $(W_-, L|_{W_-}, \psi|_{W_-})$ is a $K$-chain with compact vanishing set. As the construction in section 4, we can build a cap. By gluing on the cap, we obtain a compact $K$-chain. Suppose that the new compact $K$-chain gives a bordism between

\[(M_R, E_R, \phi_R) \sim (M'_R, E'_R, \phi'_R),\]

where both $M_R$ and $M'_R$ consist of closed orbifolds. By bordism (in compact sense) invariance of index map, we have that

\[ [\text{Index}(M_R, E_R, \phi_R)]^\gamma = [\text{Index}(M'_R, E'_R, \phi'_R)]^\gamma \in \mathbb{Z}. \tag{4.1.5}\]

For any fixed $\gamma$, when we choose $R$ large enough, we have that

\[ [\text{Index}(M_R, E_R, \phi_R)]^\gamma = [Q_{\text{TOP}}(M, E, \phi)]^\gamma \]
and

$$[\text{Index}(M'_R, E'_R, \phi'_R)]^\gamma = [Q_{\text{TOP}}(M', E', \phi')]^\gamma.$$ 

This completes the proof.  

Proposition 4.1.5. Let $(\hat{\mathcal{M}}, \hat{E}, \hat{\phi})$ be a bundle modification of $(\mathcal{M}, E, \phi)$. We have that

$$Q_{\text{TOP}}(\mathcal{M}, E, \phi) = Q_{\text{TOP}}(\hat{\mathcal{M}}, \hat{E}, \hat{\phi}),$$

Proof. Suppose that $\mathcal{M}$ is modified by a principle bundle $\mathcal{P}$ with fiber $\mathcal{N}$. We assume that $(W, L, \psi)$ gives a bordism between

$$(\mathcal{M}, E, \phi) \sim \bigsqcup_{k=1}^{\infty} (\mathcal{M}_k, E_k, \phi_k),$$

where $\mathcal{M}_k$ are closed orbifolds. As the construction of vector bundles on $W$, we can extend the principle bundle $\mathcal{P}$ on $\mathcal{M}$ to $W$, denoted by $\mathcal{P}_W$. We define the bundle modification (with respect to $\mathcal{P}_W$) of $W$ to be

$$\hat{W} = P_W \times_H N.$$ 

Let $\hat{L}$ and $\hat{\psi}$ be the pullback of $L$ and $\psi$ to $\hat{W}$. Then $(\hat{W}, \hat{L}, \hat{\psi})$ forms a $K$-chain, giving a bordism between

$$(\hat{\mathcal{M}}, \hat{E}, \hat{\phi}) \text{ and } \bigsqcup_k (\hat{\mathcal{M}}_k, \hat{E}_k, \hat{\phi}_k).$$

Hence,

$$Q_{\text{TOP}}(\hat{\mathcal{M}}, \hat{E}, \hat{\phi}) = \sum_k \text{Index}(\hat{\mathcal{M}}_k, \hat{E}_k).$$

On the other hand, because $(\hat{\mathcal{M}}_k, \hat{E}_k)$ is a bundle modification of $(\mathcal{M}_k, E_k)$, we have that

$$\text{Index}(\mathcal{M}_k, E_k) = \text{Index}(\hat{\mathcal{M}}_k, \hat{E}_k).$$

This completes the proof.  

By the two propositions above, we can conclude that the quantization map $Q_{\text{TOP}}$ is a well-defined map from $\hat{K}(G)$ to $\hat{R}(G)$.

Theorem 4.1.6. The quantization map $Q_{\text{TOP}}$ gives an isomorphism:

$$Q_{\text{TOP}} : \hat{K}(G) \longrightarrow \hat{R}(G).$$
Proof. It is clear that

\[ Q_{\text{TOP}} \circ P_{\text{TOP}} : \tilde{R}(G) \longrightarrow \tilde{R}(G) \]

equals to the identity map. Therefore, \( Q_{\text{TOP}} \) is surjective.

Since every \( K \)-cycle is equivalent to a discrete \( K \)-cycle, it is enough to prove the injectivity of compact \( K \)-cycles. This immediately follows from the fact that geometric \( K \)-homology is isomorphic to analytic \( K \)-homology [BHS07, BOOSW10]. \( \square \)

### 4.2 Quantization Commutes with Reduction

In the previous section, we give the definition of quantization for general \( K \)-cycles. This framework provides us a new approach to the quantization commutes with reduction theorem in non-compact setting. Let us begin with the multiplicative property.

**Theorem 4.2.1.** Let \((M, E, \phi_1)\) and \((N, F, \phi_2)\) be two pre-quantum data. When \( N \) is compact and \( \phi_1 \) is proper, we have that

\[ Q_{\text{TOP}}(M, E, \phi_1) \times Q_{\text{TOP}}(N, F, \phi_2) = Q_{\text{TOP}}(M \times N, E \boxtimes F, \hat{\phi}_1 + \hat{\phi}_2), \]

where \( \hat{\phi}_1 \), \( \hat{\phi}_2 \) are the pullbacks of \( \phi_1 \) and \( \phi_2 \) respectively.

**Proof.** Consider the triple \((M \times N \times [0,1], \hat{E} \otimes \hat{F}, \hat{\phi})\), where \( \hat{E} \) and \( \hat{F} \) are the pullbacks of \( E \) and \( F \), and the map \( \phi \) is defined to be

\[ \phi(m,n,t) = \phi_1(m) + t\phi_2(n). \]

By assumption, it is easy to check that \((M \times N \times [0,1], \hat{E} \otimes \hat{F}, \phi)\) satisfies the properness and compatibleness conditions. Thus, this \( K \)-chain gives a bordism between

\[ (M \times N, E \boxtimes F, \hat{\phi}_1) \text{ and } (M \times N, E \boxtimes F, \hat{\phi}_1 + \hat{\phi}_2). \quad (4.2.1) \]

By Theorem 3.5.1, let us assume that there is a bordism

\[ (M, E, \phi_1) \sim \bigsqcup_k (M_k, E_k, \rho_k), \quad (4.2.2) \]
where all the $M_k$ are closed. Since $N$ is compact, (4.2.2) induces another bordism:

$$(M \times N, E \boxtimes F, \hat{\phi}_1) \sim \bigsqcup_k (M_k \times N, E_k \boxtimes F, \hat{\rho}_k),$$

where $\hat{\rho}_k$ are the pullbacks of $\rho_k$. By the bordism invariance of $Q_{\text{TOP}}$, we have

$$Q_{\text{TOP}}(M \times N, E \boxtimes F, \hat{\phi}_1) = \sum_k \text{Index}(M_k \times N, E_k \boxtimes F).$$

Meanwhile, we have that

$$\text{Index}(M_k \times N, E_k \boxtimes F) = \text{Index}(M_k, E_k) \times \text{Index}(N, F).$$

Hence, we get

$$Q_{\text{TOP}}(M \times N, E \boxtimes F, \hat{\phi}_1) = Q_{\text{TOP}}(M, E, \phi_1) \times Q_{\text{TOP}}(N, F, \phi_2). \quad (4.2.3)$$

The theorem follows from (4.2.1) and (4.2.3).

\[\square\]

**Theorem 4.2.2.** Let $(M, E, \mu)$ be pre-quantum data. If the moment map $\mu$ is proper and 0 is a regular value, then

$$[Q_{\text{TOP}}(M, E, \mu)]^G = Q_{\text{TOP}}(M_0, E_0),$$

where $M_0 = \mu^{-1}(0)/G$ and $E_0 = (E|_{\mu^{-1}(0)})/G$.

One can find the proof of Theorem 4.2.2 in [Mei98, TZ98, Par01] etc. Here, we just sketch the proof.

First, we have a decomposition of the vanishing set $M^\mu$ [Par01, Lemma 6.3]:

$$M^\mu = \sum_{\gamma \in \Gamma} G.(M^\gamma \cap \mu^{-1}(\gamma)),$$

where $\Gamma$ is a discrete set in $t_+$. For each $\gamma \in \Gamma$, let $U_\gamma$ be a small open $G$-invariant neighborhood of $G.(M^\gamma \cap \mu^{-1}(\gamma))$ in $M$. By Theorem 3.2.3, we have

$$Q_{\text{TOP}}(M, E, \mu) = \sum_{\gamma \in \Gamma} Q_{\text{TOP}}(U_\gamma, E|_{U_\gamma}, \mu|_{U_\gamma}).$$

In accord with [Par01], one can show that for all $\gamma \neq 0$,

$$[Q_{\text{TOP}}(U_\gamma, E|_{U_\gamma}, \mu|_{U_\gamma})]^G = 0.$$
Thus,

$$[Q_{\text{TOP}}(M, E, \mu)]^G = [Q_{\text{TOP}}(U_0, E|_{U_0}, \mu|_{U_0})]^G.$$ 

Since 0 is a regular value, we have that $U_0 \cong \mu^{-1}(0) \times g^*$. As in section 4, we can build a cap $(W, L, \psi)$. By gluing on the cap, we can get a compact $K$-cycle $(N, F)$. Remember that the construction of the cap (in particular, the line bundle $L$) is not unique. In fact, we can build $(W, L, \psi)$ in a way such that

$$\text{Index}(N, F) = [Q_{\text{TOP}}(M, E, \mu)]^G.$$ 

In this case, $(N, F)$ turns out to be a bundle modification of $(M_0, E_0)$ (for example, when $G = S^1$, $N$ is a $\mathbb{C}P^1$-fiber bundle over $M_0$). Therefore,

$$Q_{\text{TOP}}(M_0, E_0) = [Q_{\text{TOP}}(M, E, \mu)]^G.$$ 

As a result, by taking $(N, F, \phi_2)$ to be $(O_{\gamma}, E_{\gamma}, \iota_{\gamma})$, Theorem 1.5.4 follows from Theorem 4.2.1 and 4.2.2.
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