THE NATURE OF SECONDARY MATHEMATICS TEACHERS’ EFFORTS TO MAKE IDEAS OF SCHOOL ALGEBRA ACCESSIBLE

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by

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ABSTRACT

This study was conducted to investigate and to understand secondary mathematics teachers’ efforts to make targeted mathematical ideas (TMIds) of school algebra accessible and comprehensible to students. Influenced by TIMSS researchers’ use of close-up and wide-angle lenses (Hiebert et al., 2003) to examine and discuss data (e.g., video, lesson artifacts) about lessons taught by teachers from many countries, this study made use of close-up and several wide-angle lenses to examine and to understand data from the efforts of seven middle school and high school teachers. Data include approximately 60 hours of video recordings of the teachers teaching lessons (i.e., 3 to 5 lessons per teacher), observation-related interviews, and problem-solving interviews, as well as lesson and interview artifacts.

Initial examination of data was done using the lens of a conceptual framework whose development was based upon research supporting two perspectives of teaching: Teaching is an application of knowledge for teaching mathematics and teaching is communication of mathematical ideas. These perspectives are reflected in the four constructs that make up this initial framework: decompressing, trimming, bridging, and framing. The first three constructs originated in the Knowledge of Algebra for Teaching (KAT) and are described by KAT researchers as ways in which teachers use their KAT. Framing emerged from research literature in fields for which communication is an object of study. Loosely speaking, framing is the ways in which teachers make TMIds accessible and comprehensible to students.
Data analysis was conducted in four rounds, each round examining the data using a different lens. In Round 1, the data are examined using the initial conceptual framework as a lens (i.e., close-up lens) to identify instances of framing, and resulted in the identification of more than fifty instances of framing. After the instances of framing were identified, the framings were sorted and re-examined for patterns that cut across the framings. Round 2 resulted in the identification of 11 characteristics of framing. The 50+ instances of framing were also sorted into framings that reflected the use of decompressing, trimming, bridging, and nonmathematical elements and/or approaches. The fourth set of framings was examined more closely to determine how the nonmathematical elements were employed. This round of analysis (#3) led to the identification of three additional teaching practices: quasi-decompressing, quasi-trimming, and quasi-bridging; and the modification of the initial conceptual framework guiding this study, the Conceptual Framework for Framing. In the fourth and final round of data analysis, the instances of framing were revisited, examined using the lens of the Conceptual Framework for Framing, and coded according to what teaching practice(s) are being used in the framing. The three teachers’ tables of coded framings reflect how different these teachers are with respect to which teaching practices were emphasized during data collection. The tables of coded framings for each teacher provided a concise way of representing each teacher’s efforts to make TMIds accessible and comprehensible to students, and facilitated side-by-side comparisons of which teaching practices may be emphasized.
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ACKNOWLEDGEMENTS

My late-parents, Masako and Shiro Shimizu, gave me Keiko for a middle name. Depending upon the how *keiko* is written using Japanese characters (i.e., kanji), *keiko* has different meanings. According to Mom and Dad, my name means *blessed one*. As I reflect upon my life, I see that my parents’ hopes have come to fruition many times, and in many ways. I have been blessed with many opportunities. With respect to my career in mathematics education, I have been blessed with opportunities to work with individuals who have helped me continue to learn and continue to grow. I am truly grateful.

Interestingly, *keiko*, written using different kanji, means *study*. While my name does not mean *study*, this meaning of *keiko* is as much a part of my life experiences and who I am as is *blessed one*. I owe a great deal to my teachers, Drs. M. Kathleen Heid, Glendon W. Blume, and Rose Mary Zbiek, who made sure that I was always studying, learning and growing. They introduced me to the world of mathematics education research and scholarship. With their help I learned to navigate in this world and to see teaching and learning through eyes that are very different from the eyes I had developed in 21 years as a high school mathematics teacher. I am especially indebted to Dr. Heid for her patience, her support, her encouragement, and her belief in me. Her high expectations and persistent demands to think harder and to dig deeper pushed me to accomplish more than I had thought I could.

I am also grateful for the encouragement and support I have received from members of my various families. My sister Sharon and friend Laurena provided love,
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Finally, I am thankful for the ten teachers who agreed to participate in my study and my pilot studies, and the educators who helped me find them. Without their help, this dissertation would not have been possible.

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Chapter 1

Rationale

In 1983, the National Commission on Excellence in Education (NCEE) published a report entitled, *A Nation at Risk*, which called for major reforms in education. The Commission stated:

If an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might have viewed it as an act of war. As it stands, we have allowed this to happen to ourselves. We have even squandered the gains in student achievement made in the wake of the Sputnik challenge. Moreover, we have dismantled essential support systems which helped make those gains possible. We have, in effect, been committing an act of unthinking, unilateral educational disarmament. … This report, the result of 18 months of study, seeks to generate reform of our educational system in fundamental ways and to renew the Nation's commitment to schools and colleges of high quality throughout the length and breadth of our land. (pp. 5–6)

Thirteen years later, members of the National Governors Association (NGA), and education, business, and community leaders gathered at the 1996 National Education Summit to begin “a national effort to establish high academic standards, assessments and accountability and improve the use of school technology as a tool to reach high standards” (Achieve, 1996, p. 1). Included in this effort was the establishment of
Achieve, a bipartisan, nonprofit organization, to assist states in raising academic standards, improving assessments, and strengthen accountability (Achieve, 1996).

Whether the 1996 National Education Summit is a product of the call for reform from *A Nation at Risk* is unclear; however, the belief in the need for change is consistent. Since that meeting, state policymakers have passed legislation raising the standards for high school graduation (American Diploma Project Network [ADPN], 2008). The Common Core State Standards for Mathematics (CCSS-M) were written and made public in 2010 (NGA Center for Best Practices & Council of State School Officers [CCSSO], 2010). These efforts, according to Achieve, fall under the agenda of preparing college- and career-ready high school graduates (2010). Achieve’s website (2012b) reports that 45 states, the District of Columbia (DC), 4 U.S. territories and the U.S. Department of Defense (DoD) schools have adopted the CCSS-M for their respective state standards.2

In a separate education reform effort from “preparing college- and career-ready high school graduates” as characterized by Achieve, Congress passed the No Child Left Behind (NCLB) Act of 2001. NCLB includes requirements that all states create standards-based assessments to be given to all students in Grades 3 through 8 and at least once in high school; and that schools are held accountable for student achievement (Bush, 2002).

---

1 This effort matches the first of five recommendations made in *A Nation at Risk* (NCEE, 1983).

2 This effort matches part of the second of five recommendations made *A Nation at Risk* (NCEE, 1983).
According to NCLB, schools whose tests results show adequate yearly progress\(^3\) may be eligible for bonus federal funding, whereas, schools whose test scores do not show adequate yearly progress are subject to corrective actions. The nature of the corrective action depends upon the number of years the school had not made adequate yearly progress.

With the adoption of CCSS-M as their standards for mathematics education, under NCLB legislation, the DoD schools and schools in 45 states, D.C., and 4 U.S. territories are now held accountable to help students meet standards described in the CCSS-M. What seems to be missing in this effort to prepare 21\(^{st}\) century college- and career-ready high school graduates is attention to what mathematics teachers are doing to help students meet the CCSS-M. With respect to mathematics, what also seems to be missing is attention to what teachers are doing to help students comprehend the ideas of school mathematics. This is an effort that, speaking as an educator with 21 years of experience teaching at the secondary level, I believe is critical to help students meet the CCSS-M. Teachers are directly involved in working with students.

Most of what we know about what goes on in U.S. classrooms nationally comes to us from surveys (Weiss, 2005), for example, the National Survey of Science and Mathematics Education (NSSME). The data from such surveys are teacher self-report data and are informative, but they are not enough to help us understand what goes on in classrooms and not enough to help us examine the quality of instruction (Weiss, 2005).

\(^3\) According to Bush (2002), adequate yearly progress is defined by each state.
Weiss asserts that researchers need to go into classrooms to observe instruction and to interview teachers. Furthermore, “the need for information on the nature and quality of K–12 mathematics and science lessons is particularly acute given the current emphasis on reform” (Weiss, Pasley, Smith, Banilower, & Heck, 2003b, p. 1). For this reason, Weiss et al. conducted the *Looking Inside the Classroom Study*.

One of the premises upon which researchers for the *Looking Inside the Classroom Study* based their assessment of lesson quality is the notion that “it is important that lessons engage students in doing the intellectual work, with the teacher helping to ensure that they are, in fact, making sense of the key concepts being addressed” (Weiss, Pasley, Smith, Banilower, & Heck, 2003a, p. 8). Using a classroom observation instrument based upon a 5-point rating scale on 29 indicators, Weiss et al. (2003) examined each of the 364 mathematics and science lessons and found that, with respect to lesson content,

- 20% of the lessons effectively address the need to engage students intellectually with mathematics/science content and 55% of the lessons failed to address this same need (p. 4); and
- 16% of the lessons received high ratings for helping students make sense of the mathematics/science content and 66% of the lessons were rated as inadequate in helping students in sense-making (p. 8)

---

4The instrument was used as part of the core evaluation of NSF’s Local Systemic Change Initiative by Horizon Research, Inc. Horizon Research demonstrated the classroom observation instrument to be valid for use in assessing the quality of the design and implementation of mathematics and science lessons (Weiss et al., 2003a).
These statistics indicate a clear need for improvement in helping students engage intellectually with mathematics/science content and helping students make sense of mathematics/science content. The results suggest two questions. What are teachers whose lessons received high ratings doing? What are teachers whose lessons received low ratings doing?

Weiss et al. (2003a) do offer characterizations of what teachers are doing with content; however, because their study is a large-scale study of elementary, middle, and high school mathematics and science lessons, their characterizations are general in nature, and the examples of mathematics they provide at each level (i.e., elementary, middle, and high school) of mathematics are relatively few in number. More information is needed. To improve what mathematics teachers do with mathematics content, understanding what mathematics teachers do with mathematics content is important.

Understanding What Mathematics Teachers Do to Engage Students Intellectually With Mathematics Content, and What Mathematics Teachers Do to Help Students Make Sense of Mathematics Content: Narrowing the Scope

A study aimed at understanding what mathematics teachers do with mathematics content is ambitious and much too broad. Narrowing the scope is necessary. I chose to narrow the scope initially based upon my professional experiences and interests, the subject area of my study, and the unit of analysis for my data.

My professional experiences. I chose to narrow the scope of my study to studying the work of secondary mathematics teachers. It is a natural choice given my 21 years of experience as a secondary school mathematics teacher. I have a greater
understanding of the work of secondary mathematics teachers than the work of elementary teachers. In addition, over the course of 21 years, I observed the growing importance of school mathematics in the lives of my students with respect to high school and college graduation. When I began teaching in California in the early 1980s, courses such as Prealgebra and General Math could be used to meet the mathematics course requirement for graduation. Now all students in California must take and pass Algebra 1 to graduate from a public high school (California Education Code Section 51224.5). In addition, California’s college freshmen who do not pass the Entry Level Mathematics (ELM) placement test are required to take one or more remedial mathematical courses (e.g., College Algebra). Students in this situation are paying, in time and dollars, for courses that do not count toward their bachelor’s degrees.

Subject area. For several reasons, I chose to narrow the scope to one subject area within school mathematics: school algebra.

Increased mathematics requirements for high school graduation. In the most recent decade, state policymakers passed legislation resulting in raising the standards for high school graduation (ADPN, 2008). Depending upon the state, the legislation explicitly identified specific mathematics courses students must take, for example, Algebra 1, Geometry, and Algebra 2 (or the equivalent); and/or increased the number of years of mathematics that students must complete to 3 or 4 years (Achieve, 2012c; Reys et al., 2007). According to Achieve, Inc. (2012a), students residing in one of 23 states or

---

5 A test that assesses students in 3 areas: Number and Data, Algebra, and Geometry.

http://www.ets.org/csu/about/elm/elm_topics
the District of Columbia who want to earn a high school diploma are required\(^6\) or will soon be required\(^7\) to complete successfully Algebra 1, Geometry, and Algebra 2 (or the equivalent). The 23 states and the District of Columbia have legislated high school graduation requirements (mathematics and nonmathematics) identified by the American Diploma Project Network (ADPN)\(^8\) as *college- and career-ready graduation requirements* (Achieve, 2012). These requirements reflect a rigorous and broad curriculum, grounded in the core academic disciplines, but also consisting of other subjects that are part of “a well-rounded education” (Achieve, 2012a). Such a curriculum includes mathematics and English content that are aligned to meet the demands of college and career (Achieve, 2012a). The twenty-three states and D.C. account for “nearly half of the students in the U.S.” (Achieve, 2012a, p. 1).

---

\(^6\) Arkansas, Delaware, District f Columbia, Georgia, Indiana, Kentucky, Michigan, Mississippi, Oklahoma, South Dakota, Texas, and Utah.

\(^7\) Alabama (as of the class of 2013), Arizona (as of the class of 2013), Florida (as of the class of 2016), Hawaii (as of the class of 2016), Iowa (as of the class of 2015), Minnesota (as of the class of 2015), Nebraska (as of the class of 2015), New Mexico (as of the class of 2013), North Carolina (as of the class of 2013) Ohio (as of the class of 2014), Tennessee (as of the class of 2013), and Washington (as of the class of 2013).

\(^8\) The ADPN is a partnership of Achieve, Inc., state governors, state education officials, postsecondary leaders and business executives that was formed at the 2001 National Education Summit.
The twenty-three states identified by Achieve, Inc., however, are not the only states that have made school algebra important to prospective high school graduates. In a report published by the Education Commission of the States on the graduation requirements of all fifty states, Zinth (2012) reports that, in addition to the twenty-three states whose high school graduates have or will have earned a “college- and career-ready” diploma, Idaho, Illinois, New Jersey, and New York have legislated Algebra 1 as a high school graduation requirement; and Louisiana and Virginia have legislated Algebra 1, Geometry, and Algebra 2 as graduation requirements. This additional information indicates that a majority of U.S. high school students have to take at least Algebra 1 to graduate from high school—a major reason for choosing to limit my study to the teaching of school algebra.

More students are taking school algebra. Furthermore, empirical data obtained from the National Assessment of Educational Progress (NAEP)\(^9\) spanning 1978 to 2008\(^{10}\) indicates that not only are students supposed to take Algebra 1 and/or Algebra 2 for graduation, greater percentages of students are taking these courses. Table 1-1 reflects the increases in 13-year old students taking Algebra 1 in middle school. Table 1-2 reflects increases in 17-year old students having completed Algebra 2.

---

\(^9\) The data were obtained from NAEP using NAEP’s Data Explorer, an online tool which makes it possible for visitors to examine and explore the data collected as part of the NAEP assessments and to create reports using subsets of these data.

\(^{10}\) [http://nces.ed.gov/nationsreportcard/lttdata/](http://nces.ed.gov/nationsreportcard/lttdata/)

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<th>Year</th>
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<th>Regular math\textsuperscript{12}</th>
<th>Pre-algebra</th>
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#: Rounds to zero. *: Original assessment format. (Note: %-ages may not sum to totals because of rounding.)


\textsuperscript{12}I was unable to find a definition for “regular mathematics;” however, I presume “regular mathematics” is another name for “General Mathematics.”

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<th>Geometry</th>
<th>Algebra 2</th>
<th>Calculus</th>
<th>Other\textsuperscript{14}</th>
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<td>16</td>
<td>14</td>
<td>39</td>
<td>5</td>
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<tr>
<td>1986*</td>
<td>18</td>
<td>18</td>
<td>16</td>
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</tr>
<tr>
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<td>15</td>
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<tr>
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<td>13</td>
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<td>17</td>
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</tr>
<tr>
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<td>15</td>
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<tr>
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<td>19</td>
<td>1</td>
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</table>

– Not available. * Original assessment format. (Note: %-ages may not sum to totals because of rounding.)

Table 1-1 describes what mathematics class participating 13-year old students took each year NAEP data was collected from 1986 to 2008. Based upon the assumption that most children enter 1\textsuperscript{st} grade when they are 6 years old, the data describes the general enrollment trend of 8\textsuperscript{th} grade students taking algebra over 22 years. The data indicate that


\textsuperscript{14} I was unable to find a definition of “other” from NAEP; however, it is possible for “other” to include Geometry and Algebra 2. These two courses are offered in the middle school where one of the teachers participating in my study teaches.
the percentage of 8th graders taking algebra in 2008 is approximately twice the percentage of 8th graders taking algebra in 1986. Table 1-2 describes the highest mathematics class taken by 17-year old students, most of whom would be high school seniors from 1978 to 2008. The data indicate that in 1978, roughly 80% of the 17-year olds completed Algebra 1 or higher. In 2008, percentage of the 17-year olds completing at least Algebra 1 increased to roughly, 97 percent. The data also indicate that the percentage of seniors who had taken at least Algebra 2 increased from approximately 43 percent to 71 percent from 1978 to 2008. Thus, the NAEP data, reflecting a trend of higher and higher percentages of students are taking school algebra, supports my claim that more students are taking school algebra as part of their secondary school program.

Diversity of students taking school algebra. Except for those schools whose student body population is composed entirely of students who plan to go to college, these changes in graduation requirements mean that mathematics teachers are responsible to help a larger group\(^{15}\) of students learn ideas in school algebra. The greater percentages of a student body being required to complete Algebra 1 and Algebra 2 in order to graduate from high school suggest that more teachers are challenged with the task of supporting the learning of mathematics by students who, in years prior to the legislation increasing graduation requirements, would have graduated having passed, perhaps even struggled to pass, courses (e.g., General Mathematics, Prealgebra, vocational mathematics) that would not have met college entrance requirements. In other words, teachers are now required to

\(^{15}\)Data capturing the increased percentages of 13-year-olds and 17-year-olds taking algebra since the late 1970s are provided in chapter 2.
support the learning of Algebra 1 and Algebra 2 students whose mathematical interests and grasp of mathematics addressed in prerequisite courses reflect a greater diversity. This diversity suggests greater challenges to teachers in making ideas of school algebra accessible and comprehensible to students. This observation is supported by research (e.g., Filloy & Rojano, 1984, 1989; Filloy, Rojano, & Solares, 2010; Herscovics & Linchevski, 1994) which identifies ideas (e.g., variable operating with unknown quantities) first encountered in secondary school algebra which students find cognitively demanding.

My experiences in teaching high school mathematics in the late 1980’s when General Math 1, General Math 2, and High School Arithmetic (all yearlong courses) were eliminated from my school district’s high school curriculum supports this observation. My challenges to helping students move from the concrete world of arithmetic to the more abstract world of algebra increased greatly.

Increasing the challenges to teachers who are charged with helping students meet the mathematics requirements for graduation is States’ adoptions of the CCSS-M as the standards by which school districts’ mathematics curricula and materials, and State-level assessments are to be aligned (Achieve, 2010). As of 2011, 45 states and D.C. have adopted the CCSS-M (Achieve, 2012b). The descriptions of the mathematical practice standards in the CCSS-M reveal an emphasis on students achieving mathematics proficiency, which, according to the authors of Adding it Up, “is necessary for anyone to learn mathematics successfully” (Kilpatrick, Swafford, & Findell, 2001, p. 116). This claim is based upon the observation that the description of each standard for mathematical practice begins with “mathematically proficient students” (NGA &
CCSSO, 2010) and proceeds to describe what mathematically proficient students do or can do. Thus, a third reason for studying the work of teachers teaching school algebra is my professional desire to help teachers meet the challenges of helping an academically diverse population of students become mathematically proficient. Helping teachers begins with understanding what they currently do.

**What teachers do to help students learn is important.** From the perspective of sociocultural theory, a critical aspect of helping students achieve mathematical proficiency is what teachers do to make ideas of mathematics accessible and comprehensible to students. At the heart of sociocultural theory are the understandings that “human cognition and learning as social and cultural rather than individual phenomena” (Kozulin, Gindis, Ageyev, & Miller, 2003, p. 1) and that “learning is a necessary and universal aspect of the process of developing culturally organized, specifically human, psychological functions” (Vygotsky, 1978a, p. 90). Vygotsky’s sociocultural theory puts forth the idea that higher mental functions are achieved through mediated activity (Kozulin, 1985; Vygotsky, 1978b, 1986) in which the concept of mediation emphasizes “the role played by humans and symbolic intermediaries placed between the learner and the material to be learned” (Kozulin et al., 2003, pp. 2-3). In other words, from theoretical as well as pragmatic perspectives, the teacher and what the teacher does are important to student learning.

A critical aspect of helping students achieve mathematical proficiency is what teachers do to make ideas of mathematics accessible and comprehensible to students. The importance of what teachers do to make mathematics accessible and comprehensible is explained from a sociocultural perspective by Vygotsky’s concept of zone of proximal
development (ZPD), a concept he described in more than one way. Vygotsky (1986) defines ZPD as “the distance between the actual developmental level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance or in collaboration with more capable peers” (p. 86). This conception of ZPD presupposes that interaction between learners and an adult or more capable peer on a given task helps learners accomplish—without help—what in the beginning could be accomplished only with assistance (Kozulin et al., 2003). This conception suggests the importance of examining what teachers do in their interactions with students that help students learn mathematics content. The particular form of interaction that is of interest in this study is what teachers do to make mathematics content accessible and comprehensible so that whatever mathematics content is being addressed in the interaction is in students’ respective ZPDs.

Vygotsky also spoke of ZPD as a dynamic interaction between opposing forces (Vygotsky, 1978), a psychological space in which a child’s (i.e., student’s) less systematic and highly contextual knowledge meets the adult’s (i.e., teacher’s) specialized, highly structured, and logically organized knowledge (Kozulin et al., 2003). According to Vygotsky, the dialectical tension in this psychological space leads to changes in a student’s thinking (Kozulin, 1985, 2003). In the context of this study, “changes in a student’s thinking” (Kinard & Kozulin, 2005) is interpreted as the student’s learning of and thinking about the mathematical content being addressed. This conception of ZPD suggests that if a student is not engaged in working with and thinking about those ideas, he or she will not learn those ideas. Both conceptions of ZPD provide support for the
importance of examining and gaining understandings of how teachers make school algebra accessible and comprehensible.

**Unit of analysis.** This study examining how teachers teach school algebra was partly inspired by Weiss et al.’s (2003a) *Looking Inside the Classroom Study* which examined the nature and quality of mathematics and science lessons. Their findings established a research-based need to understand what teachers are doing to help students to engage intellectually in and to make sense of mathematics content. Given that teachers often address several ideas within a lesson, I chose to examine what teachers do to make targeted mathematical ideas (TMIds) accessible and comprehensible. For purposes of this study, *mathematical ideas* are defined to be the subject matter of school algebra as defined by Cooney, Davis, and Henderson (1975): algebraic facts, concepts, skills, generalizations, and procedures. Mathematical ideas are said to be *targeted*, if they are the focus of discussion or explanation by teachers during interviews or during classroom observations.

Thus, based upon my professional experiences and interests, subject area, and unit of analysis, the general goal of understanding what mathematics teachers do to engage students intellectually with, and to help students make sense of mathematics content, is narrowed to following research question.

How do certificated secondary mathematics teachers attempt to make targeted mathematical ideas (TMIds) of school algebra accessible and comprehensible to students?
How is This Study Different from Other Studies?

The idea of examining what teachers do to make ideas of mathematics accessible and comprehensible to students is not new. However, researchers (e.g., Baxter & Williams, 2010; Doyle, 1983; Stein & Lane, 1996) have usually focused their attentions on examining specific strategies (e.g., use of tasks, questioning, use of examples, explaining and scaffolding) that support students learning school mathematics.

Researchers’ collective discussions of teachers’ teaching suggest that each teaching strategy influences the aspect or aspects of a mathematical idea to which students attend. In other words, the questions teachers ask, the tasks teachers create for students to do, and the examples teachers present all have the potential of bringing attention to one or more aspects of mathematical ideas. Teachers’ explanations of a mathematical idea can be seen as providing students with a lens or lenses through which to view that idea. For example, in teaching students the mathematical idea of how to solve equations of the form, \( ax + b = cx + d \) for \( x \), for which \( a, b, c, d \) are real numbers, a teacher could talk about the solving process as “maintaining balance” (i.e., what you do to one side of the equation you must do to the other to maintain a “balanced” equation).

A teacher could also talk about the solution of this equation as the \( x \)-value of the intersection of two lines, defined by the rules, \( y = ax + b \) and \( y = cx + d \); and that this intersection can be found graphically. By providing students with a lens through which to view a mathematical idea, the teacher’s actions can be seen as highlighting or emphasizing a particular aspect of that mathematical idea.

Researchers’ decisions to examine a teacher’s use of a particular teaching strategy and the publications discussing the corresponding findings provide the field with views of
teaching using lenses focused on specific teaching strategies. Teachers can have multiple strategies for a mathematical idea. Teachers can have different strategies, and which strategy is employed can depend upon the type of subject matter and other factors (e.g., availability of technology, needs of students). A researcher who enters a classroom with the intent to study a particular strategy is likely not to attend to and to document other strategies which make ideas accessible and comprehensible.

For this study, I began each observation and interview not knowing what strategies teachers would implement or discuss. I assumed that at some point I would observe teachers implementing strategies described in existing research (e.g., asking questions, offering explanations), but informed by my preliminary and pilot studies and by my experiences working with other secondary mathematics teachers, I also assumed that the teachers would reveal other strategies as well.

**Overview of This Study**

This study examines a different slice of the work of teaching than has been investigated by researchers who study strategies of teaching. It is a study of teaching using a lens that is not restricted to a particular teaching strategy (e.g., use of cooperative/collaborative learning groups) or specific teaching practice (e.g., questioning, use of wait time). It is a study of a critical aspect of engaging students in doing mathematics and maintaining their engagement in the doing of mathematics: what teachers do to make TMI ds accessible and comprehensible to students. This study does not examine efforts such as providing students with mathematics textbooks or offering students extra help through after-school tutoring, each of which can be interpreted as
attempts to make mathematics more accessible and comprehensible as well. Providing students with textbooks and offering students extra help are general teaching practices that do not address particular mathematical ideas. The focus of this study is what teachers do with mathematical ideas with the goal of making those ideas accessible and comprehensible to students. Thus, once again, as part of an effort to understand what teachers are doing currently, this study examines the following research question:

How do certificated secondary mathematics teachers attempt to make targeted mathematical ideas (TMIds) accessible and comprehensible to students?
Chapter 2

Literature Review

As a starting point to understanding what secondary teachers do in their efforts to make ideas of school algebra accessible and comprehensible to students, research literature was reviewed and examined for this study with four goals in mind. The goals are as follows:

1. To develop a conceptual framework with which collect and analyze data about teaching school algebra,
2. To gain insights about teaching practices that reflect teachers’ efforts to make mathematics accessible and comprehensible to students,
3. To gain insights about what research identifies as topics in school algebra in which students struggle, and
4. To gain insights from studies for which researchers have visited classrooms and observed teachers teaching.

This chapter is divided into four main sections, one for each goal.

Making TMIds Accessible and Comprehensible to Students: Developing a Framework to View What Teachers Do

The conceptual framework developed for this study represents the convergence of two perspectives of teaching. The first perspective, mathematics teaching as an application of knowledge of mathematics for teaching emerged from mathematics education research. The second perspective emerged from examining examples of
teaching that provided evidence that teachers also use knowledge that is outside of their knowledge of mathematics to make TMIDs accessible and comprehensible.

**Perspective:** Teaching is an application of knowledge of mathematics for teaching. In his often-cited article that introduces the concept of pedagogical content knowledge (PCK) to the education community, Shulman (1986) identified “the need for a more coherent theoretical framework” (p. 9) of teacher knowledge that identifies “domains and categories of content knowledge in the minds of teachers” (p. 9). He states that it was a need that became apparent as he pursued answers to the question, “How does the successful college student transform his or her expertise in the subject matter into a form that high school students can comprehend?” (p. 9). Shulman’s question is related to the question governing this study, concerning how secondary mathematics teachers make TMIDs accessible and comprehensible to students. Both Shulman’s question and the research question governing this study identify making subject matter accessible and comprehensible to students as a goal. However, they differ in that Shulman’s question focuses on understanding teacher knowledge, and it reflects a perspective that teaching is an application of knowledge. The research question for this study focuses on understanding what teachers do.

More than 20 years later, mathematics education researchers (Ball, Hill, & Bass, 2005; and Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2006) have developed two conceptual frameworks that describe knowledge for teaching school mathematics. The Mathematics Knowledge for Teaching (MKT), pictured in Figure 2-1, consists of six domains of knowledge and has been used to develop assessments to measure the MKT of elementary school teachers (Hill, Schilling, & Ball, 2004). The MKT framework
identifies six forms of knowledge teachers might draw upon when making ideas accessible and comprehensible, but provides no insights as to how teachers might use their MKT.

*Figure 2-1. Domain map for mathematical knowledge for teaching (Hill & Ball, 2008, p. 377)*

The Knowledge of Algebra for Teaching (KAT) conceptual framework, has been used to develop assessments of the KAT of secondary school algebra teachers (Floden & McCrory, 2007; Floden, McCrory, Ferrini-Mundy, Reckase, & Senk, 2009). **KAT is composed of two dimensions: mathematical content knowledge and mathematical uses of knowledge in teaching** that they also call *teaching practices* (McCrory, Floden, Ferrini-Mundy, Reckase, & Senk, 2012). The *mathematical content knowledge dimension is composed of three types of mathematical knowledge teachers draw upon as they carry out the work of teaching: school knowledge, advanced knowledge, and teaching knowledge*. KAT researchers describe these types of mathematical knowledge
as “knowing what they will teach,” “knowing more advanced mathematics that is relevant to what they will teach,” and “knowing mathematics that is particularly relevant for teaching and would not typically be taught in undergraduate mathematics courses,” (p. 595), respectively. The mathematical uses of content knowledge (or teaching practices) dimension identifies three ways in which teachers use their mathematical content knowledge as they carry out the work of teaching decompressing, trimming, and bridging.

**Decompressing, trimming, and bridging.** Given that making TMIds accessible and comprehensible to students is an important part of the work of teachers, these three teaching practices seem to be promising lenses through which to view what teachers do in this aspect of teaching. This promise is underscored by the fact that decompressing, trimming, and bridging have their origins in analysis of teachers’ teaching of school algebra and of teachers talking about teaching. More specifically, KAT researchers report that decompressing, trimming, and bridging emerged from their analysis of videos of teaching and interviews with teachers (McCrory, Ferrini-Mundy, Floden, Reckase, & Senk, 2009). The videos of teaching that KAT researchers analyzed are publically available videos\(^\text{16}\) and not ones they created for their own research. Because KAT researchers did not create the videos of teaching for their own research, interviews with

\(^{16}\) Videocases for Mathematics Professional Development (Seago, Mumme, & Branca, 2004), the Annenberg/CPB Teaching Math Video Library (WGBH Boston, 1996), and the Third International Math and Science Study (Hiebert et al., 2003) public release video.
the teachers in the videos were not part of this data. KAT researchers examined video lessons for the mathematics that teachers taught and for how teachers used mathematics “to do something” (McCrory et al., 2012, p. 594). For example, they observed a teacher making connections between a table of values and an algebraic expression. KAT researchers classified this teacher’s actions as bridging. KAT researchers also interviewed algebra teachers intended to get insights regarding teachers’ understandings of “particular problems students have in learning algebra” (McCrory et al., 2012, p. 592). For example, one of the interview tasks they asked teachers to do was to examine a set of nine equations, discuss similarities and differences among the equations, and explain the order in which the equations might occur in their respective classes. In one teacher’s discussion of the equation, $9x - 3y = 12$, KAT researchers noted that this teacher’s use of solve was influenced by students’ understandings. According to this teacher, her students understand solve to mean “they’re looking for one answer that is $x$ is equal to something or $y$ is equal to something” (p. 593). Thus, when converting equations such as $9x - 3y = 12$ to slope-intercept form, she stated, “I made it a point, I never say solve for $y$. I always say put the problem in slope-intercept form. When I say solve, I mean that’s for one answer” (p. 593). KAT researchers characterized this instance of the teacher’s use of knowledge as “bridging (across meanings of language)” (p. 593) and “decompressing (what a word means in a given context)” (p. 593).

17 A. $y - 3x = 4$.  B. $f(x) = 3x - 4$.  C. $9x - 3y = 12$.  D. $3x - 4 = 12$.

E. $3(x - 2) + 2 = 4x - 4(x - 3)$.  F. $4(x - 1) - x = 3x - 4$.  G. $g(x, y) = 3x - y$.

H. $3x^2 + x - 2 = 0$.  I. $y = 3x^2 + x - 2$. (McCrory et al. 2012, p. 593)
McCrory et al. (2012) do not define what they mean by trimming, decompressing, and bridging. Instead they state that each of their constructs parallels teaching practices identified in the work of other researchers, and elaborate each of the respective descriptions/definitions “to fit the particular case of secondary school algebra instruction” (p. 601) and offer examples of what they see as decompressing, trimming, and bridging, respectively. McCrory et al. (2012) state

- “our decompressing parallels Ball and Bass’ (2000b), as well as Cohen’s (2004) category of unpacking, but shifts the content focus from arithmetic to algebra” (p. 601);
- “our trimming resembles Bruner’s (1960) idea of teaching ‘intellectually honest’ (p. 33) simpler versions of advanced concepts that would be revisited in later study; we identify particular concepts in secondary school algebra that are revisited in linear algebra, abstract algebra, and calculus. Trimming is also related to an idea from the PISA framework (Organization for Economic and Cooperative Development, 2010). The framework identifies ‘trimming away the reality’ (p. 86) as a feature of the problem-solving process that entails getting to the mathematical essence of a real world problem” (p. 601); and
- “our bridging is a version of the curricular ideas of horizontal and vertical curriculum alignment presented by Ralph Tyler (1949)” (p. 601).

In order to communicate what is meant by decompressing, trimming, and bridging in this study, it is important to specify definitions. KAT researchers’ descriptions of these three mathematical uses of knowledge and the corresponding examples serve as the bases upon which definitions of decompressing, trimming, and bridging used for this
study are inferred. The KAT researchers’ description of decompressing in conjunction with examples such as attending to different meanings of similar symbols (e.g., \(ab\) as single object and \(ab\) as “\(a\) times \(b\)” ) lead to a definition of decompressing used in this study: Decompressing means breaking a TMId into its constituent ideas to reveal mathematical meaning. The definition of trimming used in this study, choosing and emphasizing some set of constituent ideas of a TMId that reflects and contains the core ideas (i.e., the most important ideas) of the subject matter being trimmed, is primarily based upon the KAT researchers’ statement that trimming involves scaling down or up, intentionally omitting or adding detail, or modifying levels of rigor, and it also involves recognizing mathematics that has been trimmed too much, namely, instances in which important details or special cases are missing (p. 604).

The definition of bridging used in this study (i.e., making connections between a TMId and other mathematical ideas) is primarily based upon KAT researchers’ statement that the category of bridging includes algebra teachers’ efforts to “connect and link mathematics across topics, courses, concepts, and goals, including connecting the ideas of school algebra to those of abstract algebra and real analysis, and linking one area of school mathematics to another” (p. 606–607).

From a practical perspective, there seems to exist private as well as public aspects of teachers’ mathematical uses of knowledge in teaching. In the few seconds teachers have in deciding how to respond to an unexpected student question or deciding whether to take advantage of a teachable moment, teachers can engage in decompressing, trimming, and/or bridging and choose not to share any of their thinking or share only
selected aspects of their thinking. Consider, for example, a teacher’s explanation of a TMId. An explanation that is comprehensible to students, presumably an audience that has less knowledge/understanding of the idea being explained, depends upon a teacher’s capacity to take apart his or her understandings of the TMId (Ball & Bass, 2000). In addition, the explanation depends upon decisions, for example, about what components from the teacher’s unpacked (i.e., decompressed) knowledge of the TMId to highlight for students (i.e., trimming), and what connections, if any, to make (i.e., bridging). This study examines decompressing, trimming, and bridging that teachers make public as they teach their students, when discussing how they make TMIds accessible and comprehensible to students during interviews, and in their classroom and/or lesson artifacts (e.g., worksheets, tests, quizzes, PowerPoint presentations).

Using decompressing, trimming, and bridging as lenses through which to understand how teachers make TMIds accessible and comprehensible to students goes beyond the fact that these three practices appear in the KAT conceptual framework. These processes are important in helping students learn new mathematical ideas. Consider for example, a teacher who has decided to teach the vertical line test (a common test used in secondary school for determining whether a graph is that of a function) to his or her Algebra 2 students. The products of this teacher’s decompressing of “vertical line test” (VLT) could include some or all of the following:

- knowing what the VLT is (i.e., if a vertical line intersects a graph of points \((x,y)\) on the Cartesian rectangular coordinate plane in more than one point anywhere on the graph, then the graph is not the graph of \(y\) as a function of \(x\)).
• knowing that the underlying basis for the vertical line test is that a function is a special kind of mapping, in which each element of one set (i.e., the domain) is mapped to exactly one element of another set (i.e., the range).

• knowing that a function can be defined as a mapping of a set of x values to a set of y values such that each x value is mapped to exactly one y value. In such a mapping, y is said to be a function of x.

• knowing that the VLT is used to determine whether a graph drawn on the Cartesian xy-coordinate plane is or is not a representation of y as a function of x.

• knowing that it is because of a special kind of mapping that defines y as a function of x that a vertical line cannot intersect the graph of the ordered pairs that define the mapping in two or more points.

• knowing that not all functions can be graphed and so the vertical line test is not directly applicable to all functions.

Students who have not previously encountered the VLT would be hard-pressed to decompress this TMId, without the assistance of outside resources.

When trimming a TMId, teachers decide what products of decompressing are more important to emphasize and act on those decisions by making those products the focus of class discussion and/or attention. It is important to study teachers’ trimming because a teacher’s trimming has an impact on what students have opportunities to learn and what the students learn. When bridging, teachers make and/or help students make connections between and/or among ideas. Bridging is important to supporting conceptual understanding, which, according to Kilpatrick et al. (2001), “refers to an integrated and
functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods” (p. 118).

If, for example, a teacher trims the vertical line test so that it is the only means students have to determine whether a relationship between two sets is or is not a function, then it is unlikely that this teacher’s students would be able to determine whether a relation between club member and dues owed described in Figure 2-2 is or is not a function. The relationship described in Figure 2-2 is not one that can be graphed on the rectangular Cartesian coordinate plane. In other words, it is possible for a student to have an understanding of function that is restricted to graphs on the rectangular Cartesian coordinate plane.

If we let $x =$ club member’s name and $y =$ amount owed, is $y$ a function of $x$?

<table>
<thead>
<tr>
<th>Name</th>
<th>Owed</th>
<th>Name</th>
<th>Owed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sue</td>
<td>$17</td>
<td>Iris</td>
<td>6</td>
</tr>
<tr>
<td>John</td>
<td>6</td>
<td>Eve</td>
<td>12</td>
</tr>
<tr>
<td>Sam</td>
<td>27</td>
<td>Henry</td>
<td>14</td>
</tr>
<tr>
<td>Ellen</td>
<td>0</td>
<td>Louis</td>
<td>6</td>
</tr>
</tbody>
</table>

*Figure 2-2.* An example of an exercise asking students whether the relation represented in a table is a function. Source: Dubinsky and Harel (1992).

An important part of KAT researchers’ characterization of decompressing, trimming, and bridging is that these are uses of *mathematical* knowledge in teaching (McCrorry et al., 2012) and that as uses of mathematical knowledge in teaching, they are mathematical practices (McCrorry, Ferrini-Mundy, Floden, Reckase, & Senk, 2009). Data
from my pilot studies, my teaching experiences, and my work with other teachers indicate that teachers make use of nonmathematical knowledge as well as mathematical knowledge when making TMIIs accessible and comprehensible to students. For purposes of clarity in communication, in this study mathematical is defined using the following definitions from WolframAlpha “of or pertaining to the nature of mathematics” and “characterized by the exactness or precision of mathematics” (http://www.wolframalpha.com/input/?i=mathematical&a=*C.mathematical-*Word-).

The definition of nonmathematical is a negation of these definitions.

Another lens is needed. Decompressing, trimming, and bridging do not account for teaching strategies/approaches such as

- teaching the quadratic formula to the tune of “Pop Goes the Weasel” (http://www.regentsprep.org/Regents/math/algtrig/ATE3/quadsongs.htm);
- telling students that they will go to math prison if they divide by zero (Emily, PS Interview—pilot study); and
- factoring quadratic trinomials using the “tic-tac-toe method” (http://www.youtube.com/watch?v=HNZruRmmjJM), a procedure (refer to Figure 2-3. A more detailed explanation is provided in Appendix D) that uses a graphics organizer in the shape of a tic-tac-toe game board to help students organize the computations involved in factoring quadratic trinomials.
None of these examples solely involves revealing mathematical meaning, or mathematically modifying the complexity of mathematical ideas, or connecting mathematical ideas with other mathematical ideas. Teaching the quadratic formula to the tune of “Pop Goes the Weasel” is a way of making the quadratic formula easier for some students to remember and it brings attention to the formula. However, this is not an example of the use of mathematical knowledge. Telling students that “if they divide by zero, they will go to math prison” brings attention to the fact that dividing by zero is problematic; however, the means by which this attention is attracted uses knowledge beyond mathematics content knowledge. A critical feature the tic-tac-toe method is a graphics organizer that is in the shape of a tic-tac-toe game board (i.e., “#”). The graphic organizer provides the factored places to record specific information. Although the method can, if correctly followed, lead to transforming a factorable quadratic trinomial into its completely factored form, its reliance on the graphic organizer makes it a teaching practice that makes use of nonmathematical elements.
Perspective: Teaching is a way of communicating mathematics. The goal of this study is to understand what underlies what teachers are doing. Teaching as an application or use of mathematical content knowledge (i.e., school knowledge, advanced knowledge, and teaching knowledge) is insufficient to understanding what teachers are doing. This perspective is based upon the assumption that mathematics is the means by which TMIIds are made accessible and comprehensible to students. What the three examples of teaching provided previously that are not explained by decompressing, trimming, and bridging seem to have in common is that they communicate something about a TMIId and can be seen to be efforts to solve a teaching dilemma (e.g., “how can I help students remember ___?” and “how can I help students to arrive at a correct answer?”). The tune “Pop Goes the Weasel” operates as a mnemonic to help students remember the quadratic formula. Telling students that they will go to math prison if they divide by zero communicates that division by zero is something students are not supposed to do. Teaching students the tic-tac-toe method of factoring quadratic trinomials can be seen as a way of helping students produce a correctly factored quadratic. The graphic organizer in the tic-tac-toe method can be seen as a way to help students remember a procedure. These examples of teaching practices and others that emerged from my pilot studies that are not explained by decompressing, trimming, and bridging lead to the consideration of teaching from the perspective of communication of aspect(s) of TMIIds. This consideration resulted in examination of research literature in fields (e.g., media and discourse) for which communication is an object of study in search of research-based ways of describing and explaining the nonmathematical and not-
strictly-mathematical approaches teachers use to make TMIds accessible and comprehensible.

**Pedagogical framing.** Frequently appearing in literature in the fields where communication is a research focus (e.g., Tannen, 1993b; Bateson, 1972/2006; Gitlin, 1982; van de Sande & Greeno, 2010) are the notions of *frame* and *framing*. Gitlin (1982), a researcher in the area of media communications, for example, describes frames as “principles of selection, emphasis and presentation composed of little tacit theories about what exists, what happens and what matters.” (p. 6). Bateson (1972/2006), credited with the introduction of the terms *frame* and *framing* (Tannen, 1993b) defines frame to be a psychological concept that delineates a class or set of messages or meaningful actions. Of the descriptions and definitions of frame and framing encountered, Bateson’s and Gitlin’s notions, when combined, explain what decompressing, trimming, and bridging could not. Seen through the lens of Bateson’s and Gitlin’s definitions of frame and framing.

- Teaching students the quadratic formula to the tune of Pop Goes the Weasel is an example of the use of presentation (the tune) that communicates that the quadratic formula is important to know in part because of the novelty of singing a song in an algebra class.

- Telling students that division by zero is a math sin is also an example of the use of a presentation (a characterization of a TMId as an unforgivable transgression) that communicates that division by zero is problematic.

- Teaching students how to factor quadratic trinomials using the tic-tac-toe method is an example of a selection of a method, among many factoring
methods, that provides students with a graphic organizer that is intended to assist students in producing a factored form of a given quadratic.

In addition, Bateson’s definition and Gitlin’s description pertain to teachers’ efforts to make TMIds accessible and comprehensible to students that are explained by decompressing, trimming, and bridging. Recall that decompressing means breaking a TMId into its constituent ideas to reveal mathematical meaning. Breaking a TMId into its constituent ideas, depending upon how this action is done, can be a presentation. In revealing meaning, a teacher communicates messages about the TMId. When a teacher engages in trimming she makes selections and emphasizes some aspect or set of aspects of a TMId. In bridging, a teacher communicates a connection between a TMId and one or more mathematical ideas. In other words, decompressing, trimming, and bridging are forms of framing.

An important quality of Bateson’s and Gitlin’s frame and framing is that they are not restricted to a particular language or field, or to a mode of communication (e.g., symbolic, oral, written, verbal, pictorial). Mathematics teachers’ framings, however, differ from those of social scientists in that the ideas being framed are those of school mathematics, the context is classroom practice, and the communication emphasis is that of accessibility and comprehensibility of TMIds. For this reason I infer a pedagogical form of framing from Bateson’s definition and Gitlin’s description. In the context of teaching school algebra, pedagogical framing is communicating and/or presenting a set of messages about one or more TMIds. When a teacher engages in pedagogical framing, he or she adjusts, organizes, stages and/or situates one or more TMIds to support learning and to promote development of learning of students. From this point onward, however, I
will use the term, *framing* to mean *pedagogical framing*. Given that my study is about teaching, contextually speaking, the framings are all pedagogical in nature.

**A conceptual framework for framing.** Emerging from this literature review, decompressing, trimming, bridging and framing emerged as very promising lenses through which to view how teachers make TMIds accessible and comprehensible to students. The relationships among these constructs are represented in Figure 2-4. Decompressing, trimming, and bridging are represented as distinct constructs because they emerged from KAT researchers’ descriptions, and each of these mathematical uses of knowledge was discussed separately.

![Figure 2-4. Conceptual framework for framing (initial).](image)

**Insights from research literature on framing: Frames of reference.** Besides setting the foundation for the conceptual framework guiding this study, Bateson’s frame and framing offer an explanation of how, for example, person A punching person B on his shoulder can be witnessed by two observers and interpreted by one as an aggressive act and by the other as a playful act. Their frames of reference are different. According
to Tannen (1993a) no communicative move can be understood without a lens (i.e., a frame) through which to view the action. In the context of a mathematics classroom, I infer from Tannen (1993a) that a student needs a lens (i.e., a frame) through which to view his or her teacher’s messages about a TMId in order to understand the mathematical idea.

Consider, for example, introducing students to the following symbolic statement:

$$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}.$$  

It is possible for students to read this mathematical statement, interpret it, and understand what it means. For those students who struggle to understand what it means, a teacher might use his or her students’ familiarity with student populations within a 4-year high school as a means by which to give meaning to this statement. A teacher could say that $$\bigcup_{i=1}^{n} A_i$$ is like the student body population attending W High School, a 4-year high school. In one scenario, the teacher could identify three sets that make up the student body population (i.e., $$\bigcup_{i=1}^{3} A_i$$): 9th and 10th grade students (i.e., $$A_1$$), 11th and 12th grade students (i.e., $$A_2$$), and students on a W High School varsity sports team (i.e., $$A_3$$). So, 

$$\{x \in U \mid x \in A_i \text{ for at least one } i = 1, 2, 3, 4\}$$ means that if a student (i.e., $$x$$) is a member (i.e., $$\in$$) the student body at W High School (i.e., $$U$$), then the student is a member of at least one of these three groups. In this scenario, according to Tannen (1993a) the teacher’s framing of a TMId provides students with a frame of reference (i.e., a lens) with which to view, interpret, and/or understand a TMId. For other TMIds, a teacher’s
framing may be a means to help students change/modify their respective frames of reference to frames of reference that are needed for students to understand the TMld in ways that are consistent with the ways in which the teacher wants the students to understand the TMld. For example, many students say that the expression, \(-x\), represents a negative number. A teacher might ask students to consider the following two situations:

1. If \(x\) is a positive number, \(-x\) can be rewritten as “\(-\) (positive number),” which represents a negative number.
2. If \(x\) is a negative number, \(-x\) can be rewritten as “\(-\) (negative number),” which does not represent a negative number.

This notion of a teacher’s framing as providing a lens to students or as modifying students’ lenses is a research perspective; one that is not necessarily held by the teacher. Although it is possible for teachers to view their teaching from the perspective of providing students with a lens or modifying a lens held by students, data from my pilot studies and this study neither supported nor refuted such a possibility. When participants were asked what their goals and objectives for the upcoming lessons were, participants always talked about what they were going to do (e.g., derive the quadratic formula for students), or what they were going to have students do (e.g., graph quadratic functions of the form, \(y = ax^2\) for selected values of \(a\) and note relationships between values of \(a\) and the shape of the corresponding parabolas). Their discussions revolved around what was going to take place during the lesson with respect to subject matter. There was no
discussion of providing students with a frame of reference or modifying students’ frames of reference.

**Insights from research literature on framing: Framing as “media packages.”**

Gamson and Modigliani (1989) discuss *media packages*, “a set of interpretive packages that give meaning to an issue” (p. 3) and identify five specific methods, which they call *framing devices*, by which members of the media help their audience members think about or view an issue. The framing devices are as follows: metaphors, which connect two or more ideas; and exemplars, catchphrases, depictions, and visual images (e.g., political cartoons), which are methods that bring attention to an idea. Gamson and Modigliani’s work is informative because their framing devices identified ways in which teachers could communicate or present a set of messages about one or more TMIds. This information was helpful during the data collection because it increased my awareness of what teachers could do. As a result, part of my data collection protocol included an examination of teachers’ bulletin boards, posters, displays, and anything else in the room visible to students. As a result, during one of my pilot studies, I noted that Emily had an enlargement of a cartoon (see Figure 2-5) posted on her front bulletin board. Seeing this posted cartoon eventually led—during that day’s postobservation interview—to a discussion of division by zero, why it is problematic, and how she makes the concept accessible to her students.

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18 Emily is a pseudonym for a teacher who participated in one of my pilot studies.
Insights from research literature about framing: Conceptual framing. In their study of interactions between teacher and student(s), and among students in which one party offers an explanation to another, Greeno and van de Sande (2007, 2010) note episodes in which “there was an initial lack of alignment in understanding, followed by alignment that was sufficient for the purposes of the participants’ activities” (2010, p. 72). They hypothesize that the initial lack of alignment is due to differences between the frames of reference held by the person(s) offering the explanation and the recipient(s) of the explanation. Unlike the work of Gamson and Modigliani (1989) which identifies what strategies/approaches teachers could use to make TMIIs accessible and

for an explanation to be communicated successfully, the person giving the explanation and the person receiving it need to have framings that are aligned, at least to the extent that the one who needs to integrate the information in a new understanding has the resources needed to do that (p.69).

Informed by Greeno and van de Sande (2007, 2010), as part of the data analysis process, I looked for evidence of efforts on the part of a teacher to frame a TMId in such a way that the teacher’s framing was somehow made to align with his or her students’ frames of reference.

The existence of such an effort was found in comments made by a teacher participating in a study reported in McCrory et al. (2012). As part of a discussion of $9x - 3y = 12$, this teacher revealed that when talking to students, she made a point never to say, “solve for $y$.” She stated that she always says, “put the problem in slope-intercept form” (p. 593) because her students understand solve to mean that they are to look “for one answer that is $x$ is equal to something or $y$ is equal to something” (p. 593). McCrory et al. concluded that this teacher modified her use of the term solve to accommodate her students’ understandings. McCrory et al.’s conclusions are consistent with Greeno and van de Sande’s (2010) discussions of conceptual framing. With respect to the term solve this teacher aligned her framing (i.e., her use of the term solve) to be closer to that of her students’ framing (i.e., their understandings of what solve tells them to do). In this particular instance the teacher seems to be cognizant of her students’ understandings of
mathematical language, and in an effort to avoid confusion on the part of her students, she seems to have made a point of using specific terms only in specific situations.

**Ideas and Insights From Research on Teaching**

The focus of this study is on understanding what teachers do to make TMIds accessible and comprehensible to students. A search of literature was conducted to gain insights about when and how teachers might attempt or are likely to attempt to make TMIds accessible and comprehensible. Because this study does not focus on specific teaching strategies, what follows are discussions that reflect a collection teaching practices, each of which informed this study in some way.

A natural place to begin gaining such insights is by examining what research says about teachers’ mathematical talk. In fact, “one of the most powerful pedagogical moves a teacher can make is one that supports making detail explicit in mathematical talk, in both explanations given and questions asked” (Franke, Kazemi, & Battey, 2007, p. 232). Following the discussions of research addressing teachers’ mathematical talk are discussions of research addressing teaching strategies.

**Teachers’ mathematical talk: Explanations and explaining.** Perhaps the simplest responses to the question, “How to teachers attempt to make TMIDs accessible and comprehensible to students?” are teachers give explanations for TMIDs, and teachers explain TMIDs. Although one would be hard pressed to disagree with such statements, from a research perspective, a means of identifying when a teacher is or is not explaining and whether something a teacher says is or is not an explanation is needed for the analysis process. Achinstein (1977) provides such a means. *Explaining*, according to
Achinstein, is an illocutionary act that “is typically performed by uttering (or writing) words in certain contexts with certain intentions” (p. 1) and is performed under certain conditions. For the purposes of my study, Achinstein’s conditions are most informative. He states, \( S \) [i.e., some person] explains \( q \) by uttering \( u \) only if

1. \( S \) utters \( u \) with the intention that his utterance of \( u \) renders \( q \) understandable,
2. \( S \) believes that \( u \) expresses a correct answer to \( Q \) (\( Q \) is a question which “presupposes” \( q \)), and
3. \( S \) utters \( u \) with the intention that his utterance of \( u \) renders \( q \) understandable solely by producing the recognition that \( u \) expresses a correct answer to \( Q \) (Achinstein, 1977, p. 2)

\( S \), in the context of my study is the teacher, \( q \) is the TMId, and \( u \) is the explanation.

Besides indicating that an explanation is an utterance with a purpose (i.e., to make \( q \) understandable), these conditions also afford a less abstract definition of explanation.

Expressed in the context of my study, an explanation is an answer to a question about a TMId that makes the TMId understandable to students. The question (\( Q \)) being asked, according to Achinstein (1977), begins with any of the following interrogative pronouns: who, what, when, where, and how. As informed by my pilot studies, questions can be asked explicitly. Sometimes the question or questions is inferred by the teacher from assessments of student work (e.g., written work, oral discussions with other students) or comments to the teacher.

**Teachers’ mathematical talk: Questions and questioning.** Research literature (Franke et al., 2007; Wood, 1995, 1998) describes questioning patterns that can take
place in classrooms: the initiation-response-evaluation (IRE) pattern, a *funneling pattern*, and a *focusing pattern*.

**IRE and funneling patterns.** In the IRE pattern, a teacher begins the pattern with a teacher-initiated question. A student (or students) responds to the question. The teacher evaluates the response. In the funneling pattern, the interaction begins with a teacher-initiated question but differs in how the teacher responds to an incorrect student response. The teacher asks the student to answer a sequence of explicit questions. The interaction ends when the student provides a correct answer. Using the language of Stein et al. (1996), the questions make little cognitive demand on students but lead to students stating the correct answer. What follows is an example of a teacher engaging in a funneling questioning pattern with Jim, who was asked to give an answer to $9 + 7$, which can be seen as part of lesson on addition of single digit numbers:

Jim: 14

Teacher: OK. 7 plus 7 equals 14, 8 plus 7 is just adding one more to 14, which makes ___? (voice slightly raising)

Jim: 15

Teacher: And 9 is one more than 8. So 15 plus one more is ___?

Jim: 16 (Wood, 1995, p. 176–177)

Jim can answer all of his teacher’s questions without understanding how to add $9 + 7$ (i.e., the TMId). Jim correctly adds 1 to 14, which is his incorrect original response; and then correctly adds 1 to 15 and arrives at the answer to $9 + 7$. However, it is not necessarily clear to Jim how the sequence of addition problems to which he correctly responded is related to how to compute $9 + 7$. 
In the context of my study, this example and research literature on funneling patterns suggests a need to examine the relationship between a teacher’s questions/questioning and the TMId in the analysis process. There may be a mismatch or a disconnect between what mathematics is being targeted for students to what is learn and the questions are asked to answer.

**Focusing pattern.** In a focusing pattern, the interaction is situated in classrooms in which instruction is designed so that students are expected to participate actively in classroom dialogue with the teacher and with other students. Teachers using this pattern of questioning ask questions that are intended to direct students’ attentions to important features of what students are working on (e.g., an equation to solve, a word problem), but then step out of the conversation to allow for students to make sense of the solution for themselves (T. Wood, 1995).

A line of questioning following a focusing pattern is more complex than the IRE and the funneling lines of questioning. Focusing questions are directed at what the teacher sees as important features of the TMId that students need to address in order to complete a given task. This characteristic of focusing questions suggests that focusing questions are responses to some combination of students’ written work, discussion, or questions about a given task. Capturing all facets of the interaction (i.e., the task, the TMIds being addressed in the task, the question[s], the impetus for the question[s], and referents for the question[s]) could be a challenge due to issues associated with proximity, especially if instruction is being conducted in a setting other than the large group. For example, students’ the written work could be out of camera view or all questions in the interaction may not be picked up by audio recording devices. For this
reason, recognizing and making note of a teacher’s use of focusing questions during the observation is important so that the teacher can be asked to talk about what features of the TMId were being made accessible and his or her reasoning behind the questions.

**Teaching strategies: Scaffolding.** Research examining various forms of scaffolding afforded several insights into what teachers do to make TMIds accessible and comprehensible. D. Wood, Bruner, and Ross (1976) is cited by researchers (e.g., Bliss, Askew, & Macae, 1996; Pressley, Hogan, Wharton-McDonald, & Mistretta, 1996, Sherin, Reiser, & Edelson, 2004) as having introduced the notion of scaffolding as a way to support student learning. Scaffolding, according to Wood et al. (1976) who examined the interactions between a tutor and a learner, are interventions by the tutor to help a learner’s efforts to achieve a goal (e.g., solve a problem, complete a task), which would be beyond the learner’s unassisted efforts. Because the learner would not have been able to achieve the goal without the interventions, the tutor’s interventions (collectively described as scaffolding) are considered ways in which the tutor made the goal accessible to the learner and are pertinent to this study of teaching in school algebra classrooms.

Although it can be argued that teaching a student in a one-to-one situation (i.e., tutoring) is different from teaching a classroom full of students, applicability of the notion of scaffolding to classroom instruction is seen by other researchers (e.g., Williams & Baxter, 1996; Nathan & Knuth, 2003; and Speer & Wagner, 2009). Subsequent to Wood et al.’s (1976) use of the notion of scaffolding to describe tutor interventions, other forms of scaffolding (e.g., instructional, social, and analytic) have been used to describe what takes place in the classroom. *Instructional scaffolding* (Pressley, Hogan, Wharton-McDonald & Mistretta, 1996), for example, is “providing help to students on an as-
needed basis, enough that the child can make progress but not more than that, with instructional support reduced as student competence increases” (p. 138). It was used by Pressley et al as part of their study of scaffolding as a form of classroom instruction and the challenges to using scaffolding effectively.

With respect to my study, two of the six tasks in Wood et al.’s (1976) framework known as the “Scaffolding” Process afforded the most insights. The “Scaffolding” Process consists of recruitment, reduction in degrees of freedom, direction maintenance, marking critical features, frustration control, and demonstration. Of the six tasks in the “scaffolding” process, three tasks (i.e., recruitment, direction maintenance, and frustration control) pertain to behavior management and are therefore not pertinent to my study. The remaining three tasks (i.e., reduction of degrees of freedom, marking critical features, and demonstration) reflect ways in which the tutor (i.e., Ross) made the problem-solving task (i.e., building a pyramid out of a given set of wooden blocks) accessible to the tutees in the Wood study. Wood et al. describe reduction of degrees of freedom as reducing the “number of constituent acts required to reach solution” (p. 98), marking critical features as “mark[ing] or accentuat[ing] certain features of the task that are relevant” (p. 98) and demonstration as showing some part or all of a procedure. All three scaffolding tasks are ways of bringing tutees’ attention to what needs to be done to build the pyramid and are consistent with trimming. With respect to my study, these scaffolding tasks suggest a need for me to examine my data for teachers’ use of both direct methods of making TMIds accessible (e.g., identifying critical features of TMIds, and demonstrating) and indirect methods of making TMIds accessible (e.g., removing distractions from TMIds).
**Analytic scaffolding.** Baxter and Williams (2010) described social and analytic scaffolding in mathematics classrooms as part of their study of teaching practices of two teachers who teach in reform-oriented classrooms. Insights gained about social scaffolding are not included in this review because social scaffolding addresses the establishment of classroom social norms (e.g., helping students learn to work together) which are important to helping students learn TMIds but are background support. Analytic scaffolding is discussed here because it addresses directly the scaffolding of mathematical ideas. In fact, Williams and Baxter’s (1996) original description of analytic scaffolding is “the scaffolding of mathematical ideas for students” (p. 24). In a later study, Williams and Baxter offered more details about analytic scaffolding.

Analytic scaffolding refers to the support offered to students by materials, teachers, or one another, in building mathematical understanding. Analytic scaffolding might include physical manipulatives, models, metaphors, representations, explanations, or justifications that allow students to better understand mathematical tasks and solutions (Baxter & Williams, 2010, p. 11). Their more detailed description of analytic scaffolding is helpful in that it identifies specific means by which TMIds can be made accessible, and serves as a reminder for my study to include the examination of data for teacher’s use of metaphors, physical manipulatives, models, representations, explanations or justifications and how they use them.

The main impact of the review of literature on scaffolding on my study lies in two areas. It resulted in a broader view of what is and can be identified as support of student learning. Wood et al. (1976) discusses scaffolding as controlling factors. Baxter and
Williams (2010) discuss scaffolding as support involving the use of materials (e.g., concrete objects/manipulatives), and methods (e.g., use of metaphors, representations, and explanations). Other researchers describe scaffolding as the support that a teacher provides, gradually lessens, and eventually removes as students develop their respective understandings. Given that framing is an aspect of what teachers do when scaffolding student learning of TMIds, this broader view primarily impacted data analysis in what is identified as an instance of framing. Wood et al.’s (1976) discussion of controlling factors led to the identification of, for example, a teacher’s hierarchical organization of factoring procedures and use of stationery (discussed in chapter 4) as examples of framing.

**Teaching strategies: Teachers’ use of problems and problem solving.**

Grugnetti and Jaquet’s (1996) discussion of problem solving in their chapter on “current trends affecting the teaching of mathematics at the secondary level” (p. 616) provided insights on conditions under which students solving problems might have the potential of making school mathematics accessible and comprehensible. They state that “problem solving places the student in the role of actor in the construction of his/her knowledge” (p. 616) and define *problem* as follows:

> a new activity which is meaningful to the students and which must be sufficiently close to the current knowledge to be assimilated and yet must be sufficiently different in order to force them to transform their methods of thinking and working (p. 616).

Grugnetti and Jaquet’s argument suggests that lessons structured around students solving problems have the potential of making mathematics accessible and comprehensible, if the
problems on which students work reflect properties identified in their definition of problem. The problems need to be accessible so that students can engage in the solution process but be challenging enough for students to be forced to modify how they think and work. Their argument suggests a need to examine my data for participants’ thinking with respect to the problems they choose/design for students to work on.


1. the products students are to formulate, such as an original essay or answers to a set of test questions;

2. the operations that are to be used to generate the product, such as memorizing a list of words or classifying examples of a concept; and

3. the “givens” or resources available to student while they are generating a product, such as a model of a finished essay supplied by the teacher or a fellow student (p. 161)

In other words, academic tasks are defined by what students are asked to produce (e.g., answers, solutions, explanations) and the paths students are allowed to take in order to address what they are being asked to produce. Examining the tasks teachers ask students to do is important because of the influence tasks have in directing students’ attention to particular aspect of TMIs and how to process information. A worksheet of 20 factoring exercises, all of which are of the form, \( x^2 + bx + \left(\frac{b}{2}\right)^2 \) (\( b \) is even and \( b \in \mathbb{Z} \)), asks students to attend to different aspects of factoring from those addressed by a worksheet of 20
factoring exercises, of the form, $x^2 + bx + c$ ($b, c \in \mathbb{Z}$) for which $c$ may or may not be equal to $\left(\frac{b^2}{2}\right)$. The former worksheet reflects an emphasis on a single method. The latter worksheet reflects an emphasis on being able to factor a variety of quadratics and, if quadratics that are not factorable over the set of integers are included in the worksheet, being able to determine when a quadratic is prime over the integers. Both worksheets may suggest the importance of memorization and/or automaticity.

**Teaching strategies: Using mathematical tasks.** Stein et al. (1996) studied teachers’ implementations of what they call *mathematical tasks*. Mathematical tasks are similar to Doyle’s academic tasks in that mathematical tasks direct students’ attentions to what students are expected to produce, how they are to produce the product(s), and what tools/resources with which to produce the product(s); however, they differ “in terms of duration or length of the task” (p. 460). Stein et al. (1996) define mathematical task as a classroom activity, the purpose of which is to focus students’ attention on a particular mathematical idea. An activity is not classified as a different or new task unless the underlying mathematical idea toward which the activity is oriented changes. (p. 460)

From Stein et al.’s (1996) examination of set-up and of implementation of mathematical tasks, task features are as follows:

- the existence of multiple-solution strategies;
- the extent to which the task lends itself to multiple representations; and
- the extent to which the task demands explanations and/or justifications from students. (p. 461)
In addition, we learn about cognitive demands of a task (i.e., “the kind of thinking processes entailed in solving the task” (p. 461) as identified by the teacher before implementation begins and what actually takes place as students engage in the task. Stein et al., noted that a task that begins as cognitively demanding may, during the course of implementation, maintain its level of demand or it may decline in demand. In the context of my study, Stein et al.’s (1996) work suggests the need for me to identify the features of the task that teachers ask students to do and to attend to what is actually demanded of students over the course of the lesson.

**Teaching strategy: Using examples.** Examples are said to play a central role in mathematics and the learning of mathematics (Zavlasky, 2010). Insights as to how examples are used in teaching are provided by Michener (1978), who identifies four different types of examples: start-up examples, reference examples, model examples and counterexamples. Start-up examples are used to “motivate basic definitions and results, and setting up useful intuitions” (p. 5). Reference examples “provide a common point of contact through which many results and concepts are linked together” (p. 6). Model examples “suggest and summarize expectations and default assumptions about results and concepts” (p. 7). Counterexamples, which are used to show a statement is false, “sharpen distinctions between concepts” (p. 7). Michener’s descriptions of these four types of examples are consistent with KAT researchers’ discussions of trimming, and bridging. All four types of examples seem to be used to emphasize or bring attention to whatever these examples are examples of. Reference examples, which are said to provide a “common point” of contact through which to link results and concepts, provide opportunities for teachers to engage in bridging.
More specific information as to how examples can be used is provided by Zodik and Zavlasky (2008), who examine teachers’ choice of examples. They identify six considerations that teachers make when choosing and creating examples. They are as follows:

1. Start with a familiar case.
2. Attend to students’ errors.
3. Draw attention to relevant feature.
4. Convey generality by “random choice.”
5. Include uncommon cases.
6. Keep unnecessary work to a minimum.

As part of a teacher’s thinking process for selecting and creating examples, it seems that these considerations are consistent with teachers deciding what to emphasize or what not to emphasize, which are qualities of trimming. Zodik and Zavlasky’s (2008) work suggests that teachers’ selection/generation of examples can be thoughtful and deliberate.

Gaining insights about participants’ thinking behind their choice in examples might afford information about the reasoning that led to and/or supports the mathematical and/or nonmathematical approaches to making TMI ds accessible and comprehensible to students. This inference led to a more thoughtful effort to probe participants’ thinking for their reasoning behind the examples I observed being used in class during the observation-related interviews.

**What Does Research Say About Areas of Student Struggle in School Algebra?**

From a mathematical perspective, more students taking algebra courses in order to graduate from high school means that more students are expected to move from working
with numbers to working with symbolic representations of mathematical objects. It is a transition that many students find difficult and which leads to them claiming that, “Algebra is hard.” Much research exists supporting their claim.

The purpose of the search of literature addressing areas of student struggles in school algebra is to gain insights about what I might be seeing in classroom observations. I assumed that an increased awareness of where students struggle in school algebra would assist my efforts to collect more complete data (i.e., a more focused video recording of the teacher’s oral, written, and gestural work as he or she tried to make accessible those ideas identified by research as problematic to students). The literature search and review revealed five areas in which students struggle.

**Area of struggle #1: Renegotiating their understandings of the letters of the alphabet.** When students make the transition from arithmetic to algebra, students have to renegotiate their understandings of letters of the alphabet. Letters, when used in the context of mathematics, have new uses, have different meanings and are referenced using different terms (Kuchemann, 1978; Usiskin, 1988) in addition to the general use of letters comprising words in everyday language. In linear and quadratic equations, such as \(3x - 7 = 21\) and \(x^2 + 5x + 6 = 0\), respectively, the letter \(x\) is called an *unknown*. The \(x\) represents one or more numerical solutions to a given equation. In functions such as \(f(x) = \sin x\), \(x\) is called an *argument* or the *independent variable*. In school algebra, \(x\) represents an element of the real numbers or some subset of the real numbers. In a formula, such as the formula for the area of a rectangle, \(A = lw\), the letters represent quantities that have a specified relationship. In the case of \(A=lw\), there are three
quantities: \( l \) and \( w \), which are \textit{length} and \textit{width}, respectively, of a rectangle; and \( A \), which is area, the product of \( l \) and \( w \). In identities, such as \( a + b = b + a \), the letters are called \textit{variables}, in which each represents an infinite set of numerical values. Sometimes a letter represents a particular number (e.g., \( e \), Euler’s number; or \( i \), which equals \( \sqrt{-1} \)). Thus, from the perspective of students, these new uses and meanings for letters mark students’ beginning relationship with abstract concepts and with symbolic representations of mathematical ideas, and therefore, students’ relationships with higher levels of mathematics.

**Area of struggle #2: Dealing with multiple interpretations and meanings of symbolic representations.** Sfard and Linchevski (1994) address different perspectives of algebraic expressions: What students see for example, in the expression, \( 5x \), differs. Some students may have an understanding of algebraic expressions as computational processes (e.g., \( 5x \) is “five times \( x \)” ) but not as abstract objects (e.g., \( 5x \) represents numbers that are multiples of 5, if \( x \) is replaced by an integer). Sfard and Linchevski (1994) call the former understanding an \textit{operational perspective}, and the latter understanding a \textit{structural perspective}.

Sajka (2003) describes struggles with symbolism as they relate to the concept of function in his case study of Kasia, a secondary student. In discussing Kasia’s understandings, Sajka reveals her challenges, two of several, with the notion that, for example, \( f(x) = x^2 + 3 \) communicates two things at the same time: how to compute values of a function (i.e., start with a number, square it, and add 3) and the function, \( f \), a special mapping of a set of \( x \)-values to a set of \( y \)-values, \( f(x) \). The meaning of what is
being communicated depends upon context. The multiple aspects of symbolism can be sources of confusion to novice and/or struggling algebra students.

**Areas of struggle #3 and #4:** Solving linear equations (i.e., \( ax + b = cx + d \); for which \( a, b, c \) and \( d \) are constants) and systems of linear equations in two unknowns (i.e., \( ax + by = c \) and \( dx + ey = f \); for which \( a, b, c \) and \( d \) are constants).

With respect to solving equations, Filloy and Rojano (1984, 1989) and Filloy, Rojano, and Solares (2010) identify two places in school algebra curriculum at which students may struggle. Area of struggle #3 occurs when students encounter linear equations of the form, \( ax + b = cx + d \) (where \( a, b, c, \) and \( d \) are constants), and find that such equations are not readily solved using purely arithmetic means such as counting procedures or inverse operations (Filloy & Rojano, 1984, 1989). For example, in solving \( x + 6 = 15 \), a student can start with 6 and count up to 15, raising a finger with count, and noting 9 raised fingers when he or she finished counting. In solving \( 2x - 3 = 11 \), using inverse operations, a student will “add 3” (i.e., the inverse of subtracting 3); and then “divide by 2” (i.e., the inverse of multiplying by 2) to get an answer of 4. In equations of the form \( ax + b = cx + d \) (where \( a, b, c, \) and \( d \) are constants), students have to operate with unknown quantities. For such operations to make sense to students, researchers state that these equations need to be provided with some meaning (Filloy & Rojano, 1989), which in turn means that students need to change their conceptions of equation or the equality of numbers. Area of student struggle #4, according to Filloy et al. (2010), also occurs when students encounter systems of two linear equations in two unknowns, for example:

\[
ax + by = c \\
dx + ey = f
\]
(x and y are unknowns; and a, b, c, d, e, and f are known constants), and the students are required to work with equations in which one unknown is represented in terms of the other, such as:

\[ x = \frac{c-by}{a}, \quad a \neq 0. \]

Once again students have to perform operations using unknown quantities, and once again students need to change their conceptions of equation or the equality of numbers.

Both of these mathematical ideas, solving linear equations that have variables on both sides of the equal sign, and solving a system of linear equations using a substitution method, according to Filloy and Rojano (1984, 1989) and Filloy et al. (2010) require students to go beyond using arithmetic strategies to solve equations. Students need to build new meanings and new senses for arithmetic objects and operations (Filloy et al., 2010) as they learn the algebraic procedures involved in solving these equations.

**Insights From Large Scale Studies on Mathematics Classroom Practice**

A review of literature revealed two large-scale studies about teaching that examined teaching practices in U.S. secondary mathematics classrooms: Looking Inside the Classroom: A Study of K–12 Mathematics and Science Education in the United States ([Inside the Classroom Study]; Weiss, Pasley, Smith, Banilower, & Heck, 2003), and the 1998–2000 Third International Mathematics and Science Study Video Study (TIMSS 1999 Video Study; Hiebert et al., 2003). Weiss et al. (2003) state that “the Inside the Classroom study was designed to provide snapshots of what transpires inside the nation’s mathematics and science classrooms and the factors that shape that instruction” (p. 1). Hiebert et al. (2003) state that the purpose of the TIMSS 1999 Video
Study was “to investigate and describe teaching practices in eighth-grade mathematics and science in a variety of countries” (Hiebert et al., 2003, p. 1). Both set out to capture teachers teaching mathematics lessons. Each study offers the field a different view of teaching. For my study, each study offered different considerations for data analysis.

**Looking Inside the Classroom Study.** The *Looking Inside the Classroom Study* was a large-scale, nationwide study that examined the quality of 364 mathematics and sciences conducted by elementary, middle, and high school teachers. Although my study is not an examination of lesson quality, their findings that (a) engaging students with the content of a lesson and (b) helping students make sense of content of lessons are two factors that distinguish effective lessons from ineffective lessons suggest a need to consider—as part of the data analysis process—the nature of the engagement with a TMId that is emphasized in a teacher’s framing and to consider the extent to which making sense of a TMId is a part of a teacher’s framing. Weiss et al. (2003a) offered several examples in which teachers had students engage in procedures without regard to meaning. What follows is one such example.

The mathematics content in an 8th grade algebra class was the simplification of radical expressions. Said the observer, “Although the teacher’s content was accurate, the students were engaged only in following the procedures. There was no sense-making of concepts—only understanding of the procedures to solve the problems” (Weiss et al., 2003b, p. 73).

**The TIMSS 1999 Video Study.** The TIMSS 1999 Video Study is a large-scale study of 638 lessons from seven countries: Australia, Czech Republic, Hong Kong SAR, Japan, Netherlands, Switzerland, and United States (Hiebert et al., 2003). Through the
use of questionnaires, researchers gathered “information about the professional background of the teacher, the nature of the mathematics course in which the lesson was filmed, the context and goal of the filmed lesson, and the teacher’s perceptions of its typicality” (p. 164). Of the many messages resulting from TIMSS researchers’ data analysis, of particular interest is Hiebert et al.’s promotion of viewing teaching through two lenses: “a wide-angle lens” (p. 120) and a “close-up lens” (p. 121) to view general and specific aspects of teaching. The wide-angle lens revealed that “eighth-grade mathematics lessons across the seven countries share some general features” (p. 120). For example, lessons generally included some large-group and small-group/individual sections; “teachers in all countries talked more than students, at a ratio of at least 8:1 teacher to student words” (p. 120); and lessons included both review of content as well as attention to new content. The close-up lens was used to consider how the general features were combined and carried out during the lesson. The close-up lens revealed “particular differences among countries in mathematics problems and how they are worked on” (p. 121). Hiebert et al. (2003) used “looking through both lenses” to explore, for example, what they term, lesson signatures (i.e., a pattern of features that characterize teaching) of each country.

Influenced by Hiebert et al.’s use of two lenses to examine and discuss their data, I used multiple lenses to analyze my data as well. I wanted to view both general and specifics aspect of how teachers make TMI ds accessible. I wanted to see, if possible, how these aspects of a teacher’s teaching practice relate. Using a close-up lens I studied approximately 60 hours of video recordings of classroom observations, observation-related interviews, and problem-solving interviews to study how my participants
endeavored to make TMI ds accessible and comprehensible to their students. As a result, I identified a small collection of different approaches participants used in teaching, for example, how to factor quadratic trinomials, to simplify exponential expressions, to graph quadratic functions, and to evaluate composite functions. Using a wide-angle lens, in conjunction with viewing video data repeatedly and with scrutiny of classroom and interview artifacts, I examined the collection of approaches teachers used to make TMI ds accessible and comprehensible. As a result, I developed a conceptual framework for how teachers make TMI ds of school algebra accessible and comprehensible that accounts for uses of mathematical approaches as well as nonmathematical approaches. Using the lens provided by the newly developed conceptual framework, another wide-angle lens, I examined the instances of framing and coded each instance accordingly. The analyses of the data will be discussed in greater detail in chapter 3.
Chapter 3

Research Methods

To address the research question, “How do certificated secondary mathematics teachers attempt to make targeted mathematical ideas (TMIds) accessible and comprehensible to students?” I conducted an instrumental multicasere study (Merriam, 2009; Stake, 2005). The focus of the study (i.e., the instrument) is a particular type of classroom event: teachers making TMIds accessible and comprehensible to students. To gain insights into the nature of how teachers make TMIds accessible and comprehensible to students, multiple classroom observations and multiple interviews of secondary mathematics teachers (i.e., cases) were conducted. The multiple-case aspect of this study is based upon the assumption that each teacher’s teaching activity is contingent upon many factors, which include the mathematical ideas being targeted, the teachers’ respective mathematical backgrounds, and the students for whom the mathematical ideas are being made accessible and comprehensible. Thus, to gain insights about teachers’ framing, data were gathered from as diverse a group of teachers as possible. For the purpose of more efficient communication, the term framing will be used when talking teaching efforts to make TMIds accessible and comprehensible in general and the term decompressing, trimming, or bridging will be used when addressing the specific construct.

Participants

Background information. Pilot studies conducted for this study confirmed that a teacher’s framing of a TMId can involve multiple components and that those components
are often enacted quickly and simultaneously. A teacher’s framing of zeros of quadratic functions, for example, can involve symbolic and/or graphic representations recorded on the board, and oral descriptions or explanations that are connected to the written work only by pointing gestures or the fact that what is written and what is said, from the perspective of an observer, seem to be coordinated. In order to document the many facets of how a teacher frames a TMId, I made the use of video and audio recorders to capture the data a requirement for data collection. This decision had direct impact on participant recruitment. It was difficult to locate secondary mathematics teachers who were willing to participate in a study conducted by someone they did not know, who were willing to be video recorded during interviews and as they taught their students, and who worked for school districts that would allow nondistrict personnel to video record teachers teaching their students.

Recruitment and participant “selection.” The sample of the seven teachers participating in this study is a convenience sample. The teachers were originally contacted by mediators who agreed to act on my behalf advertising the need for algebra teachers to participate in this study to teachers in their respective geographic areas, and to assure potential volunteers that this study was a study of teaching and not an evaluation of teachers. The mediators are mathematics educators who are professional developers, student teaching supervisors, or in-service teachers who are active members of local mathematics education professional development groups. Besides agreeing to assist me in finding participants, what the mediators have in common is that they each know many middle school and high school mathematics teachers.
Phase I of recruitment. To gain access to practicing secondary mathematics teachers, I contacted supervisors of mathematics student teachers I have met since coming to The Pennsylvania State University and former colleagues who are currently in-service mathematics teachers and active participants in professional development groups, and asked for their assistance in finding in-service Algebra 1 teachers who met five criteria.

1. The teacher has a secondary mathematics or high school mathematics teaching credential.
2. The teacher has an undergraduate or graduate degree with a major or minor in mathematics or mathematics education.
3. The teacher has at least 3 years of full-time mathematics teaching experience.
4. The teacher, at the time of data collection, will be teaching at least one Algebra 1 class using a nonintegrated curriculum published by one of the major publishers (i.e., Pearson Education, Inc., Houghton, Mifflin Harcourt Pub. Co., and McGraw-Hill Education), whose early development was not funded by state, federal, or foundation grant monies (e.g., a California Postsecondary Education

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19Pearson Education, Inc. includes Prentice-Hall; Addison-Wesley; Scott Foresman. http://www.pearsonschool.com


Commission [CPEC] grant, National Science Foundation [NSF] grant, Carnegie Foundation Grant).

5. The teacher would be willing and able to be video and audio recorded during classroom observations and during interviews.

The corresponding reasons for these criteria 1 through 4 are as follows:

1. Possession of a teaching certificate means that each participant has completed at least one set of professional requirements (e.g., some combination of proof of a bachelor’s degree, teacher education coursework, practical experience, professional examination) as specified by a state-level Department of Education.

2. This explicit statement of an academic major/minor was included because not all secondary school teachers certified to teach mathematics have the sufficient mathematics coursework preparation. Some states such as California have several routes by which an individual can become a certificated secondary mathematics teacher (State of California, 2009).

3. Research describes the first 1 to 3 years of teaching as a period marked by intensive learning and anxiety (Feiman-Nemser, Schwille, Carver, & Yusko, 1999). It is a stressful time for beginning teachers, all of whom are charged with the same responsibilities and held to the same expectations for effectiveness as their veteran colleagues (Bartell, 2005). This requirement restricts my study to experienced teachers who are likely to have had experience establishing foundations for their classroom social norms and sociomathematical norms.

4. Researchers have stated that textbooks have a great deal of influence on teaching practices (Clement, 2001; Kilpatrick, Swafford, & Findell, 2001). The type of
textbook published by one of the major publishers are used by the majority of teachers in the United States. See Table 3-1.

Table 3-1. Percentage of Sample Held by Market-Leading Programs\textsuperscript{22} for Algebra 1 (Resnick, 2008).

<table>
<thead>
<tr>
<th>Grades 6–8</th>
<th>Title</th>
<th>Publisher</th>
<th>Parent company</th>
<th>2008</th>
<th>2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra 1: Concepts &amp; Skills</td>
<td>McDougal-Littell</td>
<td>Houghton Mifflin</td>
<td></td>
<td>9.9</td>
<td>3.2</td>
</tr>
<tr>
<td>Algebra 1</td>
<td>Glencoe</td>
<td>McGraw-Hill</td>
<td></td>
<td>8.1</td>
<td>11.9</td>
</tr>
<tr>
<td>Algebra 1</td>
<td>Holt</td>
<td>Harcourt</td>
<td></td>
<td>7.7</td>
<td>&lt;2</td>
</tr>
<tr>
<td>Algebra 1</td>
<td>Prentice Hall</td>
<td>Pearson</td>
<td></td>
<td>7.4</td>
<td>10.4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grades 9 – 12</th>
<th>Title</th>
<th>Publisher</th>
<th>Parent company</th>
<th>2008</th>
<th>2005</th>
</tr>
</thead>
<tbody>
<tr>
<td>Algebra 1</td>
<td>Glencoe</td>
<td>McGraw-Hill</td>
<td></td>
<td>16.7</td>
<td>16.3</td>
</tr>
<tr>
<td>Algebra 1</td>
<td>Prentice Hall</td>
<td>Pearson</td>
<td></td>
<td>12.9</td>
<td>12.8</td>
</tr>
<tr>
<td>Algebra 1</td>
<td>Holt</td>
<td>Harcourt</td>
<td></td>
<td>9.1</td>
<td>2.1</td>
</tr>
<tr>
<td>Algebra 1: Explorations &amp;</td>
<td>McDougal Littell</td>
<td>Houghton Mifflin</td>
<td></td>
<td>7.2</td>
<td>11.7</td>
</tr>
<tr>
<td>Applications</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The student teaching supervisors were able to locate teachers who met the first four criteria and were willing to participate in the study. However, the teachers all teach in

\textsuperscript{22} Although none of the textbooks in Table 3-1 were developed by NSF funded projects, the results of the survey from which this table is drawn do include \textit{Everyday Mathematics} and \textit{Connected Math}. Since neither of these curricula are Algebra 1 curricula, they were not included in Table 3-1.
districts that have policies that restrict nondistrict personnel from bringing video cameras into classrooms. One of the former colleagues was able to recruit two teachers who met all five criteria.

**Phase II of recruitment.** Rather than use alternative methods to collect classroom observation data (e.g., copious field notes), I decided to pursue looking outside of Pennsylvania for participants in whose classrooms a video recorder could be used to collect data. Gathering data that is as complete as possible was set as a priority. In addition, I decided to expand selection criteria #1 to include teachers who are certified to teach middle school mathematics, to include teachers who teach Algebra 1 and/or Algebra 2, and to allow teachers who do not have an undergraduate/graduate major or minor in mathematics or mathematics education to participate in the study.

Under the new selection criteria, another mathematics educator was able to provide me with contact information for four teachers. Of these four teachers, three participated in the study and one dropped out before data collection began. Recruitment efforts that continued during data collection on my behalf resulted in four additional volunteers. Two of the volunteers went on to participate in the study. One volunteer’s offer was not accepted because of scheduling conflicts. Another volunteer’s offer was not accepted because she did not have middle school, high school, or secondary certification and because the curriculum being used in her class was not published by one of the major publishers.

**Results of recruitment and “selection.”** As a result of recruitment and “selection” process, data from seven teachers from three different states—one in the Midwest, one in the South and one in the West—were collected for this study. With
respect to a number of factors, these seven teachers represent a diverse group of individuals. Some of the factors include: number of years of teaching experience, certification, grade-levels of the students in their respective observed classes, and the geographic location of their school. What follows is a table summarizing those factors that are pertinent to this study.

Table 3-2. Summary of Participants’ (Pseudonyms) Professional Backgrounds at the Time of Data Collection

<table>
<thead>
<tr>
<th>Participant</th>
<th>Participant’s professional background</th>
<th>Class observed</th>
<th>School</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ann</td>
<td>• Bachelor’s degree in Mathematics</td>
<td>Honors Algebra 2</td>
<td>• High School</td>
</tr>
<tr>
<td></td>
<td>• Enrolled in a Master’s program in</td>
<td>10th grade</td>
<td>• Suburban School District</td>
</tr>
<tr>
<td></td>
<td>Secondary Education</td>
<td></td>
<td>• In the Midwest</td>
</tr>
<tr>
<td></td>
<td>• Secondary Certification in Mathematics</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• 8 years of teaching experience</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bonnie</td>
<td>• Bachelor’s degree in Mathematics</td>
<td>Honors Algebra 2</td>
<td>• High School</td>
</tr>
<tr>
<td></td>
<td>• Master’s degree in ??? (Bonnie claims not to remember.)</td>
<td>10th grade</td>
<td>• Urban School District</td>
</tr>
<tr>
<td></td>
<td>• Secondary Certification in Mathematics</td>
<td></td>
<td>• In the South</td>
</tr>
<tr>
<td></td>
<td>• 30+ years of teaching experience</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Delia</td>
<td>• Bachelor’s degree in Business</td>
<td>Algebra 1B 9th</td>
<td>• High School</td>
</tr>
<tr>
<td></td>
<td>Administration-Human Resources</td>
<td>grade only</td>
<td>• 4x4 Block Schedule</td>
</tr>
<tr>
<td></td>
<td>• Master’s degree in Teaching</td>
<td></td>
<td>• Suburban</td>
</tr>
<tr>
<td></td>
<td>Mathematics, and in Reading and</td>
<td></td>
<td>• In the South</td>
</tr>
<tr>
<td></td>
<td>Writing</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Middle School Certification</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• 8 years of teaching experience</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Faith</td>
<td>• Bachelor’s degree in Mathematics</td>
<td>Algebra 1</td>
<td>• Middle School</td>
</tr>
<tr>
<td></td>
<td>• Master’s degree in Mathematics</td>
<td>7th and 8th</td>
<td>• Suburban</td>
</tr>
<tr>
<td></td>
<td>Education</td>
<td>grade</td>
<td>• In the West</td>
</tr>
<tr>
<td></td>
<td>• Secondary Certification</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>• 3 years of teaching experience</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Name</td>
<td>Education</td>
<td>Certification</td>
<td>Experience</td>
</tr>
<tr>
<td>------</td>
<td>-----------</td>
<td>---------------</td>
<td>------------</td>
</tr>
<tr>
<td>Gary</td>
<td>Bachelor’s degree in K–12 Physical Education&lt;br&gt;Master’s degree in Educational Leadership&lt;br&gt;Middle School Certification in Mathematics&lt;br&gt;12 years of teaching experience (all in mathematics)</td>
<td>Algebra 1&lt;br&gt;8th grade</td>
<td>Middle School&lt;br&gt;Suburban&lt;br&gt;In the South</td>
</tr>
<tr>
<td>Hildi</td>
<td>Bachelor’s degree Physics&lt;br&gt;Secondary Certification in Mathematics&lt;br&gt;4 years of teaching experience (all in mathematics)</td>
<td>Algebra 2&lt;br&gt;10th grade</td>
<td>High School&lt;br&gt;Urban School District&lt;br&gt;In the West</td>
</tr>
<tr>
<td>Kate</td>
<td>Bachelor’s degree in Drama with a minor in Mathematics&lt;br&gt;Master’s degree in Mathematics Education&lt;br&gt;Secondary Certification in Mathematics, with additional coursework in mathematics&lt;br&gt;16 years of teaching experience (primarily in high school mathematics, 2 years in middle school mathematics, and high school drama)</td>
<td>Algebra 1&lt;br&gt;9th, 10th, and 11th grade</td>
<td>High School&lt;br&gt;Alternating Block Schedule&lt;br&gt;Suburban&lt;br&gt;In the West</td>
</tr>
</tbody>
</table>

**Data and Data Collection**

To address the research question, data about each participant were collected from three sources: classroom observations, observation-related interviews, and problem-solving interviews. Each observation and interview was recorded using a digital video camcorder and digital audio recorders. Electronic copies were made of all lesson artifacts (e.g., lesson plans, handouts, assessments, and presentation materials), participants’ support and reference materials used in conjunction with lesson planning/preparation (e.g., textbook pages associated with observed lessons, resource materials used in/influencing observed lesson), and participants’ interview artifacts.
The clearest audio file from each classroom observation and each interview was sent to a professional transcriber, who transcribed each audio file and sent me an electronic copy of the corresponding verbatim transcript. I then annotated each transcript with screen captures of video data, and screen captures of the scans or electronic copies of the corresponding participant’s lesson artifacts, support and reference materials, or interview artifacts. During the annotation process, the transcripts were checked for correctness by comparing what the transcriber transcribed against the video and the other audio files. Based upon these comparisons, when needed, I made changes to the transcripts to create the most complete and accurate annotated verbatim transcripts possible.

**Classroom observations.** The original plan was to conduct five consecutive observations with each participant. The reasoning behind planning for five consecutive observations was based upon examination of textbooks as well as logistical considerations. Textbooks published by one of the major publishers, are revealed by examination of the tables of contents of several textbooks from the major publishers, are divided into chapters that contain seven to ten sections each. Each section involved the introduction of something new within the central focus of the chapter. Assum ing teachers who teach on a traditional schedule (i.e., 40- to 60-minute class periods) teach approximately one section per class period, and teachers who teach on a block schedule (i.e, 90-minute class periods) teachers approximately two sections per class period, I felt that I would be able to observe teachers introducing new ideas, developing ideas, and reinforcing/reviewing ideas.
Three to five observations were conducted for each participant. (See Table 3-3.) The exact number of observations for a given participant was dependent upon several calendars: my travel calendar, academic and activity calendars for each participant’s respective schools, participants’ respective State testing calendars, and each participant’s personal and lesson planning calendars. When possible, the lessons observed were consecutive lessons of the same class period. The decision to capture classroom observation data over consecutive days was influenced by research-based reasons as well as logistical reasons. By collecting classroom observation data over three to five consecutive lessons, it was hoped that I would capture a more complete development of one or more TMIds by a participant than would be captured by observing several lessons spread over a longer period of time. Also, given that each of the teachers works more than 400 miles away from my residence, collecting data over a shorter period of time for each participant was more economical.

Although only one teacher was observed five times, the analyses of the data indicate that I was able to accomplish the goals that lead to the planning of five consecutive observations per participant. I was able to capture each teacher introducing and developing new TMIds. I was able to capture each teacher reviewing and reinforcing TMIds from previous lessons. As will be shown in Chapters 4 and 5, the data collected are rich in variety of ways teachers made TMIds accessible and comprehensible to students.
Table 3-3. Summary of Observations per Participant

<table>
<thead>
<tr>
<th>Participant</th>
<th>Number of observations</th>
<th>Main lesson topics:</th>
<th>Notes:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ann</td>
<td>5</td>
<td>Parabolas, Circles, Ellipses</td>
<td>The observed lessons comprise the first half of Ann’s 2-week unit on Conic Sections.</td>
</tr>
<tr>
<td>Bonnie</td>
<td>4</td>
<td>Primarily review of simplifying rational and radical expressions.</td>
<td>Observations took place during a 4-day work week.</td>
</tr>
<tr>
<td>Delia</td>
<td>4</td>
<td>Distributive property and multiplication of pairs of linear binomials</td>
<td>Originally, 5 observations were scheduled. One observation was cancelled because of a building power outage.</td>
</tr>
<tr>
<td>Faith</td>
<td>4</td>
<td>Graphing quadratic functions: $y = ax^2 + c$ and $y = ax^2 + bx + c$</td>
<td>Because of scheduling conflicts, I was not able to be present for the lesson that took place between the 3rd and 4th observed lessons.</td>
</tr>
<tr>
<td>Gary</td>
<td>3</td>
<td>Derivation and application of the quadratic formula</td>
<td></td>
</tr>
<tr>
<td>Hildi</td>
<td>3</td>
<td>Collection of topics as part of a review for the spring semester final</td>
<td>Originally, 4 observations were scheduled. One observation was cancelled because of Hildi had an unexpected absence.</td>
</tr>
<tr>
<td>Kate</td>
<td>3</td>
<td>Completing the square and solving quadratic equations by completing the square</td>
<td>Kate teaches on an alternating block schedule. We were unable to schedule any more observations.</td>
</tr>
</tbody>
</table>

*Classroom observations: Before video recording began.* For each participant, before video recording began I walked around the room to see what was posted on the walls that might suggest how teachers make TMI'ds accessible, and what reference materials or teaching resources such as manipulatives or models were stored on bookshelves. Field notes and/or photographs were taken of classroom artifacts that might give insights about how the participant might have framed a TMI’d. Such artifacts included, but were not limited to, posters that addressed a mathematical idea, posted
student work, and collections of teacher resource materials (e.g., books, manipulatives, and models).

**Recording classroom observations:** Video. During classroom observations, I operated a digital video camcorder mounted on a stationary tripod positioned at the back of the classroom, where I remained for the duration of the class period. During large group instruction, I focused the video camcorder on the participant and his or her work at the chalkboard/whiteboard/Smartboard. Some of the participants wrote on an electronic tablet to write on as he or she led classroom discussion. In this situation, I captured what was being projected on the screen. During small group instruction or time devoted to individual seat work, the video camcorder was set to capture a wide screen capture of the room until and unless the participant worked with a group or individual students.

In this situation, the camcorder followed the participant, capturing as much of the conversations between the participant and students as possible. Often, the focus of conversation between participant and student(s) was a problem on a handout or a problem that had been written on the board or projected on a screen. However, typically, the camera was unable to capture the exact referents for the conversations. During these situations, I took field notes of gestures, unusual statements, phrases and/or questions and used this information to ask the participant to share details about the conversation during the postobservation interview. For example, during an observation of Hildi, I overheard Hildi asking a student, “Remember the spider? Err, the tarantula wasp?” (Hildi, Observation 1). Asking Hildi to share what the corresponding conversation was about resulted in Hildi sharing a metaphor she uses to help students remember how to evaluate a composite function.
Recording classroom observations: Audio. In addition to capturing classroom observation data using a video camcorder, I used digital audio recorders (four or five, depending upon the size of the classroom). I took care to place recorders in locations in the classroom that the participant identified as place(s) where he or she frequently positioned himself or herself. For example, Bonnie spends much of class time either working at the whiteboard at the front of the room, or at her document camera located on the left side of the room. In Bonnie’s classroom, one of the audio recorders was placed in the front of the room on the whiteboard marker tray, and one of the audio recorders was placed next to the document camera. I placed the remaining recorders on counters and shelves around the perimeter of the room with the idea that if Bonnie was asked a question by a student on the other side of the room from the document camera, there was the possibility that the recorder near Bonnie or the video camcorder might not capture the student’s question. The extra audio recorders placed around the room, made it possible to capture the student’s question if and when such a situation occurred.

Interviews. I conducted two types of interviews. Every classroom observation was preceded and followed by an interview. These interviews are collectively called, observation-related interviews. In addition to the observation-related interviews, I conducted problem-solving interviews. Problem-solving interviews will be discussed in greater detail later in this section.

Observation-related interviews. The number of observation-related interviews depended upon the number of classroom observations. Each set of observation-related interviews began with a “beginning interview,” the interview preceding the first observation with a participant; and an interview that took place after each observation,
which I will refer to as a “postobservation interview.” Thus, if a participant was observed 3, 4 or 5 times, we conducted 4, 5, or 6 observation-related interviews, respectively. The beginning interview was used to get acquainted with the participant, to gather logistical information (e.g., location of electrical outlets in the classroom, where I could work on campus, where in the classroom the participant anticipated positioning himself or herself during the upcoming lessons) and to gather general information about upcoming lessons.

The postobservation interviews served as the main venue for me to ask the participants directly questions about their framing of TMIds. I asked participants to share what thinking preceded a lesson (e.g., plans for framing TMIds); what thinking might have taken place while they were engaged teaching that day’s lesson; what adjustments, if any, they made during the lesson with respect to the TMId and their lesson goals and objectives; and what adjustments, if any, they planned to make in upcoming lessons. Now and then, during the observation, the participant or a student would make a statement or ask a question that made use of an unfamiliar term or involved something they learned outside the observation window. For example, one of Ann’s students said, “I factored it using the Secret Method” (Ann, Observation 3). When this occurred, in the postobservation interview, I asked clarifying questions, so that I could understand what took place (e.g., What is the “Secret Method”?). Often these clarifying questions led to the participant sharing how he or she frames other TMIds. Ann’s verification that there is a method in her class called the Secret Method involved her sharing how she frames factoring quadratic trinomials whose quadratic coefficient is not equal to one or negative one.
Observation-related interviews were conducted after school. These interviews lasted approximately 30 to 45 minutes, depending upon how many clarifying questions I asked and how much the participant had to share.

**Problem-solving interviews.** When I planned this study, I anticipated the possibility of that a participant could teach lessons that would reveal little or no framing, and lessons that are so carefully orchestrated that students play a passive role in the lesson. I, therefore, decided to use task-based interviews in which participants are asked to respond to what Ball and Bass (2003) call “mathematical problems of teaching.” The tasks consist of five classroom scenarios\(^ {23}\) in which participants were asked to respond to situations in which hypothetical students asked questions or made statements that reflect potential misconceptions about mathematical ideas students learn in Algebra 1 or Algebra 2. All scenarios are designed so that the problem can be approached mathematically from more than one direction. In responding to the potential misconceptions, the pilot studies revealed that participants could reveal how they frame the TMIds and/or their personal understandings (or lack of understanding) of the TMIds.

\(^{23}\) All scenarios can be found in Appendix B. Similar scenarios with corresponding questions and statements can be found in the collection of classroom situations gathered and developed by the Situations Project, a joint effort by the Mid-Atlantic Center for Mathematics Teaching and Learning (MAC-MTL) at The Pennsylvania State University and the Center for Proficiency in Teaching Mathematics (CPTM) at the University of Georgia.
In one scenario\textsuperscript{24}, for example, a student claims that the respective distances between two pairs of points, A(3,3) and B(7,4); and C(–2,4.5) and D(2, 5.5), are the same because of slope (see Figure 3-1). Each participant was asked to share his or her thoughts about the scenario, and then asked to share how he or she would respond. It is important to note that as part of the task design, the points in this scenario were chosen so that the distance from A to B and the distance from C to D are both equal to $\sqrt{17}$ units.

Scenario 4

You provide students with the coordinates of points A, B, C, and D and ask them to show that segments AB and CD have equal lengths.

A student graphs segments AB and CD and says, “AB and CD have the same length because they have the same slopes. To get from A to B, you go up 1 and right 4. To get from C to D you go up 1 and right 4.”

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3-1}
\caption{Scenario 4. One of five tasks used in the problem-solving interviews.}
\end{figure}

\textsuperscript{24} A more detailed statement of this scenario and the other four are included in Appendix B.
The scenario can be interpreted at least three ways. The scenario is a description of:

1. a student who uses an incorrect method, but obtains a correct answer. Cara, one of two teachers in my pilot study stated that the student is incorrect because one does not use slope to find the distance between two points. The distance formula is what is needed to show that the two segments have equal lengths.

2. a student who uses a viable method, but may not be using accurate/correct mathematical language. The student’s language suggests he or she could be making use of congruent triangles (i.e., the legs of the right triangles are correspondingly congruent to the legs of the other right triangle) or perhaps be making use of some understanding of the Pythagorean theorem (i.e., if $a_1 = a_2$ and $b_1 = b_2$, then $a_1^2 + b_1^2 = a_2^2 + b_2^2$).

3. a student who uses a viable method and an accepted “definition of slope” (i.e., “slope is rise over run”) to draw a correct conclusion. The method, however, only works because the corresponding values for the “rise” and the corresponding values for the “run” are the same. Emily, the second participant in my pilot study, was concerned that the student would think that the converse is also true. In other words, she was concerned that the student would think segments that have equal slopes means that the segments have equal length as well.

The number and duration of the problem-solving interviews were contingent upon whether the interviews were conducted after a postobservation interview and upon the participant’s mathematical knowledge/KAT. Problem-solving interviews were spread
over 2 to 4 days so that the participant was engaged in no more than 2 hours of interviews (i.e., observation-related combined with problem-solving). Pilot study data and my experiences as a research assistant for the Mid-Atlantic Center for Mathematics Teaching and Learning suggested that I needed to be cognizant of fatigue impacting the quality of responses to interview questions.

**Recording Interviews.** During the interview, the participant and I were seated side by side (see Figure 3-2). The digital video camcorder, set on a stationary tripod, was positioned and set up so that the video camcorder captured the participant’s written work. I did not have a videographer and therefore, did not have the benefits of the camcorder’s zooming features. Seated next to the participant (see Figure 3-2), I asked questions, probed his or her thinking, and encouraged the participant to provide detailed explanations. In addition to video recording, a digital audio recorder was used to record the interview. The audio file produced by the digital audio recorder was then sent to a professional transcriber, who then created a verbatim transcript.
Data processing. I began data processing by organizing all classroom observation and problem-solving interview artifacts. All paper copies of data (e.g., handouts, pages of textbooks, pages of resource materials) were scanned. Care was taken to make sure that all names and other identifying information of participants, students, and schools were covered up/removed before electronic copies were created, or blacked out/removed from copies of artifacts that the participants provided in electronic form.

Audio files, one from each classroom observation (a total of 26) and each interview (both problem-solving and observation-related interviews; a total of 47) were transcribed to create verbatim transcripts. Each of the verbatim transcripts was then annotated with screen captures from its corresponding video, and with screen captures from electronic copies of its corresponding observation and interview artifacts. One of the goals of the annotation process was to identify each and every referent so that a reader can read the annotated transcript and envision what is going on without having to view

Note: For left-handed participants, I sat on the participant’s right with the camera in front of me pointed at the participant’s written work.
the video. A second goal of the annotation process was to produce a document that is as accurate and complete as possible. Achieving these goals required repeated and focused viewing of video, and repeated listening to audio recordings. Each viewing/listening assisted me in becoming well-acquainted with the data from each participant.

Data Analysis

Data analysis began informally during the data processing stage when I viewed and re-viewed the video recordings of observation-related interviews, problem-solving interviews, and classroom observations to create annotated verbatim transcripts. It was during this time that I made notes of TMI ds being addressed and thoughts of how the TMI ds were being framed. Once a transcript was completely annotated, the transcript was analyzed for instances of framing, the first of a two-part analysis began.

The two-part analysis was inspired by Hiebert et al.’s (2003) discussions of using a close-up lens and a wide-angle lens to view specific and general aspects of teaching. In this study, the use of close-up lens (i.e., part one) and wide-angle lenses (i.e., part two) afforded specific and general views of framing, respectively. Using a close-up lens, all transcripts were analyzed for instances of framing. As a result, more than 50 instances of framing were identified. Using a wide-angle lens, the collection of 50+ instances of framing was examined and analyzed for similarities and differences among and across subsets of framings. This more general view of framing resulted in more detailed characterizations of framing and its relationships to the mathematical practices of trimming, bridging, and decompressing.
Close-up lens: Identifying instances of framing. Definitions of targeted mathematical idea, framing, trimming, bridging, and decompressing played a major role in identifying instances of framing. They served as a means of maintaining consistency. To this end, the definitions of these concepts were read and referenced several times during each transcript analysis. The definitions are as follows:

- **Framing** is communicating or presenting a set of messages or meaningful actions used to make targeted mathematical ideas (TMIds) accessible and comprehensible to students. The sets of messages and meaning actions can be the result of drawing upon strictly mathematical knowledge (i.e., trimming, bridging, and decompressing TMIds), or drawing upon a combination of nonmathematical and mathematical knowledge. For the purposes of this study, *mathematical ideas* are defined to be subject matter (Cooney, Davis, & Henderson, 1975). Mathematical ideas are said to be targeted if they are the focus of discussion or explanation.

- **Decompressing** is breaking a TMId into component ideas.

- **Trimming** is bringing attention to/emphasizing some set of component ideas.

- **Bridging** is making connections between a TMId and other mathematical ideas.

Figure 3-3 is a diagram representing relationships among these constructs as hypothesized during this part of data analysis. The three mathematical teaching practices of decompressing, trimming, and bridging are three different types of framing. A fourth category of framing is framing that involves the use of a nonmathematical element. Nonmathematical elements include, but are not limited to, the use of mnemonics to bring attention to some component(s) of a TMId, or metaphors that connect TMIds to hypothetical contexts, and use of stationery to help students organize their written work.
Figure 3-3. Framing conceptual framework (prior to data analysis).

With respect to data from observations and observation-related interviews, identification of an instance of framing began with the identification of a TMId. Once the TMId was identified, I examined the teacher’s activity related to the TMId. What follows are descriptions of my procedure when analyzing classroom observation transcripts and descriptions of my procedure when analyzing interview transcripts.

**Identifying instances of framing in classroom observations.** The data revealed that, in general, participants’ lessons could be divided into sections: (a) warm-up problem(s); (b) discussion of solutions to teacher-selected and/or student-requested problems from homework or quizzes; (c) the main part of the lesson; and (d) in-class time to work on homework. Given that so much of the participants’ lessons were devoted to explanations of solutions to problems (e.g., homework or quiz problems requested by students, example problems chosen by the participant), I often used a problem and its related work (i.e., explanation, demonstration, recorded work) as my unit of analysis. I identified the TMId within each problem and then examined the participant’s actions with
Informed by research (e.g., Baxter & Williams, 2010; Wood, Bruner, & Ross, 1976), I was aware of several methods teachers might use to frame TMIds. Such methods include reducing the number of degrees of freedom involved in a task, identifying important features, demonstrating/modeling, using manipulatives, using metaphors, using different representations, explaining, and justifying. However, during analysis my focus was to identify what each participant actually did to make TMIds accessible. I did not restrict my identification of framings to those identified in research examined in my literature review. After writing a short description of how a TMId was being framed, I went on to the next problem in the lesson and repeated the process: identify the TMId, analyze how it was being framed, and describe the framing.

For parts of a lesson that did not revolve around a problem, my identification of the TMId was informed by the participant’s statements in the previous day’s observation-related interview when he or she discussed what he or she planned to do in the upcoming lesson. These parts of the lesson were either derivations (e.g., Gary’s derivation of the quadratic formula) or demonstrations/explanations (e.g., Faith’s explanation of the concept of line of symmetry in the context of using a folded piece of paper to cut out a paper heart). In every observation-related interview, including the last observation-related interview of the data collection period, I asked the participant to identify the TMId(s) he or she planned to address in the next day’s lesson. Only one participant, Faith, provided written documentation (i.e., a written lesson plan) identifying the lesson objectives.
Identifying instances of framing in interviews. The observation-related interview following each classroom observation always began with a discussion of what happened in the observed lesson. It was during this time that participants were given an opportunity to share their perspectives about what took place in class. With respect to framing, it was my opportunity to gather additional data (e.g., background information, clarification, reasoning behind the participant’s decisions made in-the-moment) about the framings that took place in class that day, and about framings from previous lessons (observed and not observed) that supported the framings in that day’s lesson. I made a point to ask participants about anything unusual or unclear that I happened to see or hear during the lesson. I made it a point to ask for clarification about terminology (e.g., Bonnie’s AC Method, a method she uses to factor quadratic trinomials; Kate’s “rivers,” the long squiggly line she always drew through the equal sign when solving equations). Sometimes the unusual event was a passing comment made by a student or by the participant. For example, in Ann’s class a student said she used the Secret Method to factor (Ann, Observation 3), and Hildi asked a group of students if they remembered the “tarantula wasp” (Hildi, Observation 1). Ann’s and Hildi’s responses to my requests for explanation and clarification, revealed unusual ways of framing procedures: how to factor quadratic trinomials, and how to evaluate composite functions, respectively. Both of these framings are discussed in greater detail in chapter 4.

The focus of the problem-solving interviews was a set of five classroom scenarios, each of which addressed a different TMId. In each scenario, a student asks a question or makes a statement that indicates a potential misconception or some struggle with understanding the TMId. Participants’ responses to the hypothetical student in each
scenario were then examined for descriptions of what they would do to the TMId and/or with the TMId (i.e., framing) in order to help the student overcome the potential misconception or reduce/remove the struggle with understanding the TMId. The selection of these scenarios was influenced by three factors. First, all five scenarios reflect struggles or points of confusion for students that I have experienced as a classroom teacher or have had discussion about with other teachers as part of my work as a mentor teacher, as a curriculum developer, or as a contributor to the Situations Project, a joint effort of by the Mid-Atlantic Center Mathematics for Teaching and Learning at The Pennsylvania State University and the Center for Proficiency in Teaching Mathematics at the University of Georgia. Second, as part of their development of frameworks for teacher knowledge, researchers (Ball, 1988; Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2005) have identified ideas (e.g., division by zero and definition of the concept of exponent) as challenging for teachers to explain. Third, all five scenarios were written before participants were recruited. I assumed that the mathematical backgrounds of my participants could vary greatly and that some may not be teaching Algebra 2. Therefore, the five scenarios had to reflect mathematical content that could be discussed by middle school teachers as well as high school teachers.
Table 3-4. TMId in Each Problem-Solving Interview Scenario

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Targeted mathematical ideas (TMIds):</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Converting infinitely repeating decimals to fraction form.</td>
</tr>
<tr>
<td>2</td>
<td>Definition of exponents, why $a^0$ equals 1, and is the 0 in $a^n$ an exponent?</td>
</tr>
<tr>
<td>3</td>
<td>Graphing linear inequalities.</td>
</tr>
<tr>
<td>4</td>
<td>Showing that the distances between two pairs of points are equal.</td>
</tr>
<tr>
<td>5</td>
<td>Value of $\frac{5}{0}$.</td>
</tr>
</tbody>
</table>

Part of the analysis of how a TMId was being framed included classification of whether the instance of framing is an example of a mathematical practice or made use of methods that are not strictly mathematical. After this initial sorting, those instances of framing that did not involve the use of nonmathematical methods or ideas were then compared to the definitions of decompressing, trimming, and bridging. The goal of this second sorting was to determine whether the instance of framing was an example of one of these mathematical practices or none of the three. After repeated comparisons to the definitions, each of the mathematical framings were found to involve the use of decompressing, trimming, and/or bridging.

**Wide-angle lens: Characteristics of framing.** As a result of the processes described in the previous section, more than 50 instances of framing were identified. The examination of the collection began with the following general questions in mind:

- Are there any patterns that cut across the collection of framings? If so, what are they?
• Are there similarities among subsets of framings? If so, what are they?

• Are there subsets of framings that seem to be related, but differ in some way? If so, what is/are the relationship(s) and how do they differ?

During the first pass of the data, I noted that some framings involve have nonmathematical elements and have some of the characteristics of decompressing, trimming, or bridging. For example, some framings involved the participant making connections; however, the connections were between the TMId and a nonmathematical idea. Based upon this observation, I examined this subset of framings (i.e., those that have nonmathematical elements) asking the same three general questions, and one additional question.

• What would characterize the subsets of framings that have nonmathematical elements, respectively?

The definitions of trimming, bridging, and decompressing were used as references and guides as I sorted and re-sorted the 50+ instances of framings in search of broader views of framing. Resulting from this analysis are 11 claims about framing (i.e., answers to the first three sets of guiding questions), each describing a different view, and answers to the fourth set of guiding questions. The specifics about each of the 11 claims are discussed in detail in chapter 4. Answers to the fourth set of guiding questions are discussed in greater detail in chapter 5.
Chapter 4

Results: Characteristics of Framing

Emerging from the analysis of classroom observation and interview transcripts are more than 50 instances of framing (i.e., participants’ responses to the question, “How do I make this idea accessible and comprehensible to students?”). Each instance was found starting with the identification of a TMId and then an examination of what the participant did with respect to making the TMId accessible and comprehensible. Many of the TMIds were identified by the participant during a pre observation interview. Sometimes a TMId was identified by a student’s question or comment. During the problem-solving interviews, the TMIds (e.g., division by zero, graphing linear inequalities on a rectangular coordinate plane) were chosen by me prior to data collection.

To answer the broader question “How do secondary mathematics teachers attempt to make one or more targeted mathematical ideas (TMIds) accessible and comprehensible to students?” (i.e., my research question), I sorted and re-sorted my collection of 50+ framings into subsets. The sorting process began with a question that was inspired from research reviewed for this study. For example, a question based upon bridging (McCrorie et al., 2012) was “Is the teacher making one or more connections between the TMId and another mathematical idea?” Wood et al.’s (1976) description of the scaffolding process identifies six functions of tutoring. One of the functions, reduction in degrees of freedom, inspired the question, “Is the teacher somehow reducing the level of difficulty?” One of the examples of analytic scaffolding from Baxter and Williams (2010), use of
metaphors, led to the question, “Is the teacher using a metaphor?” As the sorting process continued, the questions used to sort the data came from the data themselves. For example, “Is the use of informal/unusual terminology a prominent feature of the framing?” With each sorting, I examined the subsets for patterns of similarity and for patterns of contrast across subsets, and I documented observations.

This chapter, organized into two main sections, is the product of this process of sorting, examining, re-sorting and re-examining. The first section is composed of discussions of each of 11 characteristics of framing that emerged from the data. The second section is composed of discussions of messages that cut across the 11 characteristics of framing. The 11 characteristics are as follows:

1. There are two general approaches reflected in teachers’ efforts to bring attention to a TMId: a direct approach and an indirect approach.
2. Teachers make connections between TMIds and mathematical as well as nonmathematical ideas.
3. Breaking TMIds into smaller chunks is a strategy that all of the participants in this study seem to use. The data revealed a variety of examples of how teachers orchestrate students’ engagement/interaction with those smaller chunks.
4. The role mathematics plays in framing a TMId seems to vary.
5. Some framings of procedures seem to be examples of folklore mathematics in which procedures are passed from one teacher to another; however, the mathematical explanations or justifications as to why the procedures work seem to be unknown.
6. Teachers’ framings reflect contrasting efforts to help students see/look at symbolic representations. Some teachers focus on surface features (i.e., that which can be seen with one’s eyes). Some teachers focus on seeing through or beyond the surface to symbolic structure or form (i.e., that which requires or uses some mathematical insights/understandings).

7. Teachers sometimes frame TMIds by reducing visual density/complexity of symbolic representations.

8. Teachers sometimes frame TMIds by making connections to “student-relatable contexts.” Student-relatable contexts are not necessarily real-world contexts.

9. Sometimes a teacher forces/imagines a connection between a TMId and a real world context.

10. Teachers sometimes use of local language/terminology to make TIMds accessible and comprehensible.

11. Some teachers’ framings each seem to serve multiple mathematical purposes and are components of larger framings, and some teachers’ framings serve a single mathematical purpose and are disconnected from other framings and other TMIds.

**Characteristics of Framing**

The discussions of each of the 11 characteristics of framing include descriptions and discussions of two to four examples from which the characteristic emerged. It is important to note that the examples provided are different from each other and are
intended to offer contrasting views of the characteristic for the purpose of clarifying each characteristic.

**Characteristic 1.** There are two general approaches reflected in teachers’ efforts to bring attention to a TMId: a direct approach and an indirect approach.

**Direct approach: Emphasizing the TMId or aspects of a TMId.** Two examples are provided for this type of framing. These examples document how participants emphasized the TMId directly. Example 1.1 is a discussion of Kate’s trimming of the discriminant, \(b^2 - 4ac\). Example 1.2 is a discussion of Hildi’s use of a nonmathematical means by which she brings attention to what the negative exponent of an expression means. What is interesting about these two examples, beyond how they bring attention to the TMId or aspects of the TMId, is that they present a contrast in the role that mathematics plays in the framing of TMIds or aspects of TMIds. In Example 1.1, mathematics familiar to students is used to develop students’ understanding of a TMId. In Example 1.2, mathematical meaning of the TMId is downplayed and replaced with a set of instructions that is intended to help students obtain correct answers.

**Example 1.1. Bringing attention to the same TMId in multiple ways.** Kate highlights and emphasizes the discriminant (i.e., the TMId) in 3 different ways: (a) by using \(b^2 - 4ac\) as a means to review the order of operations, (b) by telling students that \(b^2 - 4ac\) has a name and that the value of \(b^2 - 4ac\) provides insights about the graph of \(y = ax^2 + bx + c\), and (c) by teaching her students a different version of the quadratic formula, \(x = \frac{-b \pm \sqrt{D}}{2a}\) and \(D = b^2 - 4ac\).
In the first method, Kate introduced \( D = b^2 - 4ac \) several lessons before she introduced her version of the quadratic formula. Only after her students had worked with the formula for several weeks did she inform them that the formula is called, the discriminant. Initially (i.e., before my observation week), Kate used the formula \( D = b^2 - 4ac \) as a means to practice applying the order of operations. Students were given several exercises spread over several weeks for which, given sets of values for \( a, b, \) and \( c, \) respectively, students were asked to compute the value of \( D. \) The contents of Figure 4-1 are taken from a worksheet assigned to students 2 weeks before Kate’s observation week. At the time students computed values for the expression, \( b^2 - 4ac, \) they did not know that this expression is a special expression. It was presented as simply a featured part of an exercise. The contents of Figure 4-2 are taken from a worksheet assigned to students during the observation period. Students were asked once again to compute the values of \( b^2 - 4ac. \) With this exercise, however, students had to identify the values of \( a, b, \) and \( c \) from a given quadratic equation. The problem, as designed, suggests a connection between the expression \( b^2 - 4ac \) and quadratic equations. Over the several weeks of computing values for this expression served to increase student familiarity with the expression, \( b^2 - 4ac \) and computing its value.

1. Evaluate: \( b^2 - 4ac \) when.....
   a) \( a = 2 \ b = 5 \ c = 1 \)  
   b) \( a = 5 \ b = -3 \ c = -1 \)

*Figure 4-1.* Screenshot of a section of Kate’s Assignment #20, problem 1a and 1b.
4. If \( ax^2 + bx + c = 0 \), then \( D = b^2 - 4ac \). Find \( D \).

Example:

a) \( 3x^2 - 5x - 2 = 0 \)

\[ 2x^2 - 3x - 5 = 0 \]

So

\[ a = 2, \quad b = -3, \quad c = -5 \]

\[ D = b^2 - 4ac \]

\[ = (-3)^2 - 4(2)(-5) \]

\[ = 9 + 40 \]

\[ = 49 \]

Second, Kate eventually told her students that the now-familiar formula, \( D = b^2 - 4ac \), has a name, the discriminant, and that being able to compute the value of \( D \) is important because it provides information about a parabola. She said,

So I don’t want to freak you out with this formula or anything but it’s kind of important that we’re able to find [pause] \( D \) [pause], which we are later going to call the discriminant. And um, it’s going to tell you stuff about the parabola. But, you know, we’re not there yet. We’re just practicing this formula right here.”

(Observation 2)

With this statement, Kate informed students that the formula for which they have been computing values would serve as a tool to provide them information about graphs of quadratic functions. From a pedagogical perspective, Kate initiated a connection between mathematical ideas with which students were already familiar (i.e., graphs of quadratic functions) and computing the values of the quadratic formula, a TMI that students would encounter in the near future. Evidence that students were already familiar with graphs of quadratic functions was found in the previous day’s warm-up (see Figure 4-2).
4-3) and the first problem for the previous day’s worksheet (assignment #24), which students worked on in class (see Figure 4-4). These problems show that Kate’s students are familiar with the general shape of the graph of quadratic functions and the term, parabola.

Figure 4-3. Screen capture of Kate’s Observation 1 warm-up problem #3 and answer to the first question.

<table>
<thead>
<tr>
<th>Algebra 1 Assignment #24</th>
<th>Name: __________________________</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Date: __________ Period: ___</td>
</tr>
</tbody>
</table>

1. For each of the following, find the x intercepts, then sketch a graph of the parabola (must have the general shape, correct intercepts)
   a) \( y = x^2 - 4x + 3 \)
   b) \( y = x^2 + 2x - 15 \)

Figure 4-4. Screenshot of Kate’s Problem 1a and c, Assignment 24 (Observation 1).

Third, from a strictly symbolic perspective, Kate brought attention to the discriminant \( D \), in the way she expresses the quadratic formula. To the majority of students in the United States who have completed Algebra 1 or the equivalent of Algebra 1, given equations of the form, \( ax^2 + bx + c = 0 \), the quadratic formula is
given equations of the form, \( ax^2 + bx + c = 0 \), the quadratic formula consists of two parts:

\[
x = \frac{-b \pm \sqrt{D}}{2a}
\]

and \( D = b^2 - 4ac \) (Kate, Ending Interview). She brings attention to the discriminant by treating and recording \( D = b^2 - 4ac \) as a separate entity.

When asked to talk about an upcoming lesson in which Kate planned to derive the quadratic formula, she said, “The amount of symbols in the quadratic formula is going to be a bit frustrating for them [i.e., students]. But, you know, the idea is for them to be comfortable with parts” (Kate, Ending Interview). Thus, it seems that part of Kate’s development of the quadratic formula by starting with the discriminant—before the derivation of the quadratic formula—involves familiarizing students with parts of the quadratic formula. In addition, Kate said,

> It really directly speaks to—okay—square rooting this [as she traces the D in her formula]. You know. What does that mean? If it was a perfect square, well, we could have factored it. If it was less than zero, I am not going to get anything out of that. If it is equal to zero I am not adding or subtracting anything, so I am just going to get one… answer so. It helps us develop the discriminant in other ways too—err, talk about what the discriminant does—uh, what it means… when we get a number—numerical answer. (Ending Interview).

I infer from Kate’s statement that (a) the \( D \) in \( x = \frac{-b \pm \sqrt{D}}{2a} \) brings attention to the discriminant as an object in and of itself, (b) the quadratic formula is composed of parts,
and (c) the expression beneath the square root symbol can provide students with
information about graphs of quadratic functions (i.e., parabolas), and solutions to
quadratic equations.

Example 1.2. Bringing attention to aspects of a TMId as part of overcoming
errors. As part of preparing her students for an upcoming final exam, Hildi brings
attention to the negative exponent of an expression (i.e., the TMId) by providing her
students with a way to remember what the negative sign in the negative exponent tells
them. This instance of framing took place as Hildi guided students through the process of

graphing \( y = \frac{-1}{2} \left( \frac{1}{2} \right)^{-1} + 2 \) by plotting points. She began by substituting 0 for \( x \) and then
led a discussion about computing the value of \( \frac{-1}{2} \left( \frac{1}{2} \right)^{-1} + 2 \). Hildi said, “Now this is the
part that—especially like third period, they had a lot of trouble remembering what to do
when there’s a negative exponent. What is one-half to the negative one?” (Observation 1).

One student answered, “one-fourth” (Hildi, Observation 1) and another asked, “Is
it negative one-fourth?” (Hildi, Observation 1). Without acknowledging students’
responses, Hildi said,

It’s two. [pause] Right? Because the negative means flip it [as she uses her cursor
to point at the -1 exponent] [pause] Right? So we flip it and raise it to that power.
So one half [to the negative one] would be [Hildi then silently finishes recording
the “\( \frac{2}{1} + 2 \)” in \( y = \frac{-1}{2} \left[ \frac{2}{1} + 2 \right] \). (Observation 1)
Hildi asked students to pay attention to the negative one exponent because she noted that students had difficulty remembering what to do. Hildi’s framing of the exponent provided students with a way of knowing/remembering what to write when they see a negative exponent: “the negative means flip it. …So we flip it and raise it to that power” (Hildi, Observation 1) and illustrates a nonstandard use of mathematical terminology.

The numerical expression, \( \left( \frac{1}{2} \right)^{-1} \) is correctly read, “one-half raised to the negative one power” or “one-half to the negative one power.” In other words, the “−1” in this expression is a power. Hildi’s statements seem to suggest that she wants her students to note that the negative exponent is composed of two parts: the “−,” which means “flip it,” and the “number behind the negative sign” (i.e., in this case, the “1”), which is the power to which you raise the number to be flipped (i.e., \( \frac{1}{2} \)). This observation that Hildi wanted students to see two parts to a negative exponent is supported by Hildi’s descriptions of what to do to compute the value of \( \left( \frac{1}{2} \right)^{-2} \) when she computed the value of

\[
y = -\frac{1}{2} \left( \frac{1}{2} \right)^{-1} + 2 \quad \text{for } x = -1.
\]

She told her students,

Well once you flip it though, it’s not a half any more, right? Yeah. So either multiply by itself and then flip it, or flip it multiply it by itself. As long as it gets both flipped and multiplied by itself (Observation 1).

As Hildi framed it, the negative sign in an exponent has no mathematical meaning. Hildi frames the “−” as the stimulus for a stimulus-response pairing. Students are being told
that when they see a “−” in an exponent (i.e., the stimulus), they are to flip the corresponding base, and then perform what the number behind “−” indicates.

**Indirect approach: Eliminating/lessening distractions or potential distractions from TMIds.** Two examples are provided for this approach. In Example 1.3, Faith reduces the degree of difficulty for solving quadratic equations by completing the square by having students learn the procedure using equations that involve “nice numbers” (i.e., quadratic equations for which the quadratic coefficient is 1 and the linear coefficient is even). In Example 1.4, Gary has his Algebra 1 students memorize 20 fraction–decimal equivalents at the beginning of the academic year as a way of minimizing the amount of time spent on discussing fraction–decimal equivalents and increasing the amount of time spent on discussing TMIds in later lessons.

Both of these examples of framing seem to be efforts to solve teaching-related issues; however, the issues being addressed in Examples 1.3 and 1.4 are very different. The purpose of the framing discussed in Example 1.3, was to help students maintain focus on the TMId by minimizing the distractions associated with computing with fractions. In Example 1.4, the purpose of the framing was to make better use of a 42-minute algebra class period by reducing the amount of time spent on discussing fraction–decimal equivalents, a topic that typically is not part of the lesson objectives.

**Example 1.3. Reducing degree of difficulty with computation.** In teaching her students how to solve quadratic equations by completing the square, Faith stated that she began teaching the procedure by using examples with “nice numbers.” When asked what she means by “nice numbers,” Faith said,
Well starting off where there’s no, um, a term—well, its a term [sic] is one.26

Um, and then where b can be divided in half nicely as opposed to like a b term of three… I’d like to give them numbers where the numbers are going to [pause] avoid fractions and that sort of—uh situation.” (Postobservation 3 Interview).

Computation involving fractions, according to Faith, is an area of weakness for her students. Thus, Faith’s choice of values for a and b in her examples of quadratic equations to solve using the method of completing the square is a way of framing “by using nice numbers.” In this context, nice numbers are pairs of numbers: \( a = 1 \) and \( b = an \ even \ number \). Using nice numbers is a way of teaching students how to do the procedure without the distractions associated with students’ struggles with computations involving fractions. In a sense, if a teacher has his or her students complete the square using only nice numbers, the teacher has reduced the computational difficulty of his or her instruction of how to solve quadratic equations and how to determine the vertex of a quadratic function using completing the square.

By reducing computational difficulty, Faith sought to make this TMId accessible and comprehensible to her students. At the same time, however, students’ experiences with this procedure using only nice numbers could leave students with the impression that this procedure is restricted to quadratic functions for which the quadratic coefficient

26Although Faith says she starts with quadratic equations for which the “a term is one,” I am convinced that what she meant is that the value of a in \( ax^2 + bx + c = 0 \) is one. If the “a term” is one, then she would not be working with a quadratic equation and therefore, the conversation about using completing the square would make no sense.
equals one and the linear coefficient is even—a potential misconception about completing the square.

**Example 1.4: Memorizing common fraction–decimal equivalents.** Gary had his Algebra 1 students memorize what he calls “the 20 most common fraction–decimal equivalents that they’re going to use. Just so, just so they’re comfortable with them” (PS Interview). This example of framing is interesting because it reveals that a framing may serve more than one purpose. In addition to making TMIIs accessible and comprehensible, Gary’s framing of fraction–decimal equivalents is to save time. Over the course of the data collection, Gary often mentioned not having enough time to teach because class periods are only 42 minutes long (Beginning Interview). To eliminate having to discuss why, for example, 0.75 and \( \frac{3}{4} \) are equivalent answers to computations or to remind students that they are equivalent numerical expressions, Gary requires students to memorize 20 common fraction–decimal equivalents. Not having to spend time class time explaining fraction–decimal equivalents allows Gary to spend more time on his lesson objectives.

**Discussion: Connections to existing research.** This characteristic and the corresponding examples align with Wood et al.’s (1976) discussion of three of the tasks in the Scaffolding Process that support completion of the problem interviewees were asked to solve. Wood et al.’s (1976) Scaffolding Process suggests the need to be cognizant that teachers can support students’ learning using direct approaches as well as indirect approaches (e.g., marking critical features, and demonstration; and reducing degrees of freedom). Because Kate’s and Hildi’s work (i.e., Examples 1.1 and 1.2,
respectively) reflect efforts to bring attention to one or more aspects of their respective TMIds, their efforts are consistent with the marking critical features task in Wood et al.’s (1976) Scaffolding Process. Described in general, Faith’s and Gary’s work (i.e., Examples 1.3 and 1.4, respectively) and Wood et al.’s (1976) reducing degrees of freedom are efforts to remove or at least minimize distractions or obstacles from students’ focus on the task at hand. In Faith’s and Gary’s lessons, the task at hand was students learning of TMIds. In Wood et al.’s (1976) study, the task was the problem solving activity.

**Characteristic 2.** Teachers make connections between TMIds mathematical as well as nonmathematical ideas.

**Connecting TMIds to contexts that are tangible to students.** What follows are two examples of framing for this category. In Example 2.1, Hildi uses a story to make an abstract TMId (i.e., the concept of function) more concrete. In Example 2.2, Bonnie makes a statement that associates a TMId (i.e., division by zero) to a sin. In both of these examples, the framing involves making an abstract idea a more tangible to students. These examples also offer contrasting relationships between two TMIds and their respective contexts. In Example 2.1, the relationship between the TMId and the context is a metaphor. Properties of the TMId are represented by relationships between characters in the context. In Example 2.2, the relationship is an association.

**Example 2.1. Using a metaphor to connect a TMId to a nonmathematical idea.** Hildi uses a metaphor to connect the concept of function to an exclusive social event. According to Hildi, a function is like a party to which she gives only her nonscandalous friends invitations. A nonscandalous friend is a friend who brings only one date to a
party and a *scandalous friend* is a friend who brings more than one date. It is important to note that Hildi invites only her friends to the party. Her friends’ dates are welcome to attend, but the invitation goes only to her friends. According to Hildi, “you don’t want to invite scandalous people because then you can’t have a function because your function will go awry” (Ending Interview).

As Hildi told her story, she listed a set of ordered pairs, identified the *x*-coordinates as “friends” and the *y*-coordinate as “dates,” and drew a diagram in which each “friend” has exactly one “date.” For her example of a function shown in Figure 4-5, Hildi did note that two of her friends (i.e., 5 and 6) are dating the same person (i.e., 4). She accounts for two of her friends bringing the same date by saying, “Now ‘4’ is scandalous but you don’t really care because you’re just concerned about your own friends” (Hildi, Ending Interview). Each friend (i.e., *x*-value in the ordered pair or value in the *x*-column) is non-scandalous.

![Figure 4-5](image.png)

*Figure 4-5.* Screen capture of Hildi’s written work captured as she talked about her metaphor for the concept of function.

Hildi’s framing of the concept of function also involves discussing and drawing a diagram of a representation of a scandalous friend. Refer to Figure 4-6. As Hildi points to the 3, she says, “You cannot invite three to your party because they’re scandalous.
They’re dating two people, and if you have scandalous friends your function is not going to be fun” (Ending Interview).

*Figure 4-6. Screen capture of Hildi’s written work as she talked about a nonexample of a function in the context of her metaphor for the concept of function.*

Examination of Hildi’s framing of function reveals mathematics plays a key role in the framing. In this example, Hildi’s framing is the use of a clever metaphor in which a context is based upon the definition of function. Hildi’s party contains two groups of individuals, her nonscandalous friends and their dates. Her definition of nonscandalous friends contains the critical property that separates relations that are functions from relations that are not functions. Hildi’s nonscandalous friends are friends who each bring exactly one date to her party. In functions, each element of one set is mapped to exactly one element of a second set.

In addition, Hildi’s framing contains two representations of a specific function (see Figure 4-5): a set of ordered pairs, and a mapping diagram. It also contains two representations of a specific relation that is not a function (see Figure 4-6) which are a set of ordered pairs and a mapping diagram. Notable in her framing are the two-elements-to-one-element mapping in her function and a one-element-to-two-elements mapping in her relation that is not a function. Hildi uses the juxtaposition of these two framing to emphasize that in a function each element in one set (a nonscandalous friend) is mapped
to exactly one date by checking how many dates each $x$-value dates. For the set in Figure 4-5, Hildi says, “Three is only dating one person. Six is only dating one person. Five is only dating one person. Negative one is only dating one person. So none of your friends are scandalous” (Ending Interview). The fact that two nonscandalous friends bring the same date to her party does not matter. For the set shown in Figure 4-6, Hildi says,

Three is dating five. Three is dating negative one. Six is dating four. Seven is dating three. You cannot invite three [as she points to the 3 in the $x$-column] to your party because they’re scandalous. They’re dating two people and if you have scandalous friends your function is not going to be fun (Ending Interview).

The fact that one friend is scandalous (i.e., dates two people) does matter.

The metaphor of function as an exclusive social event to which only nonscandalous friends receive invitations is highly imaginative and contains the critical elements of what makes a function a function. Although Hildi never addressed this issue, there exist two assumptions in Hildi’s context. Each of Hildi’s invited nonscandalous friends brings a date and individuals who are not dates of invited nonscandalous friends do not simply show up to the party. Mathematically, Hildi’s social event is an example of an onto-mapping.

It is not too difficult to imagine students having fun listening to Hildi telling her story and agreeing that friends who bring two or more dates to a party can be problematic. However, the data does not contain information about what Hildi did after the telling of the story. It is not known whether Hildi went on to emphasize that functions are special relations that pair an independent variable value to a single dependent variable value and why such pairings are mathematically important. In
addition, it is uncertain how much of the mathematics embedded in the story is remembered by students.

**Example 2.2. Associating a TMId to a nonmathematical idea.** Bonnie tells her students that division by zero is “a math sin” (PS Interview). Her statement brings attention to the fact that division by zero is problematic. In a humorous way, she associates a mathematical rule with a shameful offense. One does not commit a sin in mathematics class.

Bonnie’s framing of division by zero took place in a problem-solving interview. When pressed for a mathematical reason why division by zero is a sin, Bonnie initially stated, “because you are not allowed to do it” (PS Interview). Pressed further, she went on to say that the “proof goes back to—what is it— defining division in terms of multiplication and the fact that you can multiply anything by zero” (PS Interview and offered an argument based upon numerical examples. Bonnie’s recorded work is captured in Figure 4-7.) She stated,

Would you think— think that five over zero is zero? Well, that means that zero times zero is five. Well, that doesn’t work. Do you think [pause] that five over zero is five? Well, that means that zero times five equals five. [pause] Well, that is not true. So it’s kind of one of those [pause] quirky little things that doesn’t fit, so therefore we say that you can’t do it. … We talk about division in terms of multiplication and we know that ten divided by two is five because two times five is ten and zero doesn’t fit in (PS Interview).
What this quote shows is that Bonnie does have a mathematical justification for her claim that division by zero is not allowed; however, this justification was not offered until I pressed her for a reason beyond “it’s a math sin.” It is not clear whether she justifies why “division by zero is a math sin” mathematically for her students or stops at “it’s a math sin.” This information is not contained in the data.

**Connecting TMIds to other mathematical ideas of school algebra.** What follows are two examples of framing for this category. In the first example, Bonnie draws analogies between the TMId (i.e., graphing an inequality on the rectangular coordinate plane) to a mathematical idea students learned in a previous lesson (i.e., graphing an inequality on a number line). In the second example, Kate connects the TMId (i.e., computing the value of \( b^2 - 4ac \) given values for \( a, b, \) and \( c \)) to mathematical ideas students will learn in a future lesson. In both of these examples mathematics familiar to students is used to make TMIds accessible and comprehensible.

**Example 2.3. Using an analogy between previously learned mathematics and a TMId.** One of the problem-solving interview tasks asked Bonnie to respond to a classroom scenario in which a hypothetical student employed a problematic approach to graphing \( 2x + 3 < y \). During the “warm-up” section of the interview I asked Bonnie to
graph $2x + 3 < y$ to give her a chance to do the task before I asked her to respond to the hypothetical student’s work. It was during this part of the interview that Bonnie shared what she tells her students about graphing linear inequalities. Bonnie uses an analogy between graphing a linear inequality on the rectangular coordinate plane (the TMId, i.e., $x + 6 < y$) and graphing a linear inequality on the number line (i.e., $x + 6 < 4$). She said,

We start with equations and we have usually one answer [as she records $x + 6 = 4$ and $x = -2$] and then we go to– [pause]. Let’s say inequalities [as she records $x + 6 < 4$]. And so then we have more answers. So we graph our answer on a number line because we have more answers [as she draws an open-ended ray on a line]. And then we go to $x$ plus six equals $y$ [as she records $x + 6 = y$], and we have two things that vary. We have more than one answer that will make this true and those all end up on a straight line [as she creates graph under $x + 6 = y$]. And so in an inequality it is like going from a point [as she points at $x = -2$] now we’ve got half the number line. Here we went from a single point [as she points at $x = -2$] now we’ve got half the number line. Here we have a line [as she traces the graph of $x + 6 = y$], so when we do an inequality it is half the coordinate plane [referent unclear] (PS Interview).

Bonnie’s recorded work for her discussion is captured in Figure 4-8.
Figure 4-8. Screen capture of Bonnie’s written work created as she connected graphing an inequality on the number line with graphing an inequality on the Cartesian coordinate plane.

Although in her written work (see Figure 4-8) Bonnie did not graph an inequality on the rectangular coordinate plane, her verbal descriptions describe what the graph of a linear inequality on this plane would look like (i.e., a half-plane). Based upon Bonnie’s statements, I created Figure 4-9 to illustrate the connections she made between the “other mathematical idea” and the TMIld. Bonnie’s framing reflects connections being drawn between objects that serve the same purpose in their respective domains (e.g., boundary point and boundary line, half line and half plane).
“Other mathematical idea” (Familiar): Graphing \( x + 6 < 4 \)

Connections

TMId (New): Graphing \( x + 6 \geq 4 \) \( y \)

| \( x + 6 = 4 \) | \( x = -2 \) | \( \Leftarrow \) Boundary equations | \( x + 6 = y \) |

"single point"

\[ \leftarrow \]

"half the number line"

\[ \leftarrow \]

"half the coordinate plane"

\[ \Rightarrow \]

\[ \Rightarrow \]

\( \Rightarrow \)

\( \Rightarrow \)

Figure 4-9. Summary of connections made by Bonnie during her explanation of graphing \( x + 6 < 4 \) and \( x + 6 \geq 4 \).

Example 2.4. Connecting a familiar mathematical idea to TMIds to be officially encountered in a future lesson. Prior to my first observation, Kate began preparing students for the quadratic formula by asking students to work the following exercise:

“Evaluate \( b^2 - 4ac \) when \( a = 2, \ b = 5, \ \text{and} \ c = 1 \)” (exercise 1a, Assignment 20; see Figure 4-1). Successful completion of this exercise involves students substituting values

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27 Bonnie does not specify the exact inequality.

28 Note: Bonnie graphed a solid line.

29 Note: Because Bonnie graphed only a solid boundary line, her regions have a solid boundary.
into the given algebraic expression, and using the order of operations to compute the value of the expression. To an observer who is familiar with school algebra, Kate—by her selection of $b^2 - 4ac$ as the expression to be evaluated—began a connection to the discriminant, to evaluating the discriminant, and to solving quadratic equations using the quadratic formula. Again, to an observer who is familiar with school algebra, Kate strengthened the connection to the quadratic formula by modifying the way in which she posed the exercise: “If $ax^2 + bx + c = 0$, then $D = b^2 - 4ac$. Find $D$.” (Assignment 24, Observation 1; see Figure 4-2). Given quadratic equations in the specified form, students then computed the value of $D$.

During the warm-up in Observation 2, Kate made an explicit connection between $b^2 - 4ac$ and future TMIIs for students when she said,

So I don’t want to freak you out with this formula or anything but it’s kind of important that we’re able to find [pause] $D$ [pause] which we are later going to call the discriminant. And um, it’s going to tell you stuff about the parabola. But [pause] you know [pause] we’re not there yet. We’re just practicing this formula right here. [Kate draws a box around $b^2 - 4ac$.] (Observation 2)

With this statement Kate connected the expression, $b^2 - 4ac$, with the term discriminant. The value of $b^2 - 4ac$ was linked to parabola, which students knew is the graph of a quadratic function (see Example 1.1, Figure 4-3).

**Discussion:** Connection to existing research. Three of the four examples presented as evidence that teachers make connections between TMIIs and nonmathematical as well as mathematical ideas also illustrate what Baxter and Williams
(2010) call analytic scaffolding, “the support offered to students by materials, teachers, or one another, in building mathematical understanding” (p. 11). When Kate (Example 2.1) explained that the value of $D = b^2 - 4ac$ will be used in future lessons to provide information about characteristics of graphs of quadratic functions (i.e., parabolas), she initiated a new meaning for, and therefore, a different understanding of $D = b^2 - 4ac$. Prior to Kate’s explanation, to students, $D = b^2 - 4ac$ was simply a formula used to practice the order of operations. Kate’s explanation indicates that in a future lesson, students will use $D = b^2 - 4ac$ as a tool. Hildi used a metaphor (a form of analytic scaffolding explicitly identified by Baxter and Williams, 2010) to introduce the concept of function to students. As part of her explanation of how a function is like a party to which only her nonscandalous friends receive invitations, Hildi used multiple representations (another form of analytic scaffolding explicitly identified by Baxter and Williams). Bonnie draws upon her students’ understanding of graphing inequalities on a number line as she explains how to graph linear inequalities on the rectangular coordinate plane. In contrast, Bonnie’s statement that division by zero is a math sin (Example 2.2) is not an example of analytic scaffolding. Beyond informing students that division by zero is not to be done, Bonnie’s framing does not serve to help students better understand the concept of division by zero.

**Characteristic 3.** Breaking TMI ds into smaller chunks is a strategy that all of the participants in this study seem to use. The data revealed a variety of examples of how teachers orchestrate students’ engagement/interaction with those smaller chunks.
Revealing/modeling procedures chunk by chunk or step by step. What follows are three examples of this category, each illustrating different ways teachers reveal or demonstrate procedures step by step. They differ in the timeframe over which the framing takes place, the extent to which students are expected to participate in the framing, how the framing is enacted in the classroom (i.e., planned event or a response to a student question/comment), and what purpose the framing serves in addition to making a TMId accessible and comprehensible. In the first example, Gary reveals how to derive the quadratic formula using completing the square, which Gary identified as a lesson objective and therefore, a planned event. In the second example, Bonnie shows students how to simplify \( \frac{\sqrt{14}}{\sqrt{35}} \), a homework exercise. This demonstration was a response to a student’s request. In the third example, Kate’s guides her students through a development of her version of the quadratic formula. Like Gary’s framing, Kate’s framing is planned. Unlike Gary’s framing, Kate’s framing is not strictly a demonstration and it takes place over several class meetings.

Example 3.1. Breaking a TMId into smaller steps/stages—a careful demonstration. The TMId, derivation of the quadratic formula, was unpacked prior to class. Students were expected to watch the demonstration. This example was chosen because it illustrates a teacher’s (i.e., Gary’s) efforts to break down a large procedure into steps to meet his lesson objective, and because his steps are clearly and easily identifiable. Gary used a PowerPoint presentation. Figure 4-10 and Figure 4-11 identify the seven major steps in his derivation. As implemented, however, the seven steps each contained two to four substeps. Gary programmed each slide to reveal each substep with
a click of his mouse. The data do not contain information about Gary’s reasons (e.g., mathematical, pedagogical, other) for choosing to divide the derivation into seven steps and corresponding substeps.

Figure 4-10. Slide sorter view (slides 1–6) of Gary’s PowerPoint presentation of the derivation of the quadratic formula (Observation 1).
During the Postobservation 1 Interview, Gary stated that his lesson objective was just to let students know that the quadratic formula is “not some mystical, magical thing that just popped out. It’s something that was derived” (Gary, Postobservation 1 Interview). Gary’s statement suggests a commitment that his students be given a mathematical rationale for the quadratic formula. Gary also revealed that although he wanted his students to know that the quadratic formula is derived from a general quadratic equation, the derivation was demonstrated with no expectation for students to master the procedure or to reproduce it (Gary, Postobservation 1 Interview). Thus, it
seems that in the observed lesson what was important to Gary is that his students see the quadratic formula being derived.

Example 3.2. Breaking a TMId into smaller steps—a carefully explained demonstration of how to do a procedure. The TMId is unpacked as students watch. Students are expected to apply the methods explained/demonstrated to similar exercises. This example was chosen because it illustrates a teacher’s (Bonnie’s) response to a student request to see how to do a homework problem (i.e., simplify $\frac{\sqrt{14}}{\sqrt{35}}$). Unlike Gary, who had no expectation that his students learn how to do what he demonstrated, Bonnie did expect students to learn how to do the procedures required in this and all other problems she assigned.

What follows is a section of Bonnie’s Observation 1 transcript. The transcript shown in Figure 4-12 is divided into sections using horizontal lines. Each section identifies a step in the simplification process as seen from my perspective as an observer. Except for the first section, which establishes the problem being explained, each transcript section (i.e., step) contains either a reminder to students of a property that Bonnie used in the simplification process, or a discussion of what Bonnie did, and ends with Bonnie recording an expression or part of an expression that corresponds to what was being said or to the conclusion of the problem. This pattern of discourse is consistent with all of the homework/quiz problems whose solutions Bonnie did for her students.

What is notable about Bonnie’s framing is the level of detail with which she explains this exercise. Furthermore, this level of detail is the level of detail with which
Bonnie explained each requested homework problem, assigned warm-up exercises, and quiz/test problems during the observation week. From this researcher’s perspective, it seems that Bonnie uses this way of framing to reteach TMIDs.

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Screen captures of Bonnie’s corresponding recorded work</th>
</tr>
</thead>
</table>
| **Step 1.**  
Student:  Can you do number thirty-four?  
Bonnie:  Did you say thirty-four? It says divide the square root of fourteen by the square root of thirty five. So there’s how you might start the problem. |
| **Step 2.**  
Remember you have that property that says you could put them under the same radical if you wanted |
| **Step 3.**  
And then you could reduce the fraction. So if I use seven to reduce with I’d get two over five. … |
| **Step 4.**  
Remember when you work with radicals you can’t leave radicals in the bottom of a fraction. So you have to rationalize—you need to turn that five into a perfect square. So what could we multiply y that would make it a perfect square?  
Student:  Um, five?  
Bonnie:  Square root of five [as she records a $\sqrt{5}$ next to the 5 in $\frac{2}{5}$]. |
Step 5. And what we do to the bottom, we have do to the top [as she records $\sqrt{5}$ next to the 2 in $\frac{2}{5}$] because we’re really just multiplying by one.

Step 6. So this becomes the square root of ten [as she records $\sqrt{10}$]

Step 7. This becomes the square root of twenty-five [as she records $\sqrt{25}$]

Step 8. which is just five [as she records 5 below $\sqrt{10}$].

Step 9. Now we can’t reduce the ten and the five because one is under the radical and one is not. So remember, you can only work with things under the radical or things outside the radical.

Figure 4-12. Bonnie's explanation of homework problem #34 (Observation 1).

Example 3.3. Unpacking TMIds — an aspect of curriculum development. In this example, components of TMIds are revealed over several lessons. Each new component is treated as a TMId in a lesson.

In the previous two examples, students were primarily observers of Gary’s and Bonnie’s unpacking of the derivation of the quadratic formula, and of how to simplify a numerical radical expression, respectively. Data about Kate reveal a more complex relationship between Kate’s unpacking of her version of the quadratic formula and
student engagement. In Gary’s class, students are expected to watch the derivation unfold, remember that a derivation exists, and apply the quadratic formula. Kate engages her students in working with components that make up her version of the quadratic formula (i.e., \( x = \frac{-b \pm \sqrt{D}}{2a}, \ D = b^2 - 4ac \)) through solving problems long before she introduces students to her version of the quadratic formula itself. Kate explained, “what we are trying to do is slowly build in the quadratic formula without shocking the system” (Postobservation 1 Interview).

In the last postobservation interview, I asked Kate to talk about her upcoming lesson on the quadratic formula and her development of the quadratic formula with students. It was during this conversation that Kate provided insights into her unpacking of the quadratic formula. Kate said,

Well, we are not introducing the quadratic formula without showing them how the quadratic formula is derived. … I do not like showing them how the quadratic formula is derived. But if I am going to show them—and it is a Standard\(^{30}\)—then I am certainly going to have them have the tools of completing the square. So one of the reasons that we are doing completing the square before quadratic formula is that it requires [an] understanding of completing the square, at some level, in order to do that. Now we will not be doing completing the square with the

\(^{30}\)Later in the interview, Kate revealed that her statement “it is a Standard” was intended to let me know that derivation of the quadratic formula is one of her state’s State Mathematics Standards and has appeared on the state’s Algebra 1 assessment.
coefficient in front of \( x \) squared, so, whereas quadratic formula does (Ending Interview).

I infer from Kate’s discussion that the results of Kate’s unpacking of her version of the quadratic formula include

1. Showing students how the quadratic formula is derived; and
2. Developing student understanding of completing the square.

As part of the discussion of the upcoming lesson on the quadratic formula, Kate provided a copy of the worksheet she would be using in the lesson. It consists of eight equations to solve using the quadratic formula. From the direction and reminder of steps at the top of the worksheet (see Figure 4-13), I infer that additional components include

3. Solving quadratic equations;
4. Identifying \( a, b, \) and \( c \) from a given quadratic equation;
5. Calculating the value of the discriminant; and
6. Using the quadratic formula to calculate the value of the variable that satisfies the initial quadratic equation.
Over the course of the 3-day observation, Kate engaged her students in four of the six listed components. Figure 4-14 and Figure 4-15 capture Kate’s projected work from the warm-up problems in Observation 2 and 3, respectively. In Observation 2, Kate engaged her students in identifying \(a\), \(b\), and \(c\) from two given quadratic equations (#4) and computing the value of the discriminant for each quadratic (#5). In Observation 3, Kate engaged students in solving a quadratic equation (#3) and using completing the square (an aspect of #2). This example is interesting because Kate decompressed the TMId over an extended period of time, and she engaged students in working with components of the TMId as she decompressed the TMId for students.

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**Figure 4-13.** Screenshot of problem 1 of Kate’s Assignment 27, a worksheet assigned the first lesson after Kate’s last observation.
Figure 4-14. Screenshot of Kate’s solutions to her Observation 2 warm-up problems.

Given the equation:

\[ x^2 + 8x = -7 \]

Adding 16 to both sides:

\[ x^2 + 8x + 16 = -7 + 16 \]

Resulting in:

\[ (x + 4)^2 = 9 \]

Figure 4-15. Reproduction of Kate’s partial solution of an Observation 3 warm up problem. (Students were asked to solve the given equation for \( x \).)

Providing students with organizational schemes through which to view or by which to work with components of a TMId. In the previous category, teachers directly revealed components/constituent parts of TMIds to their students. In this category, teachers make components/constituent parts of a TMId easier for students to observe or to use by providing students with some kind of organizational scheme. From an observer’s perspective, the organization is a strategy to help students focus their attentions on properties of TMIds.
Example 3.4. Organizing a collection of familiar procedures (i.e., TMIds) under a macro procedure. There are many procedures students have to learn in secondary school algebra. Knowing which procedure(s) to use and when to use it/them can be a challenge for many students. In this example, Bonnie chose to help her students know which of seven polynomial factoring procedures to apply by teaching them a macro procedure. The macro procedure provides students with a systematic way of examining a given polynomial, and using the information obtained from the examination to decide which of the seven factoring procedures to use.

I became aware of Bonnie’s macro procedure during Observation 2 as she explained how to do the warm-up problem. Her first two questions, “What are you supposed always [to] check for first when you are getting ready to factor?” (Bonnie, Observation 2) and “So what are you supposed to do after you look for a greatest common factor?” (Bonnie, Observation 2) were not surprising. However, I found the answer to her second question, “Well first you have to count the terms” (Observation 2) unexpected. During the Postobservation 2 Interview, Bonnie was asked to expand on what she meant by “first you count the terms.” Bonnie said,

You always check for the greatest common factor first. … And then I have them count terms. How many. [pause] So if you have two terms, it’s got to be the [pause] difference of squares or the difference of cubes [pause] or the sum of cubes. … So then if they have three terms, it would be, you know, a trinomial. I call it ‘undo FOIL,’ but—or the AC rule31, if it’s—the x squared has a, you know

31 An explanation of Bonnie’s AC Rule is provided in Appendix B.
coefficient. And then if there is four, you try to group it [pause] and factor and create a common factor. [pause] So that pretty much—if they can follow that pattern, they can pretty much factor (Postobservation 2 Interview).

As she expanded upon what she meant by “count the terms,” Bonnie created the list shown in Figure 4-16.

*Figure 4-16.* Bonnie’s list of factoring procedures (created during the Postobservation 2 Interview).

The flowchart shown in Figure 4-17 was created to facilitate analysis of Bonnie’s macro procedure for factoring. The flowchart is annotated with information she provided during the Postobservation 2 Interview as she clarified what she meant by undo FOIL and the AC rule. Undo FOIL and the AC rule are methods for factoring quadratic trinomials of the form, \( ax^2 + bx + c \). Bonnie and her students use undo FOIL when \( a = 1 \), and the AC rule when \( a \neq 1 \) (Postobservation 2 Interview).
This example of framing was chosen because it offers a counterpoint to examples 3.1 and 3.2 in which Gary and Bonnie, respectively, took a TMId and broke it down into smaller components. In this example, Bonnie takes seven separate factoring methods (i.e., components) and uses the criteria of number of terms to organize them into a hierarchy that is intended to help students factor any given polynomial. This macro procedure also focuses students’ attentions on properties of the polynomial to be
factored that help students decide which of the seven procedures to use. Bonnie herself described her list as a problem-solving strategy (Postobservation 2 Interview).

When I asked, “how would students know whether a polynomial that consists of five terms is not factorable?” Bonnie replied that they would know, “if they tried everything that they knew how to try, and it— and they couldn’t. They couldn’t create that common factor” (Postobservation 2 Interview). According to Bonnie, students would then know that the polynomial is prime. However, the more enlightening part of her response to my question is contained in the continuation of her response. With respect to polynomials with four or more terms, Bonnie went on to say, “And these and we don’t do very many of these, kind of. I mean this is something that happens but not nearly as much, you know, as if you have difference of squares” (Postobservation 2 Interview). I infer from this continuation that the TMId for this framing is not factoring any polynomial. The TMId is factoring a select set of polynomials, and few of the polynomials Bonnie gives students fall into the category of four or more terms.

Bonnie’s framing of how to decide which factoring procedure to use is designed so that students examine polynomials for visible properties (e.g., number of terms, degree of polynomial, value of quadratic coefficient). When Bonnie’s students see a certain set of visible properties, they are supposed to know which factoring method to apply. In this sense, Bonnie’s framing of factoring polynomials is similar to Hildi’s framing of a base raised to a negative exponent (see Example 1.2). When Hildi’s students see a “−” sign in an exponent, they are supposed to know that it means “flip it” (i.e., the base). Both framings identify stimuli and corresponding response.
Example 3.5. Organizing TMId(s) using a systematic written record. Faith requires that her students take notes using a special kind of stationery\(^{32}\) to help her focus students’ attentions on the TMId(s) (see Figure 4-18). For the lessons Faith conducted during the observations, on graphing quadratic functions, she presented her notes using a slightly modified version of this stationery. While the use of special stationery is not mathematical, this system of organization facilitated students’ opportunities to see—with their eyes—unpacked ideas. Figure 4-19 is a screen capture of one page of notes Faith produced during Observation 1. Even though the inscriptions are rather difficult to read, one can see that graphing quadratics has at least three components: the function itself (the left column), a table of values (the middle column), and the graph (the right column).

\(^{32}\)The stationery is based upon the Cornell note-taking system, which was developed by a professor at Cornell University (http://www.montgomerycollege.edu/Departments/enreadty/Cornell.html). This system was adopted by the cross-disciplinary team of teachers of which Faith is a member.
Figure 4-18. Sample of Faith’s note-taking stationery for students.
Figure 4-19. Screen capture of Faith’s notes presented for graphing quadratic functions during Observation 1. The handwritten sections were completed as the lesson progressed.

The organization seems to be intended to support Faith’s efforts to guide students in looking for similarities and differences among the graphs of these quadratic functions, all of which are members of the family, \( y = ax^2 \). Similarities discussed in this lesson
include: The vertex for each of these parabolas is (0,0), and the line of symmetry for each parabola is the y-axis. Differences discussed include: the “fat-/skinny-ness” of the graph, the direction the parabola opens, and the sign of the coefficient of $x^2$.

Example 3.6. Organizing TMIds using an evolving routine. In this example, Faith’s framing of how to graph quadratic functions as part of an evolving routine is the focus of discussion. During a Postobservation interview, Faith revealed that she had established this graphing routine during the fall semester when she and her Algebra 1 students plotted points to create graphs of linear functions of the form, $y = mx + b$, for which $m$ is an integer. Faith had her students compute the coordinates of five points using $\{-2, -1, 0, 1, 2\}$ as $x$-values. She said, “When we started linears [sic] we were just going to use negative two to two. So I always just gave them zero in the middle” (Faith, Postobservation 2 Interview).

When Faith and her students worked on graphing $y = mx + b$ ($m$ is a nonintegral rational number), Faith and her students began to use $\{-2k, -k, 0, k, 2k\}$ as $x$-values. The value of $k$, according to Faith is the value of the denominator of $m$. She said, When we started off it was like, I gave them their first introduction to a linear with a fraction [pause] a fractional slope and I let them use the number that they had been using which was negative two. And some of them were like how do you add—you know, negative half to three and they—they [pause] I died a little inside because they were algebra students that did not know how to add fractions. … And then they realized that if you just take these numbers negative two to two and multiply it by whatever the denominator was that gave you good numbers to pick. So they
have now that they came up with the method of how to pick numbers, from there on out, by just multiplying by whatever the denominator was (Faith, Postobservation 2 Interview).

Based upon the ways in which Faith framed how to graph quadratic functions during Observation 1 in conjunction with what she shared in her explanation of how Faith and her students graphed linear functions, Faith’s basic graphing routine (i.e., 1st generation) is as follows:

1. Set up a three-column table.
2. Enter \{-2, -1, 0, 1, 2\} in the \(x\)-column.
3. Compute the corresponding \(y\)-values.
4. Draw a set of \(x\)- and \(y\)-axes and plot points on the coordinate plane.
5. Draw a line through the points.

The 2nd generation graphing routine uses \{-2k, -k, 0, k, 2k\} for entries in the \(x\)-column. Everything else is the same.

When Faith began teaching her unit on graphing quadratic functions (Observation 1), she started by having students graph quadratic functions of the form \(y = ax^2\) for which \(a\) is an integer. For these functions, Faith and her students used her basic (1st generation) graphing routine. When Faith and her students progressed to quadratic functions of the form \(y = ax^2\) for which \(a\) is a nonintegral rational number, Faith and her students used her 2nd generation graphing routine. This time the value of \(k\) is the value of the denominator of \(a\). Figures 4-20 and 4-21 reflect Faith’s use of these two graphing routines.
Figure 4-20. Illustration of Faith’s basic (1st generation) graphing routine: \( a \in \mathbb{R} \). The \( x \)-values are \( \{-2, -1, 0, 1, 2\} \) (Observation 1 class notes).

Figure 4-21. Illustration of Faith’s 2nd generation graphing routine: \( a \in \mathbb{R}, a \neq 0 \). The \( x \)-values are 2 times \( \{-2, -1, 0, 1, 2\} \) (Observation 1 class notes).

Faith’s lesson in Observation 2 was devoted to graphing quadratic functions of the form, \( y = ax^2 + c \) (where \( a \) and \( c \) are real numbers) for which Faith and her students used Faith’s 1st and 2nd generation graphing routines. A student asked, “Do you always have to have zero in the middle? [Pause] Could it be like negative five?” (Observation 2). Faith eventually offered the student a two-part answer: first, the axis of symmetry always seems to be the \( y \)-axis (Observation 2) where the value of \( x \) is zero, and second, she likes “to consider all the cases when \( x \) could be negative, when \( x \) could be zero, and then again when \( x \) could be positive” (Observation 2).

The question the student asked, “Do you always have to have zero in the middle?” is a general question. Faith’s response is an interesting response. Mathematically, the
correct answer to the student’s question is no. Zero does not have to be the middle x-value. Faith seemed to confine her answer to quadratic functions of the form \( y = ax^2 + c \), to the convention she has made a part of her evolving routine for graphing.

Later that afternoon, I asked Faith to share what came to mind when the student asked whether zero always has to be in the middle. Faith admitted to scrambling for an answer (Observation 2) and shared,

I knew what to expect so [pause] I kind of wanted to lead them in that direction. I know that the vertex and the axis of symmetry would be where \( x \) was zero and I wanted them to have a nice symmetric parabola versus if I picked negative three to one where zero wasn’t in the middle, they would get a short side on the positive and then a big side on the negative and then maybe they are not going to see that it is a nice symmetric parabola (Postobservation 2 Interview).

Mathematically, parabolas are symmetric regardless of how much of a parabola appears in a graphical representation. Faith’s response to the student reveals her preference for visually symmetric graphs of parabolas. It seems that her preference has a strong influence on how she frames how to graph quadratic functions.

Faith’s efforts to show students how to graph quadratic functions of the form, \( y = ax^2 + bx + c \), for \( b \neq 0 \) using the new (i.e., 3\(^{rd}\) generation) graphing routine did not go very smoothly. Before discussing what happened in class, I will describe Faith’s 3\(^{rd}\) generation graphing. Again, the modification is how the \( x \)-values are determined (i.e., step 2). For this family of quadratics, the coordinate pair for the vertex is \( \left( \frac{-b}{2a}, y \right) \). In Faith’s new routine, the \( x \)-values satisfy the following conditions:
a. \( x_1 < x_2 < x_3 < x_4 < x_5 \)

b. \( x_3 = \frac{-b}{2a} \)

c. \(|x_3 - x_1| = |x_3 - x_2| \) and \(|x_3 - x_4| = |x_3 - x_5|\)

It is important to note here that Faith introduced the new conditions of her 3rd generation routine through specific examples. She did not use symbolic representations. The symbolic representations are used here as a way of describing the conditions inferred from the data. Examples illustrating the use of the new routine can be seen in Figure 4-22.

*Note:* The vertex is identified in the table with an oval.

*Figure 4-22.* Illustration of Faith’s 3rd generation graphing routine: \( x \)-coordinate of the vertex is \( \frac{-b}{2a} \) (Observation 3 class notes).
Faith began her framing of how to graph quadratic functions of the form
\[ y = ax^2 + bx + c, \quad a \neq 0,1 \text{ and } b \neq 0 \]
by presenting the general coordinates of the vertex as \( \left( \frac{-b}{2a}, y \right) \), and told her students they would discuss where the \( \frac{-b}{2a} \) comes from during the next class (Observation 3). However, her students seemed to struggle with the formula and why they were using it. Trying to engage students in graphing
\[ y = 2x^2 - 8x + 7 \]
she said,

We talked about [pause] this is the formula. Let’s work with it. And then we’ll figure out, well, where did it come from? Because you’re right—why did I take opposite B [over two A]? That’s a good question. Why did I? I want to get some practice with it before we go into [pause] why. (Faith, Observation 3).

It is important to note here that on the previous afternoon, during the Postobservation 2 Interview, when asked what her plans were for the Observation 3 lesson. Faith stated that she planned to have the class graph quadratics of the form,
\[ y = ax^2 + bx + c, \text{ for which } a \text{ and } b \text{ are nonzero and } a \text{ does not equal 1, and that she planned to introduce the class to the formula of the vertex, } \left( \frac{-b}{2a}, y \right). \]
When I asked her why the formula works, she stated explicitly,

No idea. I am sure if I sat down and figured it out I would know. But I just… I just know the formula. I wasn’t a [pause] I wasn’t a [pause] “why” kind of student, you know they gave me a formula (Postobservation 2 Interview).

During the Postobservation 3 Interview, Faith revealed that had not worked out exactly why the formula works, but that it “it has to do with taking this standard form
[i.e., \( y = ax^2 + bx + c \)] um and manipulating it into [pause] vertex [pause] form [i.e., \( y = a(x-h)^2 + k \)]” (Faith, Postobservation 3 Interview). Thus, a major reason Faith chose not to address her students’ questions about \( \frac{-b}{2a} \), was that she could not address the question. At the time of class, she had not worked out the derivation of where the formula comes from or why it works.

Faith’s framing for how to graph quadratic functions of the form 
\[
y = ax^2 + bx + c \quad (b \neq 0)
\]
includes the requirement that the graph of the five points resulting from the way the \( x \)-values are determined to be symmetric about the line \( x = \frac{-b}{2a} \).

In an effort to gain additional insight about Faith’s framing, I posed a classroom scenario. The scenario was based upon a problem Faith and her students discussed in class: Graph 
\[
y = -3x^2 - 2x - 5
\]
Faith and her students determined the coordinates of the vertex to be \( \left( -\frac{1}{3}, -6 \right) \) (Observation 4) and decided to use \{ -1, -2/3, -1/3, 0, 1/3 \} for \( x \)-values. I asked Faith to discuss what would have happened if one of her students had wanted to use \{ -2, -1, -1/3, 0, 1 \} for \( x \)-values instead. The \( x \)-values in both sets are listed in ascending order and have the \( x \)-coordinate of the vertex as the middle value. However, the set proposed by the hypothetical student would not require the class to work with fractions beyond the computation of the vertex coordinates. In a previous postobservation interview, Faith stated computation with fractions is a challenge for many of her students (Postobservation 2 Interview) and was the reason for her introduction of her 2\textsuperscript{nd} generation graphing routine. In response to my prompt, Faith said,
Then we would have had to address that they are not equidistant from the axis of symmetry. And so that we are not going to get that nice pattern that we have gotten before, where negative one and zero would not produce the same output.

(Ending Interview)

Since Faith’s response did not reveal any new information, I asked more explicit questions. Given that the task is to graph the function, “do we have to choose symmetric values to graph? … Is that part of what you are trying to maintain in your graphing technique?” (Ending Interview). After saying, “No,” Faith offered some new information. In reference to \{-1, \frac{1}{3}, 0, \frac{2}{3}\}, she said,

I was trying to keep it [pause] symmetrical [pause] because I knew that they could plug in negative one. And then one third would be the same answer. And then they could plug in zero and negative two thirds would be the same answer.

(Ending Interview)

I infer from Faith’s comments that part of her framing involves reducing the number of computations needed. By choosing \{x_1, x_2, x_3, x_4, x_5\} so that $x_1 < x_2 < x_3 < x_4 < x_5$, $|x_1 - x_5| = |x_5 - x_1|$, and $|x_3 - x_2| = |x_2 - x_3|$. Faith’s students would be able to compute fewer $y$-values. My inference is supported by a comment Faith made later in the interview.

I just— I like five points, I know. I always like to find the vertex and then on either side. But, knowing that it’s, knowing that it’s symmetrical and then if you just find one half then you already know the other half so (Ending Interview).
It seems that Faith’s framing of how to graph quadratic functions is highly influenced by benefits of line symmetry and a preference for creating graphs using five points. Her framing helps students produce “U-shaped” parabolas as opposed to “J-shaped” parabolas, and requires the computation of three y-coordinates as opposed to five.

**Characteristic 4.** The role mathematics plays in framing a TMId seems to vary.

Framing consists primarily of two components: the TMId(s), and the way in which the TMId(s) is (are) framed. A single mathematical idea can be framed in several ways. The role mathematics plays in framing a TMId seems to vary. As a means of illustrating differences in the role mathematics plays in how a TMId is framed, I will discuss different examples of framing for two TMIds: (a) expanding the product of binomials, \((ax + b)(cx + d)\), and (b) factoring quadratic trinomials of the form \(x^2 + bx + c\).

Following these discussions will be an examination of pairs (i.e., multiplication of binomials and factoring quadratic trinomials) of framings by Bonnie, Delia, and Kate to understand how the relationship between these two procedures is addressed.

*Expanding the product of binomials, \((ax + b)(cx + d)\).* In Examples 4.1 to 4.3, the expressions involved in multiplying binomials do not have contextual meaning. In Example 4.4, this procedure is situated in the context of computing areas of rectangles. When examining the product of two binomials strictly from a symbolic perspective, the distributive property of multiplication with respect to addition,\(^{33}\) the associative property of multiplication, and the commutative property of multiplication are the main

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\(^{33}\) Subsequent reference to the distributive property of multiplication with respect to addition will be shortened to “the distributive property.”
mathematical ideas underlying multiplying two binomials.

\((ax+b)(cx+d)\)  
\[= (ax+b)cx + (ax+b)d\]  
\[= acx^2 + bcx + adx + bd\]  
\[= acx^2 + (bc + ad)x + bd\]

by the distributive property
by the commutative property of multiplication & the associative property of multiplication
by the distributive property and the commutative property of multiplication

The first three examples differ in the extent to which the distributive property is brought to the attention of students.

**Example 4.1. Using FOIL.** FOIL is a mnemonic commonly used in algebra classrooms in the United States. It is used to help students remember how to multiply two binomials. Each letter in the acronym identifies which pairs of terms to multiply in order to multiply correctly two binomials (see Figure 4-23). The term FOIL was used by Ann (Observation 2), Bonnie (Observations 1 and 2), Gary (Observation 1) and their respective students to identify what to do next in their respective discussions of homework and/or practice problems. Although it is possible that the distributive property was mentioned when this TMId was introduced to students, no mention of the distributive property was made by Ann, Bonnie, and Gary as well as their respective students during classroom observations.
Example 4.2. Using the distributive property. The focus of Delia’s lesson for Observation 3 was multiplying pairs of linear binomials. Delia’s board work, as it evolved, is captured in Figure 4-24. With each practice problem, as she guided students through the procedure, Delia repeatedly used the terms distribute or distributive property.

On the previous day, when she shared what she planned to teach for this lesson, Delia said explicitly, “When we teach multiplying binomials, we do not use FOIL: the acronym, F-O-I-L.” (Postobservation 2 Interview). Delia offered two reasons for not using FOIL to frame this concept. First, this decision is a mathematics department decision (Postobservation 1 Interview). Second, as a student, Delia did not understand
FOIL. Delia said, “When I was in school it made no sense to me. The acronym did not help me remember what to do. It just doesn’t make sense– it never made sense to me” (Postobservation 2 Interview). In fact, Delia went so far as to say,

I call it a swear word in my room because kids will say to me, kids who have seen it in eighth grade will say, “Oh FOIL,”… I’m like that is a swear word. Don’t say it, unless you can explain to me what that means and the math behind it.

(Postobservation 2 Interview)

Delia never explicitly stated what she meant by “the math behind it [FOIL].” However, I infer from her work with students and her repeated use of the term, distribute, and references to the distributive property that “the math behind it” is a reference to the distributive property.

**Example 4.3. Using “the lattice/box method.”** The box or lattice method of multiplying two binomials starts with the creation of a rectangle (i.e., the box) that is divided into four quadrants. One of the two given binomials is written outside of the box along the top so that each term is above a column. The second of the two binomials is written outside of the box so that each term is recorded to the left of a row (see Figure 4-25). What is recorded in a quadrant is the product of the expressions outside the box above the quadrant and outside the box to the left of the quadrant. This method is illustrated in Figure 4-25 using the general expression. This method can be adapted to help students multiply a polynomial of $n$ terms by a polynomial of $m$ terms by starting with a lattice/box and dividing it into $n$ columns and $m$ rows. The lattice/box method, by itself, is a graphic organizer, a way of helping students keep track of the products of
terms and to identify the like terms.

Figure 4-25. Multiplying two binomials (symbolically) with a graphic organizer (a framing method discussed by Bonnie, Postobservation 3 Interview).

Faith and Bonnie provide evidence of their use of the lattice/box method as an organizer. Of the box method, Faith said,

I first introduce multiplying because I’ve noticed that the kids seem to really struggle with FOIL. Um, so we always—we've introduced multiplication we always find that oh the x squared term goes here and the constant term goes here (Postobservation 1 Interview).

Faith never explicitly identified the nature of her students’ struggles with FOIL. However, because she made a point of identifying the placement of the quadratic and the constant terms in the box, I infer that the struggle could have been about remembering which terms to multiply in the procedure, or possibly, multiplying only the first two terms and the last two terms in the binomials (i.e., \([ax+b][cx+d]=acx^2+bc\)). The box method as an organizer is an effective way to deal with these problems.

Evidence of Bonnie’s use of the “box method” appeared during Observation 2 when she projected her answers to the previous week’s quiz. Figure 2-26 shows her answer and solution method for a quiz problem, \((4p-3)(3p^2-p+2)\). During the
Postobservation 2 Interview, I asked her to talk about her solution method. Before directly addressing her use of the box method, Bonnie stated that

Typically when we do these problems, we just distribute. We take four \( p \) [as she tapped her finger on \( 4p \) in the expression] times three \( p \) to the third [as she tapped on \( 3p^3 \)], four \( p \) [as she tapped her finger on \( 4p \)] times \( p \) [as she tapped her finger on \( p \)] and four \( p \) [as she tapped her finger on \( 4p \)] times two [as she tapped her finger on \( 2 \)], and we take the negative three. (Postobservation 2 Interview)

Based upon Bonnie’s statements and her tapping motions, I infer that Bonnie was talking about multiplying the binomial and the trinomial using the distributive property on the given expression.

![Figure 4-26](image)

*Figure 4-26.* Bonnie’s answer to a quiz problem, an illustration of Bonnie’s use of the “lattice method” to multiply a binomial and a trinomial (Observation 2).

Bonnie went on to say,

Well sometimes kids get—or lose things or miss things or forget to multiply or something. So I found this, where they did this—sort of, I guess it’s sort of like lattice in a way, but you put one on one direction and one on the other direction
[see Figure 4-26]. And you’re really doing the same thing. You’re taking four $p$ times three $p$ to the third and getting your twelve $p$ to the fourth. And you’re taking four $p$ times negative $p$ and getting negative four $p$ squared. And you’re getting eight $p$. So you’re doing the same thing that you did here [as she points to $(4p-3)(3p^3-p+2)$] but you’re just kind of organized. They can make sure they have something in every box. (Postobservation 2 Interview)

What is interesting about Bonnie’s discussion of how to use the lattice/box to multiply the binomial and trinomial is that she also makes the claim that multiplying the binomial and trinomial using the lattice/box is the same as multiplying the binomial and trinomial using the distributive property. Thus, Bonnie sees a connection between the distributive property and the lattice/box method. Nevertheless, when Bonnie showed her students her answer to the quiz question, no student requested an explanation and none of classroom discussions involved the lattice/box method for the rest of the observation week.

Therefore, the data do not contain evidence for or against Bonnie drawing students’ attention to this connection for students as well.

Later in the interview Bonnie acknowledged that the lattice/box method can be used to multiply two binomials but prefers that her students use FOIL (Postobservation 2 Interview). Bonnie describes the lattice/box method as, “more or less a graphics organizer to help them organize their work so they don’t leave something out” (Postobservation 2 Interview).

As stated earlier in this section, the mathematics behind the use of FOIL and the box method includes the distributive property, the associative property, and the
commutative property of multiplication. In addition, the lattice/box method could be considered a representation (i.e., a picture) of the distributive property. Given that data about Faith and Bonnie do not include their respective introductions of the lattice/box method to their students, I do not know whether the distributive property is part of their discussion. Both of them, however, mentioned the organizational benefits for students. Strictly speaking, as an organizer, the small rectangles can be seen as spaces in which to record the product of the row heading (i.e., the expression outside the box to the left) and the column heading (i.e., the expression outside the box above the column). Using this method, teachers can help students obtain correct answers to products of pairs of polynomials, without any reference to the mathematics supporting the method.

Example 4.4. Using “generic rectangles.” Kate teaches students how to multiply binomials using a tool she calls, “generic rectangles” (Postobservation 1 Interview). What Kate and her students record when they use generic rectangles to multiply binomials is the same as what Faith and Bonnie record when they use the “box method” (see Figure 4-27). However, to Kate, the expressions that she recorded inside each of the four smaller rectangles are representations of area. She said,

So if I wanted to multiply \( x \) minus three, times, \( x \) plus five \([as she records, \((x-3)(x+5)\)]\), we start with just positives—because area is positive. Then we work into this generic notation of \([as she traces the outline of the largest rectangle]—this is going to represent the area. And therefore if I am trying to multiply two things like \( x \) minus 3 and \( x \) plus five, I can find the area of each
piece, and add them together to get the area of the whole thing. And that is how
we multiply (Postobservation 1 Interview).

To Faith and Bonnie, the expressions they record inside their boxes are products resulting
from multiplying two expressions.

Figure 4-27. Screen capture of Kate’s work as she explained how she teaches students
how to multiply binomials.

Kate’s explanation and her example are interesting for several reasons. Kate
referred to the rectangle she drew as a “generic notation.” She stated explicitly that “area
is positive.” However, the expression Kate chooses to use as part of her explanation
results in rectangles that have negative areas. Kate’s awareness that area is positive and
her use of area to explain how to multiply these particular binomials seem to be in
conflict. However, over the course of three class periods, Kate and her students
frequently worked with generic rectangles that contained smaller rectangles that labeled
with a negative value for area. Kate’s framing of this TMIdd may be a case of a teacher
reducing mathematical rigor in order to make this TMIdd accessible. It could be a case in
which Kate’s procedure has been operationalized so that the area context is no longer
needed by Kate and her students. At the same time, her framing may also be seen as
introducing misconceptions. In addition to communicating that area can be negative,
Kate’s framing communicates that the lengths and widths of rectangles can be negative.
The upper right small rectangle pictured in Figure 4-27, for example, has dimensions $x$ by $-3$.

**Discussion of Bonnie’s, Delia’s, and Kate’s framing of multiplication of binomials.** Based upon the snapshots of framing presented thus far under Characteristic 4, Delia’s frame (i.e., Example 4.2) is strictly symbolic and explicitly emphasizes the distributive property. Kate’s frame (i.e., Example 4.4) uses a combination of symbolic and geometric representations as it brings the concept of area to the forefront. However, as the framing seems to have been operationalized to include negative dimensions of rectangles and negative area, the concept of area seems to recede to the background. The other two frames seem to be aids for remembering what to do (i.e., Ann, Bonnie, and Gary’s in Example 4.1), and for keeping the terms organized (i.e., Faith and Bonnie’s in Example 4.3). It is not clear as to the extent to which Ann’s, Bonnie’s, Gary’s, and Kate’s students are informed of the roles of the distributive, associative, and commutative properties in how they frame multiplying binomials. It is possible that in teaching FOIL, the lattice/box method, and generic rectangles, the teacher does not inform the students of the mathematical properties in these framings.

**Factoring quadratic trinomials of the form, $x^2 + bx + c$.** Data about Bonnie, Delia, and Kate reveal the existence of relationships between how they frame how to multiply two binomials (the TMId being addressed in this set of examples) and how their respective frames relate to factoring quadratic trinomials. What follows are discussions of how Bonnie, Delia, and Kate may see their framings of the multiplication of binomials as related to factoring quadratic trinomials.
Example 4.5. Using undo-FOIL, Bonnie factors quadratics of the form,

\[ x^2 + bx + c \]

using a method she calls, undo-FOIL. During her discussion of the procedure she created to help students decide which of seven factoring procedures to apply (see Example 3.4), in reference to this family of quadratic trinomials, Bonnie shared, “I call it ‘undoing FOIL’… because you are doing FOIL in reverse—sort of” (Postobservation 2 Interview). I infer from her statement that, to Bonnie, FOIL is not an acronym. One cannot “do FOIL in reverse,” if FOIL is not perceived as a procedure.

Undo-FOIL, as Bonnie teaches it to her students, is restricted to quadratic trinomials whose quadratic coefficient equals 1. For quadratic trinomials whose quadratic coefficient does not equal to 1, Bonnie teaches a method she calls, the “AC Rule.” When asked to talk about undo-FOIL as she teaches it to students, Bonnie used \( x^2 + 7x + 12 \) to explain the procedure. Analysis of her explanation suggests that undo-FOIL has two stages: what would result in \( x^2 \) when one applies FOIL, and what would result in the 12 and the 7 when one applies FOIL.

Stage 1: Bonnie asked, “Well we say, what did, we multiply together to get this \( x \) squared. Well we multiplied [pause] \( x \) times \( x \) — that is where it came from” (Postobservation 2 Interview, 125–127). Students record “\((x \, \, \, \, ) (x \, \, \, \,)\).”

Stage 2: Bonnie said,

And then where did we—how did we, you know how did we get the twelve? Well we took the three times—we took something here and something here [referents not captured on video] to get the twelve. So we talk about the factors of twelve that will add up to seven (Postobservation 2 Interview).
Students then record +3 and +4 in the template: resulting in \((x + 3)(x + 4)\).

 Undo-FOIL as Bonnie says she was taught (Postobservation 1 Interview) was not restricted to quadratic trinomials whose quadratic coefficient is 1. In reference to factoring \(6x^2 + 11x + 3\), which Bonnie used when she explained the AC Rule of factoring, Bonnie said,

The way that I was taught to factor it was—I call it undo-FOIL and you just had to play with the factors of six and play with the factors of three until you can fit them in there and make them work. (Postobservation 1 Interview)

Later, Bonnie described undo-FOIL as she learned it as “basically, guess, check, and revise” (Postobservation 2 Interview).

Example 4.6. Factor by grouping. During the Ending interview in response to my request for information about her upcoming lessons, Delia stated the week would be devoted to factoring and proceeded to demonstrate how she planned to frame factoring trinomials. She stated that she teaches factoring by grouping and demonstrated what she meant using \(x^2 + 5x + 6\). Her work is captured in Figure 4-28. She identifies factors of 6 whose sum is 5: 2 and 3. She uses this information to partition 5x into 2x and 3x. She then factors out the greatest common factor for \(x^2 + 2x\) and \(3x + 6\), and once again for \(x(x + 2) + 3(x + 2)\) to obtain the factored form of \(x^2 + 5x + 6\).
When asked how she would teach students to factor quadratics such as

\[ 2x^2 + x - 6 \]

Delia replied,

So they’re going to take two times negative six and they’re going to have a negative twelve up here [as she creates a new table. See lower right corner of Figure 4-28.] So then they would have negative four and three or positive four and negative three. All the factors of twelve. And they’ll find the ones that add up to the [coefficient of] \( x \). So they do the same exact process. (Ending Interview)

I asked her why she multiplied the 2 and the \(-6\), Delia said, “It works. Like we didn’t derive this method. But it works every time” (Ending Interview). Pressed for more information, Delia stated that she is convinced that the procedure works, but that she did not know why the method works. Recalling that Delia does not allow her students to use the term FOIL when talking about multiplying binomials unless they could tell her “what that means and the math behind it” (Delia, Postobservation 2 Interview). It seems that in this case Delia is not following her own policy. Delia does
not allow students to say or use FOIL in her classroom unless they can tell her what FOIL means and the math behind it.

**Example 4.7. Using “generic rectangles.”** Kate uses “generic rectangles” to frame multiplying binomials (see Example 4.4), completing the square (see Figure 4-15), and factoring quadratic trinomials. All three procedures are framed in the context of dimensions and areas of rectangles. After having explained how she multiplies

\[(x - 3)(x + 5)\] using a generic rectangle, Kate explained how to factor \(x^2 + 2x - 15\). She said,

> When we are going backwards we can come back to this notion of [pause] the fact that negative fifteen had to be down here [as she records \(-15\) in the generic rectangle] and the two \(x\) had to be split here, the \(x\) squared is here [as she records \(x^2\) in the generic rectangle; see Figure 4-29]. (Kate, Postobservation 1 Interview)

By filling in two out of four rectangles in her generic rectangle with \(-15\) and \(x^2\), respectively, Kate used the generic rectangle to explain that the linear term has to be partitioned into two parts.
Figure 4-29. Screen capture of Kate’s work created as she discussed how she teaches procedures for multiplying binomials and factoring trinomials (Postobservation 1 Interview). The expanded form of the product of the binomials was used to illustrate factoring.

Kate went on to say,

The two $x$ has to be split up. You use the diamond problem to figure that out. So that is where we get the negative three $x$ and the [pause] five [pause] $x$ [as she records “–3x” and “5x” in the generic rectangle]… and they look for common factors along the side [pause] to get the [pause] factored form of that [as she points to $x^2 + 2x -15$]. (Postobservation 1 Interview)

What Kate means by “diamond problem” is explained in greater detail in the discussion for Characteristic 10. For now, we know from Kate’s explanation that a diamond problem is the means by which Kate and her students partition $2x$ into two parts, –3x and 5x. Although Kate said, “they look for common factors along the side,” she did not record information to illustrate what she meant. Her written work (see Figure 4-30) from
her explanation of how she factors quadratic trinomials for which the quadratic coefficient is a nonzero number not equal to 1 captures what she means by “common factors along the side.” The common factor for each row of her generic rectangle is recorded outside of the rectangle on the left side. The common factor for each column of her generic rectangle is recorded outside the rectangle along the top. A complete explanation of how Kate and her students factor quadratic trinomials is provided in Appendix C.

Figure 4-30. Screen capture of Kate’s written work after she finished demonstrating the use a generic rectangle to factor $2x^2 + 9x + 10$ (Postobservation 1 Interview).

Kate’s procedure for factoring quadratic trinomials of the form, $ax^2 + bx + c$, for which $a \neq 1$ differs from the case for which $a = 1$ in one step. To determine how to partition $bx$, Kate uses the factors of $ac$. When asked why she uses $ac$, Kate initially stated that she did not know, but asked if she could think about it (Postobservation 1 Interview). After a long pause (~ 45 seconds), she recorded the work captured in Figure 4-31. Kate’s response is significant for several reasons. First, her response provides
explicit evidence that although she may know that her procedure works, answering why is different. She said,

I am trying to figure out why they are related—other than it just always works. I mean I could tell you that this works [as she points to her work for $2x^2 + 9x + 10$ see Figure 4-30] because you are looking for things that multiply to be in this number [20] … and it [sic] has to add up to be the nine $x$ (Kate, Postobservation 1 Interview).

Second, Kate’s written work (see Figure 4-31) suggests that she was looking for a reason based upon a general quadratic trinomial. Third, Kate’s written work suggests that she has parts of a justification of why using $ac$ works; however, she eventually gave up saying, “I don’t feel like I am getting anywhere. [laugh] All right so, I have [pause] uh, rmns, rm [pause] ns.[pause] Hmm. [pause] I don’t know. I think that I am going to have to think about it” (Postobservation 1 Interview).

*Figure 4-31. Screen capture of Kate’s work recorded as she thought about why $ac$ is used to factor $ax^2 + bx + c$ (Postobservation 1 Interview).*
Discussion about factoring quadratic trinomials of the form \( x^2 + bx + c \). All three framings involve finding factors of \( c \) whose sum is \( b \). However, each framing asks students to think about this search differently. Bonnie’s description of undo-FOIL, as “basically, guess, check and revise” (Postobservation 2 Interview) suggests that her students are asked to treat finding factors of \( c \) that add up to \( b \) as a problem-solving exercise. Finding the factors leads immediately to what is needed to fill in the spaces in \((x + \_)(x + \_)\). Delia teaches her students to find factors of \( c \) that add up to \( b \) so that they are able to partition \( bx \) into two terms and carry out the remaining steps of the factor by grouping procedure. Kate teaches her students to find factors of \( c \) that add up to \( b \) so that they are able to partition the area represented by \( bx \) into two smaller rectangles and find the dimensions of pairs of rectangles in the generic rectangle. Delia and Kate’s framings are mathematically equivalent. Delia’s factoring of \( x^2 + 5x + 6 \) is captured in the left side of Figure 4-32. What Kate would have recorded using generic rectangles is captured in Figures 4.32b and 4.32c. The second line of Delia’s work and the filled in generic rectangle (Figure 4-32b) are both results of partitioning \( 5x \). The \( x(x + 2) \) in Delia’s third line is equivalent to finding the dimensions of the top two rectangles in Kate’s generic rectangle (see Figure 4-32c). The \( 3(x + 2) \) in Delia’s third line is equivalent to finding the dimensions of the bottom two rectangles in Kate’s generic rectangle (see Figure 4-32b). Delia’s fourth line is equivalent to finding the dimensions of the entire generic rectangle (see Figure 4-32c).
The relationship between multiplying pairs of linear binomials and factoring quadratic trinomials. Multiplying pairs of binomials and factoring quadratic trinomials are two procedures that have an inverse relationship. What follows are discussions of pairs of framings by Bonnie, Delia, and Kate and what these framings communicate about an abstract concept, the inverse relationship between the two procedures.

Example 4.8. Bonnie’s framings. Bonnie teaches her students how to multiply two linear binomials using FOIL and how to factor quadratic trinomials using two procedures, undo-FOIL and AC rule. Although the TMs being framed are related inversely, it is not clear how Bonnie’s framing of multiplying binomials (i.e., using FOIL), and her framing of factoring trinomials communicates the inverse relationship between the two TMs. Bonnie’s communication that these procedures are inversely related is, however, found in the names of the procedures, FOIL and undo-FOIL, and her
comment that when one does undo-FOIL, one is “doing FOIL in reverse—sort of” (Postobservation 2 Interview).

Example 4.9. Delia’s framing. Delia emphasizes the distributive property when she teaches students how to multiply linear binomials and teaches students to factor quadratic trinomials using a procedure called factor by grouping. Both of her framings are standard mathematical procedures. Delia sees the relationship between the two procedures as opposites. After demonstrating factor by grouping for \( 1x^2 + 5x + 6 \) (see Figure 4-28), Delia stated,

And if you, if they look at it this way [as she points to her work starting from the bottom row moving up to the trinomial] it’s exactly the opposite of what we’ve done in this chapter [i.e., multiply binomials] (Ending Interview).

Her statement and gestures suggest that the inverse relation between the two procedures lies in the sequence of steps one follows or possible the sequence of steps one records when showing one’s work. Delia’s framings support each other mathematically as well as procedurally.

\[
\begin{align*}
1x^2 + 5x + 6 & \\
(x^2 + 2x) + (3x + 6) & \\
(x+2)(x+3) &
\end{align*}
\]

Figure 4-33. Screenshot of Delia’s work for factoring (Postobservation 3 Interview)
annotated with arrows to indicate the “opposite” relationship between factoring quadratic trinomials and multiplying two linear binomials.

Example 4.10. Kate’s framings. Kate teaches students to factor trinomials and multiply linear binomials using generic rectangles. Her framing for both procedures is based upon an area metaphor. The mathematical basis for her framing of these two procedures is the formula for the area of rectangles, \( A = lw \). If, for a rectangle, one multiplies a given length and a given width that are represented as two linear binomials, the product is area and is represented as a quadratic trinomial. The procedure is equivalent to working with the area formula as \( lw = A \) (reading from left to right). If one is given an area of a rectangle that is represented by a quadratic trinomial that is factorable, the dimensions of this rectangle are represented as linear binomial factors. The procedure is equivalent to working with the area formula as \( A = lw \) (reading from left to right).

When Kate and her students talked about factoring, however, the connection between area of rectangles and factoring quadratic expressions was not apparent. The context of area seems to have receded to the background. In her explanation of factoring, Kate never mentioned area. Had Kate not revealed that the expressions in the quadratic trinomial represent area when she explained multiplying binomials, Kate’s framing of factoring trinomials could be seen as a different representation of what Bonnie and Delia do—teach students how to manipulate symbols.

With respect to describing the relationship between the two procedures, Kate used the term, backwards. Backwards makes sense in the context of the generic rectangles. Speaking in much abbreviated terms, when one multiplies binomials, one starts with
expressions written on the outside of the generic rectangle, and finishes with the expressions recorded in the interior of the rectangle. When one factors, one starts with four expressions written in the interior of the rectangle, and finishes with expressions they write on the exterior of the rectangle. The inverse relationship between these two procedures is embedded in the context.

**Discussion of Characteristic 4.** Data about Bonnie, Delia, and Kate reveal that although the TMIds are inversely related, their respective framings of these two procedures are not necessarily inversely related. The inverse relationship of the two TMIds is communicated in Bonnie’s names of the procedures, FOIL and undo-FOIL. In Bonnie’s class there exists a second factoring method, the AC rule. Although the AC rule can be applied to quadratic trinomials for which the quadratic coefficient equals one, it is not clear how, from the perspective of a student in Bonnie’s class, FOIL is related to the AC rule. Data about Bonnie do not contain this information. In contrast, Delia’s and Kate’s framings do communicate the inverse relationship between these two TMIds. Delia’s framings do so symbolically. Kate’s framings communicate the inverse relationship through a geometric representation.

**Characteristic 5.** Some framings of procedures seem to be examples of folklore mathematics in which procedures are passed from one teacher to another; however, the mathematical explanations or justifications as to why the procedures work seem to be unknown.

This characteristic is based upon teachers’ explanations of how to factor quadratic trinomials in standard form for which the quadratic coefficient does not equal 1, and their response to the question, “Why does it work?” Emerging from the data are two methods
(i.e., Ann’s Secret Method, and Bonnie’s AC Rule) which participants say they learned from another colleague (Ann, Postobservation 3 Interview; Bonnie, Postobservation 1 Interview) and for which participants state explicitly that they did not know why the method works (Ann, Postobservation 3 Interview; Bonnie, Postobservation 1 Interview).

**Example 5.1. The Secret Method.** According to Ann, she learned her Secret Method from another teacher (Postobservation 3 Interview). She is convinced that it works, but does not know why it works (Ann, Postobservation 3 Interview). When asked what her Secret Method is, she demonstrated and explained the method using the specific trinomial, $3x^2 + 4x + 1$.

The idea is to take that coefficient in front and multiply it by the last term. And we rewrite the equation without that three or that coefficient in front. Uh, $x$ squared plus four $x$ plus three. And then it becomes a trinomial that they’re—they like—to be able to factor. [See Equation 2 in Figure 4-34.] … So when we factor this, we get an $x$ plus one and an $x$ plus three. [See Equation 3 in Figure 4-34.] (Postobservation 3 Interview)

![Equation 2](image)

![Equation 3](image)

*Figure 4-34. Screen capture of Ann’s factoring using the Secret Method, (Postobservation 3 Interview). (Note: Equation numbers were added for reference.)*
The "secret" within Ann’s method takes place in the transition from equation 3 to equation 4 during which she compensated for making the transition from equation 1 to equation 2 (i.e., multiplying the 3 and the 1). Ann went on to say,

Well, the secret is you have to take the three. Because just taking this three and multiplying it by one, [is] not a sound mathematical principle. You—I can’t just arbitrarily change number[s] around in uh, in uh, an equation. And I tell them that. So we have to kind of go back and undo that—uh, what we did. So what happens is you reinsert that three… we’ll reinsert it both in both factors. Just, just to [pause] appease the math Gods [laugh], we’ll reinsert it twice. [See Equation 4 in Figure 4-34.] (Postobservation 3 Interview)

Ann then finished her demonstration and explanation as follows:

And then the last piece is we want to reduce each factor, if you can. So when we reduce, we get three $x$ plus one. And three will go into both of these terms, so we get $x$ …plus one. And there’s your factorization [see Equation 5 in Figure 4-34]. (Postobservation 3 Interview)

There are several interesting aspects to Ann’s Secret Method. First of all, Ann’s equations are not all equivalent, yet her method led to a correct factorization. Second, Ann is aware that this method involves the use of transformations that she sees as not mathematically sound. However, states that she has been excited about this method since she was first shown the method by another teacher. She said, “I was absolutely amazed. [laugh] I was like wow that is really cool! (Postobservation 3 Interview). Third, as evidenced by the next quote, Ann knows exactly where students make mistakes when trying to use the Secret Method.
They get to this step right here. [Ann draws an arrow next to Equation 3. See Figure 4-34.] And they forget to go back. They know what their finished factorization looks like a lot of times. And they forget to go back and reinsert the three and reduce. They, they tend to forget [as she draws a vertical line next to Equations 4 and 5. See Figure 4-34.] (Postobservation 3 Interview)

Fourth, although Ann has received complaints from the other teachers in her department for teaching this method, Ann continues to teach the method. In response to the complaints, she gives her students a certain amount of time to master the Secret Method. If her students do not demonstrate mastery, she tells them that they are not allowed to use the method (Postobservation 3 Interview). According to Ann, “And so, um, that is why we call it the Secret Method (Postobservation 3 Interview). To those students whom Ann bans from using the Secret Method Ann says, “You are back to, you know, working out factors and guess and check and looking to see if this is the prime number and the only factors are three $x$ and $x$” (Postobservation 3 Interview). Fifth, when asked, whether the Secret Method always works, Ann replied, “Yes,” but did not offer a response when asked why. When pressed for an explanation, Ann replied, “I cannot it explain it you. I don’t know why it works. I’ve never sat down—I just know that it does work….A cooperating teacher found it in a NCTM magazine and showed it to me” (Postobservation 3 Interview). It is not clear from the data whether Ann’s colleague showed her the Secret Method in the NCTM magazine or showed her the Secret Method and told Ann that

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34 A search of the NCTM’s website for the method Ann calls the Secret Method in journal articles about factoring quadratic trinomials was unsuccessful
he/she found it in the NCTM magazine. What is clear is that Ann teaches her students a factoring method that she believes works even though it involves the use of “unsound” mathematics.

**Example 5.2. The AC Rule/Method.** For quadratic trinomials for which the quadratic coefficient is not equal to 1, Bonnie frames factoring using a different approach, the AC rule, an approach she first learned from another teacher and which Bonnie says now appears in textbooks\(^{35}\) (Postobservation 1 Interview). It is important to note here, that Bonnie, herself, sees undo-FOIL as a special case of the AC rule (Postobservation 2 Interview) but that she explicitly states that she does not know why the AC rule works (Postobservation 1 Interview). Her choice to divide her framing of factoring trinomials into two framings (i.e., undo-FOIL and AC Rule) is a pedagogical choice. She found that “the kids would rather do [the] AC [rule] because to them, it’s—I think that it guides them through their thinking better” (Bonnie, Postobservation 2 Interview). When asked to explain the AC Rule, Bonnie used \(6x^2 +11x +3\) as an Example and said,

> Well, somebody discovered that if you—they call it the AC Rule because—you know—if you have ax squared plus bx plus c. Six times three is eighteen so you are looking for something that gives you a product of eighteen but a sum of eleven. [pause, as she recorded +18 and +11] So you look at the factors of

\(^{35}\) A description of factoring quadratic trinomials matching Bonnie’s AC Rule was found in Foerster (1990). Foerster, who does not use the name AC Rule, presents the method without justification.
eighteen [as she lists the factors of 18] … we are going to use two and nine [as she circles the 2 and 9]. So what we do is we re—we break eleven $x$ into two $x$ plus nine $x$ plus three [as she records $6x^2 + 2x + 9x + 3$]. And then we use partial grouping, like you factor out of the first two, factor out the two $x$, and that leaves you three $x$ plus one and then you… And then at the last two you take out the three, so you have created that common factor of three $x$ plus one. [Bonnie records $2x(3x+1)+3(3x+1)$ as she talks, then records $(3x+1)(2x+3)$ without comment] (Postobservation 1 Interview).

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**Figure 4.35.** Screen capture of Bonnie’s work after she finished factoring $6x^2 + 11x + 3$ using the AC Rule.

Bonnie’s teaching of the AC Rule replaced a method she described as “basically, guess, check and revise”\(^{36}\) (Postobservation 2 Interview). In the “guess, check, and

\(^{36}\) To factor $ax^2 + bx + c$ using the “guess, check, and revise” method, Bonnie would have students consider $a_1$ and $a_2$ such that $a_1a_2 = a$, and $c_1$ and $c_2$ such that $c_1c_2 = c$;
“revise” method, students would consider integers whose product is 6, and integers whose product is 3, use them to create two binomials and check to see whether the product is the trinomial being factored. She said she uses the AC Rule because it is “more guided” (Postobservation 2 Interview) and students have told her that they find it easier to do than guessing, checking, and revising (Postobservation 2 Interview).

Discussion of Characteristic 5. Two experienced teachers, Ann (7-year veteran) and Bonnie (30+-year veteran) use two very different methods to teach their students how to factor trinomials for which the nonzero quadratic coefficient does not equal one: the Secret Method (see Figure 4-36a), and the AC Rule (see Figure 4-36b), respectively.

Data about these two teachers reveal several similarities in background information about the use of their framings. Both teachers were shown their respective methods by a colleague. Both teachers have been teaching and using these methods for many years with the same belief: If students follow all of the steps correctly, the methods will lead to correctly factored quadratic trinomials. Neither of them seems to know why their respective framings work, but find that, for many students, their framings do make factoring trinomials accessible.

Substitute them into $(a_1x + c_1)(a_2x + c_2)$, and check to see if the product of the binomials is $ax^2 + bx + c$. If not, choose a different combination of $a_1$, $a_2$, $c_1$, and $c_2$. 

Factoring quadratic trinomials for which the quadratic coefficient is nonzero and does not equal 1: (a) Ann’s Secret Method, and (b) Bonnie’s AC Rule.

How Ann and Bonnie learned about these methods is reminiscent of the informal ways in which traditions are passed on from generation to generation—a form of folklore mathematics. These framings were shown to Ann and Bonnie by other teachers; however, it is not clear whether the colleagues who showed Ann and Bonnie their respective factoring methods presented justifications to them. Ann and Bonnie, themselves, did not offer explanations when asked why their respective methods work. What seems to be important to Ann and to Bonnie is that the method works and that if the steps are followed, students are able to produce a correct factorization. Not knowing why did not seem to be problematic.

Examination of data about the other teachers in this study reveal that Delia, Faith, Gary, and Kate also teach their students procedures for which they do not have an explanation as to why the procedures work. They are not used as examples for this characteristic because their discussions do not provide additional insights. When asked to explain why those respective procedures work, Delia and Faith stated they did not
know, Gary stated he would have to do some research, and Kate tried to produce an explanation (see Example 4.7) but was unsuccessful.

After data analysis, I revisited the Secret Method and the AC Rule in order to find mathematical explanations for these factoring methods. Proofs for these methods are included in Appendix B.

**Characteristic 6.** Teachers’ framings reflect contrasting efforts to help students see/look at symbolic representations. Some teachers focus on surface features (i.e., that which can be seen with one’s eyes). Some teachers focus on seeing through or beyond the surface to symbolic structure or form (i.e., that which requires or uses some mathematical insights/understandings).

At some point in all participants’ framings emerged an emphasis on “seeing” something about or something within symbolic representations. None of the framings discussed for this characteristic involve the use of other representations to facilitate student learning. The participants’ framings emphasize seeing surface features or seeing through or beyond the surface to see symbolic structure or form. What follows are four examples of framing, each of which emphasizes one of these two forms of “seeing.”

**“Seeing” surface features.** In Example 6.1, Bonnie directs students to pay attention to the number of terms in the denominator of a radical expression. The number of terms in the denominator, according to Bonnie, determines the method students are to use to rationalize the denominator of the expression. In Example 6.2, Hildi tells her students that the negative exponent tells students what to write down for the step in a procedure.
Example 6.1. “Seeing” a surface feature: A surface feature indicates which 
procedure to apply. In an effort to help students know which procedure to use in order to 
rationalize the denominator of a fraction that contains a radical expression in the 
denominator, Bonnie asked her students to simplify the following two expressions for the 
Observation 3 warm-up: \( \frac{\sqrt{8y}}{x} \) and \( \frac{6}{\sqrt{2} - 1} \). After students had a chance to work on 
these exercises, Bonnie told the class that she had chosen these two problems because she 
wanted students to “see” a difference in the two denominators. Bonnie said, 

> So here in the denominator you have a monomial [as she traces a circle around the 
8y] and here in the denominator you have subtraction. So you have two terms [as 
she traces a circle around \( \sqrt{2} - 1 \)]. (Bonnie, Observation 3) 

According to Bonnie and as she demonstrated applying the procedure, 

> The point is you’re still trying to get rid of that radical. And the way you get rid 
of the radical is to use that sum and difference [pause] because then the middle 
term drops out. That’s where the radical is. … This is what you’re trying to 
accomplish. Get those two things to add out to zeros. [Bonnie draws the two “/” 
marks (see Figure 4-37).] So you don’t have the radicals anymore (Observation 
3).

\[
\frac{6(\sqrt{2} + 1)}{(\sqrt{2} - 1)(\sqrt{2} + 1)} = \frac{6\sqrt{2} + 6}{\sqrt{4} - \sqrt{2} + \sqrt{2} - 1}
\]

\[\text{Figure 4-37.} \text{ Reproduction of Bonnie’s board work for an Observation 3 warm-up exercise.}\]
Of interest, with respect to framing, is that Bonnie’s highlighting visible attributes of \( \sqrt{2} - 1 \) has a mathematical rationale behind it. Bonnie brought attention to two properties of the expression \( \sqrt{2} - 1 \): “you have subtraction. So you have two terms” (Observation 3) after having reviewed with students how to simplify \( \sqrt{\frac{x^2}{8y}} \), an expression whose denominator does not have two terms. With this statement, Bonnie provided students with something to look for in the given expression that tells them that they need to use the conjugate of \( \sqrt{2} - 1 \) to simplify the given expression.

**Example 6.2. “Seeing” a surface feature: A surface feature can tell students what to write next.** In a discussion that took place during a problem-solving interview about what exponents are, and whether the 0 in \( a^0 \) is an exponent, Hildi shared that she believes that “mathematics is a language” (PS Interview) and that students have difficulty translating the symbolic language (PS Interview). Therefore, she emphasizes translating symbols. In Hildi’s terms, “when you see this …, do this” (PS Interview).

In the context of framing expressions of the form \( a^n \), to Hildi, seeing what the exponent is is important. Thus, during Observation 1 when Hildi framed \( \left( \frac{1}{2} \right)^{-1} \) as she discussed a review problem with the entire class, she wanted her students to see the “-1” and write “\( \frac{2}{1} \)” To her students she said, “It’s two, right? Because the negative [exponent] means flip it” (Observation 1). When Hildi framed \( a^{\frac{1}{2}} \), she wanted her
students to see the 1 and the 2 in \( \frac{1}{2} \) and record \( \sqrt[2]{a} \). As she talked about \( a^{\frac{1}{2}} \), Hildi shared,

It’s going to be, you know, this is the raised this power and it’s going to be rooted by this power [as she labels \( a^{\frac{1}{2}} \); see Figure 4-38]. So the way I think of this is—instead I would think of it like that [as she records “\( \sqrt[2]{a} \)’” because it’s to the one and the one and then two, but they’re invisible [as she changes \( \sqrt[2]{a} \) to \( \sqrt[2]{a^2} \)]. (PS Interview)

Figure 4-38. Screen capture of Hildi’s interpretation of \( \frac{1}{2} \) and of the exponent of \( a^\frac{1}{2} \).

From a research perspective, Hildi’s framing of exponent is interesting because the concept of exponent is framed as a set of procedures. Knowing what an exponent is means knowing what it tells you to do. If the exponent is, for example, a \(-2\), the “\(-\)” sign tells you to “flip the base” and the 2 tells you to raise it to the power of 2 (see Example 1.2). If the exponent is, for example, \( \frac{1}{2} \), the numerator tells you to raise it to the 1 power, and the denominator tells you to how to “root it.” In the case of \( \frac{1}{2} \), you “square root it.” Unlike Bonnie’s framing in Example 6.1, Hildi’s framing does not include a
mathematical justification. In Hildi’s framing, students are told what to do when they see an attribute of an exponential expression.

“Seeing through or beyond surface features.” In Example 6.3, Bonnie tries to help students see the symbolic structure/form of an expression by comparing it to a simpler related expression. In Example 6.4, Kate tries to help students see the symbolic structure/form of an expression by working with the expression itself.

Example 6.3. “Seeing through or beyond surface features”—using a simpler related expression to help students see symbolic structure/form. After Bonnie showed her class how to simplify \( \frac{x^2 + x - 6}{x^3 + 9x^2 + 27x + 27} \) to its lowest terms, a student asked, “Why did you get rid of one ‘x plus three’ and what happened to the nine x?” (Observation 2). The student’s question seems to be in reference Bonnie’s factoring of \( x^3 + 9x^2 + 27x + 27 \) (i.e., \( x^3 + 9x^2 + 27x + 27 \rightarrow x^3 + 27 + 9x^2 + 27x \rightarrow (x + 3)(x^2 - 3x + 9) + 9x(x + 3) \rightarrow (x + 3)(x^2 - 3x + 9 + 9x) \)) in particular, the transition from \( (x + 3)(x^2 - 3x + 9 + 9x) + 9x(x + 3) \) to \( (x + 3)(x^2 - 3x + 9 + 9x) \). Bonnie responded to the student’s question by beginning another explanation about how to factor the denominator of the given fraction, this time drawing an analogy between two pairs of expressions: \( (x + 3)(x^2 - 3x + 9) + 9x(x + 3) \) and \( 2x + 2y \); and \( (x + 3)(x^2 - 3x + 9 + 9x) \) and \( 2(x + y) \). These expressions and Bonnie’s factoring work are captured in Figure 4-39 in the blue and yellow boxes.
Figure 4.39. Screen capture of Bonnie’s board work of the part of her solution to a warm-up exercise (Observation 3).

To her students, Bonnie said,

Here the common factor is two [as she points to each of the 2’s in 2x+2y]. Here the common factor is this quantity [as she points at the underlined instances of (x + 3)]. So it’s the same thing. Here you would take out the two [as she records 2]. Okay. We’re going to take out the common factor which is x plus three [as she points at the (x + 3) in the bottom line of recorded work]. … And the same thing is true here. If I take this factor out, what I have left is this, x squared plus three x plus nine. And if I take the three x out of the last term, what I have left is the nine x. (Observation 2)

By using analogies, Bonnie tried to help students see the similarities in symbolic structure between pairs of expressions. Bonnie identified and mapped similar structures between 2x + 2y and (x + 3)((x^2 − 3x + 9) + 9x(x + 3)), and showed how the equivalence of 2x + 2y and 2(x + y) is parallel to saying that (x + 3)((x^2 − 3x + 9) + 9x(x + 3)) is equivalent to
\((x+3)(x^2-3x+9+9x)\). In this framing, Bonnie made use of a simpler expression that she assumed is somehow familiar to students and related it to an expression that students find difficult or challenging with which to work. Students’ “seeing” the symbolic structure of the polynomial expression goes beyond what can be seen with one’s eyes and is necessary for students to be able to move to the next step in the factoring process.

**Example 6.4.** “Seeing through or beyond surface features”—helping students see the symbolic structure/form within an expression. During Observation 3, while students worked on solving \(x^2+12x=13\) for \(x\) using by completing the square, Kate noted that students could get as far as \((x+6)^2=49\), but struggled to go beyond. Kate responded going to the front screen, covering “\(x+6\)” with her hand, and asking the question, “My hand—if I square it, it makes forty nine, what’s my hand going to have to be? [see Figure 4-40]” (Observation 3). After several students answered seven and negative seven, Kate recorded, “\(x+6=7\)” and “\(x+6=-7\)” and went on to discuss the overall process. “So when I square something—I square my hand. It makes forty nine. This—my hand could have been seven, or my hand could have been negative seven (Observation 3).

![Figure 4-40](image)

*Figure 4-40.* Screen capture of Kate’s hand placement as she says, “My hand—if I square it, it makes forty nine, what’s my hand going to have to be?” (Observation 3).
Kate’s actions can be interpreted as an effort to help students see that “$x + 6$” is a single object as opposed to three separate symbols inside of the parentheses. By asking students to think of $(x + 6)^2 = 49$ as $(\text{hand})^2 = 49$, Kate also asks students to look beyond the surface of $(x + 6)^2 = 49$ and see it as a special case of $X^2 = k$.

**Discussion of Characteristic 6.** Helping students to understand, read, and write symbolic representations of the objects of school algebra is an important part the work of secondary school mathematics teachers. This characteristic addresses two different approaches participants in this study used to carry out this work: helping students “see” surface features (i.e., that which can be seen with one’s eyes) of symbolic representations and helping students “see” through or beyond the surface.

In addition to the fact that Bonnie’s framing of rationalizing denominators (Example 6.1) and Hildi’s framing of expressions of the form, $a^n (a > 0)$ (Example 6.2) both emphasize the “seeing” of what can be seen with one’s eyes, their framings seem to have something else in common. Both framings seem to treat the symbolic representations of algebra as a type of written code. Certain parts of the code tell students what to do next.

Bonnie’s framings of factoring a polynomial (Example 6.3), and Kate’s framing of solving quadratic equations (Example 6.4) can be interpreted as efforts to help students see and understand the structure/form of a symbolic representation. Seeing and understanding the structure/form of a symbolic representation tells students what to do next. Bonnie’s framing, an explanation of why she “got rid of” an $x + 3$ and what happened to the $9x$, involves the use of analogies between the given expressions and
another expression that is composed of fewer “marks on the board” and whose
form/structure is the same as the given expression. Kate’s framing works directly with
the troublesome part of a symbolic representation. From a researcher’s perspective, by
covering up the troublesome part (i.e., \( x + k_i \)) with her hand, Kate appears to be asking
students to see the \( x + k_i \) as a single entity, an object. She seems to be asking students to
to consider \( (x + k_i)^2 = k_2 \) as \((\text{hand})^2 = k_2\) and solve the simpler equation.

**Characteristic 7.** Teachers sometimes frame TMIs by reducing visual
density/complexity of symbolic representations.

In Kate’s last postobservation interview, Kate stated that in the next lesson she
would be deriving the quadratic formula. She commented that “the amount of symbols in
the quadratic formula is going to be a bit frustrating for them” (Ending Interview). The
“amount of symbols” as a source of frustration for students prompted this characteristic.

**Example 7.1.** Another look at Kate’s framing of solving equations of the form
\( (x + k_i)^2 = k_2 \) (Example 3.4): Helping students see “\( x + k_i \)” as a unit. When Kate
placed her hand on the contents of the parentheses in \( (x + 6)^2 = 49 \) and asked, “My
hand—if I square it, it makes forty nine, what’s my hand going to have to be?”
(Observation 3) Her actions were viewed as an effort to help students see the structure of
the equation. In other words, Kate’s actions were interpreted to be an effort to help
students see that \( (x + 6)^2 = 49 \) has the same structure as \( x^2 = 49 \). This same action can be
interpreted to be an effort to reduce the visual density/complexity of the symbolic
representation.
Analysis of data about Kate reveals that the event described in Example 6.4 is not an isolated event. In a discussion about how Kate frames how to solve simple equations involving absolute value (e.g., \(|x| = 8\)), Kate recorded “| = 8 ” and said,

I try not to get them to do it by rote. But I try to get them to think about that [as she traces a blob inside the absolute value sign] as a blob. And the blob inside would have to either be eight or the blob inside would have to be negative eight, in order to make the absolute value of it eight. (Ending Interview)

Just as Kate wants students to see the contents of the parentheses in the quadratic equation as a single entity, Kate wants students to see the contents of the absolute value sign as a single entity—a blob. Both are examples of Kate’s efforts to reduce the density/complexities of the symbolic representations by trying to help students see the symbolic structure of expressions and equations.
Figure 4-41. Screen capture of Kate’s work (Ending interview) for solving absolute value functions of the form $|X| = k_2$, $k_1$ and $k_2$ are constants.

Example 7.2. Another look at Kate’s symbolic representation of the quadratic formula, $x = \frac{-b \pm \sqrt{D}}{2a}$, for which $D = b^2 - 4ac$ given that $ax^2 + bx + c = 0$: Separating an expression into smaller components, one familiar and one new. As observed previously, to the vast majority of students in the United States who have completed Algebra 1, the quadratic formula is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, given that $ax^2 + bx + c = 0$. To students who have completed Algebra 1 with Kate as an instructor, the quadratic formula is $x = \frac{-b \pm \sqrt{D}}{2a}$ and $D = b^2 - 4ac$, given that $ax^2 + bx + c = 0$ (Ending Interview). Based
upon comments from Kate about her version of the quadratic formula and its
development, and examinations of worksheets Kate had given her students to work on
during the semester, Kate seems to reduce visual density/complexity of the quadratic
formula in two ways: (1) by establishing familiarity with $D = b^2 - 4ac$ long before she
asks students to work with the quadratic formula itself, and (2) by separating the
quadratic formula into two parts.

Kate developed students’ familiarity with computing the value of the discriminant
using a sequence of problems spread over several worksheets (refer to Table 4-1).
Gradually, Kate made the problems devoted to computing the value of $b^2 - 4ac$ a little
more challenging, by asking students to do more of what they would be asked to do after
Kate guided the class through the derivation of the quadratic formula (see, Figure 4-13,
sample problem from worksheet 27). Students had been working with $b^2 - 4ac$ on their
worksheets and warm-ups, and had been asked to do the same types of problems on
quizzes for several weeks$^{37}$ before being introduced to solving equations using the
quadratic formula. Except for students who had been absent for a majority of
approximately 2 ½ weeks, students in Kate’s class would have been familiar with this
expression. Therefore, when Kate derived the quadratic formula, the new part was

$$x = \frac{-b \pm \sqrt{D}}{2a},$$

which is less complex visually and less dense symbolically, than

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$^{37}$ Classes at Kate’s school meet every other day.
Table 4-1. Sequence (by worksheet) of Practice Problems Kate Asked Students to Do That Are Related to the Quadratic Formula.

<table>
<thead>
<tr>
<th>Worksheet #</th>
<th>Problem</th>
</tr>
</thead>
</table>
| 20 (before the observation week) | 1. Evaluate: $b^2 - 4ac$ when 
  a) $a = 2 \ b = 5 \ c = 1$ |
| 21 (before the observation week) | 2. Given the standard form of $ax^2 + bx + c = 0$, identify $a$, $b$, and $c$. 
  Then calculate: $b^2 - 4ac$ 
  a) $2x^2 - 3x - 5 = 0$ |
| 24 (Observation 1) | 4. If $ax^2 + bx + c = 0$, then $D = b^2 - 4ac$. Find $D$ 
  Example: 
  a) $3x^2 - 5x - 2 = 0$ |
| 25 (Observation 2 same instructions and example, but different equations) | 2. If $ax^2 + bx + c = 0$, then $D = b^2 - 4ac$. Find $D$ 
  a) $x^2 - x - 5 = 0$ |
| 26 (Observation 3) | Solve With the Quadratic Formula 
  Reminder of Steps: 
  1. $x^2 + 6x - 7 = 0$ 
  a) Identify $a$, $b$, and $c$ 
  $a = \ b = \ c =$ 
  2. Calculate the Discriminant: $D = b^2 - 4ac$ 
  $D =$ 
  3. Finally, plug into formula $x = \frac{-b ± \sqrt{D}}{2a}$. 
  $x =$ |
| 27 (after the observation week) | Note: The bolded and underlined terms in the problem column are bolded and underlined in Kate’s worksheets. 
  By separating $x = \frac{-b ± \sqrt{b^2 - 4ac}}{2a}$ (given $ax^2 + bx + c = 0$ and $a ≠ 0$) into two smaller equations, Kate provides students with an equation, $x = \frac{-b ± \sqrt{D}}{2a}$ that is |
composed of fewer symbols and a second equation $D = b^2 - 4ac$ that is familiar. It is a
version of the quadratic formula that, according to Kate, she has been teaching students
for the last several years (Ending Interview) and she reports that her students found the
quadratic formula much less complicated and “were so much happier with it” (Ending
Interview).

Example 7.3. Another look at Bonnie’s multiplying a binomial times a trinomial
using the lattice/box method: Using graphic organizers. As discussed in Example 4.3,
Bonnie introduced the use of a lattice to help students to multiply two polynomials
because she noticed that students “lose things or miss things or forget to multiply or
something” (Postobservation 3 Interview). A possible cause for students’ struggle when
given two polynomials to multiply is that, from the perspective of the struggling student,
the expression is composed of too many components or visually too dense for students to
see the components. The lattice/box is a tool to help students unpack the given
expression.

Figure 4-42 illustrates how the lattice/box is used as a graphic organizer for the
expression discussed in Example 1.3, $(4p - 3)(3p^2 - p + 2)$. Bonnie created a
lattice/box consisting of 2 rows and 3 columns: 2 rows because $(4p - 3)$ is composed of
two terms, and 3 columns because $(3p^2 - p + 2)$ is composed of three terms. The two
terms in $(4p - 3)$ are each recorded as row headings and the three terms in $(3p^2 - p + 2)$
are each recorded as column headings. Each cell in the lattice serves as a place to record
the product of its corresponding column and row heading. The final answer is obtained
by combining like terms. By helping students identify pairs of terms, one term from each factor of \((4p - 3)(3p^2 - p + 2)\), to which to attend and what terms temporarily to ignore, the lattice/box method can be seen as reducing the visual density/complexity of the multiplication procedure.

*Figure 4-22.* Reproduction of stages of Bonnie’s work captured on video as she discussed how to use a lattice to expand \((4p - 3)(3p^2 - p + 2)\) (Postobservation 2 Interview).

**Discussion of Characteristic 7.** The reduction of the visual density/complexity of symbols in a special case of quadratic equations, in the quadratic formula, and in multiplying pairs of polynomials can be seen as efforts to help students see symbolic
representations in different ways. In two of the framings, 7.1 and 7.2, the reduction of visual density/complexity seem to be efforts to help students see mathematics within a quadratic equation and within the quadratic formula.

In Example 7.1, Kate tries to help students to see—or at least, treat—the symbols inside the parentheses and the symbols inside the absolute value sign, each as a single entity. By seeing what is inside grouping symbols (e.g., parentheses, absolute value) as a single entity, the equation is less dense. In Example 7.2, Kate separates the quadratic formula into smaller components. In doing so, she set the stage for discussions with students about mathematics related to the quadratic formula beyond computation of roots of equations. For example, when Kate told her students that $b^2 - 4ac$ is called the discriminant, she told them that the value of the discriminant will provide information about the graph of $f(x) = ax^2 + bx + c$ (Kate, Observation 2).

In contrast, the framing in Example 7.3 seems to be an effort to help students keep track of terms so that they can produce a correct answer, as opposed to helping students understand more mathematics. Bonnie provides students with a graphic organizer (i.e., the lattice/box) that provides students with places to record components of the factors in the given expression and places to record the products of pairs of components.

**Characteristic 8.** Teachers sometimes frame TMIds by making connections to “student-relatable contexts.” Student-relatable contexts are not necessarily real-world contexts.

This finding is the result of examining the nature of the nonmathematical ideas to which participants in this study connected TMIds. These nonmathematical ideas are
contexts. Some of the contexts, such as the one described in Example 8.2 (i.e., cutting out paper hearts) is truly a real-world context. Some (i.e., Examples 8.1, 8.3 to 8.5) have real-world aspects, but are not necessarily real. As a way of describing both contexts, the term *student-relatable* will be used and defined as contexts that are perceived to be meaningful to students in their real experiences or are meaningful because the contexts can be imagined by students.

The term, student-relatable, is somewhat consistent with Freudenthal’s (1981) term, *meaningful*, as defined within the following statement from his plenary address at ICME 4, “In teaching mathematizing ‘the real world’ is represented by a meaningful context involving a mathematical problem. ‘Meaningful’ means meaningful to the learners” (p. 144). The term *student-relatable contexts* as used in this paper differs from Freudenthal’s “meaningful context involving a mathematical problem” in that the teachers who used student-relatable contexts did not necessarily involve contexts that involve a mathematical problem. The student-relatable contexts discussed in this section are simply contexts to which the teacher believes students can relate.

The use of student-relatable context is consistent with realistic contexts as characterized by researchers in Realistic Mathematics Education (RME). Gravemeijer and Doorman (1999) use “problem situations that are experientially real to the student” (p. 111). Van den Heuvel-Panhuizen (2003), provides additional insights into the RME use of the term, *realistic*. According to van den Heuvel-Panhuizen (2003) *realistic* is a translation of a Dutch verb, *zich realiseren*, which means “to imagine.” Van den Heuvel-Panhuizen further explains that although connections to real life are important, RME’s use of realistic mathematics includes suitable contexts that are “‘real’ in the students’
minds” (2003, p.10). What follows are five examples of framing in which TMIs are connected to student-relatable contexts.

**Example 8.1. Using a context that has real world elements that are not truly real.** The student-relatable context in Hildi’s framing is a two-story building in which there exist unhappy tenants on the first and/or second floors, a situation that can happen in the real world. Hildi connects the context and the TMI by saying that the negative sign in the exponent is a “mood meter.” A negative exponent indicates a tenant is unhappy and the tenant needs to be moved to the other floor. What follows is Hildi’s story about unhappy tenants in a two-story building as it relates to \( \frac{x^2}{y^3} \).

<table>
<thead>
<tr>
<th>Hildi’s statements</th>
<th>Hildi’s written work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Um they’re really unhappy with their apartments because you can tell because there’s a negative [as she points at the (-2) exponent], their mood meter. Um. So in order to make this guy happy— he lives on the top floor— he doesn’t want to live on the top floor. So we move him downstairs [as she loops the ( x^2 ) and draws an arrow].</td>
<td>![Diagram 1]</td>
</tr>
<tr>
<td>In order to make this guy [as she loops ( y^3 )] happy we have to move him upstairs [as she draws an arrow]</td>
<td>![Diagram 2]</td>
</tr>
<tr>
<td>So it’d be, you know, ( y^3 ) cubed and [sic] ( x^2 ) squared because once they move, then they’re happy. (Postobservation 1 Interview)</td>
<td>![Diagram 3]</td>
</tr>
</tbody>
</table>
In the case of \( \left( \frac{1}{2} \right)^{-1} \), Hildi stated that the mood meter indicates that “the whole building is unhappy so you have to move everybody” (Postobservation 1 Interview) and so the result is \( \frac{2}{1} \).

Hildi’s student-relatable context is a story intended to make class interesting (Postobservation 3). In this example of framing, the negative sign in the exponent tells the student to move something to the other side of the fraction bar. The story appears to be a way to bring attention to the negative sign in exponents and a means to help students remember the associated procedure.

**Example 8.2. Visualizing a real-world context.** As part of Faith classroom norm, with every new lesson, Faith takes time to go over vocabulary. During Observation 1, one of the new vocabulary terms was *line of symmetry*, which Faith described as “the line that passes through the vertex and divides the parabola into two symmetric parts” (Observation 1 lesson plan). What follows is Faith’s affirmation and elaboration of a student’s statement that two symmetric parts means “Okay so when you cut through the middle basically if you like flip it or fold it, it’s going to be perfectly aligned” (Observation 1).

Faith went on to relate an experience she had with her niece—cutting out a paper heart—without actually demonstrating the procedure. Faith helped students to imagine the procedure via descriptions and gestures. Faith said,

Rather than teach her how to draw the whole heart [as Faith uses two hands to draw a shape in the air that looks like a heart], we fold the paper in half [Faith
puts her two hands together, palm to palm.] and we draw half the heart [Faith
draws a shape in the air that looks like half a heart.] and we just cut off the paper.
[Faith uses her hand as if she is using imaginary scissors to cut out half a heart
shape.] Right? And then we open it up. Boom! … When you unfold the paper,
right, you have the same thing on both sides (Observation 1).

Faith concluded,

So that’s what the line of symmetry does. If I was to take my paper and fold it
right along that line or that axis of symmetry then I would have the same picture
on both sides [Faith alternately moves her hands up and down as she says “same
picture.”] (Observation 1).

The difference between this example of framing and Example 8.1 is that the
context is truly a real-world context. Faith uses this context to situate what she means by
an axis of symmetry (i.e., the fold in the cut-out paper heart) and symmetry (i.e., “same
picture on both sides”).

Example 8.3. Using an imaginary context. While explaining the solution to a
warm-up problem in which students were instructed to compute the value of
\[ D = b^2 - 4ac \] for a given quadratic equation (see Figure 4.43), Kate asked why the
answer is -4. A student responded, “Twenty is bigger than sixteen” (Observation 2), and
Kate said, “Yeah, that’s a good reason. More zombies than ninjas?” (Observation 2). It
is this odd comment that revealed Kate’s use of student-relatable contexts to help her
students understand adding and subtracting integers.
During the postobservation interview, Kate shared that the references to ninjas and zombies is a context that she happened to stumble upon during class after multiple attempts to explain to Student D how to add and subtract integers failed. She tried repeated explanations involving manipulatives (e.g., tiles spacers which look like + and tile spacers cut to look like “−,” and two-color chips), and familiar contexts (e.g., money), but to no avail. According to Kate the key idea of helping students learn how to add and subtract integers is getting them “to think about that zero field” (Postobservation 2 Interview). When asked what she means by a zero field, she replied, “A positive and negative unit, when added together, make a zero. If you have many added together [paired off], I picture a field of zeros. That is a zero field” (personal email conversation, September 19, 2011). I infer from Kate’s description that a zero field is simply a collection of pairs of − and +. Kate’s zero field has nothing to do with the concept of field in abstract algebra nor with the concept of slope field in differential equations.

According to Kate, the context of ninjas and zombies worked for Student D because he could imagine one ninja (i.e., a good guy) “taking out” (i.e., neutralizing) one zombie (i.e., a bad guy). Thus, for Student D, this context is relatable and affords a contextual understanding of adding and subtracting integers. The context of ninjas and

\[(2) \quad x^2 - 4x + 5 = 0\]
\[a = 1 \quad b = -4 \quad c = 5\]
\[b^2 + (-4ac)\]
\[= -4 \cdot 1 \cdot 5\]
\[= -20\]

\[\boxed{-4}\]
zombies, according to Kate, did not quite work for Student D’s classmate, Student A. Student A objected saying that ninjas can be good or bad. Student A suggested super heroes and villains. Super heroes are always good and villains are always bad.

Using Student D’s context of ninjas and zombies $9 + (-16)$ means we have nine ninjas and sixteen zombies (Kate, Postobservation 2 Interview). Since one ninja “takes out” one zombie, the result of this battle is seven zombies or $-7$. When asked about $5 - (-2)$, Kate stated that minus negative two is “like somebody else taking out a couple of zombies [and that] is like having two ninjas” (Postobservation 2 Interview). Kate seems to be talking about the equivalence of $5 - (-2)$ and $5 + 2$. The result of this battle is seven ninjas or 7.

**Example 8.4. Using a context that has real world elements but is not truly real.**

Hildi uses a metaphor to teach her students the concept of function. She connects function to a social event to which only nonscandalous friends receive invitations (Ending Interview). Exclusive parties and dating are experiences that are parts of many teenagers’ social awareness either through direct personal experiences or being cognizant that people they know participate in these experiences. Hildi’s framing of the concept of function takes advantage of this awareness by describing function as a social event to which only friends who are nonscandalous (i.e., date only one person) receive invitations to attend. While it is possible for one of Hildi’s students to plan a party in the way that

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38 Details of how she makes the connection is discussed in greater detail in Chapter 5 as part of a discussion on what bridging looks like when it is not restricted to making mathematical connections between mathematical ideas.
Hildi describes, it is possible that at the time the lesson was conducted, some of Hildi’s students were not allowed to date, or were not dating. For this reason, this context is considered student-relatable but not truly real for the students.

**Example 8.5. Using a context that is imaginary but has real world elements.** In this framing, Kate created an imaginary context for a common exercise in school algebra, solving linear equations. This context is interesting for several reasons. The context is a game whose rules parallel the rules students have to learn in order to solve linear equations. The game provides contextual meaning for combining like terms, addition and multiplication properties of equality, and the distributive property. Equivalent equations are created based upon maintaining “fairness” for the players playing the game. The context provides Kate with a means to address a common mistake students make when solving equations.

My awareness of the existence of this context began with the observation that whenever Kate and her students solved a linear equation, she drew a vertical squiggly line through the equal sign. During the last postobservation interview, when I asked her if the squiggly line had significance, Kate said that she and her students call them, rivers, and that she introduced rivers when they begin learning how to solve linear equations. She then began relating a story she tells her students using \(9x - 5 + x + 7 = 2x + 9(x + 2)\) as an example.
Kate’s statements

I tell them we have two teams [as she draws a squiggly line through the equal sign] on either side of a river. So I call this a river. You are allowed to do anything to one team [as she covers the left side of the equation] that you do to the other team [as she covers the right side of the equation]. Each team has got players and you have to get all your friends together. Then the, the, the first notation is that this team is way too complicated.

That would actually be ten x by getting your friends together [as she double underlines 9x and x]. And these guys [as she circles -5 and +7] aren’t friends with them [as she points at 9x and x] but still on the same team.

That would be two [as she records +2]. And over here, I would have to use my distributive property to get nine x plus eighteen. …and these guys are friends [as she double underlines 2x and 9x]…

Kate’s written work

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39 The term, “friends,” means “like terms.” The phrase, get all your friends together means combine like terms.
And now that each, each team is simplified, then your next step is to [pause] take something away from both teams [and] to be absolutely fair to the teams on either side of the river. (Ending Interview)

In Kate’s framing of the procedures involved in solving linear equations, what seems to govern the process, after the distributive property is employed and after like terms are combined on each side of the equal sign, is “fairness” as opposed to properties of equalities. The river in Kate’s student-relatable context plays a crucial role in framing how to solve linear equations. According to Kate, the river represents “the sense that things are balanced on both sides” (Ending Interview).

In addition, the river serves as a way to help students avoid making what she has found to be a common error: adding, subtracting, multiplying and dividing quantities on the same side of the equal sign. Kate says, “This is to prevent such atrocities as this nine $x$ minus five [pause] plus $x$ plus seven equals two $x$ plus nine [pause] plus two [pause] subtract $x$, subtract $x$” (Ending Interview). What Kate was talking about is a situation pictured in Figure 4-44a in which students subtract quantities twice on the same side of the equal sign instead of subtracting quantities once on each of the opposite sides of the equal sign as shown in Figure 4-44b. According to Kate, this is an error that she sees a great deal in Algebra 1, in Geometry, and in Algebra 2.
Figure 4-14. Reproductions of Kate’s written work. (a) An example of an incorrect application of the addition property of equality. (b) An example of a correct application of the addition property of equality.

According to Kate, when she sees a student adding, subtracting, multiplying, or dividing quantities on the same side of the equal sign she reminds them of the context and one of the rules of the game. She says, “We say, is that on both sides of the river? No. So what I have done is taken away \( x \) from the same team twice. And they go, oh. You can’t do that!” (Ending Interview). Based upon these comments, Kate’s river serves a pedagogical purpose as well as a mathematical purpose. She uses the river to help students avoid errors and to correct errors.

Discussion of Characteristic 8. Data about Faith, Hildi, and Kate in this section reveal a variety of uses for student-relatable context in framing TMIs in addition to bringing attention to TMIs. Hildi states explicitly that she uses metaphors (i.e., student-relatable contexts) to make TMIs more interesting and more memorable. When asked to share what motivates her creation of the metaphors, Hildi stated, “I figure if I can make it a little different and more interesting then I can maybe that will help my students remember it. Um or at least stay awake, you know, so” (Ending Interview).

Student-relatable contexts seem to play a very different role in Kate’s classroom than they do in Hildi’s. Whereas, Hildi’s framings are intended to keep students
interested and help students remember a TMId, Kate’s use of student-relatable contexts seems to be intended to provide students with a tangible means to use and understand a procedure. With respect to the context of ninjas and zombies, Kate uses the context to make a “zero field” more real to students—more accessible to students. With respect to the context of teams on opposite sides of a river playing a game, Kate uses the context to teach students a procedure, and to avert a common student error.

The significance of the context of ninjas and zombies is that, according to Kate, it helped Student D to “see” Kate’s zero field. It seems that any context is acceptable as long as the context reflects “the idea that it is a one-on-one” (Kate, Postobservation 2 Interview) and the mathematical concept underlying the context is the concept of opposites or additive inverses (i.e., $n + -n = 0$).

The student-relatable context as used in Faith’s class and seen from an observer’s perspective, seems to be used as a way of clarifying a definition of a TMId. Faith had already given the class a definition of “axis/line of symmetry” as “the line that passes through the vertex and divide the parabola into two symmetric parts” (Observation 1 lesson plan). After presenting the definition, Faith began to talk about cutting out paper hearts with her niece according to a particular set of steps. The resulting imagined paper heart has a line of symmetry (i.e., the fold) and has two symmetric parts (i.e., per Faith, the “same shape” on either sides of the line).

**Characteristic 9.** Sometimes a teacher forces/imagines a connection between a TMId and a real world context.

In the following example, Hildi uses the life cycle of a tarantula wasp as a context to explain what happens when one evaluates a composite function. This example is not
included as part of the discussion on student-relatable contexts because the data does not contain evidence that students can imagine or relate to the life cycle of this insect. It is unlikely that Hildi’s students have encountered tarantula wasps due to the fact that Hildi’s students live in a heavily populated urban environment, and tarantula wasps, according to a webpage about tarantula wasps, http://www.desertusa.com/mag01/sep/papr/thawk.html, are creatures that live primarily in wild, desert habitat.

Example 9.1. Connecting the life cycle of a tarantula wasp to evaluating composite functions. During the Ending interview, Hildi shared that she has used the life cycle of a tarantula wasp to frame evaluating a composite function. By her own admission, the connection between this real-world event and TMIId is a product of Hildi’s creative imagination. When asked where the connection between the real-world event and the TMIId came from, Hildi replied, “I’m a little crazy. [laugh] I just made it up. [pause] Well because that’s what it reminded me of” (Ending Interview).

My awareness of this example of framing is the result of overhearing Hildi say to her students, “Remember the tarantula wasp? Right? That’s that problem, where it’s the tarantula wasp [pause] where it lays the eggs [pause] and then they’re inside [pause] and then they burst out” (Observation 2). Thus, in the postobservation interview, I asked Hildi to talk about the tarantula wasp, its purpose and what the mathematical reference is. Hildi responded,

   So kids have a lot of trouble with [pause] uh functions…. Okay so um [pause] because they have trouble with functions— they can, they can kind of get the
whole plug it in thing but then composition of functions is like whoosh for a lot of them. (Ending Interview)

What follows is (a) Hildi’s description of stages in the life cycle of the tarantula wasp, and then (b) Hildi’s discussion of how the steps of evaluating a composite function are connected to the stages of the tarantula wasp’s life cycle.

Hildi’s description of the life cycle of a tarantula wasp:

<table>
<thead>
<tr>
<th>Hildi’s statements</th>
<th>Hildi’s written work</th>
</tr>
</thead>
<tbody>
<tr>
<td>So we have the tarantula wasp and the tarantula …</td>
<td>![Image]</td>
</tr>
<tr>
<td>so the tarantula wasp has a— finds a tarantula and [pause] lays its eggs in the tarantula.</td>
<td>![Image]</td>
</tr>
<tr>
<td>lays its eggs in the tarantula….Okay [laugh] so now the tarantula, so now the tarantula has eggs in its— in its body so then the eggs [pause] hatch as you can imagine this is bad news for the tarantula—</td>
<td>![Image]</td>
</tr>
<tr>
<td>Okay so they hatch and they are this little larva [pause] or whatever they are. …</td>
<td>![Image]</td>
</tr>
<tr>
<td>They eat the tarantula from the inside and then [pause] they burst out. And our poor tarantula is no longer with us.</td>
<td>![Image]</td>
</tr>
<tr>
<td>And then they’re little tarantula wasps. [pause]</td>
<td>![Image]</td>
</tr>
</tbody>
</table>
| Much like the composition of functions. (Ending Interview) | }
When asked what this story has to do with composite functions, she repeated the story but recorded mathematical notation next to her drawings. She evaluates $g(f(x))$ at $x = 2$ for $f(x) = x - 3$ and $g(x) = x^2$. The annotated excerpts following identify what Hildi draws and references as she repeats her story. The yellow rounded rectangles on the screen captures highlight what Hildi was drawing/recording as she told her story.

<table>
<thead>
<tr>
<th>Hildi’s statements</th>
<th>Hildi’s written work</th>
</tr>
</thead>
<tbody>
<tr>
<td>Okay so we have [pause] um let’s say we have, you know, $g$ of $f$ of $x$. Okay. This is our poor tarantula [pause] with its legs.</td>
<td>![Image of Hildi's drawing]</td>
</tr>
<tr>
<td>Okay. The tarantula wasp replaces [pause] the guts with something else.</td>
<td>![Image of Hildi's drawing]</td>
</tr>
<tr>
<td>Maybe like the [pause] like a two or an $x$ or whatever [as she records $g(f(2))$]. So then [pause] instead of having, so it puts the guts—it puts the eggs in the guts. Right? So it’s like laying its eggs into the Okay. [pause] So then [pause] the eggs hatch.</td>
<td>![Image of Hildi's drawing]</td>
</tr>
<tr>
<td>So let’s say that, say that $g$ of $x$ is [pause] $x$ squared and $f$ of $x$ is $x$ minus three— or whatever.</td>
<td>![Image of Hildi's drawing]</td>
</tr>
</tbody>
</table>
Okay. So then [pause] when eggs hatch [pause] they turn into something else. Right?  [Hildi records,“g((2) –3).”] They are little larva things. So now instead of guts, [pause] the tarantula has [pause] larva.

Okay. Then when the eggs are mature and ready, they burst out [as she records, $(-1)^2$]. And become [as she records, $=1$] whatever it is, right? But when the eggs hatch [as she points at the (2) –3], we don’t have eggs anymore [as she points at $=1$]. We have something else. And when they burst out of the tarantula, we don’t have a tarantula anymore [as she points at $(-1)^2$]. You see, he died. (Ending Interview)

**Discussion of Characteristic 9.** It is important to note that an insect called a tarantula hawk wasp does exist (New Mexico Secretary of State, 2012; Williams, 2012) and Hildi’s tarantula wasp life cycle does bear resemblance to the life cycle of a real tarantula wasp as described by Williams (2012). Hildi’s and Williams’ (2012) descriptions differ in that the tarantula hawk wasp lays a single egg on the spider’s abdomen instead of laying eggs inside the tarantula’s body. In real life, new wasps do not explode from the tarantula’s body. Instead, a newly formed tarantula hawk wasps rip
apart the tarantula’s abdomen to feed directly upon the soft tissue. What Hildi seems to have done in her framing of how to evaluate composite functions is to weave a modified real-world context around a TMId. The tarantula life-cycle context does not mirror the structure of a composite function.

Hildi’s framing of evaluating composite functions is consistent with her metaphor for the concepts of negative exponent and function. In all three of these metaphors, Hildi assigned contextual meanings to a TMId and aspects of the TMId through creative storytelling. Hildi’s question, “Remember the tarantula wasp?” (Observation 2) asked in response to a student requesting help on a problem suggests a pedagogical purpose for Hildi’s metaphors. After introducing the metaphors to students, they serve as attempts to trigger student recall of a TMId or something about a TMId.

**Characteristic 10.** Teachers sometimes use of local language/terminology to make TIMds accessible and comprehensible.

Emerging from observations of three teachers’ interactions with students is the use of a mix of standard mathematical language and *local mathematical language*. Local mathematical language is a spoken and/or written language that has mathematical meaning to a teacher, his or her current and former students, and possibly teachers who inherit the teacher’s former students. However, without explanation or translation local mathematical language may have little or no meaning to outsiders. The data suggest that local language becomes a part of the classroom language in a variety of ways. Sometimes local terminology seems to reflect an organic evolution of the mathematics the teacher and students work on—a way to communicate thinking quickly. Sometimes the teacher introduces the local terminology. Sometimes the local terminology is
introduced into a teacher’s classroom by students who learned the terminology in another classroom.

**Example 10.1. Local language evolving from students and teacher discussing mathematics: diamond problem and friends.** As Kate and her students discussed what needed to be done in order to solve the warm-up problems or the problems on the daily worksheets, they used language that clearly had mathematical meaning for them, but were not terms found in the text adopted by Kate’s district for use in all Algebra 1 classes (i.e., Larson, Boswell, Kanold, & Stiff, 2008). What follows are descriptions and discussions of two of Kate’s local terms, each chosen because they illustrate different evolutions and uses of local mathematical language.

**Example 10.1a. Diamond Problem**, an example of a term whose meaning to students in Kate’s class evolved over time. The term diamond problem was a term used by Kate and her students in all three observations. In the Postobservation 1 Interview, Kate explained that diamond problems are computation problems she introduced to students early in the academic year. Diamond problems have a particular format (see Figure 4-45) in which students are given two entries in the “diamond” and asked to compute the other two entries. Kate said,

> They are a lovely way of reviewing all sorts of ideas. Um, multiplication, addition, subtraction, using positive and negative numbers. Um, uh, fractions. So we usually introduce it at the beginning because what it does is it leads really

Kate states that she encountered diamond problems when she taught Algebra 1 using College Preparatory Mathematics (CPM) curriculum.
nicely into the skills they need for solving a quadratic. (Postobservation 1 Interview)

Examples of diamond problems Kate created during this postobservation interview appear in Figure 4-46.

Figure 4-45. Diamond Problem Format ($a$ and $b$ are numbers). Note: Over time, Kate and her students stopped drawing the borders of the diamond.

Figure 4-46. Examples of diamond problems created by Kate during her Postobservation 1 Interview.

Although Kate was very clear in her explanation that a diamond problem is a kind of computation problem, none of the worksheet problems or warm-up problems assigned and/or discussed during my observations contained exercises that looked like the examples she provided. Nevertheless, Kate and several of her students talked about doing a diamond problem in each observation. At one point, a student asked, “Am I
doing the diamond problem?” (Observation 1). Kate replied, “Yes, you should start with
the diamond problem” (Observation 1).

Kate and her students’ recorded work suggest that the term diamond problem had
become a way of communicating a procedure in the context of factoring quadratic
trinomials. “Doing a diamond problem” meant that students were to create a particular
type of diamond problem, one for which the top and bottom numbers of the diamond are
known (see Figure 4-47a). Students would then determine the factors of the top number
whose sum is the bottom number and use these values to factor the given trinomial.
Example 10.1b. “Friends,” an example of a term that has local meaning that is used to help students make the transition to standard mathematical terminology. Kate’s use of the term, friends, is a byproduct of her use of a student-relatable context to teach students how to solve linear equations. As previously discussed in Example 8.5, Kate frames how to solve linear equations by connecting this TMI to two teams playing on a field that has a river flowing down the middle. It is through this context that Kate gives students an informal understanding of the concept of like terms. According to Kate, she tells students, “you are allowed to do anything to one team that you do to the other team. Each team has got players and you have to get all your friends together” (Ending Interview). In mathematical terms, “friends” are like terms. On the left team (i.e., left side of the equal sign/river), 9x and x are friends; and -5 and +7 are friends. They “get together” to form, 10x and +2, respectively in the second line. Similarly, on the right

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41 How the diamond problem is used by Kate in the factoring quadratic trinomials is explained in detail in Appendix C.
team, in the second line, $2x$ and $9x$ are friends. These friends “get together” to form $11x$ in the third line.

![Image of algebraic expression]

**Figure 4-48.** Screen capture of Kate’s work created while she explained how she engages students in solving linear equations.

**Example 10.2. Local language introduced by the teacher: Kate’s X-box, and Bonnie’s undo-FOIL and AC Rule.** The examples of local language discuss in this sections seem to be terms introduced by the teacher as a way of giving names to multistep procedures.

**Example 10.2a. Example of local terminology used in conjunction with standard language.** X-box, a term coined by one of Kate’s colleagues (Kate, Postobservation 1 Interview) is the name of a factoring method in which students begin by drawing an $X$ and a rectangle that is divided into four quadrants. What follows is an excerpt of a conversation between Kate and one of her students regarding a warm-up problem for which students had to solve $0 = 2x^2 + 9x + 10$ to graph a sketch of $y = 2x^2 + 9x + 10$.

Kate: There’s only one way we’ve learn to solve quadratics, what is that?

Student: X-box

Kate: Yeah. Factoring. Now how do we factor, we use the–

[Kate records, $0 = (\ ) (\ )$.]
we’ve been using a diamond problem, and a rectangle. In other words, we’ve been using the X-box method right? So. X-box.

[Kate draws an “X” and a box divided into four quadrants (see Figure 4-49).]

(Kate, Observation 2)

Figure 4-49. Screen capture of Kate’s inscriptions recorded just before after she said, “So. X-box” (Kate, Observation 2). The X and the rectangle are the basis for the factoring procedure’s name, X-box.

This particular excerpt provides evidence that the student’s response, “X-box,” was not simply the case of a student interjecting the name of a popular computer gaming system, but a locally meaningful mathematical response to Kate’s question, “There’s only one way we’ve learn to solve quadratics, what is that?” (Observation 2). Figure 4-50 provides evidence that the “X” (i.e., a diamond problem) and the box were used as part of the factoring process.
Example 10.2b. Examples of local terminology: Names of factoring methods. As discussed in Example 5.2, in Bonnie’s classroom, there are two categories of factoring quadratic trinomials: “undo-FOIL” and “AC Rule.” Bonnie uses these two terms to differentiate between two factoring procedures. “Undo-FOIL” is what one does when factoring quadratic trinomials of the form, \( x^2 + bx + c \), and AC Rule is what one does when factoring quadratic trinomials of the form, \( ax^2 + bx + c \) \((a \neq 0, 1)\).

Example 10.3. Local language introduced by students: doink-doink. Consider the following exchange between Faith and her students that took place during Observation 2 while they were in the process of computing the value of \(-(-6)^2 - 3\).

Faith: So we’re going to square negative six because that’s what <Student T> told me to do last time.
Student: I thought a double negative makes a positive

Student: Doink-doink!

Faith: Well if I—No! You can’t doink-doink here [as she points at 

\[ -(-6)^2 \]! ]

Faith’s response to the student’s comment provide evidence that doink-doink was not a silly interjection being made by a middle school student, but an example of local language. The conversation about doink-doink and why it should not be used when computing the value of 

\[ -(-6)^2 \] concluded with the following exchange.

Faith: Yeah if you [pause] doink-doink as you say [pause] so [pause] very [pause] joyfully, uh, you’re not following order of operation. Order of operation says, exponents before multiplication. [Faith points at the negative sign in front of 

\[ -(-6)^2 - 3 \] ] But if you’re doink-doinking—is that the plural of doink-doink?

Student: Doink-doink-eye?

Faith: then you’re doing order of operation out of order.

Student: So then I got it wrong?

Faith: If you doink-doinked first, then yes. Yes, yes you did.

One interesting point about this use of local language is that doink-doink did not originate in Faith’s classroom. In the Postobservation 2 Interview, Faith explained that doink-doink is a term her students learned in their prealgebra class. According to Faith, their prealgebra teacher says doink-doink whenever a computation involves subtracting a
negative number. For example, when their Prealgebra teacher changes \(-(-2)\) to \(+(+2)\), she says, “doink” as she changes each minus sign to a plus sign; hence, doink-doink.

The problem, according to Faith is that “every time the kids see two minus signs next to each other, they have this overwhelming urge to doink-doink, regardless of order of operation” (Postobservation 2 Interview). In addition to wanting to doink-doink for \(-(-6)^2\) Faith says some of her students would also want to doink-doink when they see \((-2)+(-2)\) (Postobservation 2 Interview).

The teacher whose students introduced doink-doink into Faith’s classroom is not one of the participants in this study and therefore, I can only hypothesize why this teacher chooses to frame subtracting negative numbers via doink-doink. Doink-doink could simply be intended to be a way of remembering what to do.

Another interesting point about Faith’s student’s introduction of doink-doink to her classroom is that it suggests the possibility of a student who has learned mathematics without also learning standard mathematical languages moving on to another mathematics classroom in which the teacher uses only standard mathematical language in his/her explanations, discussions, and instructions. This hypothetical situation suggests that framing via the use of local language might interfere in a student’s learning of mathematics in other mathematics classrooms.

**Discussion of Characteristic 10.** The data suggest that local terminology is often used in lieu of standard mathematical language as an informal way to talk about TMIIs. The excerpts of conversations between Kate and her students, and Faith and her students provide evidence that conversations involving local language can be mathematically
meaningful and facilitate student interaction with mathematical ideas. However, because students in Kate’s, Bonnie’s and Faith’s respective classes use local language when talking about TMIds, from a research perspective, one wonders to what extent the use of local language in students’ current math class helps or hinders their discussions and/or understandings of mathematics encountered in future mathematics classes in which the instructor uses standard mathematics language. The meanings conveyed by local language are contextual (e.g., friends) and/or situational (e.g., doink-doink) not the same mathematical meanings as conveyed by standard mathematical language.

Data about Kate suggest that local language can be used as a means to help students learn standard mathematical language. During Observation 1, Kate was overheard asking a student, “So if I set that equal to zero, what are the friends?” Later, when asked to explain what friends are, Kate recorded $3x^2 + 9x + 2 + 3x - 5x^2 + 1$ and proceeded to identify the sets of friends: $3x^2$ and $-5x^2$ with underlining, $9x$ and $3x$ by boxes, and 2 and 1 with circles (see Figure 4-51). According to Kate, students prefer to use the term, friends over like terms because the phrase like terms does not have a contextual meaning for them (Ending Interview).

\[
\begin{align*}
3x^2 &+ 9x + 2 + 3x - 5x^2 + 1 \\
-2x^2 &+ 11x + 3
\end{align*}
\]

Figure 4-51. Screen capture of Kate’s work for her explanation of “friends” (Postobservation 1 Interview).
When asked explicitly, whether her students would understand another teacher who says, “combining like terms” instead of “getting friends together,” Kate replied, Yes! Yeah, we use the two terms interchangeably. We try to introduce, you know, technical jargon in context. So, we do and the questions are phrased like, combining like terms. We don’t say—we usually write, combining like terms—look for friends or something like that. So we try to use both, a lot. But they should. They should! Yes. (Ending Interview)

An examination of all 42 of the worksheets, tests, and quizzes that Kate used in the observed class revealed no written use of the term, friends. The directions for all problems for which combining like terms is the focus were written using standard language.

Data from Faith’s class suggest that helping students make the transition to standard mathematical language is sometimes an inherited task and that sometimes the inheritance includes helping students correct a misconception. As described in Example 10.3, Faith’s students’ use of the term, “doink-doink,” has its origins in a prealgebra class. The term began as a sound effect made by their teacher as she converted an expression of the form, \(-k\) for \(k > 0\) to \(+k\). Unfortunately, the student who introduced the term “doink-doink” into the class discussion seems to have associated doink-doink with changing two “–” signs to two “+” signs without regard to mathematical meaning and the conditions under which doink-doink was originally

\[42\text{ Kate provided this researcher with copies of all of the worksheets, tests, and quizzes she used during the Spring semester from January to the beginning of April.}\]
introduced. Somehow this middle school student and according to Faith, a few of his classmates, started applying “doink-doink” to any expression that contains two negative signs (e.g., \(-a + \(-b\) and \(-(-c)^2\) for \(a, b, c > 0\)). Doink-doink seems to have become divorced from its original meaning. It seems to have become a part of stimulus-response action (i.e., stimulus: see any two negatives; response: make the two negatives into two positives).

**Characteristic 11.** Some teachers’ framings each seem to serve multiple mathematical purposes and are components of larger framings, and some teachers’ framings serve a single mathematical purpose and are disconnected from other framings and other TMIds.

Inherent in the notion of framing is purpose. Teachers frame to make TMIds accessible and comprehensible. However, as data analysis progressed it became apparent that teachers use framings in different ways. Some teachers’ framings serve more than one purpose. Framings make a TMId accessible and comprehensible, and are used as components for framings of other TMIds. Some teachers’ framings serve a single purpose and are unrelated to other framings. Examples of multi-purposed framing are drawn from data about Kate. Examples of singlepurposed framings are drawn from data about Hildi.

**Example 11.1. Multi-purposed framings: Revisiting Kate’s framing of the discriminant and the quadratic formula.** Examination of the worksheets Kate assigned
during the three consecutive observations revealed tasks reflecting three framings: one, familiarizing students with seeing and evaluating the discriminant (see Example 1.1); two, completing the square; and three, solving quadratic equations of the form \((x + k)^2 = k_2\) (see Example 6.4). During the End interview, Kate stated that in the next lesson she would be deriving the quadratic formula and having students solve quadratic equations using this formula. Based upon this information and an examination of the worksheet she planned to give her students, the three framings would be used in that lesson’s framings.

Kate’s framing of the discriminant, \(D = b^2 - 4ac\), as an expression whose value is important to compute (see Example 1.1) supports computing the roots of quadratic equations using the quadratic formula. Kate’s framing of completing the square that is based upon the use of the area model and generic rectangles (see Figure 4-27) supports solving quadratic equations of the form, \((x + k)^2 = k_2\). Kate’s framing of solving quadratic equations of the form, \((x + k)^2 = k_2\) (see examples 6.4 and 7.1) supports her derivation of the quadratic formula.

\(^{43}\) Kate teaches on an alternating block schedule. The three consecutive observations of Kate’s teaching and the corresponding observation-related interviews took place over a span of 6 school days in the last 2 weeks of March.
Example 11.2. Single-purposed, disconnected framings: Hildi’s framings involved in graphing

During Observation 1, Hildi stated that they would be reviewing how to graph exponential functions by “plugging in points” instead of using “shortcuts.” During the Postobservation 1 Interview, Hildi explained what she meant by “shortcuts” by identifying what each of the components in

\[ y = \frac{1}{2} \left( \frac{1}{2} \right)^{-1} + 2 \]

means. After drawing what she identified as the parent graph (see Figure 4-53) of

\[ y = \frac{1}{2} \left( \frac{1}{2} \right)^{-1} + 2, \text{ Hildi said,} \]

- the “+2” means “moves it up” (Postobservation 1 Interview).
- the “-1” in the exponent means move to the right (Postobservation 1 Interview).

Figure 4-52. Screen capture of a worksheet problem from Observation 2 that illustrates Kate’s framing of completing the square using a generic rectangle.
• the $\frac{1}{2}$ in $\left(\frac{1}{2}\right)^{x-1}$ makes the graph “swoopy left” (see Figure 4.53a).

• the “$-$” in the $\frac{1}{2}$ coefficient reflects the parent graph across the $x$-axis, which she recorded on paper as “going down” (see Figure 4.53b).

\[ y = -\frac{1}{2\left(\frac{1}{2}\right)} + 2 \] (Observation 1)

When asked to explain why she chose plotting points over using her shortcuts, Hildi said,

There's a lot to remember with that. So um, as far as the students [pause] memorizing more stuff, I'd rather have them practice a skill that was translatable.

Because if you're faced with a graph that you don't know how to do, you can always plug in points. (Postobservation 1 Interview)
Later she restated her reasoning by saying, “So instead of just going over the tricks for this one type of equation, … I’d rather have them practice a skill that’s translatable to more.”

There are several interesting aspects of Hildi’s discussion about graphing

\[ y = -\frac{1}{2} \left( \frac{1}{2} \right)^{x-1} + 2. \]

Hildi’s descriptions of the role that each of the different components of this functions plays in the transformation of \( y = 2^x \) (see bulleted list) can be taken as representatives of general transformations; however, Hildi’s comment about using “tricks for graphing this one type of equation” indicate that those descriptions were specific to the given equation. Her comments about the use of tricks for graphing one type of equation, and about students having a lot to remember/memorize suggest that Hildi views graphing exponential functions as a procedure composed of separate components. The components are connected only because they are parts of the same function. It seems that each component has its own framing (see Figure 4-53a).

Hildi’s framing of the exponent in \( y = -\frac{1}{2} \left( \frac{1}{2} \right)^{x-1} + 2 \) is of particular interest because of how she framed the exponent in \( \left( \frac{1}{2} \right)^{-1} \) during class. Hildi told her students that the \(-1\) in this expression means “flip it” (see Example 1.2). So, \( \left( \frac{1}{2} \right)^{-1} \) means \( \frac{2}{1} \). In contrast, when she talked about the exponent in the given function, she drew an arrow to the “-1” (see Figure 4-53a) and said it means “move it to the right.” Although these
frames originate from the same function and both address exponents, they are disconnected.

Evidence that Hildi might prefer her framings to be disconnected can be inferred from Hildi’s response to a scenario I posed in hopes of understanding whether “flip it” is reserved for working with fractions. Hildi had already stated that the negative coefficient of $y = -\frac{1}{2} \left( \frac{1}{2} \right)^{x-1} + 2$ reflects the parent graph across the $x$-axis. She was asked to consider that the reflection could be interpreted as “flipping” the graph across the $x$-axis. Hildi agreed that it could, but stated that this is why she chose to have her students plot points instead of going over what she called, shortcuts. It seems that Hildi sees the two interpretations for “flip it” to be confusing for her students or Hildi herself does not see the connection.

**Discussion of one teacher’s framings.** The emergence of framings that serve multiple purposes and are components of larger framings, and framings that serve a single purpose and are disconnected from other framings and other TMIds is not surprising or interesting. What is interesting is that although each of the seven teachers participating in this study demonstrated or described their use of multiple mathematical purpose framings, use of multiple mathematical purpose framings seems to be a characteristic of how one teacher—Kate—teaches. Although the data about Kate is insufficient to explain why framings that serve multiple mathematical purposes characterize her teaching, the data suggest at least one contributing factor. Kate has a strong desire for her students to “understand why math works the way it does” (Kate, Beginning Interview). When asked to say more about this goal, Kate responded,
Students should be able to explain why something works the way it does. And for me, that’s understanding. [Pause] It’s getting down to the why, and being able to explain that why. So not just being able to do it, but explaining to why they were able to make those maneuvers. Why they were able to—why it works the way it does. It’s more than just a trick or a skill-based thing. (Beginning Interview)

Kate went on to talk about “understanding why” in terms of a specific example. She said,

So a kid can take a quadratic a basic quadratic, $x^2 + 5x + 6$ [equals zero] and they can tell you $x$ is negative two, negative three. I want them to understand [pause] what they’re doing is, is using this thing called zero product property. I want them to understand that the notion behind it is that essentially taking two things being multiplied together and if those two things are going to multiply to make zero that one of them had to be zero because there’s no way other way to multiply to make zero. (Beginning Interview)

Kate’s desire for students to be able to explain why mathematics works the way it does might explain why her framings of smaller TMIds are intertwined in her framing of larger TMIds. The data do suggest that her use of local language and student-relatable contexts are ways of supporting student engagement in discussions about TMIds and possibly, opportunities to explain why. A majority of Kate’s students’ discussions took place when they worked in small groups. Unfortunately, the data about Kate does not include evidence of students explaining why to each other or to Kate.

**Overview: Characteristics of framing and their corresponding examples.**

The characteristics of framing and their corresponding examples not only document what was captured in the data, but offer different perspectives on how TMIds are made
accessible and comprehensible to students. Collectively, the discussions of these eleven characteristics of framing offer further evidence about the complexity of teaching and the variety of things teachers do to help students learn.

**Standing back: Broader insights about framing.**

Following Hiebert et al.’s (2003) advice of viewing teaching through the use of close-up lenses and the use of wide-angle lenses, to view and gain understandings of both specific and general aspects of teaching, I examined the data using a wider lens than was used to identify the eleven characteristics of framing that were discussed earlier in this chapter, and was used to identify the original 50+ instances of framing. Throughout the data analysis process four different questions repeatedly came to mind as each example of framing was examined. Where is the mathematics? What is the role of mathematics in the framing of targeted mathematical ideas? What are potential impacts of this framing on the learning of mathematics encountered in the students’ future mathematics classes? If mathematics does not play a leading role, what does seem to play the lead? The data clearly showed that mathematics does not always play a major role in the framing of the TMId. These questions were asked to gain an understanding of when and how mathematics us used in framing. The first of these questions originally came to mind before this study began as I thought about the many lesson colleagues had shared with me about how they teach specific mathematical ideas. Some teachers used mathematics to teach new mathematics (e.g., showing that the distance formula is a special case of the Pythagorean Theorem). Some teachers used other means to teach new mathematics (e.g., teaching students the quadratic formula to the tune of Pop Goes the Weasel). As the
study progressed and the concept of framing was developed as a way to view and talk about teaching, the remaining three questions came to mind. Mathematics is still the main idea, but the questions are more specific in that they address the mathematics in the framing. These questions were collectively used to examine the eleven characteristics and the examples of framing illustrating each characteristic for more general patterns and messages about teaching school mathematics.

Resulting from this examination are three messages about framing, and additional understandings of nonmathematical framings that led to a modification of the Conceptual Framework for Framing. The modified Conceptual Framework for Framing will be discussed in chapter 5. The messages are as follows:

Message 1. Mathematics sometimes plays little or no role in framing.

Message 2. Mathematics can play a critical role in making connections between TMIds and nonmathematical contexts.


**Message 1. Mathematics sometimes has little or no role in framing.** The data reveals that framings for which mathematics has little or no role were used under two conditions: (a) when the teaching emphasis seemed to be on making a TMId or some aspect of a TMId remember-able (i.e., easy to remember), or (b) when the primary emphasis seemed to be students’ production of correct answers.

**Making TMIds remember-able.** When teachers try to make a TMId or some aspect of a TMId remember-able, they try to make it easy to remember. Two examples
of teachers attempting to make a TMId remember-able include telling students that division by zero is a math sin (see Example 2.2), and teaching students to sing the quadratic formula to the tune of Pop Goes the Weasel (http://www.regentsprep.org/Regents/math/algtrig/ATE3/quadsongs.htm). In the first framing nothing about division by zero is used in the framing except to communicate, “Do not do this.” One could replace “division by zero” with “using a red pen on a math test,” say “Using a red pen on a math test is a math sin,” and communicate the same message with a different referent. The second framing tells students nothing about the quadratic formula except, perhaps that its verbal expression can be sung. Both of these framings have an appealing, fun/funny element to them, but neither is likely to help students to progress beyond remembering.

It is important to note, however, that remember-able is a relative phenomenon. FOIL (i.e., an acronym for first, outer, inner, last) is used by many teachers to help students remember which pair of terms to multiply when expanding a product of the form, \((ax + b)(cx + d)\). To Delia whose teacher apparently framed the multiplication of two binomials using FOIL, the acronym made no sense. She said, “The acronym did not help me remember what to do. It just doesn’t make sense– it never made sense to me” (Postobservation 2 Interview).

**Helping students get correct answers: See this-do this.** The data revealed framings whose primary emphasis seemed to be on helping students produce correct answers. These framings often seem to have an underlying message of “when you see this ____ , do this ____.” In Example 1.2, we learned that when you see an expression that has a base raised to a negative two exponent, “the negative means flip it. …So we flip it
In Example 6.2, Hildi advised students that when they saw $a^{\frac{1}{2}}$, (Hildi, PS Interview) they should write, $\sqrt{a}$ (see Example 6.2, Figure 4-38). Another, more complex framing that seems to be built upon this “see this-do this” message is the macro procedure discussed in Example 3.4. The macro procedure is designed so that if students see a particular set of visible features (e.g., number of terms, quadratic coefficient equal to some number other than one, type of exponents), they apply a certain factoring procedure (see Example 3.4). It is important to note here that in these framings, no justification is offered to students. These framings seem to be intended as stimulus-response descriptions.

In Example 6.1, students are told that when they are asked to simplify rational expressions which require rationalizing the denominator, as part of the simplification process, if they see that the denominator consists of two terms, they are to multiply the fraction by the number 1 in the form of a fraction with the conjugate of the original denominator as both the numerator and the denominator as opposed to multiplying by 1 in the form of a fraction with the original denominator as both the numerator and the denominator. There is an important difference between this framing and the three framings previously mentioned in this section (Examples, 1.2, 3.4, and 6.2). In this framing, the teacher justifies why they have to multiply by the conjugate. The teacher says,

The point is you’re still trying to get rid of that radical. And the way you get rid of the radical is to use that sum and difference [pause] because then the middle term drops out. That’s where the radical is” (Bonnie, Observation 3).
Helping students get correct answers: Organizers. The data revealed additional framings whose primary emphasis seemed to be on helping students produce correct answers by organizing their thinking. These framings involved the use of some kind of tool. Bonnie taught her students a macro procedure (i.e., Example 3.4) that provides students with a systematic way of deciding which of seven factoring procedure to apply. Faith and Bonnie taught their students how to multiply polynomials, using a box/lattice (i.e., Example 4.3). Both teachers used this graphic organizer to help students who multiplied incorrectly because they “lose things or miss things or forget to multiply or something” (Bonnie, Postobservation 2 Interview).

Message 2. Mathematics can play a critical role in making connections between TMIds and nonmathematical contexts. Emerging from the data are two types of framings in which teachers make concerted efforts to make connections between TMIds and contexts to which students can relate. In the first type, nonmathematical contexts are applied to TMIds. In the second type, nonmathematical contexts and TMIds are intertwined.

Contexts are applied to TMIds. Consider, for example, the framing of how to simplify the rational expression, \( \frac{x^2}{y^3} \), discussed in Example 8.1. In this example, the expression is likened to a two-story building in which both the first- and second-floor tenants, as indicated by the “–” signs in the exponents, are unhappy. To make the tenants happy, one moves each of them to the other floor. When the moves are completed, the expression is \( \frac{y^3}{x^2} \) and the tenants in the building are both happy. There does not seem to
be a mathematical basis for mapping unhappiness, an emotional state, to the “−” sign or changing the emotional state of the contents of the numerator and the denominator of an expression to the transformation of $\frac{x^2}{y^3}$ to $\frac{y^3}{x^2}$. The basis for the connections between the TMId and the context seems to be a play on words: Unhappy tenants are experiencing a negative emotion.

_Contexts and TMIds are intertwined._ Two examples (see Examples 2.3 and 8.5) of nonmathematical contexts in which mathematics plays a critical role appeared in the data. In Example 2.3, the context of an exclusive social event to which only nonscandalous friends receive invitations is used to explain the concept of function. The elements that make up the definition of function are each reflected in the context. Functions are mappings from set A to set B, such that each element of set A is mapped to exactly one element of B. In the nonmathematical context, set A is the set of nonscandalous friends, set B is the set of dates and nonscandalous friends bring exactly one date to a party. Scandalous friends would bring more than one date and are, therefore, not allowed to the party. The story told in the context is plausible.

In Example 8.5 the context of a game being played on opposing sides of a field that is divided by a river is used to teach students how to solve linear equations which have variables on both sides of an equation. When the teacher who uses this context solves linear equations, she draws a wavy line (i.e., the river) through the equal sign. One of the rules of the game includes getting the friends together. This rule is a metaphor for combining like terms. For the game to be fair, what happens to a team on one side of the river happens to the team on the other side. This rule is a way of talking about the
addition, and multiplication properties of equality. At some point in Kate’s framing of how to solve linear equations, adjustments need to be made when working with negative and fractional quantities of friends. Although data about Kate does not include what adjustments she makes in her framing of solving linear equations, I hypothesize that she either sets the context aside, or operationalizes the fairness rule so that working with negative and fractional quantities is included. What seems to be important about Kate’s framing is that most of what students need to know when solving linear equations with variables on both sides of the equal sign is reflected in this context.

In each of these framings, the context is one to which students can relate or about which students can imagine. More importantly, the design of each context is based upon its respective TMId. These framings offer a contrast to framings such as telling students that division by zero is a math sin. As stated earlier in this chapter, division by zero is connected to a major transgression; but mathematics does not play a role in the framing itself.

**Message 3. Developing students’ mathematical language, verbal and symbolic, plays a role in making TMIds of school algebra accessible and comprehensible.**

Although none of the teachers participating in this study explicitly stated that making TMIds accessible and comprehensible to students includes an effort to teach students how

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Kate, who uses this context to teach her students how to solve linear equations, did not reveal a metaphor for applying the distributive property. Kate simply said, “I would have to use my distributive property” (Ending Interview).
to speak, read, and write mathematics language, the data suggest that mathematical language plays a role in framing.

**Translating.** Hildi states explicitly that she believes that “mathematics is a language” (PS Interview) and that in her experience students have difficulty translating symbolic representations (PS Interview). One of Hildi’s strategies for addressing this difficulty is to provide translations. With respect to exponential expressions, her translations address what the exponent “tells you what to do” (PS Interview). What follows are translations for the exponents in four different exponential expressions.

- Regarding the exponent of $a^3$, “means multiply $a$ by itself three times” (Hildi, PS Interview)

- Regarding the exponents in the numerator and denominator of $\frac{x^2}{y^3}$, the “–” sign is a “mood meter” (Hildi, Postobservation 1 Interview) that indicates that the expressions in the numerator and denominator are “unhappy” and one must move the terms—without the “–” to the denominator and numerator, respectively.

- Regarding the exponent of $(\frac{1}{2})^{-1}$, “the negative means flip it (Hildi, Observation 1)

- Regarding the exponent of $(\frac{1}{2})^{-2}$, “multiply by itself $\frac{1}{2}$ and then flip it $\frac{1}{2}$, or flip it $\frac{1}{2}$ multiply it $\frac{1}{2}$ by itself $\frac{1}{2}$” (Hildi, Observation 1)
Regarding the exponent of \( \frac{1}{2} \), “It’s going to be, you know, this [1] is the “raised this power” and it’s [a] going to be rooted by this power [2]”

(Hildi, PS Interview)

From the perspective that mathematics is a language, it makes sense that translation is a strategy used in framing. At the same time, as suggested by the three translations for expressions with negative integer exponents in the collection above, there a possibility of students coming away with a collection of conceptions what negative integer exponents are.

**Informal terminology.** The use of informal terminology emerged as a common aspect of making TMIds accessible and comprehensible. Some informal terms have meanings or conveyed meanings to students that standard mathematical terms did or could not. Kate and her students say *friends* instead of *like terms*. According to Kate the phrase *like terms*, has no contextual meaning for her students (Observation 1) and the term *friends*, does. Bonnie and her students use FOIL and undo-FOIL when talking about multiplying binomials and factoring quadratic trinomials whose quadratic coefficient equals one, respectively. The “undo” prefix conveys that FOIL and undo-FOIL have an inverse relationship—what Bonnie describes as a “reverse” (Postobservation 2 Interview)—that may not be communicated by “multiplying binomials” and “factoring quadratic trinomials.”

Some informal terms are simply names. Ann teaches her students how to factor using an unusual method she calls, the Secret Method. The secret, according to Ann is one of the steps in her procedure (see, Example 5.1). Some of Faith’s 8th-grade students
use the term *doink-doink* to describe converting an expression of the form,

\[-(−n) \rightarrow +(+n)\] (see Example 10.3). All that is known about the origination of this odd term is that some of Faith’s students learned it from their 7th-grade mathematics teacher who apparently says, doink-doink as she changes the two – signs to two + signs. Neither the Secret Method nor doink-doink seems to have meaning that is intrinsic to students or convey mathematical meaning of the TMId to which they are linked. Ann has received complaints from her colleagues who teach the higher level mathematics classes about her teaching her students the Secret Method. According to Ann, some of her students factor incorrectly because they forget to do the secret step. Faith finds doink-doink problematic because several of those 8th graders who were introduced to doink-doink also changed any two negative signs that appeared in an expression to two positive signs.

The terms, themselves, are not solely to blame for students’ errors. It is quite possible that students forget an important step to an unusual factoring procedure because the important step does not make sense mathematically. Changing any two negative signs in an expression to two positive signs is also a common error among students who remember “a negative and a negative make a positive” instead of “a negative times a negative is a positive.” Nevertheless, the use of terms whose connections to mathematical meaning are weak can be seen as contributing to student errors and problematic conceptions.

*Local mathematical language.* Local mathematical language is a form of oral and/or written communication that has meaning to members of a classroom community, but may have little or no meaning to outsiders without explanation. It goes beyond the
simple use of informal mathematical terminology. When Kate or one of her students talk about getting friends together, they are talking about combining like terms (see Example 10.1b). With respect to factoring quadratic trinomials of the form, $x^2 + bx + c$, when one of Kate’s students asks if he is doing a diamond problem, he is asking if he is supposed to find factors of $c$ that add up to $b$ (see Example 10.1a). If one were to ask Kate’s students what they do to factor quadratic trinomials of the form, $ax^2 + bx + c$ ($a \neq 1$), they are likely to reply that they use generic rectangles, or X Box (see Example 10.2a).

As a way of helping students learn and operate with abstract ideas of school algebra, the use of informal terminology and developing a local mathematical language seem to be logical teaching strategies. Questions arise when one considers the impact of the use of informal mathematical language for students, when students take state-mandated tests, and when students take higher level mathematics classes from another teacher who uses standard mathematical language or a different local language? The use of informal/local mathematical language may make TMIIds accessible and comprehensible, but to what extent does the use of local/informal mathematical language hinder students’ understanding of those same ideas? Would, for example, students who know $3x$ and $9x$ as friends be able to understand instructions directing them to combine like terms? To what extent do students who were amused and entertained by the story of an exclusive party to which only nonscandalous friends receive invitations understand what a function is? To what extent do nonmathematical contexts, informal terminology, and local languages—all intended to make a TMIId accessible and comprehensible—help in the long term?
This chapter is the first of two chapters that are devoted to discussing the results of analyzing data about seven teachers’ teaching and the same teachers’ talking about their teaching. The data were analyzed in four rounds, each round with a different lens and goal. In the first round, the goal was to identify instances of framing in the data. In the second and third rounds, the overall goal was to learn about different aspects of framing. In the second round, 11 characteristics of framing were identified from the 50+ instances of framing that were identified in the first round. In the third round, the 11 characteristics and their corresponding examples were examined to identify the nature of the roles that mathematics plays in framing. Products of this analysis are three messages about framing and an addition of three new constructs to the Conceptual Framework for Framing. It seems odd to ask what the role of mathematics is in the teaching of school algebra, yet the data revealed that the answer to this question is not a straightforward one. Data exist indicating that mathematics can play a key role in a teacher’s framings. At the same time, data exist indicating that mathematics can play little or no role in a teacher’s framing.

Chapter 5 is composed of two parts. In the first part, the six constructs (i.e., the specific teaching practices) in the modified Conceptual Framework for Framing will be defined and discussed. In the second part, insights gained by looking at the data through the lens of the modified Framework will be shared.
Chapter 5

Results: A Modified Conceptual Framework for Framing

The conceptual framework (shown in Figure 5-1) guiding this study’s data collection and data analysis involves four constructs: framing, and the three mathematical teaching practices (decompressing, trimming, and bridging). The constructs of framing, decompressing, trimming, and bridging were directly observed through participants’ overt use with students or through discussions of how they make TMIds accessible and comprehensible with the interviewer, or indirectly through examination of lesson handouts. In the original framework, framing consists of mathematical teaching practices (i.e., decompressing, trimming, and bridging) and framings that involve nonmathematical approaches.

![Figure 5-1. Conceptual framework for framing (prior to data analysis).](image-url)
Identification of the New Constructs

After the data were examined to identify characteristics of framing in the second round of data analysis, the more than 50 instances of framing identified in the first round were sorted into four groups: Framings that are examples of decompressing, framings that are examples of trimming, framing that are examples of bridging, and framings that involved nonmathematical elements. Each example of framing was compared to the definitions inferred from KAT researchers’ descriptions of these practices and to the examples of decompressing, trimming, and bridging provided by KAT researchers. Each framing in the fourth group (i.e., those not identified as decompressing, trimming, or bridging) was examined to identify its nonmathematical property/properties or quality/qualities. The examinations resulted in the realization that the framings that have nonmathematical elements could be sorted into subsets of framings that involved breaking down a TMId (i.e., a quality of decompressing), bringing attention to some aspect/component of a TMId (i.e., a quality of trimming), or making one or more connections (i.e., a quality of bridging). It is important to note that some framings involved the use of two or more teaching practices. In such a case, the framing was placed into the subset that characterized the primary teaching practice being used.

Because these three sets each reflected some of the qualities of their counterparts, decompressing, trimming, and bridging, these sets of framings were called quasi-

45 Some instances of framing involved the use of more than one teaching practice. In such cases, the framing was compared to the definitions to determine of which teaching practice the instance of framing was the best or better example.
decompressing, quasi-trimming, and quasi-bridging. The prefix, quasi, was chosen because it means seemingly; apparently, but not really; being partly; and almost. Quasi is a more accurate depiction of what the participants in this study did than does another prefix considered for use: pseudo, which means false, or artificial. Nothing the teachers participating in this study did was false or artificial. Collectively, these practices are called, quasi-mathematical teaching practices and defined as teaching practices that reflect a defining quality of mathematical teaching practices, but are also characterized by the use of nonmathematical elements or ideas.

The Conceptual Framework for Framing

Data analyses revealed that two sets of teaching practices contribute to teachers’ efforts to frame TMIIs: Three mathematical teaching practices and three quasi-mathematical teaching practices. A representation of the Conceptual Framework for Framing, modified with the addition of the quasi-mathematical teaching practices, is shown in Figure 5-2. In addition to framing, the conceptual framework consists of six constructs organized into two sets of teaching practices: Decompressing, trimming, and bridging that are collectively called mathematical teaching practices; and quasi-decompressing, quasi-trimming, and quasi-bridging that are collectively called quasi-mathematical teaching practices. In addition to the inclusion of the three quasi-mathematical teaching practices, this framework differs from the original framework (see Figure 5-1) in that the six teaching practices are represented as distinct circular regions that contribute to the overarching construct of framing. This representation allows for the possibility of future research revealing framings that are not accounted for by the
mathematical and quasi-mathematical teaching practices. In the original framework, framing is represented by a single box that is divided into four regions, a representation that suggests that framing is composed of decompressing, trimming, bridging and framings that involve nonmathematical approaches. The data neither support nor refutes such an interpretation. The more open organization of the constructs that make up the modified Conceptual Framework for Framing was designed to allow for the possibility of future research revealing framings that are not accounted for by the mathematical and quasi-mathematical teaching practices. Until further notice, this Framework will be referenced with the adjective, modified.

![Modified Conceptual Framework for Framing](image)

Figure 5-2. Modified conceptual framework for framing (subsequent to data analysis).

**Defining and Refining Definitions**

As a result of the sorting process, characteristics of the new constructs were identified and general descriptions were proposed. For example, “quasi-mathematical
practices make use of approaches that teachers believe are ones that students are likely to understand and/or follow.” In order to develop these descriptions into definitions and to refine these definitions, the help of four mathematics educators who have expertise working on research teams for the Process Project of the Mid-Atlantic Center for Teaching and Learning as well as at least five years of experience teaching school algebra. Their work as members of Process Project research teams pertinent to this part of my data analysis is qualitative data analysis of teaching and coding data, expertise in research, coding, and teaching school algebra was enlisted.

Obtaining expert panel feedback: Preparations. Developing and refining the definitions actually began with the creation of a set of working definitions in preparation for the expert panel feedback. The definitions were arranged in a two-column table so that definitions of the mathematical teaching practices were in the left column and the quasi-mathematical teaching practices were in the right column. The working definition of each mathematical teaching practice definition and its “quasi-counterpart” were placed side by side to facilitate comparing and contrasting the working definitions.

To provide the experts with a basis upon which to provide feedback on the definitions, I created a document of examples of framing. I started with the 33 examples of framing that were discussed in chapter 4, and coded them using the working definitions. These 33 examples represent the best and the clearest of the 50+ examples of framing originally identified in Round1 of data analysis. The framings that were not

46 From this point forward, these four mathematics educators will be referred to as experts.
included in the discussions in chapter 4 are similar to the framings that were discussed in chapter 4. For example, demonstrating a procedure, step by step is a practice that was used by Ann, Bonnie, Delia, Faith, and Gary in more than several lessons. Gary’s step-by-step demonstration of a procedure (see Example 3.1) was selected for discussion in chapter 4 because it most clearly illustrates breaking a procedure into steps was the most transparent of all framings of this type. Gary’s use of PowerPoint slides made identification of his steps very easy. Bonnie’s framing (see Example 3.2) was chosen for discussion in chapter 4 because the section of transcript from which her example was taken is the clearest of the sections of transcript capturing this type of framing. In addition, this kindway of framing a TMId is represents the primary means by which Bonnie makes TMs accessible and comprehensible to students and because unlike Gary’s framing of the derivation of the quadratic formula, Bonnie’s framing is that specific TMId is unplanned. It was a response to a student’s question about a homework problem.

Many of the framings reflect the use of more than one teaching practice. The document used in the feedback sessions consists of twelve examples of framing. The twelve examples were chosen according to the following guidelines: (a) each of the six teaching practices had to be represented, and (a) due to time restrictions, each example had to be relatively short (i.e., required no more than approximately 1½ pages). The documents for this activity are found in Appendix E along with all twelve framings. Two of the framings are shown in Figure 5-3 for quick reference.
### Framing A

**Teacher:** Faith

In teaching her students how to solve quadratic equations by completing the square, Faith stated that she began teaching the procedure by using examples with “nice numbers.” When asked what she means by “nice numbers,” Faith said,

Well starting off where there’s no um $a$ term—well, it’s $a$ term [sic] is one. Um and then where $b$ can be divided in half nicely as opposed to like a $b$ term of three… I’d like to give them numbers where the numbers are going to [pause] avoid fractions and that sort of—uh situation.” (Faith, Postobservation 3 Interview)

### Framing H

**Teacher:** Delia

The focus of Delia’s lesson for Observation 3 was multiplying pairs of linear binomials. Delia’s board work, as it evolved, is captured in below. With each practice problem, as she guided students through the procedure, Delia repeatedly used the terms distribute or distributive property.

![Figure 5-3. Two examples of framing presented to the expert panel.](image)

Obtaining expert feedback. Gathering the feedback took place in the course of three rounds with two experts in Round 1, one expert in Round 2 and one expert in
Round 3. The feedback sessions were conducted in the home of one of the experts in Round 1, and in the respective offices of the experts in Rounds 2 and 3. The settings of sessions were chosen to accommodate the respective schedules of all of the experts. All three rounds were conducted using the same basic guidelines.

a. The experts were each shown both versions of the Conceptual Framework for Framing to provide them with a context and told that their assistance was needed in helping me refine the definitions of the constructs.

b. The experts were each given a document containing a definition of TMId, mathematics, and each of the seven constructs (i.e., framing, decompressing, quasi-decompressing, trimming, quasi-trimming, bridging, and quasi-bridging) in the Framework to read. After they read the definitions, they were asked to share any thoughts they have about the definitions as written. At this point, I took notes but did not answer any questions about the definitions. I wanted their subsequent codings to be based upon the definitions provided so that we would have a basis for discussions about refining the given definitions.

c. The experts were then given an example of framing, asked to code the example of framing based upon the definitions they were given, and record their coding (and notes, if any) on the provided record sheet. They were told that any of the framings could reflect the use of more than one teaching practice.

d. The experts were then asked to share their coding(s) and the reasoning behind their decisions. After they shared their codings, I shared my coding(s) and my reasoning behind them. We discussed any differences. If the differences were due to lack of
clarity in a definition, I asked for input as to ways to make the definition clearer
and took notes.

e. Steps c and d were repeated until all 12 framings were coded and discussed.

f. This activity closed with a discussion about the definitions in general, what made
them clear or unclear, and what might be done to make the definitions clearer.

**Initial use of feedback.** There was one difference in implementation from round
to round. The feedback from Round 1 was used to modify some of the definitions. The
new set was used in Round 2. The feedback from Round 2 was used to make minor (i.e.,
grammatical) changes in some of the definitions. The new set was used in Round 3.

**Feedback overview.** The experts’ discussions and suggestions for major changes
revolved primarily around the definitions of decompressing and quasi-decompressing.
Figure 5-4 contains the tentative definitions used in Round 1. The two definitions are
arranged side by side to reflect the way in which the definitions were presented to the
experts. The experts in Round 1 stated that the phrase “as to reveal mathematical
meaning” led them to think of TMIds strictly as concepts, even though the definition of
TMId given to them stated explicitly that the mathematical ideas that are being targeted
are facts, concepts, skills, and generalizations. They also stated that they did not see what
made the quasi-decompressing definition “quasi.”
**Decompressing** is a mathematical teaching practice in which a TMId is broken down into components as to reveal mathematical meaning. The breaking down of a TMId is usually done directly for students by the teacher.

**Quasi-decompressing** is a teaching practice in which (1) a TMId is broken down into components, and/or (2) the components of a TMId are organized to facilitate students’ abilities to “see,” work with, or use the components of a TMId. “Easier to see” is usually taken in a literal sense: easier to see visible features.

<table>
<thead>
<tr>
<th><strong>Figure 5-4.</strong> Initial working definitions for decompressing and quasi-decompressing.</th>
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</thead>
</table>

Figure 5-5 contains my response to the feedback received in Round 1. In decompressing, the phrase, “as to reveal mathematical meaning” was replaced with reminders that the TMId could be a concept, skill or generalization. In quasi-decompressing, the “quasi-” quality of how a TMId could be broken down was made more explicit with the addition of “indirectly through the use of some kind of graphic organizer or directly through some other, sometimes mathematically questionable means.”
Decompressing is a mathematical teaching practice in which a TMId (e.g., concept, skill, generalization) is broken down into components/parts. The breaking down of a TMId is usually done directly for students by the teacher.

Quasi-decompressing is a teaching practice in which (a) a TMId is broken down into components indirectly through the use of some kind of graphic organizer, or directly through some other, sometimes mathematically questionable means, or (b) components of a decompressed TMId (usually procedures) are organized so that students are better able to work with, or use the components.

Figure 5-5. Working definitions used for Rounds 2 and 3 for decompressing and quasi-decompressing.

The expert in Round 3 did not understand why the second part of the quasi-decompressing definition (i.e., [b] components of a decompressed TMId [usually procedures] are organized so that students are better able to work with, or use the components) could not be a part of the definition of decompressing. This feedback caused me to go back to the examples of framing that led to the inclusion of this condition in the definition of quasi-decompressing for another look. In re-examining those framings, I realized that those framings also involved looking at properties of mathematical objects and that the definition of trimming (i.e., a mathematical practice in which some subset of component ideas/aspects of a TMId is addressed or emphasized) is a better match.
Based upon the feedback received from the members of the expert panel and with the theoretical origins of decompressing, trimming, and bridging in mind, changes were made to the working definitions of decompressing and quasi-decompressing. The other definitions, with the exception of grammatical changes, remained the same.

The Definitions

Although the definitions of the general constructs were not directly used in the fourth round of data analysis, the phrases mathematical teaching practice(s) and quasi-mathematical teaching practice(s) are used periodically in discussions. For this reason, these phrases are defined here. Mathematical teaching practices are teaching practices that are, relative to the teaching of school algebra, characterized by the logical reasoning, exactness, or precision of mathematics. The foci of mathematical teaching practices are mathematical objects. Quasi-mathematical teaching practices are teaching practices that reflect some of the qualities of mathematical teaching practices, but are also characterized by the use of nonmathematical elements or ideas. The foci of quasi-mathematical teaching practices are also mathematical objects.

What follows are definitions of the constructs and corresponding discussions of the specific teaching practices related to those constructs. The discussions include examples of each type of framing.

Decompressing. Decompressing is a mathematical teaching practice in which a TMId (e.g., concept, skill, or generalization) is broken down into components as to reveal mathematical meaning. The object of decompressing is a mathematical object. Emerging from the data are two ways in which participants engaged in decompressing. They
decompress a TMId (almost always a procedure) for students (a) by revealing the components step by step within a lesson, or (b) by revealing the components of a procedure across sequence of lessons.

Example:  *Decompressing within a lesson (see chapter 4, examples 3.1 and 3.2).*

Although all of the participants in this study framed at least one procedure by presenting or facilitating a presentation of a sequence of steps, data about Gary and Bonnie reflect a heavy emphasis on teaching procedures by breaking them down into small steps, and offering explanations at each and every step (and substep).

Gary’s use of decompressing is most obvious due to his use of a PowerPoint presentation to show his derivation of the quadratic formula for his students. His derivation consists of four steps, one step per slide. Gary divided each step into substeps by programming each slide to reveal each substep only after he pressed a key on his keyboard. Gary did not share how and why he decided to separate the derivation into the steps that he chose, and my examination of Gary’s PowerPoint presentation for how he chose to partition the derivation was inconclusive.

With respect to Bonnie’s use of decompressing, while it was clear that Bonnie broke procedures down into small steps, identification of the steps in her framing was not easy. After some careful examination of the videos of Bonnie’s work, each of the steps in explanation of how to do each exercise/quiz problem began with a verbal identification/description of what to do and was followed by recording what was described. What is interesting about Bonnie’s work across the observation week is that she spent a majority of her lessons showing students how to do exercises (i.e., warm-up exercises, requested homework problems, each problem on each of the quizzes and tests
she returned during the observation period). During this week, showing students how to do problems, step by step, was the primary way she taught mathematics.

**Example of decompressing across a sequence of lessons.** The clearest example of decompressing across a sequence of lessons was found in data about Kate, whose lesson objectives for the observed lessons were components that supported objectives for lessons that were to take place after my observations of Kate’s teaching was over. Immediately after my observation was done, Kate planned to teach a lesson during which she and her students would together derive the quadratic formula, which in her class is a two-part formula: \[ x = \frac{-b \pm \sqrt{D}}{2a}, \quad D = b^2 - 4ac \] (given a quadratic equation of the form, \[ ax^2 + bx + c = 0, \text{ and } a \neq 0 \]), and use it to solve quadratic equations.

In preparation for this lesson, Kate had her students practice the order of operations by computing the value of the discriminant, \( b^2 - 4ac \), for values of \( a, b, \) and \( c \) that were first given explicitly, and in the next lesson given the values of \( a, b, \) and \( c \) indirectly. The students were given a specific quadratic equation in the form, \[ ax^2 + bx + c = 0, \] shown which of the coefficients is \( a, \) which is \( b, \) and that the nonzero constant is \( c, \) and asked to compute the value of \( D = b^2 - 4ac. \) This component is discussed in chapter 4, Example 1.1.

A second component in the sequence of lessons in the development and use of the quadratic formula is a geometric development of the process of completing the square. Kate introduced students to this process by building upon their knowledge of factoring quadratic expressions, by finding the length and width of a rectangle whose area is represented by the given trinomial. This factoring process is discussed in chapter 4.
(Example 10.2a). Kate had students complete a geometric square by finding the number of unit squares needed to add to one \(x\)-by-\(x\) square, and \(b\) (\(b\) is a positive even integer) 1-by-\(x\) rectangles so that the squares and rectangles can be arranged to form a large single square. Although the students approached the process geometrically, their final product was recorded symbolically. A third component in this journey is solving quadratic equations of the form, \((x + k_1)^2 = k_2\) for which \(k_1\) and \(k_2\) are integers and \(k_2\) is positive. How Kate facilitated students solving these equations is discussed in chapter 4, Example 6.4.

What makes Kate’s framing interesting is that these components are parts of parallel lines of development that took place over three observations. The three lines of development—evaluating the discriminant, completing the square, and solving quadratic equations of the form, \((x + k_1)^2 = k_2\)—converged in the lesson when Kate and her students derived her version of the quadratic formula and used it to solve quadratic equations. Kate addressed component one in all three observed lessons, component two in all three lessons, and component three in the third observed lesson.

It is important to note that by discussing only Kate’s approach, I do not mean to imply that her approach is the only way of decompressing across lessons. Data about Gary suggests that Gary addressed the process of solving quadratic equations by completing the square in a lesson that took place prior to my data collection week with him. Since completing the square is a key aspect of deriving the quadratic formula, Gary’s lesson on solving quadratic equations by completing the square can be considered a component of deriving the quadratic formula. Thus, Gary’s teaching students how to
solve quadratic equations using completing the square and then deriving the quadratic formula might be seen as decompressing across lessons in a sequential development. However, a discussion of Gary’s work on solving quadratic equations by completing the square is not included in this chapter because it is not captured in the data.

Quasi-decompressing. Quasi-decompressing is a teaching practice in which a TMId is broken down into components (a) indirectly through the use of some kind of nonmathematical tool, or (b) directly through some other, sometimes mathematically questionable, means. The object of quasi-decompressing is a mathematical object. When a teacher engages in quasi-decompressing, precision/exactness of mathematical meaning is not necessarily a focus. What follows are discussions of two examples of quasi-decompressing and corresponding organizational tools.

Example of quasi-decompressing: Using special stationery. Faith requires all of her students to take notes using a special kind of stationery. Rather than record work on a chalkboard or whiteboard, she uses a document camera to capture the work she records on the same stationery, and project it on a screen for students to see. She has two types of stationery, one for lessons that involve the creation of graphs (see Figure 5-6) and one for all other lessons (see Figure 4-18).
Faith’s stationery plays a key role in helping her students see and describe patterns among symbolic representations and their corresponding graphs. The stationery clearly separates what appeared in the observation as a graphing procedure: start with the equation, create a table, and graph. Consider, for example, Faith’s lesson during which students were supposed to describe the relationship between the shape of the graph of $y = ax^2$ and the value of the coefficient, $a$. Given that Faith’s lesson required that students graph different functions of the form, $y = ax^2$, the stationery facilitated Faith and students creating an organized record of what they did to create the graph of each of six examples.

In addition to creating an organized record, the organization facilitates Faith’s efforts to help students identify and describe patterns in the recorded work. The organizations of the written work, set up by the stationery design, made it easy to Faith direct and focus students’ attention to looking for patterns in the graphs and to look for...
pairwise relationships between the value of the quadratic coefficient and the corresponding graph.

Example of quasi-decompressing: Using an organizer (see chapter 4, Example 4.3). Bonnie uses what she calls a lattice when she teaches her students how to multiply polynomials. The lattice, an $n$ by $m$ grid ($n$ and $m$ are the number of terms in the two polynomials, respectively) is used to help students organize their work and make sure each term (i.e., component) of one polynomial is multiplied with each term (i.e., component) of the second polynomial. The lattice is used to help students to apply a procedure correctly.

**Trimming.** Trimming is a mathematical teaching practice in which some (proper) subset of component ideas/aspects of a TMId is addressed or emphasized. Included in the subset are **some** core ideas of the TMId. Analysis of the data revealed that teachers can bring students’ attentions to some subset of component ideas/aspects directly and indirectly.

Example: Trimming a TMId directly. With her two-part form of the quadratic formula, $x = \frac{-b \pm \sqrt{D}}{2a}$ for which $D = b^2 - 4ac$, Kate brings attention to the discriminant, $b^2 - 4ac$, an expression whose value affords a knowledgeable interpreter information about the solutions to the corresponding quadratic equation, about the graph of the corresponding quadratic function, and about zeros of the corresponding function. Although Kate did not go into detail about the discriminant and its interpretations with her students during data collection, she did inform students that $b^2 - 4ac$ is called the
discriminant and that the discriminant will tell them something about the graph of the quadratic (see chapter 4, Example 1.1).

Kate’s trimming of the discriminant brings attention to the discriminant directly in several ways. She brings attention to the discriminant by making the quadratic formula a two-part formula in which the expression $b^2 - 4ac$ is extracted from the radical sign and made very prominent. Kate then tells students that the expression has a name, the discriminant, and informs them that, for a given quadratic function, the discriminant will tell them something about its graph.

**Example: Trimming a TMIId indirectly.** Faith, concerned about her students’ struggles with fractions, found ways to reduce/eliminate the distractions of having to deal with student struggles with computation (see chapter 4, Examples 1.3 and 3.6). In teaching students the procedure of completing the square, Faith chose to give her students practice problems in which the linear coefficient is a “nice number” (i.e., an even number; Example 1.3). By using an even number, her students would have an easier time correctly computing the square of half of the value of the linear coefficient, and therefore, be able to spend more time attending to the procedure of completing the square.

In her lesson on graphs of quadratic functions of the form, $y = ax^2$, Faith graphed the functions by creating a table of values and plotting the resulting points. In this lesson, Faith and her students used a method that enabled them to avoid having to plot points with nonintegral coordinates. If a coefficient is a fraction with $k$ as the denominator of the coefficient, then the $x$–values they chose to use in plotting points were $-2k$, $-k$, 0, $k$, and $2k$. The table in Figure 5-7 reflects evidence of Faith’s use of this method. By
choosing $x$-values that are multiples of the denominator of the quadratic coefficient, students’ struggles with computations involving fractions are addressed directly by making the computation as easy possible.

Figure 5-7. Faith's trimming of how to graph a quadratic function with a quadratic coefficient of $\frac{1}{2}$ (Observation 1): Use $x$-values that are multiples of 2.

**Quasi-trimming.** Quasi-trimming is a teaching practice in which a teacher brings students’ attentions to some (proper) subset of component ideas/aspects of a TMId in ways that are not characterized by the exactness or precision of mathematics. Included in the subset are core ideas of the TMId. These methods are often practical and/or imaginative as opposed to abstract and logical. Similar to trimming, a teacher may engage in quasi-trimming indirectly or directly.

Example of quasi-trimming: Using iconic pseudovariables and deictic gestures. One of the lesson objectives for Kate’s third lesson was solving quadratic equations using completing the square. The four warm-up exercises used for that lesson are shown in the screen capture in Figure 5-8. Students were asked to solve simple equations of the form $X^2 = k$, for which $X$ is an icon (e.g., ⊙, ⊙, ★, ♥) that represented a real number. After some discussion, students were able to identify two answers for each exercise: ⊙ equals
+3 and −3; ☼ equals +10 and −10; ★ equals +5 and −5, and ♥ equals +\sqrt{15} and −\sqrt{15}.

Solving these equations foreshadowed a part of the solution process students would be using when they worked on the exercises on that day’s worksheet.

![Figure 5-8. Screenshot of Kate’s warm-up exercises for Observation 3.](image)

As students worked on the day’s worksheet in their small groups, Kate moved from group to group observing what students were doing, and stopping now and then presumably to discuss students’ progress or lack of progress. Kate noted that several students were able to transform one of the practice problems from \(x^2 + 12x = 13\) to \((x + 6)^2 = 49\) but struggled to move forward.\(^{47}\) Kate then went to her computer tablet and recorded and projected the following work\(^{48}\) on screen.

\[
\begin{align*}
x^2 + 12x + 36 &= 13 + 36 \\
(x + 6)(x + 6) &= 49 \\
(x + 6)^2 &= 49
\end{align*}
\]

\(^{47}\) Confirmed in the Postobservation 3 Interview with Kate.

\(^{48}\) This is a reproduction. The video image was very dark and the screen captured could not be corrected enough for a readable image.
Kate then asked students to look at the projected work, placed her hand on the “$x + 6$” and asked the entire class, “My hand—if I square it, it makes forty nine, what’s my hand going to have to be?” (Observation 3; refer to Examples 6.4 and 7.1 in chapter 4).

Kate brought students’ attention to contents of the parentheses by placing her hand on the “$x + 6$. ” Her hand placement in conjunction with her rather unusual question seems to be an effort to remind student of the warm-up exercises and to bring attention to the structure/form of the equation. Kate’s hand placement and corresponding question is a practical and pedagogical way of helping students consider “$(x + 6)$” as a single object (i.e., “hand” or “$Ω$”), and viewing $(x + 6)^2 = 49$ as $(\text{hand})^2 = 49$ or $Ω^2 = 49$, and therefore, a particular case of $X^2 = 49$. Because the means by which Kate frames $(x + 6)^2 = 49$ as a special case of $X^2 = 49$ is concrete (i.e., she temporarily replaces an abstract mathematical object with a physical object), it is an example of quasi-trimming.

**Bridging.** Bridging is a mathematical teaching practice in which one or more connections are made between a TMId and other mathematical ideas. What follows are two examples of bridging. Both of them make use of analogies.

*Example of bridging: similarities between simple and relatively complex algebraic expressions (see chapter 4, Example 6.3).* While explaining how to simplify

\[
\frac{x^2 + x - 6}{x^3 + 9x^2 + 27x + 27}
\]

(see chapter 4, Example 6.3), Bonnie factored the denominator and obtained \((x + 3)(x^2 - 3x + 9)\) (see last line in Figure 5-9) and a student asked, “why did you get rid of one $x$ plus three and what happened to the nine $x$?” (Observation 2). In response Bonnie reminded students how to factor the simpler related expression, $2x + 2y$.
Bonnie made an analogy between the structure/form of the expressions, \(2x + 2y\) and \((x+3)(x^2-3x+9)+9x(x+3)\) by pointing out that 2 is the common factor in \(2(x+y)\) and \((x+3)\) is the common factor in \((x+3)(x^2-3x+9)+9x(x+3)\). Bonnie went on to point out the structure/form similarities between the factored form of \(2x + 2y\), i.e., \(2(x+y)\) and the factored form of \((x+3)(x^2-3x+9)+9x(x+3)\), i.e., \((x+3)(x^2-3x+9+9x)\).

**Figure 5-9.** Board work of the part of Bonnie’s solution for the day 2 (Observation 3) warm-up.

*Example of bridging:* graphing linear inequalities on the rectangular coordinate plane (see chapter 4, Example 2.3). Bonnie identifies similarities between graphing \(x + 6 < 4\) and \(x + 6 < y\). According to Bonnie graphing an inequality on the rectangular coordinate plane

‘is like going from a point [as she points at \(x = -2\) on her drawn number line] to half of a line [as she traces the ray she graphed on the number line she had
drawn]. From here we are going from the line [as she traces the line she graphed for \( x+6 = y \)] to one half [of the plane] or the other [half of the plane] [Bonnie places a hand on either side of the graph of \( x+6 = y \)].

I infer from her statements and her hand movements that she made the verbal analogy: point on the number line at \( x = -2 \) is to graph of \( x + 6 = y \) on rectangular coordinate plane, as graph of \( x < -2 \) on the number line (i.e., a half line) is to graph of \( x + 6 < y \) on the rectangular coordinate plane.

**Quasi-bridging.** Quasi-bridging is a teaching practice in which connections are made between a TMId and nonmathematical ideas(s). This category of framing contains the largest number of examples. What follows are different examples of quasi-bridgings.

*Example of quasi-bridging: Sometimes a connection is implied.* Bonnie tells her students that dividing by zero is a math sin (see chapter 4, Example 2.2). When she makes this statement she implies a connection between a problematic computation (i.e., dividing by zero) and an immoral act. The framing communicates that division by zero is simply not allowed. All that needs to be understood is that one never divides by zero. Understanding of the mathematics behind division by zero is not a part of this framing.

*Example of quasi-bridging: Sometimes connections are ones that are assigned by the teacher.* Hildi’s framing of how to simplify fractions of the form, \( \frac{x^a}{y^b} \) (for which \( a \) and/or \( b \) are positive or negative integers\footnote{Note: Hildi did not explicitly exclude zero exponents. The exclusion of zero exponents is inferred from the examples Hildi used. None of them included zero exponents.}; see chapter 4, Example 8.1) is an example of
a teacher assigning a meaning to a TMId and its components. According to Hildi, the fraction is a two-story apartment building. The numerator and the denominator are tenants of the building. The sign of the exponent serves as a “mood meter.” A negative exponent means the tenant is unhappy. A positive exponent means the tenant is happy. The fraction is simplified by resolving the unhappy tenant situation. Unhappy tenants are moved to the other floor. In Hildi’s example, $\frac{x^2}{y^{-3}}$, both tenants are unhappy, so they both have to move to the other floor. The resulting expression is $\frac{y^3}{x^4}$.

In this example, the TMId is linked to a nonmathematical context via connections that have no mathematical meaning. The goal for this framing is to help students manipulate symbols and get a correct answer. There is little mathematics students have to know and understand. The exponent (i.e., mood meter) tells you what to do with the expressions in the numerator and denominator, and therefore, help you record a correct answer.

*Example of quasi-bridging: Sometimes the connections are made by a student, and the teacher then takes advantage of the “teachable moment.”* Kate’s framing of addition and subtraction of integers as battles between ninjas and zombies (see chapter 4, Example 8.3) is an example of a teacher taking advantage of a connection seen by a student. During observations of Kate’s lessons, for example, Kate periodically made comments about “ninjas and zombies”—but always seemed to direct that comment toward a particular student. When asked to share what the reference for ninjas and zombies was, Kate talked about her struggle to help that student learn and develop an
understanding of how to add and subtract integers, and an unplanned classroom event.

After several attempts, each using a different approach to helping this student understand, for some reason, the student began to talk about a battle between ninjas and zombies.

The student stated that when one ninja (i.e., good guy and positive) and one zombie (i.e., bad guy and negative) fight each other, no one wins (i.e., zero). They both die. Kate took advantage of the student’s insights and used it to frame how to add and subtract integers $^5$.

During the post observation interview, Kate shared that she took advantage of this student’s context because the context came from the student (i.e., the student owned the context), and his context contains an element that can be interpreted as a metaphor for an important property of integers, $x + (-x) = 0, \ x \in \mathbb{Z}$. If the context did not reflect the second property, then Kate would not have used the context. Examples of other contexts she has used include superheroes and villains, which is another context proposed by a student; and debt and savings.

**Example of quasi-bridging:** Sometimes connections are about language. Every postobservation interview with Kate contains a section in which I asked Kate to clarify terms she used in class. A characteristic of Kate’s lessons is the use of local mathematical language, a spoken and/or written language that has mathematical meaning to a teacher, his or her current and former students, and possibly teachers who inherit the

$^5$ Subtraction is “taking away.” Taking away a ninja/zombie, according to Kate and her student, is like adding a zombie/ninja.
teacher’s former students that without translation or explanation may have little or no meaning to outsiders.

Interviews revealed that sometimes her local classroom language (i.e., nonmathematical terminology) is a by-product of quasi-bridging. For example, when Kate talked about how she frames how to solve linear equations, she used the phrase, “get your friends together” (see chapter 4, Example 8.5). According to Kate, “get your friends together” means “combine like terms.” When asked why she used, “get your friends together” instead of “combine like terms,” Kate stated that her students know that the term, friends, is code for “like terms” and would not have any trouble proceeding to another teacher’s class in which the phrase, “get your friends together” is not used. However, her students prefer the term friends over the phrase, like terms. According to Kate, the phrase, like terms, has no contextual meaning for her students (Postobservation 3 Interview). It somehow makes sense to Kate’s students that $9x$ and $3x$ are friends, $3x^2$ and $-5x^2$ are friends, but $9x$ and $3x^2$ are not friends.

Other examples of quasi-bridging. Other examples of quasi-bridgings that emerged from the data include as follows:

- Hildi and Kate use metaphors to connect TMIs and nonmathematical ideas.
  
  Hildi’s exclusive party nonscandalous friends (see chapter 4, Example 2.1) is her metaphor for the concept of function, and her version of the life cycle of a tarantula wasp is her metaphor for evaluating composite functions (see chapter 4, Example 9.1). Kate likens the solving of linear equations in one variable to playing a game for which the rules of game are metaphors for the rules for solving linear equations (see chapter 4, 8.5).
Faith and Kate draw upon their knowledge of students’ personal experiences to make connections between TMIIds and those experiences. Faith uses students’ familiarity with a particular way of cutting a heart from a folded piece of paper to talk about symmetry with respect to a line (i.e., the TMIId, see chapter 4, Example 8.2). Kate uses the phrase “getting friends together” as a way of talking about “combining like terms.” (see chapter 4, Example 10.1b).

Factors Affecting the Coding of Data

After the definitions of the six specific teaching practices were established, the 33 framings that were coded in preparation for feedback from the expert panel members were re-examined and coded using the refined definitions\(^{51}\). The coding of only one example of framing was affected by the refinement of the definitions. As mentioned earlier in this chapter, some framings involve the use of more than one teaching practice. In some cases, a framing may involve the use of teaching practices from the mathematical category as well as the quasi-mathematical category.

Consider, for example Kate’s framing of the discriminant (cf, chapter 4, Example 1.1). Kate introduced her students to the discriminant, \(D = b^2 - 4ac\), and its properties over the course of several lessons. Through the use of carefully designed exercises assigned to students on each lesson’s worksheet, Kate familiarized her students with the expression and how to compute its value for any given quadratic equation of the form, \(ax^2 + bx + c = 0\). She informed students that the expression has a name (i.e., the discriminant) and will be used in a future lesson to provide students information about the

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\(^{51}\)The list of 33 framings and their respective codes can be found in Appendix F.
graph of \( y = ax^2 + bx + c \). The primary teaching practice illustrated in Kate’s framing of the discriminant is trimming. However, in bringing attention to how the discriminant is used, Kate also engaged in bridging. She made a connection between the value of the discriminant for a given quadratic and graphs of quadratic functions. Thus, Kate’s teaching of the discriminant is primarily trimming that is supported by bridging.

**Grain size of data.** Kate’s framing of the discriminant also suggests that the grain size has a role in whether a framing is characterized as involving the use of multiple teaching practices. Had data been analyzed without examining the worksheets Kate assigned students, Kate’s framing of the discriminant would have been based upon comments made during a discussion of a warm-up exercise and classified as primarily bridging. The availability of Kate’s worksheets enabled a more accurate characterization of her framing of the discriminant.

**Teachers’ choices.** Teachers make choices that result in framings that involve the use of a single teaching practice or multiple teaching practices. Consider the procedure of completing the square. Data about Faith (see chapter 4, Example 1.3) and Kate (see chapter 4, Example 3.3), for example, show that they both taught lessons in which they taught their respective students how to complete the square. Faith chose to teach the procedure strictly symbolically. She trimmed the procedure by using equations with “nice numbers” (i.e., the quadratic coefficient equals 1 and the linear coefficient is even) so that students do not have to compute with fractions. All of Faith’s choices are examples of trimming. Kate also trimmed the process by using “nice numbers” in the equations she had her students solve; however, Kate chose to connect completing the
square to a geometric representation of quadratic expressions rather than conduct her lesson strictly symbolically. The expressions were interpreted as representations of areas of rectangles. Completing the square is linked to completing a geometric square. Therefore, Kate’s framing was classified as using both trimming, and bridging.

**Examining the Data using the Conceptual Framework for Framing**

As stated earlier, 33 framings from chapter 4 were reviewed and coded using the six constructs in the Conceptual Framework for Framing. These framings and their respective codes were saved on a spreadsheet to facilitate sorting the data by teaching practice, and by teacher. The goal of coding the data was to refine the definitions of decompressing, trimming, bridging, quasi-decompressing, quasi-trimming, and quasi-bridging. A table was created to keep track of the results of coding each example of framing. Unexpectedly, this table served as an additional way of examining the data to understand teachers’ efforts to make TMIds accessible and comprehensible. Using the sorting capabilities built into Microsoft Word’s table feature, I was able to sort the coded data quickly in a variety of ways (e.g., by teaching practices, by subject matter, by teacher). explore what insights might be gained about teaching school algebra by examining the data using the Conceptual Framework for Framing as a lens. Sorting the coded framings by teaching practice did not produce insights... While one can say that decompressing was used the most and quasi-trimming was used least (see Table 5-1), it is not clear what these numbers mean.

**Table 5-1. Number of Framings per Teaching Practice**

<table>
<thead>
<tr>
<th>Mathematical Teaching Practices</th>
<th>Quasi-mathematical Teaching Practices</th>
</tr>
</thead>
</table>

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The results of sorting of the coded framings by teacher provided, in contrast to sorting by teaching practices, did provide some insights. Before discussing these insights, it is important to note that there was not enough information about Ann, Delia, Faith, and Gary. Thus, there was not enough information to allow analysis of patterns. Framings from these four teachers account for only 9 of the 33 framings discussed in chapter 4 were from data about Ann, Delia, Faith, and Gary. Thus, there was not enough information to allow analysis of patterns. Framings from these four teachers account for a total of only 9 out of the 33 framings coded. Of their respective coded data. For this reason discussions about what the collections of coded framings suggest will be restricted to that of Bonnie, Hildi, and Kate.

What follows are discussions of what the coded framings revealed about Bonnie’s, Hildi’s and Kate’s efforts to make TMIds accessible and comprehensible to students. Following a general discussion of each participant’s framing is a table summarizing that participant’s coded framing. It is important to note that often more than one teaching practice was used by participants in the framing of a TMId. In such a case, the question, “If I had to choose the main teaching practice being used, which practice best identifies what is being done?” was asked. The answer to this question is noted in the table with ●. The other teaching practice(s) used in the framing is/are noted with a ○. In the event, no single teaching practice best characterizes the framing, all teaching practices identified in the framing were noted with ●.

<table>
<thead>
<tr>
<th>Decompressing</th>
<th>Trimming</th>
<th>Bridging</th>
<th>Quasi-decompressing</th>
<th>Quasi-trimming</th>
<th>Quasi-bridging</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>14</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>11</td>
</tr>
</tbody>
</table>
Of all of the teachers participating in this study, Bonnie’s lessons were the most regimented. Every lesson had the same structure: first, students took a timed arithmetic quiz; second, students completed a set of warm-up exercises as Bonnie checked each student’s homework; third, Bonnie went over solutions to the warm-up exercises; and fourth, Bonnie explained solutions to those homework problems that were requested by students. If any time remained, Bonnie passed back the previous week’s weekly quiz and explained the solution to each quiz problem (Observation 2), explained new material (Observation 3), or gave students the weekly quiz (Observation 4). During the 4-day observation, time spent on new material was approximately 10 minutes out of four 50-minute periods. Based upon this observation week, given that Bonnie spent the majority of class time going over how to do homework exercises or solutions to quiz problems, Bonnie’s primary means of teaching seems to be re-teaching. Every problem Bonnie did for her students can be described as a detailed, step by step explanation. Table 5-2, a summary of Bonnie’s coded framing, provides a visual image of Bonnie’s image. It shows that Bonnie’s framing is chiefly decompressing and trimming.

52 The purpose of the timed arithmetic quiz, according to Bonnie, was to improve students’ recall of arithmetic facts. She called the quiz an automaticity quiz.
Table 5-2. Bonnie’s Coded Framings.

<table>
<thead>
<tr>
<th>Bonnie’s Targeted Mathematical Idea (TMId)</th>
<th>Decompressing</th>
<th>Trimming</th>
<th>Bridging</th>
<th>Quasi-decompressing</th>
<th>Quasi-trimming</th>
<th>Quasi-bridging</th>
</tr>
</thead>
<tbody>
<tr>
<td>Division by zero</td>
<td></td>
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</tr>
<tr>
<td>Factoring polynomials that are visually complex</td>
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<tr>
<td>Multiplying polynomials</td>
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<tr>
<td>Factoring any given polynomial</td>
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</tr>
<tr>
<td>Factoring quadratic polynomials of the form, $x^2 + bx + c$, $b, c \in \mathbb{R}, b, c \neq 0$</td>
<td>o</td>
<td>o</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Factoring quadratic polynomials of the form, $ax^2 + bx + c$, $a, b, c \in \mathbb{R}, a \neq 1, a, b, c \neq 0$</td>
<td>o</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Rationalizing denominators</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simplifying $\frac{\sqrt{14}}{\sqrt{35}}$</td>
<td>o</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Hildi’s lessons were devoted to preparing students for the upcoming final exams. Except for a few exercises she did with the class at the beginning of class during Observation 1, the majority of class time in all three observations was devoted to students working with their group-mates on practice final exams. The questions on these exams
were all multiple-choice problems. Very little was captured on video and audio of Hildi framing TMIDs for her students as students worked on the practice final exams. The framing that was captured took place at the beginning of Observation 1 and during interviews as Hildi explained unusual comments I heard her say to students in class and as Hildi worked on tasks I had prepared for all participants in this study to do. If I were to characterize Hildi’s framing, the characterization would include using what she called stories to teach TMIDs and providing what she believes as easy-to-remember translation of symbolic mathematical language. Her stories and “easy-to-remember” translations are reminiscent of Gamson and Modigliani’s (1989) media packages and framing devices that are used by the media to bring attention to an issue and to help their respective audiences think about or view that issue. When asked what motivates her creation of her stories, she stated, “I figure if I can make it a little different and more interesting then I can maybe that will help my students remember it. Um or at least stay awake, you know, so” (Ending Interview). Table 5-3, a summary of Hildi’s coded framings, provides a visual image that reflects the fact that Hildi’s framing is dominated the use of quasi-mathematical teaching practices.
Table 5-3. Hildi’s Coded Framings.

<table>
<thead>
<tr>
<th>Hildi’s Targeted Mathematical Idea (TMId)</th>
<th>Decompressing</th>
<th>Trimming</th>
<th>Bridging</th>
<th>Quasi-decompressing</th>
<th>Quasi-trimming</th>
<th>Quasi-bridging</th>
</tr>
</thead>
<tbody>
<tr>
<td>Function</td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Negative exponents</td>
<td></td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fractional exponents</td>
<td></td>
<td></td>
<td>●</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Simplifying exponential expressions</td>
<td></td>
<td>●</td>
<td></td>
<td></td>
<td></td>
<td>●</td>
</tr>
<tr>
<td>Evaluating composite functions</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>●</td>
</tr>
</tbody>
</table>

Of all of the participants in the study, Kate’s lessons were the most complex. All lessons were structured around students working in small groups solving problems. In each lesson, students worked on review problems as well as problems that were designed to develop new mathematical ideas. The problems were designed so that new ideas were built upon familiar mathematical ideas or connected to nonmathematical, but familiar ideas. Table 5-4, a summary of Kate’s coded framings, reflects framing that is chiefly mathematical, but also involves the use of quasi-mathematical practices. The table documents Kate’s emphasis on making connections. Every example of framing involves either bridging or quasi-bridging.
Table 5-4. Kate’s Coded Framings.

<table>
<thead>
<tr>
<th>Targeted Mathematical Idea (TMId)</th>
<th>Decompressing</th>
<th>Trimming</th>
<th>Bridging</th>
<th>Quasi-decompressing</th>
<th>Quasi-trimming</th>
<th>Quasi-bridging</th>
</tr>
</thead>
<tbody>
<tr>
<td>Like terms</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The discriminant</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The quadratic formula</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
<td>●</td>
</tr>
<tr>
<td>Adding and subtracting integers</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solving linear equations</td>
<td>○</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solving quadratic equations of the form, $(x + k_1)^2 = k_3$</td>
<td>○</td>
<td></td>
<td></td>
<td>●</td>
<td>○</td>
<td></td>
</tr>
<tr>
<td>Multiplying binomials</td>
<td>○</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Solving absolute value equations</td>
<td>●</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Factoring trinomials</td>
<td>○</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Completing the square</td>
<td>○</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Perhaps the most interesting result of sorting the coded framings by teacher is the realization that each table of coded framings is, to some extent, a snapshot of one aspect (i.e., making TMId accessible and comprehensible to students) of the corresponding teacher’s teaching. Bonnie, Hildi, and Kate’s snapshots, based on only 5, 8, and 10 framings, respectively, are very small depictions of who they are as individual teachers.
However, the snapshots do indicate how very different they are in their efforts to make TMIds accessible and comprehensible to students.

**Connections: Framing and ZPD**

This study is based upon the assumption that what teachers do to make TMIds accessible and comprehensible—in particular, framing—is important. This assumption is supported by Vygotsky’s concept of ZPD which presumes that interaction between students and teachers on a given task helps students accomplish—without help—what in the beginning could be accomplished only with assistance (Kozulin et al., 2003). The results of data analysis discussed in chapters 4 and 5 provide different insights about what teachers do so that a TMId is in their students’ respective ZPD. Each example of framing data captures what a participant did so that a TMId is in students’ respective ZPD. The 11 characteristics offer general descriptions about what participants did based upon patterns found in the more than 50 examples of framing identified in the data. The messages about framing focus on the roles that mathematics played in participants’ efforts to make TMIds accessible and comprehensible. The Conceptual Framework for Framing offers the field a lens through which to view what teachers do so that a TMId is in students’ respective ZPD. The tables created as a result of coding each of Bonnie’s, Hildi’s and Kate’s framings of procedures to identify which constructs in the Conceptual Framework for Framing are being used provide overviews of what teaching practices were drawn upon most frequently so that TMIds are in students’ respective ZPD.
Chapter 6

Implications, Limitations, and Future Directions

This study was conducted to investigate and to understand how secondary algebra teachers make targeted mathematical ideas (TMIds) accessible and comprehensible to students. Classroom observation and interview data about seven secondary mathematics teachers were examined with the use of a conceptual framework that consists of four constructs: Decompressing, trimming, bridging, and framing. The first three constructs, collectively described as mathematical teaching practices, are ways in which teachers use their mathematics content knowledge to make TMIds accessible and comprehensible to students: breaking a TMId down into components (i.e., decompressing), emphasizing some proper subset of component ideas that constitutes core idea(s) of the TMId (i.e., trimming), and making one or more connections between the TMId and other mathematical idea or ideas (i.e., bridging). The fourth construct, framing, has its origins in the perspective that teaching is communication of ideas, and is defined to be the adjustment, organization, staging and/or situating of one or more TMIds to support learning and to promote the development of students’ learning. Some combination of the teaching practices of decompressing, trimming, and bridging are used by teachers when framing TMIds.

Taking the advice of Hiebert et al. (2003), who advocate examining teaching using wide-angle as well as close-up lenses to gain a clearer understanding of what takes place in mathematics classrooms, observation and interview data were examined using
multiple lenses. Four rounds of analysis were conducted, each examining the data from a different perspective and for a different purpose, and each successive round representing a view of the data using an increasingly wider wide-angle lens. In the first round, verbatim transcripts of classroom-observation and interview videos were closely examined to identify instances of framing. From this round over fifty instances were identified. In the second round, the 50+ instances of framing were sorted into smaller sets based upon questions inspired from research literature (e.g., Is the idea being broken down? Is a connection being made?). The sets of framing were examined individually and collectively for patterns of similarity and difference. As a result, 11 characteristics of framing were identified. In the third round, the eleven characteristics of framing and their corresponding examples of framing were examined to identify what role or roles mathematics plays in framing in general. This round led to the identification of three messages about framing, and the identification of three new teaching practices: quasi-decompressing, quasi-trimming and quasi-bridging. The identification of the three new teaching practices in turn led to the modification of the Conceptual Framework for Framing. In the fourth round of analysis, each of the 33 examples of framing discussed in chapter 4 was examined using the lens of the modified Conceptual Framework of Framing to identify the teaching practice or practices being used. Once coded, the framings were sorted by teaching practice and by teacher. What follows are reflections on these results and thoughts for their contribution and potential contribution to the field of mathematics education.
Contributions and Implications

This study offers the field of mathematics education a number of contributions. The three most significant contributions are (1) documentation and characterization of a variety of ways in which teachers make TMIDs accessible and comprehensible to students, (2) identification of three teaching practices that provide mathematics educators with more flexibility their discussions about what teachers do in their efforts to make TMIDs accessible and comprehensible to students, and (3) a framework with which to view, characterize, and discuss how teachers make TMIDs accessible and comprehensible to students.

Contribution: The identification of teaching practices that account for use of nonmathematical ideas to make TMIDs accessible and comprehensible. Accounting for more of what teachers do to make TMIDs accessible and comprehensible to students. Perhaps the greatest contribution this study has to offer mathematics education scholars and researchers is the identification of quasi-decompressing, quasi-trimming, and quasi-bridging (collectively called, quasi-mathematical teaching practices). The teaching practices they address can be seen as ways to help students who are beginning to learn and understand abstract ideas of school algebra to engage or maintain engagement with TMIDs.

Quasi-mathematical teaching practices combined with the mathematical teaching practices of decompressing, trimming, and bridging, and organized into the Conceptual Framework for Framing, result in a configuration of six teaching practices that allow mathematics educators greater flexibility in examining and discussing what teachers do in their efforts to make TMIDs more accessible and comprehensible to students. For
example, the data show that teachers connect new mathematical ideas to ideas with which they believe students are familiar. However, the data also shows that the ideas to which teachers make connections are often not ideas of mathematics. Similarly, the data show that teachers break down mathematical TMIIds and they bring to students’ attentions some proper subset of component ideas of TMIId(s). However, the means by which these practices are accomplished (e.g., use of stationery, use of graphic organizers) cannot always be characterized by mathematical teaching practices. The approaches used to break down TMIIds are sometimes mathematical, and sometimes nonmathematical. In addition, the means by which teachers bring students’ attentions to the subset of ideas is not necessarily mathematically precise or logical. Teachers also emphasize component ideas using approaches that are casual, loose, and/or practical. The mathematical teaching practices of decompressing, trimming, and bridging, alone cannot account for this range of teaching approaches.

**Implication:** Mathematics does not always play a significant role in making TMIIds accessible and comprehensible to students. Framing has two main components: the TMIId and its frame. For example, in teaching students the quadratic formula to the tune of Pop Goes the Weasel, the TMIId is the quadratic formula, and the frame is singing a song. This separation, in conjunction with using a lens that includes examination of both mathematical and quasi-mathematical teaching practices, led to questions about the role of mathematics in teachers’ efforts to make mathematical ideas accessible and comprehensible to students.

Consider the following brief examples of how an instance of framing is separated into the TMIId and its frame. One of the teachers in this study demonstrated the
derivation of the quadratic formula by breaking the derivation down into several small steps. The TMId is the derivation of the formula. The frame is *breaking the derivation down into steps* for purposes of demonstration. Examination of the steps revealed choices of where one step ended and another began was based upon what was inferred as changes in mathematical purpose. For example, the step “complete the square” ended when the procedure was completed, the new step, “clean up the problem” (i.e.,

transformation of the equation from $x^2 + \left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$ to

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

began.

In another example of decompressing by another teacher, prior to her lesson on the derivation of the quadratic formula, a teacher prepared her students for the derivation of the quadratic formula by framing different aspects of formula, the derivation itself, and computing values using the formula. This teacher addressed the discriminant and its computation, completing the square using both geometric and symbolic representations of quadratic expressions, solving quadratic equations of the form $(x+k)^2 = k_2$, and changing the symbolic structure of the formula itself from, given $ax^2 + bx + c = 0$,

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

to given $ac^2 + bx + c = 0$,

$x = \frac{-b \pm \sqrt{D}}{2a}$

for which $D = b^2 - 4ac$.

Like the previous teacher’s framing of the derivation of the quadratic formula, this teacher’s framing involved breaking down the derivation of the formula. In addition, this teacher broke the formula itself into two components. However, because the components were developed over time, student interaction with the mathematics of each component
was different. Each component was the focus of a lesson within a lesson (see chapter 4, examples 1.1, 3.3, and 6.4). Thus, analysis of the data suggests differences in how mathematics is used and taught within a frame. The first teacher’s framing of the derivation of the quadratic formula involved showing students components (i.e., steps) of the derivation. The second teacher’s framing of the derivation of the quadratic formula involved breaking down the derivation of her version of the quadratic formula into smaller components, with each component being the TMId of smaller lessons enacted over a period of time. Familiar mathematics was used to develop each component. For example, evaluating $b^2 - 4ac$ for given values of $a$, $b$, and $c$ was the means by which students reviewed the order of operations. In addition, students’ familiarity with an area model of factoring was used so that students could transform equations of the form $x^2 + bx = c \ (b, c \in \mathbb{Z})$ to equations of the form $(x-k_1)^2 = k_2 \ (k_1, k_2 \in \mathbb{Q})$. Later, according to the second teacher, lesson during which the derivation of her version of the quadratic formula was done\textsuperscript{53} would be a teacher-directed lesson with students being expected to contribute to identifying “next steps.” In contrast, the only expectation the first teacher had for students with respect to the derivation was that they remembered seeing the derivation.

Implication: Mathematics can play an important role in the framings that involve the use of quasi-mathematical teaching practices. The relationships between the nonmathematical element(s) in a frame and the corresponding TMId can range from being multifaceted and closely aligned to being questionable or non-existent. One

\textsuperscript{53}This lesson immediately followed the last lesson I observed for this teacher.
teacher uses a metaphor to frame the concept of function (see chapter 4, Example 2.1). A function is an exclusive social event to which only nonscandalous friends receive invitations. This metaphor works because the definition of nonscandalous friend reflects the critical property that makes a function a function. Functions map one element of one set to exactly one element of another set. Nonscandalous friends are friends that each brings only one date to a party. In contrast, this same teacher frames the “−” in the exponent of an expression such as, \( x^{-2} \), as an indicator that tells students that \( x^2 \) needs to be moved below a fraction bar to form \( \frac{1}{x^2} \). (see chapter 4, Example 8.1). In this framing, attention is brought to the inscription as opposed to what the mathematical object is, and serves as a way to help students know what to record in the next step of an exercise. In contrast to the first framing which makes use of and humorously highlights a defining property of a mathematical object, the framing does not seem to use a mathematical property or definition. It instead seems to focus on symbol manipulation and a quick way to remember what to record next.

**Contribution:** A Conceptual Framework for Framing, a framework with which to view and describe how teachers make TMIIs accessible and comprehensible to students. As illustrated at the end of chapter 5, a teacher’s efforts to make TMIIs accessible and comprehensible can be coded according to the six constructs that make up the Conceptual Framework for Framing. All 33 coded framings were initially organized into a single table whose column headings from left to right were as follows: teacher, TMIId, decompressing, trimming, bridging, quasi-decompressing, quasi-trimming, and quasi-bridging. Using this table of coded framings, messages (i.e.,
implications) about teacher’s framing were sought by sorting this table using different criteria.

**Implication:** Tables of coded framings (i.e., snapshots) provide mathematics educators a new tool with which to view, discuss and compare teachers’ efforts to make *TMI*ds accessible and comprehensible to students. As a demonstration of the potential that framings coded according to constructs in the Conceptual Framework for Framing have as a tool for mathematics educators, let us compare the snapshots of three teachers, Bonnie, Hildi, and Kate. So that the three snapshots have one factor in common, I limited the snapshots in Table 1 to these teachers’ framing of procedures. In a very concise way, the snapshots in Table 1 represent each teacher’s respective emphases with respect to the teaching practices they use, and how different their emphases are.
Table 6-1. Snapshots of Three Teachers’ Framing of Procedures

<table>
<thead>
<tr>
<th>Snapshot A Bonnie</th>
<th>Snapshot B Hildi</th>
<th>Snapshot C Kate</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Mathematical Teaching Practices</strong></td>
<td><strong>Quasi-Mathematical Teaching Practices</strong></td>
<td><strong>Mathematical Teaching Practices</strong></td>
</tr>
<tr>
<td>Decompressing</td>
<td>Trimmed</td>
<td>Bridging</td>
</tr>
<tr>
<td>1</td>
<td>○</td>
<td>○</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>○</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>○</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>○</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Snapshot A reflects an emphasis on the use of mathematical teaching practices—primarily decompressing and trimming. It represents Bonnie’s emphasis on breaking procedures are broken down into components, identifying the components and explaining them to her students. Snapshot B reflects an emphasis on the use of quasi-mathematical teaching practices: primarily quasi-decompress and quasi-bridging. It reflects the key role nonmathematical ideas/methods plays in Hildi’s teaching. Snapshot C reflects the use of both mathematical and quasi-mathematical teaching practices, with the use of
mathematical teaching practices occurring more frequently than the use of quasi-
mathematical teaching practices. Snapshot C reflects Kate’s more complex style of
teaching in which connections play a critical role. In every framing, Kate used either
bridging or quasi-bridging. Every procedure or some component of each procedure is
connected to a mathematical idea or to a nonmathematical idea. These three snapshots
show how very different these teachers are with respect to framing. In addition, the
snapshots show differences in their efforts to make TMIds accessible and comprehensible
using a means that is concise and that is clearly tied to the Conceptual Framework for
Framing.

**Implication:** Sets of collections of snapshots of specific teachers’ framings are
potential tools to discuss teachers’ use of mathematical and quasi-mathematical
teaching practices more generally. In the previous section, the work of each of three
teachers is work discussed based upon the snapshot of each teacher’s framings captured
during classroom observations and interviews. Although they are limited to the teaching
of procedures, the snapshots did provide enough information to communicate several
differences among the three teachers. With the aid of more snapshots captured during
different times in the academic year (e.g., beginning, middle and end of the fall/spring
semester), about more subject matter (e.g., concepts, generalizations), and over the course
(e.g., introduction, development, review/revisit) of the development of a TMId, the
collection of snapshots would provide a more complete picture of a teacher’s framing.
With more complete pictures, there are potentially more points for teacher educators to
discuss with in-service teachers and prospective teachers in professional development
settings and teacher preparation courses.
The snapshots in Table 1 and collections of snapshots for a teacher might can be used as starting points for scholarly discussions about teaching and learning. Using the lens of the Conceptual Framework for Framing, such discussions could include “what might teaching for mathematical proficiency and its strands (e.g., conceptual understanding and procedural fluency) as defined in Adding it Up look like?” Given the emphasis on helping students become mathematically proficient in the Common Core State Mathematics Practice Standards, these discussions are timely.

**Contribution: Documentation and characterization of a variety of framings.**

In chapter 4, 11 characteristics of framing were identified and supported with classroom observation and interview data. In addition to the serving as the basis for the identification of the three quasi-mathematical teaching practices and the development of Conceptual Framework for Framing, these characteristics and corresponding examples provide mathematics educators additional tools to use in professional development and in teacher preparation.

**Implication:** Teacher educators can use the documented framings in professional development courses/workshops and teacher preparation courses as a basis for discussion and reflection. The framings can serve as the bases to help in-service secondary mathematics teachers and prospective secondary mathematics teachers think about and discuss how mathematics is or is not used to make TMIds accessible and comprehensible to students. Framing as examined in this study is composed of two parts: the TMId, and the way in which the TMId is framed. Simply speaking, a TMId plus its frame equals an instance of framing. This view of framing facilitates the translation of the scholarly questions, “What is the role of mathematics in the teaching of school
algebra?” and “How is mathematics used in the teaching of school algebra?” into simpler forms. For a given TMId, the questions become “What is the mathematics in the frame?” and “How is the mathematics in the frame being used?” These questions in turn can be used to ask in-service and prospective secondary mathematics teachers to reflect on or discuss their own practices. Where/what is the mathematics in my framing of ___? How am I using mathematics in the framing of ___?

To facilitate discussions of answer to these questions, mathematics teacher educators can draw upon the examples of framing found in chapter 4. For example, completing the square is the TMId for lessons by two different teachers, but the frames for the procedure involve the use of different approaches. One teacher used quadratic functions that have nice numbers (i.e., the quadratic coefficient equals 1 and even linear coefficients) when she showed students how to complete the square procedure symbolically (see chapter 4, Example 1.3). Another teacher used a geometric representation of quadratic expressions (see chapter 4, Figure 4-52) that drew upon students’ existing understandings of how to factor quadratic trinomials and multiply pairs of linear binomials using what she termed, generic rectangles. The students in the second class worked on a sequence of carefully designed exercises to figure out the completing the square procedure. The mathematics involved in each frame is very different. The first frame essentially revolved around symbol manipulation. No justification of why one adds half the square of the linear coefficient was offered as part of the framing. The second frame involved manipulation of a combination of symbolic and geometric representations. The justification for adding half the square of the linear coefficient is
embedded in students’ understandings of working with generic rectangles, and that all four sides of a square are equal in length.

In addition to the discussion of what mathematics is in the frames and how mathematics is used in the frames, teacher educators can use these examples to discuss other questions. What understandings of completing the square are afforded by each frame? What understandings of school mathematics need to be in place in order for these respective frames to be successful? What understandings of TMIds to be encountered later in the course and in future courses are impacted by these respective framings? How are the understandings of mathematics students will/may encounter in later mathematics courses impacted?

The data revealed other framings in which the respective frames are less fruitful in the mathematics they have to offer. One teacher tells her students that division by zero is a math sin (see chapter 4, Example 2.2). Another tells her students that the “–” in a negative exponent tells them to “flip it [i.e., the base]” (see chapter 4, 1.2) but does not mention reciprocal and multiplicative inverse in the subsequent discussions. The answers to “what is the mathematics in the frame?” and “how is the mathematics used?” are markedly different from the answer to the same questions about the two framings for completing the square. Telling students that division by zero is a math sin is simply a way of telling students division by zero is not allowed. Telling students that the “–” means “flip it” is a quick way of telling students that have to record “\( \frac{1}{[\text{the base}]} \).” In both of these framings, it is not clear what mathematics is being used in the framings. Is the fact that division by zero is not allowed considered a mathematical property of
division by zero? Is mathematics being used when “…” is defined nonmathematically as “flip it?” These questions, which are specific to the examples of framing provided here, are related to the larger questions. What is mathematics? How is mathematics being used to make TMIIs accessible and comprehensible?

Implication: Teacher educators can use the characteristics of framing and the corresponding supporting examples in professional development courses/workshops and teacher preparation courses. The characteristics and supporting examples can serve as the bases to compare and contrast different approaches to making TMIIs accessible and comprehensible to students. This study offers the field of mathematics education description and documentation of approaches used by seven secondary mathematics teachers to make TMIIs accessible and comprehensible to students. One of the strengths of this study is the variety of framings captured in the data, described and organized by characteristic. They offer teacher educators illustrations of framing. They also offer vehicles through which to motivate discussion.

Some examples of framing describe familiar teaching moves such as breaking TMIIs down into smaller components (see Characteristic 3). Emerging from the work of the seven teachers in this study are three distinct ways in which a TMId was broken down into smaller components: pre-determined and revealed to the class in one class period; unplanned, immediate response to student questions; and long term and in the context of solving simple problems that lead to more complex problems. Each approach served different mathematical and pedagogical purposes.

Other examples of framing suggest that making connections (see characteristics #2, #8, and #9), especially those that involve quasi-bridging, is a multi-faceted and
complex teaching move. They include framings that address “making real-world connections,” a commonly heard phrase. The data shows that teachers have answered the question, “what is a real-world connection?” in their own special ways. The worlds that teachers used in their framings range in “real-ness.” Some are real-world events, such as the life cycle of a tarantula wasp which was used to frame evaluating composite functions. Some have real-world qualities, but are not really real such as a game played by two teams on a field that has a river running down the middle which was used to frame solving linear equations. Some contexts are real because a student could imagine the world, such as battles between ninjas and zombies which was used to frame how to add and subtract integers. Important questions that can be addressed in conjunction with these framings include “To what extent is mathematics made more understandable as a result of these framings?” “What are the connections to the TMIds being made?”

**Limitations of this Study**

This study was conducted to answer the question, “How do certificated secondary mathematics teachers make targeted mathematical ideas (TMIds) accessible and comprehensible to students?” Although this study was successful in answering this question, there are some aspects to this question that are beyond the scope of this study. Factors that make one or more aspects beyond the scope of this study are defined as limitations to this study. Three limitations were planned as part of the design of this study. Other limitations emerged organically during data collection and data analysis as questions about teachers’ framing that could not be answered came to the forefront.
Planned limitations. Several restrictions (e.g., teacher’s professional credentials, content area, curriculum/textbook adopted for the course, means by which data were collected) were included in the study design in order to define this study and to try to get the best possible data to answer the research question. Two restrictions, content area, and adopted curriculum/textbook, assisted in making this study more focused. One restriction, limitation of number (i.e., 3 to 5) of observations per teacher, was imposed for practical and logistical reasons. All three restrictions had some kind of impact on the nature of the answers to the research question that emerged in this study.

Content area. For several reasons discussed in Chapters 1 and 3, this study was restricted to the study of secondary school teachers teaching school algebra. The reasons included the increasing numbers of students taking school algebra due to more rigorous high school graduation requirements and research evidence supporting what many students have said for many years, “Algebra is hard.” All of the reasons explained why focusing on school algebra is beneficial. At the same time, this restriction identified the mathematical content arena from which the TMIds were drawn. Therefore, this study does not answer questions associated with how teachers frame ideas such as geometric concepts, properties of trigonometric functions and relations, and geometric proofs and proving. Additional research on framing TMIds in other content areas of secondary school mathematics is needed.

Adopted curriculum/textbook. This study was restricted to studying the work of teachers whose adopted curriculum was published by one of the major publishers (i.e., Pearson Education, Inc., Houghton, Mifflin Harcourt Pub. Co., and McGraw-Hill Education) and whose early development was not funded by state, federal, or foundation
grant monies (e.g., a California Postsecondary Education Commission [CPEC] grant, National Science Foundation [NSF] grant, Carnegie Foundation Grant). This restriction essentially means that if a teacher chose to use a textbook to make TMIDs accessible and comprehensible in his or her classroom it would likely be one that is used by a majority of teachers in the United States and have similar content organization.

What this restriction also means is that this study does not answer this research question with respect to teachers, who follow curricula such as Core Plus Mathematics Program (CPMP), and Interactive Mathematics Program (IMP) whose development began circa 1990’s and which are examples of what many mathematics educators call reform curricula. Reform curricula, developed as alternatives to traditional curricula, have very different mathematical emphases and organization than the traditional curricula. Included in the new design are multi-faceted requirements for implementation in the classroom (e.g., facilitation of student learning, use of technology, teaching mathematics through problem solving, and managing collaborative/cooperative groups). Many teachers attend professional development institutes and workshops (e.g., for CPMP, http://www.wmich.edu/cpmp/profdev.html; for IMP, http://imp.its-about-time.com) to help them understand and implement the curriculum. The more complex requirements to implement the curriculum suggest that from a research perspective, there may be additional answers to the question, “how do certificated teachers attempt to make TMIDs accessible and comprehensible to students?” that are different from the answers found in the data for this study of a group of teachers whose most frequently used form of teaching was direct instruction.
Limitation of number (i.e., 3 to 5) of observations per teacher. At the end of each day of data collection, and periodically throughout the data analysis process, as I reflected on the day’s work, several questions came to mind. For the most part, these questions could be answered had it been possible to conduct more observations. Is the framing I observed today or identified in the data typical of this teacher’s work? What if the TMId is being introduced instead of reviewed/revisited, how would this teacher frame the TMId? These data were gathered in the spring semester. The students have, or at least should have, learned more than half of the course content. Is framing different in the fall semester when so much of the content is new? Some of the teachers were preparing students for semester final exams or State assessments. To what extent do these exams and State assessments impact the framing of TMIds? What, if any, is the nature of the impact?

Limitations that revealed themselves during data collection and data analysis. The limitations being discussed in this section emerged during data collection and data analysis. But having more time to observe the teachers in this study is not necessarily enough to address the limitation.

Level of course (e.g., honors, basic, non-college preparatory). Given that framing is about making TMIds accessible and comprehensible to students, it would seem that the needs of the student audience would have an impact on how teachers make TMIds accessible and comprehensible. Each of the teachers participating in this study was observed in exactly one class. Thus, additional examples of framing captured over a long period of time would not provide insights as to what extent framing differs, if at all, if a teacher teaches a higher or a lower level of the same course. In other words, this
study cannot answer the question, “how does teachers’ framing change when teachers teach other levels (e.g., honors, regular, and basic) of the same course?”

Subject matter. Almost all of the instances of framing captured in the data involved making procedures accessible and comprehensible to students. Of the 50+ instances of framing identified in the data none of the framings involved the framing of generalizations and only six involved the framing of concepts. Thus, answers to the question how do teachers frame generalizations and concepts emerging from the data are non-existent and few, respectively. Identification of possible reasons behind the numerical dominance of framings for which the TMId is a procedure is not explored in this study. It is unknown whether this occurrence is an anomaly or a representation, generally speaking, of what aspect of school algebra teachers are making accessible and comprehensible to students. Studying what subject matter is being framed by other teachers and how the subject matter is framed is needed.

Of interest with respect to the framing of subject matter, is that of the six concepts that were framed in this study, two were framed as procedures (see chapter 4, Example 6.2). Rather than frame the respective TMIds (i.e., negative exponents and fractional exponents) so that students would know what the TMIds are, the TMIds were framed so that students would know what to do when they see them. The emphasis in both of these framings is for students to know what to write for the next step (i.e., the product of a procedure). The data does not contain enough information to conclude whether framing concepts as procedures is unique to this teacher or a way of framing concepts that is used by other teachers as well.
Possible Future Directions

Next steps for this study are inspired primarily by questions that repeatedly came to mind during data collection and data analysis. I wondered how different each teacher’s framing would be, if data collection took place during the fall semester when teachers and students do not have to contend with upcoming state assessments and/or final exams are introduced to more new content and teachers are establishing socio norms and socio-mathematical norms. I wondered what each teacher’s framings would look like over the course of an entire unit/chapter of content. I wondered to what extent a teacher’s framing differs when teaching different levels of the same course (e.g., honors and regular) and the same level of different courses (e.g., regular Algebra 1 and regular Algebra 2). To answer these questions, I am thinking about capturing more snapshots of several secondary mathematics teachers teaching school algebra over the course of a year, with some snapshot capturing framings over the course of a chapter/unit of material. Besides collecting more examples of framing, the goal here would be to determine how, if at all, teachers’ framings emphases change as the content changes and gets more difficult.

A study using existing classroom observation data. It is my hope that I would be allowed to analyze data from the Process Project of the Mid-Atlantic Center for Teaching and Learning for instances of framing and to expand my collection of framings and snapshots that capture what mathematical and/or quasi-mathematical teaching practices the teachers emphasize over a short period of time. Because the observation data was captured without the aid of video recording devices, studying the Process Project data would serve as a pilot study to gain a sense of how much detail is possible to
capture without video data. If my ability to document and characterize framing is not affected too greatly, then it may be worth conducting future studies without using video recording devices. As the early stages of this study showed, recruiting teachers who would or could allow video cameras into the classroom is extremely difficult. The Process Project data has the additional benefit of providing me with data about teachers in a different stage of their teaching careers than the teachers participating in the current study. At the time of data collection, the teachers in this study ranged in experience from 3 to over 30 years. At the time classroom observation data was collected for the Process Project, the participants were in the student teaching and then the induction phases of their respective careers. Whether there is enough data to make inferences about teachers’ framing along the lines of number of years of teaching experience is to be determined.

A long term (one full academic year) study. In addition to existing classroom observation data from the Process Project, I am aware of a secondary mathematics teacher who routinely makes video recordings of his teaching. This teacher is aware of my dissertation study, has expressed both an interest in my work and a willingness to let me use his video for study.

This teacher is of particular interest for two reasons. One, he is a veteran teacher whose teaching has been the focus of another Penn State dissertation. Two, he records every class almost every day of the academic year. Two, he is a veteran teacher whose teaching has been the focus of another Penn State dissertation. Although I have not seen any of his video data, the chance to examine Examining this teacher’s video data—one year’s worth of video data—is an opportunity that should be explored. This teacher makes video recordings of his classes as a way to help students who are absent from class.
hear and see the teacher’s explanations as well as what was recorded on the board
(Grady, personal conversation, November 2012). The video data could provide insights as to differences in framing, if any, between framings done during fall semester and spring semester; and patterns of framing, if any, over an extended period of time when TMIs are introduced, developed, reviewed, and possibly revisited. Such an examination would, to some degree, answer questions such as “to what extent does time within an academic year when a TMId is framed impact framing?” or “to what extent does purpose of framing (e.g., introduction of new TMId, reviewing TMIds for tests) beyond making TMIds accessible and comprehensible impact how a TMId is framed?” These questions cannot be answered when capturing data over a short period of time during the spring semester. Since it is unlikely that I will find another teacher who makes video recordings of his lessons on a daily basis and that I will be logistically and financially able to initiate such data collection, examining his data might help me make more informed decisions in other studies with respect to numbers of snapshots to take and when to take the snapshots. The goal is to increase the possibility of getting more informative data and different data than I was able to collect for this study.

**Further development and refinement of the Conceptual Framework for Framing.** As more studies are conducted and more data are analyzed, one question will always be revisited. Are the six teaching practices in Conceptual Framework for Framing sufficient to account for what teachers do to make TMIs accessible and comprehensible? This study started with three mathematical teaching practices and expanded the framework to include the quasi-mathematical teaching practices. I wonder, if nonmathematical practices exist, what would characteristics of nonmathematical
teaching these practices, if they exist, be? The data for this study revealed a few instances of framing that might be considered nonmathematical framings. One teacher required his students to memorize a list of twenty fraction-decimal equivalents. In this framing, the TMId is fraction-decimal equivalents and the frame is memorization, a nonmathematical skill. Another teacher required her students to use a special kind of stationery for taking notes. What was observed was the use of the stationery designed for use with graphing functions which had sections for symbolic, tabular, and graphical representations of functions. This stationery, because of its organization based upon representations of functions was coded as an example of quasi-decompressing; however, the other type of stationery used by this teacher (see chapter 4, Figure 4-18) has a generic organization. Its sections include questions, notes, lesson objective, and summary of notes. Use of this type of stationery may be an example of nonmathematical framing or it may strictly be a tool to teach students the value of organization and not a means to frame mathematics.

These two examples of framing that are possible candidates for framing are insufficient to make. More data are needed to make conclusions about what nonmathematical framings might be.

In addition to exploring whether a third category should be added to the Framework, as more instances of framing are gathered and analyzed from more classrooms and from different teachers, might there be other teaching practices to add to the mathematical teaching practices or the quasi-mathematical teaching practices? For example, mathematical modeling currently falls into the category of quasi-bridging because modeling connects a TMId and a real-world event or phenomenon. At the same time, mathematical modeling seems very different from the examples of quasi-bridging
that emerged from the data. Student-relatable contexts seemed to be developed around TMIds to make the TMIds more accessible and comprehensible. Hildi’s framing of how to evaluate composite functions by connecting it to the life cycle of a tarantula wasp is definitely not an example of mathematical modeling although her framing does draw connections between a TMId and a real-world event. This framing and framings involving making connections between TMIds and student-relatable contexts have the purpose of making Student-relatable contexts seemed to be developed around TMIds to make the TMIds more accessible and comprehensible. Because classroom use of mathematical modeling, involves the use of mathematics is used to explain or simulate real-world events/phenomenon as well as to make TMIds accessible and comprehensible, my sense is that mathematical modeling is a form of bridging. Additional study using the lens of the Conceptual Framework for Framing of the use of mathematical modeling in school mathematics classrooms is needed to articulate clearer lines between bridging and quasi-bridging when real-world contexts are involved.

A study of teachers framing concepts and generalizations and an exploration of an observation that teaching Algebra 1 and Algebra 2 seems to be about teaching procedures. The observation that a majority of the TMIds being framed by the teachers in this study are procedures was confirmed when the framings were coded, organized into a table, and sorted by subject matter. As stated earlier, it is not known whether this is an anomaly or representative of what goes on in general. As more data are collected and analyzed, the observation that Algebra 1 and Algebra 2 are about procedures will be further explored.
It is a possibility that most of the instances of framing will once again be about procedures. As data about each new teacher’s framing are gathered, exploration of two potential factors will be conducted. One, textbooks are known to have a great deal of influence on a teacher’s teaching. Examinations of the teachers’ respective textbooks could be conducted to gain understanding of how concepts, procedures, and generalizations are treated, and to count how many of each of these types of subject matter is framed. Two, framing can be interpreted as an application of subject matter/content knowledge. Part of the study could be conducted to explore teachers’ understandings of important concepts of school algebra through the use of observation-related interviews as concepts are addressed in lessons, and/or task-based interviews designed to gain insights about a teacher’s understanding of concepts such as variable, equation, slope, exponent, and polynomial. If teachers’ understanding of concepts and generalizations are as procedures, then their understanding may explain why so few concepts and generalizations were framed. Examples of the framing of two concepts as procedures were found in the data for this study. One of these framings was conducted as part of preparation for upcoming final exams. The teacher’s framing was a response to the results of a practice final: a majority of students gave incorrect responses to a problem involving negative exponents. The teacher’s emphasis was on students obtaining a correct answer rather than understanding the concept. The second framing of a concept as a procedure seemed to be part of the teacher’s belief that mathematics is a language. This teacher wanted her students to know what to do whenever they see a negative exponent or a fractional exponent. Negative exponents and fractional exponents are associated with certain actions.
A study of teachers’ teaching as teachers respond to the Common Core State Standards. Currently, many teachers are faced with having to make major changes in their teaching in order to teach to the Common Core State Standards. They are in the process of trying to figure out what they need to do to address the Mathematics Practice Standards (see Figure 6.1) as well as the Content Standards as well as the Mathematics Practice Standards (see Figure 6.1). Conducting a study to understand how teachers make TMIIs accessible and comprehensible while at the same time engaging students in the Practice Standards may be a promising contribution to the mathematics education community. Such a study would require an addition to the Conceptual Framework for Framing. As is, the Conceptual Framework for Framing is about teachers making ideas of school algebra accessible. Thus, this framework offers a way of looking at teachers’ work with the Content Standards. What is missing is a lens through which to view teachers making the ways of thinking and the intellectual habits described by the Practice Standards accessible to students who are not the mathematically proficient students described in the Practice Standards themselves. Developing such a lens is a more complex research endeavor.
Figure 6-1. Common Core State Standards for Mathematical Practices (National Governors Association Center for Best Practices, 2010).

**Conclusion**

The question that has been driving this dissertation study began as a very personal question; one I asked multiple times every day during my 21 years as a high school mathematics teacher. How do I help my students get it? As a practitioner, the answer to this question was a simple one. I did anything I could figure out that made sense to me and that I thought would make sense to my students. Through this study, I found answers to the dissertation version of my practitioner’s question, how do certificated secondary mathematics teachers make targeted mathematical ideas (TMIIs) accessible and comprehensible to students?” Examination of the work of seven secondary mathematics teachers revealed a variety of ways in which they try to make TMIIs accessible and comprehensible. The 50+ examples of framing ultimately resulted in the development of
the Conceptual Framework for Framing with its mathematical and quasi-mathematical teaching practices, a way of creating snapshots of teacher’s framing that capture which teaching practices he or she emphasizes, and the bases for scholarly discussions of answers to questions such as “what is the role of mathematics in the framing of TMIDs?” and “what does teaching for conceptual understanding look like?”

My research question, “how do certificated secondary mathematics teachers attempt to make TMIDs accessible and comprehensible to students?” has several answers. Teachers attempt to make TMIDs accessible and comprehensible to students by breaking TMId into components (i.e., decompressing and quasi-decompressing) by highlighting/emphasizing aspects of TMIds (i.e., trimming and quasi-trimming) and by making connections between TMIds and mathematical ideas (i.e., bridging) or TMIds and nonmathematical ideas (i.e., quasi-bridging). Their efforts involve the use of both direct and indirect approaches (characteristic 1). Sometimes mathematics is used as a critical means by which TMIDs are made accessible and comprehensible and sometimes mathematics is not used as the means by which TMIDs are made accessible and comprehensible. Sometimes teachers treat mathematics as a verbal and symbolic language by helping students make the transition from “English-English” to “math-English.” Their strategies include, but are not limited to, assigning rememberable contexts or meanings to inscriptions or using local language (e.g., combining like terms means “getting the friends together”). However, the question has not been answered completely. It is my plan to study the framing of more secondary mathematics teachers, to refine the Conceptual Framework for Framing, and to find a way to study how teachers
help students make progress towards becoming the mathematically proficient students as described by the eight Common Core State Mathematics Practice Standards.
Appendix A

Problem-Solving Interview Tasks

All of the five tasks in used in the problem-solving interviews are classroom scenarios in which hypothetical students ask questions, make statements, and/or provide written work that reflect a dilemma or a problematic conception of some mathematical idea. These tasks were chosen because the mathematics being discussed in each scenario can be addressed in more than one way.

Participants were presented each scenario, and asked to share how he or she would respond to the student(s). If the participant’s initial response did not include how he or she would attempt to make the mathematical idea accessible and comprehensible to the hypothetical student(s), the participant was asked follow-up questions to encourage the participant to address how he or she might help to make the mathematical idea accessible and comprehensible.
Task 1.

Scenario: In a review lesson, you have students practice how to convert fractions to decimal numbers. A student says, “I know how to start with $\frac{1}{3}$ and get $0.333... = 0. \overline{3}$. But how do you start with a decimal and get a fraction? What is the fraction for $0.555... = 0. \overline{5}$?

Commentary. The student in this scenario asks how. To address how one can obtain the fraction equivalent of an infinitely repeating decimal, a participant can show the student at least four different ways. The methods are as follows: using equivalent equations to eliminate the infinitely repeating part of the decimal number, partitioning the decimal number, using interpolation, and using the formula for the sum of a geometric series.

Possible approach 1a. Using equivalent equations to eliminate the infinitely repeating part of the decimal number. Let $S = 0.555...$

\[
\begin{align*}
S & = 0.555... \\
-10S & = -5.555... \\
-9S & = -5 \\
S & = \frac{5}{9}
\end{align*}
\]

So, $0.555...$ equals $\frac{5}{9}$.

Possible approach 1b. Partitioning the decimal number. We know that $0.333... = \frac{1}{3}$.
\[
\frac{1}{3} (0.333...) = \frac{1}{3} \left( \frac{1}{3} \right),
\]

\[
0.111... = \frac{1}{9}
\]

Since 0.555… equals five times 0.111…, \(0.555... = \frac{5}{9}\)

**Possible approach 1c: Using interpolation.** In addition to \(\frac{1}{3} = 0.333...\), algebra and prealgebra students know that \(\frac{2}{3} = 0.666...\). Using these two known fraction-decimal equivalents, the fraction for 0.555… can be found using interpolation.

Using differences of pairs of fraction-decimal equivalents, a proportion can be set up.

\[
\frac{2}{3} - \frac{1}{3} = \frac{1}{3} \quad \frac{2}{3} - \frac{0.666...}{N} = \frac{0.555...}{N} = \frac{0.111...}{N} \quad 0.666... - 0.333... = 0.333...
\]

Using differences of pairs of fraction-decimal equivalents, a proportion can be set up.

\[
\frac{2}{3} - \frac{N}{3} = \frac{0.111...}{0.333...}
\]

Since \(\frac{0.111...}{0.333...}\) equals \(\frac{1}{3}\), the proportion becomes

\[
\frac{2}{3} - \frac{N}{3} = \frac{1}{3}
\]

Simplifying,
\[
\frac{2}{3} - N = \frac{1}{9}
\]
\[
\frac{2}{3} - \frac{1}{9} = N
\]
\[
N = \frac{5}{9}
\]

**Possible approach 1d: Geometric series.** Let \(N = 0.555\ldots\), an finite geometric series, can also be expressed as \(5 \left[ 0.1 + (0.1)^2 + (0.1)^3 + \ldots \right]\) or \(5 \sum_{k=1}^{\infty} 0.1^k\). The formula for the sum of an infinite geometric series is

\[
S = \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad |r|<1
\]

Since the value of \(r\) equals 0.1 for \(N\), with a minor adjustment, we may apply this formula.

\[
N = 5 \sum_{k=1}^{\infty} 0.1^k
\]
\[
= 5 \sum_{k=0}^{\infty} 0.1^k - 5 \cdot (0.1)^0
\]
\[
= 5 \sum_{k=0}^{\infty} 0.1^k - 5
\]
\[
= 5 \left( \frac{1}{1-0.1} \right) - 5
\]
\[
= 5 \left( \frac{10}{9} \right) - 5
\]
\[
= \frac{5}{9}
\]
Task 2.

Scenario: Given: \( a^0 = 1 \). Students ask you, “Why would something you write zero times equal 1?”

**Commentary.** This scenario contains two different issues: (1) why does \( a^0 = 1 \) and (2) what is an exponent? After the first issue was addressed by the participant, the participant was asked to address the second issue. All participants offered a definition that indicates that an exponent is the number of \( a \)'s that are multiplied together. All participants agreed that the 0 in the given equation is an exponent. However, none were able to resolve the inconsistency between the definition stated and their claim that the 0 is an exponent.

**Possible approach 2a.** A response to why \( a^0 = 1 \) using a law of exponent

\[
\begin{align*}
    a^0 &= a^{m-m} \\
    &= \frac{a^m}{a^m} \\
    &= 1
\end{align*}
\]

**Possible approach 2b.** A response to why \( a^0 = 1 \) using patterns. A geometric sequence can be defined recursively as follows: \( t_{n+1} = t_n \cdot a \). In the sequence below the initial term is \( a^1 \).

\[
\begin{array}{cccccccc}
    a^0 & a^1 & a^2 & a^3 & a^4 & a^5 & a^6 & a^7 & \ldots
\end{array}
\]
Instead of multiplying by $a$ to generate the next term, one can “go backwards” by dividing by $a$. In the sequence below, the initial term is $a^4$.

\[
\begin{align*}
&\frac{1}{a} \quad \frac{1}{a} \quad \frac{1}{a} \quad \frac{1}{a} \\
&a^4, a^3, a^2, a^1, a^0
\end{align*}
\]

This sequence shows that $a^1 \div a$ is $a^0$. Since $a^1 \div a$ also equals $1$, $a^0$ equals $1$. 
Task 3.

Scenario: You provide students with the coordinates of points A, B, C, and D and ask them to show that segments AB and CD have equal lengths.

A student graphs segments AB and CD and says, “AB and CD have the same length because they have the same slopes. To get from A to B, you go up 1 and right 4. To get from C to D you go up 1 and right 4.”

Commentary. In this scenario, the student makes a correct conclusion about the lengths of segments AB and CD (i.e., $AB = CD = \sqrt{17}$ ) using a justification that reflects a problematic understanding of some kind. The participant can respond by addressing how to reach the same conclusion correctly using any of at least three approaches: (1) using the distance formula; and using right triangles for which segments AB and CD are hypotenuses, (2) using the Pythagorean Theorem, and (3) showing the triangles are congruent. The participant can choose to address the student’s definition of slope.

Possible approach 3a. How to show the segments have equal lengths using the distance formula. Given $A(3,3)$, $B(7,4)$, $C(-2,4.5)$, and $D(2,5.5)$. 
Possible approach 3b. How to show the segments have equal lengths using the Pythagorean Theorem. Given $A(3,3), B(7,4), C(-2,4.5), \text{ and } D(2,5.5)$. Right triangles are drawn for which segments $AB$ and $CD$ are hypotenuses.

\[
AB = \sqrt{(3-7)^2 + (3-4)^2} = \sqrt{16 + 1} = \sqrt{17} \\
CD = \sqrt{(-2-2)^2 + (4.5-5.5)^2} = \sqrt{16 + 1} = \sqrt{17} \\
\therefore AB = CD
\]

Possible approach 3c. How to show the segments have equal lengths using congruent triangles. Given $A(3,3), B(7,4), C(-2,4.5), \text{ and } D(2,5.5)$. Right triangles are drawn for which segments $AB$ and $CD$ are hypotenuses.

\[
(AB)^2 = 4^2 + 1^2 \quad (CD)^2 = 4^2 + 1^2 \\
AB = \sqrt{17} \quad CD = \sqrt{17} \\
\therefore AB = CD
\]
Triangles PCD and QAB are constructed so that angles P and Q are right angles. As a result, \( PC = 4 \), \( PD = 1 \), \( QA = 4 \) and \( QB = 1 \). Because \( PC = QA \), \( \overline{PC} \cong \overline{QA} \). Similarly, because \( PD = QA \), \( \overline{PD} \cong \overline{QB} \). Therefore, triangles PCD and QAB are congruent.

Because segments AB and CD are corresponding parts of congruent triangles, segments AB and CD are congruent and have equal measures.

**Possible approach 3d.** Addressing the student’s “use” of slope. A participant can interpret the student’s justification as a mis-application of a commonly used definition of slope (i.e., “slope is rise over run.”) and chose to address this issue by stating that one does not use slope to find lengths. One uses the distance formula, Pythagorean Theorem, or triangle congruence.

**Possible approach 3e.** Addressing the student’s “use” of slope. A participant could interpret the student’s justification as an issue of mathematical language and communication. The student could be applying a definition commonly used in mathematics classrooms (i.e., slope is rise over run) incorrectly, and choose to address how “slope is rise over run” is a casual, way of talking about a more precise conception of slope as a rate of change.
Task 4.

Scenario: Lee says, “I just discovered that $\frac{5}{0} = \infty$. I made a table of 5 divided by smaller and smaller numbers and noticed the answers get bigger, and bigger, and REALLY big. So, when I get to $\frac{5}{0}$, the answer is infinity. So, $\frac{5}{0} = \infty$.”

<table>
<thead>
<tr>
<th>Fraction</th>
<th>$\frac{5}{1}$</th>
<th>$\frac{5}{0.1}$</th>
<th>$\frac{5}{0.01}$</th>
<th>$\frac{5}{0.001}$</th>
<th>$\frac{5}{0.0001}$</th>
<th>$\frac{5}{0.00001}$</th>
<th>...</th>
<th>$\frac{5}{0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>After dividing</td>
<td>5</td>
<td>50</td>
<td>500</td>
<td>5,000</td>
<td>50,000</td>
<td>500,000</td>
<td>...</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>

Commentary: In this scenario, the student offers an argument that seems to suggest “$\infty$” is a REALLY big number. The participant could respond by explaining that $\frac{5}{0}$ is undefined, by addressing infinity is.

Possible approach 4a. Division by zero is undefined. Let $N = \frac{5}{0}$. By the definition of division, $N = \frac{5}{0}$ means $5 = 0 \cdot N$. Because $5 = 0 \cdot N$ is false no matter what value is used for $N$, $\frac{5}{0}$ is undefined.

In the case of $M = \frac{0}{0}$, by the definition of division, $0 = 0 \cdot M$. Because $0 = 0 \cdot M$ is true no matter what value is used for $M$, $\frac{0}{0}$ is indeterminant.

Possible approach 4b. Addressing what infinity is. Infinity is an unbounded quantity. As an unbounded quantity, something cannot equal infinity. This student’s argument is based upon the concept of limits. The table of values reflects of the limit of
\[ \frac{5}{10^{-n}} \text{ as } n \to \infty, \text{ for positive integral values of } n. \] This sequence is unbounded. To communicate this unbounded-ness, we use the symbol, \( \infty \).
Task 5.
Scenario: Students are working on some graphing exercises independently. You notice that several students struggled with graphing $2x + 3 < y$. Their graphs were similar to the adjacent graph. The boundary is graphed correctly, but the graph is shaded on the wrong side of the boundary line. When you asked them to explain their thinking, in general, they said, "'2x + 3 = y' is the boundary. The line is a dashed line because it’s ‘<.’ Shade below the line because ‘<’ tells you to shade below."

Commentary: In this scenario, students use the inequality symbol as the cue to determine which side of the boundary line to shade. This approach can be addressed in two possible ways: (1) changing the way the inequality is expressed, and (2) discussing what solutions to inequalities are.

Approach 5a. Changing the way the inequality is expressed. In this scenario, students use the inequality symbol as the basis for determining which side of the boundary line to shade. If the inequality is "<", then shade below the boundary. If the inequality is ">", then shade above the boundary. Participants can make the solution to this inequality accessible by telling them that they need to make sure that the $y$ variable is alone on the left side of the inequality sign. Only then does the inequality symbol indicate to shade above or below.
**Approach 5b.** Addressing what solutions to inequalities are. Participants can remind students that solutions to inequalities are pairs of $x$- and $y$-values that made the inequality a true statement. The shading of graphs of inequalities is a way of representing all of the pairs of $x$- and $y$-values that make the inequality true. To determine which side of the boundary line to shade, participants can use any point to test which half plane contains the solutions to the inequality. For the given example, $2x + 3 < y$, a viable test point is $(0,0)$ because it does not satisfy the boundary equation, $2x + 3 = y$. Since $2(0) + 3 < 0$ is not a true statement, the points on the plane on the same side of the line, $2x + 3 = y$, as $(0,0)$ are not solutions to the inequality, $2x + 3 < y$. 
Appendix B

Explanations for Ann’s Secret Method and of Bonnie’s AC Rule, Two

Method Used to $ax^2 + bx + c, a \neq 1$

Figure B-1 is provided for reference and a reminder of Ann’s and Bonnie’s factoring methods.

Ann’s Secret Method revisited. Ann’s Secret Method was examined with the idea that $x$’s in the first two equations represent different quantities, that the sequence of equations in Ann’s written is composed of two sets of equations, and that Ann’s written work is perhaps an abbreviation of a derivation. What follows is a discussion why Ann’s Secret Method works. The discussion starts with a general equation, which is assumed to be factorable over rational numbers.
General Case

Ann’s Example

\[ 0 = a'x^2 + b'x + c' \]  

(1.1) \hspace{0.5cm} [1] \hspace{0.5cm} 0 = 3x^2 + 4x + 1

Ann began her Secret Method by multiplying the quadratic coefficient and the constant. By multiplying equation 1.1 by \( a \), one obtains a new equation, one that is also factorable.

\[ 0 = (a'x)^2 + b'(a'x) + a'c' \]  

(1.2) \hspace{0.5cm} [2] \hspace{0.5cm} 0 = x^2 + 4x + 3

By letting \( u = a'x \), and substituting \( u \) into equation 1.2, one obtains an equation that has the same symbolic structure as Ann’s equation [2].

\[ 0 = u^2 + b'u + a'c' \]  

(1.3)

Equation 1.3 is either factorable or not. We examine each case separately.

Case 1. Equation 1.3 is factorable.

If equation 1.3 is factorable then there exists two numbers, \( b_1 \) and \( b_2 \), such that

\[ b_1 + b_2 = b' \text{ and } b_1b_2 = a'c'. \]

Therefore,

\[ 0 = (u + b_1)(u + b_2). \]  

(1.4) \hspace{0.5cm} [3] \hspace{0.5cm} 0 = (x+1)(x+3)

Case 2. Equation 1.3 is not factorable.
If equation 1.3 is not factorable, then the discriminant for this equation is either a negative number, or a non-square positive number. The discriminant for equation 1.3 is 

\[(b')^2 - 4(1)(a'c')\], which simplifies to \(b'^2 - 4a'c'\). Thus, \(b'^2 - 4a'c'\) is either (1) negative; or (2) positive, but not a square.

The values, \(a'\), \(b'\), and \(c'\) in equation 1.3 are identical to the values in equation 1.1, an equation we assume to be factorable. The discriminant for equation 1.1 is 

\[(b')^2 - 4(a')(c')\] which simplifies to \(b'^2 - 4a'c'\). Since equation 1.1 is factorable over the rational numbers, \(b'^2 - 4a'c'\) is a positive and square. However, \(b'^2 - 4a'c'\) cannot be positive and square, AND negative. In addition, \(b'^2 - 4a'c'\) cannot be positive and square, AND positive and non-square. We have arrived at a contradiction. Therefore, equation 1.3 has to be factorable.

Rewriting equation 1.4 in terms of \(x\), using \(u = a'x\), one obtains equation 1.5.

This substitution step is the mathematical basis for Ann’s transition from her equation [3] to her equation [4]—the step Ann calls “the secret” (Postobservation 3 Interview, 104–114)

\[
0 = (a'x + b_1)(a'x + b_2) \tag{1.5} \quad [4] \quad 0 = (3x+1)(3x+3)
\]

In order to obtain an equation equivalent to equation 1.1, one must divide equation 1.5 by \(a'\)

\[
0 = \frac{(a'x + b_1)(a'x + b_2)}{a'} \tag{1.6}
\]
which simplifies equation 1.7, an equation that has the same symbolic structure as Ann’s equation [5].

\[ 0 = (a'x + b_0) \left( x + \frac{b_1}{a'} \right) \quad (1.7) \]

To verify that equation 1.7 is equivalent to equation 1.1, I multiplied the binomials.

\[ 0 = (a'x + b_0) \left( x + \frac{b_1}{a'} \right) \\
= a'x^2 + b_0x + b_1x + \frac{b_0b_1}{a'} \quad (1.8) \]

Since \( b_1 + b_2 = b' \) and \( b_0b_1 = a'c' \).

\[ 0 = a'x^2 + (b_1 + b_2)x + \frac{b_0b_1}{a'} \\
= a'x^2 + b'x + c' \quad (1.9) \]

Thus, Ann’s Secret Method does work for all factorable trinomials. Based upon this derivation, Ann’s written record for each application of the Secret Method can be seen as an abbreviated version of the procedure characterized by equations 1.1 to 1.7.

**Bonnie’s AC Rule revisited.** Of the three steps in Bonnie’s AC Rule, it is the transition from \( 6x^2 + 11x + 3 \) (i.e., expression [a]) to \( 6x^2 + 2x + 9x + 3 \) (i.e., expression [b]) for which Bonnie is unable to justify. The question is why does one choose the pair of factors of the product of 6 and 3 which add up to 11 to partition the linear term? In terms of the general quadratic trinomial, \( ax^2 + bx + c \), why does one choose two
numbers whose product is \( ac \) and whose sum is \( b \) to partition \( bx \)? What follows is one possible explanation.

If \( ax^2 + bx + c \) is factorable, then there exists integers \( p, q, r, \) and \( s \) such that

\[
ax^2 + bx + c = (px + q)(rx + s).
\]  

(1.10)

Multiplying the binomials, the result is

\[
ax^2 + bx + c = prx^2 + (ps + qr)x + qs.
\]  

(1.11)

Examination of the linear coefficients on each side of equation 1.10 reveals that \( b \) is the sum of \( ps \) and \( qr \). These two numbers are composed of factors of \( pqrs \). Since \( a \) equals \( pr \) and \( c \) equals \( qs \), these relationships support that two numbers which have a product of \( ac \) and have a sum of \( b \), lead to the binomial factors of \( ax^2 + bx + c \).
Appendix C

A Detailed Description of Kate’s Use of Generic Rectangles to Factor Quadratic Trinomials

Figure C-1 is a screen capture of Kate’s work documenting what she did to factor $2x^2 + 9x + 10$. By only looking at Kate’s written work, one might interpret her method of factoring strictly as the use of a graphics organizer factor, and therefore, conclude her method as an example of quasi-decompressing. Kate, however, teaches factoring quadratic trinomials and multiplying linear binomials in the context of area. She states, “When I learned to multiply out binomials I learned to do FOIL but when I started teaching, I started teaching by using a—what we call a generic rectangle. That if I want to multiply any two things, um it could be represented as the area of a rectangle” (PS Interview). Factoring to Kate and her students is “going backwards” (Postobservation 1 Interview); starting from the area of a generic rectangle and finding its dimensions. Because she connects multiplying binomials to finding area of a rectangle, and factoring as starting with area and “going backwards” to find the dimensions of a generic rectangle, Kate’s framing is an example of quasi-bridging.
Figure C-1. Screenshot of Kate’s work for factoring quadratic trinomials using “X-box”/generic rectangles (Postobservation 1 Interview).

What follows is a step-by-step description of her factoring process.

1. Kate draws an X and what she calls a generic rectangle. She enters $2x^2$ and 10 into the quadrant shown. The placement of these terms in the generic rectangle is consistent with the placement of the quadratic and constant terms resulting from using a generic rectangle to multiply two binomials.

\[ 0 = 2x^2 + 9x + 10 \]

2. Kate uses what she and her students call a diamond problem to determine how to partition $9x$ into two parts (areas) to fill in the remaining two rectangles in the generic rectangle. Note:

Kate introduces diamond problems (discussed in Example 10.1a) early in the academic
year to help students practice computations. Ultimately, they are used to help students become skillful in finding the factors of a given number (i.e., top entry of the diamond) whose sum is a second given number (i.e., bottom entry of the diamond). See Figure C-2.

![Diamond Problem](image)

*Figure C-2. Diamond Problem formats.* Initially, students are given two entries in the diamond and asked to compute the other two entries.

To partition $9x$ into two parts, Kate places the product of the quadratic coefficient and the constant in the top of the X (i.e., what remains of the diamond template), the linear coefficient in the bottom of the X and computes what factors of 20 have a sum of 9. The factors are 4 and 5.

$$0 = 2x^2 + 9x + 10$$

4. Kate places the 5 and 4 in the X and records $5x$ and $4x$ in the two remaining rectangles in the generic rectangle. It does not matter where the $5x$ and $4x$ are placed in the rectangle.

$$0 = 2x^2 + 9x + 10$$
5. Kate finds dimensions of the top “half” of the generic rectangle by finding the greatest common factor of the two expressions (areas), $2x^2$ and $4x$. In this case, the length is $2x$. The width of the generic rectangle is, therefore, $x + 2$.

$$0 = 2x^2 + 9x + 10$$

6. Kate finds the dimensions of the rectangle whose area is $5x$. One dimension of this rectangle is $x$. So the other dimension is $+5$. This means that the dimensions of the bottom rectangle is $5$ by $x + 2$. Because $5(x + 2)$ does equal $5x + 10$, the factorization is correct.

$$0 = 2x^2 + 9x + 10$$

$$0 = (x+2)(2x+5)$$
Appendix D

The Tic-Tac-Toe Method of Factoring Quadratic Trinomials

The Method.

The tic-tac-toe method of factoring is heavily dependent upon a graphic organizer in the shape of a 3 by 3 grid (i.e., a tic-tac-toe grid). What follows is the method as I was shown by teachers I encountered at various teacher conferences. Different teachers present the method in slightly different ways. Some present the method strictly numerically. Others present the method using algebraic expressions. Because this method will be related later in this appendix to factoring quadratic trinomials by grouping, the description of the tic-tac-toe method will be presented here using algebraic expressions.

There are several relationships among the entries in the grid. Column by column, the product of the entries in the second and third row is the entry in the first row. Row by row, the product of the entries in the first and second column is the entry in the third column. The other relationships will be explained while using this method to factor both a specific quadratic trinomial, \(2x^2 + 9x + 10\) and the general quadratic trinomial, \(ax^2 + bx + c\).

The entries for row 1 are as follows: \(ax^2\), \(c\) and \(acx^2\). The remaining entries for column 3 are the factors of \(acx^2\) whose sum is \(bx\).
The remaining cell entries are the greatest common factors (gcf) of two pairs of expressions. The expressions are in row 1 in the same column and column 3 in the same row. For example, the cell in column 1, row 2 is the gcf of $ax^2$ and $p_1 x$, which are entries in row 1-column 1 and row 2-column 3.

Assuming that all of the gcfs are correctly computed, the factored forms are as follows:

$2x^2 + 9x + 10 = (x + 2)(2x + 5)$

$ax^2 + bx + c = \left[ \text{gcf} \left( ax^2, p_1 x \right) + \text{gcf} \left( c, p_2 x \right) \right] \left[ \text{gcf} \left( ax^2, p_2 x \right) + \text{gcf} \left( c, p_1 \right) \right]$
tac-toe method to the steps of factor by grouping method, discuss the factoring of 

\[2x^2 + 9x + 10\] 

using factor by grouping shown below.

\[
a. \quad 2x^2 + 9x + 10 \\
b. \quad 2x^2 + 5x + 4x + 10 \\
c. \quad x(2x + 5) + 2(2x + 5) \\
d. \quad (x + 2)(2x + 5)
\]

In factor by grouping, the linear term \((9x)\) is partitioned into two parts \((5x\) and \(4x)\) by finding the pair of factors of the product \((20)\) of the quadratic term and the constant whose sum equals the linear term. The result is expression b. The gcf \((x)\) is then factored out of the first two terms \((2x^2 \text{ and } 5x)\), and the gcf \((2)\) is factored out of the last two terms \((4x \text{ and } 10)\). The result is expression c. The two terms in this binomial have a common factor \((2x + 5)\). Factoring out the common factor results in the given quadratic in factored form (expression d).

Each of the expressions in the factoring of \(2x^2 + 9x + 10\) using the factoring by grouping method can be mapped to a term in the graphic organizer for the tic-tac-toe method of factoring. The gcf of the first two terms in expression b, is the entry in row 2-column 1. When the gcf is factored out of \(2x^2 + 5x\), the remaining factor is \(2x + 5\). The \(2x\) is the entry in row 3-column 1, and the 5 is the entry in row 2-column 2. Similarly, the gcf of the last two terms of expression b is the entry in row 3-column 2. When this gcf is factored out of \(4x + 10\), the remaining factor is \(2x + 5\). The same as what remained when \(2x\) was factored out of the first two expressions.
General case: mapping factor by grouping to the tic-tac-toe method.

Given \( ax^2 + bx + c \), the linear term is partitioned into two components. The two components, \( p_1x \) and \( p_2x \), are factors of \( acx^2 \) whose sum is \( bx \). Two expressions can result from this partitioning: case 1, \( ax^2 + p_1x + p_2x + c \); and case 2, \( ax^2 + p_1x + p_2x + c \).

Each of the terms in these expressions can be found in the tic-tac-toe grid in row 1 and column 3.

<table>
<thead>
<tr>
<th>Factor by grouping</th>
<th>Tic-Tac-Toe Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1:</td>
<td>( ax^2 + bx + c ); ( p_1p_2 = ac )</td>
</tr>
<tr>
<td>a. ( ax^2 + bx + c )</td>
<td>( ax^2 )</td>
</tr>
<tr>
<td>b. ( ax^2 + p_1x + p_2x + c )</td>
<td>( c = UV )</td>
</tr>
<tr>
<td></td>
<td>( acx^2 )</td>
</tr>
</tbody>
</table>

The next step in factor by grouping is to factor out the gcf from the first two terms (i.e., \( ax^2 \) and \( p_1x \)) and the gcf from the last two terms (i.e., \( p_2x \) and \( c \)) of expression b. Let the gcf \( (ax^2, p_1x) = T \) and gcf \( (p_2x, c) = W \). When T is factored out of \( ax^2 + p_1x \), because \( ax^2 = TV \) and \( p_1x = TU \), the resulting expression is \( T(V + U) \). When W is factored out of \( p_2x + c \), because \( p_2x = VW \) and \( c = UW \), the resulting expression is
The new quadratic expression is expression c. The two terms in expression c both contain a factor of $V + W$. Thus, the $ax^2 + bx + c$ in completely factored form is $(T + W)(V + W)$.

Factor by grouping

<table>
<thead>
<tr>
<th>Case</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>$ax^2 + bx + c$</td>
</tr>
<tr>
<td>b.</td>
<td>$ax^2 + p_1x + p_2x + c$</td>
</tr>
<tr>
<td>c.</td>
<td>$T(V + U) + W(V + U)$, $T = \text{gcf}(ax^2, p_1x)$ and $W = \text{gcf}(p_2x, c)$</td>
</tr>
<tr>
<td>d.</td>
<td>$(T + W)(V + U)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Tic-Tac-Toe Method</th>
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<tbody>
<tr>
<td>$ax^2 + bx + c$; $p_1p_2 = ac$</td>
</tr>
<tr>
<td>$ax^2 = TV$</td>
</tr>
<tr>
<td>$p_1x$</td>
</tr>
<tr>
<td>$V$</td>
</tr>
</tbody>
</table>

Similarly, case 2, $ax^2 + p_2x + p_1x + c$ is factored using $V = \text{gcf}(ax^2, p_2x)$ and $U = \text{gcf}(p_1x, c)$ to $V(T + W) + U(T + W)$. This new expressions factors to $(V + U)(T + W)$. 
Appendix E

Documents Used to Gain Feedback on the Definitions of Constructs in the Conceptual Framework for Framing with the Expert Panel

Background information:

The conceptual framework (shown in Figure E-1) that guided this study’s data collection and data analysis consists of four constructs: framing, and the three mathematical teaching practices trimming, and bridging. The constructs of framing, decompressing, trimming, and bridging were observed only through participants’ overt use with students, or discussions of how they make targeted mathematical ideas (TMIds) accessible and comprehensible with the interviewer, or indirectly through examination of lesson handouts (e.g., worksheets, tests/quizzes). Emerging from analysis of observation and interview data are three quasi-mathematical teaching practices, quasi-decompressing, quasi-trimming, and quasi-bridging. Definitions of these constructs and the other four will be discussed in the next section. As a result of the emergence of the three new teaching practices, the conceptual framework was modified as shown in Figure E-2.
Figure E-1. Conceptual Framework for Framing (prior to data analysis).

Figure E-2. Conceptual Framework for Framing (After Data Analysis).
Definitions of constructs

Mathematics is the logical study of shape, arrangement, quantity and many related concepts (James and James, 1992). Mathematics is often divided into three fields: algebra, analysis, and geometry.

Targeted mathematical idea (TMId). For purposes of this study, mathematical ideas are defined to be the subject matter of school algebra as defined by Cooney, Davis, and Henderson (1975): algebraic facts, concepts, skills, generalizations, and procedures. Mathematical ideas are said to be targeted, if they are the focus of discussion or explanation during interviews or during classroom observations.

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These definitions were provided to the experts in Round 1. The definitions provided to experts in Rounds 2 and 3 differed only in the definitions of decompressing and quasi-decompressing. The definitions for decompressing and quasi-decompressing are shown in Table E-2.
### Table E-1. Definitions of Constructs Appearing in the Conceptual Framework for Framing (used in Round 1)

<table>
<thead>
<tr>
<th>Framing</th>
<th>Mathematical teaching practices</th>
<th>Quasi-mathematical teaching practices</th>
</tr>
</thead>
<tbody>
<tr>
<td>Framing is the adjustment, organization, staging and/or situating of one or more TMIds to support learning and to promote the development of learning of students.</td>
<td>Mathematical teaching practices are teaching practices that are, relative to the teaching of school algebra, characterized by the logical reasoning, exactness or precision of mathematics. The foci of mathematical teaching practices are mathematical objects. (Note: Nonmathematical teaching practices are teaching practices that are characterized by a lack of logical reasoning, exactness and mathematical precision. Nonmathematical teaching practices may not focus on mathematical objects.)</td>
<td>Quasi-mathematical teaching practices are teaching practices that reflect some of the qualities of mathematical teaching practices, but are also characterized by the use of nonmathematical elements or ideas. (Note: The prefix, quasi- means seemingly; apparently, but not really; being partly or almost.)</td>
</tr>
<tr>
<td>Decompressing is a mathematical teaching practice in which a TMId (e.g., concept, skill, generalization) is broken down into components/parts. The breaking down of a TMId is usually done directly for students by the teacher.</td>
<td>Quasi-decompressing is a teaching practice in which (1) a TMId is broken down into components indirectly through the use of some kind of graphic organizer, or directly through some other, sometimes mathematically questionable means; or (2) components of a decompressed TMId (often procedures) are organized so that students’ are better able to work with, identify or use them.</td>
<td></td>
</tr>
<tr>
<td>Trimming is a mathematical teaching practice in which some subset of component ideas/aspects of a TMId is addressed or emphasized. Included in the subset are the core ideas of the TMId. A teacher may engage in trimming a TMId indirectly or directly. When a teacher uses indirect methods to trim a TMId, he or she lessens or removes distractions from the TMId. When a teacher uses direct methods to trim a TMId, the teacher’s efforts focus students’ attentions on the TMId.</td>
<td>Quasi-trimming is a teaching practice in which a teacher brings students’ attentions to some subset of component ideas/aspects of a TMId in ways that are not characterized by the exactness or precision of mathematics. These methods are often practical and/or imaginative as opposed to abstract and logical. Similar to trimming, a teacher may engage in quasi-trimming indirectly or directly.</td>
<td></td>
</tr>
<tr>
<td>Bridging is a mathematical teaching practice in which one or more connections are made between a TMId and other mathematical ideas.</td>
<td>Quasi-bridging is a teaching practice in which connections are made between a TMId and nonmathematical idea(s).</td>
<td></td>
</tr>
</tbody>
</table>
Table E-2. Definitions of decompressing and quasi-decompressing used in Rounds 2 and 3.

<table>
<thead>
<tr>
<th>Decompressing</th>
<th>Quasi-decompressing</th>
</tr>
</thead>
<tbody>
<tr>
<td>is a mathematical teaching practice in which a TMId (e.g., concept, skill, generalization) is broken down into components. The breaking down of a TMId is usually done directly for students by the teacher.</td>
<td>is a teaching practice in which (1) a TMId is broken down into components indirectly through the use of some kind of graphic organizer or directly through the some other, sometimes mathematically questionable, means; or (2) components of a decompressed TMId (often procedures) are organized so that students are better able to work with, identify, or use them.</td>
</tr>
</tbody>
</table>

**Framings given to members of the expert panel for coding**

What follows are the 12 examples of framing. These examples were presented, as is, for members of the panel to code based upon the provided definitions.
In teaching her students how to solve quadratic equations by completing the square, Faith stated that she began teaching the procedure by using examples with “nice numbers.” When asked what she means by “nice numbers,” Faith said, well starting off where there's no um $a$ term—well, it's $a$ term [sic] is one. Um and then where $b$ can be divided in half nicely as opposed to like a $b$ term of three… I'd like to give them numbers where the numbers are going to [pause] avoid fractions and that sort of–uh situation.” (3 Interview).
This example of framing is set in the context of graphing \( y = \frac{-1}{2} \left( \frac{1}{2} \right)^{-1} + 2 \). More specifically, Hildi and her students had just begun to create a table of values starting with \( x = 1 \), and were discussing the value of \( y = \frac{-1}{2} \cdot \frac{2}{1} + 2 \).

Hildi said, “Now this is the part that—especially like third period, they had a lot of trouble remembering what to do when there’s a negative exponent. What is one-half to the negative one?” (Observation 1).

One student answered, “one-fourth” (Hildi, Observation 1) and another asked, “Is it negative one-fourth?” (Hildi, Observation 1). Without acknowledging students’ responses, Hildi said,

It’s two. [pause] Right? Because the negative means flip it [as she uses her cursor to point at the -1 exponent][pause] Right? So we flip it and raise it to that power. So one half [to the negative one] would be [Hildi then silently finishes recording the “\( \frac{2}{1} + 2 \)” in \( y = \frac{-1}{2} \cdot \frac{2}{1} + 2 \)]. (Observation 1)

Hildi used her framing of negative exponent as a set of instructions “flip it and raise it to that power” in a slightly modified form when she went on to compute the value of

\[ y = \frac{-1}{2} \left( \frac{1}{2} \right)^{-1} + 2 \text{ for } x = -1. \] She told her students,
Well once you flip it though, it’s not a half any more, right? Yeah. So either multiply by itself and then flip it, or flip it multiply it by itself. As long as it gets both flipped and multiplied by itself (Hildi, Observation 1).
Framing

Teacher: Bonnie

TMId: graphing linear inequalities on the rectangular coordinate plane

One of the problem-solving interview tasks asked Bonnie to respond to a classroom scenario in which a hypothetical student employed a problematic approach to graphing $2x + 3 < y$. During the “warm-up” section of the interview I asked Bonnie to graph $2x + 3 < y$ to give her a chance to do the task before I asked her to respond to the hypothetical student’s work. It was during this part of the interview that Bonnie shared what she tells her students about graphing linear inequalities. Bonnie uses an analogy between graphing a linear inequality on the rectangular coordinate plane (the TMId, i.e., $x + 6 < y$) and graphing a linear inequality on the number line (i.e., $x + 6 < 4$). She said,

> We start with equations and we have usually one answer [as she records $x + 6 = 4$ and $x = 2$] and then we go to- [pause]. Let’s say inequalities [as she records $x + 6 < 4$]. And so then we have more answers. So we graph our answer on a number line because we have more answers [as she draws an open-ended ray on a line]. And then we go to $x$ plus six equals $y$ [as she records, $x + 6 = y$], and we have two things that vary. We have more than one answer that will make this true and those all end up on a straight line [as she creates graph under $x + 6 = y$]. And so in an inequality it is like going from a point [as she points at $x = -2$] to half of a line [as she traces the ray she graphed on the number line]. From here we are going from the line [as she traces the line graph of $x + 6 = y$] to one half[-plane] or the other [as she places a hand on either side of the graph of $x + 6 = y$]. So trying to sort of show them that pattern. Here we went from a single point [as she points at $x = -2$] now we’ve got half the number line. Here we have a line [as she traces
Bonnie’s recorded work for her discussion is captured in Figure E-3.

*Figure E-3.* Screen capture of Bonnie’s written work created as she connected graphing an inequality on the number line with graphing an inequality on the Cartesian coordinate plane.
Hildi uses a metaphor to connect the concept of function to an exclusive social event. According to Hildi, a function is like a party to which she gives only her nonscandalous friends invitations. A nonscandalous friend is a friend who brings only one date to a party and a scandalous friend is a friend who would bring more than one date. It is important to note here, that Hildi invites only her friends to the party. Her friends’ dates are welcome to attend, but the invitation goes only to her friends. According to Hildi, “you don’t want to invite scandalous people because then you can’t have a function because your function will go awry” (Ending Interview).

As Hildi told her story, she listed a set of ordered pairs, identified the \( x \)-coordinates as “friends” and the \( y \)-coordinate as “dates,” and drew a diagram in which each “friend” has exactly one “date.” (See Figure E-4.) She said,

Three is only dating one person. Six is only dating one person. Five is only dating one person. Negative one is only dating one person. So none of your friends are scandalous. Now four is scandalous, but you don’t really care because you’re just concerned about your own friends” (Ending Interview).

Figure E-4. Screen capture of Hildi’s written work captured as she talked about her metaphor for the concept of function.

Hildi’s framing of the concept of function also involves discussing and drawing a diagram of a representation of a scandalous friend. For the set shown in Figure E-5, Hildi says,
Three is dating five. Three is dating negative one. Six is dating four. Seven is
dating three. You cannot invite three [as she points to the 3 in the x-column] to your party because they’re scandalous. They’re dating two people and if you have scandalous friends your function is not going to be fun. (Ending Interview).

Figure E-5. Screen capture of Hildi’s written work as she talked about a nonexample of a function in the context of her metaphor for the concept of function.
Framing E
Teacher: Gary
TMIId: derivation of the quadratic formula

Gary used PowerPoint to present the derivation of the quadratic formula. Figures 9 and 10 identify the seven major steps in his derivation. As implemented, however, the seven steps each contained two to four substeps. Gary programmed each slide to reveal each substep with a click of his mouse.
Figure E-6. Slide sorter view (slides 1 – 6) of Gary’s PowerPoint presentation of the derivation of the quadratic formula (Observation 1).
Figure E-7. Slide sorter view (slides 7 – 9) of Gary’s PowerPoint presentation of the derivation of the quadratic formula (Observation 1).
Framing F
Teacher: Bonnie
TMId: multiplying polynomials

Bonnie uses a graphic organizer, a lattice, to teach their students how to multiply polynomials. What follows is a discussion of Bonnie’s framing of how to expand the product, \((4p - 3)(3p^2 - p + 2)\). Originally this problem was a quiz question. Bonnie showed the answer to this problem on her overhead projector, but since no student requested an explanation, she did not go over it in class. During the Postobservation 2 Interview, I asked her to talk about her solution method. Bonnie said,

Well sometimes kids get—or lose things or miss things or forget to multiply or something. So I found this, where they did this—sort of, I guess it’s sort of like lattice in a way, but you put one on one direction and one on the other direction [see Figure 2-26]. …You’re taking four \(p\) times three \(p\) to the third and getting your twelve \(p\) to the fourth. And you’re taking four \(p\) times negative \(p\) and getting negative four \(p\) squared. And you’re getting eight \(p\). So you’re doing the same thing that you did here [as she points to \((4p - 3)(3p^2 - p + 2)\)] but you’re just kind of organized. They can make sure they have something in every box.

(Postobservation 2 Interview)

Figure E-8 consists of reproductions of different stages of what was captured on video as she discussed the method. Video quality did not allow use of screen captures.
Figure E-8. Reproduction of stages of Bonnie’s work captured on video as she discussed how to use a lattice to expand \((4p - 3)(3p^2 - p + 2)\) (Postobservation 2 Interview).
Framing G
Teacher: Kate
TMId: multiplying binomials

Kate frames how to multiply two binomials using an area metaphor and tool she calls, “generic rectangles” (Postobservation 1 Interview). The binomials represent dimensions of a generic rectangle and the resulting trinomial represents the area of the generic rectangle. Explaining how she frames multiplying binomials, Kate said,

So if I wanted to multiply $x$ minus three, times, $x$ plus five [as she records, $(x-3)(x+5)$], we start with just positives—because area is positive. Then we work into this generic notation of [as she traces the outline of the largest rectangle]—this is going to represent the area. And therefore if I am trying to multiply two things like $x$ minus three and $x$ plus five, I can find the area of each piece, and add them together to get the area of the whole thing. And that is how we multiply. (Postobservation 1 Interview)

Figure E-9. Screen capture of Kate’s work as she explained how she teaches students how to multiply binomials.
**Framing H**

**Teacher**: Delia

**TMId**: multiplying binomials

The focus of Delia’s lesson for Observation 3 was multiplying pairs of linear binomials. Delia’s board work, as it evolved, is captured in Figure E-10. With each practice problem, as she guided students through the procedure, Delia repeatedly used the terms distribute or distributive property.

![Figure E-10](image_url)

*Figure E-10.* Screen captures of Delia’s board work taken during her explanation of how to multiply $(x + 4)(x + 3)$ (Observation 3).
Framing I
Teacher: Faith
TMId: graphing functions

Faith requires that her students take notes using a special kind of stationery to help her focus Algebra 1 students’ attentions on graphing all functions. Figure 21 is a screen capture of one page of notes Faith produced during Observation 1 in a lesson on graphing quadratic functions. (Note: Please focus on the use of the stationery not on the specifics of what is written in the sample.)
Figure E-11. Screen capture of Faith's notes presented for graphing quadratic functions during Observation 1. The handwritten sections were completed as the lesson progressed.
**Framing J**

**Teacher:** Bonnie

**TMId:** simplifying radical expressions

What follows is a section of Bonnie’s Observation 1 transcript that captures her explanation of how to simplify $\frac{\sqrt{14}}{\sqrt{35}}$. The transcript shown in figure 11 is divided into sections using horizontal lines.

<table>
<thead>
<tr>
<th>Dialogue</th>
<th>Bonnie’s recorded work</th>
</tr>
</thead>
</table>
| **Step 1.** | Student: Can you do number thirty-four?  
Bonnie: Did you say thirty-four? It says divide the square root of fourteen by the square root of thirty five. So there’s how you might start the problem. |
| **Step 2.** | Remember you have that property that says you could put them under the same radical if you wanted |
| **Step 3.** | And then you could reduce the fraction. So if I use seven to reduce with I’d get two over five. … |
| **Step 4.** | Remember when you work with radicals you can’t leave radicals in the bottom of a fraction. So you have to rationalize—you need to turn that five into a perfect square. So what could we multiply y that would make it a perfect square?  
Student: Um, five?  
Bonnie: Square root of five [as she records $\sqrt{5}$ next to the 5 in $\frac{2}{5}$]. |
| **Step 5.** | And what we do to the bottom, we have do to the top [as she records $\sqrt{5}$ next to the $\sqrt{2}$ in $\frac{2}{5}$] because we’re really just multiplying by one. |
| **Step 6.** | So this becomes the square root of ten [as she records $\sqrt{10}$] |
| **Step 7.** | This becomes the square root of twenty-five [as she records $\sqrt{25}$] |
| **Step 8.** | which is just five [as she records 5 below $\sqrt{10}$]. |
Step 9. Now we can’t reduce the ten and the five because one is under the radical and one is not. So remember, you can only work with things under the radical or things outside the radical.

Figure J-1. Bonnie's explanation of homework problem #34 (Observation 1).
Bonnie tells her students that division by zero is “a math sin” (PS Interview).
Ann factors quadratic trinomials using a procedure she calls, the “Secret Method.”

Using $3x^2 + 4x + 1$, Ann explained the procedure as follows:

The idea is to take that coefficient in front and multiply it by the last term. And we rewrite the equation without that three or that coefficient in front. Uh, $x$ squared plus four $x$ plus three. And then it becomes a trinomial that they’re—they like—to be able to factor. [See Equation 2 in figure 28.] … So when we factor this, we get an $x$ plus one and an $x$ plus three. [See Equation 3 in figure 28.]

(Postobservation 3 Interview).

Figure E-12. Screen capture of Ann’s factoring using the “Secret Method.”

(Postobservation 3 Interview). (Note: Equation numbers were added for reference.)

Although Ann has received complaints from the other teachers in her department for teaching this method, Ann continues to teach the method. In response to the complaints, she gives her students a certain amount of time to master the Secret Method. If her students do not demonstrate mastery, she tells them that they are not allowed to use
the method (Postobservation 3 Interview). According to Ann, “And so, um, that is why we call it the Secret Method (Postobservation 3 Interview). To those students whom Ann bans from using the Secret Method Ann says, “You are back to, you know, working out factors and guess and check and looking to see if this is the prime number and the only factors are three \( x \) and \( x \)” (Postobservation 3 Interview).
## Appendix F

**Instances of Framing Coded by Teaching Practice(s) Used and Sorted by Teacher**

Table F. Framings Coded by Teaching Practice(s)

<table>
<thead>
<tr>
<th>Framing Description</th>
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