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ASSESSING UNDERSTANDING OF THE CONCEPT OF FUNCTION: A STUDY COMPARING PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' RESPONSES TO MULTIPLE-CHOICE AND CONSTRUCTED-RESPONSE ITEMS

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ABSTRACT

The purpose of this study was to determine whether multiple-choice and constructed-response items assessed prospective secondary mathematics teachers' understanding of the concept of function. The conceptual framework for the study was the Dreyfus and Eisenberg (1982) Function Block. The theoretical framework was Sierpinska's (1992, 1994) Indicators of Mathematical Understanding: identification, discrimination, generalization, and synthesis. Fourteen prospective secondary mathematics teachers were recruited, nine from a land-grant university and five from a private college.

Three sets of items identified as assessing the concept of function were administered during the study. Two sets were taken from items released by the Educational Testing Service *PRAXIS Series*™, 21 multiple-choice items and 5 constructed-response items. A third set, the researcher-developed constructed-response items (RD CR), was based on a framework developed by the investigator. Four advisors in mathematics and mathematics education reviewed the framework as appropriate for creating items and recommended the inclusion of six RD CR items for the study.

Participants met with the researcher for four sessions to respond to the three sets of items in a problem-solving interview setting. They shared their process of solution aloud as they solved each item. Audio and video recordings were made of each session and then transcribed for purposes of analysis. Both quantitative and qualitative methodologies were used to analyze data: quantitative (correctness of response, subjects' rank on different type of items) and qualitative (foci of items, correctness of mathematical reasoning, mathematical reasoning involving the function concept, extended Function Block, strategies, and Sierpinska's Indicators of Mathematical Understanding).

The investigator's findings affirm the potential of the constructed-response items, particularly the RD CR items, to assess a subject's understanding of the concept of function as characterized by Sierpinska's Indicators of Mathematical Understanding, particularly generalization and synthesis. The investigator questions the potential of the multiple-choice items to assess accurately a subject's understanding of the concept of function because of the implementation of multiple-choice test-taking strategies (matching, elimination, substitution, etc.) and the use of mathematical reasoning not involving the concept of function.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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DEDICATION

To my Mom and my Dad…

My loving Parents,

My first Teachers,

My strongest Supporters,

My constant Encouragers,

My best Friends!

Chapter 1

INTRODUCTION

The mediocre teacher tells.

The good teacher explains.

The superior teacher demonstrates.

The great teacher inspires.

William Arthur Ward

There is no doubt that the experts in the field of mathematics education value the understanding of mathematics (NCTM, 1989, 1991, 1995, 2000, 2005; National Research Council, 2001; CBMS, 2001, 2012; National Governors Association for Best Practices (NGA) and Council of Chief State School Officers (CCSSO), 2010). It is the investigator's hope that the results of this study will enable those in the mathematics education community to learn more about the understanding of prospective teachers, about how a "deep conceptual understanding of mathematics" (CBMS, 2001) can be assessed, and about the characteristics of items that assess this level of understanding. In so doing, this study will contribute to the refinement of a definition of *understanding* and will further the understanding of mathematics educators regarding the instruction and assessment of understanding for prospective teachers.

This investigator has adopted a framework of understanding mathematics that is well accepted in the field of mathematics education (Sierpinska, 1994) and has used this framework to examine the understanding of mathematics of prospective teachers. The area of mathematics that will be the focus for this study is the concept of function, a unifying topic in the study of all levels of mathematics.

Recommendations for Teacher Preparation

Three professional organizations in the field of mathematics—the American Mathematical Society (AMS), the Mathematics Association of America (MAA), and/or the National Council of Teachers of Mathematics (NCTM)—have proposed recommendations for mathematics prospective teachers to encourage them to attain excellence in the field of mathematics.

In 1991, NCTM published the *Professional Standards for Teaching Mathematics.* This document states that the "education of teachers of mathematics should develop their knowledge of the content and discourse of mathematics, including

- mathematical concepts and procedures and the connections among them;
- multiple representations of mathematical concepts and procedures;
- ways to reason mathematically, solve problems, and communicate effectively at different levels of formality" (p. 132).

These directives stress the importance of knowing mathematics and such phrases as *connections among them* infer that such knowledge includes understanding.

The Mathematical Education for Teachers (referred to as MET I in this paper) states that teachers of mathematics "need a solid understanding of mathematics so that they can teach it as a coherent, reasoned activity and communicate its elegance and power" (Conference Board of the Mathematical Sciences (CBMS), 2001, p. xi). MET I makes 11 recommendations concerning the mathematics education of prospective teachers. These recommendations emphasize that "prospective teachers need … to develop a deep understanding of the mathematics they will teach" (CBMS, 2001, p. 7); that prospective teachers need to "learn mathematics in a coherent fashion that emphasizes the interconnections among theory, procedures, and applications" rather than "superficial coverage of many topics" (CBMS, 2001, p. 8); prospective teachers need to learn how to learn mathematics as well as to teach mathematics; and they need to learn how to ask good mathematical questions, as well as find solutions, and to look at problems from multiple points of view (CBMS, 2001, p. 8).

In 2005, NCTM published a position statement on highly qualified mathematics teachers that states:

> Every student has the right to be taught mathematics by a highly qualified teacher—a teacher who knows mathematics well and who can guide students' understanding and learning. A highly qualified teacher understands how students learn mathematics, expects all students to learn mathematics, employs a wide range of teaching strategies, and is committed to lifelong professional learning. (NCTM, 2005)

In order that prospective teachers have sufficient mathematics backgrounds to understand mathematics at the level to which they aspire, NCTM expects prospective secondary mathematics teachers to complete the mathematics coursework comparable to that required for a major in mathematics; middle-level mathematics teachers to complete the mathematics coursework comparable to that required for a minor in mathematics; and elementary mathematics teachers to complete the equivalent of at least three college-level mathematics courses that "emphasize the mathematical structures essential to the elementary grades (including number and operations, algebra, geometry, data analysis, and probability). Furthermore, all teachers need to know how mathematics is used in interpreting the statements, solutions, and questions of students, using such responses to build future understandings" (NCTM, 2005, pp. 1–2).

The professional mathematics education organizations are not alone in offering suggestions regarding what training is necessary to produce a highly qualified mathematics teacher, one who aspires to greatness. President George W. Bush, the Congress of the United States of America, and the National Governors Association (NGA) have demonstrated that they, too, value teacher excellence.

Shortly after taking office in 2001, President Bush announced the "No Child Left Behind" (NCLB) program, his "framework for educational reform that he described as 'the cornerstone of my Administration'" (Usiskin & Dossey, 2004, p. 4). Less than a year later, the Congress of the United States of America passed the NCLB Act of 2001. The NCLB legislation defines "highly qualified teachers" as those who have a bachelor's degree and full state certification or licensure.

On April 18, 2006, President Bush issued a Presidential Executive Order (13398) establishing the National Mathematics Advisory Panel "to help keep America competitive, support American talent and creativity, encourage innovation throughout the American economy, and help State, local, territorial, and tribal governments give the Nation's children and youth the education they need to succeed." This panel was commissioned to propose recommendations for the design of standards and assessment in promoting mathematical competence, for improving mathematics learning, for the training and professional development of teachers of mathematics, and for supporting research in mathematics education. One of their recommendations states that "teachers cannot teach what they do not know" acknowledging the importance of mathematical content knowledge for all mathematics teachers (National Mathematics Advisory Panel, 2008). The document further states that, "a critical component of this recommendation is that teachers be given ample opportunities to learn mathematics for teaching. Teachers must know in detail the mathematical content they are responsible for teaching and its connections to other important mathematics, both prior to and beyond the level they are assigned to teach" (National Mathematics Advisory Panel, 2008, p. 37).

The governors of the United States of America and their education commissioners through their representative organizations, the NGA and the Council of Chief State School Officers (CCSSO), have led the initiative to develop the *Common Core State Standards* (CCSS). The Mission Statement of the *Common Core State Standards Initiative* states that

> The Common Core State Standards provide a consistent, clear understanding of what [precollege] students are expected to learn, so teachers and parents know what they need to do to help them. The standards are designed to be robust and relevant to the real world, reflecting the knowledge and skills that our young people need for success in college and careers. With American students fully prepared for the future, our communities will be best positioned to compete successfully in the global economy. (CCSS, 2010)

CCSS (2010) provides a list of Standards for Mathematical Practice, a list of Standards for each grade level K–8, and a list of standards by topic for the study of mathematics in high school. These topics include: Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability. Assessment tools for evaluating accurately student progress toward college- and career-readiness aligned with the CCSS are currently under development and scheduled to be completed by the 2014–2015 academic year.

The professional mathematics organizations as well as the various branches and committees of the government of the United States of America agree on the importance of mathematics teachers having an understanding of mathematics. However, much research is needed on the specific mathematical knowledge and instructional skill necessary for effective teaching as well as on the means to assess this knowledge and skill, on the effects of teachers' knowledge and skill on their students learning, on what understanding of mathematics is necessary for teaching, and how that compares with the knowledge of mathematics attained by someone who majors in mathematics (Hill, Sleep, Lewis, & Ball, 2007).

Teacher Certification

In order for a teacher to receive a teaching certificate in many states, including the Commonwealth of Pennsylvania, s/he must receive a degree from an accredited institution and pass a licensure examination. According to the American Educational Research Association, the American Psychological Association, and the National Council on Measurement in Education (1999), tests used for credentialing purposes (licensure and certification) should focus on a candidate's current skill, knowledge, or competency in a given domain. *The PRAXIS Series*™, the Educational Testing Service (ETS) Educator Licensure Assessments, consists of content-specific tests, pedagogical tests, and basic skills tests used to inform decisions regarding teacher certification. The content-specific tests are designed to measure knowledge of core subject area such as mathematics. Each state-licensing agency determines its own threshold for the minimum teaching knowledge score and subject knowledge score required of entry-level teachers in that state.

These tests are most important because a teacher who possesses a certificate in a specific area can frequently attain a second certificate in a different area merely by successfully achieving a satisfactory score on a different content-specific test; no additional course work is required. In fact, some teachers who have little or no formal mathematics courses have attained a middle-grades mathematics teaching certificate (e.g., those previously certified in special education).

With the advent of CCSSM (NGA & CCSSO, 2010), new accreditation organization with significantly different standards for teacher preparation is evolving. In the past, accreditation requirements for mathematics have often been met by reporting results on tests such as the *ETS Praxis Series*™ or grades for appropriate courses. *The Mathematical Education of Teacher II* (MET II) states that "new requirements for mathematics courses will be similar in nature to the current, more detailed, accreditation requirements for methods courses. The standards for these courses have changed to include standards for mathematical practice and to reflect the content of the CCSS" (CBMS, 2012, p. 15).

This investigator believes that teaching mathematics requires much more than a bachelor's degree and certification. Teaching mathematics at any level requires that teachers "have an extensive knowledge of mathematics, including the specialized content knowledge specific to the work of teaching, as well as knowledge of the mathematics curriculum and how students learn" (NCTM, 2005, p. 1). In order to be sure that no child *is* left behind, we must be certain that all mathematics teachers understand mathematics and how students learn and understand mathematics. Our prospective mathematics teachers must have the opportunities they need to assist them as they prepare to become mathematics teachers.

Concept of Function

Mathematicians and mathematics educators agree that the study of functions is a unifying theme in the study of modern school mathematics because the concept of function is a central idea of both pure and applied mathematics at all levels. NCTM (1989, 1991, 1995, 2000) has organized the study of mathematics in four domains: Number and Operation; Algebra and Function; Geometry and Measurement; and Data Analysis, Statistics, and Probability. MET I uses the same four domains and adds a fifth domain for those who wish to teach mathematics in high school: Discrete Mathematics and Computer Science (CBMS, 2001). The *Common Core State Standards for Mathematics* (CCSSM) (NGA & CCSSO, 2010) organizes the study of mathematics in high school into six domains: Number and Quantity, Algebra, Functions, Modeling, Geometry, and Statistics and Probability, and notes (+) those standards necessary for readiness at the postsecondary level.

The Algebra Standard of the *Principles and Standards for School Mathematics* (NCTM, 2000, pp. 222 and 296) claims that instructional programs should enable all students to understand patterns, relations, and functions; to represent and analyze mathematical situations and structures using algebraic symbols; to use mathematical models to represent and understand quantitative relationships; and to analyze change in various contexts.

Beginning with an understanding of patterns, students begin to learn relationships among quantities. In the middle grades (5–8) students develop an understanding of proportional reasoning through the study of linear relationships. Students focus on learning about linear relationships and recognize that not all relationships can be described in this way.

At the middle level, the expectations focus on identifying the properties of linear and nonlinear functions; on basic understandings of linear functions, especially on understanding the meaning of intercept and slope; on modeling problems using multiple representations of linear functions, and on using graphs to analyze linear relationships (NCTM, 2000, p. 222).

Teachers of the middle grades must understand all kinds of problems dealing with the concept of rate of change. They should understand how technology can enhance an individual's understanding of the concept of function and should be competent in the use of graphing calculators and computers and should feel comfortable in using technology in their classrooms.

As students grow in their understanding, they recognize that there are classes of functions, frequently referred to as *function families*. In high school, students study these function families, including polynomial functions, logarithmic functions, exponential functions, and trigonometric functions, and learn how these functions are used to represent mathematical structures. Through the use of technology, students can deepen their understanding of function families through multiple representations such as graphs, tables, symbols, and so on. Expectations are also extended to include explicitly defined and recursively defined functions, piecewise-defined functions, composite functions and inverse functions, parametric equations, and the use of such technology as computer algebra systems, graphing calculators, spreadsheets, and dynagraphs.

At the secondary level, mathematics teachers must understand the connections between algebra, geometry, discrete mathematics, and calculus as well as technology related to the understanding of these ideas. They need to deepen their understanding of the concept of function and families of functions, particularly polynomial, exponential and logarithmic, rational, and periodic functions. Prospective secondary mathematics teachers should study the concept of function in such mathematics courses as calculus, differential equations, linear algebra, introduction to analysis, probability and statistics, and discrete mathematics. Through the use of technology while studying such mathematical topics as derivatives, integrals, multiple representations, mathematical modeling, sequences and series, teachers can enhance their own understanding of the concept of function and thus be empowered to assist their students to deepen their own understandings.

CCSSM (NGA & CCSSO, 2010) introduces the function topic in Grade 8 and claims that at this level students should be able to define, evaluate, and compare functions, and to use functions to model relationships between quantities. As students progress into high school, they are expected to interpret functions by understanding the concept of function and using function notation, interpreting functions that arise in applications in terms of the context, and analyzing functions using different representations; to build functions by modeling a relationship between two quantities and using existing functions to build new functions; to use linear, quadratic, and exponential models of functions by constructing and comparing models and solving problems and interpreting expressions for functions in terms of the situations modeled; and to use trigonometric functions by modeling periodic phenomena and proving and applying trigonometric identities. Although CCSSM (NGA & CCSSO, 2010) postpones the discussion of function to Grade 8, the topics addressed in Grades 9 through 12 are similar to those of the *Principles and Standards for School Mathematics* (NCTM, 2000).

Because *understanding* is not well-defined in the field of mathematics education at this time, this investigator has adopted a theoretical framework of mathematical understanding that is well accepted in the field. The framework upon which this study is based is Sierpinska's (1992, 1994) Indicators of Mathematical Understanding because it focuses on the understanding that a subject possesses.

Indicators of Mathematical Understanding

Because the concept of function is such a unifying and important topic in the study of all mathematics, including algebra, geometry, discrete mathematics, and calculus (NCTM, 2000), in this study the investigator will examine prospective secondary mathematics teachers' understanding of the concept of function using Sierpinska's (1994) Indicators of Mathematical Understanding (see Chapter 2 for the rationale for this choice of theoretical framework). Sierpinska's (1994) Indicators of Mathematical Understanding are identification, discrimination, generalization, and synthesis (pp. 56–60).

The initial indicator of mathematical understanding is *identification*. An individual who understands a mathematical object must be able to identify it, recognize it, and classify it. The individual may or may not be able to name the mathematical idea. These observations support Sierpinska's (1994) claim that understanding does not necessarily imply knowledge (p. 56).

The second indicator of mathematical understanding is *discrimination*. An individual who understands a mathematical object must have the ability to discriminate between two objects by identifying particularly the differences in these objects. This individual will be able to compare and contrast these objects and identify not only the similarities, but most importantly, the differences. The degree of abstraction that the individual demonstrates in the discrimination process is indicative of the depth of understanding (Sierpinska, 1994, p. 57).

A third indicator of mathematical understanding is the *generalization* operator. An individual who understands a mathematical object must have the ability to generalize, to identify one mathematical object/situation as a particular case of another more general one. This is a complicated characteristic due to the multiplicity of levels of generalizations as well as the interaction among these levels as understanding evolves. Regardless, only an individual who can identify the mathematical object will have the ability to use this object in formulating generalizations. (Sierpinska, 1994, p. 58–60).

The fourth and final indicator of mathematical understanding is *synthesis*. An individual who understands a mathematical object must have the ability to *synthesize*, to seek and find a "common link, a unifying principle, a similarity between/among several generalizations" (Sierpinska, 1994, p. 60) and identify the relationship between the new mathematical object and ones previously understood.

Purpose of the Study

In this study the investigator used these indicators in three ways:

1. to propose guidelines for the creation of constructed-response items that will elicit exemplification of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding, especially generalization and synthesis;
2. to evaluate the potential of various questions proposed in two formats (multiple choice and constructed response) to assess mathematical understanding of the concept of function; and
3. to identify the indicators of mathematical understanding evident as the subjects solve the above-mentioned items on the concept of function in a retrospective interview setting.

The investigator studied how a collection of multiple-choice items and two collections of constructed-response items assessed prospective secondary mathematics teachers' understanding of the concept of function[1]. In addition to the subjects' performance on these instruments, she used the framework provided by Sierpinska's (1992, 1994) Indicators of Mathematical Understanding to examine the indicators that these prospective teachers applied as they generated their responses to these items. Finally, she identified characteristics of those items that elicit a "deep conceptual understanding of mathematics" (CBMS, 2001) as indicated by the subjects' application of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding in their responses.

[1] The study's research questions are given in Chapter 3.

Chapter 2

REVIEW OF LITERATURE

The purpose of this study is to examine how two types of items (multiple-choice and constructed-response) assess prospective secondary mathematics teachers' understanding of the function concept. In order to situate this study with the current literature, this review will address the following topics: the importance of understanding in mathematics, the concept of function, the models of understanding from which the researcher selected her theoretical framework, the relevance of assessment items in evaluating understanding, and construct equivalence of multiple-choice and constructed-response items.

Importance of Understanding in Mathematics Education

In 2000, the NCTM published the *Principles and Standards for School Mathematics* (PSSM). The six principles for school mathematics address such themes as equity, curriculum, teaching, learning, assessment, and technology (NCTM, 2000, p. 11). The Teaching Principle states that "effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (NCTM, 2000, p. 11). Furthermore, "to be effective, teachers must know and understand deeply the mathematics they are teaching and be able to draw on that knowledge with flexibility in their teaching tasks" (NCTM, 2000, p. 17). Teachers must be seriously committed to the "development of students' understanding of mathematics" (NCTM, 2000, p. 18) as well as to the ongoing development of their own understanding of mathematics. "The teacher is responsible for creating an intellectual environment where serious mathematical thinking" (NCTM, 2000, p. 18) is encouraged strongly and nurtured daily. In this manner, the teacher promotes and supports the Learning Principle which states that, "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge" (NCTM, 2000, p. 20).

In 2001, AMS in conjunction with the MAA published *The Mathematical Education of Teachers* (referred to as *MET I* in this paper). However, much has changed since then. For example, "the attention given by the mathematics profession to the mathematical education of teachers has increased as more mathematicians and statisticians have taken increasingly active roles in teacher preparation and content-based professional development for current teachers" (CBMS, 2012, p. xi). These changes precipitated the publishing of *The Mathematical Education of Teachers II* (*MET II*) in 2012. One of the main purposes of *MET I* was to emphasize that "the mathematical knowledge needed for teaching is quite different from that required by college students pursuing other mathematics-related professions" (CBMS, 2001, p. xi). The primary recommendation offered in this document states "prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach" (CBMS, 2001, p. 7). *MET I* acknowledges that the foundation for this deep understanding of school mathematics must be laid during the formal college education of the prospective teachers recognizing "that college mathematics courses should be designed to prepare prospective teachers for the life-long learning of mathematics, rather than to teach them all they will need to know" (CBMS, 2001, p. 14). The primary recommendation proposed in *MET II* is that "prospective teachers need mathematics courses that develop a solid understanding of the mathematics they will teach" (CBMS, 2012, p. 17). *MET II* explains

> The mathematical knowledge needed by teachers at all levels is substantial yet quite different from that required in other mathematical professions. Prospective teachers need to understand the fundamental principles that underlie school mathematics, so that they can teach it to diverse groups of students as a coherent, reasoned activity and communicate an appreciation of the elegance and power of the subject. Thus, coursework for prospective teachers should examine the mathematics they will teach in depth, from a teacher's perspective. (CBMS, 2012, p. 17)

In 2010 the National Governors Association (NGA) and the Council of Chief State School Officers (CCSSO) led the initiative to develop the *Common Core State Standards for Mathematics (CCSSM)*. "These standards define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it" (NGA & CCSSO, 2010, p. 4). Asking a teacher to assess what the students understand implies that the teacher, too, must be able to understand the mathematics that they are teaching.

The *CCSSM* (2010) are organized into domains, clusters, and standards that are intended to move away from the "broad and shallow approach of current curricula to an emphasis on mastery of topics through procedural fluency and conceptual understanding" (Moursund & Sylwester, 2013, p. 38). The K–8 standards are organized by grade level with an "increased focus of content within grade levels, and an increased coherence of content from grade-to-grade" (Moursund & Sylwester, 2013, p. 38). The secondary school standards are organized within six conceptual categories: Number and Quantity, Algebra, Functions, Modeling, Geometry, Statistics and Probability. The content standards identify the critical issues or domains that teachers must address at each level, together with detailed objectives or clusters that students need to learn to demonstrate an understanding of the domains identified. In addition to the content standards, there are also standards for mathematical practice built upon the NCTM (2000) Process Standards of problem solving, reasoning and proof, communication, representation, and connections and the strands of mathematical proficiency specified in the National Research Council's report (2001):

> adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy). (NGA & CCSSO, 2010, p. 6)

The *CCSSM* Mathematics Standards for Mathematical Practice include making sense of problems and persevering in solving them, reasoning abstractly and quantitatively, constructing viable arguments and critiquing the reasoning of others, modeling with mathematics, using appropriate tools strategically, and attending to precision (NGA & CCSSO, 2010).

In NCTM's *Essential Understanding Series*, Zbiek (2011) states that teachers "must draw on a mathematical understanding that is both broad and deep. The challenge is to know considerably more about the topic than you expect your students to know and learn" (p. vii). Such a knowledge base is needed to plan learning activities, to evaluate curriculum, and to create lessons. Teachers need to anticipate the needs of their students who will demonstrate the need for remediation and lesson enhancement. Meaningful learning experiences capitalize on resolving misconceptions and learning from mistakes as well as challenging students to make meaningful connections with the mathematics they already know and understand.

Research in all areas of mathematics further confirms the importance of understanding in the learning process. For example, Ball (1990) investigated the understanding of elementary and secondary teachers' understanding of division and Simon (1993) conducted research on elementary teachers' understanding division. Simon and Blume (1994) studied prospective elementary teachers' understanding of multiplicative relationships. Heid (1988) studied the understanding of calculus students while resequencing of skills and concepts within the course. Szydlik (2000) investigated university calculus students' beliefs about mathematics and the role those beliefs play in the conceptual understanding of the limit of a function.

The value of understanding in mathematics education is widely affirmed. If students need to understand the mathematics they are learning, teachers need to understand the mathematics they are teaching. The mathematics education community must demonstrate the value it places on understanding by its support of efforts to enhance the understanding of mathematics by students and teachers alike. Ways of characterizing understanding in mathematics are addressed in the Theoretical Models of Understanding section.

The Concept of Function

The development of the function concept, one of the most fundamental topics in mathematics, has revolutionized and has transformed the study and development of mathematics. While the function concept may be a unifying one in unrelated branches of mathematics, this concept is also a most complex one (Selden & Selden, 1992). In fact, the study of function is one of the most difficult concepts for students to master perhaps due to its complexity and the numerous subnotions associated with it (Eisenberg, 1991, p. 140). Yet its importance in the understanding of mathematics cannot be disputed. The concept of function becomes increasingly important as students progress in the depth and breadth of their understanding of mathematics (Yerushalmy & Schwartz, 1993).

The study of functions is a unifying theme in the study of modern school mathematics, including algebra, geometry, discrete mathematics, and calculus (NCTM, 2000). Beginning with an understanding of patterns, students begin to learn relationships among quantities. In the middle grades, students learn about linear relationships and recognize that not all relationships can be described in this way. The expectations focus on identifying the properties of linear and nonlinear functions; on basic understandings of linear functions, especially understanding the meaning of intercept and slope; on modeling problems using multiple representations of linear functions, and on using graphs to analyze linear relationships (NCTM, 2000, p. 222). As they grow in their understanding, students recognize that there are classes of functions, frequently referred to as *function families*.

In high school, students study polynomial functions, logarithmic functions, exponential functions, and trigonometric functions and learn how these functions are used to represent mathematical structures. Students deepen their understanding of function families through multiple representations such as graphs, tables, and symbols particularly when they use dynamic tools, such as computers and graphing calculators (NCTM, 2000).

In this researcher's opinion, the study of algebra as symbolic manipulation needs to be reoriented toward the study of algebra as functional representation. Teachers should understand multiple representations, effects of parameters on various families of functions, and how to use technology to enhance their own understanding of mathematics as well as that of their students. At the secondary level, mathematics teachers should understand the connections between algebra, geometry, discrete mathematics, and calculus as well as technology related to the understanding of these ideas. They should deepen their understanding of the concept of function and families of functions, particularly polynomial, exponential and logarithmic, rational, and periodic functions.

MET I states that "for functions of one and two variables, teachers should be able to:

- recognize patterns in data that are modeled well by each important class of functions;
- identify functions associated with relationships such as $f(xy) = f(x) + f(y)$ or $f'(x) = kf(x)$ or $f(x+k) = f(x)$;
- recognize equations and formulas associated with each important class of functions and the way that parameters in these representations determine particular cases;
- translate information from one representation (tables, graphs, or formulas) to another;
- use functions to solve problems in calculus, linear algebra, geometry, statistics, and discrete mathematics;
- use calculator and computer technology effectively to study individual functions and classes of related functions. (CBMS, 2001, pp. 132–133)

At the secondary level, the expectations are extended to additional families of functions including exponential, polynomial, rational, logarithmic, periodic, and functions of two variables. Expectations are also extended to include explicitly defined and recursively defined functions, piecewise-defined functions, composite functions and inverse functions, parametric equations, and the use of such technology as computer algebra systems, graphing calculators, spreadsheets, and dynagraphs (NCTM, 2000, p. 296).

Prospective secondary mathematics teachers should study the concept of function in such mathematics courses as calculus, differential equations, linear algebra, introduction to analysis, probability and statistics, and discrete mathematics. Through the use of technology while studying such mathematical topics as limits, derivatives, integrals, multiple representations, mathematical modeling, sequences and series, teachers will enhance their own understanding of the concept of function and thus be empowered to assist their students to deepen their understandings.

Function Sense

A major goal of the secondary and collegiate curriculum should be to develop in students a *sense for functions*. However, some authors (Dreyfus, 1990; Eisenberg, 1992) have claimed that most college students, even those who have taken a fair number of mathematics courses, do not have much of an understanding of the function concept. Dreyfus (1990) attributed this to the fact that in college mathematics teaching there is greater emphasis on the mathematical content than on the development of cognitive processes (p. 115).

Eisenberg (1992) proposed that "having a sense for functions is a notion about having insights about functions" (p. 154); he claimed that although there appears to be no formal definition for *sense for functions*, one of the main components of having a well-developed sense for functions is the ability to relate their graphical and analytical representations (p. 154). Eisenberg (1992) argued that the ability to reverse the path of development (using multiple representations of functions) is another component of a sense for functions (p. 174). Yet another component of a sense for functions is the ability to recognize the arbitrary nature of functions; that is, the ability to recognize that functions do not have to be described by any specific algebraic expression or by a graph of a particular shape (Romberg, Fennema, & Carpenter, 1993, p. 5).

"The problem lies not in mathematizing the visualization, but in visualizing the mathematization" (Eisenberg, 1992, p. 172). Since the concept of function is often developed in only one way, (algebraic to graphical), having students reverse the path of development (graphical to algebraic) could be used as a measure of the extent to which they understand the development in the first place or the extent to which they possess a sense of functions.

Dreyfus and Eisenberg's Function Block

In order to understand the complexity of the concept of function, Dreyfus and Eisenberg's research on the concept of function led to their development of the Function Block, a framework upon which the concept of function can be examined and analyzed. Through their study of students' intuitions (i.e., students' innate knowledge of mathematics), they developed a model of how students grew in their understanding of the concept of function, particularly focusing on settings or multiple representations, sub-concepts, and levels of the function concept.

Through their studies, Dreyfus and Eisenberg conceptualized a Function Block, a three-dimensional block structure in which the *x*-axis represents the various settings (arrow diagrams, tables, graphs, etc.), the *y*-axis the function concepts (image, zeros, equality, etc.), and the *z*-axis a "taxonomic scale of levels of abstraction and generalization (one, two, or several variables, discrete domain, etc.). The *z*-axis in itself is multi-dimensional" (Dreyfus & Eisenberg, 1982, p. 364). The Function Block continues in all directions since the number of settings, concepts, or levels of abstraction and generalization associated with the function concept may continue to grow (see Figure **2-1**).

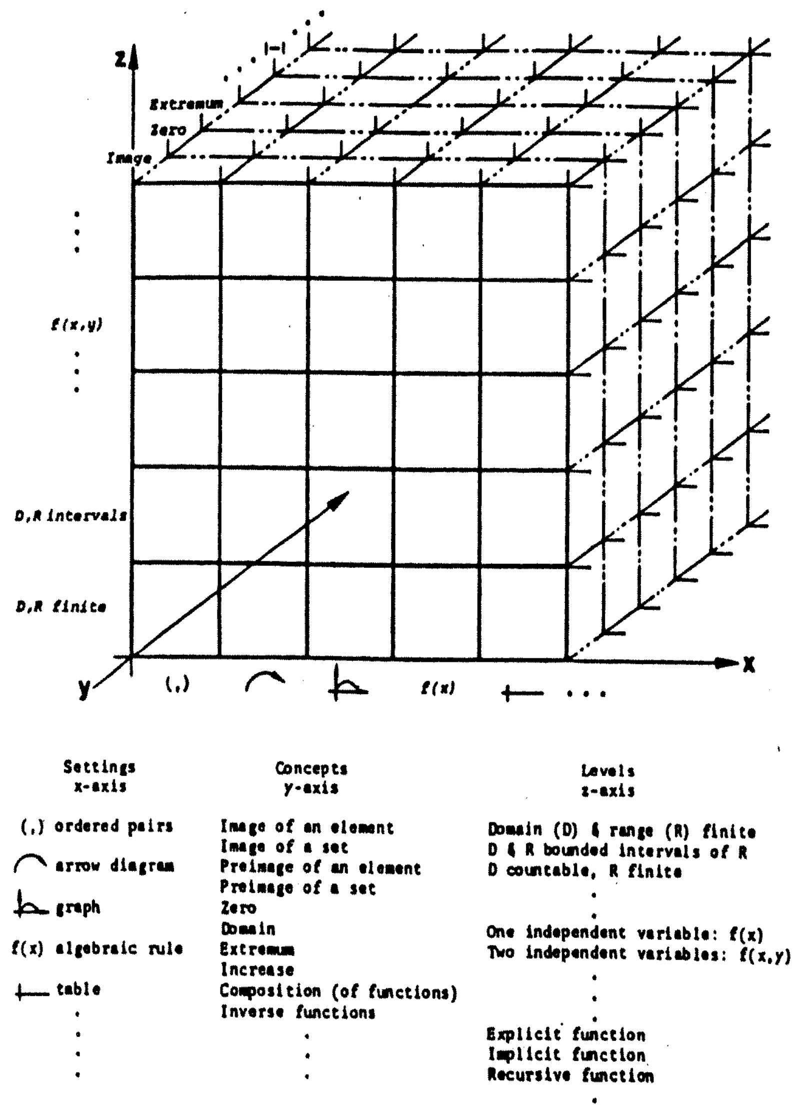

Figure **2-1**: The Function Block (Dreyfus & Eisenberg, 1982, p. 365).

This framework was built upon their study of intuitions, mathematics, and psychology of education. Transfer of learning is one of the broad underlying goals of educational programs, particularly mathematics education. Educational theorists distinguish between two types of transfer of learning, vertical transfer and horizontal transfer (Gagne, 1970). Vertical transfer contains components of generalization and abstraction; it goes beyond the information given, but remains within a particular setting. Horizontal transfer is the process of taking a concept from one setting and applying the same concept in a different setting. Fischbein (1973) claimed that "horizontal transfer builds on intuitions of adhesion, whereas vertical transfer builds on intuitions of anticipation" (p. 222). Dreyfus and Eisenberg (1984) hypothesized that an intuitively supported understanding of a concept promotes both types of transfer (p. 78).

Using this Function Block, horizontal transfer of learning (transfer of a concept learned in one setting to another setting) is represented by movement parallel to the *x*-axis. Movement parallel to the *z*-axis represents the vertical transfer of learning (transfer to levels of greater generality). Movement parallel to the *y*-axis corresponds to the learning of new concepts (Dreyfus & Eisenberg, 1982, p. 364).

In this study the Dreyfus and Eisenberg Function Block serves as a conceptual framework and enables the investigator to confirm the content validity of the instrumentation as well as the strengths and limitations in the settings, subconcepts, and levels of abstraction addressed by the manner in which the subjects chose to solve the items in the instruments.

Students' Understanding of the Concept of Function

Many scholars have affirmed the importance of understanding, particularly students' understanding of the concept of function. While the focus of this study is on studying the potential of test items to assess teachers' understanding rather than students' understanding, it is relevant to review this research as well since teachers' understanding should definitely be deeper and more comprehensive than students' understanding.

Dreyfus and Eisenberg (1982) studied students' intuitions on selected function concepts presented in diagram, graph, and table settings. They considered the students' grade, sex, setting, as well as a construct variable called Absolv, which combined the ability level of the students and the extent to which the learning environment was socially disadvantaged. They reported that pupils' intuitions on functional concepts grow with their progress through the grades and that high-level (Absolv) pupils demonstrated correct intuitions more often than low-level pupils. They found no differences in the intuitions between boys and girls in junior high school nor did they find that intuitions in concrete situations are more often correct than in abstract ones (Dreyfus & Eisenberg, 1982).

Subsequent to the study, they addressed the conjecture that “the basic mathematical concepts often have not been built from an intuitive base and that during the teaching process the students do not ‘get a feeling for’ and ‘internalize’ these concepts; that is, the concepts do not become part of them” (Dreyfus & Eisenberg, 1983, pp. 124–125). As a result, students often leave their mathematics classes with only knowledge of the mechanics. They do not learn when to apply the function concepts they were taught, let alone how to apply them. They do not think functionally, even though they have the skills to do so (Dreyfus & Eisenberg, 1983).

Dreyfus and Eisenberg (1983) also studied aspects of linearity, smoothness (differentiability), and periodicity to determine the extent to which these have been internalized by college-level students (p. 126). They found that students do not visualize the givens when they are stated algebraically, but do visualize them when they are stated graphically. In general, students seemed to feel more comfortable with the graphical exercises than with algebraic ones. However, some of the students felt that the graphical answers were not mathematical answers. This is supported by Markovits’ (as cited in Dreyfus & Eisenberg, 1983) study that revealed that students process algebraic formulae and their graphical representations as separate, unrelated phenomena. This disassociation even among college-level students “attests to the superficial nature characterizing the understanding of the function concept by a large proportion of the student population” (p. 130).

Dreyfus and Eisenberg (1984) suggested that the most effective ways to teach the mathematical notion of a function is to gather information systematically on the intuitions students have about functions and then use these intuitions for maximal transfer of learning. Since mathematical maturity influences mathematical intuitions, they questioned whether different ability groups would "elicit the same learning sequence" (Dreyfus & Eisenberg, 1984, p. 80). In Israeli junior high schools, students are assigned to ability tracks in mathematics. The subjects in this study were selected from two high-ability groups, two medium-ability groups, and two low-ability groups. Low-ability students (when still studying the same material as high-ability students) usually receive the same program of instruction as high-ability students, but at a lower conceptual level. Implicit here "is that all students learn the same way, and that they bring to a lesson the same intuitions. This, however, is not the case with mathematics in general and may not be the case with functions in particular" (Dreyfus & Eisenberg, 1984, p. 80). It thus appears that there is a need for teachers to first assess the basic intuitions and experiences various student populations have concerning functions, and then how these can be drawn upon for maximal transfer of learning. First, they gathered extensive baseline data on the intuitions various student populations have on the notion of a function and its associated concepts. Their study dealt with the initial measurements on such intuitions. The concepts selected for the purpose of the study were those of image, preimage, growth of a function, extrema, one-to-one valuedness, and membership in the domain and range. These concepts were examined in three settings: the functions were given in the form of a table, an arrow diagram, or a graph. All examples dealt with were on the same level of abstraction, namely finite domains and ranges. An additional dimension introduced was that some of the examples dealt with a concrete situation while others dealt with a mathematical abstraction (Dreyfus & Eisenberg, 1984, p. 80).

The results of this study suggested that high-ability students tended toward a graphical approach to the notion of a function, whereas low-ability students were attracted to pictorial presentations of the notions of a function and exhibited reluctance with respect to toward the graphical setting. Unsurprisingly, high-ability students scored consistently higher than low-ability students. These differences were more clearly evident on the abstract part than on the concrete part. Finally, no one setting (graphs, tables, or diagrams) seemed to be preferred by all ability groups. But within particular ability groups, preferences for one setting over another seem to appear. The implications of this for curriculum development are profound, for if these trends can be substantiated, then the idea of having one curriculum for all students should be re-examined (Dreyfus & Eisenberg, 1984, p. 84).

Each of the following studies focuses specifically on the students' understanding of the function concept and all emphasize the importance of technology in the learning process. Schwingendorf, Hawks, and Beinke (1992) studied the horizontal growth (growth in the breadth of the students' concept image) and the vertical growth (growth in the depth of the students' formal understanding) of students' conception of function. They found that students enrolled in computer-enhanced classes developed a broader concept image and sense of functions than the students in the traditional class. O'Keefe (1992) researched how the use of dynamic computer representations enhanced students' understanding of the function concept. Goldenberg, Lewis, and O'Keefe (1992) studied how dynamic representations affected the development of a process understanding of function. Yerushalmy (1997) examined seventh graders' ability to reason algebraically by studying their ability to make generalizations about multivariate functions. As students designed, described, and discussed various representations of bivariate functions, the author analyzed "their understandings of representations of quantities, relationships among quantities, and relationships among the representations of univariate and multivariate functions" (p. 431). Using the Algebra Sketchpad (Yerushalmy & Shternberg, 1993) to "mathematize situations qualitatively" and the Algebra Patterns (Yerushalmy & Shternberg, 1992) to describe patterns, the students developed these understandings (p. 433).

O'Callaghan (1998) studied the effects of the Computer-Intensive Algebra (CIA) program on students' understanding of the function concept. In order to evaluate understanding, he developed a Function Test based upon Sfard's Process/Object Theory. The O'Callaghan Function Test has four components: modeling, interpreting, translating, and reifying. Hollar and Norwood (1999) extended O'Callaghan's study by using his component competencies and the process-object framework to investigate the effects of a graphing-approach curriculum. Students studying functions in this manner demonstrated a significantly better understanding of functions on all four subcomponents of O'Callaghan's Function Test.

Zbiek and Heid (2001) investigated students' understanding of parameters using dynamic, visual representations. Using a dynamic utility called a slidergraph, students studied the effects of various parameters on different families of functions. This dynamic tool enabled students to visualize the effect a single parameter on the graph of different functions and encouraged students to generalize regarding the effect of that parameter on the specific family of functions.

Norwood (2002) studied students' ability to make connections between multiple representations of functions and their ability to use the various forms to solve quadratic equations. She found that students deepened their understanding of the concept of function through the use of multiple representations.

NCTM (1989, 2000) recommended that algebra be a unifying strand throughout the K–12 curriculum because "the functional approach to the emergence of algebraic thinking … suggests a study of algebra that centers on developing experiences with function and families of functions through encounters with real world situations whose quantitative relationships can be described by those models" (Heid, 1988, p. 239). Carraher and Schliemann (2007) examined the potential benefits of incorporating algebra in the elementary curriculum. They found that the students who participated in the program benefited from the activities and learned to

(a) think of arithmetic operations as functions rather than as mere computations on particular numbers;
(b) learn about negative numbers;
(c) grasp the meaning of variables, as opposed to instantiated values;
(d) shift from thinking about relations among particular numbers and measures toward thinking about relations among sets of numbers and measures;
(e) shift from computing numerical answers to describing and representing relations among variables;
(f) build and interpret graphs of linear and nonlinear functions;
(g) solve algebraic problems using multiple representation systems such as tables, graphs, and written equations;
(h) solve equations with variables on both sides of the equality; and
(i) interrelate different systems of representations for functions. (p. 694)

Although the incorporation of the study of algebra at the elementary level is still in its infancy, it shows great promise in its potential to help students better understand the concept of function as they progress to the secondary and collegiate levels. Zbiek (2011) suggests that in Grades 6–8 teachers and students continue to build on these ideas by recognizing that "functions provide a means for describing and understanding relationships among variables. They can have multiple representations—in algebraic symbols, situations, graphs, verbal descriptions, tables, and so on—and they can be classified into different families with similar patterns of change" (p. 44). She further suggests that in Grades 9–12, teachers and students should continue to grow in their knowledge and understanding of the function concept by recognizing how matrices and arithmetic and geometric sequences can be viewed as functions (Zbiek, 2010, p. 12); by characterizing various relationships according to the rate at which one quantity varies with respect to the other (Zbiek, 2010, p. 23); by classifying functions into various families of functions that model different real-world phenomena (Zbiek, 2010, p. 34); by adding, subtracting, multiplying, and dividing functions, by composing functions, and by studying the function's inverse if it exists (Zbiek, 2010, p. 70); and by studying relationships and change of functions through viewing multiple representation of functions (algebraic or symbolic, graphic, verbal, tabular representations) (Zbiek, 2010, p. 78).

Teachers' Understanding of the Concept of Function

"The centrality of the functions concept in school mathematics - in all mathematics - demands that teachers themselves have deep knowledge of functions, that teachers be cognizant of their students' understanding and potential pathologies in this area, and that teachers have the ability to bring their own knowledge to bear on the problem of facilitating their students' learning" (Norman, 1993, p. 169).

Few studies focus on inservice teachers' understanding of the concept of function. Norman (1992) studied 10 teachers, each working toward a master's degree in mathematics education, 8 of whom were currently teaching or had recently taught mathematics at the secondary level. Six of the eight teachers each had an undergraduate degree in mathematics. The teachers in this study favored informal definitions of function that were useful in determining whether relations were functions. They preferred graphical representations of functions to numerical or symbolic ones. Norman

(1992) reported a wide variability among teachers' conceptualizations of function and their degree of understanding of those concepts. Many teachers seemed to maintain a primarily instrumental knowledge of function; that is, an algorithmic knowledge of a concept or process. Determining what teachers understand about functions, how they interpret them, and how they apply their knowledge in the classroom is a crucial element of a better understanding of the learning and teaching of the function concept.

The teachers in Norman's (1992) study had not built strong connections between their informal definitions of function and what they viewed as the formal mathematical definition. Their concept images for functions were not very rich (pp. 223–224). The teachers readily identified standard examples of functions as such, but in more complex situations sometimes relied on inappropriate (and incorrect) tests of whether a relation was a function. They seemed particularly confused regarding notation and also had difficulties with nonpolynomial functions (Norman, 1992, p. 225). The teachers had some difficulty devising and identifying physical situations that entail functional relationships. However, they were quite knowledgeable about the evolution of the function concept throughout their textbooks and seemed comfortable with traditional approaches to the introduction and development of the function concept in instructional settings (Norman, 1992, p. 229). The results of this investigation suggest that secondary mathematics teachers would probably benefit from additional study of functions, particularly how learning in a technology-rich environment can enhance one's understanding of functions.

In many studies, the mathematics "teachers" have been undergraduate preservice or novice teachers rather than those with considerable, or ever a moderate amount of, experience teaching mathematics. The research design of these studies varied; some of the studies were case studies (Lloyd & Wilson, 1998; Wilson, 1994), others were teaching experiments (Heid, Blume, Zembat, MacCullough, MacDonald, & Seaman, 2002; Zbiek, 1998), while another was an open-ended questionnaire followed by an interview (Even, 1993).

Wilson (1994) examined the development of one preservice secondary teacher's understanding of the function concept while she participated in a mathematics methods course. The initial interviews revealed an understanding of functions as computational activities (e.g., function machines, point plotting, vertical line test). This was consistent with the teacher's procedural view of mathematics. While her understanding of the function concept deepened considerably during the study, Wilson acknowledged that the preservice teacher's personal beliefs concerning the teaching and learning of mathematics may negatively impact her ability to encourage a comparable growth in her own students' understanding of the function concept.

Lloyd and Wilson (1998), in another case study, investigated the content conceptions of an experienced high school mathematics teacher and linked his conceptions to their role in the teacher's implementation of a unit on functions in the reform curriculum of the Core-Plus Mathematics Project (1994). The teacher communicated a deep, personal understanding of the concept of function. His instruction utilized graphing utilities to emphasize the use of multiple representations to assist his students in understanding dependence patterns in data. The teacher was very articulate in communicating his ideas about the different relationships found in the representations. His discussions with his students enabled them to develop a better understanding of the complicated concept of function. However, this study focused on how a teacher taught the concept of function through effective and articulate communication with his students. Although the study characterized the teacher as an individual with a deep, personal understanding of the concept of function, the study did not assess or evaluate this understanding. Furthermore, the case study approach limited the potential for generalization.

Zbiek (1998) designed and taught a secondary-level mathematics education course with the goal of the course to "develop the students' mathematical understandings about computing tools through using such tools" (p. 186). This study explored the strategies used by prospective secondary school mathematics teachers to develop functions as mathematical models of real-world situations. Four general categories of technology use were identified:

1. Some students used a curve fitter to generate all of the tool's possible fitted functions for the data then chose the fitted function with the best goodness-of-fit value.

2. Other students also began using the curve fitter to develop the best fitting function from each of the different types that the curve fitter produced. However, these students did not rely solely on the results of the curve fitter, but also considered how the characteristics of the individual functions matched the features of the real-world data.
3. Still other students began by using the tool to create scatter plots but these students did not limit themselves to the models the tool could fit. Rather these students graphed functions on the same axes as the scatter plot other than the fitted functions. Their goal was to "generate a graph that matched the real-world situation as well as fit the data points" (Zbiek, 1998, p. 194).
4. Some students relied on the expectations about the real-world relationships with attention to a limited number of ordered pairs from the data set. They used ratios or formulas rather than curve fitters or other tools. (Zbiek, 1998, pp. 191–200).

Heid, Blume, Zembat, MacCullough, MacDonald, and Seaman (2002) studied prospective secondary mathematics teachers' understanding of the concept of function, particularly, multivariate functions. Although the subjects who participated in this study had completed a course in multivariate calculus, they initially found the visualization of the graph of bivariate functions quite challenging. However, their participation in the course enabled many of the subjects to become more comfortable with graphical representations of bivariate functions. This may be attributed to the "slicing planes" strategy through which these students improved their understanding of analytic/algebraic formulas for functions of several variables. These researchers believe that this study will draw them "closer to building viable models not only of how prospective secondary teachers come to understand new mathematics but of how they understand the concept of multivariate function" (p. 292).

Even (1993) investigated prospective secondary mathematics teachers' subject-matter knowledge and its interrelations with pedagogical content knowledge in the context of teaching the concept of function. Her study had two phases: In the first phase, 152 prospective secondary mathematics teachers completed an open-ended questionnaire concerning their knowledge about functions and in the second phase, an additional 10 prospective secondary mathematics teachers were interviewed after responding to the same questionnaire. She found that many of the subjects had a traditional conception of the functions such as described by Dreyfus and Eisenberg (1983), Markovits, Eylon, & Bruckheimer (1986, 1988), Thomas (1975), Vinner (1983), Vinner and Dreyfus (1989). In attempting to describe what a function is for students, many of the subjects chose to use their own concept image rather than an appropriate concept definition. While most of the subjects could differentiate between functions and relations that are not functions, usually by the procedural use of the vertical-line test, Even (1993) found that most of the subjects did not know why it was important to distinguish between functions and relations that are not functions. She also found that preservice teachers had problems in connecting various representational modes for functions (pp. 94–116). Although some elements of prospective secondary mathematics teachers' understanding of the function concept were addressed in this study as discussed previously, this study focused more on the subjects' limited conception of function and on how this interacted with their pedagogical content knowledge.

Although all of these studies focused on teachers' understanding of the concept of function, no research was identified that addressed how this understanding of the concept of function is assessed through standardized tests. Because the *ETS PRAXIS Series™* is designated by the Commonwealth of Pennsylvania to be part of its licensure process and because of the importance of teachers' understanding mathematics in order to teach mathematics to their students, this research focuses on how multiple-choice and constructed-response items created and released by ETS and constructed-response items developed by the investigator assess teachers' understanding of the concept of function. This research is critical in assessing the use of test items—multiple choice and constructed response—for assessment of understanding of mathematics, specifically, the concept of function. Consequently, the results of this research have the potential to influence the ways in which mathematics teachers are assessed as part of their credentialing process.

Theoretical Models of Understanding

The investigator examined the following constructs and/or frameworks that address the development of an individual's mathematical understanding for review to assess their applicability to her study: Dreyfus and Vinner's Concept Image–Concept Definition Theory, Sfard's Process/Object Theory of Learning, Pirie and Kieran's Recursive Model of Growth in Mathematical Understanding, and Sierpinska's Indicators of Mathematical Understanding.

Concept Image–Concept Definition

All mathematical concepts with the exception of primitive ones have formal definitions. Many of these definitions are presented to students in high school or college mathematics classes. Vinner (1983) described a *concept definition* as a verbal definition that accurately explains the concept in a noncircular way (p. 293). However, Vinner notes that the students do not necessarily use these concept definitions when deciding if a given mathematical object is an example or a nonexample of the concept. Rather, each student makes his/her decision based upon a *concept image*. Dreyfus (1990) described a *concept image* as a set of all the mental pictures associated in the student's mind with the concept name, together with all the properties characterizing them (p. 117). A "mental picture" is any kind of representation—picture, symbolic form, diagram, graph, etc. (Vinner & Dreyfus, 1989, p. 356). Since each student's concept image is a result of her/his experience with the examples and non-examples of the concept, the set of mathematical objects considered by the student to be examples of the concept is not necessarily the same as the set of mathematical objects determined by the definition (Vinner & Dreyfus, 1989, p. 356).

In other words, the concept image is a nonverbal entity associated in student's mind with the concept name. It can be a visual representation of the concept in case the concept has visual representations; it can be a collection of impressions or experiences. The visual representations, the mental pictures, the impressions and the experiences associated with the concept name can be translated into verbal forms. However, the verbal forms come into being at a later stage. To acquire a concept means to form a concept image for its name. To understand a concept means to have a concept image for it. Knowing a concept definition by heart does not guarantee understanding of the concept (Vinner, 1992, p.197).

Norman (1993) claims that although the definition of a particular concept may be readily absorbed by a student, the concept image must be constructed. "The definition might be taught, but it is the image that is learned" (Norman, 1993, p. 174). Vinner (1983) claims that in order to handle concepts, students need a concept image and not a concept definition. In fact, concept definitions that are introduced before the students have a concept image will remain inactive or even be forgotten. In thinking, almost always the concept image will be evoked (p. 293). Dreyfus (1990) makes the conjecture that sometimes the concept image is insufficient and may even become a cognitive obstacle. By progressing through a series of concept images whose evolution is conditional on overcoming cognitive obstacles, he claims that students construct knowledge dialectically (Dreyfus, 1990, p. 117).

The concept image–concept definition model was too limited for this study. Cooney, Davis, and Henderson (1975) have proposed various moves to teach mathematics effectively and have suggested that mathematics subject matter may be taught as concepts, skills, and/or generalizations. In this study, the investigator was interested not only in concepts, but in skills and generalizations as well. For this reason, she has chosen not to use the concept image–concept definition model as a construct for her study. However, this model was useful for analysis of data in the present study. Sierpinska (1994) referred to the concept image–concept definition model in her discussion of the identification indicator of mathematical understanding.

Process/Object Theory of Learning

In studying the genesis of mathematical objects, Sfard (1992) claims that the operational and structural modes of thinking are complementary (p. 61). When a person views a given notion as referring to a certain *process*, that person is thinking operationally. However, when a person views a given notion as referring to a certain *object*, that person is thinking structurally. The concepts of function can be conceived in two fundamentally different ways: structurally (as an *object*) or operationally (as a *process*). The concept of function would be considered a *process* when it is conceived as a computational procedure; it would be considered an *object* when it is conceived as an abstract entity (Sfard, 1992, pp. 17–18).

Mathematics deals with numbers, variables, functions, and the like, all of which may be considered objects. These objects are connected by relationships; they are parts of structures on these objects. Processes are composed of operations on these objects and thus transform the objects. Structures may or may not be preserved under these transformations. Using the concept of function as an example, a function may be considered a process that associates objects from the codomain with objects from the domain. At the middle and secondary school mathematics level, these objects are numbers. Any specific function may be considered a process that operates on these numbers by mapping them to other numbers. As the student progresses through the study of mathematics, s/he soon reaches a stage at which it is insufficient to consider a function as a process, operating on numbers. When the student encounters processes, such as differentiation, that operate on functions, it becomes necessary for that student to consider a function as an object upon which to be operated (Dreyfus, 1990, p. 118).

One reason for the complexity of mathematical knowledge is that most mathematical notions take the role of processes or of objects, depending on the problem situations and on the student's conceptualization. Learning about a concept includes many stages, starting with carrying out the operations of a process in concrete terms. As the learner becomes more familiar with a given process, the process takes the form of a series of operations that can be carried out in thought. The student has now achieved operational thinking with respect to this concept. At a further stage, the mental picture of this process crystallizes into a single entity, a new object. Once this is achieved, the student is able to think of this notion either dynamically as a process or statically as an object. In these terms, one of the most essential steps in learning mathematics is *objectification,* that is, making an object out of a process. One of the main aims of the mathematics curriculum is to develop operational thinking, thinking about a process in terms of operations on objects (Dreyfus, 1990, p. 118).

Many students initially view functions as procedural, as telling them what to do to x. Later they establish connections between representations of a function in several different settings, some of them visual. Eventually, some of them reach an abstract conception of the notion, on the basis of which operations on functions such as differentiation become possible. However, many of these students appear to fail to establish a viable connection between the different settings, and they perform better in the visual/graphical setting than in the analytic/algebraic one. The difficulties that Thomas (1975) identified occur at two important junctures in this development: visualization, that is, building the link between a visual/geometric and an analytic/algebraic setting; and objectification, that is, the transition to the conception of a function as a single mathematical entity (Dreyfus, 1990, pp. 120–121).

Sfard (1992) proposes a three-phase model of conceptual development identifying the transition from the operational to the structural mode of thinking. During the first phase, called *interiorization*, some process is performed on already familiar mathematical objects. The second phase, called *condensation*, is one in which the operation or process is converted into a more compact, self-contained unit. In this phase, a given process is dealt with automatically, that is, without considering its component steps. The condensation phase lasts as long as a new entity is conceived only operationally. The third phase, *reification*, involves the sudden ability to see something familiar in a new light. The fact that a person has interiorized and condensed a process does not mean that s/he can think about it structurally. Without the sudden "AHA!" experience called reification, his/her approach will remain purely operational (Kieran, 1993, p. 195; Sfard, 1992, p. 64). While the operational approach is indispensable for finding final answers to mathematical questions, it is the structural approach that makes all cognitive processes efficient (Sfard, 1992, p. 66).

Sfard's (1992) research demonstrates that students are confused by recursively defined functions, have difficulty with constant functions, and believe that functions must be algorithmic and reasonably simple (p. 74). She concludes that most students tend to think about functions as a process rather than as an object.

The investigator chose not to adopt Sfard's (1992) process/object theory of learning as a model for this study. This theory focuses more on the development of understanding than on the status of a teacher's understanding as a result of learning. Her three-phase model of conceptual development explaining the transition from the operational to the structural mode of thinking would not be the most appropriate model to apply in the investigator's attempt to assess what understanding the subjects have developed. Although learning may occur during the study, the focus of the research is on assessing understanding more than on learning, or constructing understanding. For these reasons, the investigator will not use it as a model in this study.

Recursive Model of Growth in Mathematical Understanding

Pirie and Kieran (1989, 1992a, 1992b, 1994) have proposed a recursive model of growth in mathematical understanding consisting of eight potential levels or modes of understanding for a specific person with respect to a specific topic (see Figure **2-2**). The process of coming to understand any topic starts at the *primitive knowing* level, the starting place for the growth of any particular understanding. The second level, *image making,* involves the individual conveying meaning of any kind to a mental representation. At the next level, *image having*, the individual is able to use the image itself, as a mental object, in mathematical learning; s/he is no longer dependent upon mental representations. At the fourth level, *property noticing*, the individual notes distinctions, combinations, or connections between images. At the next level, *formalizing*, the individual consciously thinks about the noted properties and abstracts commonalties. The sixth level, *observing*, involves observing the formalizing and organizing the observations. At the *structuring* level, the individual is able to explain his/her formal observations in a logical manner and establish interrelationship. The final level, *inventising*, infers that the individual who has a fully structured understanding has the ability to create totally new questions that have the potential of growing into a totally new topic (Pirie & Kieren, 1992a, pp. 246–247).

The investigator chose not to adopt the Pirie and Kieren (1989, 1992a, 1992b, 1994) model of growth in mathematical understanding as a model for this study. Similar to Sfard's (1992) process/object theory of learning, the Pirie and Kieran (1989, 1992a, 1992b, 1994) model of growth in mathematical understanding focuses more on the development of understanding than on the assessment of one's current understanding. This model is recursive and focuses on the subject's growth in mathematical understanding. The focus of this study is on the understanding that a subject *possesses*; it

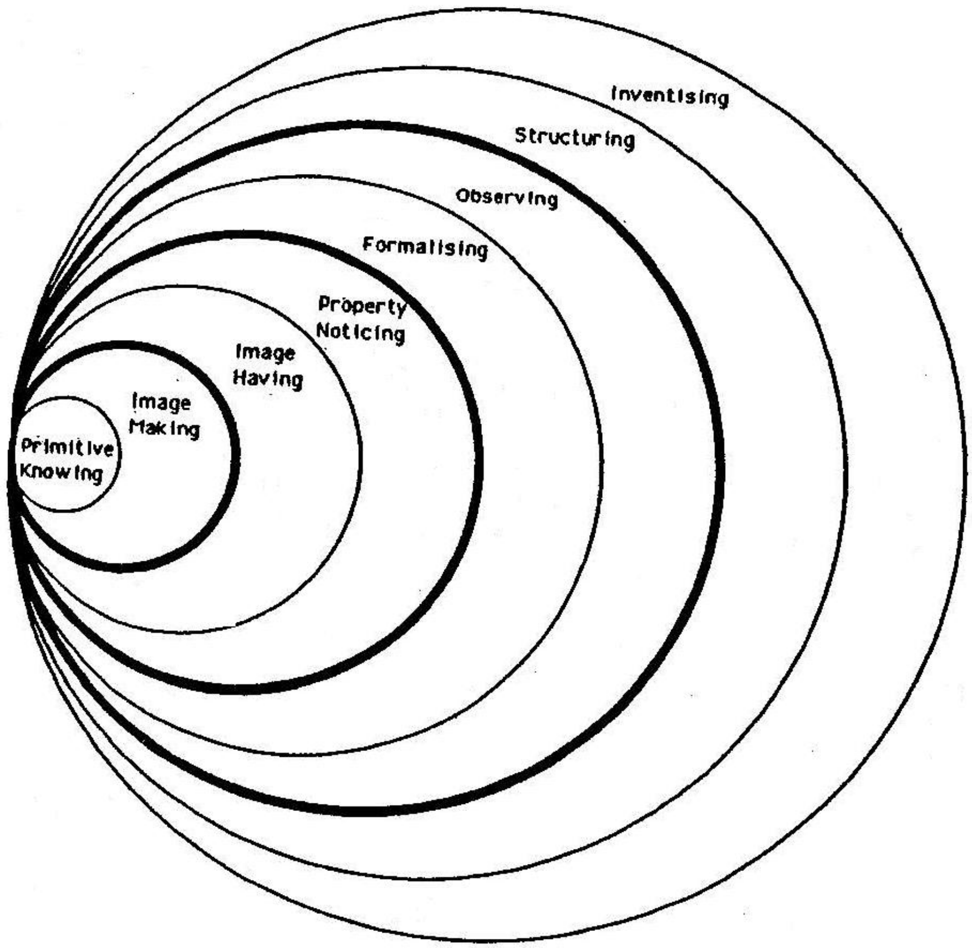

Figure **2-2**: A Diagrammatic Representation of the Recursive Model for Growth of Mathematical Understanding (Pirie & Kieren, 1992a, p. 246).

is not focused on the *growth* in this understanding. Therefore, Pirie and Kieran's model was not selected as a model for this study.

Indicators of Mathematical Understanding

Sierpinska (1992, 1994) has suggested four indicators of understanding: identification, discrimination, generalization, and synthesis (pp. 28–29). She specifically identifies these categories to understand how individuals understand the concept of function. Sierpinska (1992) suggests that in order to understand a particular mathematical concept both acts of understanding and acts of overcoming difficulties or obstacles need to be given attention (p. 27). Therefore, she claims that in order to understand how students understand the concept of function, it is necessary to study not only how the concept is understood but also how it can be misconceived. When these misconceptions are widespread and encountered by many people in various cultures or at different times, she refers to them as "epistemological obstacles" (Sierpinska, 1992, p. 28).

Identification. The initial indicator of mathematical understanding is the *identification* operator. An individual who understands a mathematical object must be able to identify it, recognize it, and classify it. The individual may or may not be able to name the mathematical idea. Such an individual will have a concept image of the given mathematical object that matches its concept definition (Vinner, 1983, 1992). However, this individual may or may not have the ability to state the mathematically precise concept definition. These observations support Sierpinska's (1994, p. 56) claim that understanding does not necessarily imply knowledge. Similarly, an individual who has the ability to state the mathematical definition of a concept accurately does not necessarily have an understanding of the concept (Vinner, 1992, p. 197).

In this category, an object is perceived as "worthy of interest and study" (Sierpinska, 1992, p. 28). Students need to be interested in what they are studying so that they will want to develop an understanding of the given concept. In the study of function, students should see how functions are connected to their everyday lives. The most fundamental acts of understanding functions according to Sierpinska (1992) are the "identification of changes observed in the surrounding world and the identification of regularities in relationships between changes" (p. 31). These acts imply that "students must become interested in variability and search for regularities before examples of well-behaved mathematical elementary functions and definitions are introduced" (Sierpinska, 1992, p. 32). When students create these connections and relationships, they realize the importance of understanding the concept of function and are interested in the topic and have the desire to learn more about it.

Discrimination. The second indicator of mathematical understanding is the *discrimination* operator. An individual who understands a mathematical object must have the ability to discriminate between two objects by identifying particularly the differences in these objects. This individual will be able to compare and contrast these objects and identify not only the similarities, but most importantly, the differences. The degree of abstraction that the individual demonstrates in the discrimination process is indicative of the depth of understanding (Sierpinska, 1994, p. 57).

In this category, the relevant properties of an object are noticed so that the similarities and the differences between objects are also noted. In order to understand the function concept, students should be able to distinguish "between two modes of mathematical thought: one in terms of known and unknown quantities, the other in terms of variable and constant quantities" (Sierpinska, 1992, p. 37). Students demonstrate the ability to be able to discriminate between number and quantity, dependent and independent variables, definitions and descriptions, and relations and functions.

Generalization. A third indicator of mathematical understanding is the *generalization* operator. An individual who understands a mathematical object must have the ability to generalize, to identify one mathematical object/situation as a particular case of another more general one. This is a most complicated characteristic due to the multiplicity of levels of generalizations as well as the interaction among these levels as understanding evolves. Regardless, only an individual who can identify the mathematical object will have the ability to generalize it. Although all four indicators, identification, discrimination, generalization, and synthesis, are important in the process of developing mathematical understanding, the ability to generalize is particularly important. Since the study of mathematics requires an individual to generalize so many ideas/objects/situations to progress to a deeper level of understanding, an individual must be able to generalize in order to grow in any understanding of mathematics (Sierpinska, 1994, pp. 58–60). This idea is further supported by the research of both Sfard and Yerushalmy. Sfard (1992) claims that it is in the process of generalizing that an individual frequently demonstrates her/his ability to reify a concept, namely, the ability to "view this new entity as a permanent object in its own right" (p. 64). Yerushalmy (1997) studied seventh graders' understanding of the concept of function through activities with multivariate functions and observed that their ability to generalize was an indication of their ability to reason algebraically (p. 431).

The generalization category "leads to an awareness of the possibility to extend the range of applications; some assumptions turn out to be irrelevant and new possibilities of interpretation are discovered" (Sierpinska, 1992, p. 28). Multiple representations of functions, such as tables, graphs, and formulae are recognized and identified. "Awareness of the limitations of each of the representations and of the fact that they represent one and the same general concept are fundamental conditions of understanding functions" (Sierpinska, 1992, p. 49). Students must be able to represent functional relationships and to use these multiple representations as tools to understand, interpret, analyze, and solve mathematical problems.

Synthesis. The fourth indicator of mathematical understanding is the *synthesis* operator. An individual who understands a mathematical object must have the ability to *synthesize*, to seek and find a "common link, a unifying principle, a similarity between/among several generalizations" (Sierpinska, 1994, p. 60) and identify the relationship between the new mathematical object and ones previously understood. Of paramount importance is the realization that only the individual student himself/herself can synthesize and deepen his/her understanding; the teacher cannot force synthesis, or any indicator or level, to occur.

The synthesis category involves the perception of links between seemingly isolated facts so that these facts are organized into consistent wholes (Sierpinska, 1992, p. 26). In this category, students exemplify a *sense* of function. Yet this ability is dependent upon the mathematical development of the students, including the development of the individual's sense of number, sense of variable, and algebraic awareness.

Although abstraction is integral to the nature of mathematics, abstraction is not an indicator of understanding since it is not an act of understanding in itself. "Abstraction is the act of detaching certain features from an object" (Sierpinska, 1990, p. 61).

Sierpinska's (1990, 1992, 1994) Indicators of Mathematical Understanding model appears to be the most appropriate framework for this study. This model can be applied in assessing the understanding that the subjects possess. It is not limited in scope as is the concept image–concept definition model nor is it limited in its applicability (e.g., both Sfard's Process/Object Theory of Learning and the Pirie and Kieran Recursive Model of Growth in Mathematical Understanding are appropriate in assessing learning, or growth in mathematical understanding). For these reasons, the investigator has adopted Sierpinska's (1990, 1992, 1994) Indicators of Mathematical Understanding as the model for the theoretical framework of this study.

Assessment Item Models

A variety of item types are used in the assessment process. In the review of literature for this study, the author examined both *multiple choice* and *constructed response* types of assessment items because these are the types of items used by ETS on *The PRAXIS Series™* Examinations. Multiple-choice items begin with an introductory sentence (frequently referred to as a *stem*) followed by a list of potential answers (frequently referred to as *alternatives, responses, or options*). Some of the literature refers to these types of items as *multiple-choice* items wheres other literature used the term *selected-response* items.

The literature also mentions *constructed-response* items, *free-response* items and *open-ended* items. Although these types of items are similar in that they do not provide a list of optional answers, these terms are not synonymous. Constructed-response items assess questions that need analytic analysis included to substantiate a response, whereas free-response items assess questions for which the response does not necessitate analytic analysis. Open-ended items answer questions that require a brief response, such as fill in the blank or completion.

Multiple-Choice Items

A multiple-choice item consists of one or more introductory sentences that include a question followed by a list of two or more suggested responses from which the subject chooses one as the correct or the best answer. The introductory part of the multiple-choice item is called the *stem*. The suggested responses are called alternatives, responses, or options. Only one of the responses is the correct or best answer; the remaining incorrect options are called *distractors* or *foils*. The purpose of the distractors is to appear as plausible answers or solutions to the problem for those subjects who do not possess sufficient knowledge. However, they should not appear to be plausible to those who have the desired degree of knowledge (Nitko, 1983, p. 190). The purpose of all multiple-choice test items is to identify or distinguish those subjects who have attained the particular level of knowledge (skill, ability, or performance) in a given discipline (Nitko, 1983, p. 191).

Although there was much discussion of multiple-choice items and tests in the literature the author reviewed, few of the examples given dealt with mathematics. There were no references specifically addressing the assessment of mathematics via multiple-choice items although an article by Romberg (1992) did provide some insights into multiple-choice items and assessment of subjects' understanding of mathematics (see **Uses**).

Uses. The uses of multiple-choice test items on many standardized tests have been based upon Bloom's (1956) *Taxonomy of Educational Objectives*: knowledge, comprehension, application, analysis, synthesis, and evaluation. Multiple-choice test items are frequently used to assess a subject's knowledge of terminology, specific facts, principles, computation, or methods and procedures; to assess a subject's comprehension of concepts, principles, generalizations, as well as his/her ability to identify the application of facts and principles, to discriminate and make correct choices, to interpret cause-and-effect relationships, to interpret new data or information, to make inferences and to reason, and to justify methods and procedures (Gronlund, 1985, pp. 171–176; Nitko, 1983, p. 193). Bloom (1956) claims that there are three kinds of synthesis based upon product: a unique communication, a plan or proposed set of operations, and a set of abstract relations (pp. 163–164). Sax (1989) claims that multiple-choice items cannot be used at the synthesis level of Bloom's taxonomy because this type of item cannot assess the subject's ability to write, say, or construct something of his/her own (p. 109). Multiple-choice items may not be used to assess an individual's ability to generalize, since in order to exemplify the generalization indicator, the subject must categorize or make a conjecture or develop a generalization; merely discriminating among generalizations given as possible choices is not sufficient. Therefore, multiple-choice items do not appear to be well suited for assessing higher levels of Bloom's taxonomy; the existence of responses seems to reduce the subject's necessity to think at a higher level. Since Sierpinska's definition of synthesis (common link, a unifying principle, a similarity between/among several generalizations) is similar to Bloom's (a set of abstract relations), the researcher believes that multiple-choice items would not be the best instrument to use to assess a subject's ability to generalize and/or to synthesize.

A description of the writing of the mathematics items for the Second International Mathematics Study (SIMS) (Weinzwieg and Wilson, 1977) referred to four levels of cognitive capability expected of students: computation, comprehension, application, and analysis. Romberg (1992) states that this classification adapted Bloom's taxonomy by replacing *knowledge* with *computation* and eliminating the higher levels of synthesis and evaluation (p. 42). However, Romberg acknowledges that these adaptations cause problems. Although computation involves knowledge of and ability to carry out a routine algorithm or procedure, knowledge of basic concepts does not fit well as either computation or comprehension. In addition, Romberg (1992) claims that the elimination of the two higher levels of Bloom's taxonomy is equivalent to admitting that important aspects of problem solving and developing a logical argument about a conjecture cannot be assessed (p. 42). "Single-answer, multiple-choice items are not reasonable at those levels" (Romberg, 1992, p. 46).

Nitko (1983) also acknowledges that multiple-choice tests do not assess an individual's ability to recollect (as opposed to recognize) information, articulate explanations and give examples, produce and express unique or original ideas, organize personal thoughts, or display thought processes or patterns of reasoning (p. 193). Multiple-choice tests may give students the impression that there is a single, correct answer to every problem (Nitko, 1983, p. 193).

Advantages. Multiple-choice tests effectively measure various types of knowledge (Gronlund, 1985, p. 177). Using a multiple-choice test format, a substantial amount of material can be assessed in a relatively short time (Nitko, 1983, p. 211; Sax, 1989, p.102). The scoring of multiple-choice tests is objective and hence can be graded easily (Sax, 1989, p. 102). Multiple-choice test items can be constructed in a manner that requires subjects to discriminate among options that vary in degree of correctness (Sax, 1989, p. 102). Multiple-choice test items are particularly amenable to item analysis to detect areas of weakness, evidence of ambiguity, item difficulty, and the extent to which the item can measure individual differences (Sax, 1989, p. 102).

In addition, multiple-choice tests are free from some of the common shortcomings characteristic of other test types. For example, in comparison with True-False tests, a subject cannot get credit merely by knowing that a statement is incorrect; s/he must also know what is correct, suggesting a greater reliability per item. Furthermore, it is not necessary to obtain statements for multiple-choice tests that are True–False without qualification (Gronlund, 1985, p. 178). Unlike Matching Tests, with multiple-choice tests there is no need for homogeneous material and the results are easier to analyze (Gronlund, 1985, p. 178). In comparison with fill-in-the-blank tests, multiple-choice test items have a limited number of options reducing the effects of guessing (Sax, 1989, p. 102). Finally, multiple-choice test items are free from response sets; that is, subjects generally do not favor a particular option when s/he does not know the correct answer (Gronlund, 1985, p. 178).

Disadvantages/Limitations. A limitation of the multiple-choice test format includes difficulty in finding a sufficient number of incorrect but plausible distractors (Gronlund, 1985, p. 179). However, the most powerful limitation of the multiple-choice items is its inability to assure that the created item will assess higher levels of thinking in mathematics, particularly the subject's ability to solve problems, to make generalizations, to write proofs, to defend conjectures, to communicate his/her ability to analyze, and to synthesize mathematical ideas.

Constructed-Response Items

A constructed-response item consists of the statement of some mathematical problem based upon a mathematical idea in which the test-taker must construct his/her response. The success or failure of the test-taker is based upon the assessor's evaluation of the mathematical logic, validity, and completeness of the test-taker's response to each item. In this study, the author uses the term *constructed-response* item as suggested by ETS; these constructed-response items frequently have multiple parts that must be answered by the application of several mathematical processes combined with some explanation or analysis of the work. The purpose of the constructed-response test is to identify those subjects who understand the mathematical ideas assessed and can demonstrate their understanding by the manner in which they answer the items posed and explain the reasoning they used.

The discussion of any type of mathematical test items in the literature is minimal. Although the literature does mention the use of free-response when addressing other disciplines, this author uses the term *constructed-response* item to describe the type of item for which the answer to a question based upon mathematical and/or scientific analysis. Even from a review of recent reports published by ETS, the author was unable to find relevant information specifically related to constructed-response items assessing mathematics (Baldwin, Fowles, & Livingston, 2008; Bennett, 2011; Fife, Graf, & Ohls, 2011; Zhang, 2013). Therefore, the information contained in the following discussion is largely based upon the author's interpretation of the ETS constructed-response tests in her study.

Uses. The uses of constructed-response test items may also be related to Bloom's (1956) *Taxonomy of Educational Objectives*. Although most of the constructed-response test items published by ETS address the lower levels of Bloom's taxonomy, the constructed-response items have the potential, in this author's opinion, of addressing the higher levels as well. In this manner, the constructed-response items would be better suited than multiple-choice items to assess a subject's understanding of mathematical ideas. This author believes that constructed-response items may be written to assess a subject's ability to make generalizations and to synthesize as well as to identify and to discriminate, in other words, to evaluate a subject's application of all four of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. The author's beliefs are supported by the National Assessment of Educational Progress (NAEP). The inclusion of a greater proportion of constructed-response items in the 2003 Mathematics Assessment is indicative of NAEP's adjusting to the shifting emphasis introduced by the release of the NCTM *Curriculum and Evaluation Standards* in 1989 (Kloosterman & Lester, 2007, p. 330).

Advantages. Constructed-response tests have the potential to assess the depth of understanding a subject has on a limited number of mathematics topics and to assess the ability of that subject to explain his/her reasoning concerning the mathematics topics addressed by the problem(s). Constructed-response items focus on a specific topic and require that the subject explain his/her answer explicitly, completely, and deeply. Multiple-choice items require the subject to identify the best answer and, although each question focuses on a specific topic, more questions may be asked, thus enabling the assessor to assess a large number of mathematical ideas. Constructed-response items frequently require more time to answer correctly, so the number of constructed-response items that can be answered in a specified time is substantially less than the number of multiple-choice items that could be answered in the same time period. This is particularly true for constructed-response items that assess a subject's ability to synthesize and to generalize. For these reasons, constructed-response items in general assess a mathematical idea in depth rather than in breadth.

However, the advantage of constructed-response test that most interests this author is its potential to assess a subject's ability to make generalizations and to synthesize; that is, to assess a subject's ability to think at a higher mathematical level. The author agrees with Nitko (1983) and Romberg (1992); she believes that multiple-choice items cannot be used effectively in the assessment of a subject's ability to make generalizations and to synthesize.

Disadvantages/Limitations. Constructed-response tests may be written in such a manner that their potential to assess a subject's ability to make generalizations and to synthesize are not utilized; in other words, constructed-response tests may be written merely to address a subject's ability to compute, comprehend, and or apply mathematical knowledge. It is challenging to develop a constructed-response item that has the potential to assess higher level thinking. Constructed-response tests are by their nature very focused and typically do not assess a large amount of material. Constructed-response items encourage each subject to express his/her understanding and are both challenging to solve and time-consuming to grade because of the unique responses and the depth of thinking encouraged.

Construct Equivalence of Multiple-Choice and Constructed-Response Items

Construct equivalence examines the differences in item statistics due to format. Traub (1993) claimed that "the true-score scales of the multiple-choice and constructed-response instruments must be equivalent for the comparison of difficulty to be meaningful" (p. 30). Rodriguez (2003) identified 67 studies investigating the construct equivalence of multiple-choice and constructed-response items. Of these 67 studies, 29 studies included 56 correlations between both formats. Five of these 56 correlations were based on tests covering mathematics, but the level of mathematics and the number of studies was not identified. Eighteen of the 56 correlations were related to postsecondary education, but the subject matter associated with these correlations was not identified. Forty-three correlations were found in journal articles, 9 in presentations, 3 in theses or dissertations, and 1 in a book. The source of the mathematics correlation was not indicated. Although Rodriguez (2003) indicates that

> when the items are constructed in both formats (multiple-choice and constructed-response) using the same stem (stem equivalent), the mean correlation between the two formats approaches unity and is significantly higher then when using non-stem-equivalent items. Construct-equivalence appears to be a function of the item design method or the item writer's intent. (p. 163)

This research suggests that multiple-choice items and constructed-response items are construct equivalent when the items are stem equivalent. Because the items studied by Rodriguez (2003) may not be comparable to the items studied in this research, readers are urged to be cautious in overgeneralizing comparability.

Conclusion

The studies of both students' and teachers' understanding of the concept of function provide evidence that the function concept is complex and challenging to teach and to learn. Middle-level and secondary-level mathematics teachers must have an understanding of the concept of function themselves as well as a willingness to continue to develop this understanding if they are to encourage a comparable growth in the students with whom they work and who they hope to inspire. However, there is currently no research addressing how this understanding of the concept of function is assessed through standardized tests.

In order to assure that prospective secondary mathematics teachers have this ability, they are tested for their qualification for licensure. The Commonwealth of Pennsylvania has adopted the *ETS PRAXIS Series™* as one of several requirements used to identify those teachers-in-training who earn certification. Each state determines a specific cutoff score below which a potential candidate will not be certified. *The PRAXIS Series™* uses two types of items, multiple choice and constructed response on the examinations. Yet how well these items measure a prospective secondary mathematics teacher's "deep conceptual understanding" (CBMS, 2001) of mathematics has not been addressed.

In the recent past, 43 states used at least one part of the *ETS PRAXIS Series™* in their teacher certification progress (Hill, Sleep, Lewis, & Ball, 2007, p. 138). The test taken at the completion of the teacher education program consists of multiple-choice and constructed-response items designed to assess general academic and pedagogical knowledge as well as subject-specific knowledge for teaching. Because ETS does not release active *PRAXIS Series™* test items, results of this study are based upon the released items found in the study guides. These items resemble those that have been used to assess teacher knowledge for generations (Hill, et al., 2007, p. 139).

Although the author was able to locate literature addressing research on the use of multiple-choice items, the research on constructed-response items is much more limited. Similarly, the research on the relationships between item type and mathematics content area is also limited. Finally, there appears to be minimal research on the construct equivalence of multiple-choice and constructed-response mathematics items.

The purpose of this study is to examine how multiple-choice and constructed-response items assess prospective secondary mathematics teachers' "deep conceptual understanding" (CBMS, 2001, 2012) of the concept of function using Sierpinska's (1992, 1994) Indicators of Mathematical Understanding as a theoretical framework. The review of literature suggests that this is an area in which such a study would be an asset.

Chapter 3

STATEMENT OF THE PROBLEM AND RESEARCH QUESTIONS

Understanding is a vital component in the teaching and learning of mathematics. CBMS (2001) states that, "prospective teachers need mathematics courses that develop a deep understanding of the mathematics they will teach" (p. 7). NCTM (2000) claims that "students must learn mathematics with understanding" and that "effective mathematics teaching requires understanding what students know and need to learn" (p. 11). Yet despite the plethora of research on mathematical understanding, Romberg admits that "there isn't a common definition of understanding" (Kieran, 1994, p. 590). Sierpinska (1994) and Simon (2002) also acknowledge that the meaning of understanding varies greatly.

The concept of function is considered by many as one of the most important concepts in all mathematics. NCTM (2000), CBMS (2001, 2012), and National Governors Association for Best Practices (NGA) and Council of Chief State School Officers (CCSSO) (2010) divide the study of mathematics into subtopics, one of which is functions. The function concept is "one of the most central ideas in the study of pure and applied mathematics" (CBMS, 2001, p. 42) and may be viewed as the unifying topic in the study of all mathematics.

Researchers have developed models of understanding to explain characteristics of understanding of mathematical topics. This research uses Sierpinska's (1992, 1994) Indicators of Mathematical Understanding as the theoretical framework to assess mathematical understanding. The framework is characterized by four indicators of mathematical understanding: identification, discrimination, generalization, and synthesis (Sierpinska, 1992, 1994).

This study includes the development of suggestions of how to create constructed-response items that have the potential of assessing mathematical understanding as defined by Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. This study focuses on how two assessment methods, multiple-choice items and constructed-response items, evaluate understanding of the concept of function by prospective secondary mathematics teachers. The study examines how these two assessment methods, demonstrated by three types of items, ETS Multiple-choice (ETS MC), ETS Constructed-response (ETS CR), and Researcher-developed Constructed-response (RD CR), evaluate prospective teachers' understanding of function as defined by Sierpinska's (1992, 1994) Indicators of Mathematical Understanding.

The focus of the current study is on how multiple-choice and constructed-response items assess prospective secondary mathematics teachers' understanding of the concept of function. The current study employs a novel theoretical framework, Sierpinska's (1992, 1994) Indicators of Mathematical Understanding, to offer suggestions for creating constructed-response items that are designed to assess prospective secondary mathematics teachers' understanding of a mathematical topic. The current study can offer insight into the nature of prospective secondary mathematics teachers' understanding of the concept of function elicited by selected multiple-choice and constructed-response items, something not currently addressed by research. This study connects the performance (i.e., quantitative score) of prospective secondary mathematics teachers on ETS MC, ETS CR, and RD CR items that assess their understanding of the function concept with the Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. Finally, the current study connects prospective secondary mathematics teachers' understanding of the concept of function as exemplified in their explanation of their solution for ETS MC, ETS CR, and RD CR items with Sierpinska's (1992, 1994) Indicators of Mathematical Understanding.

Research Questions

In order to attempt to assess the understanding of prospective secondary mathematics teachers, the investigator addressed the following research questions:

1. **Does the researcher-generated framework enable the development of items that have the potential to assess a subject's understanding of the concept of function as characterized by Sierpinska's Indicators of Mathematical Understanding (as judged by the Panel of Advisors in mathematics education and mathematics)? What are the characteristics of items that have the**

potential to assess a subject's understanding of the concept of function as characterized by Sierpinska's Indicators of Mathematical Understanding?

Does a Panel of Advisors in mathematics and mathematics education believe that the items generated by the researcher have the potential to assess a subject's understanding of the concept of function as characterized by Sierpinska's (1994) Indicators of Mathematical Understanding (identification, discrimination, generalization, and synthesis)? (See Chapter 8 for a description of this analysis, Chapter 5 for the Original Framework, and Chapter 11 for the Revised Framework.)

The purpose of the preceding research question is to characterize the nature of specific items on which subjects did or did not exhibit each of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. The investigator examined the data for patterns across items that provided evidence of subjects exhibiting any of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding (identification, discrimination, generalization, and synthesis). This information enabled the investigator to identify characteristics of items that tended to evoke a display a "deep conceptual understanding of mathematics" (CBMS, 2001) as evident in the subjects' application of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding.

2. **In what ways do the multiple-choice and constructed-response items created and released by ETS assess a subject's understanding of the concept of function as characterized by Sierpinska's Indicators of Mathematical Understanding (especially generalization and synthesis)?**
   - **2a. Which of Sierpinska's (1994) Indicators of Mathematical Understanding do the subjects display as they solve problems presented by the released ETS MC items?** (See Chapter 6 for a description of this analysis.)
   - **2b. Which of Sierpinska's (1994) Indicators of Mathematical Understanding do the subjects display as they solve problems presented by the released ETS CR items?** (See Chapter 7 for a description of this analysis.)

3. **In what ways do the items generated by the researcher from the suggestions identified assess a subject's understanding of the concept of function as characterized by Sierpinska's Indicators of Mathematical Understanding (especially generalization and synthesis)? Which of Sierpinska's (1994) Indicators of Mathematical Understanding do the subjects display as they solve problems presented by the RD CR items?** (See Chapter 9 for a description of this analysis.)

The preceding research questions involve determining which of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding each subject displays on each item. The investigator looks for patterns according to item including correctness, mathematical reasoning, mathematical reasoning involving the concept of function, use of strategies, connections with the Extended Function Block, and Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. She first studies each individual item and then looks for the same patterns across the items.

Finally, the investigator examines the results of her analyses across item types in her keystone analysis for all three research questions above.

4. **How are the three types of items (ETS MC, ETS CR, and RD CR) similar and different in their assessment of a subject's understanding of the concept of function as characterized by Sierpinska's Indicators of Mathematical Understanding (especially generalization and synthesis)? Which of Sierpinska's (1994) Indicators of Mathematical Understanding do the subjects display as they solve problems presented by the ETS MC, ETS CR, and RD CR items?** (See Chapter 10 for a description of this analysis.)

The purpose of this keystone research question was to connect the results of the subjects' performance on each item type (ETS MC, ETS CR, RD CR) with the subjects' application of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding by examining the similarities and differences exemplified by the participants. This enabled the investigator to determine whether the subjects' achievement on the different item types (ETS MC, ETS CR, RD CR) and subjects' application of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding varied. The investigator studied the relationship between performance and application of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding to determine how subjects' responses on each item type reflects a "deep conceptual understanding" (CBMS, 2001) of the concept of function.

Conclusion

The investigator believes that the information obtained from this study provides beneficial information to the mathematics education community in the following ways:

1. classifying the strengths and the weaknesses of different types of items (multiple-choice and constructed-response) for assessing mathematical understanding;
2. providing directions on how to create/develop/adapt items that will assess more accurately what subjects understand mathematically according to Sierpinska's (1992, 1994) Indicators of Mathematical Understanding (identification, discrimination, generalization, and synthesis); and
3. identifying what can be learned about items developed to assess the understanding of prospective secondary mathematics teachers concerning the concept of function particularly according to Sierpinska's (1992, 1994) Indicators of Mathematical Understanding (identification, discrimination, generalization, and synthesis) from ETS MC, ETS CR, and RD CR items.

Chapter 4

RESEARCH DESIGN AND METHODS

The Evolution of the Study

While the investigator was attempting to identify a research topic for her dissertation, she met with an individual closely associated with the teacher certification process at the university she attended. He had recently received a report on the results of students' Educational Testing Service (*ETS*) *PRAXIS Series™* exams, and he told the investigator that he was amazed that many students who had studied very little or no mathematics at the college level and had completed none of the corresponding coursework were able to pass their *ETS PRAXIS Series™* exams and become certified middle-grades mathematics teachers. This question intrigued this investigator because she firmly believes that to teach students to understand a mathematical topic, the teacher must first understand the mathematics himself or herself.

In order to investigate this question, the investigator purchased every publication ETS had available for students to use to prepare for the *ETS PRAXIS Series™* exams (ETS, 2003, 2004a, 2004b, 2004c, 2004d, 2004e, 2004f). She also searched online for comparable items. She found two types of items in her search: ETS Multiple-choice Items and ETS Constructed-response Items. Because of her interest in the concept of function and her belief that this concept is central to the understanding of mathematics, she decided to focus her study on the understanding of the concept of function by future middle-level and high school mathematics teachers. Identifying all items that ETS identified as assessing the function topic, she found 22 multiple-choice items and 5 constructed-response items. After seeking and receiving permission form ETS to use these items (see Appendices A and B), they constituted the set she used as her ETS Multiple-choice Items (ETS MC) and ETS Constructed-response Items (ETS CR) (see Appendices E and F).

Contribution of Pilot Studies

The investigator conducted three pilot studies to determine a final selection of items and mode of administration for the study to assess participants' mathematical understanding. The participants for each study varied in mathematical sophistication from undergraduates who were enrolled in their second calculus class to graduate mathematics education students who had taught, or who were qualified to teach, the same calculus courses.

The investigator began her pilot studies considering each set (ETS MC, ETS CR) of items as a "test." However, following discussions with her committee, she realized that because she gathered the ETS MC and ETS CR items by merely selecting all those released items that ETS had identified as assessing the concept of function, the study would more appropriately focus on the *items* rather than on a *test* composed of the various items. Because so few participants, particularly in the second pilot study, demonstrated the application of Sierpinska's generalization and synthesis indicators of mathematical understanding on any of the ETS items, the investigator believed additional items (Researcher-developed Constructed-response (RD CR)) were needed for the study. Following further discussions with her committee, the researcher became aware of the need for a Framework in which she could ground her study and the instrumentation.

The method of data collection also evolved as the pilot studies progressed. A significant contribution of the first pilot study was a reevaluation of the interview method and environment. The data suggested that only when the instruments (ETS MC, ETS CR, and RD CR Function Items) were administered in a one-on-one, investigator–subject environment would the investigator have the opportunity to collect data on what the subjects were thinking and understanding as they solved the item as well as to make conjectures regarding how subjects were applying Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. (A detailed explanation follows in Pilot Study I.)

Two of the instruments were refined through the pilot studies; two remained the same. The Educational Background and Future Plans questionnaire did not change; the investigator was satisfied with the information she gleaned from this questionnaire. The ETS CR remained the same as well; the investigator did not wish to alter any part of the

ETS segment of the study. There was one small change in the ETS MC: Because of the identical nature of topic and algebraic equation used in two different items (originally items 15 and 21), one item (15) was omitted and the second item (21) became item 20.

The instrument that evolved the greatest during the pilot studies was the RD CR. These items were created, adapted, and/or developed as the investigator developed the original Framework and evolved as she conducted her pilot studies. The following list provides the topic and source of each of the RD CR items:

RD CR 1: Multiple representations of functions: 1a—Adapted from Norman, 1992, p. 231 and 1b, 1c, 1d—Developed by investigator;

RD CR 2: Symbolic notation for the graph of a cubic function: Adapted from Heid, 1995, pp. 28–29;

RD CR 3: Function and rates of change: 3a—Adapted from CBMS, 2001, p. 31 and 3b—Developed by investigator;

RD CR 4: Number of roots of a quartic function with constant k: Adapted from ETS Sample Test Questions—Web XXVI – Mathematics: Proofs, Models, and Problems, Part 1 (0063);

RD CR 5: Creation of a multivariate function problem—Created by the investigator in response to her findings when writing the original framework;

RD CR 6: Inverse of functions I: Created by investigator based upon the misconceptions of the pilot study participants regarding functions and their inverses;

RD CR 7: Equality of functions: 1a and 1b: Developed by investigator and 1c: Adapted from Sfard, 1992, p. 67;

RD CR 8: Composite functions: Created by investigator to determine subjects' understanding of composite functions;

RD CR 9: Graph of function: Adapted from *PSSM*, 2000, p. 363;

RD CR 10: Recursive function: Developed by investigator. Original problem used triangular and rectangular patterns and merely asked for symbolic notation and the investigator used pentagonal pattern and asked for the symbolic notation for a recursively defined function and an explicitly defined function;

RD CR 11: Multivariate function–Box: Created by investigator to examine subjects' understanding of multivariate functions;

RD CR 12: Trigonometric function: Created by investigator to examine subjects' understanding of trigonometric functions;

RD CR 13: Inverse of functions II: Created by investigator based upon the misconceptions of the pilot study participants regarding functions and their inverses;

RD CR 14: Graph of a multivariate function: Created by investigator to examine subjects' understanding of multivariate functions;

RD CR 15: Parametric functions: Created by investigator to examine subjects' understanding of parametric functions;

RD CR 16*: Parameter exploration of trigonometric function: Created by investigator to examine subjects' understanding of parameters of trigonometric functions; and

RD CR 17*: Parameter exploration of exponential function: Created by investigator to examine subjects' understanding of parameters of exponential functions.

*RD CR 16 – 17 were added following Pilot Study II and deleted after Pilot Study III.

Pilot Study I

The purpose of the first pilot study was to establish the validity of the ETS MC (see Appendix E) and ETS CR (see Appendix F) Function Tests[1] relative to the Dreyfus and Eisenberg (1982, p. 365) Function Block and Sierpinska's (1992, 1994) Indicators of Mathematical Understanding.

The instruments for Pilot Study I were as follows:

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[1] Prior to her Proposal Meeting, the investigator viewed the set of items as a single test. For this reason, during her first three pilot studies, the sets of items are referred to as a Function Test. After her Proposal Meeting, the set of items was studied individually and not as a test. For this reason, during her third and final pilot study and her actual study, the sets of items are referred to as Function Items.

1) 22 multiple-choice items composed of questions selected from *The ETS PRAXIS Series™* Practice exams and identified by ETS as items that assessed the mathematical understanding of the concept of function; and
2) 5 constructed-response items composed of questions selected from *The ETS PRAXIS Series™* Practice exams and identified by ETS as questions that assessed the mathematical understanding of the concept of function.

The study took place over a period of 5 weeks in the spring semester, 2006. Nine participants participated in the study; eight graduate students in mathematics education who were enrolled in a research projects class were recruited to participate in this phase of the study and the instructor of the class also volunteered to participate.

The study involved two presentations and three assignments. During the initial presentation, the investigator introduced the task, requested that the participants complete the Educational Background and Future Plans (see Appendix D) questionnaire, and distributed the ETS MC (see Appendix E) and ETS CR (see Appendix F) Function Tests. The investigator instructed the participants to answer the questions contained in the ETS MC (see Appendix E) and ETS CR (see Appendix F) Function Tests prior to the next meeting.

During the second presentation, the investigator introduced the participants to the Dreyfus and Eisenberg (1982) Function Block and Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. Through an example problem (see Appendix I), the investigator discussed with the participants which topics of the Dreyfus and Eisenberg (1982) Function Block were addressed as well as which of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding she had applied in her demonstration of her solution of the example problem. The participants spent the first hour of class practicing identifying the topics of the Dreyfus and Eisenberg (1982) Function Block that were addressed by specific multiple-choice items and constructed-response items while the investigator was present to address any questions and concerns. This activity formed the basis for an assignment named the Function Block Task (see Appendix J).

The participants spent the second hour of class practicing identifying which of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding they had utilized in their solution of specific multiple-choice and constructed-response items selected from the ETS MC (see Appendix E) and ETS CR (see Appendix F) Function Tests. Again, the investigator addressed any questions that arose. This activity formed the foundation for an assignment called the Indicators of Mathematical Understanding Task (see Appendix K).

At the end of the class, the participants were instructed to complete both of the assignments: the Indicators of Mathematical Understanding Task (see Appendix K) the Function Block Task (see Appendix J).

ETS and the participants confirmed the content validity of both the ETS MC (see Appendix E) and ETS CR (see Appendix F) Function Tests relative to the Dreyfus and Eisenberg (1982, p. 365) Function Block. ETS (2003, 2004a, 2004b, 2004c, 2004d, 2004e, 2004f) identified all items in both tests as items that assessed the concept of function; the participants identified all items in both tests as items that assessed some aspect of the Dreyfus and Eisenberg (1982, p. 365) Function Block.

The participants identified the specific indicators of mathematical understanding that they implemented in their process of reaching a solution although they completed this process retrospectively. The participants indicated that they used the identification and discrimination indicators on most of the items on the ETS MC and ETS CR Function Tests. They also indicated that they used the generalization and synthesis indicators much less frequently.

Pilot Study I led the investigator to suspect that the ETS MC Function Test (see Appendix E) and the ETS CR Function Test (see Appendix F) were inadequate in assessing whether an individual had a "deep understanding of the concept of function" (CBMS, 2000). Although the ETS MC and ETS CR Function Tests did address the identification and discrimination indicators satisfactorily, neither of the tests appeared to address the generalization and/or synthesis indicators satisfactorily. Thus, according to Sierpinska's (1994) theory of understanding, these tests seemed not to assess a "deep conceptual understanding of mathematics" (CBMS, 2000).

In addition, there were levels of the concept of function for example, multivariate functions, function families, and function types) that were not addressed in either instrument (ETS MC or ETS CR Function Tests). Through discussions with her committee, she concluded that an additional instrument, such as the revised RD CR Function Test, was needed to address whether participants had a "deep conceptual understanding" (CBMS, 2000) of the concept of function.

In this first pilot study, the instrument was self-administered and the participants submitted their work and their responses. Some of the participants' responses, particularly to the Function Block Task, left the investigator wondering whether there were misconceptions present. Since she had the participants complete all tasks independently, she was unable to resolve these questions. In analyzing these data, the investigator was incapable of identifying specific instances in which the participant employed Sierpinska's Indicators of Mathematical Understanding but she had to depend on the participant's self-report for information about which indicators the participant used. The investigator was also unable to discriminate conclusively and with explicit evidence regarding precisely which of Sierpinska's Indicators of Mathematical Understanding (identification, discrimination, generalization, synthesis) were used. The participants reported what they remembered regarding how they solved each item. For these reasons, she concluded that it would be necessary in answering the research questions to administer all the instruments individually in the following pilot studies. In this way, she would be able to observe participants' reasoning as they solved each item, to identify misconceptions more readily, and to diagnose students' difficulties more efficiently. In addition, she decided that the participants in the second pilot study should be students knowledgeable about the concept of function, but not as advanced in mathematics and mathematics education as the participants in the first pilot study; participants more comparable to those who would participate in the actual study. Finally, administering the instruments individually would help her in noting specifically which indicators of mathematical understanding were evident in each participant's solution of each item.

Pilot Study II

The specific goals of the second pilot study were to determine which of Sierpinska's (1992, 1994) indicators of mathematical understanding each participant used in solving each problem on each of the instruments (ETS MC Function Test (see Appendix E) and ETS CR Function Test (see Appendix F) and the RD CR Function Test (see Appendix G)); and to assure that each item of each instrument was understandable to less mathematically sophisticated undergraduate students.

The investigator used the same instruments as she used in Pilot Study I: the ETS MC (see Appendix E) and ETS CR (see Appendix F) Function Tests. In addition, she also used a RD CR Function Test (see Appendix G). This instrument was created as a result of the data gleaned from the Pilot Study I and was developed specifically to address the limitations observed in the original two instruments.

The instruments for Pilot Study II were as follows:

1) A set of 22 multiple-choice items composed of questions selected from *The ETS PRAXIS Series™* Practice exams and identified by ETS as items that assessed the mathematical understanding of the concept of function;
2) A set of 5 constructed-response items composed of questions selected from *The ETS PRAXIS Series™* Practice exams and identified by ETS as questions that assessed the mathematical understanding of the concept of function; and
3) A set of 15 researcher-developed constructed-response items composed of items formulated specifically for this study.

Four undergraduates participated in Pilot Study II. Three of the participants were engineering students currently enrolled in a Calculus II course. One of the participants was a future education major seeking dual certification in physics and mathematics. All appeared familiar with and comfortable in discussing the concept of function.

The study took place over a period of 6 weeks in the spring semester of 2006 and involved four meetings lasting approximately 2–2.5 hours with each the participants. The investigator video recorded each of the meetings with each of the participants. During the first meeting, each participant read and signed the Informed Consent Form (see

Appendix L), completed the Educational Background and Future Plans questionnaire (see Appendix D), and took the ETS MC Function Test (see Appendix E). During the second meeting, each participant completed the ETS CR Function Test (see Appendix F). The third and fourth meetings were devoted to the RD CR Function Test (see Appendix G); during the third meeting, each participant worked on Part I, items 1–6, and at the fourth and final appointment, each participant worked on Part II, items 7–11. During the completion of each item, the investigator assessed how the participant solved the problem. In addition, she asked probing questions to determine which of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding the participants demonstrated in their solution of the problem.

The researcher analyzed each of the ETS MC and ETS CR Function Tests as well as the RD CR Function Test. She noted the strategies that each of the participants employed and the indicators of mathematical understanding (identification, discrimination, generalization, synthesis) that each participant explicitly used in attempting to find a solution.

In the ETS MC Function Test, all four participants exemplified the identification indicator in 20 of the 22 items and the discrimination indicator in 5 of the 22 items. None of the participants gave explicit evidence of using either the generalization or the synthesis indicators in any of these items.

In the ETS CR Function Test, all four participants used the identification indicator in each of the 5 items; the discrimination indicator in 3 of the 5 items; the generalization indicator in one of the 5 items; and the synthesis indicator in one of the 5 items. The participants appeared relatively successful at applying the identification and discrimination indicators.

In the RD CR Function Test, all four participants utilized the identification indicator in 8 of the 11 items; the discrimination indicator in 3 of the 11 items; the generalization indicator in 1 of the 11 items; and the synthesis indicator in 0 of the 11 items. The data were inconclusive in enabling the researcher to determine whether the participants understood certain topics such as multivariate functions well enough to be able to utilize the generalization and synthesis indicators.

Only two minor revisions in the ETS MC Function Test were made due to this second pilot study. The first revision involved the manner in which a question was presented. Item 22 involved a piecewise-defined function. The question was initially continued on the same line as the piecewise-defined function. One of the participants in this study had difficulty understanding this question since she skipped the part of the question on the same line as the piecewise-defined function. For this reason, the investigator moved the words “is equivalent” to the second line of the question.

The second revision involved reducing the number of questions in the ETS MC Function Test. Items 15 and 21 asked for the number of intersections of two functions. Both items referred to a system of two functions that were exceptionally comparable. For this reason, the investigator decided to omit item 15 in the study. There were no other revisions made to the ETS MC Function Test.

There were no revisions on the ETS CR Function Test.

There were revisions on the RD CR Function Test in order to assure that the questions were clear and understandable. These revisions included the following:

Item 6 originally, now 8: Parts g and h were added to indicate clearly the importance of the characteristics of an arbitrary function in determining its inverse.

Item 7 originally, now 9: The wording was altered from that of the original author for the purposes of clarity.

Item 10 originally, now 12: The order of the subquestions posed within this question was adjusted to place all the parts dealing with the recursively defined function first and the explicitly defined function second.

In this second pilot study, the investigator administered each instrument in a one-on-one, investigator-participant environment enabling her to identify evidence explicitly regarding which of Sierpinska’s Indicators of Mathematical Understanding (identification, discrimination, generalization, synthesis) each participant used. She employed a retrospective interview procedure; she did not intervene until the participant indicated s/he was finished or was unable to make additional progress. These changes produced data that enabled her to make more accurate conjectures regarding how participants were applying Sierpinska's (1992, 1994) Indicators of Mathematical Understanding and thus answer her research questions. For this reason, the investigator continued to use this procedure in her final pilot study.

Pilot Study III

Because the data obtained regarding the RD CR Function Items in Pilot Study II were inconclusive since the participants were unable to make adequate progress in responding to many of the problems posed, the researcher recognized the need to administer this instrument to participants who were capable and qualified to answer these items correctly and completely. The specific goals of the third and final pilot study were:

1. to determine the potential of each of the RD CR Function Items to elicit the participant's use of the generalization and/or synthesis indicators;
2. to revise the RD CR Function Items as necessary; and
3. to add additional RD CR Function Items as the data suggested were necessary.

The only instrument used in Pilot Study III was the set of the 17 researcher-developed constructed-response items composed of items formulated specifically for this study.

Five mathematics education graduate students were recruited to participate in the final pilot study. The investigator specifically recruited participants who would have a sufficient knowledge and deep understanding of the concept of function that would allow each to demonstrate the ability to employ the generalization and synthesis indicators in solving the RD CR Function Items as revised from the Pilot Study II (see Appendix G). Thus, the investigator would have evidence that the RD CR Function Items did have the potential to elicit a participant's use of the generalization and/or synthesis indicators and confirm that these items had content validity.

The study took place over a period of 8 weeks in the spring semester of 2007. The investigator video recorded each of the meetings with each of the five participants. The meetings were devoted to the revised RD CR Function Items that now had a total of 17 items (see Appendix G). During the first meeting, four of the participants worked on Part A, items 1–8, and at the second and final meeting, the same four participants worked on

Part B, items 9–17. During the meetings with the one participant whose schedule mandated four appointments, four items were attempted at each session beginning with Part A, items 1–4, next Part A, items 5–8, then Part B, items 9–12, and finally Part B, items 13–17. During the completion of each item, the investigator assessed how the participant solved the problem. In addition, she asked probing questions to determine which of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding the participants demonstrated in their solution of the problem.

The researcher analyzed each of the items from the RD CR Function Test. She was particularly interested in the clarity and completeness of each item since this was the final pilot study. Again she noted the strategies that each of the participants employed and the indicators of mathematical understanding (identification, discrimination, generalization, synthesis), focusing particularly on generalization and synthesis, that each participant explicitly used in attempting to find a solution.

She found that the participants used the identification indicator in every item on which they made any progress. However, the participants' use of the discrimination, generalization and synthesis indicators was most individualized and unpredictable. Since their level of mathematical understanding surpassed that of the preservice teachers who were the subjects of the study, she was not surprised that on the RD CR items that dealt with parameter explorations of exponential and trigonometric functions, the graduate mathematics education participants were more inclined to use the identification indicator of mathematics understanding although the investigator made the conjecture that the undergraduate preservice mathematics education participants would be more likely to use the generalization and synthesis indicators. Thus, the researcher concluded that the use of these indicators by the graduate mathematics education participants would not necessarily lend insight into the potential of the item to elicit generalization and/or synthesis in this study, since there appeared to be an interaction between the potential of the item and the level of understanding of the participants.

One major revision in the administration of the study instrumentation resulted from this pilot. From their work on other research projects, the graduate mathematics education participants suggested that the researcher focus one of the cameras on the participant and the second on the participant's work. They also suggested that the investigator have the participants use markers in recording their work on paper so that it would be more easily visible on camera.

There were revisions on the RD CR Function Items in order to assure that the questions were clear and understandable. These revisions were made during the final pilot but before the administration of the RD CR segment of the study and included the following:

Item 1ci. The ordered pair $\left(\frac{1}{2},\frac{3}{4}\right)$ in the pilot study was changed to (1, 3) in the dissertation study because the fractional order pair was difficult to read and to interpret in tabular form.

Item 2. The pilot study asked the participants to study the graph and then to
a. Identify the roots of this function.
b. Give the symbolic function notation for the graph.
The dissertation study asked the particpants to study the graph and then to give the symbolic function notation for the graph since this subsumes that they identify the roots of the given function.

Items 8 and 13. The purpose of eliminating the notation for the inverse function was to alleviate any confusion that would arise from the notation. For example, "Determine a rule for the inverse relation $f^{-1}(x)$" was changed to "Determine a rule for the inverse of function *f*."

Item 10. In the dissertation study, the researcher did not use graph paper to demonstrate the recursive pentagonal pattern as she did in the pilot study.

Item 12. The order of parts a and b were switched since the researcher believed it was more appropriate to decide whether the graph was a function before determining a symbolic description of the graph.

Item 15. The word "polar" was inadvertently omitted in the directions of the dissertation study. "Justify your conjectures" was added.

No additional items were added as a result of this pilot study; however, two items were eliminated. The researcher decided not to use the items on parameter explorations of exponential and trigonometric functions because these explorations were frequently used in the methods classes that the participants had recently taken. For this reason, this type of item would be more familiar to them than it would possibly be to other subjects who had not had these experiences.

One final contribution of this pilot study was the ordering of the items in the dissertation study. In the ETS MC Function Items segment and in the ETS CR Function Items segment the items progressed from easy to difficult. The researcher wished to order the RD CR Function Items segment in the same fashion. The item topics, and order placements in Pilot Study III and in the RD CR may be found below (see Table **4-1**).

Table **4-1**: Researcher-Developed Constructed-Response Item Topics and Placements in Dissertation Study and Pilot Study III.

Dissertation Study	Topic	Pilot Study III
1	Multiple representations of functions	1
2	Symbolic notation for the graph of a cubic function	2
3	Function and rates of change - Vase	9
4	Number of real roots of a quartic function with constant *k*	6
5	Creation of a multivariate function problem	7
6	Inverse of functions I	8
7	Equality of functions - Sfard problem adaptation	4
8	Composite functions	11
9	Graph of function - Airport	3
10	Recursive function	12
11	Multivariate function - Box	17
12	Trigonometric function	10
13	Inverse of functions II	15
14	Graph of a multivariate function	13
15	Parametric functions	14
omitted	Parameter exploration of trigonometric function	5
omitted	Parameter exploration of exponential function	16

This pilot study offered additional evidence that in order to determine precisely which of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding were present as the subjects processed each item, all instruments should be administered in a one-on-one, investigator–subject environment. The investigator employed a retrospective interview procedure where she did not intervene until the subject indicated s/he was finished or unable to continue. This enabled the investigator to have the opportunity to collect data on what the subjects were thinking and understanding as they solved the item as well as her to make conjectures regarding how subjects were applying Sierpinska's (1982, 1984) Indicators of Mathematical Understanding without influencing the subject's response as they solved the items.

The Framework[2]

There was compelling evidence from the investigator's first and second pilot studies that the items created by ETS that were selected for this study (ETS MC and ETS CR Function Items) did not assess accurately a subject's understanding of the concept of function as characterized by the application of Sierpinska's Indicators of Mathematical Understanding, particularly, the application of the indicators of generalization and of synthesis. Therefore, the investigator with the support of her committee recognized the need for an original Framework based upon the literature and her prior research for creating items that would focus on the creation of items that had the potential to assess Sierpinska's Indicators of Mathematical Understanding, particularly generalization and synthesis.

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[2] This Framework was created to:

- connect the concept of function with Sierpinska's Indicators of Mathematical Understanding;
- describe items that assess identification, discrimination, generalization, and synthesis (Sierpinska's Indicators of Mathematical Understanding) and that focus on the concept of function;
- explain how to write such items;
- develop items specifically created to focus on assessing a subject's ability to generalize and to synthesize; and
- seek affirmation from the identified Panel of Advisors that the items created do address Sierpinska's Indicators of Mathematical Understanding, particularly generalization and synthesis.

The researcher uses the word *Framework* in this dissertation in the context of the preceding definition for which it was originally created. The researcher acknowledges that *framework* may be used by other authos to convey something different from this, but because the term *framework* was used when materials were sent to the Panel of Advisors, that term will be used subsequently.

Purpose of a Framework

The purpose of such a Framework was to connect the literature on the concept of function and her research from Pilot Studies I and II to devise research-based methods for creating items that would focus on assessing Sierpinska's Indicators of Mathematical Understanding, particularly generalization and synthesis, and would have the potential to discriminate between those subjects who possess a deep conceptual understanding of the concept of function and those who do not.

The process of creating the Framework included the following:

A. connecting the concept of function with Sierpinska's Indicators of Mathematical Understanding;
B. describing items that assess identification, discrimination, generalization, and synthesis (Sierpinska's Indicators of Mathematical Understanding) and that focus on the concept of function;
C. explaining how to write such items; and
D. creating items specifically designed to focus on assessing a subject's ability to generalize and to synthesize.

It is important to note here that while Sierpinska (1994) does claim that "all four operations [indicators] are important in any process of understanding," (p. 59), however, all four operations [indicators] are not required in each item. Therefore, some items might address generalizing, in addition to identification and possibly discrimination, whereas others would address synthesizing, in addition to identification and possibly discrimination, whereas still others could address both generalizing and synthesizing, in addition to identification and possibly discrimination.

Description of Panel of Advisors[3]

The investigator sought the assistance of a Panel of Advisors to critique her Framework and the items she created, specifically seeking their comments on the

[3] The Panel of Advisors was originally called the Panel of Experts. The change in name was suggested since the researcher did not have evidence that all members of the panel were experts in the study of the concept of function although they did all have expertise in mathematics and/or mathematics education.

potential of the items she created to address Sierpinska's Indicators of Mathematical Understanding, particularly generalization and synthesis. She requested the assistance of five individuals all of whom were involved in teaching and research at the college and/or university level in the United States and had expertise in mathematics and/or mathematics education. All of these individuals had indicated an interest in the education of pre-service secondary mathematics teachers and in the concept of function. The investigator began recruiting the members for the Panel of Advisors in the Fall, 2006. She gathered the information from her Panel of Advisors by forwarding her Framework and instruments to them electronically and receiving their feedback and critiques electronically from two of the panel members, via U.S. mail from one of the panel members, and via a personal delivery from a final panel member. One individual who initially agreed to participate as a panel member did not respond by completing the required instrumentation; the data that member did provide were not included. All responses were received by December, 2007.

Because the members of the

were recruited separately and given the instruments at different times, the turnaround time was longer than originally anticipated and prohibited the researcher from incorporating their advice prior to data collection. Therefore, the researcher found it necessary to eliminate items after the data had been collected.

Framework Instruments

In addition to the Framework, the investigator developed three instruments to assist in the collection of data from the Panel of Advisors: Comments on and Critique of the Framework, Evaluation of Recommendations for Developing Constructed-Response Items, and Item Evaluation Instrument (see Appendices M, N, and O respectively).

Comments on and critique of the framework. This instrument was open ended. The investigator merely provided the members of the Panel of Advisors with the directions, "Your comments and critiques on the framework are valued and appreciated." She did this to provide the members of the Panel of Advisors with a method to critique the Framework without any specificity in order to assure that she was not unduly influencing the comments or critiques of the panel members (see Appendix M).

Evaluation of recommendations for developing constructed-response items. Using this instrument, the investigator asked the members of the Panel of Advisors to review each of the recommendations provided in the Framework and to indicate whether they believed that these recommendations were grounded in the Framework (that is, that the Framework implied these suggestions). The Panel of Advisors were asked to respond by agreeing or disagreeing (yes/no) and were also asked to provide their comments and critiques. The first 12 recommendations were general in nature addressing the structure of the items. The second 12 recommendations pertained to the evaluation of an item to assess Sierpinska's Indicators of Mathematical Understanding. The Panel of Advisors were asked to respond using a 5-point Likert scale to indicate their level of agreement and were also invited to provide their comments and critiques (see Appendix N).

Item evaluation instrument. In this instrument, the investigator requested that the members of the Panel of Advisors rate the extent to which they believed a given item had the potential to elicit generalization and/or synthesis when administered to prospective secondary mathematics teachers. With the exception of the first item, there was one response for each item; the first item requested one response for each of the four (a – d) parts of the item since each of these parts was subdivided into parts. Each question was to be answered using a 5-point Likert scale for which 1 was *not at all likely that subjects who get this answer correct will generalize and/or synthesize* and 5 was *highly likely that subjects who get this answer correct will generalize and/or synthesize.* She also requested that the members provide a rationale for their response (see Appendix O).

Framework Data Analysis

Using the responses of the members of the Panel of Advisors to these various instruments, the investigator revised her Framework (see Chapter 11) and chose the RD CR Function Items to include in her study (see Chapters 8 and 9).

The data gleaned from the Comments on and Critique of the Framework (see Appendix M) were descriptive in nature and were summarized as such. The quantitative data obtained from the Evaluation of Recommendations for Developing Constructive-response Items (see Appendix N) were similar and positive. The constructive critiques were noted and their content discussed (see Chapter 8) and then used in creating the Revised Framework (see Chapter 11). The data obtained from the Item Evaluation Instrument (see Appendix O) were summarized by item for each member of the Panel of Advisors and averaged. Due to time constraints, she was unable to incorporate the suggestions of the Panel of Advisors into the development of the RD CR Function Items; however, she used these data to identify those RD CR Function Items that she would analyze. Her decision was based upon the response of the Panel of Advisors regarding which items had the greatest potential to elicit either the generalization or the synthesis indicators of mathematical understanding (see Chapter 8).

Research Methods

Research Design

This study was built upon a quantitative research paradigm (Newman & Benz, 1998, p. 20–21). The research begins with the premise that mathematics teachers need to understand the mathematics they teach and the assumption that this understanding is determined in part by their performance on *The ETS PRAXIS™ Series*. The purpose of this study is to examine how two types of items (multiple-choice and constructed-response) assess prospective secondary mathematics teachers' understanding of the function concept. The researcher reviewed the current literature, focusing on the importance of understanding in mathematics, the concept of function, the models of understanding from which she selected her theoretical framework (SIMU), and the relevance of assessment items in evaluating understanding. The research questions were generated based on the theoretical framework and created to study how ETS MC, ETS CR, and RD CR items assess a subject's understanding of the concept of function as characterized by Sierpinska's Indicators of Mathematical Understanding (especially generalization and synthesis). The data are analyzed based upon the research questions and conclusions are determined and identified.

Although the research paradigm was quantitative, the research methodology was mixed of necessity; it was both qualitative and quantitative in nature. The review of the literature conveys that the nature of the problem as described in the research questions had not been previously studied (Cresswell, 1994, p. 9). The research questions suggested, and the pilot studies confirmed, that a subject's understanding cannot be explored deeply merely by means of a survey, a test, or the subject's responses to ETS MC, ETS CR, or RD CR Items. That part of the data that involved the subject's Educational Background and Future Plans as well as the correctness of response to the various items were analyzed quantitatively. The Educational Background and Future Plans segment was analyzed using descriptive statistics; the ETS MC items were evaluated as either correct or incorrect; and the ETS CR and RD CR items were assessed using a scoring rubric (see Appendix H) adapted from the ETS scoring rubric and consistent with the NAEP scoring rubric (Silver & Kenney, 2000). However, in order to determine what a subject understood, the researcher needed to study the subject's reasoning. In order to accomplish this, the investigator conducted interviews as she administered the ETS MC, ETS CR, and RD CR items. The interviews were then transcribed and analyzed so that the researcher could study the subject's understanding, reasoning, use of strategies, and application of SIMU. This section of the methodology was qualitative in nature.

Quantitative methodology. The investigator used descriptive statistics to analyze the Educational Background and Future Plans (see Appendix D) questionnaire. The data collected in this first instrument of the study were discrete in nature and were analyzed by compiling the data in a spreadsheet, studying the measures of central tendency, and looking for patterns of interest. These data were used to describe the educational background and future plans of participants in the study.

A quantitative methodology was also used to assess the correctness of response for the ETS MC, ETS CR, and RD CR Items. The ETS MC Items were scored as correct or incorrect while the ETS CR and RD CR Items were scored using the Scoring Rubric (see Appendix H).

Qualitative methodology. Video recordings of the interview sessions were transcribed, and the investigator reviewed the transcriptions and the videos to identify explicitly each subject's response, reasoning, use of strategies, connections with the Expanded Function Block, and application of SIMU. She noted any evidence of misconceptions using the nature of the questions as well as the completeness of the interviewee's response to determine whether understanding was exemplified. The investigator used the transcripts of the interviews and the work of the subjects as evidence of her claims of understanding. She summarized the transcripts by noting the strategies each subject used while attempting to answer each question on each of the instruments.

All questions on the ETS MC Items (see Appendix E) and the ETS CR Items (see Appendix F) were identified by ETS as assessing the concept of function and confirmed as such in Pilot Study I. She established the content validity of the RD CR Items (see Appendix G) in Pilot Study III. The investigator also confirmed the content validity by the use of the Dreyfus and Eisenberg (1982) Expanded Function Block.

Mixed methodology. After the investigator had determined the correctness of the participants' responses to each item (ETS MC, ETS CR, and RD CR Items), she then studied the participants' use of mathematical reasoning involving the function concept, the relationship between the participants' responses and mathematical reasoning, the relationship between the participants' responses and strategies, and the relationship between the participants responses and application of Sierpinska's Indicators of Mathematical Understanding (SIMU), and she concluded by studying qualitatively the relationship among the participants' responses, mathematical reasoning involving the function concept, and SIMU.

Next, the investigator conducted a cross-item analysis by studying the correctness of the participants' responses to each item for each item type (ETS MC, ETS CR, and RD CR), the participants' use of mathematical reasoning involving the function concept, the relationship between the participants' responses and mathematical reasoning, the relationship between the participants' responses and strategies, and the relationship between the participants responses and application of Sierpinska's Indicators of Mathematical Understanding (SIMU), and the relationship among the participants' responses, mathematical reasoning involving the function concept, and SIMU. She examined that data and noted any patterns in the correctness of the subjects' responses, the subjects' use of mathematical reasoning involving the function concept, the relationship between the subjects' correctness of response and reasoning, the relationship between the subjects' responses and use of strategies, the connections with the Expanded Function Block, relationship between correctness of response and use of SIMU, and concludes by studying the relationship between the participants' responses, mathematical reasoning involving the function concept, and SIMU for each item type.

In addition, the investigator found limited information in the literature on how various testing methods, specifically multiple-choice and constructed-response, assess subjects' mathematical understanding. Her research questions evolved through the pilot study process from focusing merely on the items created by ETS (ETS MC and ETS CR) that appeared not to assess accurately mathematical understanding as described by Sierpinska, to creating a framework describing how to create/develop/adapt items that had the potential to assess mathematical understanding as described by Sierpinska, particularly, subjects' application of the generalization and synthesis indicators of mathematical understanding. This segment of the study was also mixed in nature.

The analysis of the data collected from the Panel of Advisors also employed a mixed methodology. The investigator used descriptive statistics to analyze the Evaluation of Recommendations for Developing Constructed-response Items (see Appendix N) and the Item Evaluation Instrument (see Appendix O) from Panel of Advisors. The data collected in these instruments were discrete in nature and were analyzed by compiling the data in a table and looking for patterns of interest. The comments and critiques from the members of the Panel of Advisors for these two instruments as well as the Comments and Critique on Framework were descriptive in nature and analyzed similarly. These data were used to select the RD CR Items to be studied (see Chapter 8).

Instrumentation

The study used a questionnaire and three sets of mathematical function items. The purpose of the questionnaire was to obtain information on the subject's Educational Background and Future Plans. All of the mathematical items addressed the concept of function. Two of the sets of mathematical function items were authored by the ETS; one set was composed of multiple-choice function items and the second set was composed of constructed-response function items. The final set of mathematical function items was adapted, developed, or created by the researcher and consisted of constructed-response function items.

Educational background and future plans. The questionnaire, Educational Background and Future Plans, (see Appendix D) requested information on the number of semester hours the participant earned college credit for studying mathematics and mathematics education courses, the type of certification s/he was pursuing, and the grade level s/he would prefer to teach. This information was collected to identify the similarities and differences among the subjects studied to identify any exceptional information that would inform investigator in the analysis of the data.

ETS multiple-choice items. The ETS MC Items (see Appendix E) included 21 multiple-choice items. Each of these items was obtained either from an ETS publication marketed for students who wish to become certified in mathematics at the middle or secondary levels or from the ETS website created for the same purpose. Each of the items was identified by ETS as related to the concept of function.

ETS constructed-response items. The ETS CR Items (see Appendix F) included five constructed-response items. These items were obtained either from an ETS publication marketed for students who wished to become certified in mathematics at the middle or secondary levels or from the ETS website created for the same purpose. Each of the items was identified by ETS as related to the concept of function.

Researcher-developed constructed-response items. The RD CR Items (see Appendix G) was an instrument composed of 15 constructed-response questions. These questions were adapted, developed, or created by the investigator in response to limitations identified in the ETS MC Items (see Appendix E) or the ETS CR Items (see Appendix F) during the pilot studies and based upon the suggestions identified in the Framework**.** The limitations were identified either by content from the Dreyfus and Eisenberg (1982) Function Block or by depth of conceptual understanding from Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. For example, neither the ETS MC Items (see Appendix E) nor the ETS CR Items (see Appendix F) contained any questions on multivariate functions (see the Dreyfus and Eisenberg Function Block levels – z-axis). Therefore, the investigator developed items 5, 11, and 14 (see Appendix G). Similarly, most of the questions on the ETS MC Items (see Appendix E) and the ETS CR Items (see Appendix F) did not encourage subjects to generalize or synthesize (Sierpinska, 1992, 1994). Each of the questions in the RD CR Function Items (see Appendix G) had at least one part that explicitly encouraged the subject to generalize and/or to synthesize. (See Appendix G Items 1a ix, 2, 3A, 5A, 7A, 7B, 7C, 8C, 10B, 10D, 12B, 13B, 13C, 14A, 15B.)

Research Subjects

During the Fall 2006 semester, the investigator began recruiting subjects to participate in her study; she planned to recruit 15 subjects so that, allowing for attrition, she would have a minimum of 10 subjects as participants. All subjects were required to be prospective secondary mathematics teachers who were enrolled in field experiences prior to student teaching, who were currently student teaching, or who had completed student teaching. The investigator prepared a flyer that she distributed at each recruitment session (see Appendix C). She encouraged students to volunteer to participate in the proposed study by informing them that participation in the study would help them prepare for *The ETS PRAXIS Series™* examinations that all the students must pass for certification. She also informed the students that all participants who complete all four interviews in the study would be offered a stipend of $100.00.

The investigator first recruited subjects at a land-grant university who were concurrently enrolled in a secondary mathematics education course and in a field experience prior to student teaching. A total of eight students volunteered; however, when the interviews were scheduled, four withdrew. The investigator next recruited subjects at the same university who were currently student teaching. Exactly one student volunteered. Since the researcher still needed 10 more subjects, she then invited five prospective secondary mathematics teachers who had indicated an interest in her study to participate. These subjects attended a private college and were currently student teaching. This brought the number of subjects who volunteered to participate in the study to 10; however, the researcher still needed five more participants.

At the start of the Spring 2007 semester, the researcher returned to the original land-grant university and the same secondary mathematics education course from which she had recruited her first participants. Here she recruited an additional four subjects. This brought her total number of participants to 14. Since she felt confident that allowing for attrition, she would have a minimum of 10 subjects who would complete her study, she terminated the recruitment of participants (see Table **4-2**).

Table **4-2**: Distribution of Subjects Who Participated in the Study.

	Pre-student teaching	Student teaching	Post-student teaching	Total
Land-grant university	4	4 (2)	1	9 (2)
Private college	0	0	5 (1)	5 (1)
Total	4	4 (2)	6 (1)	14 (3)

Note. The number of those who participated in only part of the study is found in parentheses.

Educational background and future plans. All 14 subjects were pursuing a certificate to teach secondary mathematics. Ten of the subjects, six from the land-grant university and four from the private college, indicated a preference for teaching in grades 9–12; three subjects, two from the land-grant university and one from the private college, in grades 6–8; and one subject from the land-grant university had no preference.

The median number of mathematics and statistics credits completed by participants was 33 credits, 34 for the nine subjects from the land-grant university and 31 for the five subjects from the private college. The mean of the number of mathematics and statistics credits was 33.3 credits, 34.1 from the land-grant university and 31.8 from the private university. The number of mathematics and statistics credits earned ranged from 26–49 credits for all fourteen subjects, from 26–49 credits for the nine subjects from the land-grant university and from 26–40 credits for the five subjects from the private college. The subject (β) from the land-grant university who had earned the largest number of mathematics and statistics credits (49) appeared to be pursuing a minor in statistics since he had already earned 14 credits in statistics (15 credits are needed for a minor); the subject (γ) with the second highest number of mathematics and statistics credits (40) was from the private college.

Timetable

The instruments were administered in an order similar to that of ETS:

Session 1: Educational Background and Future Plans Questionnaire
ETS MC[4] Items
Session 2: ETS CR Items
Session 3: RD CR Items – Part I
Session 4: RD CR Items – Part II.

The investigator chose to administer the RD CR Items in sessions 3 and 4 because of the challenges she encountered in recruiting the Panel of Advisors. She was unable to

[4] When discussing the instrumentation used in the pilot studies, the investigator used the term "Function;" for example, the ETS MC Function Items, ETS CR Function Items, and RD CR Function Items. This term was dropped in the discussion of the instrumentation for the dissertation study. However, the items in the dissertation study were the revised or same items from the pilot studies (see Table **4-1**).

incorporate their suggestions prior to the administration of the RD CR Items but did use their input to identify the items believed to have the greatest potential to elicit the generalization and synthesis indicators. She used this information to identify the RD CR Items for analysis.

Each subject was scheduled to participate in a total of four data collection sessions; each session took an average of 2.5 hours to complete. During the first session, the subject was given the Informed Consent Form (see Appendix L) to read and to sign, the Educational Background and Future Plans questionnaire to complete, and the ETS MC Items to answer. During the second session, each subject was expected to complete the ETS CR Items. During the third session, each subject was asked to answer half of the RD CR Items, specifically, Part A, items 1–8. During the fourth and final session, each subject was asked to answer the other half of the RD CR Items, specifically, Part B, items 9–15. Any subject who did not complete the prescribed session was not included in the analysis of any items for that session and was not offered any part of the typical $25.00 stipend for that session.

Data Collection

Eleven of the subjects participated in a series of four test sessions lasting 2–3 hours each; three additional subjects participated only in the first two of these four sessions lasting 2–2.5 hours each (see Table **4-2**).

Each of these sessions was video-recorded so that the investigator could obtain transcriptions of the sessions. There was difficulty with both video cameras during one session (session 2) for one subject (γ). The investigator recognized and corrected the problem after the first two items had been answered. Thus, the analysis of the data for the first two ETS CR Function Items for this one subject was based solely on the subject's written work since no video or transcript was available.

At each of the sessions, the subjects were allowed to use a graphing calculator and were asked to record their work for analysis. The investigator chose to permit the use of the same materials as ETS allows students to use when taking *The ETS PRAXIS Series™* examinations.

Format

The investigator used a retrospective interview procedure with each subject while asking him/her to respond to the ETS MC Items, the ETS CR Items, and the RD CR Items. Each subject was directed to solve each problem aloud, verbalizing as much of his/her thinking as possible. Each subject was encouraged to record his/her problem-solving method on the paper provided with the question for the purpose of future analysis. Each subject was informed that they were permitted and encouraged to use his/her graphing calculator whenever and wherever s/he wished. Each subject was also told that any item at any given session could be skipped and that s/he could return to any item from the given session during that session. Once the session was ended, however, no item from that session could be reconsidered.

Each subject was informed that the interviewer/investigator would not intervene with any comments or questions until the subject indicated that s/he had reached his/her "final answer" or until the subject communicated that s/he was unable to make any more progress in solving the specific item unless the subject needed some clarification that the interviewer judged was necessary for proceeding to answer the item successfully.

During each session, the investigator listened intently as the subjects solved the item and did not intervene unless the subject had a specific question or indicated that s/he had arrived at a "final answer" or was at an impasse. The investigator then drew a line on the paper that the subject was using to record his/her work. She encouraged the subject to continue recording his/her problem-solving process below the line and/or in a writing instrument of a different color as she probed the participants for explicit evidence of their mathematical understanding to confirm her suppositions regarding the subjects' knowledge or lack of knowledge regarding the mathematics and procedures employed in the solution of the item. This strategy was particularly important to the investigator to understand the nature of the incorrect and incomplete answers. Frequently, during this intervention, the subject and investigator became aware of some misunderstanding regarding the question or some mistake that the subject inadvertently made. The subject was then given the opportunity to rework the item and demonstrate his/her ability to arrive at an answer to the item successfully. However, the analysis of each subject's work was based upon that work completed prior to the intervention.

With the permission of the subject, all sessions were videotaped so that the investigator could obtain transcriptions of the sessions.

The researcher hired a professional transcriber to provide transcriptions of those sessions that involved the solving of the ETS MC Items, the ETS CR Items, and the RD CR Items–Parts I and II. Before conducting an analysis of any of these items, the investigator reviewed each transcript while viewing the video and studying each of the subject's method of solution.

Scoring and Coding Procedures

In order to verify the scoring and coding procedures, the investigator initially reviewed each subject's taped interview, next reviewed the subject's work, and then graded each subject for each part of the item and made notes regarding the appropriate codes and meaningful comments. The investigator identified the correct answers to the items selected by confirming her responses with those given by ETS for the ETS MC and ETS CR Items. She the scored the ETS MC Items as either correct or incorrect while she used a Scoring Rubric (see Appendix H) adapted from the one used by ETS for grading the ETS CR Items and the RD CR Items–Parts I and II. She discussed each score and code, as well as all comments regarding either the scores or the codes, with another professional mathematics educator. After completing the review of all the subjects for the given item, she then examined the scores assigned, codes noted, and comments made across each part of the item. Where there were inconsistencies, she made corrections. Again, she reviewed the results of this procedure with another professional mathematics educator. Any disagreements over a given score or code were discussed and resolved prior to the completion of the analysis for that given item or section. Final decisions were reached by means of consensus because consensus coding generates more accurate categories than independent analyses (Macgillivray & Jennings, 2008).

Data Analysis

The investigator identified relevant categories of analysis, studied these categories and the relationships between/among the categories identified, and explored these relationships at multiple levels (see Table **4-3**). In this section, the researcher will define her categories of analysis, describe her methods of analysis, clarify the relationships between/among the categories identified, and explain her exploration of these relationships at each of the multiple levels.

Table **4-3**: Analysis Protocol.

Categories	Foci[b] Correctness of response[a] Mathematical reasoning[b] Mathematical reasoning involving the concept of function[b] Extended function block[b] Strategies[b] SIMU[b]
Relationships between/among categories	Correctness of response x foci[c] Correctness of response x mathematical reasoning[c] Correctness of response x mathematical reasoning involving the concept of function[c] Correctness of response x strategies[c] Correctness of response x SIMU[c] Correctness of response x mathematical reasoning involving the concept of function x SIMU[c]
Levels	Individual items (ETS MC, ETS CR, RD CR Items) Cross-item analysis (ETS MC, ETS CR, RD CR Items) Cross-type analysis (ETS MC, ETS CR, RD CR Items)

[a] Categories that are quantitative in nature.
[b] Categories that are qualitative in nature.
[c] Relationships between/among categories that are both quantitative and qualitative (or mixed) in nature.

Categories.

Foci. In this category, the investigator identifies the content focus of the given item.

Correctness of response. This category describes the accuracy with which the subject responded to the item (or part of the item). For the ETS MC Items, the investigator scored each item based upon the answer key provided by ETS. The questions with dichotomous answer options on the ETS MC Items (see Appendix E) were simply identified as correct or incorrect; those who responded correctly received a score of 1 while those who responded incorrectly received a score of 0.

For the ETS CR (see Appendix F) and the RD CR Items (see Appendix G), the investigator scored each item and each part of each item using the Scoring Rubric (see Appendix H). She scored each item and each part of each item from two perspectives: first, from the analysis of the written work only (-W) and secondly, from the analysis of the written work in conjunction with the transcript and the video (-WTV). The scores for the analysis of the written work is indicated by the number of the specific item, the part of the item [A, B, C, etc., T (total)], and –W (written) while the scores for the analysis of the written work in conjunction with the video and transcript is indicated by the number of the item, the part of the item [A, B, C, etc., T (total)], and –WTV (written, transcript, and video). For example, “2A-W” would indicate the score for the written work only for part A of item 2 while “5T-WTV” would indicate the score for the written work as well as the work demonstrated by a review of the transcript and video for the total score for item 5.

Mathematical reasoning. This category describes the mathematical thought process that the subject communicated in the interview process as s/he attempted to solve a given item. For the ETS MC Items, the mathematical reasoning was coded as correct, incorrect, or no evidence of mathematical reasoning. For the ETS CR and RD CR Items, the mathematical reasoning was coded as correct, partially correct, incorrect, or no evidence of mathematical reasoning. When coding the constructed-response items, the investigator indicated whether the code reflected her evaluation from merely studying the subject's written work or from studying the subject's written work in conjunction with their video and transcript.

Mathematical reasoning involving the concept of function. This category describes the incorporation of the concept of function during the interview as the subject attempted to solve a given item. For all item types (ETS MC, ETS CR, and RD CR Items), this category was coded as mathematical reasoning involving the concept of function or as mathematical reasoning not involving the concept of function. Because the ETS MC Items differed in the level of understanding of the concept of function that they addressed it was impossible to identify a single criterion of the function concept upon which to base this decision. Therefore, the investigator concluded that the subject's mathematical reasoning involved the function concept unless that subject provided explicit evidence that the reasoning s/he used did not involve the function concept. However, for each of the ETS CR and the RD CR Items, all subjects used mathematical reasoning involving the function concept to solve each part of each of the items.

Strategies. This category refers to the varied approaches the subject used in attempting to solve the given item. For the ETS MC Items, the investigator examined the multiple-choice test-taking strategy the subjects used to arrive at their responses. The multiple-choice test-taking strategies identified included: reasons from the question posed, matching, substitution, elimination, guessing, and a combination of these strategies as displayed by the subjects. The coding identified precisely what strategies each subject used, even when multiple strategies were employed.

When the subject reasoned from the question posed, that individual determined the answer based upon an analysis of the stem of the item separate from examining the multiple-choice responses. When the subject employed the matching strategy, s/he had to connect to facets of the specific item and base his/her conclusion on the correct answer based upon this match. When the subject used the substitution strategy, s/he used the multiple-choice options in his/her solution of the problem to determine the best answer. When the subject employed the elimination strategy, s/he arrived at his/her answer by omitting those multiple-choice options that s/he felt were not correct. The subjects who guessed at the correct answer provided no mathematical reason for their ultimate decision regarding the multiple-choice option selected.

In analyzing the ETS CR and the RD CR Items, the investigator noted that none of the subjects who made any progress in responding to these items employed a strategy comparable to multiple-choice test-taking strategies in arriving at their responses. However, the subjects did demonstrate different approaches: reasons from the question posed and guess-and-check and/or trial-and-error.

When the subject reasoned from the question posed, that individual determined the answer based upon an analysis of the item. When the subject employed the guess-and-check and/or trial-and-error, the subjects merely substituted values to determine which answer was preferable.

Extended function block. This category connects the thoughts and ideas that the subject shared with the investigator during the interview with the Extended Function Block (Dreyfus & Eisenberg, 1982). In this category, the investigator noted possible connections with the extended Dreyfus and Eisenberg (1982) Function Block for those ETS MC Items for which specific subjects employed mathematical reasoning involving the concept of function and for all of the ETS CR and the RD CR Items. These connections were based upon the investigator's subjective data analysis for which she examined potential links with the *Settings/Representations* or *x*-axis, *Subconcepts* or *y*-axis, and *Levels/Families* or *z*-axis. The settings/representations, subconcepts, and levels/families that the subject explicitly used or whose use was inferred by the subject's written work or verbal explanation were noted. The researcher did not require the subject to specifically name the *Settings/Representations*, *Subconcepts*, or *Levels/Families* because she was more interested in the subject's concept image than concept definition (Dreyfus, 1990; Vinner, 1983; Vinner & Dreyfus, 1989).

SIMU. This category refers to the explicit application of one of Sierpinska's (1984) Indicators of Mathematical Understanding: Identification, Discrimination, Generalization, and/or Synthesis.

Identification. The initial indicator of mathematical understanding is *identification*. Identification implies discovery or recognition of some mathematical object. An individual who understands a mathematical object must be able to identify it, to recognize it, and to classify it. The author distinguishes between identifying and recognizing a mathematical object based on the ability of the subject to name the object. If the individual *cannot* name the mathematical object, s/he merely *recognizes* it; if the individual *can* name the mathematical object, s/he *identifies* it. Identification is the main operation involved in any facet of mathematical understanding; one cannot discriminate, generalize, or synthesize if one cannot recognize or classify or identify it (Sierpinska, 1994, pp. 56–57). In order for a subject to exemplify the identification indicator of mathematical understanding, the author required explicit evidence of the subject's use of the following operators or mental operations: identifying, recognizing, and/or classifying mathematical objects.

Discrimination. The second indicator of mathematical understanding is *discrimination*. An individual who understands a mathematical object must have the ability to discriminate between two or more objects by identifying, not only the similarities, but particularly the differences between or among these objects. Thus, in order to discriminate, one must first be able to identify. This individual must be able to compare and contrast these objects and identify not only the similarities, but most importantly, the differences. The degree of abstraction that the individual demonstrates in the discrimination process is indicative of the depth of understanding (Sierpinska, 1994, p. 57–58). In order for a subject to exemplify the discrimination indicator of mathematical understanding, the author required explicit evidence of the subject's ability to discriminate between two or more objects by identifying similarities among, and particularly the differences between, these objects. The subject must exemplify the ability to compare and to contrast objects by identifying not only the similarities, but most importantly, the differences.

Generalization. A third indicator of mathematical understanding is *generalization*. An individual who understands a mathematical object must have the ability to generalize, to make a conjecture, to identify one mathematical object/situation as a particular case of another more general one. "The term 'situation' is used here in a broad sense, from a class of objects (material or mental) to a class of events (phenomena) to problems, theorems or statements and theories" (Sierpinska, 1994, p. 58). Only an individual who can recognize or classify or identify the mathematical object has the ability to generalize it. (Sierpinska, 1994, p. 58–60). If an individual *recalls* that one situation is a particular case of another situation, the manner in which that individual processes the information does not exemplify the generalization operator but *does* exemplify the identification operator. In order for a subject to exemplify the generalization indicator of mathematical understanding, the author required explicit evidence of the subject's ability to categorize or to make a conjecture or to develop a generalization, namely, to identify one mathematical object/situation as a particular case of another more general one.

Synthesis. The fourth and final indicator of mathematical understanding is *synthesis*. An individual who understands a mathematical object must have the ability to *synthesize*, to seek and to find a "common link, a unifying principle, a similarity between/among several generalizations" (Sierpinska, 1994, p. 60) and identify the relationship between the new mathematical object and ones previously understood. Synthesis is a "culmination of identifications, discriminations, and generalizations" (Sierpinska, 1994, p. 154). There are also degrees of the synthesis indicator, varying from a very "local" degree to a "global" degree. Because studying "global" synthesis lies beyond the scope of the study for which this framework was developed, this research will focus only on "local" synthesis.

The researcher also defined a very elementary degree of synthesis that she called *basic synthesis*. This type of synthesis was required for an individual to process any mathematical idea and was different from "local" synthesis only because this type of synthesis required no external recognition of a common link between specific generalizations. This type of synthesis occurred when an individual processed a mathematical ideas and made connections but not between generalizations. Only when the individual provided explicit evidence of the recognition of a common link between specific generalizations will the author claim evidence of "local" synthesis.

In order for a subject to exemplify the synthesis indicator of mathematical understanding, the author required explicit evidence of the subject's ability to connect generalizations. The subject's ability to connect mathematical objects that are *not* generalizations was considered an exemplification of his/her ability to make connections or of "basic" synthesis.

Relationships between/among categories.

Correctness of response and foci. The investigator studied the connections between the subjects' correctness of response and foci for the ETS MC Items only. Because of the difference in foci for the ETS CR and RD CR Items, a comparable study was not feasible. She noted the foci of the items for which all subjects responded correctly and incorrectly and discussed the mistakes made by those subjects who responded incorrectly.

Correctness of response and mathematical reasoning. Although the correctness-of-response data shed some insight on the responses of the subjects, they provided little insight into the research questions. Therefore, the subjects' work on answering each item was next analyzed by studying the reasoning they used in solving the problem and determining the correctness of the reasoning used by the subject. For the ETS MC Items, the correctness of response was coded as correct or incorrect and the mathematical reasoning was coded as correct, incorrect, or nonexistent. For the ETS CR and RD CR Items, the correctness of response was coded as correct with a score of 5; partially correct with a score of 1, 2, 3, or 4; or incorrect with a score of 0; and the mathematical reasoning was coded as correct, partially correct, incorrect, or nonexistent. When coding the constructed-response items, the investigator indicated whether the code reflected her evaluation from merely studying the subject's written work or from studying the subject's written work in conjunction with their video and transcript.

Correctness of response and mathematical reasoning involving the concept of function. As stated above, although the correctness-of-response data shed some insight on the responses of the subjects, they provided little insight into answers to the research questions. Therefore, the subjects' work on answering each item was next analyzed by studying whether the mathematical reasoning they used in solving the problem involved the concept of function because all the items used in the study from ETS were identified as items that assessed the concept of function. For the ETS MC Items, each item response was coded as correct or incorrect and the mathematical reasoning was coded as involving or not involving the concept of function. For the ETS CR and RD CR Items, the correctness of response was coded as correct with a score of 5; partially correct or with score of 1, 2, 3, or 4; or incorrect with a score of 0; and the mathematical reasoning was coded as involving or not involving the concept of function. When coding the ETS CR and RD CR Items, the investigator indicated whether the code reflected her evaluation from merely studying the subject's written work or from studying the subject's written work in conjunction with their video and transcript.

Correctness of response and use of strategies. The investigator chose to study the relationship between the correctness of response and the use of strategies in the anticipation that this might shed some light on why some subjects could score so well on the multiple-choice items but not so well on the constructed-response items. For the ETS MC Items, the correctness of response was coded as correct or incorrect and the strategy or combination of strategies (e.g., matching, substitution, elimination, guessing, or reasons from the question posed) that the subject employed was also noted. For the ETS CR and the RD CR Items, the investigator noted that none of the subjects who made any progress in responding to these items employed a strategy comparable to multiple-choice test-taking strategies in arriving at their responses. However, the subjects did demonstrate different approaches: reasons from the question posed and guess-and-check and/or trial-and-error. Therefore, for the ETS CR and RD CR Items, the correctness of response was coded as correct with a score of 5; partially correct or with score of 1, 2, 3, or 4; or incorrect with a score of 0; and the strategy or combination of strategies (e.g., reasons from the question posed, guess-and-check, and/or trial-and-error) that the subject employed was also noted.

Correctness of response and exhibition of SIMU. This analysis was not contingent upon the subjects' use of mathematical reasoning involving the function concept and therefore sometimes described understandings other than those that are related to the concept of function. The researcher examined the relationship between each subject's score or correctness of response and each subject's demonstration of SIMU: identification, discrimination, generalization, and synthesis.

For the ETS MC Items, the correctness of response was coded as correct or incorrect and then the exhibition of SIMU was noted. Similarly, for the ETS CR and RD CR Items, the correctness of response was coded as correct with a score of 5; partially correct or with score of 1, 2, 3, or 4; or incorrect with a score of 0; and then the exhibition of SIMU was noted. If the subject identified, recognized, and/or classified mathematical objects the investigator noted the identification indicator. If the subject discriminated between two or more objects by identifying similarities among, and particularly the differences between, these objects, she noted the discrimination indicator. Of particular importance was the subject's ability to compare and to contrast objects by identifying not only the similarities, but most importantly, the differences. If the subject categorized or made a conjecture or to develop a generalization, or to identify one mathematical object/situation as a particular case of another more general one, the researcher noted the generalization indicator. If the subject demonstrated the ability to connect generalizations and identified the relationship between the new mathematical object and ones previously understood, the investigator noted the local synthesis indicator.

However, if the subject demonstrated the ability to connect mathematical ideas that were not generalizations, she noted the basic synthesis indicator. In each case, the investigator discussed the specific indicators explicitly identified in relationship with the score received.

Correctness of response, use of mathematical reasoning involving the concept of function, and exhibition of SIMU. Finally, the investigator examined how the subject's use of mathematical reasoning involving the concept of function was related to the number of specific indicators (e.g., identification, discrimination, generalization, synthesis) of mathematical understanding according (Sierpinska, 1984) for each of the ETS MC Items, ETS CR Items, and the RD CR Items. Unlike the analysis of correctness of response and the application of SIMU, this analysis was contingent upon the subjects' use of mathematical reasoning involving the concept of function. For each subject whose mathematical reasoning did involve the concept of function, she examined the relationship between each subject's score or correctness of response and each subject's demonstration of SIMU: identification, discrimination, generalization, and synthesis. In this section, the investigator specifically emphasized those subjects who answered the item correctly (i.e., received a score of 1 for the ETS MC Items or a score of 4 or 5 for the ETS CR and RD CR Items) and investigated what can be learned about each item from these subjects' responses and use of indicators. If a subject answered the specific item without using mathematical reasoning involving the concept of function, the subject's data was not included in this analysis.

Levels.

Analysis of individual ETS MC, ETS CR, and RD CR items. Each of the 21 ETS MC Items (see Chapter 6), 5 ETS CR Items (see Chapter 7), and 6 selected RD CR Items (see Chapter 9), was analyzed from multiple perspectives:

1. The 5 ETS CR Items, and 6 selected RD CR Function Items were first analyzed by item with the investigator studying the correctness of subjects' responses to each part of the question (Parts A – C) based both on the subject's written response and on the

subject's written response in conjunction with the transcript and video. Each subject received a score using the Scoring Rubric (see Appendix H).

2. Because of the nature of the ETS MC Items, the correctness of response for these items was addressed in conjunction with the subjects' reasoning. The investigator was particularly interested in the correct responses of the subjects who reasoned incorrectly and the incorrect responses of the subjects who reasoned correctly. In the discussion of the relationship between the correctness of response and the subjects' reasoning the ETS CR Items and the RD CR Items, the investigator examined the responses and reasoning, where she focused on the analysis of each subject's use of mathematical reasoning in relationship to the total score that each subject received for the item using a Scoring Rubric (see Appendix H). The total score was not an average of the scores for each part of the item but rather a score for the total item based upon the Scoring Rubric (see Appendix H).
3. In analyzing the ETS MC Items, many subjects selected to solve the items without using the concept of function. Because all the ETS MC Items were identified by ETS as items that assessed the concept of function, it was important to note that a subject's correct response to a given item might not be because s/he understood the concept of function. Unless a subject provided explicit evidence that the reasoning s/he used did not involve the function concept, the investigator concluded that the subject's mathematical reasoning involved the function concept. In analyzing the ETS CR Items and the RD CR Items, all subjects who were successful in making any progress in solving each item used mathematical reasoning involving the function concept.
4. Similarly, in examining the ETS CR Items and the RD CR Items, the subjects' use of mathematical reasoning involving the function concept precluded their use of multiple-choice test-taking strategies. Thus, there was no need to examine these items from this perspective as there was when examining the ETS MC Items.
5. In analyzing the ETS MC, ETS CR, and RD CR Items, the investigator noted the connections the subjects made with the Expanded Function Block. Unlike the other sections of analysis, these connections were determined implicitly rather than explicitly. By this the investigator means that the subject did not need to specifically

mention the connection with the Expanded Function Block but needed to infer the connection by their shared thoughts and/or work.

6. When studying the ETS MC, ETS CR, and RD CR Items, the researcher examined the application of Sierpinska's Indicators of Mathematical Understanding (identification, discrimination, generalization, and synthesis). She also examined the relationship between each subject's score for each item and the specific indicators each subject exemplified.
7. In concluding this section, the investigator analyzed the ETS MC, ETS CR, and RD CR Items and noted subjects' application of SIMU, and the relationship among the correctness of the subjects' responses, their use of mathematical reasoning involving the concept of function, and their application of SIMU with specific emphasis on those subjects who answered the item correctly (i.e., received a score of 1 for the ETS MC Items or a score of 4 or 5 for the ETS CR and RD CR Items) and what we can learn about each item from these subjects' responses and use of indicators. If a subject answered the specific item without using mathematical reasoning involving the function concept, this subject's data was not included in this analysis.

Cross-item analysis within item types. Following the analysis of each individual item, the investigator analyzed all the items across the specific item type (ETS MC, ETS CR, and RD CR Items). In the Cross-item Analysis (see Chapters 6, 7, and 9 respectively), she searched for patterns using the criteria established in the analysis of each individual item by studying the following:

1. the connections between the correctness of response and foci of each item;
2. the connections between the correctness of response and the use of correct or incorrect mathematical reasoning;
3. the connections between the correctness of response and the use of mathematical reasoning involving the function concept and the use of mathematical reasoning not involving functions;
4. the connections between the correctness of response and the use of specific strategies;

5. the connections between the use of mathematical reasoning involving the function concept and the probable connections with the expanded Dreyfus and Eisenberg Function Block;
6. the connections between the correctness of response and the use of multiple-choice test-taking strategies (for the ETS MC Items only);
7. the connections between the correctness of response and application of SIMU (identification, discrimination, generalization, and synthesis); and
8. the connections among the correctness of the subjects' responses, use of mathematical reasoning involving the function concept, and use of SIMU.

Cross-type analysis across item types. In the cross-type analysis (see Chapter 10), the investigator studies the similarities and differences of the cross-item analyses for the ETS MC, ETS CR, and RD CR Items by analyzing the three types of items (ETS MC, ETS CR, and RD CR Items) from multiple perspectives:

1. In the comparison of foci, the investigator examines the focus of each item from each type of item (ETS MC, ETS CR, and RD CR Items).
2. In the analysis of the correctness of responses, the investigator studies the subjects' responses (completely correct, partially correct, incorrect) to each type of item (ETS MC, ETS CR, and RD CR Items).
3. In the analysis of correctness of responses and reasoning, the investigator notes the correctness of the subjects' answers (i.e., the total score) in conjunction with the reasoning of the subjects for each type of item (ETS MC, ETS CR, and RD CR Items).
4. In the analysis of correctness of responses and mathematical reasoning involving the function concept, the investigator notes the correctness of the subjects' answers (i.e., the total score) in conjunction with the use of mathematical reasoning involving the function concept for each type of item (ETS MC, ETS CR, and RD CR Items).
5. In the analysis of responses and strategies, the investigator studies those subjects who reasoned from the question posed as well as those who reasoned using other strategies and approaches, including all multiple-choice test-taking strategies (substitution,

elimination, matching, etc.) and guess-and-check/trial-and-error strategies, in relationship to the correctness of the responses of the subjects for each type of item (ETS MC, ETS CR, and RD CR Items).

6. In the connections between the subjects' responses and the Expanded Function Block, the investigator examines the reasoning of the subjects for each type of item (ETS MC, ETS CR, and RD CR Items) who used any facet of the function concept in answering the question or solving the problem regardless of their correctness of responses.
7. In the analysis of responses and of the subjects' application of SIMU, the investigator compares the total scores for the responses of those subjects whose reasoning exemplified one or more of Sierpinska's Indicators for each type of item (ETS MC, ETS CR, and RD CR Items) and includes a discussion of subjects' understanding by examining how the subjects employed the specific indicators in reasoning through the question and arriving at their responses.
8. In the analysis of correctness of response, mathematical reasoning involving the function concept, and SIMU, the investigator examines the indicators of mathematical understanding exemplified by subjects whose mathematical reasoning involved the function concept and whose responses demonstrated an understanding of the item for each type of item (ETS MC, ETS CR, and RD CR Items).

Chapter 5

ORIGINAL FRAMEWORK[5]

The purpose of this chapter is to share the original Framework that was sent to each of the Panel of Advisors. The chapter begins with an Overview in which the investigator explains the organization of the original Framework. The Table of Contents for the original Framework has been absorbed into the Table of Contents for the dissertation and the References for the original Framework is now contained in the References for the dissertation. The cover letter (see Appendix P) that was sent to each of the members of the Panel of Advisors and all three instruments that are described in the cover letter, the Comments and Critiques on Framework (see Appendix M), the Recommendations Evaluation of Recommendations for Creating Constructed-Response Items (see Appendix N), and the Item Evaluation Instrument (see Appendix O), are included as well in the Appendices of this dissertation. The reader should be advised that during the process of collecting and analyzing the data for this research project the investigator's understanding of this Framework evolved; therefore, a Revised Framework is also included as Chapter 11 of this dissertation. Other than this initial paragraph, the remainder of this chapter is the original Framework that was distributed to each member of the Panel of Advisors.

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[5] This Framework was created to:

- connect the concept of function with Sierpinska's Indicators of Mathematical Understanding;
- describe items that assess identification, discrimination, generalization, and synthesis (Sierpinska's Indicators of Mathematical Understanding) and that focus on the concept of function;
- explain how to write such items;
- develop items specifically created to focus on assessing a subject's ability to generalize and to synthesize; and
- seek affirmation from the identified Panel of Advisors that the items created do address Sierpinska's Indicators of Mathematical Understanding, particularly generalization and synthesis.

The researcher uses the word *Framework* in this dissertation in the context of the preceding definition for which it was originally created. The researcher acknowledges that *framework* may be used by other authos to convey something different from this, but because the term *framework* was used when materials were sent to the Panel of Advisors, that term will be used subsequently.

Overview

The purpose of this paper is to develop a framework that will be used to develop items to assess a subject's understanding of the concept of function.

The paper is organized in three sections:

1. ***Conceptual and Theoretical Models*** In this section, the author describes the conceptual and theoretical models for clarifying what "understanding the concept of function" means in the context of this study. The conceptual model on which the author builds this understanding is the Function Block (expanded from Dreyfus and Eisenberg (1982, 1983)) and the theoretical model on which the author conceptualizes this study is Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. The author uses the Function Block to identify various "dimensions" of the function concept and she chose Sierpinska's (1992, 1994) Indicators of Mathematical Understanding as the most appropriate theoretical model of understanding for this study.
2. ***Analyzing Assessment Items*** In this section, the author analyzes the advantages and disadvantages of assessing subjects' understanding of the function concept using multiple-choice and constructed-response items. She explains how to write specific items that will assess the understanding of the concept of function using Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. Finally, she develops a framework one can use to generate items with the potential to assess a subject's understanding of the concept of function based upon Sierpinska's (1992, 1994) Indicators of Mathematical Understanding.
3. ***Generating Assessment Items*** In this final section, the author uses the framework she developed to propose a set of items that she believes has the potential to assess a subject's understanding of the concept of function using Sierpinska's (1992, 1994) Indicators of Mathematical Understanding, particularly items that require subjects to generalize and synthesize.

The author's conjectures for the "potential connection to the theoretical and conceptual models" are based upon a pilot study in which five mathematics education graduate students were asked to solve each of the researcher-generated constructed-response items aloud. Once they had reached their "final answer," the author/researcher/interviewer posed questions to identify clearly which of Sierpinska's Indicators of Mathematical Understanding they had used in reaching their solution. The data from this pilot study suggest that the researcher-generated constructed-response items do have the potential to assess all of Sierpinska's Indicators of Mathematical Understanding, particularly the generalization and synthesis indicators.

The Conceptual Model

The Function Block

The concept of function is exceptionally complex; it is composed of multiple layers each containing multiple topics. The conceptual model upon which the author has ordered the various function topics is the expanded Dreyfus and Eisenberg (1982, p. 365; 1984, p.79) Function Block. Dreyfus and Eisenberg (1982, p. 365; 1984, p. 79) constructed a three-dimensional block structure in which the *x*-axis represents the various *settings* (arrow diagrams, tables, graphs, etc.) or *representations*, the *y*-axis the function *concepts* (image, zeros, equality, etc.), and the *z*-axis a "taxonomic scale of levels of abstraction and generalization (one, two, or several variables, discrete domain, etc.). The *z*-axis in itself is multi-dimensional" (Dreyfus & Eisenberg, 1982, p. 364) since the *z*-axis embodies all aspects of the concept of function that are not specific settings/representations or concepts (see Figure **5-1**). The Function Block continues in all directions since the number of settings/representations, concepts, or levels of abstraction and generalization associated with the function concept will continue to evolve and grow.

Dreyfus and Eisenberg (1984) state that they hope that "others will use the function block as a theoretical framework for understanding how students view mathematical functions" (p. 84). In the study for which this framework was developed, the author has accepted their invitation and expanded the Dreyfus and Eisenberg (1982, 1983, 1984) Function Block to exemplify those settings/representations, sub-concepts, and levels that she believes are most important in the understanding of the function concept for secondary mathematics teachers based upon the analysis of data from a pilot study conducted with a panel of mathematics education doctoral candidates (see Table **5-1**.) Like Dreyfus and Eisenberg, the author wishes to emphasize that these settings/representations, sub-concepts, and levels must *not* be considered all-inclusive; since the concept of function is ever-evolving so these settings/representations, sub-concepts, and levels will continue to evolve and grow.

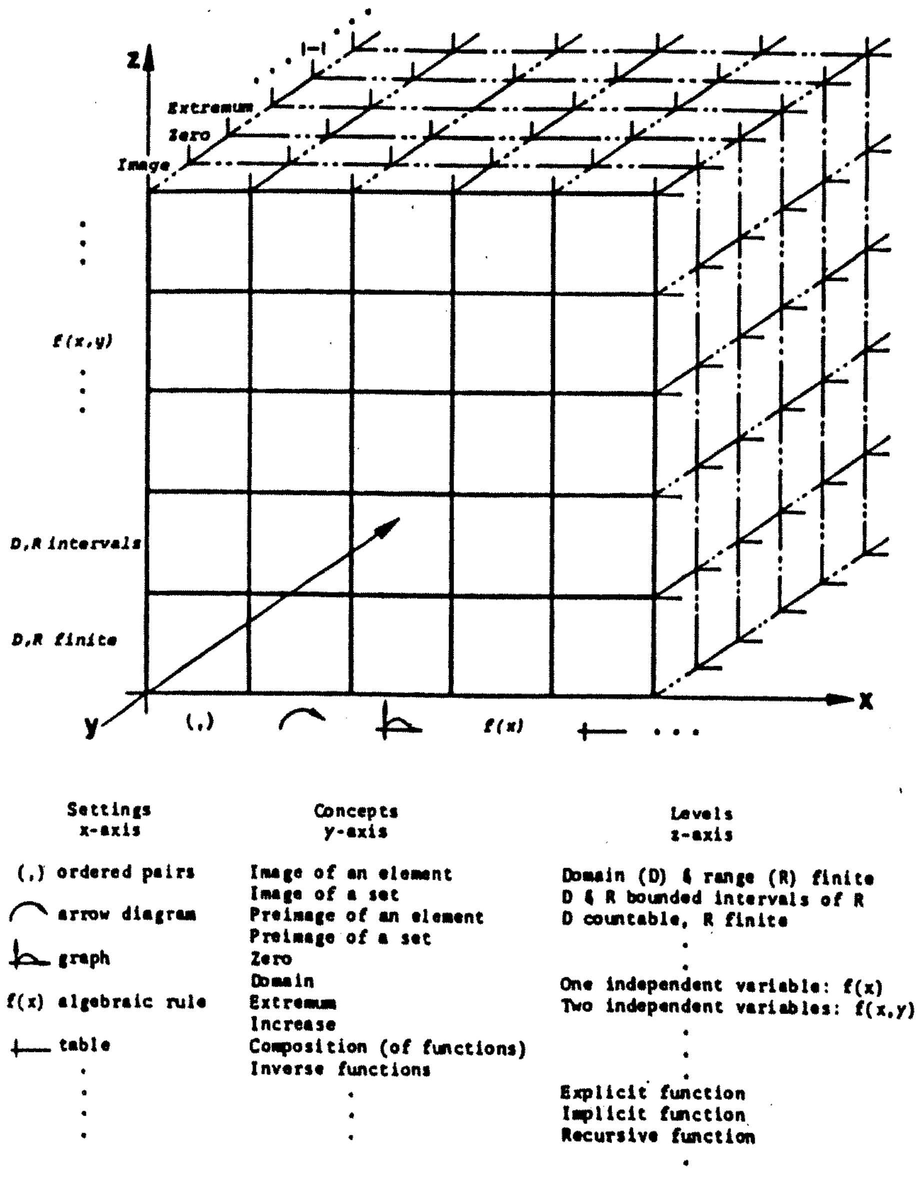


Figure **5-1:** The Function Block (Dreyfus & Eisenberg, 1982, p. 365)

Table **5-1**: The Expanded Function Block Model.

***x*-axis** **(*Settings/Representations*)**	***y*-axis** **(*Sub-concepts*)**	***z*-axis** **(*Levels/Families*)**
Arrow diagrams	Asymptotes	Domain and range types
Graphs	Bijection	Finite
Ordered pairs	One-to-one, onto	Infinite
Symbols	Boundedness	$\Re$
Tables	Composition of functions	Bounded intervals of $\Re$
Verbal explanations	Constant	Complex
Word problems	Decreasing	
	Dependent variable	Independent Variables
	Domain	Univariate functions
	Equality of functions	Bivariate functions
	Even functions	Multivariate functions (3 or more)
	Global behavior	
	Image of an element	Function Families
	Image of a set	Absolute-value function
	Increasing	Ceiling function
	Independent variable	Exponential function
	Injection (one-to-one)	Floor function
	Intercepts	Logarithmic function
	Into	Piecewise-defined function
	Inverse	Polynomial function
	Local behavior	Power function
	Mapping	Rational function
	Maximum	Staircase function
	Minimum	Trigonometric function
	Odd function	
	Parallel	Function Types
	Perpendicular	Explicitly defined functions
	Preimage of an element	Implicitly defined functions
	Preimage of a set	Recursively defined functions
	Range	Parametrically defined functions
	Rate of change	
	Restrictions	
	Surjection (onto)	
	Transformations	
	Translations	
	Univalence (single-valuedness)	
	Zeros of a function	

(Adapted from Dreyfus and Eisenberg, 1984, p. 79.)

Like the Dreyfus and Eisenberg (1982, 1983, 1984) Function Block, the author's expanded Function Block may be used to exemplify various aspects of learning. The transfer of a concept learned in one setting to another setting may be envisioned as a horizontal transfer of learning and will be represented by movement parallel to the *x*-axis (Dreyfus & Eisenberg, 1984).

Example 1. Consider the mathematical understanding of a subject[6] who comprehends how a given function can be expressed in multiple representations (symbolically, tabularly, graphically). For example, in a linear function the subject would recognize the slope of the function $y = mx + b$ as the variable *m* in the symbolic representation, as $\frac{\Delta y}{\Delta x}$ in the tabular representation, and as the $\frac{\text{rise}}{\text{run}}$ in the graphical representation. Similarly, the subject would recognize the *y*-intercept as *b* in the symbolic representation, as (0, *y*) in the tabular representation, and as the point where the graph intersects the *y*-axis in the graphic representation. Such an individual would demonstrate the understanding of a function in various settings/representations and would thus exemplify a horizontal transfer of learning or learning represented by movement parallel to the *x*-axis in the Function Block.

The transfer of learning to levels of greater generality within sub-categories may be viewed as a vertical transfer of learning and will be represented by movement parallel to the *z*-axis (Dreyfus & Eisenberg, 1984).

Example 2. Consider the mathematical understanding of a subject who comprehends the meaning of the concept of function and is able to identify those graphs in two dimensions that are graphs of univariate functions as well as those that are not. When such an individual is able to expand this understanding to graphs in three dimensions and identify those graphs that are graphs of bivariate

[6] In this paper, the author bases her examples upon the similar responses of subjects from her pilot studies. The author uses these data to exemplify the points under discussion. However, not all subjects did/would necessarily respond in the ways that the illustrative examples suggest.

functions as well as those that are not, such an individual would demonstrate the transfer of learning to a level of greater generality. This would exemplify a vertical transfer of learning or learning represented by movement parallel to the *z*-axis of the Function Block.

The learning of new concepts that are subconcepts of the concept of function will be represented by movement parallel to the *y*-axis (Dreyfus & Eisenberg, 1982, p. 364). When a subject expands his/her understanding by developing an understanding of any new concept, that subject's learning would be represented by movement parallel to the *y*-axis of the Function Block. In the study for which this framework was developed, "developing an understanding" of a concept involves the subject's ability to identify the mathematical topic, to discriminate among mathematical topics, and to use the mathematical topic in generalizing and synthesizing other mathematical topics.

The Theoretical Model

The author reviewed the following frameworks of models of mathematical understanding to assess their applicability as theoretical frameworks in the proposed study: Vinner's and Dreyfus' (1989) Concept Image–Concept Definition, Sfard's (1992) Process/Object Theory of Learning, Pirie and Kieran's (1989, 1992a, 1992b, 1994) Recursive Model of Mathematical Understanding, and Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. The Concept Image–Concept Definition model was too limited; the author is interested not only in mathematical topics that are concepts, but also in those that are generalizations. The Process/Object Theory of Learning focused more on the development of understanding *through* learning than on the assessment of understanding as a *result of* learning. Similarly, the Recursive Model of Mathematical Understanding also focuses on the growth of understanding through learning rather than on the assessment of understanding as a result of what was learned. The Indicators of Mathematical Understanding was the most appropriate model to use in assessing the understanding that an individual possesses; it is not limited in scope as is the Concept Image–Concept Definition Model nor is it limited in its applicability as are the Process/Object Theory of Learning and the Recursive Model of Mathematical Understanding.

Indicators of Mathematical Understanding

Sierpinska (1994) has proposed four basic mental operations involved in understanding mathematics: identification, discrimination, generalization, and synthesis. In this section, the author will explain her understanding concerning each of Sierpinska's basic mental operations.

In this section, the author will explain her understanding of each of Sierpinska's basic mental operations and she will connect the mathematical objects addressed in specific questions with the levels of the adapted Function Block. While the mathematical object (i.e., the mathematical idea being examined or studied) *does* determine the level of the adapted Function Block identified, the author's understandings of Sierpinska's indicators of mathematical understanding are based upon the *key* idea that the indicator is determined *not* by the mathematical object (the item or task) involved but rather by the manner in which the individual *operates* on the mathematical idea(s) s/he believes is(are) pertinent to the task. This interaction between the individual and the mathematical object determines whether there is any demonstration of understanding.

Identification. The initial indicator of mathematical understanding is *identification*. Identification implies discovery or recognition of some mathematical object. An individual who understands a mathematical object must be able to identify[7] it, to recognize it, and to classify it. The author distinguishes between identifying and recognizing a mathematical object based on the ability of the subject to name the object. If the individual *cannot* name the mathematical object, s/he merely *recognizes* it; if the individual *can* name the mathematical object, s/he *identifies* it. Sierpinska (1994) states that classifying is different from categorizing; a mathematical object included into a class is not a "particular case" of this class but is merely an element of it. For example, an individual may recognize a geometric shape as a triangle and even identify it as such. In

[7] Sierpinska (1994) describes an individual who exemplifies the identification indicator as one who must be able to identify, to recognize, and to classify a mathematical object.

this instance, the individual *classifies* the figure because s/he recognized and/or identified the figure as an element of the class of geometric figures known as triangles. However, if the individual recognizes that the geometric shape is but one of many other geometric shapes called triangles that have three sides and three angles, the individual is *categorizing*. In categorizing, a class of objects is subsumed by another class of objects. While classifying is a form of the identification indicator, categorizing is a form of the generalization operator.

Identification is the main operation involved in any facet of mathematical understanding; one cannot discriminate, generalize, or synthesize if one cannot recognize or classify or identify it (Sierpinska, 1994, pp. 56–57).

Example 3. Consider the process that an individual would use in attempting to answer the following item:

What is the range of the function $y = \dfrac{x^2 - 1}{x^2 + 1}$?

(A) $-1 < y < 1$

* (B) $-1 \leq y < 1$

(C) All real numbers except -1

(D) All real numbers

(Educational Testing Service (ETS), 2004c.)

* Indicates correct response.

In order to respond correctly to this item, the individual should understand the concept of the range. S/he could process this item by recognizing that this function is a univariate function that is expressed symbolically. The word "range" refers to the values of the dependent variable, in this case, the *y*-values. Although the problem is expressed symbolically, the problem might be more easily processed graphically. Making this transition, the individual could view the *y*-values on the graph of the given univariate function, compare this graphical representation with each of the options, and determine the option that best represents the range.

This exemplification of the identification indicator of the range of a function pertains to the *x*-axis of the expanded Function Block: Settings/Representations: Symbols and Graphs, to the *y*-axis of the expanded Function Block: Sub-Concepts: Range, and to the *z*-axis of the expanded Function Block: Levels: Family of Functions: Univariate Function (see Table **5-2.**)

Table **5-2**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for **Example 3**.[8]

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Range	Symbols and Graphs	Range	Family of Functions: Univariate Functions

Example 4. Consider the mathematical understanding of a subject who is attempting to answer the following item:

The Richter scale is a base 10 logarithmic scale used to measure the magnitude of earthquakes; namely, an earthquake measuring 7 is ten times as strong as an earthquake measuring 6. An earthquake that measures 6.8 on the Richter scales has a magnitude that is approximately what percent of an earthquake measuring 6.6?

(A) 103%

(B) 120%

* (C) 158%

(D) 200%

(Educational Testing Service (ETS), 2003, p. 70, #22)

[8] This table and those that follow are meant to summarize the example used. The table and those that follow are *not* meant to indicate all possible methods of solution.

In order to determine the answer to this item, the subject must be able to recognize a base 10 logarithmic scale and to classify it as being connected with the logarithmic family of functions. Without being able to make this identification, the subject will be unable to determine an answer to this item mathematically.

Realizing that "an earthquake measuring 7 is ten times as strong as an earthquake measuring 6," the subject could then connect the logarithmic and exponential families of functions and recognize that this is true because $\frac{10^7}{10^6} = 10^1 = 10$. Similarly, the relationship in the strengths of earthquakes measuring 6.8 and 6.6 on the Richter scale would be $\frac{10^{6.8}}{10^{6.6}} = 10^{0.2} = 1.5849$.

This exemplification of the identification indicator of the logarithmic family of functions pertains to the *z*-axis of the expanded Function Block: Levels: Family of Functions: Logarithmic and Exponential (see Table **5-3**).

Table **5-3**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for **Example 4**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Logarithmic Family of Functions			Family of Functions: Logarithmic

Since there are degrees of identification as the author described above, the individual who can recognize a mathematical idea may or may not be able to identify the mathematical idea. Even if the individual *can* identify the mathematical idea, the individual's concept definition (Vinner & Dreyfus, 1989) may not accurately reflect his/her concept image.

Example 5. Consider the mathematical understanding of a subject who is unable to give a rigorous definition of the concept of function, but can demonstrate an understanding of the concept of function by his/her identification or recognition of instances of function represented symbolically, graphically, or tabularly. This subject may have a concept image of function that does not match his/her concept definition.

This exemplification of the identification indicator in giving a definition of the concept of function pertains to the *y*-axis of the expanded Function Block: Sub-concepts: Function, while the exemplification of the identification indicator in recognizing and classifying multiple representations pertains to the *x*-axis of the expanded Function Block: Settings/Representations: Graphs, Symbols, Tables, Ordered Pairs (see Table **5-4**.)

Table **5-4**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for **Example 5**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Definition of Functions		Function	
Identification: Recognizing and Classifying Multiple Representations	Graphs, Symbols, Tables, Ordered Pairs		

Example 6. Consider the mathematical understanding of a subject who attempts to solve the following problem:

Consider the function *f* defined for all real numbers by
$f(x) = \frac{x^4}{2} - 3x^3 - x^2 + 15x + c$, where *c* is a real-valued constant.

(A) Determine a value of *c* for which the function has exactly 4 real roots. Explain how you arrived at your answer.

(B) Determine a value of *c* for which the function has no real roots. Explain how you arrived at your answer.

(C) Determine a value of *c* for which the function has exactly 2 real roots. Explain how you arrived at your answer.

(Educational Testing Service (ETS), 2004d.)

In order to solve this problem, the subject only needed to identify that the number of real roots that a function has is indicated by the number of intersections of the graph of the function and the *x*-axis. Once the subject realizes this, s/he needed only to graph the given function using various values for *c.* The answers to this constructed-response item can be attained merely by entering values into a graphing calculator and examining the resulting graphs.

This exemplification of the identification indicator in recognizing that the number of real roots that a function has is indicated by the number of intersections of the graph of the function and the *x*-axis pertains to the *x*-axis of the expanded Function Block: Settings/Representations: Graphs and Symbols; to the *y*-axis of the expanded Function Block: Sub-concepts: Translation: Vertical Shift; and to the *z*-axis of the expanded Function Block: Levels: Polynomial Functions (see Table **5-5**).

Table **5-5**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for **Example 6**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Number of Real Roots of a Function	Symbols and Graphs	Translations	Polynomial Functions

The constructed-response item addresses more axes of the expanded Function Block than any of the multiple-choice items. However, because of the manner in which this question is posed, it only requires the application of the identification indicator of mathematical understanding.

In order for a subject to exemplify the identification indicator of mathematical understanding, the author will require explicit evidence of the subject's use of the following operators: identifying, recognizing, and/or classifying mathematical objects.

Discrimination. The second indicator of mathematical understanding is *discrimination*. An individual who understands a mathematical object must have the ability to discriminate between two or more objects by identifying, not only the similarities but particularly the differences between or among these objects. Thus, in order to discriminate, one must first be able to identify. This individual will be able to compare and contrast these objects and identify not only the similarities, but most importantly, the differences. The degree of abstraction that the individual demonstrates in the discrimination process is indicative of the depth of understanding (Sierpinska, 1994, p. 57–58). One very basic degree of discrimination is the recognition that the two objects are distinct. A second degree of discrimination is the ability to recognize and to identify the similarities and differences between two objects. An even higher degree of discrimination is the ability to compare and to contrast two general ideas in an abstract relation. It should be noted that this author views Sierpinska's indicators of mathematical understanding to be specifically related to some mathematical idea; merely recognizing similarities and differences between or among non-mathematical ideas will not be viewed as an indication of the discrimination indicator in the study for which this framework was developed.

Example 7. Consider the mathematical understanding of two subjects who attempt to answer the following item:

The following functions of *x* have the same domain (*x* values). Which of the functions have the same range (*y* values)?

I. $y = x^2 + 1$

II. $y = 2|x| + 1$

III. $y = 2^x$

* (A) I and II only

(B) I and III only

(C) II and III only

(D) I, II, and III

(Educational Testing Service (ETS), 2004a, p. 11, #19.)

The first subject begins to solve this problem by determining the graph of each of the functions (I, II, III). This subject then examines each of the three graphs, identifies both the domain and range of each graph, discriminates among the domain and range of each graph, and determines which of the functions (I, II, III) have the same range.

This subject exemplifies first the identification indicator by relating the symbolic notation of each function with its graphical representation, connecting with the expanded Function Block: *x*-axis – Settings/Representations: Symbols, Graphs. Then the subject exemplifies the discrimination indicator by comparing and contrasting the graphical representations of the range of each option; again connecting with the expanded Function

Block: *x*-axis – Settings/Representations: Graphs and *y*-axis – Sub-concepts: Range (see Table **5-6**).

Table **5-6**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the First Subject in **Example 7**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Multiple Representations	Graphs and Symbols		
Discrimination: Range of the Three Options	Graphs and Symbols	Range	

The second subject analyzes each of the functions symbolically without the necessity of referring to the precise graph of each. This subject identifies the range of each of the functions (I, II, III) based upon his/her understanding of each of the families of functions (quadratic, absolute value, exponential). Thus, the second subject demonstrates an ability to discriminate in greater abstraction or at a higher level than the first subject.

This subject exemplifies first the identification indicator by classifying each function as a member of a family of functions, connecting with the expanded Function Block: *z*-axis – Levels: Family of Functions: Quadratic, Absolute Value, Exponential. Then the subject exemplifies the discrimination indicator by comparing and contrasting the range of each function; connecting with the expanded Function Block: *x*-axis – Settings/Representations: Graphs, *y*-axis – Sub-concepts: Range, and *z*-axis – Levels: Family of Functions: Quadratic, Absolute Value, Exponential (see Table **5-7**.)

Table **5-7**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the Second Subject in **Example 7**.

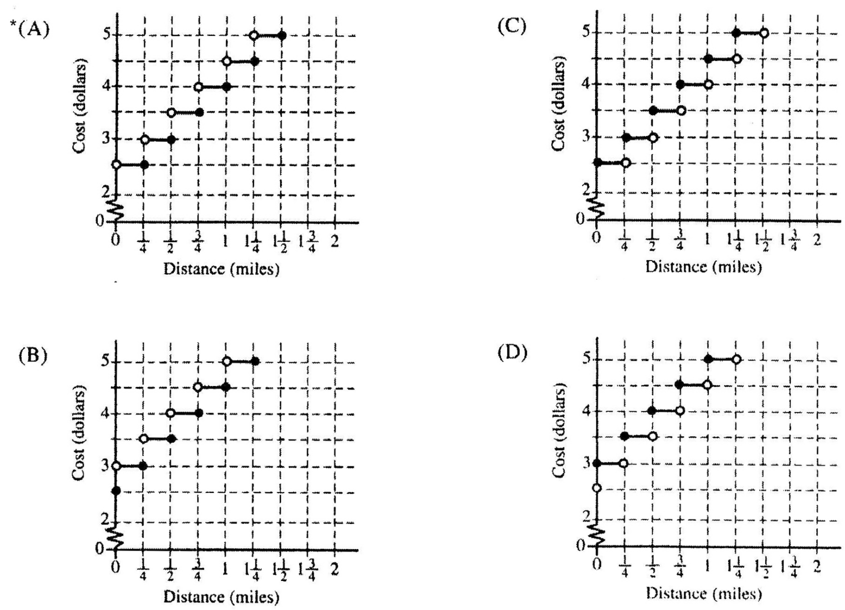
	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Functions			Families of Functions: Quadratic, Absolute Value, Exponential
Discrimination: Range of the Three Options	Graphs	Range	Families of Functions: Quadratic, Absolute Value, Exponential

Both subjects must identify the mathematical object "range," the values of the dependent variable, in this case, the *y*-values. Discrimination will not occur if the individual cannot identify the mathematical object.

This example refers to the sub-concept, range. Consider the following example that involves the identification and discrimination of a setting/representation (graph).

Example 8. Consider the process that an individual uses in attempting to answer the following item:

A taxi ride costs \$2.50 for the first $\frac{1}{4}$ mile or fraction thereof plus \$.50 for each additional $\frac{1}{4}$ mile or fraction thereof. Which of the following graphs represents the total cost of the ride as a function of distance traveled?



(Educational Testing Service (ETS), 2004a, p. 118, #16.)

The first subject begins by processing the word problem and recognizing that if the taxi ride costs \$2.50 for the first $\frac{1}{4}$ mile or fraction thereof, the graph must "begin" at the point (0, \$2.50) but that this point must *not* be included in the graph. This awareness causes the individual to eliminate two choices ((B) and (C)). The subject then recognizes that the price remains constant for the first $\frac{1}{4}$ mile or fraction thereof, causing him/her to eliminate choice (D) and select (A) as the best answer.

This subject exemplifies first the identification indicator by recognizing that the initial point (0, \$2.50) must *not* be included in the graphical representation, connecting with the expanded Function Block: *x*-axis – Settings/Representations: Word Problem, Ordered Pairs, Graphs; enabling this subject to eliminate choices (B) and (C). Then the subject exemplifies the discrimination indicator by comparing and contrasting the graphical representations of choices (A) and (D), again connecting with the expanded Function Block: *x*-axis – Settings/Representations: Graphs and finally the subject exemplifies the identification indicator by recognizing that the graph of choice (A) has the price remaining constant for the first $\frac{1}{4}$ mile or fraction thereof, connecting with the expanded Function Block: *x*-axis – Settings/Representations: Graphs (see Table **5-8**).

Table **5-8**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the First Subject in **Example 8**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Multiple Representation	Word Problem, Ordered Pairs, and Graphs		
Discrimination: Graphical Representations of the Four Choices	Graphs		
Identification: Graphical Representations	Graphs		

A second individual may reverse these observations and begin by recognizing that the price (\$2.50) remains constant for the first $\frac{1}{4}$ mile or fraction thereof and thus recognizes that choices (B) and (D) may be eliminated. Once this individual recognizes that the cost of the trip at 0 miles is *not* \$2.50, s/he also eliminates choice (C) and the best answer (A) is again apparent.

This subject exemplifies first the identification indicator by recognizing that the price ($2.50) must remains constant for the first $\frac{1}{4}$ mile or fraction thereof, connecting with the expanded Function Block: *x*-axis – Settings/Representations: Word Problem, Graphs; enabling this subject to eliminate choices (B) and (D). Then the subject exemplifies the discrimination indicator by comparing and contrasting the graphical representations of choices (A) and (C), again connecting with the expanded Function Block: *x*-axis – Settings/Representations: Graphs and finally the subject exemplifies the identification indicator by recognizing that the graph of choice (A) does *not* include the point (0, $2.50), connecting with the expanded Function Block: *x*-axis – Settings/Representations: Ordered Pairs, Graphs (see Table **5-9**).

Table **5-9**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the Second Subject in **Example 8**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Multiple Representation	Word Problems and Graphs		
Discrimination: Graphical Representations of the Four Choices	Graphs		
Identification: Graphical Representations	Ordered Pairs and Graphs		

Both of these subjects solve this problem by identifying the setting/representation (graph) (*x*-axis) and by discriminating within this setting/representation (*x*-axis).

Example 9. Consider the mathematical understanding of a subject who attempts to solve the following problem:

(A) Determine an equation of the function whose graph is formed by moving each point on the graph of the function $y = x^2$ up 4 units. Show how you arrived at your answer.

(B) Determine an equation of the function whose graph is formed by moving each point on the graph of the function $y = x^2$ to the right 3 units. Show how you arrived at your answer.

(C) Determine an equation of the function whose graph is formed by moving each point on the graph of the function $y = x^2$ to a point with the same *y*-coordinate and twice the *x*-coordinate. Show how you arrived at your answer.

(Educational Testing Service (ETS), 2003, p. 90, Item 2.)

In order to respond to this item, the subject must have the ability first to identify this problem as one that involves transformations. The subject must possess the ability to discriminate among vertical and horizontal translations, stretches and shrinks, and possess the ability either to process the item graphically by visualizing what the original graph $y = x^2$ would look like and what adjustments need to be made to the symbolic notation to make the appropriate transformation. A more abstract thinker could reason through the problem by merely recognize horizontal and vertical translations, stretches and shrinks symbolically.

This subject exemplifies first the identification indicator by recognizing that the problem involves transformations, thus connecting with the expanded Function Block: *y*-axis – Subconcepts: Transformations and with the x-axis – Settings/Representations: Symbols and Graphs. Next, the subject exemplifies the discrimination indicator by comparing and contrasting the vertical and horizontal translations and stretches and shrinks, connecting with the expanded Function Block: *y*-axis – Subconcepts: Translations and the *x*-axis – Settings/Representations: Symbols and Graphs (see Table **5-10**).

Table **5-10**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the Second Subject in **Example 9**.

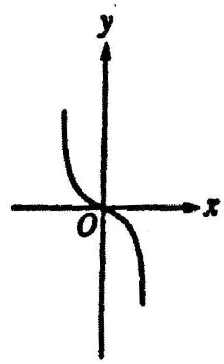
	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Recognizing Transformations	Symbols and Graphs	Transformations	
Discrimination: Comparing an Contrasting Between/Among Transformations	Symbols and Graphs	Horizontal vs. Vertical Translations	
Discrimination: Comparing an Contrasting Between/Among Transformations	Symbols and Graphs	Stretches vs. Shrinks	

Once again, it may be noted that the constructed-response item addresses more axes of the expanded Function Block than any of the multiple-choice items. However, because of the manner in which this question is posed, it only requires the application of the identification and discrimination indicators of mathematical understanding.

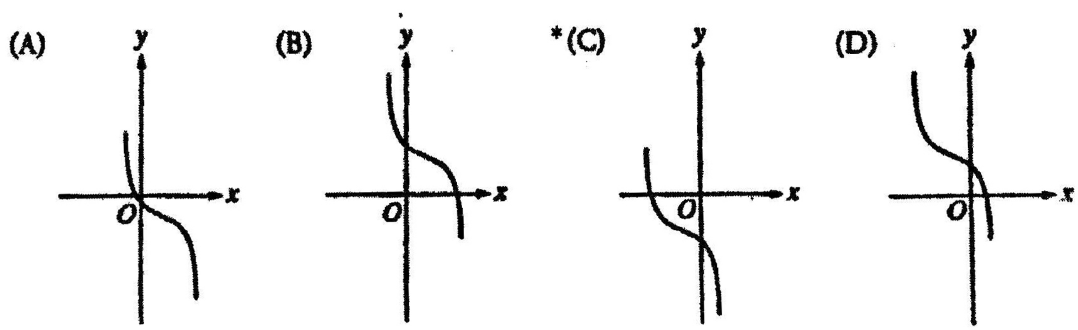
In order for a subject to exemplify the discrimination indicator of mathematical understanding, the author will require explicit evidence of the subject's ability to discriminate between two or more objects by identifying, particularly the differences, in these objects. The subject must exemplify the ability to compare and to contrast objects by identifying not only the similarities, but most importantly, the differences.

Generalization. A third indicator of mathematical understanding is *generalization*. An individual who understands a mathematical object must have the ability to generalize, to make a conjecture, to identify one mathematical object/situation as a particular case of another more general one. "The term 'situation' is used here in a broad sense, from a class of objects (material or mental) to a class of events (phenomena) to problems, theorems or statements and theories" (Sierpinska, 1994, p. 58). For example, when examining five examples of isosceles triangles the individual measures the base angles of each triangle and recognizes that the base angles are congruent. The individual recalls that in isosceles triangles the base angles are congruent. S/he then categorizes the five examples as particular cases of isosceles triangles. Or consider a second individual who makes the observation that if the base angles of each triangle are congruent, the sides opposite these angles are also congruent. In making this conjecture, the individual has generalized, since s/he has identified a mathematical object as a particular case of another more general one. The generalization indicator is most complicated due to the multiplicity of levels of generalizations as well as the interaction among these levels as understanding evolves. Regardless, only an individual who can recognize or classify or identify the mathematical object will have the ability to generalize it. (Sierpinska, 1994, p. 58–60). It must be noted that if an individual *recalls* that one situation is a particular case of another situation, the manner in which that individual processes the information does not exemplify the generalization operator but *does* exemplify the identification operator. In order to exemplify the generalization operator, the subject must categorize or make a conjecture or develop a generalization in which s/he identifies one situation as a particular case of another situation.

Example 10. Consider the mathematical understanding of a subject who is asked to solve the following multiple-choice item.



If the graph of the function $y = f(x)$ is shown above, which of the following could be the graph of $y = f(x+1) - 2$?



(Educational Testing Service (ETS), 2003, p. 66, #4)

The subject reasons that the "– 2" in the equation $y = f(x+1) - 2$ indicates a vertical shift of 2 *lower* than the original graph and immediately eliminates choices (B) and (D) since these choices indicate a *higher* vertical shift. The subject now must determine the translation indicated by the "1" in the equation $y = f(x+1) - 2$. By examining the graphs in this manner, it is obvious that the best choice is (C). Because of the concrete nature of the item, generalizing was not required; the subject discriminated among the responses to determine the best answer.

This subject exemplifies the identification indicator by recognizing the impact of the parameter *d* on the function $y = f(x) + d$ as a vertical translation, connecting to the expanded Function Block *x*-axis – Settings/Representations: Graphs, *y*-axis – Subconcepts: Translations, and *z*-axis – Levels: Family of Functions: Polynomial. The subject then discriminates among the responses connecting with the expanded Function Block *x*-axis – Settings/Representations: Graphs and *y*-axis – Sub-concepts: Translations (see Table **5-11**).

Table **5-11**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for **Example 10**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Vertical Translations	Graphs	Translations	Families of Functions: Polynomial
Discrimination: Vertical Translations of the Four Choices	Graphs	Translations	

Example 11. Consider the mathematical understanding of three subjects who are asked to solve the following problem:

Which of the following is an equation of a line that is perpendicular to the line $y = 2x + 13$?

(A) $y = -2x + 6$

* (B) $y = -\frac{1}{2}x + 3$

(C) $y = \frac{1}{2}x + 3$

(D) $y = 2x + 6$

(Educational Testing Service (ETS), 2004a, p. 120, #23.)

The first subject responds by attempting to determine how the slopes of perpendicular lines are related. S/he recalls that parallel lines have the same slope, but does not recall the relationship between the slopes of perpendicular lines. Using graph paper and a protractor, s/he creates lines that are perpendicular and then determines the slopes of all lines. After numerous examples, s/he makes the conjecture that the slopes of perpendicular lines are negative reciprocals of each other. This subject has identified one mathematical object/situation as a particular case of another more general one and thus, the first subject will have generalized.

This first subject exemplifies the identification indicator by recognizing and classifying the concept of slope and perpendicular lines pertaining to the *y*-axis of the expanded Function Block: Sub-concepts. The subject exemplifies the generalization indicator by identifying multiple examples of graphs and relating the slopes of each pair through his/her conjecture. The subject categorizes those examples whose graphs have slopes that are negative reciprocals of each other as examples of perpendicular lines. This action connects with the *x*-axis of the expanded Function Block: the *x*-axis – Settings/Representations: Symbols, Graphs (see Table **5-12**).

Table **5-12**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the First Subject in **Example 11**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Slope Perpendicular Lines		Slope Perpendicular Lines	
Generalization: Slopes of Perpendicular Lines	Graphs and Symbols	Slopes and Perpendicular Lines	

The second subject also forgets the relationship between the slopes of two lines that are perpendicular. In order to determine the best answer, this subject graphs the original equation and all four options using a graphing calculator. Observing the line that *appears* to be perpendicular to the original line, this subject chooses an answer. The second subject has discriminated, possibly incorrectly, in determining the response to this item.

This second subject exemplifies the identification indicator by recognizing and classifying the concept of slope and perpendicular lines pertaining to the *y*-axis of the expanded Function Block: Sub-concepts. The subject exemplifies the discrimination indicator by comparing and contrasting multiple examples of graphs and relating the slopes of each pair. This action connects with the *x*-axis of the expanded Function Block: the *x*-axis – Settings/Representations: Symbols and Graphs. This subject has *not* made a generalization (see Table **5-13**).

Table **5-13**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the Second Subject in **Example 11**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Slope Perpendicular Lines		Slope Perpendicular Lines	
Discrimination: Graphs of Various Perpendicular Lines	Symbols and Graphs		

The third subject recalls the fact that the slopes of perpendicular lines are negative reciprocals of each other. If the slope of the original line is 2, then the slope of the line perpendicular to the original line must be $-\frac{1}{2}$. This subject has identified a mathematical entity that has been *objectified* (Sfard, 1990). While such a response involves a different mental operator (identification rather than generalization), it actually demonstrates a higher degree of conceptual development since this subject has made the transition from thinking about the slopes of perpendicular lines as processes, that is the relationship between two particular slopes of two perpendicular lines, to thinking about the slopes of perpendicular more abstractly; that is, the relationship between *all* slopes of *any* two perpendicular lines, as objects.

The third subject has *identified* a generalization; this is an example of the identification indicator since the generalization was recalled rather than developed. This connects with the *y*-axis of the expanded Function Block: Sub-concepts (see Table **5-14**).

Table **5-14**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the Third Subject in **Example 11**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Slopes of Perpendicular Lines		Slopes Perpendicular Lines	

It is important to note that it is not the mathematical object that determines the basic mental operator used; it is the manner in which the subject *thinks about* or *processes* the mathematical object that determines the basic mental operator used.

There were no ETS constructed-response items that elicited the generalization indicator of mathematical understanding. However, the author believes that the following researcher-developed constructed-response item might possibly be solved by subjects using generalization.

Example 12.

(A) Determine a functional relationship that gives the maximum number of regions possible in a circle as a function of the number of radii.

(B) Determine a functional relationship that gives the maximum number of regions possible in a circle as a function of the number of diameters.

(C) Determine a functional relationship that gives the maximum number of regions possible in a circle as a function of the number of chords.

(Adapted from Sobel and Maletsky, 1999, p. 105)

Before attempting to solve this problem, the author believes that the subject would have to recognize and/or to identify regions, radii, diameters, and chords. If the subject is unable to identify, s/he will not be able to discriminate, generalize, or synthesize. Next, the she believes that subject is likely to collect data to help him/her determine a functional relationship. Finally, she concludes that in order for the subject to determine a pattern or functional relationship between the maximum number of regions possible and the numbers of radii, diameters, and chords, it is likely that the subject will generalize.

The author makes the following conjecture: The subject must first identify the number of regions, radii, diameters, and chords. The subject might possibly search for the functional relationship by creating a table for each part of the problem in order to connect the number of regions with the number of radii, diameters, and chords thus connecting with the expanded Function Block: *x*-axis: Settings/Representations: Tables. Finally, the author believes that the subject will recognize the pattern and make an appropriate generalization or functional relationships thus connecting with the expanded Function Block: *z*-axis: Levels: Function Families (see Table **5-15**).

Table **5-15**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the Third Subject in **Example 12**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Number of Regions and Number of Radii or Diameters of Chords	Tables		
Generalization: Determining the Functional Relationship	Tables		Function Families

In order for a subject to exemplify the generalization indicator of mathematical understanding, the author will require explicit evidence of the subject's ability to categorize or to make a conjecture or to develop a generalization, namely, to identify one mathematical object/situation as a particular case of another more general one. The recall of a generalization, as described by the third subject above, will be indicative of the exemplification of the identification indicator of mathematical understanding.

Synthesis. The fourth and final indicator of mathematical understanding is *synthesis*. An individual who understands a mathematical object must have the ability to *synthesize*, to seek and to find a "common link, a unifying principle, a similarity between/among several generalizations" (Sierpinska, 1994, p. 60) and identify the relationship between the new mathematical object and ones previously understood. Synthesis is a "culmination of identifications, discriminations, and generalizations" (Sierpinska, 1994, p. 154). There are also degrees of the synthesis indicator varying from a very "local" degree to a "global" degree. An example of "local" synthesis in mathematical understanding would pertain to the recognition of a common link between specific generalizations (e.g., recognizing the need to use a specific relationship or link to prove a specific theorem) while an example of "global" synthesis in mathematical understanding would pertain to an individual's grasping "vast domains of mathematical knowledge" such as the unifications in mathematics during the nineteenth and twentieth centuries based on the fundamental organizing idea of the function concept (Sierpinska, 1994, p. 60). Because studying "global" synthesis lies beyond the scope of the study for which this framework was developed, this research will focus only on "local" synthesis.

For the purposes of this research, the author will also define a very elementary degree of synthesis that she will call "basic" synthesis. This type of synthesis will be required for an individual to process any mathematical idea and will be different from "local" synthesis only because this type of synthesis will be automatic and will require no external recognition of a common link between specific generalizations. This type of synthesis occurs when an individual processes a mathematical idea automatically, for example, when s/he simplifies an algebraic expression or solves an algebraic equation. Only when the individual provides explicit evidence of the recognition of a common link between specific generalizations will the author claim evidence of "local" synthesis.

Example 13. Consider the mathematical understanding of two subjects who are asked to solve the following multiple-choice item:

The function $y = f(x) \bullet g(x)$, where $f(x) = x$ and $g(x) = \begin{cases} -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ 1 \text{ if } x > 0 \end{cases}$,

is equivalent to which of the following functions?

(A) $y = f(x)$

(B) $y = f(-x)$

(C) $y = -|f(x)|$

* (D) $y = f(|x|)$

(Educational Testing Service (ETS), 2003, p. 77, #50.)

Both subjects respond to this item by first recognizing or identifying *g* as a piecewise-defined function and then by defining a new function *y* as the product of the linear function $f(x)$ and the piecewise-defined function $g(x)$ in symbolic form then determine the new piecewise-defined function

$$y\text{: } y = f(x) \bullet g(x) = \begin{cases} -x \text{ if } x < 0 \\ 0 \text{ if } x = 0 \\ x \text{ if } x > 0 \end{cases}.$$

In the process of defining the new function y, both subjects employ "basic" synthesis:

If $x < 0$, then $x \bullet (-1) = -x$;
if $x = 0$, then $x \bullet 0 = 0$; and
if $x > 0$, then $x \bullet 1 = x$.

The first subject recognizes that if $x < 0$, $-x = |x|$ and thus links the following two generalizations:

1. the absolute value of a number is equal to its distance from 0 and hence is always positive; and
2. the opposite of a negative number is equivalent to the absolute value of that negative number.

(It must be understood that these are *generalizations* according to Sierpinska and thus recognized as such by this author.) The first subject realizes by reflecting on the graphical representations of each option that the only choice that would be equivalent to the newly defined function *y* would be choice D) $y = f(|x|)$. This subject exemplifies "local" synthesis in the realization and the sharing of the fact that if $x < 0,\ -x = |x|$.

This subject's methods of thought demonstrated the following mental operations: identification of linear and piecewise-defined functions connecting with the *z*-axis of the expanded Function Block: Levels: Families of Functions: Linear, Piecewise-defined and with the *x*-axis of the expanded Function Block: Settings/Representations: Symbols, Graphs; discrimination by comparing and contrasting graphical representations of the newly defined function *y* and the four options connecting with the *x*-axis of the expanded function block: Settings/Representations: Graphs; the ability to synthesize basically by automatically calculating the intervals of values of the newly defined piecewise function *y* connecting with the *x*-axis of the expanded function block: Settings/Representations: Symbols; and the ability to synthesis locally by realizing and sharing that if $x < 0,\ -x = |x|$ connecting with the *x*-axis of the expanded function block: Settings/Representations: Symbols (see Table **5-16**).

Table **5-16**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the First Subject in **Example 13**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Linear Functions Piecewise-defined Functions	Symbols and Graphs		Families of Functions: Linear and Piecewise-defined Functions
Discrimination: Newly-defined Function *y* and Four Choices	Graphs		
Synthesis: If $x < 0$, then $-x = \lvert x \rvert$.	Symbols		

The second subject does *not* recognize that if $x < 0$, $-x = |x|$ and concludes after reflecting on the graphical representation of each option that the best answer to this question is choice A) $y = f(x)$.

This subject's methods of thought demonstrated the following mental operations: identification of linear and piecewise-defined functions connecting with the *z*-axis of the expanded Function Block: Levels: Families of Functions: Linear, Piecewise-defined and with the *x*-axis of the expanded Function Block: Settings/Representations: Symbols, Graphs and discrimination by comparing and contrasting graphical representations of the newly-defined function *y* and the four options connecting with the *x*-axis of the expanded function block: Settings/Representations: Graphs (see Table **5-17**).

Table **5-17**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for the Second Subject in **Example 13**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Linear Functions Piecewise-defined Functions	Symbols and Graphs		Families of Functions: Linear and Piecewise-defined Functions
Discrimination: Newly-defined Function *y* and Four Choices	Graphs		

In answering this item in this fashion, the first subject displayed both basic and local synthesis while the second subject merely displayed basic synthesis. In other words, both subjects demonstrated basic synthesis in defining a new function *y* as the product of the linear function $f(x)$ and the piecewise-defined function $g(x)$ in symbolic form while only the first subject demonstrated local synthesis in recognizing that in the newly-defined function *y*, if $x < 0$, $-x = |x|$ and concluding $y = f(|x|)$.

Example 14. Consider the mathematical understanding of a subject who attempts to solve the following problem:

At a certain theme park, a customer can buy an admission pass for \$5.00 and then pay an additional \$0.50 for each ride, or the customer can pay \$3.00 for a pass and an additional \$0.75 for each ride.

(A) Write an equation that represents the cost *C* of admission and rides at the park for each alternative. Define any variables you use.

(B) In the *xy*-coordinate system, graph the cost on the *y*-axis versus the number of rides on the *x*-axis for each equation you wrote in part (A).

(C) Write a sentence or two summarizing what you can observe from the graphs about the relative merits of each alternative.

(Educational Testing Service (ETS), 2004a, p. 125, Item 32.)

In order to answer this item, the subject must identify the symbolic notation in part (A) to express the relation between the cost *C* of admission and the rides at the park as a linear function. In part (B), the subject must identify and create the graphs of the two linear functions identified in part (A). Without first identifying the symbolic notation of the linear functions and then identifying the graphic relationship between the two linear functions, the subject will be unable to discuss the relative merits of each alternative in part (C).

This exemplification of the identification indicator in recognizing a linear function pertains to the *x*-axis of the expanded Function Block: Settings/Representations: Word Problems, Symbols, Graphs, and Ordered Pairs and the *y*-axis of the expanded Function Block: Sub-concepts: Slope, *y*-intercept; and the *z*-axis of the expanded Function Block: Levels: Linear Function (see Table **5-18**).

Table **5-18**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for **Example 14**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Recognizing a Linear Functions	Graphs, Symbols, Ordered Pairs	Slope and *y*-intercept	Linear Functions
Synthesis: Interpretation of the Intersection of Two Linear Functions	Graphs and Verbal Explanations		Linear Functions

Example 15. Consider the mathematical understanding of a subject who is asked to respond to the following item:

A function *g* defined for all real numbers is called an even function if $g(-x) = g(x)$ for all real *x*.
A function *h* defined for all real numbers is called an odd function if $h(-x) = -h(x)$ for all real *x*.

(A) If *f* is any function defined for all real numbers, prove that $f(x) + f(-x)$ is an even function.

(B) If *f* is any function defined for all real numbers, prove that $f(x) - f(-x)$ is an odd function.

(C) Prove that any function *f* defined for all real numbers can be written as the sum of an even function and an odd function.

(Educational Testing Service (ETS), 2004d.)

In order to answer this item, the subject must be able to identify even and odd functions in a general manner; giving examples of functions that are even functions or odd functions is not sufficient. The subject must be able to think about the mathematical idea of even and odd functions in an abstract manner, that is, a manner distinct from any specific example but encompassing all possible examples, in order to be able to identify one mathematical object/situation, $f(x)+f(-x)$, as a particular case of another more general one, $F(x)$.

For example, consider the following subject's response to (A):

Suppose $F(x)=f(x)+f(-x)$ for all real numbers.

Then $F(-x)=f(-x)+f(-(-x))=f(-x)+f(x)=F(x)$.

$\Rightarrow F(x)$ is an even function $\Rightarrow$ $f(x)+f(-x)$ is an even function.

In thinking about this proof in this manner, this subject has linked the mathematical object, $F(x)$, with her understanding of the generalizations of even and odd functions as explained in the problem above and has demonstrated local synthesis. The subject must be able to think abstractly about the relationships between even and odd functions using these generalizations to write the above proofs.

This subject exemplifies the identification indicator by first recognizing and classifying even and odd functions, connecting with the *y*-axis of the expanded Function Block: Sub-concepts: Even and Odd Functions; and then the synthesis indicator, again connecting with the *y*-axis of the expanded Function Block: Sub-concepts: Even and Odd Functions (see Table **5-19**).

Table **5-19**: Connecting Sierpinska's Indicators of Mathematical Understanding With the Expanded Function Block for **Example 15**.

	Expanded Function Block		
Sierpinska's Indicators of Mathematical Understanding	***x*-axis Settings / Representations**	***y*-axis Sub-Concepts**	***z*-axis Levels**
Identification: Even Functions Odd Functions		Even Functions Odd Functions	
Synthesis: Newly-defined Function $F(x)$ and Even Functions		Even Functions Odd Functions	

Any example of generating a proof, regardless of how simple, which involves the connecting of generalizations would exemplify the synthesis indicator. Most elementary proofs such as the one above would require the recognition of a common link between specific generalizations and thus be examples of "local" synthesis rather than "global" synthesis.

In order for a subject to exemplify the synthesis indicator of mathematical understanding, the author will require explicit evidence of the subject's ability to connect generalizations. The subject's ability to connect mathematical objects that are *not* generalizations will be considered an exemplification of his/her ability to make connections or of "basic" synthesis.

All acts of synthesis must be made by the individual himself/herself; acts of synthesis cannot be made *by* the teacher *for* the individual (Sierpinska, 1994, p. 61). Thus, the individual must be comfortable with and confident in his/her ability to think abstractly, or think in a manner distinct from any specific examples but encompassing all possible examples, if s/he is to be successful in synthesizing.

Sierpinska does not include abstraction as an indicator of mathematical understanding since she believes "that abstraction does not constitute an act of understanding in itself" (Sierpinska, 1994, p. 61). Rather, "abstraction is involved in each of the four operations and each of them is somehow involved in abstraction" (Sierpinska, 1994, p. 61–62).

In this section, the author has provided a description of the framework including both the conceptual and theoretical models and the relationship between the conceptual and procedural models through examples from ETS PRAXIS SERIES™ multiple-choice and constructed-response items. The examples support the findings of her preliminary studies. They suggest that ETS multiple-choice items do not encourage subjects to demonstrate their ability to generalize and/or to synthesize because of the nature of these items, they may be more easily reduced to be solved by identification, discrimination, elimination, substitution, or mere guessing. Furthermore, these examples also suggest that the manner in which a constructed-response item is written will definitely impact the manner in which such an item will be solved. In other words, constructed-response items may be written in such a way that subjects will be inclined to generalize and/or to synthesize in order to solve the problem *or* they may be written so that subjects need only to identify or to discriminate to solve the problem. The manner in which an item is composed in conjunction with the manner in which the subject processes the item impacts the indicators of mathematical understanding evident when the problem is solved.

The following section will provide some recommendations for analyzing both multiple-choice and constructed-response items.

Analyzing Assessment Items

In this study, the author will examine two types of assessment items: multiple choice and constructed response since these are the types of items used by ETS on the *PRAXIS*^{TM} Examinations.

Multiple-Choice Items

A multiple-choice item consists of one or more introductory sentences followed by a list of two or more suggested responses from which the subject chooses one as the correct or the best answer. The introductory part of the multiple-choice item is called the *stem.* The purposes of a multiple-choice item are "to ask a item, set the task to be performed, or state the problem to be solved" (Nitko, 1983, p. 190). The suggested responses are called alternatives, responses, or options. Only one of the responses should be the correct or best answer; the remaining incorrect options are called "distractors" or foils. The purpose of the distractors is to appear as plausible answers or solutions to the problem for those subjects who do not possess sufficient knowledge. However, they should not appear to be plausible to those who in fact have the desired degree of knowledge (Nitko, 1983, p. 190). The purpose of all multiple-choice test items is to identify or distinguish those subjects who have attained the particular level of knowledge (skill, ability, or performance) identified as sufficient or necessary (Nitko, 1983, p. 191).

While there was much discussion of multiple-choice items and tests in the literature the author reviewed, less than 5% of the examples given dealt with mathematics. There were no references specifically addressing the assessment of mathematics via multiple-choice items. However, an article written by Romberg (1992) did impact the author's beliefs about assessment. The author of this paper has largely built and developed her beliefs about assessment based upon Romberg's discussion of the construction of mathematics items of the Second International Mathematics Study (SIMS) and the relationship of these items with Bloom's (1956) *Taxonomy of Educational Objectives.* This discussion enabled the author to confirm her hypothesis regarding how multiple-choice items could assess subjects' understanding of mathematics as evidenced by Sierpinska's (1994) Indicators of Mathematical Understanding.

Uses. The uses of multiple-choice test items are based upon Bloom's (1956) *Taxonomy of Educational Objectives*: knowledge, comprehension, application, analysis, synthesis, and evaluation. Multiple-choice test items are frequently used to assess a subject's knowledge of terminology, specific facts, principles, computation, or methods and procedures; to assess a subject's comprehension of concepts, principles, generalizations, as well as his/her ability to identify the application of facts and principles, to discriminate and make correct choices, to interpret cause-and-effect relationships, to interpret new data or information, to make inferences and to reason, and to justify methods and procedures (Gronlund, 1985, pp. 171–176; Nitko, 1983, p. 193). Multiple-choice test items do not appear to be well suited for assessing higher levels of Bloom's taxonomy since the responses reduce the subject's necessity to think at a higher level. In the writing of the mathematics items for the SIMS (Weinzwieg and Wilson, 1977), the behavior dimension referred to four levels of cognitive capability expected of students: computation, comprehension, application, and analysis. Romberg (1992) states that this classification adapted Bloom's taxonomy by replacing *knowledge* with *computation* and eliminating the higher levels of synthesis and evaluation (p. 42). However, Romberg acknowledges that these adaptations cause problems. While computation involves knowledge of and ability to carry out a routine algorithm or procedure, knowledge of basic concepts does not fit well as either computation or comprehension. In addition, Romberg (1992) claims that the elimination of the two higher levels of Bloom's taxonomy is equivalent to admitting that important aspects of problem solving and developing a logical argument about a conjecture cannot be assessed (p. 42). Sax (1989) claims that multiple-choice items cannot be used at the synthesis level of Bloom's taxonomy because this type of item cannot assess the subject's ability to write, say, or construct something of his/her own (p. 109). In summary, the use of Bloom's taxonomy has proven useful for assessing low-level behaviors (knowledge, comprehension, and application) but difficult for higher levels (analysis, synthesis, and evaluation). “Single-answer, multiple-choice items are not reasonable at those levels” (Romberg, 1992, p. 46).

Nitko (1983) also acknowledges that multiple-choice tests do not assess an individual’s ability to recollect (as opposed to recognize) information, articulate explanations and give examples, produce and express unique or original ideas, organize personal thoughts, or display thought processes or patterns of reasoning (p. 193). Multiple-choice tests may give students the impression that there is a single, correct answer to every problem (Nitko, 1983, p. 193).

Based upon this information as well as upon preliminary data that the author has already collected, she believes and makes the claim that multiple-choice test items in general cannot assess Sierpinska’s generalization and synthesis indicators of mathematical understanding. According to Sierpinska, the generalization indicator of mathematical understanding implies that the subject has the ability to generalize, to identify one mathematical object/situation as a particular case of another more general one. When given options as in the multiple-choice test format, the requirement to make a generalization is reduced to one of discrimination among the options. Hence, the subject discriminates rather than generalizes. Furthermore, according to Sierpinska, the synthesis indicator of mathematical understanding implies that the subject has the ability to seek and find a “common link, a unifying principle, a similarity between/among several generalizations” (Sierpinska, 1994, p. 60) and identify the relationship between the new mathematical object and ones previously understood. To exemplify this indicator, the subject must display his/her thought processes and/or patterns of reasoning as in writing a proof or defending a conjecture. This process cannot be assessed by multiple-choice test items in general (Nitko, 1985; Romberg, 1992).

Advantages. Multiple-choice tests effectively measure various types of knowledge (Gronlund, 1985, p. 177). Using a multiple-choice test format, a substantial amount of material can be assessed in a relatively short time (Nitko, 1983, p. 211; Sax, 1989, p. 102). The scoring of multiple-choice tests is objective and hence can be graded easily (Sax, 1989, p. 102). Multiple-choice test items can be constructed in a manner that requires subjects to discriminate among options that vary in degree of correctness (Sax, 1989, p. 102). Multiple-choice test items are particularly amenable to item analysis to detect areas of weakness, evidence of ambiguity, item difficulty, and the extent to which the item can measure individual differences (Sax, 1989, p. 102).

In addition, multiple-choice tests are free from some of the common shortcomings characteristic of other test types. For example, in comparison with True–False tests, a subject cannot get credit merely by knowing that a statement is incorrect; s/he must also know what is correct; there is greater reliability per item; it is not necessary to obtain statements that are True–False without qualification (Gronlund, 1985, p. 178). Unlike Matching Tests, with multiple-choice tests there is no need for homogeneous material and the results are easier to analyze (Gronlund, 1985, p. 178). In comparison with Fill-in-the-Blank Tests, multiple-choice test items have a limited number of options reducing the effects of guessing (Sax, 1989, p. 102). Finally, multiple-choice test items are free from response sets; that is, subjects generally do not favor a particular option when s/he does not know the correct answer (Gronlund, 1985, p. 178).

Disadvantages/Limitations. A limitation of the multiple-choice test format includes difficulty in finding a sufficient number of incorrect but plausible distractors (Gronlund, 1985, p. 179). However, the most powerful limitation and disadvantage of the multiple-choice test in the author's opinion is its inability to assess higher order thinking skills in mathematics, particularly the subject's ability to solve problems, to make generalizations, to write proofs, to defend conjectures, to communicate his/her ability to analyze, and to synthesize mathematical ideas.

Constructed-Response Items

A constructed-response item consists of the statement of some mathematical problem based upon an important mathematical idea in which the test taker must construct his/her response. The success or failure of the test taker is based upon the assessor's evaluation of the mathematical logic, validity, and completeness of the test-taker's response to each item. In this paper, the author uses the term "constructed-response item" as suggested by ETS; a constructed-response item frequently has multiple items that must be answered by the application of several mathematical processes combined with some explanation or analysis of the work. The purpose of the constructed-response test is to identify those subjects who truly understand the important mathematical ideas and can demonstrate their understanding by the manner in which they answer the items posed and explain the reasoning they used.

While the literature mentions the use of free-response items, none of the literature (other than the PRAXIS™ Examination Study Guides) refers to constructed-response tests. Therefore, the information contained in the following discussion is largely based upon the author's interpretation of the ETS constructed-response tests.

Uses. The uses of constructed-response test items may also be related to Bloom's (1956) *Taxonomy of Educational Objectives*. While most of the constructed-response test items published by ETS address the lower levels of Bloom's taxonomy, the constructed-response items have the potential, in this author's opinion, of addressing the higher levels as well. In this manner, the constructed-response test items would be better suited to assess a subject's understanding of mathematical ideas particularly through the application of Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. The author believes that constructed-response items may be written to assess a subject's ability to make generalizations and to synthesize as well as to identify and to discriminate.

Advantages. Constructed-response tests have the potential to assess the depth of understanding a subject has on a limited number of mathematics topics and to assess the ability of that subject to explain his/her reasoning concerning the mathematics topics addressed by the problem(s). While this type of assessment instrument lacks the breadth of a multiple-choice test, it does possess a depth not attainable in the latter format.

However, the advantage of constructed-response test that most interests this author is its potential to assess a subject's ability to make generalizations and to synthesize; that is, to assess a subject's ability to think at a higher mathematical level.

Disadvantages/Limitations. Constructed-response tests may be written in such a manner that their potential to assess a subject's ability to make generalizations and to synthesize are not utilized; in other words, constructed-response tests may be written merely to address a subject's ability to compute, comprehend, and or apply mathematical knowledge. It is challenging to write/construct a constructed-response item that has the potential to assess higher order thinking. Constructed-response tests are by their nature very focused and cannot assess a large amount of material. Constructed-response items encourage each subject to express his/her understanding and are both challenging and time-consuming to grade because of the unique responses and the depth of thinking encouraged.

Explanation of How to Write Multiple-Choice Items

Theoretically, multiple-choice items may be written to assess a subject's ability to identify, discriminate, generalize, or synthesize. However, the manner in which a subject *processes* these items will indicate the understanding the subject possesses. Due to the options presented in any multiple-choice item, all higher level items may more easily be processed as lower-level items merely by studying the responses. For this reason, most multiple-choice items merely assess a subject's ability to identify or to discriminate. There is no guarantee that a multiple-choice item *written* to assess either generalization or synthesis will force or even encourage the typical subject to process that item using those indicators. It is this author's belief that should such multiple-choice items be written the majority of students would most likely obtain the best answer by reducing these items to the identification or discrimination type rather than to attempt to solve them using generalization or synthesis.

Suggestions for Writing Multiple Choice Items

Because the most highly regarded and widely used form of objective test at present is the multiple-choice test, there are numerous suggestions from various sources regarding how best to construct this type of test. The author has grouped the suggestions into those relating to the multiple-choice test in general, those relating to the stem, those that relating to the responses, and those relating to Sierpinska's (1994) Indicators of Mathematical Understanding.

General

1. Multiple-choice test items should be based on sound, significant ideas that can be expressed as independently meaningful propositions (Ebel, 1979, p. 161).
2. Test constructors can make some multiple-choice items easier by making the stem more general and the responses more diverse; they can make items harder by making the stem more specific and the responses more similar (Ebel, 1979, p. 162; Kubiszyn & Borich, 2000, p. 100).

3. The use of vocabulary should be suited to the maturity of the students (Sax, 1989, p. 106).

Stem

4. The stem of a multiple-choice item should state or clearly imply a specific direct item or problem (Ebel, 1979, p. 161; Gronlund, 1985, p. 180; Kubiszyn & Borich, 2000, p. 100).
5. The stem of a multiple-choice item should be meaningful by itself and should present a definite problem (Gronlund, 1985, p. 180).
6. The stem of a multiple-choice item should be clear and grammatically correct and should contain elements common to each option (Sax, 1989, p, 108).
7. The stem of a multiple-choice item should include as much of the item as possible and should be free of irrelevant material (Gronlund, 1985, p. 181; Kubiszyn & Borich, 2000, p. 100).
8. The stem of a multiple-choice item should pose the essence of its item as simply and as accurately as possible (Ebel, 1979, p. 161).
9. The stem of a multiple-choice item should be expressed as concisely as possible without sacrificing clarity or omitting essential qualifications (Ebel, 1979, p. 162).
10. Use negative items or statements only if the knowledge being tested requires it. In most cases, it is more important for the student to know what a specific item of information *is* rather than what it is not (Gronlund, 1985, p. 181; Kubiszyn & Borich, 2000, p. 100).

Responses

11. All of the alternatives should be grammatically consistent with the stem of the item (Gronlund, 1985, p. 182).
12. All of the responses to a multiple-choice test item should be parallel in point of view, grammatical structure, and general appearance (Ebel, 1979, p. 162).
13. Brevity is desirable in multiple-choice item responses, but it should not be achieved at the expense of importance and significance in the item asked (Ebel, 1979, p. 162; Kubiszyn & Borich, 2000, p. 100).

14. The responses to a multiple-choice item should be expressed simply enough to make clear the essential differences among them (Ebel, 1979, p. 162).
15. An item should contain only one correct or clearly best answer (Gronlund, 1985, p. 184; Kubiszyn & Borich, 2000, p. 100; Sax, 1989, p. 107).
16. Avoid overlapping alternatives (Sax, 1989, p. 107).
17. Avoid specific determiners (Sax, 1989, p. 105).
18. Items used to measure understanding should contain some novelty, but beware of too much (Gronlund, 1985, p. 185).
19. The distractors in a multiple-choice item should be definitely less correct than the answer, but plausibly attractive to the uninformed (Ebel, 1979, p. 162; Gronlund, 1985, p. 185; Kubiszyn & Borich, 2000, p. 100; Sax, 1989, p. 106).
20. Verbal associations between the stem and the correct answer should be avoided (Gronlund, 1985, p. 187).
21. Eliminate unintentional grammatical clues (Kubiszyn & Borich, 2000, p. 100).
22. The relative length of the alternatives should not provide a clue to the answer; keep the length and form of all the answer choices equal (Gronlund, 1985, p. 188; Kubiszyn & Borich, 2000, p. 100).
23. The intended answer should be clear, concise, correct, and free of clues (Ebel, 1979, p. 162).
24. Rotate the position of the correct answer from item to item randomly so that the correct answer appears in each of the alternative positions an approximately equal number of times (Gronlund, 1985, p. 188; Kubiszyn & Borich, 2000, p. 100; Sax, 1989, p. 107).
25. Use the option "none of the above" sparingly and only when the keyed answer can be classified unequivocally as right or wrong. Do not use this option when asking for a best answer (Gronlund, 1985, p. 189; Kubiszyn & Borich, 2000, p. 100; Sax, 1989, p. 108).
26. Avoid using "all of the above" as an option (Sax, 1989, p. 108).
27. The responses to a multiple-choice item should be listed rather than written one after another in a compact paragraph.

28. Include from three to five options (two to four distractors plus one correct / best answer) to optimize testing for knowledge rather than encouraging guessing. It is not necessary to provide additional distractors for an item simply to maintain the same number of distractors for each item. This usually leads to poorly constructed distractors that add nothing to test validity and reliability (Kubiszyn & Borich, 2000, p. 100).

Indicators of Mathematical Understanding

29. *Identification* – The item must require the subject to identify, recognize, or classify a mathematical object.
30. *Discrimination* – The item must require the subject to identify, recognize, or classify two or more mathematical objects. The item must require that the subject compare and contrast these mathematical objects and particularly they must be able to identify the differences to arrive successfully at an answer.
31. To increase the degree of abstraction, require subjects to discriminate between/among mathematical objects in the stem since this is normally more indicative of the subject's depth of understanding than discrimination between/among mathematical objects in the responses.
32. To increase the degree of abstraction, require subjects to discriminate between/among mathematical objects that are expressed more in general terms than in specific quantities/numbers since this is normally more indicative of the subject's depth of understanding than discrimination between/among mathematical objects in the responses.

Since the author contends that no multiple-choice item can be created to force or even to encourage the subject to generalize or to synthesize, these two indicators are not discussed here.

Explanation of How to Write Constructed-Response Items

Constructed-response items may be written to assess a subject's ability to identify, discriminate, generalize, or synthesize. However, it is challenging to compose a constructed-response item with the potential to assess a subject's ability to make generalizations or to synthesize. Regardless of way the item is composed, it is the manner in which a subject *processes* these items that will indicate the understanding the subject possesses. Constructed-response items have a greater potential to assess a subject's understanding of higher level mathematical thinking since they can be written in such a way that the subject must make generalization(s) or synthesize to answer the item correctly.

Suggestions for Writing Constructed-Response Items

The usefulness of constructed-response items is particularly evident in the ability of this type of item to assess higher level mathematical thinking. Included in the following suggestions are some general suggestions for writing constructed-response items to assess higher level mathematical thinking and some specific suggestions for writing items to assess the subject's ability to identify, discriminate, generalize and synthesize.

While there are numerous suggestions from various sources regarding how best to construct multiple-choice items, the author has found no suggestions in the literature regarding how best to write constructed-response items, particularly those that assess a subject's ability to generalize and/or to synthesize. However, based upon the author's experience and observations she has developed some recommendations. The process she used was cyclical; as she reviewed and identified potential items, particularly those items that could be linked with the *z*-axis of the extended Function Block, she considered each item's potential to assess higher order thinking, particularly generalization and synthesis. As the author examined, analyzed, and critiqued items she believes have the potential to assess a subject's ability to generalize and/or to synthesize, she attempted to identify the relevant characteristics of these items. She was also mindful of the potential of non-traditional assessment methods to enable her to understand what the subject understood.

(See recommendation #1.) She adapted many of the researcher-developed constructed response items by applying this technique. From this cyclical process evolved the following list of recommendations. She will begin by identifying those suggestions for writing items to assess higher level thinking since she concurs with Romberg (1992) that higher order thinking skills or objectives cannot be assessed by multiple-choice items. She will then discuss how specific constructed-response items can assess the subject's ability to apply each of Sierpinska's (1994) Indicators of Mathematical Understanding.

General[9]

1. Use non-traditional, non-routine problems to assess mathematical objects with which the subjects are familiar. A non-traditional, non-routine problem will frequently have a solution that is obtained by proceeding in a manner that is "backwards." (For example, giving the subjects a graph, and requesting that they give the symbolic notation for the graph; giving the subjects a problem based on qualitative information rather than on quantitative data and asking that they construct a corresponding graph, etc.)
2. Use traditional, routine problems to assess mathematical objects with which the subjects are unfamiliar.
3. Ask the subjects to construct or to reconstruct a definition of any mathematical object that is essential to their understanding of the material to be assessed. If a subject truly understands this mathematical object, his/her definition should embody all the necessary and sufficient components of the mathematical object s/he is defining. Merely recalling a definition from memory would not be satisfactory.
4. Ask the subjects to create a word problem and to solve the word problem they create in order to determine whether they have internalized the necessary component parts of a mathematical object with which they should be familiar. By creating and solving this word problem the subjects will demonstrate if they are able to contextualize the ideas relevant to the mathematical object.

[9] The author contends that the general recommendations address the structure of the item while the indicator-specific recommendations address the manner in which the subject processes an item.

5. Ask the subjects to create and/or construct graphs, particularly graphs that are mathematical in nature and substance but do not necessarily involve numeration. When subjects create and/or construct mathematical but nonquantified (no numbers involved) graphs, the graphs display the subjects' ability to demonstrate an understanding of the "key components" of a particular graph (e.g., the relevance of the slope of the line, importance of the intercepts, the relevance of concavity)
6. Ask the subjects to interpret graphs.
7. Ask the subjects to answer questions in which a required part of the problem is inferred (i.e., not specifically stated) to determine whether they are aware of the necessity of the non-specified parts of a mathematical object (e.g., find the inverse of a function (restricted domain) or find the roots of a radical equation (extraneous roots)).
8. Require that the subjects provide the reasons for their responses when asked questions for which the answers are bivalued (e.g., true or false, yes or no, agree or disagree). When evaluating the correctness of the response, evaluate how the *reasons* the subjects give substantiate the bivalued *answer*.
9. Ask the subjects to explain their reasoning or how they arrived at the final answer.
10. Ask the subjects questions in which they *must* understand the meaning of the notation in order to respond correctly.
11. Ask the subjects questions in which they may demonstrate the depth and the breadth of their understanding of a given mathematical object (e.g., families of functions).
12. Ask the subjects a related set of questions (or a question with several subquestions) that vary in the level of abstractness, that is, that begins with a concrete or basic question with which they would feel relatively familiar and be somewhat confident of their answer, but that continues with less concrete and more abstract questions for which they need to have a definite understanding of the concept to answer correctly.

Identification

13. Require the subjects to identify, recognize, and/or classify mathematical objects in each question.

Discrimination

14. Ask the subjects to explain the relationship between specific mathematical objects to determine the degree to which s/he understands each mathematical object.
15. Ask the subjects to compare and contrast mathematical objects. The subjects must be able to identify particularly the differences. Although it is not necessary that they explicitly state these differences, they must be required to use these differences in arriving at a correct response.

Generalization

16. Ask the subjects questions that require them to search for patterns.
17. Ask the subjects questions in which they must make conjectures in which they must identify one mathematical object/situation as a particular case of another more general one.
18. Ask the subjects questions in which they must develop generalizations in which they must identify one mathematical object/situation as a particular case of another more general one.
19. Ask the subject questions in which they must categorize mathematical objects (that is, they must make a conjecture concerning a general case that subsumes several given particular cases).

Synthesis

20. Ask the subjects to justify their conjectures.
21. Ask the subjects to prove their generalizations.
22. Ask the subjects questions that have multiple methods of solutions.
23. Ask the subjects questions in which they must write proofs.

24. Ask the subjects to find a “common link, a unifying principle, a similarity between/among several generalizations” (Sierpinska, 1994, p. 60) and identify the relationship between the new mathematical objects and ones previously understood.

In this section, the author has provided some recommendations on how to create items that will assess a subject’s understanding of a concept. In the following section, she will propose specific researcher-developed constructed-response items generated to assess a subject’s understanding of the concept of function, particularly the subject’s ability to generalize and to synthesize.

Generating Constructed-Response Assessment Items

Constructed-response items may be written to assess a subject's ability to identify, discriminate, generalize, or synthesize. However, it is challenging to compose a constructed-response item with the potential to assess a subject's ability to make generalizations or to synthesize. Regardless of the way the item is composed, it is the manner in which a subject *processes* these items that will indicate the understanding the subject possesses. Constructed-response items have a greater potential to assess a subject's understanding of higher level mathematical thinking since they can be written in such a way that the subject is more likely to make generalizations or to synthesize to answer the item correctly.

The author's purpose in writing this framework was to establish suggestions and directives upon which items could be developed to assess a subject's understanding of function. Since ETS multiple-choice and constructed-response items selected for pilot studies generally assessed subjects' ability to identify and discriminate, the author has focused her attention on Sierpinska's (1994) generalization and synthesis indicators of mathematical understanding. In the foregoing discussion, the author argues that constructed-response items would provide better insight than multiple-choice items for assessing generalization and synthesis.

In this section, the author presents each item that she created or adapted.

1) Study each equation, graph, and table below. Identify those equations, graphs and tables that are examples of functions and those that are not. Give a reason for each of your answers.

a. Identify the following equations that represent functions and those that do not. Explain your reasoning.

i. $y = x^2 - 4$

ii. $4x = y^2$

iii. $xy = 8$

iv. $x^2 + y^2 = 25$

v. $p(x) = e^x$

vi. $y = \sec x$

vii. $x(y) = y^3$

viii. $f(x) = \begin{cases} 1 \text{ if } x \text{ is a rational number} \\ \text{-1 otherwise} \end{cases}$

ix. $2x^2y - \sqrt{x}\ln y = 2$

x. $y^2 = x^3$

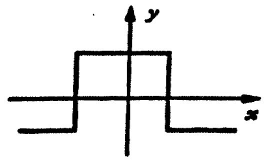
xi. $x = -1$

xii. $h(x) = \begin{cases} -x^2 + 3 \text{ if } x < \text{-2} \\ x^2 - 3 \text{ if } x > 2 \\ |x| \text{ if } -2 \le x \le 2 \end{cases}$

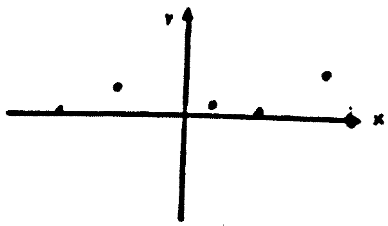
(Adapted from Norman (1992) p. 231)

b. Identify the following graphs that represent functions and those that do not. Explain your reasoning.

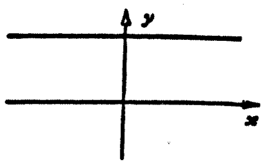
i.



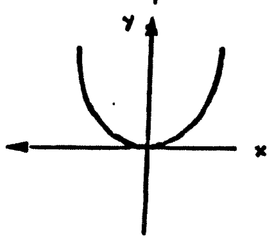
ii.



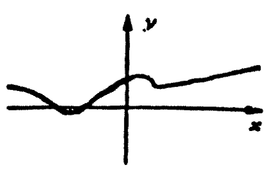
iii.



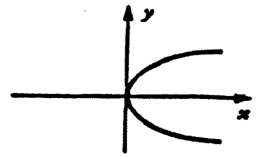
iv.



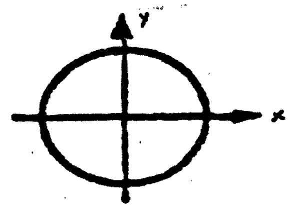
v.



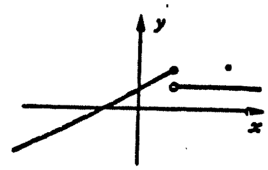
vi.



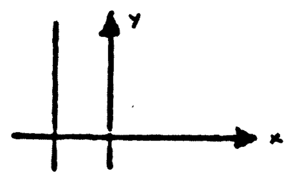
vii.



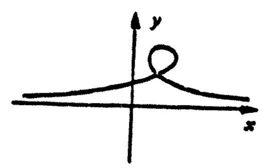
viii.



ix.



x.



c. Identify the following tables that represent functions and those that do not. Explain your reasoning.

i.

x	y
0	0
1	2
2	6
-2	2
3	4
1	3

ii.

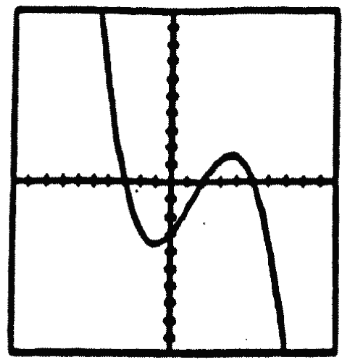
Days	Fruit
Sunday	Oranges
Monday	Bananas
Tuesday	Apples
Wednesday	Cherries
Thursday	Grapes
Friday	Apples
Saturday	Peaches

iii.

Students	Grades
Beth	B+
Joe	A-
Jacob	B+
Darla	A
Adam	B
Chris	A-

d. Define function.

2) Below is a graph of a function.



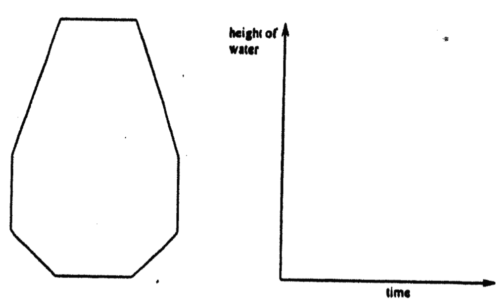
[−10, 10] by [−10, 10]

Give the symbolic function notation for the graph.
(Adapted from Heid, 1995, pp. 28–29)

3) Suppose a vase widens rapidly from its base, then has straight sides, then gradually tapers in until the top is congruent with the base.

a. If water is poured into the vase at a steady rate, what will the graph look like?

b. Explain the graph you constructed. (Adapted from CBMS, 2001, p. 31)



4) Consider the function f defined for all real numbers by $f(x) = x^4 + 5x^3 + 2x^2 - 8x + k$, where k is a real-valued constant.

a. Determine all values of *k* for which the function has exactly 4 real roots. Explain how you arrived at your answer.
b. Determine all values of *k* for which the function has exactly 3 real roots. Explain how you arrived at your answer.
c. Determine all values of *k* for which the function has exactly 2 real roots. Explain how you arrived at your answer.
d. Determine all values of *k* for which the function has exactly 1 real root. Explain how you arrived at your answer.
e. Determine all values of *k* for which the function has no real roots. Explain how you arrived at your answer.

(Adapted from ETS, 2004d.)

5)
a. Create a word problem that involves a function with two or more independent variables.
b. Identify the independent and dependent variables.
c. Solve the problem you created.
d. Explain why this problem is solved through the use of such a function.
e. What would the graph of this function look like?

6) Consider the function defined by the following rule: $f(x) = 2x^2 - 3$.

a. Sketch a graph of this function.
b. Describe the domain and range of function *f*.
c. Determine a rule for the inverse of function *f*.
d. Sketch the graph of the inverse of function *f*.
e. Describe the domain and range of the inverse of function *f*.
f. Is the inverse of function *f* also a function? Explain your answer.
g. How are the domain and range of function *f* related to the domain and range of the inverse of function *f*?

7) a. Find a function *f* for which $f(x+y) = f(x) + f(y)$. Determine whether there are other functions for which this is true. Explain your reasoning.

b. Find a function *f* for which $f(xy) = f(x)f(y)$. Determine whether there are other functions for which this is true. Explain your reasoning.

c. Find a function *f* for which $f(x+y) = f(x)f(y)$. Determine whether there are other functions for which this is true. Explain your reasoning.
(Adapted from Sfard, 1992, p. 67)

8) a. Consider the following functions:

$$g(x) = \ln x$$
$$h(x) = 3x + 1$$

Create and compare the composite functions $g(h(x))$ and $h(g(x))$.

b. Can $f(x) = \sqrt{2x-3}$ be the result of the composition of two functions? If YES, identify the two functions. If NO, explain why not.

c. Given two arbitrary functions $f(x)$ and $g(x)$, under what conditions will $f(g(x)) = g(f(x))$? Explain your answer.

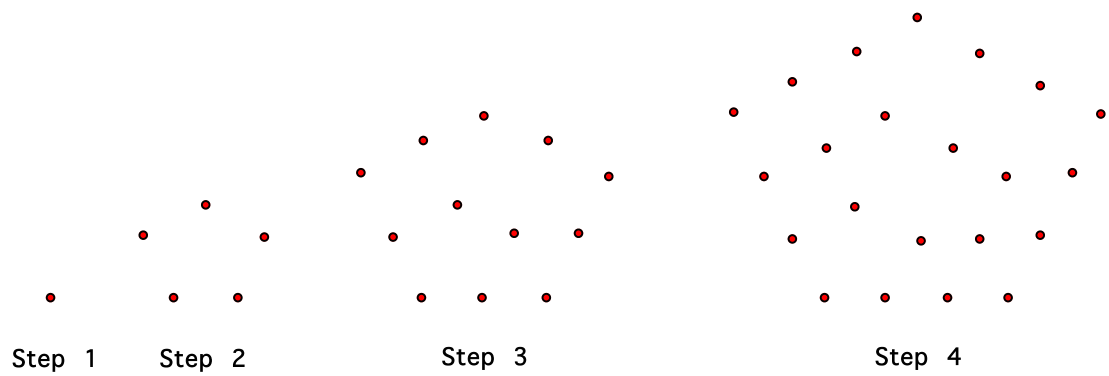
9. A flight from University Park Airport near Penn State University to Philadelphia International Airport has to circle several times before being allowed to land.

a) Plot a graph of the distance of the plane from PSU against the elapsed time from the moment of take off until landing.

b) Explain the graph you construct.

(Adapted from *PSSM*, 2000, p. 363)

10) Study the diagram below.



a. Identify the number of dots in Step 7.
b. How can this pattern be described as a recursively defined function?
c. Identify the number of dots in Step 60.
d. How can this pattern be described as an explicitly defined function?

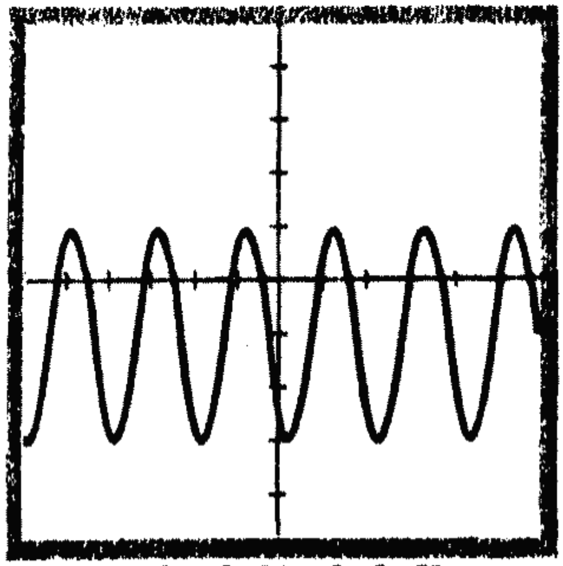
11) Solve the following problem:

A box with no lid has a volume of 500 in^3 with length l, width w, and height h.

a) Write a symbolic representation for the surface area S of the box in terms of l and w.
b) Construct a graph of this symbolic representation.
c) Determine the dimensions of the box with the minimum surface area.
d) Find the minimum surface area.

(Adapted from Demana, Waits, & Clemens, 1994, p. 654.)

12) Study the following graph.



[−2π, 2π] by [−5, 5]

a. Describe this graph. Is it a graph of a function? If YES, what kind of a function is it? If NO, explain why it is not a graph of a function.

b. Write a symbolic description of this graph.

13) a. Find the inverse of the linear function *f* on $\Re$, where $y = f(x) = mx + b$, $m \neq 0$. Note any restrictions.

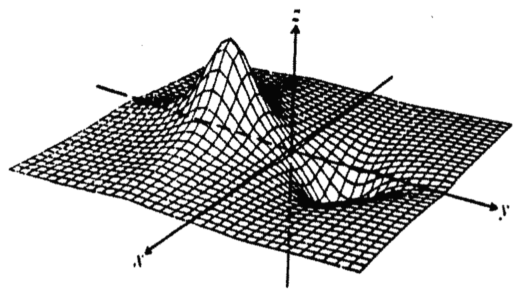
b. Find the inverse of the quadratic function *g* on $\Re$, where $y = g(x) = ax^2 + bx + c$, $a \neq 0$. Note any restrictions.

c. What are the characteristics of any arbitrary function $f(x)$ that will determine whether the function's inverse relation $f^{-1}(x)$ is also a function?

d. How can these characteristics be identified?

14) a. Given the graph below, how can one determine whether the graph is that of a function?

b. If it IS the graph of a function, identify the independent and dependent variables. Explain your reasoning. If it is NOT the graph of a function, explain why.



15) Consider the following polar functions:

I. $f(\theta) = a \sin n\theta$ II. $g(\delta) = a \cos n\delta$

a. Determine parametrically expressed function rules for the polar functions I and II above.
b. Discuss the effects of the parameters *a* and *n* in the polar functions I and II above. Justify your conjectures.

(Adapted from Demana, Waits, Clemens, 1994, p. 490)

Summary. In this section of this paper, the author has listed the items she generated according to the framework she developed. The items were generated with an awareness of the limitations of the items of the ETS CR test and focused particularly on the z-axis of the Function Block and on the Generalization and Synthesis Indicators of Mathematical Understanding. The items assess the following function-related concepts:

1. Graphs of Functions: Items 1, 2, 3, 5, 9, 11, 12, 14
2. Families of Functions: Items 2, 7, 12, 15
3. Multivariate Functions: Items 5, 11, 14
4. Inverses of Functions: Items 6, 8, 13
5. Restrictions on Functions: Items 6, 13
6. Transformations of Functions: Items 4, 12
7. Roots of Functions: Items 2, 4
8. Piecewise-defined Functions: Items 3, 9
9. Recursively and Explicitly Defined Functions: Item 10
10. Composition of Functions: Item 8
11. Parametric Functions: Item 15
12. Functions and Non-functions: Item 1

Conclusion

The purpose of this paper was to present a framework that could be used to propose suggestions for generating constructed-response items to assess a subject's understanding of the concept of function. The author described the framework for assessing a subject's understanding of the concept of function by first examining the literature, particularly the Dreyfus and Eisenberg (1982, 1983) Function Block and Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. She then provided examples based mainly upon the multiple-choice and constructed-response items created by ETS to explain what "understanding the concept of function" means in the context of the study for which this framework was developed. She described how specific items assess a subject's understanding of the concept of function using Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. Next, she examined two types of assessment items, multiple choice and constructed response, and analyzed the advantages and the limitations of both types of assessment items and provided recommendations on how to develop items that will assess a subject's understanding of the concept of function using Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. Thus, she proposed suggestions and directions to develop items that have the potential to assess a subject's understanding of the concept of function based upon Sierpinska's (1992, 1994) Indicators of Mathematical Understanding. Because the author believes that it is *not* the mathematical object that determines the basic mental operator used but rather it *is* the manner in which the subject *thinks about* or *processes* the mental object that determines the basic mental operator used, she contends there is a subject-item interaction in determining whether there is any demonstration of understanding. Finally, she proposed a set of items that she believes has the potential to assess a subject's understanding of the concept of function using Sierpinska's (1992, 1994) Indicators of Mathematical Understanding, particularly items that require subjects to generalize and synthesize.