PERTURBATIVE AND NON-PERTURBATIVE APPROACHES TO
THE QUANTUM ADS$_5 \times $XS$^5$ SUPERSTRING

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Abstract

This dissertation spans perturbative to non-perturbative approaches of testing and using integrability of the IIB superstring in the $\text{AdS}_5 \times \text{S}^5$ background.

The integrability-based solution of string theories related to $\text{AdS}_n/\text{CFT}_{n-1}$ dualities relies on the worldsheet S matrix. In chapter 2 we use generalized unitarity to construct the terms with logarithmic dependence on external momenta at one- and two-loop order in the worldsheet S matrix for strings in a general integrable worldsheet theory. We also discuss aspects of calculations as it extends to higher orders. The S-matrix elements are expressed as sums of integrals with coefficients given in terms of tree-level worldsheet four-point scattering amplitudes. Off-diagonal one-loop rational functions, not determined by two-dimensional unitarity cuts, are fixed by symmetry considerations. They play an important role in the determination of the two-loop logarithmic contributions. We illustrate the general analysis by computing the logarithmic terms in the one- and two-loop four-particle S-matrix elements in the massive worldsheet sectors of string theory in $\text{AdS}_5 \times \text{S}^5$, $\text{AdS}_4 \times \text{CP}^3$, $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ and $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$. We explore the structure of the S matrices and provide explicit evidence for the absence of higher-order logarithms and for the exponentiation of the one-loop dressing phase.

In chapter 3 we will construct the full coset space of $\frac{\text{AdS}_5 \times \text{S}^5}{\text{SO(4,1)} \times \text{SO(5)}}$ in terms of a Gross-Neveu model. After this non-perturbative transformation we have shown the theory to be UV finite at 1 loop and furthermore that it exhibits some non-local integrals of motion through a Lax connection.

The integrability of string theory in $\text{AdS}_5 \times \text{S}^5$ and of the dilatation operator of $\mathcal{N} = 4$ super-Yang-Mills theory has been used to propose an exact solution to the spectral problem in these theories. Weak coupling perturbation theory both in gauge theory and on the worldsheet has been extensively used to verify this solution.

In chapter 4 we demonstrate numerical lattice monte carlo worldsheet methods
for finding the spectrum of the $\text{AdS}_5 \times S^5$ superstring at finite values of the coupling constant and illustrate them by recovering, within numerical errors, the predictions of the BES equation for the universal scaling function. This is the first finite-coupling calculation in this theory which uses finite field theory methods.
# Table of Contents

List of Figures ix  
List of Tables xi  
Acknowledgments xii  

## Chapter 1
General Overview 1  
1.1 Motivation ........................................ 1  

## Chapter 2
Perturbative Calculation of the S-Matrix 7  
2.1 Introduction ........................................ 7  
2.2 Worldsheet perturbation theory for the  
S-matrix .............................................. 14  
2.2.1 Generalities, parametrization, symmetry restrictions ....... 15  
2.2.2 The perturbative expansion of the worldsheet S-matrix ....... 16  
2.2.3 Generalized unitarity and the worldsheet S-matrix .......... 19  
2.2.4 On higher loops, regularization, factorization and related  
issues .............................................. 23  
2.3 General expressions for one- and two-loop amplitudes .......... 28  
2.3.1 The general expression for one-loop amplitudes ............ 29  
2.3.2 The general expression for two-loop amplitudes .......... 30  
2.4 The S-matrix for strings in AdS5 × S5 .................. 35  
2.4.1 The logarithmic terms of the one-loop AdS5 × S5 S-matrix .. 37  
2.4.2 The logarithmic terms of the two-loop AdS5 × S5  
S-matrix .............................................. 38
Chapter 3
A Non-Perturbative Approach 61
3.1 Introduction ........................................ 61
3.2 Transformation ...................................... 63
  3.2.1 Gross-Neveu .................................... 63
  3.2.2 Principal chiral model .......................... 63
  3.2.3 GS model ....................................... 65
3.3 UV finiteness ........................................ 67
  3.3.1 Two Point Function Renormalization .......... 68
  3.3.2 One Loop Four Point Function ................. 69
3.4 Integrability ........................................ 70
  3.4.1 Lax Connection ................................. 70
3.5 Discussion .......................................... 72

Chapter 4
A discrete approach to the Green-Schwarz string in AdS_5 x S^5 73
4.1 Introduction .......................................... 73
4.2 A numerical approach to the Green-Schwarz string ........... 77
  4.2.1 A first pass at a discrete Green-Schwarz action in AdS_5 x S^5 77
  4.2.2 Various approaches to energy calculations .......... 80
    4.2.2.1 A conformal gauge approach ................ 81
    4.2.2.2 Operator dimensions from the boundary two-point
            functions ................................ 84
    4.2.2.3 Target space energy from a partition function ... 87
4.3 An example: the universal scaling function at finite coupling ... 88
4.3.1 The null cusp fluctuation action, discretization, and some numerical details ................................................................. 89
4.3.2 The simulation, data analysis and results ......................................................... 93
4.4 Summary and further comments ............................................................................. 98

Appendix A ................................................................................................................. 101
A.1 PSU(2, 2 | 4) ............................................................................................................ 101
A.2 Green-Schwarz String ............................................................................................... 102
   A.2.1 GS string in a general curved space ............................................................... 102
   A.2.2 GS group theory construction ....................................................................... 102
A.3 Generalized Unitarity ............................................................................................... 102

Appendix B .................................................................................................................... 104
B.1 On the definition of the tree-level S-matrix elements ............................................. 104
B.2 Beisert’s SU(2|2) spin-chain S matrix ................................................................. 105
B.3 Strong coupling expansion of the AdS_5 \times S^5 dressing phase ................................................................. 107
B.4 The AdS_5 \times S^5 worldsheet S-matrix from the spin-chain S-matrix ................. 108
B.5 Two-particle cut at L-loops for string theory in AdS_5 \times S^5 ............................... 112
B.6 One- and two-loop integrals ................................................................................... 113
B.7 s- and u-channel cuts of the one- and two-loop integrals ........................................ 115
B.8 AdS_3 \times S^3 \times S^3 \times S^1 S-matrices ....................................................................... 115

Appendix C .................................................................................................................... 120
C.1 Simulating the lattice: Algorithms ........................................................................... 120
   C.1.1 The Rational Hybrid Monte Carlo Algorithm ............................................... 120
   C.1.2 Monte Carlo methods ....................................................................................... 120
   C.1.3 The Hybrid Monte Carlo Algorithm ............................................................... 121
   C.1.4 The fermion contribution to bosonic RHMC forces ........................................ 124
   C.1.5 A summary of sources of errors ..................................................................... 125
C.2 9-point stencil .......................................................................................................... 126
   C.2.1 \rho matrices ...................................................................................................... 127
C.3 Fermions .................................................................................................................. 128
   C.3.1 Code Data Structures ..................................................................................... 128
C.4 Solving the Fermion Matrix ...................................................................................... 128
   C.4.1 Conjugate Gradient Method ........................................................................... 129
   C.4.2 Conjugate Gradient Multimass Method .......................................................... 130
List of Figures

2.1 Integrals with fields that are truncated away at the classical level. The external momentum configuration guarantees that in (a) and (b) there are $\varphi$ states crossing the cut. The one- and two-loop integrals are constants, independent of external momenta. The three-loop integral depends on both external momenta and therefore need not be a rational function. .................................................. 26

2.2 The integrals appearing in the one-loop four-point amplitudes. Tensor integrals can be reduced to them as well as to tadpole integrals, which are momentum-independent. .................................................. 28

2.3 Two-particle cuts of the one-loop four-point amplitudes .......... 29

2.4 The integrals appearing in the two-loop four-point amplitudes. Each cut in fig. 2.5 determines the coefficient of one of these integrals. There exist, of course, other two-loop four-point integrals; the structure of the Lagrangian suggests that integrals with vertices with an odd number of edges cannot appear while the integral with a six-point vertex is momentum-independent and thus it can contribute only to terms with rational momentum dependence. .................. 31

2.5 Iterated two-particle cuts of two-loop four-point amplitudes. They are all maximal cuts (in two dimensions). It is not possible to relax the cut condition on any propagator either because the corresponding tree-level amplitude does not exist or because the resulting higher-point tree amplitude has an on-shell propagator as a consequence of integrability and S-matrix factorization. As discussed in sec. 2.2 all cuts of a four-point two-loop amplitude which is a product of tree amplitudes is equivalent to a sum of the cuts shown here. ................................................................. 32

2.6 The single two-particle cuts of two-loop four-point amplitudes. They are used to determine the subleading logarithms not captured by maximal cuts. .................................................. 32
3.1 The one loop two point graph with a loop over bosonic/fermions fields ........................................... 68
3.2 The one loop two point graph with a gauge field loop, the dashed line is our gauge field propagator. .................. 68
3.3 The two point bubble graph for the gauge field propagator. .......... 68
3.4 One loop 4-point graph that is picked out in our limit of \( N_f \to \infty \). 69
3.5 The remaining 1 loop graphs not picked out by our limit \( N_f \to \infty \). 70

4.1 The value of the action as a function of the (logarithm of the) evolution time \( \tau \) for \( g = 20 \) and the distribution at sufficiently late times. The evolution of the value of the free energy along the Monte Carlo time. Each point represents the value of the free energy on the accepted field configuration at the end of each sequence of \( n_T \simeq 10 \) steps (an HMC trajectory). After some time (in this case \( \ln \tau \sim 6 \)) the state "thermalizes”, i.e. the value of the free energy on the generated field configurations follow a normal distribution. The spread of the amplitude of the thermalized state comes from picking shorter trajectories in order to increase the acceptance rate of the monte-carlo. .......................... 94
4.2 The value of the free energy and the corresponding error (and in fact of any other observable) is found by fitting a Gaussian on sufficiently many values after thermalization; the histogram is constructed from about 500 data points for \( g = 20 \). .......................... 94
4.3 The universal scaling function for the values of \( g \) in eq. (4.3.15) and their fit for the \( 10 \times 10 \) and \( 12 \times 12 \). .......................... 95
4.4 Plot of the rescaled universal scaling function from the \( 10 \times 10 \) (blue right-triangles) and \( 12 \times 12 \) (red left-triangles) and its values from the BES equation (black dots). The lattice values are artificially displaced by \( \delta g = \pm 1/4 \) for easy comparison. Clearly, the central values of the \( 12 \times 12 \) lattice is a very good approximation of the integrability results. .......................... 97

A.1 s-channel two particle cut of the one loop four point amplitude . . . 103

C.1 Sketch of the Leapfrog method: where values of fields and their conjugate momenta “leap” over each other. ...................... 122
List of Tables

4.1 The numerical values of the universal scaling function obtained from $10 \times 10$ and $12 \times 12$ lattices as well as the results of the BES equation. The latter are quoted with an uncertainty of one unit in the last digit. ............................................................ 95
4.2 Coefficients of the fit of the lattice data with the expected form of the worldsheet perturbative expansion. ......................................................... 97
4.3 Values of the universal scaling function inside the radius of convergence of $\mathcal{N} = 4$ sYM theory. ................................................................. 98
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Dedication

I dedicate my dissertation work to all of my family, especially to my daughter Jocelyn and wife Alisha.
Chapter 1

General Overview

1.1 Motivation

N=4 super-Yang-Mills (SYM) theory is a quantum field theory that is a distant relative to Quantum Chromo Dynamics (QCD) and the standard model. It is a renormalizable gauge field theory on a four dimensional Minkowski spacetime. The major differences between N=4 SYM and QCD are the extra symmetries of N=4 SYM: conformal and supersymmetry. In QCD one wishes to compute the particle masses, this is possible due to the running coupling nature of the theory. In QCD the confining nature of the theory (suggested by the negative beta-function which implies that the coupling grows at low energies) is expected to imply the existence of a massive spectrum. While technically challenging, it should in principle be possible to find it with the QCD Lagrangian as starting point. In N=4 SYM to compute the masses is not possible due to the conformal symmetry. The conformal symmetry means it has no inherent mass scale, the fields are massless. In N=4 SYM the conformal symmetry carries onto the quantum level and the \( \beta \)-function is 0 to all orders in perturbation theory [1].

Since our theory is massless and has no massive bound states either, we do not look at masses like in QCD. Local gauge-invariant composite operators – i.e. products of fields at the same space-time point – are morally analogous to bound states in non-conformal theories. Their characteristic quantity is their “scaling dimension”. While conformal field theories are finite, composite operators receive non-trivial infinite renormalization, similarly to general quantum field theories. This
renders their scale dimension a function of the coupling constant; the departure from its classical value is called “anomalous dimension”. The problem of finding all (anomalous) dimensions of all local gauge-invariant operators in N=4 super-Yang-Mills theory is known as the “spectral problem”.

Unfortunately all of this means that hadrons or mesons do not exist in N=4 SYM like they do in QCD. This begs the question if the theory cannot contain particles we know exist is it an interesting theory to study?

1. The large amount of symmetries lead to an underlying integrability, a feature not very common in four-dimensional quantum field theories [2, 3, 4].

2. The conjecture that N=4 SYM is equal to IIB superstring theory on the AdS$_5 \times$S$^5$ background [6, 7, 8].

3. QCD is asymptotically free and, if one ignores the running of the coupling and fermion masses it is scale invariant. Thus, at high energies it may be approximated by a CFT in which one folds the running coupling as external input.

4. By studying a more symmetric theory new insights, structures and techniques might be easier to find and subsequently modified and applied to less symmetric context such as QCD. A QCD example of this are the recent advances in the calculation of 1-loop amplitude with a high number of external legs [9, 10, 11].

For this work we will focus on the relationship between N=4 SYM and IIB superstring theory on the AdS$_5 \times$S$^5$ background benefited by the integrability of the theory.

The conjecture that N=4 SYM is equivalent to IIB superstring theory on the AdS$_5 \times$S$^5$ background is a realization of the holographic principle. The holographic principle is the idea that all the information inside the volume of a space is encoded on the boundary of that space. In this case all of the information of AdS$_{d+1}$ is encoded on it’s boundary as a CFT$_d$.

---

1Integrability has been demonstrated perturbatively in a weak and strong coupling expansion and it is expected to hold to all orders. Chapter 4 of this thesis ([5]) provides evidence in this direction.
IIB superstring theory on the AdS$_5 \times $S$^5$ background has a 10 dimensional target space. In this space five of the dimension are in anti de Sitter (AdS) space and the other five are rolled together into a five dimensional sphere. The non-zero cosmological constant of each of the two factors, AdS$_5$ and S$_5$, is due to Ramond-Ramond (RR) 5-form flux. The string theory is defined by a two-dimensional conformal field theory of Green-Schwarz type. It has a $\kappa$ symmetry (a fermionic local symmetry) that when combined with the target space supersymmetry makes the bosonic and fermionic degrees of freedom equal.

The flat space Green Schwarz action is:

$$S = -\frac{1}{2\pi} \int d\tau d\sigma \sqrt{-h} \Pi^i_\mu \Pi^j_\nu h^{ij} \eta_{\mu\nu}$$

$$-\frac{1}{\pi} \int d\tau d\sigma \epsilon^{ij}[\partial_i X^\mu (\bar{\theta}^1 \Gamma^\nu \partial_j \theta^1 - \bar{\theta}^2 \Gamma^\nu \partial_j \theta^2) + \bar{\theta}^1 \Gamma^\nu \partial_i \theta^1 \bar{\theta}^2 \Gamma^\nu \partial_j \theta^2] \eta_{\mu\nu}$$

where $\Pi^\mu_i = \partial_i X^\mu - i \bar{\theta}^A \Gamma^\mu \partial_i \theta^A$. $\Pi_\alpha$ is the supersymmetric invariant 1-form. The action has the following supersymmetric transformation:

$$\delta X^\mu = \bar{\epsilon} A \Gamma^\mu \theta^A$$

(1.1.2)  

$$\delta \theta^A = \bar{\epsilon}^A$$

(1.1.3)  

and has the following $\kappa$ symmetry:

$$\delta_\kappa \theta^A = \Pi^A_\mu \Gamma^i \kappa^i A$$  

$$\delta_\kappa X^\mu = i \bar{\theta}^A \Gamma^\mu \theta^A$$

(1.1.4)  

This symmetry removes half of the fermionic degrees in the action.

A brief introduction to the GS action in a curved space [12] and also an alternate geometric/group theory construction [13] is in appendix A.2.

The relations between the parameters of N=4 SYM and string theory in AdS$_5 \times $S$^5$ ([6, 7, 8]) are:

$$\lambda = 4\pi^2 T^2 \quad \frac{1}{N_c} = \frac{g_{str}}{4\pi^2 T^2}$$

(1.1.5)
An alternative form also used in literature is:

\[ \lambda = g_{YM}^2 N_c = g_{str} N_c \quad g_{YM}^2 = g_{str} \]  

(1.1.6)

where \( \lambda \) is the 't Hooft coupling and \( N_c \) is the number of colors for N=4 SYM, \( T \) is the string tension and \( g_{str} \) is the string coupling in AdS\(_5\)×S\(_5\). In Chapter 2 we will use \( \hat{g} = T \) as the coupling constant. The radius of AdS space is:

\[ R^2 = 2\pi \alpha' T = \alpha' \sqrt{\lambda} \]  

(1.1.7)

where \( \alpha' \) is known as the Regge slope.

From these relationships we get two different perturbative regions this is known as the “AdS/CFT strong/weak coupling duality”. When \( \lambda = \infty \), and \( g_{str} = 0 \) the parameter space is known as strong coupling. Even though this is known as strong coupling this is where perturbative string theory applies. This is because the coupling is strong from the N=4 SYM perspective and weak in the coupling from the worldsheet perspective. When \( \lambda = 0 \) and \( N_C \) is finite we have the weak coupling region and perturbative N=4 SYM applies. There is no perturbative overlap of the theories, strong coupling on one side of the theory relates to weak coupling in the dual. Due to this something more then perturbation theory is needed to help prove this AdS/CFT conjecture. In the “planar limit” (\( \lambda \) is fixed, \( g_{str} = 0 \), \( N_C = \infty \)) the perturbation theory is given only by planar graphs. Corrections away from the planar limit are parameterized as an expansion in \( \frac{1}{N_C} \). This limit is useful in comparing the string energy to the conformal dimension in N=4 SYM. This large N limit guarantees that on the string side solving the worldsheet theory gives the complete answer and one needs not worry about string loop corrections which are counted by \( \frac{1}{N} \). Witten, Gubser, Klebanov, and Polyakov [6, 7] give a prescription for how to relate string quantities to gauge quantities. The duality relates the string energy \( E_A \) of a string state \( |A\rangle \).

\[ H_{\text{string}}|A\rangle = E_A|A\rangle \]  

(1.1.8)
to the conformal dimension $\Delta_A$ of the local gauge invariant operator $O_A$.

$$\langle O_A(x)O_B(y) \rangle = \frac{M\delta_{AB}}{|x-y|^{2\Delta_A}}$$

(1.1.9)

which requires

$$\Delta_A(\lambda, \frac{1}{N_C}) = E_A(T, g_{str})$$

(1.1.10)

where $\Delta_A$ can be expanded as

$$\Delta_A(\lambda, \frac{1}{N_C}) = \Delta^{(0)}_A + \sum_{l=1}^{\infty} \lambda^l \sum_{g=0}^{\infty} \frac{1}{N_C^{2g}} \Delta_{l,g}$$

(1.1.11)

where $\Delta^{(0)}_A$ is the classical scaling dimension and the anomalous dimension is the rest.

Currently to date all tests of AdS/CFT have been successful. Some examples are:

1. Agreement of the underlying symmetry group the four dimensional superconformal group ($PSU(2,2|4)$).

2. “BMN plane-wave correspondence” which has been confirmed up to all loop orders [14, 15, 16, 17, 18, 19, 20, 21].

3. 3-point functions of protected operators, i.e. operators that receive no correction to their anomalous dimension.[22, 23, 24, 25, 26, 27, 28]

4. Agreement between string energy [29, 30, 31, 32] and the anomalous dimension of dual operators [33, 34, 35, 36, 37, 38].

In this dissertation we will start with a perturbative approach to deriving the scattering matrix through means of generalized unitarity, (A.3), and integrability. We will apply this approach to our target theory of $AdS_5 \times S^5$ and also to other similar Green Schwarz type strings theories that exhibit integrable structures, such as strings in $AdS_4 \times CP^3$ or in $AdS_n \times S^n \times M^{10-2n}$ supported by either an RR flux or a mixture of RR and NSNS fluxes. We will then attempt a non-perturbative approach to derive the Lax connection for the full coset space of $AdS_5 \times S^5$. Finally
we will develop a new approach for numerical string theory. Our aim is to initiate the exploration of discrete approaches to the Green-Schwarz string in \( \text{AdS}_5 \times \text{S}^5 \). We will demonstrate that numerical lattice monte carlo techniques are valid for the \( \text{AdS}_5 \times \text{S}^5 \) string and with the lattice model calculate the cusp anomalous dimension function.
Chapter 2

Perturbative Calculation of the S-Matrix

This chapter is based on the paper [39] written in collaboration with O.T. Engelung and R. Roiban

2.1 Introduction

The integrability of quantum string theory in $\text{AdS}_5 \times S^5$ has led to remarkable progress in our understanding of the spectrum of anomalous dimensions in the dual $\mathcal{N} = 4$ super-Yang-Mills theory. Similarly, the integrability of the spin chain with the dilatation operator of $\mathcal{N} = 4$ super-Yang-Mills theory as Hamiltonian offers important insight into the spectrum of string theory in $\text{AdS}_5 \times S^5$ and also provides tools for the construction of other important quantities, such as the correlation functions of gauge-invariant operators and four-dimensional scattering amplitudes. This remarkable success [40] raises the question of applying similar methods, such as algebraic curve techniques [41] or the Asymptotic Bethe Ansatz (ABA) [4], to other string theories that exhibit integrable structures, such as strings in $\text{AdS}_4 \times \text{CP}^3$ or in $\text{AdS}_n \times S^n \times M^{10-2n}$ supported by either an RR flux or a mixture of RR and NSNS fluxes, and shed light on the dual conformal field theories which, for $n = 3$ are not understood beyond their BPS sector. The essential ingredient in such an approach is the scattering matrix of worldsheet excitations around a suitably-chosen vacuum state or, alternatively, the scattering of spin-chain excita-
The assumption of quantum integrability and the symmetries of the theory go a long ways in the construction of worldsheet S matrices. The former implies that the S matrices obey a form of the Yang-Baxter equation and that higher-point S-matrix elements can be constructed by multiplying together four-point S-matrix elements. The latter implies the factorization of the S matrix into factors invariant under each of the symmetry groups of the gauge-fixed theory. All in all, the S matrices are uniquely determined up to overall phases – known as the dressing phases – and a function of the string tension. There is one such phase for each part of an S matrix that is invariant under the symmetry group.

Green-Schwarz-type supercoset sigma models for \( \text{AdS}_3 \times S^3 \), \( \text{AdS}_3 \times S^3 \times S^3 \) and \( \text{AdS}_2 \times S^2 \) can be constructed based on \( D(2, 1; \alpha) \times D(2, 1; \alpha)/(SU(1, 1) \times SU(2) \times SU(2)) \), \( PSU(1, 1|2) \times PSU(1, 1|2)/(SO(1, 1) \times U(1)) \) and respectively. As their dimension is smaller than \( d = 10 \), additional bosonic directions are required for a critical string theory. Unlike the NSR string, the worldsheet theory of the Green-Schwarz string is interacting even when the bosonic part of the target space is a product space with interactions induced by fermions which are representations of the ten-dimensional Lorenz group. Thus, these supercoset sigma models can be related to the Green-Schwarz string on \( \text{AdS}_n \times S^n \times M^{10-2n} \) with \( M = S^3 \times S^1 \), \( T^4 \) and \( T^6 \) only if there exists a non-degenerate \( \kappa \)-symmetry gauge that decouples these additional degrees of freedom.

With worldsheet diffeomorphism invariance fixed to conformal gauge, a \( \kappa \)-symmetry gauge decoupling the lone \( S^1 \) and the \( T^4 \) excitations\(^1\), was found for \( M = S^3 \times S^1 \) and \( M = T^4 \) [42]. An analogous gauge does not appear to exist for \( M = T^6 \) [43]. If instead worldsheet diffeomorphisms are fixed to a gauge in which all fields are physical – such as the static gauge or the light-cone gauge – all worldsheet excitations are coupled to each other; on shell one may nevertheless expect a decoupling similar to that seen \textit{off-shell} in conformal gauge. At the classical level, it is nevertheless possible to consistently truncate [43] all the fields orthogonal to

\(^1\)String theory with \( M = T^4 \) and no excitations along \( M \) may be interpreted as the limit \( \alpha \to 1 \) of string theory with \( M = S^3 \times S^1 \). In this limit \( M \) decompactifies to \( R^4 \times S^1 \); since worldsheet masses depend on \( \alpha \) this limit is rather subtle and from the standpoint of the S matrix it involves a nontrivial rearrangement of states. The difference in the topology of \( M \) is not observable in the absence of excitations on \( M \).
the supercoset (which are massless). While superficially this truncation may seem inconsistent at loop level, we shall argue that the integrability of the theory implies that, through two loops, the truncated states affect only the part of the S-matrix terms whose dependence on external momenta is completely rational.

Similarly to strings in AdS$_5 \times$S$^5$, these theories were quantized around a BMN-type point-like string and Asymptotic Bethe Ansatz equations were proposed. An important departure from the structure of the AdS$_5 \times$S$^5$ string is that the free worldsheet spectrum is more complex, exhibiting excitations of different masses. Moreover, masses are conserved in the scattering process which, under certain circumstances, also forbids the exchange of momentum between particles (i.e. it is reflectionless). The same steps as for string theory in AdS$_5 \times$S$^5$ [44] led to the construction of finite-gap equations [45] and led to constraints on dressing phases. ²

The S-matrices of string theory in AdS$_3 \times$S$^3 \times$S$^3 \times$S$^1$ and AdS$_3 \times$S$^3 \times$T$^4$ and of the corresponding spin chain were further analyzed from an algebraic standpoint in [53, 54, 55, 56, 57, 58] where it was emphasized that, unlike the case of strings in AdS$_5 \times$S$^5$ [59, 60, 61], the dressing phases are not determined by crossing symmetry constraints. The one-loop (worldsheet) correction to the dressing phases in AdS$_3 \times$S$^3 \times$T$^4$ were found in [62] (see also [53]) through a comparison of the one-loop correction to the energy of certain extended string configuration and the Asymptotic Bethe Ansatz predictions to these quantities; the result exposes the fact that the dressing phases are different from the original conjectures [45, 63]. A direct Feynman graph calculation [64] of the one-loop worldsheet S matrix in the near-flat space limit of AdS$_3 \times$S$^3 \times$T$^4$ confirms this conclusion.

These developments underscore the importance of a direct construction of the worldsheet S matrix in these and related integrable worldsheet theories. In AdS$_5 \times$S$^5$ such a construction would further test the crossing equation [59] as well as the assumption that the relevant dressing phase is the minimal one (i.e. that no solution of the homogeneous crossing equation needs to be included) [65, 61]. By testing factorization of the S matrix at higher loops and higher points, a direct

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²String theory in AdS$_4 \times$CP$^3$, dual to the ABJM theory [46], was constructed in supercoset form in [47, 48, 49], finite-gap equations were constructed in [50] and all-loop Asymptotic Bethe Ansatz equations were proposed [51] with specific assumptions on the dressing phase of the S matrix. As was discussed in [47] and in more detail in [52], the supercoset action does not describe string configurations with excitations only in the AdS$_4$.
evaluation of such S-matrix elements would provide a powerful test of integrability and of the existence of an integrability-preserving regularization. Moreover, in $\text{AdS}_5 \times \text{S}^5$, tree-level worldsheet calculations [66] expose the fact that the S-matrix elements depend on the choice of worldsheet (diffeomorphism and $\kappa$-symmetry) gauges. By directly constructing the S matrix in a general gauge one may demonstrate explicitly the independence of the target space spectrum on the gauge-choice parameter.

In independent developments, new and powerful methods – the generalized unitarity method [67, 68, 69] and its refinement, the method of maximal cuts [70, 71] and its further generalization to certain massive cases [72, 73] – have been developed for the calculation of scattering amplitudes in various quantum field theories. They have led to the construction with relative ease of a whole host of scattering amplitudes in three-, four- and higher-dimensional supersymmetric and non-supersymmetric gauge and gravity theories and to the hope that, in the planar limit, the entire S matrix can be found. The power of the generalized unitarity approach stems from the fact that loop-level amplitudes are constructed in terms of tree-level amplitudes, which depend only on the physical degrees of freedom. They also manifest all the symmetries of the theory, including those that exist only on shell, such as integrability. Quite generally, this method suggests that, up to potential anomalies, the symmetries of the tree-level amplitudes are inherited by loop level amplitudes. The potential anomalies can also be efficiently identified in this approach.

The direct calculation of the tree-level S matrix in $\text{AdS}_5 \times \text{S}^5$ was carried out in [66].\(^3\) Direct tree-level calculations have been carried out for a certain massive subsector of string theory in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ for general $\alpha$ in [64] and in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ with mixed RR and NSNS fluxes in [75].\(^4\) From a spin chain perspective, the all-loop symmetry-determined parts of the S matrix in these cases

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\(^3\)A direct one-loop calculation was previously attempted unsuccessfully, using worldsheet Feynman graphs, in [74]. The main problem that was encountered was non-cancellation of UV divergences. While such divergences have polynomial momentum dependence and thus can always be eliminated by a local counterterm, their existence goes against the expectation that, at least for supersymmetric ground states, the worldsheet theory does not exhibit infinite renormalization. The surviving UV divergences were also present in the resulting Bethe equations. The reason for the remaining divergences was never clarified. Similar divergences have been reported in Feynman graph calculations in $\text{AdS}_3 \times \text{S}^3 \times \text{T}^4$ in [64].

\(^4\)The bosonic S matrix in $\text{AdS}_4 \times \text{CP}^3$ was found in [76].
were discussed in [57, 56].

Here we will use generalized unitarity to find the two-dimensional cut constructible part – that is the terms with logarithmic dependence on external momenta – of the one- and two-loop four-point S-matrices for the Green-Schwarz string in $\text{AdS}_5 \times S^5$, $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\text{AdS}_3 \times S^3 \times T^4$ and also to comment on the relation between the $\text{AdS}_4 \times \text{CP}^3$ and $\text{AdS}_5 \times S^5$ S-matrices. The main ingredients of this construction are the tree-level worldsheet scattering amplitudes. As we shall see, the S matrix can be expected to obey quite generally the factorized structure following from the tree-level symmetry group and integrability. As in all higher-order calculations, regularization is necessary and, ideally, we should use regulated tree amplitudes$^5$.

The issue of ultraviolet (UV) regularization of the worldsheet theory in Green-Schwarz form is a thorny one. As a conformal field theory (perhaps with spontaneously broken conformal invariance) the theory is expected be finite to all orders in perturbation theory; an all-order argument for finiteness however relies on its symmetries, in particular on $\kappa$-symmetry. This symmetry is chiral (and has a self-dual parameter) and thus exists only in two dimensions making dimensional regularization unsuitable. This is also related to the presence of the parity-odd Wess-Zumino term in the Green-Schwarz action, which also does not exist$^6$ in dimensions other that $d = 2$.

It is not clear what is an example of regularization that preserves all symmetries of the worldsheet theory. In its absence, there are (at least) two possible approaches which, ultimately, need similar additional input to yield complete S-matrix elements. On general grounds, for any regularization, integrability as well as other classical symmetries broken by the regulator can be restored by the addition to the S matrix of matrix elements of finite local counterterms in the effective action. Their determination relies on the requirement that symmetries be realized at the quantum level. Since these counterterms contribute only rational terms to

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$^5$For example, in dimensional regularization they are the tree-level amplitudes of the $d$-dimensional theory.

$^6$Dimensional regularization can be suitably modified to be applicable in the presence of a Wess-Zumino term. For example, may prescribe some analytic continuation of the two-dimensional Levi-Civita symbol (see e.g. [77]); alternatively, one may do all algebra in two dimensions and continue only the final integrals to $d \neq 2$ (see e.g. [31] for a discussion related to the Green-Schwarz string).
the S matrix, one may simply determine their off-diagonal components from symmetry considerations (their diagonal components affect the rational part of the dressing phase and thus are not determined by symmetries). Alternatively, two-loop energy calculations [78, 31] suggest that finiteness of the theory is observed if in loop calculations one does all numerator algebra in $d = 2$; this manifestly organizes the result in terms of finite combinations of integrals which may then be evaluated in any (e.g. dimensional) regularization. Following this lead we will use two-dimensional tree amplitudes to construct generalized cuts, carry out all numerator algebra in two dimensions and regularize the resulting integrals. While this approach guarantees that the terms exhibiting imaginary parts in $d = 2$ are correctly identified, it leaves open the possibility that terms with no imaginary parts are missed$^7$. As in the previous approach, off-diagonal rational terms are determined by symmetry considerations. We will follow this approach here.

It is important to note that the specifics of two-dimensional kinematics allow for the existence of four-point loop integrals with no net momentum flowing through them$^8$. One might expect that the interpretation of generalized unitarity as a specific organization of the Feynman diagram expansion will capture at least these (cut-less) terms. Their cuts appear to be singular, however, suggesting that in the absence of a suitable IR regularization their coefficients can at best be determined by a prescription.

This chapter is organized as follows. We begin in sec. 2.2 with a discussion of the structure of the S matrix of a general integrable worldsheet theory with a product-group symmetry and of the relation between the exact (spin chain) and the worldsheet S-matrices. We shall also discuss the structure of the two-dimensional unitarity cuts to all-loop orders, regularization issues, as well as identify the two-dimensional cut-constructible parts of the S matrix. To this end the rational part of the symmetry-determined S matrix will play an important role.

Sec. 2.3 contains the general form of our results; we collect here in a compact form one- and two-loop amplitudes constructed through the generalized unitarity

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$^7$In particular, in the absence of a complete determination of terms with rational momentum dependence we cannot shed light on the fate of the divergences found in earlier attempts [74]. For the same reason we cannot address the issue pointed out in [56] that the AdS$_3 \times$S$^3 \times$S$^3 \times$S$^1$ dressing phases of [62] do not obey the generalized crossing equations constructed there, since the offending terms have a completely rational dependence on external momenta.

$^8$In higher dimensions they are regular integrals in the forward limit.
method and extract their logarithmic dependence on external momenta. We shall illustrate the application of these general expressions to string theory in \( \text{AdS}_5 \times S^5 \), \( \text{AdS}_4 \times \text{CP}^3 \), \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) and \( \text{AdS}_3 \times S^3 \times T^4 \).

In sec. 2.4 we use generalized unitarity to determine the two-dimensional cut-constructible part of the worldsheet one- and two-loop S matrix in \( \text{AdS}_5 \times S^5 \). Our results may be found in eqs. (2.4.8) and (2.4.23); we will explicitly demonstrate the exponentiation of the (logarithmic part of the) one-loop dressing phase and thus provide direct evidence for the integrability of the theory at this loop order.

Using the details of the calculations in sec. 2.4, we briefly comment in sec. 2.5 on the worldsheet S matrix in \( \text{AdS}_4 \times \text{CP}^3 \). In particular, we recover the expected result that only a reflectionless S matrix is consistent with worldsheet perturbation theory if the heavy modes of the worldsheet theory are truncated away at the classical level. We also confirm the proposed dressing phase through two-loop order.

In sec. 2.6 we determine the one-loop worldsheet S matrix in \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) in the massive sector for general \( \alpha \) parameter and confirm the (logarithmic part of the) dressing phases found in [62] – see eqs. (2.6.16) and (2.6.19).

In sec. 2.7 we find the one-loop S matrix in \( \text{AdS}_3 \times S^3 \times T^4 \) in the massive sector for a background supported by a mixture of NSNS and RR fluxes parameterized by the \( q \)-parameter of [75]; our result may be found in eqs. (2.7.18) and (2.7.19). In the \( q \to 0 \) limit we find the \( m, m' \to 1 \) limit of the \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) phase of [62]. A further near-flat space limit reproduces the one-loop S-matrix of [64]. In the \( q \to 0 \) limit we also construct the two-loop S-matrix and demonstrate the exponentiation of the one-loop dressing phase – see eqs. (2.7.27) and (2.7.35).

We close in sec. 2.8 with remarks on the implications and possible extensions of our results.

The Appendices review the conventions used for the calculation of tree-level worldsheet S-matrices pointing out the differences from the usual four-dimensional ones and collect the details of the exact spin-chain S matrices in the various theories discussed in the dissertation, the general expression for the two-particle cuts in \( \text{AdS}_5 \times S^5 \), one- and two-loop integrals and their (generalized) cuts.
2.2 Worldsheet perturbation theory for the S-matrix

Let us consider a general integrable two-dimensional quantum field theory with a factorized symmetry group $G_1 \otimes G_2$. Examples are the supercoset part of all $\text{AdS}_x \times S^x \times M^{10-2x}$ theories, where $G_i$ is either $PSU(2|2)$ or $PSU(1|1)^2$. For such a theory integrability suggests that the four-point S-matrix, $\mathcal{S}$, can be decomposed as

$$\mathcal{S} = S_{G_1} \otimes S_{G_2},$$

(2.2.1)

where $S_{G_1}$ and $S_{G_2}$ are S-matrices invariant under the groups $G_1$ and $G_2$, respectively. The excitations scattered by $S_{G_i}$ are not natural perturbative excitations of the theory; rather, the latter are bilinear in the former. One may choose to extract an overall phase in eq. (2.2.1); we will not do this but instead assign all overall phases to the two $S_{G_i}$-matrix factors. We shall denote the worldsheet coupling constant by $\hat{g} = \sqrt{\lambda/(2\pi)}$.

It is possible that the symmetry group of the S matrix has further abelian factors, apart from $G_1 \times G_2$, which assign nontrivial charges to representations of $G_1 \times G_2$. An example in this direction is the case for $M=S^3 \times S^1$ where $G_1 = G_2 = PSU(1|1)$ and the additional symmetry is a $U(1)$ factor. In such cases integrability no longer requires a factorized $\mathcal{S}$-matrix of the form (2.2.1). Since we will reduce the computation of an $\mathcal{S}$-matrix of the form (2.2.1) to a separate computation of the two factors $S_{G_i}$, the discussion in this section applies unmodified to the construction of a non-factorized S matrix as well.

9While of course physically different, this is formally reminiscent of the KLT relations of standard flat space string theory, in which tree-level supergravity scattering amplitudes are bilinears of tree-level gauge theory scattering amplitudes. The essential difference is that while the latter hold at tree level, eq. (2.2.1) holds to all orders.

10We thank B. Hoare for emphasizing this out to us.
2.2.1 Generalities, parametrization, symmetry restrictions

As usual, the expansion of the worldsheet $S$-matrix in the worldsheet coupling constant $\hat{g}^{-1}$ defines the $T$ matrix

$$S = 1 + \frac{1}{\hat{g}} i T^{(0)} + \frac{1}{\hat{g}^2} i T^{(1)} + O \left( \frac{1}{\hat{g}^3} \right) \equiv 1 + i T \ , \quad (2.2.2)$$

which contains all scattering amplitudes. Each factor $S_{G_i}$ has a similar expansion:

$$S_{G_i} = 1 + \frac{1}{\hat{g}} i T^{(0)}_{G_i} + \frac{1}{\hat{g}^2} i T^{(1)}_{G_i} + O \left( \frac{1}{\hat{g}^3} \right) \equiv 1 + i T_{G_i} \ . \quad (2.2.3)$$

We may also formally refer to the entries of $i T_{G_i}$ as "scattering amplitudes”. Integrability of the theory implies that both the $T^{(0)}$-matrix and the $T^{(0)}_{G_i}$-matrix satisfy the classical limit of the YBE.

The factorization equation (2.2.1) implies a close relation between the $L$-loop entries of $i T$ and the $l \leq L$-loop entries of $i T_{G_i}$:

$$i T^{(L)} = \sum_{l=0}^{L+1} (i T^{(l-1)}_{G_1}) \otimes (i T^{(L-l)}_{G_2}) , \quad i T^{(-1)}_{G_i} = 1 . \quad (2.2.4)$$

Consequently, if integrability is preserved, to find the $L$-loop $T$ matrix it suffices to find the $L$-loop $T$ matrix, as all the other terms in (2.2.4) are already determined at lower loops.

If we denote by capital and dotted capital letters the indices acted upon by $G_1$ and $G_2$, respectively, the tree-level factorized $T$ matrix is given in terms of the $T_{G_i}$-matrices as [66]:

$$T | \Phi_A \Phi'_B \rangle = (-)^{|[A]|+[B]+[D]} | \Phi_{CA} \Phi'_{DB} \rangle T^{CD}_{AB} + (-)^{|B|+[C]} | \Phi_{AC} \Phi'_{BD} \rangle T^{CD'}_{AB} \quad (2.2.5)$$

where $[\bullet]$ represents the grade of the argument, which is zero for a bosonic index and unity for a fermionic index. Similarly, the matrix elements of a generic term $T_i \otimes T_j \subset T_{i+j+1}$ between worldsheet states $\Phi_{AA} \Phi'_{BB}$ are related to the matrix elements of $T_i$ and $T_j$ between (fictitious) two-particle states $|AB\rangle$ and $|\hat{A}\hat{B}\rangle$

$$T | AB \rangle = | CD \rangle (T_i)^{CD}_{AB} \quad T^{CD'}_{AB} = (D' C \langle T | AB \rangle \quad (2.2.6)$$
by
\[ T | \Phi_{AA} \Phi'_{BB} \rangle \supset T_i \otimes T_j | \Phi_{AA} \Phi'_{BB} \rangle = (-)^{|i| |B| + |\hat{c}| |D|} | \Phi_{CC} \Phi'_{DD} \rangle (T_i)_{AB}^{CD} (T_j)_{AB}^{\hat{C}\hat{D}} \]

momenta of the primed states are different from momenta of the other states. If either \( T_i \) or \( T_j \) are the identity matrix, *i.e.* if \((T_i)_{AB}^{CD} = \delta_A^C \delta_B^D \) or \((T_j)_{AB}^{\hat{C}\hat{D}} = \delta_A^\hat{C} \delta_B^\hat{D} \), we recover the two terms in (2.2.5).

Since the world sheet theory is not Lorentz-invariant (by the existence of a fixed vector related to the choice of vacuum), neither the S-matrix nor the S-matrix is invariant under crossing transformations

\[ S_{\text{cross}} = C^{-1} S_{\text{st}} C \quad S_{\text{cross}} = C^{-1} S_{\text{st}} C . \quad \] (2.2.8)

Here \( S_{\text{st}} \) is the super-transpose of the S-matrix in the two labels corresponding to the crossed particles,

\[ (M_{\text{st}})_{AB} = (-)^{|A| |B| + |B|} M_{BA} , \quad \] (2.2.9)

\( C \) is the charge conjugate matrix and one is also to change the sign of the energy and momentum of the particles that are crossed. The super-transpose is necessary if hermitian conjugation and complex conjugation are defined in the same way as for regular matrices. The standard (relativistic) crossing symmetry is expected to be replaced by the generalized crossing equations suggested in [59], which relate in a nontrivial way \( S \) and \( S_{\text{cross}} \). It is not difficult to see that the two transformations (2.2.8) are consistent with the factorization (2.2.1). Through a sequence of crossing transformations (2.2.8) we shall consistently construct the \( u \)-channel cuts by relating them to \( s \)-channel without using the explicit form of the crossed S matrix.

### 2.2.2 The perturbative expansion of the worldsheet S-matrix

As mentioned previously, the worldsheet S matrix is determined by symmetries up to an overall phase denoted by \( \theta_{12} \) whose general structure in terms of spin-chain variables was discussed in [44]. Its strong coupling expansion at fixed spin-chain momenta is reviewed in Appendix B.3.
Contact with worldsheet perturbation theory is however made [79, 66] in the strong coupling expansion at fixed worldsheet momenta (the ”small momentum expansion”)

\[ p_{\text{chain}} = \frac{2\pi}{\sqrt{\lambda}} p_{\text{ws}} = \frac{1}{\hat{g}} p_{\text{ws}}. \]  

(2.2.10)

In AdS\(S_3 \times S^3 \times S^1\), AdS\(S_3 \times S^3 \times T^4\) and AdS\(S_4 \times CP^3\) the role of the coupling constant is played by a nontrivial function \(h\) whose relation to the naive worldsheet coupling \(\hat{g}\) in these cases with less-than-maximal supersymmetry is subject to finite renormalization. Taking this limit is straightforward for the state-dependent part of the S matrices and the result has the same structure as (2.2.3). We focus here only on the features added by the dressing phase to the perturbative expansion of the worldsheet S matrix. Depending on the theory one may have different dressing phases in different sectors describing the scattering of different multiplets of the symmetry group; the same discussion applies separately for each of them.

The general form of the dressing phase is included in eq. (B.3.3) with the Zhukowsky variables \(x^\pm\) defined in (B.2.7) and expanded in (B.4.6). It is not difficult to see that the leading term in the expansion of \(x^\pm\), which is independent of the indices \(\pm\), cancels out in eq. (B.3.3). It is also not difficult to construct the next order in the expansion of the right-hand side of eq. (B.3.6) for any \(\chi^{(n)}\); thus, in the small momentum expansion, \(\theta_{12}^{(n)}\) is \(O(\hat{g}^{-2})\). Extracting the leading \(\hat{g}^2\) factor, \(\hat{\theta}_{12}^{(n)} = \hat{g}^2 \theta_{12}^{(n)}\), we have (see Appendix B.4)

\[ \theta_{12} = \frac{1}{\hat{g}} \sum_{n=0}^{\infty} \frac{1}{\hat{g}^n} \hat{\theta}_{12}^{(n)}, \]  

(2.2.11)

with \(\hat{\theta}^{(1)}\) contributing at \(O(\hat{g}^{-2})\), i.e. at one-loop order. For string theory in AdS\(S_5 \times S^5\), the leading term \(\hat{\theta}_{12}^{(0)}\) is also the leading term in the small momentum expansion of the AFS phase [44]. While the AFS phase contains logarithms of worldsheet momenta, each term in its small momentum expansion (2.2.10) is a rational function of worldsheet momenta and energies.

Defining \(\hat{S}\) to be the symmetry-determined part of the S matrix dressed with
\( \hat{\theta}_{12}^{(0)} \), the S-matrix can be written as

\[
S = e^{\frac{i}{\hat{\theta}_{12}^{(0)}} \hat{S}} \, ;
\]

(2.2.12)

Its large \( \hat{g} \) expansion identifies the entries of \( S \) containing information on the loop corrections to the dressing phase:

\[
S = 1 + \frac{1}{\hat{g}} T^{(0)} + \frac{1}{\hat{g}^2} i \left( \hat{T}^{(1)} + \frac{1}{2} \hat{\theta}_{12}^{(1)} \mathbb{I} \right) + \frac{1}{\hat{g}^3} i \left( \hat{T}^{(2)} + \frac{i}{2} \hat{\theta}_{12}^{(1)} T^{(0)} + \frac{1}{2} \hat{\theta}_{12}^{(2)} \mathbb{I} \right) + \mathcal{O} \left( \frac{1}{\hat{g}^4} \right) \, .
\]

(2.2.13)

We see in particular that the one-loop correction to the dressing phase affects only the diagonal entries of the one-loop S matrix.\(^{11}\) Moreover, since the part of the S matrix that is determined by symmetries has rational dependence on momenta and so does the classical phase \( \hat{\theta}_{12}^{(0)} \), the only transcendental dependence on external momenta comes from \( \hat{\theta}_{12}^{(i)} \) with \( i \geq 1 \). A direct demonstration of the structure of the \( \mathcal{O}(\hat{g}^{-3}) \) term would give strong indication of the exponentiation of the one-loop phase.

For example, for string theory in AdS\(_5\)×S\(_5\), the expansion of the first (loop correction to the) dressing phase is \([80, 79, 81]\)

\[
\hat{\theta}_{12}^{(1)} = -\frac{1}{\pi} \frac{p^2 p'^2 (\varepsilon \varepsilon' - pp')}{(\varepsilon' p - \varepsilon p')^2} \ln \left| \frac{p'}{p_-} \right| + \text{rational} \, ,
\]

(2.2.14)

where

\[
p_{\pm} = \frac{1}{2} (\varepsilon \pm p) \, ,
\]

(2.2.15)

the mass of the worldsheet excitations is set to \( m = 1 \) and \( p_+ \) and \( p_+ \) were eliminated from the argument of the logarithm through the on-shell condition

\[
4 p_+ p_- = \varepsilon^2 - p^2 = 1 \, .
\]

(2.2.16)

In the sec. 2.4 we will find this expression for \( \hat{\theta}_{12}^{(1)} \) from a direct calculation through

\(^{11}\)We shall use the notation \( T^{(0)} \) rather than \( \hat{T}^{(0)} \) for the leading term in the small momentum expansion of \( S \) because this term is unaffected by the corrections to the dressing phase.
the generalized unitarity method; we will also demonstrate that the $O(\hat{g}^{-3})$ term in the perturbative expansion of the worldsheet S matrix has the form given by (2.2.13).

2.2.3 Generalized unitarity and the worldsheet S-matrix

The ($d$-dimensional) generalized unitarity method [67, 68, 69], its implementation in the method of maximal cuts [70, 71] and in cases of massive particles [72, 73] provide powerful tools for the construction of one- and higher-loop quantum field theory scattering amplitudes. In the presence of a suitable regularization, such as dimensional regularization, complete amplitudes can be constructed. Terms with rational dependence on momenta (and thus with no cuts) are related to terms that exhibit cuts in the presence of the regulator but disappear as the regulator is removed. We refer to them as "rational terms" or $d$-dimensional cut constructible terms. The rest, which can be determined from unregularized unitarity cuts, are referred to as two-dimensional cut-constructible terms.

Here we will use the generalized unitarity method to find the two-dimensional cut-constructible parts of the one- and two-loop corrections to the worldsheet S-matrix. It is usually the case that rational terms undetermined at some loop level lead to missing terms with logarithmic (or, in general, transcendental) dependence on momenta at the next loop level. While at one loop we do not determine explicitly rational terms, we bypass this issue by making use of the rational terms that are determined by symmetries at one loop; the rational terms proportional to the identity matrix – and thus undetermined – turn out to be irrelevant for finding all logarithms at two loops.\(^{12}\)

\(^{12}\) Since we do not evaluate directly all rational terms we will not be able to completely address questions regarding the fate of divergences that appeared in previous attempts to compute the one-loop worldsheet S-matrix in AdS$_5 \times$S$^5$ [74]. In two dimensions, the only one-loop divergent rational terms are related to tadpole integrals; following the four-dimensional construction of [82], in general theories may be determined from one-particle cuts. As we shall discuss in more detail in sec. 2.2.4, as a consequence of the factorization of six-point tree-level amplitudes in integrable theories into a product of two four-point amplitudes, single-particle cuts of one-loop four-point amplitudes have an additional hidden cut condition and thus are in fact two-particle cuts. This suggests that there are no tadpole integrals in one-loop amplitudes apart from those related to wave-function renormalization. Since the string tension should not receive infinite renormalization (which is supported by the symmetry-based determination of the off-diagonal terms with completely rational momentum dependence employed here), we may therefore say
Quite generally, one may use the generalized unitarity method to determine directly the $S$-matrix elements, which are the scattering amplitudes of the worldsheet theory. The factorization of the $T$ matrix suggests however that we ought to construct the $S$-matrix instead. Indeed, at any loop order $L$, cuts not already computed at lower loops contribute only to $T^{(L)}_{G_1} \otimes 1 + 1 \otimes T^{(L)}_{G_2}$. Since the $T$-matrix elements are substantially simpler than the entries of the $T$ matrix, we shall focus on the former.

The structure of the tree-level $S$-matrix is tightly constrained by the (assumed) integrability of the theory which, in particular, implies absence of particle production and thus that the number of incoming particles is the same as the number of outgoing ones [83]. Consequently, all components of generalized cuts should also obey this constraint. For example, for the $2 \to 2$ $S$ matrix at two loops, the three-particle cut with two external particles on each side of the cut vanishes identically. Of course, not all three-particle cuts of two-loop four-point amplitudes vanish identically; an example is the cut with one external line on one side and the other three on the other side of the cut.

At one-loop level, two-particle cuts in two dimensions play a role analogous to that of quadruple cuts in four dimensions. Indeed, since in two dimensions momenta have two components, cutting two internal lines of a one-loop amplitude completely constrains the loop momentum. Thus, two-particle cuts are maximal cuts for one-loop amplitudes; similarly, four-particle cuts are maximal cuts at two loops. Consequently, the coefficients of one-loop bubble integrals are simply given by products of tree-level amplitudes appropriately summed over all possible internal states, in close analogy to the coefficients of box integrals in four-dimensional field theories.

To construct either the $T$ matrix or, separately, the $T_{G_i}$-matrix factors it is necessary to have a spectral decomposition of the identity operator in the Hilbert space of states. We will focus here on the two-particle identity operator, for which we can write:

$$\mathbb{1} = \left| \Phi_{EE'} \Phi^\prime_{FF'} \right> \left< \Phi^\prime_{FF} \Phi_{EE} \right|$$ and $$\left< \Phi'^{AA} \Phi^{BB} | \Phi_{EE'} \Phi^\prime_{FF'} \right> = \delta_{E E'} \delta_{F F'} \delta_{A A'} \delta_{B B'}. (2.2.17)$$

that, to some extent, generalized unitarity provides a divergence-free construction of the one-loop $S$ matrix.
Here $|\Phi_{EE}\rangle$ corresponds to the state created by the field $\Phi_{EE}$ and $\langle \Phi_{EE} |$ is the conjugate of that state. We may further split the spectral decomposition (2.2.17) into $1_{G_1} \otimes 1_{G_2}$ identity operators:

$$1_{G_1} \otimes 1_{G_2} = (|EF\rangle \langle F'| E|) \otimes (|\dot{E} \dot{F}'\rangle \langle \dot{F}' \dot{E}|)$$

(2.2.18)

with scalar product: $\langle A'|B|EF\rangle = \delta_E^{B'} \delta_F^A$,

where the primes denote the fact that two excitations carry momenta different from the other two; excitations with different momenta are orthogonal, $\langle A'|E\rangle = 0$.

As usual, we interpret a generalized unitarity cut as selecting from an amplitude the parts that have a certain set of propagators present. Using the spectral decomposition in one of the two group factors and the definition (2.2.6) of the matrix elements of T, it is easy to see that the $(L_1 + L_2 + 1)$-loop component of the T matrix is:

$$(iT^{(L_1 + L_2 + 1)})_{AB'}^{CD'}_{s\text{-cut}}^{L_1 \times L_2} = (i)^2(D'C| (iT^{(L_1)}) |EF| (iT^{(L_2)}) |AB)$$

(2.2.19)

where the $(i)^2$ factor originate from the two cut propagators.

To construct the $u$-channel cut we use the crossing transformation (2.2.9) to relate it to an $s$-channel cut. The crossing transformation (2.2.9) acts consistently on both matrices $S$ and $S$. Here we will use the transformation on the latter one:

$$(-)^{[B][D]+[B]} (D'C| (iT^{(L_1 + L_2 + 1)}) |AB|)^{L_1 \times L_2}_{s\text{-cut}} = (B'C| (iT^{(L_1)}) |AD')^{L_1 \times L_2}_{s\text{-cut}}$$

(2.2.20)

In terms of matrix elements this becomes

$$(iT^{(L_1 + L_2 + 1)})^{CD'}_{AB}|^{L_1 \times L_2}_{s\text{-cut}} = (i)^2 (-)^{[B][F]+[B]} (D'C| (iT^{(L_1)}) |EF'| (-)^{[F][D]+[F]} (D'E| (iT^{(L_2)}) |AF')^{L_1 \times L_2}_{s\text{-cut}}$$

(2.2.21)

One may also understand the sign factors by formally permuting indices and bringing them to the same order as in the $s$-channel cut. We will use these relations
repeatedly in the following sections to construct loop-level worldsheet S-matrix elements in $\text{AdS}_5 \times S^5$, $\text{AdS}_4 \times \text{CP}^3$, $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\text{AdS}_3 \times S^3 \times T^4$.

The $t$-channel cut is structurally different from the other two: there is no net momentum flow across the cut. Because of this, in the product of two scattering amplitudes in this channel one encounters a kinematic singularity in the form of a factor of $\delta(0)$ or perhaps as the square of a delta function, e.g. $\delta(p_1 - p_3)^2$. This singular momentum configuration is forced on us by two-dimensional kinematics and the integrability of the theory. Clearly, some form of IR regularization is necessary to extract reliable information from this cut.

It is not obvious what such a regularization might be. Continuing external momenta to $d = 2 - 2\epsilon$ would forbid the appearance of $\delta(0)$ by allowing individual momenta to not be conserved in the scattering process by an $\mathcal{O}(\epsilon)$ amount. As discussed previously however, this regularization cannot preserve all the classical symmetries of the theory. Moreover, since integrability also requires that individual momenta be conserved in a scattering process (for any number of external lines and independently of two-dimensional kinematics), regulating $\delta(0)$ as above would also break integrability. Since the $t$-channel bubble integral is in fact a constant, we will simply ignore it and determine it together with all the other rational terms from symmetry considerations.

The description of unitarity cuts earlier in this section assumed that scattering amplitudes are normalized in the standard Lorentz-invariant way. As reviewed in Appendix B.1, worldsheet tree-level S-matrix elements are normalized slightly differently both because of the mode expansion of fields and because the on-shell conditions and momentum conservation are solved using special properties of two-dimensional kinematics. Thus, to apply the usual rules we need to compensate for the solution of the momentum conservation constraint and also adjust the normalization of the creation operators to the relativistic one for each cut leg for both the left and right side of the cut, i.e. we need to multiply by $(\sqrt{2\varepsilon})^2$ for each cut leg with energy $\varepsilon$. For example, for each two-particle cut obtained by multiplying two S-matrix elements in the standard worldsheet normalization we need to supply an additional factor of

$$J = (\sqrt{2\varepsilon} \sqrt{2\varepsilon'})^2 \left( \frac{dz}{dp} - \frac{dz'}{dp'} \right) ; \quad (2.2.22)$$
For a standard dispersion relation this is:

\[ (\sqrt{2\epsilon} \sqrt{2\epsilon'})^2 \frac{\epsilon' p - \epsilon p'}{\epsilon \epsilon'} = 4(\epsilon' p - \epsilon p') = \frac{2(m^2 p'^2 - m'^2 p^2)}{p - p'}. \] (2.2.23)

We shall denote this expression by \( J_{s,u} \), depending on the channel in which it appears. It is not difficult to see however that \( J_s = J_u \) so we will at times also simply denote it by \( J \).

### 2.2.4 On higher loops, regularization, factorization and related issues

Two-particle cuts are sufficient for one-loop calculations. In general, the 2-, 3-, \ldots, 2\( L \)-particle generalized cuts of an \( L \)-loop amplitude determine its cut-constructible part. It is interesting to examine the cuts needed for higher-loop calculations and understand their structure in view of the special properties of the theory.

Let us assume that we focus on the massive sector of the theory. Moreover, since the theory should finite both in UV and IR, we will follow the hints from the two-loop energy calculations [78, 84] and not choose an explicit regulator but instead carry out all algebra in two dimensions and organize the result in terms of finite combinations of integrals. This scheme appears to preserves the integrability of the theory and, consequently, will also preserve the absence of particle production. In particular, all nonzero amplitudes have an even number of external legs and an equal number of incoming and outgoing particles; 2\( k \)-point amplitudes are sequences of four-point scattering events. This structure places a number of constraints on the cuts that determine two-dimensional cut-constructible part of the higher-loop S-matrix elements which we discuss in the following. A regularization preserving the integrability of the theory will also preserve the absence of particle production. In particular, all nonzero amplitudes have an even number of external legs and an equal number of incoming and outgoing particles; 2\( k \)-point amplitudes are a sequence of four-point scattering events. This structure places a number of constraints on the cuts that determine the higher-loop S-matrix elements.

We first notice that any cut that is a product of tree-level amplitudes is also a maximal cut. Indeed, in an integrable quantum field theory the higher-point S-
matrix elements are given by the sum of products of four-point S-matrix elements; the internal lines connecting them are on shell and, as such, may be interpreted as cut propagators. Thus, a generalized cut of an \( L \)-loop amplitude that is a product of tree-level amplitudes is also naturally the cut of the same \( L \)-loop amplitude in which all tree-level factors are four-point tree-level amplitudes – i.e. the maximal cut of the amplitude. The coefficient of an \( L \)-loop integral that has a non-zero maximal cut is simply given – for any \( L \) – by the product of tree-level amplitudes appropriately summed over all the possible states crossing the generalized cut.

For example, the six-point tree-level amplitude factorizes as a sum of products of two four-point amplitudes; then, for the two-loop four-point amplitude, the three-particle cut with one external leg on one side and three on the other side of the cut contains a hidden on-shell condition for a propagator internal to the six-point tree-level amplitude [66] and thus it is in fact a four-particle cut.

In the computational scheme described above such cuts do not determine all terms with logarithmic dependence beyond one loop. For this purpose, and to be able to include the contribution of symmetry-determined rational terms at lower-loop orders, it is useful to consider cuts that break up an amplitude into a product of an \( L_1 \)-loop and an \( L_2 \)-loop amplitude. Integrability then implies that each such amplitudes is a sum of products of \( l_i \leq L_1 \) and \( l'_j \leq L_2 \) four-point amplitudes with \( \sum l_i = L_1 \) and \( \sum l'_j = L_2 \). In non-integrable theories, focusing on a single term in this sum requires imposing additional cut conditions. In contrast, in an integrable theory these conditions are naturally present and do not impose new restrictions on the amplitudes building up the generalized cut. In each amplitude factor we may recursively include the symmetry-determined rational terms at \( l_i \)-loop order. Based on the structure of one-loop integrals and on the structure of two-particle cuts it is possible to argue\(^\text{13}\) quite generally that \((L-1)\)-loop rational terms proportional to the identity matrix in field space – and thus undetermined by symmetries – do not contribute to \( L \)-loop terms with logarithmic dependence on momenta. One may check on a case by case basis whether rational terms which are not determined by symmetries at \( l \leq (L-2) \)-loop order make any contributions

\(^{13}\)A property of the two-dimensional one-loop bubble integrals \( \tilde{I}_s \) and \( \tilde{I}_u \) listed in Appendix B.6, is that only their difference contains logarithms; thus, the difference of the coefficients of these integrals determines the coefficient of the logarithm of external momenta.
to the logarithmic terms at $L$-loops. The argument here guarantees that no such contributions exist at two loops.

An argument similar in spirit to the one in the previous paragraph suggests that logarithmic terms at $L$-loops preserve the factorization (2.2.4) of the $\mathcal{T}$ matrix. Indeed, let us consider the cut in which one side of the cut is a four-point tree-level amplitude. Since

$$i\mathcal{T}^{(0)} = (i\mathcal{T}_{G_1}^{(0)}) \otimes 1 + 1 \otimes (i\mathcal{T}_{G_1}^{(0)}),$$

it follows that the product $(i\mathcal{T}^{(L)})(i\mathcal{T}^{(0)})$ will exhibit the factorization (2.2.4) after the cut conditions are released and the integrals are evaluated. This argument is sufficient to guarantee factorization through two-loop order, where only cuts of the type above are important. While at higher loops the factorization (2.2.4) clearly holds at the level of cuts, it is less straightforward to see it in general once cut conditions are removed. Nevertheless, consistency of cuts in all channels will relate all contributions of $\mathcal{T}^{(L)}$ to the ones that are factorized following the argument above, suggesting that it is plausible that eq. (2.2.4) indeed holds to all orders in perturbation theory.

A class of theories that feature in the context of the gauge/string duality are those containing multiplets of the symmetry group that have different masses. An interesting question which arose in the comparison of worldsheet and spin chain $S$ matrices is whether calculations in the theory truncated to some subset of fields (e.g. all fields with some subset of masses) yield the same result as calculations in the complete theory. From a Lagrangian point of view this is clearly impossible unless the desired subset is decoupled from the other fields. Generalized unitarity provides more structure since it makes use of properties of the $S$ matrix not immediately visible in the Lagrangian. To understand the terms that are missed by restricting ourselves to a subset of fields, let us start with an integrable quantum field theory with particles of different masses and ignore one of them, denoted by $\varphi$. We aim to construct the scattering amplitudes of the remaining fields from their generalized cuts; ignoring $\varphi$ means that in generalized unitarity cuts we sum over all states except $\varphi$. It is not difficult to see that, in a generic quantum field theory, these steps result in an incorrect $S$ matrix.
If the tree-level S matrix has further special properties, then the two-dimensional cut-constructible terms may be reproduced correctly. Indeed, let us assume that masses of individual particles are conserved and that the scattering of $\varphi$ (or, more generally, the scattering of fields we want to truncate away) is reflectionless, i.e. that there is no momentum exchanged between the two scattered particles. All theories with different worldsheet masses that we will discuss in later sections are of this type$^{14}$. With such assumptions we see that there is no net momentum flow across any cut putting on shell only $\varphi$ fields, such as the cuts in fig. 2.1(a) and 2.1(b). Indeed, since integrability implies that momenta of $\varphi$ fields are separately conserved, $\varphi$ fields run in the loop only if the external momenta are chosen as shown. The two integrals are momentum-independent because the integrand depends on at most one external momentum and thus disappears upon use of the on-shell condition. It is not difficult to argue in a similar way that all integrals having this property – that they have no momentum flow across some generalized cut that sets on shell only $\varphi$ fields – are independent of external momenta and thus contribute only to the rational part of the S matrix.

This argument guarantees that, at one and two loops, all logarithmic terms are correctly reproduced by calculations in the theory obtained by truncating away fields that scatter reflectionlessly off the remaining ones. At three loops and beyond it is possible construct integrals – such as the one in fig. 2.1(c) – that do not have

$^{14}$This includes the massless particles that do not decouple from the supercoset sigma model for strings in $\text{AdS}_3 \times S^3 \times T^4$ but can be consistently truncated away at the classical level [75].
any generalized cut crossed only by $\varphi$-type fields and thus the argument here does not immediately imply that such integrals are rational functions. Thus, at three loops and beyond quantum calculations in the truncated theory do not necessarily yield the complete result.

In a theory exhibiting nontrivial two-point functions the construction of the S matrix through the LSZ reduction requires that the physical pole of the two-point function of fields be identified and that its residue be correctly included in the reduction of Green’s functions to S-matrix elements. In general, this implies that the naive $L$-loop amplitudes are corrected by the addition of lower-loop amplitudes multiplied by the residue of the two-point function. In all theories we will discuss in later sections it is expected that the first correction to the dispersion relation (and hence to the two-point function) is at two loops (see e.g. [85] for a calculation in the near-flat space limit of $\text{AdS}_5 \times S^5$). Thus, through two loops, only the rational terms in the S matrix will be affected; since we determine the (off-diagonal) rational terms from symmetry considerations we will ignore the corrections to the propagator. At higher loops it is, of course, important to include such contributions.

One way to construct scattering amplitudes through the generalized unitary method is to begin with an ansatz for amplitudes in terms of Feynman-like integrals whose coefficients are subsequently determined by comparing the generalized cuts of the ansatz with the generalized cuts of amplitudes constructed in terms of lower-order amplitudes. The ansatz is based on the structure of the Feynman graphs of the theory. It is interesting to note that, in all cases we are interested in, all cuts that break up an $L$-loop amplitude into a product of tree-level amplitudes completely freeze the loop momenta and therefore cannot distinguish between scalar integrals and tensor integrals. Since rational terms are supplied separately, one may thus attempt to construct ansätze in terms of only scalar integrals; such ansätze turn out to be sufficient at one and two loops.
2.3 General expressions for one- and two-loop amplitudes

Due to the special properties of S-matrices of the Green-Schwarz string in $\text{AdS}_5 \times S^5$, $\text{AdS}_4 \times \text{CP}^3$, $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ and $\text{AdS}_3 \times S^3 \times T^4$, generalized unitarity allows us to derive compact general expressions for all logarithmic terms of the one- and two-loop amplitudes in these theories. We shall outline the derivation in this section; we will then proceed in later sections to discuss each of these theories separately, pointing out the features specific to each of them.

For a compact notation we will interpret all indices used in sec. 2.2 as multi-indices while keeping their capital letter appellation, $A, B, C, \ldots$. Each multi-index stands for the set (field label, mass of field, sector).\(^\text{15}\) Not all entries are relevant in all theories. For example, in $\text{AdS}_5 \times S^5$ all fields have the same mass and there are no left and right excitations; thus, in this case only the field label is relevant. In $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ however all entries are important. The grade $[A]$ of a multi-index is the grade of the field label ($0/1$ for bosons/fermions).

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Figure 2.3: Two-particle cuts of the one-loop four-point amplitudes

### 2.3.1 The general expression for one-loop amplitudes

The integrals that can appear in one-loop amplitudes are shown in fig. 2.2; their details depend on the worldsheet spectrum. The general structure of one-loop amplitudes is

\[
i T^{(1)} = \frac{1}{2} C_s \tilde{I}_s + \frac{1}{2} C_u \tilde{I}_u + \frac{1}{2} C_t \tilde{I}_t + \text{rational}.
\]  

(2.3.1)

with the factors of $1/2$ being the symmetry factors of bubble integrals; the expression for these integrals $\tilde{I}$ as well as their equal-mass versions are collected in Appendix B.6. The coefficients $C_{u,s}$ are tensors in field space. As discussed in the previous section, we cannot reliably determine the coefficient $C_t$ due to the kinematic singularity of the $t$-channel cut. As emphasized there, at one-loop, two-particle cuts are also maximal, as can be seen from fig. 2.3; this simply implies that the two coefficients, $C_s$ and $C_u$, have simple expressions in terms of the tree-level S matrix $i T^{(0)}$:

\[
(C_s)_{AB'}^{CD'} = (i)^2 J_s \sum_{E,F'} (i T^{(0)})_{EF'}^{CD'} (i T^{(0)})_{AB'}^{EF'},
\]

\[
(C_u)_{AB'}^{CD'} = (i)^2 J_u \sum_{E,F'} (-)^{(|B|+|F|)(|D|+|F|)} (i T^{(0)})_{EB'}^{CF'} (i T^{(0)})_{AB'}^{ED'}.
\]

(2.3.2)

The Jacobians $J_s$ and $J_u$ are the adjustment factors eq. (2.2.23) needed to transform the S-matrix elements to the relativistic normalization. We recall that the precise form of these factors depends on the dispersion relation.

The explicit form of the one-loop bubble integrals from Appendix B.6 implies
that the difference

$$\frac{C_s}{J_s} - \frac{C_u}{J_u} \quad (2.3.3)$$

governs the logarithmic dependence of one-loop amplitudes on external momenta. Noticing that only $\tilde{I}_s$ has a rational component, we can cleanly separate all logarithmic dependence on momenta by organizing $i\tilde{T}^{(1)}$ as

$$i\tilde{T}^{(1)} = \frac{1}{2} C_s \tilde{I}_s + 1 + \frac{1}{2} C_u \tilde{I}_u + i\tilde{T}^{(1)} \quad (2.3.4)$$

where

$$i\tilde{T}^{(1)} = i\tilde{T}^{(1)} + i\Phi \text{1} \quad (2.3.5)$$

$i\tilde{T}^{(1)}$ was introduced in eq. (2.2.13) as the $O(\hat{g}^{-2})$ term in the small momentum expansion of the symmetry-determined part of the S matrix dressed with the classical part of the dressing phase and $\Phi$ is the contribution of rational terms in the dressing phase which are not determined by symmetries. We will notice that, in all theories we analyze, the off-diagonal entries of $i\tilde{T}^{(1)}$ are proportional to the corresponding tree-level amplitudes.

### 2.3.2 The general expression for two-loop amplitudes

To construct the two-loop correction to the worldsheet S matrix, we begin by constructing an ansatz that contains all two-loop integrals that can appear and have a logarithmic dependence on momenta. We list in fig. 2.4 all integrals that have maximal cuts that are kinematically non-singular. The cuts determining their coefficients are shown in fig. 2.5. In addition, we will include integrals that do not have non-singular maximal cuts but have single two-particle cuts; they are just products of the s- or u-channel bubble integrals with the t-channel integral $\tilde{I}_t$. The latter factor is constant and can thus be absorbed in the coefficient of the one-loop integrals. The ansatz is therefore:

$$iT^{(2)} = \frac{1}{4} C_a \tilde{I}_a + \frac{1}{2} C_b \tilde{I}_b + \frac{1}{2} C_c \tilde{I}_c + \frac{1}{4} C_d \tilde{I}_d + \frac{1}{2} C_e \tilde{I}_e + \frac{1}{2} C_f \tilde{I}_f$$
Figure 2.4: The integrals appearing in the two-loop four-point amplitudes. Each cut in fig. 2.5 determines the coefficient of one of these integrals. There exist, of course, other two-loop four-point integrals; the structure of the Lagrangian suggests that integrals with vertices with an odd number of edges cannot appear while the integral with a six-point vertex is momentum-independent and thus it can contribute only to terms with rational momentum dependence.

\[ + \frac{1}{2} C_{s,\text{extra}} \bar{I}_s + \frac{1}{2} C_{u,\text{extra}} \bar{I}_u + \text{rational} , \quad (2.3.6) \]

where we explicitly included the symmetry factors of integrals. A sum over possible distributions of internal masses is assumed. While here we are keeping the setup general, in all our explicit two-loop calculations we shall have all masses equal. The relevant integrals are listed in Appendix B.6. This ansatz manifestly satisfies the vanishing of the three-particle cut containing five-point tree-level amplitudes.

As at one loop, the coefficients \(C\) are tensors in field space. Maximal cuts determine them in terms of the tree-level S-matrix elements or, alternatively, in terms of the tree-level S-matrix elements and one-loop integral coefficients (2.3.2):

\[
(C_a)^{CD'}_{AB'} = (i)^2 J_s \sum_{G,H'} ((i^T^{(0)})^{CD'}_{GH'} (C_s)^{GH'}_{AB'} )
\]

\[
(C_b)^{CD'}_{AB'} = (i)^2 J_s \sum_{G,H'} ((i^T^{(0)})^{CD'}_{GH'} (C_u)^{GH'}_{AB'} )
\]
Figure 2.5: Iterated two-particle cuts of two-loop four-point amplitudes. They are all maximal cuts (in two dimensions). It is not possible to relax the cut condition on any propagator either because the corresponding tree-level amplitude does not exist or because the resulting higher-point tree amplitude has an on-shell propagator as a consequence of integrability and S-matrix factorization. As discussed in sec. 2.2 all cuts of a four-point two-loop amplitude which is a product of tree amplitudes is equivalent to a sum of the cuts shown here.

Figure 2.6: The single two-particle cuts of two-loop four-point amplitudes. They are used to determine the subleading logarithms not captured by maximal cuts.

\[
\begin{align*}
(C_c)_{AB'}^{CD'} &= (i)^2 J_s \sum_{G,H'} (C_s)_{GH'}^{CD'} (i T^{(0)})_{GH'}^{AB'} \\
(C_d)_{AB'}^{CD'} &= (i)^2 J_u \sum_{G,H'} (-)^{(|B|+|H|)(|D|+|H|)} (i T^{(0)})_{GH'}^{CB'} (C_u)_{AH'}^{CD'} \\
(C_e)_{AB'}^{CD'} &= (i)^2 J_u \sum_{G,H'} (-)^{(|B|+|H|)(|D|+|H|)} (i T^{(0)})_{GH'}^{CH'} (C_s)_{AH'}^{CD'} \\
(C_f)_{AB'}^{CD'} &= (i)^2 J_u \sum_{G,H'} (-)^{(|B|+|H|)(|D|+|H|)} (C_s)_{GH'}^{CH'} (i T^{(0)})_{AH'}^{CD'}.
\end{align*}
\]

(2.3.7)

With these coefficients the three-particle cuts of two-loop amplitudes are also
reproduced.

The two remaining coefficients, $C_{s,\text{extra}}$ and $C_{u,\text{extra}}$, are found by comparing the single two-particle cuts of the ansatz with the single two-particle cut of the two-loop amplitude (see fig. 2.6). The former are given in terms of complete one-loop amplitudes $iT^{(1)}$

$$iT^{(2)}_{CD'}_{AB'}|_{s-\text{cut}} = (i)^2 J_s \sum_{G,H'} \left( (iT^{(0)})_{CD'}_{GH'} (iT^{(1)})_{AB'} + (iT^{(1)})_{CD'}_{GH'} (iT^{(0)})_{AB'} ight) \tag{2.3.8}$$

$$iT^{(2)}_{CD'}_{AB'}|_{u-\text{cut}} = (i)^2 J_u \sum_{G,H'} (-)^{(H[H')} (J_{[H][H]}) \left( (iT^{(0)})_{GH'} (iT^{(1)})_{AB'} + (iT^{(1)})_{GH'} (iT^{(0)})_{AB'} \right) \tag{2.3.9}$$

while the latter are given by

$$iT^{(2)}_{s-\text{cut}}|_{\text{eq. 2.3.6}} = \frac{C_a}{J_s^2} ((J_s \tilde{I}_s + 1) - 1) + \frac{1}{2} \left( \frac{C_b}{J_s J_u} + \frac{C_c}{J_u J_s} \right) J_u \tilde{I}_u$$

$$+ \frac{1}{2} \left( \frac{C_b}{J_s} + \frac{C_c}{J_s} \right) \tilde{I}_t + \frac{C_{s,\text{extra}}}{J_s}, \quad (2.3.10)$$

$$iT^{(2)}_{u-\text{cut}}|_{\text{eq. 2.3.6}} = \frac{C_d}{J_u^2} J_u \tilde{I}_u + \frac{1}{2} \left( \frac{C_e}{J_s J_u} + \frac{C_f}{J_u J_s} \right) ((J_s \tilde{I}_s + 1) - 1)$$

$$+ \frac{1}{2} \left( \frac{C_e}{J_s} + \frac{C_f}{J_s} \right) \tilde{I}_t + \frac{C_{u,\text{extra}}}{J_u}. \quad (2.3.11)$$

Using eq. (2.3.7) it is not difficult to see that the coefficients of $(J_s \tilde{I}_s + 1)$ and $J_u \tilde{I}_u$ are the same in eqs. (2.3.8) and (2.3.10) as well as in eqs. (2.3.9) and (2.3.11). This is a consequence of identities between the one-loop and the two-loop maximal cuts stemming from the fact that both of them are given as sums of products of tree-level amplitudes; it is also a manifestation of the consistency of the generalized unitarity method.

We can therefore immediately read off the remaining coefficients; from eqs. (2.3.8) and (2.3.9) only the terms in which $T^{(1)}$ is replaced with its rational part $\tilde{T}^{(1)}$ contribute:

$$\frac{1}{J_s} (C_{s,\text{extra}})_{CD'}_{AB'} = (i)^2 J_s \sum_{G,H'} \left( (iT^{(0)})_{GH'} (i\tilde{T}^{(1)})_{AB'} + (iT^{(1)})_{GH'} (iT^{(0)})_{AB'} \right)$$

$$+ \left( \frac{C_a}{J_s^2} \right) \frac{1}{2} \left( \frac{C_e}{J_s} \right) \frac{1}{J_s} + \frac{1}{2} \frac{C_f}{J_s} ) \tilde{I}_t, \quad (2.3.12)$$
\( \frac{1}{J_u} (C_{u, \text{extra}})^{CD'}_{AB'} = \) 
\[
(i)^2 J_u \sum_{G,H'} (-)^{(\{B\}+[H])(\{P\}+[H]}) 
\left( (iT(0))^{GH'}_{GB'} (i\tilde{T}^{(1)})^{GD'}_{AD'} + (i\tilde{T}^{(1)})^{GH'}_{GB'} (iT(0))^{GD'}_{AD'} \right) 
+ \frac{1}{2} \left( \frac{(C_e)^{CD'}_{AB'}}{J_s J_u} + \frac{(C_f)^{CD'}_{AB'}}{J_u J_s} \right) 
- \frac{1}{2} \left( \frac{(C_e)^{CD'}_{AB'}}{J_s} + \frac{(C_f)^{CD'}_{AB'}}{J_u} \right) \right) I_t .
\] (2.3.13)

The structure of one-loop integrals implies again that any logarithmic dependence on momenta that is not fixed by two-loop maximal cuts depends only on the difference

\[
\frac{C_{s, \text{extra}}}{J_s} - \frac{C_{u, \text{extra}}}{J_u} .
\] (2.3.14)

One can check that all terms proportional to the identity matrix in \( \tilde{T}^{(1)} \), \( \tilde{T}^{(1)} = \delta^{CD'}_{AB'} \), cancel out in this difference and thus all two-loop logarithmic terms can be determined unambiguously. If desired (e.g. for a three-loop calculation), the two-loop rational terms \( i\tilde{T}^{(2)} \) can be supplied separately, by expanding the symmetry-determined part of the S matrix.

Using the explicit expressions for the integrals in Appendix B.6 one can see that in theories in which all worldsheet masses are equal the coefficient of the double logarithm is given by the combination

\[
C_{\ln^2} = \frac{1}{8\pi^2 J^2} \left( -2C_a + C_b + C_c - 2C_d + C_e + C_f \right) ,
\] (2.3.15)

while the coefficient of the simple logarithm \( \ln \frac{p'}{p} \) is given by

\[
C_{\ln^1} = \frac{i}{2\pi} \left[ \frac{1}{2J^2} (2C_a - C_b - C_c) 
- \frac{1}{J} (C_{s, \text{extra}} - C_{u, \text{extra}}) - \frac{i}{8\pi J} (C_b + C_c - C_e - C_f) \right] .
\] (2.3.16)

Here we used the fact that \( J_s = J_u \).

In the following sections we shall compute the one-loop integral coefficients for AdS\(_5\times\mathrm{S}^5\), AdS\(_4\times\mathrm{CP}^3\) and AdS\(_3\times\mathrm{S}^3\times\mathrm{T}^4\) with NSNS and RR fluxes. We shall
find that the difference
\[ \frac{C_s}{J_s} - \frac{C_u}{J_u} \]
(2.3.17)
is proportional to the identity matrix in each of the different sectors of the S matrix as labeled by the third entry of the multi-index (e.g. LL and LR scattering, see footnote 15); the proportionality coefficient can be identified with the one-loop dressing phase in each sector, cf. eq. (2.2.13). We shall compute the two-loop integral coefficients for AdS$_5 \times$S$^5$, AdS$_4 \times$CP$^3$ and AdS$_3 \times$S$^3 \times$T$^4$ with RR flux and find that
\[ C_{\ln^2} = 0 \]
\[ C_{\ln^1} = \frac{i}{4\pi^2 J_2} (2C_a - C_b - C_c) \propto -\frac{1}{2} T^{(0)}. \]  (2.3.18)

The proportionality coefficient in each sector is given by the coefficient of the identity matrix in $C_s/J_s - C_u/J_u$. This demonstrates the exponentiation of the one-loop dressing phase in each sector and thus provides support for two-loop integrability in all sectors.

### 2.4 The S-matrix for strings in AdS$_5 \times$ S$^5$

In this section we will use generalized unitarity and the special properties of two-dimensional integrable quantum field theories discussed in sec. 2.2 to recover the known S matrix for string theory in AdS$_5 \times$S$^5$. In this case $G_1 = G_2 = PSU(2|2)$. The worldsheet theory contains eight bosons and eight fermions of equal mass which we shall normalize to $m = 1$. Therefore, the multi-indices relabeling the S-matrix have a single entry – the field label, which is just the fundamental representation of $PSU(2|2)$. Each of them is represented by a pair of a two-component bosonic index and a two-component fermionic index, such as $(a, \alpha)$ etc., each acted upon by an $SU(2) \subset PSU(2|2)$. The symmetry-determined part of the S matrix was found in [87] and the dressing phase conjectured in [65] was tested through two loops in the SL(2) sector. The T matrix is parametrized as
\[ T_{ab}^{cd} = A \delta_c^c \delta_b^d + B \delta_c^d \delta_b^c, \]
\[ T_{ab}^{\gamma\delta} = C \, \epsilon_{ab} \epsilon^{\gamma\delta}, \]
\begin{align}
T^{\gamma \delta}_{\alpha \beta} &= D \delta^\gamma_{\alpha} \delta^\delta_{\beta} + E \delta_\alpha^\delta \delta^\gamma_{\beta}, \\
T^{\delta}_{\alpha \beta} &= G \delta_\gamma^\delta \delta^\alpha_{\beta}, \\
T^{\gamma}_{\alpha \beta} &= H \delta_\alpha^\gamma \delta^\delta_{\beta},
\end{align}
(2.4.1)

Each of the coefficients A \ldots K has an inverse $\hat{g}$ expansion, e.g.

\begin{align}
A &= \frac{1}{\hat{g}} A^{(0)} + \frac{1}{\hat{g}^2} A^{(1)} + \ldots,
\end{align}
(2.4.2)

and similarly for all other coefficients. Here $A^{(0)}$, etc. are tree-level S-matrix elements, i.e. the entries of $T^{(0)}$ introduced in eq. (2.2.13).

The action has standard Lorentz-invariant quadratic terms but interactions break this symmetry. The tree-level S-matrix elements were found in [66]:

\begin{align}
A^{(0)}(p, p') &= \frac{1}{4} \left[ (1 - 2a) (\epsilon' p - \epsilon p') + \frac{(p - p')^2}{\epsilon' p - \epsilon p'} \right], \\
B^{(0)}(p, p') &= -E^{(0)}(p, p') = \frac{pp'}{\epsilon' p - \epsilon p'}, \\
C^{(0)}(p, p') &= F^{(0)}(p, p') = \frac{1}{2} \frac{\sqrt{(\epsilon + 1)(\epsilon' + 1)(\epsilon' p - \epsilon p' + p' - p)}}{\epsilon' p - \epsilon p'}, \\
D^{(0)}(p, p') &= \frac{1}{4} \left[ (1 - 2a) (\epsilon' p - \epsilon p') - \frac{(p - p')^2}{\epsilon' p - \epsilon p'} \right], \\
G^{(0)}(p, p') &= -L^{(0)}(p', p) = \frac{1}{4} \left[ (1 - 2a) (\epsilon' p - \epsilon p') - \frac{p^2 - p'^2}{\epsilon' p - \epsilon p'} \right], \\
H^{(0)}(p, p') &= K^{(0)}(p, p') = \frac{1}{2} \frac{pp'}{\epsilon' p - \epsilon p'} \frac{\sqrt{(\epsilon + 1)(\epsilon' + 1) - pp'}}{\epsilon' p - \epsilon p'}. 
\end{align}
(2.4.3)

Here $\epsilon = \sqrt{1 + p^2}$ denotes the relativistic energy. The parameter $a$ reflects the dependence of the S matrix on the choice of physical states selected by the gauge-fixing of two-dimensional diffeomorphism invariance. We shall see that it does not affect the logarithmic part of the dressing phase.

As noted in [66], the tree-level S matrix determined by the coefficients (2.4.3) differs from the one obtained by expanding the one in [87] by terms linear in the particle’s momenta. These terms may be accounted for by a suitable rephasing of
the S matrix in [87] (included for convenience in Appendix B.2) as

\[
\begin{align*}
\hat{A}^B &= A^B e^{i((1-2a)(p-p'))} \\
\hat{C}^B &= C^B e^{i((\frac{3}{4}+b-2a)p-(\frac{1}{4}-b-2a)p')} \\
\hat{E}^B &= E^B e^{i((\frac{1}{4}-2a)p-(\frac{1}{4}-2a)p')} \\
\hat{G}^B &= G^B e^{i((-\frac{1}{2}p+(1-2a))(p-p'))} \\
\hat{K}^B &= K^B e^{i((\frac{3}{4}-b-2a)p-(\frac{1}{4}-b-2a)p')} \\
\hat{B}^B &= B^B e^{i((1-2a)(p-p'))} \\
\hat{D}^B &= D^B e^{i((\frac{1}{2}-2a)p-(\frac{1}{2}+2a)p')} \\
\hat{F}^B &= F^B e^{i((\frac{1}{2}-b-2a)p-(\frac{1}{2}+b-2a)p')} \\
\hat{H}^B &= H^B e^{i((\frac{3}{4}+b-2a)p-(\frac{3}{4}+b-2a)p')} \\
\hat{L}^B &= L^B e^{i((\frac{3}{4}p'+(1-2a))(p-p'))}
\end{align*}
\]  

(2.4.4)

One may check that the S-matrix with these coefficients obeys the graded untwisted Yang-Baxter equation; it is therefore a particular case of the S-matrix constructed in [88]. The phases added to \( A^B, B^B, D^B, E^B, G^B, L^B \) eliminate the terms linear in momenta that are different between the spin chain and the world sheet calculations; the other phases, depending on the free parameter \( b \) may be adjusted (or eliminated) by a rephasing of external states.

### 2.4.1 The logarithmic terms of the one-loop AdS

S

\[ S \]

matrix

The one-loop amplitudes have the general form (2.3.1) in which all masses are taken to be the same, \( i.e. \ I_{s,u} \rightarrow I_{s,u} \). Using the tree-level amplitudes (2.4.3) it is not difficult to find all components of the \( C_s \) and \( C_u \) coefficients. For example,

\[
\begin{align*}
\frac{1}{J_s} (C_s)_{cd}^{ab} &= (A^{(0)} b^{(0)} + 2C^{(0)} F^{(0)}) \delta_a^c \delta_b^d + 2 (A^{(0)} B^{(0)} - C^{(0)} F^{(0)}) \delta_a^d \delta_b^c \\
\frac{1}{J_u} (C_u)_{cd}^{ab} &= A^{(0)} \delta_a^c \delta_b^d + 2 (A^{(0)} B^{(0)} + B^{(0)} - H^{(0)} K^{(0)}) \delta_a^d \delta_b^c .
\end{align*}
\]  

(2.4.5)

As mentioned in the previous section, the one-loop bubble integrals are such that the difference \( C_s/J_s - C_u/J_u \) governs the logarithmic dependence on external momenta. While the complete expressions for \( C_s \) and \( C_u \) are not immediately transparent, their difference is simple –

\[
\frac{C_s}{J_s} - \frac{C_u}{J_u} = \frac{p^2 p'^2 (\varepsilon \varepsilon' - p p')}{(\varepsilon p' - \varepsilon' p)^2} \mathbb{I} ,
\]  

(2.4.7)

\( i.e. \) it is proportional to the identity operator in field space; it is also independent of the gauge-choice parameter \( a \), as expected. Using the values of the one-loop
bubble integrals it follows that the one-loop worldsheet $S$ matrix in $\text{AdS}_5 \times \text{S}^5$ is

$$iT^{(1)} = i \left( \frac{1}{2} \left( -\frac{1}{\pi} \frac{p^2 p'^2 (\varepsilon \varepsilon' - pp')}{(\varepsilon p' - \varepsilon' p')^2} \ln \left| \frac{p'}{p_-} \right| \right) \mathbb{1} + \text{rational} \right) = i \left( \frac{1}{2} \delta_{12}^{(1)} \mathbb{1} + \text{rational} \right) .$$

(2.4.8)

We thus recover the general form of the one-loop $S$ matrix (2.2.13) with all the logarithmic terms\textsuperscript{16} given by the one-loop dressing phase in $\text{AdS}_5 \times \text{S}^5$, $\delta_{12}^{(1)}$ in eq. (2.2.14). As mentioned in sec. 2.3, we can rewrite the one-loop $S$ matrix and separate the rational part while keeping an integral representation for the logarithmic dependence:

$$iT^{(1)} = \frac{1}{2} C_s J_s (J_s I_s + 1) + \frac{1}{2} C_u J_u J_u I_u + i \tilde{T}^{(1)} ,$$

(2.4.9)

where the rational part\textsuperscript{17} of the one-loop $S$ matrix, $\tilde{T}^{(1)}$, was defined in eq. (2.3.5).\textsuperscript{18} This decomposition will be useful in the next section where we discuss the construction of the logarithmic terms in the two-loop $S$ matrix.

### 2.4.2 The logarithmic terms of the two-loop $\text{AdS}_5 \times \text{S}^5$ S-matrix

The general form of the two-loop amplitudes in $\text{AdS}_5 \times \text{S}^5$ is given by (2.3.6) with all fields having the same mass; the coefficients are given by eqs. (2.3.7) and (2.3.12). We illustrate these formulae by writing out explicitly the components contributing to $A^{(2)}$ and $B^{(2)}$; they can also be rederived easily by iterating the general

\textsuperscript{16}We note here that, in line with the fact that the integrals we used are Lorentz invariant, the argument of the logarithm is Lorentz-invariant. The coefficient of the logarithm is not, however, and can also be written in terms of Lorentz invariants and the constant time-like vector $n$ related to the choice of vacuum state: $\delta_{12} = -1/\pi (p \cdot p' - n \cdot p n \cdot p')^2 (p \cdot p')^2 \ln |p'_- / p_-|$. Such a rewriting is possible for all the other models we discuss in later sections.

\textsuperscript{17}We argued in footnote 12 that finiteness of rational terms in an integrable theory relies on the absence of tadpole integrals on external lines (sometimes known as snail graphs). One can check that the one-loop integral identified by the one-particle cut constructed from the four-point amplitude vanishes upon integration; this is in line with the expected absence of one-loop corrections to the dispersion relation of worldsheet fields in $\text{AdS}_5 \times \text{S}^5$. Thus, while we do not determine $\tilde{T}^{(1)}$ we nevertheless see that it is finite.

\textsuperscript{18}We note that, since the difference of $C_s$ and $C_u$ is independent of the gauge-choice parameter $a$, only the last term, $\tilde{T}^{(1)}$, can depend on it.
expressions for two-particle cuts in Appendix B.5:

\[
\frac{1}{J_s^2}(C_a)_{ab}^{cd} = (A_{s-cut}(1) A^{(0)} + B_{s-cut}(1) B^{(0)} + 2C_{s-cut}(1) F^{(0)}) \delta^c_a \delta^d_b \\
+ (A_{s-cut}(1) B^{(0)} + B_{s-cut}(1) A^{(0)} - 2C_{s-cut}(1) F^{(0)}) \delta^d_a \delta^c_b , \\
= (A^{(0)} A_{s-cut}(1) + B^{(0)} B_{s-cut}(1) + 2C^{(0)} F^{(1)}_{s-cut}) \delta^c_a \delta^d_b \\
+ (A^{(0)} B_{s-cut}(1) + B^{(0)} A_{s-cut}(1) - 2C^{(0)} F_{s-cut}(1)) \delta^d_a \delta^c_b , \quad (2.4.10)
\]

\[
\frac{1}{J_u^2}(C_d)_{ab}^{cd} = (A_{u-cut}(1) A^{(0)} \delta^c_a \delta^d_b \\
+ (A_{u-cut}(1) B^{(0)} + B_{u-cut}(1) A^{(0)} + 2B_{u-cut}(1) B^{(0)} - 2H^{(1)}_{u-cut} K^{(1)}) \delta^d_a \delta^c_b \\
= (A^{(0)} A_{u-cut}(1) \delta^c_a \delta^d_b \\
+ (A^{(0)} B_{u-cut}(1) + B^{(0)} A_{u-cut}(1) - 2B^{(0)} B_{u-cut}(1) - 2H^{(0)} K_{u-cut}(1)) \delta^d_a \delta^c_b , \quad (2.4.11)
\]

\[
\frac{1}{J_s J_u}(C_b)_{ab}^{cd} = (A_{u-cut}(1) A^{(0)} + B_{u-cut}(1) B^{(0)} + 2C_{u-cut}(1) F^{(0)}) \delta^c_a \delta^d_b \\
+ (A_{u-cut}(1) B^{(0)} + B_{u-cut}(1) A^{(0)} - 2C_{u-cut}(1) F^{(0)}) \delta^d_a \delta^c_b , \quad (2.4.12)
\]

\[
\frac{1}{J_s J_u}(C_c)_{ab}^{cd} = (A_{s-cut}(1) A^{(0)} \delta^c_a \delta^d_b \\
+ (A_{s-cut}(1) B^{(0)} + B_{s-cut}(1) A^{(0)} + 2B_{s-cut}(1) B^{(0)} - 2H^{(1)}_{s-cut} K^{(1)}) \delta^d_a \delta^c_b \\
= (A^{(0)} A_{s-cut}(1) \delta^c_a \delta^d_b \\
+ (A^{(0)} B_{s-cut}(1) + B^{(0)} A_{s-cut}(1) - 2C^{(0)} F_{s-cut}(1)) \delta^d_a \delta^c_b \quad (2.4.14)
\]

\[
\frac{1}{J_s J_u}(C_f)_{ab}^{cd} = (A^{(0)} A_{s-cut}(1) \delta^c_a \delta^d_b \\
+ (A^{(0)} B_{s-cut}(1) + B^{(0)} A_{s-cut}(1) - 2B^{(0)} B_{s-cut} - 2H^{(0)} K_{s-cut}(1)) \delta^d_a \delta^c_b . \quad (2.4.15)
\]

The cuts of the one-loop amplitudes are given in terms of $C_s$ and $C_u$ coefficients, e.g.

\[
iA_{s-cut}^{(1)} \delta^c_a \delta^d_b = \frac{1}{J_s}(C_s)_{ab}^{cd} \quad iA_{u-cut}^{(1)} \delta^c_a \delta^d_b = \frac{1}{J_u}(C_u)_{ab}^{cd} . \quad (2.4.16)
\]

Using the explicit expression for the tree-level S-matrix elements it is not difficult to check that

\[-2C_a + C_b + C_c - 2C_d + C_e + C_f = 0 , \quad (2.4.17)
\]
as mentioned at the end of sec. 2.3. This implies the cancellation of all the dou-
ble logarithms. This is consistent with the structure of the two-loop terms in eq. (2.2.13), since the two-loop correction to the BHL/BES dressing phase, \( \hat{\theta}^{(2)} \), does not contain any double logarithms (see Appendix B.3).

The two remaining coefficients, \( C_{s,\text{extra}} \) and \( C_{u,\text{extra}} \), are given by eq. (2.3.12) and are determined by comparing the single two-particle cuts of the ansatz with the single two-particle cut of the two-loop amplitude. Their contribution to \( A^{(2)} \) is

\[
\frac{C_{s,\text{extra}}^A}{J_s} = 2i(i)^2(A^{(0)}(i\tilde{A}^{(1)}) + B^{(0)}(i\tilde{B}^{(1)}) + C^{(0)}(i\tilde{F}^{(1)}) + F^{(0)}(i\tilde{C}^{(1)}))
+ \frac{C^A_a}{J_s^2} - \frac{1}{2} \left( \frac{C^A_b}{J_s} + \frac{C^A_c}{J_s} \right) I_t,
\]

\[
\frac{C_{u,\text{extra}}^A}{J_u} = 2i(i)^2iA^{(0)}(i\tilde{A}^{(1)}) + \frac{1}{2} \left( \frac{C^A_e}{J_sJ_u} + \frac{C^A_f}{J_sJ_u} \right)
- \frac{1}{2} \left( \frac{C^A_e}{J_s} + \frac{C^A_f}{J_s} \right) I_t. \tag{2.4.19}
\]

\( C^Y_x \) denotes the contribution of \( C_x \) to the coefficient \( Y \) in eq. (2.4.1) and \( \tilde{A}^{(1)}, \) etc. are the entries of \( \tilde{T}^{(1)} \) defined in eq. (2.3.5).

Taking their difference and reconstructing the combination on the second line of eq. (2.3.16) we observe the cancellation the diagonal term \( \tilde{A}^{(1)} \) which potentially contains rational terms undetermined by symmetries. After some amount of algebra which makes use of the explicit form of the rational terms extracted from the exact S matrix

\[
\hat{B}^{(1)} = i(A^{(0)} + D^{(0)})B^{(0)} + \frac{i}{8}aJB^{(0)}
\]

\[
\hat{C}^{(1)} = \frac{i}{2}(A^{(0)} + D^{(0)})C^{(0)} + ib(p + p')C^{(0)}
\]

\[
\hat{F}^{(1)} = \frac{i}{2}(A^{(0)} + D^{(0)})F^{(0)} - ib(p + p')F^{(0)} \tag{2.4.20}
\]

we find that the second line of eq. (2.3.16) vanishes identically. Repeating the same

---

19 Such cancellations between terms captured by different cuts is characteristic to un-ordered scattering amplitudes and was previously observed in e.g. higher-dimensional supergravity theories.

20 We note here that the off-diagonal rational terms of the one-loop amplitudes are proportional to the tree-level S-matrix elements.
steps for the coefficients of all the other tensor structures one can confirm that
\[
\frac{C_{s,\text{extra}}}{J_s} - \frac{C_{u,\text{extra}}}{J_u} + \frac{i}{8\pi J_s} (C_b + C_c - C_e - C_f) = 0 .
\] (2.4.21)

The combination of integral coefficients on first line of eq. (2.3.16) contributing to \(iA^{(2)}\) can be written as
\[
-4\pi i C_{A}^A \ln \frac{1}{\frac{J'}{J}} = -2(iA^{(0)}) \left( \frac{C_{s}^A}{J_{s}} - \frac{C_{u}^A}{J_{u}} \right) ;
\] (2.4.22)

Similar expressions hold for all the other entries. Combining them and using the value of the parenthesis above from eq. (2.4.7) we find that the AdS\(_5\)×S\(_5\) two-loop S matrix is
\[
iT^{(2)} = -\frac{1}{2} \left( -\frac{1}{\pi} \frac{p^2 p'^2 (\varepsilon \varepsilon' - pp')}{(\varepsilon' p - \varepsilon p')^2} \ln \frac{p'}{p_-} \right) T^{(0)} + \text{rational} = -\frac{1}{2} \hat{\theta}^{(1)} T^{(0)} + \text{rational} .
\] (2.4.23)

It reproduces the logarithmic terms in eq. (2.2.13) and thus gives strong support to the exponentiation of the one-loop logarithms as in eq. 2.2.12.

We have therefore demonstrated that the generalized unitarity method carried out in two dimensions, together with consequences of symmetries of the theory, can be used to efficiently determine all terms with logarithmic dependence on external momenta through at least two loop level. We shall now discuss other interesting worldsheet theories related to gauge/string dualities and in some cases find novel results regarding their scattering matrices.

### 2.5 On the S-matrix for strings in AdS\(_4\)× CP\(_3\)

A worldsheet theory that bears close similarity to the AdS\(_5\)×S\(_5\) string is that of type IIA string theory on AdS\(_4\)×CP\(_3\). This is dual to the ABJM theory [46] and there is evidence for its integrability. Quantized around a BMN-like null geodesics, its worldsheet spectrum consists of eight bosons and eight fermions four of each being light \((m^2 = 1/4)\) and four being heavy \((m^2 = 1)\). It was argued from a worldsheet perspective [89] that the heavy excitations are unstable and decay into a pair of light excitations. The bosonic worldsheet S matrix was found in [76]. In a spin-chain picture the heavy excitations are interpreted as composite and do not
exist as asymptotic states; thus, the relevant S matrix scatters only the eight light excitations, organized in two representations of $PSU(2|2) \times U(1)$, typically called the $A$ and the $B$ particles/multiplets. For our discussion we shall use the small momentum limit of this S-matrix. As discussed in sec. 2.2, the different masses of excitations and the properties of the S matrix guarantee that this truncation is perturbatively consistent through at least two loops.

In [51] a proposal was made for the Bethe equations to all-loop orders. The dressing phase for $AdS_4 \times CP^3$ was suggested to be similar to the one found for $AdS_5 \times S^5$ with the difference that the full S-matrix contained the squared dressing phase (with the two factors coming from the two factors in (2.2.1)); in the former case there is one dressing phase factor for each multiplet.

Based on these Bethe ansatz equations an S matrix was conjectured in [90]. The proposed S matrix has four sectors corresponding to the four ways one can pick the multiplets of the incoming and outgoing particles and it is reflectionless (i.e. particles of different multiplets will not exchange momentum):

\[
S_{BB}(p,p') = S_{AA}(p,p') = S_0(p,p') \hat{S}(p,p'), \tag{2.5.1}
\]
\[
S_{AB}(p,p') = S_{BA}(p,p') = \tilde{S}_0(p,p') \hat{S}(p,p'). \tag{2.5.2}
\]

The factors $S_0(p,p')$ and $\tilde{S}_0(p,p')$ are two (potentially different) dressing phases and $\hat{S}(p,p')$ is the $SU(2|2)$ invariant S matrix found in [88].

The reflectionless property of the S matrix implies that different sectors do not mix in the s- and u-channel cuts so it is sufficient to consider the cuts of a single $SU(2|2)$-invariant S matrix. Our computations in the previous section show that, at least through two loops, the dressing phase of such an S matrix would indeed have half of the logarithmic terms of the full BES/BHL phase, consistent with the expectation of [51] thus providing a non-trivial check of that proposal.

An alternative S matrix was considered in [91] and rejected as the resulting Bethe equations did not match two-loop gauge theory perturbative calculations [92]. It is also based on the $SU(2|2)$ invariant S matrix, with the only important difference that it is no longer reflectionless. We can also see from the perspective of generalized unitarity that such an S matrix is not consistent with worldsheet perturbation theory truncated to the fields with $m^2 = 1/4$. 
To this end let us consider the scattering of two $A$-type scalars into two $A$-type fermions. Since this is an off-diagonal matrix element and since (assuming integrability and $SU(2|2)$ symmetry) quantum corrections can yield logarithmic terms only in the dressing phase, eq. (2.2.13) implies that at one-loop level this matrix element should have no logarithms. For a reflectionless S matrix this is realized by the $s$- and $u$-channel cuts being equal. Allowing reflections in the S matrix changes the number of particles crossing the $u$-channel cut (from only $A$-type particles to both $A$- and $B$-type particles) while not affecting the $s$-channel cut. Thus, a reflection-containing S matrix is not consistent with worldsheet perturbation theory truncated to the light fields.

2.6 The S-matrix for strings in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$

The Green-Schwarz string in $\text{AdS}_3 \times \text{S}^3 \times \text{S}^3 \times \text{S}^1$ [93] was shown in [42] to be related to the $\mathbb{Z}_4$-graded supercoset $D(2, 1; \alpha) \otimes D(2, 1; \alpha)/(SO(1, 2) \times SO(3) \times SO(3))$ and an extra free boson, where $\alpha$ is related to the radii of the various factors by

$$\alpha = \frac{R^2_{\text{AdS}}}{R^2_{\text{S}_1^2}} = 1 - \frac{R^2_{\text{AdS}}}{R^2_{\text{S}_2^2}}. \quad (2.6.1)$$

The BMN limit was studied in [94] and it was found that the perturbative worldsheet spectrum consists of two bosons and two fermions of $m = 1$; two bosons and two fermions of $m = 0$; two bosons and two fermions of $m = \alpha$; two bosons and two fermions of $m = 1 - \alpha$. One massless mode corresponds to excitations on $\text{S}^1$ and the other to an excitation shared between the two three-spheres. Certain entries of the tree-level S matrix for particles with $\alpha$-dependent masses were found in [64].

The symmetry group of the light-cone gauge-fixed worldsheet theory is $PSU(1|1)^2$. As discussed in [55], the states with $\alpha$-dependent masses may be written as a tensor product of left excitations $(L)$, $|\phi\rangle$ and $|\psi\rangle$, and the conjugate right excitations $(R)$, $|\bar{\phi}\rangle$ and $|\bar{\psi}\rangle$; they transform (in conjugate representations) under one or the other factors of the symmetry group. The two representations are

21 The $L$ and $R$ excitations have been denoted by $+$ and $-$ in related a context [75]. We mention here again that these excitations are not directly related to left- and right- worldsheet motion.
no longer decoupled in the centrally-extended $PSU(1|1)^2$ and, while $PSU(1|1)^2$-invariant, the S matrix describes nontrivial $LL$, $RR$, $LR$ and $RL$ scattering. This sector of massive states was discussed from a spin-chain perspective in [55, 57] and independently in [58] where the symmetry-determined parts of the S-matrices were proposed. The direct tree-level calculation of [64] favors the S matrix proposed in [55, 57]. The one-loop correction to the dressing phase was found indirectly in [62], by comparing one-loop corrections to energies of semiclassical states to the energy predictions of the Asymptotic Bethe Ansatz based on the exact S matrix.

As described in [57], the proposed S-matrices scatter only states with normalized masses $m = \alpha$ and $m = (1 - \alpha)$. One of their characteristic feature is that individual masses are conserved in a scattering process. Let us consider some generic scattering process:

$$|\chi_p^{(in)} \chi_p^{(in)} \rangle \rightarrow |\chi_p^{(out)} \chi_p^{(out)} \rangle, \quad (2.6.2)$$

where we also assume that \( \frac{p_{in}}{m_{in}} > \frac{p'_{in}}{m'_{in}}, \frac{p'_{out}}{m'_{out}} > \frac{p_{out}}{m_{out}} \), where $m$ denote the normalized masses of particles. When all the masses are the same momentum and energy conservation leads to the solution \( p_{in} = p_{out}, \ p'_{in} = p'_{out} \) plus another solution that does not satisfy the assumed ordering.

When masses are not the same one can still have a similar solution to the conservation equations, \( p_{in} = p_{out}, \ p'_{in} = p'_{out}, \ m_{in} = m_{out} \neq m'_{in} = m'_{out} \) but there is also another solution where the outgoing momenta are not equal to the incoming momenta. The proposed S-matrices are reflectionless, i.e. they forbid this second possibility. The scattering of states in different multiplets of the $PSU(1|1)^2$ is also reflectionless even if they have the same mass.

The proposed S matrices do not describe the scattering of states with $m = 0$ and $m = 1$. The calculation of [64] does not shed light on the properties of the relevant S-matrix elements as the relevant matrix elements involving $m = 1$ states were evaluated only for $\alpha = 1$ (i.e. they are S-matrix elements for strings on $AdS_3 \times S^3 \times T^4$) and the $m = 0$ states were not considered\(^{23}\). In the following we shall assume that the scattering of the $m = 0$ and $m = 1$ states off states with

\(^{22}\)This condition is equivalent to assuming a definite sign for the Jacobian $J = \varepsilon'p - p'\varepsilon$.

\(^{23}\)A proposal for the inclusion of the massless degrees of freedom in the spin chain Asymptotic Bethe Ansatz for $M=S^3 \times S^1$ and $M=T^4$ was discussed in [95].
m = α and m = (1 − α) is also reflectionless and thus that they do not contribute to the one- and two-loop logarithmic terms. This is justified a posteriori, as our results are consistent with those of [62].

2.6.1 The tree-level S-matrix

Entries of the tree-level worldsheet S matrix were constructed in [64] for general α. The result reproduces the small momentum limit of the full S-matrices of [55]; an alternative S matrix was proposed in [58]; both the S matrix of Borsato, Ohlsson Sax and Sfondrini (BOSS) [55] and that of Ahn and Bombardelli (AB) [58] are included in Appendix B.8. In a notation close to that used in AdS$_5 \times$S$^5$, we parametrize the T as follows:

\[
\begin{align*}
T_{\phi\phi}^{\psi\psi} &= A_{LL}, & T_{\psi\psi}^{\phi\phi} &= D_{LL}, & T_{\phi\psi}^{\phi\psi} &= G_{LL}, \\
T_{\phi\psi}^{\psi\phi} &= H_{LL}, & T_{\psi\phi}^{\phi\psi} &= K_{LL}, & T_{\psi\psi}^{\phi\phi} &= L_{LL}, \\
T_{\phi\phi}^{\phi\psi} &= A_{LR}, & T_{\phi\psi}^{\phi\phi} &= C_{LR}, & T_{\psi\phi}^{\psi\psi} &= D_{LR}, \\
T_{\phi\psi}^{\phi\psi} &= G_{LR}, & T_{\psi\psi}^{\phi\phi} &= F_{LR}, & T_{\psi\phi}^{\psi\psi} &= H_{LR}, \\
T_{\phi\phi}^{\psi\psi} &= K_{LR}, & T_{\psi\psi}^{\phi\phi} &= L_{LR}.
\end{align*}
\]  

(2.6.3)

The lower indices denote the incoming state and the upper indices denote the outgoing state. Here we did not assign an index for the mass of the excitation; such an index can be added without difficulty. Apart from fields, the indices L and R should carry the same index related to the mass. Each component has an expansion in the appropriate worldsheet coupling constant $\hat{g}^{-1}$ similar to eq. 2.4.2, e.g.

\[
A_{LL} = \frac{1}{\hat{g}} A_{LL}^{(0)} + \frac{1}{\hat{g}^2} A_{LL}^{(1)} + \ldots .
\]  

(2.6.4)

The tree-level worldsheet S-matrix elements, $A_{LL}^{(0)}$ etc., follow from [64] or can be extracted from the small momentum expansion of the exact S matrix [55] with additional minus signs for the scattering of two fermions. The latter approach also yields the symmetry-determined rational terms at higher loops.

We will not need the explicit form of the diagonal tree-level matrix elements, so
we will not list them here. Their only property that is important for our calculation is that they are related by

$$A^{(0)} + D^{(0)} - G^{(0)} - L^{(0)} = 0,$$  \hspace{1cm} (2.6.5)

for any choice of masses and for any choice of $L$ and $R$ states.

With these preparations, the tree-level entries of the BOSS S-matrix that we will need are [64, 55]:

$$H^{(0)}_{LL}^{\text{BOSS}} = K^{(0)}_{LL}^{\text{BOSS}} = \frac{1}{2} \frac{pp'}{\varepsilon'p - p'\varepsilon} \frac{(\varepsilon + m)(\varepsilon' + m') - pp'}{\sqrt{(\varepsilon + m)(\varepsilon' + m')}},$$  \hspace{1cm} (2.6.6)

$$C^{(0)}_{LR}^{\text{BOSS}} = F^{(0)}_{LR}^{\text{BOSS}} = \frac{1}{2} \sqrt{\frac{(\varepsilon + m)(\varepsilon' + m')}{\varepsilon'p - p'\varepsilon}} \frac{(\varepsilon'p - p'm - pm')}{(\varepsilon' + m)},$$  \hspace{1cm} (2.6.7)

$$H^{(0)}_{LR}^{\text{BOSS}} = K^{(0)}_{LR}^{\text{BOSS}} = 0.$$  \hspace{1cm} (2.6.8)

Here we combined all different choices of masses in a single expression; different mass sectors correspond to different choices of $m, m' = \alpha, 1 - \alpha$; also, $\varepsilon$ and $\varepsilon'$ are the standard relativistic energies for particles of masses $m$ and $m'$, respectively.

Similarly, expanding the non-diagonal entries of the AB S matrix and combining different mass sectors we find:

$$H^{(0)}_{LL}^{\text{AB}} = H^{(0)}_{LR}^{\text{AB}} = \frac{1}{2} \frac{pp'}{\varepsilon'p - p'\varepsilon} \frac{(\varepsilon + m)(\varepsilon' + m') - pp'}{(\varepsilon + m)},$$  \hspace{1cm} (2.6.9)

$$C^{(0)}_{LR}^{\text{AB}} = F^{(0)}_{LR}^{\text{AB}} = 0,$$  \hspace{1cm} (2.6.10)

$$K^{(0)}_{LL}^{\text{AB}} = K^{(0)}_{LR}^{\text{AB}} = \frac{1}{2} \frac{pp'}{\varepsilon'p - p'\varepsilon} \frac{(\varepsilon + m)(\varepsilon' + m') - pp'}{(\varepsilon + m')}. $$  \hspace{1cm} (2.6.11)

Notice that although the functions $H_{LL}$ and $K_{LL}$ are different in eqs. (2.6.8) and (2.6.11), their product remains the same. This observation will be useful shortly and leads to the equality of the one-loop corrections in the $LL/RR$ sectors of the two S-matrices.
2.6.2 The logarithmic terms of the one-loop $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ S-matrix

Let us now illustrate the calculation of the one-loop $S$ matrix following the general discussion in sec. 2.3. Each multi-index $A, B, C, \ldots$ there stands for the triplet (field label, mass, sector), e.g. $(\phi, m, L)$; the grade of the index $[A]$ is the grade of the field label (0/1 for bosons/fermions). The general expression for one-loop four-point amplitudes is (2.3.1) with integrals having different masses. They obey the relations

$$J_s \tilde{I}_s - J_u \tilde{I}_u = -\frac{2i}{\pi} \left( \ln \left| \frac{p'}{p} \right| - \ln \left| \frac{m'}{m} \right| \right) - 1,$$

(2.6.12)

$$J_s \tilde{I}_s + J_u \tilde{I}_u + 1 = 0,$$

(2.6.13)

which implies that the logarithmic dependence of the $S$ matrix is governed by the difference $C_s/J_s - C_u/J_u$. It is instructive to consider separately the $LL$ and $LR$ sectors and find the one-loop correction in the various possible sectors. In the process we shall also expand the slightly cryptic multi-index form of the one-loop coefficients (2.3.2).

2.6.2.1 The $LL$ and $RR$ sectors

The tree-level amplitudes in the BOSS and AB $S$-matrices have the same vanishing entries; therefore the one-loop coefficients (2.3.2) have formally similar expressions:

$$\frac{1}{J_s} (C_s, LL)^{\phi\phi} = A_{LL}^{(0)2},$$

$$\frac{1}{J_u} (C_u, LL)^{\phi\phi} = A_{LL}^{(0)2} - H_{LL}^{(0)2} K_{LL}^{(0)}$$

$$\frac{1}{J_s} (C_s, LL)^{\psi\psi} = D_{LL}^{(0)2},$$

$$\frac{1}{J_u} (C_u, LL)^{\psi\psi} = D_{LL}^{(0)2} - H_{LL}^{(0)2} K_{LL}^{(0)}$$

$$\frac{1}{J_s} (C_s, LL)^{\phi\psi} = G_{LL}^{(0)2} + H_{LL}^{(0)2} K_{LL}^{(0)} + L_{LL}^{(0)2},$$

$$\frac{1}{J_u} (C_u, LL)^{\phi\psi} = D_{LL}^{(0)2} + H_{LL}^{(0)2} A_{LL}^{(0)} + L_{LL}^{(0)2}$$

$$\frac{1}{J_s} (C_s, LL)^{\psi\phi} = G_{LL}^{(0)2} + H_{LL}^{(0)2} K_{LL}^{(0)},$$

$$\frac{1}{J_u} (C_u, LL)^{\psi\phi} = D_{LL}^{(0)2} + H_{LL}^{(0)2} A_{LL}^{(0)} + L_{LL}^{(0)2}$$

$$\frac{1}{J_s} (C_s, LL)^{\phi\psi} = L_{LL}^{(0)2} + H_{LL}^{(0)2} K_{LL}^{(0)},$$

$$\frac{1}{J_u} (C_u, LL)^{\phi\psi} = A_{LL}^{(0)2} + K_{LL}^{(0)} D_{LL}^{(0)}$$

$$\frac{1}{J_s} (C_s, LL)^{\psi\phi} = L_{LL}^{(0)2} + H_{LL}^{(0)2} K_{LL}^{(0)},$$

$$\frac{1}{J_u} (C_u, LL)^{\psi\phi} = A_{LL}^{(0)2} + K_{LL}^{(0)} D_{LL}^{(0)}$$

It is easy to see that the difference $(C_s/J_s - C_u/J_u)$, governing the logarithmic dependence on external momenta, depends only on the product $H_{LL}^{(0)} K_{LL}^{(0)}$ which is...
the same for both S-matrices (2.6.8) and (2.6.11). We therefore find that

\[
\frac{C^\text{BOSS}_{s,LL}}{J_s} - \frac{C^\text{BOSS}_{u,LL}}{J_u} = \frac{C^{AB}_{s,LL}}{J_s} - \frac{C^{AB}_{u,LL}}{J_u} = \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p'} + mm')}{2(\varepsilon'p - p'\varepsilon)^2} \mathbb{1},
\]

(2.6.15)

where \(\mathbb{1}\) is the identity matrix in field space. This in turn implies that the one-loop correction to the \(LL\) sector of both S-matrices is

\[
iT^{(1)}_{LL} = i \left( \frac{1}{2} \left( -\frac{1}{\pi} \frac{p^2(p')^2(\mathbf{p} \cdot \mathbf{p'} + mm')}{2(\varepsilon'p - p'\varepsilon)^2} \left( \ln \left| \frac{p'}{p} \right| - \ln \left| \frac{m'}{m} \right| \right) \right) \mathbb{1} + \text{rational} \right).
\]

(2.6.16)

This expression reproduces the small momentum limit of the \(LL\) dressing phase of [62] for general values of \(m\) and \(m'\), see eq. (B.8.22). The additional factor of \(1/2\) is reminiscent of \(\text{AdS}_4 \times \text{CP}^3\) as the dressing phase to which we are comparing comes from a factorized S matrix.

The calculation in the \(RR\) sector is completely equivalent for both S matrices. While consistent with the BOSS S matrix, the presence of logarithms appears to contradict the conjecture [58] that in the sector with two different masses, \(m \neq m'\), there is no dressing phase. This implies that the S matrix proposed there is not the S matrix of a quantum field theory.

### 2.6.2.2 The \(LR\) and \(RL\) sectors

Due to the structure of the tree-level AB S matrix, in particular the vanishing of \(C_{LR}^{(0)}\) and \(F_{LR}^{(0)}\) as well as the fact that \(H_{LR}^{(0)} = H_{LL}^{(0)}\) and \(K_{LR}^{(0)} = K_{LL}^{(0)}\), the calculation of the one-loop S matrix in the \(LR/RL\) sectors is identical to the one in the \(LL/RR\) sectors in the previous section and \(iT^{(1),AB}_{LR} \) is given by the right-hand side of eq. (2.6.16).

Using the vanishing entries of the \(BOSS\) tree-level S matrix, the various components of the coefficients \(C_s\) and \(C_u\) describing its one-loop corrections are given.
by (2.3.2):

\[
\begin{align*}
\frac{1}{J_s}(C_{s,LR}^\text{BOSS})_{\phi\phi} &= A_{LR}^{(0)2} + C_{LR}^{(0)} F_{LR}^{(0)} \\
\frac{1}{J_s}(C_{s,LR}^\text{BOSS})_{\psi\psi} &= A_{LR}^{(0)} C_{LR}^{(0)} + C_{LR}^{(0)} F_{LR}^{(0)} \\
\frac{1}{J_s}(C_{s,LR}^\text{BOSS})_{\psi\phi} &= D_{LR}^{(0)2} + C_{LR}^{(0)} F_{LR}^{(0)} \\
\frac{1}{J_s}(C_{s,LR}^\text{BOSS})_{\phi\psi} &= D_{LR}^{(0)} F_{LR}^{(0)} + F_{LR}^{(0)} A_{LR}^{(0)} \\
\frac{1}{J_u}(C_{u,LR}^\text{BOSS})_{\phi\phi} &= A_{LR}^{(0)} \\
\frac{1}{J_u}(C_{u,LR}^\text{BOSS})_{\psi\psi} &= G_{LR}^{(0)} C_{LR}^{(0)} + C_{LR}^{(0)} L_{LR}^{(0)} \\
\frac{1}{J_u}(C_{u,LR}^\text{BOSS})_{\psi\phi} &= D_{LR}^{(0)2} \\
\frac{1}{J_u}(C_{u,LR}^\text{BOSS})_{\phi\psi} &= L_{LR}^{(0)2} - C_{LR}^{(0)} F_{LR}^{(0)} \quad \text{\(2.6.17\)}
\end{align*}
\]

From here the difference \(C_s/J_s - C_u/J_u\) that governs the S matrix' logarithmic dependence on external momenta is

\[
\frac{C_s^\text{BOSS}}{J_s} - \frac{C_u^\text{BOSS}}{J_u} = \frac{p^2(p')^2(p \cdot p' - mm')}{2(\varepsilon'p - p'\varepsilon)^2} \mathbb{1} . \quad (2.6.18)
\]

where \(\mathbb{1}\) is the identity matrix in field space. In turn this implies that the one-loop correction to the BOSS S matrix is

\[
i T_{LR}^{1,\text{BOSS}} = i \left( \frac{1}{2} \left( -\frac{1}{\pi} \frac{p^2(p')^2(p \cdot p' - mm')}{2(\varepsilon'p - p'\varepsilon)^2} \left( \ln \left| \frac{p'}{p} \right| - \ln \left| \frac{m'}{m} \right| \right) \right) \mathbb{1} + \text{rational} \right) . \quad (2.6.19)
\]

This expression reproduces the small momentum limit of the \(LR\) dressing phase of [62] for general values of \(m\) and \(m'\), see eq. (B.8.23). The additional factor of \(1/2\) has the same origin as in the \(LL\) and \(RR\) sectors.

### 2.7 The S-matrix for strings in \(\text{AdS}_3 \times S^3 \times T^4\)

\(\text{AdS}_3 \times S^3 \times T^4\) can be sourced by a mixture of RR and NSNS fluxes [8, 96, 97]. The Green-Schwarz string in such a background was constructed in [93]; the \(\text{AdS}_3 \times S^3\) part is described by the supercoset \(PSU(1,1|2) \times PSU(1,1|2)/(SU(1,1) \times SU(2))\) and it is classically integrable [98]. This model was further studied in [75] where the tree-level four-point S matrix was found for a generic mixture of both types of fluxes.

In this section we will compute the logarithmic part of the one-loop S matrix.
in the presence of both NSNS and RR fluxes. For a pure RR flux background we extract the off-diagonal rational terms from the all-loop symmetry-determined $S$ matrix found in [56] and compute the two-loop logarithmic terms.

The massive worldsheet spectrum in $\text{AdS}_3 \times S^3 \times T^4$ consists of eight modes (four bosons and four fermions) which are organized in two bifundamental representations of the light-cone gauge symmetry group $SU(1|1)^2 \times SU(1|1)^2$; they are denoted by [75]

$$|y_+\rangle = |\phi\rangle \otimes |\phi\rangle \quad |z_+\rangle = |\psi\rangle \otimes |\psi\rangle \quad |\zeta_+\rangle = |\phi\rangle \otimes |\psi\rangle \quad |\chi_+\rangle = |\psi\rangle \otimes |\phi\rangle ,$$

$$|y_-\rangle = |\bar{\phi}\rangle \otimes |\bar{\phi}\rangle \quad |z_-\rangle = |\bar{\psi}\rangle \otimes |\bar{\psi}\rangle \quad |\zeta_-\rangle = |\bar{\phi}\rangle \otimes |\bar{\psi}\rangle \quad |\chi_-\rangle = |\bar{\psi}\rangle \otimes |\bar{\phi}\rangle .$$

(2.7.1)

We slightly adjusted the notation, in particular $\phi_+ \rightarrow \phi$, $\phi_- \rightarrow \bar{\phi}$, to make it match the pure RR flux $S$ matrix of [56] where $(\phi, \psi)$ are the excitations in the $L$ sector and $(\bar{\phi}, \bar{\psi})$ are the excitations in the $R$ sector. The symmetry group implies that the $S$ matrix has the usual factorized structure (2.2.1)

$$S = S_{su(1|1)^2} \otimes S_{su(1|1)^2} .$$

(2.7.2)

As in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$, each of the two $S$-matrix factors has four sectors: $LL$, $LR$, $RL$ and $RR$ (or $(++)$, $(+)$, $(-+)$, $(--)$ in the notation of [75]).

### 2.7.1 String theory tree-level $S$-matrices

The $T$ matrix is parameterized in the same way as in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$, i.e.

$$T^{\phi\phi}_{\phi\phi} = A_{LL} , \quad T^{\psi\psi}_{\psi\psi} = D_{LL} , \quad T^{\phi\psi}_{\phi\psi} = G_{LL} ,$$

$$T^{\phi\phi}_{\psi\phi} = H_{LL} , \quad T^{\psi\psi}_{\psi\phi} = K_{LL} , \quad T^{\psi\phi}_{\psi\phi} = L_{LL} ,$$

$$T^{\phi\phi}_{\phi\psi} = A_{LR} , \quad T^{\psi\psi}_{\phi\psi} = C_{LR} , \quad T^{\psi\phi}_{\psi\phi} = D_{LR} ,$$

$$T^{\phi\phi}_{\phi\psi} = G_{LR} , \quad T^{\psi\psi}_{\psi\phi} = F_{LR} , \quad T^{\psi\phi}_{\psi\phi} = L_{LR} ,$$

(2.7.3)

with each coefficient having an expansion in the inverse string tension. For the background supported by a mixture of RR and NSNS flux the $LL$ sector of the
tree-level S matrix is \[ T^{(0)}_{LL}|\phi\phi'\rangle = \frac{1}{2}(l_1 + c)|\phi\phi'\rangle, \quad T^{(0)}_{LL}|\phi\psi'\rangle = \frac{1}{2}(l_3 + c)|\phi\psi'\rangle - l_5|\psi\phi'\rangle, \quad (2.7.4) \]

\[ T^{(0)}_{LL}|\psi\psi'\rangle = \frac{1}{2}(l_1 - c)|\psi\psi'\rangle, \quad T^{(0)}_{LL}|\psi\phi'\rangle = \frac{1}{2}(l_3 - c)|\psi\phi'\rangle - l_5|\phi\psi'\rangle, \quad (2.7.5) \]

and the LR sector is

\[ T^{(0)}_{LR}|\phi\bar{\psi}\rangle = \frac{1}{2}(l_3 + c)|\phi\bar{\psi}\rangle, \quad T^{(0)}_{LR}|\phi\bar{\phi}\rangle = \frac{1}{2}(l_2 + c)|\phi\bar{\phi}\rangle + l_4|\bar{\psi}\bar{\psi}\rangle, \quad (2.7.6) \]

\[ T^{(0)}_{LR}|\psi\bar{\phi}\rangle = \frac{1}{2}(l_3 - c)|\psi\bar{\phi}\rangle, \quad T^{(0)}_{LR}|\bar{\psi}\bar{\psi}\rangle = \frac{1}{2}(l_2 - c)|\bar{\psi}\bar{\psi}\rangle + l_4|\bar{\phi}\bar{\phi}\rangle. \quad (2.7.7) \]

In the following we shall need only the expressions for \( l_4 \) and \( l_5 \):

\[ l_4 = -\frac{pp'}{2(p + p')} \left( \sqrt{(\varepsilon_+ + (p_+))(\varepsilon'_- + (p'_-))} - \sqrt{(\varepsilon_+ - (p_+))(\varepsilon'_- - (p'_-))} \right), \quad (2.7.8) \]

\[ l_5 = -\frac{pp'}{2(p - p')} \left( \sqrt{(\varepsilon_+ + (p_+))(\varepsilon'_+ + (p'_+))} + \sqrt{(\varepsilon_+ - (p_+))(\varepsilon'_+ - (p'_+))} \right), \quad (2.7.9) \]

\[ p_\pm = p \pm q, \quad \varepsilon_\pm = \sqrt{(p \pm q)^2 + 1 - q^2}. \quad (2.7.10) \]

Here \( q \) is a measure of the ratio between the NSNS and RR fluxes; \( q \to 0 \) yields the pure RR flux theory. The other two sectors of the S matrix can be found by exchanging \( \varepsilon_+ \) and \( \varepsilon_- \) as well as barred and un-barred fields. We note that for \( q \to 0 \) the form of the S matrix in the \( LL \) and \( LR \) sectors is the same as the \( \alpha \to 1 \) limit of the \( LL \) and \( LR \) sectors of the BOSS S matrix \([55, 64]\). As before, the tree-level S-matrix coefficients are related by:

\[ A^{(0)} + D^{(0)} - G^{(0)} - L^{(0)} = 0, \quad (2.7.11) \]

in all sectors. This identity will be useful for the consistency of the construction of loop amplitudes we now discuss.
2.7.2 One-loop logarithmic terms for mixed RR / NSNS \( \text{AdS}_3 \times \text{S}^3 \times \text{T}^4 \)

The one-loop amplitudes take again the general form (2.3.1); however, because the dispersion relations in the presence of a non vanishing \( q \) are not the standard relativistic ones, the integrals \( \tilde{I} \) have a slightly different interpretation: the space-like component of the momentum of a propagator is shifted by \( \pm q \) for a field in the \( L/R \) sector. Moreover, the masses are \((1 - q^2)\). These integrals may be interpreted as regular integrals with equal masses evaluated at shifted external momenta\(^{24}\): 

\[
\tilde{I}(p, p')_{LL} = I(p + q, p' + q) \\
\tilde{I}(p, p')_{LR} = I(p + q, p' - q).
\]

Their expressions follow immediately from Appendix B.6. Due to the modified dispersion relations, the Jacobian factors (2.2.23) are modified from their usual form \( J = 4(p\epsilon' - p'\epsilon) \) to \( J_{LL} \) and \( J_{LR} \) following from (2.2.22) for the dispersion relation in eq. (2.7.10):

\[
J_{LL} = 4((p + q)\epsilon'_+ - (p' + q)\epsilon_+) \quad J_{LR} = 4((p + q)\epsilon'_- - (p' - q)\epsilon_+).
\]

Since all masses are equal, the multi-indices used in sec. 2.3 now represent the pair (field label, sector).

The non-zero integral coefficients in the \( LL \) sector are given by (2.3.2)

\[
\frac{1}{J_{s, LL}}(C_{s, LL})_{\phi\phi} = A_{LL}^{(0)2} - H_{LL}^{(0)}K_{LL}^{(0)} \\
\frac{1}{J_{s, LL}}(C_{s, LL})_{\psi\psi} = D_{LL}^{(0)2} - H_{LL}^{(0)}K_{LL}^{(0)} \\
\frac{1}{J_{s, LL}}(C_{s, LL})_{\psi\phi} = G_{LL}^{(0)} + H_{LL}^{(0)}K_{LL}^{(0)} \\
\frac{1}{J_{s, LL}}(C_{s, LL})_{\psi\phi} = L_{LL}^{(0)}K_{LL}^{(0)} + G_{LL}^{(0)} + K_{LL}^{(0)}H_{LL}^{(0)}
\]

From here follows that the difference of \( C_s \) and \( C_u \) coefficients relevant for the

\[^{24}\text{To derive these expressions it is necessary to parametrize the integrals to respect the } p \leftrightarrow p' \text{ symmetry of the graph.}\]
logarithmic terms in (2.3.3) in the $LL$ sector is

$$\frac{C_{s,LL}}{J_s} - \frac{C_{u,LL}}{J_u} = \frac{p^2(p')^2}{2(p-p')^2} (\varepsilon_+\varepsilon'_+ + (p+q)(p'+q) + (1-q^2)) \ 1 \quad (2.7.15)$$

which depends only on the product $H^{(0)}K^{(0)}$.

The nonzero integral coefficients in the $LR$ sector are given by

$$\frac{1}{J_{s,LR}} (C_{s,LR})_{\phi\phi} = A_{LR}^{(0)2} + C_{LR}^{(0)} F_{LR}^{(0)}$$
$$\frac{1}{J_{s,LR}} (C_{s,LR})_{\psi\psi} = A_{LR}^{(0)2} C_{LR}^{(0)} + C_{LR}^{(0)} D_{LR}^{(0)}$$
$$\frac{1}{J_{s,LR}} (C_{s,LR})_{\phi\psi} = D_{LR}^{(0)2} + C_{LR}^{(0)} F'_{LR}^{(0)}$$
$$\frac{1}{J_{s,LR}} (C_{s,LR})_{\psi\phi} = D_{LR}^{(0)2} F'_{LR}^{(0)} + F_{LR}^{(0)} A_{LR}^{(0)}$$
$$\frac{1}{J_{s,LR}} (C_{s,LR})_{\psi\psi} = G_{LR}^{(0)2}$$
$$\frac{1}{J_{s,LR}} (C_{s,LR})_{\psi\phi} = L_{LR}^{(0)2}$$

The corresponding difference (2.3.3) relevant for the logarithmic terms in this sector is then

$$\frac{C_{s,LR}}{J_{s,LR}} - \frac{C_{u,LR}}{J_{u,LR}} = \frac{p^2(p')^2}{2(p-p')^2} (\varepsilon_+\varepsilon'_+ + (p+q)(p'-q) - (1-q^2)) \ 1 \ , \quad (2.7.17)$$

and depends only on $C^{(0)}F^{(0)}$.

Using the integrals (2.7.12) to reconstruct the one-loop S matrix we find:

$$i T_{LL}^{(1)} = i \left( \frac{1}{2} \left( -\frac{1}{\pi} \frac{p^2(p')^2}{2(p-p')^2} (\varepsilon_+\varepsilon'_+ + (p+q)(p'+q) + (1-q^2)) \right) \right) \ 1 + \text{rat.} \right), \quad (2.7.18)$$
$$i T_{LR}^{(1)} = i \left( \frac{1}{2} \left( -\frac{1}{\pi} \frac{p^2(p')^2}{2(p+p')^2} (\varepsilon_+\varepsilon'_+ + (p+q)(p'-q) - (1-q^2)) \right) \right) \ 1 + \text{rat.} \right), \quad (2.7.19)$$

i.e. only the diagonal entries are corrected. This result is in line with the expectation (2.2.13) that at one loop all logarithmic corrections are proportional to the identity matrix. Comparison with that equation identifies the coefficients of $(i/2 \ 1)$ with the one-loop dressing phases $\theta_{LL}^{(1)}$ and $\theta_{LR}^{(1)}$ in the $LL$ and $LR$ sectors.
In the limit $q \to 0$ the momentum dependence of eqs. (2.7.18) and (2.7.19) becomes the same as that of the small momentum limit of the phase factors found in [62], eqs. (B.8.22) and (B.8.23). The near-flat space limit of these phases was also found through a Feynman graph calculation in [64]. An additional factor of 1/2 relates to the fact that the phase factor of [62] receives contributions from both factors of the factorized S matrix (2.2.1). This pattern is the same as in AdS$_5 \times$S$^5$.

### 2.7.3 Two-loop logarithmic terms in AdS$_3 \times$S$^3 \times$T$^4$ with RR flux

The two-loop logarithmic terms for mixed NSNS and RR flux background can be found as soon as the symmetry-determined part of the S matrix becomes available [99]. For the $q = 0$ case the relevant S matrix was recently suggested in [56]; up to the choice of mass scale it is the same as the S matrix in AdS$_3 \times$S$^3 \times$S$^3 \times$S$^1$ which is included in Appendix B.8.

The two-loop amplitude has the general form (2.3.6) with coefficients given by (2.3.7) and (2.3.12). As for $q \neq 0$, the multi-indices are pairs (field label, sector). We discuss separately the LL and LR sectors, focusing on the S-matrix element $A^{(2)}$. The contribution of the $C$ coefficients to it will be denoted by $C_{a}^{A_{LL}}$ and $C_{b}^{A_{LR}}$.

#### 2.7.3.1 The LL and RR sectors

Expressed in terms of cuts of the one-loop amplitudes, the six coefficients $C_{a}^{A_{LL}}$ (2.3.7) are

\[
\begin{align*}
\frac{1}{j_{2}}C_{a}^{A_{LL}} & = A_{LL}^{(1)} s \text{-cut} A_{LL}^{(0)}, \\
\frac{1}{j_{2}}C_{d}^{A_{LL}} & = A_{LL}^{(1)} u \text{-cut} A_{LL}^{(0)} - \frac{1}{2} H_{LL}^{(1)} u \text{-cut} K_{LL}^{(0)} - \frac{1}{2} H_{LL}^{(0)} K_{LL}^{(1)} u \text{-cut}, \\
\frac{1}{j_{2}}C_{b}^{A_{LL}} & = A_{LL}^{(1)} u \text{-cut} A_{LL}^{(0)}, \\
\frac{1}{j_{2}}C_{c}^{A_{LL}} & = A_{LL}^{(1)} s \text{-cut} A_{LL}^{(0)} - H_{LL}^{(1)} s \text{-cut} K_{LL}^{(0)}, \\
\frac{1}{j_{2}}C_{e}^{A_{LL}} & = A_{LL}^{(1)} u \text{-cut} A_{LL}^{(0)}, \\
\frac{1}{j_{2}}C_{f}^{A_{LL}} & = A_{LL}^{(1)} s \text{-cut} A_{LL}^{(0)} - H_{LL}^{(0)} K_{LL}^{(1)} s \text{-cut}.
\end{align*}
\]

The simplest way to see this is to solve the on shell conditions and write both expressions in terms of $p_-$ and $p_-'$.
We recall that the cuts of the one-loop coefficients are given by the relevant components of the $C_s$ and $C_u$ coefficients (2.7.14). It is trivial to check that

$$C_{\text{in}}^{\alpha LL} \propto -2C_a^{\alpha LL} + C_b^{\alpha LL} + C_c^{\alpha LL} - 2C_d^{\alpha LL} + C_e^{\alpha LL} + C_f^{\alpha LL} = 0 , \quad (2.7.21)$$

which implies that the double logarithms cancel out of the $A_{LL}^{(2)}$. The same holds for all other components of the two-loop S matrix.

We note here that eqs. (2.7.20) hold for $q \neq 0$ as well; the resulting two-loop integral coefficients $C_a \ldots C_f$ also obey the relation (2.7.21), guaranteeing the absence of double logarithms in the dressing phases of the S matrix in the mixed NSNS/RR-flux background as well.

The single-log terms in $A_{LL}$ have the same form in terms of the $C$ coefficients as given in eq. (2.3.16). The remaining coefficients $C_{s,\text{extra}}$ and $C_{u,\text{extra}}$, obtained by matching the single two-particle cuts of the ansatz onto the single two-particle cuts of the four-point amplitudes, are given by (2.3.12)

$$\frac{C_{s,\text{extra}}^{\alpha LL}}{J} = -iA_{LL}^{(0)} \frac{C_s^{\alpha LL}}{J} - \frac{1}{2} \left( \frac{C_b^{\alpha LL}}{J^2} + \frac{C_c^{\alpha LL}}{J^2} \right) J_u I_t + 2A_{LL}^{(0)} \tilde{A}_{LL}^{(1)} ,$$

$$\frac{C_{u,\text{extra}}^{\alpha LL}}{J} = -iA_{LL}^{(0)} \frac{C_u^{\alpha LL}}{J} + \frac{1}{2} \left( \frac{C_b^{\alpha LL}}{J^2} + \frac{C_c^{\alpha LL}}{J^2} \right) J_u I_t$$

$$+ 2A_{LL}^{(0)} \tilde{A}_{LL}^{(1)} - H_{LL}^{(0)} \tilde{K}_{LL}^{(1)} - K_{LL}^{(0)} \tilde{H}_{LL}^{(1)} , \quad (2.7.22)$$

with the coefficients with tilde being the entries of the rational part, $\tilde{T}^{(1)}$, of the one-loop S matrix introduced in eq. (2.3.5). The difference of the $C_{\text{extra}}$ coefficients that enters eq. (2.3.16) is then

$$\frac{C_s^{\alpha LL}}{J} - \frac{C_u^{\alpha LL}}{J} = \frac{1}{2} \left( \frac{C_e^{\alpha LL}}{J^2} + \frac{C_f^{\alpha LL}}{J^2} - \frac{C_b^{\alpha LL}}{J^2} - \frac{C_c^{\alpha LL}}{J^2} \right) J_u I_t \quad (2.7.23)$$

$$- \frac{1}{2} H_{LL}^{(0)} (2\tilde{K}_{LL}^{(1)} - (i)^2 K_{LL}^{(1)} s\text{-cut})$$

$$- \frac{1}{2} K_{LL}^{(0)} (2\tilde{H}_{LL}^{(1)} - (i)^2 H_{LL}^{(1)} s\text{-cut}) ,$$

where we used $J_s = J_u$. We notice the cancellation of the diagonal rational term $\tilde{A}_{LL}^{(1)}$ which carries the only dependence on the undetermined rational function $\Phi$. 
The relevant one-loop rational terms are\(^{36}\)

\[
\tilde{H}^{(1)}_{LL} = \hat{H}^{(1)}_{LL} = \frac{i}{2} \left( A^{(0)}_{LL} + D^{(0)}_{LL} \right) H^{(0)}_{LL} + \frac{i}{4} (1 + 4b) (p - p') H^{(0)}_{LL}, \tag{2.7.24}
\]

\[
\tilde{K}^{(1)}_{LL} = \hat{K}^{(1)}_{LL} = \frac{i}{2} \left( A^{(0)}_{LL} + D^{(0)}_{LL} \right) K^{(0)}_{LL} - \frac{i}{4} (1 + 4b) (p - p') K^{(0)}_{LL}. \tag{2.7.25}
\]

Together with the one-loop coefficients \(H^{(1)}\) and \(K^{(1)}\) they imply that the terms on the second line of (2.3.16) vanish identically and that the only contribution to the single-log terms in \(A^{(2)}_{LL}\) comes from the first line of that equation:

\[
-4\pi i C^{A_{LL}}_{in} = \frac{1}{f^2} \left( 2C^{A_{LL}}_a - C^{A_{LL}}_b - C^{A_{LL}}_c \right) = -2(i A^{(0)}_{LL}) \left( \frac{C^{A_{LL}}_s}{J_s} - \frac{C^{A_{LL}}_u}{J_u} \right); \tag{2.7.26}
\]

the expression inside the parentheses can be read off of eq. (2.7.15) in the limit \(q \to 0\).

Repeating the calculation for other entries of the two-loop S-matrix\(^ {27}\) we find similar results except that \((i A^{(0)}_{LL})\) is replaced with the tree-level value of that entry and the parenthesis is replaced with some combination of the entries of \(C_s/J_s - C_u/J_u\) whose value equals that of \(C^{A_{LL}}_s/J_s - C^{A_{LL}}_u/J_u\). Thus, the two-loop S matrix in the LL sector is

\[
2.7.3.2 \quad \text{The } LR \text{ and } RL \text{ sectors}
\]

The construction of the two-loop S matrix in the \(LR\) sector is very similar except for the specifics related to the tree-level S matrix in this sector. The six coefficients

\(^{26}\)We notice here that, similarly to the AdS\(_5 \times S^5\) S matrix, the one-loop off-diagonal rational terms are proportional to the tree-level S-matrix elements.

\(^{27}\)Both here and in the \(LR\) sector, the identity

\[
\hat{A}^{(1)} + \hat{D}^{(1)} - \hat{G}^{(1)} - \hat{L}^{(1)} + \frac{1}{2} A^{(1)}_{s-cut} + \frac{1}{2} D^{(1)}_{s-cut} - \frac{1}{2} G^{(1)}_{s-cut} - \frac{1}{2} L^{(1)}_{s-cut} = 0.
\]

is necessary for finding the off-diagonal entries of the S matrix.
determined by maximal cuts are (2.3.7)

\[
\frac{1}{f} C_{a}^{ALR} = A_{LR}^{(1)} s-cut A_{LR}^{(0)} + \frac{1}{2} C_{LR}^{(0)} F_{LR}^{(1)} s-cut + \frac{1}{2} C_{LR}^{(1)} s-cut F_{LR}^{(0)},
\]

\[
\frac{1}{f} C_{d}^{ALR} = A_{LR}^{(1)} s-cut A_{LR}^{(0)},
\]

\[
\frac{1}{f} C_{b}^{ALR} = A_{LR}^{(1)} u-cut A_{LR}^{(0)} + C_{LR}^{(1)} u-cut F_{LR}^{(0)},
\]

\[
\frac{1}{f} C_{e}^{ALR} = A_{LR}^{(1)} s-cut A_{LR}^{(0)},
\]

\[
\frac{1}{f} C_{c}^{ALR} = A_{LR}^{(1)} u-cut A_{LR}^{(0)} + C_{LR}^{(0)} F_{LR}^{(1)} u-cut,
\]

\[
\frac{1}{f} C_{f}^{ALR} = A_{LR}^{(1)} s-cut A_{LR}^{(0)}.
\]

As before, the cuts of the one-loop S-matrix elements are given in terms of the appropriate components of \(C_s\) and \(C_a\). It is easy to check that

\[
C_{ln}^{ALR} = -2C_{a}^{ALR} + C_{b}^{ALR} + C_{c}^{ALR} - 2C_{d}^{ALR} + C_{e}^{ALR} + C_{f}^{ALR} = 0
\]

and thus the double logarithms cancel out. The same holds for all other components of the S matrix.

As in the \(LL\) sector, the integral coefficients \(C_a \ldots C_f\) can also be found in a mixed RR/NSNS background and they obey the condition (2.7.29). Thus, double logarithms are also absent from the two-loop S matrix in this more general case.

The remaining two coefficients, \(C_{s,\text{extra}}^{ALR}\) and \(C_{u,\text{extra}}^{ALR}\), follow from (2.3.12):

\[
\frac{C_{s,\text{extra}}^{ALR}}{J} = -i A_{LR}^{(0)} C_{s}^{ALR} J - \frac{1}{2} i C_{s}^{(0)} C_{LR}^{F_{LR}^{(1)}} J - \frac{1}{2} i F_{LR}^{(0)} C_{s}^{(0)} C_{LR}^{C_{LR}^{(1)}} J - \frac{1}{2} \left( \frac{C_{b}^{ALR}}{J} + \frac{C_{c}^{ALR}}{J} \right) I_t
\]

\[
(2.7.30)
\]

\[
+ 2A_{LR}^{(0)} \tilde{A}_{LR}^{(1)} + C_{LR}^{(0)} \tilde{F}_{LR}^{(1)} + C_{LR}^{(1)} F_{LR}^{(0)}
\]

\[
\frac{C_{u,\text{extra}}^{ALR}}{J} = -i A_{LR}^{(0)} C_{u}^{ALR} J - \frac{1}{2} \left( \frac{C_{c}^{ALR}}{J} + \frac{C_{f}^{ALR}}{J} \right) I_t + 2A_{LR}^{(0)} \tilde{A}_{LR}^{(1)}.
\]

In their difference, which enters eq. (2.3.16), we notice again the cancellation of the diagonal one-loop rational term \(\tilde{A}_{LR}^{(1)}\), which guarantees that all the single-logarithms are independent of the rational part \(\Phi\) of the dressing phase, see eq (2.3.5).

With the off-diagonal rational terms extracted from the symmetry-determined
one-loop S matrix\(^{28}\)

\[
\tilde{C}_{LR}^{(1)} = \hat{C}_{LR}^{(1)} = \frac{i}{2} \left( A^{(0)}_{LR} + D^{(0)}_{LR} \right) C^{(0)}_{LR} + \frac{i}{4} (1 + 4b) (p + p') C^{(0)}_{LR}, \quad (2.7.32)
\]

\[
\tilde{F}_{LR}^{(1)} = \hat{F}_{LR}^{(1)} = \frac{i}{2} \left( A^{(0)}_{LR} + D^{(0)}_{LR} \right) F^{(0)}_{LR} - \frac{i}{4} (1 + 4b) (p + p') F^{(0)}_{LR}, \quad (2.7.33)
\]

we find that the second line on the right-hand side of eq. (2.3.16) vanishes identically and that

\[
-4\pi i C_{\text{in}}^{\Lambda_{LR}} = \frac{1}{2 J^2} \left( 2 C_{\alpha}^{A_{LR}} - C_{b}^{A_{LR}} - C_{c}^{A_{LR}} \right) = -2 (i A_{LR}^{(0)}) \left( \frac{C_{\alpha}^{A_{LR}}}{J_s} - \frac{C_{b}^{A_{LR}}}{J_u} \right) \quad (2.7.34)
\]

The value of the parenthesis can be read off eq. (2.7.17) in the limit \(q \to 0\).

Repeating the calculation for other entries of the two-loop S matrix we find similar results upon using all relations between the two-loop integral coefficients. Thus, the two-loop S matrix in the \(LR\) sector is

\[
i T_{LR}^{(2)} = -\frac{1}{2} T^{(0)} \theta_{LR}^{(1)} + \text{rational} \quad (2.7.35)
\]

This result, as well as eq. (2.7.27), supports the exponentiation of the one-loop dressing phase in all sectors of the theory and thus provide support for the quantum integrability of the theory through two loops. The \(RL\) sector S matrix is completely identical.

### 2.8 Concluding remarks

In this chapter we discussed the calculation of the logarithmic terms in the S-matrices of two-dimensional integrable quantum field theories using the generalized unitarity method and its refinements. The calculation of unitarity cuts was carried out in two dimensions and thus it potentially drops terms with completely rational dependence on external momenta. By supplying the off-diagonal rational lower-loop terms determined by symmetries we can recover all the higher-loop logarithms. We illustrated this approach with one- and two-loop calculations in worldsheet

\(^{28}\)As in the \(LL\) sector and in \(\text{AdS}_5 \times S^5\), the rational terms are proportional to the tree-level S-matrix elements.
theories relevant to gauge/string dualities – string theory in \( \text{AdS}_5 \times S^5 \), \( \text{AdS}_4 \times \text{CP}^3 \), \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) and \( \text{AdS}_3 \times S^3 \times T^4 \).

Using this approach we successfully recovered the known logarithmic terms in the \( \text{AdS}_5 \times S^5 \) S matrix and thus provided evidence that, in this approach, the structure of the S matrix is that implied by integrability. We also computed the logarithmic terms of the one-loop S matrix for strings in \( \text{AdS}_3 \times S^3 \times S^3 \times S^1 \) for a general value of \( \alpha \) and reproduced the dressing phase obtained by matching the one-loop energy calculation of semiclassical states with the predictions of the Asymptotic Bethe Ansatz. Our result supports the assumption that the scattering of the \( m = 1 \) and \( m = 0 \) states off states with \( m = \alpha \) and \( m = 1 - \alpha \) is reflectionless.\(^{29}\) We also discussed string theory in \( \text{AdS}_3 \times S^3 \times T^4 \) sourced by a mixture of RR and NSNS fluxes and found the logarithmic terms in the dressing phase. In the limit of vanishing NSNS flux and in the near-flat space limit we recovered the result of [64]. For vanishing NSNS flux we have also computed the logarithmic terms in the two-loop S matrix and showed that they all come from the exponentiation of the one loop dressing phases. For the mixed case one can see that the double logarithms cancel out; it should not be difficult to test this structure once a symmetry-determined S matrix becomes available.

The calculations described in this chapter can be extended to higher loops. While for strings in \( \text{AdS}_5 \times S^5 \) all necessary information is available, potential subtleties arise for other backgrounds. We argued in sec. 2.2 that through two loops, unitarity-based quantum calculations in subsectors of theories with reflectionless S-matrices capture all logarithmic terms. A more detailed analysis is necessary to ascertain whether this continues to hold at higher loops. An important part will likely be played by the structure of higher-loop integrals. Absence of additional logarithms in the dressing phase at two loops suggests that the pole structure of the S matrix is the expected one through this order and that, to this order, the spectrum of bound states is the known one. Higher-loop calculations would provide further tests in this direction.

In our calculations we relied on symmetries to determine the off-diagonal rational parts of one-loop S-matrix elements. One may continue to do so at higher

\(^{29}\)It would be interesting to check whether similarly with strings in \( \text{AdS}_4 \times \text{CP}^3 \) [89], the \( m = 1 \) states can be thought of as bound states of lighter states.
loops as well. It would of course be desirable to have an independent derivation of both diagonal and off-diagonal rational terms, the former being unconstrained by symmetries. We expect that they can be found through use of dimensional regularization, albeit in that case one would need to supply local counterterms to restore \textit{e.g.} integrability and perhaps also other symmetries.

A direct determination of all rational terms would presumably clarify the interplay between regularization and symmetries in the Green-Schwarz string and it may provide a means to discuss from an S matrix perspective the existence of integrability anomalies such as those in the bosonic $\text{CP}^{n-1}$ model [100]. It would also be an important ingredient in the unitarity-based construction of other interesting worldsheet quantities, such as worldsheet form factors [101] or correlation functions of operators, perhaps along the lines of [102]. Similar methods can be used to construct the two-point off-shell Green’s function of worldsheet fields and thus find the loop corrections to the dispersion relation as well as the corrected propagator residue needed for the determination of complete higher-loop scattering matrices.
Chapter 3

A Non-Perturbative Approach

This chapter is based on unpublished work.

3.1 Introduction

The AdS/CFT Correspondence is a conjecture that a string theory in an Anti-de-Sitter spacetime (AdS) background is partner to a quantum field theory (QFT) space with conformal spacetime symmetry. The conformally flat d-dimensional spacetime is formed on the boundary of $AdS_{d+1}$. What this means is that every observable in the CFT$_d$ will have a corresponding observable in the AdS$_{d+1}$. Most attempts (and ours) in testing this duality have focused on the CFT - N=4 and the string - IIB superstring theory on the AdS$_5 \times S^5$ background.

To prove the conjectured duality of AdS/CFT [8] one needs to find a relationship between the spectrum of quantum string energies of AdS$_5 \times S^5$ space and N=4 SYM dilatation operator. There have been many attempts at solving for these spectra. The discovered integrability of both IIB String theory on an AdS$_5 \times S^5$ background and N=4 SYM have led to new approaches to this field of research.

The GS superstring on AdS$_5 \times S^5$ space is very closely related to a nonlinear sigma model. The fields of this string live in the coset superspace $PSU(2,2|4)/SO(4,1)\times SO(5)$. The theory differs from a nonlinear sigma model from the $\kappa$ gauge symmetry and Wess-Zumino term. It is known that the Bosonic part of the space is integrable (gives an infinite symmetry algebra).
The Green Schwarz (GS) string is a highly complex theory that is difficult to directly quantize. Due to this it is very difficult to directly prove its duality with N=4 SYM. Because of its complexity most work in dealing with the GS string involves starting from expanding around some classical solution, such as string rotating in AdS5 and boosted along in S5. From this solution one then does a semi-classical quantization, which is to compute the 1 loop correction to the classical solution.[29]

Another approach to solving the GS string is the Asymptotic Bethe Ansatz (ABA)[4]. ABA is the ordinary Bethe procedure, converting a spectral problem into a system of non-linear equations to be solved, but in the asymptotic (infintely-long range) limit. In this limit finite-size effects are neglected. This procedure does not work for general states but only on states with large at least one large charge. This procedure starts by using a gauge fixed theory with $PSU(2, 2)^2$. A spring chain of $SU(1, 1|2)$ is then used to derive the nested bethe equations. It is the wish to then extend this to $PSU(2, 2|4)$. Unfortunately, this procedure does not find the full S-Matrix for $PSU(2, 2|4)$. This is because the S-Matrix does not inherently have the $PSU(2, 2|4)$ symmetry, it is spontaneously broken by the initial gauge condition.

The Asymptotic Bethe Ansatz is useful in finding a spectrum for single trace operators containing a large number (asymptotic) of fields. But for a small number of fields there exists another method. From numerical methods using short operators, we have predictions of the Konishi operator, $\sum_i Tr[\bar{\phi}_i \phi_i]$, in planar N=4 SYM, using the Y-system[103]. This spectrum should be consistent with the quantum energy spectrum of GS string in AdS$_5 \times$S$^5$.

The Pohlmeyer reduction method [104] is yet another approach to solving the GS string. This approach starts with trying to find a two-dimensional Lorentz-invariant reformulation of the AdS$_5 \times$S$^5$superstring. To do this one first performs a non-local transformation of variables from coordinates to currents which preserves the Virasoro constraints and integrability classically. This is done by solving the classical equations of motion. The hope is that the classical correspondence between the two models extends to the quantum level.

The approach we try is to start with the transformation of the Green Schwarz Superstring into a super chiral Gross-Neveu Model. This will be done using tech-
niques similar to Polyakov and Wiegmann [105],[106]. Then we will show the UV finiteness and integrability of the reduced model. This approach has many good features. It preserves Lorentz invariance and expects to be equivalent on a quantum level. Also with the reduced model all fields will have same regular kinetic terms.

3.2 Transformation

In this section we will discuss a toy model, the principal chiral model, and the transformation procedure to find the equivalent Gross-Neveu model.

3.2.1 Gross-Neveu

The Gross-Neveu model historically has been a toy model for studying quantum chromodynamics. The model is a two dimensional quantum field theory of Dirac fermions interacting through a quartic term. The model consists of N Dirac fermions $\psi_1,..,\psi_N$ with the Lagrangian:

$$L = \bar{\psi}_{a}(i\slashed{\partial} - m)\psi_{a} + g^2 \frac{N}{2} (\bar{\psi}_{a}\psi_{a})^2$$  \hspace{1cm} (3.2.1)

3.2.2 Principal chiral model

The action for the principal chiral model (PCM) is:

$$L_{\text{PCM}} = \eta^{\mu\nu}\text{Tr}(J_{\mu}J_{\nu}) \quad J_{\mu} = g^{-1}\partial_{\mu}g \quad g \in G$$  \hspace{1cm} (3.2.2)

for some group $G$.

This Lagrangian may be argued [105, 107] to be equivalent to a fermionic Lagrangian with infinitely many flavors. We start with the action

$$L = i\bar{\psi}^{j}\gamma^{\mu}(\partial_{\mu} + gA_{\mu})\psi_{j} + \text{Tr}(A^\mu A_\mu)$$  \hspace{1cm} (3.2.3)

with $A$ in the Lie algebra of some group $G$. $\psi^{i}$ with $i = 1,\ldots,N_f$ are fermions transforming in the defining representation of $G$. Integrating out $\psi$ in the path
integral leads to the effective action

$$S_{\text{eff}} = N_f \ln \det(\gamma^\mu (\partial_\mu 1 + gA_\mu)) + \int d^2 \sigma \text{Tr}(A^\mu A_\mu)$$  \hspace{1cm} (3.2.4)

$$= N_f \text{Tr} \ln(\gamma^\mu (\partial_\mu 1 + gA_\mu)) + \int d^2 \sigma \text{Tr}(A^\mu A_\mu)$$  \hspace{1cm} (3.2.5)

The trace in the first term contains implicitly an integration over all positions. This strategy was used in [105, 107] to construct S-matrices for the PCM with symmetry groups given by all ADE (SU(r+1), SO(2r), E_6, E_7, E_8) Lie groups and in [106] to propose a solution of the $O(3) = SU(2)/U(1)$ non-linear sigma model.

The determinant (or the Tr ln) may be computed exactly. It is however not completely necessary to do so. Rather, it suffices to notice that the variation of the effective action with respect to the vector field $A$ is

$$\frac{\delta W}{\delta A} \equiv \frac{\delta}{\delta A} \ln \det(\gamma^\mu (\partial_\mu 1 + gA_\mu)) = J^1$$  \hspace{1cm} (3.2.6)

the quantum version of the current for which $A$ is gauge field if we wanted to gauge that symmetry. Classically this current is:

$$J_\mu^a = i \bar{\psi} \gamma_\mu T^a \psi; \hspace{1cm} (3.2.7)$$

since 2d fermions are chiral, there is also an axial version of this current $- J_{\mu,\text{axial}} = \epsilon_{\mu\nu} J_\nu$. Classically they are both conserved; it is possible to choose a regularization such that the vector current remains conserved quantum mechanically, i.e.

$$\partial^\mu J_\mu + g[A^\mu, J_\mu] = 0$$  \hspace{1cm} (3.2.8)

With this choice, the axial current is no longer conserved, but rather has an anomaly arising from a simple bubble diagram:

$$\epsilon^{\mu\nu} (\partial_\mu J_\nu + g[A_\mu, J_\nu]) \propto \epsilon^{\mu\nu} F_{\mu\nu}$$  \hspace{1cm} (3.2.9)

where $F$ is the field strength of $A - F = dA + A \wedge A$. This is the so-called “anomaly equation” and its right-hand side of this equation is the chiral anomaly of this model. The conservation and the anomaly equation determine, in principle $J$ in
terms of $F$ and its covariant derivatives. In the large $N_f$ limit, any nonzero value of $J = \delta W/\delta A$ will lead to a very suppressed path integral. Thus, we would like to have $J = 0$ or, equivalently,

$$ F = 0 \quad (3.2.10) $$

That is, $A$ should be pure gauge, i.e.

$$ A = g^{-1} dg \quad (3.2.11) $$

where $g$ is an element of $PSU(2, 2|4)$. Plugging this back in (3.2.5) and making use of the fact that if $A$ is pure gauge then $W$ is independent of it, we find (following Polyakov and Wiegman) that

$$ S_{eff} = \eta^{\mu\nu} \text{Tr}[(g^{-1} \partial_\mu g)(g^{-1} \partial_\nu g)] \quad (3.2.12) $$

Integrating out $A$ in (3.2.3) gives (here $g$ is the coupling constant and should not be confused with $g$ in eqs 3.2.11 and 3.2.12 where it stands for an element of $PSU(2, 2|4)$)

$$ \mathcal{L}_{GN} = i \bar{\psi}^j \gamma^\mu \partial_\mu \psi_j + g^2 \bar{\psi}^j \gamma^\mu T^a \psi_j \bar{\psi}^k \gamma_\mu T_{a\alpha} \psi_k \quad (3.2.13) $$

Since both eq. (3.2.2) and the one above are obtained from the same action (3.2.3) by integrating out different fields, it follows that they should be equivalent (though not order by order in perturbation theory)

### 3.2.3 GS model

Now that we have done the toy model let us do the same to the GS model in $AdS_5 \times S^5$:

$$ \mathcal{L}_{GS} = \frac{1}{2} h^{\mu\nu} (L^a_\mu L^b_\nu) g_{ab} + \frac{1}{2} \epsilon^{\mu\nu} (L^a_\mu L^b_\nu) g_{a\dot{b}} \quad (3.2.14) $$

Here $L$ are components of the left-invariant 1-form on $PSU(2, 2|4)$. This group has a $\mathbb{Z}_4$ automorphism under which the left-invariant 1-form $J = g^{-1} dg$ breaks
up in four pieces with eigenvalues $\pm 1$ and $\pm i$. The part with eigenvalue $+1$ is the left-invariant 1-form on $SO(4,1) \times SO(5)$; $L^a$ are the components with eigenvalue $-1$ and $L^\alpha$ and $L^{\hat{\beta}}$ are the components with eigenvalues $\pm i$, respectively. Thus, the action above has fields in the coset $PSU(2,2|4)/SO(4,1) \times SO(5)$

We will repeat the same procedure as before, with the only difference that now the term that allows us to integrate out $A$ will not contain all components of $A$. The components that are missing act as auxiliary fields imposing the constraint that certain components of the symmetry generators annihilate the states.

More precisely:

$$\mathcal{L} = i\bar{\psi}^j \gamma^\mu (\partial_\mu + g(A^{(1)}_\mu + A^{(i)}_\mu + A^{(-1)}_\mu + A^{(-i)}_\mu)) \psi_j$$

$$+ h^{\mu\nu} \text{STr}(A^a_\mu A^b_\nu) g_{ab} + \epsilon^{\mu\nu} \text{STr}(A^\alpha_\mu A^{\hat{\beta}}_\nu) g_{\alpha\hat{\beta}}$$

(3.2.15)

Here $h^{\mu\nu}$ is the bosonic metric and $\epsilon^{\mu\nu}$ is the fermionic. $\psi^j$ with $j = 1, \ldots, N_f$ are fields in the fundamental representation of $PSU(2,2|4)$, they are fields corresponding to the oscillators in the oscillator construction of representations of this group. Each one of them has 8 components, the first 4 are complex chiral bosons and the last 4 are complex fermions, similar to those of a generic bosonic group in the previous section. $A^{(1)}_\mu, A^{(i)}_\mu, A^{(-1)}_\mu, and A^{(-i)}_\mu$ are the components of $A$ with definite charge under the $\mathbb{Z}_4$ automorphism of $PSU(2,2|4)$.

It has been shown in [108] that the free theory of this supertwistor construction reproduces the states of the string theory in AdS$_5 \times$S$^5$. The interactions in this free theory come from the vector fields.

This supertwistor model has a complete oscillator construction of the unitary irreducible representations of $SU(2,2|4)$. From this fock space it is possible to reproduce states that correspond to particular theories. They include the states of d=4 SYM theory, massless and KK states of AdS$_5$ supergravity, and the descendants on AdS$_5$ of the standard massive string states, which form intermediate and long supermultiplets,[108] 2

Integrating out the $\psi$-s and taking $N_f \to \infty$ will force again $A = A^{(1)} + A^{(i)} + A^{(-1)} + A^{(-i)}$ to be pure gauge; replacing this in the effective action yields

2For example $Z=0, P=1$ doubleton contain the supergravity states in AdS$_5 \times$S$^5$ or, equivalently the protected operators in N=4 SYM theory.
the GS action (3.2.14).

After integrating out \( A^{(i)} \), \( A^{(-1)} \) and \( A^{(-i)} \) we are left with:

\[
\mathcal{L} = i \bar{\psi}^j \gamma^\mu (\partial_\mu + g A^{(1)}_\mu) \psi_j + h^{\mu\nu} \bar{\psi}^j \gamma_\mu T^a \psi_k \gamma_\nu T^b \psi_j g_{ab} + \epsilon^{\mu\nu} \bar{\psi}^j \gamma_\mu T^a \psi_k \gamma_\nu T^b \psi_j g_{a\beta} \tag{3.2.16}
\]

Even though in (3.2.15) all indices are constructed with some arbitrary worldsheet metric, in (3.2.17), this arbitrary metric cancels out in all terms on the second line. It survives only in the first term, which may be used to define a Virasoro constraint.

The component \( A^{(1)} \) remains as a Lagrange multiplier, which sets to zero the \( SO(4,1) \times SO(5) \) current. This is a reflection of the fact that in the coset construction the \( A^{(1)} \) current is unphysical. Specifically in this approach we keep all of the groups/states in the numerator and set all of the states belonging to the denominator of the coset to zero.

An interesting thing to note about this Lagrangian is that both the fermions and bosons have a linear kinetic term. One would assume that this quadratic boson term could then be ignored by rewriting as a total derivative. But the bosons are complex so the this quadratic term is not a total derivative.

### 3.3 UV finiteness

Constructing the Feynman propagators for this lagrangian in the usual manner we get \( \frac{1}{(p^2 + i\epsilon)} \) for the bosons and fermions and just a constant 1 propagator for the gauge fields \( A \). We will treat all the \( A \)'s the same expect we will set any term with \( \text{STr}[A^{(1)} A^{(1)}] \) coefficient to be 0. There are two nice features that help to eliminate divergences in our graphs. Since the symmetry group is \( PSU(2,2|4) \) there are vanishing dual Coxeter number \( (f^{AB} c f_{ABD} = 0) \) which will eliminate some of the divergences. While being in two dimensions is the other. In two dimensions a trace of gamma matrices will come with a factor of \( (D - 2) \) which will help regularize our divergent graphs.
### 3.3.1 Two Point Function Renormalization

The one loop correction to the bosonic and fermionic propagators are the following graphs.

![Diagram 3.1](image1.png)

Figure 3.1: The one loop two point graph with a loop over bosonic/fermions fields

The integral for this graph 3.1 is:

\[
\langle \psi_\mu(p)\psi_\nu(-p) \rangle = \int_{-\infty}^{\infty} d^n p\gamma^\mu \frac{\phi}{(p^2 + i\epsilon)} \gamma_\mu T^a T_a
\]

(3.3.1)

This integral is trivially zero due to it being an odd function.

![Diagram 3.2](image2.png)

Figure 3.2: The one loop two point graph with a gauge field loop, the dashed line is our gauge field propagator.

This graph 3.2 has a very similar integral is also zero because it is an odd function.

Classically the gauge field, \( A^{(1)} \), is non-propagating. The correction to the gauge field propagator is the following graph.

![Diagram 3.3](image3.png)

Figure 3.3: The two point bubble graph for the gauge field propagator.
\[
\langle A_\mu(p)A_\nu(-p) \rangle = \int_{-\infty}^{\infty} d^np\text{Tr}\left[ \gamma_\mu \frac{\not{p}}{(p^2 + i\epsilon)}\gamma_\nu \frac{(\not{p} - \not{k})}{((p - k)^2 + i\epsilon)} \right] 
\] (3.3.2)

This graph is UV finite only in the special case of two dimensions due to the gamma matrix relations.

\[
\gamma_\mu \gamma_\rho \gamma_\nu \gamma_\rho = (D - 2)\gamma_\mu \gamma_\nu
\] (3.3.3)

From this we see that all of the 2-point functions are finite at 1-loop.

### 3.3.2 One Loop Four Point Function

There are many Feynman diagrams at the one loop order. But from our construction our two actions are only equivalent in the limit \(N_f \to \infty\). Due to this limit only one graph will matter. The reason for this is that only one graph is non-zero and comes with a factor \(N_f\), thus will be picked out when we take our limit.

The integral for this graph will be UV finite due to dimensional regularization. The momentum part of this graph comes with a divergence that gets canceled due to the fact we are in two dimensions from the trace of the Dirac matrices.

![Image of One Loop Four Point Function](image)

Figure 3.4: One loop 4-point graph that is picked out in our limit of \(N_f \to \infty\)

The integral to this graph depends only on the bubble graph to the gauge field propagator 3.3.

The other one loop graphs that exists are shown in figure ???. As you can see none of these graphs have a loop over the fermions so they do not come with a factor of \(N_f\).
3.4 Integrability

The classical Yang-Baxter equation is:

\[(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)\]  \hspace{1cm} (3.4.1)

If \(R\) is the scattering matrix and obeys this relationship the system is said to be integrable. Another method for determining classical integrability is showing that the equations of motion can be put into a zero-curvature form. That is, if one can construct a Lax connection \(L(\sigma, \tau, z)\) where \(\sigma\) and \(\tau\) are the coordinates and \(z\) is the spectral parameter, such that \(dL + L \wedge L = 0\). From this one can define the monodromy operator as the integral of the Lax connection. Using this one can construct an infinite set of conserved charges. It has been shown that non-linear sigma models on squashed spheres maintain their integrability [109], [110]. The main importance shown is that the coefficient of the deformation does not appear in the \(R\)-matrix and it is this that preserves integrability. In our case we have the presence of a nontrivial coefficient for \(\text{STr}[A^{(1)}A^{(1)}]\) which will later set to 0.

3.4.1 Lax Connection

We attempt to find the Lax Connection that can be used to find our integrable charge. From the equations of motion \(\epsilon^{\mu\nu}\partial_\mu J_\nu + [A^\mu, J_\nu] = \epsilon^{\mu\nu}F_{\mu\nu}\) which classically equals zero and \(\partial^\mu J_\mu + [A^\mu, J_\mu] = 0\), we obtain:
Equations (3.4.2), (3.4.3) are known as the Maurer-Cartan equations. The equations of motion for $A$ give us:

\[ J_1 = A_1 \quad J_{-1} = A_{-1} \]
\[ J_i = *A_i \quad J_{-i} = *A_{-i} \]  \hspace{1cm} (3.4.4)

Using these substitutions, and the general property for differential forms in two-dimensions $k \wedge *l + *k \wedge l = 0$ we obtain the following equations from (3.4.2).

\[ dJ_1 + \lambda J_1 \wedge J_1 + J_1 \wedge \lambda J_1 + J_{-1} \wedge J_{-1} + J_{-1} \wedge J_1 = 0 \]  \hspace{1cm} (3.4.5)
\[ dJ_{-1} + J_{-1} \wedge J_1 + J_1 \wedge J_{-1} + \lambda J_1 \wedge J_{-1} + J_{-1} \wedge \lambda J_1 = 0 \]  \hspace{1cm} (3.4.6)
\[ dJ_i + \lambda J_1 \wedge J_i + J_i \wedge \lambda J_1 + *J_i \wedge J_1 + J_1 \wedge *J_i 
     + J_{-1} \wedge J_{-1} + J_{-1} \wedge J_{-1} + *J_{-i} \wedge J_{-1} + J_{-1} \wedge *J_{-i} = 0 \]  \hspace{1cm} (3.4.7)
\[ dJ_{-i} + \lambda J_1 \wedge J_{-i} + J_{-i} \wedge \lambda J_1 + J_{-1} \wedge J_i + J_i \wedge J_{-1} 
     + *J_i \wedge J_{-1} + J_{-1} \wedge *J_i + *J_{-i} \wedge J_{-1} + J_{-1} \wedge *J_{-i} = 0 \]  \hspace{1cm} (3.4.8)

Using the same substitutions and properties, we obtain the following equations from (3.4.3).

\[ d*J_1 - 2(J_i \wedge J_{-i} + J_{-i} \wedge J_i) = 0 \]  \hspace{1cm} (3.4.9)
\[ d*J_{-1} - 2(J_i \wedge J_{i} + J_{-i} \wedge J_{-i}) + (\lambda - 1)(J_1 \wedge *J_{-1} + *J_{-1} \wedge J_1) = 0 \]  \hspace{1cm} (3.4.10)
\[ d*J_i - (J_i \wedge J_1 + J_1 \wedge J_i) + \lambda(J_1 \wedge *J_i + *J_i \wedge J_1) - (J_{-1} \wedge J_{-1} 
     + J_{-1} \wedge J_{-1}) = 0 \]  \hspace{1cm} (3.4.11)
\[ d*J_{-i} + \lambda(J_1 \wedge *J_{-i} + *J_{-i} \wedge J_1) + J_{-1} \wedge *J_i + *J_i \wedge J_{-1} \]
\[ -(J_i \wedge J_{-1} + J_{-1} \wedge J_i) - (J_{-i} \wedge J_1 + J_1 \wedge J_{-i}) = 0 \] \hspace{1cm} (3.4.12)

The ansatz for the Lax connection is:

\[ L = a_0 J_1 + a_1 J_i + a_2 J_{-1} + a_3 J_{-i} + b_0 \ast J_1 + b_1 \ast J_i + b_2 \ast J_{-1} + b_3 \ast J_{-i} \] \hspace{1cm} (3.4.13)

We check the for the flatness of L and find: \( a_0 = \frac{a_2^2}{2} \) and all other coefficients are 0. This solution is unique to this reduced model. This procedure has been done for the GS string and found to have all of the components of the symmetry current present. In the GS case it does not appear possible to eliminate the other currents due to the Mauer-Cartan equations. The Maurer-Cartan equations enforce the grade \((\pm i)\) components are in the Lax operator, so this solution does not appear in the regular GS case.

### 3.5 Discussion

We have shown in this chapter the transformation of our original model into a new super chiral Gross-Neveu model. After this transformation we have shown the theory to be UV finite at 1 loop and that it exhibits some non-local integrals of motion. It appears from this method we may only be able to access a subset of the total integrals. In this GN model only the bosonic integrals of motion are visible.

We had also attempted a nested Bethe ansatz brute force method to construct the S-Matrix and directly check this in the Yang-Baxter equation. Unfortunately, we were not able to directly solve for a S-matrix that satisfied this integrability equation without the fermionic fields. Interestingly enough the problem terms in the S-Matrix in the brute force method are removed in this algebraic method by setting those currents to 0 from the Maurer-Cartan equations. This might be further proof that we can only access a specific region of the full GS model with this transformation.
Chapter 4

A discrete approach to the Green-Schwarz string in AdS$_5 \times S^5$

This chapter is based on [5] written in collaboration with R. Roiban.

4.1 Introduction

Superstring theory in AdS$_5 \times S^5$ is described by a complicated interacting two-dimensional field theory of Green-Schwarz type which is expected to be conformally invariant; solving it exactly appears to be a difficult problem. From a conformal field theory perspective one is interested in finding vertex operators labeled by PSU(2,$|4$) quantum numbers $\hat{C} = (E; S_1, S_2; J_1, J_2, J_3) \equiv (E; C)$, their two-dimensional anomalous dimensions $h(\sqrt{\lambda}, \hat{C})$ and their correlation functions. The marginality condition, identifying the physical states of the corresponding string theory, determines the energy of the state – or the dimension of the dual $\mathcal{N} = 4$ super-Yang-Mills (sYM) theory operator – in terms of its charges and the 't Hooft coupling, $E = \Delta = E(\sqrt{\lambda}, C)$. In the static gauge the string energy $E$ has the more direct worldsheet interpretation as the worldsheet energy.

The complete worldsheet theory is classically integrable [2]; together with the integrability of the planar dilatation operator of the dual gauge theory this suggests that the worldsheet theory is integrable at finite values of the coupling. $^1$ Assuming

$^1$Classical integrability was also shown in the pure spinor formalism in [111]; in [112] it was argued that the theory is also quantum-integrable.
all-orders integrability Asymptotic Bethe Equations (ABA) [65], Thermodynamic
Bethe Ansatz (TBA) equations [113] and TBA equations in $Y$ variables [114] have
been formulated for the spectrum of target space energies of long and general string
states, respectively.

A remarkable feature of the worldsheet fluctuations around the BMN vacuum
is that their exact dispersion relation

$$\epsilon^2 = 1 + \frac{\lambda}{\pi^2} \sin^2 \frac{\pi p_{ws}}{\sqrt{\lambda}}$$ (4.1.1)

bears a distinct similarity with that of a scalar particle on a space with a discrete
space-like direction (with spacing $a = 2\pi/\sqrt{\lambda} = 1/2g$) with the important differ-
ence that $a$ priori the momentum $p_{ws}$ is not discrete. While this structure is not
immediately manifest on the worldsheet and its consistency has been checked only
in certain limits [115, 85, 116], it can be derived from various perspectives in the
dual gauge theory [20, 21, 87].

The integrability-based predictions for the energies of certain long strings and
long operators (i.e. strings/operators carrying at least one large quantum number)
have been tested through four loops at weak gauge theory coupling [117, 118] and
through two loops at strong coupling (weak worldsheet coupling expansion) [84, 78].
For short strings and short operators, the results of TBA/Y-system equations
[119, 103] have been confirmed through five loops at weak gauge theory coupling
[120] and through one loop at strong coupling [121, 122, 123]. While extensive and
quite suggestive that integrability does indeed hold to all orders, such tests cannot
definitively answer questions like:

1. Is it the target space energy operator that is diagonalized by the Bethe ansatz
   or is it another operator that differs from it at finite coupling or at sufficiently
   high order in weak/strong coupling perturbation theory?

2. Strong coupling perturbation theory is at best an asymptotic series which
   in certain cases [36] is known not to be Borel summable; is it possible to
   recover the weak gauge theory coupling results for anomalous dimensions of
   local gauge-invariant operators from worldsheet calculations?

3. The worldsheet theory should be a finite quantum field theory; how sensitive
are the results obtained in this theory on the regularization scheme?

4. How can one find the target space energy of short string states that cannot be (formally) described as limits of classical string solutions but rather have only a description in terms of vertex operators (*e.g.* chargeless states)?

Our aim in the present chapter is to initiate the exploration of discrete approaches to the Green-Schwarz string in $\text{AdS}_5 \times S^5$. Using techniques of lattice field theory\(^2\) to evaluate numerically the energy of a particular long string state at finite values of the 't Hooft coupling, in this chapter we shall present evidence that the answers to the first two questions above are positive. While we shall not explicitly address the scheme dependence of the worldsheet calculations, we shall see that our results are consistent with the scheme chosen in worldsheet perturbative calculations.

We shall also describe strategies for finding the spectrum of generic string states both in this framework as well as in worldsheet perturbation theory.\(^3\)

Lattice field theory \cite{130} is used extensively to study certain finite-coupling aspects of QCD as well as of condensed matter systems that can be analyzed in Euclidean setting. One constructs a discrete action on a four-dimensional square lattice which in the continuum limit becomes the desired action and evaluates the path integral numerically though Monte Carlo techniques. The partition function is arguably the simplest quantity to evaluate. Discrete forms of operators can also be placed on the lattice and the corresponding path integral yields their correlation functions. Here we will follow this strategy, discuss possible square-lattice discretizations of the Green-Schwarz string in $\text{AdS}_5 \times S^5$ and describe calculations that yield (in principle) quantities of interest for the dual gauge theory.

The properties of the theory are both a source of simplifications and complications. On the one hand, all fields are scalars (albeit some of them are anticom-

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\(^2\)In the context of the gauge/string duality lattice field theory was used to study the Plane Wave Matrix Model \cite{124} and the 16-supercharge $0 + 1$-dimensional sYM theory \cite{125}. Two-dimensional sigma models were studies in \cite{126}.

\(^3\)Discrete approaches to the worldsheet theory have been proposed \cite{127} as a systematic means of identifying the perturbation theory of the target space effective field theory from a worldsheet standpoint. In the context of the AdS/CFT correspondence this approach has been discussed in \cite{128, 129}.
muting\(^4\)) so their discretization is to some extent straightforward. On the other, Grassmann-odd fields can appear more than quadratically. Since it is difficult to simulate numerically anticommuting fields, they are usually integrated out; here however it is necessary to first linearize the terms that are more than quadratic and this potentially leads to a proliferation of auxiliary fields.

Calculations then proceed by integrating out the Grassmann-odd fields and exponentiating the resulting determinant such that the number of propagating fields in the continuum limit is correct. Absence of anomalies makes this much more straightforward than in four dimensions.

To illustrate the discrete approach to the Green-Schwarz string in AdS\(_5\) × S\(_5\) we shall evaluate the energy of the long folded string for \(1 \leq g \leq 40\) by computing the partition function of the discretized AdS light-cone gauge action in the background of the null cusp solution and reproduce within reasonable accuracy the results of the BES equation. This is the first finite-coupling calculation in this theory using field theory methods. We shall use the standard Rational Hybrid Monte Carlo (RHMC) algorithm [133, 134, 135] for an efficient treatment of the fermion determinant contribution (R) and for the evaluation of the path integral (HMC). The numerical code runs on a GPU and uses the CUSP libraries to speed up the calculation of the fermion contribution [136].

We begin in section 4.2 with a discussion on the discretization of the various forms of the Green-Schwarz action and brief comments on Monte Carlo methods for the evaluation of path integrals while relegating more details to Appendix C.1. We also outline three possible approaches to determining the worldsheet spectrum and other interesting quantities from numerical calculations. In section 4.3 we discuss in detail the example of the calculation of the universal scaling function. We outline the construction of the discrete version of the AdS light-cone action in the null cusp background and point out the discrete derivative required for a stable fermionic contribution, illustrate the evaluation of the partition function for one value of the coupling constant, discuss the dominant source of errors in the simulation and present our results. In section 4.4 we summarize our conclusions.

\(^4\)From this perspective these fields resemble the topologically-twisted fermions proposed in [131] as a way to realize supersymmetric field theories on the lattice, see [132] for a review.
4.2 A numerical approach to the Green-Schwarz string

Lattice field theory provides a means to evaluate observables of the Green-Schwarz string in $\text{AdS}_5 \times S^5$ that are accessible on a Euclidean worldsheet, such as energies of string states or dimensions of the dual gauge-invariant operators, correlation functions of worldsheet operators, expectation values of Wilson loops, etc. In this section we discuss general features of possible approaches to the discretization of the Green-Schwarz action in $\text{AdS}_5 \times S^5$ as well as review and extend possible approaches to the calculation of target space energies of string states that may be implemented in this framework.

4.2.1 A first pass at a discrete Green-Schwarz action in $\text{AdS}_5 \times S^5$

To simulate a quantum field theory on a Euclidean lattice one begins with the continuum Euclidean action and discretizes it in a way that preserves as many of the continuum symmetries as possible. Lattice fields are assigned to either links (such as gauge fields) or to nodes (such as scalar fields). Under gauge transformations the link variables transform non-locally, depending on the beginning and end of the link, e.g. $L_{n,m} \mapsto U_n L_{n,m} U_m^\dagger$ while the node variables transform locally. Further considerations in the construction of the discrete action is the error introduced by the finite size lattice and the speed of the convergence of the continuum limit, the restoration of broken symmetries in the continuum limit, the existence of nontrivial renormalization and in particular of quantum-generated terms proportional to inverse lattice spacing which spoil the properties of the classical continuum limit, the proper treatment of fermions, the reality of the fermion contribution to the partition function (known as the fermion sign/phase problem [137]), the existence of anomalies and the solution to the fermion doubling problem etc.

Many of the issues present in standard matter-coupled gauge theories are absent in the Green-Schwarz string in curved space $^5$. For example, from the worldsheet standpoint all fields are scalars and thus they are uniformly assigned to

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$^5$This is true also in flat space, but that theory is effectively free.
lattice sites. Also, for a real bosonic background the fermionic contribution to
the partition function is real and consequently there cannot be a phase ambiguity. Moreover, since the worldsheet theory is expected to be finite (and conformal
in an appropriate gauge) one does not expect that renormalization is necessary
and thus no counterterms need to be included in the action; however, composite
operators whose correlation functions we may be interested in computing should
receive infinite normalization so they should require counterterms. Last but not
least, no two-dimensional anomalies are present and thus the doubling problem of
fields with linear quadratic terms can be resolved without resorting to the usual
four-dimensional techniques (see e.g. [138] for an introductory review).

The fact that $\kappa$ symmetry is different from a standard gauge symmetry – in that
it acts nonlinearly and does not have an independent gauge field – suggests that
it must be treated differently from usual local symmetries. A possible approach
– which we will adopt here and use in the explicit calculation in section 4.3 – is
to discretize the gauge-fixed action. Since $\kappa$ symmetry is related to worldsheet
supersymmetry such an approach potentially breaks it. One may however test
whether this is the case and, if necessary, correct for such effects.

There are several (classes of) actions that one might consider as the starting
point for the construction of a lattice action:

1. Poincaré patch conformal gauge actions are typically simpler but they require
the presence [139, 140, 13] of an extended string background. One may in
principle avoid this by choosing a light-cone type $\kappa$-symmetry gauge while
maintaining conformal gauge for two-dimensional diffeomorphisms. While
the resulting action cannot be used for perturbative calculations (unless one
further chooses a bosonic background) due to the absence of a free-fermion
quadratic term, it may nevertheless be useful for lattice calculations.

2. Actions in which all constraints are solved are an appealing starting point
for a construction of a discrete action as one needs not worry about the
restoration of the corresponding symmetries in the continuum limit. Examples are the AdS-light-cone gauge [141] and the uniform light-cone gauge

$^6$Technical difficulties may arise in the latter case due to the Nambu square-root form of the
3. One may consider discrete projective light-cone gauge actions already put forward in the literature [127, 129, 128].

The appropriate choice depends on the observable to be computed and on the computational strategy. In the next subsection we will discuss approaches to the calculation of target space energies that make use of the first two types of actions and in section 4.3 we will use a discretized AdS-light-cone action.

The discretization of the bosonic action is straightforward: since all fields are scalars they are assigned to lattice sites

$$\phi(\tau, \sigma) \mapsto \phi(m,n)$$

and their derivative in the direction of a unit two-dimensional vector $\vec{v}$ is replaced by a finite difference

$$\partial_{\vec{v}}\phi \mapsto \frac{1}{2a}(\phi(m,n)+\vec{v} - \phi(m,n)-\vec{v}) \, ,$$

where $a$ is the lattice spacing. The error introduced by this replacement is $O(a)$; depending on the desired precision for the calculation more involved discrete approximations for the continuum derivatives may be necessary. In Appendix C.2 we include details on a nine-point approximation used in the calculation in section 4.3.

A common treatment of Grassmann variables is to integrate them out analytically; if higher-point fermion interactions are present they are first linearized by introducing an appropriate set of auxiliary fields. The resulting determinant is either evaluated directly or exponentiated in terms of commuting pseudo-fermions,

$$\int D\Psi e^{-\int d^2\xi \psi M \psi} = (\det M)^{1/2} = (\det MM^\dagger)^{1/4}$$

$$= \int D\zeta D\bar{\zeta} e^{-\int d^2\xi (MM^\dagger)^{-1/4}\zeta} \, .$$

In general, care must be taken to avoid double-counting the fermionic contribution in the continuum limit. In our case one can show that the Pfaffian of $M$ is real and thus it is sufficient to exponentiate the operator $MM^\dagger$; since it contains a Klein-Gordon operator (in general up to a field-dependent prefactor which is expected
to effectively acquire a vacuum expectation value), its discretization is similar to that of regular commuting scalars and is free of unwanted doublers [143].

The strategy for computing correlation functions in lattice field theory is to approximate the path integral in terms of finitely many field configurations which sample the entire phase space. A very efficient algorithm for generating these field configurations is the RHMC algorithm [133, 134, 135], which we shall review in some detail in Appendix C.1. With a specific rational approximation for the fractional power of the fermion matrix (4.2.4), one defines an evolution Hamiltonian by adding to the discrete action the squared conjugate momentum for each field, randomly generates some field configuration and then generates further ones by repeating the following steps:

1. randomly generates momenta for all fields from a gaussian distribution

2. evolves the field configuration with the evolution Hamiltonian (Molecular dynamics) for “time” interval $\Delta \tau$

3. accepts the initial or the final field configuration stochastically, through a Metropolis acceptance test (which, in some sense, “decides” whether the final field configuration can be the result of a quantum mechanical evolution). This step eliminates errors introduced at step 2) due to various approximations.

It has been shown in [133] that this algorithm leads to field configurations which cover the entire phase space of the system and can be used to construct the partition function or correlation functions of operators.

### 4.2.2 Various approaches to energy calculations

The semiclassical expansion has been extensively used to study the worldsheet perturbative expansion of energies of “long” strings in $\text{AdS}_5 \times S^5$ — *i.e.* strings with large quantum numbers and thus dual to “long” sYM operators with large canonical dimensions (see *e.g.* [40] for a review). It was suggested in [144] that similar techniques may also be applied to “short” strings, provided that the corresponding state can be obtained by analytic continuation from a long string state. A

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7This Hamiltonian is the evolution operator along some fictitious (Monte Carlo) "time" direction $\tau$. 

strategy that uses the conformal dimension of worldsheet vertex operators and potentially yields the energy of generic string states, has been discussed in [145]. Here we briefly review these techniques and add another one based on the worldsheet construction of two-point functions of local gauge theory operators and phrase them such that they are amenable to numerical calculations based on discretized worldsheet actions in $\text{AdS}_5 \times S^5$.

4.2.2.1 A conformal gauge approach

In conformal gauge the worldsheet theory for strings in $\text{AdS}_5 \times S^5$ is a conformal field theory – albeit one which is neither factorizable nor particularly suited to a perturbative treatment due to the special features of the Green-Schwarz fermions. Conformal invariance determines the form of the two-point function of local worldsheet operators; using it we may extract target space information from a (direct numerical) evaluation of the two-point function of fairly general local worldsheet operators.

Indeed, a general local operator $W(\xi)$ can be expanded in the basis of local operators with definite worldsheet dimension $h_n$ as

$$W(\xi) = \sum_n c_{W,n} V_n(\xi) .$$

The general form of the two-point function of the basis elements

$$\langle V_m(\xi_1) V_n(\xi_2) \rangle = \delta_{m,n} \frac{c_V}{|\xi_1 - \xi_2|^{2h_n}}$$

implies that the two-point function $\langle W(\xi_1) W(\xi_2) \rangle$ is

$$\langle W(\xi_1) W(\xi_2) \rangle = \sum_n \frac{c_n c_W^n}{|\xi_1 - \xi_2|^{2h_n}} ,$$

where the normalization factor $c_V$ and the conformal dimensions $h_n$ depend on the worldsheet coupling constant and the charges of the operator $V_n$ under various target space symmetries. Assuming some way of determining the worldsheet two-point function of two operators $W(\xi)$, the lowest worldsheet conformal dimension that appears in (4.2.6) may be extracted from the large distance asymptotics.
More generally, given a sufficiently precise determination of \( \langle W(\xi_1)W(\xi_2) \rangle \), the dimensions \( h_n \) of other vertex operators that enter the decomposition (4.2.4) may in principle be extracted through a Fourier analysis; the coefficients \( c_n \) and \( c_{W,n} \) need not be known.

Then, the condition that \( V_n \) are exactly marginal,

\[
h_n = 2 ,
\]

(4.2.7)
determines the target space energy of the corresponding string state in terms of its charges and the 't Hooft coupling.

To carry out such a program it is useful to choose an operator \( W \) with definite charge under the target space symmetries which also depend explicitly on the target space energy or boundary conformal dimension; the classical bosonic vertex operators discussed in conformal gauge in [146, 147, 145] are possible candidates.

As discussed in section 4.2, from the perspective of a numerical calculation of the two-point function (4.2.6), purely bosonic approximate vertex operators are very useful because fermions can be integrated out analytically. Bosonic vertex operators are naturally constructed in terms of the embedding coordinates

\[
Y_a Y^a = Y_+ Y_+^* - Y_\times Y_\times^* - Y_\gamma Y_\gamma^* = 1, \quad Z_k Z_k^* = Z_x Z_x^* + Z_y Z_y^* + Z_z Z_z^* \quad (4.2.8)
\]

where \( Y_+ = Y_0 + i Y_5, \ Y_\times = Y_1 + i Y_2, \ Y_\gamma = Y_3 + i Y_4, \ Z_x = Z_1 + i Z_2, \ Z_\times = Z_3 + i Z_4, \ Z_z = Z_5 + i Z_6 \) and the relation to the Poincaré patch is the usual one

\[
Y_m = \frac{x_m}{z}, \quad Y_4 = \frac{1}{2z}(-1 + z^2 + x^m x_m), \quad Y_5 = \frac{1}{2z}(1 + z^2 + x^m x_m) . \quad (4.2.9)
\]

In the embedding coordinates the conformal gauge bosonic action is

\[
S = \frac{\sqrt{\lambda}}{4\pi} \int d^2 \sigma \left( - \partial Y_a \cdot \partial Y^a + \partial Z_k \cdot \partial Z_k + \text{fermions} \right) . \quad (4.2.10)
\]

To describe vertex operators it is useful to do a Euclidean rotation both on the worldsheet and in the target space

\[
t_e = it , \quad Y_0e = i Y_0 , \quad x_0e = i x_0 , \quad (4.2.11)
\]
so that $Y^M Y_M = -Y_5^2 + Y_6^2 + Y_4 Y_4 + Y_4^2 = -1$. Then, unintegrated vertex operators have the form

$$V \sim (Y_+)^{-\Delta} \left[ (\partial^s Y)^r \cdots (\partial^m Z)^n + \cdots \right] \equiv (Y_+)^{-\Delta} U(Y, Z, \ldots). \quad (4.2.12)$$

where $\Delta$ is constant related to the dimension of the dual gauge theory operator and

$$K(x, z) = k_\Delta (Y_+)^{-\Delta} = k_\Delta \left( z + z^{-1} x^m x_m \right)^{-\Delta}, \quad \text{extand} \quad Y_+ \equiv Y_5 + \mathbb{I}. \quad (4.2.13)$$

is the usual bulk-to-boundary propagator in AdS space. 8

On a discrete worldsheet the unintegrated vertex operator is placed at a lattice site

$$V(\xi) \mapsto V_{(m,n)} \quad (4.2.14)$$

and depends on as many adjacent sites as are used to define the discrete derivative of a field (the number of such derivatives appearing in $V$ is related to the string level). The two-point function of such operators (specified by the parameter $\Delta$ and a choice of $U(Y, Z, \ldots)$)

$$\langle V_{(m_1,n_1)} V_{(m_2,n_2)} \rangle = \frac{1}{Z} \int D[x_{(m,n)}, z^M_{(m,n)}, \theta_{(m,n)}, \eta_{(m,n)}] e^{-S_{\text{discrete}}} V_{(m_1,n_1)} V_{(m_2,n_2)} \quad (4.2.15)$$

may then be computed through Monte Carlo techniques which we will review in Appendix C.1 as a function of $\Delta$, the symmetry charges, the worldsheet coupling and the lattice separation $d_{12}^2 = (m_1 - m_2)^2 + (n_1 - n_2)^2$. 9 Extracting from it the worldsheet conformal dimensions $h_n$ (or perhaps only the smallest one) 10 as

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8 As explained in [146, 147], the structure of the vertex operator follows closely that of flat space vertex operators, with $(Y_+)^{-\Delta}$ being the Euclidean analog of $e^{-iEt}$ and $U(Y, X, \ldots)$ carrying the dependence on the string level, symmetry transformations, etc.

9 In computing this correlation function it may be necessary to add to $V_{(m,n)}$ counterterms proportional to the inverse lattice spacing; this is due to the fact that in the continuum theory operators are expected to receive infinite renormalization responsible for their two-dimensional anomalous dimension.

10 It is possible to argue that, in the context of our discussion, one can unambiguously follow one worldsheet dimension from weak to strong worldsheet coupling. Indeed, it was suggested in [146] that the dimensions of gauge theory operators obey a non-intersection principle -- i.e. that there should be no level crossings for states with the same quantum numbers as $\lambda$ changes...
a function of $\Delta$, $\lambda$ and identifying the curve with the equation (4.2.7) yields the target space energy $\Delta = \Delta(\lambda, \ldots)$.

Since the discrete action is not conformally invariant, the extraction of the worldsheet conformal dimensions $h_n$ requires a careful consideration of the continuum limit. \(^1\) In turn, the curve with equation (4.2.7) yields the target space energy $\Delta = \Delta(\lambda, \ldots)$.

By not imposing the Virasoro constraint from the outset as well as by not focusing from the outset on target space quantities, the outcome of this algorithm potentially provides us with a wealth of information on conformal field theories of Green-Schwarz type. For the goal of finding the energies of string states these features make it however computationally intensive due to the large number of intermediate steps necessary to find $\Delta$ from the worldsheet two-point function. It is therefore important to identify strategies that require fewer/simpler auxiliary quantities.

4.2.2.2 Operator dimensions from the boundary two-point functions

Integrated vertex operators \([147, 148]\), labeled by a point on the boundary of AdS space, are dual to local gauge-invariant operators of the boundary theory. They provide a means to construct the two-point function of general local operators in the boundary theory from which one may extract directly the eigenvalues of the dilatation operator.

A general local gauge-invariant operator $O(x)$ in the boundary theory can always be decomposed in a basis of local operators $O_n(x)$ with definite (anomalous) dimensions

$$O(x) = \sum_n c_{O,n}(\lambda)O_n(x).$$

(4.2.16)

from small to large values. Then, the general structure of the worldsheet anomalous dimension \([147, 145]\)

$$h_n = \Delta_n(\Delta_n - c) + F_n(\sqrt{\lambda}, Q),$$

together with the positivity of $\Delta$ and $h$ imply that the worldsheet (anomalous) dimensions should also obey a non-intersection principle.

\(^1\)It is possible that the leading term in the large distance expansion is less sensitive to the details of the discretization and may be used to extract the smallest worldsheet dimension even on a finite-size lattice.
Using the form of the two-point functions of $\mathcal{O}_n(x)$ dictated by conformal invariance,

$$\langle \mathcal{O}(x)\mathcal{O}(y) \rangle = \frac{c_\mathcal{O}(\lambda)}{|x-y|^{2\Delta_\mathcal{O}}}.$$  \hspace{1cm} (4.2.17)

($c_\mathcal{O}(\lambda)$ is a potentially nontrivial normalization factor) it follows that the two-point function of the operators $O(x)$ is

$$\langle O(x)O(y) \rangle = \sum_n c_n(\lambda)c_{O,n}^2(\lambda)\frac{1}{|x-y|^{2\Delta_n}}.$$  \hspace{1cm} (4.2.18)

Thus, assuming that it is possible to find the exact two-point function of $O(x)$, it is then possible to extract the dimensions of the operator of lowest dimension in its spectral decomposition (4.2.16) from the large-distance behavior of the two-point function (4.2.18); the dimensions of the other operators in (4.2.16) may also be extracted through a Mellin transform or by simply fitting the two-point function to an expression of the type (4.2.18).

It is in general difficult to identify the vertex operator corresponding to a specified gauge theory operator; to carry out the program above however it suffices to know the two-point function of some operator $O$. Thus, one can simply pick any desired integrated vertex operator $V(x)$ and, if its two-point function can be computed, it can in principle be used to extract the dimension of the gauge theory operators with nonvanishing overlap with its dual $O_V$. It is intuitively clear that, to extract information from a numerical (and thus inherently not exact) evaluation of the two-point function $\langle V(x)V(y) \rangle$, it is useful to have as few dominant terms as possible on the right-hand side of eq. (4.2.18). One may, of course, target operators with a specific PSU(2, 2|4) quantum numbers $C = (S_1, S_2; J_1, J_2, J_3)$ by choosing a worldsheet operator with the same charges. \hspace{1cm} \footnote{Operators with identical charge vectors $C$ may be further distinguished by hidden local charges such as those related to the integrable structure of the planar theory.}

Further super-selection sectors are introduced by the gauge theory engineering dimension and the string level. At least at small and large 't Hooft coupling they guarantee that the dimensions $\Delta$ differ by $\mathcal{O}(1)$ and $\mathcal{O}(\lambda^{1/4})$ quantities respectively and thus the features of the two-point function are expected to be well-separated in Mellin space.
The integrated vertex operator associated to the unintegrated vertex operator (4.2.12) is [147, 148]

\[ V(x) = \int d^2\xi \ V(x(\xi) - x; ...) = \int d^2\xi \ [K(x(\xi) - x, z(\xi))]^{-\Delta} U[x(\xi) - x, Z(\xi)] ; \quad (4.2.19) \]

Its form on the discrete worldsheet is obtained by using the discrete fields and replacing the integral with a sum over all lattice sites,

\[ V(x) = \frac{1}{a^2} \sum_{m,n} V_{(m,n)}(x_{(m,n)} - x; ...) . \quad (4.2.20) \]

The boundary two-point function of the dual operator \( O_V(y) \) is then given by

\[ \langle O_V(x)O_V(y) \rangle = \langle V(x)V(y) \rangle = \frac{1}{Z} \int D[x_{(m,n)}, z^M_{(m,n)}, \theta_{(m,n)}, \eta_{(m,n)}] \ e^{-S_{\text{discrete}}V(x)V(y)} \quad (4.2.21) \]

and, as in the case of the two-point function in the previous section, may then be computed through Monte Carlo techniques.\(^{13}\)

For each value of the ’t Hooft coupling this algorithm requires the evaluation of a single auxiliary function of one variable – the boundary separation of operators \(|x_1 - x_2|\) – and the dimensions of operators in the boundary theory follow directly from it. This two-point function may be computed either with a discretized conformal gauge action or with a discretized action in a physical gauge, such as the light-cone gauge. In the latter case however it may be useful to choose a slightly different worldsheet operator \( V(x) \) than described above.

Indeed, in this gauge one of the Cartan generators of the boundary rotation group is spontaneously broken and thus it cannot be used to label \( V \). Moreover, the worldsheet field \( x^- \) is nonlocal (with local derivatives) and thus not easy to use as an argument of \( V \) in eq. (4.2.19). A straightforward resolution is to simply not include the bulk-to-boundary factor \( K \) in \( V \) and choose a function \( U \) which

\(^{13}\)The comments in footnote 9 apply to this calculation as well.
depends on \( x^\pm \) only through their derivatives:

\[
V_{lc}(x_\perp) = \int d^2\xi \ U[x_\perp(\xi) - x_\perp, Z(\xi)] \to \sum_{m,n} U_{(m,n)}(x_{\perp(m,n)} - x_\perp; \ldots) . \tag{4.2.22}
\]

While this cannot be called “vertex operator”, it is nevertheless dual to some local boundary operator labeled by the two coordinates \( x_\perp \) transverse to the light-cone. Interpreting the absence of \( x^\pm \) as they having been set to zero, the dimensions of operators in the boundary theory may be extracted from an expansion analogous to eq. (4.2.18) which now depends only on \( |x_{\perp,1} - x_{\perp,2}| \).

### 4.2.2.3 Target space energy from a partition function

The time-honored approach to perturbative calculations of worldsheet quantum corrections to the energy of long string states is as the worldsheet vacuum energy in the background of the classical solution describing the state \([29]\) (see \([40]\) for a review and a complete list of references). It has been argued in \([149]\) and further elaborated on in \([150, 78]\) that, for single-charge string states, this is also the worldsheet free energy. More precisely, for single-charge solutions for which the AdS global time is related to the worldsheet time as \( t = \kappa \tau \) the target space energy is also given by

\[
E = -\frac{1}{\kappa} \ln Z . \tag{4.2.23}
\]

More involved expressions relate the partition function and the energy of multi-charge states for which all parameters may be interpreted as chemical potentials for various charges \([150, 78]\). It was moreover argued in \([145, 121, 122, 123]\) that similar semiclassical techniques capture correctly the energy of short string states which can be interpreted as the small charge continuation of long string states.

The evaluation of the partition function of a theory is arguably the simplest lattice field theory calculation and thus provides a good testing ground for the applicability of this framework to the Green-Schwarz string. Thus, a strategy – which we will illustrate in section 4.3 – is to first find the complete continuum action of fluctuations \( \tilde{\Phi} \) around the desired classical solution \( \Phi_{\text{cl}} \) and discretize it
as discussed in sec. 4.2.1. 14 As before, the action $S_{E,\text{discrete}}$ is the sum over all lattice sites of the discrete Lagrangian. Then we evaluate the partition function

$$Z = \int D\Phi(m,n)e^{-S_{E,\text{discrete}}}$$

(4.2.24)

as a function of the parameters of the solution (i.e. for many choices of those parameters and then construct an interpolating function) and the ’t Hooft coupling; from it one extracts the energy of the corresponding string state either though (4.2.23) or though the more involved relations derived in [150, 78].

This approach should be insensitive to the details (such as gauge choices) of the continuum action. However, only long string states whose Euclidean action is real can be considered; this is due to the fact that, in the Monte Carlo evaluation of path integrals, the corresponding probability measure $\exp(-S_E)$ is required to be real. Many interesting solutions – such as the spin-$S$ folded string (with and without angular momentum on $S^5$), various circular string solutions whose analytic continuation to small charges describe members of the Konishi multiplet, etc. – satisfy these restrictions. In the next section we discuss in detail an application of this strategy.

4.3 An example: the universal scaling function at finite coupling

The universal scaling function is expected to be the solution to the BES equation; this equation reproduces the first few orders in the weak [117, 118] and strong [149, 84] coupling expansions that have been computed directly:

$$f(\lambda)|_{\lambda \to 0} = 8g^2 \left[ 1 - \frac{\pi^2}{3} g^2 + \frac{11\pi^4}{45} g^4 - \left( \frac{73\pi^6}{315} + 8\zeta_3 \right) g^6 + \ldots \right]$$

(4.3.1)

$$f(\lambda)|_{\lambda \to \infty} = 4g \left[ 1 - \frac{3\ln 2}{4\pi g} - \frac{K}{16\pi^2 g^2} + \ldots \right] , \quad g = \sqrt{\frac{\lambda}{4\pi}}.$$  (4.3.2)

14A priori this action may exhibit position dependence inherited from the classical background (if the classical solution is inhomogeneous). This simply translates into an explicit dependence of the Lagrangian on the lattice site (and on the lattice spacing) apart from that of fields.
Further terms in these expansions as well as the value of the universal scaling function at finite values of the coupling can be obtained from the BES equation\(^\text{15}\). We will reproduce these values (within numerical errors) at various values of \(g \in [1, 40]\) by numerically evaluating the worldsheet partition function in the background of the null cusp solution. Smaller values of \(g\), inside the radius of convergence of sYM perturbation theory are, in principle accessible but are currently limited by the accuracy of our simulation.

We take the continuum worldsheet theory to be given by AdS-light-cone gauge action Wick-rotated to a Euclidean worldsheet [141, 84]:

\[
S = g \int dt \int_{0}^{\infty} ds \mathcal{L}_E, \quad (4.3.3)
\]

\[
\mathcal{L}_E = \dot{x}^2 + (z^M + i\eta_i \rho^{MN} N \eta^i \eta^j)^2 + i(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i - \text{h.c.}) - z^{-2}(\eta^2)^2 + z^{-4}(x^M x^M + z^M z^M) + 2i \left[ z^{-3} \eta^i \rho^{MN}_i \eta^j \eta^j \rho^{MN}_{ij} \eta^i N \eta^N \right] (4.3.4)
\]

It has manifest \(U(1) \times SO(6) \simeq U(1) \times SU(4)\) symmetry. The fermions are complex \(\theta^i = (\theta_i)^\dagger\), \(\eta^i = (\eta_i)^\dagger\) \((i = 1, 2, 3, 4)\) and transform in fundamental representation of \(SU(4)\); the matrices \(\rho^{MN}_{ij}\) are the off-diagonal blocks of six-dimensional gamma matrices in chiral representation and \((\rho^{MN})_i^j = (\rho^{[M} \rho^{N]}_i^j)\) and \((\rho_{i[M} \rho^{N]}_{j]) = (\rho^{i[N} \rho^{M]}_{j})\) are the \(SO(6)\) generators. The fields \(z^M\) are neutral under \(U(1)\), \(\theta^i\) and \(\eta^i\) have opposite charges and the charge of \(\eta_i\) is half the charge of \(x\).

### 4.3.1 The null cusp fluctuation action, discretization, and some numerical details

The classical solution of (4.3.4) dual to the null cusp and the action for fluctuations around it were constructed in [84] (\(\tilde{z}\) is the norm of the six-component fluctuation vector \(\tilde{z}^M\)):

\[
\mathcal{L}_{cusp} = \left| \partial_t \tilde{x} + \frac{1}{2} \tilde{x} \right|^2 + \frac{1}{\tilde{z}^2} \left| \partial_s \tilde{x} - \frac{1}{2} \tilde{x} \right|^2 + \left( \partial_t \tilde{z}^M + \frac{1}{2} \tilde{z}^M + \frac{i}{\tilde{z}^2} \tilde{\eta}_i (\rho^{MN})_i^j \tilde{\eta}^j \tilde{z}_N \right)^2 + \frac{1}{\tilde{z}^4} \left( \partial_s \tilde{z}^M - \frac{1}{2} \tilde{z}^M \right)^2
\]

\(^{15}\)While strong coupling perturbation theory is not summable [36], the first three terms in this expansion are a good numerical approximation to the exact function for \(g \geq 1\).
\[ + i(\tilde{\theta}^i \partial_t \tilde{\theta}_i + \tilde{\eta}^i \partial_t \tilde{\eta}_i + \tilde{\theta}_i \partial \tilde{\theta}^i + \tilde{\eta}_i \partial \tilde{\eta}^i) - \frac{1}{\tilde{z}^2} (\psi^2)^2 \]  
\[ + 2i \left[ \frac{1}{\tilde{z}^2} \tilde{\eta}_i (\rho^M)_{ij} \tilde{z}^M (\partial_s \tilde{\theta}_j - \frac{1}{2} \tilde{\theta}_j - \frac{i}{\tilde{z}} \tilde{\eta}_j (\partial_s x - \frac{1}{2} \tilde{x})) \right. 
\[ \left. + \frac{1}{\tilde{z}^2} \tilde{\eta}_i (\rho^M)_{ij} \tilde{z}^M (\partial_s \tilde{\eta}_j - \frac{1}{2} \tilde{\eta}_j + \frac{i}{\tilde{z}} \tilde{\theta}_j (\partial_s x^* - \frac{1}{2} \tilde{x}^*)) \right] . \]  

As discussed there, the universal scaling function is proportional to the worldsheet free energy

\[ Z_{\text{string}} = e^{-W(g)} , \quad W(g) = \frac{1}{2} f(\lambda) V = \frac{1}{8} f(\lambda) \int dt \int ds \, 1 . \]  

The various numerical factors are related to the coordinate transformation and field redefinition between the long folded (GKP) string in global AdS coordinates and the null cusp solution in the Poincaré patch in light-cone gauge [84].

To integrate out the fermions the quartic terms are linearized by introducing a scalar and an \( \text{SO}(6) \) vector auxiliary fields:

\[ - \frac{1}{\tilde{z}^2} (\tilde{\phi}^2)^2 \mapsto \frac{1}{2} \tilde{\phi}^2 + \sqrt{2} \tilde{\phi} \tilde{\eta}^2 \]  
\[ - \frac{1}{\tilde{z}^4} (\tilde{z}_N \tilde{\eta} \rho^{MN} \tilde{\eta})^2 \mapsto \frac{1}{2} (\tilde{\phi}_M)^2 + \sqrt{2} \tilde{\phi}_M \tilde{z}_N \tilde{\eta} \rho^{MN} \tilde{\eta} . \]  

The resulting quadratic fermion matrix \( M_{\text{cusp}} \) can be read without difficulty; for completeness we include it in appendix C.3. Integrating out the fermions and exponentiating the resulting determinant leads to the action

\[ \mathcal{L} = |\partial_t \tilde{x} + \frac{1}{\tilde{z}^2} \tilde{x}|^2 + \frac{1}{\tilde{z}^4} |\partial_s \tilde{x} - \frac{1}{\tilde{z}} \tilde{x}|^2 + (\partial_t \tilde{z}^M + \frac{1}{2} \tilde{z}^M)^2 + \frac{1}{\tilde{z}^4} (\partial_s \tilde{z}^M - \frac{1}{2} \tilde{z}^M)^2 
\[ + \frac{1}{2} \tilde{\phi}^2 + \frac{1}{2} (\tilde{\phi}_M)^2 + \zeta^t (M_{\text{cusp}}^\dagger M_{\text{cusp}})^{-1/4} \zeta ; \]  

it contains 15 real bosonic fields (8 physical and 7 auxiliary) as well as 16 more complex bosons \( \zeta \) used to exponentiate the fermion determinant, cf. eq. (4.2.4) \footnote{In that equation the formal variable \( \psi \) stands for \( \psi \equiv (\tilde{\theta}^i, \tilde{\theta}_i, \tilde{\eta}^i, \tilde{\eta}_i) \) with \( i = 1, \ldots, 4 \).}.

Its discretization, as reviewed in section 4.2.1, is relatively straightforward. Using the block structure of \( M_{\text{cusp}} \) (C.3.2) one can easily convince oneself that \( M_{\text{cusp}}^\dagger M_{\text{cusp}} \) contains the Klein-Gordon operator as a free limit. As in the case of
bosons its discrete version converges to the right continuum limit and is free of unwanted doublers. Moreover, its determinant is real and positive and thus its fourth-root is unambiguous. As described in Appendix C.1, we will approximate $(M^{\dagger}_{\text{cusp}} M_{\text{cusp}})^{-1/4}$ as a rational function of $(M^{\dagger}_{\text{cusp}} M_{\text{cusp}})$ [151]:

$$(M^{\dagger}M)^{-\frac{1}{4}} = \alpha_0 + \sum_{i=1}^{P} \frac{\alpha_i}{M^{\dagger}M + \beta_i}. \quad (4.3.10)$$

The standard choice $P = 15$ leads in our case to an error $O(10^{-5})$ for $g \in (10^{-7}, 10^{3})$.

In the discretization of the Lagrangian (4.3.9) we could, in principle, use the simplest two-site version of the discrete derivative (4.2.2), whose error is $O(a^2)$. It turns out however that, for lattice sizes accessible to us this approximation is too coarse and leads to numerically-unstable fermion contributions. We found that stability is achieved only for an error of $O(a^8)$. We must therefore use a nine point stencil for the derivative

$$\partial_x f(x) = \frac{4}{5} (f(x + a) - f(x - a)) - \frac{1}{5} (f(x + 2a) - f(x - 2a)) + \frac{4}{105} (f(x + 3a) - f(x - 3a)) - \frac{1}{280} (f(x + 4a) - f(x - 4a)) + O(a^8). \quad (4.3.11)$$

For completeness we include its derivation in Appendix C.2. It is also trivial to choose a discretization that preserves the manifest global $SO(6)$ symmetry of (4.3.9).

To construct the action (4.3.9) from the action in eq. (4.3.5) we introduced a number of Lagrange multiplier fields; to recover the original action one is to use the saddle-point approximation to integrate them out. While this is harmless in the continuum theory, strictly speaking the partition function of the original theory is recovered only up to factors of the determinant of the unit operator. While these factors are expected to be unity in the continuum theory, in the presence of a regulator they may be nontrivial, albeit coupling constant independent. Since the initial partition function is such that it equals unity if the action were zero, we will eliminate the potential extra factors by dividing by the partition function
with $S_E = 0$. We have checked that the corresponding subtraction term in the free energy decreases as the lattice spacing is decreased, consistent with it being an artifact of the discretization.

Even though the original action $(4.3.9)$ is defined on an infinite worldsheet, simulating it on a lattice requires placing it in finite volume. It has been suggested in [152] that the finite-size effects due to placing the theory on a cylinder of length $L$ translate into $\delta f \sim 1/L^2$ corrections to the universal scaling function (and to $1/L = 1/\ln S$ corrections to the energy of the long folded string). It is easy to argue that such effects are of the same order as the minimal finite-volume error (if the lattice has equal sides) and thus cannot be extracted from a calculation of the type we will describe here.

To estimate a lower bound on the finite-volume effects vis-à-vis the fact that we are interested in the value of the (free) energy we use the uncertainty relations. Taking the time uncertainty to be the same as the length of the (Euclidean) time direction of our lattice, $\Delta t = T$, the uncertainty in the energy is

$$\Delta E \geq \frac{1}{2T}. \quad (4.3.12)$$

By dividing out the length of the time direction in $(4.3.6)$, the uncertainty in the energy is related to the uncertainty in the universal scaling function as

$$\Delta E = \frac{V_2}{8T} \Delta f(g) ; \quad (4.3.13)$$

It thus follows that the error on the universal scaling function is bounded from below by

$$\Delta f(g) \geq \frac{4}{V_2}. \quad (4.3.14)$$

Other sources – such as statistical, discretization, etc – add to this estimate. If the length of the space-like and Euclidian time-like direction are of the same order this error is of the same order as the expected finite volume correction to the universal scaling function, $\delta f \sim 1/L^2$. One may attempt to distinguish them by considering an asymmetric lattice with the time-like direction much larger than the space-like direction. We shall not pursue this here.
4.3.2 The simulation, data analysis and results

To simulate the discretized action (4.3.9) on a lattice we employed the RHMC algorithm reviewed in appendix C.1. We used $17 \times 10$ and $12 \times 12$ lattices with volume $V_2 = \pi^2$ and evaluated the worldsheet free energy $W$ and the universal scaling function $f(g)$ (cf. eq. (4.3.6)) for several values of the coupling

$$g \in \{1, 2, 5, 10, 15, 20, 30, 40\}.$$  \hspace{1cm} (4.3.15)

To avoid potential problems with constant spinors in the regime when fermions are effectively light\(^{18}\) we will use anti-periodic boundary conditions for fermions while all bosons are taken to be periodic. The fermion boundary conditions are captured by the detailed structure of the matrix $(M^\dagger_{\text{cusp}} M_{\text{cusp}})$ in the discretized theory and arises from the derivatives on fermions at the edges of the lattice. It is interesting to note that if the fermion boundary conditions are chosen to be periodic the simulation does not appear to converge.

The universal scaling function for each value of $g$ is the result of an independent simulation. Following the RHMC algorithm \cite{133, 134, 135} reviewed in Appendix C.1, field configurations are generated by starting with a random field configuration and evolving it along a fictitious time direction. At the end of every $n_T$-step ”time” sequence\(^{19}\) the resulting field configuration is kept or rejected whether or not it can be interpreted as the result of an actual quantum mechanical evolution of the system and momenta are re-generated. It was shown in \cite{133} that the resulting field configurations sample the complete phase space of the system. After a sufficiently long evolution the values of the free energy (or of any other observable) follow a Gaussian distribution; the free energy and its error are extracted as the mean value and the standard deviation of this distribution, respectively.

Figure ?? shows, for $g = 20$, the evolution of the value of the action along the evolution towards thermal equilibrium and figure ?? shows the corresponding

\(^{17}\)The number of lattice sites and the lattice volume are chosen such that the simulation runs sufficiently fast while still having reasonably small discretization errors.

\(^{18}\)This occurs at small values of the ’t Hooft coupling: the free momentum space action looks like $S \sim g(p^2 + m^2)$; for fixed $gp^2$ the mass term is irrelevant at small values of $g$.

\(^{19}\)The number of steps $n_T$ and the length of each step are tunable parameters chosen to decrease the thermalization time, see Figure ??.
Figure 4.1: The value of the action as a function of the (logarithm of the) evolution time $\tau$ for $g = 20$ and the distribution at sufficiently late times. The evolution of the value of the free energy along the Monte Carlo time. Each point represents the value of the free energy on the accepted field configuration at the end of each sequence of $n_T \simeq 10$ steps (an HMC trajectory). After some time (in this case $\ln \tau \sim 6$) the state "thermalizes", i.e. the value of the free energy on the generated field configurations follow a normal distribution. The spread of the amplitude of the thermalized state comes from picking shorter trajectories in order to increase the acceptance rate of the monte-carlo.

Figure 4.2: The value of the free energy and the corresponding error (and in fact of any other observable) is found by fitting a Gaussian on sufficiently many values after thermalization; the histogram is constructed from about 500 data points for $g = 20$.

Gaussian fit.\textsuperscript{20}

The results of the simulations of the action (4.3.9) on a $10 \times 10$ and $12 \times 12$

\textsuperscript{20}Here and for the other values of $g$ we carry out the fit using the Maximum-likelihood Fitting of Univariate Distributions in the statistical data analysis package R [153]. See also http://stat.ethz.ch/R-manual/R-devel/library/MASS/html/fitdistr.html
Figure 4.3: The universal scaling function for the values of $g$ in eq. (4.3.15) and their fit for the $10 \times 10$ and $12 \times 12$. lattice are listed in the second and third two columns of table 4.1 and plotted in figures ?? and ??, respectively. On the scale of the plot the two data sets are practically indistinguishable. Inspecting the numerical values reveals small variations in the position of the central value as well as a reduction in their absolute errors. This reduction is consistent with the expectation that the discretization errors are smaller on finer lattices. In the fourth column of table 4.1 we include the values of the universal scaling function obtained by solving the BES equation for the same values of $g$.\footnote{We thank D. Volin for a numerical solution of the BES equation based on the algorithm described in [154].}

<table>
<thead>
<tr>
<th>$g$</th>
<th>$10 \times 10$ lattice</th>
<th>$12 \times 12$ lattice</th>
<th>BES equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.328 ± 0.97</td>
<td>3.335 ± .89</td>
<td>3.3066</td>
</tr>
<tr>
<td>2</td>
<td>8.127 ± 0.96</td>
<td>7.385 ± .91</td>
<td>7.3246</td>
</tr>
<tr>
<td>5</td>
<td>19.694 ± 0.92</td>
<td>19.373 ± .91</td>
<td>19.3332</td>
</tr>
<tr>
<td>10</td>
<td>39.652 ± 0.92</td>
<td>39.380 ± .92</td>
<td>39.3359</td>
</tr>
<tr>
<td>15</td>
<td>59.878 ± 0.93</td>
<td>59.306 ± .93</td>
<td>59.3369</td>
</tr>
<tr>
<td>20</td>
<td>79.804 ± 0.95</td>
<td>79.313 ± .94</td>
<td>79.3429</td>
</tr>
<tr>
<td>30</td>
<td>120.628 ± 1.00</td>
<td>119.408 ± .97</td>
<td>119.4052</td>
</tr>
<tr>
<td>40</td>
<td>160.480 ± 1.05</td>
<td>159.422 ± .98</td>
<td>159.5485</td>
</tr>
</tbody>
</table>

Table 4.1: The numerical values of the universal scaling function obtained from $10 \times 10$ and $12 \times 12$ lattices as well as the results of the BES equation. The latter are quoted with an uncertainly of one unit in the last digit.

The sources of errors are well-understood, see Appendix C.1.5 for a brief sum-
mary. The lower bound on the finite volume error estimated in (4.3.14) becomes here

\[ \Delta f(g) \geq \frac{4}{V^2} = \frac{4}{\pi^2} \simeq 0.4 . \]  

(4.3.16)

This accounts for about 50\% of the reported error in table 4.1. At large values of the coupling constant the extra error is statistically-dominated due to a slow thermalization time and low acceptance rate of the RHMC algorithm; this can presumably be justified by the fact that at large values of \( g \) the partition function is dominated by a single classical field configuration rather than by a distribution of field configurations. At small values of \( g \) the error induced by the fermion determinant provides the bulk of the extra error; this is a consequence of the fact that fermions become effectively light in this regime. It is difficult to estimate the precise effects of the discretization and of the finite lattice spacing; by comparing the values in the second and third columns we see that the error decreases with the decrease of the lattice spacing, in agreement with expectations and implying that even finer lattices will yield higher-precision results. We also ran these simulations for a smaller volume \(^{22}\) and found a substantial increase of the error estimate. This is consistent with an increase in the lower bound on the finite-volume uncertainty estimate (4.3.16).

In figure 4.4 we have plotted the three data sets rescaled by a factor of \((4g)\) such that at large values of \( g \) the graph asymptotes to one; to facilitate the comparison we also artificially shifted the plot along the horizontal axis by \( \delta g = -1/4 \) for the \( 10 \times 10 \) lattice data and by \( \delta g = +1/4 \) for the \( 12 \times 12 \) lattice data while keeping fixed the values obtained from the BES equation. We notice a very good agreement between the BES (black dots) and the central values of the \( 12 \times 12 \) lattice results (red triangles) while still being in fair agreement within the error bars with the \( 10 \times 10 \) lattice results (blue triangles).

While the reported absolute errors for universal scaling functions are relatively large, we can try to make contact with worldsheet perturbation theory by fitting the lattice data in table 4.1 onto the known form of the worldsheet perturbation

\(^{22}\)Increasing the lattice volume at fixed lattice spacing is computationally very expensive.
Figure 4.4: Plot of the rescaled universal scaling function from the $10 \times 10$ (blue right-triangles) and $12 \times 12$ (red left-triangles) and its values from the BES equation (black dots). The lattice values are artificially displaced by $\delta g = \pm 1/4$ for easy comparison. Clearly, the central values of the $12 \times 12$ lattice is a very good approximation of the integrability results.

<table>
<thead>
<tr>
<th>Origin</th>
<th>$a_0$</th>
<th>$a_1$</th>
<th>$a_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10 \times 10$ lattice</td>
<td>$4.031 \pm 0.007$</td>
<td>$-0.667 \pm 0.188$</td>
<td>-</td>
</tr>
<tr>
<td>$12 \times 12$ lattice</td>
<td>$4.001 \pm 0.002$</td>
<td>$-0.662 \pm 0.041$</td>
<td>$-0.015 \pm 0.066$</td>
</tr>
<tr>
<td>perturbation theory</td>
<td>$4.000 \pm 0.000$</td>
<td>$-0.662 \pm 0.001$</td>
<td>$-0.023 \pm 0.001$</td>
</tr>
</tbody>
</table>

Table 4.2: Coefficients of the fit of the lattice data with the expected form of the worldsheet perturbative expansion.

The resulting coefficients and their errors are listed in table 4.2. It is true that the universal scaling function is almost linear with $g$ at large $g$, but we regard this as a prediction of the BES equation rather than a feature of the universal scaling function that can be inferred on physical grounds. Not surprising, the quality of the fit degrades as one attempts to extract higher-order coefficients.

$$f(g) = a_0 g + a_1 + \frac{a_2}{g} + \ldots. \quad (4.3.17)$$
(e.g. an estimate for the two-loop coefficient cannot be extracted reliably from the $10 \times 10$ lattice). This is consistent with the observation that the first two terms in worldsheet perturbation theory provide a good approximation for the universal scaling function for $g > 1$. The fit may be slightly improved by assuming an expansion in $\sqrt{1 + 16g^2}$ rather than in $g$; this accounts for the expected $g_* = 1/4$ radius of convergence of sYM perturbation theory. In particular, the central value of $a_2$ becomes very close to the results of perturbation theory at the expense of a slightly poorer fit for $a_1$.

<table>
<thead>
<tr>
<th>Origin \ $g$</th>
<th>$10^{-1}$</th>
<th>$(4\sqrt{\pi})^{-1}$</th>
<th>$4^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$12 \times 12$ lattice</td>
<td>$0.014 \pm 0.67$</td>
<td>$0.095 \pm 0.69$</td>
<td>$0.581 \pm 0.82$</td>
</tr>
<tr>
<td>BES equation</td>
<td>$0.077 \pm 10^{-3}$</td>
<td>$0.150 \pm 10^{-3}$</td>
<td>$0.427 \pm 10^{-3}$</td>
</tr>
</tbody>
</table>

Table 4.3: Values of the universal scaling function inside the radius of convergence of $\mathcal{N} = 4$ sYM theory.

We have also evaluated the universal scaling function for values of $g$ at or below the expected radius of convergence of $\mathcal{N} = 4$ sYM perturbation theory, $g_* = 1/4$. The results are included in table 4.3. Since expected values for $f(g)$ are close to or below the expected lower bound of the error of the simulation on a lattice of volume $V_2 = \pi^2$, (4.3.16), the results cannot be statistically significant; we nevertheless note that, close to $g = g_*$, where the value of the cusp anomaly is larger than the error’s lower bound, the central value is relatively close to the result of the BES equation. It should be possible – albeit nontrivial – to reduce the error bar on such data points.

### 4.4 Summary and further comments

In this chapter we computed the energy of the folded string in $\text{AdS}_5 \times \text{S}^5$ (and thus the universal scaling function) at finite values of the ’t Hooft coupling using the Green-Schwarz string in $\text{AdS}_5 \times \text{S}^5$ by discretizing the worldsheet theory in the relevant background and computing numerically its partition function. Our results reproduce the predictions of the Asymptotic Bethe Ansatz within the accuracy of our simulation and thus strongly support the expectation that the Asymptotic
Bethe Ansatz yields the long string spectrum, or the spectrum of long operators in the dual $\mathcal{N} = 4$ sYM theory for all values of the coupling. We have attempted to find the universal scaling function inside the radius of convergence of $\mathcal{N} = 4$ sYM theory; while our simulation is not sufficiently precise for this purpose, there is no conceptual obstacle.\(^{23}\)

Our calculation can in principle be extended to other long string states as well as to short string states that have a description as the small charge continuation of long string states. The computational complexity depends on the details of the state and is strongly correlated with the presence or absence of massless fermions. We have also discussed strategies for finding the spectrum of general string states that are not in this class from the two-point functions of certain worldsheet (vertex) operators. By computing the two-point function of worldsheet fluctuations in long string background it should be possible to find information on their spectra at finite values of the coupling.

The sources of errors in such calculations are well-understood; it turns out however that the single dominant source — related to the need to use a lattice of finite extent in both space-like and (Euclidean) time-like directions — is the most difficult to overcome. A volume increase at fixed lattice spacing requires an increase in the number of lattice sites which increases the duration of the calculation. A second — but not less important — consequence of the increase in the number of lattice sites is the rapid increase in the size of the fermion matrix ($16^2$ times faster than the increase in the number of lattice sites, due to the number of independent fermions); this in turn leads to either an increase in the time needed to find a solution to the systems (C.1.11) (necessary for the generation of field configurations) or a decrease in the accuracy of the solution.\(^{24}\)

It therefore appears that perhaps the most efficient improvement of our simulation it to employ a more efficient treatment of the fermion matrix. At the analytical level one may consider factorizing it into simpler (e.g. upper-triangular and lower-triangular) factors and exponentiating each factor separately. At the

\(^{23}\)These methods are however inappropriate for studying the regime $\lambda < 0$, whose existence is the reason for a finite radius of convergence for the planar theory.

\(^{24}\)Perhaps more dramatically, as the size of the matrix increases so does the memory needed to store it. One of the most expensive operations in the GPU calculations is transferring the data back and forth; if the matrix is sufficiently large such that it cannot be fully stored on the GPU, most of the advantage of the GPU speedup is lost.
level of the numerical calculation one could attempt to use a Fourier-accelerated (R)HMC algorithm [155], which may reduce the thermalization time. One can also attempt to use a more efficient algorithm for solving the systems (C.1.11) (e.g. one that accepts a preconditioner – the numerical analog of the analytic factorization such as multigrid [156]). Moreover, to alleviate the issues related to the size of the fermion matrix one could attempt to use a parallel GPU solver.

The physically very interesting case of short string states provides a natural partial solution to the difficult consequences of a larger lattice volume. Indeed, the worldsheet of such strings is a cylinder and thus increasing the volume requires increasing only one of the two dimensions of the lattice. Consequently, the increase in the number of lattice sites occurs at a much slower rate allowing in principle for smaller finite-volume errors while simultaneously keeping the discretization and statistical errors under control. It would be very interesting analyze states on the first excited string level using the methods proposed in this chapter.
A.1 PSU(2, 2 | 4)

The N=4 superconformal group, PSU(2, 2|4) is the symmetry group of both N=4 SYM and AdS\(_5\times S^5\). The algebra \(\mathfrak{psu}(2, 2|4)\) is best explained through a 16x16 block matrix. The matrix can be broken into four 4x4 blocks of complex numbers we will call A,B,C,D.

\[
X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A.1.1}
\]

Blocks A,D are considered even and blocks B,C are known as odd. \(X\) obeys the graded commutator, Lie bracket, \([.,.]\) defined as:

\[
[X,Y] = XY - (-1)^{XY} YX \tag{A.1.2}
\]

where \(Y\) is just a different analog of \(X\).

The supertrace, \(\text{STr } X := \text{Tr} A - \text{Tr} D\). Setting this \(\text{STr } X\) to 0 and realizing that the identity supermatrix commutes with all over matrices moves us from the superalgebra \(\mathfrak{gl}(4 \mid 4, \mathbb{C})\) to \(\mathfrak{psl}(4 \mid 4, \mathbb{C})\). To get the real version of this group or \(\mathfrak{psu}(2, 2 \mid 4)\) we impose a hermicity condition on the matrices.

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} HA^\dagger H^{-1} & -iHC^\dagger \\ -iB^\dagger H^{-1} & D^\dagger \end{pmatrix} \tag{A.1.3}
\]

Here \(H\) is a hermitian matrix of signature (2,2).
A.2 Green-Schwarz String

A.2.1 GS string in a general curved space

The GS string in a general curved space [12] is the following:

$$ S = \int d^2 \xi [\sqrt{-g}g^{ij}E_i^a E_j^b \Phi \eta_{ab} + \frac{1}{2} \epsilon^{ij} E_i^B E_j^A B_{AB}] $$

(A.2.1)

Where $E_i^A \equiv \partial_i z^M E_M^A = (E_i^a, E_i^a, \bar{E}_i^a)$, $\Phi$ is an arbitrary function of the scalar fields and $B_{AB}$ is two-form of the WZ term generalized to a curved space.

A.2.2 GS group theory construction

AdS$_5 \times$S$_5$ is a coset space the various terms that enter the action have geometric/group theory constructions in terms of left-invariant 1-forms $J = g^{-1} dg$. From this one can write the action in terms of projections of J onto eigenvectors of $Z_4$ symmetry.[13]

$$ S = \int J_{ab} \wedge * J_{ab} - J_{\bar{a} \bar{b}} \wedge * J_{\bar{a} \bar{b}} $$

$$ \pm \frac{1}{2} (E^{1/2} J_{ab} \wedge J_{\bar{a} \bar{b}} - E^{-1/2} J_{\bar{a} \bar{b}} \wedge J_{ab}) $$

(A.2.2)

A.3 Generalized Unitarity

The unitarity method is a powerful method to obtain loop-level scattering amplitudes. Loop amplitudes are calculated directly from on-shell tree amplitudes. Where on-shell means that the intermediate cut momenta are replaced by their masses, $p_i^2 = m_i^2$. By using the on-shell method and constructing loop amplitudes from tree-level symmetries from tree-level amplitudes can be carried over to loop level. This method may be compared to the Feynman diagrammatic methods. The main difference between the methods is that Feynman diagrams are computed off
shell and are inherently gauge dependent. These features make it difficult to carry
tree-level symmetries to higher loops. The generalized unitarity method is a more
modern approach to the unitarity method. In the generalized unitarity method
[67, 68, 69] multiple internal lines are placed on-shell which subdivides the loop
amplitude further.

To give a brief overview of how to use the method here is an example of a
s-channel two particle cut of the one loop four point amplitude.

![Figure A.1: s-channel two particle cut of the one loop four point amplitude](image)

In this cut we put two internal lines, $p$ and $p'$, on-shell, $p^2 = m^2$, $p'^2 = m'^2$.

\[
C_s = \sum_{\text{states}} = A_{\text{tree}}(-p', 1, 2, p)A_{\text{tree}}(-p, 3, 4, p') \tag{A.3.1}
\]

Here the sum runs over all possible physical states in the theory.

For a more in depth review of this rich subject see [158].
B.1 On the definition of the tree-level S-matrix elements

The worldsheet tree-level S-matrix elements are computed in a slightly different normalization than in usual four-dimensional calculations. In this appendix we summarize the relevant definitions, which hopefully will make transparent the translation to four-dimensional conventions.

- Fields are expanded in creation and annihilation operators as, (φ is a generic real scalar field)

$$\phi = \int \frac{dk}{(2\pi)\sqrt{2k_0}} \left( a_k e^{ik \cdot x} + a_k^\dagger e^{-ik \cdot x} \right)$$  \hspace{1cm} (B.1.1)

i.e. with the measure missing a factor of \((\sqrt{2k_0})^{-1}\) compared to the standard relativistic normalization.

- The commutation relations are missing a factor of \(2k_0\) compared to the standard relativistic normalization

$$[a_k, a_{k'}^\dagger] = 2\pi \delta(k - k')$$  \hspace{1cm} (B.1.2)

This is of course a consequence of the previous item.

- States are defined in the usual way, e.g.

$$a_{k_{1,1}}^\dagger a_{k_{1,2}}^\dagger |0\rangle$$  \hspace{1cm} (B.1.3)
• Momentum conservation $\delta^{(2)}(\sum_k p_k)$ was solved as [81] in the presence of the on-shell condition for external states

$$\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\delta(p_1 + p_2 - p_3 - p_4) = \frac{\varepsilon_1\varepsilon_2}{\varepsilon_2 p_1 - \varepsilon_1 p_2}(\delta(p_1 - p_3)\delta(p_2 - p_4) + \delta(p_1 - p_4)\delta(p_2 - p_3))\text{[B.1.4]}$$

This expression assumes that $p_1 > p_2$; otherwise the Jacobian factor is negative.

## B.2 Beisert’s SU(2|2) spin-chain S matrix

Beisert’s spin-chain S matrix [87] is defined by its action on two-particle states:

$$S^B |\phi_a \phi_b\rangle = A^B |\phi'_a \phi_0\rangle + B^B |\phi'_a \phi_0\rangle + \frac{1}{2} C^B \epsilon_{ab} \epsilon^{\alpha\beta} |Z^- \psi'_a \psi_\beta\rangle, \quad \text{(B.2.1)}$$

$$S^B |\psi_\alpha \psi_\beta\rangle = D^B |\psi'_a \psi_\beta\rangle + E^B |\psi'_a \psi_\beta\rangle + \frac{1}{2} F^B \epsilon_{ab} \epsilon^{\alpha\beta} |Z^+ \phi'_a \phi_\beta\rangle, \quad \text{(B.2.2)}$$

$$S^B |\phi_\alpha \phi'_b\rangle = G^B |\phi'_a \phi'_b\rangle + H^B |\phi'_a \phi'_b\rangle, \quad \text{(B.2.3)}$$

$$S^B |\psi_\alpha \phi'_b\rangle = K^B |\psi'_a \phi'_b\rangle + L^B |\phi'_b \psi_\alpha\rangle. \quad \text{(B.2.4)}$$

Its coefficients are fixed by the requirement that it exhibits $PSU(2|2)$ symmetry; they are

$$A^B = S^0_{pp'} \frac{x^+_{p'} - x^-_p}{x^+_p - x^+_p},$$

$$B^B = S^0_{pp'} \frac{x^+_{p'} - x^-_p}{x^+_p - x^+_p} \left(1 - 2 \frac{1 - \frac{1}{x^+_p x^+_p}}{1 - \frac{1}{x^+_p x^+_p}} \frac{x^+_p - x^-_p}{x^+_p - x^-_p} \right),$$

$$C^B = S^0_{pp'} \frac{2 \gamma_p \gamma_{p'}}{x^+_p x^+_p} \frac{1}{1 - \frac{1}{x^+_p x^+_p}} \frac{x^-_{p'} - x^-_p}{x^-_{p'} - x^-_p},$$

$$D^B = -S^0_{pp'},$$

$$E^B = -S^0_{pp'} \left(-1 - 2 \frac{1 - \frac{1}{x^+_p x^+_p}}{1 - \frac{1}{x^+_p x^+_p}} \frac{x^+_p - x^-_p}{x^+_p - x^-_p} \right),$$

$$F^B = -S^0_{pp'} \frac{2 \gamma_p \gamma_{p'} x^-_{p'} x^-_{p'}}{x^+_p x^+_p} \frac{(x^+_p - x^-_p)(x^+_p - x^-_p)}{1 - \frac{1}{x^+_p x^+_p}} \frac{x^+_p - x^-_p}{x^+_p - x^-_p}.$$
where

\[ \gamma_p = |x_p^+ - x_p^-|^{1/2} \]  

(B.2.6)

and

\[ x_p^\pm = \frac{\pi e^{\pm k_p}}{\sqrt{\lambda \sin^2 \frac{p}{2}}} \left( 1 + \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} \right). \]  

(B.2.7)

are the Zhukowsky variables.

The overall phase factor \( S^0 \) is undetermined by the algebraic construction. The one that correctly reproduces the semiclassical string spectrum via Bethe equations is

\[ S^0_{pp'} = \frac{1 - \frac{1}{x_p^+ x_p^-}}{1 - \frac{1}{x_{p'} x_{p'}}} e^{i \theta(p, p')} \]  

(B.2.8)

with the dressing phase \( \theta \) given to the leading order in \( 1/\sqrt{\lambda} \) by [44]

\[ \theta(p, p') = \frac{\sqrt{\lambda}}{2\pi} \sum_{m, m' = \pm} m m' \hat{\chi}(x_p^m, x_{p'}^{m'}) , \]

\[ \hat{\chi}(x, y) = (x - y) \left[ \frac{1}{xy} + \left( 1 - \frac{1}{xy} \right) \ln \left( 1 - \frac{1}{xy} \right) \right]. \]  

(B.2.9)
B.3 Strong coupling expansion of the $\text{AdS}_5 \times \text{S}^5$ dressing phase

The general form of the dressing phase in terms of higher local conserved charges is [159]:

\[
\theta_{12} = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s} (\sqrt{x})(q_r(x_{1}^{\pm})q_s(x_{2}^{\pm}) - q_r(x_{2}^{\pm})q_s(x_{1}^{\pm})) \quad (B.3.1)
\]

\[
q_r(x^{\pm}) = \frac{i}{r-1} \left( \frac{1}{(x^{+})^{r-1}} - \frac{1}{(x^{-})^{r-1}} \right) \quad (B.3.2)
\]

This may be rewritten as [80]

\[
\theta_{12} = \chi(x_{1}^{+}, x_{2}^{+}) - \chi(x_{1}^{+}, x_{2}^{-}) - \chi(x_{1}^{-}, x_{2}^{+}) + \chi(x_{1}^{-}, x_{2}^{-}) \\
- \chi(x_{2}^{+}, x_{1}^{+}) + \chi(x_{2}^{+}, x_{1}^{-}) + \chi(x_{2}^{-}, x_{1}^{+}) - \chi(x_{2}^{-}, x_{1}^{-}) \quad (B.3.3)
\]

\[
\chi(x_{1}, x_{2}) = \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} \frac{-c_{r,s}}{(r-1)(s-1)} \frac{1}{x_{1}^{r-1}x_{2}^{s-1}} \quad (B.3.4)
\]

The coefficients $c_{r,s}$ depend on the coupling constant $\hat{g}$ as

\[
c_{r,s} = \sum_{n=0}^{\infty} \hat{g}^{1-n} c_{r,s}^{(n)} \quad \rightarrow \chi(x_{1}, x_{2}) = \sum_{n=0}^{\infty} \hat{g}^{1-n} \chi^{(n)}(x_{1}, x_{2}) \\
\theta_{12} = \sum_{n=0}^{\infty} \hat{g}^{1-n} \theta_{12}^{(n)} \quad (B.3.5)
\]

\[
\theta_{12}^{(n)} = \chi^{(n)}(x_{1}^{+}, x_{2}^{+}) - \chi^{(n)}(x_{1}^{+}, x_{2}^{-}) - \chi^{(n)}(x_{1}^{-}, x_{2}^{+}) + \chi^{(n)}(x_{1}^{-}, x_{2}^{-}) \\
- \chi^{(n)}(x_{2}^{+}, x_{1}^{+}) + \chi^{(n)}(x_{2}^{+}, x_{1}^{-}) + \chi^{(n)}(x_{2}^{-}, x_{1}^{+}) - \chi^{(n)}(x_{2}^{-}, x_{1}^{-}) \\
. \quad (B.3.6)
\]

For string theory in $\text{AdS}_5 \times \text{S}^5$ the coefficients $c_{r,s}^{(0)}$ were found in [44] and $c_{r,s}^{(1)}$ in [160]. An all-loop proposal was put forward in [65].

The functions $\chi^{(n)}$ to various loop orders are:

\[
\chi^{(0)}(x_{1}, x_{2}) = -\frac{1}{x_{2}} + \left( \frac{1}{x_{2}} - x_{1} \right) \ln \left( 1 - \frac{1}{x_{1}x_{2}} \right) \quad (B.3.7)
\]

\[
\chi^{(1)}(x_{1}, x_{2}) = -\frac{1}{2\pi} \text{Li}_{2} \frac{\sqrt{x_{1}} - 1/\sqrt{x_{2}}}{\sqrt{x_{1}} - \sqrt{x_{2}}} - \frac{1}{2\pi} \text{Li}_{2} \frac{\sqrt{x_{1}} + 1/\sqrt{x_{2}}}{\sqrt{x_{1}} + \sqrt{x_{2}}}
\]
\[\chi^{(2)}(x_1, x_2) = + \frac{1}{2\pi} \text{Li}_2 \sqrt{x_1 + 1/\sqrt{x_2}} + \frac{1}{2\pi} \text{Li}_2 \sqrt{x_1 - 1/\sqrt{x_2}} \]  
\[\chi^{(3)}(x_1, x_2) = - \frac{x_2}{24(x_1x_2 - 1)(x_2^3 - 1)} \]  
\[\chi^{(4)}(x_1, x_2) = - \frac{x_2^3 + 4x_2^5 - 9x_1x_2^6 + x_2^7 + 3x_1^2x_2^7 - 3x_1x_2^8 + 3x_1^2x_2^9}{720(x_1x_2 - 1)^3(x_2^3 - 1)^5} \]  
\[\chi^{(2k+1)}(x_1, x_2) = 0 \]  

An all-order integral representation of the dressing phase was put forward in [161].

We emphasize here that the only logarithmic terms in the dressing phase have a one-loop origin. This will nevertheless lead to logarithmic terms at higher loops in worldsheet perturbation theory. The relation between the spin chain strong coupling expansion and worldsheet perturbation theory was mentioned in sec. 2.2 will be reviewed in Appendix B.4.

**B.4 The AdS$_5 \times$S$^5$ worldsheet S-matrix from the spin-chain S-matrix**

The symmetry of the world sheet theory is $PSU(2|2)^2$; together with the expected integrability they imply that the world sheet $S$-matrix factorizes as

\[S = S \otimes S \]  

where each factor transforms under a copy of $PSU(2|2)$; neither factor is the $S$ matrix of any obvious excitations of the worldsheet theory. The four-point tree-level worldsheet $S$ matrix in AdS$_5 \times$S$^5$ was computed in [66] and this factorization was verified.

Since only SU(2) $\times$ SU(2) $\subset$ PSU(2|2) is a manifest symmetry of the gauge-fixed worldsheet theory, $S$ may be parametrized in terms of ten unknown functions of
the momenta \( p \) and \( p' \) of the two incoming particles:\(^1\).

\[
S^\delta_{ab} = A \delta^\delta_a \delta^\delta_b + B \delta^\delta_a \delta^\beta_b , \quad S^\gamma_{ab} = C \epsilon_{ab} \epsilon^\gamma ,
\]
\[
S_{\alpha\beta} = D \delta^\gamma_{\alpha\beta} + E \delta^\beta_{\alpha\beta} , \quad S_{\gamma\delta} = F \epsilon_{\alpha\beta} \epsilon^{\gamma\delta} , \quad (B.4.2)
\]
\[
S_{\gamma\delta} = G \delta^\delta_{\alpha\beta} , \quad S_{\alpha\beta} = H \delta^\gamma_{\alpha\beta} , \quad S^\gamma_{ab} = I \delta^\delta a \delta^\beta b .
\]

As described in [66], in the comparison between the worldsheet and the spin chain we are interested in the coefficients of \( P_g P_{pp'} S_B \), where \( P_g \) is the graded permutation operator and \( P_{pp'} \) exchanges the excitation momenta. Furthermore, to find the S matrix for the full \( PSU(2,2|4) \) theory we use the relation

\[
S = \frac{1}{A^B} S^B \otimes S^B \quad S^{CC\bar{D}D}_{AABB}(p,p') = \frac{1}{A^B} (S^B)^{CD}_{AB}(p,p') S^{\bar{C}\bar{D}}_{AB}(p,p') . \quad (B.4.3)
\]

This is because the \( PSU(2|2) \) S matrix was defined in [87] as the physical scattering matrix of the fields \( \Phi_{A_1} \); in addition to \( S \) for the left \( PSU(2|2) \) indices, the scattering of this field receives contribution from \( S_{11} = A^B \).

Consequently we can relate the above coefficients to those of \( S \) used in Beisert’s S matrix

\[
A = \frac{1}{2 \sqrt{A^B}} (A^B - B^B) , \quad B = \frac{1}{2 \sqrt{A^B}} (A^B + B^B) , \quad C = \frac{i}{2 \sqrt{A^B}} C^B ,
\]
\[
D = \frac{1}{2 \sqrt{A^B}} (-D^B + E^B) , \quad E = \frac{1}{2 \sqrt{A^B}} (-D^B - E^B) , \quad F = \frac{i}{2 \sqrt{A^B}} F^B ,
\]
\[
G = \frac{1}{\sqrt{A^B}} G^B , \quad H = \frac{1}{\sqrt{A^B}} H^B , \quad K = \frac{1}{\sqrt{A^B}} K^B . \quad (B.4.4)
\]

The worldsheet perturbative expansion of the S matrix is obtained from the spin-chain S matrix in the small momentum expansion [79, 66], defined as the large-\( \lambda \) expansion at fixed string moment \( p_{\text{string}} \sim \sqrt{\lambda} p_{\text{chain}} \):

\[
p \rightarrow \frac{2\pi p}{\sqrt{\lambda}} \quad p_{\text{chain}} = \frac{2\pi}{\sqrt{\lambda}} p_{\text{string}} = \frac{1}{g} p_{\text{string}} . \quad (B.4.5)
\]

\(^1\)These definitions are similar but not identical to those of [87]. The relationship between the two definitions is given in equation B.4.4 below.
The matrix elements in (B.2.5) depend on $\hat{g} \propto 1/\sqrt{\lambda}$ only through $x_\pm^\alpha$. In the small momentum expansion these variables (B.2.7) become ($\varepsilon = \sqrt{1 + p^2}$):

$$x_\pm = \frac{1 + \varepsilon}{p} \left( 1 \pm \frac{ip}{2\hat{g}} - \frac{p^2(2 + 2\varepsilon + 3p^2)}{24(1 + \varepsilon + p^2)\hat{g}^2} \right) + \frac{ip^5}{48(1 + \varepsilon + p^2)\hat{g}^3} + \mathcal{O}\left(\frac{1}{\hat{g}^4}\right).$$

(B.4.6)

Plugging these expressions into eq. (B.3.6) we notice that the term independent of the upper index $\pm$ of the Zhukowsky variables cancels out. Terms linear in $1/\hat{g}$ have a similar fate and the first nonzero terms are proportional to $1/\hat{g}^2$ leading to eq. (2.2.11)

$$\theta_{12} = \frac{1}{\hat{g}} \sum_{n=0}^{\infty} \frac{1}{\hat{g}^n} \hat{\theta}_{12}^{(n)}. \quad (B.4.7)$$

Each term $\hat{\theta}_{12}^{(n)}$ is given in terms of the derivatives of the functions $\chi^{(n)}$:

$$\hat{\theta}_{12}^{(n)} = -(1 + \varepsilon)(1 + \varepsilon') \left( \partial_2 \partial_y \chi^{(n)}(x, y) - \partial_2 \partial_x \chi^{(n)}(y, x) \right) \bigg|_{x=1+\varepsilon, y=1+\varepsilon'} + \chi^{(x \leq n-1)} \text{-contrib's}, \quad (B.4.8)$$

where the contributions of $\chi^{(x \leq n-1)}$ involve three or more derivatives. We notice here that since the coefficients of the transcendental functions in $\chi^{(0)}$ and $\chi^{(1)}$ depend on at most one of the two arguments $x_1$ and $x_2$ the degree of transcendentality of $\theta^{(n)}$ is lower than that of $\chi^{(0)}$ and $\chi^{(1)}$ by (at least) one unit. In particular, $\theta^{(0)}$ receives only in rational contributions in worldsheet perturbation theory while $\theta^{(1)}$ contributes only logarithmic contributions (cf. (2.2.14)). Since $\chi^{(n \geq 2)}$ are themselves rational, they also receive only rational contributions in worldsheet perturbation theory.

An alternative parametrization of $S$ is obtained by extracting a phase factor form $S$, that is $S \mapsto \tilde{S}_{1/2}^\prime \tilde{S}$. Then $S$ becomes

$$S = \tilde{S}_{0} \tilde{S} \otimes \tilde{S}, \quad (B.4.9)$$
The coefficients entering the tensor decomposition (B.4.2) are

\[ \tilde{A} = \frac{x'_- - x_-}{x'_- - x_+} \frac{1 - \frac{1}{x'_- x_+}}{1 - \frac{1}{x'_+ x_+}}, \]

\[ \tilde{B} = \frac{x'_+ - x_-}{x'_- - x_+} \left( 1 - \frac{x'_+ - x_-}{x'_+ - x_-} \right) \frac{1 - \frac{1}{x'_- x_+}}{1 - \frac{1}{x'_+ x_+}}, \]

\[ \tilde{C} = \frac{i \gamma_p \gamma_{p'}}{x_+ x_+} 1 - \frac{x'_- - x_-}{x'_- - x_+} \frac{1 - \frac{1}{x'_- x_+}}{1 - \frac{1}{x'_+ x_+}} e^{i \frac{p'}{2}}, \]

\[ \tilde{D} = \frac{x'_+ - x_-}{x'_- - x_+} \frac{1 - \frac{1}{x'_+ x_+}}{1 - \frac{1}{x'_- x_+}} e^{i \frac{(p' - p)}{2}}, \]

\[ \tilde{E} = 1 - \frac{x'_+ - x_+}{x'_- - x_+} \frac{1 - \frac{1}{x'_+ x_+}}{1 - \frac{1}{x'_- x_+}} e^{i \frac{(p' - p)}{2}}, \]

\[ \tilde{F} = \frac{-i (x_+ - x_-)(x'_+ - x'_-)}{\gamma_p \gamma_{p'} x_- x'_-} \frac{1 - \frac{1}{x'_- x_+}}{1 - \frac{1}{x'_+ x_+}} x'_+ - x_+ e^{-i \frac{p'}{2}}, \]

\[ \tilde{G} = \frac{x'_+ - x_+}{x'_- - x_+} e^{-i \frac{p'}{2}}, \quad \tilde{H} = \frac{\gamma_p}{\gamma_{p'}} \frac{x'_+ - x'_-}{x'_- - x_+} e^{i \frac{(p' - p)}{2}}, \]

\[ \tilde{L} = \frac{x'_- - x_-}{x'_- - x_+} e^{i \frac{p'}{2}}, \quad \tilde{K} = \frac{\gamma_{p'}}{\gamma_p} \frac{x_+ - x_-}{x'_- - x_+}, \] (B.4.10)

with \( x_\pm \equiv x_\pm(p) \), \( x'_\pm \equiv x_\pm(p') \),

\[ \gamma_p = |x_+ - x_-|^{1/2}, \quad \gamma_{p'} = |x'_+ - x'_-|^{1/2}. \]

and

\[ \tilde{S}_0 = \frac{1 - \frac{1}{x'_+ x_+}}{1 - \frac{1}{x'_- x_+}} x'_+ - x_+ e^{i \theta(p, p')} . \] (B.4.11)

As before, \( p, p' \) stand for the spin-chain momenta.
B.5 Two-particle cut at L-loops for string theory in AdS$_5 \times$S$^5$

We collect here the explicit forms s- and u-channel cuts of the L-loop four-point S-matrix of the AdS$_5 \times$S$^5$ string. Here $L = L_1 + L_2 + 1$; the S-matrix factor at $L_1$-loops $\text{--- i.e. the left factor in each term -- carries the incoming particles}$.

- The s-channel two-particle cuts are:

$$iT^{(L)}_{ab}(p, p')|_{s\text{-cut}} = \left( A^{(L_1)} A^{(L_2)} + B^{(L_1)} B^{(L_2)} + 2C^{(L_1)} F^{(L_2)} \right) \delta_a^c \delta_b^d$$

$$+(A^{(L_1)} B^{(L_2)} + B^{(L_1)} A^{(L_2)} - 2C^{(L_1)} F^{(L_2)}) \delta_a^d \delta_b^c \quad \text{(B.5.1)}$$

$$iT^{(L)}_{\alpha\beta}^{\gamma\delta}(p, p')|_{s\text{-cut}} = (D^{(L_1)} D^{(L_2)} + E^{(L_1)} E^{(L_2)} + 2F^{(L_1)} C^{(L_2)}) \delta_\alpha^\gamma \delta_\beta^\delta$$

$$+(D^{(L_1)} E^{(L_1)} + E^{(L_1)} D^{(L_2)}) \delta_\alpha^\gamma \delta_\beta^\delta \quad \text{(B.5.2)}$$

$$iT^{(L)}_{a\beta}^{\gamma\delta}(p, p')|_{s\text{-cut}} = (H^{(L_1)} K^{(L_2)} + G^{(L_1)} G^{(L_2)}) \delta_\beta^\gamma \delta_a^\delta \quad \text{(B.5.3)}$$

$$iT^{(L)}_{a\beta}^{\gamma\delta}(p, p')|_{s\text{-cut}} = (G^{(L_1)} H^{(L_2)} + H^{(L_1)} L^{(L_2)}) \delta_\beta^\gamma \delta_a^\delta \quad \text{(B.5.4)}$$

$$iT^{(L)}_{ab} C^{(L_2)} + C^{(L_1)} D^{(L_2)} - E^{(L_2)} \right) \epsilon_{ab} \epsilon^{\gamma\delta} \quad \text{(B.5.5)}$$

$$iT^{(L)}_{ab} C^{(L_2)} + D^{(L_1)} - E^{(L_1)} \right) F^{(L_2)} \epsilon_{\alpha\beta} \epsilon^{cd} \quad \text{(B.5.6)}$$

$$iT^{(L)}_{ab} C^{(L_2)} + L^{(L_1)} L^{(L_2)} \delta_\alpha^\gamma \delta_b^\delta \quad \text{(B.5.7)}$$

$$iT^{(L)}_{ab} C^{(L_2)} + L^{(L_1)} K^{(L_2)} \delta_\alpha^\gamma \delta_b^\delta \quad \text{(B.5.8)}$$

A factor of $(i)^4$ was set to unity; the first $(i)^2$ is from the two cut propagators and a second factor of $(i)^2$ is due to the fact that scattering amplitudes are $iT$ while the coefficients A, B... parametrize $T$, cf. eq. (2.4.1).

- The u-channel two-particle cuts of an L-loop amplitude: as before, $L = L_1 + L_2 + 1$; the S-matrix factor at $L_1$-loop order $\text{--- i.e. the left factor in each term -- carries the first lower index and the second upper index}:$

$$iT^{(L)}_{ab}(p, p')|_{u\text{-cut}} = A^{(L_1)} A^{(L_2)} \delta_a^c \delta_b^d$$

$$+(A^{(L_1)} B^{(L_2)} + B^{(L_1)} A^{(L_2)} - 2C^{(L_1)} F^{(L_2)}) \delta_a^d \delta_b^c$$

$$iT^{(L)}_{\alpha\beta}^{\gamma\delta}(p, p')|_{u\text{-cut}} = D^{(L_1)} D^{(L_2)} \delta_\alpha^\gamma \delta_\beta^\delta$$

$$+(D^{(L_1)} E^{(L_2)} + E^{(L_1)} D^{(L_2)} + 2E^{(L_1)} E^{(L_2)} - 2K^{(L_1)} H^{(L_2)} \delta_\alpha^\gamma \delta_\beta^\delta$$

$$iT^{(L)}_{a\beta}^{\gamma\delta}(p, p')|_{u\text{-cut}} = -C^{(L_1)} F^{(L_2)} + G^{(L_1)} G^{(L_2)} \delta_\beta^\gamma \delta_a^\delta \quad \text{(B.5.9)}$$
\[ i T^{\gamma d} (p, p') \big|_{u-cut} = (H^{(L_1)} D^{(L_2)} + A^{(L_1)} H^{(L_2)} + 2B^{(L_1)} H^{(L_2)} + 2H^{(L_1)} E^{(L_2)}) \delta^d_{\alpha} \delta^\gamma_{\beta} \]
\[ i T^{\gamma \delta} (p, p') \big|_{u-cut} = (G^{(L_1)} C^{(L_2)} + C^{(L_1)} L^{(L_2)}) \epsilon_{ab} \epsilon^{\gamma \delta} \]
\[ i T^{\gamma d} (p, p') \big|_{u-cut} = (F^{(L_1)} G^{(L_2)} + L^{(L_1)} F^{(L_2)}) \epsilon_{\alpha \beta} \epsilon^{\gamma d} \]
\[ i T^{\gamma \delta} (p, p') \big|_{u-cut} = (-F^{(L_1)} C^{(L_2)} + L^{(L_1)} L^{(L_2)}) \delta^\gamma_{\alpha} \delta^{\delta b} \]

As in the s-channel cuts, a factor of \((i)^4\) of the same origin as there was set to unity.

Clearly, to reconstruct the S-matrix element one is to consider all choices of \(L_1\) and \(L_2\) such that \(L_1 + L_2 = L - 1\). Generalized cuts can be constructed iteratively, by using (generalized) cuts in place of \(A^{L_1}, B^{L_1}, \ldots\).

### B.6 One- and two-loop integrals

The one-loop massive bubble integral in two dimensions shown in fig. 2.2(a) and 2.2(b) were computed previously in [101] using the techniques of [162]; the third integral carried no momentum dependence. The first two integrals may be compactly written as

\[ I(p, p') = \int \frac{d^2q}{(2\pi)^2} \frac{1}{(q^2 + 1 + i\varepsilon)((q + p + p')^2 + 1 + i\varepsilon)} \quad \text{(B.6.1)} \]

or, explicitly,

\[ I(p, p') = \frac{i}{2\pi m^2 p_-^2 - p'_-^2} \left\{ \begin{array}{ll}
\ln\left( \frac{p_-'}{p_-} \right) - i\pi & \text{for } 0 < p_- < p'_- \text{ or } p'_- < p_- < 0 \\
\ln\left( \frac{p_-'}{p_-} \right) & \text{for } p_- < 0 < p'_- \text{ or } p'_- < 0 < p_-
\end{array} \right. \quad \text{(B.6.2)} \]

where

\[ p_\pm = \frac{1}{2}(\varepsilon \pm p) \quad \text{and} \quad p_+ p_- = \frac{1}{4}. \quad \text{(B.6.3)} \]

Assuming that \(p_\pm, p'_\pm > 0\) and \(p_- < p'_-\) the integrals that enter all ampli-
tudes one- and two-loop amplitudes in worldsheet theories with excitations of equal masses are \[85\]:

\[
\begin{align*}
I_s &= \frac{1}{J_s} \left( -\frac{i}{\pi} \ln \frac{p_+^l}{p_-} - 1 \right) \\
I_u &= \frac{1}{J_u} \left( \frac{i}{\pi} \ln \frac{p_+^l}{p_-} + 0 \right) \\
I_t &= \frac{i}{4\pi} \\
I_a &= \left( \frac{1}{J_s} \left( -\frac{i}{\pi} \ln \frac{p_+^l}{p_-} - 1 \right) \right)^2 \\
I_d &= \left( \frac{1}{J_u} \left( \frac{i}{\pi} \ln \frac{p_+^l}{p_-} + 0 \right) \right)^2 \\
I_b &= \frac{1}{16\pi^2} \left( \frac{4}{J_u^2} \ln^2 \frac{p_+^l}{p_-} + \left( -\frac{8i\pi}{J_u^2} + \frac{2}{J_u} \right) \ln \frac{p_+^l}{p_-} + \text{rational} \right) \\
I_c &= \frac{1}{16\pi^2} \left( \frac{4}{J_u^2} \ln^2 \frac{p_+^l}{p_-} - \frac{2}{J_u} \ln \frac{p_+^l}{p_-} + \text{rational} \right) \\
I_e &= \frac{1}{16\pi^2} \left( \frac{4}{J_u^2} \ln^2 \frac{p_+^l}{p_-} - \frac{2}{J_u} \ln \frac{p_+^l}{p_-} + \text{rational} \right) \\
I_f &= \frac{1}{16\pi^2} \left( \frac{4}{J_u^2} \ln^2 \frac{p_+^l}{p_-} - \frac{2}{J_u} \ln \frac{p_+^l}{p_-} + \text{rational} \right) \\
\end{align*}
\] (B.6.4)

Integrals with different masses assuming that \(\frac{p}{m} > \frac{p'}{m'}\):

\[
\tilde{I}_s = \frac{-i}{4\pi(p\varepsilon - p'\varepsilon)} \left( \ln \left| \frac{p_+^l}{p_-} \right| - \ln \left| \frac{m'}{m} \right| - i\pi \right) \\
\tilde{I}_u = \frac{+i}{4\pi(p\varepsilon - p'\varepsilon)} \left( \ln \left| \frac{p_+^l}{p_-} \right| - \ln \left| \frac{m'}{m} \right| \right) \\
\] (B.6.5) (B.6.6)

where

\[
\varepsilon = \sqrt{p^2 + m^2}. \\
\] (B.6.7)
B.7 \( s \)- and \( u \)-channel cuts of the one- and two-loop integrals

In the construction of the two-loop \( C_{s,\text{extra}} \) and \( C_{u,\text{extra}} \) integral coefficients it is necessary to take of the ansatz (2.3.6). We list here the two-particle cuts of the integrals that appear in the text.

\[
\begin{align*}
I_s|_{s-\text{cut}} &= \frac{2}{4(p\epsilon' - p'\epsilon)} \\
I_u|_{s-\text{cut}} &= 0 \\
I_a|_{s-\text{cut}} &= \frac{2 \times 2}{4(p\epsilon' - p'\epsilon)} I_s \\
I_b|_{s-\text{cut}} &= \frac{1}{4(p\epsilon' - p'\epsilon)} (I_u + I_t) \\
I_c|_{s-\text{cut}} &= \frac{1}{4(p\epsilon' - p'\epsilon)} (I_u + I_t) \\
I_d|_{s-\text{cut}} &= 0 \\
I_e|_{s-\text{cut}} &= 0 \\
I_f|_{s-\text{cut}} &= 0 \\
I_s|_{u-\text{cut}} &= 0 \\
I_u|_{u-\text{cut}} &= \frac{2}{4(p\epsilon' - p'\epsilon)} \\
I_a|_{u-\text{cut}} &= 0 \\
I_b|_{u-\text{cut}} &= 0 \\
I_c|_{u-\text{cut}} &= 0 \\
I_d|_{u-\text{cut}} &= \frac{2 \times 2}{4(p\epsilon' - p'\epsilon)} I_u \\
I_e|_{u-\text{cut}} &= \frac{1}{2(p\epsilon' - p'\epsilon)} (I_s + I_t) \\
I_f|_{u-\text{cut}} &= \frac{1}{2(p\epsilon' - p'\epsilon)} (I_s + I_t)
\end{align*}
\]

(B.7.1)

In \( I_a \) and \( I_d \) one factor of 2 comes from the two chained bubbles and the second from the two solutions to the cut conditions.

B.8 \( \text{AdS}_3 \times \mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^1 \) S-matrices

• The S matrix of Borsato, Ohlson Sax and Sfondrini

The S matrix proposed by Borsato, Ohlson Sax and Sfondrini assigns different amplitudes depending on whether the scattered states are a left-mover and a right-mover or are two of the same kind (where the left- and right-movers are representations of one of the other \( PSU(1|1) \) factors of the \( PSU(1|1)^2 \) symmetry group of the gauge-fixed theory). There are also differences depending on the masses of the two states but for the most part this is contained within the Zhukowsky variables and we can write down the S matrix in terms of general coefficients without
having to specify what the masses are. For the \( LL \) sectors it is defined as:

\[
S_{\text{BOSS}}^{\phi\phi'} = A_{LL}^{\text{BOSS}} |\phi\phi'\rangle, \quad S_{\text{BOSS}}^{\psi\psi'} = G_{LL}^{\text{BOSS}} |\psi\psi'\rangle + H_{LL}^{\text{BOSS}} |\phi\psi'\rangle, \quad (B.8.1)
\]

\[
S_{\text{BOSS}}^{\psi\phi'} = D_{LL}^{\text{BOSS}} |\psi\phi'\rangle, \quad S_{\text{BOSS}}^{\phi\phi'} = K_{LL}^{\text{BOSS}} |\phi\phi'\rangle + I_{LL}^{\text{BOSS}} |\phi\psi'\rangle, \quad (B.8.2)
\]

with the \( RR \) sectors behaving in a completely equivalent way. For the \( LR \) sectors the \( S \) matrix is defined as:

\[
S_{\text{BOSS}}^{\phi\tilde{\phi}'} = A_{LR}^{\text{BOSS}} |\tilde{\phi}'\tilde{\phi}\rangle + C_{LR}^{\text{BOSS}} |\tilde{\psi}'\tilde{\psi} Z^-\rangle, \quad S_{\text{BOSS}}^{\phi\psi'} = G_{LR}^{\text{BOSS}} |\tilde{\psi}'\tilde{\phi}\rangle, \quad (B.8.3)
\]

\[
S_{\text{BOSS}}^{\psi\tilde{\psi}'} = D_{LR}^{\text{BOSS}} |\tilde{\psi}'\tilde{\psi}\rangle + F_{LR}^{\text{BOSS}} |\tilde{\phi}'\tilde{\phi} Z^+\rangle, \quad S_{\text{BOSS}}^{\psi\phi'} = L_{LR}^{\text{BOSS}} |\tilde{\phi}'\tilde{\psi}\rangle, \quad (B.8.4)
\]

and again the \( RL \) sectors are similar to this.

For the \( L_1 L_1 \) sector – \textit{i.e.} the \( LL \) scattering of excitation of mass \( m_1 = \alpha \) – the coefficients are given by:

\[
A_{L_1 L_1}^{\text{BOSS}} = S_{L_1 L_1} \frac{x_p^+ - x_p^-}{x_p^+ - x_p^-}, \quad D_{L_1 L_1}^{\text{BOSS}} = -S_{L_1 L_1}, \quad (B.8.5)
\]

\[
G_{L_1 L_1}^{\text{BOSS}} = S_{L_1 L_1} \frac{x_p^+ - x_p^+}{x_p^+ - x_p^+}, \quad H_{L_1 L_1}^{\text{BOSS}} = S_{L_1 L_1} \frac{x_p^+ - x_p^+}{x_p^+ - x_p^+} \bar{\eta}_p, \quad (B.8.6)
\]

\[
K_{L_1 L_1}^{\text{BOSS}} = S_{L_1 L_1} \frac{x_p^+ - x_p^-}{x_p^+ - x_p^-} \bar{\eta}_p, \quad L_{L_1 L_1}^{\text{BOSS}} = S_{L_1 L_1} \frac{x_p^- - x_p^+}{x_p^- - x_p^+} \eta_p, \quad (B.8.7)
\]

The \( R_1 R_1 \) sector is exactly the same with \( S_{L_1 L_1} = S_{R_1 R_1} \) and so are the \( L_2 L_2 / R_2 R_2 \) sectors except there the mass appearing in Zhukowsky variables are different \( m_2 = 1 - \alpha \) and the dressing phase factor could be different as well.

The coefficients of the \( L_1 L_2 \) sectors \( S \) matrix – \textit{i.e.} the \( LL \) scattering of excitation of different masses – are given by:

\[
A_{L_1 L_2}^{\text{BOSS}} = S_{L_1 L_2}, \quad D_{L_1 L_2}^{\text{BOSS}} = -S_{L_1 L_2} \frac{x_p^+ - x_p^-}{x_p^+ - x_p^-}, \quad (B.8.8)
\]

\[
G_{L_1 L_2}^{\text{BOSS}} = S_{L_1 L_2} \frac{x_p^+ - x_p^+}{x_p^+ - x_p^+}, \quad H_{L_1 L_2}^{\text{BOSS}} = S_{L_1 L_2} \frac{x_p^+ - x_p^-}{x_p^+ - x_p^-} \bar{\eta}_p, \quad (B.8.9)
\]

\[
K_{L_1 L_2}^{\text{BOSS}} = S_{L_1 L_2} \frac{x_p^+ - x_p^-}{x_p^+ - x_p^-} \bar{\eta}_p, \quad L_{L_1 L_2}^{\text{BOSS}} = S_{L_1 L_2} \frac{x_p^- - x_p^+}{x_p^- - x_p^+}, \quad (B.8.10)
\]
Again the $R_1 R_2$ sector is the same with $S_{L_1 L_2} = S_{R_1 R_2}$ and the $L_2 L_1 / R_2 R_1$ sectors only differ by change of masses and dressing phase factor.

In the $L_1 R_1$ sector the coefficients are:

$$A_{L_1 R_1}^{BOSS} = S_{L_1 R_1} \frac{1 - \frac{1}{x_p x'_{p'}}}{1 - \frac{1}{x_p x'_{p'}}}, \quad G_{L_1 R_1}^{BOSS} = -S_{L_1 R_1} \frac{\eta_p \eta_{p'}}{x_p x'_{p'}} 1 - \frac{i}{x_p x'_{p'}}. \quad (B.8.11)$$

$$D_{L_1 R_1}^{BOSS} = -S_{L_1 R_1} \frac{1 - \frac{1}{x_p x'_{p'}}}{1 - \frac{1}{x_p x'_{p'}}}, \quad F_{L_1 R_1}^{BOSS} = S_{L_1 R_1} \frac{\eta_p \eta_{p'}}{x_p x'_{p'}} 1 - \frac{i}{x_p x'_{p'}}. \quad (B.8.12)$$

$$C_{L_1 R_1}^{BOSS} = S_{L_1 R_1}, \quad L_{L_1 R_1}^{BOSS} = S_{L_1 R_1} \frac{1 - \frac{1}{x_p x'_{p'}}}{1 - \frac{1}{x_p x'_{p'}}}. \quad (B.8.13)$$

The $L_2 R_1 / L_1 R_2 / L_2 R_2$ sectors behave in exactly the same way with the possibility of a different dressing phase.

The coefficients of the $R_1 L_1$ sector are given by:

$$A_{R_1 L_1}^{BOSS} = S_{R_1 L_1} \frac{1 - \frac{1}{x_p x'_{p'}}}{1 - \frac{1}{x_p x'_{p'}}}, \quad C_{R_1 L_1}^{BOSS} = -S_{R_1 L_1} \frac{\eta_p \eta_{p'}}{x_p x'_{p'}} 1 - \frac{i}{x_p x'_{p'}}. \quad (B.8.14)$$

$$D_{R_1 L_1}^{BOSS} = -S_{R_1 L_1} \frac{1 - \frac{1}{x_p x'_{p'}}}{1 - \frac{1}{x_p x'_{p'}}}, \quad F_{R_1 L_1}^{BOSS} = S_{R_1 L_1} \frac{\eta_p \eta_{p'}}{x_p x'_{p'}} 1 - \frac{i}{x_p x'_{p'}}. \quad (B.8.15)$$

$$C_{R_1 L_1}^{BOSS} = S_{R_1 L_1}, \quad L_{R_1 L_1}^{BOSS} = S_{R_1 L_1}. \quad (B.8.16)$$

The $R_2 L_1 / R_1 L_2 / R_2 L_2$ sectors are the same with the possibility of different dressing phase.

There are further relations between the different scalar factors which are implied by the crossing equation [59]; we will not review them since they are not important for our calculation.

Phases can be added to make the S matrix satisfy the untwisted Yang-Baxter equations; it is this S matrix we will be using in the article:

$$\hat{A}^{BOSS} = A^{BOSS} e^{i \tau (p - p')} \quad \hat{C}^{BOSS} = C^{BOSS} e^{-2i p' + i\theta (p + p')}$$
\[
\begin{align*}
\hat{D}^\text{BOSS} & = D^\text{BOSS} \\
\hat{G}^\text{BOSS} & = G^\text{BOSS} e^{-i\theta'} \\
\hat{K}^\text{BOSS} & = K^\text{BOSS} e^{-i\theta''} \\
\hat{F}^\text{BOSS} & = F^\text{BOSS} e^{i\theta(p+p')} \\
\hat{H}^\text{BOSS} & = H^\text{BOSS} e^{i\theta(p'-p)} + i\theta' \\
\hat{L}^\text{BOSS} & = L^\text{BOSS} e^{i\theta'} 
\end{align*}
\] (B.8.17)

The one-loop dressing phases in both mixed and unmixed sectors were extracted in [62] by comparing the one-loop corrections to the energy of classical string solutions with the predictions of the Asymptotic Bethe Ansatz:

\[
\begin{align*}
\theta_{LL} & = \theta(x_p, x_{p'}) = -\frac{\hat{\alpha}(x_p)\hat{\alpha}(x_{p'})}{\pi (x_p - x_{p'})^2} \ln \left( \frac{x_p + 1}{x_p - 1} \right) + \text{rational}, \quad (B.8.18) \\
\theta_{LR} & = \tilde{\theta}(x_p, x_{p'}) = -\frac{\hat{\alpha}(x_p)\hat{\alpha}(x_{p'})}{\pi (1 - x_p x_{p'})^2} \ln \left( \frac{x_p + 1}{x_p - 1} \right) + \text{rational}. \quad (B.8.19)
\end{align*}
\]

The factors \(\hat{\alpha}\) are:

\[
\hat{\alpha}(x_p) = \frac{2m}{h} \frac{x^2}{x^2 - 1}, \quad x_p = \frac{p}{\varepsilon - m} \quad (B.8.20)
\]

Here \(h\) is plays the role of coupling constant. Its relation to the gauge theory coupling constant is not fixed by integrability or symmetries. At the level of the tree level worldsheet theory it is

\[
h = \frac{\sqrt{\lambda}}{2\pi} \quad (B.8.21)
\]

it potentially receives regularization scheme-dependent corrections and \(m\) is formally the mass parameter. Keeping \(m\) and \(m'\) and expanding in the small momentum limit, the one-loop phase factors are:

\[
\begin{align*}
\theta_{LL} & = \theta(x_p, x_{p'}) = -\frac{1}{2\pi h^2} \frac{p^2(p')^2 |p \cdot p' + mm'|}{(\varepsilon' p - p' \varepsilon)^2} \left( \ln \left( \frac{p_+'}{p_-} \right) - \ln \left( \frac{m'}{m} \right) \right) + \text{rational}, \quad (B.8.22) \\
\theta_{LR} & = \tilde{\theta}(x_p, x_{p'}) = -\frac{1}{2\pi h^2} \frac{p^2(p')^2 |p \cdot p' - mm'|}{(\varepsilon' p - p' \varepsilon)^2} \left( \ln \left( \frac{p_+'}{p_-} \right) - \ln \left( \frac{m'}{m} \right) \right) + \text{rational}. \quad (B.8.23)
\end{align*}
\]

We notice that these phases correspond to twice the computed expressions in
The S matrix of Ahn and Bombardelli

The S matrix proposed in [58] is somewhat simpler in that it does not depend as much on what representations the scattered states are in.

The $L_1L_1$ sector of the S matrix can be written as

\[
S^{AB}|\phi\phi'\rangle = A_{LL}^{AB}|\phi\phi'\rangle, \quad S^{AB}|\phi\psi\rangle = G_{LL}^{AB}|\phi\phi'\rangle + H_{LL}^{AB}|\phi\psi\rangle, \\
S^{AB}|\psi\psi'\rangle = D_{LL}^{AB}|\psi\psi'\rangle, \quad S^{AB}|\psi\phi\rangle = K_{LL}^{AB}|\psi\phi'\rangle + L_{LL}^{AB}|\psi\phi\rangle,
\]

with the coefficients given by:

\[
A_{L_1L_2}^{AB} = S_{L_1L_2}, \quad D_{L_1L_2}^{AB} = -S_{L_1L_2} \frac{x_p^- - x_p^+}{x_{p'}^- - x_{p'}^+}, \\
G_{L_1L_2}^{AB} = S_{L_1L_2} \frac{x_{p'}^+ - x_p^+}{x_{p'}^+ - x_p^+}, \quad H_{L_1L_2}^{AB} = S_{L_1L_2} \frac{x_{p'}^+ - x_p^- \omega_p}{x_{p'}^+ - x_p^+ \omega_{p'}}, \\
K_{L_1L_2}^{AB} = S_{L_1L_2} \frac{x_p^+ - x_{p'}^- \omega_p}{x_p^+ - x_{p'}^- \omega_{p'}}, \quad L_{L_1L_2}^{AB} = S_{L_1L_2} \frac{x_{p'}^- - x_p^-}{x_{p'}^- - x_p^-}
\]

where $\omega_p$ and $\omega_{p'}$ are chosen to be 1 and $S_{XY}$ are dressing phases. The $L_2L_2/R_1R_1/R_2R_2$ sectors are exactly the same with the appropriate changes of masses. The $L_1R_1/L_2R_1/R_1L_2/R_2L_1$ are also similar, but with a different dressing phase, while the $L_1L_2/L_2L_1/R_1R_2/R_2R_1$ and the $L_1R_2/L_2R_1/R_1L_2/R_2L_1$ sectors are also the same except that the dressing phase is set to be 1.
C.1 Simulating the lattice: Algorithms

C.1.1 The Rational Hybrid Monte Carlo Algorithm

In this appendix we review the structure of the Rational Hybrid Monte Carlo (RHMC) algorithm [133, 134, 135] which we used to simulate the Green-Schwarz string. It differs from the standard Hybrid Monte Carlo (HMC) algorithm [163] in the treatments of the fermion contribution for which it uses a rational approximation for the fractional power of the quadratic fermion matrix:

\[
(M^\dagger M)^{-\frac{1}{4}} = \alpha_0 + \sum_{i=1}^{P} \frac{\alpha_i}{M^\dagger M + \beta_i}
\]  

(C.1.1)

with real \(\beta_i\). This approximation is obtained though the Remez algorithm [151, 135] implemented e.g. in \texttt{alg_remez} which is part of the library \texttt{RHMC-on-GPU}.

C.1.2 Monte Carlo methods

A method to evaluate high-dimensional integrals such as the path integrals necessary to evaluate expectation values of operators in quantum field theories with fields \(\phi\) and Euclidean action \(S_E[\phi]\)

\[
\langle \mathcal{O}(\phi) \rangle = \frac{1}{Z} \int D\phi \, \mathcal{O}(\phi) \, e^{-S_E[\phi]}
\]  

(C.1.2)
is to randomly generate a sequence of field configurations with probability \( P(\phi) = 1/Z \exp(-S_E[\phi]) \) and then construct the “time” average

\[
\overline{O} = \frac{1}{T} \sum_{i=1}^{T} O(\phi_i) .
\]  

(C.1.3)

In the limit \( T \to \infty \) the statistical expectation value \( \langle O(\phi) \rangle \) and the time-average \( \overline{O} \) are equal up to corrections \( \mathcal{O}(T^{-1/2}) \). A similar method is used to construct the partition function except that the normalization factor of the probability, \( P(\phi) = \mathcal{N} \exp(-S_E[\phi]) \), is chosen on physical grounds. A useful algorithm uses a Markov process which generates a new field configuration \( \phi' \) from the old configuration \( \phi \) with probability \( P_M(\phi \rightarrow \phi') \) which samples the entire configuration space and satisfies

\[
P_S(\phi) P_M(\phi \rightarrow \phi') = P_S(\phi') P_M(\phi' \rightarrow \phi) .
\]  

(C.1.4)

These conditions guarantee convergence to a unique distribution \( P_S \).

It is convenient to split the generation of the new field configuration in two steps: (1) one generates a new field configuration from the old one by some method and with some probability \( P_C \) and (2) one chooses between the newly generated configuration and the old one (to be called “new configuration” if chosen) with some probability \( P_A \). Any information about the initial state will be lost after a sufficiently large number of steps. In this construction it is important that correlations between successive configurations be minimal; moreover, for the process to sample sufficiently quickly large parts of configuration space, it is useful to have a relatively large acceptance probability \( P_A \), of the order of 50 – 90%.

### C.1.3 The Hybrid Monte Carlo Algorithm

An elegant method which realizes these ideas is the Hybrid Monte Carlo (HMC) algorithm proposed in ref. [163]; a deterministic molecular dynamics evolution is used to generate new field configurations and a stochastic Metropolis acceptance test is used to select the configurations which are retained; see e.g. [164] for a thorough discussion of this algorithm and its variations.
Denoting by the index $\phi$ and $\zeta$ the bosonic fields and the pseudo-fermions (cf. eq. (4.2.4)), ref. [163] postulates the Hamiltonian

$$H_\tau = \frac{1}{2} \pi_\phi^2 + \pi_\zeta \pi_\zeta + S_\phi + S_\zeta,$$  \hspace{1cm} (C.1.5)

which describes the evolution along some fictitious direction $\tau$. All fields are assumed to depend on this fictitious coordinate. The new fields $\pi_\phi$ and $\pi_\zeta$ are interpreted as momenta conjugate to bosons and pseudo-fermions. The partition function with this Hamiltonian is the partition function with the action $S_b + S_\zeta$ up to a normalization factor from integrating out $\pi_\phi$ and $\pi_\zeta$, This factor may be found by computing the partition function with $S_\phi + S_\zeta = 0$

![Figure C.1: Sketch of the Leapfrog method: where values of fields and their conjugate momenta “leap” over each other.](image)

Given some initial field configuration $(\phi, \zeta)$ and some randomly generated momenta $\pi_\phi$ and $\pi_\zeta$ a new field configuration is constructed deterministically by solving Hamilton’s equations of motion for $H_\tau$:

$$\partial_\tau \pi_\phi = -\frac{\partial H_\tau}{\partial \phi} \equiv F_\phi \quad \partial_\tau \pi_\zeta = -\frac{\partial H_\tau}{\partial \zeta} \equiv F_\zeta \quad \text{(C.1.6)}$$

$$\partial_\tau \phi = \frac{\partial H}{\partial \pi_\phi} = \pi_\phi \quad \partial_\tau \zeta = \frac{\partial H}{\partial \pi_\zeta} = \bar{\pi}_\zeta, \quad \text{(C.1.7)}$$

where $F_\phi$ and $F_\zeta$ are bosonic and fermionic “forces”. These equations are integrated with an energy-conserving symplectic (i.e. phase space-area preserving) integrator that uses the standard leapfrog scheme: we update the values of fields and their conjugate momenta at staggered time steps, such that one is leaping over the other,
see figure C.1. For the bosons $\phi$ the evolution over $\delta\tau$ is given by

$$
\pi_{\phi,\tau+\delta\tau/2} = \pi_{\phi,\tau} + F_{\phi,\text{initial}} \frac{\delta\tau}{2}, \quad \phi_{\tau+\delta\tau} = \phi_{\tau} + \pi_{\phi,\tau+\delta\tau/2} \delta\tau \quad (C.1.8)
$$

$$
\pi_{\phi,\tau+\delta\tau} = \pi_{\phi,\tau+\delta\tau/2} + F_{\phi,\text{updated}} \frac{\delta\tau}{2}, \quad (C.1.9)
$$

The evolution of $\zeta$ and $\pi_{\zeta}$ is similar. This integration method introduces only an $O(\delta\tau^2)$ error.\(^1\)

To eliminate this artificial error the field configuration obtained after some number $n_T$ of time steps – known as an HMC trajectory – undergoes a Metropolis acceptance step \([163]\): a number $n_M \in [0,1]$ is randomly generated and the final configuration is accepted if $n_M < e^{-\delta H_{\tau}}$ where $\delta H_{\tau}$ is the change in the value of the Hamiltonian (C.1.5) between the initial and final field configuration of that trajectory. If $e^{-\delta H_{\tau}} < n_M$ then the initial field configuration is accepted as the new field configuration contributing to (C.1.3). At the beginning of each HMC trajectory – i.e. after each Metropolis test – momenta are refreshed based on a Gaussian distribution (centered at zero and with unit standard deviation), which will keep the simulation ergodic. Since in the limit $\delta\tau \to 0$ the integrator preserves the energy, the only reason $e^{-\delta H}$ might not equal unity is the presence of some error due to the finite time-step $\delta\tau$. The inclusion of the Metropolis acceptance step renders the HMC algorithm exact, with results independent of step size \([163]\) (reversibility of the dynamics is important for proving this).

\(^1\)To see this \([165]\) one compares the infinitesimal evolution with $H_{\tau} = T(\pi_{\phi,\pi_{\zeta}}) + S(\phi,\zeta)$ with the step-wise evolution operator (C.1.9),

$$
e^{-\frac{1}{2}\delta\tau S} e^{-\delta\tau T} e^{-\frac{1}{2}\delta\tau S} .
$$

Using the Baker-Campbell-Hausdorff relation

$$
e^{X} e^{Y} = e^{C(X,Y)} C(X,Y) = C_1(X,Y) + C_2(X,Y) + C_3(X,Y) + \ldots
$$

$$
C_1 = X + Y \quad C_2 = \frac{1}{2}[X,Y] \quad C_3 = \frac{1}{12}([[[X,Y],Y] - [[X,Y],X])
$$

and the fact that the action of $S$ and $T$ is given by the Poisson brackets of momenta and fields, the two evolution operators can be easily related:

$$
e^{-\frac{1}{2}\delta\tau S} e^{-\delta\tau T} e^{-\frac{1}{2}\delta\tau S} \simeq \exp\{-\delta\tau (S + T) - \delta\tau^3 ([S,T],S) + [S,T],T) + O(\delta\tau^5)\} .
$$

The antisymmetry of the Poisson bracket leads to the cancellation of the contribution of the commutator $C_2$ in the Baker-Campbell-Hausdorff formula. Thus, the two evolution operators differ by terms suppressed by a factor of $\delta\tau^2$. 

C.1.4 The fermion contribution to bosonic RHMC forces

Due to the rational approximation (C.1.1) of \((M_{\text{string}}^\dagger M_{\text{string}})^{-\frac{1}{4}}\), the fermionic contribution to the bosonic forces takes a relatively simple form, which suggests an efficient way to evaluate it. Denoting as before a generic boson by \(\phi\) and the pseudo-fermions by \(\zeta\), the pseudo-fermion contribution to the bosonic forces \(F^\zeta\) is:

\[
F^\zeta = -\frac{\partial S^\zeta}{\partial \phi} = \sum_{i=1}^{P} \alpha_i \left( \frac{1}{M^\dagger M + \beta_i} \right) \frac{\partial}{\partial \phi} \left( M^\dagger M \right) \left( \frac{1}{M^\dagger M + \beta_i} \right) \\
= \sum_{i=1}^{P} \left[ \alpha_i \left( \frac{1}{M^\dagger M + \beta_i} \right) \frac{\partial M}{\partial \phi} \left( \frac{1}{M^\dagger M + \beta_i} \right) + \alpha_i \left( \frac{1}{M^\dagger M + \beta_i} \right) \frac{\partial M^\dagger}{\partial \phi} \left( \frac{1}{M^\dagger M + \beta_i} \right) \right] \\
= \sum_{i=1}^{P} \alpha_i \left( (Ms_i)^\dagger \frac{\partial M}{\partial \phi} s_i + s_i^\dagger \frac{\partial M^\dagger}{\partial \phi} Ms_i \right), \quad (C.1.10)
\]

where \(s_i = 1/(M^\dagger M + \beta_i)\zeta\) are interpreted as the solutions to the matrix equation

\[
(M^\dagger M + \beta_i)s_i = \zeta, \quad i = 1, \ldots, P. \quad (C.1.11)
\]

In the discretized theory \(\zeta\) stands for the vector of values of the field \(\zeta\) at all lattice sites, \(M^\dagger M\) is a matrix of size (nr. of fermion components \(\times\) nr. of lattice sites)\(^2\). For example, for the cusp anomaly calculation in section 4.3 on the \(12 \times 12\) lattice, \(M^\dagger M\) has size \((16 \times 12^2) \times (16 \times 12^2)\) and \(\zeta\) stands for a \((16 \times 12^2)\)-component vector comprising all the components \((16)\) pseudo-fermions at all lattice sites \((12^2)\).

The systems (C.1.11) are solved using a multi-mass conjugate gradient solver – in particular \texttt{cg.m} routine which is part of the CUSP library [136] – the which allows for the solution of all \(P\) systems in a single solve. As long as all the shifts \(\beta_i\) are positive one can solve for the entire family of solutions at the cost of solving for a single unshifted system: one solves for the shifted system with slowest convergence\(^2\) and then the other shifted solutions can be found by an additional multiplication step [166]. The standard conjugate gradient solver is described in [167].

\(^{2}\beta_i=0\) is the slowest-converging system.
With the same notation the fermionic forces are:

\[ F_\zeta = -\frac{\partial S_\zeta}{\partial \zeta} = -\alpha_0 \frac{\partial}{\partial \zeta} (\zeta^\dagger \zeta) - \sum_{i=1}^{P} \frac{\partial}{\partial \zeta} (\zeta^\dagger \frac{\alpha_i}{M^\dagger M + \beta_i} \zeta) \]

\[ = -\alpha_0 \zeta^\dagger - \sum_{i=1}^{P} \alpha_i s_i^\dagger \]  \hspace{1cm} (C.1.12)

and are determined by the solution to the same system (C.1.11). Even though the fermionic matrix, \( M^\dagger M \), is not guaranteed to be symmetric, its hermiticity guarantees that the multi-mass conjugate-gradient method will yield a solution to (C.1.11) \([168, 167]\).

### C.1.5 A summary of sources of errors

The RHMC algorithm is considered to be an exact algorithm to machine precision \([163]\). Nevertheless, errors are introduced through the approximations made in the construction of the simulation; they have both a statistical and a systematic origin. We summarize here the ones relevant for the Green-Schwarz string in \( \text{AdS}_5 \times S^5 \) space:

- **Statistical error** – arises because the path integral (with or without additional operator insertions) is approximated in terms of a finite number of field configurations. In our calculations we employed \( \mathcal{O}(500) \) independent configurations. Statistical errors may be reduced by increasing the number of field configurations.

- **Discretization errors** – arise because of the finite lattice spacing and in the extrapolation to the continuum limit. In two dimensions the approach to the continuum limit is accompanied by a quadratic increase in the number of lattice sites which in turn leads to an increase in the computational cost at a higher rate. These errors may be reduced by employing higher-point approximations for field derivatives.

- **Other approximation errors** – arise because of the Remez algorithm used to construct a rational approximation of the inverse fractional power as well as because of the numerical errors of the multi-mass conjugate-gradient solver.
The former may be reduced by using more terms in eq. C.1.1. The latter may potentially be reduced by improving the treatment of the fermion matrix, such as using a preconditioner and other algorithms for solving the linear systems (C.1.11).

- Finite volume errors – all simulations are carried out on lattices of finite extent. Since the Compton wave length of a massive particle is proportional to its inverse mass, finite volume effects are larger for particles of smaller masses. In the context of our calculations we have estimated these errors in sec. 4.3.1.

C.2 9-point stencil

We want to construct a discrete approximation to the derivative of some function \( f(x) \) such that the error is of the order \( \mathcal{O}(a^8) \) where \( a \) is the lattice spacing. This requires a nine-point stencil using \( f(x \pm a), f(x \pm 2a), f(x \pm 3a), f(x \pm 4a) \). We expand then in Taylor series around \( a = 0 \) to \( \mathcal{O}(a^8) \) and then solve the resulting system for \( \partial_x f(x) \):

\[
f(x \pm a) = f(x) \pm a f^{(1)}(x) + \frac{1}{2} a^2 f^{(2)}(x) \pm \frac{1}{3!} a^3 f^{(3)}(x) + \frac{1}{4!} a^4 f^{(4)}(x)
\]

\[
\pm \frac{1}{5!} a^5 f^{(5)}(x) + \frac{1}{6!} a^6 f^{(6)}(x) \pm \frac{1}{7!} a^7 f^{(7)}(x) + \mathcal{O}(a^8) \quad (C.2.1)
\]

\[
f(x \pm 2a) = f(x) \pm 2a f^{(1)}(x) + 2a^2 f^{(2)}(x) \pm \frac{4}{3} a^3 f^{(3)}(x) + \frac{2}{3} a^4 f^{(4)}(x)
\]

\[
\pm \frac{2^5}{5!} a^5 f^{(5)}(x) + \frac{2^6}{6!} a^6 f^{(6)}(x) \pm \frac{2^7}{7!} a^7 f^{(7)}(x) + \mathcal{O}(a^8) \quad (C.2.2)
\]

\[
f(x \pm 3a) = f(x) \pm 3a f^{(1)}(x) + \frac{9}{2} a^2 f^{(2)}(x) \pm \frac{9}{2} a^3 f^{(3)}(x) + \frac{27}{8} a^4 f^{(4)}(x)
\]

\[
\pm \frac{3^5}{5!} a^5 f^{(5)}(x) + \frac{3^6}{6!} a^6 f^{(6)}(x) \pm \frac{3^7}{7!} a^7 f^{(7)}(x) + \mathcal{O}(a^8) \quad (C.2.3)
\]

\[
f(x \pm 4a) = f(x) \pm 4a f^{(1)}(x) + 8a^2 f^{(2)}(x) \pm \frac{32}{3} a^3 f^{(3)}(x) + \frac{32}{3} a^4 f^{(4)}(x)
\]

\[
\pm \frac{4^5}{5!} a^5 f^{(5)}(x) + \frac{4^6}{6!} a^6 f^{(6)}(x) \pm \frac{4^7}{7!} a^7 f^{(7)}(x) + \mathcal{O}(a^8) \quad (C.2.4)
\]
Requiring that
\[
\alpha \partial_x f(x) + \mathcal{O}(a^8) = e(f(x + a) - f(x - a)) - b(f(x + 2a) - f(x - 2a)) \\
- c(f(x + 3a) - f(x - 3a)) - d(f(x + 4a) - f(x - 4a) + 9)
\]

constrains the coefficients on the right-hand side to be a solution of the system
\[
e - 8b - 27c - 64d = 0 \\
e - 32b - 243c - 1024d = 0 \quad (C.2.10) \\
e - 128b - 2187c - 16384d = 0
\]
The solution \( e = 224, b = 56, c = - \frac{32}{3}, d = 1 \) leads to \( \alpha = 280 \) and to the nine-point stencil quoted in eq. (4.3.11).

### C.2.1 \( \rho \) matrices

We include here the explicit form of the matrices \( \rho^M \) entering the AdS light-cone gauge action. They are off-diagonal blocks of the six-dimensional Dirac matrices in chiral representation:
\[
\rho^M_{ij} = -\rho^M_{ji}, \quad (\rho^M)^{il}_{ij} \rho^N_{lj} + (\rho^N)^{il}_{ij} \rho^M_{lj} = 2\delta^{MN}_{ij} \delta^i_j, \quad (\rho^M)^{ij} \equiv -(\rho^M)^{ji} \quad (C.2.11) \\
\rho^{MN}_{ij} = \frac{1}{2}[(\rho^M)^{il}_{ij} \rho^N_{lj} - (\rho^N)^{il}_{ij} \rho^M_{lj}]. \quad (C.2.12)
\]

One can choose the following explicit representation for the \( \rho^M_{ij} \) matrices
\[
\rho^1_{ij} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \rho^2_{ij} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \rho^3_{ij} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},
\]
\[
\rho^4_{ij} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \rho^5_{ij} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \rho^6_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]
C.3 Fermions

After the introduction of the auxiliary fields \( \phi \) and \( \phi_M \) the fermion Lagrangian is quadratic:

\[
L_{\text{fermions}} = \psi^T M \psi .
\] (C.3.1)

For the AdS light-cone gauge action \( \psi \equiv (\tilde{\theta}^i, \tilde{\theta}^i, \tilde{\eta}^i, \tilde{\eta}^i) \) and in the background of the null cusp solution \( M \) is given by:

\[
M_{\text{cusp}} = \begin{pmatrix}
0 & i\partial_t 1_4 & -i(\partial_s + \frac{1}{2})\rho^M \tilde{z}^M & 0 \\
i\partial_t 1_4 & 0 & 0 & -i(\partial_s + \frac{1}{2})\rho^M_{1\Gamma} \tilde{z}^M \\
i\rho^M \tilde{z}^M (\partial_s - \frac{1}{2}) & 0 & -\rho^M \tilde{z}^M (\partial_s \tilde{x} - \tilde{\tilde{x}} - \tilde{x} \partial_s) & i\partial_t 1_4 + A^\dagger \\
0 & i\rho^M_{1\Gamma} \tilde{z}^M (\partial_s - \frac{1}{2}) & i\partial_t 1_4 + A & -\rho^M \tilde{z}^M (\partial_s \tilde{x}^* - \tilde{\tilde{x}}^* - \tilde{x}^* \partial_s)
\end{pmatrix}
\] (C.3.2)

with \( A \) given by

\[
A = \frac{1}{\tilde{z}^4} \tilde{z}^M \rho^{MN} \tilde{z}^N + \frac{1}{\tilde{z}^2} \tilde{z} + \frac{i}{\tilde{z}^2} \tilde{z}^N \rho^{MN} (\partial_s \tilde{z}^M) .
\] (C.3.3)

C.3.1 Code Data Structures

Each site of our lattice will contain 15 Bosons, and a 16x16 Fermion matrix. The fermions will be stored in a global matrix. The global fermion matrix that is \( N \times N \) where each element is a 16 x 16 matrix and \( N \) is the total number of lattice sites. This matrix is very large (16 \( N \times 16 \) N) so for calculation purposes will be stored as a sparse matrix. The fermions will be anti-periodic in the lattice and the bosons periodic.

C.4 Solving the Fermion Matrix

There are two methods for finding a solution to a linear system of equations: iteratively and directly. To solve our Fermion Matrix system:
\( (M_1^t M + \beta_i) s_i = \Phi \)  \hspace{1cm} (C.4.1)

We will use an iterative Krylov subspace solver. We use an iterative solver because our matrix is very large and sparse so significant speed up can be obtained iteratively.

\[ x^{(i)} = Ax^{(i-1)} + B \]  \hspace{1cm} (C.4.2)

There are two major types of iterative methods stationary and Krylov subspace. In the stationary method A and B are independent of the iteration \( (i) \). The stationary methods only converge for a small class of matrices, mainly diagonally dominant matrices. In a Krylov subspace solver A and B will be dependent or change with each iteration. The solver will choose an approximate solution that will get better at each iteration by finding an approximate solution to a higher order Krylov subspace. The standard conjugate gradient algorithm can be found in [167].

Now the system we are trying to solve is a shifted (multimass) system. As long as all the shifts are positive one can solve for the whole family of solutions at the cost of solving for a single unshifted system. In the CG-M algorithm one solves for the worst (slowest convergence) shifted system\(^3\) and can find all the other shifted solutions by an additional multiplication step.[166]

**C.4.1 Conjugate Gradient Method**

For the Conjugate Gradient method we start with a basis of \( p_k \) that are conjugate to each other in \( n \) directions. We expand our solution, \( x' \), of \( Ax=b \) in this basis.

\[ x' = \sum_{i=1}^{N} \alpha_i p_i \]  \hspace{1cm} (C.4.3)

\[ b = Ax' = \sum_{i=1}^{N} \alpha_i A p_i \]  \hspace{1cm} (C.4.4)

\(^3\beta_i=0 \) is the worst possible
We now need the coefficients $\alpha_i$

\[ p_k^T b = p_k^T A x' = \sum_{i=1}^{N} \alpha_i p_k^T A p_i \quad (C.4.5) \]

because the $p_k$ form a conjugate basis only one term survives.

\[ p_k^T b = \alpha_k p_k^T A p_k \quad (C.4.6) \]
\[ \alpha_k = \frac{p_k^T b}{p_k^T A p_k} \quad (C.4.7) \]

### C.4.2 Conjugate Gradient Multimass Method

The multimass Conjugate Gradient (CG-M) method from [166].

The crucial step from this paper is realizing that solving

\[ (A + \sigma)x - b = 0 \quad (C.4.8) \]

is the same procedure as solving:

\[ P_n^\sigma (A + \sigma) = c P_n (A) \quad (C.4.9) \]

Where $P_n (A)$ is the polynomial constructed in the Krylov space method. One can then derive the same recurrence relations as the CG method using these polynomials with a few additional terms.

The shifted algorithm directly from [166] is:

\[ x^\sigma_0 = 0, r_0 = p^\sigma_0 = b, \beta_{-1} = \zeta^\sigma_{-1} = \zeta^\sigma_0 = 1, \alpha^\sigma_0 = 0 \quad (C.4.10) \]

for each $i$ iteration we calculate:

\[ \beta_i = \frac{-r_i^T r_i}{p_i^T A p_i} \quad (C.4.11) \]
\[ \beta^\sigma_i = \beta_i \frac{\zeta^\sigma_{n+1}}{\zeta^\sigma_n} \quad (C.4.12) \]
\[ \zeta_{i+1} = \frac{\zeta_i \beta_i}{\beta_i \alpha_i (\zeta_i \beta_i - \zeta_i) + \zeta_i \beta_i - 1 (1 - \sigma \beta_i)} \]  
(C.4.13)

\[ x_{i+1} = x_i - \beta_i \sigma_i A p_i \]  
(C.4.14)

\[ r_{i+1} = r_i + \beta_i A p_i \]  
(C.4.15)

\[ \alpha_{i+1} = \frac{r_{i+1}^T r_{i+1}}{r_i^T r_i} \]  
(C.4.16)

\[ \alpha_{i+1}^\sigma = \frac{\zeta_i \beta_{i+1}^\sigma}{\zeta_i \beta_{i+1}} \]  
(C.4.17)

\[ p_{i+1}^\sigma = \zeta_i r_{i+1} + \alpha_i^\sigma p_i^\sigma \]  
(C.4.18)

This repeats until \( r_i \) is sufficiently small.

The only differences from this algorithm and the regular CG is the addition of the \( \sigma \) terms. And for whatever reason the author decided to switch \( \alpha \) and \( \beta \) from the standard notation of the CG method.

## C.5 Remez Algorithm

The Remez algorithm is an iterative approach to approximating a function. The algorithm finds the approximate solution by creating a system of linear equations and the iteratively solving for the best coefficients using a minimization procedure.

In our case we will be approximating:

\[ (M^\dagger M)^{-\frac{1}{2}} = \alpha_0 + \sum_{i=1}^{15} \frac{\alpha_i}{(M^\dagger M) + \beta_i} \]  
(C.5.1)

The \( \alpha \) and \( \beta \) coefficients will directly come from the AlgRemez code[151, 135, 169] with an overall error of \( 2 \times 10^{-5} \).
Bibliography


[54] ABBOTT, M. C. (2013) “Comment on Strings in AdS3 x S3 x S3 x S1 at One Loop,” JHEP, 1302, p. 102, 1211.5587.


140


Vita

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