DEVELOPMENT AND ANALYSIS OF AN ICE CRYSTAL
SCATTERING DATABASE FOR REMOTE SENSING
APPLICATIONS AND CLOUD MODEL EVALUATION

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Abstract

An ice crystal (ice dendrites, plates and columns) scattering database at radar wavelengths (W-, Ka-, Ku- and X-bands) is created to facilitate radar retrievals and cloud model evaluations of ice cloud properties. The ice crystals in the database are modeled as clusters of closely-packed tiny spheres and their scattering properties calculated with the Generalized Multi-particle Mie-method (GMM). Radar observables, such as effective radar reflectivity factor at $hh$- and $vv$-polarizations ($Z_{hh}$ and $Z_{vv}$), differential reflectivity ($Z_{dr}$), specific differential phase ($K_{DP}$) and specific attenuation at $h$- and $v$-polarizations ($A_h$ and $A_v$), are created from the database using gamma size distributions with various parameters. For the four radar wavelengths considered, ice water content retrievals based purely on radar backscattering cross sections are highly uncertain while $K_{DP}$ shows greater promise.

Using the concepts of resonance and internal electric field strength across an ice crystal, a modified Rayleigh-Gans (MRG) theory is developed that captures most of the variability in the single ice crystal scattering properties within the database. Radar observables are also estimated using the MRG theory and compared to GMM results. $Z_{hh}$, $Z_{vv}$, $Z_{dr}$ and $K_{DP}$ estimated using the MRG theory show good agreement with GMM results. The errors for $A_h$ and $A_v$ are large but not important because attenuation at radar wavelengths is generally small for ice crystals. As such, this MRG theory may be useful in radar applications.

In the representing ice crystals as clusters of tiny spheres with air gaps between them, the dielectric constant of the tiny spheres/air gaps mixtures is smaller than that of solid ice. To make the mixture electromagnetically equivalent to solid ice, the dielectric constant of the tiny spheres in the clusters is artificially increased by an amount dictated by the Maxwell-Garnett approximation. This approach leads to computations that more accurately represent the scattering properties of the original solid ice crystals as compared to increasing the thicknesses of the clusters of tiny spheres so that their total masses match that of the original solid ice crystals.
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4.1 Summary of different ways that ice crystal scattering properties are computed. Method VI uses clusters of tiny spheres with the same overall shape as the solid ice crystals with ice as the sphere dielectric constant. Because the volume fraction \( f \) of the cluster of tiny spheres is smaller than 1 for Method VI and no correction to the dielectric constant is applied, the calculated backscattering cross sections based on Method VI are multiplied by \( 1/f^2 \) and the amplitude scattering matrix elements by \( 1/f \) to compensate for the mass difference. .......................... 71
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Chapter 1

Introduction

Clouds and precipitation are not only important for daily life, but also key determinants of climate. The recent Intergovernmental Panel on Climate Change report (IPCC, [3]) indicates that the largest uncertainties in predicting climate change are related to cloud feedbacks. Important cloud types in these feedbacks include those that contain ice crystals, including both cirrus and mixed phase clouds. Cirrus clouds have large spatial extents and a wide range of ice water contents, thereby producing highly variable surface and atmospheric radiative heating rates. Mixed-phase clouds often have these same two characteristics, but also include intricate interactions between the liquid and ice phases. To make progress in understanding these cloud types observations of their ice crystal properties are essential.

One region particularly sensitive to climate change via changes in cloud properties and the energy budget is the Arctic (Shupe and Intrieri [4]). A large percentage of Arctic clouds is mixed-phase clouds. Ice crystals in these clouds grow into various shapes, sizes and masses. The properties of these ice crystals in turn determine their potential for future growth and fall speeds, hence interactions with other ice crystals and cloud drops, which have a bearing on the lifetimes of these clouds (see Part IV of Lamb and Verlinde [5]). To understand the properties of Arctic clouds in-situ data and field experiments to collect them are necessary (e.g. the Mixed-Phase Arctic Cloud Experiment, or MPACE; Verlinde et al. [6]). These in-situ field campaigns provide detailed information on cloud properties but unfortunately are rare given their expense. Remote sensing of cloud properties, including with cloud radars, provides the potential for continuous retrievals of cloud properties over large spatial and temporal
domains. Polarimetric radars, in particular, are potentially of particular value in inferring particle types as well as ice water contents (Aydin and Tang [7], Botta et al. [2]).

Understanding the underlying mechanisms for the growth of the myriads of ice crystal properties in nature requires the use of cloud resolving models. However, different cloud models show large differences in these properties (Morrison et al. [8]). Most cloud models in use today track only two pieces of information regarding modeled ice crystals: mass and maximum dimension. Sulia and Harrington [9] argue that this is insufficient for accurate modeling of the evolution of different populations of ice crystals; they obtain more realistic model results by tracking mass and the two dimensions of an underlying spheroidal shape for ice crystals. Evaluating cloud models requires conversion of the modeled cloud and precipitation properties to observables, leading to the need of forward models that serve as a link between models and observations. The research in this thesis is focused on radar observables and how best to interpret and apply them in evaluations of cloud models.

The cornerstone of both radar-based retrievals and forward modeling of ice crystal properties is the single scattering properties of a single ice crystal. Exact solutions only exist for several special shapes, such as spheres, spheroids, and spherical core-shell structures. Mie theory provides scattering properties for spheres. The T-matrix method (Mishchenko et al. [10]) is accurate and fast for spheroids.

Calculating scattering properties for other types of ice crystals is complicated because they come in a variety of different shapes (columns, plates, dendrites, aggregates, etc.), maximum dimensions, thicknesses and masses. Several accurate numerical techniques for computation of scattering properties by ice crystals with these irregular shapes have been developed and are now widely used. For example, the Discrete Dipole Approximation (DDA, Purcell and Pennypacker [11], Draine and Flatau [12], Yurkin and Hoekstra [13], Petty and Huang [14]), in which the particles are modeled by clusters of dipoles, is an accurate method for arbitrary shaped particles. With the help of the fast Fourier transform (Goodman et al. [15]), the time requirement of DDA calculations is greatly reduced. But using the fast Fourier transform version of the DDA approximation requires locating the dipoles on a periodic lattice, making it inefficient in computer memory for calculating the scattering properties of sparse particles, such as aggregates, because of the large
amount of empty space within them. In such cases DDA codes without fast Fourier transform are used to reduce the memory requirements in the expense of increasing CPU time (Petty and Huang [14]). The finite difference time domain method (FDTD, Yee [16], Taflove [17], Aydin and Walsh [18]) is another accurate numerical method, but has difficulties when modeling particles with one dimension larger than the wavelength and another dimension much smaller than the wavelength. The Generalized Multi-particle Mie (GMM) method (Xu [19], Xu and Gustafson [20], Grecu and Olson [21], Botta et al. [2]) is an accurate numerical method for clusters of spheres, making it ideal for melting ice particles with water drops anchored to an underlying ice structure. Ice crystals and aggregates can also be modeled using clusters of tiny spheres that resemble their shapes.

Though accurate, these methods are generally time-consuming. Furthermore, sometimes there is insufficient information to warrant an exact calculation, such as lack of knowledge of the exact shape of an ice crystal. As a result, computationally efficient and less accurate alternative methods are sometimes used to calculate scattering properties of ice crystals with complicated shapes. However, these other methods must be used with caution.

One widely used approximate method is modeling ice crystal aggregates as spheres/spheroids with effective dielectric constants (e.g. Matrosov [22], Haynes et al. [23], Hogan [24]) or modeling melting ice aggregates with concentric spheres (Fabry and Szyrmer [25]) to simplify computations. However, recent results of Botta et al. [26] question this approach. They modeled the complicated shapes of ice crystal aggregates with different shapes, sizes and masses as collections of thousands of tiny (about two orders of magnitude smaller than the wavelength), non-overlapping, closely packed spheres. They subsequently used the GMM method to calculate the backscattering cross sections of these aggregates. They learned that aggregates with similar masses and maximum dimensions can have backscattering cross sections that vary by tens of dBs. Modeling aggregates as spheroids with effective dielectric constants is incapable of capturing this variability and can lead to errors as large as tens of dBs. This is also shown in recent work of Petty and Huang [14] and Tynella et al. [27]. This is not surprising because this approximate method redistributes the mass of an aggregate to that of a spheroid. Regions that were previously air gaps become filled with material with an effective dielectric constant. If the dimension of
the particle along the propagation direction of the incident wave is comparable to or larger than the wavelength, the phases of the backscattered electromagnetic waves from different parts of the aggregate and the spheroid show large differences due to different path lengths, leading to large differences in the backscattering cross sections. This raises the possibility that modeling ice crystals using spheres and even spheroids is insufficient for scattering applications.

Another approximate method is based on the Rayleigh-Gans theory (Westbrook et al. [28]). In the Rayleigh-Gans theory the interference effects between the scattered electromagnetic waves from different parts within an ice crystal are considered, while the interactions between different parts of the ice crystal are ignored. Bohren and Singham [29] argue that the dielectric constant of ice is too large for Rayleigh-Gans theory to be applicable to ice crystals. Petty and Huang [14] show that the Rayleigh-Gans theory leads to 2-7 dB underestimation of backscattering cross sections for aggregates made of columns or dendrites. In this thesis, it is shown that the Rayleigh-Gans theory may lead to errors greater than 50%, depending on the orientation of the particle and the polarization of the incident wave. This thesis extends the Rayleigh-Gans theory by considering the interactions between different parts of an ice crystal and estimating the internal electric field across it that result. This approach leads to estimates for the first and higher order components in the scattering-order formulation of the coupled-dipole method described in Singham and Bohren [30].

Modeling single ice crystal scattering properties of hundreds of ice crystals with accurate methods such as GMM and DDA are now possible with today's computer resources, though they are still time consuming. Databases of single scattering properties of ice crystals with exact shapes are of value in studies of the applicability of approximate methods by comparing results from the two. For example, Liu [31] created a database of microwave single-scattering properties for 11 different types of particles using the DDA method. Hong et al. [32] created scattering database in the millimeter and submillimeter wave range of 1001000 GHz for nonspherical ice particles. Botta et al. [2] modeled dendrites using closely packed tiny spheres and then applied the GMM method to them. These sets of calculations were subsequently compared to results from simpler, approximate methods.

To link single ice crystal scattering properties to the total scattering properties
of all of the ice crystals in a certain volume, i.e. to radar observables, many physical properties of the ice crystals inside the volume must be known, including their sizes, masses, shapes, and orientations. For retrieval problems, these properties are unknown. As a result many assumptions about particle physical properties are made. For example, gamma distributions are often assumed to adequately describe the size distributions of ice crystals and ice crystals are usually considered to fall with their maximum dimensions in a horizontal plane with known canting angles. Total scattering properties of the ice crystals inside a volume are then calculated based on these assumptions together with the single scattering properties of each ice crystal in the volume. These assumptions lead to uncertainties in the retrievals that in turn need to be characterized.

Databases of ice crystal scattering properties also serve another purpose: they can be used in search of observables that are sensitive to the masses but not the shapes of particles. If such observables exist, they will greatly facilitate retrieval of total mass contents of ice crystals. Aydin and Tang [7] show that the specific differential phase (KDP) is sensitive to ice water content, but not to other properties, for plates and columns. In this thesis, this idea is tested on dendrites with the help of a database started by Botta et al. [2] and extended to include the amplitude scattering matrix elements in the forward scattering direction. Based on these single ice crystal scattering properties, radar observables for dendrites based on realistic particle size distributions and four specially concocted mass-dimensional relationships are also shown and assessed for ice water content information. Radar observables are tested for a primary dependence on ice water content and not ice-crystal shape and orientation.

This thesis focuses mostly on single scattering properties of ice crystals dendrites, plates and columns at radar wavelengths and their variability. One point of emphasis within the research is on understanding which ice crystal properties drive the variability in their scattering properties. Such understanding can serve as a guide in the development of ice crystal observations necessary for the improvement of cloud models.

The contents of the chapters to follow are now briefly summarized.

Chapter 2 begins by introducing an ice crystal scattering database started by Botta et al. [2] who modeled ice crystal dendrites as clusters of closely packed tiny
ice spheres resembling the shape of the ice crystals and then used the Generalized Multi-particle Mie method (GMM) to calculate single ice crystal dendrite scattering properties. This work extends their database to columns and plates. Moreover, this extension includes two additional upgrades to the database. First, the amplitude scattering matrix for each ice crystal computation is incorporated within the database, allowing calculation of any radar observable such as specific differential phase ($K_{DP}$) and specific attenuation ($A_h$). And second, Ku-band calculations are incorporated into the database to go along with those for the X-, Ka- and W-bands already in it.

Single ice crystal scattering properties are subsequently calculated from the amplitude scattering matrices. To represent the random nature of falling ice crystals they are rotated around the z-axis (which is perpendicular to the horizontal plane) in increments of 30° for dendrites/plates and 10° for columns and the resulting scattering properties averaged. To explain the characteristics of the scattering properties that result, two concepts resonance and internal electric field strength are introduced.

Radar observables for dendrites (those for plates, columns and aggregates are not considered in this thesis) are obtained from the single ice crystal scattering property database by utilizing four specially concocted mass-dimensional relationships and gamma size distributions. The four different mass-dimensional relationships represent sparse thin, dense thin, sparse nominal and dense nominal dendrites, where the thicknesses of the nominal dendrites are the reference thicknesses based on in Pruppacher and Klett [1] (p51-52) while the thicknesses of the thin dendrites are half of the reference thicknesses. Gamma distributions constrained by values from the literature are used to build size distributions of dendrites. The radar observables that result are analyzed in terms of their utility for retrieving ice water content via millimeter-wavelength radar remote sensing.

Chapter 3 contains a model a modified Rayleigh Gans theory that explains the large variability within the database using the concepts of resonance across a dendrite and internal electric field strength variations within it. As is demonstrated in Chapter 3, resonance effects come from the phase differences between backscattered waves from different parts of an ice crystal via path length differences. This is captured by the Rayleigh-Gans theory, whose results are shown first. Subsequently, the internal electric field strengths within dendrites are shown to demonstrate the internal electric field strength variations within a particle and their impact on backscattering
cross sections. Combining these two concepts, a modified Rayleigh-Gans theory is developed that explains most of the variability in the database.

The modified Rayleigh-Gans theory only works for particles with one spatial dimension small compared to the wavelength, which is the case if a particle itself is small compared to the wavelength. For large particles standing-wave-like structures are formed by the internal electric field strengths within the particles, invalidating an underlying assumption of the modified Rayleigh-Gans theory and leading to failure of the method. Applying the same mass-dimensional relationships and particle size distributions used in Chapter 2, radar observables for particles small compared to the wavelength are estimated using the modified Rayleigh-Gans theory and compared with the radar observables from the database presented in Chapter 2. These comparisons show fair agreement, satisfying accuracy requirements for radar applications.

Throughout this work ice crystals are modeled as clusters of closely packed tiny ice spheres resembling the shape of the original ice crystals. The air gaps between the tiny ice spheres make the effective dielectric constant of the mixture of ice spheres and air smaller than that of the original solid ice crystal, leading to differences in scattering properties between the clusters and the original ice crystals. Chapter 4 introduces a method to compensate for the air gaps by artificially increasing the dielectric constant of the tiny ice spheres such that the tiny sphere/air gap mixture is electromagnetically equivalent to the solid ice crystal. Scattering property differences between the modeled ice crystals with the dielectric constant adjustment to the tiny spheres and the original ice crystals are assessed. This method is useful in improving the cluster representations of solid ice crystals used in Chapter 2 and Chapter 3 when computing their scattering properties.

Chapter 5 discusses future work that is a natural follow on to the results presented in this thesis.
Botta et al. [2] created a dendritic ice crystal scattering database by constructing dendrites with different sizes, thicknesses, branch widths, and masses followed by computation of their scattering properties using the GMM method. Using different particle size distributions, they modeled radar signal returns for a variety of different situations. This work extends their database to columns and plates. Moreover, this extension includes two additional upgrades to the database. First, the amplitude scattering matrix for each ice crystal computation is incorporated within the database, allowing calculation of any radar observable, such as specific differential phase ($K_{DP}$) or specific attenuation ($A_h$ or $A_v$), the definitions of which are given in Section 2.4.3. And second, Ku-band calculations are incorporated into the database to go along with those for the X-, Ka- and W-bands already in it.

Methods for constructing ice crystal morphologies contained in the database are first described. Then the methods used to compute the amplitude scattering matrices for each ice crystal are considered. Using the amplitude scattering matrices in the database, the radar observables of particular interest in our studies are computed. The salient features in these radar observables both for single ice crystals and distributions of them are highlighted and their relevance to radar remote sensing presented.

### 2.1 Ice crystal characteristics

Ice crystal dendrites, plates and columns are generated using about 2,000 to 50,000 tiny spheres packed into face-centered cubic (FCC) lattices. The dendrites and plates
Table 2.1. Properties of ice crystal dendrites, plates and columns composed of clusters of tiny spheres. Reference thicknesses for the ice crystals are based on maximum dimensions and dimensional relationships based on crystal types found in Pruppacher and Klett [1, p.51-52] (see 2.2). The actual thicknesses of the ice crystals are determined by this reference thickness multiplied by the Thickness variation in the table. Dendrites with the same maximum dimension and thickness also vary in core sizes and branch sizes, as well as the numbers, locations, and widths of the sub-branches. Note that the thinnest (0.5 thickness variation) plates have only 15 realizations.

<table>
<thead>
<tr>
<th>Crystal type</th>
<th>Maximum dimension (mm)</th>
<th>Number of different maximum dimensions</th>
<th>Crystal type</th>
<th>Thickness variation</th>
<th>Total number of crystals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dendrites</td>
<td>0.50-5.65</td>
<td>11</td>
<td>P1e</td>
<td>0.5, 1</td>
<td>412</td>
</tr>
<tr>
<td>Plates</td>
<td>0.10-3.27</td>
<td>16</td>
<td>P1a</td>
<td>0.5, 1, 2, 3</td>
<td>63</td>
</tr>
<tr>
<td>Columns</td>
<td>0.18-4.51</td>
<td>15</td>
<td>N1e</td>
<td>1, 2, 4</td>
<td>45</td>
</tr>
</tbody>
</table>

Table 2.2. Dimensional relationships based on crystal types found in Pruppacher and Klett [1, p.51-52]. Here \( h \) is the thickness of dendrites and plates; \( d \) is the maximum dimension of dendrites and plates, and the thickness of columns; \( L \) is the maximum dimension (length) of columns. These values are in centimeters.

<table>
<thead>
<tr>
<th>Crystal type</th>
<th>Dimensional relationship, ( h ) (cm), ( d ) (cm), ( L ) (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dendrites</td>
<td>( h=9.022 \times 10^{-3}d^{0.374} )</td>
</tr>
<tr>
<td>Plates</td>
<td>( h=1.41 \times 10^{-2}d^{0.437} )</td>
</tr>
<tr>
<td>Columns</td>
<td>( d=3.527 \times 10^{-2}L^{0.437} )</td>
</tr>
</tbody>
</table>

have three layers of tiny spheres (orders of magnitude smaller than the wavelength) along their thickness while the columns have seven spheres along their thickness. The geometrical properties of the ice crystals are summarized in Table 2.1. These properties are carefully chosen to cover a wide range of mass-maximum dimension relationships found in the literature (e.g. Kajikawa [33], Mitchell [34], Mitchell et al. [35], Heymsfield and Kajikawa [36]). Ice crystal maximum dimensions range from a minimum to a maximum value in equally partitioned steps in logarithmic space within these ranges. Reference thicknesses of the ice crystals are based on different thickness-size relationships for different crystal types in Pruppacher and Klett [1] and are summarized in Table 2.2. The actual thicknesses of the ice crystals in the database are these reference thicknesses multiplied by the thickness variations in Table 2.1 in order to increase the range of variability. Only one plate and one column are generated for each maximum dimension and thickness combination, while
multiple dendrites with the same maximum dimension and thickness combination are generated with different widths, core sizes, branch widths, and sub-branch numbers and locations (e.g. Figure 2.1). (See Figure 3.1 in Chapter 3 for an example of a dendrite composed of 2659 tiny spheres. Botta et al. [2, Figure 1] contains examples of their constructed dendrites compared with real dendrites; their appendix contains detailed information on dendrite geometries.) The masses of these ice crystals are equal to the total masses of the tiny spheres that compose them. As such, the masses and maximum dimensions of the ice crystals overlap with a range of representative mass-dimensional relationships in the literature (see Figure 2.2 for dendrites, Figure 2.3 for plates and Figure 2.4 for columns.)

To compute the scattering properties of ice crystals in the database they are modeled as clusters of tiny spheres resembling the morphology of the original ice
Figure 2.2. Mass versus maximum dimension of dendrites in the database (solid dots) compared with mass versus maximum dimension relationships found in the literature (lines).
Figure 2.3. Same as Figure 2.2, but for plates.

Figure 2.4. Same as Figure 2.2, but for columns.
Table 2.3. Dielectric constants of solid ice at 0 °C for W-, Ka-, Ku-, and X-band wavelengths.

<table>
<thead>
<tr>
<th>Band (wavelength in mm)</th>
<th>W (3.19)</th>
<th>Ka (8.40)</th>
<th>Ku (22.4)</th>
<th>X (31.86)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real part</td>
<td>3.1682</td>
<td>3.1683</td>
<td>3.1686</td>
<td>3.1688</td>
</tr>
<tr>
<td>Imaginary part (10⁻⁴)</td>
<td>3.2586</td>
<td>6.5053</td>
<td>13.044</td>
<td>16.777</td>
</tr>
</tbody>
</table>

Figure 2.5. Incident angle ξ, rotation angle ζ, scattering polar angle θ, scattering azimuth angle φ, and polarization states H and V of radiation illuminating an ice crystal. Vertical incidence corresponds to a 90° incident angle while side incidence corresponds to a 0° incident angle.

Crystals. However, an infinite lattice of closely packed spheres occupies only 74% of the volume. For dendrites, plates, and columns this volume fraction is even smaller because they are made of only several layers of tiny spheres for which boundary effects become significant. As such, it is the scattering properties of clusters of tiny spheres with dendrite-like shapes that are calculated using the GMM method. Botta et al. [2] chose to increase the thickness of each cluster of tiny spheres in order to match the total mass of the cluster to that of the original ice crystal. For hexagonal plates Botta et al. [2] compared this approach with the results of Aydin and Walsh [18] using the FDTD method and noted that at vertical incidence the backscattering cross sections matched to within 2 dB. The database shown in this chapter follows the approach of Botta et al. [2]. Chapter 4 introduces another method to compensate for the air gaps between the tiny spheres in a cluster by increasing the dielectric constant of the tiny spheres such that the tiny spheres/air gaps mixture is electromagnetically equivalent to that of solid ice.
Each ice crystal dendrite, plate and column is illuminated by both horizontally and vertically polarized radiation (see Figure 2.5) at W-band (3.19 mm), Ka-band (8.40 mm), Ku-band (22.40 mm) and X-band (31.86 mm) wavelengths. The dielectric constants of solid ice at 0°C are used (Table 2.3). As is shown in Figure 2.5, the illumination ranges from perpendicular (side incidence, incident angle $\xi=0^\circ$) to parallel (vertical incidence, incident angle $\xi=90^\circ$) to the ice crystal symmetry axis as well as in 10° steps in-between. Canting is not considered in construction of the database; that is, ice crystals are falling with their maximum dimensions in the horizontal plane. To represent the random nature of falling ice crystal rotation angles $\zeta$, dendrites and plates are rotated around the z-axis by 0° and 30° rotation angles and columns are rotated around the z-axis every 10° from 0° to 90°, inclusive. Scattering properties are calculated for each realization, i.e. each ice crystal, each wavelength, each incident angle $\xi$, and each rotation angle $\zeta$. Two polarization states, the traditional $h$-polarization (usually horizontal) and the $v$-polarization (perpendicular to both the $h$-polarization and propagation directions), are considered.

### 2.2 Amplitude scattering matrix

The database contains amplitude scattering matrices for each realization. The amplitude scattering matrix for an ice crystal scattering computation contains the relationship between the incident radiation and that scattered in some direction specified by the angle pair $(\theta, \varphi)$, where $\theta$ is the scattering polar angle [i.e. the angle between the incident direction and the scattering direction, from 0° (forward scattering) to 180° (backward scattering)] and $\varphi$ the scattering azimuth angle (measured within a plane perpendicular to the incident direction from 0° to 180°). The incident wave is decomposed into components parallel ($E_{\parallel i}$) and perpendicular ($E_{\perp i}$) to the scattering plane, which is the plane defined by the incident and scattering directions. The scattered wave is also decomposed into components parallel ($E_{\parallel s}$) and perpendicular ($E_{\perp s}$) to the scattering plane. Based on the notation of Bohren and Huffman [37, section 3.2] and setting the origin of the coordinate system to the center of the particle, the relationship between the amplitudes of the components of
the incident plane wave and the scattered wave at the distance $r$ from the particle is

\[
\begin{pmatrix}
E_{||}\n
\end{pmatrix}
\left(\begin{array}{c}
E_{\perp}\n
\end{array}\right) = \exp(ikr)
\begin{pmatrix}
S_2(\theta, \varphi) & S_3(\theta, \varphi) \\
S_4(\theta, \varphi) & S_1(\theta, \varphi) 
\end{pmatrix}
\begin{pmatrix}
E_{||i}\n
\end{pmatrix}
\left(\begin{array}{c}
E_{\perp i}\n
\end{array}\right)
\]

(2.1)

where $S_i$ (i=1, 2, 3, 4) are the elements of the amplitude scattering matrix and are functions of the scattering polar angle $\theta$ and the scattering azimuth angle $\varphi$. In the database $\theta$ ranges from 0° to 180° in 1° steps whereas $\varphi$ ranges from 0° to 355° in 5° steps. For each incident plane wave impinging on an ice crystal 12,960 amplitude scattering matrices are generated and stored within the database.

Of all the scattering directions, the two most important ones for radar applications are the forward scattering direction (i.e. along the same direction as the incident radiation, or 0° scattering polar angles) and the backward scattering direction (i.e. opposite in direction to the incident radiation, or 180° scattering polar angles). For these two scattering directions the scattering plane is not well-defined because the incident and scattered directions are co-linear along the z-axis. Here we choose the plane containing the traditional $h$-polarization direction (usually horizontal, see Figure 2.5) as the scattering plane. Thus $h$-polarization corresponds to parallel polarization and $v$-polarization (perpendicular to both the $h$-polarization and propagation directions) corresponds to perpendicular polarization. For both of these two scattering directions, the scattering azimuth angle $\varphi$ can be dropped. Single ice crystal scattering properties, or radar observables of interest, are calculated from the amplitude scattering matrices for these two scattering directions.

2.3 Single ice crystal scattering characteristics

Although both modeling and retrieval studies mostly consider the total scattering properties of all of the particles of different sizes, masses, shapes and orientations in some volume, single particle scattering properties are the cornerstone of these studies. The single ice crystal scattering properties emphasized in this study are the backscattering cross sections for $hh$-polarization ($\sigma_{hh}$, send $h$-polarization radiation and receive $h$-polarization radiation) and $vv$-polarization ($\sigma_{vv}$, send $v$-polarization radiation and receive $v$-polarization radiation), the backscattering cross section ratio $\sigma_{hh}/\sigma_{vv}$, and the forward scattering parameters $Im\{S_2(0^\circ)-S_1(0^\circ)\}$, $Re\{S_1(0^\circ)\}$ and
Re\{S_2(0^\circ)\} , which are related to the specific differential phase (between $h$- and $v$-polarized waves, $K_{DP}$) and the specific attenuation of $h$- and $v$-polarized waves ($A_h$ and $A_v$), respectively. These single ice crystal scattering properties are linked directly to the forward and backward amplitude scattering matrices. In dealing with these amplitude scattering matrices one must pay attention to the conventions used by different authors because they are not always the same.

The backscattering cross section $\sigma$ ($\sigma_{hh}$ or $\sigma_{vv}$) of a particle is defined as

$$\sigma = \lim_{r \to \infty} 4\pi r^2 \frac{|E_s|^2}{|E_i|^2},$$

(2.2)

where $E_i$ is the electric field of the incident plane wave and $E_s$ is the electric field of the scattered wave in the backward direction. Using the amplitude scattering matrix given in 2.2, $\sigma_{hh}$ and $\sigma_{vv}$ are

$$\sigma_{hh} = \frac{4\pi}{k^2} |S_1(180^\circ)|^2, \quad \sigma_{vv} = \frac{4\pi}{k^2} |S_2(180^\circ)|^2.$$  

(2.3)

The backscattering cross section ratio is just the ratio of the two.

In this database, canting is not considered. The ice crystals fall horizontally and their orientations in the horizontal plane are random. Their scattering properties should be averaged over all the possible orientations of ice crystals in the horizontal plane (rotation angles $\zeta$ in Figure 2.5). Based on the symmetry of dendrites and plates, $0^\circ$ and $30^\circ$ rotation angles $\zeta$ are reasonable representatives and the scattering properties of these crystal types at these two rotation angles are simply averaged. Columns are rotated from $0^\circ$ rotation angle (perpendicular to $h$-polarization direction) to $90^\circ$ rotation angle (parallel to $h$-polarization direction) in $10^\circ$ steps, where each rotation angle represents a $0^\circ$ rotation angle interval. For example, a $90^\circ$ rotation angle represents an interval from $85^\circ$ to $95^\circ$. Because of symmetry, the scattering properties need to be averaged over a quarter of the horizontal plane, from $0^\circ$ to $90^\circ$ rotation angles. The $0^\circ$ and $90^\circ$ rotation angle intervals both contribute $5^\circ$ intervals to this quarter while the other representative rotation angles contribute $10^\circ$ intervals to this quarter. As such, the $0^\circ$ and $90^\circ$ scattering properties must first be halved and then averaged with those from $10^\circ$ to $80^\circ$. These rotationally averaged results for $\sigma_{hh}$, $\sigma_{vv}$, $\sigma_{hh}/\sigma_{vv}$, $Im\{S_2(0^\circ)-S_1(0^\circ)\}$ , $Re\{S_1(0^\circ)\}$ , and $Re\{S_2(0^\circ)\}$ of dendrites,
Figure 2.6. Path lengths of scattering wave from different parts of a column for (a) the forward scattering direction and (b) the backward scattering direction.

Before discussing salient characteristics of single ice crystal scattering properties contained in the database, several concepts are helpful in understanding them and are now introduced. The first concept is that of resonance. Assume a particle, a column for example, is illuminated by an incoming plane electromagnetic wave propagating in the direction of the positive z-axis (Figure 2.6). The incident electromagnetic wave must travel through different path lengths to reach different parts of the column with different z-coordinates. If an antenna is placed along the negative z-axis (Figure 2.6a), the scattered electromagnetic wave collected by the antenna must travel backwards...
to the antenna. Thus the difference in total path length from two parts of the ice crystal is twice the difference in their z-coordinates, leading to different phases for the two scattered waves when they reach the antenna. If the phase differences of the backscattered waves from all parts of an ice crystal sum to zero, the total resultant backscattered wave from the ice crystal would be very small. This is known as resonance. However, if the transmitting antenna is placed along the negative z-axis and the receiving antenna is placed along the positive z-axis (Figure 2.6b), all of the waves forward scattered to it have almost the same phase because all of the path lengths are almost the same. Thus no resonance occurs in the forward scattering direction.

The second concept is that of (self) interactions amongst different parts of an ice crystal. When the polarization direction of the incident radiation is along the maximum (minimum) dimension of an ice crystal, the internal electric field inside the particle is larger (smaller) than that of incident wave, leading to larger (smaller) amplitudes of the scattered waves. This effect will be discussed in more detail in Chapter 3.

The third concept pertains to the difference between columns and dendrites/plates. Dendrites, plates and columns are all considered to fall with their maximum dimensions parallel to the horizontal plane (xy-plane). When dendrites and plates are rotated around the z-axis, their lengths along the propagation direction at side incidence do not vary much. However, the lengths of columns along the propagation direction at side incidence change by a large amount when rotated around the z-axis, from very long (propagation direction parallel to the column maximum dimension) to very short (propagation direction perpendicular to the maximum dimension). This leads to large changes in the strength of the resonance if the column length is comparable to the wavelength. Figure 2.7 shows an example of the backscattering cross section of a column with a 2.12 mm maximum dimension horizontally illuminated by W-band (3.19 mm) wavelength radiation.

Note the large variation in the strength of the resonance as the column is rotated. When averaging over all rotation angles, the most significant contributions come from column orientations for which the resonance is small, in this case, the larger rotation angles. As a result, although resonance is strong for a column at certain rotation angles, resonance is not strong for results averaged over rotation angles. On
Figure 2.7. $\sigma_{hh}$ versus rotation angle (the angle between the incident radiation and the symmetry axis) of a column with its maximum dimension lying in the horizontal plane. The maximum dimension of the column is 2.12 mm; the thickness of the column is 0.15 mm; the wavelength of the incident radiation is 3.19 mm. The incident radiation also lies within the horizontal plane. A rotation angle of 0° corresponds to the incident radiation along the maximum dimension of the column, while a rotation angle of 90° corresponds to the incident radiation perpendicular to the maximum dimension of the column.

The contrary, for plates and dendrites, the mass distribution along the propagation direction does not change much compared to the wavelength after rotating around the z-axis because of the morphology of dendrites and plates. Therefore, if the resonance is strong for a plate or a dendrite for one orientation, the resonance is still strong after rotating the dendrite or plate about the z-axis. Therefore, strong resonances are observed in some of the horizontally averaged results for dendrites and plates.

These three concepts explain most of the characteristics of the rotationally averaged scattering properties in the database, as we now briefly describe.

The rotationally averaged backscattering cross sections $\sigma_{hh}$ and $\sigma_{vv}$ of dendrites and plates show clear resonances when the lengths of these dendrites and plates along the propagation direction are about $2/3$ of the wavelength (see Figure 2.8 for plates
**Figure 2.8.** $hh$-polarization backscattering cross sections ($\sigma_{hh}$) of plates at W-band (3.19 mm) wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$ versus maximum dimension.

**Figure 2.9.** Same as Figure 2.8, but for columns.

$\sigma_{hh}$ results; dendrites show similar behavior and not shown, same below). At this length the phase differences of the backscattered waves from all parts of an ice crystal almost sum to zero. This feature is particularly clear for the results of plates because the geometry of plates is well defined. It is not as clear for dendrites because dendrites have complicated morphologies.

The rotationally averaged backscattering cross sections $\sigma_{hh}$ and $\sigma_{vv}$ of columns do not show clear resonances (see Figure 2.8 for column $\sigma_{hh}$ results). This is evidence for the third concept discussed above.

The characteristics shown in Figure 2.8 through Figure 2.19 are discussed below. The $h$-polarization direction is always parallel to the horizontal plane, while the angle
Figure 2.10. $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, which is related to specific differential phase ($K_{\text{DP}}$), of plates at W-band (3.19 mm) wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus maximum dimension.

Figure 2.11. Same as Figure 2.10 but for columns.

between the $v$-polarization direction and the horizontal plane equals $90^\circ$ minus the incident angle. For example, at side incidence ($0^\circ$ incident angle) the $v$-polarization direction is perpendicular to the horizontal plane. Chapter 3 illustrates that when the polarization direction is along the maximum (minimum) dimension of an ice crystal the internal electric strength within the ice crystal is large (small), leading to large (small) amplitudes of the scattered radiation. For dendrites and plates the $h$-polarization direction is always along their maximum dimensions for all incident angles while the $v$-polarization direction is along their minimum dimensions at side incidence ($0^\circ$ incident angle) and along their maximum dimensions at vertical
Figure 2.12. Backscattering cross section ratios \((\sigma_{hh}/\sigma_{vv})\) of plates at W-band (3.19 mm) wavelengths for angles of incidence of 0°, 30° and 60° versus maximum dimension.

Figure 2.13. Same as Figure 2.12 but for columns.

incidence (90° incident angle). As such, for dendrites and plates the rotationally averaged forward scattering parameter \(Im\{S_2(0°)-S_1(0°)\}\) decreases with increasing incident angle (see Figure 2.10 for plates). So do the backscattering cross section ratios \((\sigma_{hh}/\sigma_{hh})\) outside of resonance regions (see Figure 2.12 for plates). The forward scattering parameter \(Re\{S_1(0°)\}\) does not change with incident angle (see Figure 2.14 for plates) while \(Re\{S_2(0°)\}\) clearly increases with incident angle (see Figure 2.16 for plates).

For columns it is a bit more complicated. For small columns, the \(h\)-polarization direction has a higher probability to lie along the maximum dimension of the columns as compared to the \(v\)-polarization direction, leading to similar characteristics as
**Figure 2.14.** $Re\{S_1(0^\circ)\}$, which is related to specific attenuation at $h$-polarization ($A_h$), of plates at W-band (3.19 mm) wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus maximum dimension.

**Figure 2.15.** Same as Figure 2.14, but for columns.

for dendrites and plates (Figure 2.11, Figure 2.13 and Figure 2.15). For large columns the forward scattering properties show similar characteristics as for dendrites and plates, while the backscattering cross section ratios ($\sigma_{hh}/\sigma_{hh}$) at $60^\circ$ incident angle for the W-band are still large, though a bit smaller than for a $0^\circ$ incident angle. This is because the backscattering cross sections are dominated by the columns with large rotation angles (the maximum dimension more perpendicular to the incident direction) because the columns with small rotation angles experience strong resonances (Figure 2.6). At the large rotation angles the $v$-polarization direction is more perpendicular to the maximum dimensions of the columns while
Figure 2.16. $Re\{S_2(0^\circ)\}$, which is related to specific attenuation at $v$-polarization ($A_v$), of plates at W-band (3.19 mm) wavelengths for angles of incidence of 0°, 30°, 60° and 90° versus maximum dimension.

Figure 2.17. $Im\{S_2(0^\circ)-S_1(0^\circ)\}$, which is related to specific differential phase ($K_{DP}$), of plates at W-band (3.19 mm) wavelengths for angles of incidence of 0°, 30° and 60° versus mass.

the $h$-polarization directions are more parallel to the maximum dimensions of the columns, leading to large backscattering cross section ratios ($\sigma_{hh}/\sigma_{hh}$) even at large incident angles.

The rotationally averaged forward scattering parameters $Im\{S_2(0^\circ)-S_1(0^\circ)\}$ (see Figure 2.17 for plates), $Re\{S_1(0^\circ)\}$ (see Figure 2.18 for plates) and $Re\{S_2(0^\circ)\}$ (see Figure 2.19 for plates) of dendrites, plates and columns do not exhibit resonances and almost linearly increase with the mass of the ice crystals. This characteristic of these parameters may be useful in ice water content retrievals.
2.4 Linking single particle scattering properties to radar observables

In the previous section rotationally averaged single scattering properties of dendrites, plates and columns were discussed. Radar observables, such as radar reflectivity factor \(Z\), differential reflectivity \(Z_{dr}\), specific differential phase \(K_{DP}\) and specific attenuation \(A_h\) or \(A_v\), are related to the scattering properties of all of the particles in a radar sample volume. In order to link single ice crystal scattering properties to radar observables, particle size distributions are employed. In this section only radar
observables of dendrites are computed and analyzed.

To test the sensitivity of radar observables to differences in dendrites, dendrites with different shapes, thicknesses and masses are sorted into four dendritic class mass-dimensional relationships that represent sparse thin, dense thin, sparse nominal, and dense nominal dendrites, where the thicknesses of the nominal dendrites are the reference thicknesses based on Pruppacher and Klett [1, p.51-52] while the thicknesses of the thin dendrites are half of the reference thicknesses. As a first test of the value of radar observables in ice crystal retrievals, they are computed for gamma size distributions of dendrites within each of the four dendritic classes and then analyzed for their value in retrieving ice water contents.

2.4.1 Four dendritic class mass-dimensional relationships

In the database many realizations of dendrites with the same maximum dimensions are created to increase variability, as is discussed in Section 2.1. To obtain radar observables and test their sensitivity to dendrite shapes, four different subsets of dendrites are used, representing sparse thin, dense thin, sparse nominal, and dense nominal dendrites. These four classes form the four mass-dimensional relationships that are used in the sensitivity study (Figure 2.20).

Two of the four classes are for dendrites with thicknesses equal to the reference thickness based on Pruppacher and Klett [1, p.51-52], while the remaining two classes have half of those thicknesses. To separate the two classes for a given dendritic thickness, for each maximum dimension one third of the dendrites with the most mass are classified as dense dendrites and one third with the least mass are classified as the sparse dendrites. If the number of realizations of nominal or thin dendrites at a given maximum dimension is less than 9, only three dendrites are placed into each of the dense and sparse classes. By placing multiple ice crystals into each of the four classes, the variability of averaged scattering properties will be reduced, leading to more representative results for each class.
Figure 2.20. Four subsets of dendrites representing sparse thin (red), dense thin (green), sparse nominal (blue), and dense nominal (cyan) dendrites. The blue dots show nominal dendrites with the reference thicknesses based on Pruppacher and Klett [1, p.51-52] while the green dots show thin dendrites with half the reference thicknesses. The solid lines show the mean masses for each of the four dendritic ice crystal classes versus their maximum dimensions; the dashed lines show the upper and lower bounds of the masses versus maximum dimension for each class of crystals.

2.4.2 Ice crystal size distributions

Truncated gamma distributions are widely used to represent populations of ice crystals. Following the convention of Botta et al. [2], the gamma distribution

$$N(D) = N_i \frac{1}{c D_n} \left( \frac{D}{D_n} \right)^{\nu-1} \exp \left( -\frac{D}{D_n} \right)$$  \hspace{1cm} (2.4)
is determined by three parameters: the total number concentration of ice crystals \( N_t \), the scaling diameter \( D_n \), and the shape factor \( \nu \). The normalization constant \( c \) is defined as

\[
c = \int_{D_{\text{min}}}^{D_{\text{max}}} \left( \frac{D}{D_n} \right)^{\nu-1} \exp \left( -\frac{D}{D_n} \right) \frac{dD}{D_n}
\]

Here \( D_{\text{min}} \) and \( D_{\text{max}} \) are the minimum and maximum ice crystal sizes in the distribution. A \((\nu, D_n)\) pair determines the shape of a size distribution while \( N_t \) scales the magnitude of the size distribution.

In this study the values of \( D_n \) and \( \nu \) are chosen to vary in the ranges from 0.05 mm to 1.25 mm and 1 to 7, respectively. These values, as well as those below for the other gamma distribution parameters, are from Botta et al. [2]. To generate physically realistic particle size distributions, two additional constraints are necessary when generating \((\nu, D_n)\) pairs. The first one is on the size at which the distribution has its peak value \( D_{\text{peak}} \):

\[
\alpha D_{\text{min}} \leq D_{\text{peak}} \leq \beta D_{\text{max}}
\]

with \( \alpha = 1.05 \) and \( \beta = 0.5 \). The second constraint is that the peak value of the distribution should be three orders of magnitude larger than its value at a diameter of 3 mm (Delanoe et al. [38]). When \( \nu \leq 1 \), \( N(D) \) monotonically decreases with \( D \), which does not satisfy constraint 2.6. When \( \nu > 1 \), \( D_{\text{peak}} = (\nu - 1)D_n \). Figure 2.21 shows valid \((\nu, D_n)\) pairs for ice crystal size distributions based on the mass-dimensional relationship for the dense nominal dendritic class.

The ice water content (IWC) is defined as

\[
IWC = \int_{D_{\text{min}}}^{D_{\text{max}}} m(D)N(D)dD
\]

where \( m(D) \) is the mass of ice crystals as a function of the maximum dimension \( D \). \( N_t \) determines the magnitude of the size distribution, which in turn determines the IWC for one \((\nu, D_n)\) pair. The constraints on IWC and \( N_t \) are

\[
IWC_{\text{min}} \leq IWC \leq IWC_{\text{max}}, \tag{2.8}
\]

\[
N_{t,\text{min}} \leq N_t \leq N_{t,\text{max}}, \tag{2.9}
\]
where $IWC_{\text{min}} = 10^{-4} \text{g m}^{-3}$, $IWC_{\text{max}} = 1 \text{g m}^{-3}$, $N_{t,\text{min}} = 1 \text{m}^{-3}$, and $N_{t,\text{max}} = 10^5 \text{g m}^{-3}$. Values of $N_t$ that satisfy both Eq. 2.8 and 2.9 are used to generate ice crystal size distributions.

2.4.3 Radar observables for distributions of dendrites from the four dendritic classes

Radar observables, such as radar reflectivity factor ($Z$), differential reflectivity ($Z_{\text{dr}}$), specific differential phase ($K_{\text{DP}}$), and specific attenuation ($A_h$ or $A_v$), are related to the scattering properties of all of the particles in a radar sample volume. To link these radar observables to the scattering properties of dendritic ice crystals the mass-dimensional relationships for the four dendritic classes and the gamma distribution are combined to create radar observables for the four classes.
2.4.3.1 Reflectivity factors $Z_{hh}$ and $Z_{vv}$

The $hh$- and $vv$-polarization effective reflectivity factors, $Z_{hh}$ and $Z_{vv}$, are

$$Z_{hh} = \frac{\lambda^4}{0.93\pi^5} \int_{D_{min}}^{D_{max}} \frac{\sigma_{hh}(D)N(D)dD}{\sigma_{hh}(D)} \quad \text{(mm}^6 \text{m}^{-3}) \quad (2.10)$$

$$Z_{vv} = \frac{\lambda^4}{0.93\pi^5} \int_{D_{min}}^{D_{max}} \frac{\sigma_{vv}(D)N(D)dD}{\sigma_{vv}(D)} \quad \text{(mm}^6 \text{m}^{-3}) \quad (2.11)$$

where $\lambda$ is the wavelength and the bar indicates an average over shapes and orientations at each maximum dimension $D$.

2.4.3.2 Differential reflectivity $Z_{dr}$

The differential reflectivity $Z_{dr}$ (Seliga and Bringi [39]) is obtained from the ratio between $Z_{hh}$ and $Z_{vv}$:

$$Z_{dr} = 10 \log 10 \left( \frac{Z_{hh}}{Z_{vv}} \right) \quad \text{(dB)} \quad (2.12)$$

Note that $Z_{dr}$ is not necessarily equal to the integral of $\sigma_{hh}/\sigma_{vv}$ over the size distribution.

2.4.3.3 Specific differential phase $K_{DP}$

When a radar wave propagates through hydrometers, the $h$- and $v$-polarized waves may have different phase speeds if the hydrometer particles are nonsymmetric and have a preferred orientation. This leads to a phase difference between the $h$- and $v$-polarized waves. The rate of change of this phase difference relative to the distance from the radar is the specific differential phase. Based on Aydin [40], whose amplitude scattering matrix is defined differently than in Bohren and Huffman [37], the specific differential phase over a size distribution is

$$K_{DP} = \int_{D_{min}}^{D_{max}} 1000 \frac{180}{\pi} \frac{2\pi}{k^2} Im \{ S_2(D,0^\circ) - S_1(D,0^\circ) \} N(D)dD \quad \text{(deg km}^{-1}) \quad (2.13)$$

where $S_i(D,0^\circ)$ ($i=1,2$) is the average of the amplitude scattering matrices (in meter units) in the forward direction over all the dendrites with maximum dimension $D$ and
all their possible orientations. Here, \( \text{Im}\{\} \) means the imaginary part of the complex quantity.

### 2.4.3.4 Specific attenuation \( A_h \) or \( A_v \)

Based on Aydin [40], the specific attenuation (attenuation per kilometer) over a distribution is

\[
A_h = \int_{D_{\text{min}}}^{D_{\text{max}}} 8686 \frac{2\pi}{k^2} \text{Re}\{S_1(D,0^\circ)\} N(D) dD \quad (\text{dB km}^{-1})
\]

\[
A_v = \int_{D_{\text{min}}}^{D_{\text{max}}} 8686 \frac{2\pi}{k^2} \text{Re}\{S_2(D,0^\circ)\} N(D) dD \quad (\text{dB km}^{-1})
\]

where \( \text{Re}\{\} \) indicates the real part of the complex quantity.

### 2.5 Radar observables inferred from the database

Employing the four dendritic class mass-dimensional relationships, the gamma distribution together with all realistic combinations of \((N_t, \nu, D_n)\), and single scattering properties of dendrites from the four dendritic classes, radar observables for a radar sample volume containing distributions of ice crystals for each one of the four dendritic classes are computed. Ice water content \((IWC)\) is a key cloud microphysical quantity and is thus emphasized in the sensitivity analysis that now follows. Figure 2.22 through Figure 2.26 show ice water content \((IWC)\) versus each radar observable. Each color in the figures corresponds to one of the four dendritic class mass-dimensional relationships. Therefore, the variability within each color shows the sensitivity of \(IWC\) to different ice particle size distributions via different combinations of \((N_t, \nu, D_n)\); the separation between the colors shows the sensitivity of \(IWC\) to the different dendritic ice crystal classes via differences in their mass-dimensional relationships. Based on these figures, there is large variability in \(IWC\) for given values of \(Z_{\text{hh}}\) and \(Z_{\text{dr}}\) whereas the \(IWC\) shows much less variability for given \(K_{\text{DP}}\) and \(A_h\).

From a retrieval perspective the distributions of ice crystal sizes within a radar sample volume are unknown and the mass distribution versus maximum dimension for each type of crystal is also unknown. If the variation of \(IWC\) for a radar
Figure 2.22. Ice water content (IWC) versus hh-polarization radar reflectivity factor (Z_{hh}) at vertical incidence for sparse thin (red), dense thin (green), sparse nominal (blue) and dense nominal (cyan) dendrites computed from simulated particle size distributions for each of the four dendritic classes.

Figure 2.23. Same as Figure 2.22, but for side incidence.
Figure 2.24. Same as Figure 2.22, but for differential reflectivity ($Z_{dr}$) at side incidence.

Figure 2.25. Same as Figure 2.22, but for specific differential phase ($K_{DP}$) at side incidence.
Figure 2.26. Same as Figure 2.22, but for $hh$-polarization specific attenuation($A_h$) at side incidence.

observable is small, then the radar observable has potential to provide information on $IWC$; otherwise, the radar observable cannot be used directly. For example, for any fixed value of $Z_{hh}$, the $IWC$ varies by orders of magnitude. Thus $Z_{hh}$ alone is not appropriate for $IWC$ retrieval.

As it turns out, the specific differential phase ($K_{DP}$) has a much smaller variability versus $IWC$ for each of the four dendritic class mass-dimensional relationships. Over the four mass-dimensional relationships, $K_{DP}$ shows about a 20% variability in $IWC$. Thus $K_{DP}$ may be of value for $IWC$ retrieval. Although the $hh$-polarization specific attenuation ($A_h$) exhibits much smaller variability compared with $Z_{hh}$ or $Z_{dr}$, the values of $A_h$ are quite small and perhaps too difficult to measure. For example, a typical $IWC$ is 0.2 g m$^{-3}$, leading to values of $A_h$ that are less than 0.05 dB km$^{-1}$ at W-band and much smaller at Ka-, Ku- and X-band.

As such, out of all the radar observables discussed above, $K_{DP}$ is the most promising radar observable for $IWC$ retrieval. This is also shown by Aydin and Tang [7] for hexagonal plates and columns.
A modified Rayleigh-Gans theory

As discussed in Chapter 2, most of the variability in the database of single ice crystal scattering properties can be explained by the concepts of resonance and internal electric field strength. In this chapter these two concepts are defined quantitatively and then used to model explicitly the variability in the database. Based on these two concepts, a modified Rayleigh-Gans theory, which can be used to estimate scattering properties of ice crystals, is developed. Single ice crystal scattering properties estimated using this method generally show small differences compared with GMM method results. Applying the four dendritic class mass-dimensional relationships and particle size distributions introduced in Chapter 2, computations of radar observables via the modified Rayleigh-Gans theory are even closer to the GMM method results than for the single ice crystal scattering properties. As such, the modified Rayleigh-Gans theory presented here may be of use in radar applications.

3.1 Development of a modified Rayleigh-Gans theory

This section of the paper was published in the Journal of Quantitative Spectroscopy and Radiative Transfer in 2013 (see Lu et al., 2013: Modeling variability in dendritic ice crystal backscattering cross sections at millimeter wavelengths using a modified Rayleigh-Gans theory. J. Quant. Spectrosc. Radiat. Transf., http://dx.doi.org/10.1016/j.jqsrt.2013.05.008). This paper shows the development
of a modified Rayleigh-Gans theory and uses it to estimate single ice crystal scattering properties. In keeping with the Pennsylvania State University policies (The Graduate School Thesis Guide, p.15, previously published work and first author requirements sections) Yinghui Lu initiated and performed the work published in this paper under the supervision of his co-authors. Copyright policy of the Journal of Quantitative Spectroscopy and Radiative Transfer (http://www.elsevier.com/journal-authors/author-rights-and-responsibilities) permits use of this paper in Yinghui Lus doctoral thesis.

### 3.1.1 Introduction

Ice crystals in the atmosphere grow into various shapes, sizes and masses. These properties of ice crystals in turn determine their potential for future growth and fall speeds, hence interactions with other ice crystals and cloud drops, which have a bearing on the lifetimes of clouds (see Part IV of Lamb and Verlinde [5]). These properties of ice crystals also determine their visible and infrared properties which impact the energy budget of the atmosphere. Millimeter-wave radar signals from ice crystals, such as backscattered powers and differential reflectivity, are valuable in probing ice crystal properties within clouds optically thick at visible and infrared wavelengths. Cloud models can utilize a forward model to calculate ice crystal radiation properties and map model outputs of ice crystal properties into radar signals. For these applications, both the number fraction of each type of ice crystal within a model volume and the forward model used to calculate the radiation properties of ice crystals from model outputs of ice crystal shapes, sizes and masses are critical.

Most cloud models in use today track only two pieces of information regarding modeled ice crystals: mass and maximum dimension. Sulia and Harrington [9] argue that this is insufficient for accurate modeling of the evolution of different populations of ice crystals; they obtain more realistic model results by tracking mass and the two dimensions of an underlying spheroidal shape for ice crystals. Treating ice crystals as spheroids in computation of their backscattering cross sections at millimeter wavelengths has a history within the atmospheric radiation community (e.g. Matrosov [22]), however recent results of Botta et al. [26] question this
approach. They modeled the complicated shapes of ice crystal aggregates with different shapes, sizes and masses as collections of thousands of tiny (about two orders of magnitude smaller than the wavelength), non-overlapping, closely packed spheres. They subsequently used the Generalized Multi-particle Mie method (GMM, Xu [19], Xu and Gustafson [20], a numerical method that computes the scattering properties of clusters of non-overlapping spheres) to calculate the backscattering cross sections of these aggregates. They learned that aggregates with similar masses and maximum dimensions can have backscattering cross sections that vary by tens of dBs. Modeling aggregates as spheroids with effective dielectric constants, which is often used in practice today, is incapable of capturing this variability and can lead to errors as large as tens of dBs. Botta et al. [2] extended the Botta et al. [26] results to dendrites by creating 412 different realizations of a dendrite using from 2659 to 49879 tiny spheres packed into three layers of a face-centered cubic (FCC) lattice. These dendrites have eleven different maximum dimensions from 0.5 mm to 5.5 mm, equally partitioned in logarithmic space. The thickness of the dendrites was based on the thickness-size relationship for P1e crystals from Pruppacher et al. [1] together with variations of 1/2 of that thickness to increase the range of variability. Dendrites with the same maximum dimension and thickness have different widths, core sizes, branch widths, sub-branch numbers and locations. (See Figure 3.1 for an example of a dendrite composed of 2659 tiny spheres. Botta et al. [2, Figure 1] contains examples of their constructed dendrites compared with real dendrites; their appendix contains detailed information on dendrite geometries.) The masses of the dendrites are determined by, hence overlap with, a range of representative mass-dimensional relationships (Botta et al. [2, Figure 2]). Because the closely packed spheres occupy only 74% of the overall dendrite volume, the scattering properties of clusters of tiny spheres with dendrite-like shapes are calculated. Botta et al. [2] show that it is reasonable to increase the thickness of a GMM model dendrite to match the mass and maximum dimension of a real dendrite.

Each dendrite is illuminated by both horizontally and vertically polarized radiation at W-band (3.19 mm), Ka-band (8.40 mm) and X-band (31.86 mm) wavelengths. The dielectric constant of solid ice at 0°C is used: $3.1682 + i3.2586 \times 10^{-4}$ for W-band, $3.1683 + i6.5053 \times 10^{-4}$ for Ka-band and $3.1688 + ii1.6777 \times 10^{-3}$ for X-band. The illumination ranges from perpendicular (side
Figure 3.1. Example of a dendrite in the database of Botta et al. [2]. This dendrite is made of three layers composed of 2659 tiny spheres arranged in a hexagonal close-packed lattice.

Figure 3.2. Angle of incidence $\theta$ and polarization states H and V of radiation illuminating an ice crystal. Vertical incidence corresponds to a 90° incident angle while side incidence corresponds to a 0° incident angle.
incidence) to parallel (vertical incidence) to the ice crystal symmetry axis with several angles in-between (Figure 3.2). The large spread in backscattering cross sections that result from Botta et al.'s [2] GMM calculations is illustrated for side incidence, vertical incidence and two angles in-between in Figure 3.3 for $hh$-polarization (i.e. illuminating with $h$-polarization waves and measuring $h$-polarization returns) backscattering cross sections $\sigma_{hh}$. Figure 3.4 contains the results for $vv$-polarization backscattering cross sections $\sigma_{vv}$. These two figures show that for dendrites with the same maximum dimensions and incident directions of illumination the spread in the backscattering cross sections can be tens of dBs, especially for dendrites larger than half of the wavelength. These results of Botta et al. [26, 2] suggest that while spheroids may be appropriate for modeling the microphysical evolution of ice crystals according to the ideas of Sulia and Harrington [9], they are not appropriate for modeling individual ice crystal backscattering cross sections.

Botta et al. [26, 2] do not attempt to characterize what it is about the aggregate and dendrite structures that lead to the observed variability. The aim of this study is to develop a physically intuitive model that predicts the dendrite backscattering cross sections produced by Botta et al. [2]. Such a model provides insights into the dendrite physical properties that must be modeled so that computing accurate backscattering
cross sections for them is possible. Section 3.1.2 shows that the Rayleigh-Gans theory, which considers interference effects across the tiny spheres into which each ice crystal is decomposed, captures most of the variability in the database. Section 3.1.3 illustrates that the internal electric field within a dendrite is different from the incident electric field because of the interactions between the tiny spheres into which an ice crystal is decomposed and explains much of the residual variability not accounted for by the Rayleigh-Gans theory. In Section 3.1.4 a modification is made to the Rayleigh-Gans theory by weighting each term in the form factor by appropriately normalized internal electric field strength, yielding better agreement with the GMM results. Section 3.1.5 provides a brief summary and discussion.

### 3.1.2 The Rayleigh-Gans theory

To capture the variability in the dendrite backscattering cross section database of Botta et al. [2] we first adopt the Rayleigh-Gans theory (e.g. Bohren and Huffman [37]; Westbrook et al. [28]). In our application of the Rayleigh-Gans theory (see Section 3.1.6) a dendrite is decomposed into small volumes treated as spheres much smaller than the wavelength. Interactions between the spheres are ignored. (That is, each small sphere experiences only the incident electric field.) Assuming the incident
wave propagates along the positive-z direction, the backscattering cross section of a dendrite is (see Bohren and Huffman [37, Chapter 6] and Westbrook et al. [28, Section 4])

\[
\sigma = \sigma_r \cdot f, \quad \sigma_r = \frac{9k^4}{4\pi} \left| \frac{\varepsilon - 1}{\varepsilon + 2} \right|^2 V^2, \quad f = \left| \frac{1}{V} \sum_m [V_m \exp(i2kz_m)] \right|^2, \quad (3.1)
\]

where \( k = 2\pi/\lambda \), \( \lambda \) is the wavelength in free space, \( \varepsilon \) is the dielectric constant of solid ice, \( V \) is the total volume of all the tiny spheres, \( f \) is the form factor and \( V_m \) and \( z_m \) are the volume and z-coordinate of the \( m^{th} \) tiny sphere. The cross section \( \sigma_r \) \((\propto M^2\lambda^{-4})\) serves as a reference that accounts for the total mass (\( M \)) and wavelength contributions to the backscattering cross sections. The form factor accounts for deviations of the dendrite backscattering cross sections caused by interference effects between the backscattered electromagnetic waves from the small spheres into which the dendrites are decomposed; it is a weighted (by volume of the small spheres) average that accounts for relative phase differences of the backscattered electric field from each of the small spheres to an observer far away from the dendrites. When the dimension of a dendrite is much smaller than the wavelength, the form factor \( f \) is about 1. The thicknesses of the dendrites are much smaller than a millimeter, so that for vertically incident waves the form factor \( f \) is also about 1. The distribution of mass of the dendrite along the propagation direction determines the form factor.

Figure 3.5 shows the comparison between the backscattering cross sections estimated using the Rayleigh-Gans theory and that from Botta et al. [2]. The Rayleigh-Gans theory captures most of the variability in the backscattering cross sections, but there is a noticeable difference between the results for \( hh \)- and \( vv \)-polarization states and a dependence on incident angle for \( vv \)-polarization. Ignoring the interactions among the small spheres leads to errors in estimating the electric field at the location of each small sphere, resulting in errors in the estimates of the backscattering cross sections.

### 3.1.3 Internal electric field strength

The internal electric field within a dendrite can be estimated through a physically intuitive, iterative-based method. A sphere much smaller than the wavelength can
Figure 3.5. Differences (in dB) between estimated backscattering cross sections using the Rayleigh-Gans (RG) theory and those calculated using the GMM method as a function of the form factor ($f$ in Eq. 3.1).

be approximated as a dipole (see Bohren and Huffman [37, Chapter 5]). Consider the electric field produced by a dipole at the locations of its neighbors (Figure 3.6). If a neighboring sphere is along the axis of symmetry of the dipole (i.e. along the polarization direction of the incident electric field illuminating the tiny sphere, as is shown at points $A$ in Figure 3.6), the electric field at the location of the neighbor is larger than that of the incident wave because the electric field produced by the dipole enhances that of the incident wave. If the neighboring sphere lies within a plane perpendicular to the dipole symmetry axis (i.e. perpendicular to the electric field of the incident radiation as is shown at points $B$ in Figure 3.6), the electric field at its location is smaller than that of the incident wave. There are cones centered on the symmetry axis of the dipole such that if neighbors fall within the cones the
Figure 3.6. Electric field lines near a dipole induced by incident radiation with an electric field upward. The dipole electric field enhances (weakens) the incident electric field inside (outside) the shaded cones.

local electric field of the dipole enhances the internal electric field at these neighbors (upper and lower shaded regions in Figure 3.6); outside of the cones the electric field of the dipole weakens the internal electric field at these neighbors.

How much can the electric field at the location of a sphere change due to interaction between the sphere and its neighbors? Figure 3.7 answers this question by showing how much the electric field at the center of a 0.01-mm-radius sphere centered on the origin changes relative to the incident electric field because of the presence of a neighboring 0.01-mm radius sphere. For this example the incident radiation has a wavelength of 8.40 mm with its electric field parallel to the y-axis. The white region within the figure has a radius of 0.02 mm, illustrating the region from which the center of the neighboring sphere is excluded by the sphere at the origin. The gray shade shows the altered electric field strength relative to the electric field strength of incident radiation when the neighboring sphere is at that location. Note that although the scattered radiation from the neighbor has three components, only the component along the polarization direction of the incident wave is shown. Considering
Figure 3.7. Enhancement of the electric field at the origin by a nearby sphere (treated as a dipole). The shading shows how much the electric field of a 0.01-mm radius sphere centered on the origin is enhanced by the electric field of another 0.01-mm radius sphere centered elsewhere. The dash lines separates the region where the neighboring sphere enhances (the darker region) and decreases (the lighter region) the electric field at the origin.

In three-dimensional space, the regions that strengthen and weaken the field at the origin are separated by two cones. Neighbors inside these two cones increase the electric field at the origin while neighbors outside the two cones decrease the electric field at the origin. Notice that the magnitude of the increase and decrease in field strength drops very quickly with increasing distance from the origin. Thus, if the neighboring spheres lie mostly within the cones and close to the sphere, the electric field strength would be largest for the sphere at the origin.

Consider millimeter-wavelength radiation incident along the symmetry axis of a prolate and an oblate spheroid with its electric field perpendicular to it. Decomposing these spheroids into thousands of tiny spheres, we find that, on average, spheres in the oblate spheroid have more neighbors inside the cones of enhancement while less do so.
Figure 3.8. The internal electric field strength throughout the medium layer of one realization of a dendrite estimated via our approximate method. The propagation direction is the z-axis (vertical) and the polarization direction is the y-axis (horizontal); the wavelength of the radiation is 8.40 mm.

for the spheres in the prolate spheroid. The average internal electric field strength, hence backscattering cross section, is therefore higher for the oblate spheroid, less for the prolate spheroid. Raindrops are oblate spheroids with their major axes lying in the horizontal plane and their minor axes vertical to it. After decomposing these raindrops into lots of small spheres, it is clear that on average more small spheres are within the cones of enhancement of each other for horizontally polarized incident electromagnetic fields while less are so for vertically polarized fields. This leads to overall larger internal electric fields across raindrops for horizontal polarization, yielding larger backscattering cross sections. This agrees with the observation that the backscattering cross sections of raindrops much smaller than the wavelength for horizontal polarization are larger than those for vertical polarization at side incidence.

Figure 3.8 illustrates the internal electric field strength estimated via the approximate method (see Section 3.1.7) throughout the middle of three layers that compose one realization of a dendrite. (Dendrites are composed of three layers of tiny spheres in the study of Botta et al. [2].) The propagation direction is the z-axis
(vertical) and the polarization direction is the y-axis (horizontal); the wavelength of the radiation is 8.40 mm. The highest internal electric field strengths are for locations where lots of nearby spheres are inside their cones of enhancement while fewer neighboring spheres are outside of it (points 1 in the figure). The lowest internal electric field strengths are for locations where their cones of enhancement lack nearby spheres but with many neighboring spheres located outside of them (points 2 in the figure).

Figure 3.9 is similar to Figure 3.8 except that the polarization direction is the x-axis (perpendicular to the paper, not shown). Note that the color scale is different from that in Figure 3.8. In this case most spheres throughout the middle layer of the dendrite have neighbors that are outside their cones of enhancement with the few exceptions being the neighbors along the polarization direction in the surrounding two layers. Thus almost all of the tiny spheres have internal electric field strengths relative to the incident field that are less than 1, indicating a decrease in the incident electric field strength because of interactions among the tiny spheres. The edges have relative field strengths closer to 1 because of the lack of neighboring spheres outside of their cones of enhancement.
This internal electric field strength can explain the characteristics of the variability shown in Figure 3.5. For $hh$-polarization the polarization direction is always parallel to the major axes of the dendrites, as is shown in Figure 3.8, so that interactions mostly increase the internal electric field strength, hence increase the backscattering cross section. For $vv$-polarization, the polarization direction changes from parallel (corresponding to Figure 3.8) to perpendicular (corresponding to Figure 3.9) to the plane the dendrites lie in so the interactions change from mostly increasing to mostly decreasing the internal electric field strength, leading to the incident angle-dependent characteristics of the backscattering cross sections. These findings show that interactions among the tiny spheres can lead to significant variations in the electric field strength throughout the dendrites and can change the backscattering cross sections by several dBS compared to the Rayleigh-Gans theory. It is worthwhile pointing out that the extreme aspect ratio of dendrites leads to this significant effect. For those dendrites with smaller aspect ratios, the internal electric field strength would be closer to 1 because neighbors are more likely to be located both inside and outside the cones of enhancement.

### 3.1.4 A modified Rayleigh-Gans theory

To account for interactions between the small spheres in the GMM approach a modification must be made to the Rayleigh-Gans theory (see Section 3.1.6). This is done by first estimating the internal electric field strength throughout a dendrite (using the method described in Section 3.1.7) and then using these estimates to weight the terms in the Rayleigh-Gans theory. The modified Rayleigh-Gans theory then becomes (see Section 3.1.7)

$$\sigma = \sigma_r \cdot f, \quad f_{MRG} = \left| \frac{1}{V} \sum_m [G_m V_m \exp (i2kz_m)] \right|^2,$$

where $G_m$ is the ratio of the internal electric field strength as a result of interactions between the tiny spheres to that of the incident wave and the subscript $MRG$ indicates modified Rayleigh-Gans theory. As is pointed out in Section 3.1.7, this modification works best when the smallest dimension of the local structure of an ice crystal (for example, thickness of a dendrite, width of a column and branch width in a tree-shaped
Figure 3.10. Estimated backscattering cross sections using the modified Rayleigh-Gans (MRG) theory compared with those calculated using the GMM method.

Figure 3.10 shows comparisons between the modified Rayleigh-Gans theory and the GMM-computed backscattering cross sections for all the dendrites, all the incident directions and both of the two polarization states. Figure 3.11 shows that the difference between the modified Rayleigh-Gans theory and GMM are mostly within 0.5 dB. The errors tend to be largest when the form factor from the Rayleigh-Gans theory is very small. All cases for which the error is larger than 1 dB have form factors on the order of $10^{-3}$. To test the accuracy of the GMM method for these highly resonant dendrites with form factors close to 0 seven cases with error larger than 1 dB are recalculated using the discrete dipole approximation (DDA) implemented by Loke et al. [41]. The locations of the dipoles in the DDA calculations are the same as those of the tiny spheres composing the dendrites; the polarizability of the dipoles in the DDA calculations is $\alpha = a^3(\varepsilon - 1)/(\varepsilon + 2)$, where $a$ and $\varepsilon$ are the radius and dielectric constant of the tiny spheres.

The differences between GMM and DDA results for these seven cases are quite small (maximum of about 2.9% or 0.12 dB; see Table 3.1), indicating that the GMM
**Figure 3.11.** Differences (in dB) between estimated backscattering cross sections using the modified Rayleigh-Gans (MRG) theory and those calculated using the GMM method as a function of the form factor ($f$ in Eq. 3.1) to facilitate comparison with Figure 3.5.

| Table 3.1. Comparison between DDA and GMM backscattering cross sections for seven dendrites with small form factors. Units: $10^{-6}$ mm$^2$. |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $\sigma_{hh, \text{GMM}}$       | 0.518           | 0.744           | 0.471           | 0.345           | 1.458           | 0.254           | 0.282           |
| $\sigma_{hh, \text{DDA}}$       | 0.517           | 0.763           | 0.470           | 0.345           | 1.450           | 0.261           | 0.285           |
| $\sigma_{vv, \text{GMM}}$       | 0.995           | 3.341           | 0.951           | 0.824           | 3.628           | 4.331           | 10.025          |
| $\sigma_{vv, \text{DDA}}$       | 0.981           | 3.294           | 0.937           | 0.812           | 3.574           | 4.265           | 9.853           |

method is accurate for the small form factor cases. One possible reason for these large errors of the modified Rayleigh-Gans theory when the form factor is close to 0 is that the backscattered electric fields from all the small spheres almost cancel. For these cases small errors in estimation of the internal electric field strength can lead to large relative errors.

To demonstrate this concept, consider the idealized ice crystals in Figure 3.12 whose lengths are half the wavelength of the incident radiation (W-band wavelength of 3.19 mm). The incident radiation propagates vertically and the polarization direction is horizontal as is shown in the figure. The two spherical globules shown in Figure 3.12a are fixed one-quarter wavelength apart. The only difference from Figure 3.12a through Figure 3.12d is that the upper globule is flattened from a sphere to an oblate spheroid with aspect ratio of 0.2, which increases the internal electric field strength inside the upper globule. This configuration is carefully chosen so that the form
Figure 3.12. Idealized ice crystals concocted to demonstrate the importance of internal electric field strength contributions to backscattering cross sections for cases with small form factors. Flattening the top spherical globule (panel a) in small increments to the oblate spheroid (panel d) leads to the results in Figure 3.13. The propagation direction is parallel to the columnar ice crystals and has a wavelength of 3.19 mm.

Figure 3.13. Backscattering cross sections of the idealized ice crystals in Figure 3.12 as the top spherical globule in a) is flattened into the oblate spheroid in d). The backscattering cross sections are plotted against the aspect ratio of the oblate spheroid. The annotations indicate the particles shown in Figure 3.12.

The form factor for all of these particles is small and almost the same, corresponding to the form factors with the greatest differences in Figure 3.11. Figure 3.13 shows that the GMM-calculated backscattering cross sections of these particles have a spread of about 20 dB. Thus, when the form factor is small, errors in estimating the internal electric field strength in just one part inside the particle may lead to several dB errors.
in estimates of backscattering cross sections.

In meteorological radar applications ice crystals come in a variety of different sizes, shapes and orientations within a radar sample volume. Those crystals with the largest relative errors in the estimates of their backscattering cross sections have small backscattering cross sections because of small form factors, and thus have a small contribution to the total backscattering cross section of the radar sample volume. For example, the sum of the backscattering cross sections of all the cases in the Botta et al. [2] database estimated through the modified Rayleigh-Gans theory differs from the corresponding GMM value by only 0.074 dB. These results indicate that the modified Rayleigh-Gans theory, while an approximate method compared to more accurate numerical methods such as the discrete dipole approximation (DDA, Draine and Flatau [12]), the finite-difference time-domain method (FDTD, Yee [16], Taflove and Hagness [17]) and the T-matrix method (Mishchenko et al. [42]), may aid in estimation of the total radar backscattering cross sections of relatively large sample volumes and all of the ice crystals within them. Unfortunately, these results indicate that the total backscattering cross sections themselves may not be related in any simple way to ice mass within these radar sample volumes.

3.1.5 Discussion

The large variability within the database of Botta et al. [2] indicates that providing just the maximum dimension and mass of dendrites is insufficient in estimation of their backscattering cross sections. The modified Rayleigh-Gans theory captures most of the variability within the database of Botta et al. [2], indicating that both the relative phase of the backscattered electric fields from all the small spheres that compose a dendrite and the interactions between them contribute to the backscattering cross section. That is, to estimate accurately ice crystal backscattering cross sections knowledge of the distribution of mass along both the propagation and polarization directions is necessary, requiring knowledge of detailed ice crystal structure from which the form factor and internal field strength can be estimated.

For ice microphysical models to better inform the interpretation of radar returns at millimeter wavelengths these models must be able to predict or diagnose the distribution of mass within different ice crystals. The development of such models,
in turn, depends on the availability of observations of the mass distribution within ice crystals. The Cloud Particle Imager (CPI) developed by the Stratton Park Engineering Company (SPEC), Inc., provides images from which ice crystal structure can begin to be gleaned but the images are one-dimensional with no accompanying mass estimates. Techniques for observing the three-dimensional structure of ice crystals (e.g. Garrett et al. [43]), together with characterization of ice crystal mass and distribution of mass within the ice crystal, will ultimately be necessary to advance the microphysical models in terms of their use in interpretation of millimeter-wavelength radar returns.

3.1.6 Appendix A. The Rayleigh-Gans theory

Take the propagation direction to be in the positive z-axis direction with the origin of the Cartesian coordinate system at the center of mass of the ice crystal. Based on Bohren and Huffman [37, Eq. 6.1 and Eq. 6.5] the small sphere regime for the backward scattered electric field observed at point \((0, 0, -z')\) from the \(m\)th tiny sphere located at \((x_m, y_m, z_m)\), where \(z'\) is much larger than the dimensions of the ice crystal, is

\[
E_{s,m} = \frac{1}{z'} \frac{3k^2}{4\pi} \left( \frac{\varepsilon - 1}{\varepsilon + 2} \right) V_m \exp[i2k(z_m + z')] E_0 \hat{e}_i, \tag{3.3}
\]

where \(\varepsilon\) is the dielectric constant of solid ice, \(V_m\) is the volume of the \(m\)th sphere, wavenumber \(k = 2\pi/\lambda\), and \(\hat{e}_i\) is a unit vector along the polarization direction. Note that the time harmonic term \(\exp(-i\omega t)\) is omitted. The total backscattered electric field is the vector sum of all the backscattered electric fields from all of the tiny spheres into which a dendrite is decomposed:

\[
E_s = \sum_m E_{s,m} = \frac{1}{z'} \frac{3k^2}{4\pi} \left( \frac{\varepsilon - 1}{\varepsilon + 2} \right) \sum_m \left\{ V_m \exp[i2k(z_m + z')] \right\} E_0 \hat{e}_i. \tag{3.4}
\]

The backscattering cross section is

\[
\sigma = 4\pi z'^2 \left| \frac{E_s}{E_0} \right|^2 = \frac{9k^4}{4\pi} \left| \frac{\varepsilon - 1}{\varepsilon + 2} \right|^2 V^2 \left| \frac{1}{V} \sum_m \left[ V_m \exp(i2kz_m) \right] \right|^2. \tag{3.5}
\]
3.1.7 Appendix B. Estimation of internal electric field strength

Take each of the small spheres into which a dendrite is decomposed to be a dipole. A sphere with radius $a$ illuminated by a plane wave has a dipole moment $p$ given by (Bohren and Huffman [37, Eq. 5.15])

$$p = 4\pi\varepsilon_0 a^3 \frac{\varepsilon - 1}{\varepsilon + 2} E,$$  \hspace{1cm} (3.6)

where $\varepsilon_0$ is the vacuum permittivity and $E$ is the external (incident) electric field at the location of the small sphere. Jackson [44] derived an expression for the electric field around an electric dipole excited by $E$:

$$E_s = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{k^2}{r} (\hat{r} \times p) \times \hat{r} + \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) [3\hat{r} (\hat{r} \cdot p) - p] \right\} e^{ikr}. \hspace{1cm} (3.7)$$

Using a 3x1 matrix notation for vectors in Cartesian coordinates leads to

$$\hat{r} = \frac{1}{r} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad p = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}, \hspace{1cm} (3.8)$$

After some tedious algebra Eq. 3.7 can be written as

$$E_s = \frac{1}{4\pi\varepsilon_0} \left\{ \frac{k^2}{r} (I_3 - \hat{r}\hat{r}^T) + \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) (3\hat{r}\hat{r}^T - I_3) \right\} e^{ikr} p, \hspace{1cm} (3.9)$$

where $I_3$ is the 3x3 identity matrix and the superscript $T$ indicates the transpose of a matrix. This notation is similar to the dyadics used in Lakhtakia and Mulholland [45]. Substituting Eq. 3.6 into Eq. 3.9 leads to

$$E_s = A(r) E,$$  \hspace{1cm} (3.10)

where

$$A(r) = a^3 \frac{\varepsilon - 1}{\varepsilon + 2} \left\{ \frac{k^2}{r} (I_3 - \hat{r}\hat{r}^T) + \left( \frac{1}{r^3} - \frac{ik}{r^2} \right) (3\hat{r}\hat{r}^T - I_3) \right\} e^{ikr}. \hspace{1cm} (3.11)$$

The electric field at the location of each dipole is the sum of the incident electric
field and the scattered electric fields from all of the other dipoles. The total electric
field at the location of the \(m^{th}\) dipole is

\[
E_m = E_{\text{inc},m} + \sum_{n \neq m} A_{n,m} E_n, \tag{3.12}
\]

where \(A_{n,m} = A(r_{n,m})\), \(r_{n,m}\) is the location vector from the \(n^{th}\) sphere to the \(m^{th}\)
sphere and \(E_{\text{inc},m}\) is the electric field of the incident wave at the location of the \(m^{th}\)
sphere. The discrete dipole approximation (DDA, Lakhtakia and Mulholland [45])
method solves this linear equation to get the electric field at the location of each
dipole from which the scattering properties of the whole dendrite can be calculated.
However, solving this linear equation is time consuming. In the approximate method
for this study an iterative approach is used to estimate the electric field at the location
of each sphere.

In the near field of a dipole the most significant part of \(A(r)\) in Eq. 3.11 is the term
with the factor of \(1/r^3\). Thus decreases rapidly with increasing \(r\), which means that
only the dipoles close to the \(m^{th}\) dipole have a significant contribution to the electric
field at the \(m^{th}\) dipole. Therefore, we consider only the contributions to the electric
field at the location of the \(m^{th}\) dipole from neighboring dipoles within a distance \(r_0\)
of it. In this work \(r_0\) is empirically set to 41 times the tiny sphere radius (or 20
diameters plus a radius). However, since \(\iiint_V (3 \hat{r} \hat{r}^T - I_3) = 0\) when \(V\) is a sphere
centered on the origin, longer range interactions become important because the short
range interactions cancel out. Therefore, including only short range interactions is
appropriate only if some, but not all, of the neighboring regions \(r \ll \lambda\) of a sphere are
occupied by other spheres. Dendrites are an example for which the above argument
is valid. Based on these ideas, the total electric field at each dipole can be estimated
through the following steps:

1. Each dipole is illuminated by the incident radiation, so \(\mathbf{E}_m\) changes from 0 to
   \(\mathbf{E}_m = \mathbf{E}_{\text{inc},m}\).

2. The scattered electric field from each dipole influences its neighboring dipoles.
The total scattered electric field at the location of the \(m^{th}\) dipole from its neighbors
\[ \Delta_1 E_m = \sum_{r_{n,m} < r_0} A_{n,m} E_{\text{inc},n} \approx \left( \sum_{r_{n,m} < r_0} A_{n,m} \right) E_{\text{inc},m} = A_m E_{\text{inc},m}. \] (3.13)

Thus the electric field at the location of the \( m^{th} \) dipole becomes \( E_{\text{inc},m} + \Delta_1 E_m \).

(3) The changes in the electric fields in step 2, i.e. \( \Delta_1 E_n \) at all dipole locations, subsequently influence the electric field at the location of the \( m^{th} \) dipole. Similar to step 1, the influence of this part is

\[ \Delta_2 E_m = \sum_{r_{n,m} < r_0} A_{n,m} \Delta_1 E_n \approx \left( \sum_{r_{n,m} < r_0} A_{n,m} \right) \Delta_1 E_m = A_m^2 E_{\text{inc},m}. \] (3.14)

(4) Similarly, the \( j^{th} \) iteration leads to \( \Delta_j E_m = A_m^j E_{\text{inc},m} \). The total electric field at the \( m^{th} \) dipole is

\[ E_m = E_{\text{inc},m} + \sum_{j=1}^{\infty} \Delta_j E_m \approx \sum_{j=0}^{\infty} A_m^j E_{\text{inc},m} = (I_3 - A_m)^{-1} E_{\text{inc},m}. \] (3.15)

Equation 3.15 provides an estimate of the internal electric field of an ice crystal, from which the scattered electric field of the ice crystal can be calculated. For polarization along \( \hat{e} \) the amount that interactions amplify the incident electric field is

\[ G_m = \hat{e}^T (I_3 - A_m)^{-1} \hat{e} \] (3.16)

where \( G_m \) contains both amplitude and phase changes.

### 3.2 Applying the modified Rayleigh-Gans theory to particles small compared to the wavelength

The modified Rayleigh-Gans theory introduced in the previous section produces estimates for the backscattering cross sections of dendrites at radar wavelengths with differences mostly within 0.5 dB compared with the GMM results. Sinham and Bohren [30] developed a scattering-order formulation of the coupled-dipole method.
The Rayleigh-Gans theory captures the zero-order of the scattering-order formulation while the modified Rayleigh-Gans theory added higher orders in the formulation through an estimation method. In this section this modified Rayleigh-Gans method is generalized in three ways. First, the polarizability of a tiny sphere is changed to the polarizability of a dipole. With this change the modified Rayleigh-Gans theory formulation in Section 3.1 now has the capability to estimate scattering properties of solid ice crystals. Second, the formulation in Section 3.1 is extended from calculating backscattering cross sections to calculating amplitude scattering matrices. Third, a method is developed and implemented in the theory for efficiently calculating rotationally averaged scattering properties. Scattering properties of the ice crystals in the scattering database generated using the GMM method are computed using this updated method and the results compared. Radar observables are also estimated based on the same four dendritic class mass-dimensional relationships and particle size distributions introduced in Chapter 2 and the results are compared with those in Chapter 2.

### 3.2.1 Generalization of the modified Rayleigh-Gans theory

For a dipole with polarizability $\alpha$, its dipole moment $P$ when excited by the total electric field at its location $E_{\text{tot}}$ is given by $P = E_{\text{tot}}$ for non-magnetic (i.e. magnetic permeability of $\mu_0$) materials. The updated method retains the basic ideas of the modified Rayleigh-Gans theory in Section 3.1 but changes the polarizability of a tiny sphere, i.e. $\alpha_{\text{sphere}} = 4\pi\varepsilon_0 a^3(\varepsilon - 1)/(\varepsilon + 2)$ where $a$ is the radius of the sphere, to the polarizability of a dipole. Solid ice crystals can then be divided into small volumes and reasonable polarizabilities assigned to these small volumes. With this change the method can be extended to estimating the scattering properties of solid ice crystals.

In this new formulation Eq. 3.11 for $A(r)$ is written, with the aid of Eq. 3.13, as

$$A_m = \sum_{r_{m,n} < r_0} \frac{\alpha_n}{4\pi\varepsilon_0} \left[ \frac{k^2}{r_{m,n}} \begin{pmatrix} I_3 - \hat{r}_{m,n} \hat{r}_{m,n}^T \end{pmatrix} + \left( \frac{1}{r_{m,n}^3} - \frac{ik}{r_{m,n}^2} \right) \left( 3\hat{r}_{m,n} \hat{r}_{m,n}^T - I_3 \right) \right] e^{ikr_{m,n}},$$

(3.17)

where $I_3$ is the 3x3 identity matrix, $\alpha_n$ is the polarizability of the $n^{th}$ dipole, $r_{m,n}$ is the location vector from the $n^{th}$ dipole to the $m^{th}$ dipole, $\hat{r}_{m,n}$ and $r_{m,n}$ are the unit vector and length of $r_{m,n}$, $\varepsilon_0$ is the electric permittivity of vacuum, $k = 2\pi/\lambda$, ...
λ is the wavelength in vacuum, the superscript $\mathbf{T}$ indicates matrix transpose, and $r_0$ is the range of self-interactions. If the distance of the $m^{th}$ and $n^{th}$ dipoles exceeds $r_0$, the self-interaction is not considered for these two dipoles. Note that the time dependence $e^{i\omega t}$ is assumed for every field.

Estimating amplitude scattering matrices in all directions requires knowledge of dipole moments of all dipoles. Based on Eq. 3.15 and Eq. 3.17, and the fact that $\mathbf{P}_m = \mathbf{E}_m^{\text{tot}}$, where $\mathbf{E}_m^{\text{tot}}$ is the total electric field at the $m^{th}$ dipole, the dipole moment $\mathbf{P}_m$ of the $m^{th}$ dipole after self-interactions is given by

$$\mathbf{P}_m = \alpha_m (\mathbf{I}_3 - \mathbf{A}_m)^{-1} \mathbf{E}_m^{\text{inc}},$$

where $\alpha_m$ is the polarizability of the $m^{th}$ dipole, $\mathbf{E}_m^{\text{inc}}$ is the incident electric field at the $m^{th}$ dipole. Therefore, once the factor $\alpha_m (\mathbf{I}_3 - \mathbf{A}_m)^{-1}$ has been computed, the dipole moment of each dipole can be calculated using Eq. 3.18. The scattered field, from which the amplitude scattering matrix can be extracted, is then given by (see Draine and Flatau [12, Eq. 10])

$$\mathbf{E}^{\text{ sca}} = \frac{k^2 \exp(ikr)}{4\pi \varepsilon_0 r} \sum_{m=1}^{N} \exp(-ik\hat{r} \cdot \mathbf{r}_m)(\hat{r}\hat{r}^\mathbf{T} - \mathbf{I}_3)\mathbf{P}_m,$$

where the origin of the coordinate system is the center of the particle, $\mathbf{r}$ is the location vector of the observer, $\hat{r}$ and $r$ are the unit vector and length of $\mathbf{r}$, respectively, and $\mathbf{r}_m$ is the location vector of the $m^{th}$ dipole.

To efficiently perform rotational averages for different incident angles, one must have an efficient method for calculating the factor $\alpha_m (\mathbf{I}_3 - \mathbf{A}_m)^{-1}$ for different incident directions. Changing incident directions is equivalent to rotating an ice crystal. Rotation of ice crystals is therefore used here to represent changes in the incident direction. Let $\mathbf{M}$ denote the rotation matrix such that a location vector $\mathbf{r} = (x, y, z)$ changes to $\mathbf{M} \cdot \mathbf{r}$ after the rotation. Under this rotational transformation,

$$\alpha_m \rightarrow \mathbf{M} \alpha_m \mathbf{M}^\mathbf{T},$$

$$(\mathbf{I}_3 - \hat{r}_{m,n} \hat{r}_{m,n}^\mathbf{T}) \rightarrow (\mathbf{I}_3 - \mathbf{M} \hat{r}_{m,n} \hat{r}_{m,n}^\mathbf{T} \mathbf{M}^\mathbf{T}) = \mathbf{M} (\mathbf{I}_3 - \hat{r}_{m,n} \hat{r}_{m,n}^\mathbf{T}) \mathbf{M}^\mathbf{T},$$

$$(3\hat{r}_{m,n} \hat{r}_{m,n}^\mathbf{T} - \mathbf{I}_3) \rightarrow \mathbf{M} (3\hat{r}_{m,n} \hat{r}_{m,n}^\mathbf{T} - \mathbf{I}_3) \mathbf{M}^\mathbf{T},$$
where the right arrows indicate that the former changes to the latter after rotation. Note that the distance \( r_{m,n} \) does not change after rotation. Using the property \( M^T = M^{-1} \) of the rotation matrix yields

\[
\alpha_m (I_3 - A_m)^{-1} \rightarrow M \alpha_m (I_3 - A_m)^{-1} M^T.
\]

From Eq. 3.20, the factor \( \alpha_m (I_3 - A_m)^{-1} \) after the rotation of an ice crystal can be efficiently calculated from \( M \), meaning that \( \alpha_m (I_3 - A_m)^{-1} \) only needs to be calculated once. This one calculation has a time requirement roughly on the order of \( N^2 \), i.e. \( O(N^2) \), whereas for other particle orientations the time requirement is only \( O(N) \) with use of Eq. 3.20, which is efficient.

### 3.2.2 Computational considerations

As stated in Section 3.1, the term in Eq. 3.17 with \( 1/r_{m,n}^3 \) dominates when \( r_{m,n} \) is much smaller than the wavelength. As this term decreases rapidly with increasing \( r_{m,n} \), only dipoles close by lead to large magnitudes for this term. Furthermore, the basic assumption underlying Eq. 3.13 is that the electric fields at a dipole and its neighbors are almost the same. This requires limiting (self) interactions to neighboring dipoles at close range. To this end a threshold distance \( r_0 \) is set on the range of interactions between dipoles for both efficiency and accuracy reasons. In practice, the threshold is about 10 times the local minimum dimension (e.g. the thickness of a dendrite).

When \( V \) is a sphere of material centered on the \( m^{th} \) dipole, \( \oint V (3r_{m,n} \hat{r}_{m,n} - I_3) = 0 \). The short range interactions almost cancel, leaving the longer range interactions as the important ones. For this case the assumption that the electric fields at a dipole and its neighbors are almost the same becomes poorer at larger ranges, especially if \( r_0 \) is comparable to the wavelength. Because of this limitation, the modified Rayleigh-Gans theory presented in Section 3.1 works best for ice crystals with large local aspect ratios and with local minimum dimensions much smaller than the wavelength, such as ice crystal dendrites, plates, and thin columns at millimeter wavelengths.

This approach does not require that the dipoles be located in a periodic lattice, as is required in DDA implementations using the fast Fourier transform. The dipoles can be arbitrarily located, making it simple to model ice crystals with sparse structures. For \( N \) randomly located dipoles, the memory requirements to store \( r_m, A_m, \) and
\(\alpha_m(I_3 - A_m)^{-1}\) are all of order \(N\), i.e. \(O(N)\). To calculate the particle scattering properties at one orientation from scratch, the time requirement to calculate \(A_m\) is smaller than \(O(N^2)\) because of the threshold \(r_0\). The time requirement to calculate \(P_m\) is \(O(N)\). After that, the time requirement to calculate the scattering properties of the ice crystals for other orientations is only \(O(N)\), which is efficient.

### 3.2.3 Comparison between the modified Rayleigh-Gans theory and the GMM method

To test the accuracy of this modified Rayleigh-Gans theory it is used to compute \(\sigma_{hh}\), \(Z_{dr}\), \(Im\{S_2(0^\circ) - S_1(0^\circ)\}\) and \(Re\{S_1(0^\circ)\}\) for dendrites, plates and columns in the database and the results are compared with those from the GMM method. Based on the \(\sigma_{hh}\) comparisons between the two methods shown in Section 3.1, most of the large differences occur when the form factors from the Rayleigh-Gans theory are less than 0.05. Thus only the cases with form factors larger than 0.05 (over 90% of all the cases) are compared for \(\sigma_{hh}\) and \(\sigma_{hh}/\sigma_{vv}\). Figure 3.14 through Figure 3.16 show the comparisons between the modified Rayleigh-Gans theory and the GMM method for \(hh\)-polarization backscattering cross sections (\(\sigma_{hh}\)), backscattering cross section ratios (\(\sigma_{hh}/\sigma_{vv}\)), \(Im\{S_2(0^\circ) - S_1(0^\circ)\}\) , which relates to specific differential phase, and \(Re\{S_1(0^\circ)\}\) , which relates to \(hh\)-polarization specific attenuation. For these comparisons averaging over rotation angle is performed. When the form factor is greater than about 0.05, the differences between the backward scattering parameters \(\sigma_{hh}\) and \(\sigma_{hh}/\sigma_{vv}\) from the two methods are mostly less than 1 dB. The differences in \(Im\{S_2(0^\circ) - S_1(0^\circ)\}\) are generally less than 15%, whereas the differences in \(Re\{S_1(0^\circ)\}\) are large. The large differences in \(Re\{S_1(0^\circ)\}\) are not surprising given that the specific attenuation of ice crystals is small and not seemingly useful in retrievals with todays technology (see Section 2.5 in Chapter 2). As a result, \(Re\{S_1(0^\circ)\}\) is not considered further.
Figure 3.14. Differences between a) $hh$-polarization backscattering cross sections ($\sigma_{hh}$), b) backscattering cross section ratios ($\sigma_{hh}/\sigma_{vv}$), c) $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, and d) $\text{Re}\{S_1(0^\circ)\}$ estimated using the modified Rayleigh-Gans theory and the GMM method for single dendritic ice crystals. Note that the scattering angle ($0^\circ$) is dropped in the figure labels for amplitude scattering matrix elements. The $\sigma_{hh}$, $\sigma_{hh}/\sigma_{vv}$, and $\text{Re}\{S_1(0^\circ)\}$ results contain all incident angles, all rotation angles and all four wavelengths. Rotational averaging is performed. The $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$ results contain all rotation angles and all four wavelengths but incident angles range from $0^\circ$ (side incidence) to 70° since KDP for 80° and 90° incidence angles give little information. The results for $\sigma_{hh}$ and $\sigma_{hh}/\sigma_{vv}$ when the form factor is small (< 0.05) are not shown.

3.2.4 Scattering properties averaged over orientation and particle size distribution

The comparisons above are for single ice crystal scattering properties for each incident angle, rotation angle, and radar wavelength. When estimating radar observables, averaging over ice crystals with different sizes, shapes and orientations is necessary. As illustrated previously, the $hh$- and $vv$-backscattering cross sections and their ratios have the largest percentage errors when their form factors are small. But when the form factors are small, the backscattering cross sections themselves are small. Thus these large relative differences between the two methods only lead to relatively small absolute differences. As a result, the large relative error results have little impact.
Figure 3.15. Same as Figure 3.14, but for plates.

Figure 3.16. Same as Figure 3.14, but for columns.
on the averaged results. Other seemingly random differences between the modified Rayleigh-Gans theory and GMM method are also reduced after averaging. Thus the differences in radar observables between the two methods are smaller for distributions of ice crystals, thereby extending the range of validity of the modified Rayleigh-Gans theory in retrievals and forward simulations.

Figure 3.17 through Figure 3.19 compare radar observables computed with the modified Rayleigh-Gans theory and the GMM method after incorporating the four dendritic class mass-dimensional relationships and particle size distributions used in Chapter 2. Note that all dendrites are included. The differences are generally less than 0.25 dB for $Z_{hh}$, 0.40 dB for $Z_{dr}$, and 6% for $K_{DP}$. All three radar observables estimated using the modified Rayleigh-Gans theory are biased low. By accounting for this bias, the differences between these two methods are further reduced.
Figure 3.18. Same as Figure 3.17, but for radar differential reflectivity ($Z_{dr}$) at side incidence.

Figure 3.19. Same as Figure 3.17, but for specific differential phase ($K_{DP}$) at side incidence.
3.3 Particles large compared to the wavelength

The modified Rayleigh-Gans theory works best for particles with a minimum dimension small compared with the wavelength. This includes ice crystals small compared to the wavelength and ice crystals with large aspect ratios whose small dimension is small compared to the wavelength. For ice crystals large compared to the wavelength, including thick ones, the modified Rayleigh-Gans theory leads to larger errors. When ice crystals are large compared to the wavelength for all dimensions, longer range interactions become most significant and the phase differences between different parts of an ice crystal should be taken into account when considering self interactions. Longer range interactions are not well captured in the modified Rayleigh-Gans theory because in this theory the interactions are estimated assuming that the electric fields at neighboring parts of an ice crystal are the same as the electric field at the part in consideration.

To show the differences in the structures of the internal electric field strengths between particles small and large compared to the wavelength, the internal electric field strengths within two cylinders, one large and the other small compared to the wavelength, are calculated using the DDA method (ADDA code by Yurkin and Hoekstra [13]). The large cylinder has a base diameter of 1 mm and a length of 10 mm. The small cylinder has a base diameter of 0.01 mm and a length of 0.1 mm. Both cylinders have their symmetry axis along the z-axis. The incident radiation propagates in the positive z-axis direction with its electric field polarization direction along the y-axis. The wavelength of the incident wave is 3.19 mm.

Figure 3.20 shows the internal electric field strength within the small cylinder. The total internal electric field strength has mostly a y-component that corresponds to the polarization direction of the incident radiation. The x- and z-components of the internal electric field strength are much smaller. Because the largest dimension of the cylinder is perpendicular to the polarization direction, the internal electric field strength is smaller compared to the incident electric field strength, which agrees with the results shown in 3.20.

Figure 3.21 shows that the internal electric field strength within the large cylinder is clearly different from that in Figure 3.20. The x-component of the internal electric field remains small. However, the y-component of the internal electric field
Figure 3.20. Internal electric field strength obtained using the DDA method for the cylinder small compared to the wavelength. The cylinder has a base diameter of 0.01 mm and a length of 0.1 mm. The propagation direction of the incident radiation is along the positive z-axis; the polarization direction is along the y-axis; and the wavelength is 3.19 mm. The colors show the internal electric field strength normalized with respect to the incident electric field strength. $|E|$ shows the total normalized internal electric field strength. $|E_x|$, $|E_y|$, and $|E_z|$ show the x-, y-, and z- components of the normalized internal electric field strength, respectively.

shows a standing-wave-like structure with increasing magnitude along the propagation direction; the standing wave is almost axis-symmetric with respect to the symmetry axis of the cylinder. The z-component of the internal electric field also exhibits a standing-wave-like structure with an increase in magnitude along the propagation direction. Unlike the y-component, the z-component is not axis-symmetric about the symmetry axis, but rather is mirror-symmetric about the yz-plane that passes through the symmetry axis and increases in magnitude with increasing distance to the plane. These standing-wave-like structures are also reported by Muinonen et al. [46]. These structures are not modeled by the near field interactions in the
modified Rayleigh-Gans theory. As a result, a new approximate method that extends the modified Rayleigh-Gans theory to these standing-wave-like structures is needed in order to model them and the scattering properties of large (compared to the wavelength) ice crystals.

3.4 Relationship between the modified Rayleigh-Gans theory and the scattering-order formula

The scattering-order formula and its associated infinite series introduced by Singham and Bohren [30] explicitly accounts for interactions between different parts of a particle. The infinite series in Eq. 3.15 can be interpreted as an estimate of the
scattering orders in Singham and Bohren [30]. The estimate represented by Eq. 3.15 works for many types of ice crystals because they are often sparse and have at least one dimension smaller than the wavelength. As a result, the convergence problem discussed in Singham and Bohren [30] is unlikely to occur in the present applications for which near field interactions dominate.

The problems illustrated in Section 3.3 for large, compact ice crystals, i.e. those ice crystals for which the modified Rayleigh-Gans theory does not work, are likely the result of convergence issues discussed in Singham and Bohren [30]. For these large ice crystals long range interactions are significant, perhaps providing the physical basis for the failure of the infinite series in the scattering orders in Singham and Bohren [30] to converge for them. A better understanding of the spatial distribution of the internal electric field within large, compact ice crystals is necessary for extending a modified Rayleigh-Gans like theory to them.
Dielectric constant adjustments in computations of the scattering properties of solid ice crystals using the Generalized Multi-particle Mie (GMM) theory

Throughout this work ice crystals are modeled as clusters of closely packed tiny ice spheres resembling the shape of the original ice crystals. The air gaps between the tiny ice spheres make the effective dielectric constant of the mixture of ice spheres and air smaller than that of the original solid ice crystal, leading to differences in scattering properties between the clusters and the original ice crystals. These differences are reduced by artificially increasing the dielectric constant of the tiny ice spheres such that the tiny spheres/air gaps mixture is electromagnetically equivalent to solid ice crystals. This method is useful in improving the cluster representations of solid ice crystals used in Chapter 2 and Chapter 3 when computing their scattering properties.

The rest of this chapter is a paper submitted to Journal of Quantitative Spectroscopy and Radiative Transfer (2013) with some notational modifications to stay consistent with other chapters in this thesis. In keeping with the Pennsylvania State University policies (The Graduate School Thesis Guide, p.15, previously...
published work and first author requirements sections) Yinghui Lu initiated and performed the work published in this paper under the supervision of his co-authors. Copyright policy of the Journal of Quantitative Spectroscopy and Radiative Transfer (http://www.elsevier.com/journal-authors/author-rights-and-responsibilities) permits use of this paper in Yinghui Lus doctoral thesis.

4.1 Introduction

Modeling scattering properties of ice crystals at radar wavelengths is important for radar retrievals but is difficult, especially at shorter wavelengths, because of their complicated morphologies. Botta et al. [26, 47], Petty and Huang [14] and Tyynel et al. [27] show that modeling ice aggregates using spheres and spheroids with an effective dielectric constant and then using the Mie scattering formulation and T-matrix method to compute their scattering properties leads to large errors. Applying accurate numerical methods, such as the Finite Difference Time Domain method (FDTD, Yee [16], Taflove [17]) and Discrete Dipole Approximation (DDA, Draine and Flatau [12], Yurkin and Hoekstra [48]), to ice crystals with realistic morphologies is a more reasonable approach.

Botta et al. [2] modeled the complicated morphologies of dendritic ice crystals with clusters of tiny ice spheres and then applied the Generalized Multi-particle Mie method (GMM, Xu [19], Xu and Gustafson [20]), an accurate method for calculating the scattering properties of clusters of spheres, to explore the variability of dendritic ice crystal scattering properties at millimeter wavelengths. However, in this approach the spheres occupy only 74% of the original ice crystal volumes, with gaps between the spheres accounting for the remaining 26%. Botta et al. [2] chose to match the total mass of each cluster of tiny spheres that represents a dendrite with the mass of the original dendrite by making the cluster a little bit thicker. For hexagonal plates Botta et al. [2] compared this approach with the results of Aydin and Walsh [18] using the FDTD method and noted that at vertical incidence the backscattering cross sections matched to within 2 dB. Though such an approach may be acceptable for calculating backscattering cross sections, the impact of making the cluster thicker on polarimetric parameters, such as the ratio of backscattering cross sections at horizontal ($h$) and vertical ($v$) polarizations and propagation differential phase shift between $h$- and
Another way to compensate for gaps in the cluster representation of a solid ice crystal is by artificially increasing the density or dielectric constant of the tiny spheres that compose a cluster so that the mixture of tiny spheres and gaps is electromagnetically equivalent to the original solid ice crystal. To match the density of the original solid ice crystal, the density ($\rho_{\text{spheres}}$) of the material in the tiny spheres must be $1/f$ times the ice density ($\rho_{\text{ice}}$) of the original crystal, where $f$ is the total volume fraction of all the tiny spheres. Estimating the appropriate dielectric constant ($\varepsilon_{\text{spheres}}$) of the tiny spheres is not as straightforward. Consider the cluster of tiny spheres with gaps as a mixture of some (unknown) material with dielectric constant $\varepsilon_{\text{spheres}}$ and air for which the average dielectric constant is that of solid ice ($\varepsilon_{\text{ice}}$). The Maxwell Garnett (MG) function and the Bruggemans effective-medium (EM) function are two widely used methods for estimating average dielectric functions (Bohren and Battan [37]). The Maxwell Garnett function deals with spherical inclusions in a background material (see Bohren and Huffman [37, Chapter 8]) while for Bruggemans effective-medium function the two components are interchangeable. These two approaches are used to estimate $\varepsilon_{\text{spheres}}$ and the resulting scattering properties from GMM when using $\varepsilon_{\text{spheres}}$ to represent the tiny spheres are compared with scattering properties obtained from other methods applied to the original solid ice crystals.

### 4.2 Methods

For the clusters of tiny spheres in the GMM approach it is natural to use the Maxwell Garnett function with the tiny spheres in a cluster as inclusions and air in the gaps as the background material. Solving for $\varepsilon_{\text{spheres}}$ in the Maxwell Garnett (MG) function (see Bohren and Battan [49, Eq. 1]) yields

$$\varepsilon_{\text{spheres}} = \varepsilon_{\text{air}} \left[ 1 + \frac{3}{f \varepsilon_{\text{ice}} + 2 \varepsilon_{\text{air}} - \varepsilon_{\text{air}} - 1} \right], \quad (4.1)$$

where $\varepsilon_{\text{air}}$ is the dielectric constant of air and $f$ is the volume fraction of the tiny spheres.

Treating air in the gaps as the inclusions and the tiny spheres as the
Table 4.1. Summary of different ways that ice crystal scattering properties are computed. Method VI uses clusters of tiny spheres with the same overall shape as the solid ice crystals with ice as the sphere dielectric constant. Because the volume fraction $f$ of the cluster of tiny spheres is smaller than 1 for Method VI and no correction to the dielectric constant is applied, the calculated backscattering cross sections based on Method VI are multiplied by $1/f^2$ and the amplitude scattering matrix elements by $1/f$ to compensate for the mass difference.

<table>
<thead>
<tr>
<th>Method</th>
<th>Ice crystal model, computation technique</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>solid ice crystal, T-matrix</td>
<td>$\varepsilon_{\text{ice}}$</td>
</tr>
<tr>
<td>II</td>
<td>solid ice crystal, DDA</td>
<td>$\varepsilon_{\text{ice}}$</td>
</tr>
<tr>
<td>III</td>
<td>cluster of spheres, GMM</td>
<td>Eq. 4.1</td>
</tr>
<tr>
<td>IV</td>
<td>cluster of spheres, GMM</td>
<td>Eq. 4.2</td>
</tr>
<tr>
<td>V</td>
<td>cluster of spheres, GMM</td>
<td>Eq. 4.3</td>
</tr>
<tr>
<td>VI</td>
<td>cluster of spheres, GMM (mass adjusted)</td>
<td>$\varepsilon_{\text{ice}}$</td>
</tr>
</tbody>
</table>

Background material in the Maxwell Garnett function leads to

\[
\begin{align*}
\varepsilon_{\text{spheres}} = & \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \\
a = 2f - 2, \\
b = f\varepsilon_{\text{ice}} - 2f\varepsilon_{\text{air}} + 2\varepsilon_{\text{ice}} - \varepsilon_{\text{air}}, \\
c = (1 - f)\varepsilon_{\text{ice}}\varepsilon_{\text{air}}.
\end{align*}
\]

Note that there is one solution larger than $\varepsilon_{\text{ice}}$ because $\varepsilon_{\text{air}} < \varepsilon_{\text{ice}}$. Solving Bruggemans effective-medium function (Bohren and Battan [49, Eq. 2]) for $\varepsilon_{\text{spheres}}$ leads to

\[
\varepsilon_{\text{spheres}} = \frac{(3f - 2)\varepsilon_{\text{air}} + 2\varepsilon_{\text{ice}}}{\varepsilon_{\text{air}} + (3f - 1)\varepsilon_{\text{ice}}}. 
\]

Using these estimates for $\varepsilon_{\text{spheres}}$ the scattering properties of ice crystals are computed in a number of different ways (Table 4.1).

The incoming radiation is taken to propagate in a horizontal plane with the $h$-polarization direction lying inside the horizontal plane and perpendicular to the propagation direction; the $v$-polarization direction is vertical to the horizontal plane. The scattering properties of interest are the backscattering cross section $\sigma_{hh}$ (illuminating with $h$-polarized waves and measuring $h$-polarized returns), the backscattering cross section ratio ($\sigma_{hh}/\sigma_{vv}$), and forward scattering parameters $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$ and $\text{Re}\{S_1(0^\circ)\}$, which are related to the specific differential phase (between $h$- and $v$-polarized waves) $K_{\text{DP}}$ and the specific attenuation at $h$-polarization.
\( A_h \), respectively.

Based on Aydin [40],

\[
K_{DP} = 10^3 \left( \frac{180}{\pi} \right) \left( \frac{2\pi}{k} \right) n_0 \text{Im}\{ < S_2 - S_1 > \} \text{ (deg km}^{-1} \text{ )} \quad (4.4)
\]

and

\[
A_h = 8686 \left( \frac{2\pi}{k} \right) n_0 \text{Re}\{ < S_1 > \} \text{ (dB km}^{-1} \text{ )}, \quad (4.5)
\]

where \( k = 2\pi/\lambda \), \( \lambda \) is the wavelength (in meters) in free space, \( S_1 \) and \( S_2 \) are amplitude scattering matrix elements in the forward direction (in meters), \( \text{Re}\{\} \) and \( \text{Im}\{\} \) are the real and imaginary parts of a complex number, the brackets \( <> \) indicate an ensemble average, and \( n_0 \) is the total number of particles per cubic meter. In the current work only the scattering properties of single ice crystals are evaluated, so we focus on single ice crystal scattering properties \( \sigma_{hh}, \sigma_{hh}/\sigma_{vv}, \text{Im}\{S_2-S_1\} \text{ and Re}\{S_1\} \).

### 4.3 Results

To test the applicability of the six methods in Table 4.1, scattering properties are calculated at an X-band wavelength (31.86 mm) for two sets of spheroids with their axes of symmetry perpendicular to the horizontal plane. The equal-volume sphere diameters of the first and second set of spheroids are 0.22 mm and 11.06 mm, respectively, representing ice crystals much smaller and comparable to the wavelength. The aspect ratios \( c/a \) (\( c \) and \( a \) are the semi axes along and perpendicular to the axis of symmetry, respectively) are \( 1/n \) (\( n=10, 9, \cdots, 2 \)) and \( n \) (\( n=1, 2, \cdots, 10 \)). To illustrate how the scattering properties change with aspect ratio, the symmetry axes of both prolate (aspect ratio greater than 1) and oblate (aspect ratio less than 1) spheroids are vertical and not horizontal as is usually the case when representing falling columnar ice crystals.

Figure 4.1 and Figure 4.2 show the scattering properties of the two sets of spheroids calculated using the methods listed in Table 4.1. The differences in \( \sigma_{hh} \) and \( \sigma_{hh}/\sigma_{vv} \) of DDA and GMM\(_{III}\) (subscript indicates the method in Table 4.1) compared with the T-matrix method are quite small, generally less than 0.3 dB. The largest difference in GMM\(_{III}\) occurs at an aspect ratio of 0.11 for the large spheroid results illustrated...
Figure 4.1. Scattering properties of spheroids with an equal-volume sphere diameter of 0.22 mm versus aspect ratio: (a) $\sigma_{hh}$, (b) $\sigma_{hh}/\sigma_{vv}$, (c) $\text{Im}\{S_2(0^\circ) - S_1(0^\circ)\}$, and (d) $\text{Re}\{S_1(0^\circ)\}$ at an X-band wavelength (31.86 mm). The solid black lines show the T-matrix results. DDA and GMM$_{III}$ (spheres calculated from Eq. 4.1 representing the mixture of 74% tiny sphere inclusions and 26% air background material mixture in Maxwell Garnett function) results fall on top of the T-matrix results and thus are not shown. DDA differences relative to the T-matrix results (gray dotted lines) are defined as (DDA-Tmatrix) in (a) and (b), whereas in (c) and (d) the differences are defined as (DDA-Tmatrix)/|Tmatrix|. GMM$_{III}$ differences relative to the T-matrix method are shown in black dotted lines. GMM$_{IV}$ ($\varepsilon_{\text{spheres}}$ calculated from Eq. 4.2 representing the 26% air inclusions and 74% sphere background material mixture in the Maxwell Garnett function, squares), GMM$_{V}$ (spheres calculated from Eq. 4.3 representing 74% ice volume fraction in Bruggemans effective-medium function, diamonds), and GMM$_{VI}$ ($\varepsilon_{\text{ice}}$ with a mass compensation obtained by multiplying by 1/0.742, circles; see caption of Table 4.1) results are also shown in (a). The lower left oblate spheroid (c/a < 1) and the upper right prolate spheroid (c/a > 1) in (a) indicate the orientation of the spheroids and the polarization directions.

in Figure 4.2, where $\sigma_{hh}/\sigma_{vv}$ exceeds 15 dB and the difference reaches 2.5 dB. The reason for this big difference is that in resonance regions the scattering properties are very sensitive to crystal shape (see Lu et al. [50, Figure 12]). When the aspect ratio of an ice crystal becomes too extreme, the cluster of tiny spheres representing it is unable to sufficiently model crystal shape, leading to large errors in $\sigma_{hh}/\sigma_{vv}$ in resonance regions. The results for GMM$_{IV}$, GMM$_{V}$, and GMM$_{VI}$ are also shown.
for small spheroids in Figure 4.1a. They show large differences compared to the T-Matrix method results, implying that only the Maxwell Garnett function with the tiny spheres as the inclusions (with dielectric constants obtained using Eq. 4.1) and air as background material leads to reasonable results. As a consequence, methods IV, V, and VI are not considered further. The percentage differences for $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$ and $\text{Re}\{S_1(0^\circ)\}$ of DDA and GMM_{III} compared with the T-matrix method are less than 6\% (Figure 4.1c and 4.1d).

To test GMM_{III} for a broader range of sizes and in resonance regions, the scattering properties of spheroids with aspect ratios of 0.6 and 3 versus equal-volume sphere diameters ranging from 0.2 mm to 10 mm at a Ka-band wavelength (8.40 mm) are shown in Figure 4.3 and Figure 4.4. DDA and GMM_{III} results still agree with T-matrix results reasonably well to within 1.2 dB for $\sigma_{hh}$ and $\sigma_{hh}/\sigma_{vv}$, where the extreme differences occur near resonance minima in $\sigma_{hh}$, approximately 20\% for $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$ and 5\% for $\text{Re}\{S_1(0^\circ)\}$. The large percentage differences (DDA-Tmatrix)/|Tmatrix| and (GMM_{III}-Tmatrix)/|Tmatrix| in Figure 4.3c and Figure 4.4c are where the T-matrix results are close to zero. The actual differences
Figure 4.3. Scattering properties for oblate spheroids with an aspect ratio of 0.6 versus equal-volume sphere diameter: (a) $\sigma_{hh}$, (b) $\sigma_{hh}/\sigma_{vv}$, (c) $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, and (d) $\text{Re}\{S_1(0^\circ)\}$ at a Ka-band wavelength (8.40 mm). The T-matrix results are illustrated by solid black lines. The DDA and GMM results fall on top of the T-matrix results and are not shown; instead, differences of DDA (gray dotted lines) and GMM (black dotted lines) from the T-matrix results are shown. In (c) the differences exceed 100%, but those large errors are because of dividing by very small T-matrix results near resonances. The dot-dash line in (c) indicates $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}=0$ and the boxes and double arrows in (c) show where T-matrix results are very close to zero and the percentage differences are large. The percentage differences are limited to a range from -20% to +20% to better illustrate the differences in regions where the T-matrix results are not close to zero. The lower left oblate spheroid in (a) indicates the orientation of the oblate spheroids relative to the polarization directions.

are also close to zero so that they will not influence any quantities based on them. Thus the y-axes for the percentage errors in Figure 4.3c and Figure 4.4c are limited to -20% to +20% in order to show meaningful percentage errors where the T-matrix values are not close to zero. Figures 4.1 and 4.4 show that the results from DDA are generally closer than those of GMM to the T-matrix results, more so near resonance regions. Hence, it is worthwhile comparing GMM with DDA in modeling hexagonal plates and columnar crystals.

Figure 4.5 shows comparisons of GMM and DDA results for hexagonal plates...
and columnar crystals with a circular cross section. For these results the plates have 16 different maximum dimensions ranging from 0.10 mm to 3.20 mm; they are equally partitioned in logarithmic space with the thickness-size relationship for the P1a crystal type in Pruppacher and Klett [1, p.51-52]. The plates lie in the horizontal plane with their symmetry axes perpendicular to it; the maximum dimension of the plates is parallel to the direction of radiation propagation. The columns have 15 different lengths ranging from 0.18 mm to 4.17 mm, again equally partitioned in logarithmic space but with a thickness-size relationship for the N1e crystal type in Pruppacher and Klett [1, p.51-52]. The columns also lie in the horizontal plane with their symmetry axis perpendicular to the direction of propagation of the radiation.

For hexagonal plates, the volume fraction of a cluster of three-layered spheres is 69%; for columns modeled with a single-string of touching spheres, it is 67%. Figure 4.5 shows that results for plates from GMM_III agree to within 0.6 dB for backscattering cross section ($\sigma_{hh}$) and 1 dB for $\sigma_{hh}/\sigma_{vv}$, less than 11% for $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$ and 12% for $\text{Re}\{S_1(0^\circ)\}$ when compared against DDA results at a W-band wavelength (3.19 mm). The differences are even smaller for the columnar crystals: less than 0.3 dB for $\sigma_{hh}$ and $\sigma_{hh}/\sigma_{vv}$, and less than 8% for $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$ and 3% for....
Figure 4.5. Scattering properties of hexagonal plates (black lines) and columns with circular cross sections (gray lines) versus maximum dimension: (a) $\sigma_{hh}$, (b) $\sigma_{hh}/\sigma_{vv}$, (c) $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, and (d) $\text{Re}\{S_1(0^\circ)\}$ at a W-band wavelength (3.19 mm). DDA results are shown by the solid lines; GMM$^\text{III}$ differences with respect to DDA are shown by the dotted lines. The differences in (a) and (b) are defined as (GMM-DDA), whereas in (c) and (d) they are defined as (GMM-DDA)/|DDA|.

In Figure 4.6 a comparison was made of the scattering parameters for hexagonal plates using the GMM$^\text{III}$ method and the GMM method of Botta et al. [2] in which the plates are modeled slightly thicker to provide the same ice mass for a given maximum dimension (similar to the GMM$^\text{VI}$ method in Table 4.1). Compared with the DDA method the GMM$^\text{III}$ method shows improvements over the GMM$^\text{VI}$ method for all four parameters discussed above. The differences in $\sigma_{hh}$ over the size range in Figure 4.5 are 0.6 - 2.2 dB with GMM$^\text{VI}$ and 0.3 - 0.6 dB with GMM$^\text{III}$. Similarly, the differences in $\sigma_{hh}/\sigma_{vv}$ decrease from a range of 1.1 - 1.9 dB down to 0.6 - 1 dB; for $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$ the differences reduce from 23 - 27% to 8 - 11%; and the $\text{Re}\{S_1(0^\circ)\}$ differences reduce from 21 - 39% to 7 - 12%.
Figure 4.6. Scattering properties of hexagonal plates versus maximum dimension: (a) $\sigma_{hh}$, (b) $\sigma_{hh}/\sigma_{vv}$, (c) $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, and (d) $\text{Re}\{S_1(0^\circ)\}$ at a W-band wavelength (3.19 mm) side incidence. The DDA results are shown in solid lines; the GMM method of Botta et al. [2], in which the plates are modeled slightly thicker to provide the same ice mass for a given maximum dimension, differences with respect to DDA are shown in dotted lines.

4.4 Summary and Conclusions

Solid ice crystals are modeled using tiny non-overlapping spheres with air gaps between them in scattering computations with the GMM method. The dielectric constants for the tiny spheres are obtained by setting the effective-medium dielectric constant inferred from the Maxwell Garnett function (with the tiny spheres as inclusions and air as the background) to be equal to that of solid ice. This approach compensates for the mass difference between a solid ice crystal and its GMM representation while maintaining its shape, including the aspect ratio. The differences between GMM using this method and the T-matrix method for small ice spheroids are about 0.3 dB in backscattering cross section ($\sigma_{hh}$) and $\sigma_{hh}/\sigma_{vv}$ and less than 6% in $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$ (relating to specific differential phase) and $\text{Re}\{S_1(0^\circ)\}$ (relating to specific attenuation). For large ice spheroids, the differences are mostly within 1.2 dB (and often better) in $\sigma_{hh}$ and $\sigma_{hh}/\sigma_{vv}$ (in one instance where
this ratio exceeds 15 dB the difference is about 2.5 dB), 20% in $Im\{S_2(0^\circ)-S_1(0^\circ)\}$ except for cases with $Im\{S_2(0^\circ)-S_1(0^\circ)\}$ very close to zero and 6% in $Re\{S_1(0^\circ)\}$. The magnitudes of the differences are sensitive to how well the shapes of solid ice crystals are modeled in resonance regions. The DDA results for the same spheroids are generally closer than those of GMM to the T-matrix results. For hexagonal plates the differences between the GMM and DDA methods are mostly within 0.6 dB for $\sigma_{hh}$ and 1 dB for $\sigma_{hh}/\sigma_{vv}$ (with the largest differences in both parameters occurring near resonance minima in $\sigma_{hh}$), 11% for $Im\{S_2(0^\circ)-S_1(0^\circ)\}$ and 12% for $Re\{S_1(0^\circ)\}$. For columns the differences are within 0.3 dB for $\sigma_{hh}$ and $\sigma_{hh}/\sigma_{vv}$, 8% for $Im\{S_2(0^\circ)-S_1(0^\circ)\}$ and 4% for $Re\{S_1(0^\circ)\}$. Overall, the proposed approach for computing the scattering parameters of solid ice crystals with the GMM method is viable away from resonances and produces more accurate results for the scattering parameters compared to adjusting the shape of the crystal to compensate for the difference in mass.
Aggregates have more complicated morphologies than dendrites, plates and columns. The next step in development of the database is to incorporate the scattering properties of aggregates to go along with those of dendrites, plates and columns.

The scattering properties of ice crystals large (in all directions) compared to the wavelength cannot be calculated accurately by the modified Rayleigh-Gans theory introduced in Chapter 3. Continuing the work introduced in Section 3.3, characteristics of the internal electric field strengths within ice crystals large compared to the wavelength will be studied. The goal is either to extend the modified Rayleigh-Gans theory to model accurately the scattering properties of large ice crystals or to develop an alternative method for doing so. If successful, such a theory should provide insights into the physical properties of ice crystals that drive their scattering characteristics.

The modified Rayleigh-Gans theory links ice crystal morphologies to their scattering properties. For ice crystal plates and columns, of which the morphologies are well defined, analytical approximate relations between their physical (e.g. maximum dimensions and thicknesses) and scattering properties may exist. For dendrites, functions may exist that describe the links between the mass distributions within them and their scattering properties. If these relationships exist, they can be applied to forward modeling of radar observables of ice crystals produced by ice microphysical models. These relationships will be studied further.

One of the most important cloud microphysical quantities is ice water content. Results from this research indicate that retrievals of ice water contents based solely
on radar backscattering cross sections come with large errors. Implementation of other radar observables, such as $K_{DP}$, for more accurate retrievals of ice water content will be investigated.
Appendix A

Scattering properties versus crystal maximum dimension
Figure A.1. \( hh \)-polarization backscattering cross sections (\( \sigma_{hh} \)) of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of 0°, 30°, 60°, and 90° versus maximum dimension.
Figure A.2. $hh$-polarization backscattering cross sections ($\sigma_{hh}$) of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$ versus maximum dimension.
Figure A.3. $hh$-polarization backscattering cross sections ($\sigma_{hh}$) of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of 0°, 30°, 60°, and 90° versus maximum dimension.
Figure A.4. $vv$-polarization backscattering cross sections ($\sigma_{vv}$) of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$ versus maximum dimension.
Figure A.5. $vv$-polarization backscattering cross sections ($\sigma_{vv}$) of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$ versus maximum dimension.
Figure A.6. vv-polarization backscattering cross sections ($\sigma_{vv}$) of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$ versus maximum dimension.
Figure A.7. Backscattering cross section ratios ($\sigma_{hh}/\sigma_{vv}$) of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus maximum dimension.
Figure A.8. Backscattering cross section ratios ($\sigma_{hh}/\sigma_{vv}$) of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus maximum dimension.
Figure A.9. Backscattering cross section ratios ($\sigma_{hh}/\sigma_{vv}$) of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus maximum dimension.
Figure A.10. $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, which is related to specific differential phase ($K_{\text{DP}}$), of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus maximum dimension.
Figure A.11. $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, which is related to specific differential phase ($\text{KDP}$), of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus maximum dimension.
Figure A.12. $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, which is related to specific differential phase ($K_{DP}$), of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus maximum dimension.
Figure A.13. \(Re\{S_1(0^\circ)\}\), which is related to attenuation at \(h\)-polarization (\(A_h\)), of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of 0°, 30°, 60° and 90° versus maximum dimension.
Figure A.14. $\text{Re}\{S_1(0^\circ)\}$, which is related to attenuation at $h$-polarization ($A_h$), of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus maximum dimension.
Figure A.15. $Re\{S_1(0^\circ)\}$, which is related to attenuation at $h$-polarization ($A_h$), of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus maximum dimension.
Figure A.16. $Re\{S_2(0^\circ)\}$, which is related to attenuation at $v$-polarization ($A_v$), of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of 0°, 30°, 60° and 90° versus maximum dimension.
Figure A.17. $\Re\{S_2(0^\circ)\}$, which is related to attenuation at $v$-polarization ($A_v$), of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus maximum dimension.
Figure A.18. $Re\{S_2(0^\circ)\}$, which is related to attenuation at $v$-polarization ($A_v$), of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus maximum dimension.
Scattering properties versus crystal mass
Figure B.1. $hh$-polarization backscattering cross sections ($\sigma_{hh}$) of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$ versus mass.
Figure B.2. $hh$-polarization backscattering cross sections ($\sigma_{hh}$) of plates at W-, Ka-, Ku-
and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$ versus mass.
Figure B.3. $hh$-polarization backscattering cross sections ($\sigma_{hh}$) of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of 0°, 30°, 60°, and 90° versus mass.
Figure B.4. \( \sigma_{vv} \)-polarization backscattering cross sections (\( \sigma_{vv} \)) of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of 0\(^\circ\), 30\(^\circ\), 60\(^\circ\), and 90\(^\circ\) versus mass.
Figure B.5. $vv$-polarization backscattering cross sections ($\sigma_{vv}$) of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$, and $90^\circ$ versus mass.
Figure B.6. \(\sigma_{vv}\)-polarization backscattering cross sections \((\sigma_{vv})\) of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of 0\(^\circ\), 30\(^\circ\), 60\(^\circ\), and 90\(^\circ\) versus mass.
Figure B.7. Backscattering cross section ratios ($\sigma_{hh}/\sigma_{vv}$) of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus mass.
Figure B.8. Backscattering cross section ratios ($\sigma_{hh}/\sigma_{vv}$) of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of 0°, 30° and 60° versus mass.
Figure B.9. Backscattering cross section ratios ($\sigma_{hh}/\sigma_{vv}$) of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus mass.
Figure B.10. $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, which is related to specific differential phase ($K_{DP}$), of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus mass.
Figure B.11. $\text{Im}\{S_2(0^\circ)-S_1(0^\circ)\}$, which is related to specific differential phase ($K_{DP}$), of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$ and $60^\circ$ versus mass.
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Figure B.13. $\text{Re}\{S_1(0^\circ)\}$, which is related to attenuation at $h$-polarization ($A_h$), of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus mass.
Figure B.14. $\text{Re}\{S_1(0^\circ)\}$, which is related to attenuation at $h$-polarization ($A_h$), of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus mass.
Figure B.15. $Re\{S_1(0^\circ)\}$, which is related to attenuation at $h$-polarization ($A_h$), of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus mass.
Figure B.16. $Re\{S_2(0^\circ)\}$, which is related to attenuation at $v$-polarization ($A_v$), of dendrites at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus mass.
Figure B.17. $\text{Re}\{S_2(0^\circ)\}$, which is related to attenuation at $\nu$-polarization ($A_\nu$), of plates at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus mass.
Figure B.18. $Re\{S_2(0^\circ)\}$, which is related to attenuation at $v$-polarization ($A_v$), of columns at W-, Ka-, Ku- and X-band wavelengths for angles of incidence of $0^\circ$, $30^\circ$, $60^\circ$ and $90^\circ$ versus mass.


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Publications

[1]. Yinghui Lu, Eugene Clothiaux, Kültegin Aydin, Giovanni Botta and Johannes Verlinde, Modeling variability in dendritic ice crystal backscattering crosssections at Millimeter wavelengths using a modified Rayleigh–Gans theory, J Quant Spectrosc Radiat Transfer (2013), http://dx.doi.org/10.1016/j.jqsrt.2013.05.008