PNEUMATIC TRANSPORT MODELING OF AIR/GAS DRILLING HYDRAULICS IN HORIZONTAL AND DEVIATED WELLBORES

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by
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Abstract

Optimal design of any drilling program requires a strong knowledge of drilling hydraulics. Field drilling technicians and drilling engineers designing air/gas drilling projects both typically need to know (1) the time it takes to circulate out a pill of formation material at any given depth in order to reduce drilling time (thereby reducing the cost of drilling); and (2) the optimum rate at which to flow the circulation fluid to ensure the hole is maintained free of cuttings at the minimum capacity (and thereby cost) of surface equipments. While there have been substantial studies and some established techniques for wellbore hydraulics design/analysis for conventional mud drilling, this is not the case for air/gas drilling.

To this end, a transient numerical pneumatic transport model has been developed to study air/gas drilling hydraulics. This model is based on a mono-dispersed particulate two-fluid model (TFM). The mathematical representation of the problem involves the solution to coupled non-linear hyperbolic PDEs and therefore a detailed review of monotone and higher resolution schemes for hyperbolic systems is presented. Two hyperbolic model equations: (1) The benchmark Euler equation (2) The mono-dispersed particulate Two Fluid Model (TFM) equation, are presented along with analytical expressions for their decomposed characteristics. The Euler equation is used to demonstrate the efficacy of the numerical schemes discussed. The most efficient numerical scheme is then applied to solve the TFM equation. Numerical results compared to experimental data were satisfactory.

The experimental work of Temple was used to evaluate the model for annular vertical wellbore air drilling flow conditions. Flow characteristic curves (FCC) that cover both dilute and dense phase flow conditions were generated. Simulation results showed a good match with experiments. While matching the data points in the dilute phase flow region was less precise towards the extreme end of the larger particles, the optimum flow velocity was still accurately predicted at 9.1 %AAD.

Unlike vertical flow, air drilling experiments for horizontal flow pneumatic
transport are difficult to come by. This applies to industry field data as well as they are usually incomplete and typically labeled as confidential information. Therefore horizontal pneumatic transport studies for food processing applications were used to evaluate the performance of the current model for horizontal and highly-deviated well systems. The area ratio correlations were incorporated into the TFM to account for the lower-side deposition and re-entrainment phenomenon peculiar to horizontal wells. While we were able to show a match in the FCC plots in the dilute flow region, it was difficult to establish a match for the dense phase region of the FCC plot. Predicted pressure drops in dense flow region of the FCC plot for horizontal systems were significantly lower than experiment. This was due to the inability of the current model formulation to account for slug flow regime typical of horizontal pneumatic transport systems transitioning from dilute to dense flow conditions.

Overall, it was observed that the model is capable of predicting volumetric rate requirements and circulation time for hole cleaning in vertical wells. The model is also capable of estimating cutting circulation time in horizontal and deviated well sections while in dilute flow conditions only. However, the model was unable to accurately simulate the inherent transient behaviors expected under dense horizontal pneumatic transport. More studies need to be conducted in the area of constitutive relations specific to horizontal pneumatic transport flow regimes and empirical relations for the prediction of critical velocities. These will provide the capability of upgrading the current model for the prediction of fluid volumetric rate requirements in horizontal/highly-deviated well sections particularly under dense flow conditions.

The transient model was also used to predict circulation time for dispersed slugs of cuttings from the bottom of a well with both vertical and horizontal sections. The model results can only be trusted to predict accurately when the flow in the horizontal section is maintained within the dilute flow region. In order to ensure that the flow in the horizontal section is maintained above flow condition, we developed correlations for optimum circulation velocity using three distinct data sets for both vertical and horizontal pneumatic transport systems in the literature. Our results showed a relative error less than 10% for most data points and RMSRD less that 5.1%.
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List of Symbols

$d$ or $D$ diameter (m)

$f$ friction factor

$g$ Acceleration due to gravity

$p$ Pressure (Pa or atms)

$v$ velocity (m/s)

$C_d$ drag coefficient

$F$ Force per unit volume

$G$ Modulus of elasticity (Pa)

$H$ Conduit height (m)

$M_g$ Molecular mass of the gas phase ($M_{air} = 28 : 97 \frac{kg}{kmol}$)

$R$ Universal Gas constant ($8.314E3 \frac{J}{kmol.K}$)

$Re$ Fluid Reynolds number

$Re_r$ Relative (or Particle) Reynolds number

$T$ Temperature (K)

$W_s$ Solids mass rate $\frac{kg}{s}$ (or $\frac{g}{s}$)
0.1 Greek Letters

\( \beta \)  Inter-phase friction coefficient

\( \varepsilon \)  Void fraction

\( \epsilon \)  Roughness factor

\( \rho \)  Density \( (\frac{kg}{kmol}) \)

\( \gamma \)  Adiabatic index (or Heat capacity ratio, \( \gamma = \frac{c_p}{c_v} \))

\( \mu \)  Gas viscosity (Pa.s)

0.2 Subscripts

\( d \)  fluid-particle drag effect on gas phase

\( g \)  gas phase

\( p \) or \( s \)  particulate (or solids) phase

\( r \)  relative (or solids)

\( 0 \)  grid cell/block no. 0 (boundary block)

\( gg \)  gravitational effect on gas phase

\( gs \)  gravitational effect on particulate phase

\( nc \)  number of grid cells

\( wg \)  wall friction effect on gas phase

\( ws \)  wall friction effect on particulate phase

0.3 Superscripts

\( n \)  Time step \( n \)

\( k \)  Iteration level

\( o \)  Initial value
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Dedication

This dissertation is dedicated to Africans of good faith around the world.
Introduction

1.1 Underbalanced Drilling

Underbalanced drilling involves drilling through potentially producing formations with annular bottom-hole pressures below formation pore-pressure. Under this condition, there is the likelihood of an influx of formation fluids into the wellbore. Hence, specialized pressure-management equipments such as a rotary head, which provides a pressure seal, is used to divert produced hydrocarbon fluids away from the rig floor during the drilling process. This is unlike conventional drilling procedures where wellbore pressure is maintained above formation pressure and the invasion of drilling fluids into the formation is expected.

![Figure 1.1. Underbalanced drilling versus conventional](PetroWiki)
Over 30% of the world’s onshore drilling operations are carried out underbalanced (Lyons [1]). Most of these are in areas where formation pressures are relatively low and/or the risk of lost circulation is relatively high. The major advantage of underbalanced drilling is the reduction of drilling time as a result of the higher rate of penetration (ROP) achieved due to lower hydrostatic head. The increase in ROP ultimately translates to significant reduction in the cost of drilling. Other advantages of underbalanced drilling methods include:

- increased bit life;
- minimal damage to sensitive pay zones;
- instantaneous and continuous detection of hydrocarbon fluid bearing zones and water ingress while drilling;
- better control of lost circulation-prone formations
- reduced probability of differential sticking (so long as water influx is minimal);
- easier well deviation control where low weight on bit is necessary;
- and cleaner cores for lab tests.

Underbalanced drilling techniques fall into four categories based on the choice of drilling fluid. These include (1) air/gas, (2) mist, (3) foam (surfactant) and (4) aerated mud. Of these four, aerated mud drilling is the closest to conventional drilling. Air or Nitrogen is injected into the drilling mud in an effort to reduce the effective mud weight (density) of the fluid column and thereby control bottom hole pressure. A well known disadvantage of this approach is mainly corrosion of downhole tools due to the reaction with the injection gas. In foam drilling a stiff foam is formed from a mixture of surfactants, water and air. The stiff foam then serves as the circulation fluid as it is pumped down into the wellbore annulus through the drill-pipe. The major advantage of foam drilling is its significant cutting carrying capacity and reduced optimum annular velocity requirement. While surfactant drilling reducing surface compression equipment costs, it requires large mud pits to be built in order to ensure enough settling time for circulated foam before
re-injection. Additionally, water influx from the formation can potentially breakdown the foam and reduce it’s cutting carrying capacity.

Mist drilling is similar to air/gas drilling with the exception that some amount of liquids are injected in the flowing gas stream as droplets. Mist drilling is known to reduce ROP relative to air drilling as it increases bottom hole pressure. While mist drilling reduces the risk of downhole fires without the need for Nitrogen, it is the least recommended form of underbalanced drilling due to the possibility of differential sticking, which can lead to the loss of the BHA. Air/gas drilling is the oldest, most economical and the easiest to implement of all four underbalanced drilling categories. It is also the most prevalent and the technique of interest in this study. The circulation fluid may be any gaseous fluid such as compressed air, Nitrogen, Carbon-dioxide and so on. For fear of repetition, we would refer to air/gas drilling as air drilling going forward.

Fluid circulation path when drilling with air is somewhat similar to that of conventional drilling fluids. Compressed gas is circulated from the compressor to the kelly or swivel, down through the drill string and out into the wellbore annulus through the bit nozzles. The compressed gas then serves primarily as a circulation fluid for transporting crushed formation materials (or drill-cuttings) up and out of the wellbore annulus during the drilling process as shown in Fig. 1.2. Drill-cuttings are separated from the gas by cyclone separators as they arrive at the surface or simply allowed to drop into cuttings pits. Additional tools such as the downhole hammer (DHH), a fluid-activated percussive drilling bit, may also be used to improve ROP and hole alignment accuracy when drilling with gas. Furthermore, bit nozzle sizes may be selected to suit down-hole conditions based on compressor capacity at the surface and gas velocity requirements downhole.

Despite the advantages of gas drilling, it is prudent to avoid air drilling through geological formations under the following conditions:

- high formation pressures - due to the risk of a blow out;
- high water influx - due to the inherent difficulty of cuttings transport in the presence of excess water;
- and severe sloughing - due to the likely-hood of hole instability.

On the other hand, excellent candidates for air drilling operations include:
Figure 1.2. Vertical well during air drilling operations

- tight sand and shale formations (like the Marcellus shale) - due to ultra-low permeability;
- hard rock formations with low liquids ingress - due to reduced fear of hole instability in the absence of an over-balanced mud column;
- formations with severe lost-circulation problems - due to reduced fear of fluid ingress into the formation;
- and reactive/sensitive formations - due to the reduced risk of precipitation or sloughing.

There is no single unique bit type, bit nozzle size, drill pipe size, downhole BHA
architecture, surface compressor capacity combination. All of these are dependent on availability and drilling engineers design limits and discretion. Therefore, we will make no emphasis on these individual components of the wellbore and BHA in this study. Instead our focus is on the flow conditions that will ensure efficient hole cleaning. The task of the wellbore and BHA design is beyond the scope of this study.

1.2 Hole Cleaning

Drilling fluids serve many purposes which include:

- circulation of excavated formation materials (or drill cuttings) out of the wellbore;
- maintaining the desired hydrostatic head within the wellbore;
- cooling, cleaning and lubricating the bit;
- and transmitting hydraulic power to the bottom-hole assembly (BHA).

However, the most important of these is its role as a circulation fluid for suspending and transporting formation material (or cuttings) out of the wellbore. Hole cleaning is the most important aspect of the entire drilling process. Inefficient hole cleaning is detrimental to rate of penetration (ROP) but could also quickly lead to pack-offs, stuck-pipe, excessive pump pressure, lost circulation and eventually the loss of a well. Severe accumulation of solids (or “pack-off”) can lead to non-productive time (NPT) on the rig and from an economic standpoint, considering the high daily capital cost (or burn rate) of rig leases, an increase in NPT significantly cuts into the profitability of a well. Consequently, hole cleaning is one of the most studied phenomena in the petroleum industry.

The optimum volumetric rate requirement for transporting cuttings along proposed/designed well-paths must be estimated accurately prior to the commencement of drilling operations. Fluid volumetric flow rate information helps with decision making and logistics for the procurement of surface equipments capable of handling drilling projects ahead of drilling operations. This process helps to ensure efficient hole-cleaning while drilling and prevents the over- or under-design of
necessary surface equipments/facilities. An under-design of rig surface equipments may lead to inefficiencies like:

- cuttings accumulation and prohibitive cuttings bed growth;
- unnecessary regrinding of excavated cuttings;
- slugging;
- and ultimately choking if enough accumulation of cuttings occur.

An over-design reduces the economic advantage of using air as hydraulic fluid among other inauspicious consequences like:

- excessive drill-string vibrations;
- hole washouts;
- and bit ice-balling.

Down-hole vibrations have been known to cause severe failures of the drilling assembly. They are also known to cause irreparable damage to down-hole measurement equipment on the bottom hole assembly (BHA).

The prevention of these problems lies in the optimization of fluid volumetric rates. Optimizing the volumetric rate is about striking a balance between maintaining minimum annular pressure drop while simultaneously maintaining dilute flow conditions everywhere possible along the wellbore. Fig. 1.3 shows the typical curve of annular pressure drop as a function of air velocity at a fixed solid mass feed rate also known as the flow characteristics curve (FCC) or Zenz plot. In this study we will refer to it as a FCC plot. Between points AB, fluid-wall viscous/frictional resistance contribution to the pressure drop in annulus predominates. In this section, annular pressure drop is directly proportional to gas velocity. Therefore, as gas velocity decreases, pressure drop decreases and vice-versa. Region (AB) is also known as the dilute flow region. The flow characteristics behavior of the fluid mixture while within region AB is similar to the behavior of single phase gas flow. If wellbore gas velocity is maintained within region AB, drill-cuttings will continue to exit the wellbore without significant accumulation. However, if gas velocity within the wellbore is set below a critical velocity (point B), solid accumulation
within the wellbore begins to influence pressure drop. At annular air velocities below point B, there exists an inverse relationship between annular gas velocity and annular pressure drop. Therefore, as gas velocity falls lower below the critical value (point B), the rate of solid accumulation increases and so does pressure drop.

At point A in Fig. 1.3, the effect of cloud particle drag and weight on pressure drop are minimal while the contribution of frictional resistance to pressure drop due to the conduit wall boundaries dominates. As superficial gas velocity is reduced, the role of the dominating forces are switched. Eventually the pressure drop due to head loss and cloud particle drag becomes dominant over pressure drop due to wall friction. This occurs as a transition into region BC occurs within which the inverse relationship between gas velocity and pressure drop exists in the annulus. Region BC is known as the dense flow region.

Annular fluid velocity must always be maintained above the choking velocity (Point C) as shown in Fig. 1.3 to ensure that cuttings remain fluidized. As
accumulation persists in region BC, a well may experience slug flow followed by choking as air velocity is further reduced. Slugging is usually characterized by severe oscillations in annular pressure drop and can therefore be identified and used as a key performance indicator (KPI) for real-time drilling optimization and the prevention of the choking phenomena.

It is advantageous to keep the pressure drop along the wellbore to a minimum even while trying to maintain dilute flow conditions in order to save on capital cost of surface equipments. However, in order to maintain optimal hole-cleaning efficiency, superficial gas velocity must be maintained above critical gas velocity.
Figure 1.5. Flow characteristics curve at constant solid volume flux and varying particle density [Adewumi and Tian, 1992]

Figure 1.6. Flow characteristics curve at varying rate of penetration [Adewumi and Tian, 1992]
along the wellbore at all times. In essence, the optimal annular air velocity is slightly to the right of the minimum annular pressure drop (or critical air velocity) at point B.

Fig. 1.4 shows the various multiphase flow conditions expected across the vertical flow characteristics curve with varying solid mass influx from single phase gas flow \( W_{s0} \) to the higher solid influx rates \( W_{s1} < W_{s2} < W_{s3} \). The pressure drop at \( W_{s0} \) is primarily due to frictional losses with a minute contribution from hydrostatic head. However, when solid particles are introduced into the pipeline at a constant feed rate, we observe an additional rise in pressure due to fluid-particle drag, particle wall interaction (solids friction) and an additional hydrostatic head due to the mass of solids. Between line AB at \( W_{s1} \), the effects of frictional resistance to flow predominates the pressure drop. However between line BC hydrostatic head and fluid-particle drag predominates. As illustrated in Fig. 1.4, the flow characteristics curves shift upward and to the right as solid mass influx rate \( W_s \) increases.

Fig. 1.5 shows the shift in the flow characteristics curve for a constant solid volumetric rate while varying particle densities. An increase in solid density, which corresponds to an increase in mass flux, results in a shift in the flow characteristics curve in the upper-right direction. A similar phenomenon is shown in Fig. 1.6 with varying rate of penetration (ROP), which also translates to varying solid mass influx.

The dense flow region (negative slope) of the flow characteristics curves for vertical flow in Figs. 1.3 and 1.4 is inherently unstable due to slugging. On the other hand, pneumatic/dilute region (positive slope) of the curve is more stable and shows relatively lower changes in annular pressure drop for equivalent changes in gas velocity. Therefore, for drilling purposes, any gas velocity below point B is undesirable.

### 1.3 Horizontal and Deviated/Directional Wells

Directional drilling is the deviation of a wellbore along a designed well path to a target sub-surface location at a known lateral and vertical distance from the well pad at the drilling site. Even horizontal and deviated wells begin with a
vertical section, which is usually drilled underbalanced. Typically, heavier drilling fluids are introduced into the wellbore prior to an angle build-up from the vertical section to the desired inclination in deviated/horizontal wells in order to maintain hydrostatic pressure above formation pore pressure, which will in turn keep the formation fluids in place. However, natural gas discoveries in tight sand and shale formations are proving to be excellent avenues to drill the entire length of a wellbore underbalanced using air. This is due to the low permeabilities present in these geological formations within which the natural gas is trapped and is released only after these formations are hydraulically fractured. There are well known natural gas companies operating in the Appalachian region (Mercurius, Devonian and Utica shales) that have recognized this fact and are currently taking advantage of the possibility of drilling to target depth by air in the development of their tight sand and shale assets more efficiently. Some of the benefits include significant savings in drilling operational cost and an increase in ultimate recovery from each well [2].
Unlike vertical wells (see Fig. 1.2), it is next to impossible to maintain homogeneous flow conditions when drilling lateral well sections by air. As is illustrated in Fig. 1.7, flow segregation tends to occur as heavier formation materials settle to the lower side of the wellbore. This therefore creates inhomogeneous flow whereby both dilute and dense phase flow conditions exists side by side along the wellbore.

Saltation velocity is defined as the gas velocity (in a horizontal pipe run) at which the particles of a homogeneous flow of solids begins to fall out of the gas stream. This critical velocity is sometimes used as a basis for choosing the design gas velocity in horizontal pneumatic conveying systems. While some formation materials may be suspended and carried along in the upper side of the wellbore, larger diameter cuttings accumulate along the lower side of the wellbore when gas velocity is below saltation velocity.

Fig. 1.8 shows flow conditions encountered when superficial gas velocity ($U_g$) is reduced at constant solid mass flow rate ($W_s$). A similar behavior is observed when $W_s$ is increased under constant $U_g$. The flow conditions are defined as follows:

a. Fully dispersed homogeneous particulate phase flow

d. Surging flow (saltation and re-entrainment begins)

c. Suspended flow with stationary bed (a stable solids bed is present)

d. Slug flow (accompanied by pressure fluctuations)

e. Packed flow or choking.

Flow conditions b and d are inherently unstable while a, c and possibly e are stable. The generic characteristics flow curves in Fig. 1.9 indicates the regions where each flow condition (a-e) is likely to form. At velocities greater than point a, it is expected that solids transport will be fully suspended and homogeneous. Between region a-b, surging flow is likely to develop. This is where saltation and re-entrainment begins prior to the formation of a stable solids bed. Between b-c, a stationary bed that reduces the flow area and permits high enough velocity to maintain suspended solids flow above it is formed. Between c-d, slug flow begins to occur along with severe pressure surges. Between d-e, dense phase packed flow may occur for the case of fine particles flowing within a relatively large conduit.
Figure 1.8. Flow regimes and transitions in horizontal flow

Figure 1.9. Generic flow characteristic curves
diameter. However, choking may also occur between d-e for the case of large (or coarse) particles in a relatively small conduit diameter.

The transition from dilute to dense phase flow conditions are usually much sharper in horizontal flow relative to vertical flow. Fig. 1.10 a & b are characteristics curves for both vertical and horizontal flow for the same conduit and solids.

![Characteristic curves for both vertical and horizontal flows](image)

**Figure 1.10.** Characteristic curves for both vertical and horizontal flows

Over the years, a number of scientists have made efforts to model vertical transport of solids in wellbores. They include Martin [3], [4], Angel [5], Gray [6], [7], McCray and Cole [8], B. Guo and R. Lee. [9], W. Kexiong et al. [10], Schoeppl and Sapre [11], Ikoku et al. [12], Machado and Ikoku [13], Puon and Ameri [14], Adewumi and Arastoopour [15], Tian and Adewumi [16], [17], Adewumi and Tian [18], [19], [20] and Zhu et al. [21]. However, despite the extensive body of studies published in this area, cuttings transport is still one of the least understood areas of study in the oil and gas industry. This is partially due to the fact that most of these studies look at cuttings transport as a steady state phenomena. However, it is well known that the circulation process is anything but steady. Typically drillers stop to circulate out pills of formation material intermittently during the drilling
process, which is transient in nature.

Furthermore, due to the conventional approach of only using gas drilling techniques for vertical well sections, there have been very limited studies on air drilling in horizontal well sections. However, as is usually the case in drilling, these techniques are already in application in field operations despite the limited understanding of the process involved. While this technique is not recommended for most wells, it is important that we develop an understanding of horizontal pneumatic transport of cuttings during the drilling process in order to optimize the circulation rates in horizontal well sections drilled with gas where applicable.

1.4 Modeling Horizontal/Deviated Well Systems

The ability to design efficient and effective hole cleaning operations while drilling with air as circulation fluid requires the capability to predict flow conditions at critical points along the wellbore. Additionally, the ability to reduce circulation time without compromising hole cleaning efficiency requires the capability to predict circulation time of excavated rock materials of all sizes along the entire length of the wellbore including horizontal well sections. These capabilities become even more difficult to achieve when dealing with deviated or horizontal wells versus vertical wells due to the possibility of deposition and re-entrainment of drill cuttings on the low side of the well.

Most of the air drilling hydraulic design models in the literature have been developed for application to vertical well systems. This is due to the fact that most air drilling applications have historically been designed for vertical well sections only. Therefore, the inherent flow regimes variations present horizontal pneumatic transport systems have neither been emphasized nor considered in air model developments to date. Traditional models have performed reasonably well in vertical well sections without much need for additional constitutive relations due to the symmetric and annular nature of the flow regimes in vertical pneumatic transport systems (See Fig. 1.4). This implies that simple disperse phase flow regime assumption may be sufficient in capturing all flow conditions across the FCC plot using one set of constitutive relations for wall friction factors, fluid-particle interactions, particle-particle interaction and gravitational effects.
However, the current interest in a more innovative drilling technique that include the use of air as drilling fluid for horizontal well sections in tight sand and low permeability shale formations requires a reevaluation of the standard assumptions in previous studies. While the fully dispersed particulate system flow assumption is applicable under very dilute flow regimes in horizontal flow systems, the prevailing flow profile quickly becomes asymmetric as conditions change from dilute flow to dense flow (See Figs. 1.8 and 1.9). Hence, the dispersed flow assumption is insufficient for modeling all flow regimes in horizontal flow pneumatic transport systems. For this reason, additional constitutive relations that account for the effects of the prevailing flow regime must be taken into consideration. This will require detailed experimental studies were new constitutive relations that account for flow conditions under various flow regimes are developed and is unfortunately beyond the scope of the current study.

However, in this study we will try to circumvent the need for distinct constitutive relations for each flow regime in modeling horizontal pneumatic systems by modeling only continuous regions under dispersed flow. This will require accounting for reduction in flow area due to saltation or cuttings bed formation as solid deposition occurs in the lower side of a horizontal or deviated well system.

1.5 Objectives

Like many drilling developments, pneumatic cutting transport (or air drilling) is in extensive use, especially for vertical well sections, though not well understood. However, accurate predictions of rate requirements and circulation time for hole cleaning while drilling with air is of paramount importance. The goals of this study is as follows:

1. Develop a better understanding of vertical and horizontal pneumatic transport of cuttings under transient flow conditions

2. Develop the capacity to predict rate requirements and circulation time for hole cleaning while drilling both the vertical and lateral well sections using air.
A wellbore drilling hydraulics model capable of capturing the flow conditions present in both vertical and horizontal well sections will be developed. The model developed will include the capability of accounting for the effect of flow stratification typical of horizontal and deviated wellbores. Ultimately, these models will aid in the prediction of volumetric rate requirements and circulation time for hole cleaning in vertical and horizontal well sections drilled with air as drilling fluid. In the following chapter will provide some literature review on the subject of Pneumatic transport modeling, cuttings-bed modeling and solutions to hyperbolic PDEs.
Chapter 2

Background

2.1 Pneumatic Transport Modeling

Pneumatic transport is a process by which bulk materials (or solids) are transported using gas flow as the conveying medium. As it relates to underbalanced drilling hydraulics, formation materials (cuttings) excavated during the drilling process are conveyed out of the wellbore through the annulus from down-hole to the surface. This is necessary for keeping the wellbore clean and free of crushed formation material to prevent regrinding of crushed rock; provide easy access for the drill-bit to contact new formation; prevent the growth of cuttings bed; and ultimately to prevent stuck pipe scenarios. Modeling these phenomenological process on a field scale is not an easy task and requires a strong background in mathematics and computational physics.

Several efforts have been made beginning in the early 1950’s to tackle this problem for vertical wells. Martin [3],[4] made the first attempt to use a mathematical model to predict the volumetric requirement for dry air drilling operations. His publications in the Hughes Tool Co. Bulletins, No. 23 and 23-A had once served as the industry standard for determining volumetric rate requirements. His work was developed based on the Weymouth formula which is an empirical equation for gas flow in horizontal pipes. Although it was an important and timely contribution to the drilling industry, the formulation failed to provide a mechanistic description of the multiphase flow conditions involved in the displacement of cuttings with a drilling fluid. As a result, the model usually under-predicts air rate requirements.
However, despite its shortcomings, this theoretical development challenged others to consider the application of mechanistic models in predicting the required volumetric flow rates for air drilling operations.

Angel [5] proposed one of the most widely used method for estimating the benchmark minimum rate requirement to date. His work was a significant improvement to that of Martin due to the inclusion of the effect of solid mass rate through the rate of penetration (ROP) parameter. He also included depth correction for temperature and applied the hydraulic diameter of the wellbore annulus to the Weymouth equation. Despite these improvements, like Martin, his approach was also based on the assumption that the circulating fluid and the drill-cuttings are a homogeneous fluid mixture having the fluid flow properties of a perfect gas. It is worth noting that he warned that for lack of a better approach, “...the volumes determined by this approach should be regarded as the minimum volumes required to produce a desired equivalent annular velocity”. Angels model also assumed that the optimum circulation rates corresponds to the condition whereby the annular velocity equivalent to 3000 ft/min (15.24 m/s) at standard atmospheric conditions regardless of the geometric properties of the solids phase, arguing that this was the minimum velocity necessary for effective cuttings transport. However, Gray [6],[7] has shown that the optimal air velocity should instead be determined relative to the critical slip velocity of the averaged particle diameter of the transported solids. Grey also argued that the minimum annular velocity assumption of 3000 ft/min may only be sufficient to lift small particles.

Over the years many other empirical and semi-empirical studies based on improving Martin and Angels’ models have been presented in the literature (McCray and Cole [8], B. Guo and R. Lee. [9], W. Kexiong et al. [10]). The advantage of empirical and semi-empirical modeling approaches is that their solutions are quick and easy. However, the major shortcoming of these models is the assumption that the gas-solids mixture is a homogeneous system experiencing no slip. Models developed with this assumption preclude the possibility of cuttings accumulation within the wellbore. Therefore, the no slip assumption significantly limits the accuracy of their predictive capability especially as the prevailing flow conditions move from the dilute flow towards the dense phase flow as explained in Sec. 1.2. Hence, results obtained from such models will only be accurate for very dilute particle
transport and will increase in accuracy as cuttings concentration (or solids feed rate) within the wellbore draws closer to zero.

A more accurate approach for modeling cuttings displacement is by solving continuum-based conservation equations for multiphase flow along with associated constitutive relations. The simplest version of this is known as the homogeneous equilibrium model (HEM), where the mass and momentum conservation equations are used to describe the flow of a homogeneous mixture. These equations are solved with the assumption that flow properties (such as velocities, pressures, viscosities and so on) of all components of the fluid mixture may be represented by single mixture equivalents. This assumption also implies that there is no slip between phases that constitute the flowing fluid mixture and any mass or energy transfer between phases occur rapidly. Although, HEM is superior to models based on the empirical Weymouth equation, it also precludes the possibility of modeling variations in solids concentration along the wellbore, which is a very important parameter in cuttings transport. While the accuracy of HEMs increases as the density difference between phases decreases, HEM is unattractive for modeling cuttings transport with air/gas as drilling fluid.

The separated flow model (SFM) is another approach based on solving the macroscopic conservation equations. In SFM, the slip condition is introduced and therefore momentum exchange between individual flowing phases and the channel (or viscous effects) can be modeled separately using distinct phase velocities. Typically, the momentum conservation equations for all phases that constitute the flowing fluid are summed up in order to avoid the development of constitutive relations for modeling momentum transfer between phases (or interfacial force effects). Empirical correlations are then used to model slip velocity (or slip ratio as a function of solid concentration), the frictional pressure drop, and other fluid flow properties. Additional constitutive relations that relate volume concentrations to mass flow rates (or slip velocity) are also required. Schoeppel and Sapre [11] pioneered this approach with a gas volumetric rate prediction model based on critical slip. Ikoku et al. [12], Machado and Ikoku [13] and Puon and Ameri [14] have also applied variations of SFM in modeling cuttings transport in wellbores using air/gas as drilling fluid. The SFM approach is a significant improvement to HEM and empirical models, however, SFMs are not free of inadequacies. The
fundamental draw back of SFM lies in the assumption of equal pressures and temperatures between phases. For two-phase air-solids flow, the assumption that the normal stresses on both phases will be equal is far from accurate. However, the error from this assumption is typically lumped into the empirical equations used to model momentum transfer between the phases and the conduit wall.

An improvement to SFM is the two fluid model (TFM). Here each phase is treated as a separate fluid having its own set of continuum-based conservations equations. Each phase has its own velocity, pressure and temperature and differences in phase velocities are implicitly induced. Separate equations of state may be used to relate phase-densities to pressures and local constitutive relations may be used to relate the pressure of one phase to another. It is also typical that pressure of two different but closely related phases (e.g. both liquid) be assumed equal. Differences in temperature are also implicitly induced by energy transfer between phases. TFM has the advantage that actual transport processes can be rigorously defined, however it requires constitutive relations developed from experimental data that are very difficult to obtain. Adewumi and Arastoopour [15] was one of the first to introduced TFM in modeling steady state cuttings transport with air/gas as drilling fluid. Other models based on this approach include Tian and Adewumi [16], [17], Adewumi and Tian [18], [19], [20]. More recently Zhu et al. [21] applied the TFM in modeling transient cuttings transport in a short section of a wellbore. While TFM is the best possible description of multiphase flow systems on a macroscopic (or continuum) scale, when applied to pneumatic transport systems having non-uniform particle geometries (among other properties), it suffers from the assumption that the individual/distinct solid particles present in the system have to be averaged out and lumped into one or at best a few “particulate phases”. Each solids/particulate phase represent a group of particles in the system with common intrinsic properties that allow them to the defined as one distinct continuous phase in the multiphase system.

One way to alleviate the assumption of a continuum description for the solids phase with discrete representations of individual particles having distinct properties is known as the discrete particle model (DPM) approach. Here collision between individual solid particles are followed along in space and time within the domain of interest using the Lagrangian formulation of the conservation equations.
However, the Eulerian description may still be adopted for the continuous (fluid) phase. Lain et al. [22], Tsuji [23], Lain and Sommerfeld [24], Ibrahim et al. [25] and Zhu [26] have all applied DPM approach for pneumatic conveying. Despite the improvements in accuracy that DPM provides, its application to practical field projects has been limited. This is due to DPMs prohibitive computational expense which is due to the large number of grid points that are required in resolving flow conditions even for small system sizes (20,000 particles or less). Hence, DPM is not very feasible for modeling cuttings transport in wellbores.

Another possible approach for modeling multiphase pneumatic transport of cuttings is direct numerical simulation (DNS). Here the Eulerian-Lagrangian description discussed is formulated and the entire spatial scale of turbulence is resolved down to the smallest dissipative scales (Kolmogorov scale). This is the most accurate way of modeling any CFD problem, however it is also prohibitively expensive for modeling just about any problem of practical importance based on the current computational resources available at most research institutes. Therefore, the use of DNS models has mostly been limited to validation of key portions of less computationally demanding models.

In this study, the TFM approach is adopted in modeling cuttings transport since it balances reasonable computational demand with an acceptable level of accuracy. Closure relationships in the form of empirical constitutive equations are used to account for momentum exchange due to (1) fluid-wall and solids-wall interactions (2) fluid-solids interaction and (3) hydrostatic head.

2.2 Saltation and Re-entrainment Modeling

Significant studies have been conducted on hole cleaning in both vertical and horizontal wells using conventional drilling mud (see reviews by Azar and Sanchez [27], Mishra [28] and Xiaofeng et al. [29]). However, publications on hole cleaning with air/gas have been limited to the case of vertical wells only. This is because the standard practice is to switch to water/oil-based mud before building angle into a horizontal sections. However, recent developments in shale fracking has lead to an increase in natural gas production from shallow low-permeability consolidated sand and shale pay zones. This is fueling the need for a better understanding of
flow condition within highly deviated and horizontal wells drilled on air/gas.

The main difference between vertical and horizontal circulation process using air/gas is the formation of cuttings beds on the lower side of the wellbore. There are three ways of modeling the cuttings bed using the TFM approach: (1) Assume a two/three layer modeling approach where the cuttings bed is either assumed to be stationary or having both a stationary layer and fluidized layer, while the dilute phase is assumed to be a homogeneous gas-like phase. (2) Assume we have two TFMs flowing side by side with mass, momentum and energy transfer terms for coupling due to interaction between the two systems. One models the dense slurry-like phase and the other models the dilute phase. (3) Assume the stationary bed may be modeled as a change in channel area and use TFM to account for the fluidized phase.

The first approach suggested is similar to that used by most researchers (see Li et al. [30]) for cuttings bed modeling using conventional drilling fluids where the cuttings bed can be very stable. However, several studies have shown that cutting beds are much more unstable when drilling with low-viscosity under turbulent flow conditions [31], [32], [33], [34]. Besides, it is expected that the deposition and re-entrainment process will be fast-paced and erratic while drilling with air/gas.

The second approach which is based on two separate but coupled TFMs is theoretically possible. However, not only does this approach increase the complexity of the overall modeling process, it also leaves a lot of numerical redundancies along majority of the wellbore length where and when the flow remains dilute. This is true for majority of the vertical section of a wellbore, but also true for sections along the deviated or horizontal zones where the flow condition favors re-entrainment of cuttings, which as mentioned before is expected to be erratic.

The third approach is based on incorporating stationary bed models into TFMs as a change in conduit flow area. This is a representative approach since it will allow the transport of suspended particles above a stationary cutting bed. Stationary beds are formed due to deposition of cuttings under horizontal flow conditions as illustrated in the previous chapter. Here the model can also be transient with the capability of modeling both dilute and dense phase flow conditions while accounting for the presence of cuttings beds along the horizontal section of a well.

In this study, two methods were investigated for modeling the formation of the
cuttings beds. They are:

1. Critical Velocity Approach

2. Area Ratio Approach

2.2.1 Critical Velocity Approach

This approach is based upon the use of cumulative distribution functions with bounds that are represented by distinctly identifiable points of deposition and re-entrainment (also known as critical velocities). Deposition and re-entrainment of particles under pneumatic transport are dependent on following critical velocities respectively:

1. Saltation velocity

2. Pick-up velocity

In deviated or horizontal well sections, there may exist dense phase flow conditions along side suspended flow at various sections within the wellbore once velocity of the carrier phase drops below a critical value. This thereby induces flow segregation as illustrated in Figs. 1.7 and 1.8. This critical velocity is known as the “saltation velocity”. The accumulation of solids creates what is known as cuttings beds at the lower side of the wellbore thereby reducing the area available to flow and in effect permitting suspended flow above a stationary cuttings bed. If the local velocity of the carrier phase is raised above a critical point re-entrainment of particles will be initiated and the cuttings bed will begin to diminish in size. This critical velocity is known as the “pick-up velocity”. However, most pneumatic conveying designers refer to “pick-up velocity as” the velocity needed to keep particles in suspension at the feed point.

There are a number of empirical correlations in the literature for saltation and pick-up velocities. For saltation velocity, there those due to Rizk [35], [36], Matsumoto [37], Thomas [38], Geldart and Ling [39], Cabrejos and Klinzing [40] and Tashiro et al. [41]. The correlation due to Rizk [36] is the most widely accepted in industry. However, there has been comparative studies like that of Jones and Leung [42] that have shown the correlation due to Thomas [38] is more accurate
than that of Rizk and many others when compared over a wide range of data sets. That being said, it is important to point out that even Thomas’s correlation showed a root mean square relative deviation (RMSRD) of 44 % (RMSRD<60 % for all others). Klinzing et al. [43] suggests that further experimental work may lead to a better understanding of the mechanism of saltation. They also suggest that progress in theoretical prediction of saltation velocity will have to await further developments in areas like:

1. Detailed description of the fluid velocity profile near the wall with particle flow;

2. Effect of particle size on fluid turbulence;

3. Effect on solid mass rate on slip velocity; and

4. Effect of particle shape on saltation.

Correlations available in the literature for pick-up velocity include those due to Cabrejos and Klinzing [40], [44], Cabrejos et al. [45], Villareal and Klinzing [46], Kalman et al. [47], Rabinovich and Kalman [48], Goy et al. [49] and Gomes and Mesquita [50]. Kalman et al. [47] identified three regions that display distinctly different pick-up velocity behavior using Geldarts classical particle classification [51]. A summary of these three distinct pick-up characteristics are as follows:

a. Zone 1 (Coarse particle group): pick-up velocity increases as the particle diameter increases due to an increase in the gravitational effects;

b. Zone 2 (Mid-size particle group): pick-up velocity increases as the particle diameter decreases due to an increase in van der Waals (cohesive) forces; and

c. Zone 3 (Fine particles group or powders): pick-up velocity remains constant due to such high cohesive forces that cause particles to move as agglomerates.

Kalman et al. [47] reported a bound of ±30 % deviation from 90% of measured experimental data from correlations of all three zones. Rabinovich and Kalman [48] also obtained correlations for these three zones and reported a maximum error of ±30 % from all measured experimental data. The more recent correlation of
Gomes and Mesquita [50] show a much better match with experimental data over a wide range of pipe and particle dimension and properties.

Cabrejos and Klinzing [40] also encountered a unique non-linear relationship between saltation and pick-up velocities. Unlike pick-up velocity where a unique critical fluid velocity needs to be reached for re-entrainment to occur, crawling flow (moving bed) is sometimes observed for some particles types before a stationary bed is formed due to saltation. This observation was said to likely be due to the particle shape-factor [43].

### 2.2.2 Area Ratio Approach

Another way of modeling cuttings bed involves using cutting bed area ratio correlations. Along this line, several studies have been conducted on hole cleaning using conventional fluids along with cuttings bed formation and modeling. Some of these include Ozbayoglu et al. [52], [53], [54], Sorgun [55] and Xiaofeng et al. [29]. Ozbayoglu et al. [53], [54], and Sorgun [55] both developed correlations for cuttings bed formation from experimental data that included the use of Newtonian fluids. Their area ratio correlations were dependent upon the following parameters: (1) cuttings concentration (2) fluid velocity (3) inclination angle (4) hydraulic diameter (5) fluid density (6) pipe rotation speed (7) cuttings diameter and (8) gravitational effects.

In this study, both critical velocity and the area ratio approaches were considered. While the former is the better of the two approaches it’s implementation was unsuccessful. This was mostly due to poor accuracy of the saltation velocity correlations, which would sometimes predict unrealistically estimates that were above pick-up velocity estimates.

### 2.3 Numerical Methods for Hyperbolic Systems

The TFM developed in this study consists of a set of coupled, non-linear, hyperbolic PDE’s. Both the gaseous and particulate phases are modeled as independent continuous phases coupled together with interaction forces that are a function of localized flow parameters. The hyperbolic nature of the TFM in the context of
the current application appears in two forms. First is the discontinuity in flow profile due to compression of the gaseous phase, which is of little interest in the current study. The other is the discontinuity associated with the propagation of a continuity wave due to the particulate (solids) phase. This allows us to track changes in volume fraction (or sand concentration) along the pipe annulus. Both of these discontinuities have to be resolved accurately.

In petroleum engineering, hyperbolic PDE’s show up in pipeline pressure transient analysis, pneumatic cuttings transport modeling, streamline reservoir simulation, slimtube simulation and so on. Analytical solutions to such models are available only for limited and simplified cases, hence the need for reliable numerical methods. The characteristic discontinuous solutions of hyperbolic PDE’s lead to computational challenges. The application of naive finite difference techniques where derivatives are approximated by finite differences at grid points typically breakdown at such discontinuities. This motivated the development of techniques that approximate the integral form of conservation equations across grid cells known as finite volume methods. In addition to the issue of discontinuities, two important concerns when dealing with numerical solutions are: (1) physical accuracy/resolution and (2) computational requirements/hardware limitations. Both are related in the sense that higher physical resolution increases computational requirements. Depending on the size of the problem one will have to be traded for the other.

A number of techniques have been developed in the field of computational fluid dynamics (CFD) that help tackle some of these issues. One such technique involves the introduction of monotone schemes that are based on the solution to the physics of discontinuities or the Riemann problem. Some of these include upwind schemes such as flux vector splitting methods, which utilize information of the direction of propagating waves and Godunov-type schemes, which solves the Riemann problem between grid cells. These schemes are generalized versions of the one-sided single point upwind (SPU) schemes typically used in streamtube and streamline simulation studies. Despite improvements, these first-order monotone schemes suffer from smearing at discontinuities. This smearing phenomena is also known as dissipation or numerical diffusion. It is important to distinguish numerical diffusion from fluid dynamic (or physical) diffusion, which results from the movement
of systems towards equilibrium under prevailing concentration gradients. Rather, smearing or numerical diffusion is instead only related to how well the sharpness of a discontinuous front can be captured numerically. In other words, numerical diffusion or smearing is purely a form of numerical error.

Higher-order schemes were introduced to help solve the problem of numerical diffusion. By including higher-order terms in Taylor series expansions when approximating differential equations with discrete expressions, better estimates of the continuous solutions may be obtained. Consequently, higher-order schemes permit the relaxation of the number of grid cells required for adequate numerical resolution while reducing smearing, or numerical diffusion, at discontinuities. Though higher-order schemes reduce numerical diffusion, they may suffer from non-physical oscillations at discontinuities also referred to as numerical dispersion. Just like the case of the difference between fluid dynamic (or physical) diffusion and numerical diffusion there is a clear distinction between fluid dynamic (or physical) dispersion and numerical dispersion. Fluid dynamic dispersion is the fluid mixing that occurs due to tortuosity and geological heterogeneities in porous media while numerical dispersion is a numerical artifact in the form of oscillations near sharp discontinuities. Numerical dispersion is also sometimes referred to as the "Gibbs Phenomenon".

In an effort to resolve the issue of numerical dispersion when using higher-order schemes, Total Variation Diminishing (TVD) methods were introduced. These methods ensure that slopes at discontinuities are limited through the aid of slope/flux monotonicity preserving limiters to ensure that solutions remain physical/non-oscillatory. In the same vein, Essentially Non-Oscillatory (ENO) and Weighted-ENO (WENO) schemes were also introduced. These numerical schemes utilize adaptive stencils as opposed to limiters to control the growth of the total variation of the numerical solutions.

A general account of numerical schemes for solutions to hyperbolic systems may be found in the works of Toro [56], Barth and Deconinck [57], LeVeque [58] and Trangenstein [59]. However, here we will review certain centered monotone numerical schemes as well as upwind schemes that take into consideration the nature of the physics of first-order hyperbolic conservation laws. We also discuss the application of higher resolution TVD and ENO schemes for both linear and
non-linear hyperbolic systems. The vast amount of numerical schemes available in the literature for hyperbolic equations cannot all be covered here. However, the techniques discussed in this review are sufficient for familiarization with the difficulties involved in obtaining solutions to hyperbolic problems. These techniques serve as essential building blocks for solving the hyperbolic terms in any convection dominated system of PDE’s as they pose the most stringent restrictions on the discretization of such systems.

2.4 Summary of Literature Review

In this chapter, we presented the various Pneumatic transport modeling approach available in literature. We also discuss feasibility of two different approached for saltation and re-entrainment modeling. Finally, we provided some background on solutions to hyperbolic PDEs. In the chapter that follow, we will present the two-fluid model (TFM) developed in this study. This will include the method used in this study to account for flow stratification through saltation and re-entrainment of solids in the particulate phase in a manner that ensures mass and momentum conservation. Finally, we will discuss how we numerically account for the effect of temperature variation on fluid viscosity in the isothermal system given the temperature profile in a well.
The Mathematical Model

3.1 Model Description

Cuttings transport by a pneumatic fluid through the annulus of a wellbore is expressed here using an unsteady, variable area, two-fluid model (TFM). The space-time dependent variation in flow area along the wellbore permits the simplicity of a 1D description of the multi-phase flow conditions and yet accounts for the complex geometries expected along the horizontal and inclined portions of the wellbore attributed to the continuous deposition and re-entrainment of cuttings. Please see Appendix A for detailed derivation of the conservation equations.

3.1.1 Pneumatic Transport Model

A quasi one-dimensional TFM transport system may be expressed as follows:

\[ U_t + F(U)_x = B(U), \quad (3.1) \]

where:

\[
U = \begin{bmatrix}
\varepsilon \rho_\text{g} A \\
(1 - \varepsilon) \rho_\text{s} A \\
\varepsilon \rho_\text{g} v_\text{g} A \\
(1 - \varepsilon) \rho_\text{s} v_\text{s} A \\
\end{bmatrix}, \quad F(U) = \begin{bmatrix}
\varepsilon \rho_\text{g} v_\text{g} A \\
(1 - \varepsilon) \rho_\text{s} v_\text{s} A \\
\varepsilon \rho_\text{g} v_\text{g}^2 A + p_\text{g} A \\
(1 - \varepsilon) \rho_\text{s} v_\text{s}^2 A + p_\text{s} A \\
\end{bmatrix},
\]
\[
\mathbf{B}(\mathbf{U}) = \begin{bmatrix}
0 \\
0 \\
p_g \frac{\partial A}{\partial x} + F_{gg} A + F_{fg} A + F_d A \\
p_s \frac{\partial A}{\partial x} + F_{gs} A + F_{fs} A - F_d A
\end{bmatrix}
\]

Eq. 4.83 contains mass and momentum conservation for both the dispersed \((s)\) and carrier \((g)\) phases. \(\mathbf{U}(x,t)\) is the vector of conserved variables, \(\mathbf{F}(\mathbf{U})\) is the physical flux vector and \(\mathbf{B}\) is a vector of sources terms. Subscripts \(x\) and \(t\) are used to denote partial derivatives with respect to the two independent variables. Subscripts \(g\) and \(s\) represents the gas and dispersed solids phases respectively. \(\rho, v, A\) and \(p\) are density, velocity, cross-sectional flow area and pressure. \(\varepsilon\) is the gas volume fraction (or voidage) and it is constrained by \(\varepsilon_{\text{min}} < \varepsilon < 1\) where \(\varepsilon_{\text{min}}\) is the void fraction when the conduits is fully packed with cuttings. The source term, \(\mathbf{B}\), contains empirical relations for determining force per unit volume due to gravity \((F_{gg} \text{ and } F_{gs})\), wall friction \((F_{fg} \text{ and } F_{fs})\) and fluid-particle drag on the fluid phase \((F_d)\). Since solid density \((\rho_s)\) is a known constant we have seven unknowns and four equations. However, we may express the pressures of both phases in terms of the other primitive variables.

The gas phase pressure may be related to gas density (and temperature) by the following EOS:

\[
p_g = \frac{ZRT}{M_g} \rho_g.
\]

\(R, M_g\) and \(T\) are the universal gas constant \((8.314 \times 10^3 \text{ J} / \text{kmol}K)\), molecular mass of the gas phase and fluid temperature. For this isothermal system, pressure of the carrier phase is directly proportional to its density with the proportionality constant \(\frac{RT}{M_g}\). \(M_g\) for any gas may be determined as a function of the molecular mass of air \((M_{\text{air}} = 28.97 \text{ kg} / \text{kmol})\) and specific gravity \((\text{SG} = \frac{M_g}{M_{\text{air}}})\).

The solids phase pressure gradient may be expressed in terms of an empirically defined modulus of elasticity, \(G(\varepsilon)\), which is determined as a function of the local value of the voidage \((\varepsilon)\). The pressure gradient of the particulate (solids) phase represents the normal stresses due to particle-particle interactions. Its inclusion helps to prevent the solid phase from reaching unrealistically low values of volume fractions. In addition its inclusion helps in maintaining strict hyperbolicity of
the system of equations, which though not necessary for stability is a desirable characteristic for numerical purposes. The modulus of elasticity, $G(\varepsilon)$, is expressed as follows:

$$\nabla p_s = G(\varepsilon) \nabla \varepsilon_s$$  \hspace{1cm} (3.3)

A number of empirical correlations for the modulus of elasticity, $G(\varepsilon)$ has been proposed and several comparative studies [60], [61] have been carried out to test their efficacy. Rietma and Musters [62], [63] measured solid modulus in a tilted fluidized bed experiment in order to quantify inter-particle stresses. An empirical expression was later fitted to their data by Gidaspow and Ettahadieh [64]. The expression is as follows:

$$G(\varepsilon) = 10^{-8.76\varepsilon+5.43}$$  \hspace{1cm} (3.4)

Eq. 4.83 may therefore be expressed with 4 unknowns ($\varepsilon$, $\rho_g$, $v_g$ and $v_s$) as follows:

$$U_t + F(U)_x = B(U),$$  \hspace{1cm} (3.5)

where:

$$U = \begin{bmatrix} \varepsilon \rho_g A \\ (1 - \varepsilon) \rho_s A \\ \varepsilon \rho_g v_g A \\ (1 - \varepsilon) \rho_s v_s A \end{bmatrix}, \quad F(U) = \begin{bmatrix} \varepsilon \rho_g v_g A \\ (1 - \varepsilon) \rho_s v_s A \\ \varepsilon \rho_g v_g^2 A + a^2 \rho_g A \\ (1 - \varepsilon) (\rho_s v_s^2 + G(\varepsilon)) A \end{bmatrix},$$

$$B(U) = \begin{bmatrix} 0 \\ 0 \\ \rho_g \frac{\partial A}{\partial x} + F_{gy} A + F_{f_g} A - F_d A \\ (1 - \varepsilon) \frac{\partial [G(\varepsilon) A]}{\partial x} + F_{gs} A + F_{f_s} A + F_d A \end{bmatrix}$$

$a$ is the constant speed of sound determined as follows:

$$a = \sqrt{\frac{dp_g}{d\rho}} = \sqrt{\frac{ZRT}{M_g}}$$  \hspace{1cm} (3.6)
where:

\[ Z = EOS(p, T, \gamma) \]  \hspace{1cm} (3.7)

The Dranchuk and Abou-Kassem [65] equation of state (EOS) was used to determine the real gas compressibility factor \( Z \) as a function of pressure, temperature and specific gravity \( \gamma \). Now we have five unknowns but four equations. The additional equation needed to close the system is the area ratio model and is introduced in section 3.2.

### 3.1.2 Characteristic analysis

When Eq. 4.87 is expressed in it’s quasi-linear form as in Eq. 4.2 its corresponding flux Jacobian matrix \( A(U) \) is:

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-v_g^2 + \frac{a^2}{\varepsilon} & \frac{a^2 \rho_g}{\varepsilon \rho_s} & 2v_g & 0 \\
0 & -v_s^2 + \frac{G(\varepsilon)}{\rho_s} & 0 & 2v_s
\end{bmatrix}
\]  \hspace{1cm} (3.8)

The eigenvalues and eigenvectors of Eq. 4.87 are as follows:

\[
\Lambda = \begin{bmatrix}
v_g - \frac{a}{\sqrt{\varepsilon}} & 0 & 0 & 0 \\
0 & v_g + \frac{a}{\sqrt{\varepsilon}} & 0 & 0 \\
0 & 0 & v_s - \sqrt{\frac{G(\varepsilon)}{\rho_s}} & 0 \\
0 & 0 & 0 & v_s + \sqrt{\frac{G(\varepsilon)}{\rho_s}}
\end{bmatrix}
\]  \hspace{1cm} (3.9)
\[
\kappa_1 = \begin{bmatrix} 1 & 0 \\ \nu_g - \frac{a}{\sqrt{\varepsilon}} & 0 \end{bmatrix} \quad \kappa_2 = \begin{bmatrix} 1 & 0 \\ \nu_g + \frac{a}{\sqrt{\varepsilon}} & 0 \end{bmatrix}
\]
\[
\kappa_3 = \begin{bmatrix}
1 \\
\frac{\varepsilon(\sqrt{G(\varepsilon)} + \sqrt{\rho_s(v_s - v_g)})^2 - a^2 \rho_s}{\nu_s}
\end{bmatrix}
\]
\[
\kappa_4 = \begin{bmatrix}
1 \\
\frac{\varepsilon(\sqrt{G(\varepsilon)} + \sqrt{\rho_s(v_s - v_g)})^2 - a^2 \rho_s}{\nu_s + \sqrt{G(\varepsilon)}}
\end{bmatrix}
\]

Eq. 4.87 is strictly hyperbolic, inhomogeneous, non-linear and does not satisfy the homogeneity property.

### 3.1.3 Constitutive Relations

The gravitational forces per unit volume is determined as follows:

\[
F_{gg} = \varepsilon \rho_g g \cos \theta, \quad F_{gs} = (1 - \varepsilon)(\rho_s - \rho_g)g \cos \theta.
\]  

(3.11)

\(g\) is the gravitational acceleration \((9.8 \text{ m/s}^2)\) and \(\theta\) is the angle of inclination from a vertical wellbore (where \(\theta = 0\)). The viscous force per unit volume due to the conduit wall friction is determined as follows:

\[
F_{fi} = \frac{4}{D_h} \left( \frac{1}{2} \rho v_i^2 \right) f_i \quad i = g, s
\]  

(3.12)
The parameters $D_h$ and $f_i$ are the hydraulic diameter and dimensionless friction factor respectively. The gas phase friction factor, $f_g$, is a function of pipe roughness ($\epsilon$) and Reynolds number ($\text{Re}$):

$$\text{Re} = \frac{\epsilon \rho_g v_g D_h}{\mu_g}$$

where $\mu_g$ in Eq. 4.91 is the gas viscosity. $f_g$ is estimated from the fanning equation (Eq. 3.14) for laminar flow ($\text{Re} < 2100$) and Chen’s [66] friction factor equations (Eq. 3.15) for turbulent flow conditions ($\text{Re} > 2100$).

$$f_g = \frac{16}{\text{Re}}$$

$$\frac{1}{\sqrt{f_g}} = -4 \log \left[ \frac{\epsilon}{3.7065} - \frac{5.0452}{\text{Re}} \log \left( \frac{\epsilon}{2.8257} + \frac{5.8506}{\text{Re}^{0.898}} \right) \right] \quad (3.15)$$

The term $\epsilon$ in Eq. 3.15 represents pipe roughness. The dispersed phase friction factor, $f_s$, is determined by the empirical correlation of Yang [67]:

$$f_s = \begin{cases} 
0.00315 \left( 1 - \epsilon \right)^{\frac{U_t}{v_g - v_s}} \left( 1 - \epsilon \right)^{-0.979} & \text{for vertical flow} \\
0.0293 \left( 1 - \epsilon \right)^{\frac{v_g}{\sqrt{gD_h}}} \left( 1 - \epsilon \right)^{-1.15} & \text{for horizontal flow} 
\end{cases} \quad (3.16)$$

where terminal velocity ($U_t$) is defined as:

$$U_t = \sqrt{\frac{4 \left( \rho_s - \rho_g \right) g d_p}{3 C_d \rho_g}}$$

The fluid-particle drag force ($F_d$) is estimated as in Eq. 3.18 The inter-phase friction coefficient ($\beta$) is estimated using the equation of drag in particle-laden fluid [68],[69],[70] shown in Eq. 3.19.

$$F_d = \beta (v_g - v_s) \quad (3.18)$$
\[ \beta = \frac{3}{4} \frac{C_d (1 - \varepsilon) \rho_g |v_g - v_s|}{d_p} g(\varepsilon) \]  

The parameters \( d_p \) and \( C_d \) are the weighted average particle diameter and dimensionless single sphere drag coefficient respectively. The voidage function, \( g(\varepsilon) \), accounts for the effects of neighboring particles on the drag of individual particles. For Wen and Yu [68], \( g(\varepsilon) = 2.65 \). Di Felice [69] developed an improved correlation for the voidage function using the empirical correlation of Richardson and Zaki [71], which relates the fluidized flux velocity to the void fraction under varying flow regimes. Di Felice’s voidage function is of the form:

\[ g(\varepsilon) = \varepsilon^\eta \]  

where:

\[ \eta = 3.7 - 0.65 \exp \left[ -\frac{1}{2} (1.5 - \log \text{Re}_r)^2 \right] \]

Single sphere drag coefficient (\( C_d \)) is estimated with the correlation of Turton and Levenspiel [72]:

\[ C_d = \begin{cases} \frac{24}{\text{Re}_r} + \frac{4.152}{\text{Re}_r^{0.44}} + \frac{0.433}{1 + 16300 \text{Re}_r^{0.86}} & \text{if } \text{Re}_r < 1 \times 10^{-5} \\ 0.44 & \text{if } \text{Re}_r > 1 \times 10^{-5} \end{cases} \]  

where:

\[ \text{Re}_r = \frac{\varepsilon \rho_g |v_g - v_s| d_p}{\mu_g} \]

### 3.2 Cuttings Bed Modeling

After testing several cuttings bed area ratio models, the model of choice used in this study is that of Ozbayoglu [53]. It appeared to produce the most reasonable results and was developed with sound and consistent dimensionless parameters, unlike that of Sorgun [55] where cuttings concentration was defined with dimensions of \( L^{-1} \).
Ozbayoglu’s cuttings bed area ratio correlation in SI units is as follows:

\[
\frac{A_b}{A_o} = C_0 \frac{26.31(1 - \varepsilon)^{0.2108} \rho^{0.034}}{(\frac{v^2}{D_{ho}})^{0.2933}(\frac{D_{ho} \rho_y v_s}{\mu})^{0.1023}}
\]  

(3.22)

where \(\frac{A_b}{A_o}\), \(A_o\), and \(D_{ho}\) are the cuttings bed area ratio, wellbore cross sectional area and wellbore hydraulic diameter respectively. \(C_0\) is an empirical constant we introduced to fit experimental data. Eq. 3.22 is used to update the flow cross sectional area systematically after each time step. At every timestep, the local slip velocities are compared to critical slip velocity, which in this study is assumed to be the individual particle terminal velocity. If the local slip velocity is below critical, Eq. 3.22 is used to update the flow cross sectional area. However, if local slip velocity is above critical, Eq. 3.22 is only applied if the flow area in the local grid cell is below maximum i.e. cell contains some solid deposits. The adjustments are made in a manner that ensures that all conserved variables (\(U\)) remain conserved after the adjustment at every time step.

### 3.2.1 Iterative Time-Dependent Flow Area Adjustment

In order to ensure that all conserved variables remain conserved after flow area adjustments there is a need to adopt an iterative approach. Therefore, the five primitive variables within every grid block are adjusted at each time step \((n)\) as follows:

\[
V^k = V^{k-1} - J^{-1} \Delta R
\]

(3.23)

where:

\[
V = [\varepsilon, \rho_g, v_g, v_s, A]^T; \quad \Delta R(V^{k-1}) = \begin{bmatrix} U(V^{k-1}) - U^n \\ A(V^{k-1}) - A^{k-1} \end{bmatrix}; \quad J = \frac{dR}{dV}
\]

The superscript \(k\) represents the iteration level and the Jacobian matrix \((J)\) is defined as follows:
\[ J = \begin{bmatrix}
\rho_g A & \varepsilon A & 0 & 0 & \varepsilon \rho_g \\
-\rho S A & 0 & 0 & 0 & (1 - \varepsilon) \rho_S \\
\rho_g v_g A & \varepsilon v_g A & \varepsilon \rho_g A & 0 & \varepsilon \rho_g v_g \\
-\rho_S v_s A & 0 & 0 & (1 - \varepsilon) \rho_S A & (1 - \varepsilon) \rho_S v_s \\
J_{1,5} & J_{2,5} & J_{3,5} & 0 & 1
\end{bmatrix} \]  

(3.24)

where:

\[
J_{1,5} = \frac{0.4668 A_o \theta^{0.034}}{(1 - \varepsilon)^{0.7892} \left( \frac{v_g^2}{D_h} \right)^{0.2933} \left( \frac{D_h \rho_g v_g}{\mu} \right)^{0.1023}}
\]

\[
J_{2,5} = \frac{0.2265 A_o (1 - \varepsilon)^{0.2108} \theta^{0.034}}{\rho_g \left( \frac{v_g^2}{D_h} \right)^{0.2933} \left( \frac{D_h \rho_g v_g}{\mu} \right)^{0.1023}}
\]

\[
J_{3,5} = \frac{A_o (1 - \varepsilon)^{0.2108} \theta^{0.034} v_g \left( \frac{0.2265}{D_h \left( \frac{v_g^2}{D_h} \right)^{0.1023}} + \frac{1.2990}{\left( \frac{D_h \rho_g v_g}{\mu} \right)^{0.1023}} \right)}{D_h \left( \frac{v_g^2}{D_h} \right)^{1.2933}}
\]

3.3 1-D Limitation

On major assumption in this study is that the wellbore may be modeled as a one-dimensional conduit. The implications of this assumption include the inability to model asymmetric physical phenomena that occur in the radial direction. However, given dimensions and flow velocities in the axial direction relative to the radial and azimuthal directions in most wellbores during drilling, this assumption is justifiable and has been adopted by many in the field of drilling hydraulics [30] and pneumatic transport [64].

3.4 Inclined Sections

The local inclination angle (\( \theta \)) relative to the vertical axis is used as a weighting variable to determine the weighted forcing functions or weighted friction factor (\( f_W \)) in the case of wall viscous effects within each grid cell as a function of both
vertical \((f_V)\) and horizontal \((f_H)\) friction factors as follows:

\[
f_W = f_V \cos \theta + f_H (1 - \cos \theta).
\]

\[(3.25)\]

3.5 Irregular Hole Size

Most wellbores have undulations in hole size along its length. These are due to hole washouts and formation instabilities. However, these irregularities in hole size are accounted for in the forcing functions through terms like the roughness factor \((\epsilon)\) and lumped effects in correlations for solid-wall friction factor. However, these may not be sufficient to capture the true effect of these irregularities.

3.6 Temperature Dependence of Viscosity

As can be observed in Eq. 4.83, we have assumed isothermal flow conditions. However, temperature conditions, on which the viscosity of the carrier phase is dependent, varies spatially within a wellbore. Additionally, the viscosity of the carrier phase determines the fluid-particle drag or the cutting carrying capacity of the carrier phase. However, a variable local viscosity accommodation was made in the numerical development of the model described in Eq. 4.83. Eq. 3.26 is used to determine the local air viscosity \((\mu)\) as a function of local temperature (in Kelvin).

\[
\mu[Pa.s] = 10^{-5} \left( -2 \times 10^{-6} T^2 + 0.006 T + 0.1866 \right).
\]

\[(3.26)\]

This simple but effective accommodation permits accounting for the effect of temperature variation on the viscosity of the carrier phase given the temperature profile in a wellbore. Eq. 3.26 is only applicable to air. New correlations need to be developed for other gases.
3.7 Summary

In this chapter, we have described the governing equations that represent our mono-dispersed particulate Two Fluid Model (TFM) along with analytical expressions for its decomposed characteristics. We have also presented corresponding constitutive relations used to account for momentum exchange due to (1) fluid-wall and solids-wall interactions (2) fluid-solids interaction and (3) hydrostatic head. The variable flow area approach used to model particle saltation and re-entrainment was also presented. Finally, temperature dependence of viscosity was discussed and an empirical expression for local air viscosity as a function of temperature is presented. In the following chapter we discuss numerical analysis for hyperbolic systems.
Chapter 4

Numerical Analysis

4.1 Hyperbolic Systems

A coupled non-linear inhomogeneous system of hyperbolic conservation laws in one spatial dimension can be expressed as:

\[ U_t + F(U)_x = B \]  \hspace{1cm} (4.1)

where:

\[ U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad F(U) = \begin{bmatrix} f_1(x,t,u_1,u_2,\ldots,u_m) \\ f_2(x,t,u_1,u_2,\ldots,u_m) \\ \vdots \\ f_m(x,t,u_1,u_2,\ldots,u_m) \end{bmatrix} \]

\( U(x,t) \) is the vector of conserved variables, \( F(U) \) is the physical flux vector, \( B \) is a vector of sources terms that may be linearly or non-linearly dependent on \( U \), \( m \) is the number of conservation equations and subscripts represent partial derivatives with the corresponding independent variable. If the elements of \( F \) and \( B \) are linearly dependent on \( U \) the system is linear, otherwise it’s non-linear.

Eq.4.1 can also be expressed in quasi-linear form by applying chain rule to the second term. The resulting equation is as follows:

\[ U_t + A(U)U_x = B \]  \hspace{1cm} (4.2)
where:

\[ A(U) = \begin{bmatrix}
    a_{11} = \frac{\partial f_1}{\partial u_1} & \cdots & a_{1m} = \frac{\partial f_1}{\partial u_m} \\
    \vdots & \ddots & \vdots \\
    a_{m1} = \frac{\partial f_m}{\partial u_1} & \cdots & a_{mm} = \frac{\partial f_m}{\partial u_m}
\end{bmatrix} \]

If the elements \((a_{ij})\) of the flux-Jacobian matrix \((A)\) are independent of \(U\), the system is said to be linear, otherwise quasi-linear. For Eq. 4.1 to be considered hyperbolic at some point \((x,t)\), \(A\) must have \(m\) real eigenvalues \((\lambda_j)\) and a corresponding set of linearly independent eigenvectors \((\kappa_j = [k_{j1}, \ldots, k_{jm}]^T)\) where \(j = 1, \ldots, m\). That is, Eq. 4.1 is hyperbolic if \(A\) has real eigenvalues and is diagonalisable. \(A\) is diagonalisable if it can be expressed as Eq. 4.3. The eigenvalues of \(A\) are the solutions to the characteristic polynomial expressed in Eq. 4.4. Their corresponding right eigenvectors must satisfy Eq. 4.5.

\[ A = K\Lambda K^{-1} \quad (4.3) \]

where:

\[ K = [\kappa_1, \ldots, \kappa_m] = \begin{bmatrix}
    k_{11} & \cdots & k_{1m} \\
    \vdots & \ddots & \vdots \\
    k_{m1} & \cdots & k_{mm}
\end{bmatrix} \]

\[ |A - \lambda I| = 0 \quad (4.4) \]

\[ A\kappa = \lambda\kappa \quad (4.5) \]

If the eigenvalues are all distinct, the system is said to be strictly hyperbolic.

### 4.1.1 The Riemann Problem

The Riemann problem for a homogeneous hyperbolic system of \(m\) equations can be expressed as the initial value problem (IVP):

\[ U_t + F(U)_x = 0, \quad -\infty < x < \infty, \quad t > 0, \quad (4.6) \]
with

\[ \text{I.C.: } U(x, 0) = U^0(x) = U_s, \quad S = \begin{cases} L & \text{if } x < x_0 \\ R & \text{if } x > x_0 \end{cases} \]

The initial data shows a discontinuity at \( x = x_0 \). However, consider the simpler case of a linear constant coefficient hyperbolic system:

\[ U_t + AU_x = 0 \quad -\infty < x < \infty \quad t > 0 \quad (4.7) \]

with

\[ \text{I.C.: } U(x, 0) = U^0(x) = U_s, \quad S = \begin{cases} L & \text{if } x < x_0 \\ R & \text{if } x > x_0 \end{cases} \]

Eq. 4.7 can be decoupled into its canonical (or characteristic) form and solved by characteristics if hyperbolic. First we define the set of characteristic variables \( (W) \) as:

\[ W = K^{-1}U. \quad (4.8) \]

Defining \( U \) in terms of \( W \) and substituting \( U = KW \) into Eq. 4.7 and multiplying from the left by \( K^{-1} \) gives Eq. 4.9 where \( \Lambda = K^{-1}AK \).

\[ W_t + \Lambda W_x = 0, \quad (4.9) \]

with

\[ \text{I.C.: } W(x, 0) = W^0(x) = W_s, \quad S = \begin{cases} L & \text{if } x < x_0 \\ R & \text{if } x > x_0 \end{cases} \]

Eq. 4.9 can also be expressed as \( m \) scaler equations as follows:

\[ (w_i)_t + \lambda_i(w_i)_x = 0, \quad \text{for } i = 1, \ldots, m \quad (4.10) \]

with

\[ \text{I.C.: } w_i(x, 0) = w^0_i(x) = w_{iS}, \quad S = \begin{cases} L & \text{if } x < x_0 \\ R & \text{if } x > x_0 \end{cases} \]
Using characteristics, the solution to Eq. 4.9 or 4.10 is the similarity solution:

\[ w_i(x, t) = w_i^0(x_0) = w_i^0(x - \lambda_it) = w_iS, \]  

(4.11)

for \( i = 1, \ldots, m \)

\[ S = \begin{cases} 
L & \text{if } x - \lambda_it < x_0 \\
R & \text{if } x - \lambda_it > x_0
\end{cases}. \]

By the transformation \( U = KW \) we then obtain the solution in terms of the conserved variables (\( U \)) as follows:

\[ U(x, t) = \sum_i w_i^0(x_0) \kappa_i = \sum_i w_i^0(x - \lambda_it) \kappa_i = \sum_i w_iS \kappa_i, \]  

(4.12)

\[ S = \begin{cases} 
L & \text{if } x - \lambda_it < x_0 \\
R & \text{if } x - \lambda_it > x_0
\end{cases}. \]

This solution implies that there are \( m \) characteristic waves emanating from \( x_0 \) on the x-t plane, each carrying a jump discontinuity in \( U \) and moving with the speeds, \( \lambda_i (i = 1, \ldots, m) \). We may view the solutions of the decoupled system (i.e. \( w_i^0(x - \lambda_it) \)) as coefficients of an eigenvector expansion of \( U \). Therefore we can also expand \( U(x, 0) \) for both left and right states as follows:

\[ \text{I.C.: } U^0(x) = KW^0(x) = KW_S = \sum_i w_iS \kappa_i, \]  

(4.13)

with

\[ S = \begin{cases} 
L & \text{if } x < x_0 \\
R & \text{if } x > x_0
\end{cases}. \]

Finally, the solution can then be expressed as a function of the initial state as follows:

\[ U(x, t) = \sum_{i=I+1}^m w_{iL} \kappa_i + \sum_{i=1}^I w_{iR} \kappa_i \quad \text{for } i = 1, \ldots, m \]  

(4.14)

where \( I \) is the maximum value of index \( i \) where \( x - \lambda_it > x_0 \) with the eigenvalues
ordered as $\lambda_1 < \lambda_2 < \ldots < \lambda_m$. Eq. 4.14 can also be written as:

$$U(x, t) = KW^0(x_0) = KW_S, \quad S = \begin{cases} L & \text{if } x - \lambda_i t < x_0 \\ R & \text{if } x - \lambda_i t > x_0 \end{cases} \quad (4.15)$$

### 4.2 Numerical Methods

The presence of shocks or discontinuities not only pose strict restrictions on the mathematical formulation of hyperbolic problems but also on the discretization techniques and numerical schemes necessary to resolve the physical phenomena accurately. Point wise approximations of the differential form of the conservation equations typically breakdown at discontinuities. This is because conservation laws of the form in Eq. 4.1 are derived from the integral form (Eq. 4.16) with the assumption that $u$ and $f(u)$ are sufficiently smooth, which is not the case at discontinuities.

$$\frac{d}{dt} \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t) dx = -f(u(x, t)) \bigg|_{x_{i-1/2}}^{x_{i+1/2}} dt \quad (4.16)$$

Eq. 4.16 implies that the rate of change of the substance $(u)$ at any given time $(t)$ is only due to the fluxes at the edges $x_{i \pm \frac{1}{2}}$. Finite volume methods are based on the integral form and offer a better approximation of the physical system. Integrating Eq. 4.16 from $t^n$ to $t^{n+1}$ gives:

$$\int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^{n+1}) dx - \int_{x_{i-1/2}}^{x_{i+1/2}} u(x, t^n) dx = -\int_{t^n}^{t^{n+1}} f(u(x, t)) \bigg|_{x_{i-1/2}}^{x_{i+1/2}} dt \quad (4.17)$$

We cannot evaluate the time integral on the right hand side exactly since $f(u(x_{i \pm \frac{1}{2}}, t)$ varies with time. Thus, we divide through by $\Delta x$ to obtain Eq. 4.18 where $F_{i \pm \frac{1}{2}}$ is an approximation to the averaged flux in time interval $t^n$ to $t^{n+1}$.

$$U_{i}^{n+1} = U_{i}^{n} + \frac{\Delta t}{\Delta x} \left( F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}} \right) \quad (4.18)$$
where:

\[
U^n_i = \frac{1}{\Delta x} \int_{x_i-\frac{1}{2}}^{x_i+\frac{1}{2}} u(x, t^n) \, dx
\]

\[
F_{i\pm\frac{1}{2}} \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_i\pm\frac{1}{2}, t)) \, dt
\]

\(F_{i\pm\frac{1}{2}}\) may also be referred to as the time average of the physical flux at the cell interfaces \(x_{i\pm\frac{1}{2}}\). \(U^n_i\) is the spatial average of the solution in the grid cell \(x_i \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]\) at time \(t^n\). Eq. 4.18 is a fully discretized finite volume formulation and applicable to both scalar and systems of conservation laws. However, we may also adopt a semi-discrete formulation by integrating Eq. 4.1 only along its spatial dimension to obtain the ODE in Eq. 4.19. Eq. 4.19 is typically the formulation of choice for ENO and WENO schemes because its semi-discrete form permits the use of higher-order time integration methods in propagating the solution in time.

\[
\frac{d}{dt} U(t) = R_i(U) \tag{4.19}
\]

where

\[
R_i(U) = \frac{1}{\Delta x} \left( F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}} \right).
\]

Eq. 4.19 remains a finite volume formulation if \(U(t)\) is taken to be the average values of the conserved variables in each grid cell \(x_i \in [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]\) at time \(t\). Note however, that the numerical fluxes \((F_{i\pm\frac{1}{2}})\) in Eq. 4.19 are different from those of Eq. 4.18. Instead of time averages, \(F_{i\pm\frac{1}{2}}\) in Eq. 4.19 are evaluated at time, \(t\). In ENO and WENO schemes, the solution of the semi-discrete scheme in Eq. 4.19 is advanced in time by TVD Runge-Kutta methods.

In order to remain within the domain of dependence in every block at each time step, the following time step limitation is applied along with the explicit formulation in Eq. 4.18:

\[
\Delta t = CFL \frac{\Delta x}{(\lambda^n_\Omega)_{max}} \tag{4.20}
\]

where: \((\lambda^n_\Omega)_{max}\) is the maximum \(\lambda\) across the spatial domain at \(t^n\) and the Courant-Friedrichs-Lewy (CFL) number is a dimensionless quantity that can be regarded as the ratio of \((\lambda^n_\Omega)_{max}\) to the grid speed \((\Delta x/\Delta t)\).

Additionally, there is the choice of variables, i.e. primitive (non-conservative) or conservative variables. For problems with shocks or discontinuities, conserved
variables must be used in order to ensure shock capturing accuracy. Hou and Le Floch [73] have shown that non-conservative formulations do not converge to the correct values when discontinuities are present in the solution. They give wrong jump conditions and therefore wrong shock speeds and strengths.

4.2.1 Monotone Schemes

Any choice of numerical flux that satisfies the consistency condition will result in a conservative scheme. The consistency property is met if for a constant value of $U$ whereby $U_i = U_{i+1} = U$,

$$F_{i+1/2} = F_{i+1/2}(U_i, U_{i+1}) = F(U) \quad (4.21)$$

A seemingly reasonable but naive choice will be to take the arithmetic average of the fluxes between two neighboring cells as follows:

$$F_{i+1/2} = F_{i+1/2}(U_i, U_{i+1}) = \frac{1}{2}(F_i + F_{i+1}) \quad (4.22)$$

This scheme is second order accurate in space and unconditionally unstable regardless of the size of the time step adopted. Another centered scheme that’s similar but conditionally stable for $0 < CFL < 1$ is the Lax-Friedrichs method.

$$F_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}) + \frac{1}{2} \frac{\Delta x}{\Delta t} (U_i - U_{i+1}) \quad (4.23)$$

The additional term on the right hand side is known as the numerical viscosity term. It artificially smears (or spreads) the solution at discontinuities thereby affecting how accurately the shape of the shock front is captured. A scheme that follows a similar form but reduces the effect of the numerical viscosity is that of Rusanov who replaced the grid speed ($\frac{\Delta x}{\Delta t}$) in Eq. 4.23 with the local spectral radius of $A$ or maximum of the absolute value of the local wave speeds $(\lambda_{n[i,i+1]}^{\text{max}})^\text{max}$. Rusanov’s numerical flux function is as follows:

$$F_{i+1/2} = \frac{1}{2}(F_i + F_{i+1}) + \frac{1}{2} (\lambda_{n[i,i+1]}^{\text{max}})^\text{max} (U_i - U_{i+1}) \quad (4.24)$$
where:
\[
(\lambda_{[i,i+1]}^n)_{\text{max}} = \max_{[i,i+1]} \left( \max_j (|\lambda_j(U_i)|), \max_j (|\lambda_j(U_{i+1})|) \right)
\]
for \( j = 1, \ldots, m \)

Another approach that reduces smearing even further is one based on identifying the upwind direction before applying the appropriate difference scheme. Such numerical schemes are called flux vector splitting (FVS) schemes and are classified under the family of upwind schemes. However, FVS schemes are limited in their applicability in that in addition to hyperbolicity, the flux term must also satisfy the homogeneity property:

\[
F(U) = A(U)U
\]  
(4.25)

The flux vector is then split into positive and negative fluxes as follows:

\[
U_t + F^+(U)_x + F^-(U)_x = 0
\]  
(4.26)

where:

\[
F^+(U) = A^+(U)U, \quad F^-(U) = A^-(U)U
\]

The split flux Jacobian is obtained by separating the positive and negative eigenvalues into \( \Lambda^+ \) and \( \Lambda^- \) respectively as follows:

\[
A^+ = K\Lambda^+K^{-1}, \quad A^- = K\Lambda^-K^{-1}
\]  
(4.27)

where:

\[
\lambda_i^+ = \frac{1}{2}(\lambda_i + |\lambda_i|), \quad \lambda_i^- = \frac{1}{2}(\lambda_i - |\lambda_i|) \quad \text{for} \quad i = 1, \ldots, m
\]

The positive and negative fluxes are approximated by backward differencing and forward differencing respectively. The corresponding conservative numerical flux for the conservative scheme of Eq. 4.18 is:

\[
F_{i+\frac{1}{2}} = F_{i+\frac{1}{2}}^+ + F_{i+\frac{1}{2}}^-
\]  
(4.28)

This natural approach to flux splitting is attributed to Steger and Warming.
and it is stable for $0 < \text{CFL} < 1$. There are also other FVS methods like that of van Leer [75] and Kinetic-FVS schemes [76], [77]. Another method that utilizes information from the physics of the problem is the Godunov scheme [78], [79]. It is one of the most successful techniques for solving systems of non-linear hyperbolic PDE’s. The key to its success is that Godunov-type schemes solve the Riemann problem at discretized cell edges (i.e. $RP(U_{i-1}, U_i)$ and $RP(U_i, U_{i+1})$). This can either be done exactly or by a suitable approximation. The Godunov numerical flux for the conservative scheme of Eq. 4.18 is:

$$F_{i+\frac{1}{2}} = F(U_{i+\frac{1}{2}}(0)) \quad (4.29)$$

where:

$$U_{i+\frac{1}{2}}(0)$$

is the solution to $RP(U_i, U_{i+1})$.

For a linear constant coefficient system like that of Eq. 4.30 the Godunov scheme can be determined from the Riemann solution obtained in section 4.1.1.

$$Ut + AUx = 0 \quad (4.30)$$

The Godunov numerical flux for the linear constant coefficient equation (Eq. 4.30) according to Eq. 4.29 can be written as:

$$F_{i+\frac{1}{2}} = AU_{i+\frac{1}{2}}(0) = A(KW^0(x_0)) = AKW_S, \quad (4.31)$$

$$S = \begin{cases} 
  i & \text{if } \lambda_j > 0 \\
  i+1 & \text{if } \lambda_j < 0 
\end{cases} \quad \text{for } j = 1, \ldots, m$$

With some linear algebra manipulations we may express Eq. 4.31 as:

$$F_{i+\frac{1}{2}} = A^+U_i^n + A^-U_{i+1}^n. \quad (4.32)$$

Eqs. 4.31 and 4.32 can also be expressed in term of left and right fluxes as follows:

$$F_{i+\frac{1}{2}} = \frac{1}{2}(F_i + F_{i+1}) + |A|(U_i - U_{i+1}) \quad (4.33)$$
\[ F_{i+\frac{1}{2}} = \frac{1}{2}(F_i + F_{i+1}) + \sum_{j=1}^{m} \alpha_j |\lambda_j| \tilde{\kappa}_j \]  

(4.34)

where:

\[ |A| = A^+ - A^- \]

\[ |\lambda_j| = \lambda_j^+ - \lambda_j^- \]

\[ \alpha_j = (w_i - w_{i+1})_j. \]

Note that the fact that the Riemann problem is solved exactly to obtain the numerical flux for Godunov scheme does not imply that it will produce an exact solution for the linear constant coefficient system (Eq. 4.30). Eqs. 4.33 and 4.34 indicate the presence of an artificial viscosity term that will lead to some smearing of shock fronts. Additionally, one can see that Eq. 4.32 is equivalent to Eq. 4.28 for the linear constant coefficient problem. There are other Godunov-type schemes that use approximate Riemann solvers to obtain the numerical flux function for non-linear systems. Examples are the Roe scheme [80],[81],[82], Osher scheme [83],[84],[85], Harten-Lax-van Leer (HLL) scheme [86],[87],[88] and HLL-Contact (HLLC) scheme [89]. Perhaps the most widely accepted of these approximate Riemann solvers is that of Roe. For this reason, it has enjoyed an extensive application and improvements that have resolved a number of its shortcomings since it was first introduced. Roe’s method solves non-linear systems of the form in Eq. 4.6 by approximating it with a linearized cell-edge-average constant-coefficient system of the form:

\[ U_t + \tilde{A} U_x = 0, \quad 0 < x < L, \quad t > 0 \]  

(4.35)

I.C.: \( U(x, 0) = U^0(x) = \begin{cases} 
U_L & \text{if } x < x_0 \\
U_R & \text{if } x > x_0 
\end{cases} \)

where:

\[ \tilde{A} = \tilde{A}(U_L, U_R) \] is evaluated at cell edges
The Riemann problem for Eq. 4.35 is then solved exactly at inter-cells and the resulting numerical flux for the conservative scheme of Eq. 4.18 is:

\[ F_{i+\frac{1}{2}} = \frac{1}{2} (F_i + F_{i+1}) + |\tilde{\mathbf{A}}|(U_i - U_{i+1}) \]  

(4.36)

where:

\[ \tilde{\mathbf{A}} = \tilde{\mathbf{A}}(U_i, U_{i+1}) \]

The integral form of Eq. 4.35 is necessary in obtaining the solution of the Riemann problem for Roe’s numerical flux. This is due to the fact that \( \tilde{\mathbf{A}} = \tilde{\mathbf{A}}(U_L, U_R) \) is evaluated as a function of both \( U_L \) and \( U_R \) as opposed to the case of the constant coefficient matrix, \( \mathbf{A} \), in the Riemann problem of Eq. 4.30 discussed earlier. Eq. 4.36 is also expressed as:

\[ F_{i+\frac{1}{2}} = \frac{1}{2} (F_i + F_{i+1}) + \frac{1}{2} \sum_{j=1}^{m} \tilde{\alpha}_j |\tilde{\lambda}_j| \tilde{\kappa}_j \]  

(4.37)

where \( \tilde{\kappa}_j, \tilde{\lambda}_j \) and \( \tilde{\alpha}_j \) are the \( j^{th} \) eigenvalue, eigenvector and wave strength of \( \tilde{\mathbf{A}} \). In addition to hyperbolicity, \( \tilde{\mathbf{A}} \) must satisfy both the consistency and conservation properties:

\[ \tilde{\mathbf{A}}(U, U) = \mathbf{A}(U) \]  

(4.38)

\[ \mathbf{F}(U_R) - \mathbf{F}(U_L) = \tilde{\mathbf{A}}(U_R - U_L) \]  

(4.39)

Roe developed a method for determining the cell-edge-average matrix, \( \tilde{\mathbf{A}} \), by introducing an intermediate variable \( \mathbf{Q} \), such that \( \mathbf{U} \) and \( \mathbf{F} \) can be expressed in terms of \( \mathbf{Q} \) with the aid of transformation matrices as follows:

\[ \Delta \mathbf{U} = \tilde{\mathbf{B}}(\mathbf{Q}) \Delta \mathbf{Q}, \]  

(4.40)

\[ \Delta \mathbf{F} = \tilde{\mathbf{C}}(\mathbf{Q}) \Delta \mathbf{Q} \]  

(4.41)

From Eqs. 4.40 and 4.41 we obtain the following:

\[ \tilde{\mathbf{A}} = \tilde{\mathbf{C}}\tilde{\mathbf{B}}^{-1}. \]  

(4.42)
Roe and Pike [82] later presented a more efficient approach to determining \( \tilde{\kappa}_i, \tilde{\lambda}_i \) and \( \tilde{\alpha}_i \). It involves obtaining what is known as the Roe-Pike averages, which are then used to evaluate \( \tilde{\kappa}_i, \tilde{\lambda}_i \) and \( \tilde{\alpha}_i \) or \( \tilde{A} \). One major shortcoming of the Roe scheme is that it computes unphysical rarefaction shocks, which violate the entropy condition. A number of corrections (see [90],[91]) have been proposed to fix this issue. The numerical flux for Roe scheme with entropy correction typically takes the form:

\[
F_{i+\frac{1}{2}} = \frac{1}{2}(F_i + F_{i+1}) + \frac{1}{2} \sum_{j=1}^{m} \tilde{\alpha}_j \psi(\tilde{\lambda}_j) \tilde{\kappa}_j \tag{4.43}
\]

Harten and Hyman [90] presented the following methods for obtaining the entropy fix, \( \psi(\tilde{\lambda}_j) \):

\[
\psi^{HH1}(\tilde{\lambda}_j) = \begin{cases} 
\delta_j & \text{if } |\tilde{\lambda}_j| < \delta_j \\
|\tilde{\lambda}_j| & \text{if } |\tilde{\lambda}_j| \geq \delta_j 
\end{cases} \tag{4.44}
\]

where:

\[
\delta_j = \max\{0, \tilde{\lambda}_j - \lambda_j(u_i), \lambda_j(u_{i+1}) - \tilde{\lambda}_j\}
\]

and

\[
\psi^{HH2}(\tilde{\lambda}_j) = \begin{cases} 
\frac{\tilde{\lambda}_j^2 + \delta_j^2}{2\delta_j} & \text{if } |\tilde{\lambda}_j| < \delta_j \\
|\tilde{\lambda}_j| & \text{if } |\tilde{\lambda}_j| \geq \delta_j 
\end{cases} \tag{4.45}
\]

where:

\[
\delta_j = \max\{0, \tilde{\lambda}_j - \lambda_j(u_i), \lambda_j(u_{i+1}) - \tilde{\lambda}_j\}.
\]

The correction has the effect of smoothing out the rarefaction shock. It has been suggested by the work of Harten [92] that the parameter \( \delta_j \) may be chosen somewhat arbitrarily. One form for \( \delta_j \) that is applied to the Euler equation in the open source code, Gryphon, which seems to do a better job with the smoothening process is:

\[
\delta_j = \delta = v_i - v_{i+1} \tag{4.46}
\]
where \( v \) is the fluid velocity.

The disadvantage of the monotone schemes presented is that they experience smearing or numerical diffusion at discontinuities. However, one monotone scheme that doesn’t suffer from smearing is the random choice method (RCM) [93],[94],[95],[96]. However, like the Godunov method, it requires the exact solution to the Riemann problem. RCM is based on a random sample of state in the local Riemann solution using a sequence of random numbers as opposed to an integral average of the solution to the Riemann problem. The statistical property of the random sample determines the accuracy of the method. Unlike Godunov’s method, there are no approximate Riemann solvers based on RCM.

One monotone scheme that stems from RCM is the First-Order Centered (ForCE) scheme. It is obtained by replacing the stochastic integrations in the development of RCM with deterministic ones. The ForCE numerical flux function for the conservative scheme of Eq. 4.18 is:

\[
F_{i+\frac{1}{2}} = \frac{1}{2} \left[ F^n_{i+\frac{1}{2}} + \frac{1}{2}(F^n_i + F^n_{i+1}) \right] + \frac{1}{4} \frac{\Delta x}{\Delta t} (U^n_i - U^n_{i+1})
\]

where:

\[
F^n_{i+\frac{1}{2}} = F(U^n_{i+\frac{1}{2}})
\]

\[
U^n_{i+\frac{1}{2}} = \frac{1}{2} (U^n_i + U^n_{i+1}) + \frac{\Delta x}{2\Delta t} (F^n_i - F^n_{i+1}).
\]

The resulting scheme turns out to be an arithmetic average of the Lax-Friedrichs and the second-order accurate Richtmyer scheme. However, the scheme produces no spurious oscillations and instead displays only the typical properties of first-order schemes.

### 4.2.2 Higher-Order Monotonicity Preserving Schemes

Higher-order schemes, which include those of Lax-Wendroff (LW) [97], Richtmyer [98], MacCormack [99], weighted-average flux (WAF) [100], [101] and those based on MUSCL-reconstruction [102], [103], [104], [105], [106], better approximate the continuous solution but are prone to spurious oscillations at discontinuities. In an effort to combine the benefits of higher-order schemes with the monotone nature of first-order schemes, higher-order monotonicity preserving schemes were developed.
Schemes based on ensuring the TVD property or ENO-type schemes [107], help to eliminate or limit spurious oscillations typical of higher-order schemes.

The theory of TVD methods is based on one dimensional scalar conservation laws (see Harten [92]). Its extension to non-linear multidimensional problems is purely empirical. However, numerous applications of TVD methods to practical problems over the years have demonstrated the validity of this extension. A numerical scheme is TVD if:

\[ TV(U^{n+1}) \leq TV(U^n), \forall n \] (4.48)

where:

\[ TV(U) = \sum_i |U_{i+1} - U_i| \] and \( i \) represents grid cells

Here we present two approaches for constructing TVD methods: (1) Flux Limiting and (2) Variable Extrapolation (MUSCL).

<table>
<thead>
<tr>
<th>Limiter Name</th>
<th>Limiter</th>
<th>( \lim_{r \to \infty} \phi )</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>van Leer</td>
<td>( \phi_{vl}(r) = \frac{r +</td>
<td>r</td>
<td>}{1 +</td>
</tr>
<tr>
<td>MC***</td>
<td>( \phi_{mc}(r) = \max[0, \min(2r, 0.5(1 + r), 2)] )</td>
<td>2</td>
<td>van Leer, 1977</td>
</tr>
<tr>
<td>van Albada</td>
<td>( \phi_{va}(r) = \frac{r^2 + r}{r^2 + 1} )</td>
<td>1</td>
<td>van Albada et al., 1982</td>
</tr>
<tr>
<td>Minmod</td>
<td>( \phi_{mm}(r) = \max[0, \min(1, r)] )</td>
<td>1</td>
<td>Roe, 1982</td>
</tr>
<tr>
<td>SuperBee</td>
<td>( \phi_{sb}(r) = \max[0, \min(2r, 1), \min(r, 2)] )</td>
<td>2</td>
<td>Roe, 1983</td>
</tr>
<tr>
<td>Osher***</td>
<td>( \phi_{os}(r) = \max[0, \min(r, \beta)] )</td>
<td>( \beta )</td>
<td>Chakravathy &amp; Osher, 1983</td>
</tr>
<tr>
<td>Sweby</td>
<td>( \phi_{sw}(r) = \max[0, \min(\beta r, 1), \min(r, \beta)] )</td>
<td>( \beta )</td>
<td>Sweby, 1984</td>
</tr>
<tr>
<td>Ospre</td>
<td>( \phi_{op}(r) = \frac{1.5(r^2 + r)}{(r^2 + r + 1)} )</td>
<td>1.5</td>
<td>Waterson &amp; Deconinck, 1995</td>
</tr>
</tbody>
</table>

\[ 1 \leq \beta \leq 2 \quad \text{Asymmetric Limiter} \quad \beta \quad \text{Monotonized Central (MC)} \]

Table 4.1. Example of limiters from literature

4.2.2.1 A Flux Limiting Approach

A general flux-limited TVD scheme can be achieved by combining a monotone scheme and a higher-order scheme with the aid of a flux/slope limiter function (\( \phi \)). For the conservative scheme of Eq. 4.18 the generalized flux limited scheme is expressed as follows:

\[ F_{i+\frac{1}{2}} = F_{i+\frac{1}{2}}^{low} + \phi_{i+\frac{1}{2}}(F_{i+\frac{1}{2}}^{high} - F_{i+\frac{1}{2}}^{low}) \] (4.49)
The slope/flux limiter is defined as follows:

$$
\phi_{i+\frac{1}{2}} = \min\left(\min_j(\phi(r^{L}_{i+\frac{1}{2}})), \min_j(\phi(r^{R}_{i+\frac{1}{2}}))\right)
$$

(4.50)

with:

$$
r^{L}_{i+\frac{1}{2}} = \frac{U_{i+1} - U_i}{U_{i+1} - U_i}, \quad r^{R}_{i-\frac{1}{2}} = \frac{U_{i+1} - U_i}{U_{i+1} - U_i};
$$

$$
r^{L}_{i+\frac{1}{2}} = \frac{U_{i+2} - U_{i+1}}{U_{i+2} - U_i}, \quad r^{L}_{i-\frac{1}{2}} = \frac{U_{i-2} - U_{i-1}}{U_{i-2} - U_{i-1}};
$$

$$
F_{i+\frac{1}{2}}^{\text{low}} \text{ can be any of the monotone schemes discussed in section 4.2.1. } F_{i+\frac{1}{2}}^{\text{high}} \text{ can be any higher-order scheme such as the second-order accurate Richtmyer method.}
$$

The numerical flux of the Richtmyer method for the conservative scheme of Eq. 4.18 is:

$$
F_{i+\frac{1}{2}} = F(U_{i+\frac{1}{2}})
$$

(4.51)

where:

$$
U_{i+\frac{1}{2}} = \frac{1}{2}(U_i + U_{i+1}) + \frac{1}{2}\frac{\Delta t}{\Delta x}(F_i - F_{i+1})
$$

It can be inferred from Eq. 4.49 that when $\phi$ is zero or one, the scheme reverts back to the basic low-order monotone scheme or higher-order scheme respectively.

The idea is to maintain higher order accuracy in regions where the solution is smooth and to gradually revert back to the lower-order scheme at discontinuities to avoid the spurious oscillations. Table 4.1 contains some well known limiters. $\phi$ is constrained to be greater than or equal to zero.

### 4.2.2.2 Variable Extrapolation (MUSCL) Approach

In the formulation of Eq. 4.18 we have assumed piece-wise constant averages across grid cells. However, with variable extrapolation we instead used piece-wise linear averages to approximate the state variables within each grid cell. Hence, each grid cell will have two different values at its edges as opposed to one piece-wise constant value across the cell. This procedure is also known as MUSCL-reconstruction. The linear extrapolation provides second-order accuracy in the spatial domain and parabolic extrapolation provides third-order accuracy. The
the corresponding flux function for the conservative scheme of Eq. 4.18 is as follows:

\[ F_{i + \frac{1}{2}} = F(\bar{U}_R^i, \bar{U}_L^{i + 1}) \]  \hspace{1cm} (4.52)

Note that the numerical flux \( F_{i + \frac{1}{2}} \) at the cell edge, \( x_{i + \frac{1}{2}} \), is no longer evaluated at the neighboring states as in \( F_{i + \frac{1}{2}} = F_{i + \frac{1}{2}}(U_i, U_{i+1}) \) but is instead evaluated at the new states \( \bar{U}_R^i \) and \( \bar{U}_L^{i + 1} \) as shown in Eq. 4.52. \( \bar{U}_R^i \) and \( \bar{U}_L^{i + 1} \) are the evolved values of the extrapolations (\( U_R^i \) and \( U_L^{i + 1} \)) over a half step as follows:

\[
\bar{U}_R^i = U_R^i + \frac{1}{2} \frac{\Delta t}{\Delta x} (F(U_L^i) - F(U_R^i)) \\
\bar{U}_L^{i + 1} = U_L^{i + 1} - \frac{1}{2} \frac{\Delta t}{\Delta x} (F(U_R^i) - F(U_L^{i + 1}))
\]  \hspace{1cm} (4.53)

A linear form of the extrapolations is as shown in Eq. 4.54. A slope limiter is used to eliminate the oscillations that are produced as a result of the increase in the order of approximation.

\[
\begin{align*}
U_R^i &= U_i + \frac{1}{2} \phi_{i + \frac{1}{2}} (U_{i+1} - U_i) \\
U_L^{i + 1} &= U_i - \frac{1}{2} \phi_{i - \frac{1}{2}} (U_i - U_{i-1})
\end{align*}
\]  \hspace{1cm} (4.54)

The slopes (\( r \)) and limiters (\( \phi \)) are as defined in Eq. 4.50 and the limiters are available in Table 4.1. The symmetric limiters exhibit the symmetry property:

\[
\frac{\phi(r)}{r} = \phi\left(\frac{1}{r}\right)
\]  \hspace{1cm} (4.55)

Symmetry ensures that forward and backward facing gradients are handled in the same manner (see Hirsch [108]). Also, note that \( r_{i + \frac{1}{2}} = (r_{i - \frac{1}{2}})^{-1} \).

### 4.2.2.3 ENO and WENO schemes

ENO schemes were first introduced in the late 80’s [107], [109] and predicated on the idea that a wider interpolation stencil yields higher order accuracy. However, the non-oscillatory nature of ENO schemes is achieved by adaptively choosing the appropriate stencil to ensure that the function interpolated remains smooth within the stencil. This choice is based on the absolute value of the divided differences (or undivided in the case of uniform grids) of the averaged piece-wise primitive
variables around the block of interest. As a result, ENO reconstructions are uniformly higher-order accurate right up to the discontinuity unlike the previously discussed TVD methods. Weighted-ENO (WENO) schemes make the following improvements to ENO schemes:

- Eliminates the need for the many logical if statements in the adaptive stencil choosing process
- Eliminates errors that result from stencil change as a result of round off error perturbation near zero divided differences in smooth regions
- Allows a bias towards the stencils with the smoother solutions as opposed to having one fixed set of weights for all candidate stencils considered.

The semi-discrete formulation of Eq. 4.19 is typically utilized in the development of ENO and WENO schemes [110],[111],[112],[113] [114]. The numerical fluxes \( F_{i \pm \frac{1}{2}} = F(U(x_{i \pm \frac{1}{2}}, t)) \) in Eq. 4.19 are replaced by the monotone numerical fluxes \( F(U^{+}_{i \pm \frac{1}{2}}, U^{-}_{i \pm \frac{1}{2}}) \) where \( U^{-}_{i \pm \frac{1}{2}} \) and \( U^{+}_{i \pm \frac{1}{2}} \) are reconstructed by the ENO or WENO methods using the cell averaged values \( (U_i) \) for a finite volume formulation. Numerical solutions to Eq. 4.19 are advanced in time by means of TVD Runge-Kutta time integrators [110]. A common version is the third-order TVD Runge-Kutta method:

\[
\begin{align*}
U_i^{n+\frac{1}{3}} &= U_i^n + \Delta t R_i(U_i^n) \\
U_i^{n+\frac{2}{3}} &= U_i^n + \frac{1}{4}U_i^{n+\frac{1}{3}} + \frac{1}{4}R_i(U_i^{n+\frac{1}{3}}) \\
U_i^{n+1} &= U_i^n + \frac{2}{3}U_i^{n+\frac{2}{3}} + \frac{2}{3}R_i(U_i^{n+\frac{2}{3}})
\end{align*}
\] (4.56)

### 4.2.2.4 WENO Reconstruction

The WENO reconstruction of \( U_{i \pm \frac{1}{2}}^- \) is obtained as follows:

- First reconstruct the \( k \)-th degree polynomial \( p_j(x) \), associated with each of the stencils \( S_j \in [x_{i-j}, \ldots, x_{i-j+k}] \) where \( j = 0, \ldots, k \), and \((2k)\)-th degree polynomial \( Q(x) \), associated with the larger stencil \( T \in [x_{i-k}, \ldots, x_{i+k}] \) such that:

\[
U_{i+l} = \frac{1}{\Delta x_{i+l}} \int_{I_{i+l}} p_j(x) dx \quad l = -j, \ldots, -j + k
\]
\[ U_{i+l} = \frac{1}{\Delta x_{i+l}} \int_{I_{i+l}} Q(x) dx \quad l = -k, \ldots, k \]

where:

\[ I_{i+l} \in [x_{i+l-\frac{1}{2}}, x_{i+l+\frac{1}{2}}] \]

- Determine linear weight \( \gamma_0, \ldots, \gamma_k \) from the relation:

\[ Q(x_{i+\frac{1}{2}}) = \sum_{j=0}^{k} \gamma_j p_j(x_{i+\frac{1}{2}}) \tag{4.57} \]

- Compute the smoothness indicator, \( \beta_j \), for each stencil, \( S_j \), which measures how smooth the function \( p_j(x) \) is in the target cell \( I_i \). \( \beta_j \) is defined as follows:

\[ \beta_j = \sum_{l=1}^{k} \int_{I_i} \Delta x_i^{2l-1} \left( \frac{\partial^l}{\partial x^l} p_j(x) \right)^2 dx \tag{4.58} \]

Note that the smaller \( \beta_j \) is, the smoother the function \( p_j(x) \) is in the target cell, \( I_i \).

- Compute the non-linear weights \( \omega_j \) based on the smoothness indicators, \( \beta_j \) as follows:

\[ \omega_j = \frac{\alpha_j}{\sum_j \alpha_j} \tag{4.59} \]

with:

\[ \alpha_j = \frac{\gamma_j}{\sum_j (\varepsilon + \beta_j)^2} \]

\( \varepsilon \) could be any small number such as \( 1 \times 10^{-15} \).

- Finally, the WENO reconstructed value at the right edge of the target cell, \( I_i \), is given as:

\[ U_{i+\frac{1}{2}} = \sum_{j=0}^{k} \omega_j p_j(x_{i+\frac{1}{2}}). \tag{4.60} \]
Note that the reconstruction at the left edge \((U_{i-\frac{1}{2}}^+)\) of the target cell is mirror symmetric with respect to \(x_i\) of the above procedure. Additionally, it has been shown that use of conserved variables in the reconstruction process results in oscillations even for simple test problems \([109],[115]\). Hence for systems of conservation laws the reconstruction procedure should be performed in primitive variables.

For a 5th-order WENO reconstruction on a uniform grid (i.e. \(\Delta x_i = \Delta x\)), \(k = 2\) and \(p_j(x), Q(x), \beta_j\) and \(\gamma_j\) are defined as follows:

\[
\begin{align*}
p_0(x_{i+\frac{1}{2}}) &= \frac{1}{6}(2W_i + 5W_{i+1} - W_{i+2}) \\
p_1(x_{i+\frac{1}{2}}) &= \frac{1}{6}(-W_{i-1} + 5W_i + 2W_{i+1}) \\
p_2(x_{i+\frac{1}{2}}) &= \frac{1}{6}(2W_{i-2} - 5W_{i-1} + 11W_i)
\end{align*}
\]

where: \(W\) is the vector of primitive variables.

\[
Q(x_{i+\frac{1}{2}}) = \frac{1}{30}W_{i-2} - \frac{13}{60}W_{i-1} + \frac{47}{60}W_i + \frac{9}{20}W_{i+1} - \frac{1}{20}W_{i+2}
\]

\[
\begin{align*}
\beta_0 &= \frac{13}{12}(W_i - 2W_{i+1} + W_{i+2})^2 + \frac{1}{4}(3W_i - 4W_{i+1} + W_{i+2})^2 \\
\beta_1 &= \frac{13}{12}(W_{i-1} - 2W_i + W_{i+1})^2 + \frac{1}{4}(W_{i-1} - W_{i+1})^2 \\
\beta_2 &= \frac{13}{12}(W_{i-2} - 2W_{i-1} + W_i)^2 + \frac{1}{4}(W_{i-2} - 4W_{i-1} - 3W_i)^2.
\end{align*}
\]

Substitution of Eqs. 4.61 & 4.62 into Eq. 4.57 we solve for \(\gamma_j\) and obtain:

\[
\begin{align*}
\gamma_0 &= \frac{3}{10}, & \gamma_1 &= \frac{6}{10}, & \gamma_2 &= \frac{1}{10}
\end{align*}
\]

### 4.2.3 Extension to Inhomogeneous and Multidimensional Problems

Solving inhomogeneous problems of the form in Eq. 4.1 may be done in three ways. One is a direct extension of the finite volume formulation in Eq. 4.18. If the source term is a flux term it is included in the space and time integrations used to develop Eq. 4.18 \([116],[117]\). However, if the source term contains spatial averages that are simply functions of the unknown variables as in Eq. 4.1, it may
be included in Eq. 4.18 as follows:

\[ U_{i}^{n+1} = U_{i}^{n} + \frac{\Delta t}{\Delta x} \left( F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}} \right) + \Delta t B_{i}^{n}. \]  

(4.64)

A second approach to handling the source term is the splitting approach. Here Eq. 4.1 is first solved as a homogeneous problem (\( B = 0 \)) and then the solution to the homogeneous system is used as an initial condition for the simplified ODE problem (Eq. 4.66) as follows:

\[
\begin{align*}
& U_{t} + F(U)_{x} = 0 \\
& I.C. : U(x, t) = U^{n} \\
& \implies \mathbf{U}^{n+1} \\
& U_{t} = B \\
& I.C. : U(x, t) = U^{n+1} \\
& \implies \mathbf{U}^{n+1} \quad \text{(4.65)}
\end{align*}
\]

(4.65)

\[
\begin{align*}
& U_{t} + F(U)_{x} = 0 \\
& I.C. : U(x, t) = U^{n} \\
& \implies \mathbf{U}^{n+1} \\
& F(U)_{x} = B \\
& I.C. : U(x, t) = U^{n+1} \\
& \implies \mathbf{U}^{n+1} \quad \text{(4.66)}
\end{align*}
\]

A third approach follows a similar pattern with a different ODE problem (Eq. 4.68) in the second step as follows:

\[
\begin{align*}
& U_{t} + F(U)_{x} = 0 \\
& I.C. : U(x, t) = U^{n} \\
& \implies \mathbf{U}^{n+1} \\
& F(U)_{x} = B \\
& I.C. : U(x, t) = U^{n+1} \\
& \implies \mathbf{U}^{n+1} \quad \text{(4.67)}
\end{align*}
\]

(4.67)

\[
\begin{align*}
& U_{t} + F(U)_{x} + G(U)_{y} + H(U)_{z} = 0 \\
& \text{with} \\
& I.C. : U(x, t) = U^{n} \quad \text{(4.69)}
\end{align*}
\]

For multidimensional problems, there are two main methods for handling the additional dimensions: (1) unsplit and (2) split methods. Consider the three dimensional initial value problem:

\[ U_{t} + F(U)_{x} + G(U)_{y} + H(U)_{z} = 0 \]  

(4.69)

with

\[ I.C. : U(x, t) = U^{n} \]

With the unsplit method, the solution is evolved in one step through the fol-
lowing formulation:

\[
U_{i}^{n+1} = U_{i}^{n} + \frac{\Delta t}{\Delta x} \left( F_{i-\frac{1}{2}}^{n} - F_{i+\frac{1}{2}}^{n} \right) + \frac{\Delta t}{\Delta y} \left( G_{i-\frac{1}{2}}^{n} - G_{i+\frac{1}{2}}^{n} \right) + \frac{\Delta t}{\Delta z} \left( H_{i-\frac{1}{2}}^{n} - H_{i+\frac{1}{2}}^{n} \right)
\]  
(4.70)

With the split method, the solution may be evolved over three steps as follows:

\[
\begin{align*}
U_t + F(U)_x &= 0 & \implies U^{n+\frac{1}{3}} \\
I.C. : U(x,t) &= U^n \\
U_t + G(U)_y &= 0 & \implies U^{n+\frac{2}{3}} \\
I.C. : U(x,t) &= U^{n+\frac{1}{3}} \\
U_t + H(U)_z &= 0 & \implies U^{n+1} \\
I.C. : U(x,t) &= U^{n+\frac{2}{3}}
\end{align*}
\]  
(4.71, 4.72, 4.73)

The time step for a multidimensional problem of the form in Eq. 4.69 may be determined as follows:

\[
\Delta t = CFL \times \min \left[ \frac{\Delta x}{(\lambda_{ijk}^n)_{max}}, \frac{\Delta y}{(\lambda_{ijk}^n)_{max}}, \frac{\Delta z}{(\lambda_{ijk}^n)_{max}} \right]
\]  
(4.74)

where:

\[(\lambda_{ijk}^n)_{max} = \max_{\Omega} |\lambda_{ijk}^n|\]

### 4.2.4 Implicit FVM formulation

In this review, it is assumed that the numerical flux functions \( F_{i\pm\frac{1}{2}}^{n} = F_{i\pm\frac{1}{2}}^{n} \) in Eq. 4.18 are evaluated at time level \( n \) so that the formulation is explicit. However Eq. 4.18 may also be expressed in the implicit form whereby \( F_{i\pm\frac{1}{2}} = F_{i\pm\frac{1}{2}}^{n+1} \) [134], [118]. The benefit of implicit formulations are many, but one major one is the stability of the system even with the use of larger time steps since implicit methods are not restricted by the CFL condition. However, there are disadvantages such as a reduction in accuracy with large time steps and the need to construct large Jacobian matrices.
4.3 Test Problems

In order to illustrate the application of some of the numerical schemes discussed, we solve the initial boundary value problem (IBVP) of the form:

\[ U_t + F(U)_x = B, \quad 0 < x < L, \quad t > 0, \quad (4.75) \]

with

\[
\begin{align*}
\text{I.C.:} & \quad U(x, 0) = U^0(x), \\
\text{B.C.:} & \quad U(0, t) = U_L(t), \quad U(L, t) = U_R(t)
\end{align*}
\]

Considering that most of the numerical methods discussed were developed with the Euler equation in mind, it is appropriate that we first consider their application to the Euler equation. In this section we present the various transport equations in detail along with their decomposed characteristics. The initial and boundary conditions will be presented in Sec. 4.4.

4.3.1 Euler Equations

The Euler equations are a set of conservation equations that govern the inviscid flow of fluids. They are a sub-model of the Navier-Stoke equations without the effects of viscosity or heat exchange. This equation and its variations are used in transient natural gas flow simulations [134], [119], [120], [121]. The one-dimensional isentropic Euler equations in conservation form are as follows:

\[ U_t + F(U)_x = 0, \quad (4.76) \]

where:

\[
U = \begin{bmatrix} \rho \\ \rho v \\ \rho E \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho v^2 + p \\ v(\rho E + p) \end{bmatrix}
\]

The variable \( \rho, v, E \) and \( p \) are are density, velocity, specific total energy and pressure respectively. Eq. 4.76 contains three conservation equations and four unknowns. For closure the following ideal gas equation of state (EOS) is used to
relate pressure to internal energy and density:

\[ p = \rho (\gamma - 1) e \]  \hspace{1cm} (4.77)

where:

\[ e = E - \frac{1}{2}v^2, \quad \gamma = \frac{c_p}{c_v} \]

The parameters \( e \) and \( \gamma \) are specific internal energy and adiabatic index. The adiabatic index is the ratio of specific heat at constant pressure \( (c_p) \) to specific heat at constant volume \( (c_v) \). When the adiabatic index is assumed constant, the system is said to be isentropic. Given the isentropic EOS, the speed of sound within the medium can be expressed as:

\[ a = \sqrt{\frac{\gamma p}{\rho}} \]  \hspace{1cm} (4.78)

4.3.1.1 Characteristic Analysis

When Eq. 4.76 is expressed in its quasi-linear form as in Eq. 4.2, its corresponding flux Jacobian matrix \( A(U) \) is:

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
\frac{1}{2}(\gamma - 3)v^2 & -(\gamma - 3)v & \gamma - 1 \\
\frac{1}{2}(\gamma - 2)v^2 - \frac{a^2 v}{\gamma - 1} & \frac{3 - 2\gamma}{2}v^2 + \frac{a^2}{\gamma - 1} & \gamma v
\end{bmatrix}
\]  \hspace{1cm} (4.79)

However it is often expressed in terms of specific total enthalpy \( (H) \) as follows:

\[
A = \begin{bmatrix}
0 & 1 & 0 \\
\frac{1}{2}(\gamma - 3)v^2 & -(\gamma - 3)v & \gamma - 1 \\
v\left[\frac{1}{2}(\gamma - 1)v^2 - H\right] & H - (\gamma - 1)v^2 & \gamma v
\end{bmatrix}
\]  \hspace{1cm} (4.80)

where:

\[ H = E + \frac{p}{\rho} \]

By solving the characteristic polynomial in Eq. 4.4 and the expression in Eq.
4.5 we obtain the eigenvalues and eigenvectors of $A(U)$.

\[
\Lambda = \begin{bmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{bmatrix} = \begin{bmatrix}
v - a & 0 & 0 \\
0 & v & 0 \\
0 & 0 & v + a
\end{bmatrix}
\] (4.81)

\[
K = [\kappa_1, \kappa_2, \kappa_3] = \begin{bmatrix}
1 & 1 & 1 \\
v - a & v & v + a \\
H - va & \frac{1}{2}v^2 & H + va
\end{bmatrix}
\] (4.82)

The Euler equation is strictly hyperbolic. Additionally, it satisfies the homogeneity property (Eq. 4.25) and therefore all of the numerical methods described in section 4.2.1 including flux vector splitting methods (Eq. 4.28) can be applied in solving this system of equations.

### 4.3.2 Pneumatic Transport Equations

Applications of pneumatic transport modeling cuts across both upstream and downstream sectors of the petroleum industry. In the petrochemical sector, pneumatic transport models are employed in circulating fluidized bed (CFB) riser simulations [122],[123],[124],[125]. Upstream, pneumatic transport models are used to describe cuttings circulation for injection rate requirement predictions in air drilling processes [126], [127]. There are also cuttings circulation studies that use other forms of drilling fluids with a similar set of equations [128],[129]. A one dimensional two-phase Eulerian-Eulerian transport equation is as follows:

\[
U_t + F(U)_x = B(U),
\] (4.83)

where:

\[
U = \begin{bmatrix}
\varepsilon \rho_g \\
(1 - \varepsilon) \rho_s \\
\varepsilon \rho_g v_g \\
(1 - \varepsilon) \rho_s v_s
\end{bmatrix},
\]

\[
F(U) = \begin{bmatrix}
\varepsilon \rho_g v_g \\
(1 - \varepsilon) \rho_s v_s \\
\varepsilon \rho_g v_g^2 + p_g \\
(1 - \varepsilon) \rho_s v_s^2 + p_s
\end{bmatrix},
\]
\[
\mathbf{B}(\mathbf{U}) = \begin{bmatrix}
0 \\
0 \\
F_{gg} + F_{wg} - F_d \\
F_{gs} + F_{ws} + F_d
\end{bmatrix}
\]

Subscripts \( g \) and \( s \) represent the gas and dispersed solids phases respectively. \( \rho, v \) and \( p \) are density, velocity, and pressure. \( \varepsilon \) is the gas volume fraction, which is constrained by \( \varepsilon_{\min} < \varepsilon < 1 \). The source term \( \mathbf{B} \) contains empirical relations for determining force per unit volume of gravity (\( F_{gg} \) \& \( F_{gs} \)), wall friction (\( F_{wg} \) \& \( F_{ws} \)) and fluid-particle drag (\( F_d \)). Eq. 4.83 contains mass and momentum conservation for both the dispersed and carrier phase. Since solid density (\( \rho_s \)) is a known constant we have six unknowns and four equations. Gas pressure is related to gas density and temperature by the following EOS:

\[
p_g = \frac{RT}{M_g} \rho_g
\]

\( R, M_g \) and \( T \) are the universal gas constant (\( 8.314 \times 10^3 \frac{J}{kmolK} \)), molecular mass of the gas phase (\( M_{air} = 28.97 \frac{kg}{kmol} \)) and fluid temperature. For this system, pressure of the carrier phase is directly proportional to its density with the proportionality constant \( \frac{RT}{M_g} \). The solid pressure is expressed in terms of a modulus of elasticity, \( G(\varepsilon) \), which is assumed to be a function only of the local value of the voidage. The solid pressure gradient represents the normal stresses due to particle-particle interactions. Its inclusion helps to prevent the solid phase from reaching unrealistic low values of void fraction in addition to helping to ensure strict hyperbolicity, which though not necessary for stability is a desirable characteristic. It is expressed as follows:

\[
\nabla p_s = G(\varepsilon) \nabla \varepsilon
\]

A number of empirical correlations for the modulus of elasticity, \( G(\varepsilon) \) has been proposed and several comparative studies \([60], [61]\) have been carried out to test their efficacy. In this paper we adopt one proposed by Gidaspow and Ettehadieh
Eq. 4.83 can be expressed as:

\[ U_t + F(U)x = B(U), \]  

where:

\[ U = \begin{bmatrix} \varepsilon \rho_g \\ (1 - \varepsilon) \rho_s \\ \varepsilon \rho_g v_g \\ (1 - \varepsilon) \rho_s v_s \end{bmatrix}, \quad F(U) = \begin{bmatrix} \varepsilon \rho_g v_g \\ (1 - \varepsilon) \rho_s v_s \\ \varepsilon \rho_g v_g^2 + a^2 \rho_g \\ (1 - \varepsilon)(\rho_s v_s^2 + G(\varepsilon)) \end{bmatrix}, \]

\[ B(U) = \begin{bmatrix} 0 \\ 0 \\ F_{gg} + F_{wg} - F_d \\ (1 - \varepsilon)G(\varepsilon)x + F_{gs} + F_{ws} + F_d \end{bmatrix} \]

\( a \) is the constant speed of sound determined as follows:

\[ a = \sqrt{\frac{dp_g}{d\rho}} = \sqrt{\frac{RT}{M_g}} \]  

The gravitational force per unit volume is determined as follows:

\[ F_{gg} = \varepsilon \rho_g g, \quad F_{gs} = (1 - \varepsilon)(\rho_s - \rho_g)g \]  

\( g \) is the gravitational acceleration (9.8 m/s\(^2\)). The viscous force per unit volume is determined as follows:

\[ F_{wi} = \frac{4}{D_h} \left( \frac{1}{2} \rho v_i^2 \right) f_i \quad i = g, s \]  

The parameters \( d \) and \( f \) are the pipe diameter and dimensionless friction factor respectively. For the gas phase friction factor, \( f_g \), is a function of pipe roughness.
(ε) and Reynolds number (Re):

\[
Re = \frac{\varepsilon \rho_g v_g d}{\mu_g}
\]  

(4.91)

where \( \mu_g \) in Eq. 4.91 is the gas viscosity. \( f_g \) is estimated from the fanning (Eq. 4.92) for laminar flow (Re < 2100) and Chen’s [66] friction factor equations (Eq. 4.93) for turbulent flow conditions (Re > 2100).

\[
f_g = \frac{16}{Re}
\]  

(4.92)

\[
\frac{1}{\sqrt{f_g}} = -4 \log \left[ \frac{(\varepsilon D_h)}{3.7065} - \frac{5.0452}{Re} \log \left( \frac{1.1098}{2.8257} + \frac{5.8506}{Re^{0.898}} \right) \right]
\]  

(4.93)

The dispersed phase friction factor, \( f_s \), is determined by the empirical correlation of Konno and Saito [130]:

\[
f_s = 0.0285 \sqrt{g d} / v_s
\]  

(4.94)

The fluid-particle drag force (\( F_d \)) is estimated as follows:

\[
F_d = \beta (v_g - v_s)
\]  

(4.95)

\( \beta \) is the inter-phase friction coefficient and is estimated using the equation for drag on a single sphere in particle-laden fluid [68],[69],[70].

\[
\beta = \frac{3}{4} \frac{C_d \varepsilon (1 - \varepsilon) \rho_g |v_g - v_s|}{d_p} g(\varepsilon).
\]  

(4.96)

The parameters \( d_p \) and \( C_d \) are the weighted average particle diameter and dimensionless single sphere drag coefficient respectively. Note that the voidage function, \( g(\varepsilon) \), accounts for the effects of neighboring particles on the drag of individual particles. For Wen and Yu [68], \( g(\varepsilon) = \varepsilon^{-2.65} \) (see also Di Felice[69]). The single sphere drag coefficient (\( C_d \)) is estimated with the correlation of Turton
and Levenspiel [72]:

\[
C_d = \begin{cases} 
\frac{24}{Re_r} + \frac{4.152}{Re_r^{0.844}} + \frac{0.433}{1+16300Re_r^{-1.09}} & \text{if } Re_r < 1 \times 10^{-5} \\
0.44 & \text{if } Re_r > 1 \times 10^{-5}
\end{cases}
\]

(4.97)

where:

\[
Re_r = \frac{\varepsilon \rho_g |v_g - v_s| d_p}{\mu_g}
\]

4.3.2.1 Characteristic analysis

When Eq. 4.87 is expressed in its quasi-linear form as in Eq. 4.2 its corresponding flux Jacobian matrix \( A(U) \) is:

\[
A = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-v_g^2 + \frac{a^2}{\varepsilon} & \frac{a^2 \rho_g}{\varepsilon \rho_s} & 2v_g & 0 \\
0 & -v_s^2 + \frac{G(\varepsilon)}{\rho_s} & 0 & 2v_s
\end{bmatrix}
\]

(4.98)

The eigenvalues and eigenvectors of Eq. 4.87 are as follows:

\[
\Lambda = \begin{bmatrix}
v_g - \frac{a}{\varepsilon^2} & 0 & 0 & 0 \\
0 & v_g + \frac{a}{\varepsilon^2} & 0 & 0 \\
0 & 0 & v_s - \sqrt{\frac{G(\varepsilon)}{\rho_s}} & 0 \\
0 & 0 & 0 & v_s + \sqrt{\frac{G(\varepsilon)}{\rho_s}}
\end{bmatrix}
\]

(4.99)
Eq. 4.87 is strictly hyperbolic, inhomogeneous, non-linear and does not satisfy the homogeneity property.

4.4 Validation

We examine and compare the performance of the various numerical schemes discussed in section 4.2 at solving the Euler equations as a benchmark problem. More specific and suitable numerical schemes will then be selected to solve the pneumatic transport equations.

4.4.1 Euler Equations

The Euler equations were solved for $\gamma = 1.4$, $x \in [0, 1]$ and the following initial conditions:
**I.C. Test Case 1:**

\[
\rho, p, v = \begin{cases} 
1, 1, 0 & \text{for } x < 0.5 \\
0.125, 0.1, 0 & \text{for } x > 0.5
\end{cases}
\]

**I.C. Test Case 2:**

\[
\rho, p, v = \begin{cases} 
1, 1000, 0 & \text{for } x < 0.5 \\
1, 0.01, 0 & \text{for } x > 0.5
\end{cases}
\]

For both cases, transmissive boundary conditions were used for a spatial domain divided into \( nc \) grid cells.

**B.C's.:**

\[
U_0^n = U_1^n, \quad U_{nc+1}^n = U_{nc}^n
\]

Test case 1 is the well known Sod’s shock tube problem [131]. The solution contains a right moving shock and contact discontinuity and a left moving rarefaction wave. Test case 2 is a more severe case with a stronger shock. The solution to test case 1 at \( t = 0.25 \text{ sec(s)} \) using the highly dissipative but monotone Lax-Friedrichs (LxF) scheme is shown in Fig. 4.1. Fig. 4.1 a-d are the density, velocity, pressure and specific total energy profiles respectively. The figures show that the numerical solution improves as the number of grid cells increases from 100 to 1000. However, even when the number of grid cells is increased to 300, LxF scheme is still unable to resolve the various discontinuities (shock, contact and rarefaction) in the solution completely. This dissipative behavior is typical of most first-order accurate schemes. However, despite the numerical diffusion, shock locations are predicted accurately.

Fig. 4.2 shows the solution to test case 1 at \( t = 0.25 \text{ sec(s)} \) using the Richtmyer scheme, which is a member of the family of second-order accurate Lax-Wendroff schemes. Even with 100 grid cells the sharpness of the various discontinuities are much better resolved. However, the drawback of the Richtmyer scheme is the spurious oscillations present in the solution. This is especially pronounced near the shock and contact discontinuity. As discussed earlier, these spurious oscillations—
also referred to as numerical dispersion—are typical of higher-order approximations of the numerical flux function.

Fig. 4.3 shows a comparison between the monotone schemes discussed in Section 4.2 for test case 1 using 100 grid cells. As expected, it can be observed that the monotone schemes with the best performance are the upwind schemes. Particularly, the Godunov-type scheme of the Roe performs best followed by the flux vector splitting method of Steger and Warming. Although the upwind schemes perform significantly better than the highly dissipative LxF scheme in resolving the shock and rarefaction waves, they do not quite resolve the contact discontinuity. They also have trouble matching the leading edge of the rarefaction wave and the shock. Hence the needs for higher-order accurate TVD and WENO schemes that are expected to resolve discontinuities better.

Fig. 4.4 shows the result of test case 1 using both the higher-order monotonicity preserving Flux LImited Centered (FLIC) and Slope LImited Centered (SLIC) schemes using 100 grid cells. FLIC is a combination of the ForCE and Richtmyer schemes using the general flux limiter function defined in Eq. 4.49. SLIC is as defined in Eq. 4.52. The minmod limiter, which is the most dissipative limiter in table 4.1 was applied in all higher-resolution schemes in this study. The results show significant improvement to the monotone schemes in Fig. 4.3 including that of Roe. Both FLIC and SLIC are equally good but FLIC appears to have a slight edge over the SLIC scheme.

Both SLIC and FLIC are higher-order methods associated with centered monotone schemes. However, it is expected that higher-order methods associated with upwind schemes can better resolve discontinuities. Figs. 4.5 shows the result for test case 1 using the higher-order flux limited (HOFL) and slope limited (HOSL) methods with Roe’s numerical flux function as the choice of monotone scheme. As expected both HOSL-Roe and HOFL-Roe show improvements over SLIC and FLIC respectively. This is a result of the superiority of the monotone scheme applied, which in this case is the upwind scheme of Roe as opposed to the centered ForCE. The HOFL-Roe scheme performs slightly better than the HOSL-Roe. Although the HOFL- and HOSL-Roe schemes perform much better in capturing the shock and rarefaction waves, they appear to have trouble capturing the contact discontinuity.
Figure 4.1. Shock tube solution, Test Case 1 at $t = 0.25$ sec(s) and CFL = 0.5 using Lax-Friedrichs Scheme

Figure 4.2. Shock tube solution, Test Case 1 at $t = 0.25$ sec(s) and CFL = 0.5 using Richtmyer scheme with 100 grid cells
Figure 4.3. Shock tube solution, Test Case 1 at $t = 0.25$ sec(s) and CFL = 0.5 comparing monotone schemes using 100 grid cells

Figure 4.4. Shock tube solution, Test Case 1 at $t = 0.25$ sec(s) and CFL = 0.5 comparing the SLIC and FLIC schemes using 100 grid cells
Figure 4.5. Shock tube solution, Test Case 1 at $t = 0.25$ sec(s) and CFL = 0.5 comparing the HOSL- and HOFL-Roe schemes using 100 grid cells

Figure 4.6. Shock tube solution, Test Case 1 at $t = 0.25$ sec(s) and CFL = 0.5 comparing the WENO-ForCE and WENO-Roe schemes using 100 grid cells
Fig. 4.6 shows the result of test case 1 using the 5th order WENO reconstruction applied using the ForCE (WENO-ForCE) and Roe (WENO-Roe) schemes. Both WENO schemes performed much better in resolving the contact discontinuity than the HOSL- and HOFL-Roe schemes. However, the HOSL- and HOFL-Roe schemes both perform just as well or even slightly better in resolving both the shock and rarefaction waves. It is important to note that there is no significant difference in the results between WENO-ForCE and WENO-Roe scheme considering the additional computational requirement of the Roe scheme relative to ForCE. WENO schemes, therefore, are not very sensitive to the choice of the monotone numerical flux function.

Despite the improvement in results from the higher-order schemes relative to the monotone schemes we can still observe that the higher-order schemes have trouble capturing the contact discontinuity completely unless the number of grid cells is increased to about 300 cells. In order to emphasize this, we apply the HOFL-Roe scheme and WENO-Roe schemes to the severe test case (Test Case 2). Fig. 4.7 shows a comparison of the LxF, Roe, HOFL-Roe and WENO-Roe schemes on Test case 2 at 0.035 sec(s). One can observe clearly the inability of these schemes to capture the contact discontinuity from the density and specific
total energy profiles (Fig. 4.7a & d).

Figure 4.8. Luo’s experimental setup for CFB Riser Simulation

4.4.2 Pneumatic Transport Equations

Here we simulate the entrance region \((x \in [0, 0.92])\) of a Circulating Fluidized Bed (CFB) riser. Fig. 4.8 shows the CFB riser simulation setup used by Luo [132].
The numerical solution for the pneumatic transport equations were solved using the WENO-ForCE scheme for the following initial and boundary conditions:

\[ \begin{align*}
    [\varepsilon, \rho_g, v_g, v_s]_0 &= \begin{cases} 
        \varepsilon_{in}, \frac{p_{o}^{0} M_g}{RT}, \frac{v_{sg}}{(1-\varepsilon_{in})\rho_s A}, & \text{for } x = 0 \\
        1, \frac{p_{out} M_g}{RT}, v_{sg}, 0, & \text{for } x > 0 
    \end{cases} \\
    [\varepsilon, v_g, v_s]_0 &= \begin{cases} 
        \varepsilon_{in}, \frac{v_{sg}}{(1-\varepsilon_{in})\rho_s A}, & \text{for } x = 0 \\
        \frac{W_s}{\frac{\partial \varepsilon}{\partial x} + \frac{\partial v_g}{\partial x} + \frac{\partial v_s}{\partial x}} \bigg|_{nc+1} = 0; & \text{for } x > 0 \\
        \frac{\partial p}{\partial x} \bigg|_{0} = 0; & \rho_g = \rho_g(p) \\
    \end{cases}
\end{align*} \]

**Figure 4.9.** CFB riser simulation at pseudo-steady state using the WENO-ForCE (solid lines) and LxF (dash lines) schemes with both 100 and 300 grid cells. Experimental values of velocities were not measured.

I.C.:

\[ [\varepsilon, \rho_g, v_g, v_s] = \begin{cases} 
    [\varepsilon_{in}, \frac{p_{o}^{0} M_g}{RT}, \frac{v_{sg}}{(1-\varepsilon_{in})\rho_s A}, & \text{for } x = 0 \\
    1, \frac{p_{out} M_g}{RT}, v_{sg}, 0, & \text{for } x > 0 
\end{cases} \]

B.C.’s:

\[ [\varepsilon, v_g, v_s]_0 = \begin{cases} 
    [\varepsilon_{in}, \frac{v_{sg}}{(1-\varepsilon_{in})\rho_s A}, & \text{for } x = 0 \\
    \frac{W_s}{\frac{\partial \varepsilon}{\partial x} + \frac{\partial v_g}{\partial x} + \frac{\partial v_s}{\partial x}} \bigg|_{nc+1} = 0; & \text{for } x > 0 \\
    \frac{\partial p}{\partial x} \bigg|_{0} = 0; & \rho_g = \rho_g(p) \\
\end{cases} \]
where:
\[
\begin{bmatrix}
\varepsilon_{in} \\
v_{sg} \\
W_s \\
p_{out}
\end{bmatrix}
= 
\begin{bmatrix}
0.97556 \\
11.2 \text{ m}^2/\text{s} \\
34.81 \text{ kg/m}^2/\text{s} \\
1.068 \text{ atm}
\end{bmatrix}
\]

\(v_{sg}\) and \(W_s\) are the superficial gas velocity and solid mass flux respectively at the riser inlet. \(H\) represents the height of the pipe and \(g\) is acceleration due to gravity. Fig. 4.9 shows the results using 100 and 300 grid cells compared with experimental data [133] for both LxF and WENO-ForCE schemes. Figs. 4.9 a-b are gas velocity, solid velocity, gas pressure, and gas volume fraction (or voidage) profiles, respectively. The WENO scheme performed significantly better than the monotone LxF scheme. The poor result obtained from the LxF schemes may be attributed to the difficulty in resolving the sharp discontinuity at the inlet of the riser. This type of erroneous result will be expected from most of the first-order monotone schemes for problems of this nature.

### 4.5 Summary of Numerical Analysis

We have presented a detailed review of hyperbolic systems along with a generalized solution methodology for such systems. A number of numerical experiments and validations were conducted to obtain solutions to the physical phenomena represented by both the Euler system and the TFM Pneumatic Transport equations. Analytical expressions for their decomposed characteristics were presented along with validated numerical solutions. We have been able to show significant improvements in numerical simulation results obtained using higher-order schemes relative to first-order monotone schemes. The 5th-order WENO schemes in particular produced the most satisfactory results and will therefore be applied in our numerical simulations in the chapters that follow.

The following are conclusions that can be drawn from the review and numerical simulations presented here:

1. A fundamental understanding of the physical phenomena and mathematical description of a transport model helps in the identification of the best numerical technique for an optimized numerical model.
2. Higher-order monotonicity preserving schemes can reduce computational requirements through a reduction in the number of grid cells necessary to resolve discontinuities. This ultimately reduces computation time.

3. Monotone upwind schemes like FVS, Godunov and Roe, though more computationally demanding, perform better than central difference schemes like the LxF, Rusanov and ForCE. This is also true for high-order methods with the exception in the case of ENO/WENO reconstruction, which tend to be less sensitive to the choice of numerical scheme.

4. When the fully non-linear form of conservation equations are discretized using conservative formulations with minimal to no linearizing assumptions made to the physical flux terms, shock front locations are predicted accurately even in the presence of significant numerical diffusion.

5. First-order monotone schemes serve as essential building blocks for higher-order schemes but should not be applied directly to practical problems as they are likely to produce erroneous results.

6. Adoption of much of what has been built upon in the CFD community will help prevent re-inventing the wheel with regards to numerical techniques for hyperbolic problems in petroleum engineering.

In the following chapter, we will discuss the application of the two-fluid model (TFM) developed here. The TFM will be applied to air drilling hydraulics for the prediction of solids circulation time and optimum flow velocity. Experimental measurements will be used to demonstrate the capability and limitations of the current TFM.
Chapter 5

Results and Discussion

Drilling is the most capital intensive part of the entire oil and gas extraction process. In addition, hole cleaning is the most important aspect of the entire drilling process. Therefore, it is imperative that drilling engineers complete a thorough hydraulic design ahead of the drilling process. Furthermore, rock excavation and cutting circulation occur intermittently during drilling and varies depending on the drilling technique adopted by the driller. Hence, the ability to predict the circulation time sufficient to keep the wellbore free of excavated formation materials at various depths is necessary for optimizing the hole cleaning process. This information will aid drilling technicians in allotting the optimum circulation time necessary to maintain a wellbore annulus free of excavated rock materials at varying depths (and thereby reducing the cost of drilling). In other words, both drillers (or Drilling Technicians) on the field and drilling engineers designing underbalanced drilling projects both need to know the following:

1. the time it takes to circulate out pills of formation material at any given depth in order to reduce drilling time (thereby reducing the cost of drilling); and

2. the optimum rate at which to flow the circulation fluid to ensure the hole is maintained free of formation materials at minimum capacity (and thereby cost) of surface equipments.

The model validated in the previous chapter is used to address these two concerns in the sections that follow.
5.1 Circulation Time Prediction

Field drilling engineers/drillers need to ensure they allocate enough time for fluid circulation (or hole cleaning) in order to ensure successful landing of a well at the pre-designed Target Depth (TD). To this end, the model discussed in Ch. 3 is used to conduct a case study whereby sand pill circulation from the bottom of a well up to the well surface in three well geometry types are numerically simulated. Table 5.1 shows general flow parameters and particle properties used in these three case studies. The three well types are of vertical, horizontal and deviated geometries each 28.4 m in length. Fig. 5 shows the three conduit geometry types drawn as a pipe with a representative hydraulic diameter.

At initial condition, it is assumed that nothing is flowing in the pipe i.e. the volume fraction of the dispersed phase present in the conduit is set to zero (therefore voidage \( \varepsilon = 1 \)). At \( t = 0^+ \), 0.1875 kg of sand is then introduced from the bottom end of the simulated conduit at 1.875 kg/s within the first 0.1 seconds. The flow profiles and circulation times are recorded as the sand-pill (or slug) is gradually transported out of the opposite end of the conduit.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Atmospheric Temperature</td>
<td>293 K</td>
</tr>
<tr>
<td>Air Viscosity</td>
<td>1.8E-5 Pa.s</td>
</tr>
<tr>
<td>Hydraulic Diameter</td>
<td>61 mm</td>
</tr>
<tr>
<td>Conduit Length</td>
<td>28.4 m</td>
</tr>
<tr>
<td>Inlet Velocity</td>
<td>28.5 m/s</td>
</tr>
<tr>
<td>Particle Density</td>
<td>1,320 kg/m³</td>
</tr>
<tr>
<td>Particle Diameter</td>
<td>3,860 µm</td>
</tr>
</tbody>
</table>

**Table 5.1. Pneumatic transport flow parameters**

### 5.1.1 Vertical Conduit

At $t = 0$, we assume no flow condition in the simulated vertical wellbore conduit. At $t = 0^+$, a gas velocity of 28.5 $m/s$ is used to push 0.1875 $kg$ of 3,860 $µm$ diameter sand into the conduit from the bottom end of the conduit at 1.875 $kg/s$ within the first 0.1 seconds. The sand-pill transport is tracked and the total circulation time (i.e. elapsed time after all solid particles exit the 28.4 meter long conduit) is recorded.

Fig. 5.2 shows several flow profiles all plotted at 0.5 second intervals. Fig. 5.2 (a) shows interstitial velocity of the continuous (gas) phase; Fig. 5.2 (b) shows pressure profile across the conduit; Fig. 5.2 (c) shows both dispersed phase velocity profile and the slip velocity ($v_g - v_s$) profile in solid and dotted lines respectively; Fig. 5.2 (d) shows volume fraction of the continuous phase (solid lines) and the fraction of the conduit cross-sectional area available to flow after any deposition (dotted lines). For vertical flow systems, the fraction of the conduit cross-sectional area available to flow remains the total pipe area.

The results show a rapid increase in gas velocity profile across the entire length of the conduit to the specified inlet gas velocity of 28.5 $m/s$ within the first 0.1 seconds. This gas velocity is translated across the entire length of the conduit with a pressure wave propagating at speed of sound within the medium. The gas velocity stabilizes relatively quickly as can be observed in Fig. 5.2 (a). The pressure at the inlet also rises dramatically from 1 $atm$ before stabilizing at about 1.07 $atm$ as shown in (Fig. 5.2 (b)). This is as a consequence of resistant forces due to (1) gravity, (2) wall viscous effects and (3) fluid-particle interaction. It can be observed that the pressure profile is still not stabilized after 0.5 seconds due to diminishing pressure waves traveling back and forth between the inlet and
outlet boundaries of the conduit. The velocity and volume fraction of the dispersed (solids) phase close to the inlet can be observed to have increased from their initial value of zero in Figs. 5.2 (c) and (d) respectively. The dotted lines in Fig 5.2 (c) is the local slip velocity, which is the difference between the continuous (gas) phase and the dispersed (solids) phase velocities. These initial 0.5 s interval changes may be attributed to the introduction of the solids phase at the constant inlet gas velocity from the inlet (bottom) of the vertical conduit.

Following the initial rise in solids velocity and concentration, the solids may be observed propagating along the vertical conduit toward the exit over time. An important and noticeable profile feature observed is the spread of the solids concentration across the conduit as the sand-pill propagates upward. At \( t = 0.5 \) s, the solids concentration profile, as shown in Fig. 5.2 (d), spans a 2.6 m interval with a mean and standard deviation of 0.0186 (or 0.981 for voidage) and 0.393 m respectively. However, after \( t = 2.5 \) s it had spread across a 3.9 m interval with a mean and standard deviation of 0.0126 (or 0.987 for voidage) and 0.818 m respectively. The spread highlights the effect of the implicit slip velocity formulation and indicates that fluid-particle interactions are appropriately accounted with the inclusion normal stresses within the particulate phase. This very different from
results observed under a constant slip velocity assumption every solid particle is assumed to be moving at a specific slip relative to the fluid phase.

Most importantly, circulation time, which may also be referred to as the time it takes the sand pill to completely exit the conduit, was found to be approximately 3 seconds. This is different from the time it takes the entire fluid system to reach steady state (ss) condition. As can be observed in Fig. 5.2, steady state tolerance was achieved after 4.1 seconds. For the vertical transport case, the forces opposing the induced linear momentum of the particulate phase associated with the specified inlet gas velocity are: (1) the gravitational force of the particulate phase (2) viscous forces due to conduit wall friction and; (3) the viscous force due to fluid-particle drag. Of these three, the gravitational force associated with the particulate phase and the fluid-wall interaction viscous forces are the two dominant resistant force for vertical transport. It is also important to note that while the fluid-particle drag force has a negative effect on the gas phase, its effect on the particulate phase is in the opposite direction. This force is what provide the positive linear momentum for the particulate phase and permits its movement out of the vertical conduit.

5.1.2 Horizontal Conduit

Similar to the vertical well geometry type, the simulated sand-pill transport is tracked across a horizontal conduit and steady state time is recorded. Fig. 5.3 shows the flow profiles plotted at 0.5 second intervals for a horizontal well geometry. At $t = 0$, the horizontal conduit is assumed to be empty. At $t = 0^+$, 0.1875 kg of 3,860 µm diameter sand was introduced at 28.5 m/s from inlet at 1.875 kg/s within the first 0.1 seconds.

The profile plots are ordered similar to that of Fig. 5.2. Fig. 5.3 (a) shows interstitial velocity of the continuous (gas) phase; Fig. 5.3 (b) shows pressure profile across the conduit; Fig. 5.3 (c) shows both dispersed phase (solids) velocity profile and the slip velocity ($v_g - v_s$) profile in solid and dotted lines respectively; Fig. 5.3 (d) shows volume fraction of the continuous phase (solid lines) and the fraction of the conduit cross-sectional area available to flow (dotted lines) after any deposition.

The gas phase velocity profile rises to the inlet specification of 28.5 m/s and
fluctuates around the same value for most of the run time as can be observed in Fig. 5.3 (a). Pressure at the inlet also rises to about 1.05 atm as the solids are being introduced at the inlet as in Fig. 5.3 (b). The pressure spikes observed in along the well are due to the deposition and re-entrainment of sand particles, which tends to reduce total flow area. As can be observed in Figs. 5.3 (b) and (d) there is a significant spike in pressure relative to the rest of the conduit length in regions where some corresponding sand deposition is located. The dotted lines in Fig. 5.3 (d) correspond to solids deposition. The deposition profile can be observed at any instance around the sand-pill with a corresponding shape on the pressure profile. Solids velocity also rises along with the front of the sand-pill and propagates along the pipe towards the exit as illustrated with the solid lines in Fig. 5.3 (c). The slip velocity (dotted lines) is almost a shifted inversion of the solids velocity profile. Fig. 5.3 (d) shows the propagating sand-pill concentration in terms of the void fraction. The spreading of the sand-pill observed in the case of the vertical conduit is even more visible in the pressure profile of the horizontal transport case as particles that deposited and re-entrained into the flow shortly behind the propagating sand-pill.

As a result of the asymmetric nature of the solids deposition phenomena, it is
difficult to model all flow regimes encountered in horizontal transport systems with the assumption that the particulate phase is fully dispersed. The current model takes into account deposition and re-entrainment as a change in flow area. This is due to the fact that a major assumption in the current model is that the particulate phase is fully dispersed in the gaseous phase. Therefore stationary cuttings bed that forms along the conduit due to deposition is treated as a reduction in flow area. Correspondingly, re-entrainment of particles into the flowing stream is treated as an increase in flow area. At any instance, area above the stationary bed becomes the new flow area containing both the gaseous and fully dispersed particulate phases. The effect of the reduction in flow area therefore appears in the pressure profile due to compression of the gaseous phase at local area reductions. A detailed discussion of how mass conservation is maintained during deposition and re-entrainment is in Sec. 3.2 of Ch. 3. The effect of reduction in flow area on the pressure profile can be better observed in numerical studies of partial blockages in pipelines [134].

Results indicate that total circulation time in the horizontal conduit is 2.3 seconds. Given that both the horizontal and vertical conduits are of equal length, one may conclude that circulation time is less for horizontal pneumatic transport under dilute flow relative to vertical. The steady state time recorded for the horizontal conduit simulation was 3.5 seconds. The delay between total circulation time and steady state condition can be explained using the pressure profile at \( t = 2 \) seconds in Fig. 5.3. It can be observed that the particulate phase had begun exiting the conduit and a backward moving pressure wave is propagating towards in inlet of the horizontal conduit. This wave later diminishes and stabilizes as the entire particulate phase exits the conduit completely soon after.

For the horizontal sand pill transport case, the forces opposing the drag force on the particulate phase from the induced linear momentum associated with the specified inlet velocity are: (1) the viscous forces due to conduit wall friction and; (2) the viscous force due to fluid-particle drag on the gaseous phase. The fluid-wall viscous forces determine the final pressure profile at steady-state in the absence of the particulate phase.
5.1.3 Deviated Conduit

The deviated well as shown in Fig. 5 (c) is divided into three sections. The first and last sections of the deviated conduit are perfectly horizontal and Vertical 11.35 meter (40 % of total length) segments respectively. The bend between them is 5.68 meters in length (20 % of total length).

![Graph of Velocity and Pressure Profiles](image)

**Figure 5.4.** L-shaped/Deviated pipe transport of 0.1875 kg sand introduced at 1.875 kg/s

Fig. 5.4 shows the profile plots for the case of the deviated conduit. Similar to the horizontal flow case, we can observe saltation at the 0.5 second mark (in green). The reduction in flow area due to deposition also resulted in a pressure spike were there is some deposition of particles. It can be noted that at the 1 second mark (in red), the deposition profile spans a much less region than the region occupied by fully dispersed particulate flow. It appears that the fully-dispersed flow is moving ahead of the region with sand deposition. This indicates some retention of deposited solids at the mid section, which is later re-entrained into the gas stream. This is different from what we observed at the 0.5 second mark where the region experiencing deposition was overlayed over the dispersed flow region. The spike in pressure generated in the horizontal section is eliminated as the sand pill is transported into the vertical section of the deviated conduit. This is due to the
fact that there is no deposition occurring in the vertical section.

It can also be observed in Fig. 5.4 (d) that the distance between the green (0.5 second mark) and red (1 second mark) lines is much larger that the distance between the sky-blue (1.5 second mark) and purple (2 second mark). This shows that the particulate phase is transported faster in the horizontal section that the vertical under the tested flow condition. The recorded circulation time and steady state time are 2.6 and 3.5 seconds respectively.

<table>
<thead>
<tr>
<th>Well Geometry</th>
<th>Circulation Time (s)</th>
<th>Steady State Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertical</td>
<td>3.0</td>
<td>4.1</td>
</tr>
<tr>
<td>Horizontal</td>
<td>2.3</td>
<td>3.5</td>
</tr>
<tr>
<td>Deviated</td>
<td>2.6</td>
<td>3.5$^+$</td>
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</tbody>
</table>

Table 5.2. Summary of circulation time prediction studies

5.1.4 Computational Time

Most of the simulations from our numerical studies were completed in less than thirty minutes on Penn State University Research Computing and Cyberinfrastructure (RCC) computers. Computational time varied depending on flow conditions and how quickly the system reached steady state. However, we explored parallel computing in order to further reduce computation time and found that cost of parallelization outweighed its benefits as it increased the computation time instead. We therefore resorted back to basic 03 optimization.
5.2 Optimum Circulation Rate Prediction

5.2.1 Vertical Flow

A number of underbalanced hole cleaning studies were conducted at Penn State’s Drilling Hydraulics Research Center (DHRC) between 1990 and 2003. They include the experimental studies of Supon et al. [135], [136], Hagar et al. [137] and Temple et al. [138]. These studies are of significant relevance to our current study due to the nature of their experimental setup, which simulates annular wellbore drilling hydraulics.

The most recent DHRC experiment is that of Temple [138]. Temple set up a closed pneumatic flow-loop system as shown in Fig. 5.5. Additional information about the setup may be found in Table 5.3. In this setup, cuttings are continually supplied into the wellbore from its base by an L-Valve and then transported upward through the wellbore annulus and out into a cyclone separator before being collected in the downcomer/hopper ready for re-introduction into the wellbore annulus. The cuttings are carried up the wellbore annulus by air introduced at high flow rates from the top of the drillpipe and out into the wellbore annulus through small nozzles at the drill bit.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>Drill Collar Inner Diameter</td>
<td>77.9 mm</td>
</tr>
<tr>
<td>Drill Pipe Outer Diameter</td>
<td>42.2 mm</td>
</tr>
<tr>
<td>Wellbore Length</td>
<td>2.74 m</td>
</tr>
<tr>
<td>Atmospheric Temperature</td>
<td>288.1 K</td>
</tr>
<tr>
<td>Air Viscosity</td>
<td>1.760E-5 Pa.s</td>
</tr>
</tbody>
</table>

Table 5.3. Temple’s experimental setup parameters

<table>
<thead>
<tr>
<th>Particle Size Range (U.S. Mesh)</th>
<th>Particle Size Range (microns)</th>
<th>Weighted Avg. Size (microns)</th>
<th>Dominant Size (microns)</th>
<th>Difference (microns)</th>
</tr>
</thead>
<tbody>
<tr>
<td>40 - 70</td>
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<td>400</td>
<td>53.5</td>
</tr>
<tr>
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<td>853 - 422</td>
<td>713.5</td>
<td>825</td>
<td>111.5</td>
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<tr>
<td>16 - 20</td>
<td>1200 - 853</td>
<td>852.5</td>
<td>950</td>
<td>97.5</td>
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<tr>
<td>12 - 20</td>
<td>1680 - 853</td>
<td>1214</td>
<td>1600</td>
<td>386</td>
</tr>
</tbody>
</table>


\(^{a}\)Particle density remained constant \((\rho_s=3.150 \text{ kg/m}^3)\)

\(^{b}\)Difference between dominant particle size and weighted average size.

Table 5.4. Temple’s particle diameters
Temple generated a number of flow characteristics curves (FCC) using this experimental setup from which optimum circulation rates can be extracted. In this study we have painstakingly reconstructed these FCC plots using the current transient numerical model and compared our results to the measured FCC data from Temple’s experiments. While the actual experiment had distributed particles sizes, single valued weighted average particle diameters were used in our computational study. These particle diameters along with the corresponding solids injection rates \((W_s)\) that were simulated are tabulated in Tables 5.4 and 5.5 respectively. Figs. 5.6 - 5.13 show both experimental and simulation results of FCC plots for a wide range of particle sizes and solid mass injection rates \((W_s)\).

Figs. 5.6 and 5.7 show FCC plots that were obtained using a weighted average
Figure 5.6. FCC - Temple’s data \(d_p=346.5 \mu m\) \(W_s=15.14\) g/s

Figure 5.7. FCC - Temple’s data \(d_p=346.5 \mu m\) \(W_s=30.15\) g/s
Figure 5.8. FCC - Temple's data $d_p=713.5$ µm $W_s=15.11$ g/s

Figure 5.9. FCC - Temple's data $d_p=713.5$ µm $W_s=20.07$ g/s
Figure 5.10. FCC - Temple’s data \(d_p=852.5 \, \mu m \) \(W_s=25.09 \, g/s\)

Figure 5.11. FCC - Temple’s data \(d_p=852.5 \, \mu m \) \(W_s=30.03 \, g/s\)
**Figure 5.12.** FCC - Temple’s data $d_p=1214$ µm $W_s=25.25$ g/s

**Figure 5.13.** FCC - Temple’s data $d_p=1214$ µm $W_s=30.08$ g/s
<table>
<thead>
<tr>
<th>Avg. Size (microns)</th>
<th>Particle Weighted size (microns)</th>
<th>Particle Dominant size (microns)</th>
<th>Standard Deviation for $W_s$ (g/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>346.5</td>
<td>400</td>
<td>15.14</td>
<td>0.24</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30.15</td>
<td>0.26</td>
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<tr>
<td>713.5</td>
<td>825</td>
<td>15.11</td>
<td>0.28</td>
</tr>
<tr>
<td></td>
<td></td>
<td>20.07</td>
<td>0.26</td>
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<tr>
<td>852.5</td>
<td>950</td>
<td>25.09</td>
<td>0.3</td>
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<td>30.03</td>
<td>0.27</td>
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<tr>
<td>1214</td>
<td>1600</td>
<td>25.25</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>30.08</td>
<td>0.38</td>
</tr>
</tbody>
</table>

Table 5.5. Temple’s experimental test runs

particle diameter of 346.5 microns introduced at the constant solids mass rates of 15.14 g/s and 30.15 g/s respectively. It can be observed that the simulated results show a very good match with measured experimental data. This is despite the fact that Temple reported an error margin of $\pm 57.7 \ Pa$ in his pressure drop measurements for these set of experimental runs.

Figs. 5.8 and 5.9 were obtained using the weighted average particle diameter of 713.5 microns introduced at the constant solids mass rates of 15.11 g/s and 20.07 g/s respectively. The simulated results also show a good match with measured experimental data. Recorded error in pressure drop measurements for these set of experimental runs was $\pm 56.9 \ Pa$, while the maximum difference between simulation and measured pressure drop under dilute phase flow is about 30 $\ Pa$ for this set of runs.

Figs. 5.10 and 5.11 were obtained using a weighted average particle diameter of 852.5 microns at 25.09 g/s and 30.03 g/s respectively. The simulated results show an under-prediction of pressure drop for dilute phase flow conditions. However, recorded error for pressure drop measurements for these set was $\pm 110 \ Pa$ and the maximum difference in the pressure drop between the simulated results and measured data is 80 $\ Pa$. Therefore, the results are within the margin of error.

Figs. 5.12 and 5.13 were obtained using a weighted average particle diameter of 1214 microns at solids mass rates of 25.25 g/s and 30.08 g/s respectively. The simulated results for this set of runs showed the most deviation from experimental data. Despite the fact that the recorded error for pressure drop measurements
Figure 5.14. Weight percent particle size distribution before and after Temple’s experiments for this set of data was ±113 Pa, the maximum difference between simulated and measured pressure drop is as high as 170 Pa.

However, an inspection of the recorded particle sample weight % and cumulative wt. % distributions for each run as shown in Fig. 5.14 provides some insight into the disparities between the simulation results and experiment. Fig. 5.14 (a-d) correspond to the sample particle size distributions by weight for the 346.5, 713.5, 852.5 and 1214.0 microns weighted avg. particle diameters before and after each experiment. Figs. 5.14 (e-h) are their corresponding cumulative weight percent.
Unlike the 346.5, 713.5, 852.5 (Figs. 5.14 a-c) micron weighted average particle sizes, there is a much larger variation in particle size distribution profile in the 1214 micron weighted avg. particle size (Fig. 5.14 d). No single particle size exceeded 50% of the total weight in the 1214 micron weighted avg. particle size according to the measurements recorded before the experiment. In addition, there is a vivid difference in the size distribution before and after the experiments for the 1214 micron weighted average particle size relative to the other three weighted average particle sizes and this might have affected the weighted average size estimate conducted before the experiment. The variable in size distribution before and after the experiment must have been due to particles attrition during the experiment.

In essence, the 713.5 and 852.5 micron particle sizes could be well represented as a mono-dispersed fluid-particle mixture simply using the weighted avg. particle size. However, the wide range of particle size distribution in the 1214 micron particle sample must have made it difficult to represent it as a mono-dispersed fluid-particle mixture. While Figs. 5.14 (d) and (h) also show a relatively wide particle distribution for the 346.5 micron weighted average particle size, this was for the smallest particle size and it can be noted from table 5.4 that the difference between the dominant and the weighted avg. particle size was 53.5 microns as opposed to 386 microns (over 700% larger) for the 1214 weighted avg. particle size.

Table 5.6 shows optimum velocity predictions from both experiment and simulation runs obtained from the FCC plots in Figs. 5.6 - 5.13. Surprisingly, the maximum error observed from the optimum velocity predictions (16.1%) corresponds to the 346.5 micron weighted average particle size run at the constant solids mass rate of 15.14 g/s shown in Fig. 5.6 despite the reasonably good FCC plot match. This is due to the fact that optimum velocity is only determined by the point of minimum pressure drop on the FCC plot, which indicates the point of transition from dilute to dense flow conditions. Also, at lower particle sizes, the minimum pressure point is not as well defined on the FCC plot relative to the larger particles FCC plots. This is why the relative errors obtained from the largest particle size (1214 microns) are more consistent (8.47 and 8.83%) and are closest to the % AAD value from all the datapoints (9.2%).

The most accurate optimum velocity prediction corresponds to the 713.5 micron
particle size, which was run at a solids mass rate of 20.07 g/s shown in Fig. 5.9. However, the experimental measurement for the set of runs conducted using the 713.5 micron samples are questionable due to the fact that they show a decrease in optimum velocity from 9.5 to 9.4 m/s with a corresponding increase in mass rate. This implies an inverse relationship between optimum velocity and solids mass rate, which is inconsistent with the general trend of the remaining dataset. Overall, it is safe to say that the current model over-predicts optimum velocity on average by about 10%, which is a reasonably conservative estimate. Fig. 5.15 shows the general trend of optimum velocity prediction for a range of particle sizes. As expected, it can be observed that the optimum flow velocity increases as particle size increases.

<table>
<thead>
<tr>
<th>$d_p$ (µm)</th>
<th>$W_s$ (g/s)</th>
<th>$U_{g_{opt}}$ ($\frac{m}{s}$)</th>
<th>Exp.</th>
<th>Sim.</th>
<th>% Rel. Err.</th>
</tr>
</thead>
<tbody>
<tr>
<td>346.5</td>
<td>15.14</td>
<td>6.2</td>
<td>7.2</td>
<td>-16.13</td>
<td></td>
</tr>
<tr>
<td>346.5</td>
<td>30.15</td>
<td>7.2</td>
<td>8.2</td>
<td>-13.89</td>
<td></td>
</tr>
<tr>
<td>713.5</td>
<td>15.11</td>
<td>9.5</td>
<td>9.4</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>713.5</td>
<td>20.07</td>
<td>9.4</td>
<td>9.45</td>
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</tr>
<tr>
<td>852.5</td>
<td>25.09</td>
<td>9.98</td>
<td>11.2</td>
<td>-12.22</td>
<td></td>
</tr>
<tr>
<td>852.5</td>
<td>30.03</td>
<td>10.2</td>
<td>11.5</td>
<td>-12.75</td>
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<td>1214</td>
<td>30.08</td>
<td>12.0</td>
<td>13.06</td>
<td>-8.83</td>
<td></td>
</tr>
</tbody>
</table>

% AAD = 9.2%

Table 5.6. Optimum velocity predictions

Figs. 5.16 (a-c) show FCC plots with different particle sizes at constant solids mass rates plotted together. As expected, it can be observed that there is an upper-right shift in the FCC profile as particle size is increased at a fixed solids feed rate. Figs. 5.16 (d-f) show the same results from simulation alone. Fig. 5.17 show FCC plots for different solids mass rates at constant particle sizes plotted together. As expected, it can be observed that there is an upper-right shift in the FCC profile as solids mass rate is increased.

The results obtained here indicate a significant improvement to numerical studies conducted previously on pneumatic transport of cuttings [16], [17], [18], [19], [20]. Adewumi and Tian [19] observed an inverse relationship between particle size and pressure drop as the pneumatic system tends towards dilute flow conditions
(see Fig. 5.18). However, this result is contradictory to Temple’s experimental studies which instead shows the opposite effect (i.e. the larger the particle diameter, the larger the pressure drop even under dilute flow conditions). Other vertical pneumatic transport experiments have indicated a similar trend (See also Klinzing et al. [43], Narimatsu and Ferreira [139] and Welschof [140]). However, it is fair to note one possible explanation for the numerical results observed by Adewumi and Tian might have been their use of a simplified pseudo 2-dimensional wall viscous effect relationship in their one dimensional transport equation formulation. This must have diminished the contribution of viscous effects to pressure drop and instead made prominent only fluid-particle drag effects, which does have an inverse relationship with particle diameter.

In the current study, we have applied experimentally validated friction coefficient correlations and the Fanning equation to account for wall viscous effects. In
Figure 5.16. FCC comparison at constant solids loading

this model formulation, it is that expected that wall viscous effects will tend to dominate as flow conditions become more dilute and the systems moves toward pure fluid flow. Figs. 5.16 (a-c) show both Temple’s experimental FCC results and numerical simulation from the current study obtained by varying particle diameters. Figs. 5.16 (d-f) also show numerical results only. Numerical results from the current study indicate that while pressure drop will tend to merge under dilute flow conditions, the relationship never becomes inverse. The merging of the pressure drop is due to the diminishing effect of increasing velocity on pressure drop due to fluid-particle viscous effects and the dominance of fluid-wall viscous forces.

Figs. 5.17 (a-d) show FCC plots comparing Temple’s experimental measurements to results from the current study that illustrate the effect of changes in solids
Figure 5.17. FCC comparison at constant particle diameter feed rate, which is equivalent to changes in ROP while drilling. As expected, the results indicate an upward and rightward movement of the FCC with increasing solids feed rate (or ROP).

One clear deficiency in the current model can be observed in 5.17 (c) and (d) where numerical results deviate significantly from Temple’s experimental measurements particularly for larger particle sizes (852.5 and 1214 microns) flow under dilute conditions.
Figure 5.18. FCC plots showing the effect of particle size on pressure drop predictions under dilute flow conditions. Cutting density ($\rho_s$) = 2650 kg/m$^3$, Cutting diameter ($d_p$) = 228, 911 and 1822 µm, System Hydraulic diameter ($D_h$) = 7.13 in [Adewumi and Tian (1992)]
5.2.2 Horizontal Flow

Unfortunately, there are no horizontal pneumatic cuttings transport experimental data available in the literature. A study conducted by Hagar et al. [137] on pneumatic cuttings transport in deviated wellbore only considered low angle deviations of up to 14 degrees from vertical. They were able to develop few correlations for determining optimum velocity as a function of deviation angle among other parameters. They concluded that as wellbore angle (i.e. inclination angle relative to the vertical position) increases, the gas velocity that correspond to the minimum pressure point on the FCC plot increases.

However, a study conducted by Welshcof [140] on industrial grain transportation in food processing plants includes experimentally-generated horizontal FCC plots. Welshcof’s experiment is one of the most referenced horizontal pneumatic transport studies in the literature. Welshcof’s data was used in this study due to lack of horizontal flow data for annular flow wellbore cuttings transport. Some of the relevant experimental conditions for Welshcof’s experiment including particle properties are cataloged in table 5.7.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>Pipe Diameter</td>
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</tr>
<tr>
<td>Pipe Length</td>
<td>28.4 m</td>
</tr>
<tr>
<td>Atmospheric Temperature</td>
<td>291 K - 295 K</td>
</tr>
<tr>
<td>Air Viscosity</td>
<td>1.8E-5 Pa.s</td>
</tr>
<tr>
<td>Particle Density</td>
<td>1,320 kg/m³</td>
</tr>
<tr>
<td>Particle Diameter</td>
<td>3,860 µm</td>
</tr>
</tbody>
</table>

Table 5.7. Welshcof’s experiment parameters

The current model was run under Welshcof’s flow conditions for three different solid feed rates (i.e. 0.129 kg/s, 0.297 kg/s and 0.595 kg/s). Figs. 5.19 - 5.21 show both simulated and experimental FCC profiles for Welshcof’s horizontal pneumatic transport system. While the model performed fairly within the dilute flow region, it was unable to produce the expected results for dense flow conditions. Therefore, the transition velocity between dilute and dense flow conditions could not be predicted.

This poor performance is due to the limitation of the dispersed flow model
formulation developed in the current study. As shown in Fig. 1.8, there are several transitions between flow regimes as fluid velocity is decreased under constant mass rate conditions. States a to c are under dilute flow conditions while d and e are dense flow conditions. The current model is only capable of simulating segregated flow using the area ratio approach to model cuttings bed formation but it is unable to simulate slug flow. Therefore, we are unable to model the dense phase flow condition which begins with slug flow in the case of horizontal pneumatic transport. Under slug flow condition, there will be portions of the horizontal pipe where the entire conduit cross-section is filled with deposited cuttings. This will lead to a build up of pressure behind the slug until there is enough pressure to make way for the fluid to move freely again through the pipe. This phenomenon is observed along with severe oscillations in pressure drop.

![Graph](image)

**Figure 5.19.** Horizontal FCC - Welschof’s data $d_p=3860 \, \mu m$, $\rho_s = 1320 \, kg/m^3$ and $W_s=129 \, g/s$

The use of the critical velocity approach along with a dynamic two-layer two-fluid model may have provided the capability of modeling slug flow in addition
to the other flow regimes expected in horizontal pneumatic transport. However, this approach is not feasible as of yet due to a lack of suitable empirical correlations/models for critical velocities that can serve as constitutive relations.

As an alternate method for predicting optimum fluid circulation velocity for horizontal pneumatic cutting transport, we developed correlations for optimum velocities using experimental data. These will be discussed in chapter 6.
Figure 5.21. Horizontal FCC - Welschof’s data $d_p = 3860 \, \mu m$, $\rho_s = 1320 \, kg/m^3$ and $W_s = 595 \, g/s$
Chapter 6

Correlation Development

While the current model evaluated in Ch. 5 is capable of predicting circulation time in horizontal and deviated wellbore under dilute flow, it is important to ensure that the flow in the horizontal sections are maintained within the dilute flow region when using the current model for circulation time predictions in horizontal and deviated wellbores. This therefore requires the development of the capacity to predict the point of minimum pressure on the FCC plot a priori. These predictions are necessary for maintaining efficient pneumatic transport of cuttings in wellbores while drilling with air/gas.

As illustrated in Ch. 5, the current model is not capable of predicting the point of minimum pressure on the FCC plot for horizontal pneumatic transport systems. This is due to its inability to simulate dense flow conditions in horizontal/highly-deviated pneumatic systems. Consequently, the current model is incapable of predicting the point of minimum pressure in the FCC plot for horizontal/highly-deviated well sections. A simple but crude solution to this problem in the absence of better physics-based mechanistic models is the use of empirical correlations.

Empirical correlations available in pneumatic transport literature for predicting the velocity corresponding to the minimum pressure on the FCC plot, sometimes erroneously referred to as saltation velocity [43], includes those of Rizk [35], [36], Matsumoto [37], Thomas [38], Geldart and Ling [39], Cabrejos and Klinzing [40] and Tashiro et al. [41]. The correlation developed by Rizk is the most widely accepted in industry. Notwithstanding, comparative studies like that of Jones and Leung [42], have shown that the correlation due to Thomas is more accurate.
when compared over a wide range of data sets. That being said, it is important to point out that even Thomas’ correlation showed a root mean square relative deviation (RMSRD) of 44 % (RMSRD<60 % for all others). Klinzing et al. [43] suggests that further experimental work may lead to a better understanding of the mechanism of saltation velocity. This is because saltation actually begins long before the minimum pressure point on the FCC plot (see Figs. 1.8 and 1.9).

In this study we developed correlations for optimum flow velocity based on the velocity corresponding to the minimum pressure on FCC plots. This is similar to the assumption of Rizk and others for the definition of the saltation velocity in pneumatic transport systems. Experimental datasets for both vertical ([138], [139], [43]) and horizontal ([140], [141], [43]) pneumatic transport systems respectively were used to develop correlations for optimum velocity estimation.

From our numerical studies, we had observed that certain parameters were major contributors to the shape and location of the flow characteristic curves (FCC) and thereby the optimum velocity ($v_{opt}$). These parameters include:

- Solids mass flow rate ($W_s$)
- Weighted average particle diameter ($d_p$)
- Single particle terminal ($U_t$)
- Acceleration due to gravity ($g$)
- Relative density ($\rho_s - \rho_g$)
- Carrier phase density ($\rho_g$)
- Carrier phase viscosity ($\mu$)

From this set of parameters, it was understood that we needed to develop correlations of the following relation:

$$f(v_{opt}, W_s, d_p, U_t, g, \rho_s - \rho_g, \rho_g, \mu) = 0.$$\hspace{1cm}(6.1)

According to the Buckingham II theorem, a physically meaningful equation involving $n$ physical variables can be rewritten in terms of a set of $p = n - k$
dimensionless parameters constructed from the original variables, where \( k \) is the number of fundamental physical dimensions present in all \( n \) physical variables. From Eq. 6.1 we may observe that we have eight physical variables and three fundamental physical dimensions (mass, length and time) from which all \( n = 8 \) physical variables may be defined. Therefore, we can express Eq. 6.1 in terms of \( p = 5 \) dimensionless parameters as follows:

\[
f(\pi_0, \pi_1, \pi_2, \pi_3, \pi_4) = 0.
\] (6.2)

The experimental datasets were fitted to the form in Eq. 6.3 by regression analysis in Minitab\textsuperscript{®}, a statistics package originally developed at the Pennsylvania State University in 1972. See Table 6.1 for coefficients \( a_0 - a_4 \) in Eq. 6.3. Both correlations were developed and tested using three distinct data sets each for both vertical and horizontal pneumatic transport systems. Figures 6.1 and 6.2 are the parity plot and relative error for the vertical pneumatic transport systems. A similar set of result was obtained for horizontal pneumatic transport systems as can be observed in Figures 6.3 and 6.4. These correlations showed relative errors less than 10 % for most data points and RMSRD less that 5.1 %. The parity plots also indicate excellent matches for all three data sets for both vertical and horizontal systems.

\[
\frac{v_{opt}}{U_t} = a_0 \pi_1^{a_1} \pi_2^{a_2} \pi_3^{a_3} \pi_4^{a_4}
\] (6.3)

where:

\[
\pi_1 = \frac{W_s}{(\rho_s - \rho_g)U_t d_p^2}
\]
\[
\pi_2 = \frac{(\rho_s - \rho_g)U_t d_p}{\mu g}
\]
\[
\pi_3 = \frac{U_t^2}{g d_p}
\]
\[
\pi_4 = \frac{(\rho_s - \rho_g)}{\rho_g}
\]

The newly developed correlation was compared to well known published corre-

<table>
<thead>
<tr>
<th></th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
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<td>Vertical Flow</td>
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<td>Horizontal Flow</td>
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<td>0.1647</td>
<td>-0.6568</td>
<td>2.843</td>
<td>0</td>
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</table>

Table 6.1. Optimum velocity correlation coefficients
Figure 6.1. Parity Plot for vertical flow $U_{opt}$ prediction

Figure 6.2. Correlation error for $U_{opt}$ prediction in vertical wells
Figure 6.3. Parity plot for horizontal flow $U_{opt}$ prediction

Figure 6.4. Correlation error for $U_{opt}$ prediction in horizontal wells
Figure 6.5. Optimum velocity prediction for vertical wells

Figure 6.6. Optimum velocity prediction for horizontal wells
lations. Tables 6.2 and 6.3 show the result of a comparison of Eq. 6.3 with those of Rizk and Thomas. It is clear than the new correlations outperforms both that of Rizk and Thomas for the tested range of conditions.

Eq. 6.3 has been used to generate the optimum velocity charts in Appendix B and C for air drilling systems. It is important to note that the figures in Appendix B and C were developed with a fluid viscosity and density of $1.8 \times 10^{-5} \text{ Pa.s}$ and $1.2 \text{ kg/m}^3$ respectively. For deep wellbores where formation temperature varies significantly from atmospheric condition, it will be prudent to use Eq. 6.3 instead of the charts in Appendix B and C. Otherwise, safety factor adjustments for fluid viscosity differences should be employed. For optimum velocity predictions in deviated well geometries, both the vertical and horizontal correlations developed may be applied and weighted by wellbore inclination angle.

<table>
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<th>Relative Error</th>
</tr>
</thead>
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<td>Eq. 6.3</td>
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<tr>
<td>Temple et al. [138]</td>
<td>0.031</td>
</tr>
<tr>
<td>Narimatsu and Ferreira [139]</td>
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<td>Klinzing et al. [43]</td>
<td>0.010</td>
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**Table 6.2.** Correlation comparison for vertical systems

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<tr>
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<td>Eq. 6.3</td>
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<td>Welschof [140]</td>
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<tr>
<td>Rinoshika and Suzuki [141]</td>
<td>0.085</td>
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<tr>
<td>Klinzing et al. [43]</td>
<td>0.030</td>
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</table>

**Table 6.3.** Correlation comparison for horizontal systems
Chapter 7

Conclusion

The purpose of this study was to (1) develop a better understanding of vertical and horizontal pneumatic transport of cuttings under transient flow conditions and (2) develop the capacity to predict rate requirements and circulation time for hole cleaning while drilling vertical, lateral and deviated well sections using air/gas. In this effort, a transient variable area wellbore drilling hydraulics model based on the two-fluid dispersed flow formulation was developed. A Riemann solver based on the higher-order weighted essentially non-oscillatory (WENO) scheme was developed to solve the coupled system of hyperbolic conservation equations. The numerical scheme was validated using Sod’s shock tube experiment [131] as well as Luo’s vertical pneumatic transport experiment [132].

Numerical experiments for vertical, horizontal and deviated wellbores were then conducted to determine relative circulation time in these three well geometry types. The performance of variable area two-fluid dispersed flow model in predicting optimum flow velocity requirements was then evaluated using Temple’s experiments [138] for vertical wellbore cutting transport systems. While there is data collection among operators in industry that currently drill horizontal sections by air, these data are not being shared publicly. Therefore, Welshcof’s horizontal pneumatic grain transport data [140] was also used to evaluate the performance of the model in predicting optimum circulation rate for horizontal systems due to lack of horizontal/deviated cuttings transport data in public literature.

The model proved to be capable of predicting rate requirements and simulating inherent transient behaviors expected in vertical well sections. Optimal velocity in
vertical wells were predicted to within 16 % relative error with Average Absolute Deviation (AAD) of 9.2 %. The model also proved capable of predicting circulation time in both vertical and horizontal systems under dilute flow conditions. However, while the model accounted for flow stratification due to solid deposition common in horizontal pneumatic transport systems through the variable area formulation, it was unable to accurately simulate the inherent transient behaviors expected under dense horizontal flow conditions. Hence, the current two-fluid model is unable to predict circulation time in horizontal/highly-deviated well sections particularly under dense flow conditions. This also means that the model is unable to predict optimum circulation velocity for horizontal/highly-deviated well sections due to its inability to identify the transition point from dilute to dense flow on the FCC plot.

The inability of the current model to simulate appropriately the transition from segregated flow to slug flow regimes (c-d in Fig. 1.8), which occurs in horizontal pneumatic transport systems inherently makes it difficult to generate the flow characteristic curves for horizontal flow in its entirety. This shortcoming renders the current two-fluid model incapable of predicting optimum gas circulation velocity under horizontal and highly deviated pneumatic transport systems. However, this model may still be used to predict cutting circulation time for horizontal flow condition while in dilute phase flow conditions (a-c in Fig. 1.8).

For predicting optimum circulation velocity in horizontal and highly deviated wellbores, correlations were developed. Both vertical and horizontal pneumatic transport data were used in the development of these correlations. Three distinct data sets for both vertical and horizontal pneumatic transport systems available in public literature were used. The newly developed correlations show a relative error less than 10 % for most data points and RMSRD less that 5.1 %. The developed correlations show an order of magnitude improvement to similar empirical models available in pneumatic transport literature. For optimum velocity predictions in deviated well geometries, both the vertical and horizontal correlations developed may be applied and weighted by inclination angle.

The isothermal flow assumption is not a limitation of the model due to the variable local viscosity accommodation. This permits local dynamic viscosity estimation as a function of profile temperature. However, it requires knowledge of
temperature profile in the wellbore a priori. The findings in this research can be applied for the following:

1. Circulation time predictions for air/gas drilling operations in order to streamline cutting circulation process in vertical, horizontal and deviated wellbores.

2. Optimum velocity (or volumetric rate) prediction for drilling design purposes, which will include surface compressor, drill-pipe, bit and BHA selection.
Recommendations

8.1 Pneumatic Transport Modeling

While the current model is suitable for the prediction of volumetric rate requirements and circulation time for pneumatic hole cleaning in vertical wells, there is still room for improvement. The discrepancies between simulation and experimental results observed for larger particle size with broader particle size distributions could be improved using a multi-particle system instead of the mono-dispersed model adopted in this study. That way systems with broader particle size distributions may be modeled as dual-dispersed systems having both coarse and fine particulate phases flowing side by side under pneumatic transport. This is expected to help improve pressure drop estimates under dilute flow conditions for larger particulate systems with broad particle size distributions.

The current numerical model could be improved significantly if the critical velocity approach is used to model cuttings bed formation and growth in conjunction with a dynamic two-layer approach since it will help to appropriately characterize the transition between stratified and slug flow regimes (See Fig. 1.8). This will involve the implementation of a two layer model that is updated dynamically as a function of saltation and pick-up (or re-entrainment) velocities. Perhaps such a model will be capable of predicting optimum rate requirements in horizontal/highly-deviated wellbores.

The 1D limitations as discussed in Ch. 3 can also be alleviated by exploring 2D and 3D drilling hydraulic models. 2D and 3D pneumatic transport models are
expected to better capture the asymmetric physical phenomena present in horizontal and deviated well sections like saltation and re-entrainment. However, while higher dimension models will eliminate the need to develop constitutive relations specific to 1D pneumatic transport models, they will also raise the computational requirement. Therefore, parallelization and Compute Unified Device Architecture (CUDA) computing capabilities would be essential in the development of such models.

The current model makes accommodations for accounting for the effect of temperature variation on viscosity of the gas phase given the wellbore temperature profile. However, the inclusion of an energy conservation will help to account for the effect on temperature variation on the viscosity of the carrier phase implicitly and therefore eliminate the need to have downhole measurement tools for wellbore temperature profile information.

For future studies, it is highly recommended that the pneumatic transport model equations are expressed and solved in dimensionless form. However, this will require ingenuity in the scaling of auxiliary equations. The studies of Shook et al. [142] and Li and Lake [143] might be helpful for appropriate scaling parameter selection.

### 8.2 Experimental Studies

The following experiments would be beneficial for future numerical studies in pneumatic transport modeling of air drilling hydraulics in highly deviated and horizontal well systems:

- Experimental studies that will help to define constitutive relations specific to flow regimes in horizontal and inclined pneumatic transport

- Experiments to define critical velocities with emphasis on the saltation velocity (not necessarily minimum pressure point on the FFC plot) for annular flow under pneumatic transport

- Experiments from that help to define ways to account for the effect of rough undulating well geometries and circumferences due to hole washout and bit cutting mechanism
For both vertical and horizontal flow systems, it would be good to develop experiments that help to account for the effect of added contribution of a additional particulate phases to pressure drop in a multi-particulate pneumatic transport model.
Appendix A

Conservations Laws

A.1 Mass Balance

For a fixed control volume ($V_c$) the conservation of mass may be written as follows:

\[
\begin{bmatrix}
\text{Rate of accumulation of phase "i" in } V_c \\
\end{bmatrix}
= \begin{bmatrix}
\text{Net rate of phase "i" transported into } V_c \\
\text{Net rate of generation of phase "i" in } V_c \\
\end{bmatrix}
\] (A.1)

Left hand side:

\[
\begin{bmatrix}
\text{Rate of accumulation of phase "i" in } V_c \\
\end{bmatrix}
= \frac{\partial}{\partial t} \int_{V_c} \rho_i \, dV
\] (A.2)

Right hand side:

\[
\begin{bmatrix}
\text{Net rate of phase "i" transported into } V_c \\
\end{bmatrix}
= \oint_{\partial V_c} \rho_i \vec{v}_i \cdot (\vec{n}) \, dA
\] (A.3)
and

\[
\begin{bmatrix}
\text{Net rate of} \\
\text{generation of} \\
\text{phase “i” in } V_c
\end{bmatrix} = \int_{V_c} S_i \, dV 
\]  \hspace{1cm} (A.4)

Eq. A.1 may therefore be expressed as:

\[
\frac{\partial}{\partial t} \int_{V_c} \rho_i \, dV = - \oint_{A} \rho_i \vec{v}_i \cdot \mathbf{n} \, dA + \int_{V_c} S_i \, dV 
\]  \hspace{1cm} (A.5)

Figure A.1. Control volume

Provided the a general variable area control volume as in Fig. A.1, dV may be expressed as \(A(x, t)dx\). Therefore, Eq. A.5 may be expressed as:

\[
\frac{\partial}{\partial t} \int_{x} \rho_i A(x, t) \, dx = \left[ -\rho_i v_i A \bigg|_{x_1}^{x_2} + \rho_i v_i A \bigg|_{x_2}^{x_1} \right] + \int_{x} S_i A(x, t) \, dx 
\]  \hspace{1cm} (A.6)

If we assume \(x \neq x(t)\) and take the limit as the \(\Delta x = x_2 - x_1\) approaches zero Eq. A.6 becomes:

\[
\frac{\partial}{\partial t} \int_{x} (\rho_i A_i) \, dx = - \int_{x} \frac{\partial}{\partial x} (\rho_i v_i A_i) \, dx + \int_{x} S_i A_i \, dx 
\]  \hspace{1cm} (A.7)
or

\[ \int x \left[ \frac{\partial}{\partial t} (\rho_i A_i) + \frac{\partial}{\partial x} (\rho_i v_i A_i) - S_i A_i \right] \, dx = 0 \quad (A.8) \]

The integrands in the equation must satisfy the overall equation. This gives the usual differential equation:

\[ \frac{\partial}{\partial t} (\rho_i A_i) + \frac{\partial}{\partial x} (\rho_i v_i A_i) = S_i A_i \quad (A.9) \]

If there is no mass transfer between phases or other forms of a source/sink, \( S_i = 0 \).

### A.2 Momentum Balance

The linear momentum conservation equation is developed from Newtons second law as follows:

\[
\begin{bmatrix}
\text{Rate of change of momentum of material volume } V(t) \\
\text{on } V(t)
\end{bmatrix} = \begin{bmatrix}
\text{Sum of forces acting on } V(t)
\end{bmatrix} \quad (A.10)
\]

Left hand side:

\[
\begin{bmatrix}
\text{Rate of change of momentum of material volume } V(t) \\
\end{bmatrix} = \frac{d}{dt} \int_{V(t)} \rho v \, dV \quad (A.11)
\]

Applying Reynolds Transport Theorem to the time evolving volume integral above we obtain the following:

\[
\frac{d}{dt} \int_{V(t)} \rho v \, dV = \frac{\partial}{\partial t} \int_{V_c} \rho v \, dV + \int_{A} \rho v \, \vec{v} \cdot \vec{n} \, dA \quad (A.12)
\]
From Eq. A.12 we gather that:

\[
\begin{bmatrix}
\text{Rate of change} \\
\text{of momentum of} \\
\text{material volume } V(t)
\end{bmatrix} = \begin{bmatrix}
\text{Rate of change} \\
\text{of momentum} \\
\text{in } V_c
\end{bmatrix} + \begin{bmatrix}
\text{Net} \\
\text{momentum flux} \\
\text{in } V_c
\end{bmatrix} \quad (A.13)
\]

Therefore we may express the rate of change of linear momentum of phase “i” in \( V_c \) as follows:

\[
\begin{bmatrix}
\text{Rate of change} \\
\text{of momentum of} \\
\text{phase “i” in } V_c
\end{bmatrix} + \begin{bmatrix}
\text{Net} \\
\text{momentum flux of} \\
\text{phase “i” in } V_c
\end{bmatrix} = \begin{bmatrix}
\text{Surface forces} \\
\text{acting on} \\
\text{phase “i” in } V_c
\end{bmatrix} + \begin{bmatrix}
\text{Body forces} \\
\text{acting on} \\
\text{phase “i” in } V_c
\end{bmatrix} + \begin{bmatrix}
\text{Interfacial forces} \\
\text{acting on} \\
\text{phase “i” in } V_c
\end{bmatrix} + \begin{bmatrix}
\text{Momentum change} \\
\text{in phase “i” due} \\
\text{to mass transfer}
\end{bmatrix} \quad (A.14)
\]

Right hand side:

\[
\begin{bmatrix}
\text{Surface forces} \\
\text{acting on} \\
\text{phase “i” in } V_c
\end{bmatrix} = \oint_A P_i \cdot (-n) \, dA + \int_{V_c} F_{if} \, dV \quad (A.15)
\]

\[
\begin{bmatrix}
\text{Body forces} \\
\text{acting on} \\
\text{phase “i” in } V_c
\end{bmatrix} = \int_{V_c} \rho_i g \sin \theta \, dV \quad (A.16)
\]
\[
\begin{bmatrix}
\text{Interfacial forces} \\
\text{acting on} \\
\text{phase “i”} \\
in \mathcal{V}_c
\end{bmatrix} = \int_{\mathcal{V}_c} F_{di} \, dV \quad (A.17)
\]

\[
\begin{bmatrix}
\text{Momentum change} \\
in \mathcal{V}_c \text{ due} \\
to mass transfer \\
in \mathcal{V}_c
\end{bmatrix} = \int_{\mathcal{V}_c} F_{Si} \, dV \quad (A.18)
\]

Eq. A.14 may therefore be expressed as:

\[
\frac{\partial}{\partial t} \int_{\mathcal{V}_c} \rho_i v_i \, dV + \oint_A \rho_i v_i \vec{v}_i \cdot \mathbf{n} \, dA = \\
\oint_A P_i \cdot (-\mathbf{n}) \, dA + \int F_{if} \, dV + \int \rho_i g \sin \theta \, dV + \int F_{di} \, dV + \int F_{Si} \, dV \quad (A.19)
\]

Using the variable area control volume in Fig. A.1, we may express Eq. A.14 above as:

\[
\frac{\partial}{\partial t} \int_{x} \rho_i v_i A(x, t) \, dx + \left[ -\rho_i v_i^2 A \bigg|_{x_1} + \rho_i v_i^2 A \bigg|_{x_2} \right] = \\
\left[ -(P_i A) \bigg|_{x_1} + (P_i A) \bigg|_{x_2} \right] + \int \frac{\partial A}{\partial x} \, dx + \\
\int_{x} F_{if} A(x, t) \, dx + \int \rho_i g A(x, t) \sin \theta \, dx + \int F_{di} A(x, t) \, dx + \int F_{Si} A(x, t) \, dx
\]

If we assume \( x \neq x(t) \) and take the limit as the \( \Delta x = x_2 - x_1 \) approaches zero Eq. A.19 becomes:

\[
\frac{\partial}{\partial t} \int_{x} (\rho_i v_i A_i) \, dx + \int \frac{\partial}{\partial x} (\rho_i v_i^2 A_i) \, dx = -\int \frac{\partial}{\partial x} (P_i A_i) \, dx + \int P_i \frac{\partial A}{\partial x} \, dx + \\
\int_{x} F_{fi} A_i \, dx + \int \rho_i g A_i \sin \theta \, dx + \int F_{di} A_i \, dx + \int F_{Si} A_i \, dx
\]

or
\[
\int x \left[ \frac{\partial}{\partial t} (\rho_i v_i A_i) + \frac{\partial}{\partial x} (\rho_i v_i^2 A_i) + \frac{\partial}{\partial x} (P_i A_i) - \frac{\partial A_i}{\partial x} - F_{fi} A_i - \rho_i g A_i \sin \theta - F_{di} A_i - F_{Si} A_i \right] dx = 0
\]
(A.20)

The integrands in the equation must satisfy the overall equation. This gives the usual differential equation:

\[
\frac{\partial}{\partial t} (\rho_i v_i A_i) + \frac{\partial}{\partial x} (\rho_i v_i^2 A_i + P_i A_i) = P_i \frac{\partial A_i}{\partial x} + F_{fi} A_i + \rho_i g A_i \sin \theta + F_{di} A_i + F_{Si} A_i
\]
(A.21)

If there is no mass transfer between phases or other forms of a source/sink, \( F_S = 0 \).
Appendix B

Optimum Velocity Charts: Vertical

B.1 Vertical Pneumatic Transport

Figure B.1. Optimum velocity for vertical transport of particles with 200 micron weighted average diameter
Figure B.2. Optimum velocity for vertical transport of particles with 300 micron weighted average diameter

Figure B.3. Optimum velocity for vertical transport of particles with 400 micron weighted average diameter
Figure B.4. Optimum velocity for vertical transport of particles with 500 micron weighted average diameter

Figure B.5. Optimum velocity for vertical transport of particles with 600 micron weighted average diameter
Figure B.6. Optimum velocity for vertical transport of particles with 700 micron weighted average diameter

Figure B.7. Optimum velocity for vertical transport of particles with 800 micron weighted average diameter
Figure B.8. Optimum velocity for vertical transport of particles with 900 micron weighted average diameter

Figure B.9. Optimum velocity for vertical transport of particles with 1000 micron weighted average diameter
Figure B.10. Optimum velocity for vertical transport of particles with 1100 micron weighted average diameter.

Figure B.11. Optimum velocity for vertical transport of particles with 1200 micron weighted average diameter.
Figure B.12. Optimum velocity for vertical transport of particles with 1300 micron weighted average diameter

Figure B.13. Optimum velocity for vertical transport of particles with 1400 micron weighted average diameter
Figure B.14. Optimum velocity for vertical transport of particles with 1500 micron weighted average diameter

Figure B.15. Optimum velocity for vertical transport of particles with 1600 micron weighted average diameter
Figure B.16. Optimum velocity for vertical transport of particles with 1700 micron weighted average diameter

Figure B.17. Optimum velocity for vertical transport of particles with 1800 micron weighted average diameter
Figure B.18. Optimum velocity for vertical transport of particles with 1900 micron weighted average diameter.

Figure B.19. Optimum velocity for vertical transport of particles with 2000 micron weighted average diameter.
Figure B.20. Optimum velocity for vertical transport of particles with 2100 micron weighted average diameter

Figure B.21. Optimum velocity for vertical transport of particles with 2200 micron weighted average diameter
Figure B.22. Optimum velocity for vertical transport of particles with 2300 micron weighted average diameter

Figure B.23. Optimum velocity for vertical transport of particles with 2400 micron weighted average diameter
Figure B.24. Optimum velocity for vertical transport of particles with 2500 micron weighted average diameter

Figure B.25. Optimum velocity for vertical transport of particles with 2600 micron weighted average diameter
Figure B.26. Optimum velocity for vertical transport of particles with 2700 micron weighted average diameter

Figure B.27. Optimum velocity for vertical transport of particles with 2800 micron weighted average diameter
Figure B.28. Optimum velocity for vertical transport of particles with 2900 micron weighted average diameter

Figure B.29. Optimum velocity for vertical transport of particles with 3000 micron weighted average diameter
Figure B.30. Optimum velocity for vertical transport of particles with 3100 micron weighted average diameter

Figure B.31. Optimum velocity for vertical transport of particles with 3200 micron weighted average diameter
Figure B.32. Optimum velocity for vertical transport of particles with 3300 micron weighted average diameter

Figure B.33. Optimum velocity for vertical transport of particles with 3400 micron weighted average diameter
Figure B.34. Optimum velocity for vertical transport of particles with 3500 micron weighted average diameter
Appendix C

Optimum Velocity Charts: Horizontal

C.1 Horizontal Pneumatic Transport

Figure C.1. Optimum velocity for horizontal transport of particles with 200 micron weighted average diameter
**Figure C.2.** Optimum velocity for horizontal transport of particles with 300 micron weighted average diameter

**Figure C.3.** Optimum velocity for horizontal transport of particles with 400 micron weighted average diameter
Figure C.4. Optimum velocity for horizontal transport of particles with 500 micron weighted average diameter

Figure C.5. Optimum velocity for horizontal transport of particles with 600 micron weighted average diameter
Figure C.6. Optimum velocity for horizontal transport of particles with 700 micron weighted average diameter

Figure C.7. Optimum velocity for horizontal transport of particles with 800 micron weighted average diameter
**Figure C.8.** Optimum velocity for horizontal transport of particles with 900 micron weighted average diameter

**Figure C.9.** Optimum velocity for horizontal transport of particles with 1000 micron weighted average diameter
Figure C.10. Optimum velocity for horizontal transport of particles with 1100 micron weighted average diameter

Figure C.11. Optimum velocity for horizontal transport of particles with 1200 micron weighted average diameter
Figure C.12. Optimum velocity for horizontal transport of particles with 1300 micron weighted average diameter

Figure C.13. Optimum velocity for horizontal transport of particles with 1400 micron weighted average diameter
Figure C.14. Optimum velocity for horizontal transport of particles with 1500 micron weighted average diameter

Figure C.15. Optimum velocity for horizontal transport of particles with 1600 micron weighted average diameter
Figure C.16. Optimum velocity for horizontal transport of particles with 1700 micron weighted average diameter

Figure C.17. Optimum velocity for horizontal transport of particles with 1800 micron weighted average diameter
Figure C.18. Optimum velocity for horizontal transport of particles with 1900 micron weighted average diameter

Figure C.19. Optimum velocity for horizontal transport of particles with 2000 micron weighted average diameter
Figure C.20. Optimum velocity for horizontal transport of particles with 2100 micron weighted average diameter

Figure C.21. Optimum velocity for horizontal transport of particles with 2200 micron weighted average diameter
Figure C.22. Optimum velocity for horizontal transport of particles with 2300 micron weighted average diameter

Figure C.23. Optimum velocity for horizontal transport of particles with 2400 micron weighted average diameter
Figure C.24. Optimum velocity for horizontal transport of particles with 2500 micron weighted average diameter

Figure C.25. Optimum velocity for horizontal transport of particles with 2600 micron weighted average diameter
Figure C.26. Optimum velocity for horizontal transport of particles with 2700 micron weighted average diameter

Figure C.27. Optimum velocity for horizontal transport of particles with 2800 micron weighted average diameter
**Figure C.28.** Optimum velocity for horizontal transport of particles with 2900 micron weighted average diameter

**Figure C.29.** Optimum velocity for horizontal transport of particles with 3000 micron weighted average diameter
Figure C.30. Optimum velocity for horizontal transport of particles with 3100 micron weighted average diameter

Figure C.31. Optimum velocity for horizontal transport of particles with 3200 micron weighted average diameter
Figure C.32. Optimum velocity for horizontal transport of particles with 3300 micron weighted average diameter

Figure C.33. Optimum velocity for horizontal transport of particles with 3400 micron weighted average diameter
Figure C.34. Optimum velocity for horizontal transport of particles with 3500 micron weighted average diameter
## Appendix D

### Experimental datasets

#### D.1 Vertical Experimental datasets

<table>
<thead>
<tr>
<th>$d_p$ (m)</th>
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Table D.1. Temple (1996) [$\rho_s = 3,150 \text{ kg/m}^3; D_H = 0.0357 \text{ m}$]

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Table D.2. Klingzing et al. (2010) [$\rho_s = 1,050 \text{ kg/m}^3; D_H = 0.0526 \text{ m}$]
**Table D.3.** Narimatsu and Ferreira (2001) \[\rho_s = 2,500 \text{ kg/m}^3; D_H = 0.0534 \text{ m}\]

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**D.2 Horizontal Experimental Datasets**

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**Table D.4.** Welschof (1962) \[\rho_s = 1,320 \text{ kg/m}^3\]

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**Table D.5.** Rinoshika and Suzuki (2010) \[\rho_s = 952 \text{ kg/m}^3\]

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<td>2.39E-03</td>
<td>0.345556</td>
<td>7.87</td>
<td>14.9</td>
</tr>
</tbody>
</table>

**Table D.6.** Klingzings et al. (2010) \[\rho_s = 1,050 \text{ kg/m}^3; D_H = 0.0526 \text{ m}\]
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- Flow assurance