SOLUTION OF THE LOSSY NONLINEAR TRICOMI EQUATION WITH
APPLICATION TO SONIC BOOM FOCUSING

A Dissertation in
Acoustics
By
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Abstract

Sonic boom focusing theory has been augmented with new terms that account for mean flow effects in the direction of propagation and also for atmospheric absorption/dispersion due to molecular relaxation due to oxygen and nitrogen. The newly derived model equation was numerically implemented using a computer code. The computer code was numerically validated using a spectral solution for nonlinear propagation of a sinusoid through a lossy homogeneous medium. An additional numerical check was performed to verify the linear diffraction component of the code calculations. The computer code was experimentally validated using measured sonic boom focusing data from the NASA sponsored Superboom Caustic and Analysis Measurement Program (SCAMP) flight test. The computer code was in good agreement with both the numerical and experimental validation. The newly developed code was applied to examine the focusing of a NASA low-boom demonstration vehicle concept. The resulting pressure field was calculated for several supersonic climb profiles. The shaping efforts designed into the signatures were still somewhat evident despite the effects of sonic boom focusing.
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List of Symbols

\( A_n \) absorption coefficient from the spectral solution used in the numerical validation

\( B_n \) coefficient used in the splitting method for the absorption and x-wind terms

\( B \)
\( A \) parameter of nonlinearity

\( c \) (total) speed of sound [m/s]

\( c_0 \) ambient speed of sound [m/s]

\( c_{\infty,v} \) frozen speed of sound for the \( v \)-th relaxation component [m/s]

\( C_{p,v} \) specific heat at constant pressure for the \( v \)-th relaxation component [J/(kg-K)]

\( C_{v,v} \) specific heat at constant volume for the \( v \)-th relaxation component [J/(kg-K)]

\( D_n \) dispersion coefficient from the spectral solution used in the numerical validation

\( f_{ac} \) characteristic acoustic frequency [Hz]

\( f_0 \) fundamental frequency [Hz]

\( F \) incoming acoustic waveform

\( \tilde{F}_n \) \( n \)-th harmonic of the Fourier transform of the incoming waveform, \( F \)

\( G \) outgoing acoustic waveform

\( i \) \( \sqrt{-1} \)

\( L \) Lagrangian density [kg/(m\cdot s^2)]

\( m_v \) dispersion parameter for the \( v \)-th relaxation component

\( M_{ac} \) characteristic acoustic Mach number
$M_X$  ratio of the wind speed, $v_{0x}$, in the $x$-direction to the ambient speed of sound

$M_Z$  ratio of the wind speed, $v_{0z}$, in the $z$-direction to the ambient speed of sound

$p$  pressure [Pa]

$p'$  acoustic pressure perturbation [Pa]

$p_{ac}$  characteristic acoustic pressure [Pa]

$\hat{p}_n$  $n$-th harmonic of the Fourier transform of the pressure field at a particular $z$-coordinate [Pa]

$\tilde{p}_n$  $n$-th harmonic of the complex pressure for the spectral solution used in the numerical validation

$p_0$  ambient pressure [Pa]

$Pr$  Prandtl number

$R_{cel}$  radius of curvature of the ray tangent to the caustic due to the local (effective) speed of sound gradient in the $z$-direction [m]

$R_{den}$  characteristic distance due to the local density gradient in the $z$-direction [m]

$R_{rot}$  relative radius of curvature between the radius of curvature for the ray tangent to the caustic and the radius of curvature of the line defined as the intersection of the caustic surface and the plane directed by both the tangent ray and the caustic normal [m]

$R_{XZ}$  radius of curvature of the line defined as the intersection of the caustic surface and the plane directed by both the tangent ray and the caustic normal [m]

$s$  entropy per unit mass [J/(kg-K)]

$s_{fr}$  frozen entropy per unit mass [J/(kg-K)]

$s'_{fr}$  acoustic entropy perturbation per unit mass [J/(kg-K)]

$s_{fr,0}$  ambient frozen entropy per unit mass [J/(kg-K)]

$t$  time [s]

$\tilde{t}$  dimensionless retarded time

$T$  temperature [K]

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\(T'\) acoustic temperature perturbation [K]
\(T_0\) ambient temperature [K]
\(T'_v\) vibrational temperature associated with the \(v\)-th relaxation component [K]
\(T'_v'\) acoustic vibrational temperature perturbation associated with the \(v\)-th relaxation component [K]
\(\overline{v}\) fluid velocity vector [m/s]
\(\overline{v}'\) acoustic velocity perturbation vector [m/s]
\(\overline{v}_0\) ambient fluid velocity vector [m/s]
\(v_{0x}\) component of wind in the \(x\)-direction to the caustic [m/s]
\(v_{0z}\) component of wind in the direction normal to the caustic [m/s]
\(x\) distance from the caustic in the direction tangent to the caustic [m]
\(\overline{x}\) dimensionless distance from the caustic in the direction tangent to the caustic
\(y\) distance from the caustic in the direction laterally away from the caustic [m]
\(z\) distance from the caustic in the direction normal to the caustic [m]
\(\overline{z}\) dimensionless distance from the caustic in the direction normal to the caustic
\(\overline{\alpha}\) dimensionless absorption coefficient
\(\beta\) coefficient of nonlinearity
\(\gamma\) specific heat ratio
\(\delta\) diffusivity of sound [m\(^2\)/s]
\(\delta_D\) diffraction boundary layer thickness [m]
\(\Delta\) discretized step size or increment
\(\varepsilon\) dimensionless diffraction parameter
\(\xi\) argument of the Airy function
\(\eta\) ratio of the nonlinear effects relative to the diffraction effects
维度化扩散系数对于第 ν 个松弛成分

κ 热导率 [W/(m-K)]

λ_{ac} 特征声波长 [m]

μ 剪切粘度 [kg/(m-s)]

μ_B 块粘度 [kg/(m-s)]

ν 动态粘度 [m^2/s]

ρ 密度 [kg/m^3]

ρ' 声密 [kg/m^3]

ρ_0 环境密度 [kg/m^3]

σ 假时间变量

τ_ν 松弛时间对于第 ν 个松弛成分 [s]

τ_ν 维度化松弛时间对于第 ν 个松弛成分

ω_n 第 n 个谐波的维度化角度频率

\bar{ω}_n 维度化角度频率在第 n 个谐波用于验证的谱解决方案

∇ 梯度算子

Subscripts

0 环境量

1 第二个位置在该特定轴的计算域

2 第三个位置在该特定轴的计算域

ac 特征声学量

B 块

cel 与声速梯度相关

den 与密度梯度相关
\( fr \)  frozen state

\( j \)  index associated with the \( \bar{Z} \) axis position in the computational grid

\( m \)  index of harmonic number

\( max \)  maximum \( \bar{Z} \) or \( \bar{t} \) location in the computational grid

\( min \)  minimum \( \bar{Z} \) or \( \bar{t} \) location in the computational grid

\( M \)  index of the maximum \( \bar{Z} \) location

\( n \)  harmonic number

\( NL \)  nonlinear

\( p \)  subscript for specific heat at constant pressure

\( ray \)  associated with the ray propagation path

\( TV \)  thermoviscous

\( tot \)  associated with the total relative radius of curvature

\( v \)  \( v \)-th relaxation component, subscript for specific heat at constant volume

\( X \)  associated with the direction tangent to the caustic

\( XZ \)  associated with the plane created by both the ray tangent to the caustic and the caustic normal

\( Z \)  associated with the direction normal to the caustic

\( \infty \)  frozen speed of sound subscript

**Superscripts**

\( ' \)  perturbed acoustic quantity

\( k \)  pesudotime iteration step index
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Chapter 1

Introduction

1.1 Background

Knowledge of the focusing of wave phenomena has existed for several decades. First developed were caustic corrections to ray theory [1], and later catastrophe theory [2], both of which have been applied to understand the physics and mathematics of focused waves in several disciplines: electromagnetic waves [3] underwater acoustics [1] and optics [4] to only name a few [5]. And, with Chuck Yaeger breaking the sound barrier in the Bell X-1 on October 14, 1947, a new acoustic source for the application sound wave focusing was born years before the physics of sonic booms were understood. The advancements of aircraft and propulsion technology marked the advent of a new era in acoustic sound sources. Supersonic aircraft allowed people to hear sonic booms and also the amplification of the sonic boom due to the focusing effects associated with acceleration of a vehicle through and past Mach 1.

Aspects of sound propagation combined with diffraction theory to develop what is now an important part of sonic boom theory. The nonlinear Tricomi equation, derived by Guiraud [6], Gill and Seebass [7] and Hayes [8], describes the pressure field in the vicinity of a caustic and provides us insight into the physics surrounding sonic boom focusing. The Tricomi equation is named after Francesco Giacomo Tricomi, an Italian mathematician well known for his work in mixed-type partial differential equations [9]. He was born on May 5, 1897 in Naples, Italy, and died on November 21, 1978 in Turin, Italy. He graduated from the University of Naples in 1918. In 1923 he published his famous paper that described a partial differential equation of the form:
\[ u_{xx} - xu_{yy} = 0 \]  \hspace{1cm} (1.1.1)

Equation 1.1.1 became known as the “Tricomi equation.” He was also involved in studies pertaining to integral equations, differential operators and functional transforms to only name a few. His lifetime of impressive research resulted in over 300 papers. As it turned out, Tricomi was ahead of his time, publishing work that would eventually be used, in part, towards sonic boom focusing prior to the invention of supersonic aircraft.

The typical noise footprint created by a supersonic aircraft can potentially cover a significant area on the ground (Figure 1.1) [10]. As the aircraft passes through Mach 1 and beyond Mach cutoff, the primary sonic boom carpet begins with a caustic that intercepts the ground. The cutoff Mach number (in a windless atmosphere) is defined as the ratio of the speed of sound at the ground to the speed of sound at the flight altitude. For an aircraft flying above the tropopause (11,000 m) in the standard atmosphere [11], this corresponds to a Mach number of 1.15 [12]. However, the Mach cutoff can vary due to changes in atmospheric conditions and the orientation of the aircraft flight path with respect to the wind direction [13]. The carpet widens until the aircraft reaches its cruise segment of flight. The focus boom exists along the caustic line that intercepts the ground. A shadow zone exists prior to the caustic line and outside the lateral edges of the primary sonic boom carpet. Sound penetrates into the shadow zone through scattering and diffraction. The secondary sonic boom carpet occurs due to atmospheric refraction. Sonic booms propagate from the top of the supersonic vehicle and refract downward to the ground. They also reflect from the ground due to the primary carpet booms, propagate upwards and are refracted downward to the ground. The size and extent of all of these regions is determined by the altitude and Mach number of the vehicle and also the atmospheric conditions.
Figure 1.1 - Depiction of a typical sonic boom carpet exposure at the ground [10].

Figure 1.2 - Nominal primary sonic boom carpet for a New York – San Francisco flight through a winter seasonal average atmosphere [14].
As an example, Figure 1.2 shows an estimate for the primary sonic boom footprint for a potential route between New York City, NY and San Francisco, CA. There are two contours shown, light blue for a westerly heading and dark blue for an easterly heading. The atmosphere used for determining the estimate was comprised of winter average atmospheres for approximately 90 locations across the continental United States [14]. The winter average atmosphere was chosen because it provided the widest primary carpet out of all four seasons. The nominal width is approximately 90 km for the primary sonic boom carpet in Figure 1.2. The cruise Mach number was 1.8 and the cruise altitude was 51,000 ft (15,545 m).

As mentioned above, an aircraft accelerating from subsonic to supersonic speeds will create a focus boom. The depiction in Figure 1.3 is of an aircraft accelerating (in the standard atmosphere) from Mach 0.96 to Mach 1.35 at a Mach rate of 0.004 Mach/s at a constant altitude of 45,000 ft (13,715 m). Acoustic rays (blue lines) propagate away from the Mach cone perpendicular to the cone surface and form a wavefront (black line). Once the aircraft is supersonic, shocks develop near the aircraft and form a Mach cone. As the aircraft accelerates, the Mach cone angle becomes smaller due to the increase in Mach number. The continuum of rays that comprise the wavefront converge as the wavefront propagates to the ground. There is amplification of the acoustic waveform that results from these rays that converge prior to when they intercept. In the literature, this has been quantified in terms of the Blokhintsev invariant scaling factor, defined as [15]:

$$B = \sqrt{\frac{(\rho c)A_{\text{ref}}}{(\rho c)_{\text{ref}} A}}$$  \hspace{1cm} (1.1.2)

where $A$ is the ray tube area, $\rho$ is the ambient air density, $c$ is the equilibrium speed of sound, and $\text{ref}$ denotes a reference quantity at a fixed reference location corresponding to the initial propagation of the acoustic source. The version presented above in Eq. (1.1.2) is the expression for a windless atmosphere. Once the rays converge and intersect the ray tube area reaches zero and a caustic forms in the wavefront. A caustic is not a physical process but merely a mathematical byproduct of the ray acoustics formulation for
determining the wavefront amplitude. Observe in Figure 1.3 that above the caustic (dashed red line) two rays exist at any location in the illuminated zone. Below the caustic in the shadow zone no rays are present. Ray acoustics predicts infinite pressure amplitude where the rays intersect and therefore is not suitable for determining the amplitude of the wavefront at the location where focusing occurs. Diffraction effects must be included to predict a finite pressure amplitude in the neighborhood of the focusing region, both in the illuminated and shadow zone.

![Figure 1.3 - Illustration of the rays (blue lines) that create a wavefront (black line) due to acceleration of an aircraft from subsonic to supersonic flight speeds in the standard atmosphere. Above the caustic (red dashed line) is the illuminated zone and below the caustic is the shadow zone.](image)

The focus boom created at the caustic intercept to the ground is typically the loudest component of the sonic boom carpet. Hence, quantifying the amplification that occurs due to focusing is important for understanding the impact of supersonic operations. The diagram in Figure 1.4 shows a typical sonic boom focusing prediction process. The prediction begins with the near-field pressure waveform that serves as the initial acoustic disturbance. The near-field pressure is calculated or adjusted for the aircraft altitude and Mach number and the atmospheric conditions. The near-field signature is propagated along the ray path at the initial aircraft trajectory information (altitude, Mach, Mach rate, etc.) and atmospheric conditions. The far-field propagation models account for
nonlinearity, atmospheric stratification, geometric spreading and atmospheric absorption and dispersion [16-19]. Mean flow atmospheric effects are accounted for in [16, 18, 19]. For an accelerating aircraft, focusing of the rays will eventually occur at the ground. The far-field propagation model will cease its calculations at some point prior to the onset of the focusing where the diffraction effects are dominant. The acoustic waveform for the far-field propagation will then serve as the input waveform for some type of focus boom prediction model. A brief overview will now be presented that provides a summary of the previous work of others regarding their focus boom prediction models and their approach to obtaining a finite pressure amplitude in the focusing region.

Figure 1.4 – Schematic of a typical sonic boom focusing prediction process.

1.2 Previous Work

1.2.1 Plotkin and Cantril, “Prediction of Sonic Boom at a Focus” (1976)
Plotkin and Cantril [20] discuss in detail the three-dimensional geometry associated with sonic boom focusing. They also discuss the definition of a caustic, the types of caustics and ways they can form as a result of a maneuvering supersonic aircraft. Diffraction theory is included in their work, both from an analytical perspective and through numerical implementation in a computer program. The authors provided a brief
comparison of previous sonic boom theory predictions to flight test data that was provided by Wanner et al. [21]. They also provide the results from sample calculations of their computer program for sonic boom focusing. The authors cite three definitions of a caustic: 1) the ray tube area formed by infinitesimally adjacent rays goes to zero. 2) adjacent rays converge and intersect. 3) the caustic is actually a three-dimensional surface formed by the ensemble of rays that occur due to the first two definitions.

Figure 1.5 – Illustration of three types of caustics caused by supersonic flight operations (from [20]): (a) the caustic formed by a turn, (b) the caustic formed by an acceleration and (c) the caustic formed from flight below Mach cutoff.

The authors describe the common types of caustics formed by typical maneuvers of a supersonic aircraft (Figure 1.5). For a supersonic turn, the rays on the inside of the turn
focus and form a caustic. The rays on the outside of the turn diverge and do not form a caustic. For a supersonic acceleration, rays generated at a later instant in time catch up and overtake rays generated from an earlier instant in time forming a caustic surface. Supersonic flight below the cutoff Mach number in an atmosphere with a sound speed that decreases with an increase in altitude creates a caustic surface at some location above the ground. The rays emanating from a supersonic aircraft travelling slower than Mach cutoff will refract upward before reaching the ground.

Plotkin and Cantril provide analytical expressions for the focusing geometry in a plane of symmetry valid for the zero-degree azimuth. For the two-dimensional case, they derive the equation for the curvature of the caustic. The authors cite that a challenge to their approach is when the rays focus horizontally leading to the radius of curvature of the caustic is infinity due to their choice of coordinate system. However, the “challenge” mentioned by the authors is not a real one since they could reformulate their expressions with a localized coordinate system with the ray tangential and ray normal directions. They also mathematically define the three-dimensional focus condition. Rays are represented as a function of azimuth angle and position. The caustic conditions are given in terms of parametric relations that can mathematically describe the caustic surface comprised of the ensemble of adjacent rays which cross. That is, the authors developed analytical formulas to determine the caustic location in three dimensions.

The authors make the connection, for the linear case, between the Tricomi equation and various focusing conditions similar to Eq. (1.1.1) except the authors use the acoustic potential instead of pressure and account for the curvature associated with the speed of sound gradient.

\[ \Phi_{zz} + \frac{z}{R} \Phi_{xx} = 0 \]  

(1.2.1)

The benefit to using a formulation like Eq. (1.2.1) is that it can be used for curved rays and a “straight” caustic line, a straight ray and a curved caustic line or both curved rays
and caustic lines. The value of $R$ is the total relative radius of curvature shown by the authors as:

$$\frac{1}{R} = \frac{1}{R_c} - \frac{1}{R_r} \quad (1.2.2)$$

where $R_r$ is the radius of curvature of the ray propagation path and $R_c$ is the radius of curvature of the caustic surface as it intersects the plane that shares the caustic normal vector and the vector corresponding to the ray that is tangent to the caustic. The solution to Eq. (1.2.1) is given by Airy functions [22] and is solved for each frequency. The formulation provides a smooth transition from the illuminated zone to the shadow zone and can depict the linear diffraction behavior in the vicinity of a caustic.

The authors also include a discussion of the nonlinear effects and distortion of a wavefront that occurs as a result of sonic boom focusing. The mathematical incorporation of these effects results in a nonlinear Tricomi equation. The authors show this from Gill and Seebass [7] as

$$\left( y + \varphi_x \right) \varphi_{xx} - \varphi_{yy} = 0 \quad (1.2.3)$$

where they use the velocity potential, $\varphi$, instead of pressure, the $x$-direction is the direction of the propagation and the $y$-direction is normal to the $x$-direction. Equation (1.2.3) was obtained by a series of variable transformations that account for the local geometry in the vicinity of the caustic and the assumption that the local speed of sound follows a linear sound speed gradient.

Guiraud [6] and Gill and Seebass [7] had solved Eq. (1.2.3) for the case when the incoming waveform is a step shock. They use the scaling law derived by Guiraud [6] which provides a relationship between the peak pressure and its corresponding focus location so that they scale in terms of the strength of a nonlinearity parameter. The nonlinear parameter is a ratio between the nonlinear and diffraction effects. The
pressure, time and space variables were scaled by constants so that the solution to the problem is invariant. The authors incorporated the solution by Gill and Seebass into a focus boom prediction code. They apply the solution for focusing at a step shock to each shock in an N-wave and for the portion of the waveform after the shock they subtract the deviation of the expansion portion of the waveform from the step shock value. Using this method, the authors are able to provide a time-history prediction of an N-wave and are not limited to only predicting the peak focusing amplitude.

The authors compare computed predictions to measured data from ballistic projectiles and flight test data. The first comparison used data collected from an experiment where a projectile was fired into a localized gradient of carbon dioxide to refract the propagation path of the shock waves. They compare the peak pressures of a prediction to an observation made from a microphone measurement. There is good agreement (after the authors corrected calculations made by the references they obtained the measurement data from) considering the challenge of capturing the “actual” peak focus location in the pressure field. The second comparison used flight test data by Wanner et al. [21] from a Mirage aircraft. The authors compared the peak pressures in a series of microphone locations to the power law decay relative to the caustic location. There was very good agreement in the vicinity of the caustic as well as at the peak pressure location. The authors cite that better agreement for the prediction was achieved by accounting for the reduction in amplitude as a result of the finite thickness of the shock.

1.2.2 Auger and Coulouvrat, “Numerical Simulation of Sonic Boom Focusing” (2002)

Auger and Coulouvrat [23] and Auger [24] developed a different approach to sonic boom focus prediction. They provided a formulation of the nonlinear Tricomi equation and steps of a pseudospectral algorithm for the corresponding numerical implementation. They also provided examples of numerical validations and convergence of the algorithm. A benefit of their algorithm is that it is not limited to N-waves but is applicable to any type of waveform, even waveforms with multiple shocks that are in close enough
proximity to interact with each other. To demonstrate this, the authors conducted numerical simulations of Concorde sonic boom focusing.

The authors present a form of the nonlinear Tricomi equation that models the field in the vicinity of the caustic in terms of the acoustic pressure.

\[
\frac{\partial^2 \bar{P}}{\partial \bar{z}^2} - \bar{z} \frac{\partial^2 \bar{P}}{\partial \bar{t}^2} + \frac{\eta}{2} \frac{\partial^2 \bar{P}^2}{\partial \bar{t}^2} = 0
\]  

(1.2.4)

where \( \eta \) is a ratio of the nonlinear effects relative to the diffraction effects, \( \bar{P} \) is the dimensionless pressure, \( \bar{z} \) is the dimensionless distance from the caustic, and \( \bar{t} \) is the dimensionless time. Their depiction of the focused pressure field is shown in Figure 1.6. A value of \( \bar{z} = 1 \) corresponds to a distance of one diffraction boundary thickness and serves as upper boundary for specifying the incoming waveform. Note that in their depiction an incoming waveform is shaped like an “N” while the outgoing waveform has a “U” shape. For the linear case of Eq. (1.2.4), the classical “N” shape of the sonic boom (referred to as an “N-wave”) is modified into a “U” (hence, referred to as a “U-wave”) as a result of the Hilbert transform. The derivation for Eq. (1.2.4) began with determining an eikonal function from evaluating the propagation through the caustic in terms of the caustic and ray geometry. In Auger [24], the eikonal function was a rigorous derivation that included the influence of winds in determining the total radius of curvature. The curvilinear coordinates allow the rays to “appear the same way” to the caustic as they propagate through it. That is, mathematically the formulation makes curved rays with a straight caustic, straight rays and a curved caustic, or both curved rays and a curved caustic to all be evaluated in a similar mathematical manner in the vicinity of the caustic. The authors provide boundary conditions for Eq. (1.2.4) that correspond to the pressure field shown in Figure 1.6. Their formulation matches geometric acoustics on the illuminated side, models diffraction in the vicinity of the caustic and the evanescent decay in the shadow zone. Thus, their approach computes more than just the solution at the peak pressure location. It provides a full time-history of the focused pressure waveform as a function of the distance away from the caustic. Their algorithm solves Eq.
by adding an unsteady term with the help of a pseudotime variable and iterating until the solution converges (Eq. 1.2.5).

\[
\frac{\partial^2 \bar{p}}{\partial \bar{\sigma} \partial \bar{t}} = \frac{\partial^2 \bar{p}}{\partial \bar{z}^2} - \bar{z} \frac{\partial^2 \bar{p}}{\partial \bar{t}^2} + \frac{\eta}{2} \frac{\partial^2 \bar{p}^2}{\partial \bar{t}^2}
\]

(1.2.5)

where the additional term on the LHS is the unsteady term and \( \bar{\sigma} \) is the pseudotime variable. The pseudotime variable here serves an iterative purpose not to be confused with physical time. The addition of the unsteady term is similar to the propagation term found in the KZK equation [25] where there is a physical coordinate in place of the pseudotime variable. The technique of adding both the pseudotime variable and unsteady term is also known as dual time stepping [26] and is also implemented in solving computational fluid dynamics problems [27].

Figure 1.6 – Depiction of the focused pressure field [23, 24]. Two rays exist on the illuminated side above the caustic. No rays exist on the shadow side below the caustic.
The authors use a split-step method where the diffraction terms are solved in the frequency domain and the nonlinearity is solved in the time domain. The diffraction terms are discretized to second order and solved using the Thomas algorithm for each frequency. The nonlinear term is solved using a shock-capturing scheme from McDonald and Ambrosiano [28]. The authors provide numerical validation by comparing the linear solution of their scheme to the analytical solution to the linear Tricomi equation. There is excellent agreement. The authors also demonstrate that their code is independent of the starting solution. The authors also show numerical convergence with grid refinement by increasing the grid density in both dimensions and show an N-wave case where the peak pressures for the front and rear shocks converge to consistent values. Lastly, the authors conduct a numerical check of their code’s ability to match the Guiraud scaling law [6]. The results are favorable and the authors attribute deviations in their code’s validation due to the fact that the input waveforms are not the idealized step shocks that Guiraud’s scaling law requires.

The authors exercise their formulation and code to predict focusing from the Concorde during an acceleration maneuver through the standard atmosphere. They obtained near-field estimates by using the Whitham F-function [29]. The Whitham F-function uses the integral of the second derivative of the cross sectional area of a “slender” supersonic body of revolution to obtain an estimate of the resulting pressure perturbation. The computed near-field was propagated using nonlinear geometrical theory. Their propagation provided the incoming waveform for their nonlinear Tricomi simulation at one diffraction boundary layer thickness from the caustic intercept at the ground. Their simulation included different altitudes, Mach rates and incoming pressure amplitudes, all of which play a role in the ratio of nonlinear effects relative to the diffraction effects. The results of their parametric study show little variation on the resulting maximum pressure. However, their study did show that focusing of an incoming waveform with two stair-stepped shocks in the front part of the signature has a lower amplification factor than for an N-wave. Additionally, the authors point out that the method employed by Plotkin and Cantril cannot handle more complex incoming waveforms because the interaction of the shocks in the vicinity of the caustic is not properly modeled.
1.2.3 Marchiano, Coulouvrat and Grenon, “Numerical Simulation of Shock Wave Focusing at Fold Caustics, with Application to Sonic Boom” (2003)

Marchiano, Coulouvrat and Grenon [30] made improvements on the formulation and computations of sonic boom focusing. Their approach modified the formulation of Auger and Coulouvrat by changing the field variable of (dimensionless) pressure into the acoustic potential. They defined the acoustic potential as:

\[ \bar{p} = \frac{\partial \phi}{\partial t} \Leftrightarrow \phi = \int_{-\infty}^{t} \bar{p}(t')dt' \quad (1.2.6) \]

and transformed Equation 1.2.4 into the following:

\[ \frac{\partial^2 \phi}{\partial \sigma^2} = \frac{\partial^2 \phi}{\partial t^2} - \zeta \frac{\partial^2 \phi}{\partial t^2} + \eta \frac{\partial}{\partial t} \left[ \frac{(\partial \phi)}{(\partial t)} \right]^2 = 0 \quad (1.2.7) \]

where \( \eta \) is the same ratio nonlinearity to diffraction effects from Eqs. (1.2.4) and (1.2.5).

The authors cite that their modified formulation is more suitable to handle shocks because the boundary condition associated with the incoming waveform does not contain the derivative of the incoming waveform but contains the waveform itself. Their computational algorithm proceeds in the same manner as Auger and Coulouvrat with the addition of the pseudotime variable and the unsteady term as shown in Eq. (1.2.8).

\[ \frac{\partial^2 \phi}{\partial \sigma^2} = \frac{\partial^2 \phi}{\partial \sigma^2} - \zeta \frac{\partial^2 \phi}{\partial \sigma^2} + \eta \frac{\partial}{\partial \sigma} \left[ \frac{(\partial \phi)}{(\partial \sigma)} \right]^2 \quad (1.2.8) \]

Their approach also uses a splitting method, solving for the nonlinearity and diffraction in separate computational steps. The authors solve for nonlinearity in the time domain and diffraction in the frequency domain. Additionally, the acoustic potential formulation lends itself to solving the nonlinear computational step using the shock “fitting” method.
implemented by Hayes et al. [31]. The Hayes method solved the inviscid Burgers equation with the Poisson solution for propagation along a ray path. Shock positions and amplitudes are determined by the connection between the multi-valued Poisson solution and its corresponding series of maximum values. It has the benefit of an exact solution for the nonlinear step and does not have a Courant-Friedrichs-Lewy (CFL) condition [23]. That is, the iteration step size is not restricted by the signal content or how much the waveforms distort as a result of the nonlinear contributions.

Marchiano, Coulouvrat and Grenon conduct several numerical checks to validate their analytical formulation and computational algorithm. First, they prove their solutions iteratively converge down to machine precision in significantly fewer iterations compared to the method by Auger and Coulouvrat. Next, they show convergence of the maximum pressure amplitudes with respect to the temporal discretization. Their results show the maximum pressure approaches a constant value with an increase in the number of points for the time-wise axis. They also observe that their numerical simulations for N-wave focusing are consistent with flight test values and behave quantitatively as expected when the strength of the nonlinearity is varied. Lastly, they perform the same Guiraud scaling law check that Auger and Coulouvrat had performed using a rectangular window as the incoming waveform. The agreement between their code predictions and the Guiraud scaling law is excellent. While there were no comparisons to flight test measurements in this publication, comparisons were made to laboratory-scale experiments conducted in water tank [32] using their formulation. The predictions were in very good agreement with the water-tank measurements. Additionally, the authors’ nonlinear Tricomi model equation and computational code were exercised to conduct numerical simulations for studying the statistical variability of sonic boom focusing due to diverse meteorological conditions [33].

The authors apply their nonlinear Tricomi focusing capability to numerical simulations of sonic boom focusing of a supersonic transport conceptual design. The near-field pressure was obtained by CFD and coupled to far-field propagation down to the vicinity of the caustic along the ray that eventually is tangent to the caustic at the ground. The near-
field pressure signature evolves from a three-dimensional flow field to a one-dimensional propagation as one increases the distance away from the aircraft. Since the far-field propagation is assumed to be one-dimensional, the authors varied the distance at which they matched the near-field and far-field computations. That is, they vary the distance at which the near-field pressure is extracted from the CFD solutions and examine the resulting focused signatures at the maximum pressure location. They also vary the spacing between the two front shocks to reduce the focused pressure amplification. Their results suggest that reduction of focused signatures are possible if multiple, smaller shocks are included in both the front and rear parts of the incoming waveform.

1.2.4 Kandil and Zheng, “Prediction of Superboom Problem using Computation Solution of Nonlinear Tricomi Equation” (2005)

Kandil and Zheng [34] solve the acoustic potential formulation of Marchiano and Coulouvrat with three different computational/discretization schemes: a frequency domain scheme, a time-domain finite difference scheme and a time-domain finite difference scheme with an overlapping grid. They also present a conservative form of the unsteady nonlinear Tricomi equation as a fourth computational implementation. They apply their schemes to N-waves and to shaped waveforms. However, the authors do not provide numerical or experimental validation efforts for their code as previous authors have done. They only compare the solutions from their four schemes one-to-another and assume that since there is qualitative agreement that their prediction codes are “validated.”

The frequency domain and time domain methods are carried out with a splitting method, solving for the diffraction and nonlinear terms in separate steps. The authors treat the numerical schemes with an ADI approach similar to CFD and heat transfer solution techniques. Kandil and Zheng provide four ways to solve the unsteady nonlinear Tricomi equation: the diffraction terms step can be solved in either the frequency domain or the time domain; and, the nonlinear term step takes the form of an inviscid Burgers equation and the authors can solve the step with a “shock-capturing” finite differencing scheme or with an exact shock fitting scheme [31]. They use the same boundary condition for the
upper part of the computational domain as Marchiano and Coulouvrat. However, the authors only specify a zero-pressure value for the boundary condition at the lower part of the computational domain instead of leveraging the linear Tricomi solution as Auger and Coulouvrat did. They iterate using a pseudotime increment of 0.001 for 20,000 iterations until the unsteady pressure term reduces to approximately $10^{-6}$.

It is difficult to assess which of Kandil’s schemes provided the best results. The authors state their maximum peak overpressures but also state the associated “percent error” for their prediction relative to the time-domain finite difference scheme. However, they do not justify or explain their decision for choosing the time-domain scheme as the “baseline” to compare to the others. Additionally, in many of the figures for their results, there is possibly evidence of numerically induced reflections from the upper boundary. Kandil and Zheng exercise their numerical schemes for an incoming N-wave, a Concorde type of signature, and also ramp and flat-top type of waveforms. The authors comment that the results of their study of shaped signatures show the front part of the focused signature is more readily influenced by the shaping efforts. Their study also showed that the flat-top waveform resulted in the lowest peak overpressure. However, they based their assessment of shaping efforts on only looking at the peak overpressure for their predictions. Additionally, they make the claim that their codes and methodology are “validated” based on the fact that the solutions for their first three computational schemes are in close agreement for predictions of Concorde focusing.

A contribution of Kandil’s and Zheng’s work is they present the nonlinear Tricomi equation in conservative form and solve for the pressure using a finite volume scheme. The main benefit to this formulation is that a splitting method is not required. The authors cite an improvement in computational execution by a factor of four. The conservative form solutions are in agreement with the other three numerical schemes for the N-wave focusing case.

Sescu and Afjeh [35] present a method for computing the nonlinear Tricomi equation. Their method can deal with two issues that make solving the nonlinear Tricomi equation a difficult task. First, the default domain of the nonlinear Tricomi equation is mixed hyperbolic/elliptic. The authors formulate the problem in conservative form in a manner that ensures the entire computational domain is hyperbolic. Second, their method deals with discontinuities in the solution in two ways. They define a new boundary condition on the upper z-boundary that removes the derivative of the incoming waveform, which can cause challenges when a discontinuity exists at the boundary. They also employ a nonlinear limiter to control oscillations that may occur in the solution domain.

For the upper boundary condition, the authors leverage the fact that the upper boundary can be located far enough away to separate the N- and U-waves. They divide the upper boundary condition into two parts, for the negative and positive time axis. For the negative time axis they specify the pressure and use the incoming waveform. For the positive time axis they use the radiation condition. This is a new way to prescribe the upper boundary condition for this type of problem. The authors specify zero pressure for the lower boundary condition. This may not provide the most accurate answers near the caustic if the lower boundary is not extended far enough away from the caustic.

The authors present a conservation law form of the nonlinear Tricomi equation similar to the approach by Kandil and Zheng. This formulation has a pseudo-time variable (also used in the work from previous authors) but includes a new dependent variable. The authors use an eigenvalue/eigenvector solution to ensure their formulation is hyperbolic. They use a Galerkin method to solve their conservative law formulation with Legendre polynomials as the basis. They note that their method requires a limiting scheme to reduce oscillations due to discontinuities in the solution domain and use a total variation bounded limiting scheme. They solve for the resulting set of ODE’s by a second-order total-variation diminishing Runge-Kutta method. They show that their solutions typically converge after 25,000 – 30,000 iterations down to an error of $10^{-7}$. 
The authors compute results using second and third order Galerkin schemes and, for the linear case, compare the solutions to the linear analytical solution. Their solutions agree well, with no oscillations in the solution and limited levels of reflections from the boundaries. However, their solutions show a noticeable amount of rounding at the corners of the N-wave and U-waves. They also compare their method to several finite difference schemes when solving for the function values at the nodes. One of the methods is a Runge-Kutta discontinuous Galerkin scheme. The discontinuous Galerkin method is used for spatial discretization and a Runge-Kutta scheme is used to march the iterations in pseudotime. They note that their RKDG method is more computationally intensive but more accurate than the classical fourth order schemes they selected. They also compare their method to an upwind third order (Weighted Essentially Non-Oscillating (WENO) scheme. The use of the WENO scheme provides higher accuracy but also mitigates oscillatory behavior that would otherwise occur in the vicinity of steep gradients in the computational domain [36]. Their results show that their RKDG method and the WENO scheme are similar in accuracy and performance. However, looking at their comparison plot of the two methods, the WENO scheme appears to have more rounding in the peaks of the U-wave.

One of the drawbacks to using the focus boom prediction method developed by Sescu and Afjeh is the application of the “limiters” hinders their code’s ability to accurately capture a shock’s rise time (the time it takes for the shock to rise from its minimum value to its maximum value). The shock structure can be modified depending on the amount of the “limiter” applied to overcome the oscillatory behavior at the shocks. Thus, the implementation of their numerical losses to mitigate amplitude excursions associated with sharp pressure gradients prevents one from accurately modeling the rise profile and shock structure for shocks in their solution.

1.3 Description of Atmospheric Loss Mechanisms

Detailed shock structure is not preserved for the focus boom calculation methods described in the work of the previous authors. There are no loss mechanisms to prevent “over steepening” of shocks and a model equation should include real gas effects to
prevent this [37-39]. The primary loss mechanisms in air for the range of human hearing are thermoviscous effects and the molecular relaxation of diatomic oxygen and diatomic nitrogen [11]. Thermoviscous effects include absorption due to heat conduction and viscosity. The acoustic perturbations are dissipated as a result of the viscosity in the air and thermal conduction in the medium resulting from the passage of an acoustic disturbance [40]. The result of this dissipation causes the thermoviscous attenuation. Molecular relaxation is caused by molecular collisions that occur due to a departure from thermodynamic equilibrium resulting from a disturbance from the equilibrium state (for the case here, the passage of an acoustic wave) [40, 41]. These collisions excite an internal mode of vibration and result in the sharing of energy that causes absorption and dispersion. For the frequency range encompassed by typical human hearing, molecular relaxation of nitrogen and oxygen contribute a significant amount of absorption and dispersion [11]. The quantity, $\tau_r$, is the relaxation time and is the characteristic time associated with how long it takes to return to thermodynamic equilibrium [42]. The relaxation time (in air) is greatly influenced by the relative humidity. In fact, the absorption and dispersion of the air is a function of temperature, pressure and molecular concentration of water vapor as shown in Eq. (1.3.1) [11].

$$\alpha = 8.686 f^2 \left[ 1.84 \times 10^{-11} \left( \frac{p_{ref}}{p_0} \right) \left( \frac{T_0}{T_{ref}} \right)^{1/2} \right]$$

$$+ \left( \frac{T_0}{T_{ref}} \right)^{-5/2} \left\{ 0.01275 \exp \left( \frac{-2239.1}{T_0} \right) \left[ \frac{f_O}{f_O^2 + f^2} \right] + 0.1068 \exp \left( \frac{-3352.0}{T_0} \right) \left[ \frac{f_N}{f_N^2 + f^2} \right] \right\}$$

(1.3.1)

where $\alpha$ is the atmospheric absorption in decibels per meter, $p_{ref}$ and $T_{ref}$ are the reference pressure and temperature, respectively, $p_0$ and $T_0$ are the ambient pressure and temperature, respectively, $f_O$ and $f_N$ are the relaxation frequencies for oxygen and nitrogen, respectively, and $f$ is the oscillation frequency of the pressure disturbance. Expressions for $f_O$ and $f_N$ below are also found in [11]:
\[
    f_O = \left( \frac{p_0}{p_{\text{ref}}} \right) \left( 24 + \frac{40400h(0.02 + h)}{0.391 + h} \right) \]  

(1.3.2)

\[
    f_N = \left( \frac{p_0}{p_{\text{ref}}} \right) \left( \frac{T_0}{T_{\text{ref}}} \right)^{-1/2} \left( 9 + 280h \exp \left\{ -4.170 \left( \frac{T_0}{T_{\text{ref}}} \right)^{-1/3} - 1 \right\} \right) \]  

(1.3.3)

where \( h \) is the molecular concentration of water vapor (in percent). Figure 1.7(a) is a plot of atmospheric absorption as a function of frequency for 15 degrees Celsius, 60% relative humidity and 101.325 kPa. The dashed black line is the absorption due to thermoviscous effects and is proportional to frequency squared as indicated in Eq. (1.3.1). The blue line is the absorption due to the thermoviscous effects plus the effects of nitrogen and oxygen relaxation. The cyan circle and the red circle indicate where the relaxation frequencies are for nitrogen and oxygen relaxation, respectively, where the relaxation frequency is given by \( f_v = 1/(2\pi\tau_v) \) and \( v \) is \( O \) for oxygen and \( N \) for nitrogen. Below the nitrogen relaxation frequency, the contributions to absorption are mostly nitrogen relaxation, oxygen relaxation and thermoviscous effects. Between the nitrogen relaxation frequency and the oxygen relaxation frequency, the absorption is mainly due to thermoviscous effects and oxygen relaxation. Also observe that oxygen and nitrogen relaxation contribute to absorption in a frequency squared manner when below their respective relaxation frequency. Above the oxygen relaxation frequency, most of the absorption is due to thermoviscous effects. A similar trend is visible in regards to dispersion when examining the contributions of the relaxation components to the ambient speed of sound.

Figure 1.7(b) is a plot of the speed of sound as a function of frequency (blue line) for the same atmospheric conditions provided for Figure 1.7(a). The dashed black line denotes the equilibrium speed of sound. Observe the monotonic increase in speed in the proximity of the relaxation frequency. As the frequency of the pressure oscillations increase, the dispersive component reaches its frozen speed of sound. It is referred to the ‘frozen’ speed of sound because the rate of oscillations occurs so rapidly above those frequencies the gas does not recover to an equilibrium state [15, 40]. The difference
between the contributions of molecular relaxation to absorption compared to dispersion is that for dispersion the relaxation frequency is centered at the corresponding speed of sound increment for the particular relaxation component. Additionally, while nitrogen relaxation contributes slightly more to absorption than oxygen relaxation in the illustrated frequency range, it is oxygen relaxation that adds a greater increment of dispersion than does nitrogen relaxation [15, 40].

![Graphs showing atmospheric absorption and dispersion as a function of frequency.](image)

**Figure 1.7** – Atmospheric absorption (a) and dispersion (b) as a function of frequency. Conditions are 15 degrees C, 60% relative humidity and 101.325 kPa.

### 1.4 Scope of the Thesis

Current codes and models for one-dimensional sonic boom propagation along rays account for atmospheric loss mechanisms that influence the detailed shock profiles in sonic booms as previously mentioned. The model equations described in Section 1.2 account only for nonlinearity and diffraction. They are also of common origin in that they assume a linear sound speed profile and the model equations incorporate the variables in terms of the local caustic geometry. However, none of the formulations
include terms for the absorption and dispersion that occurs as a result of propagation in the atmosphere. Including absorption and dispersion will provide more consistent matching with the geometric acoustics waveform that serves as a boundary condition (from Section 1.2.2 and 1.2.3) for the diffraction in the focal zone. Accounting for atmospheric loss mechanisms would allow more accurate modeling of the shock rise profiles and high frequency content present in sonic booms that are important for the calculation of objective metrics [43, 44] and this will be demonstrated later in this thesis for sonic boom focusing. Research has shown that human response to sonic booms heard outdoors can be correlated well with the Perceived Loudness and A-weighted sound exposure levels [43, 44]. The Perceived Loudness [45] is the Stevens Mark VII Perceived Loudness (PL) and the SEL_\text{A}, SEL_\text{C} and SEL_\text{Z} are the A-weighted, C-weighted and un-weighted sound exposure levels, respectively [46, 47]. The PL is a loudness metric that accounts for the nonlinearity in human hearing, in that the loudness levels vary with both frequency and amplitude content of the signal. This is different than the A-weighting and C-weighting, which are only a function of frequency [47]. Referring back to Figure 1.7(a), observe that one order of magnitude increase in frequency from 1 kHz to 10 kHz results in two orders of magnitude of an increase in absorption. Thus, as one attempts to increase the frequency range of the focus boom prediction, the loss mechanisms may no longer be negligible.

This thesis will: 1) augment existing sonic boom focusing theory and develop a model equation that includes absorption and dispersion, 2) show that the loss mechanisms provide better matching to the geometric acoustics boundary condition, 3) include an analysis with a frequency range much higher than the previous work of others, 4) compare focus boom predictions consisting of a pressure time-history to large-scale experimental data, and 5) predict the focused sonic boom of a conceptual supersonic business jet designed for a quiet sonic boom at its cruise condition and also compute loudness metrics for its predicted focus boom.
Chapter 2

Mathematical Derivation and Formulation

2.1 Derivation of a lossy nonlinear wave equation for a homogeneous medium with uniform mean flow

We can assume the medium is homogeneous by leveraging the findings from [24]. There, Auger made the observation that the inhomogeneous terms are third order with respect to the diffraction parameter he uses to rank order terms for his derivation of the nonlinear Tricomi equation. However, we will change the assumption of a homogeneous medium later in section two of this chapter.

Let \( \frac{D_0}{Dt} = \frac{\partial}{\partial t} + \bar{v}_0 \cdot \nabla \) and the equations of motion for a fluid described in [15] are:

\[
\begin{align*}
\text{Mass:} & \quad \frac{D_0 \rho}{Dt} + \rho \nabla \cdot \bar{v} = 0 & \quad (2.1.1) \\
\text{Momentum:} & \quad \rho \frac{D_0 \bar{v}}{Dt} = -\nabla p + \mu \nabla^2 \bar{v} + (\mu_B + \frac{1}{3} \mu) \nabla (\nabla \cdot \bar{v}) & \quad (2.1.2) \\
\text{Energy:} & \quad T \rho \frac{D_0 S_{fr}}{Dt} + \sum_v \rho C_{v,v} \frac{D_0 T_v}{Dt} - k \nabla^2 T = 0 & \quad (2.1.3)
\end{align*}
\]
Relaxation:  \[
\frac{D_0 T_v}{Dt} = \frac{1}{\tau_v} (T - T_v) \tag{2.1.4}
\]

State:  For the equations of state, assume that pressure and temperature are functions of density and entropy, \( p = p_0 \left( \rho_0, s_{fr,0} \right) \) and \( T = T_0 \left( \rho_0, s_{fr,0} \right) \).

\[
dp = \left. \frac{dp}{d\rho} \right|_{\rho_0, s_{fr,0}} \rho + \left. \frac{d^2 p}{d\rho^2} \right|_{\rho_0, s_{fr,0}} \frac{d\rho^2}{2} + \left. \frac{dp}{ds} \right|_{\rho_0} ds_{fr} \tag{2.1.5}
\]

\[
dT = \left. \frac{dT}{dp} \right|_{\rho_0, s_{fr,0}} dp + \left. \frac{dT}{ds} \right|_{s_{fr,0}} ds_{fr} \tag{2.1.6}
\]

It is assumed in Eq. (2.1.5) and in Eq. (2.1.6) that the expansion assumes entropy is of second order as indicated in [15, 37, 48] where the “smallness” of perturbations is determined according to the amplitude of the acoustic Mach number, \( \left| \bar{v} \right| / c_0 \) which is assumed to be much less than one. According to [48], the acoustic Mach number is 0.01 for a 154 dB pressure wave in air.

Now, we perturb the material derivative as \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \left( \bar{v}_0 + \bar{v}' \right) \cdot \nabla = \frac{D_0}{Dt} + \bar{v}' \cdot \nabla \) and also perturb Eqs. (2.1.1) – (2.1.4) with:

\[
\rho \Rightarrow \rho_0 + \rho' \quad \bar{v} \Rightarrow \bar{v}_0 + \bar{v}' \quad p \Rightarrow p_0 + p' \quad T \Rightarrow T_0 + T' \quad s_{fr} \Rightarrow s_{fr,0} + s'_{fr} \tag{2.1.7}
\]

The steps for the derivation here follow the same steps given in Chapter 3 of [48] in arriving at a lossy nonlinear wave equation. That is, the equations of mass, momentum, energy and state are perturbed and manipulated to determine an augmented Westervelt type of equation. Starting with the perturbation in Eq. (2.1.1):
\[
\frac{D(\rho_0 + \rho')}{Dt} + (\rho_0 + \rho')\nabla \cdot (\vec{v}_0 + \vec{v}') = 0
\]

Expanding the above equation results in:

\[
\frac{\partial \rho_0}{\partial t} + \frac{\partial \rho'}{\partial t} + \vec{v}_0 \cdot \nabla \rho_0 + \vec{v}_0 \cdot \nabla \rho' + \vec{v}' \cdot \nabla \rho_0 + \vec{v}' \cdot \nabla \rho' + \\
\rho_0 \nabla \cdot \vec{v}_0 + \rho_0 \nabla \cdot \vec{v}' + \rho' \nabla \cdot \vec{v}_0 + \rho' \nabla \cdot \vec{v}' = 0
\]

In the above equation, terms with a spatial derivative of an ambient quantity have been omitted since we assume the medium is homogeneous. Also, we omit terms with partial derivatives with respect time of ambient quantities since we assume the ambient conditions are steady-state. This leaves us with:

\[
\frac{D_0 \rho'}{Dt} + \vec{v}' \cdot \nabla \rho' + \rho_0 \nabla \cdot \vec{v}' + \rho' \nabla \cdot \vec{v}' = 0 \tag{2.1.8}
\]

Now perturb Eq. (2.1.2):

\[
(\rho_0 + \rho')\frac{D(\vec{v}_0 + \vec{v}')}{Dt} = -\nabla (p_0 + p') + \mu \nabla^2 (\vec{v}_0 + \vec{v}') + (\mu_B + \frac{1}{3} \mu) \nabla (\nabla \cdot [\vec{v}_0 + \vec{v}'])
\]

Expanding the above equation results in:

\[
\rho_0 \frac{D_0 \vec{v}_0}{Dt} + \rho_0 \vec{v}' \cdot \nabla \vec{v}_0 + \rho_0 \frac{D_0 \vec{v}_0}{Dt} + \rho_0 \vec{v}' \cdot \nabla \vec{v}_0 + \rho_0 \frac{D_0 \vec{v}'}{Dt} + \rho_0 \vec{v}' \cdot \nabla \vec{v}' + \rho' \frac{D_0 \vec{v}'}{Dt} + \\
\rho' \vec{v}' \cdot \nabla \vec{v}' = -\nabla p_0 - \nabla p' + \mu \nabla^2 \vec{v}_0 + \mu \nabla^2 \vec{v}' + (\mu_B + \frac{1}{3} \mu) \nabla (\nabla \cdot \vec{v}_0) + (\mu_B + \frac{1}{3} \mu) \nabla (\nabla \cdot \vec{v}')
\]

In the above equation, we can neglect third order quantities and also neglect spatial and time derivatives of ambient quantities as we did before to get:
\[
\rho_0 \frac{D_0 \vec{v}'}{Dt} + \rho_0 \vec{v}' \cdot \nabla \vec{v}' + \rho' \frac{D_0 \vec{v}'}{Dt} = -\nabla p' + \mu \nabla^2 \vec{v}' + (\mu_B + \frac{1}{2} \mu) \nabla (\nabla \cdot \vec{v}')
\] (2.1.9)

From Chapter 3 in [48] they use the vector calculus identity \((\vec{v}' \cdot \nabla)\vec{v}' = \frac{1}{2} \nabla (\nabla \cdot \vec{v}')^2 - \vec{v}' x \nabla x \vec{v}'\) (where ‘\(x\)’ denotes the cross product) but also mention that the \(\vec{v}' x \nabla x \vec{v}'\) term is negligible in this case since we assume we are not near boundaries or surfaces, so we are left with \((\vec{v}' \cdot \nabla)\vec{v}' = \frac{1}{2} \nabla (\nabla \cdot \vec{v}')^2\). In [40] and [48] it is discussed that in air we are dealing with a “weakly thermoviscous” medium for our frequency range of interest. Additionally, [48] provides a rigorous explanation that “exploits” the “weakly thermoviscous” assumption and the fact that we are not propagating not near any surface or boundary which allows us to then substitute \(\nabla (\nabla \cdot \vec{v}') \approx \nabla^2 \vec{v}'\) in the last term on the right hand side of Eq. (2.1.9).

Also discussed in [48] is that the viscous terms on the right hand side of Eq. (2.1.9) are second order in acoustic Mach number. We can leverage this assumptions and use first order relations to substitute into all of the second order terms since the error associated with that type of substitution is third order [37, 48, 49]. Thus, we can use

\[
\rho_0 \frac{D_0 \vec{v}'}{Dt} = -\nabla p' , \quad \rho' = \frac{p'}{c_0^2} , \quad \nabla \cdot \vec{v}' = \frac{-1}{\rho_0 c_0^2} \frac{D_0 p'}{Dt}
\]

in Eq. (2.1.9) to obtain the following:

\[
\rho_0 \frac{D_0 \vec{v}'}{Dt} + \frac{1}{2} \rho_0 \nabla v'^2 + \left( \frac{p'}{c_0^2} \right) \left( \frac{-\nabla p'}{\rho_0} \right) = -\nabla p' + (\mu_B + \frac{1}{2} \mu) \nabla \left( \frac{-1}{\rho_0 c_0^2} \frac{D_0 p'}{Dt} \right)
\]

Expanding the above equation and rearranging the coefficients leads to:

\[
\rho_0 \frac{D_0 \vec{v}'}{Dt} + \frac{1}{2} \rho_0 \nabla v'^2 + \frac{-\nabla p'^2}{2 \rho_0 c_0^2} = -\nabla p' - \frac{(\mu_B + \frac{1}{2} \mu)}{\rho_0 c_0^2} \nabla \left( \frac{D_0 p'}{Dt} \right)
\]

Consolidating the second and third terms in the above equation results in:
where \( L = \frac{1}{2} \rho_0 v^2 - \frac{p'^2}{2 \rho_0 c_0^2} \) is the Lagrangian density [37, 48].

Next, rewrite Eq. (2.1.8) to also get a Lagrangian density term by substituting first order equations as done above into the second order terms. That is, we use \( \frac{\rho_0 D_0 v'}{Dt} = -\nabla p' \), \( \nabla \rho' = \nabla p'/c_0^2 \) and \( \nabla \rho' = -\frac{\rho_0}{c_0^2} \frac{D_0 v'}{Dt} \) to obtain:

\[
\frac{D_0 \rho'}{Dt} + v' \left( -\frac{\rho_0}{c_0^2} \frac{D_0 v'}{Dt} \right) + \rho_0 \nabla \cdot v' + \left( \frac{p'}{c_0^2} \right) \left( -\frac{1}{\rho_0 c_0^2} \frac{D_0 p'}{Dt} \right) = 0
\]

Expanding and rearranging the terms in the above equation leads to:

\[
\frac{D_0 \rho'}{Dt} + \rho_0 \nabla \cdot v' = \frac{\rho_0}{2 c_0^2} \frac{D_0 v'^2}{Dt} + \frac{1}{2 \rho_0 c_0^4} \frac{D_0 p'^2}{Dt} + \frac{1}{2 \rho_0 c_0^4} \frac{D_0 p'^2}{Dt} - \frac{1}{2 \rho_0 c_0^4} \frac{D_0 p'^2}{Dt}
\]

The first and last terms in the right hand side of the above equation can then be consolidated to become a term that includes the Lagrangian density.

\[
\frac{D_0 \rho'}{Dt} + \rho_0 \nabla \cdot v' = \frac{1}{\rho_0 c_0^4} \frac{D_0 p'^2}{Dt} + \frac{1}{c_0^2} \frac{D_0 L}{Dt}
\]

Next, we use the following relations in Eq. (2.1.5) and solve for \( dp \).

\[
\frac{dp}{d\rho}_{s_{i,0}} = c_0^2 \quad \frac{d^2 p}{d\rho^2}_{s_{i,0}} = \frac{c_0^2}{\rho_0 A} \quad \frac{dp}{ds}_{\rho_0} = \frac{p_0}{C_v}
\]
\( \gamma = \frac{C_p}{C_v} \quad p_0 = \frac{c_0^2 \rho_0}{\gamma} \quad \frac{dp^2}{c_0^2} = d\rho^2 \)

Substituting the above into Eq. (2.1.5)

\[
dp = c_0^2 d\rho + \frac{c_0^2 B}{\rho_0 A} \frac{dp^2}{2} + \frac{p_0}{C_v} ds_{fr}
\]

Using the first order relations in the second term on the right hand side, which is second order, and also substituting for the ambient pressure in the third term on the left hand side leads to:

\[
dp = c_0^2 d\rho + \frac{1}{2 \rho_0 c_0^2} \frac{B}{A} dp^2 + \frac{\rho_0 c_0^2}{C_p} ds_{fr}
\]

Rearranging the above to solve for \( d\rho \) results in:

\[
d\rho = \frac{dp}{c_0^2} - \frac{1}{2 \rho_0 c_0^2} \frac{B}{A} dp^2 - \frac{\rho_0}{C_p} ds_{fr} \quad (2.1.12)
\]

For Eq. (2.1.6) use the relations \( \frac{dT}{dp} \bigg|_{s_{fr},0} = \frac{1}{\rho_0 C_p} \) and \( \frac{dT}{ds} \bigg|_{\rho_0} = \frac{T_0}{C_p} \) to get:

\[
dT = \frac{1}{\rho_0 C_p} dp + \frac{T_0}{C_p} ds_{fr} \quad (2.1.13)
\]

We need the perturbed forms of Eqs. (2.1.12) and (2.1.13), and, note that the entropy term in Eq. (2.1.13) is negligible according to the assumptions from [15, 40, 48].
\[ \rho' = \frac{p'}{c_0^2} - \frac{1}{2\rho_0 c_0^4} \frac{B}{A} \rho'^2 - \frac{\rho_0}{C_p} s'_{fr} \]  

(2.1.14)

\[ T' = \frac{1}{\rho_0 C_p} p' \]  

(2.1.15)

The perturbation of the relaxation equation, Eq. (2.1.4), is the next part of the derivation with perturbations for \( T_v \Rightarrow T_0 + T' + T'_v \) and \( T \Rightarrow T_0 + T' \):

\[ \frac{D}{Dt}(T_0 + T' + T'_v) = \frac{1}{\tau_v} (T_0 + T' - T_0 - T' - T'_v) \]

Expanding the above equation results in:

\[ \frac{D_0 T_0}{Dt} + \bar{v}' \cdot \nabla T_0 + \frac{D_0 T'}{Dt} + \bar{v}' \cdot \nabla T' + \frac{D_0 T'_v}{Dt} + \bar{v}' \cdot \nabla T'_v = \frac{-1}{\tau_v} T'_v \]

Multiplying the above by \( \tau_v \) and removing time and spatial derivatives of the ambient quantities gives:

\[ \tau_v \frac{D_0 T'}{Dt} + \tau_v \bar{v}' \cdot \nabla T' + \tau_v \frac{D_0 T'_v}{Dt} + \tau_v \bar{v}' \cdot \nabla T'_v = -T'_v \]

Solving for \( T'_v \) and using the assumption that the second and fourth terms in the left hand side of the above equation are third order then leads to:

\[ T'_v = \frac{-\tau_v \frac{D_0 T'}{Dt}}{1 + \tau_v \frac{D_0}{Dt}} \]  

(2.1.16)
Next, we perturb Eq. (2.1.3) for $T'$ and $T_v'$. Note that there are separate perturbations for the two temperatures: $T \Rightarrow T_0 + T'$ and $T_v \Rightarrow T_v'$ because of how they each contribute to the energy equation [15, 42]. We can also neglect spatial and time derivatives of ambient quantities as we did for the continuity and momentum equations and neglect terms higher than second order. Also, arguments are provided in [37] and Chapter 3 in [48] to support the assumption that entropy perturbations are of second order since we are not near any boundaries or walls.

\[
(\rho_0 + \rho')(T_0 + T') \frac{D(s_{fr,0} + s'_{fr})}{Dt} + \sum_v (\rho_0 + \rho')C_{v,v} \frac{DT_v'}{Dt} - \kappa \nabla^2 (T_0 + T') = 0
\]

Expanding the above leads to:

\[
(\rho_0 T_0 + \rho' T_0 + \rho' T_0 + \rho' T') \left( \frac{D_0 s_{fr,0}}{Dt} + \nabla' \cdot \nabla s_{fr,0} \right) + \\
(\rho_0 T_0 + \rho' T_0 + \rho' T_0 + \rho' T') \left( \frac{D_0 s'_{fr}}{Dt} + \nabla' \cdot \nabla s'_{fr} \right) + \\
\sum_v (\rho_0 + \rho')C_{v,v} \left( \frac{D_0 T_v'}{Dt} + \nabla' \cdot \nabla T_v' \right) - \kappa \nabla^2 T_0 - \kappa \nabla^2 T' = 0
\]

Removing the time and spatial derivatives of the ambient quantities and omitting third order and higher terms results in:

\[
\rho_0 T_0 \frac{D_0 s'_{fr}}{Dt} + \sum_v \rho_0 C_{v,v} \frac{D_0 T_v'}{Dt} - \kappa \nabla^2 T' = 0 \tag{2.1.17}
\]

Now, insert Eq. (2.1.16) into Eq. (2.1.17) to obtain:
\[
\rho_0 T_0 \frac{D_0 s'_{fr}}{Dt} - \sum_v \rho_0 C_{v,v} \tau_v \frac{D_0^2 T'}{Dt^2} \left( \frac{1}{1 + \tau_v} \right) - \kappa \nabla^2 T' = 0
\]  
(2.1.18)

If Eq. (2.1.14) is solved for the entropy, 
\[
s'_{fr} = \frac{C_p}{\rho_0} \left( \frac{p'}{c_0^2} - \frac{1}{2 \rho_0 c_0^4 \frac{B}{A} p'^2} - \rho' \right),
\]
and inserted, along with Eq. (2.1.15), into Eq. (2.1.18) the result is:

\[
\frac{\rho_0 T_0 C_p}{\rho_0} \left[ \frac{D_0^2}{Dt^2} \left( \frac{p'}{c_0^2} \right) - \frac{D_0}{Dt} \left( \frac{1}{2 \rho_0 c_0^4 \frac{B}{A} p'^2} \right) - \frac{D_0 \rho'}{Dt} \right] - \sum_v \rho_0 C_{v,v} \tau_v \frac{D_0^2}{Dt^2} \left( \frac{p'}{\rho_0 C_p} \right) \left( \frac{1}{1 + \tau_v} \right) - \kappa \frac{\rho_0 C_p}{\rho_0 C_p} \nabla^2 p' = 0
\]

So that,

\[
T_0 C_p \left[ \frac{1}{c_0^2} \frac{D_0 p'}{Dt} - \frac{1}{2 \rho_0 c_0^4 \frac{B}{A}} \frac{D_0 p'^2}{Dt^2} - \frac{D_0 \rho'}{Dt} \right] - \sum_v \rho_0 C_{v,v} \tau_v \frac{D_0^2}{Dt^2} \left( \frac{p'}{\rho_0 C_p} \right) \left( \frac{1}{1 + \tau_v} \right) - \kappa \frac{\rho_0 C_p}{\rho_0 C_p} \nabla^2 p' = 0
\]

This gives,

\[
\frac{D_0 p'}{Dt} = \frac{1}{c_0^2} \frac{D_0 p'}{Dt} - \frac{1}{2 \rho_0 c_0^4 \frac{B}{A}} \frac{D_0 p'^2}{Dt^2} - \sum_v \frac{C_{v,v} \tau_v}{C_p T_0} \frac{D_0^2}{Dt^2} \left( \frac{p'}{\rho_0 C_p} \right) \left( \frac{1}{1 + \tau_v} \right) - \kappa \frac{\rho_0 C_p}{\rho_0 C_p} \nabla^2 p' \quad (2.1.19)
\]

Next, take \( \frac{D_0}{Dt} \) of Eq. (2.1.11) and subtract from it the divergence of Eq. (2.1.10):
\[
\frac{D_0^2 \rho'}{Dt^2} + \rho_0 \nabla \cdot \frac{D_0 \mathbf{v}'}{Dt} - \frac{1}{\rho_0 c_0^4} \frac{D_0^2 p'^2}{Dt^2} - \frac{1}{\rho_0 c_0^2} \frac{D_0^2 L}{Dt^2} - \rho_0 \nabla \cdot \frac{D_0 \mathbf{v}'}{Dt} - \nabla^2 p' - \frac{(\mu_B + \frac{4}{3} \mu)}{\rho_0 c_0^2} \nabla^2 \left( \frac{D_0 p'}{Dt} \right) - \nabla^2 L = 0
\]

Simplifying the above equation results in:

\[
\frac{D_0^2 \rho'}{Dt^2} - \nabla^2 p' - \frac{1}{\rho_0 c_0^4} \frac{D_0^2 p'^2}{Dt^2} - \frac{(\mu_B + \frac{4}{3} \mu)}{\rho_0 c_0^2} \nabla^2 \left( \frac{D_0 p'}{Dt} \right) - \frac{1}{c_0^2} \frac{D_0^2 L}{Dt^2} - \nabla^2 L = 0 \quad (2.1.20)
\]

Next, take \( \frac{D_0}{Dt} \) of Eq. (2.1.19) and insert into Eq. (2.1.20) for the density term to obtain an expression that only includes the pressure perturbation and the Lagrangian density:

\[
\frac{1}{c_0^2} \frac{D_0^2 p'}{Dt^2} - \nabla^2 p' - \frac{1}{\rho_0 c_0^4} \frac{D_0^2 p'^2}{Dt^2} - \frac{1}{\rho_0 c_0^2} \frac{D_0^2 L}{Dt^2} - \sum_v \frac{C_v \tau_v}{C_p T_0} \frac{D_0^3 p'}{Dt^3} \quad (2.1.21)
\]

Use the following relations according to [15, 37, 40, 48] and insert them in Eq. (2.1.21):

\[
\beta = 1 + \frac{1}{2} \frac{B}{A} \quad \Pr = \frac{\mu c_p}{\kappa} \quad \frac{(\gamma - 1) C_{v,v}}{C_p} = m_v = \frac{C_{v,v}}{c_0^2} - \frac{C_{v,v}}{c_0^2} \frac{\mu_B}{\mu} + \frac{\gamma - 1}{\Pr}
\]

and the result is:
\[
\n\nabla^2 p' - \frac{1}{c_0^2} \frac{D_0^2 p'}{Dt^2} + \frac{\beta}{\rho_0 c_0^4} \frac{D_0^2 p'^2}{Dt^2} + \frac{\delta}{c_0^2} \nabla^2 \frac{D_0 p'}{Dt} + \sum_v m_v \tau_v \frac{D_0^3 p'}{Dt^3} = -\frac{1}{c_0^2} \frac{D_0^2 L}{Dt^2} - \nabla^2 L
\]

\hspace{1cm} (2.1.22)

For the thermoviscous term only in Eq. (2.1.22), we can make the same substitution that Blackstock had made in Chapter 9 of [40] for his derivation of a wave equation in a thermoviscous medium. We substitute \( \nabla^2 p' = \frac{1}{c_0^2} \frac{D_0^2 p'}{Dt^2} \) into the thermoviscous term, \( \frac{\delta}{c_0^2} \nabla^2 \frac{D_0 p'}{Dt} \), of Eq. (2.1.22) to obtain \( \frac{\delta}{c_0^4} \frac{D_0^3 p'}{Dt^3} \). Equation (2.1.22) then becomes:

\[
\nabla^2 p' - \frac{1}{c_0^2} \frac{D_0^2 p'}{Dt^2} + \frac{\beta}{\rho_0 c_0^4} \frac{D_0^2 p'^2}{Dt^2} + \frac{\delta}{c_0^4} \frac{D_0^3 p'}{Dt^3} + \sum_v m_v \tau_v \frac{D_0^3 p'}{Dt^3} = -\frac{1}{c_0^2} \frac{D_0^2 L}{Dt^2} - \nabla^2 L
\]

\hspace{1cm} (2.1.23)

Equation (2.1.23) is the same equation derived by Cleveland in his thesis for the augmented Westervelt equation, except the time derivatives in Cleveland’s derivation have been replaced here by convective time derivatives. This is as expected since the uniform flow scenario assumed here could have been obtained by a change of referential in Cleveland’s model equation that uses a referential where the “observer” moves with the uniform flow.

### 2.2 Derivation of the lossy nonlinear Tricomi equation

Figure 2.1 is an illustration of the coordinate system in the vicinity of the caustic and shows the continuum of rays that form the caustic. The illuminated zone is above the caustic; the shadow zone is below the caustic. There are two rays that pass through any location in the illuminated zone and no rays propagate in the shadow zone. Sound travels
into the shadow zone because of diffraction, which is not predicted by geometric acoustics. Also note that the incoming ray has not yet propagated through the caustic and the outgoing ray has already propagated through the caustic.

Figure 2.1 – Continuum of rays that form a caustic [15, 30]. The illuminated zone is above the caustic. The shadow zone is below the caustic.

Starting from Eq. (2.1.22), one can transform the augmented convective Westervelt equation into a lossy nonlinear Tricomi equation. The coordinate system is put in terms of the local caustic geometry by using curvilinear coordinates. The derivation here for the lossy nonlinear Tricomi equation follows the same physical coordinates, variables, notation and eikonal equation as in Auger’s thesis [24] to obtain a nonlinear Tricomi equation.

Auger analyzed the confluence of rays and the formation of the caustic similar to the depiction in Figure 2.1 and derived an eikonal equation. The benefit to the eikonal equation used by Auger [24] is that the sonic boom focusing which actually occurs in four dimensions (time and space) can be described by only one physical dimension, the
distance from the caustic in the normal direction, along with a time coordinate. The eikonal function, \( \psi \), from [24] is:

\[
\psi = \frac{x}{c_0} \left( 1 + \frac{z}{R_{XZ}} \right) \pm \frac{1}{c_0} \sqrt{\frac{8z^3}{9R_{tot}}} \tag{2.2.1}
\]

where \( R_{XZ} \) is the radius of curvature of the caustic and \( R_{tot} \) is the total relative radius of curvature between the radius of curvature of the caustic and \( R_{cel} \), the radius of curvature of the ray that intercepts the caustic (Figure 2.3). The plus and minus signs in Eq. (2.2.1) correspond to the incoming ray and outgoing ray. \( R_{tot} \), \( R_{cel} \), and \( R_{XZ} \) are related by [24]:

\[
\frac{1}{R_{XZ}} = -\frac{1}{R_{tot}} - \frac{1}{R_{cel}} \tag{2.2.2}
\]

A purpose of the eikonal equation is to identify the locations of the wavefront that all have a common phase value [3]. For example, the wavefront from Figure 1.3 is plotted as it reached the ground compared in Figure 2.2 to the eikonal function from Eq. (2.2.1). The wavefront is represented by the colored dots shaded by the time at which that portion of the wavefront was generated. The blue line is the eikonal function using the radii curvature values at the ground appropriate for that particular wavefront. The agreement is excellent between the eikonal function and the propagated wavefront.

The first term on the right hand side of Eq. (2.2.1) leads to the dimensionless retarded time, \( \tilde{t} \), which has been normalized by the characteristic acoustic frequency, \( f_{ac} \) [24]:

\[
\tilde{t} = f_{ac} \left( t - \frac{x(1 + z/R_{XZ})}{c_0} \right) \tag{2.2.3}
\]
Figure 2.2 – Comparison of a propagated wavefront at the ground (colored dots) to the eikonal function (blue line) in Eq. (2.2.1). The wavefront, comprised of successive rays generated at certain time intervals, is colored according to the initial time a particular ray was generated.

The second term in the right hand side of Eq. (2.2.1) introduces the diffraction boundary layer thickness, $\delta_D$, given by Eq. (2.2.4). The significance of the diffraction boundary layer is that between the caustic and that distance from the caustic the effects of diffraction are not negligible.

$$
\delta_D = \left(\frac{c_0^2 R_{tot}}{2f_{ac}^2}\right)^{1/3}
$$

(2.2.4)
Equation (2.2.5) below is a diffraction smallness parameter which is the ratio of the characteristic acoustic wavelength, $\lambda_{ac}$, compared to the diffraction boundary layer thickness. The parameter, $\varepsilon$, is assumed to be small (on the order of 0.1 according to [24]) and will serve as a means of ranking terms when deriving the nonlinear Tricomi equation and determine which terms to include or exclude.

$$\varepsilon = \frac{\lambda_{ac}}{\delta_D} = \left( \frac{2c_0}{f_{ac} R_{tot}} \right)^{1/3}$$  \hspace{1cm} (2.2.5)

For the derivation by Auger [24], terms of third order and higher in $\varepsilon$ are excluded. The derivation here will follow suit with one difference that will be explained later when we also retain the loss terms. The coordinate variables and their derivatives are non-dimensionalized in terms of the caustic geometry and characteristic acoustic parameters. The natural choice for non-dimensionalizing the $z$-coordinate is to take the ratio of the $z$-coordinate to the diffraction boundary layer thickness.

$$z = \frac{2f_{ac}^2}{c_0^2 R_{tot}} \left( \frac{2c_0}{f_{ac} R_{tot}} \right)^{1/3} \Rightarrow \frac{z}{\delta_D} = \bar{z}$$  \hspace{1cm} (2.2.6)

Equation (2.1.22) was derived assuming a homogenous medium. This will be changed in what follows and will now assume the medium is “locally” heterogeneous in the vicinity of the caustic by assuming the speed of sound changes as a function of vertical distance, $\bar{z}$, away from the caustic using the expression from [24].

$$c(\bar{z}) = (c_0 + \bar{u}_0 \cdot \vec{n}) \left( 1 + \varepsilon^2 \frac{R_{tot}}{2R_{cel}} \bar{z} \right)$$  \hspace{1cm} (2.2.7)
In Eq. (2.1.22) there are terms that include the ambient speed of sound in the denominator. The reciprocals of $c_0$ are approximated in Eqs. (2.2.5) – (2.2.7) to second order in $\varepsilon$ as:

\[
\frac{1}{c} \approx \frac{1}{c_0} \left(1 - \varepsilon^2 \frac{R_{tot}}{2R_{cel}} \right) \quad (2.2.8)
\]

\[
\frac{1}{c^2} = \frac{1}{c_0^2} \left(1 - \varepsilon^2 \frac{R_{tot}}{R_{cel}} \right) \quad (2.2.9)
\]

\[
\frac{1}{c^4} = \frac{1}{c_0^4} \left(1 - 2\varepsilon^2 \frac{R_{tot}}{R_{cel}} \right) \quad (2.2.10)
\]

where $R_{cel}$ is defined as:

\[
R_{cel} = \left(\frac{1}{c} \frac{dc}{dz}\right)^{-1} \quad (2.2.11)
\]

The density is also assumed to vary linearly with the vertical distance, $\bar{z}$, away from the caustic using the expression from [24].

\[
\rho(\bar{z}) = \rho_0 \left(1 + \varepsilon^2 \frac{R_{tot}}{2R_{den}} \bar{z} \right) \quad (2.2.12)
\]

Additionally, there are terms that also have the density in the denominator and can be approximated to second order in $\varepsilon$ as:

\[
\frac{1}{\rho} \approx \frac{1}{\rho_0} \left(1 - \varepsilon^2 \frac{R_{tot}}{2R_{den}} \bar{z} \right) \quad (2.2.13)
\]
where $R_{den}$ is the radius of curvature associated with the change in density [24]:

$$R_{den} = \left( \frac{1}{\rho} \right)^{-1}$$  \hspace{1cm} \text{(2.2.14)}$$

The derivatives are also normalized and put in terms of the caustic geometry using the approach by Auger [24]. The partial derivative with respect to time becomes:

$$\frac{\partial}{\partial t} = f_{ac} \frac{\partial}{\partial \tilde{t}}$$  \hspace{1cm} \text{(2.2.15)}$$

The partial derivatives with respect to $x$ are modified to reflect a preferred propagation in the $x$-direction. Thus, the partial derivatives with respect to $x$ are transformed to partial derivatives with respect to (dimensionless) time [24]:

$$\frac{\partial}{\partial x} = -\frac{f_{ac}}{c_0} \left( 1 + \epsilon^2 \frac{R_{tot}}{2R_{xz}} \right) \frac{\partial}{\partial \tilde{t}}$$  \hspace{1cm} \text{(2.2.16)}$$

$$\frac{\partial^2}{\partial x^2} = \frac{f_{ac}^2}{c_0^2} \left( 1 + \epsilon^2 \frac{R_{tot}}{R_{xz}} \right) \frac{\partial^2}{\partial \tilde{t}^2}$$  \hspace{1cm} \text{(2.2.17)}$$

Note that the eikonal function obtained by Auger did not include any terms associated with the $y$-axis. This led Auger to determine that the partial derivatives with respect to $y$ are third order with respect to the diffraction smallness parameter, $\epsilon$ [24]. In other words, he determined that diffraction effects for the zero-degree azimuth ($y = 0$ plane) can be neglected.

$$\frac{\partial}{\partial y} = O(\epsilon^3)$$  \hspace{1cm} \text{(2.2.18)}$$
The partial derivatives with respect to $z$ now become [24]:

\[
\frac{\partial}{\partial z} = \varepsilon f ac \frac{\partial}{c_0 \partial \bar{z}} \tag{2.2.20}
\]

\[
\frac{\partial^2}{\partial z^2} = \varepsilon^2 f^2 ac \frac{\partial^2}{c_0^2 \partial \bar{z}^2} \tag{2.2.21}
\]

The components of wind, $v_{ox}$ and $v_{oz}$ in the $x$- and $z$-directions, are also normalized as [24]:

\[
M_X = \frac{v_{ox}}{c_0} \tag{2.2.22}
\]

\[
M_Z = \frac{v_{oz}}{c_0} \tag{2.2.23}
\]

Equations (2.2.15) through (2.2.23) can now be used to transform the convective time derivatives as [24]:

\[
\frac{D_0}{Dt} = f_{ac} \left(1 - M_X \right) \frac{\partial}{\partial t} + f_{ac} M_Z \varepsilon \frac{\partial}{\partial \bar{z}} \tag{2.2.24}
\]

\[
\frac{D^2_0}{Dt^2} = f_{ac}^2 \left(1 - 2M_X + M_X^2 \right) \frac{\partial^2}{\partial t^2} + 2 f_{ac}^2 M_Z \varepsilon \frac{\partial^2}{\partial t \partial \bar{z}} \tag{2.2.25}
\]

\[
\frac{D^3_0}{Dt^3} = f_{ac}^3 \left(1 - 3M_X + 3M_X^2 \right) \frac{\partial^3}{\partial t^3} + 3 f_{ac}^3 M_Z \varepsilon \frac{\partial^3}{\partial t^2 \partial \bar{z}} \tag{2.2.26}
\]
The next step in the derivation is to insert Eqs. (2.2.2) through (2.2.26) into Eq. (2.1.23) from the previous section. Auger had shown that the diffraction effects in the $y$-direction are negligible [24] and that assumption is carried out here as well. It is also assumed that the propagation has a “preferred” direction oriented in the $x$-axis (Eqs. 2.2.16, 2.2.17). These two assumptions, combined with the eikonal equation derived by Auger, allow one to transform the two-dimensional focusing scenario in Figure 2.1 into a formulation that models the focusing in the vicinity of the caustic as a function of only one physical coordinate, the vertical distance, $\bar{z}$, normal to the caustic.

\[
\frac{f_{ac}^2}{c_0^2} \left( 1 + e^2 \frac{R_{tot}}{R_{XZ}} \right) \frac{\partial^2 p'}{\partial t^2} + e^2 \frac{f_{ac}^2}{c_0^2} \frac{\partial^2 p'}{\partial \bar{z}^2} - \frac{f_{ac}^2}{c^2} \left( \frac{\partial^2 p'}{\partial t^2} - 2M_X \frac{\partial^3 p'}{\partial t^3} + 2M_Z e \frac{\partial^2 p'}{\partial \bar{t} \partial \bar{z}} + M_X^2 \frac{\partial^2 p'}{\partial \bar{t}^2} \right) \\
+ \frac{\delta tv f_{ac}^3}{c^4} \left( \frac{\partial^3 p'}{\partial t^3} - 3M_X \frac{\partial^3 p'}{\partial t^3} + 3M_X^2 \frac{\partial^3 p'}{\partial \bar{t}^3} + 3M_Z^2 \frac{\partial^3 p'}{\partial \bar{t}^2 \partial \bar{z}} \right) \\
+ \frac{f_{ac}^3}{c^2} \sum_m \tau_m \left[ \frac{\partial^3 p'}{\partial t^3} - 3M_X \frac{\partial^3 p'}{\partial t^3} + 3M_X^2 \frac{\partial^3 p'}{\partial \bar{t}^3} + 3M_Z \frac{\partial^3 p'}{\partial \bar{t}^2 \partial \bar{z}} \right] \\
+ \frac{\beta f_{ac}^2}{\rho c^4} \left[ 1 - M_X \frac{\partial}{\partial \bar{t}} + eM_Z \frac{\partial}{\partial \bar{z}} \right] \\
+ \frac{\beta^2 f_{ac}^2}{\rho c^4} \left[ 1 - 2M_X + M_X^2 \frac{\partial^2 p''}{\partial \bar{t}^2} + 2M_Z e \frac{\partial^2 p''}{\partial \bar{t} \partial \bar{z}} \right] = 0
\] (2.2.27)

The Lagrangian density terms from equation 2.1.23 in the previous section were eliminated by the following analysis. In Auger’s thesis, he used a formal order analysis with respect to the diffraction parameter to assess the acoustic perturbation velocity vector. One can also use a “slow” scale similar to that used by Hamilton-Blackstock [48] and Cleveland [37] when deriving a parabolic propagation equation from a 3-D hyperbolic wave equation. The acoustic perturbation velocity in the $x$-direction is assumed to be similar to that of a quasi-plane wave. Thus, we have $v_x = p'/(\rho_0 c_0)$ since the propagation through the caustic occurs “primarily” in the $x$-direction. Similarly, the
propagation in the z-direction is not nearly as dominant so we have \( v_z' = \epsilon \frac{p'}{(\rho_0 c_0)} \) for the acoustic perturbation velocity in the z-direction. The expressions for the x- and z-direction perturbation velocities were those obtained by Auger, and, when inserted into the expression for the Lagrangian density term, yield:

\[
\frac{\rho_0}{2} \frac{p'^2}{\rho_0^2 c_0^2} + \frac{\rho_0}{2} \frac{\epsilon^2 p'^2}{\rho_0^2 c_0^2} - \frac{p'^2}{2 \rho_0 c_0^2} \approx 0 + O(\epsilon^4)
\]

The second term in the above equation is assumed to be negligible since the acoustic Mach number, \( M_{ac} = \frac{p_{ac}}{\rho_0 c_0^2} \), is on the order of \( \epsilon^2 \), making the second term on the order of \( \epsilon^4 \). Note that the characteristic acoustic pressure, \( p_{ac} \), is typically determined by the maximum acoustic pressure at the \( \bar{z} = 1 \) location. Continuing simplifications of equation (2.2.27) gives:

\[
\left(1 + \epsilon^2 \frac{R_{tot}}{R_{XZ}} \right) \frac{\partial^2 p}{\partial t^2} + \epsilon^2 \frac{\partial^2 p}{\partial z^2} - \left(1 - \epsilon^2 \frac{R_{tot}}{R_{cel}} \right) \left( \frac{\partial^2 p}{\partial t^2} - 2M_X \frac{\partial^2 p}{\partial t^2} + 2M_Z \epsilon \frac{\partial^2 p}{\partial t \partial z} + M_X^2 \frac{\partial^2 p}{\partial t^2} \right)
\]

\[
+ \frac{\delta_{IV} f_{ac}}{c_0^2} \left(1 - 2 \epsilon^2 \frac{R_{tot}}{R_{cel}} \right) + \left(1 - \epsilon^2 \frac{R_{tot}}{R_{cel}} \right) \sum_v \frac{m_v f_{ac} \tau_{ac}}{1 + f_{ac} \tau_{ac}} \frac{1}{(1 - M_X) \frac{\partial}{\partial t} + M_Z \epsilon \frac{\partial}{\partial z}}
\]

\[
x \left( \frac{\partial^3 p'}{\partial t^3} - 3M_X \frac{\partial^3 p'}{\partial t^3} + 3M_Z \epsilon \frac{\partial^3 p'}{\partial t^2 \partial z} + 3M_X^2 \frac{\partial^3 p'}{\partial t^3} \right)
\]

\[
+ \frac{\beta}{\rho_0 c_0^2} \left(1 - \epsilon^2 \frac{R_{tot}}{2 R_{den}} \right) \left(1 - 2 \epsilon^2 \frac{R_{tot}}{R_{cel}} \right) \left(1 - 2M_X + M_X^2 \right) \frac{\partial^2 p'^2}{\partial t^2} + 2M_Z \epsilon \frac{\partial^2 p'^2}{\partial t \partial z} \right) = 0
\]

Let \( \alpha = \delta f_{ac} / c_0^2 \), \( \tau_v = f_{ac} \tau_v \) and \( \bar{\theta}_v = \tau_v m_v \). Even though the absorption and dispersion terms are on the order \( \epsilon^5 \), those terms are retained because at higher frequencies they come within two orders of magnitude of the nonlinear and diffraction
terms. That is, when considering the magnitude of the absorption and dispersion terms in the frequency domain, they are no longer negligible for high frequencies. “High” frequencies are considered as those approaching 10 kHz in the upper end of the human hearing range (~10kHz to ~20kHz). Refer back to Figure 1.7 where the absorption increases by two orders of magnitude for one order of magnitude increase in frequency as one approaches these high frequencies. Thus, the influence of absorption and dispersion may no longer be negligible due to the increased frequency range desired for the focus boom predictions presented later in this thesis.

Now, continue eliminating additional terms of order $\varepsilon^3$ and higher but retaining the absorption and dispersion terms for the reasons stated above:

$$
\left(1 + \varepsilon^2 \frac{R_{tot}}{R_{XZ}} \right) \frac{\partial^2 p'}{\partial t^2} + \varepsilon^2 \frac{\partial^2 p}{\partial z^2} - \\
\left(1 - \varepsilon^2 \frac{R_{tot}}{R_{cel}} \right) \frac{\partial^2 p'}{\partial t^2} - 2M_X \frac{\partial^2 p'}{\partial t^2} + 2M_Z \varepsilon \frac{\partial^2 p'}{\partial t \partial z} + M_X \frac{\partial^2 p'}{\partial t^2} + \\
+ \alpha + \sum_v \frac{\partial_v}{1 + \tau_v \left(1 - M_X \frac{\partial}{\partial t} + M_Z \varepsilon \frac{\partial}{\partial z} \right)} \left(\frac{\partial^3 p'}{\partial t^3} - 3M_X \frac{\partial^3 p'}{\partial t^3} \right) + \\
\frac{\beta}{\rho_0 c_0^2} \left(1 - 2M_X + M_X \frac{\partial^2 p}{\partial t^2} \right) = 0
$$

The $M_X$ terms in the last two terms in the above equation can be neglected because $M_X$ is assumed to be first order in $\varepsilon$ as noted by Auger in [24]. The first and third terms in the above equation are consolidated by the use of Eq. (2.2.2). These further simplifications then result in:
\[
\epsilon^2 \frac{\partial^2 p'}{\partial \bar{z}^2} - \epsilon^2 \frac{\partial^2 p'}{\partial \bar{t}^2} + 2M_x \frac{\partial^2 p'}{\partial \bar{t}^2} - 2M_x \epsilon \frac{\partial^2 p'}{\partial \bar{t} \partial \bar{z}} - M_x^2 \frac{\partial^2 p'}{\partial \bar{t}^2} + \left( \bar{\alpha} + \sum_v \frac{\bar{\theta}_v}{1 + \bar{r}_v} \right) \frac{\partial^3 p'}{\partial \bar{t}^3} + \frac{\beta}{\rho_0 c_0^2} \frac{\partial^2 p'^2}{\partial \bar{t}^2} = 0
\]

Dividing the above equation by \(\epsilon^2\) yields the lossy nonlinear Tricomi equation (LNTE):

\[
\frac{\partial^2 p'}{\partial \bar{z}^2} - \frac{\partial^2 p'}{\partial \bar{t}^2} - 2M_x \frac{\partial^2 p'}{\partial \bar{t}^2} + \left( \frac{2M_x - M_x^2}{\epsilon^2} \right) \frac{\partial^2 p'}{\partial \bar{t}^2} + \left( \frac{\bar{\alpha}}{\epsilon^2} + \sum_v \frac{\bar{\theta}_v}{1 + \bar{r}_v} \right) \frac{\partial^3 p'}{\partial \bar{t}^3} + \frac{\beta}{\rho_0 c_0^2} \frac{\partial^2 p'^2}{\partial \bar{t}^2} = 0 \tag{2.2.28}
\]

It should be recognized that the LNTE is only dependent on the dimensionless time, \(\bar{t}\), and the dimensionless distance from the caustic, \(\bar{z}\). The first, second, third and sixth terms in the above equation are identical to the nonlinear Tricomi equation derived in [24]. However, the final formulation in [24] did not include an \(x\)-wind term. The \(x\)-wind term derived here is similar to a term in the general KZK equation derived by Coulouvrat in [50]. Additionally, if the acoustic pressure, \(p\), is normalized by the characteristic acoustic pressure, \(p_{ac}\), Eq. (2.2.28) can be rewritten as:

\[
\frac{\partial^2 p'}{\partial \bar{z}^2} - \frac{\partial^2 p'}{\partial \bar{t}^2} - 2M_x \frac{\partial^2 p'}{\partial \bar{t}^2} + \left( \frac{2M_x - M_x^2}{\epsilon^2} \right) \frac{\partial^2 p'}{\partial \bar{t}^2} + \frac{\beta}{\rho_0 c_0^2} \frac{\partial^2 p'^2}{\partial \bar{t}^2} + \left( \frac{\bar{\alpha}}{\epsilon^2} + \sum_v \frac{\bar{\theta}_v}{1 + \bar{r}_v} \right) \frac{\partial^3 p'}{\partial \bar{t}^3} = 0 \tag{2.2.29}
\]
The parameter $\eta = 2\beta M_{ac}/\varepsilon^2$ used here in front of the nonlinear term (the last term on the left hand side) is a ratio of the nonlinear effects relative to the diffraction effects [23, 24, 30] as discussed in Section 1.2.

The boundary conditions associated with equation (2.2.28) are as follows. It is assumed the acoustic medium is undisturbed for large negative and large positive dimensionless time, which is equivalent to:

$$p'(\tau \to \pm\infty, z) = 0$$  \hspace{1cm} (2.2.30)

The pressure field decays exponentially and goes to zero as $\bar{z}$ approaches negative infinity. For the domain that extends infinitely away from the caustic, the boundary condition is:

$$p'(\tau, \bar{z} \to -\infty) = 0$$  \hspace{1cm} (2.2.31)

The boundary condition for positive $\bar{z}$ must match geometrical acoustics for the incoming waveform and outgoing waveform. The pressure field is the sum of the incoming and outgoing waves assuming sufficient distance away from the caustic.

$$p'(\tau, \bar{z} \to \infty) \approx \bar{z}^{-1/4} \left[ F\left( \frac{\bar{\tau}}{3} - \frac{2}{3} \bar{z}^{3/2} \right) + G\left( \frac{\bar{\tau}}{3} + \frac{2}{3} \bar{z}^{3/2} \right) \right]$$  \hspace{1cm} (2.2.32)

where $F$ is the incoming waveform and $G$ is the outgoing waveform, both of which are also normalized by the characteristic acoustic pressure. The challenge with the boundary condition in equation (2.2.32) is that the outgoing waveform is unknown. The boundary condition can be rewritten in terms of a radiation condition that only includes the incoming waveform [23, 24, 30, 34, 35].
\[
\frac{\partial P'}{\partial t} + z^{-1/4} \frac{\partial P'}{\partial z} + z^{-5/4} P' = 2F'(r \frac{1}{3} z^{3/2})
\] (2.2.33)

where the prime applied to \( F \) indicates the derivative with respect to the argument. Details for how one obtains Eq. (2.2.33) from Eq. (2.2.32) are discussed in the Appendix.

This chapter has introduced the mathematical formulation and associated boundary conditions of the LNTE. The next chapter will describe the discretization of the model equation and the boundary conditions for implementation in a computer code.
Chapter 3

Numerical Implementation and Validation

3.1 Numerical Discretization

In order to solve Eq. (2.2.28) we introduce a pseudotime variable, $\tilde{\sigma}$, and an unsteady term \[23, 24, 30\]. As the solution is iterated in pseudotime, the unsteady term goes to zero and the solution to Eq. (2.2.28) is obtained.

\begin{equation}
\frac{\partial^2 p'}{\partial \tilde{\sigma} \partial t} = \frac{\partial^2 p'}{\partial \tilde{\sigma}^2} - \tilde{z} \frac{\partial^2 p'}{\partial t^2} + \left( \frac{2M_x - M_x^2}{\epsilon^2} \right) \frac{\partial^2 p'}{\partial \tilde{t}^2} - \frac{2M_z}{\epsilon^2} \frac{\partial^2 p'}{\partial \tilde{t} \partial \tilde{z}}
\end{equation}

\begin{equation}
+ \left( \frac{\alpha}{\epsilon^2} + \sum_{\nu} \frac{\tilde{\nu}}{1 + \tilde{\nu}} \frac{\partial \tilde{\nu}}{\partial \tilde{t}} \right) \frac{\partial^3 p'}{\partial \tilde{t}^3} + \frac{\beta}{\epsilon^2 \rho_0 c_0^2} \frac{\partial^2 p'}{\partial \tilde{t}^2}
\end{equation}

The unsteady Eq. (3.1.1) is solved computationally in a finite domain using a splitting method \[51-53\] that accounts for the effects of diffraction, nonlinearity and absorption/dispersion in separate steps. The form of the mixed derivative in the unsteady term $\frac{\partial^2 p'}{\partial \tilde{\sigma} \partial t}$ is chosen based on the discussion presented in Section 1.2.2 describing the approach used in \[23\] and \[24\]. That is, the variable, $\tilde{\sigma}$, serves to iteratively evolve the solution instead of as a physical time or a physical distance as discussed in Section 1.2.2. The diffraction and absorption/dispersion terms are solved in the frequency domain and the nonlinear term is solved in the time domain. The frequency domain steps are solved
for each harmonic component in the spectrum corresponding to a $\bar{z}$-coordinate. The nonlinear solution step is also solved for each $\bar{z}$-coordinate but in the time domain.

At the onset of the numerical algorithm, the computational domain is transformed to the frequency domain using an FFT. The first step in the splitting method solves for the diffraction and the $z$-wind component term.

\[
\frac{i\omega_n \hat{\partial}_n}{\hat{\partial}_{\bar{\sigma}}} \bar{p}_n = \omega_n^2 \bar{p}_n + \frac{\hat{\partial}_n^2 \bar{p}_n}{\hat{\partial}_{\bar{z}}^2} - i\omega_n \frac{2M_z}{\varepsilon} \hat{\partial}_n \bar{p}_n \tag{3.1.2}
\]

where $n$ denotes the index corresponding the $n$-th frequency component. Equation (3.1.2) is discretized as fully implicit, second order in space and first order in pseudotime. Equation (3.1.3) then is solved using a Tridiagonal Matrix Algorithm (TDMA) algorithm [54].

\[
\frac{\hat{p}_{n,j}^{k+1} - \hat{p}_{n,j}^{k}}{\Delta \bar{\sigma}} = -i\omega_n \bar{z}_j \hat{p}_{n,j}^{k+1} - i \omega_n^2 \bar{p}_{n,j}^{k+1} + \frac{\hat{p}_{n,j}^{k+1}}{\Delta \bar{z}^2} - \frac{2M_z}{\varepsilon} \left( \frac{\hat{p}_{n,j+1}^{k+1} - \hat{p}_{n,j-1}^{k+1}}{2\Delta \bar{z}} \right) \tag{3.1.3}
\]

where $k$ is the index of the $k$-th iteration or pseudotime step. The next step solves for the absorption and the x-wind component terms.

\[
\frac{i\omega_n \hat{\partial}_n}{\hat{\partial}_{\bar{\sigma}}} \bar{p}_n = -\frac{\omega_n^2}{\varepsilon^2} (2M_x - M_x^2) \hat{p}_n - \frac{i\omega_n^3}{\varepsilon^2} \left( \bar{\alpha} + \sum_v \frac{\bar{\theta}_v}{1 + i\omega_n \bar{z}_n} \right) \hat{p}_n \tag{3.1.4}
\]

Equation (3.1.4) is solved using the exact solution [48, 53]:

\[
\hat{p}_n^{k+1} = \hat{p}_n^k \exp (B_n \Delta \bar{\sigma}) \tag{3.1.5}
\]
where $B_n$ is:

$$
B_n = \frac{i\omega_n}{\epsilon^2} \left( 2M_x - M_x^2 \right) - \omega_n^2 \left( \bar{\alpha} + \sum_v \frac{\theta_{v}}{1 + i\omega_n \bar{\alpha}_v} \right)
$$

(3.1.6)

At this step, the computational domain is transformed back into the time domain using an IFFT and the nonlinear term is solved for as Eq. (3.1.7).

$$
\frac{\partial p'}{\partial \sigma} = \frac{\beta}{\epsilon^2 \rho_0 c_0^2} \frac{\partial p'^2}{\partial t}
$$

(3.1.7)

The nonlinear step in Eq. (3.1.7) is solved using the Poisson solution [37, 48]:

$$
p^{k+1} = p^k \left( \bar{\sigma}_k + \frac{2\beta}{\epsilon^2 \rho_0 c_0^2} \Delta \bar{\sigma}^k \right)
$$

(3.1.8)

where $k$ index of the current pseudotime value. In the nonlinear solution step, the time base is adjusted according to Eq. (3.1.8), then the time and pressure values are linearly interpolated back to uniform time spacing. The solution domain is transformed back to the frequency domain via FFT and the solution process begins again with the first computational step solving for the diffraction.

The boundary conditions are also discretized and made compatible with the numerical algorithm. The boundary condition from Eq. (2.2.30) is applied as

$$
p^{k+1}(\tilde{r}_{\min,\max}, \tilde{z}) = 0
$$

(3.1.9)
In the shadow zone, the pressure field decays exponentially towards zero. As the pressure amplitudes get smaller, the nonlinear effects can be neglected. The analytical solution to the linear lossless Tricomi equation contains a form of the Airy function. For the lower boundary condition, Auger [23, 24] leveraged the asymptotic expression for the Airy function [22, 55] and the spectral solution of the lossless linear Tricomi equation:

\[
\hat{p}_n(z) = \sqrt{2\pi} [1 + i \text{sgn}(\omega_n)] |\omega_n|^{1/6} A i(-|\omega_n|^{2/3} z) \hat{F}_n
\]  

(3.1.10)  

\[
A i(\xi \to \infty) \approx \frac{\exp(-\frac{2}{3} \xi^{3/2})}{2\sqrt{\pi} \xi^{1/4}}
\]  

(3.1.11)  

where \( \text{sgn} \) is the signum function, \( A i \) is the Airy function (see Figure 3.1 [55]) and \( \xi \) is the argument of the Airy function.

Equations (3.1.10) and (3.1.11) were combined and manipulated to obtain the boundary condition for the lower part of the computational domain [23, 24]. More details on the
how this was accomplished are found in the Appendix. Equation (2.2.31) becomes for a finite computational domain:

\[
\frac{\partial \hat{p}_n}{\partial \bar{z}} - \left( \frac{1}{4} \left| \bar{z}_{\text{min}} \right|^{-1} + \left| \nu_n \right| \left| \bar{z}_{\text{min}} \right|^{1/2} \right) \hat{p}_n = 0
\]  

(3.1.12)

Equation (3.1.12) is discretized with a first order discretization in space.

\[
\left[ 1 + \Delta \bar{z} \left( \frac{1}{4} \left| \bar{z}_{\text{min}} \right|^{-1} + \left| \nu_n \right| \left| \bar{z}_{\text{min}} \right|^{1/2} \right) \right] \hat{p}_{n,1}^{k+1} - \hat{p}_{n,2}^{k+1} = 0
\]  

(3.1.13)

The boundary condition in equation (2.2.33) for the maximum \( z \)-coordinate in the frequency domain is

\[
\frac{\partial \hat{p}_n}{\partial \bar{z}} + \left( i \nu_n \bar{z}_{\text{max}}^{1/4} + \bar{z}_{\text{max}}^{-1} \right) \hat{p}_n = \bar{z}_{\text{max}}^{3/4} 2i \nu_n \exp \left( \frac{2}{3} i \nu_n \bar{z}_{\text{max}}^{3/4} \right) \hat{F}_n
\]  

(3.1.14)

where \( \hat{F}_n \) is the \( n \)-th harmonic of the Fourier transform of \( F \). The boundary condition corresponding to \( \bar{z}_{\text{max}} \) from Eq. (3.1.14) is discretized with a second order discretization in space:

\[
\frac{3 \hat{p}_{n,M}^{k+1} - 4 \hat{p}_{n,M-1}^{k+1} + \hat{p}_{n,M-2}^{k+1}}{2 \Delta \bar{z}} + \left( i \nu_n \bar{z}_{M}^{1/2} + \bar{z}_{M}^{-1} \right) \hat{p}_{n,M}^{k+1} =
\]

\[
\bar{z}_{M}^{1/4} 2i \nu_n \exp \left( \frac{2}{3} i \nu_n \bar{z}_{M}^{3/4} \right) \hat{F}_n
\]  

(3.1.15)

Note that Eqs. (3.1.13) and (3.1.15) are the lower and upper boundary conditions corresponding to the implementation of the diffraction step in Eq. (3.1.3).
The pseudotime step size, $\Delta \bar{\sigma}$, is determined to prevent numerical “shocks” from occurring due to the nonlinear effects. That is, a CFL condition is enforced to prevent multi-valued waveforms from developing and to also reduce the numerical error associated with the splitting method. The maximum pressure gradient in the time-wise axis is monitored for each $\bar{z}$-coordinate and the pseudotime step size is governed by [17, 37]:

$$
\Delta \bar{\sigma}_{NL}^k = \min \left[ \frac{\bar{t}}{2\beta} \frac{\max (\bar{p})^k}{\rho_0 c_0^2 \rho^2} \right] \quad \text{for all } \bar{z} 
$$

The initial guess for the pressure field is a zero value solution. Next, the computational code iterates to the linear Tricomi equation solution with losses. Then, the code iterates to the nonlinear lossy solution. Iteration of the computational solution occurs until suitable convergence has been achieved. Several parameters are used for determining convergence. The change in the solution between successive pseudotime steps is monitored at several locations in the computational domain. The code performs a global search over all of the $\bar{z}$ locations and determines the $\bar{z}$-coordinate that showed the largest change in pressure value during the nonlinear solution step for all of the time-values at that $\bar{z}$-coordinate. The convergence parameter for that $\bar{z}$ location is the maximum difference in the solution between the current and the next iterations divided by the pseudotime step size. The normalization by the pseudotime step size was chosen because the pseudotime increment can vary and the solution difference per unit pseudotime places the amount of change per iteration in the same relative value from one iterative step to the next. Additionally, the code also performs a global search over all of the $\omega$ frequencies and finds the frequency with the largest change in spectral amplitude during the diffraction solution step for all of the $\bar{z}$ locations at that frequency. The convergence parameter for that $\omega$ frequency is the maximum difference in the solution from the current iteration to the next iteration, again normalized by the pseudotime step increment, for the spectral amplitudes at that $\omega$ for all of the corresponding $\bar{z}$ locations.
An additional convergence parameter is used at the top edge of the domain at $\bar{z}_{\text{max}}$ to monitor the evolution of the outgoing waveform. The convergence parameter for the $\bar{z}_{\text{max}}$ location is the maximum value of the difference of the solution at $\bar{z}_{\text{max}}$ between the current and previous iterations divided by the pseudotime step size. Lastly, the pseudotime step value itself is a convergence parameter that correlates to the largest slope (with respect to the time-wise direction) in the solution domain. For the nonlinear solution step, the pseudotime step increment is determined based on a search for the maximum pressure gradient, with respect to dimensionless time, in the solution. Thus, the pseudotime step increment is also an indicator for convergence since it represents how the maximum slope in the solution domain evolves in pseudotime and evolves to a constant value. Figure 3.2 is an example of the how the convergence parameters change in pseudotime during calculation of N-wave focusing. Observe that near a value of six in pseudotime the convergence parameters have reduced by four orders in magnitude.

**Figure 3.2 - Example of the evolution of the convergence parameters as a function of pseudotime.**
3.2 Numerical Validation

Two numerical validation checks were conducted to assess the performance of the code developed to solve the lossy nonlinear Tricomi equation. The first validation check examines the computational performance of the diffraction effects. The second check involves validation of the nonlinear, absorption and dispersion effects. Retaining only the diffraction terms yields the linear lossless Tricomi equation. Removing the wind and diffraction terms yields a Burgers-type equation.

The solution to the linear lossless Tricomi equation has the analytical form of Eq. (3.2.1) [24].

\[ p(\bar{\tau}, \bar{z}) = \text{IFFT}\left(\sqrt{2\pi}\left[1 + i\text{sgn}(\omega)\right]\omega^{1/6}\text{Ai}\left(-|\omega|^{2/3}\bar{z}^{\frac{1}{3}}\right)\hat{F}\right) \]  

(3.2.1)

where ‘IFFT’ indicates the inverse Fourier transform, sgn is the signum function and Ai is the Airy function.

![Figure 3.3 - Incoming N-wave for the lossless linear diffraction validation check.](image-url)
The incoming N-wave used for this validation part is shown in Figure 3.3. The N-wave has a peak amplitude of 50 pascals, a duration of 0.15 seconds and the shocks have a finite rise time due to nonlinear lossy propagation through the standard atmosphere.

Figure 3.4 - Results of the LNTE code with only the diffraction terms included in the computations (blue line) compared to the analytical solution of the lossless linear Tricomi equation (red line) at four different $\bar{z}$ locations: a) $\bar{z} = 1.0$, b) $\bar{z} = 0.5$, c) $\bar{z} = 0$, and d) $\bar{z} = -0.5$.

Figure 3.4 shows the comparison of the analytical solution from Eq. 3.2.1 to the Tricomi code output when using the N-wave from Figure 3.3 as the incoming signature. Only the linear diffraction terms were enabled in the execution of the Tricomi code for this validation check. The agreement is very good in the illuminated zone (Figure 3.4a and
Figure 3.4b), the shadow zone (Figure 3.4d) and on the caustic line (Figure 3.4c). It should be noted that if the incoming N-wave had had shocks with zero rise time (perfect discontinuity) there would be a singularity in the solution for Eq. (3.2.1). However, because the incoming waveform has finite thickness in the shocks, these singularities are not visible in the solution for Figure 3.4.

The second validation check propagates a sinusoid through a homogenous medium. Nonlinear, absorption and dispersion effects are included in the propagation, making it suitable for modeling by a lossy Burgers equation. A spectral solution for this type of equation exists in Hamilton and Blackstock [48]. They solve the Burgers equation in the frequency domain using a Runge-Kutta scheme with a finite number of harmonics. The number of harmonics recommended for the solution is dependent on the strength of the nonlinearity compared to the absorption and dispersion effects. Robin Cleveland used this check for validation of his THOR code in his dissertation [37]. The Hamilton-Blackstock solution example only includes one relaxation process and the atmosphere for sonic boom propagation includes two relaxation processes. Salamone [56] modified the Hamilton-Blackstock solution to include two relaxation processes (oxygen and nitrogen relaxation) for numerical validation of his augmented KZK code. The version of the lossy Burgers equation used for validation in this work is slightly different than that presented in [37] and [48] due to the coefficients in the terms for Eq. (3.1.1).

\[
\frac{d\tilde{p}_n}{d\overline{\sigma}} = \frac{2\beta}{\varepsilon^2} \frac{i\overline{\omega}_n}{\rho_0 c_0^2} \frac{1}{4} \left( \sum_{m=1}^{n-1} \tilde{p}_m \tilde{p}_{n-m} + 2 \sum_{m=n+1}^{M} \tilde{p}^*_m \tilde{p}_{m-n} \right) \left( A_n + iD_n \right) \tilde{p}_n \quad (3.2.2)
\]

\[
A_n = \frac{\overline{\omega}_n^2}{\varepsilon^2} \left( \alpha + \sum_v \frac{\overline{\vartheta}_v}{1 + (\overline{\omega}_n \overline{\tau}_v)^2} \right) \quad (3.2.3)
\]

\[
D_n = -\frac{\overline{\omega}_n^3}{\varepsilon^2} \sum_v \frac{\overline{\vartheta}_v \overline{\tau}_v}{1 + (\overline{\omega}_n \overline{\tau}_v)^2} \quad (3.2.4)
\]
where $\bar{\omega}_n = 2\pi f_0 n / f_{ac}$ is the dimensionless angular frequency, $f_0$ is the fundamental frequency of the input sinusoidal waveform, $A_n$ is the absorption coefficient, $D_n$ is the dispersion coefficient and $\tilde{\rho}_n$ is the complex pressure at the $n$-th harmonic, and $M$ is the number of harmonics used in carrying out the Runge-Kutta solution. An additional difference between the expression in Eq. 3.2.2 and that presented in [56] is that the pseudotime variable takes the role of a propagation “distance.” For this numerical validation check, the fundamental frequency of the sine wave is the relaxation frequency for nitrogen, which corresponds to 262.13 Hz at ambient conditions of 284.5 K, relative humidity of 70% and ambient pressure of 101.325 kPa. The initial amplitude, $p_{ac}$, of the sine wave was 12.09 pascals with $M=100$ harmonics. The total relative radius of curvature was 40,000 meters.

Figure 3.5 shows a comparison of the LNTE code (solid blue line) to the spectral solution (red dashed line) at four different pseudotimes. The pseudotimes in each subplot of Figure 3.5 correspond to different shock formation “distances”, where one shock formation “distance” used in this validation is the pseudotime value the propagation would need to transpire for a shock to form in the solution for the lossless Burgers equation. The upper left plot of Figure 3.5 corresponds to ½ a shock formation “distance,” the upper right plot corresponds to one shock formation “distance,” the lower left plot corresponds to two shock formation “distances,” and the lower right plot in Figure 3.5 corresponds to three shock formation “distances.” The agreement is excellent between the LNTE code and the spectral solution.
Figure 3.5 - Results of lossy nonlinear propagation of a sinusoid through a homogeneous medium with two relaxation processes. The LNTE code output (solid blue line) is compared to a spectral solution (dashed red line) after propagation for a pseudotime of four different shock formation “distances.”

Grid convergence was checked by initiating a solution on a coarse mesh and successively refining the grid examining convergence to a final solution. All solutions were iterated to a pseudotime value of 5. Table 3.1 shows the grid parameters for this convergence check. Observe that even though the number of points in the time-wise axis did not change for the last two grid configurations, the corresponding $\sigma$ associated with that dimension was reduced.
Figures 3.6 and 3.7 show the Tricomi solution for the N-wave focusing with the grid discretizations in Table 3.1. Figure 3.6 shows a close up of the time history at the location of the peak amplitude associated with the focusing of the front shock. Figure 3.7 shows a close up of the time history at the location corresponding to the peak amplitude of the rear shock focusing. Both plots show how the shocks converge with respect to both the rise time and amplitude as the grid is refined. Additionally, observe how the Gibbs-like oscillations before and after the shock also reduce in amplitude as the grid is refined.

Figure 3.6 - Comparison of the front shock from N-wave focusing as the grid resolution is increased to finer values for the location corresponding to the maximum peak pressure of the front shock.
Table 3.1 - Parameters for demonstrating grid discretization convergence.

<table>
<thead>
<tr>
<th>sample rate (kHz)</th>
<th>z-axis points</th>
<th>time-axis points</th>
<th>$d\bar{z}$</th>
<th>$d\bar{t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.120</td>
<td>3000</td>
<td>4096</td>
<td>0.000667</td>
<td>0.00160</td>
</tr>
<tr>
<td>10.240</td>
<td>6000</td>
<td>8192</td>
<td>0.000333</td>
<td>0.00080</td>
</tr>
<tr>
<td>17.067</td>
<td>10000</td>
<td>16384</td>
<td>0.000200</td>
<td>0.00048</td>
</tr>
<tr>
<td>20.480</td>
<td>12000</td>
<td>16384</td>
<td>0.000167</td>
<td>0.00040</td>
</tr>
</tbody>
</table>

Figure 3.7 - Comparison of the rear shock from N-wave focusing as the grid resolution is increased to finer values for the location corresponding to the maximum peak pressure of the rear shock.
3.3 Additional Numerical Discussions

A computational exercise was conducted to compare the influence of the addition of the absorption and dispersion terms in Eq. (2.2.28). Figure 3.8 is a comparison of the LNTE code solution run with losses (blue line) versus without the loss terms (green line) when the incoming waveform is an N-wave. The red line, shown as a reference, is the incoming waveform scaled and phased according to Eq. (2.2.32) for the function, $F$. The example chosen here in Figures 3.8 and 3.9 is for the same incoming N-wave used in the grid convergence study from the previous section. The $\bar{z}$ location in the LNTE solution chosen for the comparison, $\bar{z} = 0.5$, was selected because it is far enough away from the caustic for diffraction and focusing effects to dominate the front part of the incoming waveform but also far enough away from the upper boundary for propagation effects to be observed in the front shock of the incoming waveform. Observe that the LNTE solution without losses (green line) has a much steeper rise for the front shock than the LNTE solution with losses included (blue line). In fact, the front shock pressure rise and amplitude profile for the LNTE solution with the losses included closely resembles the incoming waveform (red line), which was propagated through an atmosphere with the same absorption and dispersion mechanisms included. The lossy nonlinear Tricomi equation provides a pressure field solution that is more consistent with the matching of the incoming signature from geometric acoustics when losses are accounted for in the incoming waveform.
Figure 3.8 - Comparison at $\bar{z} = 0.5$ of the LNTE code solutions with and without losses included in the computational solution. The blue line is the LNTE code with losses included and the green line is the LNTE solution with the absorption and dispersion terms excluded in the numerical computations. The red line is the incoming N-wave scaled and phased according to Eq. (2.2.32) for the function, $F$. 
Figure 3.9 - Comparison at approximately $\bar{z} = 0.15$ of the LNTE code solutions with all losses included, without losses and with only thermoviscous losses included in the computational solution for the case of an incoming N-wave. The blue line is the LNTE code with losses included, the green line is the LNTE solution with the absorption and dispersion terms excluded, and the red line is the LNTE with only the thermoviscous term in the numerical computations.

Figure 3.9 shows a comparison of the LNTE computations near $\bar{z} = 0.15$, which is the location corresponding to the peak pressure amplitude of the front shock for three different scenarios: the LNTE solution with losses, the LNTE solution with all absorption and dispersion terms disabled, and also the LNTE solution with only the thermoviscous term enabled. First, observe that the LNTE solution with only thermoviscous losses is very similar to the LNTE solution with no absorption or dispersion terms included. A
close look at the comparison for the pressure rise of the front shock (left side of Figure 3.9) shows that the LNTE solution with losses has a lower peak amplitude than the other two scenarios. A close look at the rear shocks (right side of Figure 3.9) shows the LNTE solution with losses (blue line) has slightly longer rise times and slightly lower pressure amplitudes than the other two LNTE solution scenarios. The difference is also evident in the loudness levels, in that the LNTE solution with losses is about 1 dB lower than without losses. The loudness levels for the LNTE solution with losses is 122.9 PL(dB) and 106.0 SEL_A(dB). The loudness levels for both the LNTE solution without losses and with only thermoviscous losses are 123.8 PL(dB) and 107.1 SEL_A(dB).

Figure 3.10 – Incoming shaped sonic boom for investigating the influence of the loss mechanisms in the LNTE for the case when the incoming waveform is not an N-wave.
Next, a similar comparison is made to understand the effects of including the absorption and dispersion terms for the case when the incoming waveform is not an N-wave. Instead, the incoming waveform is a shaped signature (Figure 3.10) that has reduced pressure amplitudes and multiple smaller shocks in an attempt to mitigate the loudness levels typically associated with sonic booms that are N-waves.

Figure 3.11 - Comparison at approximately $\tilde{z} = 0.04$ in the LNTE code solutions with all losses included, without losses and with only thermoviscous losses included in the computational solution for the case of the incoming shaped sonic boom in Figure 3.10. The blue line is the LNTE code with losses included, the green line is the LNTE solution with the absorption and dispersion terms excluded, and the red line is the LNTE with only the thermoviscous term in the numerical computations.
Observe the front part of the signature in Figure 3.10 has been flattened and reduced in pressure amplitude (this characteristic is typically referred to as a “flat-top”). Additionally, the shaped waveform is comprised of smaller shocks in the middle and rear parts of the signature instead of the larger front and rear shocks of the typical N-wave. Figure 3.11 shows an excerpt of the focused-pressure field solution for the focusing of the signature in Figure 3.10 at approximately \( \bar{z} = 0.04 \) which is the \( \bar{z} \) location with the highest PL value for each respective computational scenario. The same three scenarios from Figure 3.9 were also computed for the shaped sonic boom signature. The left side of Figure 3.11 is a close-up of the middle shock, and the right side of Figure 3.11 is a close-up of the rear shocks. Again we see that when only the thermoviscous losses are included the results are similar to the LNTE solution with the absorption and dispersion terms disabled. Also visible are the Gibbs-like oscillations near the onset of the sharp pressure rise of the middle shock for the case where the losses are not included and also for the case where only the thermoviscous term is included. However, these oscillations are not present for the LNTE solution with all losses included. The LNTE solution with the absorption and dispersion terms shows a much lower amplitude for the shocks, especially for the middle shock. There is also a significant reduction in the loudness levels as a result of including the loss mechanisms. The PL for the LNTE solution with the losses included was 103.4 dB compared to the LNTE solution without losses or the LNTE solution with only the thermoviscous term where the PL was 109.1 dB.

A benefit to exercising the LNTE code with the absorption and dispersion terms and also with only the thermoviscous term is that the influence of the dispersion due to the molecular relaxation terms can also be discussed. Note that the position of the shocks for the LNTE with all loss mechanisms (blue line) in Figure 3.11 and in the left hand side of Figure 3.10 are slightly to the left of the LNTE solution where only the thermoviscous term was included in the solution (red line). The presence of dispersion increases the speed of sound with an increase in frequency as discussed in Section 1.3. The influence of the dispersion in the cases here causes a slight temporal shift in the position of the shocks in a manner similar to [38, 39, 48].
Lastly, it is relevant to discuss the relative amount of losses due to the addition of the atmospheric loss mechanisms compared to any “numerical losses” inherent in the LNTE code due to the numerical implementation. First, the numerical algorithm presented in Section 3.1 is based on a splitting method, the use of which lends itself to a solution that is not “perfectly exact.” That is, despite the fact the pseudotime step increment is chosen conservatively the original form of Eq. (3.1.1) is not “fully” retained due to the split-step approach. Also, the nonlinear solution step the linear interpolation of the time base from the non-uniform time discretization back into a uniform time discretization is slightly lossy as discussed in Cleveland’s dissertation [37]. Lastly, the computational step implemented for the diffraction step in Eq. (3.1.3) is not a lossless computational step. Fully implicit numerical schemes exhibit some element of numerical dissipation [57]. One should bear in mind that the potential “numerical losses” due to the fully implicit scheme are initially present in the spatial domain because Eq. (3.1.3) is operated across the entire $\xi$–axis at only one specific frequency at a time. Any potential numerical dissipation associated with each iterative step in the LNTE code is coupled due to the split step approach comprised of the numerical dissipation from the diffraction step in the frequency domain along the $\xi$–axis and the numerical dissipation from the nonlinear solution step in the time domain. However, in Figures 3.9 and 3.11 note that the LNTE solution with only the thermoviscous term included (red line) is very similar to the LNTE solution without any losses (green line). Thus, it can be said that the numerical dissipation for the LNTE numerical implementation is on the order of the thermoviscous losses.

This chapter has shown the results of implementing the LNTE with a computer code and checked the results of the validation cases against analytical solutions. Excellent agreement was observed between the LNTE code and the validation cases. The influence of the loss mechanisms in the model equation was shown to modify the peak amplitude and slope of shocks in the vicinity of the caustic and provide a better matching to the geometric acoustics boundary condition. The presence of the dispersion terms has also
been shown to produce a slight temporal shift of the shock positions. Additionally, the LNTE solutions without losses in both Figures 3.8 and 3.9 show numerical oscillations following the front shock, thus the loss terms have the advantage of providing numerical stabilization based on physics instead of entirely based on artificially added losses (for instance, as in [35]). This chapter has also shown the numerical dissipation is on the order of the thermoviscous losses in the LNTE implementation. The next step in the validation of the LNTE code is to compare focus boom predictions to actual flight test measurements from an accelerating F-18. That is, the comparison to acoustic focusing measurements will occur at the “real-world” scale and not in a laboratory. Chapter 4 will describe the flight test experiment itself and Chapter 5 will describe the focus boom prediction methodology and comparisons of the predictions to microphone measurements in the vicinity of the caustic intercept to the ground.
Chapter 4

SCAMP Overview

4.1 SCAMP Background

The Superboom Caustic Analysis and Measurement Program (SCAMP) spanned a two and a half year period from May 2010 until January 2013 at the American Institute of Aeronautics and Astronautics Aerospace Sciences Meeting at Grapevine, Texas. There were two phases in the program with follow-on efforts suggested by NASA to conduct additional analysis of low-boom aircraft focusing. The objectives of SCAMP were to [58]:

- develop analytical and numerical models for focus boom prediction
- collect flight test measurements of focused sonic booms for validation of the focus boom prediction codes
- utilize the focus boom codes to predict and evaluate sonic boom focusing of potential low-boom supersonic aircraft concepts

The above objectives were accomplished by a team of collaborators from across the country. Each participating organization held certain roles and responsibilities that enabled the success of the project.

- Wyle
  - Project lead
  - Flight test planning, coordination and execution
  - Gill-Seebass focus boom prediction method in PCBoom
  - Reconstruction of the pseudospectral focus boom prediction method by
Kandil [34]
  o Lossy propagation of the near-field signatures to the far-field
  o Computation of the caustic geometry (total relative radius of curvature)

• Boeing
  o Near-field CFD of the NASA F-18
  o Provided microphones, data recording equipment for the flight test measurements
  o Performance analysis and near-field CFD of a potential low-boom supersonic aircraft concept

• Pennsylvania State University
  o Development of the Lossy Nonlinear Tricomi Equation
  o Participation in the flight test measurements

• Gulfstream Aerospace
  o Development of the Lossy Nonlinear Tricomi Equation
  o Participation in the flight test measurements
  o Performance analysis and near-field CFD of a potential low-boom supersonic aircraft concept

• Central Washington University
  o Development of the Nonlinear Parabolic Equation method for focus boom prediction

• Eagle Aeronautics
  o Consultant for the flight test measurements and focus boom analysis

• NASA (not a member of the contracting team but was still a collaborating organization)
  o Facilitated the flight test planning and preparations
  o Orchestrated and choreographed the pilot preparations, flight test logistics and execution
o Provided the F-18 flight test aircraft, the TG14 motorized sailplane and support of aircraft operations
o Participation in the flight test measurements
o Performance analysis and near-field CFD of a potential low-boom supersonic aircraft concept
o Auralization of the LNTE signatures at the NASA Langley Interior Effects Room
o Meteorological data collection for the upper-air GPS sondes and ground weather stations

Along with the contractual members of the project, there were additional organizations that participated in the flight test measurements:

• Cessna – microphone measurements from a tethered blimp
• Metrolaser – ground to air Schlieren photography
• Seismic Warning Systems – buried accelerometers to support development of a sonic boom resistant earthquake warning system
• Northrop Grumman – microphone and data recording support for a portion of the primary microphone array
• Nagoya University, Japan – participation in the flight test measurements

Phase I of SCAMP included site selection, pilot preparation and flight test planning for the focus boom measurements. The first phase of the program also included development of “alpha” versions of the Gill-Seebass code, the NPE code and the pseudospectral reconstruction of Kandil’s code for use in SCAMP. The intent of developing “alpha” versions of the code was to provide preliminary prediction capability that would later be improved based on comparisons to the experimental data. The comparison effort would then facilitate a “beta” version of the code that includes any
improvements and modifications to the “alpha” code. Phase I concluded with the execution of the flight test focus boom measurements.

Phase II of SCAMP included comparisons of “alpha” code predictions for all four focusing codes and analysis of the low-boom concept vehicles. The “beta” version of the codes were also developed and comparisons to measured data updated to reflect the newer software version. The low-boom concept vehicle analyses were also rerun with the “beta” version of the codes. Upon review and discussion of the results of original Phase II tasks with NASA, SCAMP Phase II was extended to include low-boom analysis for a variety of acceleration scenarios for each of the respective supersonic aircraft concepts. SCAMP activities concluded at the 2013 AIAA Aerospace Sciences Meeting with a dedicated session at the conference. Eight technical papers were presented at the session [59 - 66] and one AIAA Journal manuscript (closely affiliated with this thesis) was published [67]. During the authorship of [67], additional coding modifications were made for the LNTE code. These improvements resulted in a “delta” version of the LNTE code which was used for all of the computations in this thesis.

4.2 SCAMP Flight test Description

The SCAMP flight test took place at the Cuddeback Gunnery Range near Ridgecrest, California (Figure 4.1) in May of 2011. A NASA Dryden F-18 performed unsteady supersonic maneuvers that generated focused sonic booms at the ground. These unsteady flight operations consisted of pitching the nose of the aircraft downward at a constant rate while accelerating at a constant rate and also included level-altitude accelerations.

The primary microphone array consisted of 81 microphones spaced 125 ft. (38.1 m) apart along a "fairly" straight line (Figure 4.2). Microphone data were recorded with four
data acquisition systems. Three microphones were also placed on the cabling of a tethered blimp elevated to a height of 3000 ft. (914.4 m) above the ground. Weather stations were located at three positions along the array: one at each end and one in the middle. GPSsonde weather balloons were released at certain instances during the test day to capture the upper-air atmospheric parameters required to model the test conditions in the prediction codes. The trajectory of the aircraft was time-synchronized along with the acoustic and meteorological data to also provide the required input data for prediction of the sonic boom focusing. Onsite preparation prior to the test took 5 days, the test itself required two weeks and the overall manpower to support the test was over 50 people.

![Figure 4.1 – SCAMP test location near Ridgecrest, California, USA.](image)
Figure 4.2 – Microphone layout and data recording station locations for the SCAMP test. “NGC” stands for Northrop Grumman, “Dryden” indicates NASA Dryden, “LaRC” indicates NASA Langley, “GAC” indicates Gulfstream Aerospace Corporation and “PSU” indicates Pennsylvania State University.

The supersonic aircraft used for this flight test was a NASA Dryden F-18, serial number 852 (Figure 4.3). This aircraft has been used for several NASA sponsored sonic boom flight tests. The aircraft conducted choreographed maneuvers to vary the total relative radius of curvature from between ~100,000 ft. to ~260,000 ft. (~30,000 m to ~80,000 m). Most of the flyovers captured the undertrack focusing, with some additional passes measuring non-zero azimuth focusing. Table 4.1 lists the seven types of steady or maneuvering flyovers for the SCAMP test. Note that maneuvers E, F and G do not have a total relative radius of curvature because the aircraft was in steady, level flight when passing over the array. As mentioned previously, the choreographed maneuvers consisted of pitching the nose of the aircraft downward and/or accelerating the aircraft, and simultaneously ensuring the caustic intercept for the untrack azimuth coincided with the microphone array.
The term “choreographed” was used to describe the maneuvers because timing of executing the flight operation was needed for focus boom placement at the microphone array. Table 4.2 (from [58]) lists the “as-flown” maneuver types for each pass of the 13 flights. All passes were flown with the microphone array capturing the zero degree azimuth except for passes with an ‘o’ next to the letter indicating the maneuver type (the zero degree azimuth corresponds to directly underneath the aircraft). The ‘o’ denotes that the aircraft flew over the array to capture the sonic boom focusing at a non-zero azimuth. That is, the test points with the ‘o’ are designed such that the array captures the portion of the caustic intercept at the ground at an offset away from the undertrack location of the flight path. Several of the same types of maneuvers were flown multiple times to examine repeatability. Maneuvers A – D were the intended flight test maneuvers while maneuvers E – G were steady, level flyovers intended for ‘calibration’ to check the near-field CFD and far-field propagation at the ground.
Table 4.1 – Aircraft maneuver descriptions.

<table>
<thead>
<tr>
<th>Maneuver</th>
<th>Description</th>
<th>Total Relative Radius of Curvature (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>Level altitude, Mach rate = 0.0035 Mach/s</td>
<td>79,250</td>
</tr>
<tr>
<td>B</td>
<td>Level altitude, full power acceleration</td>
<td>73,150</td>
</tr>
<tr>
<td>C</td>
<td>0.8 g pushover, Pitch rate = -0.25 degrees/s, Mach rate = 0.0035 Mach/s</td>
<td>42,675</td>
</tr>
<tr>
<td>D</td>
<td>0.6 g pushover, Pitch rate = -0.50 degrees/s, Mach rate = 0.0035 Mach/s</td>
<td>30,480</td>
</tr>
<tr>
<td>E</td>
<td>Steady, level flyover, Mach 1.17</td>
<td>N/A</td>
</tr>
<tr>
<td>F</td>
<td>Steady, level flyover, Mach 1.21</td>
<td>N/A</td>
</tr>
<tr>
<td>G</td>
<td>Steady, level flyover, Mach 1.30</td>
<td>N/A</td>
</tr>
</tbody>
</table>

Table 4.2 – As-flown test matrix for the SCAMP flight test [58].

<table>
<thead>
<tr>
<th>Date</th>
<th>Flight</th>
<th>Pass 1</th>
<th>Pass 2</th>
<th>Pass 3</th>
<th>Pass 4</th>
<th>Pass 5</th>
<th>Pass 6</th>
<th>Pass 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 May</td>
<td>1261</td>
<td>E</td>
<td>E</td>
<td>F</td>
<td>F</td>
<td>G</td>
<td>G</td>
<td>-</td>
</tr>
<tr>
<td>10 May</td>
<td>1262</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>10 May</td>
<td>1263</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>-</td>
</tr>
<tr>
<td>11 May</td>
<td>1264</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>11 May</td>
<td>1265</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>12 May</td>
<td>1266</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>-</td>
</tr>
<tr>
<td>12 May</td>
<td>1267</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
<td>B</td>
</tr>
<tr>
<td>12 May</td>
<td>1268</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>-</td>
</tr>
<tr>
<td>16 May</td>
<td>1269</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>16 May</td>
<td>1270</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>20 May</td>
<td>1272</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>20 May</td>
<td>1273</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>-</td>
</tr>
<tr>
<td>20 May</td>
<td>1274</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>Å</td>
<td>-</td>
</tr>
</tbody>
</table>

The variations in the total radius of curvature were achieved by controlling the Mach rate and the pitch rate of the aircraft. Pitch rate variations provide a wider range in the total radius of curvature than the Mach rate variations alone. The main challenge with executing the desired flight conditions is that the NASA F-18 is not equipped, by default, with a Mach meter or a pitch rate meter. The NASA Dryden engineers developed a Mach rate display for the cockpit and, combined with the readout of the G loading for an estimate of the pitch rate, were able to provide sufficient pilot guidance to execute the
desired test matrix. Practice runs were first performed by the pilots in a flight simulator, with ray tracing calculated from the simulated flight trajectory. Refinements of the pilot motion and handling were made based on the simulator results prior implementing the maneuvers with the actual aircraft. Execution for the maneuvers involved the forward pilot adjusting the throttle to control speed and Mach rate, while the rear pilot controlled the flight path angle and pitch rate.

An example of an acceleration maneuver is illustrated in Figure 4.4 [58]. The x- and y-axis values have been removed due to International Traffic in Arms (ITAR) restrictions on the data. There were 70 supersonic flight passes, with approximately 67% of the passes capturing the caustic intercept to the ground at the microphone array.

The location selected for the SCAMP flight test was the Cuddeback Gunner Range near Ridgecrest, California. The location was chosen because it is sufficiently far away from populated areas yet close enough to Edwards Air Force Base where the aircraft is based.
The test site was required to be far away from populated areas due to the loud pressure levels associated with a focus boom. Preliminary levels were predicted to be on the order of 8 to 12 psf (400 to 600 Pa) and could potentially cause damage in residential areas, solar farms, etc., if exposed to such high overpressures [28].

Additionally, the location is within the high altitude supersonic corridor (Figure 4.5), the portion of the airspace that allows supersonic operations. The downside to conducting measurements in such a remote area is that it is an hour drive from Ridgecrest where the support crew resided during the SCAMP test. However, a time savings element for the test was that all of the recording equipment and hardware were allowed to remain at the test site at the end of each day. This was possible because NASA provided 24-hour security at the test site.
Figure 4.6 – Microphone placement at the ground [58].

An example of the microphone placement at the test site is shown in Figure 4.6. Note that the microphone is placed on a ground board to present a hard surface that is common for each microphone in the array. The board is made of ½” plywood with dimensions of approximately 18” x 18” (46 x 46 cm). It is recognized that presence of the ground board causes edge diffraction that is not compensated for after the focus booms are recorded.

The microphone layout was determined by balancing the desired spatial resolution needed to capture the peak focus amplitudes with the available measurement resources for the test. The initial layout for the microphone array was a straight linear array. However, during the process of obtaining a permit from the local Bureau of Land Management, a burrowing desert owl nest was found in the proximity of the east end of the array. The SCAMP team was required to shift certain microphones to be at least 250 ft. (76.2 m) away from the owl nest. The target placement for the caustic was near
microphones 20. The microphones east of that location were Bruel & Kjaer 4193’s and have low frequency measurement capability down to approximately 0.1 Hz and up to approximately 20 kHz. Microphones 00 through 20 were GRAS microphones with low frequency measurement capability down to around 0.5 Hz up to approximately 20 kHz. The reason for placing the GRAS microphones on the west end of the array was because the desired placement of the caustic would put the shadow zone on the western end of the array. The evanescent waves would not require the same microphone performance capability as the illuminated zone for capturing the pressure signatures.

![Figure 4.7 – Motoglider used for the SCAMP flight test [58].](image)

A motorized glider aircraft was also instrumented at a wing tip (Figure 4.7) with a Bruel & Kjaer 4193 microphone mounted with a Bruel & Kjaer UA-0386 nose cone wind screen designed for acoustic measurements made in airflow. The motoglider flew perpendicular to the array to allow a larger time window to capture the focus boom in the vicinity of the caustic under track. The goal was to measure the focus above the array +/- 5 degrees azimuth angle from the undertrack focus intercept at the ground.
The upper-air meteorological data was measured by GPSsonde (Figure 4.8). Figures 4.9 – 4.11 are the upper air profiles captured for Cases A, C and D, respectively [58]. The lowest elevation for the upper-air measurements do not go to zero but instead go down to approximately 853 m (2800 ft) due to the ground elevation of the test site. The lapse rate for each of the temperature profiles is fairly similar (Figure 4.12), with Case C having a small temperature inversion in close proximity to the ground and Case A is nearly isothermal close to the ground. The relative humidity profiles show zero values above approximately 9100 m (30,000 ft) because the relative humidity levels were too low for the humidity sensor to detect a value. The zonal and meridional winds do not exhibit as clear a gradient or trend as a function of altitude. This meteorological data will be used later for far-field propagation predictions at the ground as described in the next chapter.
Figure 4.9 – Flight 1266 atmospheric profile (Case A).

Figure 4.10 – Flight 1264 atmospheric profile (Case C).
Figure 4.11 – Flight 1265 atmospheric profile (Case D).

Figure 4.12 – Comparison of temperature profiles for Cases A, C and D.
Chapter 5

Comparison of Focus Boom Predictions to SCAMP Measurements

5.1 Focus boom prediction methodology

Focus boom prediction at the ground incorporates inputs from several sources. Figure 5.1 is a flow/block diagram that illustrates the required components to determine a focus boom at the ground. These elements are also shown graphically in Figure 5.2. The overall focus boom prediction methodology begins with near-field CFD computed at the altitude and Mach number associated with the aircraft flight conditions. Even though the aircraft is actually flying at unsteady conditions (such as an acceleration), it is assumed the flow-field near the aircraft is “quasi-steady” enough to use CFD to determine the near-field pressure. The aircraft flight trajectory and upper-air atmospheric conditions are also required to determine the ray path and focusing geometry. The near-field signature is propagated through the atmosphere with the nonlinear and atmospheric loss mechanisms included so that there is appropriate matching between the far-field propagation and the prediction in the Tricomi domain.
Figure 5.1 – Focus boom prediction flowchart. The red boxes indicate the required inputs to execute the Tricomi code.

Figure 5.2 – Graphical illustration of the focus boom prediction scheme. Observe the spatial extent required to determine the inputs to the Tricomi code.
For the purposes of the predictions for SCAMP (and this thesis), the components for the focus boom predictions were obtained from multiple organizations. NASA Dryden supplied the supersonic aircraft and measurement of the upper-air conditions. Boeing performed the near-field CFD using their BCFD program which solves the Euler equations with grid adaptation on the aircraft body and in the flow field regions to better resolve pressure gradients in the solution [58]. Wyle computed the ray tracing of the aircraft trajectory and the far-field propagation down to the Tricomi domain using PCBoom6. The starting signature for the PCBoom6 propagation was the direct output of the Boeing near-field CFD (no multiple matching was performed [68]). The focusing geometry at the caustic intercept to the ground was also provided by PCBoom6 for the focus boom calculations [58]. Table 5.1 lists the flight conditions for the CFD analysis of each case examined for SCAMP (and this thesis). Figures 5.3 and 5.4 are sample CFD results for the “Case C” analysis representing solutions for a computational domain consisting of nearly 40 million grid points. The near-field analysis for Case D, Case A and Case C were each extracted at a distance corresponding to three body lengths away from the aircraft (152 ft, 46.3 m) and propagated to the ground using PCBoom at the relevant aircraft trajectory conditions specified in Table 5.1, which included accounting for the appropriate angle of attack, $\alpha$.

### Table 5.1 – Flight conditions for CFD analysis of SCAMP cases.

<table>
<thead>
<tr>
<th>Condition</th>
<th>CFD Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case D: Flt 1265, Pass 2</td>
<td>Mach 1.302, 38,000', $\alpha = 1.27^\circ$</td>
</tr>
<tr>
<td>Case A: Flt 1265, Pass 4</td>
<td>Mach 1.159, 34,000', $\alpha = 2.05^\circ$</td>
</tr>
<tr>
<td>Case C: Flt 1264, Pass 4</td>
<td>Mach 1.227, 42,000', $\alpha = 2.29^\circ$</td>
</tr>
</tbody>
</table>
Figure 5.3 – Portion of the CFD solution for Case C (courtesy of Boeing) [58].

Figure 5.4 – CFD cylinder from Case C at three body lengths (152 ft, 46.3 m) (courtesy of Boeing) [58].
Figures 5.5, 5.10 and 5.16 are the outputs from the PCBoom propagation for Case D, Case A and Case C, respectively. Since the supersonic aircraft is a fighter jet, it is expected that the far-field signatures are N-waves. Observe the finite rise time for each shock due to the inclusion of the loss mechanisms in the PCBoom propagation. The addition of the loss terms in the Tricomi formulation, as shown in Figure 3.7, will allow proper matching of the acoustic pressure waveform between the far-field propagation and the onset of the diffraction region due to focusing. After the LNTE calculations are performed, the entire pressure field is multiplied by a scale factor of 1.9 to account for ground reflection. This simple approach in addressing the presence of the ground in the predictions is justified by the microphone measurement to which the predictions are compared to. The flat, hard plywood boards wrapped in a thin plastic moisture barrier presented a highly reflective surface minimizing the more complex impedance effects on sonic boom amplitude [69]. Thus, for this thesis and for SCAMP the pressure field solution was multiplied by 1.9 to account for the ground reflection present in the microphone measurements.

5.2 Comparison to Measured Data

Figure 5.6 is the pressure field calculated from the Tricomi code for Case D. The plot is color shading according to the pressure amplitude in the solution. The horizontal axis is the dimensionless retarded time, $\tilde{t}$, and the vertical axis is the dimensionless distance from the caustic, $\tilde{z}$. Observe the incoming waveform beginning at the upper left portion of the solution, the outgoing waveform in the upper right and the amplitude decay in the shadow zone. Additionally, one can see two regions of amplification, one corresponding to the front shock and the other to the rear shock. Figures 5.7 – 5.9 are comparisons of measurements from the SCAMP flight test to predictions using the LNTE with inputs provided by PCBoom. Examining Figure 5.7, one can see the incoming N-wave and the outgoing U-wave for both the prediction and the measurement. The $\tilde{z}$ position in this figure is close to the upper boundary condition edge of the Tricomi domain. The amplitudes of the incoming N-wave and outgoing U-wave are in fair agreement between prediction and measurement. For the N-wave, the front shock is overpredicted by 19% and the rear shock is underpredicted by 27%. However, the period of the predicted N-
wave is shorter than the measurement by 24%, indicating the near-field CFD prediction requires improvement. The outgoing U-wave front and rear shock are over predicted by 23% and 13%, respectively. The underprediction of the incoming N-wave duration by the CFD resulted in an underprediction of the U-wave duration by 40%. The comparison in Figure 5.8 is in the vicinity of the $\bar{z}$ location for the peak overpressure in the solution. This figure also shows that the period between the measurement and the prediction does not agree (underpredicted by 26%) due to the incoming N-wave being too short. Additionally, the front shock is underpredicted by 9% and the rear shock is overpredicted by 30%. Figure 5.9 shows a comparison of measurement to prediction in the shadow zone. The period of the waveform is underpredicted by 22% but the amplitude of the positive and negative peaks are in fair agreement. The front shock is overpredicted by 40% and the rear shock is underpredicted by 4%.

![Figure 5.5 – Incoming waveform from PCBoom for Case D, flight 1265, pass 2.](image-url)
Figure 5.6 – Tricomi code pressure field solution for Case D, flight 1265, pass 2.

Figure 5.7 – Case D comparison of the LNTE prediction at \( \bar{z} = 0.915 \) to the SCAMP measurement at microphone 52.
Figure 5.8 – Case D comparison of the LNTE prediction at $\bar{z} = 0.13$ to the SCAMP measurement at microphone 41.

Figure 5.9 – Case D comparison of the LNTE prediction at $\bar{z} = -0.29$ to the SCAMP measurement at microphone 35.
Figure 5.11 is the pressure field prediction for Case A and has similar characteristics as Case D in regards to the incoming and outgoing waveform location, the regions of amplification and the amplitude decay in the shadow zone. Figure 5.12 is a comparison between the measurement and the prediction at the upper edge of the Tricomi solution. The incoming N-wave has the front shock amplitude underpredicted by only 1% and the rear shock is overpredicted by only 8%. However, the duration of the incoming N-wave is slightly shorter than in the measurement by 6%. The period of the outgoing U-wave is also underpredicted by 10% and the predicted amplitude of the front and rear shocks do not agree exactly with the measurement (front and rear shock over predicted by 35% and 25%). The comparison in Figure 5.13 corresponds to the microphone location that measured the highest peak pressure in the front shock. The predicted waveform is slightly shorter in time than the measured waveform by 8%. The amplitude of the rear shocks are in fair agreement, overpredicted by about 10%, but the front shock is overpredicted by 17%. Figure 5.14 shows a comparison corresponding to the microphone location that measured the highest pressure rise for the rear shock. The agreement for the shock amplitudes is excellent between the prediction and the measurement with the front and rear shocks overpredicted by only 7% and 6%, respectively. The prediction duration is shorter in duration than the measurement by 15%. Figure 5.15 is a comparison at the microphone location that is the farthest west position in the array. The evanescent wave predicted by the code is in very good agreement with the measurement. The waveform duration is underpredicted by only 1% and the front and rear shock amplitudes are overpredicted by 3% and 8%, respectively.
Figure 5.10 – Incoming waveform from PCBoom for Case A, flight 1266, pass 4.

Figure 5.11 – Tricomi code pressure field solution for Case A, flight 1266, pass 4.
Figure 5.12 – Case A comparison of the LNTE prediction at $\bar{z} = 1.0$ to the SCAMP measurement at microphone 49.

Figure 5.13 – Case A comparison of the LNTE prediction at $\bar{z} = 0.19$ to the SCAMP measurement at microphone 17.
Figure 5.14 – Case A comparison of the LNTE prediction at $\bar{z} = 0.065$ to the SCAMP measurement at microphone 12.

Figure 5.15 – Case A comparison of the LNTE prediction at $\bar{z} = -0.21$ to the SCAMP measurement at microphone 0.
Figure 5.17 shows the pressure field solution for Case C and Figures 5.18 – 5.22 are more detailed time histories comparing predictions to the SCAMP recordings. Figure 5.18 is a comparison at the microphone location closest to the upper edge of the Tricomi solution. There is good agreement between the measurement and the prediction for the incoming N-wave. The front shock and rear shock are underpredicted by 18% and 16%, respectively. The outgoing U-wave front and rear shocks were also slightly underpredicted, at 14% for the front shock and 16% for the rear shock. The predicted N-wave and U-wave periods are slightly less than the microphone measurement, with the N-wave duration at 1% and the U-wave duration at 4%. Figure 5.19 is a comparison near the middle of the illuminated zone for the Tricomi solution. There is excellent agreement between the prediction and the measurement for both the incoming N-wave and the outgoing U-wave. The front shock amplitude is only overpredicted by 4% and the rear shock amplitude is only underpredicted by 2%. The predicted N-wave duration is within 1% of the measured waveform duration. The front and rear shock amplitude for the outgoing U-wave are only underpredicted by 5% and 7%, respectively, and the U-wave duration is underpredicted by only 4%. Figure 5.20 is a comparison corresponding to the microphone location in the array that captured the highest peak overpressure of the front shock. The agreement is very good, with only a slight overprediction of the front shock amplitude by 3% and a slight overprediction in the period of the waveform by 1%. The rear shock amplitude, however, was overpredicted by 20%. Figure 5.21 is a comparison in the vicinity of the highest peak of the rear shock. The rear shock amplitude is well-captured and is overpredicted by only 3%, but the front shock is underpredicted by 21%. The waveform duration is also well-captured and is underpredicted by only 2%. Figure 5.22 is a comparison in the shadow zone. The prediction underestimates the amplitude of both the positive and negative peak pressures at 14% and 23%, respectively. The duration is in good agreement and was underpredicted by only 2%.
Figure 5.16 – Incoming waveform from PCBoom for Case C, flight 1264, pass 4.

Figure 5.17 – Tricomi code pressure field solution for Case C, flight 1264, pass 4.
Figure 5.18 – Case C comparison of the LNTE prediction at $\bar{z} = 0.97$ to the SCAMP measurement at microphone 71.

Figure 5.19 – Case C comparison of the LNTE prediction at $\bar{z} = 0.67$ to the SCAMP measurement at microphone 67.
Figure 5.20 – Case C comparison of the LNTE prediction at $\bar{z} = 0.15$ to the SCAMP measurement at microphone 60.

Figure 5.21 – Case C comparison of the LNTE prediction at $\bar{z} = 0.085$ to the SCAMP measurement at microphone 59.
For the Case C and Case A predictions, in general, there is good agreement in the shape of the waveforms and a lot the amplitude mismatch is mostly due to the atmospheric turbulence in the vicinity of the shocks in the measured data. However, in Case D the poor agreement in waveform duration was primarily due to the incoming waveform being well below the duration of the measured waveform. This was shown to be the case by comparing the incoming waveform from PCBoom (which received its input from the CFD output) to a microphone far downstream from the caustic intercept to the ground (Figure 5.23). Observe that the incoming waveform is also much shorter in duration that what was measured by the microphone. Thus, the fact that the Tricomi code underpredicted the duration of the waveforms in the pressure field was due to the incoming waveform provided to it. However, recall that the input to the PCBoom prediction is dependent on the CFD output that is input to PCBoom, thus illustrating the dependence of the LNTE predictions on all of the incoming data and input parameters provided to it.

**Figure 5.22 – Case C comparison of the LNTE prediction at** $\bar{z} = -0.22$ **to the SCAMP measurement at microphone 55.**
Tables 5.2 through 5.5 summarize the relative agreement of the shock amplitudes and waveform durations between the measured data and the LNTE predictions. Referring back to Figures 5.1 and 5.2, the LNTE prediction is highly dependent on the input elements, especially the incoming waveform. This is definitely reflected in the resulting predictions. In Table 5.2, note that Case A and Case C had better agreement for the incoming N-wave and the resulting predictions were much closer to the measured data than for Case D. Additionally, the duration of the outgoing U-wave near the edge of the diffraction boundary layer was consistently underpredicted. This could possibly be attributed to the fact that the ray associated with the outgoing waveform has a slightly lower Mach number than the ray associated with the incoming waveform since it is generated earlier in the acceleration maneuver. The difference in the Mach number and the resulting ray propagation path would result in a slightly longer signature for the measured U-wave since the Tricomi formulation assumes a common incoming waveform
for each ray that comprises the illuminated zone. With that thought in mind, the N-wave measured at each microphone on the illuminated side of the caustic intercept comes from a different ray, each with a slightly different Mach number because the aircraft is accelerating.

**Table 5.2 – Summary of LNTE agreement percentage with measured data for the incoming N-wave and the outgoing U-wave.**

<table>
<thead>
<tr>
<th>Case</th>
<th>Incoming N-wave Percentage Under/Over Prediction</th>
<th>Outgoing U-wave Percentage Under/Over Prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Front Shock Amplitude</td>
<td>Rear Shock Amplitude</td>
</tr>
<tr>
<td>Case A</td>
<td>-1%</td>
<td>8%</td>
</tr>
<tr>
<td>Case C</td>
<td>-18%</td>
<td>-16%</td>
</tr>
<tr>
<td>Case D</td>
<td>19%</td>
<td>-27%</td>
</tr>
</tbody>
</table>

**Table 5.3 – Summary of LNTE agreement percentage for the waveform corresponding to the front shock focusing.**

<table>
<thead>
<tr>
<th>Case</th>
<th>Waveform at Peak Front Shock Focusing Percentage Under/Over Prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Front Shock Amplitude</td>
</tr>
<tr>
<td>Case A</td>
<td>17%</td>
</tr>
<tr>
<td>Case C</td>
<td>3%</td>
</tr>
<tr>
<td>Case D</td>
<td>-9%</td>
</tr>
</tbody>
</table>

**Table 5.4 – Summary of LNTE agreement percentage for the waveform corresponding to the rear shock focusing.**

<table>
<thead>
<tr>
<th>Case</th>
<th>Waveform at Peak Rear Shock Focusing Percentage Under/Over Prediction</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Front Shock Amplitude</td>
</tr>
<tr>
<td>Case A</td>
<td>7%</td>
</tr>
<tr>
<td>Case C</td>
<td>-21%</td>
</tr>
</tbody>
</table>
Table 5.5 – Summary of LNTE agreement percentage for the waveform comparisons in the evanescent zone.

<table>
<thead>
<tr>
<th>Waveform in the Evanescent Zone Percentage Under/Over Prediction</th>
<th>Front Shock Amplitude</th>
<th>Rear Shock Amplitude</th>
<th>Duration</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A</td>
<td>3%</td>
<td>8%</td>
<td>-1%</td>
</tr>
<tr>
<td>Case C</td>
<td>-14%</td>
<td>-23%</td>
<td>-2%</td>
</tr>
<tr>
<td>Case D</td>
<td>40%</td>
<td>-4%</td>
<td>-22%</td>
</tr>
</tbody>
</table>

Figure 5.24 – N-waves measured under calm, low-wind conditions (left side) compared to N-waves measured under high-wind, gusty conditions (right side) (Figure 10 from [70]).

In general, there is good agreement in the shape of the waveforms for all three cases. The predicted amplitude for the focusing of the front and rear shocks matches well with the measured data. However, some of the amplitude mismatch is due to the atmospheric turbulence in the vicinity of the shocks in the measured data. The atmospheric turbulence in the microphone measurements is observed in the form of “small” pressure...
perturbations in and around the more gross features of the sonic booms [70]. Figure 5.24 (Figure 10 from [70]) shows the notional variations observed in sonic booms that result from propagation in different wind velocity conditions. The left side of Figure 5.24 is an example of N-waves measured under calm, low-wind conditions. The right side of Figure 5.24 is an example of N-waves measured under gusty, high-wind conditions. The propagation through the gusty, high-wind conditions produced pressure perturbations in and around the front and rear shocks due to the influence of the atmospheric turbulence present in the gusty wind conditions. These types of pressure perturbations are not visible in the LNTE predictions because the effects of the atmospheric turbulence are not included in the LNTE model.

Figure 5.25 – Illustration to depict the SCAMP microphone arrangement relative to the actual passage of a wavefront due to a maneuvering aircraft and a temperature gradient with a decreasing sound speed with an increase in altitude. The evanescent wave measure by each microphone corresponds to a different instant of the wavefront propagation.
A possible reason for the differences in the comparisons in the shadow zone can be caused by the fact that for the “actual” flight test experiment, the evanescent wave that reaches the ground isn’t actually generated from the caustic intercept at the ground as modeled here. The evanescent wave that reaches the ground in the shadow zone is actually generated up above the ground from the edge of the wavefront as the caustic is formed. The depiction in Figure 5.25 illustrates this. The curved blue lines represent the wavefront propagating from the supersonic aircraft at different instances in time. The evanescent wave in the shadow zone is always beneath the wavefront as it travels through the atmosphere. Thus, the evanescent wave measured by each microphone in the shadow zone had emanated from a different location in the sky from the edge of the wavefront as it propagated to the ground.

The SCAMP report [58] includes comparisons for the other focus boom prediction codes in SCAMP to the measured flight test data. Although it is beyond the scope of this thesis to provide a thorough summary of the results for the comparisons of the other codes, it can be said that they were in fair agreement with the microphone data. However, the LNTE results from this chapter provided the best agreement to the measured data for all of the codes exercised as part of SCAMP.

When attempting predictions of actual flight measurements, the uncertainty associated with each of the input data leaves the LNTE prediction susceptible to only be as accurate as the information input to the code. For the SCAMP comparisons in this chapter it was shown that the input data for Case A and Case C were adequate for the LNTE code to make sonic boom focusing predictions. The input data for Case D was adequate with the exception of the incoming waveform that was too short in duration. The LNTE code has been both numerically (Chapter 3) and experimentally validated for the important aspects of sonic boom focusing. That is, the LNTE code can model the pressure field in the illuminated zone and also the diffraction near the caustic and in the shadow zone. The next chapter will exercise the LNTE code to predict sonic boom focusing for a NASA conceptual low-boom demonstration vehicle so that focusing of signatures that are not N-waves can be examined.
Chapter 6

Focus Boom Analysis of a NASA Conceptual Low-boom Demonstration Vehicle

6.1 Motivation

The intent of the work summarized in this chapter exercises the focus boom prediction methodology from Chapter 5 for a proposed vehicle that does not generate an N-wave and has “shaping” features in the propagated waveform. Most supersonic vehicles are initially designed for these shaping features to be present during cruise conditions. However, the Mach number and altitude that correspond to focusing conditions are typically not close to the cruise conditions. The byproduct of the aircraft operating away from the low-boom design conditions typically increases the pressure amplitude of most portions of the waveform in an adverse manner. An additional source of amplification is that the converging rays due to the acceleration maneuver of the aircraft trajectory cause the Blokhintzev invariant scaling factor [15, 48] to be larger than for steady-state aircraft operations. Thus, the observed focusing of the pressure field in the vicinity of the caustic is greatly influenced by the amplification already present in the incoming signal due to the off-design conditions and unsteady aircraft maneuvering.

The goals of this chapter are two-fold: 1) to examine whether the shaping features designed for cruise operations persist under focusing conditions, and 2) to vary the aircraft trajectory and examine the influence of the climb profile changes on corresponding changes in the sonic boom focusing.
6.2 Vehicle and Trajectory Description

NASA has a conceptual design for a low-boom demonstration aircraft (Figure 6.1) [58, 71]. The demonstrator concept would have a length of 127 ft. (38.7 m), a weight of 36,000 lbs. (16,143 kg), a cruise Mach of 1.6, a cruise altitude of 45,000 ft. (13,716 m) and a supersonic range of 1000 nm (1852 km). Such a vehicle would not serve as a potential commercial aircraft but only demonstrate that it is possible to design an aircraft that has a quieter sonic boom during supersonic operations. The low-boom shaping features were designed for the cruise segment portion of the flight operations. Four different climb profiles were considered for focus boom analysis. The variations in climb criteria provide a range of Mach numbers, Mach rates and altitudes corresponding to the onset of focusing at the ground during the climb to the cruise altitude. The conditions corresponding to the focusing at the ground for each profile are summarized in Table 6.1. The focusing conditions were obtained by performing PCBoom ray tracing along the zero-degree azimuth at one-second intervals along the climb profile, identifying the ray linked to caustic intercept to the ground and then noting the altitude, Mach number, etc. for that ray. The atmosphere for this analysis was the ANSI standard atmosphere [11] for temperature and relative humidity without winds (Figure 6.2).

Table 6.1 – Trajectory conditions corresponding to focus boom generation.

<table>
<thead>
<tr>
<th>NASA Case #</th>
<th>Mach</th>
<th>Mach Rate (Mach/s)</th>
<th>Altitude (ft)</th>
<th>Altitude (m)</th>
<th>Climb Angle (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.146</td>
<td>0.00134</td>
<td>33,144</td>
<td>10,102</td>
<td>0.642</td>
</tr>
<tr>
<td>2</td>
<td>1.173</td>
<td>0.00138</td>
<td>35,821</td>
<td>10,918</td>
<td>0.000</td>
</tr>
<tr>
<td>3</td>
<td>1.168</td>
<td>0.00103</td>
<td>38,154</td>
<td>11,629</td>
<td>0.001</td>
</tr>
<tr>
<td>4</td>
<td>1.157</td>
<td>0.00066</td>
<td>40,524</td>
<td>12,352</td>
<td>0.003</td>
</tr>
</tbody>
</table>
Figure 6.1 – NASA low boom demonstrator concept (courtesy of NASA) [58].

Figure 6.2 – ANSI upper atmospheric conditions for temperature, pressure and relative humidity [11].
6.3 Focus Boom Analysis

The focus boom analysis followed the same prediction methodology presented in Chapter 5 for comparison to the measured SCAMP data. The near-field CFD for these cases, however, corresponds to the altitudes and Mach numbers in Table 6.1 for the vehicle illustrated in Figure 6.1. It is understood that even though the aircraft is undergoing unsteady maneuvering, the accelerations are “slow enough” for a steady CFD analysis to provide sufficient near-field pressure predictions. Figure 6.3 is a sample near-field CFD contour slice for the zero-degree azimuth. The black line is the pressure values extracted at approximately one body length for PCBoom propagation to the far-field. The near-field pressure signatures from CFD were directly matched to the onset of the far-field propagation from PCBoom like the prediction process from Chapter 5.

![Figure 6.3 – Sample near-field CFD of the NASA low boom demonstrator concept (courtesy of NASA) [58]. The black line is the pressure values at one body length from the vehicle.](image)
The PCBoom ray tracing also provides the caustic geometry and the extent of the focal zone at the ground. Table 6.2 summarizes these elements for the four cases examined here. Note that the fourth column is the distance between the upper and lower edges of the diffraction boundary layer as it intercepts with the ground. That is, the Tricomi footprint distance is twice the distance between where the rays corresponding to $\bar{z} = 1$ and $\bar{z} = 0$ propagate to the ground. The caustic geometry and orientation at the ground are dependent on the trajectory and flight conditions of the aircraft as it generated the ray corresponding to $\bar{z} = 0$. That ray’s elevation angle once it reaches the ground and the atmospheric conditions. As the caustic line becomes more grazing to the ground, the projection of the Tricomi domain to the ground spreads out farther and increases the region on the ground exposed to the diffraction boundary layer.

Table 6.2 - NASA focus boom conditions for each climb profile case.

<table>
<thead>
<tr>
<th>Case #</th>
<th>$f_{ac}$ (Hz)</th>
<th>$R_{tot}$ (m)</th>
<th>Tricomi Footprint Distance (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>9.20</td>
<td>98236</td>
<td>3798</td>
</tr>
<tr>
<td>2</td>
<td>9.00</td>
<td>89950</td>
<td>3124</td>
</tr>
<tr>
<td>3</td>
<td>9.06</td>
<td>89451</td>
<td>3403</td>
</tr>
<tr>
<td>4</td>
<td>8.82</td>
<td>87854</td>
<td>5472</td>
</tr>
</tbody>
</table>

Figures 6.5, 6.9, 6.13 and 6.17 are plots showing the incoming waveform for cases 1 – 4, respectively, as propagated by PCBoom. These far-field predictions are the “direct” waveforms and do not have a ground reflection factor included since these propagations do not go all the way to the ground. Observe that the front portion of the waveform has a “flat-top” shape to it prior to the onset of the expansion portion of the signature. Additionally, instead of one shock in the rear part of the waveform, there is an intermediate shock in the middle of the expansion portion of the signature and two closely spaced shocks at the very rear portion. These features of the waveform are typically designed into the aircraft with the intent of reducing the peak over pressure and metrics levels of a sonic boom.
The four NASA cases were initially computed with 4,000 points in the $\bar{z}$-axis and then recomputed with 8,000 points in the $\bar{z}$-axis. The pressure field solutions were very similar for both of these $\bar{z}$-axis grid discretizations and also produced identical metrics results. Thus, it was assumed that 8,000 grid points in the $\bar{z}$-axis was sufficient for grid convergence. The iterative convergence history also showed trends similar to what was shown in Chapter 3 for the N-wave case. The iterative convergence history as a function of pseudotime for Case 1 is shown in Figure 6.4 and is representative of the convergence history for all of the NASA cases analyzed. The two Figures 6.6, 6.10, 6.14 and 6.18 are the Tricomi solution pressure field solutions. The results presented here are for a computational domain from $\bar{z} = -1$ to $\bar{z} = 1$ with 8,000 points in the $\bar{z}$-axis and 32,768 points in the $\bar{t}$-axis. The values in Figures 6.6, 6.10, 6.14 and 6.18 do not have a ground reflection factor included. One can observe the superposition of the incoming wave at the upper left and the outgoing wave at the upper right in a manner consistent with the N-wave focusing from Chapter 5. However, there are some differences in the resulting pressure fields for shaped signatures when compared to those of N-waves, specifically, the $\bar{z}$ locations of the peak pressure amplification.

The metrics used for this analysis were also used in [44] to help characterize the annoyance to sonic booms heard outdoors, specifically, low-amplitude sonic booms. The analysis here follows suit by including the PL, $\text{SEL}_A$, $\text{SEL}_C$ and $\text{SEL}_Z$. Additionally, the minimum and maximum pressure levels in the signature are also included in the analysis to assess the overall amplification of the positive and negative peak pressure levels in the solution. The plots shown in Figures 6.7, 6.11, 6.15 and 6.19 are metrics calculations as a function of $\bar{z}$ (dimensionless distance from the caustic) and include a ground reflection factor of 1.9 applied to the solution prior to computing the metrics. Observe that the $\bar{z}$ location corresponding to the peak overpressure does not coincide with the $\bar{z}$ location for the maximum $\text{SEL}_A$ and PL. In fact, the peak pressure levels are decreasing from their maximum values as one gets closer to $\bar{z} = 0$ yet the loudness levels are still increasing. The maximum overpressures typically occur at a $\bar{z}$ between 0.17 to 0.19 while the maximum loudness levels typically occur at a $\bar{z}$ between 0.03 to 0.04. Table 6.3 summarizes the maximum levels for each metric and Table 6.4
summarizes the location at which each of the levels in Table 6.3 occur. Observe that there is approximately 15 dB difference between the PL and SEL\textsubscript{A} values in Table 6.3 and similar locations in maximum PL and maximum SEL\textsubscript{A} in Table 6.4. This is consistent with the findings from [9] where the correlation to human response for the PL and SEL\textsubscript{A} were very similar for shaped sonic booms.

Table 6.3 – Maximum metric levels for each focus boom case.

<table>
<thead>
<tr>
<th>Case #</th>
<th>Minimum Pressure (Pa)</th>
<th>Maximum Pressure (Pa)</th>
<th>Maximum Perceived Loudness (dB)</th>
<th>Maximum SEL\textsubscript{A} (dB)</th>
<th>Maximum SEL\textsubscript{C} (dB)</th>
<th>Maximum SEL\textsubscript{L} (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-76.6</td>
<td>100.7</td>
<td>111.1</td>
<td>95.9</td>
<td>110.6</td>
<td>118.6</td>
</tr>
<tr>
<td>2</td>
<td>-72.7</td>
<td>102.0</td>
<td>106.7</td>
<td>91.5</td>
<td>109.5</td>
<td>118.3</td>
</tr>
<tr>
<td>3</td>
<td>-67.0</td>
<td>73.1</td>
<td>104.1</td>
<td>89.1</td>
<td>107.7</td>
<td>116.8</td>
</tr>
<tr>
<td>4</td>
<td>-65.8</td>
<td>68.3</td>
<td>103.0</td>
<td>88.2</td>
<td>107.5</td>
<td>116.5</td>
</tr>
</tbody>
</table>

Table 6.4 – \( \overline{Z} \) locations corresponding to the metrics levels in Table 6.3.

<table>
<thead>
<tr>
<th>Case #</th>
<th>( \overline{Z} ) location for min. pressure</th>
<th>( \overline{Z} ) location for max. pressure</th>
<th>( \overline{Z} ) location for max. PL</th>
<th>( \overline{Z} ) location for max. SEL\textsubscript{A}</th>
<th>( \overline{Z} ) location for max. SEL\textsubscript{C}</th>
<th>( \overline{Z} ) location for max. SEL\textsubscript{L}</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5294</td>
<td>0.1893</td>
<td>0.0313</td>
<td>0.0313</td>
<td>0.0773</td>
<td>0.2198</td>
</tr>
<tr>
<td>2</td>
<td>0.2303</td>
<td>0.1693</td>
<td>0.0293</td>
<td>0.0288</td>
<td>0.1108</td>
<td>0.2288</td>
</tr>
<tr>
<td>3</td>
<td>0.2453</td>
<td>0.1763</td>
<td>0.0333</td>
<td>0.0338</td>
<td>0.1123</td>
<td>0.2213</td>
</tr>
<tr>
<td>4</td>
<td>0.2683</td>
<td>0.1658</td>
<td>0.0363</td>
<td>0.0378</td>
<td>0.1023</td>
<td>0.2223</td>
</tr>
</tbody>
</table>

Figures 6.8, 6.12, 6.16, and 6.20 are comparisons of pressure time-histories between the waveform at the peak overpressure (blue line) and the maximum Perceived Loudness level (red line). Their metrics values are shown in Table 6.5. Note that the waveforms with the higher peak pressures are higher by a factor of approximately one and a half to two, yet the signatures with the higher loudness levels are so by approximately 6 - 8 dB. In all four figures, the middle shock of the waveform with the higher loudness levels is considerably steeper than the other shocks in both waveforms and is likely the main contributor to the higher loudness levels. Additionally, the rear shocks are much steeper than in the incoming waveform but are not as steep as the middle shock.
Figure 6.21 shows a Power Spectral Density (PSD) comparison of spectra at certain $z$ locations for Case 4. It is very obvious that the spectrum corresponding to $z = 0.04$ has the largest values across a majority of the frequency spectrum, and hence, also illustrates why that $z$ location has the highest loudness levels. In fact, the spectrum at $z = 0.17$ is somewhat similar to the spectrum at the upper boundary ($z = 1.0$). Lastly, the spectrum at $z = -0.20$ in the shadow zone primarily has low frequency content and the change in spectral characteristics between the illuminated region and the shadow zone can be attributed to diffraction. This is as expected, since the exponential decay with distance away from the caustic is a function of frequency (Eqs. 3.1.10 and 3.1.11). That is, the higher the frequency, the faster the decay will occur as one moves farther into the shadow zone.

Table 6.5 – Comparison of metrics levels for the signatures plotted in Figures 6.7, 6.11, 6.15 and 6.19.

<table>
<thead>
<tr>
<th>Case #</th>
<th>Perceived Loudness at Location of Max. Pressure (dB)</th>
<th>Maximum Perceived Loudness (dB)</th>
<th>SELA at Location of Max. Pressure (dB)</th>
<th>Maximum SELA (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>104.2</td>
<td>111.1</td>
<td>88.0</td>
<td>95.9</td>
</tr>
<tr>
<td>2</td>
<td>101.4</td>
<td>106.7</td>
<td>85.1</td>
<td>91.5</td>
</tr>
<tr>
<td>3</td>
<td>98.4</td>
<td>104.1</td>
<td>82.0</td>
<td>89.1</td>
</tr>
<tr>
<td>4</td>
<td>91.6</td>
<td>103.0</td>
<td>81.4</td>
<td>88.2</td>
</tr>
</tbody>
</table>

6.4 Observations

It does appear that the shaped “flat-top” feature in the front half of the incoming waveforms does lead to less amplification at the front shock than typically observed for N-wave focusing. When removing the ground reflection factor, the effects of focusing for these four low-boom cases cause amplification on the order of 2-3 times the peak overpressure. This is in contrast to the N-wave cases presented in Chapter 5 where the focusing amplified the peak overpressure of the incoming waveform by a factor of approximately 3 to 3.5. Even though the flat-top shape from the incoming waveform is absent in the focused signatures, the “rippled” feature from the incoming waveform following the front shock is still somewhat evident. This would lead to one to think there should be optimism that aircraft shaping efforts, combined with aircraft operations, are
possible means to influence the amplification and increased loudness levels due to sonic boom focusing. Table 6.6 compares the peak overpressure and maximum Perceived Loudness with the corresponding Tricomi footprint distance. Based on this analysis, there seems to be a trade between reducing the loudness and peak pressure with keeping the focus region at the ground from increasing. Specifically, Case 4 has the lowest peak over pressure and lowest loudness, but also has the largest focus zone at the ground. From these four cases examined, Case 3 has the best balance between managing the size of the focus region at the ground and reducing the metrics values.

The variations in the trajectories for Case 1 through Case 4 (Table 6.1) did little to change the focus parameters (Table 2). The characteristic frequencies were approximately 9 Hz and the total radius of curvatures were on the order of 90,000 m. However, the variations in maximum PL levels for Cases 1 through 4 varied between ~103 dB and ~111 dB (Table 6.6). Looking at the incoming signatures in Figures 6.5, 6.9, 6.13, 6.17, the waveforms reduce in amplitude in conjunction with the reduction in amplitude for the PL values in Table 6.6. Referring to Table 6.1, the ray associated with the caustic intercept at the ground for Case 1 propagated from a lower altitude and has a faster Mach rate than Case 4, which propagated from a higher altitude and had a slower Mach rate. These trajectory differences contribute to observed differences in the far-field signature that is input to the Tricomi code. That is, when the onset of the focusing occurs at a lower altitude and faster acceleration, the incoming waveform tends to be louder (and vice versa). For the four cases examined, the reduced amplitude of the incoming waveforms has more of an influence on reducing the maximum PL in the focusing region than the variation in the focusing geometry.

The above discussion only made mention of the metrics correlated to human response for sonic booms heard outdoors. The levels in the PSD spectra below 20 Hz in Figure 6.21 in combination with the peak temporal pressures predicted from the four NASA cases show that these levels may likely produce entirely different human reactions when heard indoors. The human response to sonic booms heard indoors is the subject of ongoing
research [72 - 75] and the indoor human response to focus booms are opportunities left for future work.

Table 6.6 – Comparison between signature amplitude, loudness and the size of the focal zone at the ground.

<table>
<thead>
<tr>
<th>Case #</th>
<th>Maximum Pressure (Pa)</th>
<th>Maximum Perceived Loudness (dB)</th>
<th>Tricomi Footprint Distance (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100.7</td>
<td>111.1</td>
<td>3797.9</td>
</tr>
<tr>
<td>2</td>
<td>102.0</td>
<td>106.7</td>
<td>3124.0</td>
</tr>
<tr>
<td>3</td>
<td>73.1</td>
<td>104.1</td>
<td>3403.1</td>
</tr>
<tr>
<td>4</td>
<td>68.3</td>
<td>103.0</td>
<td>5472.0</td>
</tr>
</tbody>
</table>

Figure 6.4 – Iterative progression of the convergence parameters for NASA Case 1.
Figure 6.5 – Incoming waveform for Case 1, Mach 1.166, mdot=0.00134.

Figure 6.6 – Tricomi code pressure field contour for Case 1, Mach 1.166, mdot=0.00134.
Figure 6.7 – Metrics plots for Case 1, Mach 1.166, \( \text{mdot}=0.00134 \).

Figure 6.8 – Case 1 comparison of waveforms corresponding to the maximum peak pressure (blue line) and the maximum Perceived Loudness (red line).
Figure 6.9 – Incoming waveform for Case 2, Mach 1.175, mdot=0.00138.

Figure 6.10 – Tricomi code pressure field contour for Case 2, Mach 1.175, mdot=0.00138.
Figure 6.11 – Metrics plots for Case 2, Mach 1.175, mdot=0.00138.

Figure 6.12 – Case 2 comparison of waveforms corresponding to the maximum peak pressure (blue line) and the maximum Perceived Loudness (red line).
Figure 6.13 – Incoming waveform for Case 3, Mach 1.169, mdot=0.00103.

Figure 6.14 – Tricomi code pressure field contour for Case 3, Mach 1.169, mdot=0.00103.
Figure 6.15 – Metrics plots for Case 3, Mach 1.169, \( \dot{m}=0.00103 \).

Figure 6.16 – Case 3 comparison of waveforms corresponding to the maximum peak pressure (blue line) and the maximum Perceived Loudness (red line).
Figure 6.17 – Incoming waveform for Case 4, Mach 1.161, \(\text{mdot}=0.00066\).

Figure 6.18 – Tricomi code pressure field contour for Case 4, Mach 1.161, \(\text{mdot}=0.00066\).
Figure 6.19 – Metrics plots for Case 4, Mach 1.161, \( \text{mdot}=0.00066 \).

Figure 6.20 – Case 4 comparison of waveforms corresponding to the maximum peak pressure (blue line) and the maximum Perceived Loudness (red line).
Figure 6.21 – Comparison of Power Spectral Density (PSD) spectra at different $z$ locations for Case 4.
Chapter 7

Conclusions and Recommendations for Future Work

7.1 Summary and Conclusions

A new model that augments the nonlinear Tricomi equation was derived and numerically implemented with a computer code. For the purposes of the NASA SCAMP project (and this thesis), the LNTE code was coupled with PCBoom for focus boom prediction. A large-scale flight test sponsored by NASA provided a vast set of data for experimental validation of the focus boom code and the overall prediction methodology. The LNTE code showed very good agreement with the numerical validation and the LNTE code and the prediction methodology agreed very well with the experimental data. However, the experimental validation did show that the output of the Tricomi code is highly dependent on the inputs. That is, the prediction capability of the LNTE code is limited by the fidelity of the input information. Additionally, atmospheric turbulence was evident in most of the SCAMP recordings and did pose a challenge in certain microphone comparisons.

The research presented in this thesis showed that including loss mechanisms makes a difference in the rise profiles in shocks that are present in sonic boom focusing. The work in this thesis also showed that overprediction of the amplitudes of the signatures can occur if the loss mechanisms are not included (Section 3.3). This research showed that including the atmospheric loss mechanisms plays a role in modeling the profile and rise time of the shocks for the incoming and outgoing waves in the vicinity of the caustic. By
including the loss mechanisms, the shock profiles modeled by the LNTE are more consistent with the physics included in the propagation of the incoming waveform. The work in this thesis provided predictions of sonic boom focusing that encompassed a frequency range much higher than previously published.

The focusing of shaped sonic boom signatures was also examined in this thesis. The results show that when the incoming waveform is shaped (less of an N-wave), the amplification and increased loudness due to the focusing effects are reduced relative to that of N-wave focusing. Additionally, when shaping features are introduced into the incoming waveform the location in the solution corresponding to the peak loudness does not necessarily coincide with the location of the peak overpressure (as is the trend for N-wave focusing). However, a trade was recognized from the results in Chapter 6 where the cases corresponding to lower metrics values also had larger regions on the ground over which the focusing occurs. That is, the diffraction boundary layer as it intercepts the ground was typically larger for cases where the aircraft accelerated more slowly. The slower accelerations at the higher altitudes trended towards in lower metrics values but at the expense of a larger area at the ground corresponding to the focusing zone.

7.2 Recommendations for Future Work

Modifications could be made to improve the code’s computational efficiency and accuracy. The iterative convergence approach presented in this thesis examined the primary aspects of the solution domain and provides guidance towards convergence. However, an improvement can be made to change the iterative convergence scheme from a “relative” convergence (reduction in magnitude of the convergence parameters by 4 to 5 orders of magnitude) to an approach that is more “absolute” in nature. If one examines the unsteady term from Eq. (3.1.1), one can discretize it directly and monitor its change in value with pseudotime:
\[
\frac{\partial^2 p'}{\partial \sigma \partial t} \Rightarrow \left( \frac{\partial^2 p'}{\partial \sigma \partial t} \right)_{i,j}^k = \frac{\left( \frac{\partial p'}{\partial \sigma} \right)_{i+1,j}^k - \left( \frac{\partial p'}{\partial \sigma} \right)_{i-1,j}^k}{2\Delta \xi}
\]

where \( k \) denotes the value at the current pseudotime step, \( k-1 \) denotes the value at the previous pseudotime step, \( j \) is the index for the \( \bar{z} \)-axis location and \( i \) is the index for the \( \bar{t} \)-axis location. The benefit to this proposed convergence scheme is that it monitors the Frobenius norm of the LHS of Eq. (3.1.1) as it trends to zero. The disadvantage is that the memory requirements would double since the solution from the previous iteration is required as a part of Eq. (7.2.1). The code must be executed on a system with a significant amount of RAM or the code must be modified to write/read the solution from the previous iteration as part of the convergence evaluation scheme.

Memory requirements can potentially be reduced in two ways. First, using a higher order discretization scheme can improve the resolution of the higher wavenumbers [76]. Instead of using second order discretization for the diffraction solution steps and the corresponding boundary conditions, one could use fourth order discretization. Additionally, the \( \bar{z} \)-axis discretization was carried out with a constant \( \Delta \bar{z} \) increment. A non-uniform discretization could be used so that a larger \( \Delta \bar{z} \) increment is applied where a reduced resolution is required. That is, in the shadow zone where the pressure amplitude decays rapidly away from the caustic, the resolution can be more coarse than in the illuminated zone where sharper pressure gradients exist.

A limitation of the Tricomi formulation used in this thesis is that it is valid “primarily” in the undertrack, or zero-degree azimuth, plane. It would be important to quantify the
extent to which the assumption is appropriate. Also, the Tricomi formulation also assumed that the caustic intercept at the ground is not at grazing incidence to the ground. The analysis presented in Chapter 6 showed that where the acceleration rate is reduced, the caustic becomes more grazing to the ground and ground impedance effects may no longer be negligible. The formulation of the problem would need to be revised to incorporate those effects.

The comparisons to measured data showed the amplitudes in the evanescent zone were underpredicted. It is possible that applying the “pressure doubling” to the entire Tricomi solution is not applicable in the shadow zone. That is, physically the pressure doubling only applies in the illuminated zone which would then lead to higher amplitudes at the interface to the shadow zone than predicted by the LNTE formulation. Future work could entail including the finite ground impedance effects, similar to the approach in [77], and further augmenting the LNTE formulation.

Test participants at the SCAMP test “heard” several phenomena that are also not accounted for or included in the LNTE predictions. For several of the flight passes a “rumble” noise was heard just prior to the arrival of the focus boom events. Several possible theories were discussed for the source of the rumble: jet noise, wave propagation in the ground, and so forth. Test participants in the shadow zone, but close to the caustic, also heard a “sheet metal” noise primarily consisting of high frequencies that decay rapidly and coincided with the amplitude excursions in the evanescent wave. The sound was very directional, emanating from overhead, and seemed to occur as the caustic portion of the wavefront approached and then moved away from the listeners. It is possible this occurred due to high frequency diffraction not accounted for in the model. These anomalies could possibly be topics for future investigation.
According to [24], the influence of the wind was to shift the origin of the formulation away from the ray intercept at the caustic. This same assumption was utilized for the present work and for SCAMP. However, the influence of the wind can be further investigated for Doppler effects or other effects similar to the approach in [78]. That is, it is possible that terms that were neglected in the LNTE derivation could be included to provide additional effects of wind for each specific term in the lossy nonlinear Tricomi equation. An additional item for inclusion of a future model equation could also account for the wind gradients in a similar manner to [50, 79].

The influence of propagation through atmospheric turbulence was observed in all of the SCAMP microphone recordings. Alternative models and prediction methodologies can be utilized to incorporate terms that include the effects of atmospheric turbulence into focus boom predictions. For example, one could account for turbulence using terms given by Salomons [80] and applying them in the frequency domain, or, by applying “filter functions” [81] in the time domain as a post-processing step after the solution is computed.

Accurate focus predictions have not yet been made at the lateral edge of the caustic intercept to the ground. The focusing scenario is no longer two-dimensional and would require accounting for diffraction in all three dimensions. This is because there is an additional shadow zone that extends laterally away from the primary sonic boom carpet. The focusing conditions at this location must match the focusing as one migrates to the zero-degrees azimuth and also match the conditions for creeping waves generated in the shadow zone as one moves forward along the carpet edge in the direction of the aircraft flight path away from the caustic region at the ground. A possible model equation capable handling this type of focusing is the HOWARD formulation [82].
Appendix

Explanation of the Upper and Lower Boundary Conditions

A.1 Upper boundary condition

This section of the appendix describes how one obtains the radiation condition for the upper boundary, Eq. (2.2.33), from the expression for the sum of the incoming and outgoing waveforms, Eq. (2.2.32). Equation (2.2.32) is repeated below as Eq. (A.1.1) for convenience.

\[ p'(\tilde{t}, z \to \infty) \approx \tilde{z}^{-1/4} \left[ F\left(\tilde{t} + \frac{2}{3} \tilde{z}^{3/2}\right) + G\left(\tilde{t} - \frac{2}{3} \tilde{z}^{3/2}\right) \right] \quad \text{(A.1.1)} \]

The outgoing waveform, \( G \), is not known prior to the start of the computations. Hence, the motivation here is to obtain an equation for the upper boundary condition that does not require knowledge of the outgoing waveform. The first step is to take the derivative of Eq. (A.1.1) with respect to \( \tilde{t} \).

\[ \frac{\partial p'}{\partial \tilde{t}} \approx \tilde{z}^{-1/4} \left[ F'\left(\tilde{t} + \frac{2}{3} \tilde{z}^{3/2}\right) + G'\left(\tilde{t} - \frac{2}{3} \tilde{z}^{3/2}\right) \right] \quad \text{(A.1.2)} \]

The prime symbol next to \( F \) and \( G \) denotes a derivative with respect to the argument.
The next step is to take the derivative of Eq. (A.1.1) with respect to \( \bar{z} \).

\[
\frac{\partial p'}{\partial \bar{z}} \approx -\frac{1}{4} \bar{z}^{-5/4} \left[ F'\left(\bar{t} + \frac{2}{3} \bar{z}^{3/2}\right) + G\left(\bar{t} - \frac{2}{3} \bar{z}^{3/2}\right) \right] + \bar{z}^{-1/4} \left[ \bar{z}^{1/2} F'\left(\bar{t} + \frac{2}{3} \bar{z}^{3/2}\right) - \bar{z}^{1/2} G'\left(\bar{t} - \frac{2}{3} \bar{z}^{3/2}\right) \right]
\]  

(A.1.3)

An examination of the \( G \) and \( G' \) terms for Eqs. (A.1.1), (A.1.2) and (A.1.3) leads one to determine the appropriate coefficients and variables required to eliminate the \( G \) and \( G' \) terms in those equations. If one multiplies Eq. (A.1.1) by \( \bar{z}^{-5/4} / 4 \) and Eq. (A.1.2) by \( \bar{z}^{1/2} \) and sums Eqs. (A.1.1) through (A.1.3), the result is Eq. (A.1.4).

\[
\bar{z}^{1/2} \frac{\partial p'}{\partial \bar{t}} + \frac{\partial p'}{\partial \bar{z}} + \bar{z}^{-1} \frac{p'}{4} = \bar{z}^{1/4} 2F'\left(\bar{t} + \frac{2}{3} \bar{z}^{3/2}\right)
\]  

(A.1.4)

Equation (A.1.4) can also be expressed as:

\[
\bar{z}^{1/4} \frac{\partial p'}{\partial \bar{t}} + \bar{z}^{-1/4} \frac{\partial p'}{\partial \bar{z}} + \bar{z}^{-5/4} \frac{p'}{4} = 2F'\left(\bar{t} + \frac{2}{3} \bar{z}^{3/2}\right)
\]  

(A.1.5)

Equation (A.1.5) is the same as Eq. (2.2.33).

### A.2 Lower boundary condition

The intent of this section in the appendix is to explain how one can obtain Eq. (3.1.12) from Eqs. (3.1.10) and (3.1.11) as was derived in [14]. Equation (3.1.10) is restated below as Eq. (A.2.1)
\[
\hat{p}_n(\bar{z}) = \sqrt{2\pi} [1 + i \text{sgn}(\omega_n)] |\omega_n|^{1/6} Ai(-|\omega_n|^{2/3} \bar{z}) \hat{F}_n
\] (A.2.1)

One can consolidate Eq. (A.2.1) by replacing the portion of the expression that does not include the Airy function into a separate coefficient, \( a_n(\omega_n) \).

\[
a_n(\omega_n) = \sqrt{2\pi} [1 + i \text{sgn}(\omega_n)] |\omega_n|^{1/6} \hat{F}_n
\] (A.2.2)

From [12], the asymptotic expression for the Airy function for large positive arguments was given by Eq. (3.1.11), restated as Eq. (A.2.3).

\[
\text{Ai}(\xi \to \infty) \approx \frac{\exp\left(-\frac{2}{3} \xi^{3/2}\right)}{2\sqrt{\pi} \xi^{1/4}}
\] (A.2.3)

If we allow \( \xi = |\omega|^{2/3} |z| \) and substitute Eqs. (A.2.2) and (A.2.3) into Eq. (A.2.1), we obtain:

\[
\hat{p}_n(\bar{z}) = \frac{a_n(\omega_n)}{2\sqrt{\pi}} |\omega_n|^{-1/6} |\bar{z}|^{-1/4} \exp\left(-\frac{2}{3} |\omega_n| |\bar{z}|^{3/2}\right)
\] (A.2.4)

While we could actually use Eq. (A.2.4) as the lower boundary condition, it could be further simplified. First, take the derivative of Eq. (A.2.4) with respect to \( \bar{z} \).

\[
\frac{d \hat{p}_n(\bar{z})}{d\bar{z}} = \frac{a_n(\omega_n)}{2\sqrt{\pi}} |\omega_n|^{-1/6} |\bar{z}|^{-5/4} \exp\left(-\frac{2}{3} |\omega_n| |\bar{z}|^{3/2}\right) + \frac{a_n(\omega_n)}{2\sqrt{\pi}} |\omega_n|^{5/6} |\bar{z}|^{-1/4} \exp\left(-\frac{2}{3} |\omega_n| |\bar{z}|^{3/2}\right)
\] (A.2.5)
Rearranging the terms of the above equation can show that the RHS of Eq. (A.2.5) includes Eq. (A.2.4).

\[
\frac{d\hat{p}_n(z)}{dz} = \left[ \frac{1}{4} |z|^{-1} + |\omega_n| |z|^{1/2} \right] a_n(\omega_n) \frac{a_n(\omega_n)}{2\sqrt{\pi}} |\omega_n|^{-1/6} |z|^{-1/4} \exp\left(-\frac{2}{3}|\omega_n|^3/|z|^{3/2}\right)
\]  

(A.2.6)

Equation (A.2.6) can be rewritten using Eq. (A.2.4) so that we obtain [14]:

\[
\frac{d\hat{p}_n(z)}{dz} = \left[ \frac{1}{4} |z|^{-1} + |\omega_n| |z|^{1/2} \right] \hat{p}_n(z)
\]  

(A.2.7)

Equation (A.2.7) is the same as Eq. (3.1.12).
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Vita

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