MODELS OF NONCOOPERATIVE GAMES

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by
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Abstract

This thesis is divided into three parts.
In the first part, motivated by Stackelberg differential games, we consider a “non-classical” control system where the dynamics depends also on the spatial gradient of the feedback control function. Given a probability measure on the set of initial states, we seek feedback controls which minimize the expected value of a cost function. A relaxed system is considered, and compared with the ”nonclassical” one. Necessary conditions for optimality and the continuous dependence of expected minimum cost are discussed, for both systems.
The second part is concerned with Stackelberg solutions of feedback type for a differential game with random initial data. The existence of a Stackelberg equilibrium solution is proved, provided that the control is restricted in a finite dimensional space of admissible functions. An example shows that, for a wide class of systems, where the minimal cost for the leading player would correspond to an impulsive control function, and thus cannot be exactly attained.
In the last part of the thesis we consider a continuum model of the limit order book in a stock market, regarded as a noncooperative game for n players. Motivated by the necessary conditions for a Nash equilibrium, we introduce a two-point boundary value problem for a system of discontinuous ODEs, and prove that this problem always has a unique solution, Under some additional assumptions we then prove that this solution actually yields a Nash equilibrium.
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Dedication

To my parents
Chapter 1

Introduction

Game theory is a study of strategic decision making. It is a theory of conflict and cooperation between rational decision makers and has been widely recognized as an important tool in economics, finance, engineering, and psychology, as well as logic and biology. Differential games are dynamic game models used to study systems that evolve in continuous time and where the system dynamics can be described by differential game.

The games studied in game theory are well-defined mathematically. To be fully defined, a game must specify the following elements: the players of the game, the information and actions available to each player at each decision point, and the payoffs for each outcome. Game theorists use the different types of equilibriums concept to analyze the outcome of the strategic interaction of several decision makers. The concept of equilibrium is important because different types of equilibriums may lead to different strategies. Equally important is the development of suitable concepts to describe and understand conflicting situations. It turns out, for instance, that the role of information—what one player knows relative to others—is crucial in such problems.

One of the most popular equilibria is Nash equilibrium [27]. It was named after John Forbes Nash, Jr. for his famous contribution to non-cooperative games, and he won Nobel prize for that in economics in 1994. Nash equilibrium is a solution concept of a non-cooperative game involving two or more players, in which each player is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing only their own strategy unilaterally. If
each player has chosen a strategy and no player can benefit by changing strategies while the other players keep theirs unchanged, then the current set of strategy choices and the corresponding payoffs constitute a Nash equilibrium.

Another important type of equilibrium is Stackelberg equilibrium. A Stackelberg equilibrium is used for two-player noncooperative decision making problem formalized as a hierarchical combination of two optimization problems. The lower level decision maker, called the follower, select a strategy optimizing his/her own objective function, depending on the strategy of the upper level decision maker, called the leader. The leader may decide his/her strategy, optimizing his/her objective function, relative to the decision of both players by knowing the rational reaction of the follower. The Stackelberg game is first proposed by Heinrich von Stackelberg with his pioneering work on static game [41]. The field of application of Stackelberg strategies is large and includes, for example, economy [4], social behaviors, marketing [24], engineering [7]. Differential game Stackelberg strategy was introduced in [19], and a discrete-time version of Stackelberg game was defined in [21].

The information structure in the game is the set of all available information for the players to make their decisions. There are mainly two types of strategy structure, open-loop structure and closed-loop (feedback) structure. Open-loop information structure is used for the situation that no measurement of the state of the system is available and the players are committed to follow a predetermined strategy based on their knowledge of the initial state, the system’s model and the cost functional to be minimized. Necessary condition for open-loop information structure are well known [43]. For the closed-loop information structure case, each player can observe the state of the system and thus can adapt his strategy to the system’s evolution, which in general is much more harder than the the other information structures and has been an open problem for a long time. The main difficulty [6] comes from the partial derivative of the leader’s control with respect to the measurement of the state in the expression of the rational reaction set of the follower.

In deterministic Stackelberg differential games, it is easy to obtain the necessary condition for the optimal strategies with open-loop type, and it is even possible to derive the optimal strategies explicitly for the linear-quadratic problems [19]. However, in a differential game that characterizes a dynamic situation, it is not
very reasonable to assume that the players have access to only open-loop information. When players have access to state of the system, the closed-loop Stackelberg solution has not yet been obtained and remained as an unsolved problem because of the partial derivative of the leader’s control. To overcome the difficulty, researchers have taken two different ways. One way is to assume specific structures [30] for the closed-loop strategies of the players and to introduce the Stackelberg solution only with regard to the parameters that are defined in those structures. The second approach is consistent with the whole rational reaction set of the follower for a given control of the leader and leads to a nonclassical problem which can not be solved using the usual Pontryagin Minimum Principle. A variational method is proposed in [33] to solve this kind of nonclassical problem. Moreover, in [30] it is shown that this technique does not lead to a solution for all initial states. The difficulty is bypassed by assuming that there is an uniform distribution on the initial data and replacing the optimization criterion with its expected value over the initial state.

This thesis deals with the theory and application of noncooperative differential games. A noncooperative game [6] is a strategic situation in which decision makers cannot make binding agreements to cooperate. In a noncooperative game, the players can independently pursue their own best interests. Differential games are related closely with optimal control problems. In an optimal control problem there are a single control $u(t)$ and a single payoff function to be maximized; while two players differential game theory generalizes this to two controls $u(t), v(t)$ and two different payoffs, one for each player. Each player attempts to control the state of the system so as to achieve his goal, and the system responds to the inputs of both players.

Differential game theory put together the ideas of game theory and optimal control. It involves a dynamic decision process evolving in continuous time, with more than one decision maker, each with his own cost function and possibly having access to different information. In chapter 3 and 4, a special class of differential game problems which motivated by Stackelberg solution is discussed and reformulated as a nonclassical optimal control problem. Not surprisingly, the analytical tools of differential games has been influenced by those of optimal control (e.g. maximum principles, Hamilton-Jacobi-Bellman equations of dynamic programming [11], state
space analysis). However, differential game theory has long since transcended its origins in one-person dynamic optimization and moved on to become a subclass in its own right of the broader field of optimal control.

In this thesis, we are interested in three aspects of optimal control problems derived from differential games:

(i) **Existence of optimal controls.** Under a suitable convexity assumption, optimal solutions can be constructed following the direct method in the Calculus of Variations, i.e., as limits of minimizing sequences, relying on compactness and lower semi-continuity properties. In some special cases, however, the existence of optimal control can still be proved, using a variety of more specialized techniques.

(ii) **Necessary conditions for the optimality of a control.** The ultimate goal of any set of necessary conditions is to isolate a hopefully unique candidate for the minimum. The major result in this direction is the celebrated Pontryagin Maximum Principle [11], which extends to control systems the Euler-Lagrange and the Weierstrass necessary conditions for a strong local minimum in the Calculus of Variations. These first order conditions have been supplemented by several high order conditions, which provide additional information in a number of special cases.

(iii) **Sufficient conditions for optimality.** For some special classes of optimal control problems, one finds a unique control $u^*(\cdot)$ which satisfies Pontryagin’s necessary conditions. In this case, $u^*(\cdot)$ provides the unique solution to the optimization problem.

This thesis has four chapters which are organized as follows:

In chapter 2 we give a brief introduction on the concepts in game theory that are particularly relevant for two differential games with two players. Two basic solution techniques for optimal control problem are introduced in this chapter: the Hamilton-Jacobi-Bellman equation and Pontryagin’s maximum principle. It is shown that optimal solutions can be represented in many different ways and that the choice of the representation, also called the strategy, depends on the informational assumptions of the model. Two types of strategies are discussed in the following thesis: 1) open loop strategies, 2) Markovian strategies (or, equivalently, feedback type). Two types of solutions are introduced in this chapter, one is Nash equilibrium solution, the other is Stackelberg equilibrium solution.

Chapter 3 is concerned with problems of optimal feedback control with “nonclassi-
cal” dynamics \( \dot{x} = f(t, x, u, Du) \), where the evolution of the state \( x \) depends also on the Jacobian matrix \( Du = (\partial u_1/\partial x_j) \) of the feedback control function \( u = u(t, x) \). Problems of this kind arise naturally in connection with Stackelberg solutions to differential games in closed-loop form [29, 33]. Given a probability measure \( \mu \) on the set of initial states, we seek feedback control \( u(\cdot) \) which minimizes the expected value of a cost functional. After introducing a basic framework for the study of these problems, this paper focuses on three main issues: (i) necessary conditions for optimality, (ii) equivalence with a relaxed feedback control problem in standard form, and (iii) dependence of the expected minimum cost on the probability measure \( \mu \).

Chapter 4 is about Stackelberg solutions for a differential game with deterministic dynamics but random initial data, where the leading player can adopt a strategy in feedback form: \( u_1 = u_1(t, x) \). We study the existence of an optimal strategy for the leading player, within a family of feedback strategies depending on finitely many parameters. The necessary conditions for the optimality of a feedback strategy are also provided.

The last chapter is concerned with a continuum model of limit order book in a stock market, viewed as a noncooperative game for \( n \) players. Agents offer various quantities of an asset at different prices \( p \in [0, P] \), competing to fulfill an incoming order, whose size \( X \) is not known a priori. Players can have different payoff functions, reflecting different beliefs about the fundamental value of the asset and probability distribution of the random variable \( X \). In [10] the existence of a Nash equilibrium was established by means of a fixed point argument. Our main goal is to compute the Nash equilibrium, determining the optimal bidding strategies of the various agents who submit limit orders. The main issue discussed in the this chapter is whether this equilibrium can be obtained from the unique solution to a two-point boundary value problem, for a suitable system of discontinuous ODEs. The statement is true for the case when there are exactly 2 players, or when all \( n \) players assign the same exponential probability distribution to the random variable \( X \).
Basic Concepts of Game Theory

This chapter introduces those concepts in game theory that are particularly relevant for two differential games with two players. In a differential game each player maximize or minimize his objective function subject to a number of constraints which include a differential equation to describing the evolution of the state of the game. Optimization problems of this type are known are optimal control problems and are widely used in economic theory and management science. Two basic solution techniques for optimal control problem are introduced in this chapter: the Hamilton-Jacobi-Bellman equation and Pontryagin’s maximum principle. It is shown that optimal solutions can be represented in many different ways and that the choice of the representation, also called the strategy, depends on the informational assumptions of the model. Two types of strategies are discussed in the following thesis: 1) open loop strategies, 2) Markovian strategies (or, equivalently, feedback type). Section 3 dealt with differential games in which all players make their moves simultaneously which leads to Nash equilibrium, while section 4 turn to a class of differential games in which one player has priority of moves over the other player resulting to Stackelberg equilibrium.

2.1 Games with two players

The basic setting of two players game (Player A and Player B) is as follows

- The two sets of strategies: A and B, available to the players.
• The two payoff functions: \( \Phi^A : A \times B \mapsto \mathbb{R} \) and \( \Phi^B : A \times B \mapsto \mathbb{R} \).

If the first player chooses a strategy \( a \in A \) and the second player chooses \( b \in B \), then the payoffs achieved by the two players are \( \Phi^A(a, b) \) and \( \Phi^B(a, b) \), respectively. The goal of each player is to maximize his own payoff. Throughout the following, our basic assumption will be

\( (A1) \) The sets \( A \) and \( B \) are compact metric spaces. The payoff function \( \Phi^A, \Phi^B \) are continuous function from \( A \times B \) into \( \mathbb{R} \).

In general, one cannot simply speak of an “optimal solution” of the game. Indeed, one player’s strategy may strongly influence the other player’s payoff. An outcome that is optimal for one player can be very bad for the other one. Various concepts of solutions have been considered in the literature, depending on the information available to the players and on their willingness to cooperate. Three types of solutions are discussed below.

**I - Pareto optimality.** A pair of strategies \((a^*, b^*)\) is said to be Pareto optimal if there exists no other pair \((a, b) \in A \times B\) such that

\[
\Phi^A(a, b) > \Phi^A(a^*, b^*) \quad \text{and} \quad \Phi^B(a, b) \geq \Phi^B(a^*, b^*) \quad (2.1.1)
\]

or

\[
\Phi^B(a, b) > \Phi^B(a^*, b^*) \quad \text{and} \quad \Phi^A(a, b) \geq \Phi^A(a^*, b^*) . \quad (2.1.2)
\]

In other words, at equilibrium it is not possible to strictly increase the payoff of one player without strictly decreasing the payoff of the other.

**II - Stackelberg equilibrium.** In this case, the game can be reduced to a pair of optimization problems, solved on after the other. In connection with the strategy a adopted by the first player, the second player needs to maximize his payoff function \( b \mapsto \Phi^B(a, b) \). He will thus choose a strategy

\[
b^* = \arg\max_{b \in B} \Phi^B(a, b) . \quad (2.1.3)
\]
By compactness, at least one optimal strategy $b^*$ certainly exists. Assume that this optimal strategy is unique, and that the map $a \mapsto b^* \doteq \beta(a)$ is continuous. In turn, at equilibrium Player A needs to maximize the continuous function $a \mapsto \Phi^A(a, \beta(a))$ over the compact set $A$. This problem has at least one solution, namely

$$a_S \doteq \underset{a \in A}{\text{argmax}} \Phi^A(a, \beta(a)). \quad (2.1.4)$$

The point $(a_S, b_S) \doteq (a_s, \beta(a_s))$ is called the **Stackelberg equilibrium** of the game.

**III - Nash equilibrium.** The pair of strategies $(a^*, b^*)$ is a Nash equilibrium of the game if, for every $a \in A$ and $b \in B$, one has

$$\Phi^A(a, b^*) \leq \Phi^A(a^*, b^*), \quad \Phi^B(a^*, b) \leq \Phi^B(a^*, b^*). \quad (2.1.5)$$

In other words, no player can increase his payoff by single-mindedly, changing his strategy, as long as the other player sticks to the equilibrium strategy.

One can characterize a Nash equilibrium solution in terms of the best reply map. Namely, for a given choice $b \in B$ of player B, consider the set of best possible replies of player A:

$$R^A(b) \doteq \left\{ a \in A; \Phi^A(a, b) = \max_{\omega \in A} \Phi^A(\omega, b) \right\}. \quad (2.1.6)$$

Similarly, for a given choice $a \in A$ of player A, consider the set of best possible replies of player B:

$$R^B(a) \doteq \left\{ b \in B; \Phi^B(a, b) = \max_{\omega \in B} \Phi^B(a, \omega) \right\}. \quad (2.1.7)$$

By the assumption (A1), the above sets are non-empty. However, in general they need not be single-valued. From the definition it now follows that a pair of strategies $(a^*, b^*)$ is a Nash equilibrium if and only if

$$a^* \in R^A(b^*), \quad b^* \in R^B(a^*). \quad (2.1.8)$$
2.2 Differential games

Consider a differential game for two players. Let \( x \in \mathbb{R}^N \) describe the state of the system, which evolves according to the differential equation

\[
\dot{x}(t) = f(t, x, u_1, u_2), \quad t \in [0, T], \tag{2.2.9}
\]

with initial data

\[
x(0) = x_0. \tag{2.2.10}
\]

Here the upper dot denotes a derivatives w.r.t. time. The functions \( u_1(\cdot), u_2(\cdot) \) are the controls implemented by the two players. We assume that they satisfy the pointwise constraints

\[
u_1 \in U_1, \quad u_2 \in U_2, \tag{2.2.11}
\]

for some given sets \( U_1, U_2 \subseteq \mathbb{R}^m \). For \( i = 1, 2 \), the goal of the \( i \)-th player is to maximize his own payoff, namely

\[
J_i \equiv \psi_i(x(T)) - \int_0^T L_i(t, x(t), u_1(t), u_2(t))dt. \tag{2.2.12}
\]

Here \( \psi_i \) is a terminal payoff, while \( L_i \) accounts for a running cost.

In order to completely describe the game, it is essential to specify the information available to the two players. Indeed, the strategy adopted by a player depends on the information available to him at each time \( t \). Therefore, different information structures result in vastly different game situations. Here we shall consider different cases concerning the information that each player has, regarding: (i) the current state of the system \( x(t) \), or (ii) the control \( u(\cdot) \) implemented by the other player.

**CASE 1 (open loop strategies):** Apart from the initial data, Player \( i \) cannot make any observation of the state of the system, or of the strategy adopted by the other player. In this case, his strategy must be open loop, i.e. it can only depend on time \( t \in [0, T] \). The set \( \mathcal{S}_i \) of strategies available to the \( i \)-th player will thus consist of all measurable functions \( t \mapsto u_i(t) \) from \( [0, T] \) into \( U_i \).

**CASE 2 (Markovian strategies):** Assume that, at each time \( t \in [0, T] \), Player \( i \) can observe the current state \( x(t) \) of the system. However, he has no additional
information about the strategy of the other player. In this case, his strategy can be Markovian (or, equivalently, of feedback type), i.e. it can depend both on time $t$ and on the current state $x$. The set $\mathcal{S}_i$ of strategies available to the $i$-th player will thus consist of all measurable functions $(t,x) \mapsto u_i(t,x)$ from $[0 \times T] \times \mathbb{R}^n$ into $U_i$.

**CASE 3 (hierarchical play):** Player 1 (the leader) announces his strategy in advance. This can be either open loop $u_1 = u^*_1(t)$, or feedback $u_1 = u^*_1(t,x)$. In this case, the game yields an optimal control problem for Player 2 (the follower). Namely

$$\maximize: \quad \psi_2(x(T)) - \int_0^T L_2\left(t, x(t), u^*_1(t,x(t)), u_2(t)\right)dt,$$

subject to

$$\dot{x}(t) = f(t, x, u^*_1(t,x), u_2(t)), \quad x(0) = x_0, \quad u_2(t) \in U_2. \quad (2.2.14)$$

Notice that in this case the knowledge of the initial point $x_0$ together with the evolution equation (2.2.14) provides Player 2 with complete information about the state of the system for all $t \in [0, T]$.

### 2.3 Nash equilibrium for differential games

#### 2.3.1 Open loop strategies

We first consider the game (2.2.9)-(2.2.12), in the case where the strategies implemented by the players must be functions of time alone.

**Definition (open-loop Nash equilibrium).** A pair of control functions $t \mapsto (u^*_1(t), u^*_2(t))$ is a Nash equilibrium for the game (2.2.9)-(2.2.12) within the class of open-loop strategies if the following holds.

(i) The control $u^*_1(\cdot)$ provides a solution to the optimal control problem for Player 1:

$$\max_{u_1} \left\{ \psi_1(x(T)) - \int_0^T L_1\left(t, x(t), u_1(t), u^*_2(t)\right)dt \right\}, \quad (2.3.15)$$
for the system with dynamics

\[ x(0) = x_0 \in \mathbb{R}^n, \quad \dot{x}(t) = f(t, x, u_1(t), u_2^*(t)), \quad u_1(t) \in U_1, \quad t \in [0, T]. \]

(ii) The control \( u_2^*(\cdot) \) provides a solution to the optimal control problem for Player 2:

\[ \max_{u_2} \left\{ \psi_2(x(T)) - \int_0^T L_2\left(t, x(t), u_1^*(t), u_2(t)\right) dt \right\}, \]

for the system with dynamics

\[ x(0) = x_0 \in \mathbb{R}^n, \quad \dot{x}(t) = f(t, x, u_1^*(t), u_2(t)), \quad u_2(t) \in U_2, \quad t \in [0, T]. \]

To find Nash equilibrium solutions, we thus need to simultaneously solve two optimal control problems. The optimal solution \( u_1^*(\cdot) \) of the first problem enters as a parameter in the second problem, and vice versa. The famous Pontryagin Maximum Principle (PMP) is discussed below in order to solve the optimal control problem. In general, the PMP yields a (highly nonlinear) system of O.D.E’s for the optimal trajectory and for a corresponding adjoint vector, which must then be solved with appropriate boundary conditions.

Assuming that all functions \( f, \psi_1, \psi_2, L_1, L_2 \) are continuously differentiable, necessary conditions for optimality are provided by the Pontryagin Maximum Principle (see [11])

**Pontryagin Maximum Principle (PMP)**  
Let the control \( u_1^*(\cdot) \) be optimal for the problem (2.3.15)-(2.3.16), and let \( x^*(\cdot) \) be the corresponding optimal trajectory. Then there exists an absolutely continuous adjoint vector \( t \mapsto q(t) \in \mathbb{R}^N \) such that the following holds:

\[ q(T) = \nabla \psi_1(x^*(T)), \]

\[ \dot{q}(t) = -q(t) \frac{\partial f}{\partial x}(t, x^*(t), u_1^*(t), u_2^*(t)) + \frac{\partial L_1}{\partial x}(t, x^*(t), u_1^*(t), u_2^*(t)), \]

(2.3.19)
\[ q(t) \cdot f(t, x^*(t), u_1^*(t), u_2^*(t)) - L_1(t, x^*(t), u_1^*(t), u_2^*(t)) \]

\[ = \max_{\omega \in U_1} \left\{ q(t) \cdot f(t, x^*(t), \omega, u_2^*(t)) - L_1(t, x^*(t), \omega, u_2^*(t)) \right\}. \]  

(2.3.20)

The above ODEs are meant to hold for a.e. \( t \in [0, T] \). Notice that \( q, x, f \) are all vectors in \( \mathbb{R}^N \). Using coordinates, the products in (2.3.19)-(2.3.20) take the form

\[ \left( q \cdot \frac{\partial f}{\partial x} \right)_i = \sum_{j=1}^{N} q_j \frac{\partial f_j}{\partial x_i}, \quad q \cdot f = \sum_{i=1}^{N} q_i f_i. \]

Similar optimality conditions can be stated for the problem (2.3.17)-(2.3.18).

Based on the PMP, we now describe a procedure for finding a pair of open-loop strategies \( t \mapsto (u_1^*(t), u_2^*(t)) \) yielding a Nash equilibrium. Toward this goal, we shall assume that the maximum in (2.3.20) is attained at a unique point \( u_1^* \), and a similar condition holds for \( u_2^* \). More precisely, we assume

(A2) \( \text{For any } (t, x) \in [0, T] \times \mathbb{R}^N \text{ and every pair of vectors } (q^1, q^2) \in \mathbb{R}^N \times \mathbb{R}^N, \text{ there exists a unique pair } (u_1^*, u_2^*) \in U_1 \times U_2 \text{ such that} \)

\[ u_1^* = \arg\max_{\omega \in U_1} \left\{ q^1 \cdot f(t, x, \omega, u_2^*) - L_1(t, x, \omega, u_2^*) \right\}, \]  

(2.3.21)

\[ u_2^* = \arg\max_{\omega \in U_2} \left\{ q^2 \cdot f(t, x, u_1^*, \omega) - L_2(t, x, u_1^*, \omega) \right\}, \]  

(2.3.22)

The corresponding map will be denoted by

\[ (t, x, q^1, q^2) \mapsto \left( u_1^*(t, x, q^1, q^2), u_2^*(t, x, q^1, q^2) \right). \]  

(2.3.23)

Assume that (A2) holds, in order to find a Nash equilibrium, by the Pontryagin
Maximum Principle one needs to solve the system of ODEs

\[
\begin{align*}
\dot{x} &= f(t, x, u_1^*, u_2^*), \\
\dot{q}^1 &= -q^1 \frac{\partial f}{\partial x}(t, x, u_1^*, u_2^*) + \frac{\partial L_1}{\partial x}(t, x, u_1^*, u_2^*), \\
\dot{q}^2 &= -q^2 \frac{\partial f}{\partial x}(t, x, u_1^*, u_2^*) + \frac{\partial L_2}{\partial x}(t, x, u_1^*, u_2^*),
\end{align*}
\]

with initial and terminal conditions

\[
\begin{align*}
x(0) &= x_0, \\
q^1(T) &= \nabla \psi_1(x(T)), \\
q^2(T) &= \nabla \psi_2(x(T)).
\end{align*}
\]

Notice that in (2.3.24) the variables $u_1^*, u_2^*$ are functions of $(t, x, q^1, q^2)$, defined by (2.3.21)-(2.3.22). The system (2.3.24) consists of three ODEs in $\mathbb{R}^N$. This needs to be solved with the mixed boundary data (2.3.25). Here the value of variable $x$ (the state of the system) is explicitly given at the initial time $t = 0$. On the other hand, since $x(T)$ is not a priori known, the values for $q^1, q^2$ (the adjoint variables) are only determined by two implicit equations at the terminal time $t = T$. Together with the strong nonlinearity of the maps $u_1^*, u_2^*$ in (2.3.23), this makes the problem (2.3.24)-(2.3.25) hard to solve, in general.

### 2.3.2 Markovian strategies

We consider here the case where both players can observe the current state of the system. Their strategies will thus be functions $u_i = u_i(t, x)$ of time $t$ and of the state $x$.

Observe that, in the open-loop case, the optimal controls $u_i = u_i^*(t)$ strongly depend on the initial data $x_0$ in (2.2.10). On the other hand, in the Markovian case, it is natural to look for optimal feedback strategies $u_i = u_i^*(t, x)$ that are optimal for the problem (2.2.9)-(2.2.12), simultaneously for any choice of initial data

\[
x(\tau) = y,
\]

(2.3.26)
with \( \tau \in [0, T] \), \( y \in \mathbb{R}^N \). In the following, we say that a control \( (t, x) \mapsto u(t, x) \in U \) is an optimal feedback for the optimization problem
\[
\max_u \left\{ \psi(x(T)) - \int_\tau^T L(t, x, u) \, dt \right\},
\]
(2.3.27)
with dynamics
\[
\dot{x}(t) = f(t, x, u), \quad u(t) \in U, \quad t \in [0, T]
\]
(2.3.28)
if, for every initial data \( (\tau, y) \in [0, T] \times \mathbb{R}^N \), every Carathéodory solution of the Cauchy problem
\[
\dot{x}(t) = f(t, x, u(t, x)), \quad x(\tau) = y
\]
(2.3.29)
is optimal.

**Definition (feedback Nash equilibrium).** A pair of control functions \( (t, x) \mapsto (u_1^*(t, x), u_2^*(t, x)) \) is a Nash equilibrium for the game (2.2.9)-(2.2.12) with the class of feedback strategies if the following holds.

(i) The control \( (t, x) \mapsto u_1^*(t, x) \) provides an optimal feedback in connection with the optimal control problem for Player 1:
\[
\max_{u_1} \left\{ \psi_1(x(T)) - \int_0^T L_1(t, x(t), u_1, u_2^*(t, x(t))) \, dt \right\},
\]
(2.3.30)
for the system with dynamics
\[
\dot{x}(t) = f(t, x, u_1, u_2^*(t, x)), \quad x(0) = x_0 \in \mathbb{R}^n, \quad u_1(t) \in U_1, \quad t \in [0, T].
\]
(2.3.31)

(ii) The control \( (t, x) \mapsto u_2^*(t, x) \) provides an optimal feedback in connection with the optimal control problem for Player 2:
\[
\max_{u_2} \left\{ \psi_2(x(T)) - \int_0^T L_2(t, x(t), u_1^*(t, x), u_2) \, dt \right\},
\]
(2.3.32)
for the system with dynamics

\[ \dot{x}(t) = f(t, x, u_1^*(t, x), u_2), \quad x(0) = x_0 \in \mathbb{R}^n, \quad u_2(t) \in U_2, \quad t \in [0, T]. \quad (2.3.33) \]

Assume that the pair of feedback controls \((u_1^*, u_2^*)\) provides a Nash equilibrium. Given an initial data \((\tau, y) \in [0, T] \times \mathbb{R}^N\), call \(t \mapsto x^*(t; \tau, y)\) the solution of

\[ \dot{x}(t) = f(t, x, u_1^*(t, x), u_2^*(t, x)), \quad x(\tau) = y. \quad (2.3.34) \]

We can then define the corresponding value function \(V_1, V_2\) as

\[ V_i(\tau, y) = \psi_i(x^*(T; \tau, y)) - \int_{\tau}^{T} L_i \left( t, x^*(t), u_1^*(t, x^*(t)), u_2^*(t, x^*(t)) \right) dt. \quad (2.3.35) \]

Here \(V_i(\tau, y)\) is the total payoff achieved by Player \(i\) if the game starts at \(y\) at time \(\tau\). Let the assumption \((A1)\) hold. On a region where \(V_1, V_2\) are \(C^1\), by the dynamic programming principle they satisfy the system of Hamilton-Jacobi PDEs (see [11])

\[
\begin{cases}
V_{1,t} + \nabla V_1 \cdot f(t, x, u_1^*, u_2^*) = L_1(t, x, u_1^*, u_2^*), \\
V_{2,t} + \nabla V_2 \cdot f(t, x, u_1^*, u_2^*) = L_2(t, x, u_1^*, u_2^*). 
\end{cases} \quad (2.3.36)
\]

This system is closed by the equations

\[ u_i^* = u_i^*(t, x, \nabla V_1, \nabla V_2), \quad i = 1, 2, \quad (2.3.37) \]

introduced at (2.3.23), and complemented by the terminal conditions

\[ V_1(T, x) = \psi_1(x), \quad V_2(T, x) = \psi_2(x). \quad (2.3.38) \]

2.4 Stackelberg solutions for differential games

In contrast with the previous section, we now assume that the strategies of the players are not chosen simultaneously, but in two stages. Namely, Player 1 (the leader) announces his strategy in advance, say \(u_1(\cdot)\). In a second stage, Player 2 (the follower) chooses his strategy \(u_2(\cdot)\) in order to maximize his own payoff given the strategy \(u_1(\cdot)\) already chosen by the first players.
2.4.1 Open loop strategies

We first consider the game (2.2.9)-(2.2.12), in the case where the strategies implemented by the players are open loop, i.e. are functions of time alone. Given any admissible control $u_1^*: [0,T] \mapsto U_1$ for the first player, we denote by $R_2(u_1^*)$ the set of best replies for the second player. More precisely, $R_2(u_1)$ is the set of all admissible control function $u_2: [0,T] \mapsto U_2$ for Player 2, which achieve the maximum payoff in connection with $u_1^*$. Namely, they solve the optimal control problem

$$\maximize: \psi_2(x(T)) - \int_0^T L_2(t, x(t), u_1^*(t), u_2(t)) dt,$$  \hspace{1cm} (2.4.39)

subject to

$$\dot{x}(t) = f(t, x, u_1^*(t), u_2(t)), \quad x(0) = x_0, \quad u_2(t) \in U_2.$$  \hspace{1cm} (2.4.40)

In the following, given two control functions $t \mapsto u_1(\cdot), t \mapsto u_2(\cdot)$, we denote by $t \mapsto x(t, u_1, u_2)$ the solution of the Cauchy problem

$$\dot{x}(t) = f(t, x, u_1(t), u_2(t)), \quad x(0) = x_0.$$  \hspace{1cm} (2.4.41)

Definition (open-loop Stackelberg equilibrium). We say that a pair of control functions $t \mapsto (u_1^*(t), u_2^*(t))$ is a Stackelberg equilibrium for the game (2.2.9)-(2.2.12) within the class of open-loop strategies if the following holds.

(i) $u_2^* \in R(u_1^*)$,

(ii) Given any admissible control $u_1(\cdot)$ for Player 1 and every best reply $u_2(\cdot) \in R(u_1)$ for Player 2, one has

$$\psi_1(x(T, u_1, u_2)) - \int_0^T L_1(t, x(t, u_1, u_2), u_1(t), u_2(t)) dt$$

$$\leq \psi_1(x(T, u_1^*, u_2^*)) - \int_0^T L_1(t, x(t, u_1^*, u_2^*), u_1^*(t), u_2^*(t)) dt.$$  \hspace{1cm} (2.4.42)

In other words, Player 1 has to calculate the best reply of Player 2, and choose his
control function \( u^*_1(\cdot) \) in order to maximize his own payoff \( J_1 \). We are here taking the optimistic view that, if Player 2 has several best replies to a strategy \( u_1(\cdot) \), he will choose the one most favorable to Player 1. Let \( t \mapsto x^*(t) \) be the trajectory of the system determined by the controls \( u^*_1, u^*_2 \). Since \( u^*_2(\cdot) \) is an optimal reply for Player 2, the Pontryagin Maximum Principle yields the existence of an adjoint vector \( q^*_2(\cdot) \) such that

\[
\begin{aligned}
\dot{x}^*(t) &= f(t, x^*(t), u^*_1(t), u^*_2(t)), \\
q^*_2(t) &= -q_2 \frac{\partial f}{\partial x}(t, x^*(t), u^*_1(t), u^*_2(t)) + \frac{\partial L_2}{\partial x}(t, x^*(t), u^*_1(t), u^*_2(t)),
\end{aligned}
\]

with boundary conditions

\[
\begin{aligned}
x^*(0) &= x_0, \\
q^*_2(T) &= \nabla \psi_2(x^*(T)),
\end{aligned}
\]

Moreover, the following optimality conditions holds:

\[
\begin{aligned}
u^*_2(t) &\in \arg\max_{\omega \in U_2} \left\{ q^*_2(t) \cdot f(t, x^*(t), u^*_1(t), \omega) - L_2(t, x^*(t), u^*_1(t), \omega) \right\} \text{ for a.e. } t \in [0, T].
\end{aligned}
\]

We now take the side of the first player. To derive a set of necessary conditions for optimality, our main assumptions will be as follows

\[ \textbf{(A2)} \quad \text{For each } (t, x, u_1, q_2) \in [0, T] \times \mathbb{R}^N \times U_1 \times \mathbb{R}^N \text{ there exists a unique optimal choice } \nu^*_2 \in U_2 \text{ for Player 2, namely} \]

\[
u^*_2(t, x, u_1, q_2) = \arg\max_{\omega \in U_2} \left\{ q_2(t) \cdot f(t, x(t), u_1(t), \omega) - L_2(t, x(t), u_1(t), \omega) \right\}. \quad (2.4.46)
\]

The optimization problem for Player 1 will be formulated as an optimal control problem in an extended state space, where the state variables are \((x, q_2) \in \mathbb{R}^N \times \mathbb{R}^N\).

\[
\text{maximize: } \psi_1(x(T)) - \int_0^T L_1 \left( t, x(t), u_1(t), \nu^*_2(t, x(t), u_1(t), q_2(t)) \right) dt
\]

\[
\quad (2.4.47)
\]
for the system on $\mathbb{R}^{2N}$ with dynamics

$$
\begin{cases}
    \dot{x}(t) = f(t, x, u_1, w_2(t, x, u_1, q_2)), \\
    \dot{q}_2(t) = -q_2 \frac{\partial f}{\partial x}(t, x, u_1, w_2(t, x, u_1, q_2)) + \frac{\partial L_2}{\partial x}(t, x, u_1, w_2(t, x, u_1, q_2)),
\end{cases}
$$

(2.4.48)

and with boundary conditions

$$
x(0) = x_0, \quad q_2(x(T)) = \nabla \psi_2(x(T)).
$$

(2.4.49)

This is a standard problem in optimal control. Notice, however, that the state variables $(x, q_2)$ are not given both at time $t = 0$. Instead, we have the constraint $x = x_0$ valid at $t = 0$ and another constraint $q_2 = \nabla \psi_2(x(T))$ valid at $t = T$. In order to apply the PMP, we need to assume that all functions in (2.4.47)-(2.4.49) are continuously differentiable w.r.t. the new state variables $x, q_2$. More precisely

(A3) \quad For every fixed $t \in [0, T]$ and $u_1 \in U_1$, the maps

$$
\begin{align*}
(x, q_2) &\mapsto L_1\left(t, x, u_1, w_2(t, x, u_1, q_2)\right), \\
(x, q_2) &\mapsto L_2\left(t, x, u_1, w_2(t, x, u_1, q_2)\right), \\
(x, q_2) &\mapsto -q_2 \frac{\partial f}{\partial x}(t, x, u_1, w_2(t, x, u_1, q_2)) + \frac{\partial L_2}{\partial x}(t, x, u_1, w_2(t, x, u_1, q_2))
\end{align*}
$$

(2.4.50)

are continuously differentiable.

### 2.4.2 Markovian strategies

We consider here the case where both players can observe the current state of the system. Their strategies will thus be functions $u_i = u_i(t, x)$ of time $t$ and of the state $x$. The optimality condition for Player 2 is similar to (2.4.45). To derived a set of necessary conditions for optimality, we assume that (A2) holds for each $(t, x, u_1(t, x), q_2) \in [0, T] \times \mathbb{R}^N \times U_1 \times \mathbb{R}^N$, i.e. there exists a unique optimal choice
$u_2^\flat \in U_2$ for Player 2, namely

$$u_2^\flat(t, x, u_1(t, x), q_2) = \arg\max_{\omega \in U_2} \left\{ q_2(t) \cdot f(t, x(t), u_1(t, x), \omega) - L_2(t, x, u_1(t, x), \omega) \right\}. \quad (2.4.51)$$

The optimization problem for Player 1 will be formulated as an optimal control problem in an extended state space, where the state variables are $(x, q_2) \in \mathbb{R}^N \times \mathbb{R}^N$.

$$\text{maximize: } \psi_1(x(T)) - \int_0^T L_1(t, x(t), u_1(t), u_2^\flat(t, x(t), u_1(t), x(t), q_2(t))) dt \quad (2.4.52)$$

for the system on $\mathbb{R}^{2N}$ with dynamics

$$\begin{cases} 
\dot{x}(t) = f(t, x, t, x, u_1(t, x), q_2) , \\
\dot{q}_2(t) = -q_2 \left( \frac{\partial f}{\partial x}(t, x, u_1(t, x), u_2^\flat) + \frac{\partial f}{\partial u_1}(t, x, u_1(t, x), u_2^\flat) \cdot \frac{\partial u_1}{\partial x}(t, x) \right) \\
\quad \quad + \frac{\partial L_2}{\partial x}(t, x, u_1(t, x), u_2^\flat) + \frac{\partial L_2}{\partial u_1}(t, x, u_1(t, x), u_2^\flat) \cdot \frac{\partial u_1}{\partial x}(t, x), 
\end{cases} \quad (2.4.53)$$

and with boundary conditions

$$x(0) = x_0 , \quad q_2(x(T)) = \nabla \psi_2(x(T)). \quad (2.4.54)$$

Here the second equation in (2.4.53) is different from the second equation in (2.4.48) because we also need to take the derivative of feedback control $u_1(t, x)$ with respect to $x$. This leads to a standard problem in optimal control, since dynamics in (2.4.53) contains $\frac{\partial u_1(t, x)}{\partial x}$. In the next chapter, we study a nonclassical feedback control problem motivated by this nonstandard problem (2.4.52)-(2.4.54). In chapter 3, we study this Stackelberg solutions of feedback type with random initial data.
Chapter 3

Nonclassical Problem of Optimal Feedback Control

The chapter is concerned with problems of optimal feedback control with “nonclassical” dynamics \( \dot{x} = f(t, x, u, Du) \), where the evolution of the state \( x \) depends also on the Jacobian matrix \( Du = (\partial u_i / \partial x_j) \) of the feedback control function \( u = u(t, x) \). Given a probability measure \( \mu \) on the set of initial states, we seek feedback controls \( u(\cdot) \) which minimize the expected value of a cost functional. Various relaxed formulations of this problem are introduced. In particular, three specific examples are studied in section 2, showing the equivalence or non-equivalence of these approximations. After introducing a basic framework for the study of these problems, this paper focuses on three main issues: (i) necessary conditions for optimality, (ii) equivalence with a relaxed feedback control problem in standard form, and (iii) dependence of the expected minimum cost on the probability measure \( \mu \).

3.1 Introduction

We consider a control problem where the dynamics has the “nonclassical” form

\[
\dot{x} = f(t, x, u, Du).
\]

(3.1.1)

Here \( x \in \mathbb{R}^n \) while the feedback control \( u = u(t, x) \) takes values in \( \mathbb{R}^m \). Notice that the right hand side depends also on the Jacobian matrix \( Du = (\partial u_i / \partial x_j) \) of
the control function \( u = u(t, x) \). Problems of this kind arise naturally in connection with Stackelberg solutions to differential games in closed-loop form [29, 33]. Together with (5.2.2) we consider the relaxed system

\[
\dot{x} = f(t, x, u, v),
\]

(3.1.2)

where \( u \in \mathbb{R}^m \) and \( v \in \mathbb{R}^{m \times n} \) are now regarded as independent controls. Clearly, every solution of (5.2.2) yields a solution of (5.2.3), simply by choosing \( v = Du \). On the other hand, given an initial data

\[
x(0) = \bar{x},
\]

(3.1.3)

let \( t \mapsto x^*(t) \) be the solution of the Cauchy problem (5.2.3)-(5.2.4) corresponding to the open-loop measurable controls \( u(t), v(t) \). If we choose

\[
u^*(t, x) = u(t) + v(t) \cdot (x - x^*(t))
\]

(3.1.4)

for all \( x \) in a neighborhood of \( x^*(t) \), then \( x^*(\cdot) \) provides a solution also to the original equation (5.2.2). Given a cost functional such as

\[
J(\bar{x}, u) = \int_0^T L(t, x(t), u(t)) \, dt,
\]

(3.1.5)

for any initial datum \( \bar{x} \) the infimum cost is thus the same for trajectories \( x(\cdot) \) of (5.2.2) or (5.2.3). The main difficulty in the study of this minimization problem lies in the fact that the control \( v \) is unbounded and comes at zero cost. Therefore, an optimal solution may exist only within an extended class of impulsive controls [31, 37, 39].

As in [13], our main goal is to understand what happens in the case where the initial data is not assigned in advance, and one seeks a feedback \( u = u(t, x) \) which is optimal in connection with a whole collection of possible initial data. Motivated by problems related to differential games [8, 29, 30, 33], we consider a system whose state is described by a pair of scalar variables \((x, \xi) \in \mathbb{R} \times \mathbb{R}\). For simplicity, we also assume that the control variable \( u(t) \in U \subseteq \mathbb{R} \) is one-dimensional. Let the
system evolve in time according to the ODEs

\[
\begin{align*}
\dot{x} &= f(t, x, \xi, u), \\
\dot{\xi} &= g(t, x, \xi, u, u_x),
\end{align*}
\]

(3.1.6)

where \(f, g\) are smooth functions. We assume that the initial data for the variable \(x\) is distributed according to a probability distribution \(\mu\) on \(\mathbb{R}\), while the variable \(\xi\) satisfies

\[
\xi(0) = h(x(0)),
\]

(3.1.7)

for some smooth function \(h : \mathbb{R} \mapsto \mathbb{R}\). We seek a feedback control \(u = u(t, x)\) which minimizes the expected value of the cost:

\[
J(u, \mu) \doteq E^\mu \left[ \int_0^T L(t, x(t), \xi(t), u(t, x(t))) \, dt \right].
\]

(3.1.8)

Here \(E^\mu\) denotes the conditional expectation w.r.t. the probability distribution \(\mu\) on the set of initial data. We shall always assume that the functions \(f, g, h, L\) and the measure \(\mu\) satisfy the following assumptions.

**(A1)** The functions \(f, g\) are continuous in all variables, and continuously differentiable w.r.t. \(x, \xi, u\) with globally bounded derivatives. The function \(h\) is continuously differentiable.

**(A2)** The function \(L\) is non-negative and continuous.

**(A3)** The probability measure \(\mu\) has bounded support.

Even with these assumptions, in general an optimal feedback may not exist, within the class of \(C^2\) functions. Indeed, it is quite possible that the optimal control will have impulsive character, or be discontinuous w.r.t. the space variable \(x\). To bypass all difficulties stemming from the possible lack of regularity, we consider the family \(\mathcal{U}\) of all \(C^2\) functions \(u : [0, T] \times \mathbb{R} \mapsto \mathbb{R}\). For each feedback control \(u \in \mathcal{U}\) the
equations (3.1.6) uniquely determine a flow on \( \mathbb{R}^2 \). We denote by

\[
t \mapsto \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = \Psi^{u}_t \begin{pmatrix} \bar{x} \\ \bar{\xi} \end{pmatrix}
\]

the solution of the Cauchy problem

\[
\frac{d}{dt} \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = \begin{pmatrix} f \left( t, x(t), \xi(t), u(t, x(t)) \right) \\ g \left( t, x(t), \xi(t), u(t, x(t)), u_x(t, x(t)) \right) \end{pmatrix}, \tag{3.1.9}
\]

with initial data

\[
\begin{pmatrix} x(0) \\ \xi(0) \end{pmatrix} = \begin{pmatrix} \bar{x} \\ \bar{\xi} \end{pmatrix} = \begin{pmatrix} \bar{x} \\ h(\bar{x}) \end{pmatrix}. \tag{3.1.10}
\]

Here \( \bar{x} \in \mathbb{R} \) is a random variable, distributed according to the probability measure \( \mu \). Let \( \mu(t) \) be the corresponding probability distribution at time \( t \), defined as the push-forward of \( \mu \) through the flow \( \Psi^{u}_t \). This means

\[
\mu(t)(\Psi^{u}_t(A)) = \mu(A)
\]

for every Borel set \( A \subset \mathbb{R}^2 \). The cost functional in (3.1.8) can be equivalently written as

\[
J(u, \mu) = \int_0^T E^{\mu(t)} L \left( t, x, \xi, u(t, x) \right) dt, \tag{3.1.11}
\]

where \( E^{\mu(t)} \) denotes expectation w.r.t. the probability distribution \( \mu(t) \). We then consider

**Problem 1.** Determine

\[
J(\mu) \doteq \inf_{u \in \mathcal{U}} J(u, \mu). \tag{3.1.12}
\]

Describe a sequence of feedback controls \( (u_n)_{n \geq 1} \) achieving the infimum in (3.1.12).

As shown by some examples in [13], the infimum in (3.1.12) may not be stable w.r.t. perturbations of the probability distribution \( \mu \). A related question is to determine the value

\[
J^w(\mu) \doteq \liminf_{d(\bar{\mu}, \mu) \to 0} \inf_{u \in \mathcal{U}} J(u, \bar{\mu}), \tag{3.1.13}
\]
where

\[ d(\tilde{\mu}, \mu) = \sup \left\{ \left| \int \phi \, d\tilde{\mu} - \int \phi \, d\mu \right| ; \quad \phi \in C^1, \quad |\nabla \phi| \leq 1 \right\} \]

is the Kantorovich-Wasserstein distance between two probability measures. One can think of \( J^w \) as the lower semicontinuous regularization of \( J \) w.r.t. the topology of weak convergence of measures.

In the case where \( \mu \) is absolutely continuous with density \( \phi \) w.r.t. Lebesgue measure, it is also natural to consider

\[ J^s(\mu) \overset{\text{def}}{=} \liminf_{\|\tilde{\phi} - \phi\|_{L^1} \to 0} \inf_{u \in \mathcal{U}} J(u, \tilde{\mu}), \tag{3.1.14} \]

where \( \tilde{\mu} \) is the probability measure having density \( \tilde{\phi} \). In other words, \( J^s \) is the lower semicontinuous regularization of \( J \) w.r.t. a strong topology, corresponding to \( L^1 \) convergence of the densities.

In addition, by replacing \( u_x \) with an independent control function \( v \), from (3.1.9) one obtains the relaxed system

\[
\frac{d}{dt} \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix} = \begin{pmatrix} f(t, x(t), \xi(t), u(t, x(t))) \\ g(t, x(t), \xi(t), u(t, x(t)), v(t, x(t))) \end{pmatrix}.
\tag{3.1.15}
\]

Let \( \mathcal{J}(\mu, u, v) \) be the corresponding cost defined at (3.1.8), with dynamics given at (3.1.15). We then define

\[ J^{relax}(\mu) \overset{\text{def}}{=} \inf_{u, v \in \mathcal{U}} \mathcal{J}(u, v, \mu). \tag{3.1.16} \]

**Remark 1.** In general, the optimal control \( u = u(t, x) \) which minimizes the expected cost (3.1.8) subject to the dynamics (3.1.9) will strongly depend on the probability distribution \( \mu \) on the initial data. On the other hand, since the dynamics (3.1.15) does not involve derivatives of the control functions \( u, v \), the optimal value can be achieved pointwise for each initial data \( x(0), \xi(0) \). In this case, the same pair of feedback controls \( (u^*, v^*) \) can be optimal for every probability distribution \( \mu \) on the initial data.
From the above definitions, it is immediately clear that
\[ J_{\text{relax}}(\mu) \leq J(\mu) , \quad J^w(\mu) \leq J^s(\mu) \leq J(\mu) . \] (3.1.17)

The main goal of our analysis is to understand in which cases equalities hold, and point out the obstructions encountered in the opposite cases.

In the section 2, three basic examples were studied. The goal of the present paper is two-fold: deriving necessary conditions for optimality and determining in which cases the equalities hold in (5.3.33).

In Sections 3 and 4, assuming that an optimal feedback exists and is sufficiently smooth, we derive a set of necessary conditions. More precisely, in Section 3 we single out a situation where these necessary conditions take the form of a second order, scalar elliptic PDE. In Section 4 we study the general case.

The remainder of the paper is devoted to the study of the relations between the values \( J, J^s, J^w \), and \( J_{\text{relax}} \) in (5.3.33). In Section 5 we prove that the equivalence \( J_{\text{relax}}(\mu) = J(\mu) \) holds whenever the probability measure \( \mu \) consists of finitely many point masses. Since every probability measure can be approximated by finitely many point masses (in the topology of weak convergence), this yields the inequality \( J^w(\mu) \leq J_{\text{relax}}(\mu) \) under very general assumptions. In Section 6 we show that \( J^w(\mu) = J^s(\mu) \) holds whenever the cost function \( L \) is bounded.

Finally, in Section 7 we prove a general result relating \( J^s(\mu) \) with \( J_{\text{relax}}(\mu) \). As suggested by Example 3 in [13], the result is based on a crucial controllability condition for an augmented system of ODEs, obtained by adjoining to (3.1.15), (3.1.10) the corresponding evolution equations for \( \partial x / \partial \bar{x} \) and \( \partial \xi / \partial \bar{x} \). As shown by Theorem 5 in Section 8, this controllability condition can be easily achieved, as soon as the partial derivatives \( f_x, f_\xi, g_v \) do not vanish.

All the present results refer to initial value problem with random initial data. They can be regarded as preliminary steps toward the analysis of two-point boundary value problems, where (3.1.7) is replaced by a terminal condition of the form
\[ \xi(T) = h(x(T)) , \] (3.1.18)

which is more appropriate in connection with Stackelberg solutions to differential
3.2 Examples

3.2.1 A case of shrinking funnels

Example 1. Consider the optimization problem

\[ \text{minimize: } J(u) = \mathbb{E}^\mu \left[ \int_0^T [x^2(t) + \xi^2(t) + u^2(t)] dt \right]. \tag{3.2.1} \]

for the system with dynamics

\[ \begin{cases} \dot{x} = u, \\ \dot{\xi} = \xi u_x. \end{cases} \tag{3.2.2} \]

Here \( u = u(t, x) \) can range over the entire real line \( \mathbb{R} \). As initial condition, assume that \( \xi(0) \equiv 1 \) while \( x(0) \) is uniformly distributed on the interval \([0, 1]\). Of course, this means that \( \mu \) is the measure with density \( \phi = \chi_{[0,1]} \) (the characteristic function of the unit interval) w.r.t. Lebesgue measure.

In this case, the corresponding relaxed problem, with \( u_x \) replaced by an independent control function \( v \), is decoupled. Indeed, it yields two independent problems:

\[ \text{minimize: } J_1(u) = \int_0^T [x^2(t) + u^2(t)] dt, \quad \text{with dynamics } \dot{x} = u, \tag{3.2.3} \]

\[ \text{minimize: } J_2(v) = \int_0^T \xi^2(t) dt, \quad \text{with dynamics } \dot{\xi} = \xi v. \tag{3.2.4} \]

The first is a standard linear-quadratic optimal control problem. The optimal feedback is linear w.r.t. \( x \), namely

\[ u^*(t, x) = \frac{e^{t-T} - e^{T-t}}{e^{t-T} + e^{T-t}} \cdot x. \tag{3.2.5} \]

The second problem is solved by an unbounded impulsive control \( v \) that instantly steers the component \( \xi \) to the origin.
Returning to the original problem (3.2.1)-(3.2.2), call \( \phi(t, \cdot) \) the density of the probability distribution \( \mu(t) \). This function satisfies
\[
\phi_t + u \phi_x = - \phi u_x.
\]
Calling \( \xi(t, x) \) the value of \( \xi(t) \) along a characteristic, i.e. at the point \( (t, x(t)) \), the second equation in (3.2.2) yields
\[
\xi_t + u \xi_x = \xi u_x.
\]
Together, these two equations yield
\[
(\phi \xi)_t + u(\phi \xi)_x = 0.
\] (3.2.6)

In the following, for \( y \in [0, 1] \) we shall denote by \( t \mapsto x(t, y) \) the particular solution of the Cauchy problem
\[
\dot{x} = u(t, x), \quad x(0) = y.
\] (3.2.7)

Expressing the feedback control in terms of this new variable: \( u(t, y) \equiv u(t, x(t, y)) \), we obtain
\[
u_x(t, x(t, y)) = u_y(t, y) \cdot \phi(t, y) = \frac{u_y(t, y)}{\xi(t, y)}.
\]
The problem can thus be reformulated as
\[
\text{minimize: } \int_0^T \int_0^1 \left[ x^2(t, y) + \xi^2(t, y) + u^2(t, y) \right] dy dt
\] (3.2.8)
subject to
\[
\begin{cases}
\dot{x} = u, & x(0, y) = y, \\
\dot{\xi} = u_y, & \xi(0, y) = 1.
\end{cases}
\] (3.2.9)

Since the evolution equation does not depend explicitly on \( x, \xi \), the adjoint equa-
tions are
\[
\begin{align*}
    \dot{\lambda}_1 &= - \frac{\partial L}{\partial x} = -2x, \\
    \dot{\lambda}_2 &= - \frac{\partial L}{\partial \xi} = -2\xi,
\end{align*}
\]
(3.2.10)

Hence
\[
\begin{align*}
    \lambda_1(t, y) &= \int_t^T 2x(\tau, y) \, d\tau, \\
    \lambda_2(t, y) &= \int_t^T 2\xi(\tau, y) \, d\tau.
\end{align*}
\]
(3.2.11)

The maximality condition yields
\[
u(t, \cdot) = \arg\min_{\omega(\cdot)} \int_0^1 \left[ \lambda_1(t, y) \omega(y) + \lambda_2(t, y) \omega_\gamma(y) + \omega^2(y) \right] \, dy.
\]
(3.2.12)

**Figure 3.1.** Left: the optimal trajectories for the standard linear-quadratic optimization problem with dynamics (3.2.9) and cost (3.2.3) independent of $\xi$. Center: the presence of a cost depending on $\xi$ renders more profitable a control where $u_x$ is large and negative. Hence the optimal solution, obtained by solving (3.2.16), should be supported on a smaller interval. Right: if we allow small gaps in the support of the probability distribution $\mu$ on the initial data, then the minimum cost becomes arbitrarily close to the minimum cost for relaxed problem (3.2.3)-(3.2.4).

Assume that, for a fixed time $t$, the function $u = u(t, y)$ provides the minimum in (3.2.12). Then, for every smooth function $\varphi : [0, 1] \mapsto \mathbb{R}$, setting $u^{(e)}(y) = \ldots$
\[ u(t, y) + \varepsilon \varphi(y) \text{ one should have} \]

\[
0 = \frac{d}{d\varepsilon} \int_0^1 \left[ \lambda_1(t, y) u^{(\varepsilon)}(y) + \lambda_2(t, y) u_y^{(\varepsilon)}(y) + (u^{(\varepsilon)})^2(y) \right] dy \bigg|_{\varepsilon=0}
\]

\[
= \int_0^1 \left[ \lambda_1(t, y) \varphi(y) + \lambda_2(t, y) \varphi_y(y) + 2u(t, y)\varphi(y) \right] dy
\]

\[
= \int_0^1 \left[ \lambda_1(t, y) - \lambda_{2,y}(t, y) + 2u(t, y) \right] \varphi(y) dy + \lambda_2(t, 1)\varphi(1) - \lambda_2(t, 0)\varphi(0).
\]

Since the function \( \varphi \) can be arbitrary, this yields the Euler-Lagrange equations

\[
u(t, y) = \frac{-\lambda_1(t, y) + \lambda_{2,y}(t, y)}{2}.
\]

(3.2.13)

Together with the boundary conditions

\[
\lambda_2(t, 0) = \lambda_2(t, 1) = 0.
\]

(3.2.14)

Differentiating (3.2.13) w.r.t. \( t \) and using (3.2.10), we obtain

\[
u_t(t, y) = \frac{\lambda_{2,y}(t, y) - \lambda_{1,t}(t, y)}{2} = x - \xi_y.
\]

(3.2.15)

Using the identities

\[
x_{tt}(t, y) = u_t(t, y), \quad x_y(t, y) = \frac{1}{\phi(t, y)} = \xi(t, y),
\]

we eventually obtain the PDE

\[
x_{tt} + x_{yy} - x = 0.
\]

(3.2.16)

This is a linear elliptic equation, to be solved on the rectangle \([0, T] \times [0, 1] \). From (3.2.9) and the terminal conditions in (3.2.10), using (3.2.13) one obtains the boundary conditions

\[
x(0, y) = y, \quad x_t(T, y) = u(T, y) = 0.
\]

(3.2.17)
Moreover,

\[
\begin{aligned}
  x_y(t,0) &= \xi(t,0) = -\frac{1}{2}\lambda_{2,t}(t,0) = 0, \\
  x_y(t,1) &= \xi(t,1) = -\frac{1}{2}\lambda_{2,t}(t,1) = 0,
\end{aligned}
\]  

(3.2.18)

because of (3.2.11) and (3.2.14).

By standard PDE theory, the linear elliptic boundary-value problem (3.2.16), (3.2.17), (3.2.18) has a unique solution. Particular solutions of (3.2.16) satisfying (3.2.18) can be obtained by separation of variables. For every integer \( k \geq 0 \) and coefficients \( A_k, B_k \), one has the solution

\[
X_k(t) = A_k e^{\sqrt{1+k^2\pi^2} t} + B_k e^{-\sqrt{1+k^2\pi^2} t}, \quad Z_k(y) = \cos k\pi y, \quad (3.2.19)
\]

Imposing the additional boundary conditions (3.2.17) one obtains a representation of the solution as a Fourier series:

\[
\begin{aligned}
  x(t,y) &= \sum_{k=1}^{\infty} \left( \frac{e^{\sqrt{1+k^2\pi^2} t - 2T} - e^{-\sqrt{1+k^2\pi^2} t}}{2(1 - (-1)^k)} \right) 2 \frac{(-1)^k - 1}{k^2 \pi^2} \cos k\pi y, \\
  \xi(t,y) &= x_y(t,x) = \sum_{k=1}^{\infty} \left( \frac{e^{\sqrt{1+k^2\pi^2} t - 2T} - e^{-\sqrt{1+k^2\pi^2} t}}{2(1 - (-1)^k)} \right) 2 \frac{1 - (-1)^k}{k\pi} \sin k\pi y. 
\end{aligned}
\]  

(3.2.20)

Estimates on this solution can be obtained by comparison with upper and lower solutions [36]. Differentiating (3.2.16) w.r.t. \( y \) one obtains a boundary value problem for \( x_y \), namely

\[
(x_y)_{tt} + (x_y)_{yy} - x_y = 0 \quad (t,y) \in [0,T] \times [0,1],
\]  

(3.2.21)

with the boundary conditions

\[
\begin{aligned}
  x_y(0,y) &= 1, \\
  (x_y)_t(T,y) &= 0, \\
  x_y(t,1) &= 0, \\
  x_y(t,0) &= 0,
\end{aligned}
\]  

(3.2.22)
A standard comparison argument here yields the lower bound

\[ x_y(t, y) \geq 0 \quad \text{for all } t, y. \]

One can also consider the above problem on the half line, for \( t \in [0, +\infty[. \) In this case, (3.2.20) reduces to

\[ x(t, y) = \frac{e^{-t}}{2} + \sum_{k=1}^{\infty} e^{-\sqrt{1+k^2} t} \cdot \frac{2((-1)^k - 1)}{k^2 \pi^2} \cos k \pi y. \quad (3.2.23) \]

As \( t \to \infty \), this solution approaches zero. Indeed,

\[ \|x(t, \cdot)\|_{L^\infty([0,1])} \to 0, \quad \|\xi(t, \cdot)\|_{L^\infty([0,1])} \to 0, \quad \|u(t, \cdot)\|_{L^\infty([0,1])} \to 0. \quad (3.2.24) \]

We now study what happens if we allow arbitrarily small gaps in the support of the initial probability distribution \( \mu \). For any \( n \geq 2 \), let \( \mu_n \) be the probability distribution with density

\[ \phi_n(x) = \begin{cases} \frac{n}{n-1} & \text{if } x \in [a_i, b_i] = \left[ \frac{i-1}{n}, \frac{i}{n} - \frac{1}{n^2} \right], \quad \text{for some } i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2.25) \]

Clearly, \( \lim_{n \to \infty} \|\phi_n - \phi\|_{L^1} = 0. \) We claim that, as \( n \to \infty \), the costs of the corresponding optimal solutions approach the minimum cost for (3.2.3). To prove this, consider any of the above intervals \([a_i, b_i] \subset [0,1]\). Let \( x_i(t, y) \) be the solution of the linear elliptic boundary value problem

\[ x_{tt} + x_{yy} - x = 0 \quad (t, y) \in [0, T] \times [a_i, b_i], \quad (3.2.26) \]

with boundary conditions

\[ \begin{cases} x(0, y) = y, \\
x_{t}(T, y) = 0, \end{cases} \quad \begin{cases} x_y(t, a_i) = 0, \\
x_y(t, b_i) = 0. \end{cases} \quad (3.2.27) \]
The solution to this boundary value problem can again be expressed as a Fourier series:

\[ x_i(t, y) = \left( \frac{e^{-2T} + e^{-t}}{e^{-2T} + 1} \right) \frac{b_i + a_i}{2} \]

\[ + \sum_{k=1}^{\infty} \left( \frac{e^{\lambda_k(t-2T)} + e^{-\lambda_k t}}{e^{-2\lambda_k T} + 1} \right) \frac{2(-1)^k - 1}{k^2 \pi^2} \cos \frac{k \pi (y - a_i)}{b_i - a_i}, \]  

where

\[ \lambda_k = \sqrt{1 + \frac{k^2 \pi^2}{(b_i - a_i)^2}} \geq 1. \]  

In connection with the intervals \([a_i, b_i]\) defined at (3.2.25), for \(i = 1, \ldots, n\) consider the funnels

\[ \Gamma_i \doteq \{(t, x) ; \ t \in [0, T], \ x = x_i(t, y) \text{ for some } y \in [a_i, b_i]\}. \]  

We claim that these funnels are pairwise disjoint. Indeed, consider the function

\[ z(t, y) = \frac{e^{-2T} + e^{-t}}{e^{-2T} + 1} y \]  

where \(t \mapsto z(t, y)\) is determined as the unique solution to the two-point boundary value problem

\[ \ddot{z}(t) - z(t) = 0 \quad \text{for} \ t \in [0, T], \quad z(0) = y, \quad \dot{z}(T) = 0. \]  

Then \(z_y\) provides a solution to the elliptic boundary value problem

\[ w_{tt} + w_{yy} - w = 0 \quad (t, y) \in [0, T] \times [a_i, b_i], \]  

with boundary conditions

\[ \begin{cases} \displaystyle w(0, y) = 1, \\ w_t(T, y) = 0, \end{cases} \quad w(t, a_i) = w(t, b_i) = \frac{e^{-2T} + e^{-t}}{e^{-2T} + 1}. \]  

On the other hand, the partial derivative \(x_{i,y}\) provides a solution to the same
equation (3.2.32), but with boundary conditions
\[
\begin{aligned}
\{ & w(0, y) = 1, \\
& w_t(T, y) = 0, \\
& w(t, a_i) = (t, b_i) = 0. \\
\end{aligned}
\] (3.2.34)

By comparison, we obtain
\[
x_y(t, y) \leq z_y(t, y) \quad \text{for all } (t, y) \in [0, T] \times [a_i, b_i].
\] (3.2.35)

When \( y = (a_i + b_i)/2 \), from (3.2.28) it follows \( x_i(t, y) = z(t, y) \) for all \( t \in [0, T] \).

Since the estimates (3.2.35) hold for every \( i = 1, \ldots, n \), we conclude
\[
x_{i-1}(t, b_{i-1}) \leq z(t, b_{i-1}) < z(t, a_i) \leq x_i(t, a_i),
\]
proving that the funnels \( \Gamma_1, \ldots, \Gamma_n \) remain disjoint.

We can now define a feedback control \( u_n \) by setting
\[
u_n(t, x) = x_{i,t}(t, y) \quad \text{if } (t, x) \in \Gamma_i \text{ for some } i \in \{1, \ldots, n\},
\] (3.2.36)
and extending \( u_n \) in a smooth way to the entire domain \([0, T] \times \mathbb{R}\).

**Proposition 1.** The above construction yields
\[
\lim_{n \to \infty} J(\mu_n, u_n) = J^{relax}(\mu).
\] (3.2.37)

Therefore, for this example one has \( J^s(\mu) = J^{relax}(\mu) \).

**Proof.** Writing the Euler-Lagrange equations for (3.2.3) and observing that the infimum cost for (3.2.4) is zero, we compute
\[
J^{relax}(\mu) = \int_0^1 \int_0^T \left( z^2(t, y) + z_t^2(t, y) \right) dt \, dy,
\] (3.2.38)
where \( z \) is the function in (3.2.30). On the other hand,
\[
J(\mu_n, u_n) = \frac{n}{n-1} \sum_{i=1}^{n} \int_{a_i}^{b_i} \int_0^T \left( x_{i,t}^2(t, y) + x_{i,t}^2(t, y) + x_{i,y}^2(t, y) \right) dt \, dy,
\] (3.2.39)
where, for \( y \in [a_i, b_i] \), the quantity \( x_i(t, y) \) is given by (3.2.28). We observe that each function \( x_i \) provides the global minimizer to the variational problem

\[
\text{minimize:} \quad J_i(w) = \int_{a_i}^{b_i} \int_{0}^{T} \left( w^2(t, y) + w_t^2(t, y) + w_y^2(t, y) \right) dt \, dy \tag{3.2.40}
\]

among all functions \( w \in W^{1, 2}([0, T] \times [a_i, b_i]) \) such that

\[
w(0, y) = y \quad \text{for all} \quad y \in [a_i, b_i]. \tag{3.2.41}
\]

Consider the functions \( w_i \) defined as follows. For \( y \in [a_i, b_i] \), let

\[
w_i(t, y) = \begin{cases} nt \left( \frac{1}{n}, \frac{a_i + b_i}{2} \right) + (1 - nt) y, & \text{if } t \in [0, n^{-1}], \\
z \left( t, \frac{a_i + b_i}{2} \right), & \text{if } t \in [n^{-1}, T]. \tag{3.2.42}
\end{cases}
\]

For every \( n \geq 1 \) and \( i \in \{1, \ldots, n\} \), it is easy to check that these functions satisfy the uniform bounds

\[
w_i(t, y) \in [0, 1], \quad w_{i,y}(t, y) \in [0, 1], \quad |w_{i,t}(t, y)| \leq M, \tag{3.2.43}
\]

for some uniform constant \( M \). Hence the following estimates hold:

\[
\int_{a_i}^{b_i} \int_{0}^{T} w_{i,y}^2(t, y) \, dt \, dy \leq \int_{a_i}^{b_i} \int_{0}^{1/n} dt \, dy = \frac{b_i - a_i}{n}, \tag{3.2.44}
\]

\[
\int_{a_i}^{b_i} \int_{0}^{1/n} \left( w_i^2(t, y) + w_{i,t}^2(t, y) \right) dt \, dy \leq \frac{(1 + M^2)(b_i - a_i)}{n}. \tag{3.2.45}
\]

Using the above inequalities and recalling that \( \sum_{i=1}^{n} (b_i - a_i) = (n - 1)/n \), since
each \( x_i : [0, T] \times [a_i, b_i] \mapsto \mathbb{R} \) provides a global minimizer, we obtain

\[
J(\mu_n, u_n) \leq \frac{n}{n-1} \sum_{i=1}^{n} \int_{a_i}^{b_i} \int_{0}^{T} \left( w_i^2(t, y) + w_{i,t}^2(t, y) + w_{i,y}^2(t, y) \right) \, dt \, dy
\]

\[
\leq \frac{n}{n-1} \sum_{i=1}^{n} \frac{(2 + M^2)(b_i - a_i)}{n}
\]

\[
+ \frac{n}{n-1} \sum_{i=1}^{n} (b_i - a_i) \int_{1/n}^{T} \left( z^2\left(t, \frac{a_i + b_i}{2}\right) + z_t^2\left(t, \frac{a_i + b_i}{2}\right) \right) \, dt
\]

\[
\leq \frac{(2 + M^2)}{n} + \sum_{i=1}^{n} (a_{i+1} - a_i) \int_{0}^{T} \left( z^2\left(t, \frac{a_i + b_i}{2}\right) + z_t^2\left(t, \frac{a_i + b_i}{2}\right) \right) \, dt
\]

\[
= A_n + B_n.
\]

Letting \( n \to \infty \), we clearly have \( A_n \to 0 \). On the other hand, \( B_n \) is an approximate Riemann sum for the integral (3.2.38). Hence \( \lim_{n \to \infty} B_n = J_{\text{relax}}(\mu) \). From (3.2.46) it follows

\[
\liminf_{n \to \infty} J(\mu_n, u_n) \leq J_{\text{relax}}(\mu),
\]

The converse inequality is clear.

**Remark 2.** In this example, the presence of gaps in the probability distributions \( \mu_n \) is essential. Indeed, if we used the feedback controls \( u_n \) in connection with the original probability \( \mu \), uniformly distributed on \([0, 1]\), the cost \( J(\mu, u_n) \) would be very large. This is because, for initial data \( b_i < x(0) < a_{i+1} \), along the trajectory \( t \mapsto x(t) \) one can have \( u_x(t, x(t)) >> 1 \). This forces

\[
\xi(t) = \exp \left( \int_{0}^{t} u_x(s, x(s)) \, ds \right)
\]

to be very large, producing a large cost in (3.2.1). Although the probability of the initial data falling outside the intervals \( \bigcup_{1 \leq i \leq n} [a_i, b_i] \) is very small, these few initial data determine a big increase in the expected cost in (3.2.1).
This example illustrates a case where

\[ J^{relax}(\mu) = J^s(\mu), \quad \text{but} \quad J^s(\mu) \neq J(\mu). \]

The problems (3.2.3) and (3.2.8)-(3.2.9) both have regular solutions, but the minimization problem (3.1.14) does not. A minimizing sequence \( u_n \) should have the form

\[ u_n(t, x) = u^*(t, x) + \tilde{u}_n(t, x) \]

where \( u^* \) is the optimal linear feedback in (3.5.4), while \( \|\tilde{u}_n\|_{C^0} \to 0, \|\tilde{u}_n\|_{C^1} = O(1) \).

Remark 3. This first example suggests a general strategy for proving the equivalence \( J^{relax}(\mu) = J^s(\mu) \). Namely:

(i) Let \( u_\varepsilon, v_\varepsilon \) be \( C^2 \) feedbacks which achieve an almost optimal cost, in connection with the relaxed problem (3.1.11), (3.1.15). In other words, assume \( J(\mu; u_\varepsilon, v_\varepsilon) \leq J^{relax} + \varepsilon \).

(ii) Split the support of the initial distribution \( \mu \) into several small, disjoint intervals \([a_i, b_i]\), separated by small gaps. For each \( i \), let \( t \mapsto x_i(t) \) be the solution of (3.1.15) with initial data \( x_i(0) = (a_i + b_i)/2 \).

(iii) Define the linear feedback control

\[ u^i(\varepsilon)(t, x) = u_\varepsilon(t, x_i(t)) + v_\varepsilon(t, x_i(t)) \cdot (x - x_i(t)), \]

and let \( \mathcal{F}_i \) be the set of all solutions to the ODE

\[
\begin{cases}
\dot{x} = f(t, x_i(t), \xi(t), u_i^\varepsilon(t, x)), \\
\dot{\xi} = g(t, x(t), \xi(t), u_i^\varepsilon(t, x), v_\varepsilon(t, x_i(t))),
\end{cases}
\]

with initial data \( x(0) \in [a_i, b_i] \).

(iv) If the funnels

\[ \Gamma_i \doteq \left\{(t, x(t)) ; \ t \in [0, T], \ x(\cdot) \in \mathcal{F}_i \right\}, \quad i = 1, \ldots, n, \]
do not overlap, then one can define a new feedback by setting

\[ \tilde{u}(t, x) = u^i(t, x) \quad \text{if} \quad (t, x) \in \Gamma_i, \quad (3.2.47) \]

and extending \( \tilde{u} \) in a smooth way on \( \mathbb{R}^2 \setminus \bigcup_i \Gamma_i \). By choosing the intervals \([a_i, b_i]\) sufficiently small, the cost provided by this feedback control \( u^* \) can be rendered arbitrarily close to \( J(\mu, u_\varepsilon, v_\varepsilon) \).

Here the fact that the funnels \( \Gamma_i \) remain disjoint is essential. In the next section we look at a case where this property fails, and the two infimum costs \( J^{relax} \) and \( J^s \) do not coincide.

3.2.2 A case of expanding funnels

Example 2. We now consider the problem of minimizing the same quadratic functional as in (3.2.1), but subject to the dynamics

\[
\begin{aligned}
\dot{x} &= u, \\
\dot{\xi} &= -\xi u_x.
\end{aligned}
\] (3.2.1)

Because of the negative sign in the second equation, we now have \( \phi(t, x) = \xi(t, x) \) for all \( t, x \). Using again the variable \( y \) to label characteristics, consider the problem (3.2.8), for a system with dynamics

\[
\begin{aligned}
x_t &= u, \\
\xi_t &= -\xi^2 u_y, \\
x(0, y) &= y, \\
\xi(0, y) &= 1.
\end{aligned}
\] (3.2.2)

The evolution of the dual variables is determined by

\[
\begin{aligned}
\lambda_{1,t} &= -2x, \\
\lambda_{2,t} &= 2\xi \lambda_2 u_y - 2\xi, \\
\lambda_1(T, y) &= 0, \\
\lambda_2(T, y) &= 0.
\end{aligned}
\] (3.2.3)
Therefore
\[
\lambda_1(t, y) = \int_t^T 2x(\tau, y) \, d\tau, \quad \lambda_2(t, y) = \int_t^T \exp \left( - \int_t^\tau 2\xi(s, y) \, u_y(s, y) \, ds \right) \, 2\xi(\tau, y) \, dy.
\]
(3.2.4)

The maximality condition takes the form
\[
\begin{align*}
\lambda_1(t, y) & = \begin{argmin}_{\omega(y)} \int_0^1 \left[ \lambda_1(t, y) \omega(y) - \lambda_2(t, y) \xi^2(t, y) \omega_y(y) + \omega^2(y) \right] \, dy. \quad (3.2.5)
\end{align*}
\]

Assume that, for a fixed time \( t \), the function \( u = u(t, y) \) provides the minimum in (3.2.12). Then, for every smooth function \( \varphi : [0, 1] \to \mathbb{R} \), setting \( u(\varepsilon)(y) = u(t, y) + \varepsilon \varphi(y) \) one should have
\[
0 = \left. \frac{d}{d\varepsilon} \int_0^1 \left[ \lambda_1(t, y) \, u(\varepsilon)(y) - \lambda_2(t, y) \xi^2(t, y) \, u_y(\varepsilon)(y) + (u(\varepsilon))^2(y) \right] \, dy \right|_{\varepsilon=0}
\]
\[
= \int_0^1 \left[ \lambda_1(t, y) \, \varphi(y) - \lambda_2(t, y) \xi^2(t, y) \, \varphi_y(y) + 2u(t, y) \varphi(y) \right] \, dy
\]
\[
= \int_0^1 \left[ \lambda_1(t, y) + (\lambda_2 \xi^2)(t, y) + 2u(t, y) \right] \varphi(y) \, dy + \lambda_2(t, 1) \xi^2(t, 1) \varphi(1)
\]
\[
- \lambda_2(t, 0) \xi^2(t, 0) \varphi(0).
\]

Since the function \( \varphi \) can be arbitrary, this yields the Euler-Lagrange equations
\[
u(t, y) = - \frac{\lambda_1(t, y) + (\lambda_2 \xi^2)(t, y)}{2}. \quad (3.2.6)
\]

together with the boundary conditions
\[
(\lambda_2 \xi^2)(t, 0) = (\lambda_2 \xi^2)(t, 1) = 0. \quad (3.2.7)
\]

Observe that (3.2.2) and (3.2.3) yield
\[
(\lambda_2 \xi^2)_t = (2\xi \lambda_2 u_y - 2\xi) \xi^2 - \lambda_2 \xi^2 u_y = -2\xi^3. \quad (3.2.8)
\]
Differentiating both sides of (3.2.6) w.r.t. \( t \) and using (3.2.8) one obtains
\[
    u_t(t, y) = -\lambda_1(t, y) + \frac{(\lambda_2 \xi^2)_t (t, y)}{2} = x + (\xi^3)_y. \tag{3.2.9}
\]

Using the identities
\[
x_{tt}(t, y) = u_t(t, y), \quad x_y(t, y) = \frac{1}{\phi(t, y)} = \frac{1}{\xi(t, y)}, \tag{3.2.10}
\]
we thus recover the PDE
\[
x_{tt} + \frac{3x_{yy}}{(x_y)^4} - x = 0. \tag{3.2.11}
\]
This is a nonlinear elliptic equation, to be solved on the rectangle \([0, T] \times [0, 1]\).
From (3.2.2) and the terminal conditions in (3.2.3), using (3.2.6) one obtains the boundary conditions
\[
x(0, y) = y, \quad x_t(T, y) = u(T, y) = 0. \tag{3.2.12}
\]
Moreover,
\[
\begin{align*}
x_y(t, 0) &= \frac{1}{\xi(t, 0)} = +\infty, \\
x_y(t, 1) &= \frac{1}{\xi(t, y)} = +\infty,
\end{align*} \tag{3.2.13}
\]
because \(0 = (\lambda \xi^2)_t (t, 0) = -2\xi^3(t, 0), \quad 0 = (\lambda \xi^2)_t (t, 1) = -2\xi^3(t, 1)\).

**Figure 3.2.** Left: the optimal trajectories for the standard linear-quadratic optimization problem with dynamics (3.2.9) and cost (3.2.3) independent of \( \xi \). Center: for the system (3.2.1), the presence of a cost depending on \( \xi \) renders more profitable a control where \( u_x \) is positive. Hence the optimal solution should be supported on a larger interval. Right: allowing gaps in the support of \( \mu \) does not provide a way to achieve a lower cost, because in this case the funnels \( \Gamma_i \) determined by near-optimal feedback controls would overlap.
In contrast with the optimal solution in Example 1, letting $T \to \infty$ the optimal trajectories do not converge to zero. Rather than (3.2.24), we expect that the solution $x(t, y), \xi(t, y)$ will approach a steady state $(\bar{x}(y), \bar{\xi}(y))$. Because of the identity $\xi \cdot x_y \equiv 1$, the function $\bar{x}(\cdot)$ should provide a global minimizer to the variational problem

$$\minimize: \int_0^1 \left[ x^2(y) + \left( \frac{1}{x'(y)} \right)^2 \right] dy,$$

the minimum being sought among all absolutely continuous, non-decreasing functions $x : [0, 1] \mapsto \mathbb{R}$, with free boundary conditions. The Euler-Lagrange equations for this problem yield

$$x - \frac{3x''}{(x')^4} = 0,$$

$$\lim_{y \to 0^+} x'(y) = \lim_{y \to 1^-} x'(y) = +\infty.$$

A solution to the above equations is found to be implicitly determined by

$$\frac{x(y)}{2} \sqrt{\frac{2\sqrt{3}}{\pi}} - x^2(y) + \frac{\sqrt{3}}{\pi} \arcsin \frac{\sqrt{\pi} x(y)}{\sqrt{2\sqrt{3}}} = \sqrt{3} \left( y - \frac{1}{2} \right).$$

Observe that this function satisfies $x(y) = -x(1 - y)$ for $y \in [0, 1]$. As $t \to +\infty$, the measures $\mu^{(t)}$ approach a probability distribution $\mu^\infty$ which is symmetric w.r.t. the origin. Notice that, as $T \to +\infty$, in Example 1 the minimum cost remains bounded, while in Example 2 it approaches $+\infty$.

In this case, we could again consider the funnels $\Gamma_i$, defined as in (3.2.29), where now $x = x(t, y)$ is the solution of (3.2.11) on $[0, T] \times [a_i, b_i]$, with boundary conditions

$$\begin{cases} x(0, y) = y, & \quad x_y(a, t) = +\infty, \\ x_T(t, y) = 0, & \quad x_y(b, t) = +\infty. \end{cases}$$

However, these funnels now overlap with each other, and the definition (3.2.36) is not meaningful. For this example, we thus expect

$$J(\mu) = J^s(\mu), \quad \text{but} \quad J^s(\mu) \neq J^\text{relax}(\mu).$$
3.2.3 A case where the width of the funnels can be controlled

Example 3. We now consider a case where \( f \) depends also on \( \xi \), and one can use this additional variable in order to control the width of the funnels \( \Gamma_i \), preventing their overlap. Consider again the optimization problem (3.2.1), but assume that the state of the system evolves according to

\[
\begin{align*}
\dot{x} &= u + \xi, \\
\dot{\xi} &= u_x,
\end{align*}
\]  

(3.2.1)

with initial data

\[
x(0) = y, \quad \xi(0) = h(y).
\]  

(3.2.2)

As before, we assume that \( y \) is a random variable uniformly distributed on the interval \([0, 1]\). Otherwise stated, the probability measure \( \mu \) has density \( \phi = \chi_{[0,1]} \) w.r.t. Lebesgue measure. The corresponding relaxed system is

\[
\begin{align*}
\dot{x} &= u + \xi, \\
\dot{\xi} &= v.
\end{align*}
\]  

(3.2.3)

By (3.2.1) and (3.2.3), to achieve a global minimum one should have

\[
u = \xi = \frac{x_t}{2}.
\]  

(3.2.4)

For each fixed \( y \), writing the Euler-Lagrange equations we find that the optimal solution \( t \mapsto x(t, y) \) to the relaxed problem solves the two-point boundary value problem

\[
\ddot{x} - 2x = 0 \quad \text{for} \quad t \in [0, T], \quad x(0, y) = y, \quad \dot{x}(T, y) = 0.
\]  

(3.2.5)

The optimal solution is thus found to be

\[
\hat{x}(t, y) = \frac{e^{\sqrt{2}(t-2T)} + e^{-\sqrt{2}t}}{1 + e^{-2\sqrt{2}T}} y.
\]  

(3.2.6)
For this relaxed solution, the corresponding variables $u, \xi$ are given by

$$\hat{u}(t, y) = \hat{\xi}(t, y) = \frac{\hat{x}_t(t, y)}{2} = \frac{e^{\sqrt{2}(t-2T)} - e^{-\sqrt{2}t}}{1 + e^{-2\sqrt{2}T}} \frac{y}{\sqrt{2}}. \quad (3.2.7)$$

On the other hand, for a fixed $y$, the map $\hat{v}(\cdot, y)$ should formally be given by the distributional derivative of the map $t \mapsto \hat{\xi}(t, y)$. This is a measure containing a point mass of size

$$\hat{\xi}(0+, y) - h(y) = \frac{e^{-2\sqrt{2}T} - 1}{e^{-2\sqrt{2}T} + 1} \frac{y}{\sqrt{2}} - h(y)$$

at the origin, while its restriction to the open set $\{t > 0\}$ is absolutely continuous w.r.t. Lebesgue measure, with density

$$\hat{v}(t, y) = \frac{1}{2} \hat{x}_tt(t, y) = \hat{x}(t, y) = \frac{e^{\sqrt{2}(t-2T)} + e^{-\sqrt{2}t}}{1 + e^{-2\sqrt{2}T}} \frac{y}{1 + e^{-2\sqrt{2}T}} \frac{y}{\sqrt{2}}. \quad (3.2.8)$$

From the above analysis, we conclude that the infimum among all costs $J(\mu, u, v)$, with $u, v \in C^2$, is provided by

$$J_{relax}^{\mu} = \int_0^1 \int_0^T \left( \hat{x}^2(t, y) + \frac{1}{2} \hat{x}_t^2(t, y) \right) dt dy. \quad (3.2.9)$$

A minimizing sequence of feedback controls $(u_\nu, v_\nu)_{\nu \geq 1}$ is provided by

$$u_\nu(t, x) = u(t, x) = \frac{e^{\sqrt{2}(t-2T)} - e^{-\sqrt{2}t}}{e^{\sqrt{2}(t-2T)} + e^{-\sqrt{2}t}} \frac{x}{\sqrt{2}},$$

$$v_\nu(t, x) = \begin{cases} \nu \cdot \left( \frac{e^{-2\sqrt{2}T} - 1}{e^{-2\sqrt{2}T} + 1} \frac{x}{\sqrt{2}} - h(x) \right) & \text{if } t \in [0, \nu^{-1}], \\
                        x & \text{if } t \in [\nu^{-1}, T]. \end{cases}$$

By performing a suitable cut-off, followed by a mollification, we achieve $u_\nu, v_\nu \in C^2$. This preliminary analysis shows that, for any $\varepsilon > 0$, there exists smooth feedback
controls \( u^* = u^*(t, x) \) and \( v^* = v^*(t, x) \) such that

\[
J(\mu, u^*, v^*) \leq J^{relax}(\mu) + \varepsilon, \quad (3.2.10)
\]

and, calling \( x = x^*(t, y) \), \( \xi = \xi^*(t, y) \) the corresponding solutions of (3.2.2)-(3.2.3), one has

\[
\|x^*\|_{C^2([0, T] \times [0, 1])} \leq M_0, \quad \|
\]

\[
\|\xi^*\|_{C^2([0, T] \times [0, 1])} \leq M_0, \quad (3.2.11)
\]

\[
\|u^*\|_{C^2([0, T] \times \mathbb{R})} \leq M_1, \quad \|v^*\|_{C^2([0, T] \times \mathbb{R})} \leq M_1,
\]

\[
x_y^*(t, y) \geq \rho_0 > 0, \quad \|h(y)\|_{C^2([0, 1])} \leq M_2, \quad (3.2.12)
\]

for some constants \( M_0, M_1, M_2, \rho_0 \), possibly depending on \( \varepsilon \).

**Proposition 2.** In the above example one has \( J^s(\mu) = J^{relax}(\mu) \).

**Proof.** Given \( \varepsilon > 0 \), let \( (u^*, v^*) \) be a pair of generalized feedback controls for which all the estimates (3.2.10)–(3.7.13) hold. To prove Proposition 2, we need to show that there exists a measure \( \tilde{\mu} \) with density \( \tilde{\phi} \) satisfying \( \|\tilde{\phi} - \phi\|_{L^1} \leq \varepsilon \) and a feedback control \( \tilde{u} \in C^2 \) such that

\[
J(\tilde{u}, \tilde{\mu}) \leq J(u^*, v^*, \mu) + \varepsilon. \quad (3.2.13)
\]

1. Consider the augmented system of ODEs

\[
\begin{cases}
\dot{x} = u + \xi, & x(0, y) = y, \\
\dot{\xi} = v, & \xi(0, y) = h(y), \\
\dot{\eta} = \eta v + z, & \eta(0, y) = 1, \\
\dot{z} = w\eta, & z(0, y) = h'(y).
\end{cases} \quad (3.2.14)
\]

Here we think of \( \eta = x_y \) and \( z = \xi_y \) as an additional variable, while \( v = u_x \), \( w = v_x = u_{xx} \) are additional controls. Notice that the last two ODEs in (3.2.14)
follow from

\[(x_y)_t = u_y + \xi_y = u_x x_y + \xi_y, \quad (\xi_y)_t = v_y = u_x x_y.\]

2. For \( n \) large, consider the probability distribution \( \mu_n \) having density

\[
\phi_n(x) = \begin{cases} 
\frac{n}{n-1} & \text{if } x \in \left[ \frac{i-1}{n}, \frac{i}{n} - \frac{1}{n^2} \right] = [a_i, b_i], \text{ for some } i \in \{1, \ldots, n\}, \\
0 & \text{otherwise}.
\end{cases}
\] (3.2.15)

For \( i = 1, \ldots, n \), denote by \( t \mapsto x_i(t) \doteq x^*(t, a_i) \), \( t \mapsto \xi_i(t) \doteq \xi^*(t, a_i) \) the components of the solution of (3.2.2)-(3.2.3) with \( y = a_i \). As a first attempt, one may construct the feedback \( \tilde{u} \) by setting

\[
\tilde{u}(t, x) \doteq u^*(t, x_i(t)) + v^*(t, x_i(t)) \cdot (x - x_i(t)) \tag{3.2.16}
\]

for \( x \approx x_i(t) \). For \( y \in [a_i, b_i] \), call \( t \mapsto (\tilde{x}(t, y), \tilde{\xi}(t, y)) \) the solution of (3.2.1)-(3.2.2), with \( u = \tilde{u} \) given by (3.2.16). Observe that this construction yields

\[
\tilde{x}(t, a_i) = x_i(t) \doteq x^*(t, a_i), \quad \tilde{\xi}(t, a_i) = \xi_i(t) \doteq \xi^*(t, a_i), \quad t \in [0, T], \quad 1 \leq i \leq n.
\]

Introducing the tubes

\[
\Gamma_i \doteq \left\{ (t, \tilde{x}(t, y)) ; \quad t \in [0, T], \quad y \in [a_i, b_i] \right\}, \tag{3.2.17}
\]

one may hope to define \( \tilde{u}(t, x) \) by (3.2.16) for \( (t, x) \in \Gamma_i \), and extend \( \tilde{u} \) in a smooth way on the complementary set \( ([0, T] \times IR) \setminus \cup_i \Gamma_i \). Notice that (3.7.13) implies \( x_1(t) < x_2(t) < \cdots < x_n(t) \), so that the centers of these tubes do not cross each other. Unfortunately, in the present situation there is no guarantee that the tubes \( \Gamma_i \) remain disjoint for all \( t \in [0, T] \). We thus need to refine our construction, relying on a global controllability property of the system (3.2.14). On a small time interval \( [0, \delta] \), we will construct a feedback \( \tilde{u} \) such that the corresponding solution
of (3.2.1)-(3.2.2) satisfies

\[ \left| \tilde{x}(\delta, y) - x_i(\delta) \right| < \epsilon, \quad \left| \tilde{\xi}(\delta, y) - \xi_i(\delta) \right| < \epsilon, \quad \text{for all } y \in [a_i, b_i]. \] (3.2.18)

If (3.2.18) holds, with \( \epsilon \ll n^{-1} \) suitably small, then for \( t \in [\delta, T] \), and \( x \approx x_i(t) \) the definition (3.2.16) will provide a feedback with the desired properties. To achieve (3.2.18) we shall construct a feedback such that \( \tilde{x}_y(\delta, y) \approx 0 \) and \( \tilde{\xi}_y(\delta, y) \approx 0 \), for all \( y \in [a_i, b_i] \).

Figure 3.3. On the initial time interval \([0, \delta]\) a feedback control is implemented such that all initial points \( y \in [a_i, b_i] \) are steered inside a very small neighborhood of the point \( x_i(\delta) = x(\delta, a_i) \). Since at time \( t = \delta \) we have \( x_y \approx 0 \) and \( \xi_y \approx 0 \) inside each tube \( \Gamma_i \), this guarantees that for \( t \in [\delta, T] \) the tubes \( \Gamma_i \) remain mutually disjoint.

3. Let \( \delta > 0 \) be given. Relying on the controllability of the ODE (3.2.14), for \( 0 \leq t \leq \delta \) and \( i \in \{1, \ldots, n\} \), we construct control functions \( u_i(\cdot), v_i(\cdot), w_i(\cdot) \) such that the solution of the Cauchy problem

\[
\begin{align*}
\dot{x} &= u_i + \xi, \\
\dot{\xi} &= v_i, \\
\dot{\eta} &= \eta v_i + z, \\
\dot{z} &= \eta w_i, \\
x(0) &= a_i, \\
\xi(0) &= h(a_i), \\
\eta(0) &= 1, \\
z(0) &= h'(a_i).
\end{align*}
\] (3.2.19)
satisfies \( x(t) = x_i(t) = x^*(t, a_i) \) for all \( t \in [0, \delta] \) and moreover
\[
\xi(\delta) = \xi_i(\delta) = \xi^*(\delta, a_i), \quad \eta(\delta) \approx 0, \quad z(\delta) \approx 0.
\]

Solutions of (3.2.19) are more conveniently found using the variables
\[
X = \ln \eta, \quad Y = \frac{z}{\eta}, \quad (3.2.20)
\]
which evolve according to
\[
\begin{align*}
\dot{X} &= \frac{\dot{\eta}}{\eta} = v_i + Y, \\
\dot{Y} &= \frac{\dot{z}}{\eta} - \frac{z \dot{\eta}}{\eta^2} = w_i - Y^2 - Y v_i \doteq \tilde{w}_i,
\end{align*}
\]
\[
\begin{align*}
X(0) &= 0, \\
Y(0) &= h'(a_i). \quad (3.2.21)
\end{align*}
\]

We regard \( \tilde{w}_i \) as an independent control function. Clearly, we can assign the controls \( v_i(\cdot), \tilde{w}_i(\cdot) \) arbitrarily, then compute the solution of (3.2.21) and define the control \( w_i(t) = \tilde{w}_i(t) + Y^2(t) + Y(t)v_i(t) \).

To achieve (3.2.18), we use the controls
\[
v_i(t) \doteq \begin{cases} 
-1 + \frac{\xi_i(\delta) - \xi_i(0)}{\delta} & 0 \leq t < \frac{\delta}{2}, \\
1 + \frac{\xi_i(\delta) - \xi_i(0)}{\delta} & \frac{\delta}{2} \leq t \leq \delta,
\end{cases} \quad (3.2.22)
\]
\[
\tilde{w}_i(t) \doteq -\frac{1}{\delta^3}, \quad 0 \leq t \leq \delta, \quad (3.2.23)
\]
while the control \( u_i \) is defined as
\[
u_i(t) \doteq u^*(t, x_i(t)) + \xi_i(t) - \left[ \xi_i(0) + \int_0^t v_i(s) \, ds \right] \quad t \in [0, \delta]. \quad (3.2.24)
\]
The corresponding solution of (3.2.21) is

\[
\begin{align*}
X(t) &= \int_0^t v_i(s) \, ds + \int_0^t \left( h'(a_i) - \frac{s}{\delta^3} \right) \, ds, \\
Y(t) &= h'(a_i) - \frac{t}{\delta^3}.
\end{align*}
\]  

(3.2.25)

In particular, at \( t = \delta \) one has

\[
\begin{align*}
X(\delta) &= \xi_i(\delta) - \xi_i(0) + h'(a_i)\delta - \frac{1}{2\delta}, \\
Y(\delta) &= h'(a_i) - \frac{1}{\delta^2}.
\end{align*}
\]  

(3.2.26)

Going back to original variables \( \eta, z \), one obtains

\[
\begin{align*}
\eta(\delta) &= \exp \left( X(\delta) \right) = \exp \left( \xi_i(\delta) - \xi_i(0) + h'(a_i)\delta - \frac{1}{2\delta} \right), \\
z(\delta) &= Y(\delta) \eta(\delta) = \left( h'(a_i) - \frac{1}{\delta^2} \right) \exp \left( \xi_i(\delta) - \xi_i(0) + h'(a_i)\delta - \frac{1}{2\delta} \right).
\end{align*}
\]  

(3.2.27)

Moreover, by the definition of \( \tilde{w} \) in (3.2.21), the control \( w_i = \tilde{w}_i + Y^2 + Y v_i \) is given by

\[
\begin{align*}
w_i(t) &= \left\{ \begin{array}{ll}
- \frac{1}{\delta^3} + \left( h'(a_i) - \frac{t}{\delta^3} \right)^2 + \left( -\frac{1 + \xi_i(\delta) - \xi_i(0)}{\delta} \right) \left( h'(a_i) - \frac{t}{\delta^3} \right), & 0 \leq t < \frac{\delta}{2}, \\
- \frac{1}{\delta^3} + \left( h'(a_i) - \frac{t}{\delta^3} \right)^2 + \left( \frac{1 + \xi_i(\delta) - \xi_i(0)}{\delta} \right) \left( h'(a_i) - \frac{t}{\delta^3} \right), & \frac{\delta}{2} \leq t \leq \delta.
\end{array} \right.
\end{align*}
\]  

(3.2.28)

By (3.2.22), (3.2.23), and (3.8.21), for \( \delta > 0 \) sufficiently small we have the estimates

\[
\begin{align*}
|v_i(t)| &\leq \frac{2}{\delta}, \\
|Y(t)| &\leq \frac{2}{\delta^2}, \\
|w_i(t)| &\leq \frac{5}{\delta^2},
\end{align*}
\]  

for all \( t \in [0, \delta] \).

(3.2.29)

Moreover, by (3.2.24), the solution \((x(\cdot), \xi(\cdot))\) of (3.2.19) satisfies

\[
\dot{x}(t) = u_i(t) + \xi(t) = u^*(t, x_i(t)) + \xi_i(t) = \dot{x}_i(t), \quad \text{for all } t \in [0, \delta].
\]  

4. On a suitable neighborhood of each trajectory \( t \mapsto x_i(t) \), we then define the
feedback control \( \tilde{u} \) as

\[
\tilde{u}(t, x) = \begin{cases} 
  u_i(t) + (x - x_i(t)) \cdot v_i(t) + \frac{(x - x_i(t))^2}{2} \cdot w_i(t), & t \in [0, \delta], \\
  u^*(t, x_i(t)) + (x - x_i(t)) \cdot v^*(t, x_i(t)), & t \in [\delta, T].
\end{cases}
\]

The corresponding solution of (3.2.1) with initial data (3.2.2) will be denoted by \( t \mapsto (\tilde{x}(t, y), \tilde{\xi}(t, y)) \). We can then extend \( \tilde{u} \) in a smooth way (w.r.t. the \( x \)-variable) on the complement of the set \( \bigcup_{1 \leq i \leq n} \Gamma_i \). Notice that, by choosing \( n = n(\delta) >> \delta^{-1} \), we can achieve the convergence

\[
\|\tilde{u} - u^*\|_{L^\infty([\delta, T] \times \mathbb{R})} \to 0 \quad \text{as} \quad \delta \to 0.
\]

For every \( i \in \{1, \ldots, n\} \), the above construction yields

\[
\tilde{x}(t, a_i) = x_i(t) \quad t \in [0, T].
\]

\[
|\tilde{\xi}(t, a_i)| = \left| \xi_i(0) + \int_0^t v_i(s) \, ds \right| \leq M_0(1 + \delta) + 1 \quad t \in [0, \delta],
\]

\[
\tilde{\xi}(\delta, a_i) = \xi_i(\delta).
\]

We claim that, by choosing \( \delta > 0 \) sufficiently small one can achieve

\[
0 < \tilde{x}_y(\delta, a_i) < \delta, \quad |\tilde{\xi}_y(\delta, a_i)| < \delta,
\]

\[
0 < \tilde{x}_y(t, a_i) < x^*_y(t, a_i) \quad \text{for all} \quad t \in [0, \delta].
\]

To prove (3.2.33) we observe that the functions \( \eta(t) = \tilde{x}_y(t, a_i) \) and \( z(t) = \tilde{\xi}_y(t, a_i) \) satisfy the system of ODEs (3.2.19). Hence the bounds (3.2.33) are an immediate consequence of (3.2.27), because \( e^{-1/2\delta} << \delta \) for \( \delta > 0 \) small.

Next, by (3.7.12) we have \( \|x^*_y\|_{C^1} \leq \|x^*\|_{C^2} \leq M_0 \). Since \( x^*_y(0, a_i) = 1 \), this implies

\[
x^*_y(t, a_i) \geq 1 - M_0 t.
\]
For $t \in [0, \delta/2]$ by (3.8.21) we have

$$X(t) = -\frac{1 + \xi_i(\delta) - \xi_i(0)}{\delta} t + h'(a_i) t - \frac{t^2}{2\delta^3} \leq -2M_0 t \leq \ln \left( x^*_y(t, a_i) \right),$$  \hspace{1cm} (3.2.35)

provided that $\delta > 0$ is small enough. On the other hand, for $y \in [\delta/2, \delta]$, we obtain

$$X(t) = \int_{\delta/2}^{\delta} v_i(s) \, ds + \int_{\delta/2}^{t} v_i(s) \, ds + \int_{0}^{t} \left( h'(a_i) - \frac{s}{\delta^3} \right) \, ds$$

$$\leq 1 + |\xi_i(\delta) - \xi_i(0)| + M_2 \delta - \frac{1}{8\delta} \leq -2M_0 t. \hspace{1cm} (3.2.36)$$

Therefore, always assuming that $\delta > 0$ is sufficiently small, for all $t \in [0, \delta]$ we have

$$0 \leq \bar{x}_y(t, a_i) = \exp \left( X(t) \right) \leq x^*_y(t, a_i).$$

An entirely similar estimate can be proved for every initial point $y \in [a_i, b_i]$. Provided that $\delta > 0$ is sufficiently small, we thus conclude

$$0 < \bar{x}_y(\delta, y) < \delta, \hspace{1cm} |\bar{\xi}_y(\delta, y)| < \delta, \hspace{1cm} \text{for all } y \in [a_i, b_i], \hspace{1cm} (3.2.37)$$

$$\bar{x}_y(t, y) \leq x^*_y(t, y) \hspace{1cm} \text{for all } t \in [0, \delta], y \in [a_i, b_i]. \hspace{1cm} (3.2.38)$$

5. The estimate (3.2.38) shows that on the initial time interval $[0, \delta]$ the tubes $\Gamma_i$ defined as in (3.7.23) do not overlap with each other. Next, we check that the tubes $\Gamma_i$ remain disjoint also for $t \in [\delta, T]$.

The first inequality in (3.7.13) implies

$$x_{i+1}(t) - x_i(t) \geq \frac{\rho_0}{n} \hspace{1cm} \text{for all } i \in \{0, \ldots, n - 1\}. \hspace{1cm} (3.2.39)$$

Observe that, by (3.2.1) and (3.7.4),

$$\frac{\partial}{\partial t} \left( |\bar{x}_y(t, y)| + |\bar{\xi}_y(t, y)| \right) \leq \left( v^*(t, x_i(t)) + 1 \right) \cdot \left( |\bar{x}_y(t, y)| + |\bar{\xi}_y(t, y)| \right).$$
Therefore the bound (3.7.12) on $v^*$ together with (3.2.33) yields

$$\tilde{x}_y(t, y) \leq e^{(M_1+1)T} \left( |\tilde{x}_y(\delta, y)| + |\tilde{\xi}_y(\delta, y)| \right) \quad \text{for all } t \in [\delta, T]. \quad (3.2.40)$$

For $y \in [a_i, b_i]$ and $t \in [\delta, T]$ we have

$$\tilde{x}(t, y) - x_i(t) \leq \int_{a_i}^{y} \tilde{x}_y(t, z) d\tilde{z} \leq (b_i - a_i) \cdot \sup_{y \in [0,1]} \tilde{x}_y(t, y) \leq \frac{\rho_0}{n}, \quad (3.2.41)$$

provided that

$$e^{(M_1+1)T} \cdot \sup_y \left( |\tilde{x}_y(\delta, y)| + |\tilde{\xi}(\delta, y)| \right) \leq \rho_0.$$

Recalling (3.2.33) we can now choose $\delta > 0$ small enough so that $e^{(M_1+1)T} 4\delta < \rho_0$. Then we choose $n = n(\delta) >> 1/\delta$ large enough so that, by continuity, the estimates

$$\left( |\tilde{x}_y(\delta, y)| + |\tilde{\xi}(\delta, y)| \right) < 4\delta < \frac{\rho_0}{e^{(M_1+1)T}}$$

remain valid for every $y \in [a_i, b_i], i = 1, \ldots, n$. By (3.2.39) and (3.2.41), this implies that the tubes $\Gamma_i$ remain mutually disjoint also for $t \in [\delta, T]$.

6. It is clear that the sequence of densities $\phi_n$ in (3.7.17) converges to $\phi = \chi_{[0,1]}$ as $n \to \infty$. Having chosen $\delta > 0$ and $n = n(\delta) >> \delta^{-1}$ as before, let $\tilde{u} = \tilde{u}(t, x)$ be a feedback control satisfying (3.7.4) on each tube $\Gamma_i$, extended in a smooth way outside the union $\bigcup_{i=1}^{n} \Gamma_i$. It remains to show that, as $\delta \to 0$, the expected cost for the feedback $\tilde{u}$ approaches $J(u^*, v^*, \mu)$. Indeed, on the initial interval $[0, \delta]$, by (3.2.32) all functions $\tilde{x}, \tilde{\xi}, \tilde{u}$ remain uniformly bounded as $\delta \to 0$. Therefore

$$\frac{n}{n-1} \cdot \sum_{i=1}^{n} \int_{0}^{\delta} \int_{a_i}^{b_i} \left( \tilde{x}^2(t, y) + \tilde{\xi}^2(t, y) + \tilde{u}^2(t, \tilde{x}(t, y)) \right) dy dt \leq C\delta. \quad (3.2.42)$$

To see what happens on the remaining interval $[\delta, T]$, consider the quantities

$$A_n = \frac{n}{n-1} \cdot \sum_{i=1}^{n} \int_{\delta}^{T} \int_{a_i}^{b_i} \left( \tilde{x}^2(t, y) + \tilde{\xi}^2(t, y) + \tilde{u}^2(t, \tilde{x}(t, y)) \right) dy dt$$
\[ B_n = \frac{1}{n} \sum_{i=1}^{n} \int_0^T \left( (x^*)^2(t,a_i) + (\xi^*)^2(t,a_i) + (u^*)^2(t,x_i(t)) \right) dy dt. \]

Recalling (3.2.31) we have \(|A_n - B_n| \to 0\) as \(\delta \to 0\) and \(n = n(\delta) \to \infty\). Moreover,

\[
\lim_{n \to \infty} B_n = \int_0^T \int_0^1 \left( (x^*)^2(t,y) + (\xi^*)^2(t,y) + (u^*)^2(t,x^*(t,y)) \right) dy dt = J(u^*,v^*,\mu). 
\]

This completes the proof. \(\square\)

**Remark 4.** We point out a fundamental difference between the first two examples and this last one. Namely, consider the system of four ODEs, obtained by adding to (3.1.15) two additional equations for the variables \(\alpha(t) = x_y(t,y)\) and \(\beta(t) = \xi_y(t,x)\).

\[
\begin{cases}
\dot{x} = f, \\
\dot{\xi} = g, \\
\dot{\alpha} = (f_x + f_u v) \alpha + f_{\xi} \beta, \\
\dot{\beta} = (g_x + g_u v + g_{u_x} w) \alpha + g_{\xi} \beta.
\end{cases}
\] (3.2.43)

Here \(v = u_x\) and \(w = u_{xx}\) are regarded as independent control functions. In the first two examples this system is not controllable. Indeed, no matter what controls are implemented, in Example 1 we always have \(\xi(t) - \alpha(t) \equiv 0\), while in Example 2 one has \(\xi(t) \cdot \alpha(t) \equiv 1\). On the other hand, in Example 3 there is no functional relation between \(\xi\) and \(\alpha\). We expect that this controllability property of the extended system (3.2.43) should play a key role, determining the equality between the minimal costs \(J^s(\mu)\) and \(J^{relax}(\mu)\).

### 3.3 An alternative formulation

Assume that the initial probability \(\mu\) on the initial point \(\bar{x} \in \mathbb{R}\) is absolutely continuous with density \(\bar{\phi}\) w.r.t. Lebesgue measure. Given a smooth feedback control \(u = u(t,x)\), consider the density \(\phi(t,\cdot)\), obtained by solving the linear transport equation

\[
\phi_t + (f \phi)_x = 0, \quad \phi(0,\cdot) = \bar{\phi}. \] (3.3.1)
For $y \in [0, 1]$, consider the characteristic $t \mapsto x(t, y)$ such that

\[
\dot{x}(t, y) = f(t, x, \xi, u(t, x)), \quad x(0, y) = \bar{x}(y),
\]

where the non-decreasing map $y \mapsto \bar{x}(y)$ is implicitly defined by

\[
\int_{-\infty}^{\bar{x}(y)} \bar{\phi}(s) \, ds = y.
\]

Notice that $\bar{x}(y)$ is well defined for a.e. $y \in [0,1]$. Moreover, as long as $\phi$ remains bounded, the map $(t, y) \mapsto (t, x(t, y))$ satisfies

\[
x_y(t, y) = \frac{1}{\phi(t, y)}.
\]

In terms of the "Lagrangian" variable $y$, the optimization problem takes the form

\[
\text{minimize: } \int_0^T \int_0^1 L\left(t, x(t, y), \xi(t, y), u(t, y)\right) \, dy \, dt,
\]

subject to

\[
\begin{aligned}
\dot{x} &= f(t, x, \xi, u), \\
\dot{\xi} &= g(t, x, \xi, u, \phi u_y), \\
\dot{\phi} &= -\phi^2 \frac{d}{dy} f(t, x, \xi, u),
\end{aligned}
\]

\[
\begin{aligned}
x(0, y) &= \bar{x}(y), \\
\xi(0, y) &= h(\bar{x}(y)), \\
\phi(0, y) &= \bar{\phi}(y).
\end{aligned}
\]

In this formulation, the infimum is sought over all smooth control functions $u = u(t, y)$ defined on the domain $[0, T] \times [0, 1]$. We remark that this formulation is meaningful as long as $x_y > 0$, so that the map $(t, y) \mapsto (t, x(t, y))$ is invertible.

Since $x, \xi, u$ are now regarded as functions of $t, y$, one has

\[
\frac{d}{dy} f(t, x, \xi, u) = f_x x_y + f_\xi \xi_y + f_u u_y.
\]
Setting \( v = u_x = \phi u_y \), the corresponding relaxed system takes the form

\[
\begin{align*}
\dot{x} &= f(t, x, \xi, u), \\
\dot{\xi} &= g(t, x, \xi, u, v), \\
\dot{\phi} &= -\phi \left( f_x + \phi f_{\xi y} + f_u v \right).
\end{align*}
\]

\[x(0, y) = \bar{x}(y), \quad \xi(0, y) = h(\bar{x}(y)), \quad \phi(0, y) = \bar{\phi}(y).\]  

(3.3.7)

We now investigate in which cases the optimal solution of (3.3.5)-(3.3.6) can be determined by solving a second order PDE.

Assume that, for each \( y \in [0, 1] \) and every choice of the feedback control \( u \), the cost function \( L \) can be expressed in terms of the variables \( t, x, y, x_t, x_y \), say

\[ L(t, x(t, y), \xi(t, y), u(t, y)) = \tilde{L}(t, y, x(t, y), x_t(t, y), x_y(t, y)). \]  

(3.3.8)

Then the optimization problem (3.3.5) can be written as

\[ \text{minimize : } \int_0^T \int_0^1 \tilde{L}(t, y, x(t, y), x_t(t, y), x_y(t, y)) \, dy \, dt. \]  

(3.3.9)

Assuming that \( \tilde{L} \) is continuously differentiable w.r.t. \( x, x_t, x_y \), the Euler-Lagrange equation for (3.3.9) takes the form

\[ \frac{d}{dt} \frac{\partial \tilde{L}}{\partial x_t} + \frac{d}{dy} \frac{\partial \tilde{L}}{\partial x_y} = - \frac{\partial \tilde{L}}{\partial x}. \]  

(3.3.10)

Writing out the total derivatives w.r.t. \( t, y \), one obtains a second order PDE, namely

\[ \frac{\partial^2 \tilde{L}}{\partial (x_t)^2} x_{tt} + 2 \frac{\partial^2 \tilde{L}}{\partial x_t \partial x_y} x_{ty} + \frac{\partial^2 \tilde{L}}{\partial (x_y)^2} x_{yy} = \mathcal{K}, \]  

(3.3.11)

where \( \mathcal{K} = \mathcal{K}(t, y, x, x_t, x_y) \) collects all lower order terms. The equation (3.3.11)
should be solved on the domain $[0,T] \times [0,1]$, together with the boundary conditions

$$
\begin{align*}
\begin{cases}
    x(0, y) &= \bar{x}(y), & \frac{\partial \tilde{L}}{\partial x_t}(T, y) &= 0 & \text{for all } y \in [0,1], \\
    \frac{\partial \tilde{L}}{\partial x_y}(t, 0) &= 0, & \frac{\partial \tilde{L}}{\partial x_y}(t, 1) &= 0 & \text{for all } t \in [0,T].
\end{cases}
\end{align*}
$$

(3.3.12)

**Remark 2.** One expects that the above boundary value problem will be well posed provided that the PDE (3.3.11) is elliptic. This holds if

$$
\frac{\partial^2 \tilde{L}}{\partial (x_t)^2} > 0, \quad \frac{\partial^2 \tilde{L}}{\partial (x_t)^2} \cdot \frac{\partial^2 \tilde{L}}{\partial (x_y)^2} > \left( \frac{\partial^2 \tilde{L}}{\partial x_t \partial x_y} \right)^2.
$$

(3.3.13)

**Remark 3.** In general, the running cost $L$ cannot be expressed as a function of $t, x, x_t, x_y$ as in (3.3.8). However, there are a few special cases where this assumption is satisfied. To understand what conditions are needed, assume that $f = f(t, x, u)$ is independent of $\xi$. This leads to the system

$$
\begin{align*}
\begin{cases}
    \dot{x} &= f(t, x, u) \\
    \dot{\xi} &= g(t, x, \xi, u, v) \\
    \dot{\phi} &= -\phi(f_x + f_u v)
\end{cases}
\quad \begin{cases}
    x(0, y) = \bar{x}(y) \\
    \xi(0, y) = h(\bar{x}(y)) \\
    \phi(0, y) = \bar{\phi}(y).
\end{cases}
\end{align*}
$$

(3.3.14)

Here we regard $v = u_x = \phi u_y$ as an independent control variable. Assume that $f_u(t, x, u) \neq 0$ for every $t, x, u$. By the implicit function theorem, we can then recover $u$ as a function of $t, x$, and $x_t = f(t, x, u)$.

Next, observe that $x_y$ and $\phi$ are always functionally dependent, because of (3.3.4). To obtain a representation of the form $\xi = \xi(t, x, y, x_y)$, we thus need to express $\xi$ as

$$
\xi = \Psi(t, x, y, \phi).
$$

(3.3.15)
Differentiating (3.3.15) w.r.t. time and using (3.3.14) one obtains

\[ \Psi_t + \Psi_x f - \Psi \phi (f_x + f_u v) = g(t, x, \Psi, u, v). \tag{3.3.16} \]

Further differentiations w.r.t. \( u, v \) yield

\[
\begin{cases}
\Psi_x f_u - \Psi \phi (f_{ux} + f_{uu} v) = g_u, \\
-\Psi \phi f_u = g_v. 
\end{cases} \tag{3.3.17}
\]

Solving for \( \Psi \phi \) and then for \( \Psi_x \) we obtain

\[
\begin{cases}
\Psi \phi = -\frac{g_v}{\phi f_u}, \\
\Psi_x = \frac{1}{f_u} \left( g_u - \frac{g_v}{f_u} (f_{ux} + f_{uu} v) \right). 
\end{cases} \tag{3.3.18}
\]

Notice that the right hand sides can depend on \( t, x, y, \) and \( \xi = \Psi \), but not on \( u, v \). This yields a further restrictive condition on the system (3.3.14).

Two cases admitting the representation (3.3.8) were considered in [13]. Another example is the following.

**Example 1.** Consider the optimization problem (3.1.6)–(3.1.8), with

\[
f(x, u) = -x + u, \quad g(x, \xi, u, v) = \xi (v - 1) + xv - u, \quad h(x) \equiv 1, \\
L(x, \xi, u) = x^2 + \xi^2 + u^2,
\]

and let the initial data be uniformly distributed on the interval \([0, 1]\).

In terms of the “Lagrangian” coordinate \( y \in [0, 1] \), this problem can be reformulated as

\[
\text{minimize:} \quad \int_0^1 \int_0^T \left( x^2(t, y) + u^2(t, y) + \xi^2(t, y) \right) dt \, dy
\]
subject to

\[
\begin{cases}
\dot{x} = -x + u, \\
\dot{\xi} = \xi(\phi u_y - 1) + x\phi u_y - u, \\
\dot{\phi} = \phi(1 - \phi u_y),
\end{cases}
\]

\[
\begin{cases}
x(0) = y, \\
\xi(0) = 1, \quad \phi(0) = 1.
\end{cases}
\]

(3.3.19)

We claim that the identities

\[
\begin{cases}
u(t, y) = x_t(t, y) + x(t, y), \\
\xi(t, y) = (y + 1)x_y(t, y) - x(t, y).
\end{cases}
\]

(3.3.20)

hold for all \( t, y \). Indeed, the first identity follows from the first ODE in (3.3.19). Because of the initial data in (3.3.19), it is clear that the second identity is valid at \( t = 0 \), for any \( y \in [0, 1] \). To prove that the equality still holds for all \( t \in [0, T] \), for any choice of the control functions \( u, u_y \), we check that both sides have the same derivative w.r.t. time. This is true because of (3.3.4) and the ODEs in (3.3.19),

We can now write

\[
\bar{L}(x, x_t, x_y) = x^2 + \left((y+1)x_y - x(t, y)\right)^2 + (x_t - x)^2 = x^2 + \xi^2 + u^2 = L(x, \xi, u).
\]

The Euler-Lagrange equation (3.3.10) take the form

\[
(y + 1)^2 x_{yy} + x_{tt} + 2(y + 1)x_y - 4x = 0.
\]

(3.3.21)

Observe that in this case (3.3.19) yields

\[
f_u = 1, \quad g_u = -1, \quad g_v = \xi + x = \frac{y + 1}{\phi}.
\]
The two identities in (5.3.3) take the form

\[
\begin{align*}
\Psi_\phi &= -\frac{g_v}{\phi f_u} = y + 1, \\
\Psi_x &= \frac{1}{f_u} \left( g_u - \frac{g_v}{f_u} (f_{ux} + f_{uu} u) \right) = -1.
\end{align*}
\]

(3.3.22)

where the right hand sides do not depend on \(u, v\).

### 3.4 Necessary conditions for optimality

In this section we derive some necessary conditions for the optimal solution of (3.1.6)–(3.1.8). In general, these necessary conditions will take the form of a system of PDEs, which can reduced to a single scalar equation only in the special case considered in the previous section.

Consider again the problem

\[
\begin{align*}
\text{minimize: } & \int_0^T \int_0^1 L(t, x(t, y), \xi(t, y), u(t, x(t, y))) \, dy \, dt, \\
\text{subject to } & \begin{alignat*}{2}
\dot{x} &= f(t, x, \xi, u), \\
\dot{\xi} &= g(t, x, \xi, u, u_x), \\
x(0, y) &= \bar{x}(y), \\
\xi(0, y) &= h(\bar{x}(y)).
\end{alignat*}
\end{align*}
\]

(3.4.1)

(3.4.2)

(3.4.3)

In addition to (A1)–(A3), we now assume that the integrand  \(L\) in (3.4.1) is continuously differentiable w.r.t. \(x, \xi, u\). We regard (3.4.1)-(3.4.3) as a problem of optimal control on the infinite dimensional space whose elements are couples of functions \((x, \xi) : [0, 1] \mapsto \mathbb{R} \times \mathbb{R}\). The infimum is sought over all control functions \(u : [0, T] \times \mathbb{R} \mapsto \mathbb{R}\) which are \(C^2\) w.r.t. \(x\), so that the corresponding evolution of the variables \(x, \xi\) in (3.3.6) is well defined.

For a given control \(u = u(t, x)\), consider a family of perturbed solutions of (4.3.2),
defined as
\[
\begin{cases}
  x^\varepsilon(t, y) &= x(t, y) + \varepsilon X(t, y) + o(\varepsilon), \\
  \xi^\varepsilon(t, y) &= \xi(t, y) + \varepsilon Z(t, y) + o(\varepsilon).
\end{cases}
\] (3.4.4)

Linearizing (4.3.2) around the reference trajectory \( t \rightarrow (x(t, y), \xi(t, y)) \), we obtain a linear equation for the first order perturbations \( X, Z \), namely
\[
\begin{pmatrix}
  \dot{X} \\
  \dot{Z}
\end{pmatrix} =
\begin{pmatrix}
  f_x + f_u u_x & f_\xi \\
  g_x + g_u u_x + g_v u_{xx} & g_\xi
\end{pmatrix}
\begin{pmatrix}
  X \\
  Z
\end{pmatrix},
\] (3.4.5)

Let the couple of functions \((P, Q) : [0, 1] \mapsto \mathbb{R} \times \mathbb{R}\) evolve according to the dual system
\[
\begin{pmatrix}
  \dot{P} \\
  \dot{Q}
\end{pmatrix} =
- \begin{pmatrix}
  f_x + f_u u_x & g_x + g_u u_x + g_v u_{xx} \\
  f_\xi & g_\xi
\end{pmatrix}
\begin{pmatrix}
  P \\
  Q
\end{pmatrix} - \begin{pmatrix}
  L_x + L_u u_x \\
  L_\xi
\end{pmatrix},
\] (3.4.6)

with terminal data
\[
P(T, y) = 0, \quad Q(T, y) = 0 \quad \text{for all } y \in [0, 1]. \tag{3.4.7}
\]

Observe that, for any solution \( \begin{pmatrix} X \\ Z \end{pmatrix} \) of (4.3.6) and any solution \( \begin{pmatrix} P \\ Q \end{pmatrix} \) of (4.3.9), one has
\[
\frac{d}{dt} \int_0^1 \left[ X(t, y) P(t, y) + Z(t, y) Q(t, y) \right] dy = - \int_0^1 \left[ (L_x + L_u u_x) X + L_\xi Z \right] dy.
\] (3.4.8)

**Theorem 1 (necessary conditions for optimality).** Let \( u = u(t, x) \) be an optimal feedback control for the problem (3.4.1)–(3.4.3), and let \((x, \xi) : [0, T] \times [0, 1] \mapsto \mathbb{R} \times \mathbb{R}\) be the corresponding optimal solution. Assume that \( u \) is piecewise continuous w.r.t. \( t \) and twice continuously differentiable w.r.t. \( x \). Moreover, assume that \( L \) is continuously differentiable w.r.t. \( x, \xi, u \). Let the couple of dual functions \((P, Q) : [0, T] \times \mathbb{R} \mapsto \mathbb{R} \times \mathbb{R}\) be the solutions of (4.3.8)-(4.3.9).

Then, for a.e. \( t \in [0, T] \), the feedback control \( x \mapsto u(t, x) \) provides a global mini-
mizer to the functional

\[
J(t, \omega(\cdot)) = \int_0^1 P(t, y) \cdot f\left(t, x(t, y), \xi(t, y), \omega(t, x(t, y))\right) dy \\
+ \int_0^1 Q(t, y) \cdot g\left(t, x(t, y), \xi(t, y), \omega(t, x(t, y)), \omega_x(t, x(t, y))\right) dy \\
+ \int_0^1 L\left(t, x(t, y), \xi(t, y), \omega(t, x(t, y))\right) dy.
\]

(3.4.9)

**Proof.** Assume that the above minimality condition does not hold. Then there exists a time \( \tau \in ]0, T] \) at which \( u(\cdot) \) is continuous, and a control function \( \omega : \mathbb{R} \mapsto \mathbb{R} \) such that

\[
J\left(\tau, \omega(\cdot)\right) < J\left(\tau, u(\tau, \cdot)\right).
\]

(3.4.10)

We then construct a family of “needle variations” of \( u \) by setting

\[
u_\varepsilon(t, x) = \begin{cases} \\
\omega(x) & \text{if } t \in [\tau - \varepsilon, \tau], \\
u(t, x) & \text{if } t \notin [\tau - \varepsilon, \tau].
\end{cases}
\]

(3.4.11)

The corresponding first order perturbations \( X, Z \) in (4.3.5) satisfy

\[
X(t, y) = Z(t, y) = 0 \quad \text{for all } t < \tau,
\]

while

\[
X(\tau, y) = \lim_{\varepsilon \to 0^+} \frac{x^\varepsilon(\tau, y) - x(\tau, y)}{\varepsilon} \\
= f\left(\tau, x(\tau, y), \xi(\tau, y), \omega(x(\tau, y))\right) - f\left(\tau, x(\tau, y), \xi(\tau, y), u(\tau, x(\tau, y))\right),
\]

(3.4.12)
Differentiating the total cost w.r.t. $\varepsilon$ at $\varepsilon = 0^+$, and using (4.3.10), the boundary condition (4.3.9), and finally (4.3.17)-(4.3.18), we obtain

\[
\frac{d}{d\varepsilon} \left[ \int_0^T \int_0^1 L\left( t, x^\varepsilon(t, y), \xi^\varepsilon(t, y), u^\varepsilon(t, x^\varepsilon(t, y)) \right) dy dt \right]_{\varepsilon = 0^+} = \int_0^T \int_0^1 \left( \left( L_x + L_u u_x \right) X + L_\xi Z \right) dy dt + \int_0^1 L\left( \tau, x(\tau, y), \xi(\tau, y), \omega(x(\tau, y)) \right) dy
\]

\[
- \int_0^1 L\left( \tau, x(\tau, y), \xi(\tau, y), u(\tau, x(\tau, y)) \right) dy
\]

\[
= - \int_\tau^T \frac{d}{dt} \left[ \int_0^1 (XP + ZQ) dy \right] dt + \int_0^1 \left[ L(\tau, x, \xi, \omega) - L(\tau, x, \xi, u(\tau)) \right] dy
\]

\[
= \int_0^1 \left( P(\tau, y)X(\tau, y) + Q(\tau, y)Z(\tau, y) \right) dy + \int_0^1 \left[ L(\tau, x, \xi, \omega) - L(\tau, x, \xi, u) \right] dy
\]

\[
= J\left( \tau, \omega(\cdot) \right) - J\left( \tau, u(\tau, \cdot) \right) < 0.
\]
a smooth positive function, then from the optimality condition

\[ J(t, u(t, \cdot)) = \min_\omega J(t, \omega(\cdot)) \]  \hfill (3.4.15)

one can derive an ODE satisfied by \( u(t, \cdot) \), for a.e. time \( t \). Toward this goal, let \( \phi = \phi(t, y) \) be the solution to

\[ \phi_t = -\phi^2 (f_x x_y + f_\xi \xi_y + f_u u_y), \quad \phi(0, y) = \bar{\phi}(y) \overset{\equiv}{=} \frac{1}{x_y(0, y)}, \]

so that \( x_y(t, y) \cdot \phi(t, y) \equiv 1 \).

Given a time \( t \) and functions \( x, \xi, P, Q : [0, 1] \mapsto \mathbb{R} \), a global minimizer \( u(t, \cdot) \) of (4.3.12) must then satisfy

\[ 0 = \int_0^1 (P f_u w + Q g_u w + Q g_v w_x + L_u w) \, dy \]

\[ = \int_0^1 (P f_u + Q g_u - (Q g_v \phi)_y + L_u) w \, dy \]  \hfill (3.4.16)

for every function \( w \in C^1_c([0, 1]) \). Notice that in (4.3.21) we integrated by parts, using the identity \( \omega_x = \phi(t, y) \omega_y \). Since \( w \) is arbitrary, the above necessary condition yields

\[ P f_u + Q g_u - (Q g_v \phi)_y + L_u = 0. \]  \hfill (3.4.17)

At a given time \( t \), it is understood that the left hand side of (4.3.22) should be computed at the point \( (t, x(t, y), \xi(t, y), u(t, x(t, y))) \), for any \( y \in [0, 1] \). Moreover, by (3.4.3), (4.3.9) and choosing \( \omega \) which does not vanish on the boundary, we obtain the boundary conditions as follows

\[ (Q g_v)_{|y=0} = (Q g_v)_{|y=1} = 0, \]

\[ x(0, y) = \bar{x}(y), \quad L_u_{|t=T} = 0. \]  \hfill (3.4.18)

As shown in [13], in some special cases the equations (4.3.22)-(4.3.23) yield a scalar, elliptic boundary value problem for \( x = x(t, y) \).
Example 2. Consider the optimization problem

\[
\text{minimize: } \int_0^T \int_0^1 \left( x^2 + \xi^2 + u^2 \right) dy \, dt , \tag{3.4.19}
\]

subject to

\[
\begin{align*}
\dot{x} &= u , \quad x(0,y) = y , \tag{3.4.20} \\
\dot{\xi} &= \xi u_x , \quad \xi(0,y) = 1 .
\end{align*}
\]

We are here assuming that the initial datum \(x(0)\) is uniformly distributed on the interval \([0,1]\). By (3.4.20) we have the identity \(\xi(t,y) \equiv x_y(t,y)\). Moreover, the dual system (4.3.8) takes the form

\[
\begin{align*}
\dot{P} &= -u_x P - \xi u_{xx} Q - 2x - 2uu_x , \\
\dot{Q} &= -u_x Q - 2\xi . \tag{3.4.21}
\end{align*}
\]

The necessary conditions (4.3.22)-(4.3.23) in this case take the form

\[
\begin{align*}
P - Q_y + 2u &= 0 ,
\end{align*}
\]

\[
\begin{align*}
\left. \left( \frac{\partial P}{\partial y} \right) \right|_{y=0} &= \left. \left( \frac{\partial Q}{\partial y} \right) \right|_{y=1} = 0 ,
\end{align*}
\]

\[
\begin{align*}
x(0,y) &= y , \quad u(T,y) = 0 . \tag{3.4.22}
\end{align*}
\]

By (3.4.20)-(3.4.21), one has

\[
\dot{Q} \xi + Q \dot{\xi} = (-u_x Q - 2\xi) \xi + Q \xi u_x = -2\xi^2 .
\]

Therefore, differentiating the first two boundary conditions in (3.4.22) one obtains

\[
-2\xi^2 \bigg|_{y=0} = -2\xi^2 \bigg|_{y=1} = 0 .
\]

Moreover, differentiating w.r.t. \(y\) the second equation in (3.4.21) we obtain

\[
\dot{Q} = -u_{xx} Q_x y - u_x Q_y - 2\xi_y . \tag{3.4.23}
\]
By (3.4.20)-(3.4.23) it follows

\[ 0 = \dot{P} - \dot{Q}_y + 2\dot{u} = u_x(Q_y - P - 2u) - 2x + 2\xi_y + 2\dot{u} = -2x + 2x_{yy} + 2x_{tt}, \]

Therefore, assuming sufficient regularity, the component \( x = x(t, y) \) of the optimal solution will satisfy the linear elliptic equation

\[ x_{yy} + x_{tt} - x = 0, \]

with boundary condition

\[
\begin{aligned}
x_y(t, 0) &= x_y(t, 1) = 0, \\
x(0, y) &= y, \\
x_t(T, y) &= 0.
\end{aligned}
\] (3.4.24)

**Example 3.** Consider the same optimization problem (3.4.19), but subject to

\[
\begin{aligned}
\dot{x} &= u, \\
\dot{\xi} &= -\xi u_x, \\
x(0, y) &= y, \\
\xi(0, y) &= 1.
\end{aligned}
\] (3.4.25)

In this case we have the identity \( x_y \equiv \xi^{-1} \). Moreover, \( P \) and \( Q \) satisfy

\[
\begin{aligned}
\dot{P} &= -u_xP + \xi u_xQ - 2x - 2uu_x, \\
\dot{Q} &= u_xQ - 2\xi.
\end{aligned}
\] (3.4.26)

The necessary condition (4.3.22) now becomes

\[ P + (Q\xi^2)_y + 2u = 0, \] (3.4.27)

which yields the nonlinear elliptic PDE

\[ x_{tt} + \frac{3x_{yy}}{(x_y)^4} - x = 0, \]
with the boundary condition

\[
\begin{aligned}
x_y(t, 0) &= x_y(t, 1) = +\infty, \\
x(0, y) &= y, \quad x_t(T, y) = 0.
\end{aligned}
\]  

(3.4.28)

3.5 An equivalence result, for a discrete probability distribution

Throughout the following, we consider the optimization problem introduced at (3.1.6)–(3.1.8), where the initial data \(x(0)\) are distributed according to a probability measure \(\mu\). We always assume that the assumptions (A1)–(A3) listed in the Introduction remain valid. The main result of this section gives a condition for the equivalence of the infimum costs in (3.1.12) and (3.1.16).

**Theorem 2.** Let (A1)-(A2) hold. If the probability measure \(\mu\) consists of finitely many point masses, then

\[ J(\mu) = J_{\text{relax}}(\mu). \]

**Proof.** 1. Let the probability measure \(\mu\) be supported on the finite set \(\{y_1, \ldots, y_n\}\).

By the definition (3.1.16), given any \(\varepsilon > 0\) one can find two smooth feedback controls \(u(t, x), v(t, x)\) such that

\[ J(u, v, \mu) < J_{\text{relax}}(\mu) + \varepsilon. \]  

(3.5.1)

For each \(i = 1, \ldots, n\), call \(t \mapsto (x_i(t), \xi_i(t))\) the solution to the Cauchy problem

\[
\begin{aligned}
\dot{x} &= f(t, x, \xi, u(t, x)), \\
\dot{\xi} &= g(t, x, \xi, u(t, x), v(t, x)), \\
x(0) &= y_i, \\
\xi(0) &= h(y_i).
\end{aligned}
\]  

(3.5.2)

By the regularity assumption (A1), each solution \((x_i(t), \xi_i(t))\) is well defined.

2. In the special case where the trajectories \(t \mapsto x_i(t)\) do not intersect, say with

\[ |x_i(t) - x_j(t)| \geq 3\delta > 0 \quad \text{for all} \quad t \in [0, T], \quad i \neq j, \]  

(3.5.3)
the proof is straightforward. Indeed, one can define a feedback control $u^*$ by setting

$$u^*(t, x) = u(t, x_i(t)) + (x - x_i(t)) \cdot v(t, x_i(t)) \quad \text{if } |x - x_i(t)| \leq \delta. \quad (3.5.4)$$

The function $u^*$ is then extended in a smooth way outside the disjoint tubes

$$\Gamma_i \doteq \{(t, x); \ |x - x_i(t)| \leq \delta\}.$$

It is now immediate to check that this definition yields $J(\mu, u^*) = J(\mu, u, v)$. In the remainder of the proof we deal with the case where two or more trajectories intersect, so that (3.5.3) fails.

3. By the regularity assumptions, there exist constants $M_0, M_1$ such that

$$|x_i(t)| < M_0, \quad |\xi_i(t)| < M_0, \quad t \in [0, T], \quad i \in \{1, \ldots, n\}, \quad (3.5.5)$$

$$|u(t, x)| < M_1, \quad |v(t, x)| < M_1 \quad \text{whenever } |x| \leq M_0. \quad (3.5.6)$$

Moreover, let $M_2$ and $M_3$ be Lipschitz constants such that

$$|f(t, x, \xi, u) - f(t, \tilde{x}, \tilde{\xi}, \tilde{u})| \leq M_2(|x - \tilde{x}| + |\xi - \tilde{\xi}| + |u - \tilde{u}|),$$
$$|g(t, x, \xi, u, v) - g(t, \tilde{x}, \tilde{\xi}, \tilde{u}, \tilde{v})| \leq M_2(|x - \tilde{x}| + |\xi - \tilde{\xi}| + |u - \tilde{u}| + |v - \tilde{v}|),$$

$$|u(t, x) - u(t, \tilde{x})| \leq M_3|x - \tilde{x}|,$$
$$|v(t, x) - v(t, \tilde{x})| \leq M_3|x - \tilde{x}|,$$  \quad (3.5.7)

whenever $t \in [0, T]$, $|x| \leq M_0$, and $|u|, |\tilde{u}|, |v|, |\tilde{v}| \leq M_1$.

4. Fix $\varepsilon > 0$ and let $t \mapsto Z(t)$ be the solution to the scalar ODE

$$\dot{Z} = (2M_2 + 4M_2M_3) Z + \varepsilon, \quad Z(0) = 0. \quad (3.5.9)$$

Consider the tubes

$$\Gamma_i = \{(t, x, \xi); \ t \in [0, T], \ |x - x_i(t)| \leq Z(t), \ |\xi - \xi_i(t)| \leq Z(t)\}. \quad (3.5.10)$$
By choosing $\varepsilon > 0$ sufficiently small, by (3.5.5) it follows

$$|x| \leq M_0, \quad |\xi| \leq M_0 \quad \text{whenever} \quad (t, x, \xi) \in \Gamma_i.$$ 

Next, consider two absolutely continuous maps $t \mapsto x(t), t \mapsto \xi(t)$, taking the initial values

$$x(0) = x_i(0) = y_i, \quad \xi(0) = \xi_i(0) = h(y_i). \quad (3.5.11)$$

If $x(\cdot)$ and $\xi(\cdot)$ satisfy the differential inequalities

$$\left| \dot{x}(t) - f(t, x(t), \xi(t), u(t, x(t))) \right| \leq \varepsilon,$$
$$\left| \dot{\xi}(t) - g(t, x(t), \xi(t), u(t, x(t)), v(t, x(t))) \right| \leq \varepsilon, \quad (3.5.12)$$

then

$$\frac{d}{dt} |x(t) - x_i(t)| \leq M_2(1 + M_3)|x(t) - x_i(t)| + M_2|\xi(t) - \xi_i(t)| + \varepsilon,$$
$$\frac{d}{dt} |\xi(t) - \xi_i(t)| \leq M_2(1 + 2M_3)|x(t) - x_i(t)| + M_2|\xi(t) - \xi_i(t)| + \varepsilon.$$ 

Calling $z(t) \doteq \max\{|x(t) - x_i(t)|, |\xi(t) - \xi_i(t)|\}$, we obtain

$$\dot{z}(t) \leq 2M_2z(t) + 4M_2M_3z(t) + \varepsilon.$$ 

Recalling (3.5.9)-(3.5.10), by a comparison argument we conclude that $z(t) \leq Z(t)$ and hence

$$(t, x(t), \xi(t)) \in \Gamma_i \quad \text{for all} \quad t \in [0, T].$$

5. We now introduce an inductive algorithm which constructs a feedback control $u^* = u^*(t, x)$ with the following properties:

(i) There exists a finite partition $0 = t_0 < t_1 < t_2 < \cdots < t_N = T$ such that $u^*$ is smooth restricted to each domain $]t_{k-1}, t_k[ \times \mathbb{R}.$
(ii) For each \( i = 1, \ldots, n \), the Cauchy problem

\[
\begin{cases}
\dot{x} = f(t, x, \xi, u^*(t, x)), \\
\dot{\xi} = g(t, x, \xi, u^*(t, x), u^*_x(t, x)),
\end{cases}
\]

\[
x(0) = y_i, \\
\xi(0) = h(y_i),
\]

(3.5.13)

has a unique solution \( t \mapsto (x^i(t), \xi^i(t)) \), satisfying \((t, x^i(t), \xi^i(t)) \in \Gamma_i \) for all \( t \in [0, T] \).

By the previous step, to achieve (ii) it suffices to achieve the inequalities

\[
\left| u^*(t, x^i(t)) - u(t, x^i(t)) \right| \leq \varepsilon, \\
\left| u^*_x(t, x^i(t)) - v(t, x^i(t)) \right| \leq \varepsilon.
\]

(3.5.14)

Figure 3.4. For \( t \in [t_0, t_1] \) the first partition of the set \{1, 2, 3\} is \( J_1,1 \cup J_1,2 \cup J_1,3 = \{1\} \cup \{2\} \cup \{3\} \). We define \( t_1 \) as the smallest time \( t > 0 \) such that \( |x^2_2(t) - x^3_3(t)| \leq 3\delta \). For \( t \in [t_1, t_2] \), the points \( x^2_2(t) \) and \( x^3_3(t) \) must be clustered together. The second partition is thus \( J_2,1 \cup J_2,2 = \{1\} \cup \{2, 3\} \). This works up to the first time \( t_2 \) at which \( |x^2_3 - x^1_1| \leq 3\delta \). For \( t \in [t_2, t_3] \) the third partition consists of the single set \( J_{3,1} = \{1, 2, 3\} \). Finally, \( t_3 \) is the first time \( t > t_2 \) where \( |x^1_1(t) - x^3_3(t)| \geq 5\delta \). Hence we need to put \( x_3 \) into a separate cluster. For \( t \in [t_3, t_4] \) the fourth partition is \( J_{4,1} \cup J_{4,2} = \{1, 2\} \cup \{3\} \).

6. The construction of the feedback control \( u^* \) satisfying (3.5.14) will be achieved by induction on the time intervals \( [t_{k-1}, t_k] \), relying on a dynamic clustering algorithm (see Fig. 3.4). For each \( k = 1, \ldots, N \) we shall define a partition \( J_{k,1} \cup \cdots \cup J_{k,\nu(k)} \) of the set \{1, \ldots, n\}. Roughly speaking, two indices \( i, j \) will be assigned to the same equivalence class if the points \( x^i_1(t) \) and \( x^j_2(t) \) are close to each other for
Consider a small threshold parameter $\delta > 0$, whose precise value will be determined later. At this stage we only assume that $4\delta < \min\{|y_i - y_j|; \ i \neq j\}$.

Set $t_0 = 0$ and define

$$t_1 \doteq \inf \left\{ t \in [0, T] ; \ |x_i(t) - x_j(t)| \leq 3\delta \text{ for some } i \neq j \right\}. \quad (3.5.15)$$

On the initial interval $[t_0, t_1] \times \mathbb{R}$ we let $u^*$ be any smooth feedback satisfying (3.5.4). Moreover, we let $\{1\} \cup \{2\} \cup \cdots \cup \{n\}$ be the corresponding partition.

7. If $t_1 = T$ we are done. Otherwise, assume that the feedback $u^*$ has been constructed up to some restarting time $t_{k-1}$. Looking at the points $x^*_1(t_{k-1}), \ldots, x^*_n(t_{k-1})$, a new partition is defined as follows.

We say that two indices $i, j$ lie in the same equivalence class if and only if there exists a chain of points $x^*_{\ell(0)}(t_{k-1}), x^*_{\ell(1)}(t_{k-1}), \ldots, x^*_{\ell(m)}(t_{k-1})$, with $\ell(0) = i$, $\ell(m) = j$, and

$$|x^*_{\ell(p)}(t_{k-1}) - x^*_{\ell(p-1)}(t_{k-1})| \leq 4\delta \quad \text{for all } p = 1, \ldots, m.$$

The equivalence classes of the above relation yield the desired partition

$$J_{k,1} \cup \cdots \cup J_{k,\nu(k)} = \{1, \ldots, n\}.$$

On the interval $[t_{k-1}, t_k]$, the feedback $u^*$ is defined as follows. For each $\ell = 1, \ldots, \nu(k)$, let $[a_{\ell}(t), b_{\ell}(t)]$ be the smallest interval containing all points $x^*_i(t)$ with $i \in J_{k,\ell}$. In other words,

$$a_{\ell}(t) \doteq \min_{i \in J_{k,\ell}} x^*_i(t), \quad b_{\ell}(t) \doteq \max_{i \in J_{k,\ell}} x^*_i(t) \quad t \in [t_{k-1}, t_k].$$
We then set \( j(\ell) = \min \{ i ; \; i \in J_{k,\ell} \} \) and define the trajectory \( t \mapsto (x^*_{j(\ell)}(t), \xi^*_{j(\ell)}(t)) \) as the solution of the Cauchy problem

\[
\begin{align*}
\dot{x} &= f(t, x, \xi, u(t, x)), \\
\dot{\xi} &= g(t, x, \xi, u(t, x), v(t, x)),
\end{align*}
\]

(3.5.16)

Finally, we define

\[
u^*(t, x) = u(t, x^*_{j(\ell)}(t)) + (x - x^*_{j(\ell)}(t)) \cdot v(t, x^*_{j(\ell)}(t)) \quad \text{if} \quad x \in [a_{\ell}(t) - \delta, b_{\ell}(t) + \delta].
\]

(3.5.17)

The feedback \( u^* \) is then extended to a smooth function on a domain of the form \([t_{k-1}, t_k] \times \mathbb{R}\). Here \( t_k \) is defined as the first time \( t > t_{k-1} \) at which one of the following occurs:

(i) Two points belonging to distinct chains get within a distance \( \leq 3\delta \) from each other. That means: \( a_{\ell'}(t) - b_{\ell}(t) \leq 3\delta \) for some \( \ell' \neq \ell \).

(ii) A gap of size \( \geq 5\delta \) appears within one of the chains. That means: there exists some index \( \ell \) and a point \( x \in [a_{\ell}(t), b_{\ell}(t)] \) such that the open interval \( ]x, x + 5\delta[ \) does not contain any of the points \( x^*_i(t), i = 1, ..., n \).

If the above cases never happen, we set \( t_k = T \) and the induction procedure terminates.

Calling \( M \) an upper bound on all speeds \( |\dot{x}^*_i(t)| \), it is clear that the length of each time interval \([t_{k-1}, t_k] \) satisfies

\[
t_k - t_{k-1} \geq \delta/2M.
\]

(3.5.18)

Indeed, if the minimum distance between distinct chains at time \( t_{k-1} \) is \( \geq 4\delta \), it takes at least a time \( \delta/2M \) for this distance to become \( \leq 3\delta \). Similarly, if every two consecutive points in a chain are at a distance \( \leq 4\delta \), it takes at least a time \( \delta/2M \) to open up a gap of size \( 5\delta \).

8. In this step we estimate the differences \( u^* - u \) and \( u^*_x - v \). Assume \( x \in [a_{\ell}(t), b_{\ell}(t)] \). By construction this implies \( |x - x^*_{j(\ell)}(t)| \leq 5n\delta \). Therefore, recalling
(3.5.6) and (3.5.8) we obtain

\[ |u^*(t, x) - u(t, x)| \leq |u^*(t, x) - u^*(t, x_j(t))| + |u(t, x_j(t)) - u(t, x)| \]
\[ \leq 5n\delta \left( \|u_x\|_{L^\infty} + \|v\|_{L^\infty} \right) \leq 5n\delta (M_3 + M_1). \]  

(3.5.19)

Similarly, from (3.5.8) it follows

\[ |u^*_x(t, x) - v(t, x)| = |v(t, x_j(t)) - v(t, x)| \leq M_3 \cdot 5n\delta. \]  

(3.5.20)

We remark that the feedback \( u^* \) constructed in the previous steps is smooth w.r.t. the variable \( x \) but possibly discontinuous at the times \( t_1 < \ldots < t_{N-1} \). However, it is clear that we can slightly modify \( u^* \) in a neighborhood of the times \( t_k \) (by a suitable mollification) and obtain a new feedback \( \tilde{u}^* \) which is smooth w.r.t. both variables \( t, x \) and yields similar estimates.

9. For any given \( \varepsilon > 0 \), choosing \( \delta > 0 \) sufficiently small the previous construction yields a feedback control \( u^* \) with the properties stated in step 5. More precisely:

\[ \max\{ |x^*_i(t) - x_i(t)|, |\xi^*_i(t) - \xi_i(t)| \} \leq z(t) \leq Z(T) = \frac{e^{2M_2 + 4M_2M_3}T - 1}{2M_2 + 4M_2M_3} \varepsilon, \]  

(3.5.21)

\[ |u^*(t, x^*_i(t)) - u(t, x^*_i(t))| \leq \varepsilon, \quad |u^*_x(t, x^*_i(t)) - v(t, x^*_i(t))| \leq \varepsilon. \]  

(3.5.22)

We now observe that

\[ \left| J(u^*, \mu) - J(u, v, \mu) \right| \]
\[ \leq \max_{1 \leq i \leq n} \int_0^T \left| L(t, x^*_i(t), \xi^*_i(t), u^*(t, x^*_i(t))) - L(t, x_i(t), \xi_i(t), u(t, x_i(t))) \right| dt. \]  

(3.5.23)

Since the cost function \( L \) is continuous w.r.t. all variables, and the feedbacks \( u, u^* \) are uniformly Lipschitz continuous, by choosing \( \varepsilon > 0 \) sufficiently small the bounds (3.5.21)-(3.5.22) imply that the integrand in (3.5.23) can be rendered arbitrarily small. This establishes the inequality \( J(\mu) \leq J^{\text{relax}}(\mu) \). The converse inequality is trivial. \( \square \)
Corollary 1. For the problem (3.1.6)–(3.1.8), let the assumptions (A1)–(A3) hold. Then $J^{\text{relax}}(\mu) = J^{w}(\mu)$.

Indeed, in the topology of weak convergence, every probability measure can be approximated by a measure consisting of finitely many point masses.

3.6 An equivalence result, for bounded costs

The next result describes a simple case where the minimum cost $J(\mu)$ is lower semicontinuous w.r.t. strong convergence $\tilde{\phi} \to \phi$ in $L^1$ of the densities of the probability measures.

Theorem 3. In addition to (A1)–(A3), assume that the cost function $L = L(t,x,\xi,u)$ is globally bounded. Then $J(\mu) = J^s(\mu)$.

Proof. Assume that $L$ satisfies the uniform bound $|L(t,x,\xi,u)| \leq C$. Let $\phi$ be the density of the probability measure $\mu$. By definition, for every $n \geq 1$, we can find a smooth feedback control $u_n(t,x)$ and an absolutely continuous measure $\mu_n$ with density $\phi_n$ such that

$$J(u_n,\mu_n) < J^s(\mu) + \frac{1}{n}, \quad ||\phi_n - \phi|_{L^1} \leq \frac{1}{n}. \quad (3.6.24)$$

In the following, $t \mapsto (x_n(t,y), \xi_n(t,y))$ denotes the solution to

$$\begin{cases}
\dot{x} = f(t,x,\xi,u_n(t,x)), \\
\dot{\xi} = g(t,x,\xi,u_n(t,x), u_{n,x}(t,x)),
\end{cases} \quad (3.6.25)$$

with initial data

$$x(0) = y, \quad \xi(0) = h(y). \quad (3.6.26)$$
The upper bound on $L$ together with the second inequality in (3.6.24) imply

$$\left| J(u_n, \mu) - J(u_n, \mu_n) \right|$$

$$= \left| E^\mu \left[ \int_0^T L(t, x_n(t), \xi_n(t), u_n(t, x_n(t))) dt \right] \right.$$

$$- E^{\mu_n} \left[ \int_0^T L(t, x_n(t), \xi_n(t), u_n(t, x_n(t))) dt \right] \right|$$

$$= \left| \int \int_0^T L \left( t, x_n(t, y), \xi_n(t, y), u_n(t, x_n(t, y)) \right) \left( \phi(y) - \phi_n(y) \right) dt dy \right|$$

$$\leq 2C \int_0^T \int |\phi(y) - \phi_n(y)| dy dt \leq \frac{2CT}{n}. \quad (3.6.27)$$

By (3.6.24) and the first inequality in (3.6.24) it follows

$$J(\mu) \leq J(u_n, \mu) \leq J(u_n, \mu_n) + \frac{2CT}{n} < J^s(\mu) + \frac{1}{n} + \frac{2CT}{n}.$$ 

Letting $n \to \infty$, we conclude that $J(\mu) \leq J^s(\mu)$. The converse inequality is trivial.

\[\square\]

### 3.7 Nearly optimal feedback strategies

In this section we study conditions for which the inequality

$$J^s(\mu) \leq J^{relax}(\mu) \quad (3.7.1)$$

holds. Toward a proof of (3.7.1), the main idea is as follows (see Fig. 3.5). Let $(u^*, v^*)$ be nearly optimal relaxed control pair, so that (3.5.1) holds. By inserting some gaps in the support of the probability measure $\mu$, we first construct a feedback control $u^\flat : [0, \delta] \times \mathbb{R} \mapsto \mathbb{R}$ that steers all the mass into small neighborhoods $B(y_i, \rho_i)$ of finitely many points $y_1, \ldots, y_N$. Starting with a probability measure supported on the finite set $\{y_1, \ldots, y_N\}$, Theorem 2 then provides the existence of
a feedback \( u^\sharp : [\delta, T] \times \mathbb{R} \mapsto \mathbb{R} \) which, on the subinterval \([\delta, T]\) achieves almost the same cost as \((u^*, v^*)\). If the radii \( \rho_i > 0 \) are sufficiently small, and the cost during the short initial interval \([0, \delta]\) can be rendered arbitrarily small, then the feedback control

\[
\tilde{u}(t, x) = \begin{cases} 
 u^\flat(t, x) & \text{if } t \in [0, \delta], \\
 u^\sharp(t, x) & \text{if } t \in [\delta, T],
\end{cases}
\]

achieves a cost very close to \( J(u^*, v^*, \mu) \).

As before, given a probability measure \( \mu \) on the initial data \( x(0) \), we wish to minimize the functional \( J(u, \mu) \) at (3.1.8), for the system described at (3.1.9)-(3.1.10). We always assume that the conditions (A1)–(A3) stated in the Introduction are satisfied. Together with the relaxed system (3.1.15), we also consider the aug-
mented system

\[
\begin{align*}
\dot{x} &= f(t,x,\xi,u), & x(0,y) &= y, \\
\dot{\xi} &= g(t,x,\xi,u,v), & \xi(0,y) &= h(y), \\
\dot{\eta} &= (f_x + fu)v\eta + f_xz, & \eta(0,y) &= 1, \\
\dot{z} &= (g_x + guv +gvw)\eta + gz, & z(0,y) &= h'(y).
\end{align*}
\] (3.7.3)

This is obtained from (3.1.15) adding two evolution equations for the additional variables \( \eta = x_y \) and \( z = \xi_y \). Here \( v = u_x \) and \( w = v_x = u_{xx} \) are regarded as additional, independent control functions. As it will become clear by subsequent analysis, to implement the technique outlined in Fig.3.5, one needs to find a control functions \( v, w \) such that, at time \( t = \delta \), the solution of (3.7.3) satisfies

\[
x(\delta) \approx x(0), \quad \xi(\delta) \approx \xi(0), \quad \eta(\delta) \approx 0, \quad z(\delta) \approx 0.
\]

As a basic assumption, we shall thus need a controllability property for the augmented system (3.7.3).

In the following analysis, for a given initial point \( y_0 \) we consider a feedback control of the form

\[
u(t,x) = u_0(t) + (x - y_0)v_0(t) + \frac{(x - y_0)^2}{2}w_0(t) \quad \text{for} \quad t \in [0,\delta], \quad (3.7.4)
\]

where \( u_0, v_0, w_0 \) are measurable functions of time. For \( y \) in a neighborhood of \( y_0 \),
the augmented system (3.7.3) becomes

\[
\begin{align*}
\dot{x} &= f(t, x, \xi, u_0), \\
\dot{\xi} &= g(t, x, \xi, u_0, v_0 + (x - y_0)w_0), \\
\dot{\eta} &= (f_x + f_u(v_0 + (x - y_0)w_0))\eta + f\xi z, \\
\dot{z} &= (g_x + g_u(v_0 + (x - y_0)w_0) + g_v w_0)\eta + g\xi z, \\
\end{align*}
\]

(3.7.5)

Our main assumption can now be formulated in terms of the controllability of the system (3.7.5).

\textbf{(H)} Given any \( \varepsilon_0 > 0 \) there exists \( \delta_0 \in ]0, \varepsilon_0] \) such that the following holds. For each \( y_0 \in \text{Supp}(\mu) \) there exist open-loop controls \( u_0, v_0, w_0 \in L^\infty([0, \delta_0]) \) and \( \tau_0 \in ]0, \delta_0] \) such that the solution \( t \to (x(t, y_0), \xi(t, y_0), \eta(t, y_0), z(t, y_0)) \) of (3.7.5) with initial data

\[
x(0) = y_0, \quad \xi(0) = h(y_0), \quad \eta(0) = 1, \quad z(0) = h'(y_0) \quad (3.7.6)
\]

satisfies the following properties:

\[
\begin{align*}
x(t, y_0) &= y_0, \quad |\xi(t, y_0)| < M^*, \\
|u_0(t)| &< M^*, \quad |v_0(t)| < M^*, \quad \text{for all } t \in [0, \delta_0] \quad (3.7.7)
\end{align*}
\]

and

\[
\eta(t, y_0) < -1 \quad \text{for } t \in [0, \tau_0], \quad \eta(t, y_0) < 1 \quad \text{for } t \in [\tau_0, \delta_0], \quad (3.7.8)
\]

where \( M^* \) is a constant independent of \( y_0 \) and \( \delta_0 \). Moreover, at time \( t = \delta_0 \) one has

\[
\xi(\delta_0, y_0) = h(y_0), \quad 0 < \eta(\delta_0, y_0) < \varepsilon_0, \quad |z(\delta_0, y_0)| < \varepsilon_0. \quad (3.7.9)
\]
Notice that, if (3.7.7)-(3.7.9) hold, then by continuity there exists a \( \rho > 0 \) such that, for every initial point \( y \in B(y_0, \rho) \), the solution of (3.7.5) with the same initial data as in (3.7.3) satisfies

\[
0 \leq \eta(\delta_0, y) \leq \varepsilon_0, \quad |z(\delta_0, y)| \leq \varepsilon_0, \quad |(x(t, y) - y_0) w_0(t)| \leq \varepsilon_0, \quad \eta(t, y) \leq 1,
\]

\[
|x(t, y)| \leq R + 1, \quad |\xi(t, y)| \leq M^*, \quad |u(t, y)| \leq M^*, \quad |v(t, y)| \leq M^*,
\]

for all \( t \in [0, \delta_0] \).

**Theorem 4.** Let the conditions (A1)-(A3) hold, and let the probability measure \( \mu \) on the initial data be absolutely continuous w.r.t. Lebesgue measure. If the assumption (H) holds, then \( J^*(\mu) \leq J^{relax}(\mu) \).

**Proof.** Let \( \varepsilon > 0 \) be given, and let \( u^* = u^*(t, x) \in C^2 \) and \( v^* = v^*(t, x) \in C^2 \) be nearly optimal feedback controls for the relaxed system (3.1.15) such that

\[
J(u^*, v^*, \mu) \leq J^{relax}(\mu) + \varepsilon.
\]

As before, let \( x^*(t, y), \xi^*(t, y) \) be the corresponding solution. By (A3), we can assume that the probability measure \( \mu \), with density \( \phi(\cdot) \) w.r.t. Lebesgue measure, is supported inside a bounded interval \([-R, R]\). Choose constants \( M_i \) such that

\[
\|x^*\|_{C^2([0,T] \times [-R,R])} \leq M_0, \quad \|\xi^*\|_{C^2([0,T] \times [-R,R])} \leq M_0 \quad (3.7.12)
\]

\[
\|u^*\|_{C^2([0,T] \times R)} \leq M_1, \quad \|v^*\|_{C^2([0,T] \times R)} \leq M_1 \quad (3.7.13)
\]

\[
\|h(y)\|_{C^2([-R,R])} \leq M_2,
\]

\[
|f(t, x, \xi, u) - f(t, \tilde{x}, \tilde{\xi}, \tilde{u})| \leq M_3(|x - \tilde{x}| + |\xi - \tilde{\xi}| + |u - \tilde{u}|),
\]

\[
|g(t, x, \xi, u, v) - g(t, \tilde{x}, \tilde{\xi}, \tilde{u}, \tilde{v})| \leq M_3(|x - \tilde{x}| + |\xi - \tilde{\xi}| + |u - \tilde{u}| + |v - \tilde{v}|).
\]

To prove Theorem 4, we need to show that there exist a measure \( \tilde{\mu} \) with density \( \tilde{\phi} \)
satisfying \( \| \tilde{\phi} - \phi \|_{L^1} \leq \varepsilon \) and a feedback control \( \tilde{u} \in C^2 \) such that

\[
J(\tilde{u}, \tilde{\mu}) \leq J(u^*, v^*, \mu) + \varepsilon. \tag{3.7.15}
\]

1. Let \( \varepsilon_0 > 0 \) be given, and let \( \delta_0 \in [0, \varepsilon_0] \) be as in the statement of the assumption (H). Then for each \( y_0 \in Supp(\mu) \subseteq [-R, R] \) by assumption (H) we can find open-loop controls \( u_0, v_0, w_0 \in L^\infty([0, \delta]) \) and a radius \( \rho(y_0) \in [0, \varepsilon_0] \) such that (3.7.7)–(3.7.10) hold. By compactness, we can cover \( Supp(\mu) \) with finitely many open balls \( B(y_i, \rho_i) \), where

\[
-R \leq y_1 < y_2 < \cdots < y_n \leq R, \quad \rho_i = \rho(y_i).
\]

We now choose disjoint intervals \([a_i, b_i] \subseteq [-R, R] \cap B(y_i, \rho_i)\), with \( y_i \in [a_i, b_i] \) for each \( i = 1, \ldots, n \), and such that

\[
\mu \left( \bigcup_{i=1}^{n} [a_i, b_i] \right) > 1 - \varepsilon_0. \tag{3.7.16}
\]

Let \( \tilde{\mu} \) be the probability distribution having density

\[
\tilde{\phi}(x) = \begin{cases} 
\left( \mu \left( \bigcup_{i=1}^{n} [a_i, b_i] \right) \right)^{-1} \phi(x) & \text{if } x \in \bigcup_{i=1}^{n} [a_i, b_i], \\
0 & \text{otherwise}.
\end{cases} \tag{3.7.17}
\]

By (3.7.16), the above definition yields

\[
\| \tilde{\phi} - \phi \|_{L^1} = \mu \left( R \setminus \bigcup_{i=1}^{n} [a_i, b_i] \right) + \left( \mu \left( \bigcup_{i=1}^{n} [a_i, b_i] \right) \right)^{-1} \int_{\bigcup_{i=1}^{n} [a_i, b_i]} \phi(x) \, dx \leq 2\varepsilon_0. \tag{3.7.18}
\]

This can be rendered arbitrarily small by a suitable choice of \( \varepsilon_0 > 0 \).

Together with \( \tilde{\mu} \), consider the probability distribution \( \mu^\dagger \) supported on the finite set \( \{y_i; \ 1 \leq i \leq n\} \). This is defined by

\[
\mu^\dagger \left( \{y_i\} \right) = \tilde{\mu}([a_i, b_i]) = \left( \mu \left( \bigcup_{i=1}^{n} [a_i, b_i] \right) \right)^{-1} \mu([a_i, b_i]). \tag{3.7.19}
\]
2. For each \( i \), let \( t \mapsto u_i(t), v_i(t), w_i(t) \) be open loop controls for which the conditions (3.7.7)-(3.7.10) hold, with \( y_0 \) replaced by \( y_i \). For \( t \in [0, \delta_0] \) and \( x \in [a_i, b_i] \), we then define the feedback control \( u^\flat \) as

\[
u^\flat(t, x) = u_i(t) + (x - y_i) \cdot v_i(t) + \frac{(x - y_i)^2}{2} \cdot w_i(t) \tag{3.7.20}
\]

The control \( u^\flat \) is then extended in a smooth way (w.r.t. the variable \( x \)) outside the union of the intervals \([a_i, b_i]\). As stated in assumption (H), for all \( t \in [0, \delta_0] \) we now have

\[
x^\flat(t, y_i) = y_i, \quad |\xi^\flat(t, y_i)| \leq M^*, \quad |u_i(t)|, |v_i(t)| \leq M^*,
\]

\[
0 < x^\flat_y(t, y) \leq 1, \quad |(x^\flat(t, y) - y_i) w_i(t)| \leq \varepsilon_0, \quad \text{for all } y \in [a_i, b_i], \tag{3.7.21}
\]

where \( M^* \) is a constant independent of \( y_i \). Moreover, at time \( t = \delta_0 \) one has

\[
\xi^\flat(\delta_0, y_i) = h(y_i), \quad 0 < x^\flat_y(\delta_0, y) < \varepsilon_0, \quad |\xi^\flat_y(\delta_0, y)| < \varepsilon_0, \quad \text{for all } y \in [a_i, b_i]. \tag{3.7.22}
\]

From the relations \( x^\flat(t, y_i) = y_i \) and \( 0 < x^\flat_y(t, y) \leq 1 \), it follows

\[
\Gamma_i \doteq \left\{(t, x^\flat(t, y)); \quad t \in [0, \delta_0], \quad y \in [a_i, b_i]\right\} \subseteq [0, \delta_0] \times [a_i, b_i]. \tag{3.7.23}
\]

In particular, the trajectories \( t \mapsto x(t, y) \) starting at a point \( y \in [a_i, b_i] \) do not depend on how the feedback \( u^\flat \) is extended outside the interval \([a_i, b_i]\).

For future use we also observe that, if \( y \in [a_i, b_i] \), then at time \( t = \delta_0 \) the above conditions yield

\[
|x^\flat(\delta_0, y) - y_i| \leq \varepsilon_0(b_i - a_i), \quad |\xi^\flat(\delta_0, y) - h(y_i)| \leq \varepsilon_0(b_i - a_i). \tag{3.7.24}
\]

3. Motivated by the proof of Theorem 2, we shall construct a feedback \( u^\sharp : [\delta_0, T] \times IR \mapsto IR \) which is nearly optimal for a probability distribution supported inside small neighborhoods of the finitely many points \( y_1, \ldots, y_n \). For \( t \in [\delta_0, T] \), we shall denote by \( t \mapsto (x^\sharp(t, y, \zeta), \xi^\sharp(t; y, \zeta)) \) the solution of (3.1.6) corresponding
to the feedback $u^*$, with initial data
\[ x(\delta_0) = y, \quad \xi(\delta_0) = \zeta. \]

Moreover, we write
\[ (x^\sharp(t), \xi^\sharp(t)) = \left( x^\sharp_i(t; y_i, h(y_i)), \xi^\sharp_i(t; y_i, h(y_i)) \right). \]

We shall also use the constant
\[ M^\sharp = \exp \left\{ M_3(2 + M_1)T \right\}. \quad (3.7.25) \]

The time interval $[\delta_0, T]$ will be divided into finitely many subintervals, inserting the times $\delta_0 = t_0 < t_1 < \cdots < t_N = T$ by an inductive procedure. Assume that a feedback $u^*$ has been constructed for $t \in [\delta_0, t_{k-1}]$. At time $t_k - 1$, we define a partition of the set $\{1, \ldots, n\}$ as follows. Two indices $i, j$ are in the same equivalence class if and only if there exists a chain of points $x^\sharp_\ell(0)(t_{k-1}), x^\sharp_\ell(1)(t_{k-1}), \ldots, x^\sharp_\ell(m)(t_{k-1})$, with $\ell(0) = i$, $\ell(m) = j$, and
\[ |x^\sharp_\ell(p)(t_{k-1}) - x^\sharp_\ell(p-1)(t_{k-1})| \leq 4\varepsilon_0 M^\sharp \cdot \left( b_\ell(p) - a_\ell(p) + b_\ell(p-1) - a_\ell(p-1) \right) \]
for all $p = 1, \ldots, m$. The equivalence classes of the above relation yield the desired partition
\[ J_{k,1} \cup \cdots \cup J_{k,\nu(k)} = \{1, \ldots, n\}. \]

On the interval $[t_{k-1}, t_k]$, the feedback $u^*$ is defined as follows. For each $\ell = 1, \ldots, \nu(k)$, define the interval $[a_\ell(t), b_\ell(t)]$ by setting
\[ a_\ell(t) = \min_{i \in J_{k,\ell}} \left\{ x^\sharp_i(t) - \varepsilon_0 M^\sharp (b_i - a_i) \right\}, \quad b_\ell(t) = \max_{i \in J_{k,\ell}} \left\{ x^\sharp_i(t) + \varepsilon_0 M^\sharp (b_i - a_i) \right\}. \]

We then set $j(\ell) = \min\{i; \ i \in J_{k,\ell}\}$ and let $t \mapsto (x^\sharp_{j(\ell)}(t), \xi^\sharp_{j(\ell)}(t))$ be the solution
of the Cauchy problem
\[
\begin{align*}
\dot{x} &= f(t, x, \xi, u^*(t, x)), \\
\dot{\xi} &= g(t, x, \xi, u^*(t, x), v^*(t, x)), \\
x(t_{k-1}) &= x_{j_{k-1}}^x(t_{k-1}), \\
\xi(t_{k-1}) &= \xi_{j_{k-1}}^\xi(t_{k-1}).
\end{align*}
\]
(3.7.26)

Finally, we define
\[
u^*(t, x, \xi) = u^*(t, x, \xi) + \left(x - x_{j_{k-1}}^x(t)ight) \cdot \left(v^*(t, x_{j_{k-1}}^x(t)) - x_{j_{k-1}}^x(t)\right)
\]
if \(x \in [a_\ell(t), b_\ell(t)].\)
(3.7.27)

The feedback \(u^*\) is then extended to a smooth function on a domain of the form \([t_{k-1}, t_k] \times \mathbb{R}\).

The subsequent time \(t_k\) is defined as the first time \(t > t_{k-1}\) at which one of the following occurs:

(i) Two distinct chains become close to each other. Namely, there exist two indices \(i, j\) belonging to distinct equivalence classes at time \(t_{k-1}\), such that
\[
\left| x_{\ell_i}^x(t) - x_{\ell_j}^x(t) \right| \leq 3\varepsilon_0 M^2 \cdot (b_i - a_i + b_j - a_j).
\]

(ii) In one of the chains, a wide gap appears. Namely, there exist two indices \(i, j\) belonging to the same equivalence class at time \(t_{k-1}\), but such that at time \(t\) there exist no chain of points \(x_{\ell_0}^x(t), x_{\ell_1}^x(t), \ldots, x_{\ell_m}^x(t)\) with \(\ell(0) = i, \ell(m) = j\), and
\[
\left| x_{\ell_{p}}^x(t) - x_{\ell_{p-1}}^x(t) \right| < 5\varepsilon_0 M^2 \cdot \left(b_{\ell(p)} - a_{\ell(p)} + b_{\ell(p-1)} - a_{\ell(p-1)}\right)
\]
for all \(p = 1, \ldots, m\).

If none of the above cases ever happens, we set \(t_k = T\) and the induction procedure terminates. As in the proof of Theorem 2, it is clear that the length of the time intervals \([t_{k-1}, t_k]\) is uniformly positive. Hence the induction procedure must terminate after finitely many steps.

4. In this step we estimate the dependence of solutions on their initial data at \(t = \delta_0\). The definition (3.7.27) together with the bounds (3.7.13)-(3.7.14) imply that, for \(x, x' \in [a_\ell(t), b_\ell(t)]\) and \(\xi, \xi' \in \mathbb{R}\), the functions \(f, g\) are Lipschitz continuous
w.r.t. $x, \xi$. Namely

$$\left| f(t, x, \xi, u^\sharp(t, x)) - f(t, x', \xi', u^\sharp(t, x')) \right| \leq M_3(|x - x'| + |\xi - \xi'| + M_1|x - x'|),$$

$$\left| g(t, x, \xi, u^\sharp(t, x), u^\sharp_x(t, x)) - g(t, x', \xi', u^\sharp(t, x'), u^\sharp_x(t, x')) \right|$$

$$\leq M_3(|x - x'| + |\xi - \xi'| + M_1|x - x'|).$$

Assume that, at time $t = \delta_0$, we choose initial data $y, \zeta$ such that

$$|y - y_i| \leq \varepsilon_0(b_i - a_i), \quad |\zeta - h(y_i)| \leq \varepsilon_0(b_i - a_i). \quad (3.7.28)$$

Then Gronwall’s lemma yields the bound

$$\max \left\{ |x(t; y, \zeta) - x_i(t)|, |\xi(t; y, \zeta) - \xi_i(t)| \right\}$$

$$\leq e^{M_3(2 + M_1)(t - \delta_0)} \cdot \max \{|y - y_i|, |\zeta - h(y_i)|\} \leq M^2 \cdot \varepsilon_0(b_i - a_i). \quad (3.7.29)$$

By our previous construction this guarantees that, for every $t \in [\delta_0, T]$, one has

$$x^\sharp(t; y, \zeta) \in \bigcup_\ell [a_\ell(t), b_\ell(t)].$$

Therefore, this solution does not depend on the way in which the feedback $u^\sharp$ is extended outside the intervals $[a_\ell(t), b_\ell(t)]$. In this connection it is also important to observe that, by (3.7.24), the feedback $u^\flat$ constructed on the initial time interval $t \in [0, \delta_0]$ steers every initial data

$$x(0) = \bar{y} \in [a_i, b_i], \quad \xi(0) = h(\bar{y}), \quad (3.7.30)$$

to a point $(y, \zeta) = (x^\flat(\delta_0), \xi^\flat(\delta_0))$ which satisfies the bounds (3.7.28).

5. We now define the feedback control $\tilde{u}$ by putting together the controls $u^\flat$ and $u^\sharp$, according to (3.7.2). We claim that, by choosing $0 < \delta_0 \leq \varepsilon_0$ sufficiently small, the expected total cost can be made arbitrarily close to $J(u^*, v^*, \mu)$. For convenience we shall use the notation $J = J_{[0, \delta_0]} + J_{[\delta_0, T]}$, where the first term accounts for the
cost on the initial time interval $[0, \delta_0]$, while the second term accounts for the cost on $[\delta_0, T]$. With this notation, since $L \geq 0$, one has

$$J(\tilde{u}, \tilde{\mu}) - J(u^*, v^*, \mu) \leq J_{[0, \delta_0]}(\tilde{u}, \tilde{\mu}) + \left( J_{[\delta_0, T]}(\tilde{u}, \tilde{\mu}) - J_{[\delta_0, T]}(u^*, \mu) \right) + \left( J_{[\delta_0, T]}(u^*, v^*, \mu) - J_{[\delta_0, T]}(u^*, v^*, \mu) \right).$$

(3.7.31)

The next steps will provide bounds on the four terms on the right hand side of (3.7.31), showing that they all approach zero as $\varepsilon_0 \to 0$ (and hence $\delta_0 \to 0$ as well). This will achieve the proof.

6. Recalling (3.7.10), the cost determined by the feedback control $\tilde{u}$ on the initial interval $[0, \delta_0]$ can be estimated by

$$J_{[0, \delta_0]}(\tilde{u}, \tilde{\mu}) = \int_0^{\delta_0} \int_{-R}^R L \left( t, x(t, y), \xi(t, y), u^b(t, x(t, y)) \right) \tilde{\phi}(y) dy dt$$

$$\leq \delta_0 \cdot \sup \left\{ L(t, x, \xi, u) : t \in [0, \delta_0], \ |x| \leq R + 1, \ |\xi| \leq M^*, \ |u| \leq M^* \right\}.$$  

(3.7.32)

Clearly, the right hand side of (3.7.32) goes to zero as $\delta_0 \to 0$.

7. To estimate the next term, observe that our construction yields

$$b_\ell(t) - a_\ell(t) \leq 7\varepsilon_0 M^2 \sum_i 2(b_i - a_i) \leq 7\varepsilon_0 M^2 \cdot 4R.$$  

(3.7.33)

This can be made arbitrarily small by a suitable choice of $\varepsilon_0 > 0$. Denote by $t \mapsto (\tilde{x}(t, y), \tilde{\xi}(t, y))$ the solution to the Cauchy problem

$$\begin{cases}
\dot{x}(t) = f(t, x(t), \xi(t), \tilde{u}(t, x(t))), & x(0) = y, \\
\dot{\xi}(t) = g(t, x(t), \xi(t), \tilde{u}(t, x(t)), \tilde{u}_x(t, x(t))), & \xi(0) = h(y).
\end{cases}$$  

(3.7.34)

Setting $d_i = \varepsilon_0(b_i - a_i)$ and recalling the notation introduced in step 3, we now
have

\[
J_{[\delta_0, T]}(\tilde{u}, \tilde{\mu}) - J_{[\delta_0, T]}(\tilde{u}, \mu^\dagger)
\]

\[
= \int_{\delta_0}^{T} \int_{-R}^{R} L\left(t, \bar{x}(t, y), \bar{\xi}(t, y), \tilde{\bar{u}}(t, \bar{x}(t, y))\right) \bar{\phi}(y) \, dy \, dt
\]

\[
- \sum_{i=1}^{n} \mu^\dagger(\{y_i\}) \int_{\delta_0}^{T} L\left(t, \bar{x}(t, y_i), \bar{\xi}(t, y_i), \tilde{\bar{u}}(t, \bar{x}(t, y_i))\right) \, dt
\]

\[
\leq \max_i \sup_{y \in B(y_i, \rho_i)} \sup_{\zeta \in B(h(y_i), \xi)} \int_{\delta_0}^{T} \left| L\left(t, x^\sharp(t; y, \zeta), \xi^\sharp(t; y, \zeta), u^\sharp(t, x^\sharp(t; y, \zeta))\right) - L\left(t, x^\sharp_i(t), \xi^\sharp_i(t), u^\sharp(t, x^\sharp_i(t))\right) \right| \, dt.
\]

The last inequality holds because \( \bar{x}(\delta_0, y_i) = y_i, \bar{\xi}(\delta_0, y_i) = h(y_i) \), and moreover

\[
|\bar{x}(\delta_0, y) - y| \leq \varepsilon_0(b_i - a_i), \quad |\bar{\xi}(\delta_0, y) - h(y_i)| \leq \varepsilon_0(b_i - a_i)
\]

for every \( y \in B(y_i, \rho_i) \). Notice that, for \( t \in [\delta_0, T] \),

\[
\begin{align*}
\bar{x}(t, y_i) &= x^\sharp_i(t), \\
\bar{\xi}(t, y_i) &= \xi^\sharp_i(t),
\end{align*}
\]

where \( \zeta = \bar{\xi}(\delta_0, y) \). Thanks to (3.7.28)-(3.7.29) and the continuity of the cost function \( L(\cdot) \), the right hand side of (3.7.35) can be made arbitrarily small by a suitable choice of \( \varepsilon_0 > 0 \).
8. The third term on the right hand side of (3.7.31) is estimated by

\[
J_{\delta_0, T}(\tilde{u}, \mu^\dagger) - J_{\delta_0, T}(u^*, v^*, \mu^\dagger)
= \sum_{i=1}^{n} \mu^\dagger\{y_i\} \int_{\delta_0}^{T} \left[ L\left(t, \tilde{x}(t, y_i), \tilde{\xi}(t, y_i), \tilde{u}(t, \tilde{x}(t, y_i))\right) - L\left(t, x^*_i(t), \xi^*_i(t), u^*(t, x^*_i(t))\right)\right] dt
\leq \max_i \int_{\delta_0}^{T} \left| L\left(t, x^*_i(t), \xi^*_i(t), u^*(t, x^*_i(t))\right) - L\left(t, x^*_i(t), \xi^*_i(t), u^*(t, x^*_i(t))\right)\right| dt.
\]

(3.7.36)

Here \(x^*_i(t) = x^*(t, y_i), \xi^*_i(t) = \xi^*(t, y_i)\). By the construction of \(u^*(t, x)\) in (3.7.27), the differences \(|x^*_i(t) - x^*_i(t)|, |\xi^*_i(t) - \xi^*_i(t)|,\) and \(|u^*(t, x^*_i(t)) - u^*(t, x^*_i(t))|\) all approach zero as \(\varepsilon_0 \to 0\), uniformly w.r.t. \(t\). By the continuity of \(L(\cdot, \cdot)\), the right hand side of (3.7.36) can be made arbitrarily small by a suitable choice of \(\varepsilon_0 > 0\). Indeed,

\[
|x^*_i(t) - x^*_i(t)| \leq |x^*_i(t) - x^*(t; y_i, h(y_i))| + |x^*(t; y_i, h(y_i)) - x^*_i(t)|, \quad (3.7.37)
\]

where \(t \mapsto (x^*(t; y, \zeta), \xi^*(t; y, \zeta))\) denotes the solution of (3.1.15) corresponding to controls \(u^*, v^*\), with initial data

\[
x(\delta_0) = y, \quad \xi(\delta_0) = \zeta.
\]

By the construction of \(u^*(t, x)\) and by Theorem 2, \(|x^*_i(t) - x^*(t; y_i, h(y_i))| \to 0\) as \(\varepsilon_0 \to 0\). The term \(|x^*(t; y_i, h(y_i)) - x^*_i(t)|\) also approaches zero as \(\varepsilon_0 \to 0\), because \(x^*(t, y)\) is smooth as assumed in (3.7.12). A similar argument can be applied to \(|\xi^*_i(t) - \xi^*_i(t)|\). Finally, the difference between the control values can be bounded by

\[
|u^*(t, x^*_i(t)) - u^*(t, x^*_i(t))| \leq |u^*(t, x^*_i(t)) - u^*(t, x^*_i(t))| + |u^*(t, x^*_i(t)) - u^*(t, x^*_i(t))|.
\]

(3.7.38)

By (3.7.12), (3.7.37) and the construction of \(u^\dagger\), both terms on the right hand side of (3.7.38) approach zero as \(\varepsilon_0 \to 0\).
9. The last term on the right hand side of (3.7.31) can be estimated by

\[ J_{[\delta_0,T]}(u^*, v^*, \mu) - J_{[\delta_0,T]}(u^*, v^*, \mu^1) \leq \sum_{i=1}^{n} \mu^1\{y_i\} \int_{\delta_0}^{T} L\left(t, x^*(t, y_i), \xi^*(t, y_i), u^*(t, x^*(t, y_i))\right) dt \]

\[ - \int_{\delta_0}^{T} \int_{a_i}^{b_i} L\left(t, x^*(t, y), \xi^*(t, y), u^*(t, x^*(t, y))\right) \tilde{\phi}(y) dy dt \]

\[ + \int_{\delta_0}^{T} \int_{-R}^{R} L\left(t, x^*(t, y), \xi^*(t, y), u^*(t, x^*(t, y))\right) \left| \tilde{\phi}(y) - \phi(y) \right| dy dt \]

\[ \leq \max_i \sup_{y \in B(y_i, \rho_i)} \int_{0}^{T} \left| L\left(t, x^*(t, y_i), \xi^*(t, y_i), u^*(t, x^*(t, y_i))\right) - L\left(t, x^*(t, y), \xi^*(t, y), u^*(t, x^*(t, y))\right) \right| dt \]

\[ + \left\| \tilde{\phi} - \phi \right\|_{L^1} \cdot \sup_{t \in [0,T], y \in [-R,R]} L\left(t, x^*(t, y), \xi^*(t, y), u^*(t, x^*(t, y))\right) . \]

(3.7.39)

By (3.7.18) and the assumption $0 < \rho_i < \varepsilon_0$, the regularity of the functions $f, g, L$ and of the feedback controls $u^*, v^*$ implies that the right hand side of (3.7.39) approaches zero as $\varepsilon_0 \to 0$.

10. According to the four previous steps, each of the terms on the right hand side of (3.7.31) can be rendered arbitrarily small by choosing $0 < \delta_0 < \varepsilon_0$ small enough. In particular, given any $\varepsilon > 0$ there exists a feedback control $\tilde{u}$ and a probability distribution $\tilde{\mu}$ with density $\tilde{\phi}$ such that

\[ \left\| \tilde{\phi} - \phi \right\|_{L^1} \leq \varepsilon, \quad J(\tilde{u}, \tilde{\mu}) \leq J(u^*, v^*, \mu) + \varepsilon. \]

(3.7.40)

Observe that our construction yields a control $\tilde{u}$ which is smooth w.r.t. $x$ but only measurable w.r.t. $t$. This regularity issue can be easily fixed by taking a suitable mollification w.r.t. both variables $t, x$. In this way we obtain a $C^\infty$ control function.
such that
\[ J(\hat{u}, \hat{\mu}) < J(\bar{u}, \bar{\mu}) + \varepsilon. \]
Together with (3.7.11) and (3.7.40), this proves the theorem.

3.8 On the controllability assumption

In the statement of Theorem 4, the controllability assumption (H) on the augmented system (3.7.3) played a key role. In this section we seek some easily verifiable conditions which guarantee that (H) holds. As before, we assume the initial probability distribution \( \mu \) has bounded support.

Theorem 5. Let \( f, g \in C^2 \), \( h \in C^1 \) and let the cost function \( L \) be continuous. Assume that, for each \( y_0 \in \text{Supp}(\mu) \) there exists values \( U_0, V_0 \) such that
\[
\begin{align*}
  f(0, y_0, h(y_0), U_0) &= 0, & g(0, y_0, h(y_0), U_0, V_0) &= 0, \\
  f_u(0, y_0, h(y_0), U_0) &\neq 0, & f_\xi(0, y_0, h(y_0), U_0) &\neq 0, \\
  g_v(0, y_0, h(y_0), U_0, V_0) &\neq 0.
\end{align*}
\]
(3.8.1)
(3.8.2)
(3.8.3)

Then the condition (H) holds.

Proof. 1. Let \( \varepsilon_0 > 0 \) be given. Fix any point \( y_0 \in \text{Supp}(\mu) \) and let \( U_0, V_0 \) be such that (3.8.1)–(3.8.3) hold. We claim that these values can be chosen so that they remain uniformly bounded, as \( y_0 \) ranges over \( \text{Supp}(\mu) \). Indeed, for every point \( y_0 \in \text{Supp}(\mu) \), by the implicit function theorem and by continuity we can find a radius \( r(y_0) \in [0, \varepsilon_0] \) and maps \( y \mapsto U(y), y \mapsto V(y) \) defined for \( |y - y_0| < r(y_0) \), such that
\[
\begin{align*}
  |U(y) - U_0| < 1, & \quad |V(y) - V_0| < 1, \\
  f(0, y, h(y), U(y)) = 0, & \quad g(0, y, h(y), U(y), V(y)) = 0, \\
  f_u(0, y, h(y), U(y)) \neq 0, & \quad f_\xi(0, y, h(y), U(y)) \neq 0, \\
  g_v(0, y, h(y), U(y), V(y)) \neq 0.
\end{align*}
\]
(3.8.4)
As \(y_0\) ranges over the compact set \(\text{Supp}(\mu)\), the open balls \(B(y_0, r(y_0))\) provide an open covering. By taking a finite subcovering, our claim is proved.

In addition, the above finite covering argument yields the existence of a constant \(\rho_1 > 0\) such that
\[
\begin{align*}
|f_u(0, y, h(y), U(y))| &> \rho_1 \\
|f_\xi(0, y, h(y), U(y))| &> \rho_1 \\
|g_v(0, y, h(y), U(y), V(y))| &> \rho_1
\end{align*}
\]
for all \(y \in \text{Supp}(\mu)\). \(3.8.5\)

2. In connection with the augmented control system (3.7.3) and initial data (3.7.6), we need to construct controls \(u_0, v_0, w_0 : [0, \delta_0] \mapsto \mathbb{R}\) for some \(\delta_0 \in [0, \varepsilon_0]\), such that the conditions (3.7.7)–(3.7.9) hold. The main idea of the proof is as follows. By (3.8.2)–(3.8.3) and the implicit function theorem, there exist controls \(u_0(t) \approx U_0\) and \(v_0(t) \approx V_0\) implicitly defined by
\[
f(t, y_0, h(y_0), u_0(t)) = 0, \quad g(t, y_0, h(y_0), u_0(t), v_0(t)) = 0. \quad (3.8.6)
\]
Inserting these feedback controls in (3.7.5) with initial data (3.7.6), the first two equations in (3.7.5)–(3.7.6) yield
\[
x(t) \equiv y_0, \quad \xi(t) \equiv h(y_0).
\]
Hence the last two equations reduce to
\[
\begin{align*}
\dot{\eta} &= \left(f_x + f_u v_0(t)\right)\eta + f_\xi z, \\
\dot{z} &= \left(g_x + g_u v_0(t) + g_v w_0(t)\right)\eta + g_\xi z, \quad \begin{cases}
\eta(0) = 1, \\
z(0) = h'(y_0).
\end{cases}
\end{align*}
\]
(3.8.7)

Here all coefficients are evaluated at the point
\[
(t, x, \xi, u, v) = (t, y_0, h(y_0), u_0(t), v_0(t)) \approx (t, y_0, h(y_0), U_0, V_0).
\]
We regard (3.8.7) as a linear dynamical system for \((\eta, z)\), with control \(w_0(\cdot)\) entering in a nonlinear way. Thanks to the controllability properties of (3.8.7), we can
choose \( w_0 \) so that at a given small time \( t = \delta_0 \) one has \( \eta(\delta_0) \approx 0, \ z(\delta_0) \approx 0. \) Unfortunately, this construction may not guarantee the additional condition \( \dot{\eta} \leq -1 \) for all \( t \in [0, \tau_0] \) for some \( \tau_0 < \delta_0. \) To fulfill this additional requirement, we need to modify the control \( v_0. \) More precisely, on a small initial interval \([0, \tau_0]\) with \( \tau_0 \ll \delta_0, \) we choose \( v_0(t) = V_1, \) where \( V_1 \) is defined implicitly by

\[
 f_x(0, y_0, h(y_0), U_0) + f_u(0, y_0, h(y_0), U_0)V_1 + f_\xi(0, y_0, h(y_0), U_0)h'(y_0) = -2. \tag{3.8.8}
\]

By (3.8.5) and the assumption (A1), \( V_1 \) satisfies the uniform bound

\[
 |V_1| \leq \frac{2 + \|f_x\|_\infty + \|f_\xi\|_\infty \|h\|_{C^1}}{\rho_1}.
\]

Let \( \hat{\xi}_0 = \xi(\tau_0) \) be the value at time \( t = \tau_0 \) of the solution to

\[
 \dot{\hat{\xi}} = g(t, y_0, \xi(t), u_0(t, \xi), V_1), \quad \xi(0) = h(y_0).
\]

On the remaining interval \([\tau_0, \delta_0]\) we then choose the control \( t \mapsto v_0(t) \) so that the affine function

\[
 t \mapsto \xi(t) = \frac{\delta_0 - t}{\delta_0 - \tau_0} \hat{\xi}_0 + \frac{t - \tau_0}{\delta_0 - \tau_0} h(y_0) \tag{3.8.9}
\]

is a solution to the ODE

\[
 \dot{\xi}(t) = g(t, y_0, \xi(t), u_0(t, \xi), v_0(t)). \tag{3.8.10}
\]

Notice that, by choosing \( 0 < \tau_0 \ll \delta_0, \) we can render the difference \( |\hat{\xi}_0 - h(y_0)| \) as small as we like. In turn, the time derivative \( \dot{\xi}(t) \) of the right hand side of (3.8.9) can be made arbitrarily small. By the implicit function theorem, there exists a unique function \( v_0 : [\tau_0, \delta_0] \mapsto \mathbb{R}, \) ranging in a small neighborhood of \( V_0, \) which satisfies (3.8.10). The above construction already achieves the identity and the bounds in (3.7.7), together with the terminal requirement \( \xi(\delta_0) = h(y_0). \)

3. In the two previous steps we constructed the controls \( u_0(t), v_0(t), \) for \( t \in [0, \delta_0]. \) In turn, these determine the functions \((x(t, y_0), \xi(t, y_0))\), on the same time interval \([0, \delta_0]. \) In this step we construct a suitable control \( w_0(t) \) such that \( \eta(\delta_0, y_0) \approx 0, \ z(\delta_0, y_0) \approx 0 \) and \( \dot{\eta}(t, y_0) < -1 \) for \( t \in [0, \tau_0]. \) To fix the ideas, in (3.8.5) we shall
assume that
\[ f_\xi(0, y, h(y), U(y)) > \rho_1, \quad \text{for all } y \in \text{Supp}(\mu), \quad (3.8.11) \]
and the case \( f_\xi < -\rho_1 \) is similar. By definition, the map \( t \mapsto (\eta(t, y_0), z(t, y_0)) \) is the solution of the Cauchy problem
\[
\begin{align*}
\dot{\eta} &= \left( f_x + f_u v_0 \right) \eta + f_\xi z, \\
\dot{z} &= \left( g_x + g_u v_0 + g_v w_0 \right) \eta + g_\xi z,
\end{align*}
\]
\[ \eta(0, y_0) = 1, \quad z(0, y_0) = h'(y_0). \quad (3.8.12) \]

Here the right hand side of the ODEs is computed at point \( (t, y_0, \xi(t), u_0(t), v_0(t)) \).

We can rewrite (3.8.12) as a linear system of ODEs for \( \eta, \zeta \), with coefficients depending on the control \( w_0(\cdot) \), namely
\[
\begin{align*}
\dot{\eta} &= f_1(t) \eta + f_2(t) z, \\
\dot{z} &= \left( g_1(t) + g_2(t) w_0(t) \right) \eta + g_3(t) z,
\end{align*}
\]
\[ (3.8.13) \]

where \( f_1 = f_x + f_u v_0, f_2 = f_\xi, g_1 = g_x + g_u v_0, g_2 = g_v, g_3 = g_\xi \). By (3.7.13), (3.7.14), (3.8.5) and (3.8.11), the functions \( f_i, g_i \) satisfy the bounds
\[
\begin{align*}
|f_1(t)| &\leq M_3(1 + M^*), & \frac{\rho_1}{2} &\leq f_2(t) \leq M_3, \\
|g_1(t)| &\leq M_3(1 + M^*), & |g_3(t)| &\leq M_3, \quad \text{for all } t \in [0, \delta_0].
\end{align*}
\]
\[ (3.8.14) \]
\[
\frac{\rho_1}{2} \leq |g_2(t)| \leq M_3, \quad t \in [\tau_0, \delta_0]. \quad (3.8.15)
\]

Solutions of (3.8.13) are more conveniently found using the variables
\[ X = \ln \eta, \quad Y = \frac{z}{\eta}, \quad (3.8.16) \]
which evolve according to
\[
\begin{aligned}
\dot{X} &= f_1(t) + f_2(t) Y , \\
\dot{Y} &= g_1(t) + g_2(t) w_0(t) + (g_3(t) - f_1(t)) Y - f_2(t) Y^2 \doteq \tilde{w}_0(t) , \\
X(0) &= 0 , \\
Y(0) &= h'(y_0) .
\end{aligned}
\] (3.8.17)

For \( t \in [0, \tau_0] \), we defined \( w_0(t) \doteq 0 \). Choosing \( \tau_0 \) small, the difference \( |z(t) - h'(y_0)| \) will remain small for all \( t \in [0, \tau_0] \). By (3.8.8) and continuity, the inequalities
\[-3 < \eta(t, y_0) < -1\]
remain valid for all \( t \in [0, \tau_0] \). Hence \( \eta(t, y_0) < 1 - t \) and
\[X(t) = \ln \eta(t, y_0) \leq \ln (1 - t) , \quad t \in [0, \tau_0] .\] (3.8.18)

Concerning \( Y(\cdot) \), choosing \( \tau_0 > 0 \) small enough we achieve
\[|Y(\tau_0)| = \left| \frac{z(\tau_0, y_0)}{\eta(\tau_0, y_0)} \right| \leq \| h(y) \|_{C_1} + 1 = M_2 + 1 .\] (3.8.19)

For \( t \in [\tau_0, \delta_0] \), we regard \( \tilde{w}_0 \) in (3.8.17) as an independent control function. Since \( g_2 \neq 0 \) by (3.8.15), we can assign the control \( \tilde{w}_0(\cdot) \) arbitrarily, then compute the solution of (3.8.17) and define the corresponding control
\[w_0(t) = \frac{\tilde{w}_0(t) - g_1(t) - (g_3(t) - f_1(t)) Y(t) + f_2(t) Y^2(t)}{g_2(t)} .\]

To achieve the properties (3.7.8)-(3.7.9), we simply use the control
\[\tilde{w}_0(t) \doteq - \frac{1}{\tau_0^3} , \quad t \in [\tau_0, \delta_0] .\] (3.8.20)
The corresponding solution of (3.8.17) is

\[
\begin{align*}
X(t) & = X(\tau_0) + \int_{\tau_0}^{t} f_1(s) \, ds + \int_{\tau_0}^{t} f_2(s) Y(s) \, ds, \\
Y(t) & = Y(\tau_0) - \frac{t - \tau_0}{\tau_0^3},
\end{align*}
\]

where \( \tau_0 \leq t \leq \delta_0 \). (3.8.21)

By (3.8.14), (3.8.15) and (3.8.19), for \( t \in [\tau_0, \delta_0] \) we have the following estimate

\[
X(t) = X(\tau_0) + \int_{\tau_0}^{t} \left( f_1(s) + f_2(s) Y(s) \right) \, ds \\
\leq \ln(1 - \tau_0) + \int_{\tau_0}^{t} \left( M_3(1 + M^*) + f_2(s) \left( Y(\tau_0) - \frac{s - \tau_0}{\tau_0^3} \right) \right) \, ds \\
\leq \ln(1 - \tau_0) + \int_{\tau_0}^{t} M_3(2 + M_2 + M^*) - \frac{\rho_1(s - \tau_0)}{2\tau_0^3} \, ds \\
= \ln(1 - \tau_0) + M_3(2 + M_2 + M^*) (t - \tau_0) - \frac{\rho_1(t - \tau_0)^2}{4\tau_0^3}. 
\]

(3.8.22)

In particular, at \( t = \delta_0 \) one has

\[
X(\delta_0) \leq \ln(1 - \tau_0) + M_3(2 + M_2 + M^*) (\delta_0 - \tau_0) - \frac{\rho_1(\delta_0 - \tau_0)^2}{4\tau_0^3}, \\
Y(\delta_0) = Y(\tau_0) - \frac{\delta_0 - \tau_0}{\tau_0^3}. 
\]

(3.8.23)

Going back to original variables \( \eta, z \), one obtains

\[
\begin{align*}
\eta(\delta_0, y_0) & = \exp \left( X(\delta_0) \right) \\
\quad \leq \exp \left( \ln(1 - \tau_0) + M_3(1 + M_2 + M^*) (\delta_0 - \tau_0) - \frac{\rho_1(\delta_0 - \tau_0)^2}{4\tau_0^3} \right), \\
z(\delta_0, y_0) & = Y(\delta_0) \exp \left( X(\delta_0) \right) = \left( Y(\tau_0) - \frac{\delta_0 - \tau_0}{\tau_0^3} \right) \exp \left( X(\delta_0) \right).
\end{align*}
\]

(3.8.24)
By (3.8.24), choosing $0 < \tau_0 << \delta_0 << 1$ sufficiently small, we achieve

$$0 < \eta(\delta_0, y_0) < \delta_0, \quad |z(\delta_0, y_0)| < \delta_0. \quad (3.8.25)$$

4. It remains to check that our construction yields $\eta(t, y_0) < 1$ for all $t \in [\tau_0, \delta_0]$. By (3.8.22), one has

$$X(t) \leq \ln(1 - \tau_0) + M_3(2 + M_2 + M^*)(t - \tau_0) - \frac{\rho_1(t - \tau_0)^2}{4\tau_0^3}. \quad (3.8.26)$$

The right hand side of (3.8.26) is less than zero for all $t \in [\tau_0, \delta_0]$, provided that $\tau_0$ was chosen sufficiently small. Therefore, always assuming that $0 < \tau_0 << \delta_0 \leq \varepsilon_0$ and $\varepsilon_0$ sufficiently small, we have

$$\eta(t, y_0) = \exp(X(t)) < 1, \quad t \in [\tau_0, \delta_0]. \quad (3.8.27)$$

5. In the previous steps, the initial time interval $[0, \delta_0]$ was chosen depending on the particular choice $y_0 \in Supp(\mu)$. However, for the previous construction it is clear that the size of $\delta_0$ depends only on the constant $\rho_1 > 0$ in (3.8.5) and on the norms $\|f\|_{C^2}, \|g\|_{C^2}, \|h\|_{C^1}$. Hence we can choose
Chapter 4

Stackelberg Solutions of Feedback Type for Differential Games with Random Initial Data

The chapter is concerned with Stackelberg solutions for a differential game with deterministic dynamics but random initial data, where the leading player can adopt a strategy in feedback form: \( u_1 = u_1(t, x) \). The first main result provides the existence of a Stackelberg equilibrium solution, assuming that the family of feedback controls \( u_1(t, \cdot) \) available to the leading player are constrained to a finite dimensional space. A second theorem provides necessary conditions for the optimality of a feedback strategy. Finally, in the case where the feedback \( u_1 \) is allowed to be an arbitrary function, an example illustrates a wide class of systems where the minimal cost for the leading player corresponds to an impulsive dynamics. In this case, a Stackelberg equilibrium solution does not exists, but a minimizing sequence of strategies can be described.

4.1 Introduction

Consider a differential game for two players. Let \( x \in \mathbb{R}^n \) describe the state of a system, which evolves according to the differential equation

\[
\dot{x}(t) = f(t, x(t), u_1(t), u_2(t)) \tag{4.1.1}
\]
with initial condition
\[ x(0) = \bar{x}. \] (4.1.2)

Here the upper dot denotes a derivative w.r.t. time. The functions \( u_1(\cdot), u_2(\cdot) \) are the controls implemented by the two players, taking values inside admissible sets \( U_1, U_2 \subseteq \mathbb{R}^m \), respectively. For \( i = 1, 2 \), the goal of the \( i \)-th player is to minimize his own cost, given by
\[ J_i = \int_0^T L_i(t, x(t), u_1(t), u_2(t)) \, dt. \] (4.1.3)

According to the definition of Stackelberg equilibrium [6, 8, 22], we assume that Player 1 (the leader) announces his feedback strategy in advance, say \( u_1 = u_1(t, x) \). Given the initial point \( \bar{x} \), Player 2 (the follower) then chooses his strategy \( u_2 = u_2(t; u_1, \bar{x}) \) in order to minimize his own cost \( J_2 \).

We remark that, in a standard control problem, an optimal feedback control yields the minimum cost in connection with every initial data \( \bar{x} \). However, this important property cannot be achieved for a Stackelberg solution to a differential game. Namely, the optimal feedback strategy for the leading player usually depends heavily on the initial data [29, 33]. To obtain a meaningful mathematical problem, in connection with a large set of initial data, in this paper we shall thus consider a probability distribution \( \mu \) on the set of initial data \( \bar{x} \in \mathbb{R}^n \).

**Definition 1.** Define \( \mathcal{R}_2(u_1, \bar{x}) \) as the set of best replies for Player 2. These are the controls \( u_2 : [0, T] \mapsto \mathbb{R}^m \) which yield the minimum cost for the optimization problem
\[
\text{minimize: } \int_0^T L_2(t, x(t), u_1(t, x(t)), u_2(t)) \, dt,
\]
subject to: \( \dot{x} = f(t, x(t), u_1(t, x(t)), u_2(t)), \quad x(0) = \bar{x} \). (4.1.5)

In the following, given a probability measure \( \mu \) on the set of initial data, we denote by \( E^- \) the expected value of a quantity depending on \( \bar{x} \).

**Definition 2.** A feedback \( u_1^* \) is an optimal strategy for the leading player if for \( \mu \)-a.e. initial data \( \bar{x} \) one can choose a best reply \( u_2(\cdot; u_1^*, \bar{x}) \in \mathcal{R}_2(u_1^*, \bar{x}) \) in such a
way that
\[
E^u[J(u_1; \bar{x})] = \int \left[ \int_0^T L_1(t, x(t), u^*_1(t, x(t)), u_2(t, u^*_1, \bar{x})) \, dt \right] d\mu(\bar{x})
\]
\[
\leq \int \left[ \int_0^T L_1(t, x(t), u_1(t, x(t)), u_2(t, u_1, \bar{x})) \, dt \right] d\mu(\bar{x}),
\]
\[\text{(4.1.6)}\]
for every other feedback \(u_1 \in \mathcal{F}\) and any selection of best replies \(u_2(\cdot; u_1, \bar{x}) \in \mathcal{R}_2(u_1, \bar{x})\).

Notice that, if Player 2 has several best replies all yielding the same minimum cost, we are here assuming that he chooses the one most favorable to Player 1.

In Section 2 we assume that, for each \(t \in [0, T]\), the feedback \(u_1(t, \cdot)\) depends on a finite set of parameters. For example, \(u_1(t, \cdot)\) is a polynomial in \(x\), or a piecewise affine function. Under natural convexity hypotheses on the cost functions \(L_1, L_2\), our first main result provides the existence of an optimal strategy for the leading player.

Section 3 is devoted to necessary conditions for the optimality of a strategy of the leading player. As it is well known in the literature, one here encounters a fundamental difficulty. Namely, replies of the second player that satisfy the Pontryagin necessary conditions are not necessarily optimal. It thus makes sense to consider a wider set of “weakly optimal replies”.

**Definition 3.** Given a feedback control \(u_1 = u_1(t, x)\) for the first player and an initial data \(\bar{x} \in \mathbb{R}^n\), we call \(\mathcal{R}_2^w(u_1, \bar{x})\) the set of weakly optimal replies for Player 2. These are the controls \(u_2^* : [0, T] \mapsto \mathbb{R}^m\) which satisfy the Pontryagin necessary conditions for optimality:
\[
u^*_2(t) = \arg\min_{\omega \in U_2} \left\{ \xi(t) \cdot f(t, x^*(t), u_1(t, x^*(t)), \omega) + L_2(t, x^*(t), u_1(t, x^*(t)), \omega) \right\},
\]
\[\text{(4.1.7)}\]
\[
\begin{align*}
\dot{x}(t) &= f(t, x^*(t), u_1(t, x^*(t)), u_2^*(t)), \\
\dot{\xi}(t) &= -\xi(t) \cdot \left[ \frac{\partial f}{\partial x}(t, x^*(t), u_1(t, x^*(t)), u_2^*(t)) \right. \\
&\quad+ \frac{\partial f}{\partial u_1}(t, x^*(t), u_1(t, x^*(t)), u_2^*(t)) \frac{\partial u_1}{\partial x}(t, x^*(t)) \bigg] \\
&\quad- \left[ \frac{\partial L_2}{\partial x}(t, x^*(t), u_1(t, x^*(t)), u_2^*(t)) \right. \\
&\quad+ \frac{\partial L_2}{\partial u_1}(t, x^*(t), u_1(t, x^*(t)), u_2^*(t)) \frac{\partial u_1}{\partial x}(t, x^*(t)) \bigg],
\end{align*}
\]

(4.1.8)

Throughout the following, we say that a feedback \( u^* \in F \) is a weakly optimal strategy for Player 1 if it satisfies Definition 2, with \( R_2(u^*, \bar{x}) \) replaced by \( R_2^w(u^*, \bar{x}) \).

**Remark 1.** Clearly, the above definitions are meaningful only if the feedback \( u_1^* \) is sufficiently regular so that the evolution equations (4.1.8) are well defined and have at least a solution for \( \mu \)-a.e. initial data \( \bar{x} \). If this solution is unique, then \( R_2(u_1^*, \bar{x}) = R_2^w(u_1^*, \bar{x}) \).

**Remark 2.** In general, a weakly optimal strategy need not be optimal, and an optimal strategy need not be weakly optimal. The two definitions clearly coincide under the assumption that \( R_2(u_1, \bar{x}) = R_2^w(u_1, \bar{x}) \) for every admissible control \( u_1 \) and \( \mu \)-a.e. initial data \( \bar{x} \). In particular this is true if, for every \( u_1, \bar{x} \), the Pontryagin equations (4.1.7)–(4.1.9) have a unique solution.

Toward the derivation of necessary optimality conditions for the Player 1, the usefulness of this concept of weakly optimal strategy becomes clear. We sketch here the main approach.

Motivated by (4.1.7), for every \( t, x, u_1, \xi \), define the control value

\[
u_2^*(t, x, u_1, \xi) \doteq \arg\min_{\omega \in L_2} \left\{ \xi \cdot f(t, x, u_1, \omega) - L_2(t, x, u_1, \omega) \right\}.
\]

(4.1.10)
In the following, we assume that the above minimum is attained at a unique point $u^\sharp_2$, contained in the interior of $U_2$. This is certainly the case under the assumptions

**(A)** $U_2 = \mathbb{R}^m$. The function $f(t, x, u_1, u_2)$ is affine in the variable $u_2$. Moreover, the function $u_2 \mapsto L_2(t, x, u_1, u_2)$ is strictly convex and has superlinear growth:

$$\lim_{|u_2| \to \infty} \frac{L_2(t, x, u_1, u_2)}{|u_2|} = +\infty.$$  

(4.1.11)

The optimization problem for the leading player can now be written as follows.

Minimize:  $E^\mu[J_1] = \int \left[ \int_0^T L_1(t, x, u_1, u^\sharp_2(t, x, u_1, \xi)) \, dt \right] d\mu(\bar{x})$,  

(4.1.12)

for a solution to the boundary value problem

$$\begin{align*}
\dot{x} & = f, \\
\dot{\xi} & = -\xi \cdot \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u_1} u_{1,x} \right) - \left( \frac{\partial L_2}{\partial x} + \frac{\partial L_2}{\partial u_1} u_{1,x} \right), \\
x(0) & = \bar{x}, \\
\xi(T) & = 0.
\end{align*}$$

(4.1.13)

In (4.1.13) it is understood that the right hand sides of the differential equations are computed at the point $(t, x, u_1(t, x), u^\sharp_2(t, x, u_1(t, x), \xi))$.

Due to the presence of $u_{1,x}$ on the right hand side of the evolution equation (4.1.13), the problem of minimizing the expected cost (4.1.12) is non-standard. In the case where both $x$ and $\xi$ are assigned at the initial time $t = 0$ has been recently studied in [13]. See also [14] for specific examples. As anticipated in [33], if the values $u_1(t, x)$ of the feedback can be freely assigned, in most cases this leads to an ill posed system of equations.

As observed in [14], the term $u_{1,x}$ can often be considered as an additional component of the control, which can be chosen arbitrarily large and comes with no cost. In such cases, the optimal strategy for the leading player would correspond to an impulsive dynamics and can never be exactly attained.

To illustrate this point, Section 4 exhibits a class of nonlinear systems where a minimizing sequence of feedback controls $u_{1,\nu}$ is explicitly constructed, but the infimum cost for the leading player cannot be attained. Roughly speaking, in these cases there is an optimal state $x^\dag$ and an optimal control value $u_1^\dag$ which
minimize the running cost \( L_1 = L_1(x, u_1) \) for Player 1. For any \( \varepsilon > 0 \) and any \( \bar{x} \) within a bounded set of initial data, the leading player can force Player 2 to keep the system at the state \( x^\dagger \) for all times \( t \in [\varepsilon, T - \varepsilon] \). This is achieved by introducing suitable penalties whenever \( x(t) \neq x^\dagger \). As it is often the case for Stackelberg equilibria, these penalties (which would be very costly also to Player 1) are never implemented, because it is not optimal for Player 2 to deviate from the path imposed by the leader.

An introduction to the basic concepts of differential games can be found in [6, 8, 22]. See [29, 30, 33, 40] for related result on Stackelberg equilibria. For the basic theory of optimal control problems and we refer to [5, 11, 18, 23]. Patchy feedbacks, used in Section 4, were introduced in [1]. See [2, 11] for a survey.

4.2 Existence of an optimal feedback control

In this section we study the existence of an optimal strategy for the leading player, within a family of feedback strategies depending on finitely many parameters. We thus consider the system

\[
\dot{x} = f(t, x, u_1(t, x), u_2),
\]

assuming that the feedback control \( u_1(t, x) \) for the leading player has the form

\[
u_1(t, x) = \Psi(x) \cdot v(t) = \sum_{i=1}^{N} \psi_i(x) v_i(t).\]

Here \( \psi_1, \ldots, \psi_N \) are smooth functions of \( x \), while \( v_1, \ldots, v_N \) are measurable functions of time. In practice, this assumption can be justified by observing that any continuous map \( u = u(t, x) \) can be approximated by functions of the form (4.2.2) uniformly on compact sets.

In the following, we assume that

\[
v(t) = (v_1, v_2, \ldots, v_N)(t) \in U_1,
\]

where \( U_1 \) is a compact convex set in \( \mathbb{R}^N \). After the leading player has announced
his feedback control \( u_1(t, x) \) as in (4.2.2), and given an initial data
\[
x(0) = \bar{x},
\] (4.2.4)
the second player seeks to minimize his expected cost
\[
J_2(u_2) = \int_0^T L_2(t, x, u_1(t, x), u_2(t)) \, dt.
\] (4.2.5)
The minimization takes place over all measurable controls
\[
t \mapsto u_2(t) \in U_2,
\] (4.2.6)
taking values in a compact convex set \( U_2 \subset \mathbb{R}^N \).

For a given control \( t \mapsto v(t) \) for the leading player, we call \( R_2(v, \bar{x}) \) the (possibly empty) family of best replies \( t \mapsto u_2(t) \in U_2 \) for the second player. Our present goal is to provide conditions on \( f \) and on the cost functionals \( L_1, L_2 \) which guarantee the existence of a feedback \( u_1 \) for the leading player which is optimal w.r.t. a family of initial data. More precisely, given a probability distribution \( \mu \) on the set of initial data \( \bar{x} \) in (4.2.4), we seek an admissible control \( t \mapsto v^*(t) \in U_1 \) and a selection of optimal replies \( \bar{x} \mapsto u_2^*(\cdot, \bar{x}) \in R_2(v^*, \bar{x}) \) such that
\[
J_1(v^*) = \int \left[ \int_0^T L_1(t, x, u_1(t, x), u_2^*(t, \bar{x})) \, dt \right] d\mu(\bar{x})
\] (4.2.7)
achieves the global minimum. Here the minimum is sought among all measurable functions \( v : [0, T] \mapsto U_1 \), and among all selections \( \bar{x} \mapsto u_2^*(\cdot, \bar{x}) \in R_2(v, \bar{x}) \) of optimal replies.

Consider the following assumptions.

(H1) The cost functions \( L_1, L_2 \) are non-negative and continuous w.r.t. all arguments. The function \( f \) is continuous in all variables, continuously differentiable w.r.t. \( x \), and satisfies the bound
\[
|f(t, x, u_1(t, x), u_2)| \leq C \left( 1 + |x| \right) \quad \text{for all } x, t, u_2 \in U_2
\] (4.2.8)
and for all $u_1$ as in (4.2.2), with $v \in U_1$.

**(H2)** For every $t, x$, the function $f(t, x, u_1, u_2)$ is affine w.r.t. the variables $u_1, u_2$. The cost function $L_1$ is convex w.r.t. the variables $u_1, u_2$. Finally, the cost function $L_2$ has the form

$$L_2(t, x, u_1, u_2) = L_{21}(t, x)u_1 + L_{22}(t, x, u_2)$$

with $L_{22}$ convex in the variable $u_2$.

**(H3)** The set of admissible control values $U_1, U_2$ are compact, convex.

Under the above assumptions, our main theorem provides the existence of a Stackelberg equilibrium solution.

**Theorem 1.** Let the assumptions (H1)–(H3) hold, and let $\mu$ be a probability measure on the set of initial data. Assume that the leading player has at least one strategy $v(\cdot)$ with finite expected cost. Then the differential game with dynamics (4.1.1) and cost functionals (4.1.3) admits a Stackelberg equilibrium solution, among all feedback controls for the leading player having of the form (4.2.2).

**Proof. 1.** Let $t \mapsto v(t) \in U_1$ be any measurable control for the leading player, yielding the feedback $u_1 = u_1(t, x)$ in (4.2.2). Under the assumption (H1)–(H3) it is well known that, for every initial data $\bar{x}$, the set of optimal replies $u_2(\cdot, \bar{x})$ is non-empty and weakly closed. See for example [18], or Chapter 5 in [11] for details. More precisely, if $u_{2, \nu} : [0, T] \mapsto U_2$ is a minimizing sequence of control functions for Player 2, and $u_{2, \nu} \rightharpoonup u_2^*$ weakly in $L^1([0, T])$ as $\nu \to \infty$, then the corresponding trajectories satisfy $x_{\nu}(t) \to x^*(t)$ uniformly for $t \in [0, T]$. Moreover, for $i = 1, 2$

$$\int_0^T L_i(t, x^*(t), u_1(t, x^*(t)), u_2^*(t)) \, dt \leq \liminf_{\nu \to \infty} \int_0^T L_i(t, x_{\nu}(t), u_1(t, x_{\nu}(t)), u_{2, \nu}(t)) \, dt.$$ (4.2.9)

This proves that, for every fixed initial data $\bar{x}$, the set of best replies $R_2(v, \bar{x})$ is non-empty. A similar argument shows that, within this family of best replies, we can choose one that minimizes the cost to Player 1. Indeed, assume that
$u_{2,\nu} \in R_2(v, \bar{x})$ for every $\nu \geq 1$ and
\[
\lim_{\nu \to \infty} \int_0^T L_1\left(t, x_{\nu}(t), u_1(t, x_{\nu}(t)), u_{2,\nu}(t)\right) dt
= \inf_{u_2 \in R_2(v, \bar{x})} \int_0^T L_1\left(t, x(t), u_1(t, x(t)), u_2(t)\right) dt = m_1(v, \bar{x}).
\]

(4.2.10)

Extracting a subsequence, we can achieve the weak convergence $u_{2,\nu} \rightharpoonup u^*_2$ and the uniform convergence $x_\nu \to x^*$. The convexity of $L_1$ w.r.t. $u_2$ now yields
\[
\int_0^T L_1\left(t, x^*(t), u_1(t, x^*(t)), u^*_2(t)\right) dt \leq \liminf_{\nu \to \infty} \int_0^T L_1\left(t, x_\nu(t), u_1(t, x_\nu(t)), u_{2,\nu}(t)\right) dt.
\]

(4.2.11)

We conclude that, given a control $v$ for the first player, for each $\bar{x}$ there exists a best reply $u^*_2(\cdot) \in R_2(v, \bar{x})$ which is optimal for Player 1 (restricted to the set of best replies).

2. In this step we establish a lower semicontinuity property of the map $(v, \bar{x}) \mapsto m_1(v, \bar{x})$ defined at (4.2.10). Namely, consider a converging sequence of initial points $\bar{x}_k \to \bar{x}$, and a weakly convergent sequence of controls for the leading player: $v_k \rightharpoonup v$. Let $u_{2,k} = u_2(\cdot, v_k, \bar{x}_k) \in R_2(v_k, \bar{x}_k)$ be best replies of Player 2 which are optimal for Player 1 (restricted to the set of best replies). By extracting a subsequence and relabeling, we can assume the weak convergence $u_{2,k} \rightharpoonup u^*_2$ and the uniform convergence of the corresponding trajectories $x_k \to x^*$ on $[0, T]$. We claim that $u^*_2 \in R_2(v, \bar{x})$.

Indeed, consider any control $t \mapsto u^*_2(t) \in U_2$ for the second player, and call $x^*, x^*_k$ respectively the solutions of
\[
\dot{x}^*(t) = f(t, x^*(t), \Psi(x^*(t)) \cdot v(t), u^*_2(t)), \quad x^*(0) = \bar{x},
\]
\[
\dot{x}^*_k(t) = f(t, x^*_k(t), \Psi(x^*_k(t)) \cdot v_k(t), u^*_2(t)), \quad x^*_k(0) = \bar{x}_k.
\]

By the weak convergence $v_k \to v$ and by the linearity of $f$ w.r.t. $u_1, u_2$, we have
the convergence $x^k(t) \to x^\ast(t)$ uniformly on $[0,T]$. Moreover,

\[
\int_0^T \left[ L_{21}(t, x^k(t)) \Psi(x^k(t)) \cdot v(t) + L_{22}(t, x^k(t), u^k_2(t)) \right] \, dt
\]

\[
= \lim_{k \to \infty} \int_0^T \left[ L_{21}(t, x_k(t)) \Psi(x_k(t)) \cdot v_k(t) + L_{22}(t, x_k(t), u_2^k(t)) \right] \, dt
\]

\[
\geq \liminf_{k \to \infty} \int_0^T \left[ L_{21}(t, x_k(t)) \Psi(x_k(t)) \cdot v_k(t) + L_{22}(t, x_k(t), u_{2,k}(t)) \right] \, dt
\]

\[
\geq \int_0^T \left[ L_{21}(t, x^\ast(t)) \Psi(x^\ast(t)) \cdot v(t) + L_{22}(t, x^\ast(t), u^\ast_2(t)) \right] \, dt.
\]

This proves our claim.

As a consequence, given a sequence $(v_k, \bar{x}_k)$ such that $v_k \rightharpoonup v$ and $\bar{x}_k \to \bar{x}$, we have

\[
m_1(v, \bar{x}) \leq \liminf_{k \to \infty} m_1(v_k, \bar{x}_k). \tag{4.2.12}
\]

In particular, for a given control $v(\cdot)$ the map $\bar{x} \mapsto m_1(v, \bar{x})$ is lower semicontinuous, hence measurable. Therefore, for any probability measure $\mu$ on the set of initial data, the integral

\[
J_1(v) = \int m_1(v, \bar{x}) \, d\mu(\bar{x}) \tag{4.2.13}
\]

is well defined (possibly taking the value $+\infty$).

3. Next, consider a minimizing sequence of controls $v_k(\cdot)$ for Player 1, so that

\[
\lim_{k \to \infty} \int m_1(v_k, \bar{x}) \, d\mu(\bar{x}) = \inf_{v(\cdot)} \int m_1(v, \bar{x}) \, d\mu(\bar{x}). \tag{4.2.14}
\]

By possibly extracting a subsequence and reabelling, we can assume the weak convergence $v_k \rightharpoonup v^\ast$ in $L^1([0,T])$. We claim that $v^\ast$ is an optimal strategy for Player 1. Indeed, all functions $\bar{x} \mapsto m_1(v_k, \bar{x})$ are non-negative and lower semicontinuous.
By (4.2.12) and Fatou’s lemma,
\[
\int m_1(v^*, \bar{x}) \, d\mu(\bar{x}) \leq \int \liminf_{k \to \infty} m_1(v_k, \bar{x}) \, d\mu(\bar{x}) \\
\leq \liminf_{k \to \infty} \int m_1(v_k, \bar{x}) \, d\mu(\bar{x}) = \inf_{v(\cdot)} \int m_1(v, \bar{x}) \, d\mu(\bar{x}).
\]

4.3 Necessary conditions for optimality

By the analysis at (4.1.12)-(4.1.13), a weakly optimal strategy \( u_1^*(t, x) \) for the leading player must be optimal for the following problem.

Minimize:  \( J[u] = \int \left[ \int_0^T L\left( t, x(t), \xi(t), u(t, x(t)) \right) \, dt \right] \, d\mu(\bar{x}), \)  (4.3.1)

subject to:
\[
\begin{aligned}
\dot{x} &= f(t, x, \xi, u), & x(0) &= \bar{x}, \\
\dot{\xi} &= g(t, x, \xi, u, u_x), & \xi(T) &= 0.
\end{aligned}
\]  (4.3.2)

Here \( x, \xi \in \mathbb{R}^n \). Moreover, with slight abuse of notation, we rename
\[
f(t, x, \xi, u) \doteq f(t, x, u, u_2^*(t, x, u, \xi)),
\]  (4.3.3)

\[
L(t, x, \xi, u) \doteq L_1(t, x, u, u_2^*(t, x, u, \xi)),
\]

with \( u^* \) given at (4.1.10). Finally,
\[
g(t, x, \xi, u, v) \doteq -\xi \cdot \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u_1} v \right) - \left( \frac{\partial L_2}{\partial x} + \frac{\partial L_2}{\partial u_1} v \right),
\]  (4.3.4)

where \( f \) and \( L_2 \) are evaluated at the point \((t, x, u, u_2^*(t, x, u, \xi))\). We regard (4.3.1)-(4.3.2) as a problem of optimal control on the infinite dimensional space whose elements are couples of functions \((x, \xi) : \mathbb{R}^n \mapsto \mathbb{R}^n \times \mathbb{R}^n\), depending on the initial point \( \bar{x} \in \mathbb{R}^n \). The infimum is sought over all measurable control functions \( u : [0, T] \mapsto \mathcal{U} \), where \( \mathcal{U} \) is a family of admissible functions \( \omega : \mathbb{R}^n \mapsto \mathbb{R}^m \),
sufficiently regular so that the corresponding evolution of the variables $x, \xi$ in (4.3.2) is well defined. For example, we can impose that for each fixed $t$ the function $u(t, \cdot)$ be affine. In this case, $\mathcal{U}$ would be the family of all polynomial functions of degree one in the variables $x_1, \ldots, x_n$. Another natural choice is to take $\mathcal{U}$ as the family of all $C^2$ functions of the variable $x$.

In the following, we seek necessary conditions for optimality of an admissible control $u^* : [0, T] \mapsto \mathcal{U}$, for the problem (4.3.1)-(4.3.2). After a renaming of variables, these immediately yield necessary conditions for the weak optimality of a feedback control $u_1 = u_1^*(t, x)$ for the Stackelberg game.

Given a control $u = u(t, x)$, consider a family of perturbed solutions of (4.3.2) having the form

$$
\begin{align*}
\begin{cases}
  x^\varepsilon(t, \bar{x}) &= x(t, \bar{x}) + \varepsilon X(t, \bar{x}) + o(\varepsilon), \\
  \xi^\varepsilon(t, \bar{x}) &= \xi(t, \bar{x}) + \varepsilon Z(t, \bar{x}) + o(\varepsilon).
\end{cases}
\end{align*}
$$

Linearizing (4.3.2) around the reference trajectory $t \mapsto (x(t, \bar{x}), \xi(t, \bar{x}))$, we obtain a linear equation for the first order perturbations $X, Z$, namely

$$
\begin{align*}
\begin{pmatrix}
  \dot{X} \\
  \dot{Z}
\end{pmatrix} = _______ (X, Z),
\end{align*}
$$

with boundary conditions

$$
X(0, \bar{x}) = 0, \quad Z(T, \bar{x}) = 0.
$$

Next, for each fixed $\bar{x} \in \mathbb{R}^n$, let the couple of functions $(P, Q) : [0, T] \mapsto \mathbb{R}^n \times \mathbb{R}^n$ provide a solution to the dual system

$$
\begin{align*}
\begin{pmatrix}
  \dot{P} \\
  \dot{Q}
\end{pmatrix} = _______ (P, Q),
\end{align*}
$$

with boundary conditions

$$
Q(0, \bar{x}) = 0, \quad P(T, \bar{x}) = 0.
$$
By construction, for any solution \((X Z)\) of (4.3.6) and any solution \((P Q)\) of (4.3.9), integrating w.r.t. the measure \(\mu\) one obtains
\[
\frac{d}{dt} \int [X(t, \bar{x}) P(t, \bar{x}) + Z(t, \bar{x}) Q(t, \bar{x})] \, d\mu(\bar{x}) = - \int [(L_x + L_u u_x) X + L_\xi Z] \, d\mu(\bar{x}).
\] (4.3.10)

In the following discussion, we shall always assume that the functions \(f, g, L\) satisfy the following assumptions.

**(A1)** The functions \(f, g, L\) are continuous in all variables, and continuously differentiable w.r.t. \(x, \xi, u, v\).

**(A2)** Every admissible feedback control \(\omega(\cdot) \in \mathcal{U}\) is twice continuously differentiable.

**(A3)** The probability measure \(\mu\) has compact support.

In the following, given initial data \((\bar{x}, \bar{\xi})\), we denote by \(t \mapsto x(t; \bar{x}, \bar{\xi})\) the solution to the evolution equations in (4.3.2) with initial data
\[
x(0) = \bar{x}, \quad \xi(0) = \bar{\xi}.
\]
Moreover, we call \(\bar{\xi}(\bar{x}) = \xi(0)\) the initial value for \(\xi\) of the solution to the boundary value problem (4.3.2).

**Theorem 2 (necessary conditions for optimality).** Let the above assumptions (A1)-(A3) hold. Let \(u = u(t, x)\) be an optimal feedback control for the problem (4.3.1)-(4.3.2), piecewise continuous w.r.t. time. Call \(x(t, \bar{x}), \xi(t, \bar{x})\) the corresponding trajectories, which we assume remain uniformly bounded as \(\bar{x}\) ranges in the support of \(\mu\). Let the couple of dual functions \((P, Q) : [0, T] \times \mathbb{R} \mapsto \mathbb{R}^n \times \mathbb{R}^n\) provide a solution to (4.3.8)-(4.3.9) and assume that the \(n \times n\) Jacobian matrix
\[
D_\xi \xi(T; \bar{x}, \bar{\xi}) = \left( \frac{\partial \xi_i(T; \bar{x}, \bar{\xi})}{\partial \xi_j} \right)
\] (4.3.11)
is invertible, for all \(\bar{x}\) in the support of \(\mu\) and \(\bar{\xi} = \bar{\xi}(\bar{x})\).
Then, for a.e. \( t \in [0,T] \), the feedback control \( u(t,\cdot) \in U \) provides a global minimizer to the functional

\[
J(t,\omega(\cdot)) = \int P(t,\bar{x}) \cdot f\left( t, x(t,\bar{x}), \xi(t,\bar{x}), \omega(t, x(t,\bar{x})) \right) d\mu(\bar{x}) \\
+ \int Q(t,\bar{x}) \cdot g\left( t, x(t,\bar{x}), \xi(t,\bar{x}), \omega(t, x(t,\bar{x})), \omega_x(t, x(t,\bar{x})) \right) d\mu(\bar{x}) \\
+ \int L\left( t, x(t,\bar{x}), \xi(t,\bar{x}), \omega(t, x(t,\bar{x})) \right) d\mu(\bar{x}).
\]

(4.3.12)

within the family of all admissible control functions \( \omega(\cdot) \in U \).

**Proof.** 1. Assume that, for some \( \tau \in ]0,T] \) where \( u \) is continuous, the feedback \( u(\tau,\cdot) \) does not satisfy the above minimality condition. Then there exists an admissible control function \( \omega : \mathbb{R}^n \mapsto \mathbb{R}^m, \omega(\cdot) \in U \) such that

\[
J(\tau,\omega(\cdot)) < J(\tau,u(\tau,\cdot)).
\]

(4.3.13)

Consider the family of “needle variations” of \( u \), defined as

\[
u(\varepsilon)(t,x) = \begin{cases} 
\omega(x) & \text{if } t \in [\tau - \varepsilon, \tau], \\
u(t,x) & \text{if } t \notin [\tau - \varepsilon, \tau]. 
\end{cases}
\]

(4.3.14)

We claim that, for \( \varepsilon > 0 \) sufficiently small, \( J[u_\varepsilon] < J[u] \), contradicting the optimality of \( u \).

2. We claim that, for all \( \varepsilon \in ]0,\varepsilon_0] \) sufficiently small and every initial data \( \bar{x} \) in the support of \( \mu \), a solution \( (x_\varepsilon(t,\bar{x}), \xi_\varepsilon(t,\bar{x})) \) of (4.3.2) corresponding to the control \( u_\varepsilon \) does exist.

Indeed, consider the map

\[
\Phi^{\bar{x},\varepsilon} : \tilde{\xi} \mapsto \xi(T),
\]
where $\xi(T)$ is the terminal value of the solution to the Cauchy problem

$$
\begin{align*}
\dot{x} &= f(t, x, \xi, u_\varepsilon), \\
\dot{\xi} &= g(t, x, \xi, u_\varepsilon, u_{\varepsilon,x}),
\end{align*}
$$

(4.3.15)

with initial data

$$
x(0) = \bar{x}, \quad \xi(0) = \bar{\xi}.
$$

(4.3.16)

For any $\bar{x}$ in the support of the probability measure $\mu$, when $\varepsilon = 0$ and $\bar{\xi} = \bar{\xi}(\bar{x})$, by assumption we have $\xi(T) = 0$. Using the transversality assumption (4.3.11) and the implicit function theorem, we obtain a neighborhood $N_{\bar{x}}$ of $\bar{x}$ and $\varepsilon_{\bar{x}} > 0$ with the following property. For all $\varepsilon \in [0, \varepsilon_{\bar{x}}]$ and every initial point $\bar{y} \in N_{\bar{x}}$ there exists an initial value $\xi = \bar{\xi}_\varepsilon(\bar{y})$ such that the corresponding solution of (4.3.15) with initial data

$$
x(0) = \bar{y}, \quad \xi(0) = \bar{\xi}_\varepsilon(\bar{y})
$$

satisfies $\xi(T) = 0$. Since the support of $\mu$ is bounded, hence compact, it can be covered with finitely many neighborhoods $N_{\bar{x}_i}, i = 1, \ldots, \kappa$. Choosing

$$
\varepsilon_0 = \min\{\varepsilon_{\bar{x}_i}; \ i = 1, \ldots, \kappa\}
$$

our claim is proved.

3. We now estimate the difference in the costs: $J[u_\varepsilon] - J[u]$. For $t < \tau - \varepsilon$, the first order perturbations $X(t, \bar{x}), Z(t, \bar{x})$ in (4.3.5) satisfy (4.3.6), (4.3.10). Moreover, at the time $t = \tau$ where the needle variation in the control takes place, we have

$$
X(\tau+, \bar{x}) = \lim_{\varepsilon \to 0^+} \frac{x^\varepsilon(\tau, \bar{x}) - x(\tau, \bar{x})}{\varepsilon}
$$

$$
= X(\tau-, \bar{x}) + f(\tau, x(\tau, \bar{x}), \xi(\tau, \bar{x}), \omega(x(\tau, \bar{x})))
$$

$$
- f(\tau, x(\tau, \bar{x}), \xi(\tau, \bar{x}), u(\tau, x(\tau, \bar{x}))),
$$

(4.3.17)
\[
Z(\tau+, \bar{x}) = \lim_{\varepsilon \to 0^+} \frac{\xi^\varepsilon(\tau, \bar{x}) - \xi(\tau, \bar{x})}{\varepsilon}
\]

\[
= Z(\tau-, \bar{x}) + g(\tau, x(\tau, \bar{x}), \xi(x(\tau, \bar{x})), \omega_{x}(x(\tau, \bar{x})))
\]

\[
- g(\tau, x(\tau, \bar{x}), \xi(\tau, \bar{x}), u(\tau, x(\tau, \bar{x}))), u_{x}(\tau, x(\tau, \bar{x})))
\]

(4.3.18)

Differentiating the total cost w.r.t. \(\varepsilon\) at \(\varepsilon = 0^+\), and using (4.3.10), the boundary conditions (4.3.7) and (4.3.9) to eliminate boundary terms, and finally (4.3.17)-(4.3.18), we obtain

\[
\frac{d}{d\varepsilon} \int_{0}^{T} \left[ \int_{\bar{x}} \left( L(x(\tau, \bar{x}), \xi(x(\tau, \bar{x})), u_{x}(x(\tau, \bar{x}))) - L(\tau, x(\tau, \bar{x}), \xi(\tau, \bar{x}), u(\tau, x(\tau, \bar{x}))) \right) dt \right] d\mu(\bar{x}) \bigg|_{\varepsilon = 0^+} = 0
\]

(4.3.19)
because of the assumption (4.3.13). This shows that, for \( \varepsilon > 0 \) small, the feedback control \( u_\varepsilon \) in (4.3.14) achieves a lower cost, contradicting the assumption that \( u \) is optimal.

In the case where \( g \) does not depend on \( u_x \), the minimizer of the functional \( J(t, \omega(\cdot)) \) can be constructed pointwise, for each given \( \bar{x} \in \mathbb{R}^n \). The dependence on the first derivative \( u_x \) makes the problem “non-classical”.

As stated in Theorem 2, the necessary conditions are hard to implement, because at each time \( \tau \) the optimization has to be carried out w.r.t. a probability measure \( \mu \) given on the set of initial data, rather than on the probability distribution of the state at time \( \tau \). Following [1, 2], in the one-dimensional case one can write these equations in a more appealing form.

We begin by representing the probability measure \( \mu \) as the push-forward of Lebesgue measure on \([0, 1]\) by a nondecreasing map \( y \mapsto \bar{x}(y) \). This map is defined by the property

\[
\mu([-\infty, \alpha]) = \sup\{y \in [0, 1]; \bar{x}(y) \leq \alpha\} \quad \text{for every } \alpha \in \mathbb{R}.
\]

Denoting by \( x(t, y), \xi(t, y) \) the solution of (4.3.2) with boundary data

\[
x(0) = \bar{x}(y), \quad \xi(T) = 0,
\]

the expected cost in (4.3.1) can be rewritten as

\[
J[u] = \int_0^1 \left[ \int_0^T L\left(t, x(t, y), \xi(t, y), u(t, x(t, y))\right) \, dt \right] \, dy.
\]

Assume that the map \( y \mapsto x(t, y) \) is strictly increasing, for every \( t \in [0, T] \). Observe that the map \( \phi(t, y) = [x_y(t, y)]^{-1} \) can then be obtained as the solution to

\[
\phi_t = -(f_x x_y + f_\xi \xi_y + f_u u_y) \phi^2, \quad \phi(0, y) = \frac{1}{x_y(0, y)}.
\]

If \( \mathcal{U} \) is the family of all \( C^2 \) control functions and \( u \) is an optimal feedback control satisfying the assumptions of Theorem 2, then at a.e. time \( \tau \in [0, T] \) the map \( u(\tau, \cdot) \) should be a global minimizer for (4.3.12). Replacing an integration w.r.t. the
probability measure $\mu$ with an integration w.r.t. $y \in [0, 1]$, we find that the function $u(t, y) = u(t, x(t, y))$ must satisfy

$$0 = \int_0^1 \left( Pf_u w + Qg_u w + Qg_v w_x + L_u w \right) dy$$

(4.3.21)

$$= \int_0^1 \left( Pf_u + Qg_u - (Qg_v \phi)_y + L_u \right) w dy,$$

for every function $w \in C^2([0, 1])$ which vanishes at $y = 0$ and at $y = 1$. Notice that in (4.3.21) one of the terms was integrated by parts, using the identity $w_x = \phi(t, y)w_y$. Since $w$ is arbitrary, the above necessary conditions yield

$$Pf_u + Qg_u - (Qg_v \phi)_y + L_u = 0.$$  

(4.3.22)

At a given time $t$, it is understood that the left hand side of (4.3.22) should be computed at the point $(t, x(t, y), \xi(t, y), u(t, x(t, y)))$, for any $y \in [0, 1]$. Moreover, by (4.3.2), (4.3.9), and choosing a perturbation $w$ which does not vanish on the boundary, we obtain the following boundary conditions:

$$\left. (Qg_v) \right|_{y=0} = \left. (Qg_v) \right|_{y=1} = 0,$$

(4.3.23)

$$x(0, y) = \bar{x}(y), \quad \left. (Qg_u - (Qg_v \phi)_y + L_u) \right|_{t=T} = 0.$$

## 4.4 Nonexistence of optimal feedback strategies

Aim of this section is to exhibit a class of differential games where no optimal feedback control for the leading player exists, if this feedback is allowed to range over the family of all piecewise constant functions of $t, x$.

Consider a scalar system which is linear w.r.t. the control variables

$$\dot{x} = f_0(x) + f_1(x)u_1 + f_2(x)u_2,$$  

(4.4.1)
where $f_0$, $f_1$, $f_2$ are $C^1$ functions such that

$$|f_0(x)| \leq C, \quad \frac{1}{C} \leq f_1(x), f_2(x) \leq C \quad (4.4.2)$$

for some constant $C > 0$ and every $x \in \mathbb{R}$. Let the cost functions for the two players have the form

$$J_i = \int_0^T L_i(x(t), u_i(t)) \, dt \quad i = 1, 2, \quad (4.4.3)$$

with

$$L_2(x, u_2) \geq \alpha (|x| + |u_2|) - \beta, \quad \text{for all } (x, u_2) \in \mathbb{R}^2, \quad (4.4.4)$$

for some constants $\alpha, \beta > 0$. Let $(\bar{x}_1, \bar{u}_1)$ be the global minimizer for $L_1(x, u_1)$. Without loss of generality we can assume that $\bar{x}_1 = 0$, $\bar{u}_1 = 0$, and

$$L_1(0, 0) = 0 \leq L_1(x, u_1) \quad \text{for all } (x, u_1) \in \mathbb{R}^2. \quad (4.4.5)$$

Under these assumptions we claim that, for any $\bar{\eta}$ and $\varepsilon > 0$, the leading player can implement a piecewise constant “patchy” feedback $u_1 = u_1(t, x)$ such that, for every initial data $\bar{x} \in [-\bar{\eta}, \bar{\eta}]$ and any best reply $u_2(\cdot; u_1, \bar{x})$ of Player 2, its cost is $\leq \varepsilon$. As a consequence, if $\mu$ is any probability measure with bounded support, for any $\varepsilon > 0$ there exists a patchy feedback $u_1$ for the leading player which yields an expected cost $\leq \varepsilon$.

Let $C_1 > 0$ be a constant such that

$$L_2(x, 0) \leq C_1 \quad \text{for } |x| \leq 1. \quad (4.4.6)$$

Let $\bar{\eta}$ be given. It is clearly not restrictive to assume

$$\frac{\bar{\eta}}{2} > \frac{C_1 + \beta}{\alpha}. \quad (4.4.7)$$

The feedback $u_1$ is constructed as follows. Define the continuous function $\eta : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\eta(x, u) = \begin{cases} 
\frac{\bar{\eta}}{2} & \text{if } |x| + |u| \leq \frac{\bar{\eta}}{2}, \\
\alpha (|x| + |u|) - \beta & \text{otherwise.}
\end{cases} \quad (4.4.8)$$
Figure 4.1. A near-optimal strategy for Player 1. If Player 2 allows the trajectory of the system to enter the shaded area, then his strategy is not optimal. The trajectories $\gamma_1, \gamma_2, \gamma_3$ illustrate respectively the cases 1, 2, 3 in the analysis below.

$[0, T] \mapsto \mathbb{R}$ by setting (see Fig. 4.4)

$$
\eta(t) = \begin{cases} 
    \left(1 - \frac{t}{\delta}\right) \bar{\eta} & \text{if } t \in [0, \delta], \\
    0 & \text{if } t \in [\delta, T - \delta], \\
    \left(1 - \frac{T - t}{\delta}\right) \bar{\eta} & \text{if } t \in [T - \delta, T]. 
\end{cases} 
$$  

(4.4.8)

Since we are assuming that $L_1(0, 0) = 0$, we can choose the constant $\delta > 0$ small enough so that

$$
\delta C < 1, \quad \frac{1}{C} \left(\frac{\bar{\eta}}{\delta} - C\right) \geq \bar{\eta}, 
$$

(4.4.9)

and moreover

$$
\int_0^T L_1(x(t), 0) \, dt \leq \varepsilon 
$$

(4.4.10)
for every trajectory $x(\cdot)$ such that

$$|x(t)| \leq \eta(t) \quad \text{for all } t \in [0, T]. \quad (4.4.11)$$

Consider a feedback $u_1$ for the leading player having the form

$$u_1(t, x) \doteq \begin{cases} 
-K & \text{if } x < -\eta(t), \\
0 & \text{if } x \in [-\eta(t), \eta(t)], \\
K & \text{if } x > \eta(t),
\end{cases} \quad (4.4.12)$$

We claim that, if the constant $K$ is chosen large enough, then for every initial point $\bar{x} \in [-\bar{\eta}, \bar{\eta}]$, the best reply $u_2(\cdot; u_1, \bar{x})$ of Player 2 yields a trajectory satisfying (4.4.11). Hence (4.4.10) holds.

Observe that, by (4.4.2), Player 2 can steer the system along any absolutely continuous path $x(\cdot)$. Indeed, given $x(\cdot)$, his control function is determined by

$$u_2(t) = \frac{\dot{x}(t) - f_0(x(t)) - f_1(x(t))u_1(t, x(t))}{f_2(x(t))}. \quad (4.4.13)$$

In particular, if $x(t) = \eta(t)$, then the corresponding control $u_2^n$ satisfies

$$|u_2^n(t)| = \left| \frac{\dot{\eta}(t) - f_0(\eta(t))}{f_2(\eta(t))} \right| \leq \frac{\bar{\eta}\delta^{-1} + C}{C^{-1}}. \quad (4.4.14)$$

We can now find constants $C_1, C_2$, not depending of $K$, such that

$$L_2(\eta(t), u_2^n(t)) \leq C_2 \quad \text{for all } t \in [0, T]. \quad (4.4.15)$$

Next, assume that the trajectory $x(\cdot)$ does not satisfy (4.4.11). To fix the ideas, we assume $x(t) > \eta(t)$ at some time $t$. If $x(t) < -\eta(t)$, the analysis is entirely similar.

Three cases will be considered, illustrated by the paths $\gamma_1, \gamma_2, \gamma_3$ in Fig. 4.4.

**Case 1:** $x(t) > \eta(t)$ for $t \in [t_1, t_2]$ and $x(t_1) = \eta(t_1), x(t_2) = \eta(t_2)$. 

We claim that the trajectory
\[ \tilde{x}(t) = \begin{cases} \eta(t) & \text{if } t \in [t_1, t_2], \\ x(t) & \text{if } t \notin [t_1, t_2], \end{cases} \] (4.4.16)
yields a strictly lower cost to Player 2. Indeed, by (4.4.13) one has
\[ \int_{t_1}^{t_2} u_2(t) \, dt = \int_{t_1}^{t_2} \frac{\dot{x} - f_0(x) - f_1(x)K}{f_2(x)} \, dt \]
\[ \leq \frac{\vert \eta(t_2) - \eta(t_1) \vert}{C^{-1}} + (t_2 - t_1) \frac{C}{C-1} - \frac{C^{-1}K}{C} (t_2 - t_1) \] (4.4.17)
\[ \leq (\delta^{-1}C\bar{\eta} + C^2 - C^{-2}K)(t_2 - t_1). \]
If \( K \) satisfies
\[ \frac{K}{C^2} > \frac{C\bar{\eta}}{\delta} + C^2 + \frac{\beta}{\alpha} + \frac{C_2}{\alpha}, \] (4.4.18)
then by (4.4.15)
\[ \int_{t_1}^{t_2} L_2(x, u_2) \, dt \geq \int_{t_1}^{t_2} \left( \alpha (|x| + |u_2|) - \beta \right) \, dt \geq \alpha \int_{t_1}^{t_2} u_2 \, dt - \beta(t_2 - t_1) \]
\[ \geq \left( \alpha (C^{-2}K - \delta^{-1}C\bar{\eta} - C^2) - \beta \right)(t_2 - t_1) \]
\[ > C_2(t_2 - t_1) \geq \int_{t_1}^{t_2} L_2(\eta, u_2^0) \, dt, \] (4.4.19)
This shows that the trajectory (4.4.16) yields a strictly lower cost to Player 2, as claimed.

**Case 2:** \( x(t) > \eta(t) \text{ for } t \in [t_1, T] \) with \( t_1 < T - \delta \) and \( x(t_1) = \eta(t_1) = 0 \). Again, we claim that the trajectory (4.4.16) yields a lower cost to Player 2.
By (4.4.1) and (4.4.12) it follows

\[ x(t) = x(t_1) + \int_{t_1}^{t} \left( f_0(x(s)) + f_1(x(s)) K + f_2(x(s)) u_2(s) \right) ds \]

\[ \geq (-C + C^{-1}K)(t - t_1) - C \int_{t_1}^{t} |u_2(s)| \, ds. \]  

(4.4.20)

Integrating both sides of the above inequality from \( t_1 \) to \( T \) one obtains

\[ \int_{t_1}^{T} x(t) \, dt \geq \int_{t_1}^{T} \left( (-C + C^{-1}K)(t - t_1) - C \int_{t_1}^{t} |u_2(s)| \, ds \right) \, dt \]

\[ = \frac{C^{-1}K - C}{2} (T - t_1)^2 - C \int_{t_1}^{T} (T - s) |u_2(s)| \, ds \]

\[ \geq \frac{C^{-1}K - C}{2} (T - t_1)^2 - C(T - t_1) \int_{t_1}^{T} |u_2(s)| \, ds. \]  

(4.4.21)

In turn, this implies

\[ \int_{t_1}^{T} (|x(t)| + |u_2(t)|) \, dt \geq \min \left\{ \left| \frac{C^{-1}K - C}{2} (T - t_1)^2 - C(T - t_1)Z \right|, Z \right\} \]

\[ = \frac{C^{-1}K - C}{2} \frac{(T - t_1)^2}{1 + C(T - t_1)} \geq \frac{C^{-1}K - C}{2(1 + CT)} \delta^2. \]  

(4.4.22)

Comparing the second player’s costs for the trajectories \( x \) and \( \tilde{x} \) in (4.4.16), if the constant \( K \) was chosen so that

\[ \frac{\alpha \delta^2}{2C(1 + CT)} K > (C_2 + \beta)T + \frac{C}{2(1 + CT)} \alpha \delta^2, \]  

(4.4.23)

we then obtain

\[ \int_{t_1}^{T} L_2(x(t), u_2(t)) \, dt \geq \int_{t_1}^{T} \left( \alpha (|x| + |u_2|) - \beta \right) \, dt \geq \frac{C^{-1}K - C}{2(1 + CT)} \alpha \delta^2 - \beta T \]

\[ > C_2 T \geq \int_{t_1}^{T} L_2(\eta(t), u_2^0(t)) \, dt, \]
proving our claim.

**Case 3:** \( x(t) > \eta(t) \) for \( t \in [t_1, T] \) with \( t_1 \geq T - \delta \) and \( x(t_1) = \eta(t_1) \).

Observe that it is not restrictive to assume that \( |x(t)| \leq \eta(t) \) for all \( t \in [0, t_1] \).

Otherwise, as proved in Case 1, the control \( u_2 \) for Player 2 would not be optimal.

In particular, this assumption implies \( x(T - \delta) = 0 \). Consider the alternative control function

\[
u_2^*(t) = \begin{cases} 
u_2(t) & \text{if } t \in [0, T - \delta], \\ 0 & \text{if } t \in [T - \delta, T]. \end{cases}\]

(4.4.24)

and let \( x^*(\cdot) \) be the corresponding trajectory. If \( \bar{\eta}/\delta > C \), for \( t \in [T - \delta, T] \) this solution is found by solving the Cauchy problem

\[
\dot{x}^*(t) = f_0(x^*(t)), \quad x^*(T - \delta) = 0.
\]

Since \( |f_0(x)| \leq C \) and \( C\delta < 1 \), this implies \( |x^*(t)| \leq 1 \) for \( t \in [T - \delta, T] \). By (4.4.6) we conclude

\[
\int_{T-\delta}^{T} L_2(x^*(t), u_2^*(t)) \, dt = \int_{T-\delta}^{T} L_2(x^*(t), 0) \, dt \leq C_1\delta. \tag{4.4.25}
\]

On the other hand, by (4.4.4) the original trajectory yields a cost

\[
\int_{T-\delta}^{T} L_2(x(t), u_2(t)) \, dt \geq \alpha \int_{T-\delta}^{t_1} |u_2(t)| \, dt + \alpha \int_{t_1}^{T} |x(t)| \, dt - \beta \delta. \tag{4.4.26}
\]

Recalling that \( x(T - \delta) = 0 \), we obtain

\[
x(t_1) = \eta(t_1) = \frac{t_1 + \delta - T}{\delta} \bar{\eta} = \int_{T-\delta}^{t_1} \left( f_0(x(t)) + f_2(x(t))u_2(t) \right) \, dt
\]

\[
\leq C(t_1 - T + \delta) + C \int_{T-\delta}^{t_1} |u_2(t)| \, dt,
\]

By (4.4.9), this yields

\[
\int_{T-\delta}^{t_1} |u_2(t)| \, dt \geq \frac{1}{C} \left( \frac{\bar{\eta}}{\delta} - C \right) (t_1 + \delta - T) \geq (t_1 + \delta - T) \bar{\eta}. \tag{4.4.27}
\]
Moreover,
\[
\int_{t_1}^{T} |x(t)| \, dt \geq \int_{t_1}^{T} \eta(t) \, dt = \int_{t_1}^{T} \frac{t + \delta - T}{\delta} \bar{\eta} \geq (T - t_1) \frac{\bar{\eta}}{2}.
\] (4.4.28)

Using (4.4.27)-(4.4.28) in (4.4.26) and recalling the choice of \( \bar{\eta} \) at (4.4.7) and (4.4.25), we obtain
\[
\int_{T-\delta}^{T} L_2(x(t), u_2(t)) \, dt \geq \alpha \left( (t_1 + \delta - T) + (T - t_1) \right) \frac{\bar{\eta}}{2} - \beta \delta \geq \alpha \delta \frac{\bar{\eta}}{2} - \beta \delta
\]
\[
> C_1 \delta \geq \int_{T-\delta}^{T} L_2(x^\sharp(t), u^\sharp(t)) \, dt.
\]

Once again, this shows that the control \( u_2 \) does not yield the minimum cost to Player 2.

The previous analysis has shown that, for every \( \varepsilon > 0 \) and every compact set \( I \) of initial states, the leading player can design a feedback achieving a cost \( \leq \varepsilon \) for every initial point \( \bar{x} \in I \). Clearly, there is no measurable feedback that can yield exactly zero cost. Hence in this setting no optimal feedback can exist.
A Bidding Game with Heterogeneous Players

A one-sided limit order book is modeled as a noncooperative game for \( n \) players. Agents offer various quantities of an asset at different prices \( p \in [0, \overline{P}] \), competing to fulfill an incoming order, whose size \( X \) is not known a priori. Players can have different payoff functions, reflecting different beliefs about the fundamental value of the asset and probability distribution of the random variable \( X \). In [10] the existence of a Nash equilibrium was established by means of a fixed point argument.

The main issue discussed in the this chapter is whether this equilibrium can be obtained from the unique solution to a two-point boundary value problem, for a suitable system of discontinuous ODEs. Some additional assumptions are introduced, which yield a positive answer. In particular, this is the case when there are exactly 2 players, or when all \( n \) players assign the same exponential probability distribution to the random variable \( X \). A counterexample shows that these assumptions cannot be removed.

5.1 Introduction

This chapter is concerned with a continuum model of the limit order book in a stock market, viewed as a noncooperative game for \( n \) players. As in [9] our main goal is to study the existence of a Nash equilibrium, determining the optimal bidding
strategies of the various agents who submit limit orders.

We assume that an external buyer asks for a random amount of $X > 0$ of shares of a certain asset. This external agent will buy the amount $X$ at the lowest available price, as long as this price does not exceed a given upper bound $\bar{P}$. One or more sellers offer various quantities of this asset at different prices, competing to fulfill the incoming order, whose size is not known a priori.

Having observed the prices asked by his competitors, each seller must determine an optimal strategy, maximizing his expected payoff. Of course, when other sellers are present, asking a higher price for a stock reduces the probability of selling it.

The model introduced in [9] was extended in [10], assuming that agents differ from each other in various respects.

- Each agent assigns a different probability distribution to the random variable $X$, based on his own beliefs. An optimistic seller expects a large incoming order, which will fill most of the outstanding bids. A pessimistic seller will expect a small order, filling only the lowest priced bids. In the following, we denote by
  \[ \psi_i(s) = \text{Prob.}\{X > s\} \]  
  the probability distribution assigned by the $i$-th player to the random variable $X$.

- Each agent assigns a different fundamental value $p_i$ to the assets he is putting on sale. In other words, to the $i$-th agent it would be indifferent to sell his assets at unit price $p_i$ or to keep them.

Existence of a Nash equilibrium, in this more general setting, was recently proved in [10] by means of a topological technique. However, this technique did not provide information about the uniqueness of the solution, or how to construct it. Aim of the present paper is to show that, in several cases, this Nash equilibrium solution can be found by solving a a Boundary Value Problem for a system of ODEs.

Let $\kappa_i$ be the total amount of assets offered for sale by the $i$-th agent. We use the Lagrangian variable $\beta \in [0, \kappa_i]$ to label one particular asset. By a **pricing strategy** for the $i$-th seller we mean a nondecreasing map $\phi_i : [0, \kappa_i] \mapsto [0, \bar{P}]$.

To compute the expected payoff achieved by this strategy, let $\phi_j$, $j = 1 \ldots, \phi_n$,
\( j \neq i \) be the pricing strategies adopted by the other agents, and define

\[
F_j(p) = \text{meas} \left( \{ \beta \in [0, \kappa_j]; \ \phi_j(\beta) < p \} \right). \tag{5.1.2}
\]

Moreover, let

\[
\Phi_i(p) = \sum_{j \neq i} F_j(p) \tag{5.1.3}
\]

be the total amount of assets put on sale at price \( p \) by all the other agents. The expected payoff for the \( i \)-th player is then measured by

\[
J_i(\phi_i, \Phi_i) = \int_0^{\kappa_i} \left( \phi_i(\beta) - p_i \right) \cdot \psi_i \left( \beta + \Phi_i(\phi_i(\beta)) \right) d\beta. \tag{5.1.4}
\]

The integrand in (5.1.4) contains two factors. The term \( \phi_i(\beta) - p_i \) is the difference between the price at which the asset \( \beta \) is put on sale and its actual value to the \( i \)-th player. The term \( \psi_i \left( \beta + \Phi_i(\phi_i(\beta)) \right) \) is the subjective probability (according to the \( i \)-th player) that the asset \( \beta \) will be actually sold.

For future reference, we record some basic assumptions on the probability distributions (5.1.1).

(A1) All maps \( s \mapsto \psi_i(s), \ i = 1, \ldots, n \) are continuously differentiable and satisfy

\[
\psi_i(0) = 1, \quad \psi_i(+\infty) = 0, \quad \psi_i'(s) < 0 \quad \text{for all} \ s > 0, \quad (5.1.5)
\]

\[
(\ln \psi_i(s))'' \geq 0 \quad \text{for all} \ s > 0. \quad (5.1.6)
\]

**Example 1.** The assumptions (A1) are satisfied if \( \psi_i(s) = e^{-\alpha s} \) or if \( \psi_i(s) = (1 + s)^{-\alpha} \), for some \( \alpha > 0 \). On the other hand, (5.1.6) fails if \( \psi_i(s) = e^{-s^2} \).

**Definition 1.** Let \( (\phi_1^*, \ldots, \phi_n^*) \) be an \( n \)-tuple of pricing strategies such that no two players put a positive amount of assets for sale at exactly the same price. We say that these strategies provide a **Nash equilibrium** if, calling \( \Phi_1^*, \ldots, \Phi_n^* \) the corresponding functions in (5.1.2)-(5.1.3), one has

\[
J_i(\phi_i, \Phi_i^*) \leq J_i(\phi_i^*, \Phi_i^*) \tag{5.1.7}
\]
for every $i = 1, \ldots, n$ and every other pricing strategy $\phi_i : [0, \kappa_i] \mapsto [0, \overline{p}]$ for the $i$-th player.

**Remark 1.** As a basic modeling assumption, the asset will always be bought from whoever seller offers the lowest price. However, if two or more sellers put a positive amount of asset for sale exactly at the same price, one needs to specify which of the agents has selling priority. This would require an additional discussion of the model. However, in a Nash equilibrium this situation never happens, because the player that does not have priority can always improve his expected payoff by slightly reducing his price.

In the general case where agents have different payoff functions, the existence of at least one Nash equilibrium was recently established in [10]. The proof relied on a sequence of discrete approximations, combined with a topological fixed point argument. Apart from the case of players with the same payoff functions, studied in [9], the uniqueness of Nash equilibria remains an open problem.

In the present paper we seek a more explicit way to construct the functions $(F_1, \ldots, F_n)$ in (5.1.2), and hence determine the equilibrium solution. Toward this goal, we introduce a boundary value problem for a a system on $n$ ODEs, determining the functions $F_j$. These equations are obtained by adding some auxiliary inequalities to the set of necessary conditions for optimality derived in [9]. Compared with classical literature, this problem is far from standard. It consists of a system of ODEs

$$F'_i(p) = Q_i(p, F_1(p), \ldots, F_n(p)), \quad i = 1, \ldots, n, \quad (5.1.8)$$

where the right hand sides are discontinuous along the hyperplanes where $F_j = \kappa_j$.

The boundary data take the form

$$F_i(\overline{p}) = \kappa_i \quad \text{for all } i \text{ except at most one}, \quad (5.1.9)$$

together with

$$F_i(p_A) = 0 \quad i = 1, \ldots, n, \quad (5.1.10)$$

where $p_A \in [0, \overline{p}]$ is a point to be determined.
In Section 2 we give a precise description of the right hand side of (5.1.8), and study its properties. In Section 3 we prove the existence and uniqueness of the solution to the boundary value problem (5.1.8)–(5.1.10). As shown by a counterexample, this solution may not yield a Nash equilibrium, in general. This is because the ODEs in (5.1.8) are obtained by imposing some additional inequalities which are not implied by the optimality conditions. Our main result, proved in Section 4, provides various sufficient conditions in order that the solution to (5.1.8)–(5.1.10) yield a Nash equilibrium. In particular, this is always the case if either (i) there are exactly two players, or (ii) in (5.1.4) the probability distribution functions have the form $\psi_1(s) = \cdots = \psi_n(s) = e^{-\lambda s}$, while $p_1, \ldots, p_n$ can be arbitrary.

In recent literature, models of the limit order book have been studied in [3, 20, 32, 34, 38]. For a general introduction to game theory and Nash equilibria we refer to [8, 22, 44].

5.2 An algebraic problem

The main goal of this section is to study a class of functions $Q_i$ appearing on the right hand side of the ODEs (5.1.8). As a first step, we consider a set of linear equations with constraints.

Lemma 1. Given $n \geq 2$ numbers

$$0 < a_1 \leq a_2 \leq \cdots \leq a_n,$$

(5.2.1)

there exists a unique $n$-tuple $(x_1, x_2, \ldots, x_n)$ of non-negative numbers with the following properties:

$$\sum_{j \neq i} x_j \leq a_i \quad i = 1, \ldots, n,$$

(5.2.2)

$$x_i > 0 \implies \sum_{j \neq i} x_j = a_i,$$

(5.2.3)

$$\sum_{i=1}^{n} x_i > 0.$$

(5.2.4)
Proof. 1. Consider the integer

\[ m = \min \left\{ k \in \{2, \ldots, n\} : \frac{1}{k-1} \sum_{j=1}^{k} a_j \leq a_{k+1} \right\}, \tag{5.2.5} \]

where \( a_{n+1} \equiv +\infty \), so that the inequality in (5.2.5) is always satisfied when \( k = n \). It is straightforward to check that all conditions (5.2.2)–(5.2.4) are satisfied by setting

\[ x_i = \frac{1}{m-1} \left( \sum_{1 \leq j \leq m, j \neq i} a_j - (m-2)a_i \right) \quad i = 1, \ldots, m, \tag{5.2.6} \]

\[ x_{m+1} = x_{m+2} = \cdots = x_n = 0. \tag{5.2.7} \]

2. It remains to prove that the solution is unique. As a first step, we claim that any solution \((x_1, \ldots, x_n)\) must be of the form (5.2.6)-(5.2.7), for some integer \( m \in \{2, \ldots, n\} \). Indeed, assume that \( x_\ell = 0 \) but \( x_k > 0 \) for some \( \ell < k \). Then (5.2.2) implies

\[ \sum_{j \neq k} x_j < \sum_{j=1}^{n} x_j \leq a_\ell \leq a_k. \]

Hence (5.2.3) is not satisfied when \( i = k \).

3. For a given \( m \), the formulas (5.2.6)-(5.2.7) uniquely determine the \( n \)-tuple \((x_1, \ldots, x_n)\). To complete the proof of uniqueness, it remains to show that there exists at most one integer \( m \) such that this \( n \)-tuple satisfies the conditions (5.2.2)–(5.2.4).

Assume, on the contrary, that \((x_1, \ldots, x_m, 0, \ldots, 0)\) and \((x'_1, \ldots, x'_{m'}, 0, \ldots, 0)\) are two distinct solutions, with

\[ 2 \leq m < m' \leq n, \quad x'_{m'} > 0. \]

Since \((x_1, \ldots, x_m, 0, \ldots, 0)\) is a solution, by (5.2.2) it follows

\[ \frac{1}{m-1} \sum_{j=1}^{m} a_j = \sum_{j=1}^{m} x_j \leq a_{m+1}. \tag{5.2.8} \]
This implies

\[ x'_{m'} = \frac{1}{m' - 1}\left(\sum_{j=1}^{m'-1} a_j - (m' - 2)a_{m'}\right) \]

\[ = \frac{1}{m' - 1}\left(\sum_{j=1}^{m} a_j + a_{m+1} + \sum_{j=m+2}^{m'-1} a_j - (m' - 2)a_{m'}\right) \]

\[ \leq \frac{1}{m' - 1}\left((m - 1)a_{m+1} + a_{m+1} + (m' - m - 2)a_{m'-1} - (m' - 2)a_{m'}\right) \]

\[ \leq \frac{1}{m' - 1}\left((m' - 2)a_{m'-1} - (m' - 2)a_{m'}\right) \leq 0. \]

This contradicts the assumption \( x'_{m'} > 0 \), proving that the solution is unique. \( \square \)

**Corollary 1.** Given any \( n \)-tuple \( a = (a_1, a_2, \ldots, a_n) \) of strictly positive numbers (not necessarily ordered as in (5.2.1)), there exists a unique vector \( x = (x_1, \ldots, x_n) \) satisfying the conditions (5.2.2)-(5.2.4).

Indeed, we can always find a permutation \( \pi \) of the set of indices \( \{1, \ldots, n\} \) such that

\[ 0 < a_{\pi(1)} \leq a_{\pi(2)} \leq \cdots \leq a_{\pi(n)}, \quad (5.2.9) \]

and apply Lemma 1. \( \square \)

Given \( a = (a_1, a_2, \ldots, a_n) \), we shall denote by

\[ G(a) = (G_1(a), \ldots, G_n(a)) = (x_1, \ldots, x_n) \quad (5.2.10) \]

the unique solution of (5.2.2)-(5.2.4). Observe that, if \( k, \ell \) are two indices such that \( a_k = a_\ell \), then by uniqueness it follows \( G_k(a) = G_\ell(a) \). The next lemma collects some properties of the map \( G \).

**Lemma 2.** The map \( G = (G_1, G_2, \ldots, G_n) \) is Lipschitz continuous and quasi-monotone. Namely, given two \( n \)-tuples \( a = (a_1, \ldots, a_n) \) and \( \tilde{a} = (\tilde{a}_1, \ldots, \tilde{a}_n) \), if

\[ a_i = \tilde{a}_i, \quad a_j \leq \tilde{a}_j \quad \text{for all} \quad j \neq i, \quad (5.2.11) \]
then \( G_i(a) \leq G_i(\bar{a}) \).

**Proof.** 1. By Corollary 1, the map \( G \) is well defined.

As shown by the proof of Lemma 1, for any given \( a \) there exists a unique subset of indices \( I(a) \subseteq \{1, \ldots, n\} \) with cardinality \( m = \#I(a) \) such that

\[
x_i = G_i(a) = \begin{cases} 
\frac{1}{m-1} \left( \sum_{j \in I(a), j \neq i} a_j - (m-2)a_i \right) & > 0 \text{ if } i \in I(a), \\
0 & \text{if } i \notin I(a).
\end{cases}
\]

This implies the a priori bounds

\[
0 \leq x_i = G_i(a) \leq \sum_j a_j.
\]  \hspace{1cm} (5.2.13)

2. From the equations (5.2.2)-(5.2.4) it follows that the map \( G \) has closed graph. Namely, given sequences \( a' = (a'_1, \ldots, a'_n) \), \( x' = (x'_1, \ldots, x'_n) \), \( \nu \geq 1 \) such that

\[
a' \rightarrow \bar{a}, \quad x' \rightarrow \bar{x}, \quad a' = G(x') \text{ for every } \nu \geq 1,
\]

it follows that \( \bar{x} = G(\bar{a}) \). Being a locally bounded function with closed graph, \( G \) is continuous.

3. The conclusion of the lemma is clearly a consequence of the following claim:

(C) Fix any \( i \in \{1, \ldots, n\} \) and let any \( (n-1) \)-tuple of numbers \( (a_1, \cdots, a_{i-1}, a_i+1, \ldots, a_n) \) be given. Then the maps

\[
s \mapsto G_j(a_1, \cdots, a_{i-1}, s, a_{i+1}, \cdots, a_n)
\]  \hspace{1cm} (5.2.14)

are Lipschitz continuous and monotone. Namely, \( G_i \) is decreasing, while all other \( G_j \) for \( j \neq i \) are increasing w.r.t. \( s \).

A proof of (C) will be worked out in the remaining steps.

4. Up to a permutation of indices, it is not restrictive to assume that \( i = n \) and
0 < a_1 \leq a_2 \leq \cdots \leq a_{n-1}.

Call \( a(s) = (a_1, \cdots, a_{n-1}, s) \). By the continuity of \( G \) it follows that the maps \( s \mapsto G_j(a(s)) \) are all continuous.

If \( ]s^-, s^+[ \) is an open interval where the set of indices \( I(a(s)) \) remains constant, from the formula (5.2.12) it follows that the maps \( G_j \) are all uniformly Lipschitz continuous, with \( s \mapsto G_i(a(s)) \) decreasing while \( s \mapsto G_j(a(s)) \) increasing for all \( j \neq i \).

5. To complete the proof of (C), it remains to show that, as \( s \) increases, there are finitely many values \( 0 < s_1 < s_2 < \cdots < s_N \) such that the set \( I(a(s)) \) is constant on each open interval \( ]s_{k-1}, s_k[ \). In turn, for any \( i \in \{1, \ldots, n-1\} \), it suffices to show that the set \( I(a(s)) \) changes only finitely many times as \( s \) ranges in the subinterval \( ]a_i, a_{i+1}[ \). This clearly follows as a consequence of (5.2.5).

Remark 2. According to (5.2.5), if

\[
\begin{align*}
    a_k \geq \sum_{j \neq k} a_j,
\end{align*}
\]

then \( x_k = 0 \). Hence, for \( i \neq k \), the values \( x_i = G_i(a_1, \ldots, a_n) \) provide the unique solution to the system of \( (n-1) \) equations

\[
\begin{align*}
    \sum_{j \neq i, k} x_j &\leq a_i & i \in \{1, \ldots, n\} \setminus \{k\}, \quad (5.2.15) \\
    x_i &> 0 \quad \Rightarrow \quad \sum_{j \neq i, k} x_j = a_i, \quad (5.2.16) \\
    \sum_{i \neq k} x_i &> 0. \quad (5.2.17)
\end{align*}
\]

obtained from the (5.2.2)–(5.2.4) by removing \( a_k \).

Lemma 3. Consider any \( n \)-tuple \( a = (a_1, \ldots, a_n) \) of strictly positive numbers, and let \( a' = (a_1, \ldots, a_{k-1}, a_{k+1}, \ldots, a_n) \) be the \((n-1)\)-tuple obtained by removing the entry \( a_k \). For \( i = 1, \ldots, n \), let \( x_i = G_i(a) \) be the solution of (5.2.2)–(5.2.4). Moreover, for \( i \neq k \) let \( x'_i = G'_i(a') \) be the solution of the corresponding system of
(n − 1) equations obtained by removing \(a_k\). Then

\[ x_j' \geq x_j \quad \text{for all } j \neq k. \]  \hspace{1cm} (5.2.18)

Moreover, if \(x_k > 0\), then there are at least two distinct indices \(k_1, k_2 \in \{1, \ldots, n\} \setminus \{k\}\) such that

\[ x_{k_1}' > x_{k_1}, \quad x_{k_2}' > x_{k_2}. \]  \hspace{1cm} (5.2.19)

**Proof.** 1. For each \(i \neq k\), as a consequence of Remark 2 we have

\[ x_i' - x_i = \int_{a_k}^{\infty} \frac{\partial}{\partial s} G_i(a_1, \ldots, a_{k-1}, s, a_{k+1}, \ldots, a_n) \, ds. \]  \hspace{1cm} (5.2.20)

Clearly the integrand is non-negative. Recalling (5.2.6) we see that

\[ \frac{\partial}{\partial s} G_i(a_1, \ldots, a_{k-1}, s, a_{k+1}, \ldots, a_n) = \begin{cases} \frac{1}{m(s) - 1} & \text{if } x_i(s) > 0 \text{ and } x_k(s) > 0, \\ 0 & \text{otherwise}. \end{cases} \]  \hspace{1cm} (5.2.21)

Here we use the notation \(x_j(s) = G_j(a_1, \ldots, a_{k-1}, s, a_{k+1}, \ldots, a_n)\), while \(m(s)\) denotes the number of non-zero components in this solution: \(m(s) = \#\{j ; \ x_j(s) \neq 0\}\). From the representation (5.2.20) it is clear that \(x_j' \geq x_j\) for all \(j \neq k\), so that (5.2.18) holds.

2. To prove (5.2.19), we observe that it is not restrictive to assume that the \(a_i\) are arranged in increasing order, as in (5.2.1). We consider two cases.

**Case 1:** \(k \geq 3\), so that \(m(a_k) \geq k \geq 3\). In this case we have \(x_1(a_k) > 0\), \(x_2(a_k) > 0\).

By continuity, there exists \(\varepsilon > 0\) such that \(x_1(s), x_2(s) > 0\) for all \(s \in [a_k, a_k + \varepsilon]\).

By (5.2.20)-(5.2.21), this implies

\[ x_1' - x_1 \geq \int_{a_k}^{a_k + \varepsilon} \frac{\partial}{\partial s} G_1(a_1, \ldots, a_{k-1}, s, a_{k+1}, \ldots, a_n) \, ds \geq \frac{\varepsilon}{n - 1} > 0. \]

The same estimate holds for \(x_2'\). The conclusion thus holds with \(k_1 = 1, k_2 = 2\).

**Case 2:** \(k \in \{1, 2\}\). To fix the ideas, assume \(k = 2\), the case \(k = 1\) being entirely
similar. We claim that, when \( s = a_3 \),

\[
G_i(a_1, s, a_3, a_4, \ldots, a_n) > 0 \quad \text{for } i \in \{1, 2, 3\}.
\]

Indeed, the inequality introduced in (5.2.5)

\[
\frac{1}{\ell - 1} \sum_{j=1}^{\ell} a_j \leq a_{\ell+1}
\]

here cannot be satisfied when \( \ell = 2 \). Hence \( m(s) \geq 3 \). By continuity, we can find \( \varepsilon > 0 \) such that \( m(s) \geq 3 \) for \( s \in [a_3 - \varepsilon, a_3] \). For \( i \in \{1, 3\} \) we now have

\[
|x'_i - x_i| = \int_{a_2}^{\infty} \frac{\partial}{\partial s} G_i(a_1, s, a_3, \ldots, a_n) \, ds
\]

\[
\geq \int_{a_3 - \varepsilon}^{a_3} \frac{\partial}{\partial s} G_i(a_1, s, a_3, \ldots, a_n) \, ds \geq \frac{\varepsilon}{n - 1} > 0.
\]

Choosing \( k_1 = 1, k_2 = 3 \), we reach the desired conclusion. \( \square \)

**Remark 3.** The previous analysis shows that the maps \( G_i(a_1, \ldots, a_n) \) can be defined also in the case where some of the \( a_i \) take the value \(+\infty\), provided that there exist at least two distinct indices \( j \neq k \) such that \( a_j, a_k < \infty \). Indeed, one can simply define \( G_i(a) = 0 \) if \( a_i = +\infty \).

For future use, we recall here a standard comparison lemma for solutions to systems of ODEs, originally proved in [25]. We recall that a map \( G = (G_1, \ldots, G_n) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called quasimonotone if for every \( i \in \{1, \ldots, n\} \) the following holds.

If \( y_i = \tilde{y}_i \) and \( y_j \leq \tilde{y}_j \) for all \( j \neq i \), then \( G_i(t, y_1, \ldots, y_n) \leq G_i(t, \tilde{y}_1, \ldots, \tilde{y}_n) \).

**Lemma 4.** Assume that the map \( G : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is Lipschitz continuous and quasimonotone. Let \( t \mapsto y(t) = (y_1, \ldots, y_n)(t) \) and \( t \mapsto \tilde{y}(t) = (\tilde{y}_1, \ldots, \tilde{y}_n)(t) \) be two solutions of the same systems of ODEs

\[
\dot{y} = G(t, y)
\]
Then the following comparison properties holds.

(i) If

\[ y_i(t_0) \leq \tilde{y}_i(t_0) \quad \text{for all} \quad i = 1, \ldots, n, \] (5.2.24)

then

\[ y_i(t) \leq \tilde{y}_i(t) \quad \text{for all} \quad i = 1, \ldots, n, \quad t \geq t_0. \] (5.2.25)

(ii) If in addition to (5.2.24) one has the strict inequality \( y_h(t_0) < \tilde{y}_h(t_0) \) for some index \( h \in \{1, \ldots, n\} \), then \( y_h(t) < \tilde{y}_h(t) \) for all \( t \geq t_0 \).

### 5.3 The two-point boundary value problem

Given an \( n \)-tuple of pricing strategies \((\phi_1, \ldots, \phi_n)\), consider the functions

\[ F_i(p) = \text{meas}\left( \{ \beta \in [0, \kappa_i] ; \ \phi(\beta) < p \} \right) \quad i \in \{1, \ldots, n\}, \] (5.3.1)

\[ F(p) = \sum_{j=1}^{n} F_j(p). \] (5.3.2)

If \((\phi_1, \ldots, \phi_n)\) provide a Nash equilibrium in the sense of Definition 1, then the analysis in [9] has shown that the following optimality conditions must hold:

For every \( i \in \{1, \ldots, n\} \), if \( F'_i(p) > 0 \) then

\[ \sum_{j \neq i} F'_j(p) = \frac{-\psi_i(F(p))}{(p - p_i) \psi_i'(F(p))}. \] (5.3.3)

For reader’s convenience, we briefly recall how (5.3.3) is obtained. For a given \( i \in \{1, \ldots, n\} \), if \( F'_i(p) > 0 \) this means that the \( i \)-th player is putting something on sale at price \( p \), hence \( \phi_i(\beta) = p \) for some \( \beta \in [0, \kappa_i] \). The optimality of the strategy \( \phi_i \) implies that, by making a small perturbation \( \phi_i(\beta) = p + h \), the expected payoff
does not increase. Therefore

\[ 0 = \frac{d}{dh} \left[ (p + h - p_i) \cdot \psi_i \left( \beta + \sum_{j \neq i} F_j(p + h) \right) \right]_{h=0} \]

\[ = \psi_i \left( \beta + \sum_{j \neq i} F_j(p) \right) + (p - p_i) \cdot \psi'_i \left( \beta + \sum_{j \neq i} F_j(p) \right) \cdot \sum_{j \neq i} F'_j(p) \]

\[ = \psi_i(F(p)) + (p - p_i) \cdot \psi'_i(F(p)) \cdot \sum_{j \neq i} F'_j(p). \]

This yields (5.3.3). For a rigorous derivation under general assumptions we refer to [9].

In the remainder of this section we shall construct functions \( F_1, \ldots, F_n \) which satisfy these optimality conditions. In the next section, under some additional assumptions, we will prove that these functions provide a Nash equilibrium solution to the bidding game.

Without loss of generality, we assume that

\[ 0 < p_1 \leq p_2 \leq \cdots \leq p_n < \bar{P}. \]

(5.3.4)

Given \( F(p) \) as in (5.3.2), define

\[ a_i(p, F(p)) = \begin{cases} \frac{-\psi_i(F(p))}{(p - p_i) \psi'_i(F(p))} & \text{if } p > p_i \text{ and } F_i(p) < \kappa_i, \\ +\infty & \text{if } p \leq p_i \text{ or } F_i(p) = \kappa_i. \end{cases} \]

(5.3.5)

Our main goal is to prove the existence of a unique solution to the following free boundary value problem.

\[ F'_i(p) = G_i \left( a_1(p, F(p)), \ldots, a_n(p, F(p)) \right) \quad p \in [p_A, \bar{P}], \quad i = 1, \ldots, n, \]

(5.3.6)

\[ F_1(p_A) = \cdots = F_n(p_A) = 0, \]

(5.3.7)

\[ F_i(\bar{P}) = \kappa_i \quad \text{for all } i \text{ with the exception of at most one index } i^*. \]

(5.3.8)
Here $G_1,\ldots,G_n$ are the functions introduced at (5.2.10). The lowest asking price $p_A \in [0,\overline{P}]$ is regarded as a free boundary. The index $i^*$ in (5.3.8), corresponding to the unique player that puts a positive amount of assets for sale at the top price $\overline{P}$, needs also to be determined as part of the solution.

**Remark 4.** The necessary conditions for optimality proved in [9] yield the implication
\[
F_i'(p) > 0 \implies \sum_{j \neq i} F_j'(p) = \frac{-\psi_i(F(p))}{(p - p_i)\psi'_i(F(p))}.
\]
These conditions alone do not uniquely determine a system of ODEs for the functions $F_i$. For example, at each $p$ one could choose any two indices $j,k$ and set
\[
F_j'(p) = \frac{-\psi_k(F(p))}{(p - p_k)\psi'_k(F(p))}, \quad F_k'(p) = \frac{-\psi_j(F(p))}{(p - p_j)\psi'_j(F(p))}, \quad F_i'(p) = 0 \quad \text{for } i \notin \{j,k\}.
\]

Applying Lemma 1 with $a_i$ defined at (5.3.5), we can uniquely determine the values $x_i = F_i'(p)$, provided that we impose the additional inequalities
\[
\sum_{j \neq i} F_j'(p) \leq \frac{-\psi_i(F(p))}{(p - p_i)\psi'_i(F(p))}, \quad i = 1,\ldots,n, \quad (5.3.9)
\]
corresponding to (5.2.2). However, one should keep in mind that these additional inequalities do not follow from the optimality conditions. There may be Nash equilibria that do not satisfy (5.3.9), while the unique solution of the boundary value problem (5.3.6)–(5.3.8) may not yield a Nash equilibrium. This issue will be discussed in detail in Section 4.

**Theorem 1.** For $i = 1,\ldots,n$, let the quantities $\kappa_i > 0$ and the prices $p_i$ as in (5.3.4) be given, together with functions $\psi_i$ satisfying in (5.1.5)-(5.1.6). Then the boundary value problem (5.3.6)–(5.3.8) has a unique solution.

**Proof.** Because of (5.3.5), the right hand sides of the ODEs in (5.3.6) are piecewise smooth, with discontinuities occurring when $F_i = \kappa_i$. Our construction will thus be achieved by an inductive algorithm, which restarts every time where the solution reaches a discontinuity.

INITIAL STEP. Consider an initial point $q_0 \in [p_2,\overline{P}]$, whose precise value will be
determined later. We begin by solving the system of ODEs (5.3.6) with initial data (5.3.7). Since $q_0 > p_2$, we have $a_1(q_0, 0) < \infty$, $a_2(q_0, 0) < \infty$. Hence the right hand sides of the ODEs in (5.3.6) are well defined and locally Lipschitz continuous. By the analysis in Section 2, this Cauchy problem has a unique local solution defined for $p \geq q_0$. This solution can be continued up to the point

$$q_1 = \mathcal{P} \wedge \min \{ p > q_0 ; \ F_i(p) = \kappa_i \ \text{for some index } \ i \}.$$  \hspace{1cm} (5.3.10)

Here and in the sequel we use the notation $a \wedge b = \min \{a, b\}$.

INDUCTIVE STEP. Now assume that $q_0 < q_1 < \cdots < q_\nu$ have been determined, and the solution has been constructed on the interval $[q_0, q_\nu]$. If either (i) $q_\nu = \mathcal{P}$ or (ii) the set of indices $\{ i ; \ F_i(q_\nu) < \kappa_i , \ q_\nu > p_i \}$ contains less than two elements, then the construction stops.

In the opposite case, we consider the set of indices

$$I_\nu = \{ i ; \ F_i(q_\nu) < \kappa_i \}.$$  \hspace{1cm} (5.3.11)

The equations (5.3.5)-(5.3.6) now yield a system of $N_\nu = \# I_\nu$ (i.e., the cardinality of the set $I_\nu$) differential equations for the components $F_i, \ i \in I_\nu$. In addition,

$$F'_i(p) = 0 , \quad F_i(p) = \kappa_i \quad \text{for all } i \notin I_\nu.$$  \hspace{1cm} (5.3.12)

This Cauchy problem, with initial data provided by the inductive step at $p = q_\nu$, has a unique local solution, defined for $p \geq q_\nu$. This solution can be continued up to the point

$$q_{\nu+1} = \mathcal{P} \wedge \min \{ p > q_0 ; \ F_i(p) = \kappa_i \ \text{for some index } \ i \in I_\nu \}.$$  \hspace{1cm} (5.3.13)

This achieves the inductive step.

Clearly the algorithm must terminate after at most $n - 2$ steps, yielding a unique solution to (5.3.6)-(5.3.7), defined on some maximal interval $[q_0, q^*]$. In the remainder of the proof we will show that there exists a unique value for the minimum ask price $p_A$ such that, setting $q_0 = p_A$, one has $q^* = \mathcal{P}$ and (5.3.8) holds. More
precisely, writing \( q^* = q^*(q_0) \) to stress the dependence of \( q^* \) on the initial point \( q_0 \), one has

\[
p_A \doteq \inf \{ q_0 \in [0, \overline{P}] : q^*(q_0) = \overline{P} \}. \tag{5.3.14}
\]

2. In this step we prove that, for any solution of the boundary value problem (5.3.6)–(5.3.8) which is defined on the entire interval \([p_A, \overline{P}]\), all functions \( F_i \) are Lipschitz continuous with a uniform Lipschitz constant (independent of \( p_A \)). Toward this goal, choose constants \( m_0, m_1 \) such that

\[
0 < m_0 \leq \frac{-\psi_i(s)}{\psi_i'(s)} \leq m_1 \quad \text{for all} \quad i = 1, \ldots, n, \quad 0 \leq s \leq K \doteq \sum_{j=1}^{n} \kappa_j. \tag{5.3.15}
\]

Introducing the constant

\[
\delta \doteq (\overline{P} - p_n) \cdot \exp \left\{ -\frac{2Km_1}{m_0^2} \right\}, \tag{5.3.16}
\]

we claim that

\[
F'_i(p) = 0 \quad \text{for all} \quad i = 1, \ldots, n, \quad p < p_i + \delta. \tag{5.3.17}
\]

Notice that, if (5.3.17) holds, then (5.3.3) implies

\[
F'_j(p) \leq \max_i \max_{p \geq p_i + \delta} \frac{-\psi_i(F(p))}{(p - p_i)\psi'(F(p))} \leq \frac{m_1}{\delta}. \tag{5.3.18}
\]

This provides the uniform upper bound on the Lipschitz constant of all functions \( F_j, j = 1 \ldots, n \).

In the remainder of this step we thus work toward a proof of (5.3.17). To fix the ideas, fix an index \( i \) and assume

\[
p_{j-1} < p_j = p_{j+1} = \cdots = p_i = \cdots = p_k < p_{k+1}.
\]

for some indices \( j \leq i \leq k \). Assume that

\[
F'_i(p_i + \varepsilon) = G_i\left( a_1(p_i + \varepsilon), \ldots, a_n(p_i + \varepsilon) \right) > 0. \tag{5.3.19}
\]
In this case, as shown by the proof of Lemma 1, the sum of the two smallest elements in the \( n \)-tuple \( (a_1, \ldots, a_n) \) must be greater than \( a_i \). Hence,

\[
\min_2 \left( a_1(p_i + \varepsilon), \ldots, a_n(p_i + \varepsilon) \right) > \frac{a_i(p_i + \varepsilon)}{2} = \frac{-\psi_i(F(p))}{2\varepsilon \psi'_i(F(p))} \geq \frac{m_0}{2\varepsilon}. 
\] (5.3.20)

Here and in the sequel we use the notation \( \min_2(a_1, \ldots, a_n) \) to denote the second smallest element of the \( n \)-tuple \( (a_1, \ldots, a_n) \).

Next, assume that

\[
a_j(p_i + \varepsilon) > \frac{m_0}{2\varepsilon} \quad (5.3.21)
\]

for some index \( j \). For \( p > p_i + \varepsilon \), if \( a_j(p) < \infty \) then

\[
\frac{m_1}{p - p_j} \geq a_j(p) = \frac{-\psi_j(F(p))}{(p - p_j) \psi'_j(F(p))} \geq \frac{m_0}{p - p_j}. 
\] (5.3.22)

To estimate the right hand side of (5.3.22) we consider two cases.

If \( p_j \geq p_i \) then

\[
a_j(p) \geq \frac{m_0}{p - p_j} \geq \frac{m_0}{p - p_i}. 
\] (5.3.23)

On the other hand, if \( p_j < p_i \) then the assumption (5.3.21) implies

\[
\frac{m_1}{p_i + \varepsilon - p_j} \geq a_j(p_i + \varepsilon) > \frac{m_0}{2\varepsilon}.
\]

Therefore, for \( p \geq p_i + \varepsilon \) we have

\[
\frac{p - p_i}{p - p_j} \geq \frac{\varepsilon}{p_i + \varepsilon - p_j} \geq \frac{m_0}{2m_1},
\]

and

\[
a_j(p) \geq \frac{m_0}{p - p_i} \cdot \frac{p - p_i}{p - p_j} \geq \frac{m_0^2}{2m_1(p - p_i)}. 
\] (5.3.24)

From (5.3.20), (5.3.23), and (5.3.24) we conclude

\[
\min_2 \left( a_1(p), \ldots, a_n(p) \right) \geq \frac{m_0^2}{2m_1(p - p_i)} \quad \text{for all} \quad p > p_i + \varepsilon. 
\] (5.3.25)

Observing that

\[
\sum_{i=1}^{n} F_i'(p) \geq \min_2 \left( a_1(p), \ldots, a_n(p) \right)
\]
and setting $\delta_0 = \bar{P} - p_n$, from (5.3.25) we deduce

$$K \geq \int_{p_i+\varepsilon}^{\bar{P}} \sum_{i=1}^{n} F_i'(p) \, dp \geq \int_{p_i+\varepsilon}^{p_i+\delta_0} \sum_{i=1}^{n} F_i'(p) \, dp \geq \int_{p_i+\varepsilon}^{p_i+\delta_0} \frac{m_0^2}{2m_1(p-p_i)} \, dp = \frac{m_0^2}{2m_1} \ln \left( \frac{\delta_0}{\varepsilon} \right).$$

Hence

$$\varepsilon \geq \delta_0 \cdot \exp \left\{ -\frac{2Km_1}{m_0^2} \right\},$$

proving our claim (5.3.26).

3. By choosing $q_0 = \bar{P} - \varepsilon$ with $\varepsilon > 0$ sufficiently small, it is clear that the solution of (5.3.6)-(5.3.7) is well defined on $[q_0, \bar{P}]$ and satisfies $F_i(\bar{P}) < \kappa_i$ for every $i$. Hence the set on the right hand side of (5.3.14) is nonempty and the value $p_A$ is well defined.

We claim that, if the minimum asking price $p_A$ is defined as in (5.3.14), then solution of the Cauchy problem (5.3.6)-(5.3.7) satisfies the terminal condition (5.3.8) as well.

Indeed, consider a decreasing sequence of initial points $q_\nu \to p_A$, with $q^\#(q_\nu) = \bar{P}$ for each $\nu \geq 1$. Let $(F_{1,\nu}, \ldots, F_{n,\nu})$ be the corresponding solution to (5.3.6) with initial data

$$F_{1,\nu}(q_\nu) = \cdots = F_{n,\nu}(q_\nu) = 0,$$

defined on the interval $[p_\nu, \bar{P}]$. By the uniform Lipschitz continuity of the functions $F_{i,\nu}$, proved in step 2, we can extract a subsequence converging to an $n$-tuple of Lipschitz functions $(F_1, \ldots, F_n)$. It is straightforward to check that these functions provide a solution to the Cauchy problem (5.3.6)-(5.3.7) on the interval $[p_A, \bar{P}]$. Hence $q^\#(p_A) = \bar{P}$ and the infimum in (5.3.14) is actually attained as a minimum.

To prove that this solution $(F_1, \ldots, F_n)$ also satisfies the terminal condition (5.3.8), we assume that, on the contrary, $F_j(\bar{P}) < \kappa_j$ and $F_\ell(\bar{P}) < \kappa_\ell$ for two distinct indices $j \neq \ell$. We can then find a constant $\varepsilon > 0$ such that

$$F_j(\bar{P}) < \kappa_j - \varepsilon, \quad F_\ell(\bar{P}) < \kappa_\ell - \varepsilon.$$
Consider a strictly increasing sequence of initial points \( q_{0,\nu} \rightarrow p_A \). For each \( \nu \geq 1 \) let \( F_\nu = (F_{1,\nu}, \ldots, F_{n,\nu}) \) be the corresponding solution to the Cauchy problem (5.3.6) with initial data (5.3.27). By assumption, this solution is defined on some maximal interval \([q_{0,\nu}, q^*_\nu] \) with \( q^*_\nu < \bar{p} \). By quasi-monotonicity, the sequence of solutions is monotone decreasing. More precisely, for any two indices \( \mu < \nu \) and any \( i \in \{1, \ldots, n\} \) we have

\[
F_{i,\nu}(p) \leq F_{i,\mu}(p) \quad \text{for all } p \in [q_{\mu}, q^*(q_{\mu})] \cap [q_{\nu}, q^*(q_{\nu})].
\]

We now observe that the pointwise limit \( \tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_n) \), defined as

\[
\tilde{F}_i(p) = \inf_{\nu \geq 1} F_{i,\nu}(p) \quad p_A \leq p < \sup_{\nu \geq 1} q^*(q_{\nu}) \quad (5.3.29)
\]

provides a solution to the Cauchy problem (5.3.6)-(5.3.7). By uniqueness, \( \tilde{F} = F \) and hence \( q^*(q_{\nu}) \rightarrow \bar{p} \) as \( \nu \rightarrow \infty \).

A contradiction is now obtained as follows. By the analysis in step 2, the derivatives \( F'_{i,\nu} \) remain uniformly bounded. In particular, we can assume

\[
F'_{j,\nu}(p) \leq M, \quad F'_{\ell,\nu}(p) \leq M \quad (5.3.30)
\]

for some constant \( M \) and all \( \nu \geq 1 \). Choose \( \delta > 0 \) such that \( M\delta < \varepsilon/2 \). By (5.3.29) and (5.3.28), for \( \nu \) sufficiently large we have

\[
F_{j,\nu}(\bar{p} - \delta) < F_j(\bar{p} - \delta) + \frac{\varepsilon}{2} \leq \kappa_j - \frac{\varepsilon}{2}, \quad F_{\ell,\nu}(\bar{p} - \delta) \leq F_\ell(\bar{p} - \delta) + \frac{\varepsilon}{2} \leq \kappa_\ell - \frac{\varepsilon}{2}.
\]

By (5.3.30), this implies

\[
F_{j,\nu}(p) < \kappa_j, \quad F_{\ell,\nu}(p) < \kappa_\ell \quad \text{for all } p \in [\bar{p} - \delta, \bar{p}].
\]

Therefore, \( q^*(q_{\nu}) = \bar{p} \), the minimality of \( p_A \). This contradiction proves our claim, i.e. the terminal condition (5.3.8) is satisfied.

**4.** It now remains to prove that the solution to the boundary value problem (5.3.6)-(5.3.8) is unique. Clearly, as soon as the initial point \( p_A \) is chosen, the solution to the Cauchy problem (5.3.6)-(5.3.7) is unique. Consider two starting
points \( p_A < p_\tilde{A} \) and let \( F, \tilde{F} \) be the solutions to the corresponding problems

\[
\begin{aligned}
F_i'(p) &= G_i\left(a_1(p,F(p)), \ldots, a_n(p,F(p))\right) & p \in [p_A, \bar{P}], & i = 1, \ldots, n, \\
\tilde{F}_i'(p) &= G_i\left(a_1(p,\tilde{F}(p)), \ldots, a_n(p,\tilde{F}(p))\right) & p \in [p_{\tilde{A}}, \bar{P}], & i = 1, \ldots, n,
\end{aligned}
\]

(5.3.31)

\[
\begin{aligned}
F_1(p) &= \cdots = F_n(p) = 0 & p \in [0, p_A], \\
\tilde{F}_1(p) &= \cdots = \tilde{F}_n(p) = 0 & p \in [0, p_{\tilde{A}}],
\end{aligned}
\]

(5.3.32)

To prove uniqueness for the boundary value problem it suffices to show:

(C) For every \( p \in [p_A, \bar{P}] \) one has

\[
\tilde{F}_i(p) \leq F_i(p) \quad \text{for all } i \in \{1, \ldots, n\}.
\]

(5.3.33)

Moreover, there exist at least two indices \( j, k \) (possibly varying with \( p \)), such that

\[
\tilde{F}_j(p) < F_j(p), \quad \tilde{F}_k(p) < F_k(p).
\]

(5.3.34)

Indeed, if \( F = (F_1, \ldots, F_n) \) is a solution to the boundary value problem (5.3.6)–(5.3.8), then \( \tilde{F} \) cannot be a second solution, because for some indices \( j, k \)

\[
\tilde{F}_j(\bar{P}) < F_j(\bar{P}) \leq \kappa_j, \quad \tilde{F}_k(\bar{P}) < F_k(\bar{P}) \leq \kappa_k,
\]

hence the terminal condition (5.3.8) fails. The uniqueness of the solution thus follows from our claim (C).

To prove (C), we proceed by induction. For \( i = 1, \ldots, n \), define

\[
P_i \doteq \min \left\{ p \geq p_\tilde{A}; \ F_i(p) = \kappa_i \right\},
\]

(5.3.35)

\[
\tilde{P}_i \doteq \min \left\{ p \geq p_\tilde{A}; \ \tilde{F}_i(p) = \kappa_i \right\}.
\]
Rearranging these values in increasing order, we can write
\[ \{p_{\tilde{A}}, P_1, \ldots, P_n, \tilde{P}_1, \ldots, \tilde{P}_n\} = \{\tau_0, \tau_1, \ldots, \tau_N\}, \]
with
\[ p_{\tilde{A}} = \tau_0 < \tau_1 < \cdots < \tau_N = P. \]
We compare the solutions of the two Cauchy problems (5.3.31), (5.3.32). The inequalities in (5.3.33) can be proved by induction on \( \ell = 1, \ldots, N \). Indeed, they trivially hold when \( p = p_{\tilde{A}} \). Assuming that (5.3.33) holds for \( p = \tau_\ell \), by (i) in Lemma 4 and the quasi-monotonicity of the right hand sides of (5.3.31)-(5.3.32) we conclude that the same inequalities are true for \( p \in [\tau_\ell, \tau_{\ell+1}] \).

The strict inequalities in (5.3.34) will also be proved by induction on the intervals \([\tau_\ell, \tau_{\ell+1}]\).

INITIAL STEP. For every \( \tau \in [p_A, p_{\tilde{A}}] \), by Lemma 1 there are two indices \( j \neq k \) (possibly depending on \( \tau \)) such that
\[ F'_j(\tau) > 0, \quad F'_k(\tau) > 0. \]
For every \( p \in ]p_A, p_{\tilde{A}}[ \), integrating over the interval \([p_A, p]\) we conclude that there are at least two indices \( j \neq k \) such that
\[ F_j(p) > 0 = \tilde{F}_j(p), \quad F_k(p) > 0 = \tilde{F}_k(p). \]  
(5.3.36)

INDUCTIVE STEP. Assume that the inequality in (5.3.34) has been proved for all \( p \in ]p_A, \tau_\ell[, \) for some \( 0 \leq \ell < N \). We show that it remains valid on the interval \([\tau_\ell, \tau_{\ell+1}]\) as well.

For any \( p \), define the sets of indices
\[ L(p) \doteq \{i; \quad F_i(p) < \kappa_i\} = \{i; \quad p < P_i\}. \]
To achieve the inductive step, consider any \( \bar{p} \in ]\tau_{\ell-1}, \tau_\ell[. \) By the inductive hypot-
esis, there exists two indices $j \neq k$ such that

$$\tilde{F}_j(\bar{p}) < F_j(\bar{p}), \quad \tilde{F}_k(\bar{p}) < F_k(\bar{p}).$$

(5.3.37)

Two cases will be considered.

**CASE 1:** $\tau_\ell \notin \{\tilde{P}_j, \tilde{P}_k\}$. Observe that this implies

$$\tilde{F}_j(p) < \kappa_j, \quad \tilde{F}_k(p) < \kappa_k \quad \text{for all} \quad p < \tau_{\ell+1}.$$

In this case, using part (ii) of Lemma 4 we first conclude that the inequalities (5.3.37) hold for all $p \in [\bar{p}, \tau_\ell]$. A second application of Lemma 4 shows that the same strict inequalities hold also for $p \in [\tau_\ell, \tau_{\ell+1}]$.

**CASE 2:** $\tau_\ell \in \{\tilde{P}_j, \tilde{P}_k\}$. To fix the ideas, assume $\tau_\ell = \tilde{P}_j \doteq \min\{p; \tilde{F}_j(p) = \kappa_j\}$. Observe that in this case we must have $F_j(p) = \kappa_j$ for all $p \geq \tau_{\ell-1}$. Otherwise the relations

$$\tilde{F}_j(\bar{p}) < F_j(\bar{p}), \quad \tilde{F}_j(\tau_\ell) = F_j(\tau_\ell)$$

would provide a contradiction with part (ii) of Lemma 4. We are thus in the situation shown in Fig. 5.1.

**Figure 5.1.** At $p = \tau_\ell = \tilde{P}_j$ the functions $F_j$ and $\tilde{F}_j$ become equal. However, during the previous interval $[\tau_{\ell-1}, \tau_\ell]$ there are at least two other indices $h, h'$ such that $F_h' > \tilde{F}_h'$ and $F_{h'} > \tilde{F}_{h'}'$ on a set with positive measure.
We claim that there exists at least two distinct indices \( h, h' \in \{1, \ldots, n\} \) such that
\[
\tilde{F}_h(\tau_\ell) < F_h(\tau_\ell), \quad \tilde{F}_{h'}(\tau_\ell) < F_{h'}(\tau_\ell).
\] (5.3.38)

Suppose on the contrary that (5.3.38) fails. Then there exists an index \( k^* \) such that
\[
\tilde{F}_i(\tau_\ell) = F_i(\tau_\ell) \quad \text{for all } i \neq k^*.
\] (5.3.39)

To achieve a contradiction, observe that (5.3.39) implies
\[
\tilde{F}_i(p) = F_i(p) \quad \text{for all } p \in [\bar{p}, \tau_\ell], \quad i \in L(p), \quad i \neq k^*.
\] (5.3.40)

Since \( \tilde{F}_j(\bar{p}) < \tilde{F}_j(\tau_\ell) = \kappa_j \), there is a subset \( S \subset [\bar{p}, \tau_\ell] \) of positive measure such that \( \tilde{F}'_j(p) > 0 \) for every \( p \in S \). By (5.2.19) in Lemma 3, for every \( p \in S \) we can find at least one index \( h \in L(p), \ h \neq k^* \) (possibly depending on \( p \)), such that
\[
F'_h(p) > \tilde{F}'_h(p).
\] (5.3.41)

This is clearly in contradiction with (5.3.40). We thus conclude that (5.3.38) holds.

If now \( \tau_\ell = \bar{p} \), we are done. Otherwise, using again part (ii) of Lemma 4 we conclude that
\[
F_h(p) > \tilde{F}_h(p), \quad F_{h'}(p) > \tilde{F}_{h'}(p) \quad \text{for all } p \in [\tau_\ell, \tau_{\ell+1}].
\]

This completes the inductive step in the proof of our claim (C).

\[\square\]

### 5.4 Computing the Nash equilibrium

Let \((F_1, \ldots, F_n)\) be a solution of the boundary value problem (5.3.6)–(5.3.8). These functions determine a unique \( n \)-tuple of bidding strategies. Namely, for every \( p \leq \bar{p} \), the value \( F_i(p) \) determines the total amount of assets put on sale by the \( i \)-th player at price \( < p \). By (5.3.8) there can be at most one player, say the agent \( i^* \), who puts a positive amount of assets for sale exactly at the price \( \bar{p} \).

However, as explained in Remark 4, this solution to the boundary value problem does not necessarily yield a Nash equilibrium. We further illustrate this point by
an example.

Example 2. Consider a bidding game for three sellers. We assume

\[ p_1 = 1, \quad p_2 = p_3 = 4, \quad \psi_1(s) = e^{-s}, \quad \psi_2(s) = \psi_3(s) = e^{-4s}, \quad (5.4.1) \]

The values \( 0 < \kappa_1 < \kappa_2 = \kappa_3 \) and \( \overline{P} \) will be chosen later, so that the minimum ask price will turn out to be

\[ p_A = 5. \quad (5.4.2) \]

Let \((F_1, F_2, F_3)\) be the solution to the BVP \((5.3.6)\)–\((5.3.8)\) constructed in Theorem 1. Since Players 2 and 3 have the same payoff function, by uniqueness we have \(F_2(p) = F_3(p)\) for all \(p\). In a small interval of the form \([p_A, p_A + \delta]\), by \((5.3.3)\) it follows

\[
\begin{cases}
2F'_2(p) = \frac{1}{p - p_1} = G_1(p), \\
F'_1(p) + F'_2(p) = \frac{1}{4(p - p_2)} = G_2(p) = G_3(p).
\end{cases}
\]

Therefore

\[
F'_1(p) = \frac{1}{4(p - p_2)} - \frac{1}{2(p - p_1)}, \quad F'_2(p) = F'_3(p) = \frac{1}{2(p - p_1)}. \quad (5.4.4)
\]

We choose \( \kappa_1 = \varepsilon > 0 \) sufficiently small. Observe that, when \( p \approx p_A = 5 \), by \((5.4.1)\) the right hand sides of \((5.4.4)\) take the values \( F'_i(p) \approx 1/8 \). Therefore, we can uniquely determine the value

\[
q(\varepsilon) \doteq \min \{p > p_A; \quad F_1(p) = \varepsilon\} = 5 + 8\varepsilon + o(\varepsilon). \quad (5.4.5)
\]

For \( p > q(\varepsilon) \) one has

\[
F_1(p) \equiv \varepsilon, \quad F'_2(p) = F'_3(p) = \frac{1}{4(p - 4)}. \quad (5.4.6)
\]
Given $\varepsilon > 0$ sufficiently small and $\bar{P} >> 5$ we set

$$
\kappa_1 = \varepsilon, \quad \kappa_2 = \kappa_3 = \int_5^{q_1} \frac{dp}{2(p-1)} + \int_{q_1}^{\bar{P}} \frac{dp}{4(p-4)}.
$$

(5.4.7)

Notice that, (5.4.5) and (5.4.7) together imply

$$
\kappa_2 = \kappa_3 = \frac{1}{4} \ln(\bar{P} - 4) + O(\varepsilon).
$$

(5.4.8)

With the above choice of $\kappa_i$, the triple $(F_1, F_2, F_3)$ provides the unique solution to the boundary value problem.

We claim that, if $\bar{P}$ is sufficiently large, the above solution is not a Nash equilibrium, because the strategy of Player 1 is not optimal. Indeed, when Player 1 puts on sale a total amount $\kappa_1 = \varepsilon$ of assets at price $p \approx p_A = 5$, for $\varepsilon > 0$ small his expected payoff is

$$
J_1(\varepsilon) = 4\varepsilon + o(\varepsilon).
$$

(5.4.9)

On the other hand, if he puts all his assets for sale at the top price $\bar{P}$, his expected payoff is

$$
J_1^\dagger(\varepsilon) = (\bar{P} - 1) \int_0^{\varepsilon} \exp\{-\kappa_1 - \kappa_2 - s\} \, ds
$$

$$
= (\bar{P} - 1) e^{-2\kappa_1} \cdot \varepsilon + o(\varepsilon) = \frac{\bar{P} - 1}{\sqrt{\bar{P} - 4}} \cdot \varepsilon + o(\varepsilon).
$$

(5.4.10)

When $\bar{P}$ sufficiently large we have $J_1^\dagger(\varepsilon) > J_1(\varepsilon)$, hence the first strategy is not optimal.

In the remainder of this section we seek additional conditions, which guarantee that the $n$-tuple of strategies $(F_1, \ldots, F_n)$ obtained by solving the boundary value problem (5.3.6)–(5.3.8) provides a Nash equilibrium to the bidding game. Toward this goal, we shall use

**Lemma 5.** A sufficient condition in order that the strategy $\phi_i$ for the $i$-th player...
be optimal is

\[(\phi_i(\beta) - p_i)\psi_i\left(\beta + \Phi_i(\phi_i(\beta))\right) = \max_{p \in [0, P]} \left\{(p - p_i)\psi_i\left(\beta + \Phi_i(p)\right)\right\} \text{ for a.e. } \beta \in [0, \kappa_i].\]

(5.4.11)

**Proof.** Intuitively, the above statement should be clear. If \(\phi_i\) yields the maximum expected payoff from the sale of each single asset \(\beta \in [0, \kappa_i]\), then \(\phi_i(\cdot)\) is optimal.

To prove the lemma, for any admissible strategy \(\varphi : [0, \kappa_i] \mapsto [0, P]\) we simply observe that

\[
J(\varphi, \Phi_i) = \int_0^{\kappa_i} (\varphi(\beta) - p_i)\psi_i\left(\beta + \Phi_i(\varphi(\beta))\right) d\beta \leq \int_0^{\kappa_i} \max_{p \in [0, P]} \left\{(p - p_i)\psi_i\left(\beta + \Phi_i(p)\right)\right\} d\beta = \int_0^{\kappa_i} (\phi_i(\beta) - p_i)\psi_i\left(\beta + \Phi_i(\phi_i(\beta))\right) d\beta = J(\phi_i, \Phi_i).
\]

\[\square\]

To proceed further, we need to introduce an additional condition:

**(H)** For every \(i \in \{1, \ldots, n\}\) and every \(\tilde{p} < P\), the following implication holds.

If \(G_i(a_1(\tilde{p}), \ldots, a_i(\tilde{p}), \ldots a_n(\tilde{p})) > 0\) for some \(\tilde{p} \in [p_i, P]\), then for every \(p \in [\tilde{p}, P]\) one has

\[
G_i\left(a_1(p), \ldots, a_{i-1}(p), \frac{-\psi_i(F(p))}{(p - p_i)\psi_i'(F(p))} \cdot a_{i+1}(p), \ldots, a_n(p)\right) > 0. \quad (5.4.12)
\]

Roughly speaking, the assumption (H) means that, if \(i\)-th player puts some asset for sale at price \(\tilde{p}\), then he continues to put assets for sale at every price \(p \in [\tilde{p}, q_i]\), for some \(q_i\) such that \(F_i(q_i) = \kappa_i\). The only reason for which he does not put assets for sale at prices \(p > q_i\) is that he simply does not have anything more to sell. In
this case, recalling (5.3.5), we have

\[ a_i(p) = \begin{cases} 
-\psi_i(F(p)) & \text{if } p < q_i, \\
(p - p_i) \psi'_i(F(p)) & \text{if } p > q_i,
\end{cases} \quad (5.4.13) \]

**Lemma 6.** Under the same assumptions as in Theorem 1, let \( p \mapsto (F_1, \ldots, F_n)(p) \) be a solution to the boundary value problem (5.3.6)–(5.3.8). If the condition (H) holds, then the corresponding pricing strategies yield a Nash equilibrium to the bidding game with payoffs (5.1.3)-(5.1.4).

**Proof.** 1. For every \( i \in \{1, \ldots, n\} \) we need to show that the strategy \( \phi_i \) of the \( i \)-th player is a best reply to the strategies adopted by all other players. By Lemma 5, this is the case if (5.4.11) holds. Calling

\[ E(\beta) = \frac{d}{dp} \left[ (p-p_i) \psi_i(\beta + \Phi_i(p)) \right] = \psi_i(\beta + \Phi_i(p)) + (p-p_i) \psi'_i(\beta + \Phi_i(p)) \Phi'_i(p), \quad (5.4.14) \]

our conclusion will be reached by proving that

\[ \begin{cases} 
E(p) \geq 0 & \text{if } p < \phi(\beta), \\
E(p) \leq 0 & \text{if } p > \phi(\beta). 
\end{cases} \quad (5.4.15) \]

2. When \( p < \phi_i(\beta) \), since \( F_i(p) \leq \beta < \kappa_i \) and the equations (5.3.6) are satisfied, recalling (5.2.2) we have

\[ \Phi'_i(p) = \sum_{j \neq i} F'_j(p) \leq a_i(p) = \frac{-\psi_i(F(p))}{(p - p_i) \psi'_i(F(p))}. \quad (5.4.16) \]

Inserting (5.4.16) in (5.4.14) and recalling that \( \psi'_i < 0 \), for \( p < \phi_i(\beta) \) we thus
Indeed, \( \beta + \Phi_i(p) \geq F(p) \). Moreover, by the assumption (5.1.6) the map \( s \mapsto \psi_i(s)/\psi'_i(s) \) is non-increasing.

3. Next, assume \( p > \phi_i(\beta) \). Using the assumption (H) with \( \bar{p} = \phi_i(\beta) \), we obtain

\[
\Phi'_i(p) \geq \frac{-\psi_i(F(p))}{(p-p_i)\psi'_i(F(p))},
\]

with equality holding if \( F_i(p) < \kappa_i \). A similar computation as in (5.4.17) now yields

\[
E(p) \leq \psi_i(\beta + \Phi_i(p)) + (p-p_i) \cdot \psi'_i(\beta + \Phi_i(p)) \cdot \frac{-\psi_i(F(p))}{(p-p_i)\psi'_i(F(p))}
\]

\[
= \psi'_i(\beta + \Phi_i(p)) \cdot \left[ \frac{\psi_i(\beta + \Phi_i(p))}{\psi'_i(\beta + \Phi_i(p))} - \frac{\psi_i(F(p))}{\psi'_i(F(p))} \right] \leq 0.
\]

Indeed, in this second case we have \( \beta + \Phi_i(p) \leq F(p) \).

This establishes the second inequality in (5.4.15), completing the proof.

Using Lemma 6, we can give a number of sufficient conditions in order that the pricing strategies constructed in Theorem 1 provide a Nash equilibrium.

**Theorem 2.** Under the same assumptions as in Theorem 1, let \( p \mapsto (F_1, \ldots, F_n)(p) \) be a solution to the boundary value problem (5.3.6)–(5.3.8). These strategies provide a Nash equilibrium to the bidding game if any one of the following assumptions holds.

(i) \( n = 2 \).
(ii) For every $i, j \in \{1, \ldots, n\}$ and $\max\{p_i, p_j\} < p < P$, one has

$$
\frac{d}{dp} \left| \ln \frac{-\psi_i(F(p))}{(p - p_i) \psi'_i(F(p))} - \ln \frac{-\psi_j(F(p))}{(p - p_j) \psi'_j(F(p))} \right| \leq 0. \quad (5.4.20)
$$

(iii) $\psi_1(s) = \cdots = \psi_n(s) = e^{-\lambda s}$ for some $\lambda > 0$.

**Proof.** 1. In the case of two players, let $(F_1, F_2)$ be the solution to the boundary value problem (5.3.6)-(5.3.8). Then

$$
F'_1(p) = \frac{-\psi_2(F(p))}{(p - p_2) \psi'_2(F(p))} = a_2(p) > 0, \quad F'_2(p) = \frac{-\psi_1(F(p))}{(p - p_1) \psi'_1(F(p))} = a_1(p) > 0,
$$

for all $p \in ]p_A, P[$. Hence the assumption (H) in Lemma 6 trivially holds. Part (i) of the theorem is thus a consequence of Lemma 6.

2. To prove part (ii), let the inequality (5.4.20) hold. To understand the basic case, for $i = 1, \ldots, n$ define

$$
\alpha_i(p) \doteq \begin{cases} 
-\frac{\psi_i(F(p))}{(p - p_i) \psi'_i(F(p))} & \text{if } p > p_i, \\
+\infty & \text{if } p \leq p_i.
\end{cases} \quad (5.4.22)
$$

After a permutation of indices, we can assume that

$$
\alpha_1(p) \leq \cdots \leq \alpha_n(p) \quad \text{for all } p. \quad (5.4.23)
$$

Indeed, by (5.4.20), if $\alpha_i(\hat{p}) = \alpha_j(\hat{p}) < \infty$ then $\alpha_i(p) = \alpha_j(p) < \infty$ for all $p \geq \hat{p}$. Hence the difference $\alpha_i(p) - \alpha_j(p)$ can never change sign. Note that (5.4.23) implies

$$
p_1 \leq \cdots \leq p_n. \quad (5.4.24)
$$

Otherwise, if $i < j$ but $p_i > p_j$, then

$$
\lim_{p \to p_+} a_j(p) = \frac{-\psi_j(F(p_i))}{(p_i - p_j) \psi'_j(F(p_i))} < \lim_{p \to p_+} a_i(p) = +\infty,
$$
in contradiction with (5.4.23).

Next, assuming that
\[ G_i(\alpha_1(\tilde{p}), \ldots, \alpha_n(\tilde{p})) > 0 \] (5.4.25)
for some $\tilde{p}$, we claim that
\[ G_i(\alpha_1(p), \ldots, \alpha_n(p)) > 0 \quad \text{for all } p \in [\tilde{p}, P]. \] (5.4.26)

Indeed, as shown in the proof of Lemma 1, one has $G_i(\alpha_1(p), \ldots, \alpha_n(p)) > 0$ if and only if
\[ \frac{1}{k - 1} \sum_{j=1}^{k} \alpha_j(p) > \alpha_{k+1}(p) \quad k = 2, \ldots, i - 1. \] (5.4.27)

Equivalently, this holds if and only if
\[ \frac{1}{k - 1} \sum_{j=1}^{k} \frac{\alpha_j(p)}{\alpha_{k+1}(p)} > 1 \quad k = 2, \ldots, i - 1. \] (5.4.27)

By (5.4.23) and (5.4.20) it follows that, for $j \leq k + 1,$
\[ \ln \left( \frac{\alpha_j(p)}{\alpha_{k+1}(p)} \right) \leq 0, \quad \frac{d}{dp} \left( \frac{\alpha_j(p)}{\alpha_{k+1}(p)} \right) \geq 0. \]

Hence, if (5.4.27) holds for $p = \tilde{p}$, the same holds for all $p \geq \tilde{p}$.

3. The argument in the previous step shows that, if $a_i(p) = \alpha_i(p)$ for all $i = 1, \ldots, n$, then the assumption (H) in Lemma 6 is satisfied. By applying Lemma 6 we conclude that the solution $(F_1, \ldots, F_n)$ of the boundary value problem yields a Nash equilibrium.

To complete the proof of (ii), we need to consider the case where the $a_i(p)$ defined at (5.4.13) do not necessarily coincide with the $\alpha_i(p)$ in (5.4.22). This happens precisely when $p > p_i$, $F_i(p) = \kappa_i$ and $a_i(p) = \infty$.

Assume that, at some point $\tilde{p}$, one has
\[ G_i\left( a_1(\tilde{p}), \ldots, a_i(\tilde{p}), \ldots, a_n(\tilde{p}) \right) > 0. \] (5.4.28)
Clearly this implies \( a_i(\tilde{p}) = \alpha_i(\tilde{p}) < \infty \). Notice that, if \( j < i \) and \( a_j(\tilde{p}) = \infty \), then \( a_j(p) = \infty \) for all \( p \geq \tilde{p} \). For notational convenience, for any \( p \geq \tilde{p} \) define

\[
b_j(p) = \begin{cases} 
\infty & \text{if } j < i \text{ and } a_j(\tilde{p}) = \infty, \\
\alpha_j(p) & \text{otherwise.}
\end{cases}
\]

For \( p \geq \tilde{p} \) we then have

\[
G_i\left(a_1(p), \ldots, a_{i-1}(p), a_i(p), a_{i+1}(p), \ldots, a_n(p)\right) \\
\geq G_i\left(b_1(p), \ldots, b_{i-1}(p), a_i(p), b_{i+1}(p), \ldots, b_n(p)\right) > 0. \tag{5.4.29}
\]

Indeed, the first inequality is a consequence of the quasi-monotonicity of the maps \( G_i \), proved in Lemma 2. The second inequality is obtained from (5.4.28), using the arguments in step 2, after discarding the components \( j < i \) for which \( a_j(\tilde{p}) = \infty \).

Having proved that the assumption (H) holds, by an application of Lemma 6 we conclude that the solution \((F_1, \ldots, F_n)\) to the boundary value problem yields a Nash equilibrium.

4. To prove part (iii) of the theorem we show that, if \( \psi_1(s) = \cdots = \psi_n(s) = e^{-\lambda s} \), then (5.4.20) holds. Indeed, to fix the ideas assume \( p_i \geq p_j \). For \( p > p_i \) we then have

\[
\frac{d}{dp} \left| \ln \frac{1}{\lambda(p-p_i)} - \ln \frac{1}{\lambda(p-p_j)} \right| = \frac{d}{dp} \left[ \ln(p-p_j) - \ln(p-p_i) \right] = \frac{1}{p-p_j} - \frac{1}{p-p_i} \leq 0.
\]

Therefore (iii) follows as a special case of (ii).

Remark 5 (uniqueness). As proved in Theorem 1, the solution to the two-point boundary value problem (5.3.6)–(5.3.8) is always unique. Under the additional assumption (H), this solution yields a Nash equilibrium. However, this does not necessarily imply that the Nash equilibrium is unique. Indeed, in principle there may be other Nash equilibria which are not obtained by solving our system of ODEs.

In the case \( n = 2 \), the uniqueness of the Nash equilibrium follows easily from the necessary conditions (5.4.21). We conjecture that uniqueness also holds under the assumption (5.4.20) in Theorem 2. It remains an open problem to understand if
there can be multiple Nash equilibria, in cases where the assumptions (i)–(iii) in Theorem 2 fail.


[29] M. Jungers, E. Trélat, and H. Abou-Kandil, Min-max and min-min Stackel-


[32] A. Obizhaeva, and J. Wang, Optimal trading strategy and supply/demand

[33] G. P. Papavassilopoulos and J. B. Cruz, Nonclassical control problems and
Stackelberg games, *IEEE Transactions on Automatic Control*, **24** (1979), 155-
166.

[34] T. Preis, S. Golke, W. Paul, and J. J. Schneider, Multi-agent-based order book

[35] S. Predoiu, G. Shaikhet, and S. Shreve, Optimal execution in a general one-
183-212.


[37] R. W. Rishel, An extended Pontryagin principle for control systems whose

Studies* **22** (2009), 4601-4641.


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