The Pennsylvania State University
The Graduate School

PERIODIC CYCLIC HOMOLOGY AND SMOOTH DEFORMATIONS

A Dissertation in
Mathematics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2013

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Abstract

Given a formal deformation of an algebra, Getzler defined a connection on the periodic cyclic homology of the deformation, which he called the Gauss-Manin connection. We define and study this connection for smooth one-parameter deformations. Our main example is the smooth noncommutative n-torus $\mathcal{A}_\Theta$, viewed as a deformation of the algebra $C^\infty(\mathbb{T}^n)$ of smooth functions on the n-torus. In this case, we use the Gauss-Manin connection to give a parallel translation argument that shows that the periodic cyclic homology groups $HP_\bullet(\mathcal{A}_\Theta)$ are independent of the parameter Θ. As a consequence, we obtain differentiation formulas relating various cyclic cocycles on $\mathcal{A}_\Theta$.

We generalize this to a larger class of deformations, including nontrivial crossed product algebras by the group $\mathbb{R}$. The algebras of such a deformation extend naturally to differential graded algebras, and we show that they are fiberwise isomorphic as A_∞-algebras. In particular, periodic cyclic homology is preserved under this type of deformation. This clarifies and strengthens the periodic cyclic homology isomorphism for noncommutative tori, and gives another proof of Connes' Thom isomorphism in cyclic homology.

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Acknowledgments

This research was partially supported by the grant DMS1101382 from the National Science Foundation.

I would like to thank my advisor, Nigel Higson, for his suggestion of this project and his constant support, encouragement, and enthusiasm throughout. Thank you to the members of the Student GFA/NCGMP Seminar, past and present. I have thoroughly enjoyed the opportunity to learn from all of you, and I hope you can say the same about me. Thank you to all my brilliant professors at Penn State who have exposed me to more mathematics than I could possibly absorb. Going back further, I would like to thank my professors at Bucknell University, especially George Exner, who taught me how to do rigorous mathematics, and Pamela Gorkin, who guided me in my transition to graduate school.

I have been blessed with amazing family and friends who have shown me unconditional support all my life. I'd especially like to thank my mother, Margaret Yashinski, for raising me to be who I am today, and encouraging me to enter college as a math major.

Lastly, I thank my wife and best friend, Melisa, for traveling this journey with me. I truly would be lost without you.

Dedication

To my parents.

Chapter 1

Introduction

1.1 Motivation and background

1.1.1 Noncommutative tori

The motivating examples for this work are *noncommutative tori* [30]. The noncommutative 2-torus with parameter $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the universal C^*-algebra A_θ generated by two unitary elements u and v subject to the relation

$$vu = e^{2\pi i\theta} uv.$$

When $\theta = 0$, $A_0 \cong C(\mathbb{T}^2)$, the C^*-algebra of continuous functions on the 2-torus. In this way, one views A_θ as the algebra of continuous functions on some "noncommutative space," as in the philosophy of Connes [6]. The algebra A_θ is also called the *rotation algebra* with angle θ, because it can be expressed as a crossed product

$$A_\theta \cong C(\mathbb{T}) \rtimes_{R_\theta} \mathbb{Z},$$

where R_θ is the automorphism of $C(\mathbb{T})$ induced by a rotation of angle θ.

The noncommutative tori form an interesting family of algebras due to their erratic dependence on θ. For example, A_{θ_1} is isomorphic to A_{θ_2} if and only if $\theta_1 = \theta_2$ or $\theta_1 = 1 - \theta_2$. The C^*-algebra A_θ is simple if and only if θ is irrational, and A_θ has a unique tracial state if and only if θ is irrational. Another striking difference is the embeddability of A_θ into an AF-algebra when θ is irrational [27].

Rieffel [29] proved that for every $\alpha \in (\mathbb{Z}+\mathbb{Z}\theta)\cap[0,1]$, there is a projection in A_θ of canonical trace α. This is in stark constrast to the commutative torus $A_0 \cong C(\mathbb{T}^2)$, which contains no nontrivial projections.

K-theory for C^*-algebras played a large role in the classification of noncommutative tori. The six-term exact sequence of Pimsner and Voiculescu [26] allows one to calculate the K-theory groups of A_θ by taking advantage of the crossed product structure. For any θ, we have

$$K_i(A_\theta) \cong \mathbb{Z}\oplus\mathbb{Z}, \qquad i=0,1,$$

so that the K-theory of a noncommutative torus is the same as the K-theory of the commutative torus $\mathbb{T}^2$.

So we know that the K-theory of A_θ is independent of θ because it has been calculated separately for each θ. This is somewhat unsatisfactory, because the collection of C^*-algebras $\{A_\theta\}$ vary continuously in θ in some sense, and one would like to take advantage of this continuity to argue that K-theory is rigid as θ varies. Indeed, the collection $\{A_\theta\}$ form a continuous field of C^*-algebras, and if A_J denotes the sections of this field over a contractible subset $J\subset\mathbb{T}$, then the evaluation homomorphisms $A_J\to A_\theta$ induces isomorphisms at the level of K-theory [8]. This is an improvement in the sense that it gives canonical isomorphisms

$$K_\bullet(A_{\theta_1})\cong K_\bullet(A_{\theta_2}),$$

for any $\theta_1,\theta_2\in\mathbb{T}$. However, the proof still relies on the crossed product structure, and it is not clear how one could generalize to other deformations that lack this structure.

Our investigation into a more satisfactory explanation of this rigidity shall be in the context of cyclic homology, rather than K-theory. Cyclic homology was discovered independently by Connes [5] and Tsygan [34], see also [22]. It can be viewed as a noncommutative analogue of de Rham cohomology. As de Rham cohomology is defined in terms of a smooth structure, one does not typically consider a C^*-algebra here, but rather a certain dense subalgebra which is thought of as the algebra of smooth functions on the noncommutative space. Periodic cyclic homology is a variant of cyclic homology that assigns to an algebra A two vector spaces $HP_0(A)$ and $HP_1(A)$. Connes showed [5] that if one considers $C^\infty(M)$, the algebra of smooth functions on a smooth compact manifold M, then

$$HP_i(C^\infty(M)) \cong \bigoplus_{k\geq 0} H^{i+2k}_{dR}(M), \qquad i = 0, 1,$$

where $H^j_{dR}(M)$ is the j-th de Rham cohomology group of M. Here, one is considering $C^\infty(M)$ with its natural Fréchet topology and using a version of periodic cyclic homology for topological algebras.

For noncommutative tori, the appropriate dense subalgebra is a Schwarz completion $\mathcal{A}_\theta$ of the subalgebra generated by u and v, as in [5] or [23]. In the commutative case $\theta = 0$, this is exactly the Fréchet algebra $C^\infty(\mathbb{T}^2)$. As first shown by Connes on the cohomology side [5],

$$HP_i(\mathcal{A}_\theta) \cong \mathbb{C} \oplus \mathbb{C}, \qquad i = 0, 1.$$

Thus by explicit calculation for each θ, we see that $HP_\bullet(\mathcal{A}_\theta)$ does not depend on θ. One goal of this thesis is to prove this independence of θ using deformation-theoretic ideas.

1.1.2 Deformation theory

Let V be a vector space and let $\{m_t : V \otimes V \to V\}$ be a family of associative multiplications depending on a parameter t. It is natural to ask what properties, if any, the algebras $\{A_t := (V, m_t)\}$ have in common. We shall call such a family $\{A_t\}$ a *deformation of algebras*. To get some amount of control, we shall insist that the family $\{m_t\}$ has some suitable dependence on the parameter t, for example continuous, smooth, analytic, or polynomial. One way to make this rigorous is to consider vector spaces V that have a topology compatible with their linear structure.

1.1.2.1 Formal deformations

The first significant progress in the general study of deformations of algebras was made in the pioneering work of Gerstenhaber [11]. Instead of actual families of products, Gerstenhaber considered *formal deformations*. A formal deformation of an algebra A is given by a formal power series

$$a *_t b = ab + \sum_{n=1}^{\infty} t^n F_n(a,b), \qquad a, b \in A,$$

where $F_n \in \mathrm{Hom}(A \otimes A, A)$, such that $*_t$ is associative after extension by t-linearity. Since there is no topology or assumption of convergence, one cannot specialize to specific values of the parameter t, with the exception of $t = 0$, in which one recovers the original product of A. Gerstenhaber showed that such deformations are "controlled" by the Hochschild cohomology $H^{\bullet}(A, A)$. For example, the "infinitesimal" F_1 is always a Hochschild 2-cocycle, and if $[F_1] = 0 \in H^2(A, A)$, then the deformation is equivalent to a deformation for which the infinitesimal vanishes (but not the higher order terms.) By iterating this result, it follows that if $H^2(A, A) = 0$, then every formal deformation of A is equivalent to the trivial deformation

$$a *_t b = ab.$$

Such an algebra that admits no nontrivial deformations is called *(formally) rigid*.

The deformation philosophy of Gerstenhaber extends to deformations of other types of objects. For example, commutative deformations of commutative algebras are controlled by the Harrison cohomology [20]. Lie algebra deformations are controlled by the Lie algebra cohomology [25]. Deformations of a cochain complex $(C^{\bullet}, d)$ are controlled by the cohomology of the complex $\mathrm{Hom}(C^{\bullet}, C^{\bullet})$.

1.1.2.2 Smooth deformations and connections

All the types of deformations we consider will be deformations of algebraic objects in the category of vector spaces, e.g. algebras, chain complexes, and differential graded (co)algebras. Instead of considering formal deformations, we shall consider families of these structures on a single locally convex vector space X that depend smoothly in some sense on a real parameter. For example, the underlying vector space of the smooth noncommutative torus $\mathcal{A}_\theta$ is the Schwartz space $\mathcal{S}(\mathbb{Z}^2)$, and the map

$$\theta \mapsto m_\theta(x, y)$$

is smooth for all $x, y \in \mathcal{S}(\mathbb{Z}^2)$, where m_θ denotes the multiplication of $\mathcal{A}_\theta$.

Our approach to proving rigidity of a deformation (i.e. proving all objects in the deformation are isomorphic) uses the tools of connections and parallel translation. To be concrete, let us describe the case of deformations of algebras. Suppose $\{A_t = (X, m_t)\}_{t\in J}$ is a smooth one-parameter deformation of algebra structures on X depending on $t \in J$, where J is an open subinterval of $\mathbb{R}$. We can think of this collection as a bundle of algebras over the parameter space J. As a bundle of vector spaces, it is trivial, but the multiplication is changing smoothly as we pass from one fiber to another. The central object of study will be the algebra of smooth sections of this bundle $A = C^\infty(J, X)$, with product given by

$$(a_1 a_2)(t) = m_t(a_1(t), a_2(t)), \qquad a_1, a_2 \in A.$$

The algebra A also has a $C^\infty(J)$-module structure given by pointwise scalar multiplication.

By a *connection*[1] on A, we mean a $\mathbb{C}$-linear map $\nabla : A \to A$ such that

$$\nabla(f \cdot a) = f' \cdot a + f \cdot \nabla(a), \qquad \forall f \in C^\infty(J), a \in A.$$

Notice that the definition only uses the $C^\infty(J)$-module structure, and not the algebra structure. A connection allows us the possibility of linearly identifying the fibers $\{A_t\}$ of our bundle via parallel translation. To do this for all fibers, one needs the existence and uniqueness of global solutions to the equations

$$\nabla a = 0, \qquad a(t_0) = a_0$$

for every $t_0 \in J$ and $a_0 \in A_{t_0}$. As every connection has the form

$$\nabla = \frac{d}{dt} + F$$

for some $C^\infty(J)$-linear endomorphism F of A, this amounts to solving linear ordinary differential equations with values in the locally convex vector space X. Once

[1] In more traditional language, this is the covariant derivative of a connection in the only available direction of our one-dimensional base space J.

one considers spaces more general than Banach spaces, this may not be possible to do.

Now a priori, the parallel transport isomorphisms are only linear isomorphisms because ∇ has no compatibility with the product of A. To ensure that they are algebra isomorphisms, we need to choose a connection that interacts nicely with the product. The appropriate type of connection turns out to be one that is a derivation, so that

$$\nabla(a_1a_2) = \nabla(a_1)a_2 + a_1\nabla(a_2), \qquad \forall a_1, a_2 \in A.$$

For this type of connection, the parallel transport maps are necessarily algebra isomorphisms, provided they exist. Such connections do not always exist, as we should expect because there are many nontrivial algebra deformations. The problem of the existence of such a connection is a problem in Hochschild cohomology that is a smooth manifestation of Gerstenhaber's formal results. Given any connection ∇ on A, for example $\nabla = \frac{d}{dt}$, one can consider the bilinear map $E : A \times A \to A$ defined by the equation

$$\nabla(a_1a_2) = \nabla(a_1)a_2 + a_1\nabla(a_2) + E(a_1, a_2).$$

One can check that E is $C^\infty(J)$-bilinear Hochschild 2-cocycle, and moreover the class $[E] \in H^2_{C^\infty(J)}(A, A)$ is independent of the choice of ∇. Using the classification of all connections on A, it follows that A possesses a connection that is a derivation if and only if $[E] = 0$. This element E can be thought of as a smooth family of Hochschild 2-cocycles

$$E_t : A_t \otimes A_t \to A_t,$$

each of which represents the "infinitesimal" of the deformation at t. To say $[E] = 0$ is to say all directions E_t are cohomologically trivial in a way that smoothly depends on t. Flowing along the t-dependent vector fields

$$F_t : A_t \to A_t, \qquad \delta F_t = E_t,$$

if possible, identifies our deformation with the trivial deformation.

Now if $\{B_t\}_{t\in J}$ is a smooth one-parameter deformation of some other type of algebraic structure with the same underlying vector space X, we take the same general approach. We "glue" the structures fiberwise to get a $C^\infty(J)$-linear structure of the same type on the $C^\infty(J)$-module $B = C^\infty(J, X)$. Then we consider a connection on B that is compatible with the additional structure, so that its parallel transport maps respect that structure.

So we have two nontrivial steps in proving the triviality of a deformation:

(R1) Find a connection that is compatible with the additional structure of the fibers $\{B_t\}$.

(R2) Prove the existence and uniqueness of solutions to the corresponding parallel translation differential equations.

There is typically a cohomological obstruction to the existence of such a connection that parallels Gerstenhaber's formal deformation philosophy, as we saw in the case of algebras. As we have already mentioned, it is a nontrivial problem to solve the corresponding parallel translation differential equations. If this is possible, we shall say the connection ∇ is *integrable*. So we have both analytic and algebraic obstructions to this executing this method.

1.1.2.3 The Gauss-Manin connection

Suppose $\{A_t\}_{t\in J}$ is a smooth one-parameter deformation of algebras. Then the corresponding periodic cyclic chain complexes $\{C_{\mathrm{per}}(A_t)\}_{t\in J}$ form a smooth one-parameter deformation of chain complexes. The appropriate notion of a compatible connection here is a chain map. Using a calculus of Lie derivative and contraction operators on the periodic cyclic chain complex, Getzler showed that there always exists a connection ∇_{GM} for the deformation $\{C_{\mathrm{per}}(A_t)\}_{t\in J}$ that is a chain map [12]. Thus, the obstruction to (R1) always vanishes for this deformation of chain complexes. If one can prove ∇_{GM} is integrable, then one obtains parallel translation isomorphisms

$$HP_\bullet(A_{t_1}) \cong HP_\bullet(A_{t_2}), \qquad \forall t_1, t_2 \in J.$$

However, there are many deformations for which periodic cyclic homology is not preserved. For example, any algebra can be smoothly deformed into a trivial algebra by considering the deformed product

$$m_t(a,b) = t(ab)$$

and letting t go to 0. Thus, one can expect to encounter analytic difficulties in general from the resulting differential equations.

Even in somewhat simple examples, it seems rather unlikely that one could integrate the Gauss-Manin connection at the level of complexes, due to the general form that ∇_{GM} takes. Doing this would identify the periodic cyclic chain complexes of any two fibers

$$C_{\mathrm{per}}(A_{t_1}) \cong C_{\mathrm{per}}(A_{t_2})$$

as isomorphic complexes, and this is a stronger result than we desire. Instead, we could ask that ∇_{GM} be integrable at the level of homology, in the following sense. If A denotes the algebra of sections of $\{A_t\}_{t\in J}$, then the periodic cyclic chain complex $C_{\mathrm{per}}^{C^\infty(J)}(A)$ over the ground ring $C^\infty(J)$ is the chain complex of sections of the deformation $\{C_{\mathrm{per}}(A_t)\}_{t\in J}$. The Gauss-Manin connection is a chain map on $C_{\mathrm{per}}^{C^\infty(J)}(A)$, and so descends to a connection on the $C^\infty(J)$-module $HP_\bullet^{C^\infty(J)}(A)$. We'll say ∇_{GM} is integrable at the level of homology if for every $t_0 \in J$ and $[\omega_0] \in HP_\bullet(A_{t_0})$, there exists a unique solution $[\omega] \in HP_\bullet^{C^\infty(J)}(A)$ to

$$\nabla_{GM}[\omega] = 0, \qquad [\omega(t_0)] = [\omega_0].$$

Having this level of integrability is enough to construct parallel transport isomorphism

$$HP_\bullet(A_{t_1}) \cong HP_\bullet(A_{t_2}).$$

1.2 Main results

1.2.1 Integrating ∇_{GM}

In Chapter 4, we study the Gauss-Manin connection ∇_{GM} for the deformation of noncommutative tori, and prove the following result.

Theorem. *The Gauss-Manin connection associated to the deformation* $\{\mathcal{A}_\theta\}_{\theta\in\mathbb{R}}$

is integrable at the level of homology.

The proof is a little indirect. As the Gauss-Manin connection is difficult to work with on the whole periodic cyclic chain complex, we first replace it with a smaller chain equivalent complex.

Let $\mathcal{A}$ denote the algebra of sections of the deformation $\{\mathcal{A}_\theta\}_{\theta\in\mathbb{R}}$ of noncommutative 2-tori. This deformation has the property that

$$\frac{d}{d\theta}(a_1a_2) = \frac{d}{d\theta}(a_1)a_2 + a_1\frac{d}{d\theta}(a_2) + \frac{1}{2\pi i}X(a_1)Y(a_2), \qquad \forall a_1, a_2 \in \mathcal{A},$$

where X, Y are commuting derivations on $\mathcal{A}$. The obstruction class

$$[E] = [\frac{1}{2\pi i}X \smile Y] \in H^2_{C^\infty(J)}(\mathcal{A}, \mathcal{A})$$

is nontrivial, though it has a specific form that we take advantage of.

Let $\mathfrak{g}$ be the two-dimensional abelian Lie algebra spanned by X and Y. We form a $\mathfrak{g}$-invariant periodic cyclic chain complex $C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A}_\theta)$, which is easier to work with. Using the fact that the action of $\mathfrak{g}$ on $\mathcal{A}_\theta$ is the infinitesimal of an action of the Lie group $\mathbb{T}^2$ by automorphisms, we show that $C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A}_\theta)$ is chain equivalent to $C_{\mathrm{per}}(\mathcal{A}_\theta)$, so that it suffices to work in the $\mathfrak{g}$-invariant setting.

The collection of operators $\{\frac{d}{d\theta}, X, Y\}$ on $\mathcal{A}$ generate a commutative Hopf algebra $\mathcal{H}$, which naturally acts on both $C_{\mathrm{per}}(\mathcal{A})$ and $C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A})$. The action of the element $\frac{d}{d\theta}$ gives another connection $\widetilde{\nabla}$ on these complexes, and $\widetilde{\nabla}$ commutes with the boundary map in the $\mathfrak{g}$-invariant case. It turns out to be fairly straightforward to solve the differential equations to show that $\widetilde{\nabla}$ is integrable on the complex $C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A})$. Using certain homotopy formulas as in [12], we show that $\widetilde{\nabla} - \nabla_{GM}$ is nilpotent as an operator on $HP^{\mathfrak{g}}_\bullet(\mathcal{A}) \cong HP_\bullet(\mathcal{A})$. From this, we see that ∇_{GM} is integrable as well.

Now the integrability of $\widetilde{\nabla}$ is enough to prove that $HP_\bullet(\mathcal{A}_\theta)$ is independent of θ, so it may seem unnecessary to continue the argument for the integrability of ∇_{GM}. However, by doing so we gain some understanding of the deformation of the pairing between K-theory and cyclic cohomology. Let A denote the algebra of sections of any smooth one-parameter deformation of algebras $\{A_t\}$. An idempotent $P \in A$ determines a cycle $\operatorname{ch} P \in HP_0(A)$, and this gives a well-defined *Chern character* homomorphism

$$\mathrm{ch} : K_0(A) \to HP_0(A),$$

where $K_0(A)$ denotes the algebraic K-theory group of A [21, Chapter 8]. The pairing

$$\langle \cdot, \cdot \rangle : HP^0(A) \times K_0(A) \to \mathbb{C}$$

is defined in terms of the usual pairing $HP^0(A) \times HP_0(A) \to \mathbb{C}$ by

$$\langle [\varphi], [P] \rangle = \langle [\varphi], [\mathrm{ch}\, P] \rangle.$$

Now, there exists a cohomological Gauss-Manin connection ∇^{GM} on $HP^\bullet(A)$ satisfying

$$\frac{d}{dt}\langle [\varphi], [\omega] \rangle = \langle \nabla^{GM}[\varphi], [\omega] \rangle + \langle [\varphi], \nabla_{GM}[\omega] \rangle.$$

It is shown (Corollary 4.3.2) that

$$\nabla_{GM}[\mathrm{ch}\, P] = 0 \in HP_0(A)$$

for any idempotent P. Thus we obtain the differentiation formula

$$\frac{d}{dt}\langle [\varphi], [P] \rangle = \langle \nabla^{GM}[\varphi], [P] \rangle.$$

In the case of noncommutative tori, where we can compute with ∇^{GM}, this gives information about the values of this pairing without any knowledge of the group $K_0(\mathcal{A}_\theta)$.

1.2.2 An A_∞-isomorphism

Consider any smooth one-parameter deformation $\{A_t\}_{t \in J}$ whose algebra of sections A satisfies

$$\frac{d}{dt}(a_1 a_2) = \frac{d}{dt}(a_1)a_2 + a_1 \frac{d}{dt}(a_2) + \frac{1}{2\pi i} X(a_1)Y(a_2), \qquad \forall a_1, a_2 \in A, \tag{1.1}$$

where X and Y are commuting $C^\infty(J)$-linear derivations on A. This is the most crucial feature of the deformation of noncommutative tori used in Chapter 4. Of secondary importance, in that case, is the fact that X and Y come from a group action of $\mathbb{T}^2$. The proof of integrability of the Gauss-Manin connection could be generalized to all deformations having these two features. In Chapter 5, we give an improved argument that proves invariance of periodic cyclic homology under such a deformation, where we need not assume that X and Y exponentiate to give a $\mathbb{T}^2$-action.

An interesting example is when one has a Fréchet algebra B and an action $\alpha : \mathbb{R} \to \mathrm{Aut}(B)$ which is smooth in some sense. Then, as in [9], one can form the *smooth crossed product* $B \rtimes_\alpha \mathbb{R}$. For any $t \in \mathbb{R}$, we can define a rescaled action α^t by $\alpha^t(s) = \alpha(ts)$. Letting $A_t = B \rtimes_{\alpha^t} \mathbb{R}$, one can check that $\{A_t\}$ is a smooth deformation of the trivial crossed product $A_0 \cong \mathcal{S}(\mathbb{R})\widehat{\otimes}B$ into the nontrivial crossed product $A_1 = B \rtimes_\alpha \mathbb{R}$. Then this deformation has commuting derivations X and Y for which (1.1) is satisfied, but X and Y do not come from a $\mathbb{T}^2$-action.

For an arbitrary algebra A with two commuting derivations X and Y, we can form the Chevalley-Eilenberg cochain complex

$$\Omega^\bullet(A) := A \otimes \wedge^\bullet \mathfrak{g}^*,$$

where $\mathfrak{g}$ is the Lie algebra spanned by X and Y. Then $\Omega^\bullet(A)$ is a nonnegatively graded differential graded algebra (DGA) with the property that $\Omega^0(A) = A$. Working in the context of a deformation satisfying (1.1), we can form this DGA fiberwise, and so we obtain a smooth one-parameter deformation of DGAs $\{\Omega^\bullet(A_t)\}_{t\in J}$. The DGA of sections $\Omega^\bullet(A)$ of this deformation satisfies

$$\frac{d}{dt}(\omega\eta) = \frac{d}{dt}(\omega)\eta + \omega\frac{d}{dt}(\eta) + \frac{1}{2\pi i}L_X(\omega)L_Y(\eta), \qquad \forall \omega, \eta \in \Omega^\bullet(A),$$

where L_X and L_Y are Lie derivative operators coming from the natural extension of the action of $\mathfrak{g}$ to $\Omega^\bullet(A)$. On $\Omega^\bullet(A)$ there are also contraction operators ι_Z for all $Z \in \mathfrak{g}$ satisfying the Cartan Homotopy Formula

$$[d, \iota_Z] = L_Z,$$

where d is the differential on $\Omega^\bullet(A)$. It follows that L_X and L_Y are chain homotopic to zero, and so the connection $\frac{d}{dt}$ is a derivation "up to homotopy." So one could expect parallel translation maps induced by such a connection to be algebra isomorphisms "up to homotopy."

What is meant here can be made precise in the language of Stasheff's A_∞-algebras [32], see also [13]. An A_∞-algebra is a generalization of an algebra in which associativity is only required to hold up to homotopy. Additionally, an A_∞-algebra satisfies a whole sequence of "higher homotopies." A particular example of an A_∞-algebra is a DGA, and so we can view our deformation $\{\Omega^\bullet(A_t)\}_{t\in J}$ as a deformation of A_∞-algebras. The main result of Chapter 5 is that this deformation is trivial in the A_∞-category.

Theorem. *For any $t_1, t_2 \in J$, $\Omega^\bullet(A_{t_1})$ and $\Omega^\bullet(A_{t_2})$ are isomorphic as A_∞-algebras.*

This is remarkable because we cannot expect the fibers to be isomorphic as DGAs, as we can see by considering the noncommutative tori example. The isomorphism is constructed by parallel transport along a suitable connection. By adding to the connection $\frac{d}{dt}$ a suitable homotopy operator, we obtain a connection that is compatible with the A_∞-structure. That this connection exists means that the obstruction to (R1) vanishes in the A_∞-category. It is easily seen to be integrable, as it is a locally nilpotent perturbation of $\frac{d}{dt}$.

Getzler and Jones extended the theory of cyclic homology to A_∞-algebras in [13]. This shows that the periodic cyclic homology groups $HP_\bullet(\Omega^\bullet(A_t))$ are independent of t. As shown in [14], there is an isomorphism

$$HP_\bullet(\Omega^0) \cong HP_\bullet(\Omega^\bullet)$$

for any nonnegatively graded DGA $\Omega^\bullet$.

Corollary. *For any smooth one-parameter deformation $\{A_t\}_{t\in J}$ satisfying (1.1), the periodic cyclic homology groups $HP_\bullet(A_t)$ do not depend on t.*

This gives another computation of periodic cyclic homology of noncommutative tori. By considering the smooth crossed product example, it gives a proof of the Thom isomorphism [9]

$$HP_\bullet(B) \cong HP_{\bullet+1}(B \rtimes_\alpha \mathbb{R}),$$

when combined with the fact that $HP_\bullet(B) \cong HP_{\bullet+1}(\mathcal{S}(\mathbb{R})\widehat{\otimes}B)$.

It would be interesting to use the results to Chapter 5 to prove that $K_\bullet(A_t)$ is independent of t for these types of deformations. To do this, one would need a suitable version of K-theory designed for A_∞-algebras with the property that

$$K_\bullet(\Omega^\bullet) \cong K_\bullet(\Omega^0)$$

for any nonnegatively graded DGA $\Omega^\bullet$. To our knowledge, no such theory has been developed yet.

Chapter 2

Preliminaries

2.1 Locally convex topological vector spaces

We review standard facts about locally convex topological vector spaces. More details and proofs of most facts asserted can be found in [33]. See [15] and [16] for more details concerning projective and inductive tensor products.

2.1.1 Basics

All vector spaces in this thesis are over the ground field $\mathbb{C}$. Recall that a *seminorm* on a vector space X is a function $p : X \to [0, \infty)$ such that

(i) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$.

(ii) $p(cx) = |c|p(x)$ for all $c \in \mathbb{C}$ and $x \in X$.

If it is the case that $p(x) = 0$ if and only if $x = 0$, then p is a *norm*. Given a collection $\{p_\alpha\}_{\alpha \in I}$ of seminorms on X, the *locally convex topology* on X determined by $\{p_\alpha\}_{\alpha \in I}$ is the coarsest translation-invariant topology for which all of the p_α are continuous. A basis of 0-neighborhoods is obtained by taking finite intersections of sets of the form

$$p_\alpha^{-1}[0, \epsilon), \qquad \alpha \in I, \epsilon > 0.$$

Such an intersection always contains an open set of the form $p^{-1}[0, 1)$ for some continuous seminorm p, which need not be in the collection $\{p_\alpha\}_{\alpha \in I}$.

Given a locally convex topology on X, we can consider the set S of all continuous seminorms on X. By definition, the set S contains the original defining family of seminorms, but it may be larger. However, the locally convex topology determined by the set S is the same as the given topology on X. So when convenient, we may replace a defining family of seminorms by the collection of all continuous seminorms.

A linear map $F : X \to Y$ between two locally convex topological vector spaces is continuous if and only if for every seminorm q_β defining the topology on Y, there is some continuous seminorm p on X (not necessarily from the defining family) such that

$$q(F(x)) \leq p(x), \qquad \forall x \in X.$$

A locally convex topology on X is Hausdorff if and only if the generating family of seminorms $\{p_\alpha\}_{\alpha\in I}$ separates points in X. That is, for every $x \in X$, there is some $\alpha \in I$ for which $p_\alpha(x) > 0$. A net $(x_\lambda)_{\lambda\in\Lambda}$ in X converges to x if and only if

$$p_\alpha(x - x_\lambda) \to 0, \qquad \text{as } \lambda \to \infty$$

for each seminorm p_α that defines the topology. A net $(x_\lambda)_{\lambda\in\Lambda}$ in X is *Cauchy* if

$$p_\alpha(x_\lambda - x_{\lambda'}) \to 0, \qquad \text{as } \lambda, \lambda' \to \infty$$

for each seminorm p_α. The space X is *complete* if every Cauchy net converges in X.

Every Hausdorff locally convex vector space X embeds as a dense subspace into a complete Hausdorff locally convex space $\widehat{X}$ with the universal property that any continuous linear map F from X into a complete Hausdorff locally convex space Y induces a unique continuous linear map $\widehat{F} : \widehat{X} \to Y$ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "F"'] & \widehat{X} \arrow[d, "\widehat{F}"] \\ & Y \end{tikzcd}$$

commutes. The space $\widehat{X}$ is called the *completion* of X.

A complete locally convex vector space whose topology is generated by a single norm is a *Banach space*. A complete Hausdorff locally convex vector space whose topology is generated by a countable family $\{p_n\}_{n=0}^{\infty}$ of seminorms is a *Fréchet space*. By replacing p_n with ${p_n}' = \sum_{i=0}^{n} p_i$, we can always assume the topology of a Fréchet space is generated by a countable family $\{p_n\}_{n=0}^{\infty}$ of seminorms with $p_n \leq p_{n+1}$. Fréchet spaces are metrizable.

Example 2.1.1. Every vector space V can be given the locally convex topology generated by the family of all seminorms on V. Clearly, this is the finest locally convex topology on V. Among the seminorms on V are those of the form

$$p(v) = |\varphi(v)|$$

for some linear functional $\varphi : V \to \mathbb{C}$. Since the algebraic dual V^* separates points of V, this topology is Hausdorff. One can show that every Cauchy net is eventually contained in a finite dimensional subspace of V. Thus, the completeness of V with respect to this topology follows from the completeness of finite dimensional vector spaces.

Any linear map F from V into another locally convex topological vector space X is continuous with respect to this topology on V. Indeed, if q is a seminorm defining the topology on X, then $p := q \circ F$ is a seminorm on V, hence continuous, and we have

$$q(F(v)) = p(v), \qquad \forall v \in V.$$

We shall primarily work in the category LCTVS of complete Hausdorff locally convex topological vector spaces with continuous linear maps. We shall write $X \in$ LCTVS to mean that X is a complete Hausdorff locally convex topological vector space. Example 2.1.1 shows that the category of vector spaces and linear maps is a full subcategory of LCTVS.

2.1.2 Constructions in LCTVS

2.1.2.1 Direct products and bilinear maps

Given a countable collection $\{X_n\}_{n\in\mathbb{Z}} \subset \mathrm{LCTVS}$, we can form the cartesian product vector space

$$\prod_{n\in\mathbb{Z}} X_n$$

equipped with the product topology. This topology is Hausdorff and locally convex: if $\{p_\alpha^{(n)}\}_{\alpha\in I_n}$ is a defining system of seminorms for X_n, then the collection

$$\{p_\alpha^{(n)} \circ \pi_n\}$$

of seminorms where n ranges over all integers and α ranges over all elements of I_n defines this topology. Here, π_n is the canonical projection onto the factor X_n. We see that a net in $\prod_{n\in\mathbb{Z}} X_n$ converges (is Cauchy) if and only if its projection onto each X_n converges (is Cauchy) in X_n. Thus, completeness of $\prod_{n\in\mathbb{Z}} X_n$ follows from the completeness of each X_n. If each X_n is Fréchet, then $\prod_{n\in\mathbb{Z}} X_n$ is Fréchet.

Given $X, Y, Z \in \mathrm{LCTVS}$, a bilinear map $B : X \times Y \to Z$ is *jointly continuous* if it is continuous with respect to the product topology. It follows that for every continuous seminorm r on Z, there exist continuous seminorms p and q on X and Y such that

$$r(B(x, y)) \leq p(x)q(y), \qquad \forall x \in X, y \in Y.$$

The bilinear map B is *separately continuous* if for every $x_0 \in X$ and every $y_0 \in Y$, the maps

$$y \mapsto B(x_0, y), \qquad x \mapsto B(x, y_0)$$

are both continuous. Separate continuity is strictly weaker than joint continuity. However, these notions coincide in the case where X and Y are both Fréchet spaces [33, Chapter 34.2].

2.1.2.2 Direct sums

Given a countable collection $\{X_n\}_{n\in\mathbb{Z}} \subset$ LCTVS, we can form the algebraic direct sum

$$\bigoplus_{n\in\mathbb{Z}} X_n.$$

We equip the direct sum $\bigoplus_{n\in\mathbb{Z}} X_n$ with the finest locally convex topology such that the natural inclusions

$$X_k \to \bigoplus_{n\in\mathbb{Z}} X_n$$

are continuous. For any sequence $\{p_n\}_{n\in\mathbb{Z}}$, where p_n is a continuous seminorm on X_n, we define a seminorm p on $\bigoplus_{n\in\mathbb{Z}} X_n$ by

$$p(\sum_{n\in\mathbb{Z}} x_n) = \sum_{n\in\mathbb{Z}} p_n(x_n).$$

Since the sum on the left is actually finite, so is the sum on the right. The topology on $\bigoplus_{n\in\mathbb{Z}} X_n$ is the locally convex topology generated by all such seminorms. One can check that the inclusions

$$\iota_k : X_k \to \bigoplus_{n\in\mathbb{Z}} X_n$$

are continuous.

Proposition 2.1.2. *(i) Given a family of continuous linear maps $F_n : X_n \to Y$ where $Y \in$ LCTVS, there is a unique continuous linear map $F : \bigoplus_{n\in\mathbb{Z}} X_n \to Y$ such that*

$$\begin{tikzcd} X_n \arrow[r, "\iota_n"] \arrow[dr, "F_n"'] & \bigoplus_{n\in\mathbb{Z}} X_n \arrow[d, "F"] \\ & Y \end{tikzcd}$$

commutes.

(ii) A linear map $F : \bigoplus_{n\in\mathbb{Z}} X_n \to Y$ is continuous if and only if $F_n = F \circ \iota_n : X_n \to Y$ is continuous for each n.

(iii) The space $\bigoplus_{n\in\mathbb{Z}} X_n$ is complete if and only if each X_n is complete.

2.1.2.3 Quotient spaces

Let $X \in$ LCTVS and let $Y \subset X$ be a linear subspace. Then we equip the quotient vector space X/Y with the quotient topology. That is, we give X/Y the finest topology such that the quotient map

$$\pi : X \to X/Y$$

is continuous. Then one can show that the topology on X/Y is locally convex. Given a seminorm p on X, we can define a seminorm $\bar{p}$ on X/Y by

$$\bar{p}(\pi(x)) = \inf_{y \in Y} p(x+y).$$

As p varies over all continuous seminorms on X (not just from a defining family,) the locally convex topology generated by all the $\bar{p}$ coincides with the quotient topology. The space X/Y is Hausdorff if and only if Y is a closed subspace. In general, X/Y may not be complete even when Y is a closed subspace. However, if X is a Fréchet space and Y is closed, then X/Y is a Fréchet space.

So the operation of taking quotients may force us out of our category LCTVS. This will be unavoidable, as we shall eventually consider homology groups of the form

$$\ker d_n / \operatorname{im} d_{n-1}$$

for which we cannot say a priori that the image of d_{n-1} is closed.

If a continuous linear map $F : X \to Z$ vanishes on Y, then the induced linear map

$$\bar{F} : X/Y \to Z$$

is continuous.

2.1.2.4 Spaces of linear maps

For $X, Y \in$ LCTVS, we write $\operatorname{Hom}(X,Y)$ for the vector space of all continuous linear maps from X to Y. There are several ways to make $\operatorname{Hom}(X,Y)$ a topological vector space, but we shall only consider the topology of uniform convergence on bounded sets. Recall that a subset $B \subset X$ is *bounded* if and only if $p(B)$ is a bounded set of real numbers for each continuous seminorm p on X. Given a bounded subset $B \subset X$ and a continuous seminorm q on Y, we define a seminorm q_B on $\mathrm{Hom}(X, Y)$ by

$$q_B(F) = \sup_{x \in B} q(F(x)).$$

The locally convex topology defined by this collection of seminorms is Hausdorff. If X has the additional property of being *bornological*, then $\mathrm{Hom}(X, Y)$ is complete [33, Chapter 32]. We shall not go into detail here, but roughly the space X is bornological if one can recover its topology from the collection of bounded subsets of X. Examples of spaces that are bornological include Fréchet spaces and LF-spaces.

An important example is the case where $Y = \mathbb{C}$. Here, we shall write

$$X^* = \mathrm{Hom}(X, \mathbb{C}),$$

the space of continuous linear functionals on X. Equipped with the topology described above, we shall refer to X^* as the *strong dual* of X. A continuous linear map $F : X \to Y$ induces a linear map

$$F^* : Y^* \to X^*$$

in the usual way, and this map is continuous with respect to the strong dual topologies.

If X and Y are Banach spaces, then $\mathrm{Hom}(X, Y)$ is a Banach space under the operator norm

$$\|F\| = \sup_{\|x\| \le 1} \|F(x)\|.$$

This is one of the nice features of Banach spaces. For contrast, the strong dual X^* of a Fréchet space X is typically not even metrizable; it is if and only if X is a Banach space.

A subset $H \subset \mathrm{Hom}(X, Y)$ is *equicontinuous* if for every seminorm q defining the topology on Y, there is a continuous seminorm p on X such that

$$q(F(x)) \le p(x), \qquad \forall F \in H, x \in X.$$

In order to state the Banach-Steinhaus theorem, we introduce the technical notion of a barreled space. Recall that a *barrel* is a closed, balanced, absorbing, convex subset of a topological vector space [33, Definition 7.1]. A topological vector space X is *barreled* if every barrel in X contains a neighborhood of 0. Every Fréchet space is barreled. Strict inductive limits of barreled spaces are barreled, e.g. countable direct sums of Fréchet spaces are barreled. The importance of barreled spaces comes from the following theorem [33, Theorem 33.1].

Theorem 2.1.3 (Banach-Steinhaus theorem)**.** *Let* $X, Y \in \mathrm{LCTVS}$ *and let* $H \subset \mathrm{Hom}(X, Y)$*. The following are equivalent:*

(i) For every $x \in X$*,* $\{F(x) \mid F \in H\}$ *is a bounded subset of* Y*.*

(ii) H *is a bounded subset of* $\mathrm{Hom}(X, Y)$*.*

(iii) H *is equicontinuous.*

2.1.2.5 Projective tensor products

Given $X, Y \in \mathrm{LCTVS}$, the *(completed) projective tensor product* is a space $X \widehat{\otimes} Y \in \mathrm{LCTVS}$ equipped with a jointly continuous bilinear map

$$\iota : X \times Y \to X \widehat{\otimes} Y$$

which is universal in the sense that if $Z \in \mathrm{LCTVS}$ and if $B : X \times Y \to Z$ is a jointly continuous bilinear map, then there is a unique continuous linear map $\widehat{B} : X \widehat{\otimes} Y \to Z$ such that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\iota"] \arrow[dr, "B"'] & X \widehat{\otimes} Y \arrow[d, "\widehat{B}"] \\ & Z \end{tikzcd}$$

commutes.

The projective tensor product exists and is unique up to isomorphism. It can be constructed as a completion of the algebraic tensor product $X \otimes Y$ with a suitable locally convex topology. Given continuous seminorms p on X and q on Y, define the seminorm $p \otimes q$ on $X \otimes Y$ by

$$(p \otimes q)(\theta) = \inf \sum_i p(x_i)q(y_i),$$

where the infimum is taken over all ways of writing $\theta = \sum_i x_i \otimes y_i$ as a finite sum of elementary tensors. Then the *projective topology* on $X \otimes Y$ is the locally convex topology generated by the family $\{p_\alpha \otimes q_\beta\}$ of seminorms as p_α and q_β vary through generating families of seminorms for X and Y respectively. The projective topology is the strongest locally convex topology on $X \otimes Y$ such that the canonical bilinear map $\iota : X \times Y \to X \otimes Y$ is jointly continuous. Define $X \widehat{\otimes} Y$ to be the completion of $X \otimes Y$ with the projective topology. Then one can show that $X \widehat{\otimes} Y$ has the required universal property. From the construction, we see that the projective tensor product of Banach spaces is a Banach space, and the projective tensor product of Fréchet spaces is a Fréchet space.

The projective tensor product is functorial in the sense that two continuous linear maps $F : X_1 \to X_2$ and $G : Y_1 \to Y_2$ induce a continuous linear map

$$F \otimes G : X_1 \widehat{\otimes} Y_1 \to X_2 \widehat{\otimes} Y_2$$

given on elementary tensors by

$$(F \otimes G)(x \otimes y) = F(x) \otimes G(y).$$

2.1.2.6 Inductive tensor products

Given $X, Y \in$ LCTVS, the *(completed) inductive tensor product* is a space $X \bar{\otimes} Y \in$ LCTVS equipped with a separately continuous bilinear map

$$\iota : X \times Y \to X \bar{\otimes} Y$$

which is universal in the sense that if $Z \in$ LCTVS and if $B : X \times Y \to Z$ is a separately continuous bilinear map, then there is a unique continuous linear map

$\bar{B} : X \bar{\otimes} Y \to Z$ such that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "\iota"] \arrow[dr, "B"'] & X \bar{\otimes} Y \arrow[d, "\bar{B}"] \\ & Z \end{tikzcd}$$

commutes.

The inductive tensor product exists and is unique up to isomorphism. It also can be constructed as a completion of $X \otimes Y$ with respect to a particular locally convex topology. Consider a seminorm p on $X \otimes Y$ with the property that for any $x_0 \in X$ and any $y_0 \in Y$, the functions

$$y \mapsto p(x_0 \otimes y), \qquad x \mapsto p(x \otimes y_0)$$

are continuous on Y and X respectively. The locally convex topology on $X \otimes Y$ generated by all such seminorms is called the *inductive topology*. This is a nonempty family of seminorms, as it contains the seminorms defining the projective topology. The inductive topology is the strongest topology for which the canonical bilinear map $\iota : X \times Y \to X \otimes Y$ is separately continuous. We define $X \bar{\otimes} Y$ to be the completion of $X \otimes Y$ with respect to the inductive topology.

The inductive tensor product is also functorial in the sense that two continuous linear maps

$$F : X_1 \to X_2, \qquad G : Y_1 \to Y_2$$

induce a continuous linear map

$$F \otimes G : X_1 \bar{\otimes} Y_1 \to X_2 \bar{\otimes} Y_2$$

satisfying

$$(F \otimes G)(x \otimes y) = F(x) \otimes G(y).$$

From the universal property, there is a canonical continuous linear map

$$X \bar{\otimes} Y \to X \widehat{\otimes} Y.$$

If X and Y are Fréchet spaces, then this map is an isomorphism because in this case the notions of separate and joint continuity coincide.

Example 2.1.4. Let V and W be vector spaces equipped with their finest locally convex topologies. If $Z \in \mathrm{LCTVS}$, then any bilinear map $B : V \times W \to Z$ is separately continuous. Such a map induces a unique linear map $\bar{B} : V \otimes W \to Z$, which is continuous if we equip $V \otimes W$ with its finest locally convex topology. We conclude that $V \bar{\otimes} W$ is isomorphic to the algebraic tensor product $V \otimes W$ with its finest locally convex topology. Thus the algebraic tensor product is a special case of the inductive tensor product.

The inductive tensor product is named so because it respects inductive limits and so, in particular, direct sums.

Proposition 2.1.5. *If* $X, Y_n \in \mathrm{LCTVS}$*, then*

$$X \bar{\otimes} (\bigoplus_{n \in \mathbb{Z}} Y_n) \cong \bigoplus_{n \in \mathbb{Z}} (X \bar{\otimes} Y_n).$$

Proof. We shall construct inverse isomorphisms using universal properties. For any $x \in X$, the map

$$F_n^x : Y_n \to \bigoplus_{n \in \mathbb{Z}} (X \bar{\otimes} Y_n)$$

given by

$$F_n^x(y_n) = x \otimes y_n$$

is continuous. So it induces a continuous linear map

$$F^x : \bigoplus_{n \in \mathbb{Z}} Y_n \to \bigoplus_{n \in \mathbb{Z}} (X \bar{\otimes} Y_n).$$

Define the map

$$F : X \times \bigoplus_{n \in \mathbb{Z}} Y_n \to \bigoplus_{n \in \mathbb{Z}} (X \bar{\otimes} Y_n)$$

by

$$F(x, \sum y_n) = F^x(\sum y_n) = x \otimes \sum y_n.$$

We see F is bilinear and we have shown it is continuous for a fixed x. It is also continuous in x, if we fix a finite sum $\sum y_n \in \bigoplus_{n \in \mathbb{Z}} Y_n$. As F is separately continuous, it induces a continuous linear map

$$\bar{F} : X\bar{\otimes}(\bigoplus_{n\in\mathbb{Z}} Y_n) \to \bigoplus_{n\in\mathbb{Z}}(X\bar{\otimes}Y_n)$$

that satisfies

$$\bar{F}(x \otimes y_n) = x \otimes y_n$$

for $x \in X$ and $y_n \in Y_n$.

Going the other direction, the map

$$G_n : X \times Y_n \to X\bar{\otimes}(\bigoplus_{n\in\mathbb{Z}} Y_n)$$

given by

$$G_n(x, y_n) = x \otimes y_n$$

is separately continuous, so it induces a continuous linear map

$$\bar{G}_n : X\bar{\otimes}Y_n \to X\bar{\otimes}(\bigoplus_{n\in\mathbb{Z}} Y_n).$$

By the universal property of direct sum, we get a map

$$\bar{G} : \bigoplus_{n\in\mathbb{Z}}(X\bar{\otimes}Y_n) \to X\bar{\otimes}(\bigoplus_{n\in\mathbb{Z}} Y_n)$$

that satisfies

$$\bar{G}(x \otimes y_n) = x \otimes y_n$$

for $x \in X$ and $y_n \in Y_n$. Thus, we see that $\bar{F}$ and $\bar{G}$ are inverse to each other on a dense subspace, and so they are isomorphisms by continuity. $\square$

2.1.3 Locally convex algebras and modules

The terms locally convex algebra and locally convex module do not have universally accepted definitions. Both structures are defined by certain bilinear maps, and we obtain different classes of objects depending on what type of continuity we insist for the bilinear map. The only types of continuity we shall consider are joint continuity and separate continuity, which are directly related to the projective tensor product and the inductive tensor product respectively. Here and in the future, we shall use the symbol $\check{\otimes}$ as a placeholder for either $\widehat{\otimes}$ or $\bar{\otimes}$.

2.1.3.1 Locally convex algebras

By a *locally convex* $\check{\otimes}$*-algebra*, we mean a space $A \in$ LCTVS equipped with an associative multiplication m that extends to continuous linear map

$$\check{m} : A\check{\otimes}A \to A.$$

So a locally convex $\widehat{\otimes}$-algebra has a jointly continuous multiplication, whereas the multiplication of a locally convex $\bar{\otimes}$-algebra is separately continuous. Joint continuity implies that for every defining seminorm p on A, there is another continuous seminorm q such that

$$p(ab) \leq q(a)q(b), \qquad \forall a, b \in A.$$

There may be no relationship between p and q in general. In the special case where

$$p(ab) \leq p(a)p(b), \qquad \forall a, b \in A,$$

for a family of seminorms defining the topology, we say that the algebra is *multiplicatively convex* or *m-convex*.

If A is Fréchet, then there is no distinction between the choice of topological tensor product, and we shall refer to A as a *Fréchet algebra*[1].

2.1.3.2 Locally convex modules

Now suppose R is a unital commutative locally convex $\check{\otimes}$-algebra. By a *locally convex* $\check{\otimes}$*-module over* R, we mean a (left) R-module $M \in$ LCTVS for which the module map $\mu : R \times M \to M$ extends to give a continuous linear map

$$\check{\mu} : R\check{\otimes}M \to M.$$

[1] We do not insist that our Fréchet algebras are m-convex, as some authors do.

All modules will be assumed to be unital in the sense that $1 \cdot m = m$ for all $m \in M$. A *locally convex* $\check{\otimes}$*-algebra over* R is an R-algebra A which is simultaneously a locally convex $\check{\otimes}$-algebra and a locally convex $\check{\otimes}$-module over R.

If R and M are both Fréchet, then we shall refer to M as a *Fréchet R-module.* Similarly, we have *Fréchet R-algebras.*

Given two locally convex $\check{\otimes}$-modules M and N over R, we shall write

$$\mathrm{Hom}_R(M, N)$$

for the space of all continuous R-linear maps from M to N. We topologize it as a subspace of $\mathrm{Hom}(M, N)$. Continuity of the module action implies that it is a closed subspace.

2.1.3.3 Topological tensor products of locally convex modules

We shall consider topological tensor products over an algebra different from $\mathbb{C}$. As with the scalar case, we shall consider projective and inductive tensor products. A more detailed exposition can be found in [16, Chapter II].

Suppose R is a unital commutative locally convex $\widehat{\otimes}$-algebra and M and N are locally convex $\widehat{\otimes}$-modules over R. The *(completed) projective tensor product over* R of M and N is a locally convex $\widehat{\otimes}$-module $M \widehat{\otimes}_R N$ over R equipped with a jointly continuous R-bilinear map $\iota : M \times N \to M \widehat{\otimes}_R N$ which is universal in the sense that any jointly continuous R-bilinear map B from $M \times N$ into another locally convex $\widehat{\otimes}$-module P over R induces a unique continuous R-linear map

$$\widehat{B} : M \widehat{\otimes}_R N \to P$$

making the diagram

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "B"'] & M \widehat{\otimes}_R N \arrow[d, "\widehat{B}"] \\ & p \end{tikzcd}$$

commute. Similarly, if R is a locally convex $\bar{\otimes}$-algebra, then we define the *(completed) inductive tensor product over* R to be a locally convex $\bar{\otimes}$-module $M \bar{\otimes}_R N$ over R which is universal with respect to separately continuous R-bilinear maps.

In both cases, the tensor product $M\check{\otimes}_R N$ exists and is unique up to isomorphism. One can explicitly construct $M\check{\otimes}_R N$ as the completion of the quotient $(M\check{\otimes}_{\mathbb{C}} N)/K$, where K is the closure of the subspace spanned by elements of the form

$$(r \cdot m) \otimes n - m \otimes (r \cdot n), \qquad r \in R, m \in M, n \in N.$$

Given an elementary tensor $m \otimes n \in M\check{\otimes}_{\mathbb{C}} N$, we shall denote its image in $M\check{\otimes}_R N$ again by $m \otimes n$. The R-module action is given by

$$r \cdot (m \otimes n) := (r \cdot m) \otimes n = m \otimes (r \cdot n).$$

In particular, it follows from the construction that the topological tensor product of Fréchet R-modules is a Fréchet R-module.

Both tensor products have the functorial property that two continuous R-linear maps $F : M_1 \to N_1$ and $G : M_2 \to N_2$ induce a continuous R-linear map

$$F \otimes G : M_1\check{\otimes}_R N_1 \to M_2\check{\otimes}_R N_2$$

in the usual way.

A locally convex $\check{\otimes}$-module over R is *free* if it is isomorphic to $R\check{\otimes}_{\mathbb{C}} X$ for some $X \in \mathrm{LCTVS}$. Here the R-module action is given by

$$r \cdot (s \otimes x) = rs \otimes x.$$

As in the algebraic case, the free module functor is a left-adjoint to the forgetful functor.

Proposition 2.1.6. *Given $X \in \mathrm{LCTVS}$ and a locally convex $\check{\otimes}$-module M over R, there is a linear isomorphism*

$$\mathrm{Hom}(X, M) \cong \mathrm{Hom}_R(R\check{\otimes} X, M).$$

Moreover, if R, X and M are Fréchet, and either R or X is nuclear[2]*, then this isomorphism is topological.*

[2]Nuclearity is a technical condition which we shall not describe. See [33, Chapter 50] for details.

Proof. Define

$$\Phi : \mathrm{Hom}(X, M) \to \mathrm{Hom}_R(R\check{\otimes}X, M), \qquad \Psi : \mathrm{Hom}_R(R\check{\otimes}X, M) \to \mathrm{Hom}(X, M)$$

by

$$\Phi(F) = \mu(1 \otimes F), \qquad \Psi(G)(x) = G(1 \otimes x),$$

where $\mu : R\check{\otimes}M \to M$ is the module action. It is straightforward to verify that both maps are well-defined and are inverses to each other. The map Ψ is continuous. Indeed, let p be a continuous seminorm on M and let $B \subset X$ be bounded. Then the set $1 \otimes B \subset R\check{\otimes}X$ is bounded and

$$p_B(\Psi(G)) = \sup_{x\in B} p(\Psi(G)(x)) = \sup_{x\in B} p(G(1 \otimes x)) \leq p_{1\otimes B}(G).$$

The continuity of Φ is apparently more subtle. Assuming all spaces are Fréchet, it suffices to consider the projective tensor product. The map Φ factors as

$$\Phi : \mathrm{Hom}(X, M) \xrightarrow{\Phi_1} \mathrm{Hom}(R\widehat{\otimes}X, R\widehat{\otimes}M) \xrightarrow{\Phi_2} \mathrm{Hom}(R\widehat{\otimes}X, M)\ ,$$

where $\Phi_1(F) = 1 \otimes F$ and Φ_2 is composition with the module action μ. Continuity of Φ_2 follows from continuity of μ. To show that Φ_1 is continuous, we need to understand the bounded subsets of $R\widehat{\otimes}X$ in terms of the bounded subsets of R and X. The is related to the difficult "problème des topologies" of Grothendieck [15]. If either R or X are nuclear, then for every bounded subset $D \subset R\widehat{\otimes}X$, there are bounded subsets $A \subset R$, $B \subset X$ such that D is contained in the closed convex hull of

$$A \otimes B = \{r \otimes x \mid r \in A, x \in B\},$$

see [17, Theorem 21.5.8]. Let p and q be continuous seminorms on R and X respectively. Then there is a constant M such that

$$p(r) \leq M, \qquad \forall r \in A.$$

For any finite convex combination

$$\theta = \sum \lambda_i (r_i \otimes x_i), \qquad r_i \in A, x_i \in B, \qquad 0 \le \lambda_i \le 1, \qquad \sum \lambda_i = 1,$$

we have

$$\begin{aligned}(p \otimes q)((1 \otimes F)\theta) &\le \sum \lambda_i p(r_i) q(F(x_i)) \\ &\le \sum \lambda_i M q_B(F) \\ &= M q_B(F).\end{aligned}$$

So $(p \otimes q)((1 \otimes F)\theta) \le M q_B(F)$ holds for all $\theta \in D$. Thus,

$$(p \otimes q)_D(\Phi_1(F)) \le M q_B(F),$$

which completes the proof. □

The next proposition shows that the topological tensor product of free modules is free.

Proposition 2.1.7. *Given two locally convex spaces X and Y,*

$$(R \check{\otimes}_{\mathbb{C}} X) \check{\otimes}_R (R \check{\otimes}_{\mathbb{C}} Y) \cong R \check{\otimes}_{\mathbb{C}} (X \check{\otimes}_{\mathbb{C}} Y)$$

as locally convex $\check{\otimes}$-modules over R via the correspondence

$$(r_1 \otimes x) \otimes (r_2 \otimes y) \leftrightarrow r_1 r_2 \otimes (x \otimes y).$$

Proof. Once constructs inverse isomorphisms using the universal property of the topological tensor products. □

2.1.4 Spaces of smooth functions

2.1.4.1 The space $C^\infty(J) \check{\otimes} X$

Let J be a nonempty open interval of real numbers and let $X \in$ LCTVS. Consider the space $C^\infty(J, X)$ of infinitely differentiable functions on J with values in X, see

Appendix A for the definition of differentiability. Given $x \in C^\infty(J, X)$, we shall denote by $x^{(n)}$ the n-th derivative of x, which again is an element of $C^\infty(J, X)$. We equip $C^\infty(J, X)$ with the topology of uniform convergence of functions and all their derivatives on compact subsets of J. In terms of seminorms, for every continuous seminorm p on X, every compact subset $K \subset J$, and ever nonnegative integer n, define the seminorm

$$p_{K,n}(x) = \sum_{m=0}^{n} \frac{1}{m!} \sup_{t \in K} p(x^{(m)}(t)).$$

The topology on $C^\infty(J, X)$ is the locally convex topology generated by all such seminorms. If X is a Fréchet space, then $C^\infty(J, X)$ is a Fréchet space. We shall write $C^\infty(J) = C^\infty(J, \mathbb{C})$, which is a nuclear Fréchet algebra that is m-convex with respect to this family of seminorms.

We shall be interested in free locally convex $\check{\otimes}$-modules $C^\infty(J)\check{\otimes}X$. As X is complete, we have

$$C^\infty(J, X) \cong C^\infty(J)\widehat{\otimes}X,$$

see e.g. [33, Theorem 44.1]. If X is not Fréchet, then we may have

$$C^\infty(J)\bar{\otimes}X \ncong C^\infty(J)\widehat{\otimes}X.$$

Using the canonical map

$$C^\infty(J)\bar{\otimes}X \to C^\infty(J)\widehat{\otimes}X \cong C^\infty(J, X),$$

we can identify $C^\infty(J)\bar{\otimes}X$ as a subspace of smooth functions, though not topologically, as the topology on $C^\infty(J)\bar{\otimes}X$ is stronger than the topology on $C^\infty(J, X)$.

Example 2.1.8. If $X = \bigoplus_{n\in\mathbb{Z}} X_n$ is a countable direct sum of Fréchet spaces, then

$$C^\infty(J)\bar{\otimes}X \cong \bigoplus_{n\in\mathbb{Z}} (C^\infty(J)\bar{\otimes}X_n) \cong \bigoplus_{n\in\mathbb{Z}} C^\infty(J, X_n),$$

which is strictly a subspace of $C^\infty(J, X) = C^\infty(J, \bigoplus_{n\in\mathbb{Z}} X_n)$. This example will be relevant in Chapter 5.

Let $\mathrm{ev}_t : C^\infty(J) \to \mathbb{C}$ be the continuous linear map that evaluates a function at $t \in J$. More generally, we have continuous linear evaluation maps

$$\mathrm{ev}_t \otimes 1 : C^\infty(J)\check{\otimes} X \to X$$

for each $t \in J$, which we shall also denote by ev_t.

2.1.4.2 Smooth families of linear maps

We shall be interested in classifying continuous $C^\infty(J)$-linear maps

$$C^\infty(J)\check{\otimes} X \to C^\infty(J)\check{\otimes} Y.$$

Definition 2.1.9. Given two locally convex spaces X and Y, a $\check{\otimes}$*-smooth family of continuous linear maps* from X to Y is a collection of continuous linear maps $\{F_t : X \to Y\}_{t\in J}$ which vary smoothly in the sense that the formula

$$F(x)(t) = F_t(x)$$

defines a continuous map

$$F : X \to C^\infty(J)\check{\otimes} Y.$$

Notice that a $\bar{\otimes}$-smooth family is automatically a $\widehat{\otimes}$-smooth family, as can be seen by composing with the canonical map $C^\infty(J)\bar{\otimes} Y \to C^\infty(J)\widehat{\otimes} Y$. Proposition 2.1.6 implies the following result.

Proposition 2.1.10. *Let $X, Y \in$ LCTVS. There is one-to-one correspondece between $\check{\otimes}$-smooth families of continuous linear maps from X to Y and continuous $C^\infty(J)$-linear maps from $C^\infty(J)\check{\otimes} X$ to $C^\infty(J)\check{\otimes} Y$.*

For nicer spaces, we can give equivalent conditions for $\check{\otimes}$-smooth families of maps which are easier to check in practice.

Given a $\widehat{\otimes}$-smooth family $\{F_t : X \to Y\}_{t\in J}$ of continuous linear maps, it is necessary that the map

$$t \mapsto F_t(x)$$

is smooth for each $x \in X$. If X is barreled, e.g. if X is Fréchet, this is also sufficient.

Proposition 2.1.11. *(i) If $X \in \mathrm{LCTVS}$ is barreled, then $\{F_t : X \to Y\}_{t\in J}$ is a $\widehat{\otimes}$-smooth family of continuous linear maps if and only if the map*

$$t \mapsto F_t(x)$$

is smooth for every $x \in X$.

(ii) If $X = \bigoplus_{n\in\mathbb{Z}} X_n$ and $Y = \bigoplus_{n\in\mathbb{Z}} Y_n$, where each X_n, Y_n is Fréchet, then $\{F_t : X \to Y\}_{t\in J}$ is a $\bar{\otimes}$-smooth family of continuous linear maps if and only if the map

$$t \mapsto F_t(x)$$

is smooth for every $x \in X$, and for every $k \in \mathbb{Z}$, there a number N_k such that

$$F_t(X_k) \subset \bigoplus_{n=-N_k}^{N_k} Y_n, \qquad \forall t \in J.$$

Proof. By definition, if $\{F_t : X \to Y\}_{t\in J}$ is a $\widehat{\otimes}$-smooth family of maps, then

$$t \mapsto F_t(x)$$

is smooth for every $x \in X$. For the converse, we must show that the map

$$F : X \to C^\infty(J, Y), \qquad F(x)(t) = F_t(x),$$

which is well-defined by hypothesis, is continuous. For any compact $K \subset J$, any natural number n, and any $x \in X$, the set

$$\{F_t^{(n)}(x) | t \in K\}$$

is compact in Y, hence bounded. By the Banach-Steinhaus theorem, the set of maps $\{F_t^{(n)}\}_{t\in K}$ is equicontinuous. Thus for any continuous seminorm q on Y, there exists a continuous seminorm p on X such that

$$q(F_t^{(n)}(x)) \leq p(x), \qquad \forall t \in K.$$

Consequently,

$$\sup_{t\in K} q(F_t^{(n)}(x)) \leq p(x),$$

which shows F is continuous.

For part (ii), a $\bar{\otimes}$-smooth family $\{F_t : X \to Y\}_{t\in J}$ is a $\widehat{\otimes}$-smooth family, so $t \mapsto F_t(x)$ is smooth for all $x \in X$. Notice that any bounded subset $A \subset \bigoplus_{n\in\mathbb{Z}} Y_n$ must be contained in a finite subdirect sum. A $\bar{\otimes}$-smooth family $\{F_t\}$ assembles to give a continuous linear map

$$F : X \to C^\infty(J)\bar{\otimes}Y \cong \bigoplus_{n\in\mathbb{Z}} C^\infty(J)\bar{\otimes}Y_n \cong \bigoplus_{n\in\mathbb{Z}} C^\infty(J, Y_n).$$

Let $B_k \subset X_k$ denote the unit ball. Then $F(B_k)$ is bounded in $\bigoplus_{n\in\mathbb{Z}} C^\infty(J, Y_n)$, hence contained in a finite subdirect sum. Thus, the same is true for $F(X_k)$ because B_k is absorbing[3] in X_k.

Conversely, suppose

$$F_t : X_k \to \bigoplus_{n=-N_k}^{N_k} Y_n$$

is such that $t \mapsto F_t(x)$ is smooth for all $x \in X_k$. Then by part (i) these give a continuous linear map

$$F^k : X_k \to C^\infty(J)\widehat{\otimes}(\bigoplus_{n=-N_k}^{N_k} Y_n) \cong C^\infty(J)\bar{\otimes}(\bigoplus_{n=-N_k}^{N_k} Y_n),$$

where the isomorphism is because $\bigoplus_{n=-N_k}^{N_k} Y_n$ is Fréchet. Composing with the inclusion gives

$$F^k : X_k \to C^\infty(J)\bar{\otimes}Y.$$

By the universal property of the direct sum, there is a continuous linear map

$$F : X \to C^\infty(J)\bar{\otimes}Y$$

as desired. □

[3] Recall that a subset $A \subset X$ is *absorbing* if every element of X is a scalar multiple of some element of A.

2.2 Connections and parallel transport

Since we will only work with one-parameter deformations, we shall only treat connections on $C^\infty(J)$-modules where the interval J represents the parameter space. As there is only one direction to differentiate in, a connection is determined by its covariant derivative. In what follows, we shall identify the two notions, and will commonly refer to covariant differential operators as connections.

Let M be a locally convex $\check{\otimes}$-module over $C^\infty(J)$. A *connection* on M is a continuous $\mathbb{C}$-linear map $\nabla : M \to M$ such that

$$\nabla(f \cdot m) = f \cdot \nabla m + f' \cdot m \qquad \forall f \in C^\infty(J), m \in M.$$

It is immediate from this Leibniz rule that the difference of two connections is a continuous $C^\infty(J)$-linear map. Further, given any connection ∇ and continuous $C^\infty(J)$-linear map $F : M \to M$, the operator $\nabla - F$ is also a connection. So if the space of connections is nonempty, then it is an affine space parametrized by the space $\mathrm{End}_{C^\infty(J)}(M)$ of continuous $C^\infty(J)$-linear endomorphisms.

Example 2.2.1. The operator $\frac{d}{dt}(= \frac{d}{dt} \otimes 1)$ is a connection on the free module $C^\infty(J)\check{\otimes}X$. Thus any connection ∇ on $C^\infty(J)\check{\otimes}X$ is of the form

$$\nabla = \frac{d}{dt} - F$$

for some continuous $C^\infty(J)$-linear map F, which can be interpreted as a $\check{\otimes}$-smooth family of continuous linear maps $\{F_t : X \to X\}_{t\in J}$ as in Proposition 2.1.10.

An element in the kernel of a connection ∇ will be called a *parallel section* for ∇. Suppose M and N are two locally convex $\check{\otimes}$-modules over $C^\infty(J)$ with connections ∇_M and ∇_N respectively. A *parallel map* is a continuous $C^\infty(J)$-linear map $F : M \to N$ such that $F \circ \nabla_M = \nabla_N \circ F$. In particular, a parallel map sends parallel sections to parallel sections.

Proposition 2.2.2. *Given locally convex $\check{\otimes}$-modules M and N over $C^\infty(J)$ with connections ∇_M and ∇_N,*

(i) the operator $\nabla_{M\check{\otimes}N} : \nabla_M \otimes 1 + 1 \otimes \nabla_N$ is a connection on $M\check{\otimes}_{C^\infty(J)}N$.

(ii) the operator ∇_{M^*} *on* $M^* = \mathrm{Hom}_{C^\infty(J)}(M, C^\infty(J))$ *given by*

$$(\nabla_{M^*}\varphi)(m) = \frac{d}{dt}\varphi(m) - \varphi(\nabla_M m)$$

is a connection.

The definition of ∇_{M^*} ensures that the canonical pairing

$$\langle \cdot, \cdot \rangle : M^* \check{\otimes}_{C^\infty(J)} M \to C^\infty(J)$$

is a parallel map, where we consider $C^\infty(J)$ with the connection $\frac{d}{dt}$. This is just another way of saying that

$$\frac{d}{dt}\langle \varphi, m \rangle = \langle \nabla_{M^*}\varphi, m \rangle + \langle \varphi, \nabla_M m \rangle.$$

2.2.1 Parallel transport in free modules

We will be interested in identifying when we can perform parallel transport along a connection. Suppose M is a locally convex $\check{\otimes}$-module over $C^\infty(J)$ of the form

$$M = C^\infty(J)\check{\otimes}X$$

for some $X \in \mathrm{LCTVS}$. We will think of M as sections of bundle whose fiber over $t \in J$ is $M_t \cong X$. Although all the fibers are the same topological vector space, we introduce the notation M_t because we will eventually consider examples in which each M_t will contain additional structure that will depend on t.

To perform parallel transport along a connection $\nabla = \frac{d}{dt} - F$, one needs a unique parallel section through each element of each fiber. That is, for each $s \in J$ and each $x \in M_s$, we need a unique $m \in M$ satisfying the differential equation

$$\nabla m = 0, \qquad m(s) = x.$$

In this case, we can define a parallel transport operator

$$P^{\nabla}_{s,t} : M_s \to M_t$$

between any two fibers by

$$P^{\nabla}_{s,t}(x) = m(t),$$

where m is the unique parallel sections through $x \in M_s$. These operators are linear isomorphisms, as the inverse of $P^{\nabla}_{s,t}$ is $P^{\nabla}_{t,s}$. We would like them to be topological isomorphisms that vary smoothly in t and s. This motivates the following definition.

Definition 2.2.3. We shall say that a connection ∇ on a free module $M \cong C^\infty(J)\check{\otimes}X$ is *integrable* if

(i) for every $s \in J$ and every $x \in M_s$, there exists a unique $m \in M$ such that

$$\nabla m = 0, \qquad m(s) = x,$$

(ii) the parallel transport operators $\{P^{\nabla}_{s,t} : M_s \to M_t\}_{s,t\in J}$ are continuous and vary smoothly in the sense that the map $P^{\nabla} : X \to C^\infty(J \times J)\check{\otimes}X$ given by

$$P^{\nabla}(x)(s,t) = P^{\nabla}_{s,t}(x)$$

is well-defined and continuous.

Notice that the connection $\frac{d}{dt}$ on the free module $C^\infty(J)\check{\otimes}X$ is always integrable. In fact, every integrable connection can be trivialized to look like $\frac{d}{dt}1$.

Proposition 2.2.4. *If ∇ is an integrable connection on a free module M, then for each $s \in J$ there is a parallel isomorphism*

$$(M, \nabla) \cong (C^\infty(J)\check{\otimes}M_s, \frac{d}{dt} \otimes 1).$$

Proof. As above, let X denote the generic fiber of M, so that $M \cong C^\infty(J)\check{\otimes}X$ as a $C^\infty(J)$-module. Let $P^{\nabla}_{s,t} : M_s \to M_t$ denote the parallel transport maps induced by ∇. By definition of integrability, the collections $\{P^{\nabla}_{s,t} : M_s \to X\}_{t\in J}$ and $\{P^{\nabla}_{t,s} : X \to M_s\}_{t\in J}$ are $\check{\otimes}$-smooth families of continuous linear maps, and induce mutually inverse continuous $C^\infty(J)$-linear maps

$$F : C^\infty(J)\check{\otimes}M_s \to M, \qquad F^{-1} : M \to C^\infty(J)\check{\otimes}M_s$$

by Proposition 2.1.10. To verify that F is parallel, it suffices to check

$$F \circ \frac{d}{dt} = \nabla \circ F$$

on elements of the form $1 \otimes m_s \in C^\infty(J)\check{\otimes} M_s$ by $C^\infty(J)$-linearity and continuity. But this follows immediately by definition of parallel transport. That F^{-1} is parallel is automatic because F is parallel. $\square$

Now suppose X and Y are locally convex spaces and $M = C^\infty(J, X)$ and $N = C^\infty(J, Y)$. Suppose $F : M \to N$ is a continuous $C^\infty(J)$-linear map and $\{F_t : M_t \to N_t\}_{t \in J}$ is the smooth family of continuous linear maps associated to F as in Proposition 2.1.10.

Proposition 2.2.5. *Let $F : (M, \nabla_M) \to (N, \nabla_N)$ be a parallel map between free modules M and N.*

(i) If ∇_M and ∇_N are integrable, then the diagram

$$\begin{tikzcd} M_s \arrow[r, "F_s"] \arrow[d, "P_{s,t}^{\nabla_M}"'] & N_s \arrow[d, "P_{s,t}^{\nabla_N}"] \\ M_t \arrow[r, "F_t"'] & N_t \end{tikzcd}$$

commutes for all $s, t \in J$.

(ii) If ∇_M is integrable and F is a parallel isomorphism, then ∇_N is integrable.

Proof. Let X and Y denote the generic fiber of M and N respectively. Suppose ∇_M and ∇_N are integrable. Given $x \in M_s$, let $m \in M$ be the unique parallel section through x. Then $F(m)$ is the unique parallel section through $F(m)(s) = F_s(x)$. Consequently,

$$P_{s,t}^{\nabla_N}(F_s(x)) = F(m)(t) = F_t(m(t)) = F_t(P_{s,t}^{\nabla_M}(x)).$$

If F is an invertible parallel map, then ∇_N-parallel sections through $y \in N_s$ are in bijection with ∇_M-parallel sections through $F_s^{-1}(y)$. Thus, integrability of

∇_M implies the existence and uniqueness of solutions to

$$\nabla_N n = 0, \qquad n(s) = y.$$

As shown in the first part of the proof,

$$P^{\nabla_N}_{s,t} = F_t P^{\nabla_M}_{s,t} F_s^{-1}.$$

So we must show that

$$P^{\nabla_N} : Y \to C^\infty(J \times J)\breve{\otimes} Y, \qquad P^{\nabla_N}(y)(s,t) = F_t(P^{\nabla_M}_{s,t}(F_s^{-1}(y)))$$

is well-defined and continuous. But P^{∇_N} is the composition of continuous maps

$$Y \xrightarrow{\iota} N \xrightarrow{F^{-1}} M \xrightarrow{Q^{\nabla_M}} C^\infty(J \times J)\breve{\otimes} X \xrightarrow{\tilde{F}} C^\infty(J \times J)\breve{\otimes} Y,$$

where ι is the inclusion as constant functions, Q^{∇_M} is the $C^\infty(J)$-linear map induced by

$$P^{\nabla_M} : X \to C^\infty(J \times J)\breve{\otimes} X,$$

and $\tilde{F}$ is defined by the commuting square

$$\begin{array}{ccc} C^\infty(J \times J)\breve{\otimes} X & \xrightarrow{\quad\tilde{F}\quad} & C^\infty(J \times J)\breve{\otimes} Y \\ \cong\big\downarrow & & \big\downarrow\cong \\ C^\infty(J)\breve{\otimes}(C^\infty(J)\breve{\otimes} X) & \xrightarrow{1\otimes F} & C^\infty(J)\breve{\otimes}(C^\infty(J)\breve{\otimes} Y). \end{array}$$

□

Proposition 2.2.6. *Suppose ∇_M and ∇_N are integrable connections on $M \cong C^\infty(J)\breve{\otimes} X$ and $N \cong C^\infty(J)\breve{\otimes} Y$.*

(i) The tensor product connection $\nabla_{M\breve{\otimes} N}$ is integrable on $M\breve{\otimes}_{C^\infty(J)} N$.

(ii) If X is a nuclear Fréchet space, then the dual connection ∇_{M^*} is integrable on $M^* = \mathrm{Hom}_{C^\infty(J)}(M, C^\infty(J))$.

Proof. Using Proposition 2.2.4, it suffices to consider the case where ∇_M and ∇_N

are both given by $\frac{d}{dt} \otimes 1$. In this case, we observe that the isomorphism

$$M \check{\otimes}_{C^\infty(J)} N \cong C^\infty(J) \check{\otimes} (X \check{\otimes} Y)$$

is parallel with respect to the connections $\nabla_{M \check{\otimes} N}$ and $\frac{d}{dt} \otimes 1$, the latter of which is integrable. Thus, $\nabla_{M \check{\otimes} N}$ is integrable.

If X is nuclear and Fréchet, then

$$M^* \cong \mathrm{Hom}(X, C^\infty(J)) \cong C^\infty(J) \widehat{\otimes} X^* \cong C^\infty(J, X^*),$$

where $X^* = \mathrm{Hom}(X, \mathbb{C})$ is the strong dual of X. The first isomorphism is from Proposition 2.1.6, whereas the second can be found in [33, Proposition 50.5]. The result follows because this isomorphism is parallel with respect to ∇_{M^*} on M^* and $\frac{d}{dt}$ on $C^\infty(J, X^*)$, which is integrable. $\square$

2.2.2 Integrability for Banach spaces

We now consider the special case of $M = C^\infty(J) \check{\otimes} X$ where X is a Banach space. Here, both tensor products coincide, so that $M \cong C^\infty(J, X)$. In this case, any connection ∇ on M is integrable. Recall by Example 2.2.1 that $\nabla = \frac{d}{dt} - F$ for some continuous $C^\infty(J)$-linear map $F : C^\infty(J, X) \to C^\infty(J, X)$. By Proposition 2.1.10, F is given by a smooth family $\{F_t\}_{t \in J}$ of continuous linear maps on X. Thus, the initial value problem we wish to solve is

$$x'(t) = F_t(x), \qquad x(t_0) = x_0.$$

The following well-known theorem from differential equations says that there are always unique solutions.

Theorem 2.2.7 (Existence and uniqueness for linear ODE's)**.** *If X is a Banach space and $\{F_t\}_{t \in J}$ is a smooth family of continuous linear maps on X, then there is a unique global solution $x \in C^\infty(J, X)$ to the initial value problem*

$$x'(t) = F_t(x(t)), \qquad x(t_0) = x_0,$$

given by the absolutely convergent series

$$x(t) = x_0 + \sum_{n=1}^{\infty} \int_{t_0}^{t} \int_{t_0}^{s_1} \dots \int_{t_0}^{s_{n-1}} (F_{s_1} \circ \dots \circ F_{s_n})(x_0) ds_n \dots ds_1.$$

Moreover, using the explicit form of the solution x, one can show that ∇ satisfies the definition of integrability. Thus we obtain trivializations as in Proposition 2.2.4. In particular, the parallel transport map

$$P_{s,t}^{\nabla} : M_s \to M_t$$

exists for all $s, t \in J$ and is an isomorphism of topological vector spaces.

A particular notable special case is when the differential equation has "constant coefficients," that is, F_t is independent of t. The initial value problem is then

$$x'(t) = F_0(x(t)), \qquad x(t_0) = x_0$$

for some continuous linear map $F_0 : X \to X$. In this case, the explicit solution of Theorem 2.2.7 becomes

$$x(t) = \sum_{n=0}^{\infty} \frac{(t - t_0)^n}{n!} F_0^n(x_0) = \exp((t - t_0)F)(x_0),$$

where the exponential is an absolutely convergent series in the Banach algebra of continuous linear operators on X.

Remark 2.2.8. Once we start considering other classes of locally convex vector spaces, the above existence and uniqueness theorem for linear ODE's becomes false. The issue is that the infinite sum in the formula for the solution need not converge. Indeed, even in the case where X is a nuclear Fréchet space and $F_0 : X \to X$ is a continuous linear map, the infinite series $\exp(F_0)$ need not converge in $\mathrm{Hom}(X, X)$. Thus, it is not true in the Fréchet case that every connection is integrable.

2.2.3 Nilpotent perturbations of integrable connections

Suppose ∇ is an integrable connection on the free module $M \cong C^\infty(J)\check{\otimes}X$. Any other connection ∇' on M is of the form

$$\nabla' = \nabla - F$$

for some $C^\infty(J)$-linear endormorphism F of M.

Definition 2.2.9. We shall say ∇' is a *nilpotent perturbation* of the integrable connection ∇ if there is a positive integer k such that

$$F_{t_k} P^{\nabla}_{t_{k-1},t_k} F_{t_{k-1}} \dots P^{\nabla}_{t_2,t_3} F_{t_2} P^{\nabla}_{t_1,t_2} F_{t_1} = 0, \qquad \forall t_1, t_2, \dots, t_k \in J,$$

where $\nabla' = \nabla - F$.

As an example, $\nabla = \frac{d}{dt} - F$ is a nilpotent perturbation of $\frac{d}{dt}$ if and only if

$$F_{t_k} F_{t_{k-1}} \dots F_{t_2} F_{t_1} = 0, \qquad \forall t_1, t_2, \dots, t_k \in J.$$

Proposition 2.2.10. *A nilpotent perturbation $\nabla - F$ of an integrable connection ∇ is integrable.*

Proof. By Proposition 2.2.4, we may assume $\nabla = \frac{d}{dt}$. Indeed, there is a parallel isomorphism

$$G : (C^\infty(J)\check{\otimes}X, \frac{d}{dt}) \to (M, \nabla).$$

The same map gives a parallel isomorphism

$$G : (C^\infty(J)\check{\otimes}X, \frac{d}{dt} - G^{-1}FG) \to (M, \nabla - F).$$

Using the equality

$$G_t G_s^{-1} = P^{\nabla}_{s,t}$$

from Proposition 2.2.5, we see that $\frac{d}{dt} - G^{-1}FG$ is a nilpotent perturbation of $\frac{d}{dt}$.

So we need only consider a connection of the form $\nabla = \frac{d}{dt} - F$ on $M =$

$C^\infty(J)\check{\otimes}X$ where

$$F_{t_k}F_{t_{k-1}}\dots F_{t_2}F_{t_1}=0, \qquad \forall t_1,t_2,\dots,t_k\in J.$$

Given $x_0\in X$ and $s\in J$, a solution $x\in C^\infty(J)\check{\otimes}X$ to

$$\frac{dx}{dt}=F(x), \qquad x(s)=x_0$$

is the same as a fixed point of the function

$$\Phi : C^\infty(J)\check{\otimes}X \to C^\infty(J)\check{\otimes}X$$

given by

$$\Phi(x)(t)=x_0+\int_s^t F_s(x(s))ds.$$

Using the nilpotence assumption of F, we have that for any $y\in C^\infty(J)\check{\otimes}X$,

$$\Phi^k(y)(t)=\sum_{n=0}^{k-1}\int_s^t\int_s^{u_1}\dots\int_s^{u_{n-1}}(F_{u_1}\circ\dots\circ F_{u_n})(x_0)du_n\dots du_1,$$

which is independent of y. It follows that $\Phi^k(y)$ is the unique fixed point of Φ. By examing the dependence of the explicit solution on x_0 and s, one sees that the additional topological conditions are satisfied so that $\frac{d}{dt}-F$ is integrable. □

Example 2.2.11. Consider the special case of a perturbation $\nabla'=\nabla-F$ of an integrable connection ∇ where $[\nabla,F]=0$, so that F is ∇-parallel. Here, ∇' is a nilpotent perturbation if and only if $F^k=0$ for some k. One can be quite explicit about how to construct ∇'-parallel sections in terms of ∇-parallel sections. Indeed, if m is the unique ∇-parallel section through $x\in M_s$, then

$$n(t)=\exp((t-s)F)(m)(t)=\sum_{r=0}^{k-1}\frac{(t-s)^r}{r!}F_t^r(m(t))$$

is the unique ∇'-parallel section through $x\in M_s$.

2.3 Algebraic objects in LCTVS

Fix a commutative, unital locally convex $\check{\otimes}$-algebra R as the ground ring. The two cases we will consider are $R = \mathbb{C}$ and $R = C^\infty(J)$ for an open subinterval $J \subset \mathbb{R}$. In dealing with graded objects, we shall tend to stick with the inductive tensor product, as it behaves well with respect to direct sums.

2.3.1 Graded locally convex modules

We shall say that a locally convex $\check{\otimes}$-module X over R is a *graded locally convex module* if

$$X = \bigoplus_{n\in\mathbb{Z}} X_n$$

for some sequence $\{X_n\}$ of closed submodules. This is to be intepreted topologically, so that the topology on X coincides with the direct sum topology. An element $x \in X_n$ is called *homogeneous of degree n*. We say X is *trivially graded* if $X = X_0$. The ground ring R is considered as a graded module with the trivial grading. An R-linear map $F : X \to Y$ between graded modules is called *homogeneous of degree p* if $F(X_n) \subset Y_{n+p}$. Every R-linear map is a direct product of its homogeneous components. We shall write $|x|$ or $|F|$ to indicate the degree of a homogeneous element or map. Unless indicated otherwise, an isomorphism of graded locally convex modules shall mean a degree 0 isomorphism of locally convex modules. Given two homogeneous R-linear maps F and G on a graded module, the commutator shall always mean graded commutator, so that

$$[F, G] = FG - (-1)^{|F||G|} GF.$$

The inductive tensor product over R of two graded locally convex modules is naturally graded, with

$$(X \bar{\otimes}_R Y)_n = \bigoplus_{m\in\mathbb{Z}} (X_m \bar{\otimes}_R Y_{n-m}).$$

Indeed, by Proposition 2.1.5,

$$X \bar{\otimes}_R Y \cong \bigoplus_{m,n\in\mathbb{Z}} X_m \bar{\otimes}_R Y_n \cong \bigoplus_{n\in\mathbb{Z}} \left(\bigoplus_{m\in\mathbb{Z}} X_m \bar{\otimes}_R Y_{n-m} \right).$$

Given two R-linear maps

$$F : X \to X', \qquad G : Y \to Y',$$

their tensor product is the map

$$F \otimes G : X \check{\otimes}_R Y \to X' \check{\otimes}_R Y'$$

defined for homogeneous components, on homogeneous elements, by

$$(F \otimes G)(x \otimes y) = (-1)^{|G||x|} F(x) \otimes G(y).$$

To indicate that we are dealing with a graded module, we shall often write $X_\bullet$ (or $X^\bullet$) instead of just X. Given two graded locally convex modules $X_\bullet$ and $Y_\bullet$, the notation $\mathrm{Hom}(X_\bullet, Y_\bullet)$ shall mean the module algebraically spanned by the homogeneous continuous R-linear maps. In general, this is a submodule of all continuous R-linear maps from X to Y. So $\mathrm{Hom}(X_\bullet, Y_\bullet)$ is also a graded module, with the grading given by the degree of homogeneity of maps.

2.3.2 Locally convex chain complexes

A *locally convex cochain complex* is a graded locally convex $\check{\otimes}$-module $C^\bullet$ over R equipped with a degree $+1$ continuous R-linear map

$$d : C^\bullet \to C^{\bullet+1}$$

such that $d^2 = 0$. Such a map decomposes as a sequence of maps

$$d^n : C^n \to C^{n+1},$$

and we define the n-th cohomology module to be

$$H^n(C) = \ker d^n / \operatorname{im} d^{n-1}.$$

Elements of $\ker d$ are called *cocycles* and elements of $\operatorname{im} d$ are called *coboundaries*. The cohomology $H^\bullet(C)$ is a graded R-module, but it may not even be Hausdorff, as we do not know if $\operatorname{im} d$ is closed. Similarly we can define locally convex chain complexes and homology modules by changing d to have degree -1. If we reverse the grading of a chain complex, that is declare the degree n elements to have degree $-n$, then we obtain a cochain complex. We shall use the notation $C_{-\bullet}$ to indicate that we have reversed the grading of $C_\bullet$.

A degree 0 continuous R-linear map $F : C^\bullet \to D^\bullet$ between cochain complexes is a *chain map* if

$$d_D F = F d_C.$$

Two chain maps $F, G : C^\bullet \to D^\bullet$ are *continuously chain homotopic* if there is a continuous R-linear map $h : C^\bullet \to D^{\bullet-1}$ such that

$$F - G = d_D h + h d_C.$$

A chain map $F : C^\bullet \to D^\bullet$ induces a map of cohomology modules

$$F_* : H^\bullet(C) \to H^\bullet(D),$$

and chain homotopic maps induce the same map on cohomology. Two complexes are *chain homotopy equivalent* if there exist chain maps

$$F : C^\bullet \to D^\bullet, \qquad G : D^\bullet \to C^\bullet$$

such that GF is continuously chain homotopic to id_C, and FG is continuously chain homotopic to id_D. In this case, the maps F and G are called *chain homotopy equivalences*. Chain homotopy equivalences induce isomorphisms on cohomology. A complex is *contractible* if it is chain homotopy equivalent to the zero complex. In particular, the cohomology modules of a contractible complex are zero.

Given two cochain complexes $C^\bullet$ and $D^\bullet$, the graded space $\mathrm{Hom}(C^\bullet, D^\bullet)$ is naturally a cochain complex with coboundary map

$$d(F) = d_D F - (-1)^{|F|} F d_C.$$

So the 0-cocycles in $\mathrm{Hom}(C^\bullet, D^\bullet)$ are exactly the chain maps. A chain map represents the zero class in cohomology if and only if it is continuously chain homotopic to the zero map.

The tensor product $C^\bullet \bar{\otimes}_R D^\bullet$ of two cochain complexes is a cochain complex with coboundary $d_C \otimes 1 + 1 \otimes d_D$. We remark that the sign convention for the tensor product of maps is in effect here:

$$(d_C \otimes 1 + 1 \otimes d_D)(x \otimes y) = d_C(x) \otimes y + (-1)^{|x|} x \otimes d_D(y).$$

2.3.3 Locally convex differential graded algebras

A *locally convex differential graded $\bar{\otimes}$-algebra* over R is a cochain complex $(\Omega^\bullet, d)$ which is also a locally convex $\bar{\otimes}$-algebra over R such that the product

$$m : \Omega^\bullet \bar{\otimes}_R \Omega^\bullet \to \Omega^\bullet$$

is a degree 0 chain map. In other words,

$$\Omega^p \cdot \Omega^q \subset \Omega^{p+q},$$

and d is a degree $+1$ derivation,

$$d(\omega_1 \omega_2) = d(\omega_1)\omega_2 + (-1)^{|\omega_1|} \omega_1 d(\omega_2).$$

The cohomology $H^\bullet(\Omega)$ is a graded algebra.

A *map of differential graded algebras* is a chain map that is also an algebra homomorphism. Such a map induces an algebra map at the level of cohomology.

2.3.4 Locally convex differential graded Lie algebras

A *locally convex graded Lie $\bar{\otimes}$-algebra* over R is a locally convex $\bar{\otimes}$-module $\mathfrak{g}^\bullet$ over R equipped with a degree 0 continuous R-bilinear map

$$[\cdot,\cdot] : \mathfrak{g}^\bullet \bar{\otimes}_R \mathfrak{g}^\bullet \to \mathfrak{g}^\bullet$$

satisfying

(i) (Graded skew-symmetry) $[X,Y] = -(-1)^{|X||Y|}[Y,X]$,

(ii) (Graded Jacobi identity) $[X,[Y,Z]] = [[X,Y],Z] + (-1)^{|X||Y|}[Y,[X,Z]]$

for all homogeneous $X, Y, Z \in \mathfrak{g}$.

Example 2.3.1. If $X^\bullet$ is a graded locally convex $\check{\otimes}$-module over R, then $\mathrm{End}(X^\bullet)$ is a locally convex graded $\bar{\otimes}$-Lie algebra under the graded commutator.

A locally convex graded Lie $\bar{\otimes}$-algebra $\mathfrak{g}^\bullet$ is a *locally convex differential graded Lie $\bar{\otimes}$-algebra* if $\mathfrak{g}^\bullet$ is a cochain complex with a coboundary map δ that satisfies

$$\delta[X,Y] = [\delta X, Y] + (-1)^{|X|}[X, \delta Y]$$

for all homogeneous $X, Y, Z \in \mathfrak{g}$.

Example 2.3.2. If $\mathfrak{g}^\bullet$ is a locally convex graded Lie $\bar{\otimes}$-algebra and $m \in \mathfrak{g}^1$ is an element such that[4] $[m,m] = 0$, then the formula

$$\delta(X) = [m, X]$$

defines a coboundary map that makes $\mathfrak{g}^\bullet$ into a locally convex differential graded Lie $\bar{\otimes}$-algebra. That $\delta^2 = 0$ and the Leibniz rule for δ follow from the graded Jacobi identity.

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[4] Note that this is not automatic in a *graded* Lie algebra.

2.3.5 Locally convex differential graded coalgebras

A *locally convex* $\bar{\otimes}$*-coalgebra* over R is a locally convex $\bar{\otimes}$-module C over R equipped with continuous R-linear maps

$$\Delta : C \to C\bar{\otimes}_R C, \qquad \epsilon : C \to R$$

satisfying

$$(\Delta \otimes 1)\Delta = (1 \otimes \Delta)\Delta, \qquad (\epsilon \otimes 1)\Delta = 1 = (1 \otimes \epsilon)\Delta.$$

The maps Δ and ϵ are called the *coproduct* and *counit* respectively. If $C^\bullet$ is graded and Δ and ϵ are degree 0 maps, then $C^\bullet$ is a *locally convex graded* $\bar{\otimes}$*-coalgebra*. A *coalgebra homomorphism* is a degree 0 continuous R-linear map

$$F : C^\bullet \to D^\bullet$$

such that

$$\Delta_D F = (F \otimes F)\Delta_C, \qquad \epsilon_D F = \epsilon_C.$$

A *coaugmented coalgebra* is equipped with a distinguished coalgebra homomorphism $\eta : R \to C^\bullet$ such that $\epsilon\eta = 1$. Here, R is a trivially graded locally convex $\bar{\otimes}$-coalgebra over R with the structure maps determined by

$$\Delta(1) = 1 \otimes 1, \qquad \epsilon(1) = 1.$$

Given a coagumentation, we can form the linear quotient space $C^\bullet/R$. A map $F : C^\bullet \to D^\bullet$ of coaugmented graded coalgebras is required to additionally satisfy

$$F\eta_C = \eta_D.$$

All coaugmented coalgebras we encounter will have the property of *(topological) cocompleteness*, that is, $C = \varinjlim N_n$ with the inductive limit topology, where N_n is the kernel of the map

$$C \xrightarrow{\Delta^{n-1}} C^{\bar{\otimes}_R n} \longrightarrow (C/R)^{\bar{\otimes}_R n}.$$

That is, $C = \bigcup_{n=2}^{\infty} N_n$ and the topology on C is the strongest locally convex topology induced by the inclusions $N_n \to C$. Here,

$$\Delta^{n-1} = (\Delta \otimes 1^{\otimes(n-2)}) \ldots (\Delta \otimes 1)\Delta$$

is the iterated coproduct.

A *(graded) coderivation* on $C^\bullet$ is a cotinuous R-linear map $D : C^\bullet \to C^\bullet$ such that

$$\Delta D = (D \otimes 1 + 1 \otimes D)\Delta.$$

In general, we do not insist that a coderivation is homogeneous, but we shall insist they are finite sums of homogeneous components. The linear space of all graded coderivations on $C^\bullet$ shall be denoted $\mathrm{Coder}(C^\bullet)$.

Proposition 2.3.3. *The graded commutator of two graded coderivations is a graded coderivations. Thus* $\mathrm{Coder}(C^\bullet)$ *is a graded Lie R-subalgebra of* $\mathrm{End}_R(C^\bullet)$*.*

A locally convex graded $\bar{\otimes}$-coaglebra equipped with a degree $+1$ coderivation d such that $d^2 = 0$ is a *locally convex differential graded $\bar{\otimes}$-coalgebra*. Maps between differential graded coalgebras are additionally required to commute with the boundary maps.

2.4 Hochschild and cyclic homology

A good reference for Hochschild and cyclic homology is [21].

Let R be a unital locally convex $\check{\otimes}$-algebra and let A be a (possibly nonunital) locally convex $\check{\otimes}$-algebra over R. In this section, all $\check{\otimes}$-tensor products in this section are over the ground ring R, and everything in sight is an R-module. All homology theories that follow are the continuous versions of the usual algebraic theories, in that they take into account the topology of the algebra A. They also depend on the choice of topological tensor product $\check{\otimes}$. However, if we consider algebras equipped with their finest locally convex topologies as locally convex $\bar{\otimes}$-algebras, then we recover the usual algebraic theory.

Recall that the *unitization* of the algebra A is the algebra

$$\widetilde{A} = A \oplus R$$

with multiplication

$$(a_1, r_1)(a_2, r_2) = (a_1 a_2 + r_2 \cdot a_1 + r_1 \cdot a_2, r_1 r_2).$$

Then $\widetilde{A}$ is a unital locally convex $\breve{\otimes}$-algebra over R with unit $(0, 1)$, which contains A as a closed ideal. We can, and will, form the unitization in the case where A is already unital. We shall let $e \in \widetilde{A}$ denote the unit of $\widetilde{A}$, to avoid possible confusion with the unit of A, if it exists.

2.4.1 Hochschild cochains

The space of *Hochschild k-cochains* is

$$C^k(A, A) = \mathrm{Hom}_R(A^{\breve{\otimes}_R k}, A).$$

We shall identify Hochschild k-cochains with separately or jointly continuous, depending on the context, R-multilinear maps from $A^{\times k}$ into A. The coboundary map $\delta : C^k(A, A) \to C^{k+1}(A, A)$ is given by

$$\begin{aligned}\delta D(a_1, \ldots, a_{k+1}) = D(a_1, \ldots, a_k)a_{k+1} + (-1)^{k+1} a_1 D(a_2, \ldots, a_{k+1})\\ + \sum_{j=1}^{k} (-1)^{k+j-1} D(a_1, \ldots, a_{j-1}, a_j a_{j+1}, a_{j+2}, \ldots, a_{k+1}).\end{aligned}$$

One can check that $\delta^2 = 0$. The cohomology of $(C^\bullet(A, A), \delta)$ is the Hochschild cohomology of A with coefficients in A, and is denoted by $H^\bullet(A, A)$. If we wish to emphasize the ground ring R, we shall write $H^\bullet_R(A, A)$.

An important example for us is that a cochain $D \in C^1(A, A)$ satisfies $\delta D = 0$ if and only if D is a derivation, i.e.

$$D(a_1 a_2) = D(a_1)a_2 + a_1 D(a_2) \qquad \forall a_1, a_2 \in A.$$

There is an associative product

$$\smile : C^k(A, A) \otimes C^l(A, A) \to C^{k+l}(A, A)$$

given by

$$(D \smile E)(a_1, \ldots, a_{k+l}) = (-1)^{kl} D(a_1, \ldots, a_k) E(a_{k+1}, \ldots, a_{k+l}).$$

This product satisfies

$$\delta(D \smile E) = \delta D \smile E + (-1)^k D \smile \delta E,$$

so that $(C^\bullet(A, A), \delta, \smile)$ is a differential graded algebra. It is shown in [10] that $H^\bullet(A, A)$ is a graded commutative algebra with respect to this product. However, the product $\smile$ is not graded commutative in $C^\bullet(A, A)$.

The complex $C^\bullet(A, A)$ also admits the structure of a differential graded Lie algebra, after a degree shift [10]. Let $\mathfrak{g}^\bullet(A) = C^{\bullet+1}(A)$. We let $|D|$ denote the degree of an element of $\mathfrak{g}^\bullet(A)$, so that if D is a k-cochain, then $|D| = k - 1$. Given $D, E \in \mathfrak{g}^\bullet(A)$, let

$$\begin{aligned}
&(D \circ E)(a_1, \ldots, a_{k+l-1}) \\
&\quad = \sum_{i=1}^{k} (-1)^{i|E|} D(a_1, \ldots, a_{i-1}, E(a_i, \ldots, a_{i+l-1}), a_{i+l}, \ldots, a_{k+l-1}).
\end{aligned}$$

Note that $|D \circ E| = |D| + |E|$. The *Gerstenhaber bracket* is defined as

$$[D, E] = D \circ E - (-1)^{|D||E|} E \circ D.$$

One can check that $(\mathfrak{g}^\bullet(A), [\cdot, \cdot])$ is a graded Lie algebra and moreover,

$$\delta[D, E] = [\delta D, E] + (-1)^{|D|}[D, \delta E],$$

so that $\mathfrak{g}^\bullet(A)$ is a differential graded Lie algebra.

Let m denote the multiplication map for A. The equation $[m, m] = 0$ says that m is associative. The graded Jacobi identity then implies that taking a bracket with m is a differential, and indeed it is a fact that

$$\delta D = [m, D], \qquad \forall D \in \mathfrak{g}^\bullet(A).$$

That δ is a graded Lie algebra derivation also follows from the graded Jacobi identity.

2.4.2 Hochschild homology

For $n \geq 0$, the space of *Hochschild n-chains* is defined to be

$$C_n(A) = \begin{cases} A, & n = 0 \\ \widetilde{A} \check{\otimes}_R A^{\check{\otimes}_R n}, & n \geq 1 \end{cases}$$

The boundary map $b : C_n(A) \to C_{n-1}(A)$ is given on elementary tensors by

$$b(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \ldots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \ldots \otimes a_n$$
$$+ (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}.$$

More formally, b is induced by the functoriality property of the tensor product $\check{\otimes}_R$ using the continuous multiplication map $m : \widetilde{A} \check{\otimes}_R \widetilde{A} \to \widetilde{A}$. This shows that b is continuous. One can check that $b^2 = 0$, so that $C_\bullet(A)$ is a locally convex cochain complex over R. The homology of the complex $(C_\bullet(A), b)$ is called the *Hochschild homology* of A (with coefficients in A) and shall be denoted $HH_\bullet(A)$ or $HH_\bullet^R(A)$ if we wish to emphasize R.

2.4.3 Cyclic homology

We only introduce the periodic cyclic theory. Let

$$C_{\mathrm{even}}(A) = \prod_{n=0}^{\infty} C_{2n}(A), \qquad C_{\mathrm{odd}}(A) = \prod_{n=0}^{\infty} C_{2n+1}(A),$$

with the product topologies. Consider the operator $B : C_n(A) \to C_{n+1}(A)$ given on elementary tensors by

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} (-1)^{jn} e \otimes a_j \otimes \ldots a_n \otimes a_0 \otimes \ldots \otimes a_{j-1},$$

if $a_0 \in A$, and

$$B(e \otimes a_1 \otimes \ldots \otimes a_n) = 0.$$

Then it is immediate that $B^2 = 0$. Moreover, one can check that

$$bB + Bb = 0.$$

Extend the operators b and B to the periodic cyclic complex

$$C_{\mathrm{per}}(A) = C_{\mathrm{even}}(A) \oplus C_{\mathrm{odd}}(A).$$

This is a $\mathbb{Z}/2$-graded locally convex complex

$$C_{\mathrm{even}}(A) \rightleftarrows C_{\mathrm{odd}}(A)$$

with differential $b+B$. The homology groups of this complex are called the even and odd *periodic cyclic homology groups* of A, and are denoted $HP_0(A)$ and $HP_1(A)$ respectively. As before, we will write $HP^R_\bullet(A)$ if we wish to emphasize the ground ring R.

2.4.4 Dual cohomology theories

To obtain periodic cyclic cohomology, we dualize the previous notions. Let

$$C^n(A) = C_n(A)^* = \mathrm{Hom}_R(C_n(A), R)$$

be the dual R-module of $C_n(A)$. The maps

$$b : C^n(A) \to C^{n+1}(A), \qquad B : C^n(A) \to C^{n-1}(A)$$

are induced by duality, and are given explicitly by

$$\begin{aligned} b\varphi(a_0, \ldots, a_n) = \sum_{j=0}^{n-1} &(-1)^j \varphi(a_0, \ldots a_{j-1}, a_j a_{j+1}, a_{j+2}, \ldots, a_n) \\ &+ (-1)^n \varphi(a_n a_0, a_1, \ldots, a_{n-1}), \end{aligned}$$

and

$$B\varphi(a_0, \ldots, a_{n-1}) = \sum_{j=0}^{n-1} (-1)^{j(n-1)} \varphi(e, a_j, \ldots, a_{n-1}, a_0, \ldots a_{j-1}), \qquad a_0 \in A,$$

$$B\varphi(e, a_1, \ldots, a_{n-1}) = 0.$$

The cohomology of $(C^\bullet(A), b)$ is called the *Hochschild cohomology* of A (with coefficients in $A^* = \mathrm{Hom}_R(A, R)$) and will be denoted by $HH^\bullet(A)$. The periodic cyclic cochain complex is $C^{\mathrm{per}}(A) = C^{\mathrm{even}}(A) \oplus C^{\mathrm{odd}}(A)$, where

$$C^{\mathrm{even}}(A) = \bigoplus_{n=0}^{\infty} C^{2n}(A), \qquad C^{\mathrm{odd}}(A) = \bigoplus_{n=0}^{\infty} C^{2n+1}(A).$$

Then $C^{\mathrm{per}}(A)$ is a $\mathbb{Z}/2$-graded complex with differential $b + B$, and its cohomology groups are the even and odd *periodic cyclic cohomology* of A, denoted $HP^0(A)$ and $HP^1(A)$ respectively.

Since $C^{\mathrm{per}}(A) \cong C_{\mathrm{per}}(A)^*$, there is a canonical pairing

$$\langle \cdot, \cdot \rangle : C^{\mathrm{per}}(A) \times C_{\mathrm{per}}(A) \to R$$

which descends to a bilinear map

$$\langle \cdot, \cdot \rangle : HP^\bullet(A) \times HP_\bullet(A) \to R.$$

2.4.5 Noncommutative geometry dictionary

In the case where $A = C^\infty(M)$, the algebra of smooth functions on a closed manifold M with its usual Fréchet topology, the above homology groups have geometric interpretations. The Hochschild cohomology $H^\bullet(A, A)$ is the graded space of multivector fields on M. The cup product corresponds to the wedge product of multivector fields, and the Gerstenhaber bracket corresponds to the Schouten-Nijenhuis bracket. The Hochschild homology $HH_\bullet(A)$ is the space of differential forms on M. The differential B descends to a differential on $HH_\bullet(A)$, and this can be identified with the de Rham differential d up to a constant. The even (respectively odd) periodic cyclic homology can be identified with the direct sum of the even (respectively odd) de Rham cohomology groups. In a dual fashion, the Hochschild cohomology $HH^\bullet(A)$ is the space of de Rham currents and the periodic cyclic cohomology can be identified with de Rham homology.

Thus, for any, not necessarily commutative, algebra A, we can view $C^\bullet(A, A)$ and $C_\bullet(A)$ as spaces of noncommutative multivector fields and noncommutative differential forms respectively. Just as multivector fields act on differential forms by Lie derivative and contraction operations, there are Lie derivative and contraction operations

$$L, \iota : C^\bullet(A, A) \to \mathrm{End}(C_\bullet(A))$$

for any algebra A, which we shall explore in the next section.

2.5 Operations on the cyclic complex

The Cartan homotopy formula that follows was first observed by Rinehart in [31] in the case where D is a derivation, and later in full generality by Getzler in [12], see also [35], [24]. An elegant and conceptual proof of the Cartan homotopy formula can be found in [18]. Our conventions vary slightly from [12], and are like those in [35].

To simplify the notation of what follows, the elementary tensor $a_0 \otimes a_1 \otimes \ldots \otimes a_n \in C_n(A)$ will be written as $(a_0, a_1, \ldots, a_n)$. All operators that are defined in this section are given algebraically on elementary tensors, and extend to continuous linear operators on the corresponding completed topological tensor products.

All commutators of operators that follow are graded commutators. That is, if S and T are homogenous operators of degree $|S|$ and $|T|$, then

$$[S, T] = ST - (-1)^{|S||T|}TS.$$

Proofs of statements asserted in this section can be found in Appendix B.

2.5.1 Lie derivatives, contractions, and the Cartan homotopy formula

Given a Hochschild cochain $D \in C^k(A, A)$, the *Lie derivative along* D is the operator $L_D \in \mathrm{End}(C_\bullet(A))$ of degree $1-k$ given by

$$\begin{aligned} &L_D(a_0, \ldots, a_n) \\ &\quad = \sum_{i=0}^{n-k+1} (-1)^{i(k-1)}(a_0, \ldots, D(a_i, \ldots, a_{i+k-1}), \ldots, a_n) \\ &\qquad + \sum_{i=1}^{k-1}(-1)^{in}(D(a_{n-i+1}, \ldots, a_n, a_0, \ldots, a_{k-1-i}), a_{k-i}, \ldots, a_{n-i}). \end{aligned}$$

The second sum is taken over all cyclic permutations of the a_i such that a_0 is within D. In the case $D \in C^1(A, A)$, the above formula is just

$$L_D(a_0, \ldots, a_n) = \sum_{i=0}^{n}(a_0, \ldots a_{i-1}, D(a_i), a_{i+1}, \ldots, a_n).$$

To be completely precise in the above formulas, we are identifying $C^k(A, A)$ as a subspace of $\mathrm{Hom}(\widetilde{A}^{\widetilde{\otimes}_R k}, A)$ by extending by zero, so that

$$D(a_1, \ldots, a_k) = 0, \qquad \text{if } a_i = e \text{ for some } i.$$

The one exception, where we do not wish to extend by zero, is for the multiplication map m of the unitization $\widetilde{A}$. Here, the formula for L_m still gives a well-defined operator on $C_\bullet(A)$, and $L_m = b$.

Proposition 2.5.1. *If $D, E \in C^\bullet(A, A)$, then*

$$[L_D, L_E] = L_{[D,E]}, \qquad [b, L_D] = L_{\delta D}, \qquad [B, L_D] = 0.$$

So $C_{-\bullet}(A)$ and $C_{\mathrm{per}}(A)$ are differential graded modules over the differential graded Lie algebra $\mathfrak{g}^\bullet(A)$. In particular, the graded Lie algebra $H^{\bullet+1}(A, A)$ acts via Lie derivatives on both the Hochschild homology $HH_{-\bullet}(A)$ and the periodic cyclic homology $HP_\bullet(A)$.

Given a k-cochain $D \in C^k(A, A)$, the *contraction with* D is the operator $\iota_D \in$

$\mathrm{End}(C_\bullet(A))$ of degree $-k$ given by

$$\iota_D(a_0,\ldots,a_n) = (-1)^{k-1}(a_0 D(a_1,\ldots,a_k), a_{k+1},\ldots,a_n).$$

Proposition 2.5.2. *For any $D \in C^\bullet(A,A)$, $[b,\iota_D] = -\iota_{\delta D}$.*

Although ι_D interacts well with b, it does not with the differential B, and needs to be adjusted for the cyclic complex. Given $D \in C^k(A,A)$, let S_D denote the operator on $C_\bullet(A)$ of degree $2-k$ given by

$$S_D(a_0,\ldots,a_n) = \sum_{i=1}^{n-k+1} \sum_{j=0}^{n-i-k+1} (-1)^{i(k-1)+j(n-k+1)}$$
$$(e, a_{n-j+1},\ldots,a_n,a_0,\ldots,a_{i-1}, D(a_i,\ldots,a_{i+k-1}), a_{i+k}\ldots,a_{n-j}),$$

if $a_0 \in A$ and

$$S_D(e,a_1,\ldots,a_n) = 0.$$

The sum is over all cyclic permutations with D appearing to the right of a_0. Given $D \in C^\bullet(A,A)$, the *cyclic contraction with D* is the operator

$$I_D = \iota_D + S_D.$$

Theorem 2.5.3 (Cartan homotopy formula)**.** *For any $D \in C^\bullet(A,A)$,*

$$[b+B, I_D] = L_D - I_{\delta D}.$$

Theorem 2.5.3 implies that the Lie derivative along a Hochschild cocycle $D \in C^\bullet(A,A)$ is continuously chain homotopic to zero in the periodic cyclic complex. Thus, the action of $H^{\bullet+1}(A,A)$ on $HP_\bullet(A)$ by Lie derivatives is zero.

The results of this section can be summarized in another way. Consider the endomorphism complex $\mathrm{End}_R(C_{\mathrm{per}}(A))$ whose coboundary map is given by the graded commutator with $b+B$. Let

$$\mathrm{Op}(A) = \mathrm{Hom}_R(\mathfrak{g}^\bullet(A), \mathrm{End}_R(C_{\mathrm{per}}(A)),$$

and let ∂ denote the boundary map in $\mathrm{Op}(A)$. Given $\Phi \in \mathrm{Op}(A)$ and $D \in \mathfrak{g}^\bullet(A)$, we shall write $\Phi_D := \Phi(D)$. So

$$(\partial\Phi)_D = [b + B, \Phi_D] - (-1)^{|\Phi|}\Phi_{\delta D}.$$

Note that the Lie derivative L and the cyclic contraction I are elements of $\mathrm{Op}(A)$ of even and odd degrees respectively. Theorem 2.5.3 is exactly the statement

$$\partial I = L.$$

So it follows from this that $\partial L = 0$, i.e.

$$[b + B, L_D] = L_{\delta D},$$

which is roughly the content of Proposition 2.5.1.

2.5.2 Some higher operations

The Lie derivative and contraction operations of the previous section have multiple generalizations, see e.g. [12] or [35]. We shall need just one of these. For $X, Y \in C^1(A, A)$, define the operators $L\{X, Y\}$ and $I\{X, Y\}$ on $C_\bullet(A)$ by

$$\begin{aligned} L\{X,Y\}(a_0,\dots,a_n) = &\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(a_0,\dots,X(a_i),\dots,Y(a_j),\dots a_n) \\ &+\sum_{i=1}^{n}(Y(a_0),a_1,\dots,X(a_i),\dots,a_n). \end{aligned}$$

and

$$\begin{aligned} I\{X,Y\}(a_0,\dots,a_n) = &\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sum_{m=0}^{n-j}(-1)^{nm} \\ &(e,a_{n-m+1},\dots,a_n,a_0,\dots,X(a_i),\dots,Y(a_j),\dots,a_{n-m}), \end{aligned}$$

if $a_0 \in A$ and

$$I\{X,Y\}(e,a_1,\dots,a_n) = 0.$$

The following formula appears in [12], with slightly different conventions.

Theorem 2.5.4. *If X and Y are derivations, then*

$$[b+B, I\{X,Y\}] = L\{X,Y\} + I_{X\smile Y} - I_Y I_X.$$

Corollary 2.5.5. *If X and Y are derivations, then*

$$[b+B, L\{X,Y\}] = -L_{X\smile Y} + L_Y I_X - I_Y L_X.$$

2.6 Cyclic homology of differential graded algebras

Let $(\Omega^\bullet, d)$ be a (possibly nonunital) locally convex differential graded $\bar{\otimes}$-algebra over R. We define the *unitization* $\widetilde{\Omega}^\bullet$ of $\Omega^\bullet$ to be the unitization as an algebra, with the understanding that the adjoined unit e has degree 0 and satisfies $de = 0$. Then $\widetilde{\Omega}^\bullet$ is a unital locally convex differential graded $\bar{\otimes}$-algebra that contains $\Omega^\bullet$ as a closed differential graded ideal.

If $\omega \in \Omega^k$, we shall write

$$\deg \omega = k.$$

More generally, for $\omega_1 \otimes \ldots \otimes \omega_n \in \Omega^{\bar{\otimes}_R n}$, we write

$$\deg(\omega_1 \otimes \ldots \otimes \omega_n) = \deg \omega_1 + \ldots + \deg \omega_n.$$

Let $(\Omega^{\bar{\otimes}_R n})^k$ denote the closed submodule generated by elementary degree k tensors. We define the *Hochschild chain groups* to be

$$C_n(\Omega) = \begin{cases} \Omega^0 \oplus (\bigoplus_{k=1}^{\infty} (\widetilde{\Omega} \bar{\otimes}_R \Omega^{\bar{\otimes}_R k})^k), & n = 0 \\ \bigoplus_{k=-n}^{\infty} (\widetilde{\Omega} \bar{\otimes}_R \Omega^{\bar{\otimes}_R (n+k)})^k, & n \geq 1. \end{cases}$$

The Hochschild boundary $b : C_\bullet(\Omega) \to C_{\bullet-1}(\Omega)$ is given by

$$b(\omega_0 \otimes \ldots \otimes \omega_n) \\ = \sum_{i=0}^{n-1} (-1)^{\deg \omega_0 + \ldots + \deg \omega_i - i} \omega_0 \otimes \ldots \otimes \omega_i \omega_{i+1} \otimes \ldots \otimes \omega_n$$

$$+ (-1)^{(\deg \omega_0 + \ldots + \deg \omega_{n-1} - n)(\deg \omega_n - 1) + \deg \omega_n} \omega_n \omega_0 \otimes \omega_1 \otimes \ldots \otimes \omega_{n-1}$$
$$+ \sum_{i=0}^{n} (-1)^{\deg \omega_0 + \ldots + \deg \omega_{i-1} - i} \omega_0 \otimes \ldots \otimes d\omega_i \otimes \ldots \otimes \omega_n.$$

The signs are explained in Appendix B, where we also show $b^2 = 0$. The homology groups of $(C_\bullet(\Omega), b)$ are the *Hochschild homology groups* $HH_\bullet(\Omega)$ of $\Omega^\bullet$. Notice that in the case where $\Omega^\bullet$ is concentrated in degree 0, so that $d = 0$ and $\Omega^\bullet$ is just an ungraded algebra, we recover the usual Hochschild chain groups and homology. Notice also that if $\Omega^\bullet$ has positive grading, then there are negative Hochschild chain groups and homology groups.

Connes' differential $B : C_\bullet(\Omega) \to C_{\bullet+1}(\Omega)$ is modified to include signs:

$$\begin{aligned} &B(\omega_0 \otimes \ldots \otimes \omega_n) \\ &\qquad = \sum_{i=0}^{n} (-1)^{(\deg \omega_0 + \ldots + \deg \omega_{n-i} - n + i - 1)(\deg \omega_{n-i+1} + \ldots + \deg \omega_n - i)} \\ &\qquad\qquad e \otimes \omega_{n-i+1} \otimes \ldots \otimes \omega_n \otimes \omega_0 \otimes \ldots \otimes \omega_{n-i}, \end{aligned}$$

if $\omega_0 \in \Omega^\bullet$, and

$$B(e, \omega_1, \ldots, \omega_n) = 0.$$

Then

$$B^2 = 0, \qquad [b, B] = bB + Bb = 0.$$

The periodic cyclic chain complex is the $\mathbb{Z}/2$-graded complex

$$C_{\mathrm{per}}(\Omega) = C_{\mathrm{ev}}(\Omega) \oplus C_{\mathrm{odd}}(\Omega)$$

with differential $b + B$, where

$$C_{\mathrm{ev}}(\Omega) = \prod_{k \in \mathbb{Z}} C_{2k}(\Omega), \qquad C_{\mathrm{odd}}(\Omega) = \prod_{k \in \mathbb{Z}} C_{2k+1}(\Omega).$$

We denote the even and odd periodic cyclic homology groups of $\Omega^\bullet$ by $HP_0(\Omega)$ and $HP_1(\Omega)$ respectively.

Let $C^k(\Omega, \Omega) = \mathrm{Hom}(\Omega^{\bar{\otimes}_R k}), \Omega)$ be the space of Hochschild k-cochains as in the algebra case. For a homogeneous $D \in C^k(\Omega, \Omega)$, we write $\deg D$ for the degree of

D as a linear map, so that

$$D((\Omega^{\bar{\otimes} k})^n) \subset \Omega^{n+\deg D}$$

for all n. We shall write

$$|D| = \deg D + k - 1.$$

Let $\mathfrak{g}^\bullet(\Omega)$ denote the space of Hochschild cochains with the grading given by $|\cdot|$. As in the algebra case, it is a differential graded Lie algebra under the following structure. Given homogeneous $D \in C^k(\Omega, \Omega), E \in C^l(\Omega, \Omega)$, let

$$\begin{aligned}(D \circ E)&(\omega_1, \ldots, \omega_{k+l-1}) \\ &= \sum_{i=1}^{k} (-1)^{|E|(\deg \omega_1 + \ldots + \deg \omega_{i-1} - (i-1))} \\ &\qquad\qquad D(a_1, \ldots, a_{i-1}, E(a_i, \ldots, a_{i+l-1}), a_{i+l1}, \ldots, a_{k+l-1}),\end{aligned}$$

and define the *Gerstenhaber bracket* to be

$$[D, E] = D \circ E - (-1)^{|D||E|} E \circ D.$$

One can check that $\mathfrak{g}^\bullet(\Omega)$ is a graded Lie algebra under the Gerstenhaber bracket. Let $m_1 = d$, the differential of Ω, and let

$$m_2(\omega_1, \omega_2) = (-1)^{\deg \omega_1} \omega_1 \omega_2$$

be the multiplication map twisted by a sign. Set $m = m_1 + m_2$. Then $|m| = 1$ and

$$[m, m] = 2(m \circ m) = 0$$

follows because $\Omega^\bullet$ is a differential graded algebra. It follows from Example 2.3.2 that

$$\delta D = [m, D]$$

defines a differential graded Lie algebra structure on $\mathfrak{g}^\bullet(\Omega)$, so that

$$\delta[D, E] = [\delta D, E] + (-1)^{|D|}[D, \delta E].$$

Quite explicitly, if $D \in C^k(\Omega, \Omega)$, then

$$\delta D = [m_1, D] + [m_2, D],$$

where

$$\begin{aligned}[m_1, D](\omega_1, \ldots, \omega_k) &= d(D(\omega_1, \ldots, \omega_k)) \\ &\qquad + \sum_{i=1}^{k} (-1)^{|D| + \deg \omega_1 + \ldots + \deg \omega_{i-1} + i} D(\omega_1, \ldots, d\omega_i, \ldots, \omega_k)\end{aligned}$$

and

$$\begin{aligned}&[m_2, D](\omega_1, \ldots, \omega_{k+1}) \\ &\quad = (-1)^{|D| + \deg \omega_1 + \ldots + \deg \omega_k - k + 1} D(\omega_1, \ldots, \omega_k)\omega_{k+1} \\ &\qquad + (-1)^{|D|(\deg \omega_1 - 1) + \deg \omega_1} \omega_1 D(\omega_2, \ldots, \omega_{k+1}) \\ &\qquad + \sum_{i=1}^{k} (-1)^{|D| + \deg \omega_1 + \ldots + \deg \omega_i - i} D(\omega_1, \ldots, \omega_i \omega_{i+1}, \ldots, \omega_{k+1}).\end{aligned}$$

Example 2.6.1. If $D : \Omega^\bullet \to \Omega^{\bullet + \deg D}$ is a homogenous linear map, then

$$[m_1, D] = [d, D] = dD - (-1)^{\deg D} Dd = 0$$

if and only if d graded commutates with D. Also, $[m_2, D] = 0$ if and only if D is a graded derivation, that is

$$D(\omega_1 \omega_2) = D(\omega_1)\omega_2 + (-1)^{(\deg D)(\deg \omega_1)} \omega_1 D(\omega_2).$$

There is an associative cup product

$$\smile : C^k(\Omega, \Omega) \otimes C^l(\Omega, \Omega) \to C^{k+l}(\Omega, \Omega)$$

given by

$$\begin{aligned}&(D \smile E)(\omega_1, \ldots, \omega_{k+l}) \\ &\quad = (-1)^{(|E|+1)(\deg \omega_1 + \ldots + \deg \omega_k - k)} D(\omega_1, \ldots, \omega_k) E(\omega_{k+1}, \ldots, \omega_{k+l})\end{aligned}$$

for which

$$\delta(D \smile E) = (\delta D) \smile E + (-1)^{|D|+1} D \smile (\delta E).$$

The differential graded Lie algebra $\mathfrak{g}^\bullet(\Omega)$ of Hochschild cochains acts on the Hochschild chain groups $C_\bullet(\Omega)$ by Lie derivatives and contractions. Given a homogeneous $D \in C^k(\Omega, \Omega)$, let

$$\begin{aligned} &L_D(\omega_0, \ldots, \omega_n) \\ &\qquad = \sum_{i=0}^{n-k+1} (-1)^{|D|(\deg\omega_0 + \ldots + \deg\omega_{i-1} - i)} \\ &\qquad\qquad (\omega_0, \ldots \omega_{i-1}, D(\omega_i, \ldots, \omega_{i+k-1}), \omega_{i+k}, \ldots, \omega_n) \\ &\qquad\quad + \sum_{i=1}^{k-1} (-1)^{(\deg\omega_0 + \ldots + \deg\omega_{n-i} - n + i - 1)(\deg\omega_{n-i+1} + \ldots + \deg\omega_n - i)} \\ &\qquad\qquad (D(\omega_{n-i+1}, \ldots, \omega_n, \omega_0, \ldots, \omega_{k-i-1}), \omega_{k-i}, \ldots, \omega_{n-i}). \end{aligned}$$

As in the algebra case, the same remarks apply regarding extension by zero as it pertains to $D \in C^k(\Omega, \Omega)$ and the twisted multiplication map m_2, see section 2.4.2. Then

$$[L_D, L_E] = L_{[D,E]}, \qquad [L_D, B] = 0.$$

Notice that $b = L_m$, from which we see

$$[b, L_D] = L_{\delta D}, \qquad b^2 = 0.$$

The cyclic contraction is defined by

$$I_D = \iota_D + S_D,$$

where

$$\iota_D(\omega_0, \ldots, \omega_n) = (-1)^{|D|(\deg\omega_0 - 1)}(\omega_0 D(\omega_1, \ldots, \omega_k), \omega_{k+1}, \ldots, \omega_n)$$

and

$$S_D(\omega_0,\dots,\omega_n) = \sum_{i=0}^{n-k+1} \sum_{j=0}^{n-k+1-i}$$
$$(-1)^{|D|(\deg\omega_0+\ldots+\deg\omega_{i-1}-i)+(|D|+\deg\omega_0+\ldots+\deg\omega_{n-j}-n+j-1)(\deg\omega_{n-j+1}+\ldots+\deg\omega_n-j)}$$
$$(e,\omega_{n-j+1},\dots,\omega_n,\omega_0,\dots,\omega_{i-1},D(\omega_i,\dots,\omega_{i+k-1}),\omega_{i+k}\dots,\omega_{n-j})$$

if $\omega_0 \in \Omega^\bullet$, and

$$S_D(e,\omega_1,\dots,\omega_n) = 0.$$

As in the algebra case, the Cartan homotopy formula

$$[b+B, I_D] = L_D - I_{\delta D}$$

holds.

Now let's make the assumption that $\Omega^\bullet = \bigoplus_{n=0}^{\infty} \Omega^n$ is nonnegatively graded. We shall view the subalgebra Ω^0 as a differential graded algebra concentrated in degree 0 with the zero differential. Then the canonical projection

$$p : (\Omega^\bullet, d) \to (\Omega^0, 0)$$

is a map of differential graded algebras. The following result appears in [14], although with different sign conventions.

Theorem 2.6.2 ([14])**.** *If $\Omega^\bullet$ is a nonnegatively graded locally convex differential graded $\bar{\otimes}$-algebra, then the projection $\pi : \Omega^\bullet \to \Omega^0$ induces an isomorphism*

$$p_* : HP_\bullet(\Omega) \to HP_\bullet(\Omega^0).$$

Proof (sketch). First consider the case where $d = 0$. Define the degree 0 derivation

$$D : \Omega^\bullet \to \Omega^\bullet$$

on homogeneous elements by

$$D(\omega) = (\deg\omega)\omega$$

and extend linearly. Then for homogeneous $\alpha \in (\widetilde{\Omega}\bar{\otimes}_R \Omega^{\bar{\otimes}_R n})$, one has

$$L_D \alpha = (\deg \alpha)\alpha.$$

As the differentials b and B preserve degree, the periodic cyclic complex decomposes by degree into subcomplexes which are exactly the eigenspaces of L_D. But the Cartan homotopy formula

$$[b + B, I_D] = L_D$$

implies that the subcomplexes of strictly positive degree are contractible. The subcomplex of degree 0 is exactly the periodic cyclic complex of Ω^0. Using a homotopy operator constructed from I_D, one can show that

$$p : C_{\text{per}}(\Omega^\bullet) \to C_{\text{per}}(\Omega^0)$$

is a chain homotopy equivalence, whose homotopy inverse is the inclusion of complexes

$$i : C_{\text{per}}(\Omega^0) \to C_{\text{per}}(\Omega^\bullet).$$

In the case where d is nontrivial, we view the complex $C_{\text{per}}(\Omega^\bullet, d)$ as a perturbation of the complex $C_{\text{per}}(\Omega^\bullet, 0)$. Using the Homotopy Perturbation Lemma, as stated in [7], the chain homotopy equivalence

$$p_* : C_{\text{per}}(\Omega^\bullet, 0) \to C_{\text{per}}(\Omega^0)$$

can be perturbed to give a chain homotopy equivalence

$$p_* : C_{\text{per}}(\Omega^\bullet, d) \to C_{\text{per}}(\Omega^0).$$

We remark that the map p_* does not change under the perturbation, but its homotopy inverse, as well as the homotopy operator, do change drastically. □

2.7 A_∞-algebras

An A_∞-algebra is a generalization of a differential graded algebra, first introduced by Stasheff [32]. Our interest in A_∞-algebras lies not in this larger class of objects, but rather in the larger, more flexible class of morphisms naturally associated to them. The notion of A_∞-isomorphism for differential graded algebras is strictly weaker than the natural notion of differential graded algebra isomorphism. Our treatment of A_∞-algebras nearly follows Getzler and Jones [13].

2.7.1 Tensor coalgebra

If $X^\bullet$ is a graded locally convex $\bar{\otimes}$-module over R, then the *locally convex tensor coalgebra* of $X^\bullet$ is

$$T^c X = \bigoplus_{n=0}^{\infty} X^{\bar{\otimes}_R n}$$

with the direct sum topology. The grading on $T^c X$ is internal, not external, in the sense that

$$|x_1 \otimes \ldots \otimes x_n| = |x_1| + \ldots + |x_n|.$$

The coproduct $\Delta : T^c X \to T^c X \bar{\otimes}_R T^c X$ is determined by

$$\Delta(x_1 \otimes \ldots \otimes x_n) = \sum_{k=0}^{n} (x_1 \otimes \ldots \otimes x_k) \otimes (x_{k+1} \otimes \ldots \otimes x_n).$$

The empty tensor in this expression is to be interpreted as $1 \in R \cong X^{\bar{\otimes}_R 0}$. The counit $\epsilon : T^c X \to R$ is given by

$$\epsilon(x_1 \otimes \ldots \otimes x_n) = 0, \qquad n \geq 1,$$

and $\epsilon(r) = r$ if $r \in R \cong X^{\bar{\otimes}_R 0}$. There is a coaugmentation

$$\eta : R \to T^c X$$

given by mapping R isomorphically onto $X^{\bar{\otimes}_R 0}$.

Proposition 2.7.1. *The tensor coalgebra $T^c X$ is a locally convex graded*

$\bar{\otimes}$-coalgebra over R which is coaugmented and cocomplete.

Proof. The coproduct gives a separately continuous n-multilinear map

$$\Delta : X^{\times n} \to T^c X \bar{\otimes}_R T^c X,$$

and thus a continuous R-linear map

$$\Delta : T^c X \to T^c X \bar{\otimes}_R T^c X$$

using the universal properties of the inductive tensor product and the direct sum. The coassociativity and counit axioms, as well as continuity of the counit, are routine verifications. Using the coaugmentation, we see that the kernel N_n of the map

$$T^c X \xrightarrow{\Delta^{n-1}} (T^c X)^{\bar{\otimes}_R n} \longrightarrow (T^c X/R)^{\bar{\otimes}_R n}.$$

is exactly $\bigoplus_{k=0}^{n-1} X^{\bar{\otimes}_R k}$. Thus, $T^c X$ is cocomplete, as

$$\varinjlim N_n \cong T^c X,$$

both linearly and topologically. □

Remark 2.7.2. The tensor coalgebra $T^c X$ has the same underlying space as the tensor algebra, so there is an obvious multiplication on $T^c X$. However, these structures should not be considered simultaneously, as they are not compatible in the sense that $T^c X$ is not a bialgebra under these structures.

Let $\pi : T^c X \to X$ denote the canonical projection onto X. The tensor coalgebra is not actually a cofree coalgebra on X with respect to this projection, however it is in the category of coaugmented cocomplete coalgebras.

Proposition 2.7.3. *Let $C^\bullet$ be a coaugmented cocomplete locally convex graded $\bar{\otimes}$-coalgebra over R, and let $F : C^\bullet \to X$ be a degree 0 continuous R-linear map such that $F\eta_C = 0$. Then there is a unique continuous map of graded coaugmented* *coalgebras* $\tilde{F} : C^\bullet \to T^c X$ *such that*

$$\begin{tikzcd} C^\bullet \arrow[r, "\tilde{F}"] \arrow[dr, "F"'] & T^c X \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

commutes.

Proof. Notice that the composition

$$T^c X \xrightarrow{\Delta^{n-1}} (T^c X)^{\bar{\otimes}_R n} \xrightarrow{\pi^{\otimes n}} X^{\bar{\otimes}_R n}$$

is the projection onto $X^{\otimes_R n}$. This holds for all $n \geq 0$ if we make the conventions that $\Delta^0 = \mathrm{id}$ and $\Delta^{-1} = \eta\epsilon$. So we have

$$\sum_{n=0}^{\infty} \pi^{\otimes n} \Delta^{n-1} = \mathrm{id} : T^c X \to T^c X.$$

So if $\tilde{F} : C^\bullet \to T^c X$ is a map of coaugmented graded coalgebras with $\pi\tilde{F} = F$, then

$$\begin{aligned} \tilde{F} &= \sum_{n=0}^{\infty} \pi^{\otimes n} \Delta^{n-1} \tilde{F} \\ &= \sum_{n=0}^{\infty} \pi^{\otimes n} \tilde{F}^{\otimes n} \Delta^{n-1} \\ &= \sum_{n=0}^{\infty} F^{\otimes n} \Delta^{n-1}. \end{aligned}$$

This shows that such an $\tilde{F}$ is unique.

It remains to prove that such a formula gives a well-defined map of graded coaugmented coalgebras. Let $N_n^C \subset C^\bullet$ be the submodules as in the definition of cocompleteness. The maps

$$\sum_{k=0}^{n} F^{\otimes n} \Delta^{n-1} : N_n^C \to T^c X$$

are compatible with the inclusions $N_n^C \to N_{n+1}^C$ because of the condition $F\eta_C = 0$.

The induced continuous linear map on $C^\bullet \cong \varinjlim N_n$ is given by

$$\tilde{F} = \sum_{n=0}^{\infty} F^{\otimes n} \Delta^{n-1} : C^\bullet \to T^c X,$$

as desired. So $\tilde{F}$ is well-defined and it respects the coaugmentation and counit. One can check directly that $\Delta \tilde{F}$ and $(\tilde{F} \otimes \tilde{F})\Delta$ are both equal to

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (F^{\otimes m} \otimes F^{\otimes n}) \Delta^{m+n-1},$$

where the sum is actually finite when applied to an element of $C^\bullet$. $\square$

It will be of interest to consider the possible ways to make $T^c X$ a differential graded coalgebra. To do this, we first classify all possible coderivations. The following is dual to the fact that a derivation on the tensor algebra $T_a X$ is determined by a linear map $X \to T_a X$.

Proposition 2.7.4. *The map $D \mapsto \pi D$ gives a linear isomorphism*

$$\mathrm{Coder}(T^c X) \xrightarrow{\cong} \mathrm{Hom}_R(T^c X, X)$$

of graded vector spaces.

The inverse of this map sends the element $F \in \mathrm{Hom}_R(X^{\bar{\otimes}_R k}, X)$ to the coderivation

$$\prod_{n=k}^{\infty} \sum_{j=0}^{n-k} 1^{\otimes j} \otimes F \otimes 1^{n-k-j}.$$

Using this isomorphism, we can transfer the Lie bracket from $\mathrm{Coder}(T^c X)$ to $\mathrm{Hom}_R(T^c X, X)$. The bracket is given by

$$[F, G] = F \circ G - (-1)^{|F||G|} G \circ F$$

where

$$F \circ G = \sum_{j=0}^{k-1} F(1^{\otimes j} \otimes G \otimes 1^{\otimes (k-j-1)}),$$

where $F \in \mathrm{Hom}_R(X^{\bar{\otimes}_R k}, X)$. Notice that if $F \in \mathrm{Hom}_R(X^{\bar{\otimes}_R k}, X)$ and $G \in \mathrm{Hom}_R(X^{\bar{\otimes}_R l}, X)$, then $[F, G] \in \mathrm{Hom}_R(X^{\bar{\otimes}_R (k+l-1)}, X)$.

Corollary 2.7.5. *The space* $\mathrm{Hom}_R(T^c X, X)$ *is a graded Lie algebra over* R*.*

2.7.2 Locally convex A_∞-algebras

Given a graded locally convex $\bar{\otimes}$-module X over R, *suspension* sX of X is the same module as X, but with a shift in grading so that

$$(sX)^n = X^{n+1}.$$

The *bar coalgebra* of X is defined to be

$$B(X) = T^c(sX).$$

Elementary tensors in $(sX)^{\bar{\otimes}_R n}$ shall be denoted $[x_1| \dots |x_n]$ to differentiate them from elements of $X^{\bar{\otimes}_R n}$. Notice that the grading shift from the suspension propagates throughout the bar coalgebra, so that the degree of the element $[x_1| \dots |x_n]$ is $\sum_{i=1}^n \deg x_i - n$, where $\deg x_i$ is the degree of x_i viewed as an element of the graded space X. In particular if X is trivially graded, the grading on $B(X)$ is nontrivial.

Definition 2.7.6. A *locally convex A_∞-algebra* over R is a locally convex graded $\bar{\otimes}$-module A over R equipped with a degree $+1$ coderivation m on $B(A)$ for which $m^2 = 0$ and $m\eta = 0$.

Thus, $(B(A), m)$ is a locally convex differential graded $\bar{\otimes}$-coalgebra over R. For simplicity, we are only using inductive tensor products here due to their compatibility with direct sums, which are inherent to the discussion. If we consider a vector space A with its finest locally convex topology, then we recover the algebraic definition of an A_∞-algebra.

To pull apart this definition, first recall that

$$Coder(B(A)) \cong \mathrm{Hom}_R(B(A), sA),$$

so that m is given by a sequence of degree 1 maps[5]

$$m_n : (sA)^{\bar{\otimes}_R n} \to sA, \qquad n \geq 1.$$

The condition $m^2 = 0$ decomposes as an infinite sequence of identities:

$$\sum_{i+j=n} \sum_{k=0}^{i} m_i(1^{\otimes k} \otimes m_j \otimes 1^{\otimes(i-k-1)}) = 0, \qquad n \geq 1.$$

The first three identities are

$$\begin{aligned}
&m_1 m_1 = 0,\\
&m_1 m_2 + m_2(m_1 \otimes 1) + m_2(1 \otimes m_1) = 0,\\
&m_2(m_2 \otimes 1) + m_2(1 \otimes m_2) + m_1 m_3\\
&\qquad + m_3(m_1 \otimes 1 \otimes 1) + m_3(1 \otimes m_1 \otimes 1) + m_3(1 \otimes 1 \otimes m_1) = 0.
\end{aligned}$$

To make things a little easier to digest, let $\mu_n : A^{\bar{\otimes}_R n} \to A$ be defined by

$$\begin{tikzcd}
(sA)^{\bar{\otimes}_R n} \arrow[r, "m_n"] & sA \arrow[d, "s^{-1}"] \\
A^{\bar{\otimes}_R n} \arrow[u, "s^{\otimes n}"] \arrow[r, "\mu_n"] & A
\end{tikzcd}$$

Since $s : A \to sA$ is a degree -1 map, μ_n has degree $2 - n$. Aside from superficial differences due to a grading shift, m_n and μ_n differ from a sign coming from applying $s^{\otimes n}$:

$$\mu_n(a_1, \ldots, a_n) = (-1)^{\sum_{k=1}^{n-1} k \deg a_k} s m_n[a_1| \ldots |a_n].$$

The first three identities then become

$$\begin{aligned}
&\mu_1(\mu_1(a)) = 0,\\
&\mu_1(\mu_2(a_1, a_2)) = \mu_2(\mu_1(a_1), a_2) + (-1)^{\deg a_1} \mu_2(a_1, \mu_1(a_2)),\\
&\mu_2(\mu_2(a_1, a_2), a_3) - \mu_2(a_1, \mu_2(a_2, a_3))
\end{aligned}$$

[5]If we remove the condition $m\eta = 0$, there will in addition be a map $m_0 : R \to sA$, and we obtain the more general notion of a *curved locally convex A_∞-algebra.*

$$= \mu_1(\mu_3(a_1, a_2, a_3)) + \mu_3(\mu_1(a_1), a_2, a_3) + (-1)^{\deg a_1}\mu_3(a_1, \mu_1(a_2), a_3)$$
$$+ (-1)^{\deg a_1 + \deg a_2}\mu_3(a_1, a_2, \mu_1(a_3)).$$

From the first condition, we see that (A, μ_1) is a cochain complex. The second condition is that μ_1 is a derivation with respect to the bilinear map μ_2. As the third condition shows, the product μ_2 is not in general associative, but is associative up to a homotopy given by μ_3. In fact, this condition implies that the cohomology $H^\bullet(A)$ with respect to μ_1 is a graded associative algebra with product induced by μ_2. Here are some important classes of A_∞-algebras.

- If A is trivially graded, then so is $A^{\bar{\otimes}_R n}$. Then since

  $$\mu_n : A^{\bar{\otimes}_R n} \to A$$

  has degree $2 - n$, we see that $\mu_n = 0$ unless $n = 2$. The only identity which is not vacuous is that μ_2 is associative. Thus, locally convex A_∞-structures on a trivially graded space are the same thing as locally convex $\bar{\otimes}$-algebra structures on that space.

- If $\mu_n = 0$ unless $n = 1$, then we have a cochain complex with coboundary map μ_1.

- If $\mu_n = 0$ unless $n = 2$, then we have a graded associative algebra with product μ_2.

- If $\mu_n = 0$ unless $n = 1, 2$, then we have a differential graded algebra with differential μ_1 and associative product μ_2.

An A_∞-*morphism* between two locally convex A_∞-algebras A and A' is a continuous R-linear map of coaugmented differential graded coalgebras

$$f : B(A) \to B(A').$$

Thus, it is clear that locally convex A_∞-algebras with A_∞-morphisms form a category. The A_∞-morphism $f : B(A) \to B(A')$ is an A_∞-*isomorphism* if f is an isomorphism of locally convex differential graded coalgebras. From Proposition 2.7.3, an A_∞-morphism f is determined by

$$\pi f \in \mathrm{Hom}(B(A), sA'),$$

which gives a sequence of degree 0 maps

$$f_n : (sA)^{\bar{\otimes}_R n} \to sA', \qquad n \geq 1.$$

Note that since $f\eta_{B(A)} = \eta_{B(A')}$, there is no f_0. We would like to express the condition

$$fm - m'f = 0$$

in terms of the m_n, m'_n and f_n. One can check that for coderivations m, m' and a coalgebra map f, the map $fm - m'f$ is determined by the map

$$\pi(fm - m'f) : B(A) \to sA'.$$

When f is a map of differential graded coalgebras, this is the zero map. So we obtain an infinite sequence of identities, of which the first few are

$$\begin{aligned}
&f_1 m_1 = m'_1 f_1,\\
&f_1 m_2 + f_2(m_1 \otimes 1) + f_2(1 \otimes m_1) = m'_1 f_2 + m'_2(f_1 \otimes f_1),\\
&f_1 m_3 + f_2(m_2 \otimes 1) + f_2(1 \otimes m_2)\\
&\qquad + f_3(m_1 \otimes 1 \otimes 1) + f_3(1 \otimes m_1 \otimes 1) + f_3(1 \otimes 1 \otimes m_1)\\
&\qquad\qquad = m'_1 f_3 + m'_2(f_1 \otimes f_2) + m'_2(f_2 \otimes f_1) + m'_3(f_1 \otimes f_1 \otimes f_1).
\end{aligned}$$

In general, we have

$$\sum_{i+j=n+1} \sum_{k=0}^{n-j} f_i(1^{\otimes k} \otimes m_j \otimes 1^{\otimes n-k-j}) = \sum_{r=1}^{n} \sum_{i_1+\ldots+i_r=n} m'_r(f_{i_1} \otimes \ldots \otimes f_{i_r})$$

for all $n \geq 1$. To understand these identities a little better, we define maps

$\varphi_n : A^{\bar{\otimes}_R n} \to A'$ by

$$\begin{array}{ccc} (sA)^{\bar{\otimes}_R n} & \xrightarrow{f_n} & sA' \\ {\scriptstyle s^{\otimes n}}\uparrow & & \downarrow{\scriptstyle s^{-1}} \\ A^{\bar{\otimes}_R n} & \xrightarrow{\varphi_n} & A' \end{array}$$

so that

$$\varphi_n(a_1, \ldots, a_n) = (-1)^{\sum_{k=1}^{n-1} k \deg a_{n-k}} s f_n[a_1| \ldots |a_n].$$

Notice the φ_n has degree $1 - n$. In terms of the μ_n and the φ_n, the first three of these identities become

$$\begin{aligned}
&\varphi_1(\mu_1(a)) = \mu_1'(\varphi_1(a)), \\
&\varphi_1(\mu_2(a_1, a_2)) \\
&\qquad = \mu_2'(\varphi_1(a_1), \varphi_1(a_2)) + \mu_1'(\varphi_2(a_1, a_2)) + \varphi_2(\mu_1(a_1), a_2) \\
&\qquad\qquad + (-1)^{\deg a_1} \varphi_2(a_1, \mu_1(a_2)), \\
&\varphi_1(\mu_3(a_1, a_2, a_3)) + \varphi_2(\mu_2(a_1, a_2), a_3) - \varphi_2(a_1, \mu_2(a_2, a_3)) \\
&\qquad = \mu_3'(\varphi_1(a_1), \varphi_1(a_2), \varphi_1(a_3)) - \mu_2'(\varphi_2(a_1, a_2), \varphi_1(a_3)) \\
&\qquad\qquad + (-1)^{\deg a_1} \mu_2'(\varphi_1(a_1), \varphi_2(a_2, a_3)) + \mu_1'(\varphi_3(a_1, a_2, a_3)) - \varphi_3(\mu_1(a_1), a_2, a_3) \\
&\qquad\qquad - (-1)^{\deg a_1} \varphi_3(a_1, \mu_1(a_2), a_3) - (-1)^{\deg a_1 + \deg a_2} \varphi_3(a_1, a_2, \mu_1(a_3)).
\end{aligned}$$

The first condition is that φ_1 is a degree 0 chain map. The second condition is that φ_1 is a homomorphism with respect to the product μ_2 up to a homotopy given by φ_2. The third and higher conditions are harder to make sense of. However, the first two conditions are enough to ensure that φ_1 induces a map of graded algebras

$$\varphi_1 : H^\bullet(A) \to H^\bullet(A').$$

The morphism f is an A_∞-*quasi-isomorphism* if the map induced by φ_1 on cohomology is an isomorphism. One can show that if

$$f : B(A) \to B(A'), \qquad f' : B(A') \to B(A'')$$

are two A_∞-morphisms, then[6]

$$(f'f)_1 = f'_1 f_1 : sA \to sA''.$$

As a consequence, if f is an A_∞-isomorphism, then $\varphi_1 : (A, m_1) \to (A', m'_1)$ is an isomorphism of complexes. It follows that an A_∞-isomorphism is an A_∞-quasi-isomorphism. Let us consider some special cases:

- If A, A' are trivially graded, then locally convex A_∞-structures are the same as separately continuous associative multiplications, as discussed above. Since an A_∞-morphism decomposes into maps

  $$\varphi_n : A^{\bar{\otimes}_R n} \to A'$$

  of degree $1-n$, we see that $\varphi_n = 0$ for $n \neq 1$. That f is an A_∞-morphism is equivalent to φ_1 being an algebra morphism. Thus, we can view the category of ungraded locally convex $\bar{\otimes}$-algebras as a full subcategory of the category of locally convex A_∞-algebras.

- If $\mu_n, \mu'_n = 0$ unless $n = 1$, so that A and A' are just cochain complexes, then an A_∞-morphism

  $$f : B(A) \to B(A')$$

  is given by a sequence of unrelated chain maps

  $$f_n : (sA)^{\bar{\otimes}_R n} \to sA'.$$

  Here, $(sA)^{\bar{\otimes}_R n}$ is a cochain complex as it is an iterated tensor product of the complex (sA, m_1). Then f is an A_∞-quasi-isomorphism if and only if f_1 is a quasi-isomorphism of cochain complexes. Here, the notion of A_∞-morphism is more general than the usual notion of morphism of complexes. Thus, complexes do not form a full subcategory of A_∞-algebras. However, one can show that two complexes are A_∞-isomorphic if and only if they are isomorphic as complexes, so we do not obtain a different notion of equivalence by passing to A_∞-algebras.

[6] We remark that the formula for $(f'f)_n$ for $n > 1$ is more complicated.

- If we are considered graded algebras, so that $\mu_n, \mu'_n = 0$ unless $n = 2$, then an A_∞-morphism consists of an algebra map φ_1 along with some higher maps $\varphi_n : A^{\bar{\otimes}_R n} \to A'$ satisfying certain relations. However, two algebras are isomorphic as algebras if and only if they are A_∞-isomorphic if and only if they are A_∞-quasi-isomorphic.

- The case we shall most be interested in is when A and A' are differential graded algebras. If $\varphi_n = 0$ for $n \neq 1$, then φ_1 is a differential graded algebra morphism. But for a general A_∞-morphism, φ_1 is not necessarily a differential graded algebra morphism, because it is only multiplicative up to homotopy. So it may be the case that two differential graded algebras A and A' are A_∞-isomorphic, but not isomorphic. This weakening of the notion of equivalence of differential graded algebras is our main motivation for considering A_∞-algebras.

2.7.3 Cyclic homology of A_∞-algebras

Cyclic homology was first introduced for A_∞-algebras in [13].

An A_∞-algebra A is *unital* if there is an element $e \in A^0$ such that

$$m_2(e, a) = a = (-1)^{\deg a} m_2(a, e), \forall a \in A,$$

and

$$m_n(a_1, \ldots, e, \ldots, a_n) = 0, \qquad n \neq 2.$$

For any A_∞-algebra A, we define the *unitization* $\widetilde{A}$ to be A with an element e adjoined, such that the structure maps m_n are extended to satisfy the above relations.

The *space of Hochschild cochains* of a locally convex A_∞-algebra over R is

$$\mathfrak{g}^\bullet(A) := \mathrm{Hom}_R(B(A), sA) \cong \mathrm{Coder}(B(A)).$$

It is a graded Lie algebra, as described in the discussion of Corollary 2.7.5, and its bracket is called the *Gerstenhaber bracket.* Care must be taken as it pertains to the grading. A map $D \in \mathrm{Hom}_R(A^{\bar{\otimes}_R k}, A)$ shall be called a k-cochain. If D is homogeneous, then we shall write $\deg D$ to mean the integer such that

$$D((A^{\bar{\otimes}_R k})^n) \subset A^{n+\deg D}.$$

The notation $|D|$ shall refer to the degree of D as an element of $\mathfrak{g}^\bullet(A)$. So for a homogeneous k-cochain D, we have

$$|D| = \deg D + (k-1).$$

The A_∞-structure map $m \in \mathfrak{g}^1(A)$ satisfies

$$[m, m] = 2m^2 = 0.$$

So as in Example 2.3.2, the operator

$$\delta D = [m, D]$$

defines a differential on $\mathfrak{g}^\bullet(A)$ that makes it a differential graded Lie algebra.

Example 2.7.7. For a trivially graded algebra A, $\mathfrak{g}^k(A)$ is exactly the space of $(k+1)$-cochains. The differential graded Lie algebra $\mathfrak{g}^\bullet(A)$ coincides with the one defined in section 2.4.1. If $A^\bullet$ is a differential graded algebra, the differential graded Lie algebra $\mathfrak{g}^\bullet(A)$ coincides with the one defined in section 2.6.

The *Hochschild chain groups* of a locally convex A_∞-algebra over R are defined as

$$C_n(A) = \begin{cases} A^0 \oplus (\bigoplus_{k=1}^{\infty} (\widetilde{A} \bar{\otimes}_R A^{\bar{\otimes}_R k})^k), & n = 0 \\ \bigoplus_{k=-n}^{\infty} (\widetilde{A} \bar{\otimes}_R A^{\bar{\otimes}_R (n+k)})^k, & n \geq 1. \end{cases}$$

as in the differential graded algebra case in section 2.6.

The graded Lie algebra $\mathfrak{g}^\bullet(A)$ acts on the graded space $C_\bullet(A)$ by Lie derivatives exactly as given in section 2.6. Notice that the definition of the Lie derivative operators does not use anything but the linear structure. Thus we have

$$[L_D, L_E] = L_{[D,E]}, \qquad \forall D, E \in \mathfrak{g}^\bullet(A).$$

The Hochschild boundary map is defined to be $b = L_m$. Thus, it is immediate that

$$b^2 = \frac{1}{2}[L_m, L_m] = \frac{1}{2}L_{[m,m]} = 0$$

and

$$[b, L_D] = L_{[m,D]} = L_{\delta D}.$$

Connes' differential $B : C_\bullet(A) \to C_{\bullet+1}(A)$ is defined exactly as in section 2.6. Just as before

$$[L_D, B] = 0,$$

in particular,

$$[b, B] = bB + Bb = 0.$$

The periodic cyclic chain complex is the $\mathbb{Z}/2$-graded complex

$$C_{\mathrm{per}}(A) = C_{\mathrm{ev}}(A) \oplus C_{\mathrm{odd}}(A)$$

with differential $b + B$, where

$$C_{\mathrm{ev}}(A) = \prod_{k\in\mathbb{Z}} C_{2k}(A), \qquad C_{\mathrm{odd}}(A) = \prod_{k\in\mathbb{Z}} C_{2k+1}(A).$$

We denote the even and odd periodic cyclic homology groups of the A_∞-algebra A by $HP_0(A)$ and $HP_1(A)$ respectively. In [13], the Hochschild and cyclic complexes arise from a construction that is functorial in the bar coalgebra $B(A)$. It follows that periodic cyclic homology is functorial with respect to A_∞-morphisms. Now the periodic cyclic homology of a differential graded algebra, viewed as an A_∞-algebra, is the same as the periodic cyclic homology defined in section 2.6. Thus we obtain the following.

Theorem 2.7.8. *If $\Omega^\bullet$ and $\Omega'^\bullet$ are A_∞-isomorphic differential graded algebras, then $HP_\bullet(\Omega) \cong HP_\bullet(\Omega')$.*

Chapter 3

Smooth one-parameter deformations

3.1 Smooth deformations of algebras

Let $X \in$ LCTVS and let J denote an open subinterval of $\mathbb{R}$.

Definition 3.1.1. A $\check{\otimes}$-*smooth one-parameter deformation of algebras* is a $\check{\otimes}$-smooth family of continuous linear maps $\{m_t : X\check{\otimes}X \to X\}_{t\in J}$ for which each m_t is associative.

So for each $t \in J$, we have a locally convex $\check{\otimes}$-algebra $A_t := (X, m_t)$ whose underlying space is X. We shall often refer to the deformation as $\{A_t\}_{t\in J}$ when the multiplications are understood.

Given such a deformation $\{m_t\}_{t\in J}$ of multiplications on X, consider the continuous $C^\infty(J)$-linear map

$$m : C^\infty(J)\check{\otimes}(X\check{\otimes}X) \to C^\infty(J)\check{\otimes}X$$

associated to $\{m_t\}$ as in Proposition 2.1.10. Set $A = C^\infty(J)\check{\otimes}X$, so that m can be viewed as a map

$$m : A\check{\otimes}_{C^\infty(J)}A \to A$$

using Proposition 2.1.7. Associativity of m follows from associativity of the family $\{m_t\}$. Thus A is a locally convex $\check{\otimes}$-algebra over $C^\infty(J)$, which we shall refer to as the *algebra of sections* of the deformation $\{A_t\}_{t\in J}$. Explicitly, the multiplication in A is given by

$$(a_1a_2)(t) = m_t(a_1(t), a_2(t)).$$

Note that the "evaluation maps" $\epsilon_t : A \to A_t$ are algebra maps.

Proposition 3.1.2. *Associating to a deformation its algebra of sections gives a one-to-one correspondence between $\check{\otimes}$-smooth one-parameter deformations of algebras with underlying space X and parameter space J, and locally convex $\check{\otimes}$-algebra structures over $C^\infty(J)$ on $C^\infty(J)\check{\otimes}X$.*

In many deformations of interest to us, the underlying space X is a Fréchet space. Here, we need not distinguish between topological tensor products, as

$$C^\infty(J)\bar{\otimes}X \cong C^\infty(J)\widehat{\otimes}X \cong C^\infty(J, X),$$

and we shall refer to such deformations simply as smooth deformations of Fréchet algebras. In this case, it is easier to check in practice that a deformation satisfies the smoothness condition.

Proposition 3.1.3. *If X is a Fréchet space, then a set of continuous associative multiplications $\{m_t : X\widehat{\otimes}X \to X\}_{t\in J}$ is a smooth one-parameter deformation of Fréchet algebras if and only if the map*

$$t \mapsto m_t(x_1, x_2)$$

is smooth for each fixed $x_1, x_2 \in X$.

Proof. If $\{m_t\}_{t\in J}$ is a smooth one-parameter deformation, then it is immediate that $t \mapsto m_t(x_1, x_2)$ is smooth for all $x_1, x_2 \in X$.

Conversely, if $t \mapsto m_t(x_1, x_2)$ is smooth for each fixed $x_1, x_2 \in X$, then the map

$$m : X \times X \to C^\infty(J, X)$$

given by

$$m(x_1, x_2)(t) = m_t(x_1, x_2)$$

is separately continuous by Proposition 2.1.11, because Fréchet spaces are barreled.

Thus, m induces a continuous linear map

$$m : X \bar{\otimes} X \to C^{\infty}(J, X),$$

which shows that $\{m_t\}_{t\in J}$ is a $\bar{\otimes}$-smooth family of continuous linear maps. $\square$

Now, suppose A and B are the algebras of sections of two $\check{\otimes}$-smooth one-parameter deformations $\{A_t\}_{t\in J}$ and $\{B_t\}_{t\in J}$ with underlying spaces X and Y respectively.

Definition 3.1.4. A *morphism* $\{F_t\}_{t\in J}$ between two $\check{\otimes}$-smooth one-parameter deformations $\{A_t\}_{t\in J}$ and $\{B_t\}_{t\in J}$ is a $\check{\otimes}$-smooth family $\{F_t\}_{t\in J}$ of algebra homomorphisms. That is,

(i) for each $t \in J$, F_t is a continuous algebra map with respect to the algebra structures of A_t and B_t.

(ii) the maps $\{F_t\}_{t\in J}$ form a $\check{\otimes}$-smooth family in the sense of Definition 2.1.9.

Proposition 3.1.5. *There is a one-to-one correspondence between morphisms from $\{A_t\}_{t\in J}$ to $\{B_t\}_{t\in J}$ and continuous $C^{\infty}(J)$-linear algebra maps from A to B.*

Proof. Apply Proposition 2.1.10. $\square$

Two $\check{\otimes}$-smooth one-parameter deformations $\{A_t\}_{t\in J}$ and $\{B_t\}_{t\in J}$ are *isomorphic* if there are morphisms

$$\{F_t : A_t \to B_t\}_{t\in J}, \qquad \{G_t : B_t \to A_t\}_{t\in J}$$

such that

$$G_t F_t = \mathrm{id}, \qquad F_t G_t = \mathrm{id}, \qquad \forall t \in J.$$

Thus, we see that two deformations are isomorphic if and only if their algebras of sections are isomorphic as locally convex $\check{\otimes}$-modules over $C^{\infty}(J)$. A *constant deformation* is one for which the multiplications $\{m_t\}_{t\in J}$ do not depend on t. The algebra of sections of a constant deformation is of the form

$$A = C^{\infty}(J) \check{\otimes} B$$

for some locally convex $\check{\otimes}$-algebra B, where the product of A is given by

$$(f_1 \otimes b_1)(f_2 \otimes b_2) = f_1 f_2 \otimes b_1 b_2.$$

A deformation is *trivial* if it is isomorphic to a constant deformation.

3.1.1 Smooth noncommutative tori

Given an $n \times n$ skew-symmetric real valued matrix Θ, the *noncommutative torus* A_Θ is the universal C^*-algebra generated by n unitaries $u_1, \ldots, u_n$ such that

$$u_j u_k = e^{2\pi i \theta_{jk}} u_k u_j,$$

where $\Theta = (\theta_{jk})$, see [30]. In the case $\Theta = 0$, all of the generating unitaries commute, and we have $A_0 \cong C(\mathbb{T}^n)$, the algebra of continuous functions on the n-torus. It is for this reason why the algebra A_Θ has earned its name, as it can be philosophically viewed as functions on some "noncommutative torus" in the spirit of Alain Connes' noncommutative geometry [6]. We shall be interested in a dense subalgebra $\mathcal{A}_\Theta \subset A_\Theta$ which plays the role of the smooth functions on the noncommutative torus. As a topological vector space, $\mathcal{A}_\Theta$ is the Schwartz space $\mathcal{S}(\mathbb{Z}^n)$ of complex-valued sequences indexed by $\mathbb{Z}^n$ of rapid decay, defined as follows. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we shall write

$$|\alpha| = |\alpha_1| + \ldots + |\alpha_n|.$$

A sequence $x = (x_\alpha)_{\alpha \in \mathbb{Z}^n}$ is of *rapid decay* if for every positive integer k,

$$p_k(x) := \sum_{\alpha \in \mathbb{Z}^n} (1 + |\alpha|)^k |x_\alpha| < \infty.$$

The functions p_k are seminorms, and the topology on $\mathcal{A}_\Theta$ is the locally convex topology defined by these seminorms. Under this topology, $\mathcal{A}_\Theta$ is complete, and in fact is a nuclear Fréchet space [33, Chapter 51]. The identification of $\mathcal{A}_\Theta$ as a subalgebra of A_Θ is given by the map $\iota : \mathcal{A}_\Theta \to A_\Theta$,

$$\iota(x) = \sum_{\alpha \in \mathbb{Z}^n} x_\alpha u^\alpha,$$

where $u^\alpha = u_1^{\alpha_1} u_2^{\alpha_2} \dots u_n^{\alpha_n}$. Since u^α is a unitary, it has norm 1 in the C^*-algebra A_Θ, and consequently

$$\|\iota(x)\| \leq \sum_{\alpha \in \mathbb{Z}^n} |x_\alpha| = p_0(x) < \infty.$$

This show that the series defining $\iota(x)$ is absolutely convergent, and also that the inclusion ι is continuous. Then $\mathcal{A}_\Theta$ is a norm dense subalgebra of A_Θ because it contains the $*$-algebra generated by $u_1, \dots, u_n$. The multiplication is therefore given by the twisted convolution product

$$(xy)_\alpha = \sum_{\beta \in \mathbb{Z}^n} e^{2\pi i B_\Theta(\alpha-\beta,\beta)} x_{\alpha-\beta} y_\beta,$$

where

$$B_\Theta(\alpha, \beta) = \sum_{j>k} \alpha_j \beta_k \theta_{jk}.$$

One can show that for any k,

$$p_k(xy) \leq p_k(x) p_k(y), \qquad \forall x, y \in \mathcal{A}_\Theta,$$

and thus $\mathcal{A}_\Theta$ is an m-convex Fréchet algebra. In the case $\Theta = 0$, an element in $\mathcal{A}_0 = \mathcal{S}(\mathbb{Z}^n)$ represents the Fourier coefficients of a function on $\mathbb{T}^n$. The rapid decay condition implies that this function is smooth, and in fact

$$\mathcal{A}_0 \cong C^\infty(\mathbb{T}^n)$$

as Fréchet algebras.

The algebra $\mathcal{A}_\Theta$ possesses n canonical continuous derivations

$$\delta_1, \dots, \delta_n : \mathcal{A}_\Theta \to \mathcal{A}_\Theta$$

defined by

$$(\delta_j(x))_\alpha = 2\pi i \alpha_j \cdot x_\alpha.$$

Under the identification $\mathcal{A}_0 \cong C^\infty(\mathbb{T}^n)$, these are the usual partial differential operators. There is also a canonical continuous trace $\tau : \mathcal{A}_\Theta \to \mathbb{C}$ given by $\tau(x) = x_0$, which corresponds to integration with respect to the normalized Haar measure in the case $\Theta = 0$.

We view the smooth noncommutative torus $\mathcal{A}_\Theta$ as a smooth one-parameter deformation of $C^\infty(\mathbb{T}^n) \cong \mathcal{A}_0$ in the following way. For each $t \in J = \mathbb{R}$, let $A_t = \mathcal{A}_{t\Theta}$. The product in A_t is given by

$$m_t(x,y)_\alpha = \sum_{\beta \in \mathbb{Z}^n} e^{2\pi i B_\Theta(\alpha-\beta,\beta)t} x_{\alpha-\beta} y_\beta.$$

Proposition 3.1.6. *Given an $n \times n$ skew-symmetric real matrix Θ, the deformation $\{\mathcal{A}_{t\Theta}\}_{t\in\mathbb{R}}$ is a smooth one-parameter deformation of Fréchet algebras, and for x, y in the underlying space $\mathcal{S}(\mathbb{Z}^n)$,*

$$\frac{d}{dt} m_t(x,y) = \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} m_t(\delta_j(x), \delta_k(y)).$$

Proof. By Proposition 3.1.3, it suffices to show that $t \mapsto m_t(x,y)$ is smooth for each fixed $x, y \in \mathcal{S}(\mathbb{Z}^n)$. First, we will show that it is continuous. Every element $x \in \mathcal{S}(\mathbb{Z}^n)$ can be expressed as an absolutely convergent series in the natural basis $\{u_\alpha \mid \alpha \in \mathbb{Z}^n\}$. That is, given $x = \sum_{\alpha\in\mathbb{Z}^n} x_\alpha u^\alpha$, then

$$\sum_{\alpha\in\mathbb{Z}^n} p_k(x_\alpha u^\alpha) = p_k(x) < \infty$$

for all k, which says that the series is absolutely convergent in $\mathcal{S}(\mathbb{Z}^n)$. So, for fixed $x, y \in \mathcal{S}(\mathbb{Z}^n)$, the inequality

$$p_k(m_t(x,y)) \le p_k(x) p_k(y) < \infty$$

says that the absolute convergence of the series defining $m_t(x,y)$ is uniform in t. Since each partial sum in $m_t(x,y)$ is clearly continuous in t, the function $t \mapsto m_t(x,y)$ is continuous.

Next, note that for a fixed α,

$$\sum_{\beta\in\mathbb{Z}^n}\left|\frac{d}{dt}e^{2\pi iB_\Theta(\alpha-\beta,\beta)t}x_{\alpha-\beta}y_\beta\right| = \sum_{\beta\in\mathbb{Z}^n}|2\pi iB_\Theta(\alpha-\beta,\beta)||x_{\alpha-\beta}||y_\beta|$$
$$\leq \sum_{\beta\in\mathbb{Z}^n} q(\beta)|y_\beta| < \infty$$

for some polynomial $q(\beta)$. Here we have used the fact that $|x_{\alpha-\beta}|$ is bounded and $B_\Theta(\alpha-\beta,\beta)$ is a polynomial function of β. By Corollary A.2.4, we have that

$$\frac{d}{dt}[m_t(x,y)_\alpha] = \sum_{\beta\in\mathbb{Z}^n}\frac{d}{dt}e^{2\pi iB_\Theta(\alpha-\beta,\beta)t}x_{\alpha-\beta}y_\beta$$
$$= \sum_{\beta\in\mathbb{Z}^n} 2\pi iB_\Theta(\alpha-\beta,\beta)e^{2\pi iB_\Theta(\alpha-\beta,\beta)t}x_{\alpha-\beta}y_\beta.$$

By iterating and using the same argument,

$$\frac{d^r}{dt^r}[m_t(x,y)_\alpha] = \sum_{\beta\in\mathbb{Z}^n}(2\pi iB_\Theta(\alpha-\beta,\beta))^r e^{2\pi iB_\Theta(\alpha-\beta,\beta)t}x_{\alpha-\beta}y_\beta$$

for any positive integer r. Define the continuous linear map

$$\widetilde{E} : \mathcal{S}(\mathbb{Z}^n)\widehat{\otimes}\mathcal{S}(\mathbb{Z}^n) \to \mathcal{S}(\mathbb{Z}^n)\widehat{\otimes}\mathcal{S}(\mathbb{Z}^n)$$

by

$$\widetilde{E}(x\otimes y) = \frac{1}{2\pi i}\sum_{j>k}\theta_{jk}\delta_j(x)\otimes\delta_k(y).$$

Then we see that

$$\frac{d^r}{dt^r}[m_t(x,y)_\alpha] = m_t(\widetilde{E}^r(x,y))_\alpha$$

for each $\alpha\in\mathbb{Z}^n$. Now $\widetilde{E}^r(x,y)$ is a finite sum of elementary tensors, so the series defining $m_t(\widetilde{E}^r(x,y))$ converges uniformly in t by the above result. But $m_t(\widetilde{E}^r(x,y))$ is the series of r-th derivatives of the terms of the series for $m_t(x,y)$. So by Corollary A.2.4, we have

$$\frac{d^r}{dt^r}m_t(x,y) = m_t(\widetilde{E}^r(x,y)) \in \mathcal{S}(\mathbb{Z}^n).$$

This shows the deformation is smooth. □

3.1.2 Crossed products by $\mathbb{R}$

Let B be a Fréchet-algebra equipped with a group action

$$\alpha : \mathbb{R} \to \mathrm{Aut}(B)$$

such that

(i) the map $s \mapsto \alpha_s(b)$ is smooth for all $b \in B$,

(ii) for every number m and continuous seminorm p on B, there is a number k and another continuous seminorm q on B such that

$$p(\frac{d^m}{ds^m}\alpha_s(b)) \leq (1+|s|)^k q(b), \qquad \forall b \in B.$$

As in [9], we shall call such an action a *smooth action.* If additionally $p(\alpha_s(b)) = p(b)$ for each seminorm p defining the topology of B, then we shall call the action *isometric.*

Example 3.1.7. Let A be a C^*-algebra equipped with a group action

$$\alpha : \mathbb{R} \to \mathrm{Aut}(A)$$

by $*$-automorphisms. As in [4], let δ denote the densely defined derivation on A given by

$$\delta(a) = \lim_{h \to 0} \frac{\alpha_h(a) - a}{h},$$

and let B be the subalgebra of all $a \in A$ for which $\delta^n(a)$ exists for all n. Then B is an m-convex Fréchet algebra with respect to the family $\{p_n\}_{n=0}^{\infty}$ of seminorms given by

$$p_n(b) = \sum_{k=0}^{n} \frac{1}{k!}\|\delta^k(b)\|,$$

where $\|\cdot\|$ denotes the C^*-algebra norm of A. Then one can verify that the restriction of α to B is an isometric smooth action.

Following [9], given a smooth action α of $\mathbb{R}$ on B, define the *smooth crossed product* to be

$$B \rtimes_\alpha \mathbb{R} = \{f \in C^\infty(\mathbb{R}, B) \mid \sup_{s\in\mathbb{R}}(1+|s|)^m f^{(n)}(s) < \infty, \text{ for all nonnegative } m, n\}.$$

For each defining seminorm p on B, and each $m, n \geq 0$, define a seminorm on $B \rtimes_\alpha \mathbb{R}$ by

$$p_{m,n}(f) = \sum_{k=0}^{n} \int_{\mathbb{R}} (1+|s|)^m p(f^{(k)}(s))ds.$$

Then $B \rtimes_\alpha \mathbb{R}$ is complete and Hausdorff with respect to the topology defined by these seminorms, and so it is a Fréchet space. We make $B \rtimes_\alpha \mathbb{R}$ an algebra with the twisted convolution product

$$(f * g)(s) = \int_{\mathbb{R}} f(u)\alpha_u(g(s-u))du,$$

which one can check is associative.

Proposition 3.1.8. *If α is a smooth action of $\mathbb{R}$ on a Fréchet algebra B, then $B \rtimes_\alpha \mathbb{R}$ is a Fréchet algebra. If B is m-convex and α is isometric, then $B \rtimes_\alpha \mathbb{R}$ is m-convex.*

Proof. If p is a defining seminorm, choose continuous seminorms q and r and a number k such that

$$p(b_1 b_2) \leq q(b_1)q(b_2), \qquad \forall b_1, b_2 \in B$$

and

$$q(\alpha_s(b)) \leq (1+|s|)^k r(b), \qquad \forall b \in B,$$

as in the definition of smooth action. Then we first calculate

$$\begin{aligned} p_{m,0}(f * g) &= \int_{\mathbb{R}} (1+|s|)^m p((f * g)(s))ds \\ &= \int_{\mathbb{R}} (1+|s|)^m p(\int_{\mathbb{R}} f(u)\alpha_u(g(s-u))du)ds \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} (1+|s|)^m p(f(u)\alpha_u(g(s-u))duds \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}(1+|s|)^{m+k}q(f(u))r(g(s-u))duds \\
&\leq \int_{\mathbb{R}}\int_{\mathbb{R}}(1+|u|)^{m+k}(1+|s-u|)^{m+k}q(f(u))r(g(s-u))duds \\
&= q_{m+k,0}(f)r_{m+k,0}(g).
\end{aligned}$$

Combining $\frac{d}{ds}(f*g)=f*g'$, and $p_{m,n}(f)=\sum_{l=0}^{n}p_{m,0}(f^{(l)})$, we have

$$\begin{aligned}
p_{m,n}(f*g) &= \sum_{l=0}^{n}p_{m,0}(f*g^{(l)}) \\
&\leq \sum_{l=0}^{n}q_{m+k,0}(f)r_{m+k,0}(g^{(l)}) \\
&= q_{m+k,0}(f)r_{m+k,n}(g),
\end{aligned}$$

which shows that product is jointly continuous. If B is m-convex and the action is isometric, we can choose $q=r=p$ and $k=0$. Thus

$$p_{m,n}(f*g)\leq p_{m,0}(f)p_{m,n}(g)\leq p_{m,n}(f)p_{m,n}(g),$$

which shows that $B\rtimes_\alpha\mathbb{R}$ is m-convex. □

Given a smooth action α on B, let $\delta : B\to B$ be the derivation given by

$$\delta(b)=\lim_{h\to 0}\frac{\alpha_h(b)-b}{h}.$$

By the first condition of smooth action, this limit exists for each $b\in B$, and the second condition for $m=1$ and $s=0$ implies δ is continuous. Moreover,

$$\frac{d}{ds}\alpha_s(b)=\alpha_s(\delta(b))=\delta(\alpha_s(b)),\qquad \forall b\in B.$$

Then δ gives a derivation on $B\rtimes_\alpha\mathbb{R}$, which we shall also call δ, by the formula

$$(\delta f)(s)=\delta(f(s)).$$

Let ∂ be the operator on $B \rtimes_\alpha \mathbb{R}$ given by

$$(\partial f)(s) = 2\pi i s \cdot f(s).$$

Then ∂ is a continuous derivation, and ∂ commutes with δ.

Given $t \in \mathbb{R}$, let $\alpha^t : \mathbb{R} \to \mathrm{Aut}(B)$ denote the rescaled action

$$\alpha^t_s = \alpha_{ts}.$$

Then α^t is also a smooth action, so we may form the family of crossed product algebras $\{B \rtimes_{\alpha^t} \mathbb{R}\}_{t\in\mathbb{R}}$, all of which have the same underlying Fréchet space. We shall write $*_t$ for the convolution product in $B \rtimes_{\alpha^t} \mathbb{R}$.

Proposition 3.1.9. *Given a smooth action α on B,*

$$\frac{d}{dt}(f *_t g) = \frac{1}{2\pi i}(\partial f) *_t (\delta g)$$

for all B-valued Schwartz functions f and g. Consequently, $\{B \rtimes_{\alpha^t} \mathbb{R}\}_{t\in\mathbb{R}}$ is a smooth one-parameter deformation of Fréchet algebras.

Proof. We shall use Corollary A.2.4, to justify differentiating under the integral sign. Let $h_N \in C(\mathbb{R}, \mathcal{S}(\mathbb{R}, B))$ be given by

$$h_N(t)(s) = \int_{-N}^{N} f(u)\alpha_{tu}(g(s-u))du.$$

Then for each $t \in \mathbb{R}$, $h_N(t) \to f *_t g$ as $N \to \infty$. Since h_N is given by an integral over a finite domain,

$$\frac{d}{dt}h_N(t)(s) = \int_{-N}^{N} f(u)\frac{d}{dt}(\alpha_t u(g(s-u)))du.$$

We claim that in the topology of $C(\mathbb{R}, \mathcal{S}(\mathbb{R}, B))$ (uniform convergence on compact subsets of $\mathbb{R}$,) $\frac{d}{dt}h_N$ converge to the function

$$K(t)(s) = \int_{\mathbb{R}} f(u)\frac{d}{dt}(\alpha_{tu}(g(s-u)))du$$

as $N \to \infty$. Let $M > 1$ and consider t in the compact subset $[-M, M]$. Then for any continuous seminorm p on B and numbers m and n, there are continuous seminorms q, r on B and number l such that

$$\begin{aligned}
&p_{m,n}(K(t) - \frac{d}{dt}h_N(t)) \\
&\quad = \sum_{k=0}^{n} \int_{\mathbb{R}} (1+|s|)^m p(\frac{d^k}{ds^k}(K(t)(s) - h_N(t)(s))ds \\
&\quad = \sum_{k=0}^{n} \int_{\mathbb{R}} (1+|s|)^m p(\int_{|u|\geq N} f(u)\frac{d}{dt}(\alpha_{tu}(g^{(k)}(s-u)))du)ds \\
&\quad \leq \sum_{k=0}^{n} \int_{\mathbb{R}} \int_{|u|\geq N} (1+|s|)^m q(f(u))q(\frac{d}{dt}(\alpha_{tu}(g^{(k)}(s-u))))duds \\
&\quad \leq \sum_{k=0}^{n} \int_{\mathbb{R}} \int_{|u|\geq N} (1+|s|)^m q(f(u))u(1+|tu|)^l r(g^{(k)}(s-u))duds \\
&\quad \leq M^l \sum_{k=0}^{n} \int_{\mathbb{R}} \int_{|u|\geq N} (1+|s|)^m (1+|u|)^{l+1} q(f(u))r(g^{(k)}(s-u))duds \\
&\quad \leq M^l \sum_{k=0}^{n} \int_{\mathbb{R}} \int_{|u|\geq N} (1+|u|)^{m+l+1}(1+|s-u|)^m q(f(u))r(g^{(k)}(s-u))duds \\
&\quad = M^l \sum_{k=0}^{n} \int_{|u|\geq N} \int_{\mathbb{R}} (1+|u|)^{m+l+1}(1+|s|)^m q(f(u))r(g^{(k)}(s))dsdu \\
&\quad = M^l r_{m,n}(g) \int_{|u|\geq N} (1+|u|)^{m+l+1} q(f(u))du,
\end{aligned}$$

which goes to 0 as $N \to \infty$. Thus by Corollary A.2.4,

$$\begin{aligned}
\frac{d}{dt}(f *_t g)(s) &= K(t)(s) \\
&= \int_{\mathbb{R}} f(u)\frac{d}{dt}(\alpha_{tu}(g(s-u)))du \\
&= \int_{\mathbb{R}} f(u)u(\alpha_{tu}(\delta(g(s-u))))du \\
&= \int_{\mathbb{R}} \frac{1}{2\pi i}(\partial f)(u)(\alpha_{tu}(\delta(g(s-u))))du \\
&= \frac{1}{2\pi i}((\partial f) *_t (\delta g))(s).
\end{aligned}$$

By iteration, $t \mapsto f *_t g$ is infinitely differentiable. Thus, $\{B \rtimes_{\alpha^t} \mathbb{R}\}_{t\in\mathbb{R}}$ is a smooth one-parameter deformation by Proposition 3.1.3. $\square$

Of special interest is the case $t = 0$, where the action is trivial. Here, the product is

$$(f *_0 g)(s) = \int_{\mathbb{R}} f(u)g(s-u)du.$$

So as an algebra,

$$B \rtimes_{\alpha^0} \mathbb{R} \cong \mathcal{S}(\mathbb{R})\widehat{\otimes} B,$$

where $\mathcal{S}(\mathbb{R})$ is the Fréchet algebra of complex-valued Schwartz functions with the convolution product. In terms of definitions given above, $\mathcal{S}(\mathbb{R}) = \mathbb{C} \rtimes \mathbb{R}$ with the trivial action. We conclude that any smooth crossed product $B \rtimes_\alpha \mathbb{R}$ can be deformed smoothly into the trivial crossed product $\mathcal{S}(\mathbb{R})\widehat{\otimes} B$.

3.1.3 Compatible connections

Let A be the algebra of sections of a $\check{\otimes}$-smooth one-parameter deformation of algebras $\{A_t\}_{t\in J}$, and let ∇ be a connection on A. Viewing ∇ as a Hochschild cochain $\nabla \in C^1_{\mathbb{C}}(A,A)$, let $E = \delta\nabla$, so that

$$\nabla(a_1a_2) = \nabla(a_1)a_2 + a_1\nabla(a_2) - E(a_1,a_2).$$

From it's definition, it is clear that $\delta E = 0$, that is, E is a Hochschild 2-cocycle. However, from the Leibniz rule for ∇, one can check that E is $C^\infty(J)$-bilinear. So E defines a cohomology class in $H^2_{C^\infty(J)}(A,A)$. It may appear from its definition that E is a coboundary, but this is not necessarily the case because ∇ is not a $C^\infty(J)$-linear map.

Proposition 3.1.10. *The cohomology class of E in $H^2_{C^\infty(J)}(A,A)$ is independent of the choice of connection. Moreover, $[E] = 0$ if and only if A admits a connection that is a derivation.*

Proof. Let ∇ and ∇' be two connections with corresponding cocycles E and E'. Then $\nabla' = \nabla - F$ for some $F \in C^1_{C^\infty(J)}(A,A)$, and so

$$E' = \delta(\nabla - F) = E - \delta F.$$

Thus $[E] = [E']$ as elements of $H^2_{C^\infty(J)}(A, A)$.

If ∇ is a connection that is a derivation, then $E = \delta\nabla = 0$. Conversely, if ∇ is any connection on A and $[E] = 0$, then $\delta\nabla = \delta F$ for some $F \in C^1_{C^\infty(J)}(A, A)$. Thus, $\delta(\nabla - F) = 0$, and so $\nabla - F$ is a connection that is a derivation. $\square$

Proposition 3.1.11. *A $\check{\otimes}$-smooth one-parameter deformation of algebras $\{A_t\}_{t\in J}$ is trivial if and only if the algebra of sections A admits an integrable connection that is a derivation. For such a connection ∇, the parallel transport map $P^\nabla_{s,t} : A_s \to A_t$ is an isomorphism of locally convex $\check{\otimes}$-algebras for all $s, t \in J$.*

Proof. For a constant deformation, the algebra of sections is $C^\infty(J)\check{\otimes}B$, as an algebra, for some locally convex $\check{\otimes}$-algebra B. Then $\frac{d}{dt}$ is an integrable connection on $C^\infty(J)\check{\otimes}B$ which is also a derivation. If A is the algebra of sections of another deformation, and

$$F : A \to C^\infty(J)\check{\otimes}B$$

is an algebra isomorphism, then F is a parallel isomorphism

$$F : (A, F^{-1}\frac{d}{dt}F) \to (C^\infty(J)\check{\otimes}B, \frac{d}{dt}).$$

So $F^{-1}\frac{d}{dt}F$ is a connection and a derivation, and by Proposition 2.2.5, $F^{-1}\frac{d}{dt}F$ is integrable. This shows that a trivial deformation admits an integrable connection that is a derivation.

If ∇ is an integrable connection on A, then the connection $\nabla \otimes 1 + 1 \otimes \nabla$ on $A \otimes_{C^\infty(J)} A$ is integrable with parallel transport maps of the form $P^\nabla_{s,t} \otimes P^\nabla_{s,t}$ by Proposition 2.2.6. That ∇ is a derivation is equivalent to the multiplication $m : A \otimes_{C^\infty(J)} A \to A$ being a parallel map. Thus,

$$P^\nabla_{s,t} m_s = m_t(P^\nabla_{s,t} \otimes P^\nabla_{s,t})$$

by Proposition 2.2.5, which shows $P^\nabla_{s,t}$ is an algebra isomorphism. So any trivialization

$$A \cong C^\infty(J)\check{\otimes}A_s$$

constructed, as in Proposition 2.2.4 using the parallel transport maps, is an algebra isomorphism. $\square$

From this result, we see that the cohomology class $[E]$ provides an obstruction to the triviality of a deformation.

Example 3.1.12. Suppose the family of products $\{m_t\}_{t\in J}$ of a deformation has the property that

$$\frac{d}{dt}m_t(a_1,a_2)=\sum_{i=1}^{N}m_t(X_i(a_1),Y_i(a_2))$$

for all a_1, a_2 in the underlying space and some collection

$$\{X_1,\ldots,X_N,Y_1,\ldots,Y_N\}$$

of commuting operators on the underlying space, which are derivations with respect to each product m_t. Notice that both the noncommutative tori deformation and the smooth crossed product deformation both have this property. It follows that if A denotes the algebra of sections of the deformation and $\nabla=\frac{d}{dt}$, then

$$E=\delta\nabla=\sum_{i=1}^{N}X_i\smile Y_i.$$

Indeed, both $\delta\nabla$ and $\sum_{i=1}^{N}X_i\smile Y_i$ are continuous and $C^\infty(J)$-bilinear, and the equation

$$\frac{d}{dt}m_t(a_1,a_2)=\sum_{i=1}^{N}m_t(X_i(a_1),Y_i(a_2))$$

shows that they are equal on constant sections. Thus they are equal because the $C^\infty(J)$-linear span of the constant sections is dense in A.

For the noncommutative tori deformation $\{\mathcal{A}_{t\Theta}\}_{t\in J}$, one can show that

$$E=\frac{1}{2\pi i}\sum_{j>k}\theta_{jk}(\delta_j\smile\delta_k)$$

does not represent the zero class in $H^2_{C^\infty(J)}(A,A)$, provided $\Theta\neq 0$. Consequently, the deformation $\{\mathcal{A}_{t\Theta}\}_{t\in J}$ is nontrivial. This is consistent with the fact that the isomorphism class of the algebra $\mathcal{A}_{t\Theta}$ varies as t varies.

3.2 Smooth deformations of cochain complexes

Let $X^\bullet \in \mathrm{LCTVS}$ be a graded space and let J denote an open subinterval of $\mathbb{R}$.

Definition 3.2.1. A $\breve{\otimes}$-*smooth one-parameter deformation of cochain complexes* is a collection of $\breve{\otimes}$-smooth families of continuous linear maps $\{d_t^n : X^n \to X^{n+1}\}_{t\in J}$ for which $d_t^{n+1}d_t^n = 0$ for all n and all $t \in J$.

So for each $t \in J$, let $C_t^\bullet$ denote the complex which is $X^\bullet$ equipped with the coboundary map d_t. Define the *cochain complex of sections* of the deformation $\{C_t^\bullet\}_{t\in J}$ to be the locally convex cochain complex $C^\bullet$ over $C^\infty(J)$ where

$$C^n = C^\infty(J)\breve{\otimes}X^n$$

and the coboundary

$$d : C^n \to C^{n+1}$$

is given by

$$(dc)(t) = d_t(c(t)).$$

By Proposition 2.1.10, d is a continuous $C^\infty(J)$-linear map. The cohomology $H^\bullet(C)$ is a module over $C^\infty(J)$. Of course, if we consider boundary maps of degree -1, we obtain the notion of a $\breve{\otimes}$-smooth one-parameter deformation of chain complexes.

Example 3.2.2. If $\{A_t\}_{t\in J}$ is a $\breve{\otimes}$-smooth one-parameter deformation of algebras, then $\{(C_{\mathrm{per}}(A_t), b_t + B)\}_{t\in J}$ is a $\breve{\otimes}$-smooth one-parameter deformation of chain complexes. Notice that the Hochschild boundary b_t depends on the multiplication of A_t, whereas B does not. The complex of sections of $\{C_{\mathrm{per}}(A_t)\}_{t\in J}$ is isomorphic to the periodic cyclic complex $C_{\mathrm{per}}^{C^\infty(J)}(A)$, where A is the algebra of sections of $\{A_t\}_{t\in J}$.

A *morphism* between two $\breve{\otimes}$-smooth one parameter deformations of cochain complexes $\{C_t^\bullet\}_{t\in J}$ and $\{D_t^\bullet\}_{t\in J}$ is a $\breve{\otimes}$-smooth family of continuous chain maps

$$\{F_t : C_t^\bullet \to D_t^\bullet\}_{t\in J}.$$

A deformation $\{C_t^\bullet, d_t\}_{t\in J}$ is *constant* if d_t does not depend on t, and *trivial* if it is isomorphic to a constant deformation.

3.2.1 Compatible connections

Given a $\check{\otimes}$-smooth one-parameter deformation of complexes $\{C_t^\bullet\}_{t\in J}$, we shall be interested in degree 0 connections on the complex of sections $C^\bullet$ for which the boundary map is parallel. Following the algebra case, let

$$H = [d, \nabla] : C^\bullet \to C^{\bullet+1}.$$

It follows from the Leibniz rule that H is $C^\infty(J)$-linear. Recall that $\mathrm{End}_{C^\infty(J)}(C^\bullet)$ is a cochain complex whose coboundary map is given by the commutator with d. By definition, it is clear that $[d, H] = 0$, so H defines a cohomology class in $H^\bullet(End_{C^\infty(J)}(C))$. Similar to the algebra case, we need not have $[H] = 0$, because ∇ is not $C^\infty(J)$-linear.

Proposition 3.2.3. *The cohomology class $[H] \in H^\bullet(End_{C^\infty(J)}(C))$ is independent of the choice of connection ∇. Moreover, $[H] = 0$ if and only if $C^\bullet$ possesses a connection that is a chain map.*

Proof. The proof is completely analogous to that of Proposition 3.1.10. □

Proposition 3.2.4. *A $\check{\otimes}$-smooth one-parameter deformation of cochain complexes $\{C_t^\bullet\}_{t\in J}$ is trivial if and only if the complex of sections $C^\bullet$ admits an integrable connection that is a chain map. For such a connection ∇, the parallel transport map $P^\nabla_{s,t} : C_s^\bullet \to C_t^\bullet$ is an isomorphism of locally convex cochain complexes for all $s, t \in J$. In particular, the parallel transport maps induce isomorphisms*

$$(P^\nabla_{s,t})_* : H^\bullet(C_s) \to H^\bullet(C_t).$$

Proof. The proof is analogous to the proof of Proposition 3.1.11. Notice that ∇ is a chain map if and only if d is parallel with respect to ∇. So the fact that $P^\nabla_{s,t} : C_s^\bullet \to C_t^\bullet$ is a chain map follows from Proposition 2.2.5. □

So the cohomology class $[H]$ is an obstruction to the triviality of a deformation of complexes. Triviality of deformation of cochain complexes is often too strong of a result to ask for. It also lies outside of our interests, as our goal will be to understand if the cohomology groups are preserved under deformation. If ∇ is a connection on $C^\bullet$ that is also a chain map, then ∇ descends to a connection ∇_* on the $C^\infty(J)$-module $H^\bullet(C)$. Care must be taken here, as the cohomology module $H^\bullet(C)$ need not be free over $C^\infty(J)$ or even Hausdorff. In particular, we must specify what it means for ∇_* to be integrable on $H^\bullet(M)$. We will give a definition that is enough to ensure our desired results. The evaluation maps $\mathrm{ev}_t : C^\bullet \to C_t^\bullet$ induce maps $\mathrm{ev}_{t*} : H^\bullet(C) \to H^\bullet(C_t)$ in homology.

Definition 3.2.5. We shall say ∇_* is *integrable at the level of cohomology* if for every $s \in J$ and for every $c_s \in H^\bullet(C_s)$, there is a unique $c \in H^\bullet(C)$ such that

$$\nabla_* c = 0, \qquad (\mathrm{ev}_s)_* c = c_s,$$

and moreover the linear map

$$H^\bullet(C_s) \to H^\bullet(C), \qquad c_s \mapsto c$$

is continuous.

In this situation, we can define parallel transport maps

$$P_{s,t}^{\nabla_*} : H^\bullet(C_s) \to H^\bullet(C_t), \qquad P_{s,t}^{\nabla_*}(c_s) = (\mathrm{ev}_s)_* c$$

for all $s, t \in J$ by, which are isomorphisms of topological vector spaces.

3.3 The Gauss-Manin connection

The Gauss-Manin connection in cyclic homology was first introduced by Getzler in the context of formal deformations of A_∞-algebras [12]. For simplicity, we shall only work with associative algebra deformations here.

3.3.1 Gauss-Manin connection in periodic cyclic homology

Let A denote the algebra of sections of a $\check{\otimes}$-smooth one-parameter deformation of algebras $\{A_t\}_{t\in J}$. Our goal is to construct a connection on $C_{\mathrm{per}}^{C^\infty(J)}(A)$ that commutes with $b+B$. Let ∇ be a connection on A and let $E=\delta\nabla$ as in section 3.1.3. Using Proposition 2.2.2, ∇ extends to a connection on $C_{\text{per}}^{C^\infty(J)}(A)$, which is given[1] by L_∇. As in section 3.2.1, let

$$H := [b+B, L_\nabla] = L_E.$$

By Proposition 3.2.3, $C_{\text{per}}^{C^\infty(J)}(A)$ possesses a connection that is a chain map if and only if H is a boundary in the complex $\mathrm{End}_{C^\infty(J)}(C_{\text{per}}(A))$. But the Cartan Homotopy formula of Theorem 2.5.3 implies exactly this. More specifically,

$$[b+B, I_E] = L_E.$$

As I_E is a continuous $C^\infty(J)$-linear map, the *Gauss-Manin connection*

$$\nabla_{GM} = L_\nabla - I_E$$

is a connection on $C_{\text{per}}^{C^\infty(J)}(A)$ and a chain map.

Proposition 3.3.1. *The Gauss-Manin connection ∇_{GM} commutes with the differential $b+B$ and hence induces a connection on the $C^\infty(J)$-module $HP_\bullet^{C^\infty(J)}(A)$. Moreover, the induced connection is independent of the choice of connection ∇ on A.*

Proof. We have already established the first claim. For another connection ∇', let

$$\nabla'_{GM} = L_{\nabla'} - I_{E'}$$

be the corresponding Gauss-Manin connection. Then

$$\nabla' = \nabla - F, \qquad E' = E - \delta F$$

for some $C^\infty(J)$-linear map $F : A \to A$ Thus,

$$\nabla'_{GM} - \nabla_{GM} = -L_F + I_{\delta F} = -[b+B, I_F],$$

[1]Since ∇ is only $\mathbb{C}$-linear, L_∇ is an operator on $C_{\text{per}}^{\mathbb{C}}(A)$. However by the Leibniz rule, L_∇ descends to an operator on $C_{\text{per}}^{C^\infty(J)}(A)$, which is a quotient complex of $C_{\text{per}}^{\mathbb{C}}(A)$.

by Theorem 2.5.3. We conclude that the Gauss-Manin connection is unique up to continuous chain homotopy. □

Corollary 3.3.2. *If A admits a connection ∇ which is also a derivation, then the Gauss-Manin connection on $HP_\bullet^{C^\infty(J)}(A)$ is given by*

$$\nabla_{GM}[\omega] = [L_\nabla \omega].$$

Proof. In this case, $E = \delta\nabla = 0$. □

The following naturality property of ∇_{GM} says that morphisms of deformations induce parallel maps at the level of periodic cyclic homology.

Proposition 3.3.3 (Naturality of ∇_{GM}). *Let A and B denote the algebras of sections of two $\check{\otimes}$-smooth one-parameter deformations over the same parameter space J, and let $F : A \to B$ be a continuous $C^\infty(J)$-linear algebra map. Then the following diagram commutes.*

$$\begin{tikzcd}
HP_\bullet^{C^\infty(J)}(A) \arrow[r, "F_*"] \arrow[d, "\nabla^A_{GM}"'] & HP_\bullet^{C^\infty(J)}(B) \arrow[d, "\nabla^B_{GM}"] \\
HP_\bullet^{C^\infty(J)}(A) \arrow[r, "F_*"'] & HP_\bullet^{C^\infty(J)}(B)
\end{tikzcd}$$

Proof. Let ∇^A and ∇^B denote connections on A and B with respective cocycles E^A and E^B, and let $F_* : C_{\text{per}}^{C^\infty(J)}(A) \to C_{\text{per}}^{C^\infty(J)}(B)$ be the induced map of complexes. For

$$h = F_* I_{\nabla^A} - I_{\nabla^B} F_*,$$

we have

$$\begin{aligned}
[b + B, h] &= F_*[b + B, I_{\nabla^A}] - [b + B, I_{\nabla^B}]F_* \\
&= F_*(L_{\nabla^A} - I_{E^A}) - (L_{\nabla^B} - I_{E^B})F_* \\
&= F_*\nabla^A_{GM} - \nabla^B_{GM} F_*,
\end{aligned}$$

which shows that the diagram commutes up to continuous chain homotopy. The problem is that I_{∇^A} and I_{∇^B} are not well-defined operators on the complexes

$C^{C^\infty(J)}_{\text{per}}(A)$ and $C^{C^\infty(J)}_{\text{per}}(B)$ respectively. However, one can show that by the Leibniz rule, h descends to a map of quotient complexes such that the following diagram

$$\begin{tikzcd} C^{\mathbb{C}}_{\text{per}}(A) \arrow[r, "h"] \arrow[d, "\pi"'] & C^{\mathbb{C}}_{\text{per}}(B) \arrow[d, "\pi"] \\ C^{C^\infty(J)}_{\text{per}}(A) \arrow[r, "\bar{h}"'] & C^{C^\infty(J)}_{\text{per}}(B) \end{tikzcd}$$

commutes, and consequently $[b + B, \bar{h}] = F_* \nabla^A_{GM} - \nabla^B_{GM} F_*$ as desired. □

As a simple application of Proposition 3.3.3, we get a proof of the homotopy invariance property of periodic cyclic homology by considering morphisms between trivial deformations.

Corollary 3.3.4. *(Homotopy Invariance) Let A_0 and B_0 be locally convex $\check{\otimes}$-algebras and let $\{F_t : A_0 \to B_0\}_{t\in J}$ be a $\check{\otimes}$-smooth family of algebra maps. Then the induced map*

$$(F_t)_* : HP_\bullet(A_0) \to HP_\bullet(B_0)$$

does not depend on t.

Proof. Let $A = C^\infty(J)\check{\otimes}A_0$ and $B = C^\infty(J)\check{\otimes}B_0$ be the algebras of sections corresponding to the constant deformations of with fiber A_0 and B_0 respectively. Then $\{F_t\}_{t\in J}$ is a morphism between these constant deformations, and by Proposition 3.1.5 gives a continuous $C^\infty(J)$-linear algebra map $F : A \to B$ such that

$$F(a)(t) = F_t(a(t)).$$

The complex $C^{C^\infty(J)}_{\text{per}}(A) \cong C^\infty(J)\check{\otimes}C_{\text{per}}(A_0)$ is the complex of sections of the constant deformation with fiber $C^{\mathbb{C}}_{\text{per}}(A_0)$. As the connection $\frac{d}{dt}$ is a derivation on A, the Gauss-Manin connection for A is given by $\nabla^A_{GM} = \frac{d}{dt}$, and similarly $\nabla^B_{GM} = \frac{d}{dt}$. We view a cycle $\omega \in C^{\mathbb{C}}_{\text{per}}(A_0)$ as a constant cycle in $C^{C^\infty(J)}_{\text{per}}(A)$, and then Proposition 3.3.3 implies that

$$\frac{d}{dt}[F_t\omega] = F_t \frac{d}{dt}[\omega] = 0.$$

So there is $\eta \in C^{C^\infty(J)}_{\mathrm{per}}(B)$ such that

$$\frac{d}{dt}F_t(\omega) = (b+B)(\eta(t)).$$

But, by the fundamental theorem of calculus,

$$F_{t_1}(\omega) - F_{t_0}(\omega) = \int_{t_0}^{t_1} (b+B)(\eta(s))ds = (b+B)\left(\int_{t_0}^{t_1} \eta(s)ds\right)$$

for any $t_0, t_1 \in J$. □

3.3.2 Dual Gauss-Manin connection

We define ∇^{GM} on $C^{\mathrm{per}}_{C^\infty(J)}(A)$ to be the dual connection of ∇_{GM} as in Proposition 2.2.2. In terms of the canonical pairing,

$$\langle \nabla^{GM}\varphi, \omega\rangle = \frac{d}{dt}\langle \varphi, \omega\rangle - \langle \varphi, \nabla_{GM}\omega\rangle.$$

It is straightforward to verify that ∇^{GM} commutes with $b+B$ and therefore induces a connection on $HP^\bullet_{C^\infty(J)}(A)$. The connections ∇_{GM} and ∇^{GM} are compatible with the canonical pairing in the sense that

$$\frac{d}{dt}\langle [\varphi], [\omega]\rangle = \langle \nabla^{GM}[\varphi], [\omega]\rangle + \langle [\varphi], \nabla_{GM}[\omega]\rangle,$$

for all $[\varphi] \in HP^\bullet_{C^\infty(J)}(A)$ and $[\omega] \in HP_\bullet^{C^\infty(J)}(A)$.

3.3.3 Integrating ∇_{GM}

The very fact that ∇_{GM} exists for all smooth one-parameter deformations implies that the problem of proving ∇_{GM} is integrable cannot be attacked with methods that are too general. Indeed, one cannot expect periodic cyclic homology to be rigid for all deformations, as there are plenty of finite dimensional examples for which it is not.

Example 3.3.5. For $t \in \mathbb{R}$, let A_t be the two-dimensional algebra generated by an element x and the unit 1 subject to the relation $x^2 = t \cdot 1$. Then $A_t \cong \mathbb{C} \oplus \mathbb{C}$ as an algebra when $t \neq 0$, and A_0 is the exterior algebra on a one dimensional vector space. Consequently,

$$HP_0(A_t) \cong \begin{cases} \mathbb{C} \oplus \mathbb{C}, & t \neq 0 \\ \mathbb{C}, & t = 0. \end{cases}$$

From the point of view of differential equations, one issue is that the periodic cyclic complex is never a Banach space. Even in the case where A is a Banach algebra (e.g. finite dimensional,) the chain groups $C_n(A)$ are also Banach spaces, but the periodic cyclic complex

$$C_{\mathrm{per}}(A) = \prod_{n=0}^{\infty} C_n(A)$$

is a Fréchet space, as it is a countable product of Banach spaces. The operator ∇_{GM} contains the degree -2 term $\iota_E : C_n(A) \to C_{n-2}(A)$. Thus unless $E = 0$, one cannot reduce the problem to the individual Banach space factors, as the differential equations are hopelessly coupled together.

One instance in which ∇_{GM} is integrable is when the algebra of sections A can be trivialized, as in Proposition 3.1.11.

Proposition 3.3.6. *If the algebra of sections A has an integrable connection ∇ that is a derivation, then $\nabla_{GM} = L_\nabla$ is integrable on $C_{\mathrm{per}}^{C^\infty(J)}(A)$, and $P_{s,t}^{\nabla_{GM}} : C_{\mathrm{per}}(A_s) \to C_{\mathrm{per}}(A_t)$ is the map of complexes induced by the algebra isomorphism $P_{s,t}^{\nabla} : A_s \to A_t$.*

This is not surprising in this case, as the deformation is isomorphic to a trivial one. What is interesting to note is that if we consider another connection ∇' on A, the corresponding Gauss-Manin connection ∇'_{GM} on $C_{\mathrm{per}}(A)$ need not be integrable, and in general seems unlikely to be so. However the induced connection ∇'_{GM*} on $HP_\bullet(A)$ is integrable at the level of homology by the uniqueness of the Gauss-Manin connection up to chain homotopy.

As proving integrability of ∇_{GM} at the level of the complex $C_{\mathrm{per}}(A)$ is both too difficult and, in some cases, too strong of a result, our general approach will be to find a different complex that computes $HP_\bullet(A)$ equipped with a compatible connection.

3.4 Smooth deformations of A_∞-algebras

Since we choose to only define locally convex A_∞-algebras using the inductive tensor product, we shall only use the inductive tensor product in this discussion. Let $X^\bullet \in \mathrm{LCTVS}$ be a graded space and let J be an open subinterval of $\mathbb{R}$. Then $C^\infty(J)\bar{\otimes}X^\bullet$ is a graded locally convex $\bar{\otimes}$-module over $C^\infty(J)$. So we can form the bar coalgebra $B(C^\infty(J)\bar{\otimes}X)$ over the ground ring $C^\infty(J)$, and

$$B(C^\infty(J)\bar{\otimes}X) \cong C^\infty(J)\bar{\otimes}B(X),$$

as locally convex graded $\bar{\otimes}$-coalgebras over $C^\infty(J)$. Indeed, using Proposition 2.1.5 and Proposition 2.1.7, we have

$$\begin{aligned}
C^\infty(J)\bar{\otimes}B(X) &= C^\infty(J)\bar{\otimes}\left(\bigoplus_{n=0}^{\infty}(sX)^{\bar{\otimes}n}\right)\\
&\cong \bigoplus_{n=0}^{\infty} C^\infty(J)\bar{\otimes}(sX)^{\bar{\otimes}n}\\
&\cong \bigoplus_{n=0}^{\infty}(C^\infty(J)\bar{\otimes}sX)^{\bar{\otimes}_{C^\infty(J)}n}\\
&= B(C^\infty(J)\bar{\otimes}X).
\end{aligned}$$

Definition 3.4.1. A *smooth one-parameter deformation of A_∞-algebras* is a $\bar{\otimes}$-smooth family $\{m_t \in \mathrm{Coder}(B(X))\}_{t\in J}$ of locally convex A_∞-structures on X.

The smoothness condition implies that the $\{m_t\}_{t\in J}$ give a continuous $C^\infty(J)$-linear degree $+1$ codifferential

$$m : C^\infty(J)\bar{\otimes}B(X) \to C^\infty(J)\bar{\otimes}B(X).$$

Since $C^\infty(J)\bar{\otimes}B(X) \cong B(C^\infty(J)\bar{\otimes}X)$, a smooth deformation of locally convex A_∞-algebra structures on X is equivalent to a $C^\infty(J)$-linear locally convex A_∞-structre on $C^\infty(J)\bar{\otimes}X$. As in section 2.7.2, the map m decomposes into continuous $C^\infty(J)$-linear maps

$$m_n : C^\infty(J)\bar{\otimes}(sX)^{\bar{\otimes}n} \to C^\infty(J)\bar{\otimes}sX$$

So each m_n is given by a $\bar{\otimes}$-smooth family of degree 1 maps

$$\{m_{n,t} : (sX)^{\bar{\otimes} n} \to sX\}_{t \in J}.$$

Proposition 3.4.2. *A collection $\{m_t\}_{t\in J}$ of locally convex A_∞-structures on $X^\bullet$ is a smooth one-parameter deformation of A_∞-algebras if and only if the maps*

$$\{m_{n,t} : (sX)^{\bar{\otimes} n} \to sX\}_{t \in J}$$

form a $\bar{\otimes}$-smooth family of continuous linear maps for each n.

Proof. The forward implication is immediate. For the converse, recall that the coderivation m_t on $B(X)$ determined by the family $\{m_{n,t}\}_{n=1}^{\infty}$ is given on $(sX)^{\bar{\otimes} k}$ by

$$m_t = \sum_{n=1}^{k} \sum_{j=0}^{k-n} 1^{\otimes j} \otimes m_{n,t} \otimes 1^{k-n-j} : (sX)^{\bar{\otimes} k} \to B(X).$$

So if each family $\{m_{n,t}\}_{t\in J}$ is $\bar{\otimes}$-smooth, then

$$\{m_t : (sX)^{\bar{\otimes} k} \to B(X)\}_{t\in J}$$

is a $\bar{\otimes}$-smooth family, as it is a finite sum of $\bar{\otimes}$-smooth families. Thus, the family $\{m_t\}_{t\in J}$ determine a continuous linear map

$$m : (sX)^{\bar{\otimes} k} \to C^\infty(J)\bar{\otimes} B(X).$$

By the universal property of direct sums, there is a continuous linear map

$$m : B(X) \to C^\infty(J)\bar{\otimes} B(X),$$

as desired. This shows that $\{m_t : B(X) \to B(X)\}_{t\in J}$ is a $\bar{\otimes}$-smooth family of continuous linear maps. $\square$

The case we shall be most interested in is where $X^\bullet = \bigoplus_{n\in\mathbb{Z}} X^n$, and each X^n is a Fréchet space. Here we can give a simpler form of the smoothness condition.

Proposition 3.4.3. *Suppose $X^\bullet = \bigoplus_{n\in\mathbb{Z}} X^n$, where each X^n is a Fréchet space.*

Then a collection $\{m_t\}_{t\in J}$ of locally convex A_∞-structures on $X^\bullet$ is a smooth deformation if and only if for all n and all $x_1,\ldots,x_n \in X^\bullet$, the map

$$J \to X^\bullet, \qquad t \mapsto m_{n,t}(x_1,\ldots,x_n)$$

is smooth.

Proof. By Proposition 3.4.2, we need only show that the given smoothness condition is equivalent to

$$\{m_{n,t} : (sX)^{\bar{\otimes} n} \to sX\}_{t\in J}$$

being a $\bar{\otimes}$-smooth family for each n. We shall appeal to Proposition 2.1.11 part (ii). First note that

$$(sX)^{\bar{\otimes} n} = \bigoplus_{k\in\mathbb{Z}} \bigoplus_{i_1+\ldots+i_n=k} (sX)^{i_1}\bar{\otimes}\ldots\bar{\otimes}(sX)^{i_n}$$

decomposes as a direct sum of Fréchet spaces. Moreover we have

$$m_{n,t}((sX)^{i_1}\bar{\otimes}\ldots\bar{\otimes}(sX)^{i_n}) \subset (sX)^{i_1+\ldots+i_n+1}.$$

So each Fréchet direct summand maps into a Fréchet direct summand. Now if $\{m_{n,t}\}_{t\in J}$ is a $\bar{\otimes}$-smooth family, then

$$t \mapsto m_{n,t}(x_1,\ldots,x_n)$$

is smooth for any fixed $x_1,\ldots,x_n \in X^\bullet$. Conversely, the n-multilinear map

$$m_n : (sX)^{i_1} \times \ldots \times (sX)^{i_n} \to C^\infty(J)\bar{\otimes}(sX)^{i_1+\ldots+i_n+1}$$

given by

$$m_n(x_1,\ldots,x_n)(t) = m_{n,t}(x_1,\ldots,x_n)$$

is well-defined by hypothesis, and separately continuous by Proposition 2.1.11 part (i). Thus it gives a continuous linear map

$$m_n : (sX)^{i_1}\bar{\otimes}\ldots\bar{\otimes}(sX)^{i_n} \to C^\infty(J)\bar{\otimes}(sX)^{i_1+\ldots+i_n+1} \cong C^\infty(J,(sX)^{i_1+\ldots+i_n+1}),$$

which shows that

$$t \mapsto m_{n,t}(\theta)$$

is smooth for any $\theta \in (sX)^{\bar{\otimes} n}$. Thus we are done by Proposition 2.1.11 part (ii). $\square$

As in section 2.7.2, we can turn the $\{m_{n,t}\}$ into a collection of maps

$$\{\mu_{n,t} : X^{\bar{\otimes} n} \to X\}, \qquad \deg \mu_{n,t} = 2 - n.$$

The above smoothness conditions could be rephrased in terms of the $\{\mu_{n,t}\}_{t\in J}$ instead of the $\{m_{n,t}\}_{t\in J}$.

Let $\{m_t\}_{t\in J}$ be a smooth one-parameter deformation of locally convex A_∞-algebra structures on $X^\bullet$. For each $t \in J$, we shall let A_t denote the locally convex A_∞-algebra whose underlying space is $X^\bullet$ and whose A_∞-structure is given by the maps $\{m_{n,t}\}_{n=1}^\infty$. The $C^\infty(J)$-module

$$A = C^\infty(J)\bar{\otimes}X^\bullet$$

equipped with the structure maps

$$m_n(x_1, \dots, x_n)(t) = m_{n,t}(x_1(t), \dots, x_n(t))$$

shall be called the A_∞-*algebra of sections* of $\{A_t\}_{t\in J}$.

3.4.1 Compatible connections

It is advantageous to think of a smooth one-parameter defomation of locally convex A_∞-algebras $\{A_t\}_{t\in J}$ as a smooth one-parameter deformation of locally convex differential graded $\bar{\otimes}$-coalgebras $\{B(A_t)\}_{t\in J}$. The deformation $\{B(A_t)\}_{t\in J}$ is constant as a deformation of coalgebras, as the coproduct of $B(A_t)$ is always the coproduct of the bar coalgebra. However, the A_∞-structure is encapsulated entirely in the coboundary map, which is changing as t varies. From that persepective, a deformation of A_∞-algebras is a certain type of deformation of cochain complexes. If A denotes the A_∞-algebra of sections of the deformation $\{A_t\}_{t\in J}$, then the *bar coalgebra of sections* of the deformation $\{B(A_t\}_{t\in J}$ is $B(A)$, taken over the ground ring $C^\infty(J)$. As remarked earlier,

$$B(A) \cong C^\infty(J)\bar{\otimes}B(X).$$

So we are interested in connections on $B(A)$ that are compatible with the differential graded coalgebra structure. Parallel transport maps induced from such a connection will be differential graded coalgebra isomorphisms, which by definition are A_∞-algebra isomorphisms. Such a connection $\nabla : B(A) \to B(A)$ must respect both the (coaugmented) coalgebra structure and the cochain complex structure. Thus, such a connection should be a degree 0 coderivation and a chain map. Recall that the Hochschild differential graded Lie algebra (over the ground ring R) is

$$\mathfrak{g}^\bullet_R(A) = \mathrm{Coder}_R(B(A)) \cong \mathrm{Hom}_R(B(A), sA),$$

with differential

$$\delta D = [m, D].$$

So the type of connection we seek satisfies,

$$\nabla \in \mathfrak{g}^0_{\mathbb{C}}(A), \qquad \delta\nabla = 0,$$

that is, ∇ is a Hochschild 0-cocycle. Additionally, we insist that $\nabla\eta = 0$ for the coaugmentation map η. We shall refer to such a Hochschild 0-cocycle $D \in \mathfrak{g}^0(A)$ as an A_∞-*derivation*. An A_∞-derivation is given by a sequence of degree 0 maps[2]

$$D_n : (sA)^{\bar{\otimes}n} \to sA, \qquad n \geq 1.$$

That $\delta D = [m, D] = 0$ implies a sequence of identities involving Gerstenhaber brackets of the A_∞-structure maps $\{m_n\}$ with the $\{D_n\}$. By removing the suspensions, we can view each D_n as a map

$$\partial_n : A^{\bar{\otimes}n} \to A, \qquad \deg \partial_n = 1 - n.$$

Notice that an A_∞-derivation on an ungraded associative algebra, viewed as an

[2]The compatibility $D\eta = 0$ with the coaugmentation map η implies that there is no D_0 map.

A_∞-algebra, is equivalent to a derivation.

Let $\nabla \in \mathfrak{g}^0_{\mathbb{C}}(A)$ be a connection that is not necessarily a cocycle. Then let

$$E = \delta\nabla = [m, \nabla].$$

As an operator on $B(A)$, E is $C^\infty(J)$-linear. Since

$$\delta E = \delta^2 \nabla = 0,$$

E is a Hochschild 1-cocycle in $\mathfrak{g}^1_{C^\infty(J)}(A)$. So E determines a cohomology class $[E]$ in the Hochschild cohomology module $H^1_{C^\infty(J)}(\mathfrak{g}(A))$.

Proposition 3.4.4. *The cohomology class* $[E] \in H^1_{C^\infty(J)}(\mathfrak{g}(A))$ *is independent of the choice of connection* $\nabla \in \mathfrak{g}^0_{\mathbb{C}}(A)$*. Moreover,* $B(A)$ *admits a connection that is an* A_∞*-derivation if and only if* $[E] = 0$*.*

Proof. It is analogous to Proposition 3.1.10. Any other connection $\nabla' \in \mathfrak{g}^0_{\mathbb{C}}(A)$ is of the form

$$\nabla' = \nabla - F$$

for some $F \in \mathfrak{g}^0_{C^\infty(J)}(A)$. Thus,

$$E' = \delta\nabla' = E - \delta F,$$

which shows $[E'] = [E]$.

If ∇ is an A_∞-derivation, then $E = \delta\nabla = 0$ exactly. Conversely, if

$$\delta\nabla = E = \delta F, \qquad F \in \mathfrak{g}^0_{C^\infty(J)}(A),$$

then $\nabla - F$ is a connection and an A_∞-derivation. □

Proposition 3.4.5. *The smooth deformation* $\{B(A_t)\}_{t\in J}$ *of coaugmented differential graded coalgebras is trivial if and only if* $B(A)$ *admits an integrable connection that is an* A_∞*-derivation.*

Proof. It is analogous to Proposition 3.1.11. A trivial deformation is isomorphic to a constant deformation, and $\frac{d}{dt}$ is an integrable A_∞-derivation on a constant deformation. Conversely, suppose ∇ is integrable and an A_∞-derivation on $B(A)$. The tensor product connection $\nabla \otimes 1 + 1 \otimes \nabla$ is integrable on $B(A)\bar{\otimes}_{C^\infty(J)}B(A)$ with parallel transport maps of the form

$$P^{\nabla}_{s,t} \otimes P^{\nabla}_{s,t} : B(A_s)\bar{\otimes}B(A_s) \to B(A_t)\bar{\otimes}B(A_t),$$

by Proposition 2.2.6. That ∇ is a coderivation is equivalent to the coproduct

$$\Delta : B(A) \to B(A)\bar{\otimes}_{C^\infty(J)}B(A)$$

being a parallel map. So by Proposition 2.2.5,

$$(P^{\nabla}_{s,t} \otimes P^{\nabla}_{s,t})\Delta = \Delta P^{\nabla}_{s,t},$$

that is, $P^{\nabla}_{s,t}$ is a coalgebra map. By Proposition 3.2.4, $P^{\nabla}_{s,t}$ is a chain map. It is automatic[3] that $\epsilon\nabla = 0$ because ∇ is a coderivation. It follows that $\epsilon P^{\nabla}_{s,t} = \epsilon$. Additionally, $\nabla\eta = 0$ implies $P^{\nabla}_{s,t}\eta = \eta$. Thus, $P^{\nabla}_{s,t}$ is a coaugmented differential graded coalgebra isomorphism for all $s, t \in J$. Using ∇ to construct a trivialization as in Proposition 2.2.4 shows that $B(A)$ is isomorphic to a constant deformation of coaugmented differental graded coalgebras. □

Consider a connection $\nabla \in \mathfrak{g}^0_{\mathbb{C}}(A)$, not necessarily a cocycle, on the bar coalgebra $B(A)$ that is given by a sequence of maps

$$\nabla_n : (sA)^{\bar{\otimes}n} \to sA, \qquad n \geq 1$$

as described above. It follows from the Leibniz rule for ∇ that ∇_1 is a connection on A, and ∇_n is $C^\infty(J)$-linear for $n \geq 2$. The remarkable fact about considering this more flexible class of connections on the bar coalgebra is that no new difficulties arise in showing they are integrable.

Proposition 3.4.6. *Let $\nabla \in \mathfrak{g}^0_{\mathbb{C}}(A)$ be a connection on the bar coalgebra given by maps*

$$\nabla_n : (sA)^{\bar{\otimes}n} \to sA, \qquad n \geq 1.$$

[3] For any coderivation D on a coalgebra with counit ϵ, it is always the case that $\epsilon D = 0$. This is dual to the fact that if D is a derivation on a unital algebra, then $D(1) = 0$.

Then ∇ is an integrable connection on $B(A)$ if and only if ∇_1 is an integrable connection on A.

Proof. The forward direction follows because the restriction of ∇ to $sA \subset B(A)$ is exactly ∇_1. Conversely, suppose ∇_1 is integrable. Notice that each ∇_n acts on the submodule $(sA)^{\bar{\otimes}_{C^\infty(J)} k}$ by

$$\sum_{j=0}^{k-n} 1^{\otimes j} \otimes \nabla_n \otimes 1^{k-n-j} : (sA)^{\bar{\otimes}_{C^\infty(J)} k} \to (sA)^{\bar{\otimes}_{C^\infty(J)} (k-n)}.$$

So ∇ restricts to a connection on the submodule $\bigoplus_{k=0}^{N} (sA)^{\bar{\otimes}_{C^\infty(J)} k}$ given by

$$\nabla = \nabla_1 + \ldots + \nabla_N.$$

Now ∇_1 is an integrable connection on $\bigoplus_{k=0}^{N} (sA)^{\bar{\otimes}_{C^\infty(J)} k}$ by Proposition 2.2.6. On the other hand, the fact that $\nabla_2 + \ldots + \nabla_N$ lowers the tensor power implies that ∇ is a nilpotent perturbation of ∇_1 on the submodule $\bigoplus_{k=0}^{N} (sA)^{\bar{\otimes}_{C^\infty(J)} k}$. So by Proposition 2.2.10, ∇ is integrable on $\bigoplus_{k=0}^{N} (sA)^{\bar{\otimes}_{C^\infty(J)} k}$. The fact that the parallel transport maps are compatible with the inclusions

$$\bigoplus_{k=0}^{N} (sA)^{\bar{\otimes}_{C^\infty(J)} k} \to \bigoplus_{k=0}^{N+1} (sA)^{\bar{\otimes}_{C^\infty(J)} k}$$

implies that ∇ is integrable on $B(A)$. $\square$

Chapter 4

Rigidity of periodic cyclic homology of noncommutative tori

The two main examples of smooth deformations of algebras considered in Chapter 3 were noncommutative tori and smooth crossed products by $\mathbb{R}$. Let A denote the algebra of sections of either of these deformations, and let ∇ be the canonical connection $\frac{d}{dt}$. As discussed in Example 3.1.12, both deformations have the property that

$$E := \delta\nabla = \sum_{i=1}^{N} X_i \smile Y_i,$$

where $\{X_1, \ldots, X_N, Y_1, \ldots, Y_N\}$ is some family of mutually commuting $C^\infty(J)$-linear derivations on A that also commute with ∇. Written a different way,

$$\nabla(a_1 a_2) = \nabla(a_1)a_2 + a_1\nabla(a_2) + \sum_{i=1}^{N} X_i(a_1)Y_i(a_2), \qquad \forall a_1, a_2 \in A.$$

It is this property that we shall take advantage of, so we shall abstractly study smooth deformations which satisfy it. The abelian Lie algebra

$$\mathfrak{g} = \mathrm{Span}\{X_1, \ldots, X_N, Y_1, \ldots, Y_N\}$$

acts on A as derivations. We shall first modify our homology theories to take this action into account.

4.1 The $\mathfrak{g}$-invariant complex

4.1.1 $\mathfrak{g}$-invariant chains and cochains

Suppose that $\mathfrak{g} \subset C^1(A, A)$ is a Lie subalgebra of derivations on a locally convex $\check{\otimes}$-algebra A. Then $\mathfrak{g}$ also acts on $C_\bullet(A)$ by Lie derivatives. Define the $\mathfrak{g}$-*invariant Hochschild chain group* $C^{\mathfrak{g}}_\bullet(A)$ to be the space of coinvariants of this action, that is

$$C^{\mathfrak{g}}_\bullet(A) = C_\bullet(A)/\mathfrak{g} \cdot C_\bullet(A).$$

We shall make the assumption that $\mathfrak{g} \cdot C_\bullet(A)$ is a closed submodule, so that $C^{\mathfrak{g}}_\bullet(A)$ is Hausdorff. For example, this holds in the noncommutative tori case. By Proposition 2.5.1, the operators b and B descend to operators on $C^{\mathfrak{g}}_\bullet(A)$. One can define the $\mathfrak{g}$-invariant periodic cyclic complex $C^{\mathfrak{g}}_{\mathrm{per}}(A)$ accordingly, and its homology is the $\mathfrak{g}$-*invariant periodic cyclic homology* $HP^{\mathfrak{g}}_\bullet(A)$.

Let $C^\bullet_{\mathfrak{g}}(A, A)$ denote the subspace of all Hochschild cochains D for which $[X, D] = 0$ for all $X \in \mathfrak{g}$. If $D \in C^\bullet_{\mathfrak{g}}(A, A)$, then the formula

$$0 = \delta[X, D] = [\delta X, D] + [X, \delta D]$$

shows that $C^\bullet_{\mathfrak{g}}(A, A)$ is a subcomplex because $\delta X = 0$. It's cohomology is the $\mathfrak{g}$-*invariant Hochschild cohomology* $H^\bullet_{\mathfrak{g}}(A, A)$.

Proposition 4.1.1. *For any $D \in C^\bullet_{\mathfrak{g}}(A)$, the operators L_D and I_D descend to operators on $C^{\mathfrak{g}}_\bullet(A)$.*

Proof. If $X \in C^1(A, A)$ is a derivation and $D \in C^\bullet(A, A)$, then one can verify directly that

$$[L_X, I_D] = I_{[X,D]}, \qquad [L_X, L_D] = L_{[X,D]},$$

from which the proposition follows, see Appendix B. □

Proposition 4.1.2. *If $X, Y \in C^1_{\mathfrak{g}}(A, A)$, then $L\{X, Y\}$ and $I\{X, Y\}$ descend to operators on $C^{\mathfrak{g}}_\bullet(A)$.*

Proof. For any $X, Y, Z \in C^1(A, A)$, one can verify directly the identities

$$[L_Z, I\{X, Y\}] = I\{[Z, X], Y\} + I\{X, [Z, Y]\}$$

and

$$[L_Z, L\{X,Y\}] = L\{[Z,X],Y\} + L\{X,[Z,Y]\},$$

and the result follows, see Appendix B. $\square$

4.1.2 The abelian case

Consider the special case of an abelian Lie algebra $\mathfrak{g}$ of derivations on A, so that $\mathfrak{g} \subset C^{\bullet}_{\mathfrak{g}}(A,A)$. By definition of the invariant complex, the operator L_X vanishes on $C^{\mathfrak{g}}_{\bullet}(A)$ for any $X \in \mathfrak{g}$.

One benefit of working in the $\mathfrak{g}$-invariant complex is that the cyclic contraction I_X is now a chain map when $X \in \mathfrak{g}$. Indeed,

$$[b+B, I_X] = L_X = 0$$

in $C^{\mathfrak{g}}_{\bullet}(A)$. These contraction operators obey the following algebra as operators on homology.

Theorem 4.1.3. *There is an algebra map $\chi : \Lambda^{\bullet}\mathfrak{g} \to \mathrm{End}(HP^{\mathfrak{g}}_{\bullet}(A))$ given by*

$$\chi(X_1 \wedge X_2 \wedge \ldots X_k) = I_{X_1} I_{X_2} \ldots I_{X_k}.$$

Proof. First observe that $X \mapsto I_X$ is a linear mapping. Next, we shall show that $I_X I_X$ is chain homotopic to zero. Observe that

$$0 = L_X L_X = L_{X^2} + 2L\{X,X\}$$

where X^2 denotes the composition of X with itself. Thus,

$$L\{X,X\} = -L_{\frac{1}{2}X^2}.$$

Next, notice that

$$\delta(\frac{1}{2}X^2) = X \smile X.$$

By Theorems 2.5.4 and 2.5.3,

$$-[b+B, I\{X,X\}] = -L\{X,X\} - I_{X \smile X} + I_X I_X$$

$$= L_{\frac{1}{2}X^2} - I_{\delta(\frac{1}{2}X^2)} + I_X I_X$$
$$= [b + B, I_{\frac{1}{2}X^2}] + I_X I_X,$$

proving that $I_X I_X$ is continuously chain homotopic to zero. By the universal property of the exterior algebra, the map χ exists as asserted. $\square$

There are some additional simplifications regarding the operator $L\{X, Y\}$ once we pass to $C_\bullet^{\mathfrak{g}}(A)$.

Proposition 4.1.4. *For $X, Y \in \mathfrak{g}$, the operator $L\{X, Y\}$ satisfies*

$$L\{X, Y\}(a_0, \ldots, a_n) = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} (a_0, \ldots, X(a_i), \ldots, Y(a_j), \ldots, a_n)$$

on $C_\bullet^{\mathfrak{g}}(A)$. Additionally,

$$[b + B, L\{X, Y\}] = -L_{X \smile Y}$$

on $C_\bullet^{\mathfrak{g}}(A)$.

Proof. Notice that

$$\begin{aligned} &L_Y(X(a_0), a_1, \ldots, a_n) - L_X(Y(a_0), a_1, \ldots, a_n) \\ &\quad = \sum_{j=1}^{n} (X(a_0), \ldots, Y(a_j), \ldots, a_n) - \sum_{i=1}^{n} (Y(a_0), \ldots, X(a_i), \ldots, a_n), \end{aligned}$$

using the fact that $[X, Y] = 0$. So

$$\begin{aligned} &L\{X, Y\}(a_0, \ldots, a_n) + L_Y(X(a_0), a_1, \ldots, a_n) - L_X(Y(a_0), a_1, \ldots, a_n) \\ &\quad = \sum_{i=0}^{n-1} \sum_{j=i+1}^{n} (a_0, \ldots, X(a_i), \ldots, Y(a_j), \ldots, a_n) \end{aligned}$$

gives the desired conclusion.

The formula

$$[b + B, L\{X, Y\}] = -L_{X \smile Y}$$

follows from Corollary 2.5.5 in light of the fact that $L_X = L_Y = 0$ on $C^{\mathfrak{g}}_\bullet(A)$. $\square$

4.1.3 Connections on the $\mathfrak{g}$-invariant complex

Now, we return to the scenario at the beginning of the chapter. Suppose A is the algebra of sections of a $\check{\otimes}$-smooth one-parameter deformation of algebras over J, and ∇ is a connection on A for which

$$E = \delta\nabla = \sum_{i=1}^{N} X_i \smile Y_i,$$

where $X_i, Y_i \in C^1_{C^\infty(J)}(A, A)$ are derivations that mutually commute and satisfy

$$[\nabla, X_i] = [\nabla, Y_i] = 0.$$

Let $\mathfrak{g}$ denote the abelian Lie algebra $\mathrm{Span}\{X_1, \ldots, X_N, Y_1, \ldots, Y_N\}$.

Proposition 4.1.5. *In the above situation, the Gauss-Manin Connection*

$$\nabla_{GM} = L_\nabla - I_E$$

descends to the $\mathfrak{g}$-invariant complex $C^{\mathfrak{g}}_\bullet(A)$ and therefore to a connection on the $\mathfrak{g}$-invariant periodic cyclic homology $HP^{\mathfrak{g}}_\bullet(A)$.

Proof. If $X, Y, Z \in C^1(A, A)$ and X is a derivation, then

$$[X, Y \smile Z] = [X, Y] \smile Z \; + \; Y \smile [X, Z].$$

Thus, $E \in C^2_{\mathfrak{g}}(A, A)$ and the result follows from Proposition 4.1.1. $\square$

Our main reason for working with the $\mathfrak{g}$-invariant complex is that we can define another connection on $HP^{\mathfrak{g}}_\bullet(A)$ which is easier to work with than ∇_{GM}. The connection L_∇ satisfies

$$[b + B, L_\nabla] = L_{\delta\nabla} = \sum_{i=1}^{N} L_{X_i \smile Y_i}.$$

Thus, by Proposition 4.1.4,

$$\widetilde{\widetilde{\nabla}} = L_{\nabla} + \sum_{i=1}^{N} L\{X_i, Y_i\}$$

is a connection on $C_{\mathrm{per}}^{\mathfrak{g}}$ that commutes with $b + B$ and therefore descends to a connection on $HP_{\bullet}^{\mathfrak{g}}(A)$. We emphasize that $\widetilde{\widetilde{\nabla}}$ does not commute with $b + B$ on the ordinary periodic cyclic complex $C_{\mathrm{per}}(A)$.

Remark 4.1.6. Let $\mathcal{H}$ be the Hopf algebra whose underlying algebra is the symmetric algebra $S(\mathfrak{g} \oplus \mathrm{Span}\{\nabla\})$. The coproduct $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is the unique algebra map that satisfies

$$\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i, \qquad \Delta(Y_i) = Y_i \otimes 1 + 1 \otimes Y_i,$$

$$\Delta(\nabla) = \nabla \otimes 1 + 1 \otimes \nabla + \sum_{i=1}^{N} X_i \otimes Y_i.$$

In the case where $N = 1$, these are the defining relations of the Hopf algebra of polynomial functions on the three-dimensional Heisenberg group. As an algebra, $\mathcal{H}$ acts on A, and this action is a Hopf action in the sense that for all $h \in \mathcal{H}$,

$$h(a_1 a_2) = \sum h_{(1)}(a_1) h_{(2)}(a_2),$$

where $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$. When a Hopf algebra $\mathcal{H}$ acts (say, on the left) on two spaces V and W, there is a canonical action, called the *diagonal action*, of $\mathcal{H}$ on $V \otimes W$ given by

$$h(v \otimes w) = \sum h_{(1)}(v) \otimes h_{(2)}(w).$$

The connection $\widetilde{\widetilde{\nabla}}$ on $C_n^{\mathfrak{g}}(A)$ is none other than the diagonal action of ∇ on $A^{\check{\otimes}(n+1)}$ after passing to the quotient.

Lemma 4.1.7. *On the invariant complex $C_{\bullet}^{\mathfrak{g}}(A)$, $[\nabla_{GM}, \widetilde{\widetilde{\nabla}}] = 0$.*

Proof. This is a straightforward, though tedious, computation, see Appendix B. $\square$

Proposition 4.1.8. *As operators on $HP^{\mathfrak{g}}_{\bullet}(A)$,*

$$\nabla_{GM} = \widetilde{\nabla} + \sum_{i=1}^{N} \chi(X_i \wedge Y_i).$$

Proof. We have

$$\begin{aligned}
\nabla_{GM} - \widetilde{\nabla} &= -I_E - \sum_{i=1}^{N} L\{X_i, Y_i\} \\
&= -\sum_{i=1}^{N} \Big(I_{X_i \smile Y_i} + L\{X_i, Y_i\} \Big) \\
&= -[b + B, \sum_{i=1}^{N} I\{X_i, Y_i\}] - \sum_{i=1}^{N} I_{Y_i} I_{X_i} \\
&= -[b + B, \sum_{i=1}^{N} I\{X_i, Y_i\}] - \sum_{i=1}^{N} \chi(Y_i \wedge X_i),
\end{aligned}$$

using Theorem 2.5.4. So at the level of homology,

$$\nabla_{GM} = \widetilde{\nabla} + \sum_{i=1}^{N} \chi(X_i \wedge Y_i).$$

□

Theorem 4.1.9. *As connections on $HP^{\mathfrak{g}}_{\bullet}(A)$, ∇_{GM} is integrable at the level of homology if and only if $\widetilde{\nabla}$ is integrable at the level of homology.*

Proof. By Theorem 4.1.3, $\sum_{i=1}^{N} \chi(X_i \wedge Y_i)$ is a nilpotent operator on $HP^{\mathfrak{g}}_{\bullet}(A)$. So $\widetilde{\nabla}$ is a nilpotent perturbation of ∇_{GM} at the level of homology. As in Example 2.2.11, $\widetilde{\nabla}$-parallel sections are in bijection with ∇_{GM}-parallel sections, and are related by a finite exponential sum. □

4.2 Integrating ∇_{GM} for noncommutative tori

In this section, we specialize to the noncommutative tori deformation $\{\mathcal{A}_{t\Theta}\}_{t \in J}$ for a given $n \times n$ skew-symmetric real matrix Θ with entries (θ_{jk}) and some open subinterval J of $\mathbb{R}$, see section 3.1.1. Let A denote the algebra of sections of this deformation. We view the canonical derivations $\delta_1, \dots, \delta_n$ as $C^\infty(J)$-linear derivations on A. Let $\nabla = \frac{d}{dt}$ on A, so that $[\nabla, \delta_j] = 0$ for all j and

$$E = \delta\nabla = \frac{1}{2\pi i} \sum_{j>k} \theta_{jk}(\delta_j \smile \delta_k)$$

as shown in Example 3.1.12. Thus the results of the previous section apply, where $\mathfrak{g}$ is the abelian Lie algebra spanned by $\delta_1, \dots, \delta_n$.

We shall now show that we are not losing anything in passing to $\mathfrak{g}$-invariant cyclic homology, in that the canonical map $HP_\bullet(\mathcal{A}_\Theta) \to HP_\bullet^{\mathfrak{g}}(\mathcal{A}_\Theta)$ is a chain homotopy equivalence.

First, consider the chain map $i : \bigcap_{j=1}^n \ker L_{\delta_j} \to C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A}_\Theta)$, which factors as the inclusion followed by the quotient map

$$i : \bigcap_{j=1}^n \ker L_{\delta_j} \to C_{\mathrm{per}}(\mathcal{A}_\Theta) \to C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A}_\Theta).$$

Lemma 4.2.1. *The map $i : \bigcap_{j=1}^n \ker L_{\delta_j} \to C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A}_\Theta)$ is an isomorphism of locally convex chain complexes.*

Proof. It suffices to prove that each restriction

$$i : C_m(\mathcal{A}_\Theta) \bigcap \left(\bigcap_{j=1}^n \ker L_{\delta_j} \right) \to C^{\mathfrak{g}}_m(\mathcal{A}_\Theta)$$

is an isomorphism.

Recall that for a multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, we write $u^\alpha = u_1^{\alpha_1} \cdot \ldots \cdot u_n^{\alpha_n} \in \mathcal{A}_\Theta$. If $\omega = u^{\alpha^0} \otimes \ldots \otimes u^{\alpha^m} \in C_m(\mathcal{A}_\Theta)$ for some collection $\alpha^0, \dots, \alpha^m$ of multi-indices, then

$$L_{\delta_j}\omega = (\deg_j \omega)\omega$$

where $\deg_j \omega = \sum_{i=1}^m \alpha_j^i$. Thus, $\ker L_{\delta_j}$ is just the degree 0 part of $C_{\mathrm{per}}(\mathcal{A}_\Theta)$ with respect to the grading given by $\deg_j$. Topologically, $\mathcal{S}(\mathbb{Z}^n)^{\widehat{\otimes} m} \cong \mathcal{S}(\mathbb{Z}^{nm})$, and so

$$C_m(\mathcal{A}_\Theta) \cong \widetilde{\mathcal{A}_\Theta} \otimes \mathcal{A}_\Theta^{\widehat{\otimes} m} \cong \mathcal{S}(\mathbb{Z}^{n(m+1)}) \oplus \mathcal{S}(\mathbb{Z}^{nm}).$$

The Schwartz spaces $\mathcal{S}(\mathbb{Z}^{n(m+1)})$ and $\mathcal{S}(\mathbb{Z}^{nm})$ are completed direct sums, where each direct summand is an eigenspace for L_{δ_j}. Thus, $C_m(\mathcal{A}_\Theta)$ splits as a topological direct sum of complexes

$$C_m(\mathcal{A}_\Theta) = \left(\bigcap_{j=1}^{n} \ker L_{\delta_j}\right) \bigoplus \left(\sum_{j=1}^{n} \operatorname{im} L_{\delta_j}\right).$$

So it follows that

$$C_m^{\mathfrak{g}}(\mathcal{A}_\Theta) = C_m(\mathcal{A}_\Theta)/\mathfrak{g} \cdot C_m(\mathcal{A}_\Theta) \cong \bigcap_{j=1}^{n} \ker L_{\delta_j}.$$

□

Theorem 4.2.2. *The canonical map $C_{\mathrm{per}}(\mathcal{A}_\Theta) \to C_{\mathrm{per}}^{\mathfrak{g}}(\mathcal{A}_\Theta)$ is a chain homotopy equivalence and thus induces an isomorphism $HP_\bullet(\mathcal{A}_\Theta) \cong HP_\bullet^{\mathfrak{g}}(\mathcal{A}_\Theta)$*

Proof. Continuing with notation from the previous proof, define

$$N_j(\omega) = \begin{cases} 0 & \deg_j \omega = 0 \\ \frac{1}{\deg_j \omega}\omega & \deg_j \omega \neq 0 \end{cases}$$

for homogeneous $\omega \in C_{\mathrm{per}}(\mathcal{A}_\Theta)$. Then N_j extends to a continuous operator on $C_{\mathrm{per}}(\mathcal{A}_\Theta)$. Moreover, $p_j := 1 - N_j L_{\delta_j} : C_{\mathrm{per}}(\mathcal{A}_\Theta) \to \ker L_{\delta_j}$ is the projection.

Let $M_j = \bigcap_{k=1}^{j} \ker L_{\delta_k}$, so that

$$M_0 = C_{\mathrm{per}}(\mathcal{A}_\Theta), \qquad M_n = \bigcap_{k=1}^{n} \ker L_{\delta_k} \cong C_{\mathrm{per}}^{\mathfrak{g}}(\mathcal{A}_\Theta).$$

Then for $1 \leq j \leq n$, consider the inclusion $i_j : M_j \to M_{j-1}$ and projection $p_j : M_{j-1} \to M_j$. Both maps are chain maps, and we claim they are inverses up to chain homotopy. It is immediate that $p_j i_j = 1$. Let $h_j = N_j I_{\delta_j}$. For any k, L_{δ_k} commutes with N_j and also with I_{δ_j} because

$$[L_{\delta_k}, I_{\delta_j}] = I_{[\delta_k, \delta_j]} = 0.$$

Consequently, h_j maps M_{j-1} into M_{j-1}. Using Theorem 2.5.3 and the fact that

$[b+B, N_j] = 0$, we have

$$[b+B, h_j] = N_j[b+B, I_{\delta_j}] = N_j L_{\delta_j} = 1 - i_j p_j.$$

By composing these chain equivalences, we see that the inclusion

$$i : \bigcap_{k=1}^{n} \ker L_{\delta_k} = M_n \to M_0 = C_{\mathrm{per}}(\mathcal{A}_\Theta)$$

is a chain homotopy equivalences with homotopy inverse $p = p_1 \dots p_n$.

$\square$

Remark 4.2.3. The above proof can be carried out because the action of $\mathfrak{g}$ by derivations on $\mathcal{A}_\Theta$ is the infinitesimal of an action of the Lie group $\mathbb{T}^n$ by algebra automorphisms. By the theory of Fourier series, $\mathcal{A}_\Theta$ decomposes as a completed direct sum of eigenspaces indexed by $\mathbb{Z}^n$ for the action of $\mathfrak{g}$.

Note that the same proof shows that for the section algebra A of the noncommutative tori deformation, the natural map $C_{\mathrm{per}}(A) \to C^{\mathfrak{g}}_{\mathrm{per}}(A)$ is a chain equivalence, where the theories are considered over the algebra $C^\infty(J)$.

Our goal now is to show that the connection

$$\widetilde{\nabla} = L_\nabla + \frac{1}{2\pi i} \sum_{j>k} \theta_{jk} \cdot L\{\delta_j, \delta_k\},$$

defined in section 4.1.3, is integrable on the invariant complex $C^{\mathfrak{g}}_{\mathrm{per}}(A)$.

Theorem 4.2.4. *For the the algebra of section A of the noncommutative tori deformation $\{\mathcal{A}_{t\Theta}\}_{t\in J}$, the connection $\widetilde{\nabla}$ is integrable on $C_{\mathrm{per}}(A)$ and consequently on $C^{\mathfrak{g}}_{\mathrm{per}}(A)$.*

Proof. Since $\widetilde{\nabla}$ restricts to a connection on $C_m(A)$ for each m, it suffices to prove $\widetilde{\nabla}$ is integrable on $C_m(A)$. Given $m+1$ multi-indices $\alpha^0, \dots, \alpha^m$, each of length n, we shall use the notation

$$u^{\overline{\alpha}} = u^{\alpha^0} \otimes u^{\alpha^1} \otimes \dots \otimes u^{\alpha^m} \in \mathcal{A}_\Theta^{\widehat{\otimes}(m+1)}.$$

Since $\mathcal{S}(\mathbb{Z}^n)^{\widehat{\otimes}(m+1)} \cong \mathcal{S}(\mathbb{Z}^{n(m+1)})$, every element of $\mathcal{A}_\Theta^{\widehat{\otimes}(m+1)}$ is of the form

$$\sum_{\overline{\alpha}\in\Lambda} c_{\overline{\alpha}} u^{\overline{\alpha}}$$

where $\Lambda = \prod_{i=0}^m \mathbb{Z}^n$ and $c_{\overline{\alpha}} \in \mathbb{C}$ are rapidly decaying coefficients. Likewise, an element $\omega \in A^{\widehat{\otimes}_{C^\infty(J)}(m+1)}$ is of the form

$$\omega = \sum_{\overline{\alpha}\in\Lambda} f_{\overline{\alpha}} u^{\overline{\alpha}}$$

where $f_{\overline{\alpha}} \in C^\infty(J)$ are functions that are rapidly decreasing in the sense that

$$\sum_{\overline{\alpha}\in\Lambda} f_{\overline{\alpha}}^{(k)}(t) u^{\overline{\alpha}} \in \mathcal{A}_{t\Theta}^{\widehat{\otimes}(m+1)}$$

for each k and each $t \in J$.

We shall prove that $\widetilde{\nabla}$ is integrable on $A^{\widehat{\otimes}_{C^\infty(J)}(m+1)}$ for each m, from which it will follow that $\widetilde{\nabla}$ is integrable on

$$C_m(A) \cong A^{\widehat{\otimes}_{C^\infty(J)}(m+1)} \bigoplus A^{\widehat{\otimes}_{C^\infty(J)}m}.$$

For a fixed $s \in J$ and fixed $\eta \in A^{\widehat{\otimes}_{C^\infty(J)}(m+1)}$, we must prove the existence and uniqueness of solutions to the system of differential equations

$$\widetilde{\nabla}\omega = 0, \qquad \omega(s) = \eta.$$

As above, let

$$\eta = \sum_{\overline{\alpha}\in Z} c_{\overline{\alpha}} u^{\overline{\alpha}}, \qquad \omega = \sum_{\overline{\alpha}\in Z} f_{\overline{\alpha}} u^{\overline{\alpha}}.$$

The crucial point is that

$$\frac{1}{2\pi i}\sum_{j>k} L\{\delta_j, \delta_k\} u^{\overline{\alpha}} = 2\pi i \cdot R(\overline{\alpha}) u^{\overline{\alpha}},$$

where $R(\overline{\alpha})$ is the real-valued polynomial in the multi-indices given by

$$R(\alpha^0, \ldots, \alpha^m) = \sum_{j>k} \theta_{jk} \left(\sum_{1 \leq r < s \leq m} \alpha_j^r \alpha_k^s + \sum_{r=1}^{m} \alpha_k^0 \alpha_j^r \right).$$

Thus for each $\overline{\alpha}$, the subspace of elements of the form $f_{\overline{\alpha}} u^{\overline{\alpha}}$ are invariant under $\widetilde{\nabla}$. So it suffices to solve

$$\widetilde{\nabla}(f_{\overline{\alpha}} u^{\overline{\alpha}}) = 0, \qquad f_{\overline{\alpha}}(s) = c_{\overline{\alpha}}$$

individually for each $\overline{\alpha}$. This is equivalent to solving the elementary initial value problem

$$f'_{\overline{\alpha}} + 2\pi i \cdot R(\overline{\alpha}) f_{\overline{\alpha}} = 0, \qquad f_{\overline{\alpha}}(s) = c_{\overline{\alpha}},$$

which has the unique solution

$$f_{\overline{\alpha}}(t) = \exp(-2\pi i \cdot R(\overline{\alpha})(t-s)) c_{\overline{\alpha}}.$$

If the $(c_{\overline{\alpha}})_{\overline{\alpha} \in \Lambda}$ are of rapid decay, then so are the numbers

$$(f_{\overline{\alpha}}^{(k)}(t))_{\overline{\alpha} \in \Lambda} = ((-2\pi i R(\overline{\alpha}))^k \exp(-2\pi i \cdot R(\overline{\alpha})(t-s)) c_{\overline{\alpha}}^{(k)}(t))_{\overline{\alpha} \in \Lambda}$$

for each k and $t \in J$, because $R(\overline{\alpha})$ is a real-valued polynomial in $\overline{\alpha}$. This shows that the solution

$$\omega = \sum_{\overline{\alpha} \in \Lambda} f_{\overline{\alpha}} u^{\overline{\alpha}} \in A^{\widehat{\otimes}_{C^\infty(J)}(m+1)}.$$

By examining the dependence of the solution on t, s and the initial data $(c_{\overline{\alpha}})_{\overline{\alpha} \in \Lambda}$, we see that $\widetilde{\nabla}$ is integrable on $C_{\mathrm{per}}(A)$.

The fact that $[L_{\delta_j}, \widetilde{\nabla}] = 0$ for all j shows that $\widetilde{\nabla}$ restricts to an integrable connection on $\bigcap_{j=1}^{n} \ker L_{\delta_j}$. Moreover, the topological isomorphism

$$\bigcap_{j=1}^{n} \ker L_{\delta_j} \to C_{\mathrm{per}}^{\mathfrak{g}}(A)$$

of Lemma 4.2.1 is parallel with respect to $\widetilde{\nabla}$ on each module. Thus $\widetilde{\nabla}$ is integrable on $C_{\mathrm{per}}^{\mathfrak{g}}(A)$.

$\square$

Corollary 4.2.5. *For any $n \times n$ skew-symmetric matrix Θ, there is a continuous chain homotopy equivalence*

$$C_{\mathrm{per}}(\mathcal{A}_\Theta) \to C_{\mathrm{per}}(C^\infty(\mathbb{T}^n))$$

and therefore an isomorphism

$$HP_\bullet(\mathcal{A}_\Theta) \cong HP_\bullet(C^\infty(\mathbb{T}^n)).$$

Consequently,

$$HP_0(\mathcal{A}_\Theta) \cong \mathbb{C}^{2^{n-1}}, \qquad HP_1(\mathcal{A}_\Theta) \cong \mathbb{C}^{2^{n-1}}.$$

Proof. The chain homotopy equivalence is the composition

$$C_{\mathrm{per}}(\mathcal{A}_\Theta) \longrightarrow C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A}_\Theta) \xrightarrow{P^{\widetilde{\nabla}}_{1,0}} C^{\mathfrak{g}}_{\mathrm{per}}(\mathcal{A}_0) \longrightarrow C_{\mathrm{per}}(\mathcal{A}_0),$$

where the first and last arrows are the chain equivalences from Theorem 4.2.2. Since $\widetilde{\nabla}$ is a chain map on $C^{\mathrm{per}}_{\mathfrak{g}}(A)$, the middle arrow is an isomorphism of complexes by Proposition 3.2.4.

As shown in [5], if M is a compact smooth manifold, then

$$HP_\bullet(C^\infty(M)) \cong \bigoplus_k H^{\bullet+2k}_{dR}(M, \mathbb{C}),$$

where $H^\bullet_{dR}(M, \mathbb{C})$ is the de Rham cohomology of M with values in $\mathbb{C}$. Now, $H^m_{dR}(\mathbb{T}^n, \mathbb{C})$ is a vector space of dimension $\binom{n}{m}$, and this gives the result. □

Corollary 4.2.6. *The Gauss-Manin connection ∇_{GM} is integrable at the level of homology for the deformation $\{\mathcal{A}_{t\Theta}\}_{t\in J}$ of noncommutative tori.*

Proof. Apply Theorem 4.1.9. □

We shall make explicit calculations in section 4.4 to identify the ∇_{GM}-parallel sections, however it is more interesting to do this on the cohomology side. The dual of our argument can be carried out as follows. Let

$$C^{\mathrm{per}}_{\mathfrak{g}}(A) = \mathrm{Hom}_{C^\infty(J)}(C^{\mathfrak{g}}_{\mathrm{per}}(A), C^\infty(J))$$

be the $\mathfrak{g}$-invariant periodic cyclic cochain complex. It can be identified with the subspace of $\varphi \in C^{\mathrm{per}}(A)$ such that $\varphi L_{\delta_j} = 0$ for all j. The canonical inclusion

$$C^{\mathrm{per}}_{\mathfrak{g}}(A) \to C^{\mathrm{per}}(A)$$

is a chain homotopy equivalence, as it is the transpose of the chain homotopy equivalence

$$C_{\mathrm{per}}(A) \to C^{\mathfrak{g}}_{\mathrm{per}}(A).$$

We can consider the dual connection to $\widetilde{\nabla}$ given by

$$(\widetilde{\nabla}\varphi)(\omega) = \frac{d}{dt}\varphi(\omega) - \varphi(\widetilde{\nabla}\omega).$$

By Proposition 2.2.6, or by proving it directly, $\widetilde{\nabla}$ is integrable on $C^{\mathrm{per}}_{\mathfrak{g}}(A)$, and so the periodic cyclic cohomology $HP^{\bullet}(\mathcal{A}_\Theta)$ does not depend on Θ. As in the homology case, the dual Gauss-Manin connection ∇^{GM} is a nilpotent perturbation of $\widetilde{\nabla}$, and so is integrable at the level of cohomology.

We have proved the rigidity of periodic cyclic cohomology for the deformation of noncommutative tori. It is interesting to note that the Hochschild cohomology $HH^{\bullet}(\mathcal{A}_\Theta)$ and (non periodic) cyclic cohomology $HC^{\bullet}(\mathcal{A}_\Theta)$ are very far from rigid in this deformation. As an example, $HH^0(A) = HC^0(A)$ is the space of all traces on the algebra A. Now in the simplest case where $n = 2$ and $\Theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$, it is well-known that there is a unique (normalized) trace on $\mathcal{A}_\Theta$ when $\theta \notin \mathbb{Q}$ and an infinite dimensional space of traces when $\theta \in \mathbb{Q}$. For example, every linear functional on the commutative algebra $\mathcal{A}_0 \cong C^\infty(\mathbb{T}^n)$ is a trace, and thus $HH^0(C^\infty(\mathbb{T}^n)) = C^\infty(\mathbb{T}^n)^*$ is the space of distributions on $\mathbb{T}^n$. Moreover, Connes showed in [5] that in the case $\theta \notin \mathbb{Q}$, $HH^1(\mathcal{A}_\Theta)$ and $HH^2(\mathcal{A}_\Theta)$ are either finite dimensional or infinite dimensional and non-Hausdorff depending on the diophantine properties of θ. Looking back, we conclude that there are no integrable connections on $C^{\bullet}(A)$ that commute with b, as such a connection would imply rigidity of Hochschild cohomology.

However, our connection $\widetilde{\nabla}$ does commute with b on the invariant complex $C^{\bullet}_{\mathfrak{g}}(A)$. This shows that the invariant Hochschild cohomology $HH^{\bullet}_{\mathfrak{g}}(\mathcal{A}_\Theta)$ is independent of Θ. For example, there is exactly one (normalized) $\mathfrak{g}$-invariant trace on $C^\infty(\mathbb{T}^n)$, and that corresponds to integration with respect to the only (normalized) translation invariant measure. Thus $HH^0_{\mathfrak{g}}(C^\infty(\mathbb{T}^n)) = HH^0_{\mathfrak{g}}(\mathcal{A}_\Theta) = \mathbb{C}$. Consequently, the canonical map $HH^\bullet_{\mathfrak{g}}(\mathcal{A}_\Theta) \to HH^\bullet(\mathcal{A}_\Theta)$ is not, in general, an isomorphism.

4.3 Interaction with the Chern character

4.3.1 Chern character

We shall review some basic facts about the Chern character in cyclic homology, see [21, Chapter 8] for a more detailed account.

Let A be a unital locally convex $\check{\otimes}$-algebra over the ground ring R. Given an idempotent $P \in A$, $P^2 = P$, we define the *Chern character of* P to be the element $\operatorname{ch} P \in C_{\text{even}}(A)$ given by $(\operatorname{ch} P)_0 = P$ and for $n \geq 1$,

$$(\operatorname{ch} P)_{2n} = (-1)^n \frac{(2n)!}{n!}(P^{\otimes(2n+1)} - \frac{1}{2} e \otimes P^{\otimes(2n)}).$$

Then a quick calculation shows that

$$b(\operatorname{ch} P_{2(n+1)}) = -B(\operatorname{ch} P_{2n})$$

and this implies that $(b + B) \operatorname{ch} P = 0$.

More generally, we can define $\operatorname{ch} P \in C_{\text{even}}(A)$ when P is an idempotent in the matrix algebra $M_N(A) \cong M_N(\mathbb{C}) \otimes A$. Consider the *generalized trace* $T : C_\bullet(M_N(A)) \to C_\bullet(A)$ defined by

$$T((u_0 \otimes a_0) \otimes \ldots \otimes (u_n \otimes a_n)) = \operatorname{tr}(u_0 \ldots u_n) a_0 \otimes \ldots \otimes a_n,$$

where $\operatorname{tr} : M_N(\mathbb{C}) \to \mathbb{C}$ is the ordinary trace. As shown in [21, Chapter 1], T is a chain homotopy equivalence, and so induces an isomorphism $HP_\bullet(M_N(A)) \cong HP_\bullet(A)$. So we define $\operatorname{ch} P \in C_{\text{even}}(A)$ to be the image of $\operatorname{ch} P \in C_{\text{even}}(M_N(A))$ under the map T.

Given an invertible $U \in A$, consider the cycle $\operatorname{ch} U \in C_{\text{odd}}(A)$. For $n \geq 0$, let

$$(\operatorname{ch} U)_{2n+1} = (-1)^n n! U^{-1} \otimes U \otimes U^{-1} \otimes \ldots \otimes U^{-1} \otimes U.$$

Then, one can check that

$$b(\operatorname{ch} U_{2n+1}) = -B(\operatorname{ch} U_{2n-1}),$$

so that $(b + B) \operatorname{ch} U = 0$. As in the case of idempotents, we can define $\operatorname{ch} U \in C_{\text{odd}}(A)$ for any invertible $U \in M_N(A)$ by composing with T.

For the algebraic K-theory groups $K_0(A)$ and $K_1(A)$, there are pairings

$$HP^0(A) \times K_0(A) \to \mathbb{C}, \qquad HP^1(A) \times K_1(A) \to \mathbb{C}$$

given by

$$\langle [\varphi], [P] \rangle = \langle [\varphi], [\operatorname{ch} P] \rangle, \qquad \langle [\varphi], [U] \rangle = \langle [\varphi], [\operatorname{ch} U] \rangle$$

for an idempotent $P \in M_N(A)$ and an invertible $U \in M_N(A)$.

4.3.2 Chern character and ∇_{GM}

Now let A be the algebra of sections of a $\check{\otimes}$-smooth deformation $\{A_t\}_{t \in J}$ of algebras. Then A is an algebra over both $\mathbb{C}$ and $C^\infty(J)$, and there is a surjective morphism of complexes

$$\pi : C^{\mathbb{C}}_{\text{per}}(A) \to C^{C^\infty(J)}_{\text{per}}(A).$$

Proposition 4.3.1. *If $\omega \in C^{C^\infty(J)}_{\text{per}}(A)$ is a cycle that lifts to a cycle $\tilde{\omega} \in C^{\mathbb{C}}_{\text{per}}(A)$, then*

$$\nabla_{GM}[\omega] = 0$$

in $HP^{C^\infty(J)}_\bullet(A)$.

Proof. Let $\nabla^{\mathbb{C}}_{GM} = L_\nabla - I_E$, viewed as an operator on $C^{\mathbb{C}}_{\text{per}}(A)$. By Theorem 2.5.3,

$$\nabla^{\mathbb{C}}_{GM} = L_\nabla - I_{\delta\nabla} = [b + B, I_\nabla]$$

and so $\nabla^{\mathbb{C}}_{GM}$ is the zero operator on $HP^{\mathbb{C}}_\bullet(A)$. Thus, at the level of homology, we

have
$$\nabla_{GM} \circ \pi = \pi \circ \nabla_{GM}^{\mathbb{C}} = 0$$
where $\pi : HP_\bullet^{\mathbb{C}}(A) \to HP_\bullet^{C^\infty(J)}(A)$ is the map induced by the quotient map. By hypothesis, ω is in the image of π. □

Note that the homotopy used in the previous proof does not imply that ∇_{GM} is zero on $HP_\bullet^{C^\infty(J)}(A)$. The reason is that the operator I_∇ is not a well-defined operator on the quotient complex $C_{\text{per}}^{C^\infty(J)}(A)$.

Corollary 4.3.2. *If $P \in M_N(A)$ is an idempotent and $U \in M_N(A)$ is an invertible, then*
$$\nabla_{GM}[\operatorname{ch} P] = 0, \qquad \nabla_{GM}[\operatorname{ch} U] = 0$$
in $HP_\bullet(A)$.

Proof. This is immediate from the previous proposition because the cycle $\operatorname{ch} P \in C_{\text{per}}^{\mathbb{C}}(A)$ is a lift of the cycle $\operatorname{ch} P \in C_{\text{per}}^{C^\infty(J)}(A)$, and similarly for $\operatorname{ch} U$. □

Combining this with the identity
$$\frac{d}{dt}\langle[\varphi], [\omega]\rangle = \langle\nabla^{GM}[\varphi], [\omega]\rangle + \langle[\varphi], \nabla_{GM}[\omega]\rangle,$$
we obtain the following differentiation formula.

Corollary 4.3.3. *If $P \in M_N(A)$ is an idempotent and $U \in M_N(A)$ is an invertible, then*
$$\frac{d}{dt}\langle[\varphi], [P]\rangle = \langle\nabla^{GM}[\varphi], [P]\rangle, \qquad \frac{d}{dt}\langle[\varphi], [U]\rangle = \langle\nabla^{GM}[\varphi], [U]\rangle.$$

Remark 4.3.4. Proposition 3.3.3 can be used to give an alternative proof that
$$\nabla_{GM}[\operatorname{ch} P] = 0$$
when $P \in A$ is an idempotent. Indeed, an idempotent in A is given by a collection of algebra maps
$$\{F_t : \mathbb{C} \to A_t\}_{t\in J}.$$
Smoothness of P is equivalent to $\{F_t\}_{t\in J}$ being a morphism from the trivial deformation over J with fiber $\mathbb{C}$ to $\{A_t\}_{t\in J}$. Thus by Proposition 3.1.5, it induces an

$C^\infty(J)$-linear algebra map

$$F : C^\infty(J) \to A$$

which maps 1 to P. Applying Proposition 3.3.3, we see

$$\nabla_{GM}[\mathrm{ch}\, P] = \nabla_{GM} F[\mathrm{ch}\, 1] = F\frac{d}{dt}[\mathrm{ch}\, 1] = 0.$$

4.4 Differentiation formulas for cyclic cocycles in noncommutative tori

4.4.1 Cyclic cocycles, characteristic maps, and cup products

Recall that a *(normalized) cyclic cocycle* $\varphi \in C^k(A)$ is a Hochschild cocycle such that

$$\varphi(a_0, \dots, a_k) = 0,$$

if $a_i = 1$ for some $1 \leq i \leq k$, and

$$\varphi(a_k, a_0, \dots, a_{k-1}) = (-1)^k \varphi(a_0, a_1, \dots, a_k).$$

Such a cyclic cocycle φ automatically satisfies $B\varphi = 0$, because

$$\varphi(1, a_0, \dots, a_{k-1}) = (-1)^k \varphi(a_{k-1}, 1, a_0, \dots, a_{k-2}) = 0.$$

Thus, $(b + B)\varphi = 0$, and so φ determines a cohomology class in $HP^\bullet(A)$.

Suppose that $\mathfrak{g}$ is an abelian Lie algebra of derivations on an algebra A, and suppose τ is a trace which is $\mathfrak{g}$-invariant in the sense that

$$\tau \circ X = 0, \qquad \forall X \in \mathfrak{g}.$$

Define the *characteristic map* $\gamma : \Lambda^\bullet \mathfrak{g} \to C^\bullet(A)$ by

$$\gamma(X_1 \wedge \dots \wedge X_k)(a_0, \dots, a_k) = \frac{1}{k!} \sum_{\sigma \in \mathbb{S}_k} (-1)^\sigma \tau(a_0 X_{\sigma(1)}(a_1) X_{\sigma(2)}(a_2) \dots X_{\sigma(k)}(a_k)).$$

Proposition 4.4.1. *The functional $\gamma(X_1 \wedge \ldots \wedge X_k)$ is a cyclic k-cocycle. Moreover, it is invariant with respect to the action of $\mathfrak{g}$ by Lie derivatives.*

Remark 4.4.2. The map γ is a simple case of the Connes-Moscovici characteristic map in Hopf cyclic cohomology [3]. In their work, $\mathcal{H}$ is a Hopf algebra equipped with some extra structure called a modular pair, and A is an algebra equipped with a Hopf action of $\mathcal{H}$. Assuming A possesses a trace that is compatible with the modular pair, they construct a map

$$\gamma : HP^{\bullet}_{Hopf}(\mathcal{H}) \to HP^{\bullet}(A)$$

from the Hopf periodic cyclic cohomology of $\mathcal{H}$ to the ordinary periodic cyclic cohomology of A. In our situation, $\mathcal{H} = \mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. The fact that $\mathfrak{g}$ acts on A by derivations implies that the action of $\mathcal{U}(\mathfrak{g})$ on A is a Hopf action. The compatibility condition for the trace follows from the fact that our trace is $\mathfrak{g}$-invariant. As was shown in [3],

$$HP^{\bullet}_{Hopf}(\mathcal{U}(\mathfrak{g})) \cong \bigoplus_{k = \bullet mod 2} H^{Lie}_k(\mathfrak{g}, \mathbb{C}),$$

where $H^{Lie}_k(\mathfrak{g}, \mathbb{C})$ is the Lie algebra homology of $\mathfrak{g}$ with coefficients in the trivial $\mathfrak{g}$-module $\mathbb{C}$. As $\mathfrak{g}$ is abelian, there is an isomorphism

$$H^{Lie}_k(\mathfrak{g}, \mathbb{C}) \cong \Lambda^k(\mathfrak{g}).$$

The obtained characteristic map

$$\gamma : \Lambda^{\bullet}(\mathfrak{g}) \to HP^{\bullet}(A)$$

is the map defined above.

The fact that $\gamma(X_1 \wedge \ldots X_k)$ is invariant relies on the fact that $\mathfrak{g}$ is abelian. In this case, the characteristic map factors through the inclusion $HP^{\bullet}_{\mathfrak{g}}(A) \to HP^{\bullet}(A)$, and we obtain a characteristic map

$$\gamma : \Lambda^{\bullet}(\mathfrak{g}) \to HP^{\bullet}_{\mathfrak{g}}(A).$$

Lemma 4.4.3. *Let $X_1, \dots, X_n$ be derivations on an algebra A, and let τ be a trace on A. There exists $\psi \in C^{n-1}(A)$ such that $B\psi = 0$ and*

$$\begin{aligned}
&\tau(a_0 X_1(a_1)\dots X_n(a_n)) \\
&\quad = \frac{1}{n}\sum_{j=1}^{n}(-1)^{(j-1)(n+1)}\tau(a_0X_j(a_1)\dots X_n(a_{n-j+1})X_1(a_{n-j+2})\dots X_{j-1}(a_n)) \\
&\qquad + (b\psi)(a_0,\dots,a_n).
\end{aligned}$$

Proof. Given any n derivations $Y_1, \dots, Y_n$, the $(n-1)$-cochain

$$\varphi(a_0,\dots,a_{n-1}) = \tau(Y_1(a_0)Y_2(a_1)\dots Y_n(a_{n-1}))$$

satisfies

$$(b\varphi)(a_0,\dots,a_n) = \tau(a_0Y_1(a_1)\dots Y_n(a_n) + (-1)^n a_0Y_2(a_1)\dots Y_n(a_{n-1})Y_1(a_n))$$

and $B\varphi = 0$. It follows that

$$\begin{aligned}
&\psi(a_0,\dots,a_{n-1}) \\
&\quad = \frac{1}{n}\sum_{j=1}^{n-1}(-1)^{(j-1)(n+1)}(n-j)\tau(X_j(a_0)X_{j+1}(a_1)\dots X_{j-1}(a_n))
\end{aligned}$$

satisfies the conclusions of the lemma. $\square$

Recall that for any $Z \in \mathfrak{g}$, the cyclic contraction I_Z is a chain map on the invariant complex $C^{\mathrm{per}}_{\mathfrak{g}}(A)$.

Proposition 4.4.4. *For any $Z \in \mathfrak{g}$ and $\omega \in \Lambda^\bullet\mathfrak{g}$,*

$$I_Z[\gamma(\omega)] = [\gamma(Z \wedge \omega)]$$

in $HP^\bullet_{\mathfrak{g}}(A)$.

Proof. It suffices to consider $\omega = X_1 \wedge \dots \wedge X_k$ for $X_1, \dots, X_k \in \mathfrak{g}$. Let $\varphi = \gamma(X_1 \wedge \dots \wedge X_k)$. Since φ is cyclic and normalized,

$$\varphi(1, a_1, \dots, a_k) = 0, \qquad a_1, \dots, a_k \in A,$$

and consequently $S_Z\varphi = 0$. Thus, $I_Z\varphi = \iota_Z\varphi$, and

$$\begin{aligned}(\iota_Z\varphi)(a_0,\ldots,a_{k+1}) &= \varphi(a_0 Z(a_1), a_2, \ldots a_{k+1}) \\ &= \frac{1}{k!}\sum_{\sigma\in\mathbb{S}_k}(-1)^\sigma \tau(a_0 Z(a_1) X_{\sigma(1)}(a_2) X_{\sigma(2)}(a_3)\ldots X_{\sigma(k)}(a_{k+1})) \\ &= \gamma(Z\wedge X_1\wedge\ldots\wedge X_k)(a_0,\ldots,a_{k+1}) + (b\psi)(a_0,\ldots,a_{k+1})\end{aligned}$$

for some ψ with $B\psi = 0$ by applying the previous lemma to each term in the sum. Hence, $I_Z\gamma(X_1\wedge\ldots\wedge X_k) = \gamma(Z\wedge X_1\wedge\ldots\wedge X_k) + (b+B)\psi$. □

As in the homology case (Theorem 4.1.3,) there is an algebra map

$$\chi : \Lambda^\bullet(\mathfrak{g}) \to \mathrm{End}(HP^\bullet_{\mathfrak{g}}(A))$$

given by

$$\chi(X_1\wedge\ldots\wedge X_k) = I_{X_1} I_{X_2}\ldots I_{X_k}.$$

By induction, we obtain the following result.

Corollary 4.4.5. *For any* $\omega \in \Lambda^\bullet\mathfrak{g}$*,*

$$[\gamma(\omega)] = \chi(\omega)[\tau]$$

in $HP^\bullet_{\mathfrak{g}}(A)$*.*

Remark 4.4.6. A generalization of the Connes-Moscovici characteristic map was constructed in [19]. A special case of this construction is a cup product

$$\smile : HP^p_{Hopf}(\mathcal{H}) \otimes HP^q_{\mathcal{H}}(A) \to HP^{p+q}(A),$$

where $HP^\bullet_{\mathcal{H}}(A)$ is the periodic cyclic cohomology of A built out of cochains which are invariant in some sense with respect to the action of $\mathcal{H}$. In the Connes-Moscovici picture, the properties of the trace τ ensures that it gives a cohomology class in $HP^\bullet_{\mathcal{H}}(A)$, and

$$[\omega] \smile [\tau] = \gamma[\omega]$$

for all $[\omega] \in HP^\bullet_{Hopf}(\mathcal{H})$. In our situation where $\mathcal{H} = \mathcal{U}(\mathfrak{g})$, we have that

$HP^\bullet_{\mathcal{H}}(A) = HP^\bullet_{\mathfrak{g}}(A)$ and the cup product will give a map

$$\smile : \Lambda^p\mathfrak{g} \otimes HP^q_{\mathfrak{g}}(A) \to HP^{p+q}(A).$$

Our map $\chi : \Lambda^\bullet(A) \to \mathrm{End}(HP^\bullet_{\mathfrak{g}}(A))$ composed with the canonical inclusion $HP^\bullet_{\mathfrak{g}}(A) \to HP^\bullet(A)$ coincides with this cup product. The fact that the cup product sends $\mathfrak{g}$-invariant cocycles to $\mathfrak{g}$-invariant cocycles is a consequence of the fact that $\mathfrak{g}$ is abelian.

4.4.2 Noncommutative tori

Now, let A be the section algebra of the noncommutative n-tori deformation $\{\mathcal{A}_{t\Theta}\}_{t\in J}$. More generally, we shall consider the matrix algebra $M_N(A)$, which is the section algebra of the deformation $\{M_N(\mathcal{A}_{t\Theta})\}_{t\in J}$. Using the isomorphism $M_N(A) \cong M_N(\mathbb{C}) \otimes A$, the operators $\mathrm{tr} \otimes \tau$, $1 \otimes \delta_j$, and $1 \otimes \nabla$ shall be denoted by τ, δ_j, and ∇. Then just as in the $N = 1$ case, one has

$$E := \delta\nabla = \frac{1}{2\pi i}\sum_{j>k} \theta_{jk} \cdot \delta_j \smile \delta_k.$$

The map $T : C^\bullet(A) \to C^\bullet(M_N(A))$, which is the transpose of the generalized trace, induces an isomorphism $HP^\bullet(A) \to HP^\bullet(M_N(A))$ which is parallel with respect to the Gauss-Manin connections. Thus, ∇^{GM} is integrable on $HP^\bullet(M_N(A))$.

Proposition 4.4.7. *For any $\omega \in \Lambda^\bullet\mathfrak{g}$,*

$$\widetilde{\nabla}(\gamma(\omega)) = 0.$$

Proof. Consider $\omega \in \Lambda^m\mathfrak{g}$. Notice that $\frac{d}{dt} \circ \tau = \tau \circ \nabla$ and

$$\begin{aligned}\nabla(a_0 \ldots a_m) = \sum_{j=0}^{k} & a_0 \ldots \nabla(a_j) \ldots a_m \\ & + \frac{1}{2\pi i}\sum_{j'<k'}\sum_{j>k} \theta_{jk} \cdot a_0 \ldots \delta_j(a_{j'}) \ldots \delta_k(a_{k'}) \ldots a_m.\end{aligned}$$

It follows that

$$\frac{d}{dt}\gamma(\omega)(a_0,\ldots,a_m) = \sum_{j=0}^{k}\gamma(\omega)(a_0,\ldots,\nabla(a_j),\ldots,a_m)$$
$$+\frac{1}{2\pi i}\sum_{j'<k'}\sum_{j>k}\theta_{jk}\gamma(\omega)(a_0,\ldots,\delta_j(a_{j'}),\ldots,\delta_k(a_{k'}),\ldots,a_m),$$

using the fact that $\mathfrak{g}$ is abelian and commutes with ∇. This shows $\widetilde{\nabla}(\gamma(\omega)) = 0$. $\square$

Theorem 4.4.8. *For every Θ, the map $\gamma : \Lambda^\bullet(\mathfrak{g}) \to HP^\bullet(M_N(\mathcal{A}_\Theta))$ is an isomorphism of $\mathbb{Z}/2$-graded spaces.*

Proof. It suffices to prove the $N = 1$ case as the general case follows by the fact that the diagram

$$\begin{tikzcd} \Lambda^\bullet\mathfrak{g} \arrow[r, "\gamma"] \arrow[dr, "\gamma"'] & HP^\bullet(\mathcal{A}_\Theta) \arrow[d, "T"] \\ & HP^\bullet(M_N(\mathcal{A}_\Theta)) \end{tikzcd}$$

commutes. By Theorem 4.2.4 and Proposition 4.4.7, is suffices to prove this for $\mathcal{A}_0 \cong C^\infty(\mathbb{T}^n)$. Let $s_1,\ldots,s_n$ be the coordinates in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Choosing a subset of coordinates $s_{i_1},\ldots,s_{i_m}$ determines a subtorus T of dimension m. All such subtori are in bijection with the homology classes of $\mathbb{T}^n$. The de Rham cycle corresponding to T is given by integration of a differential form over T. The cochain in $C^m(C^\infty(\mathbb{T}^n))$ corresponding to this cycle is

$$\varphi_T(f_0,\ldots,f_m) = \int_T f_0 df_1\wedge\ldots\wedge df_m,$$

and one can show $\varphi_T = \gamma(\delta_{i_1}\wedge\ldots\wedge\delta_{i_m})$ up to a scalar multiple. $\square$

Now that we have an explicit basis for $HP^\bullet(M_N(A))$, we can describe ∇^{GM} as an operator on $HP^\bullet(M_N(A))$.

Theorem 4.4.9. *For any $\omega \in \Lambda^\bullet\mathfrak{g}$,*

$$\nabla^{GM}[\gamma(\omega)] = \frac{1}{2\pi i}\sum_{j>k}\theta_{jk}\cdot[\gamma(\delta_j\wedge\delta_k\wedge\omega)].$$

Proof. The analogue of Proposition 4.1.8 in this setting is that

$$\nabla^{GM} = \widetilde{\nabla} + \frac{1}{2\pi i}\sum_{j>k}\theta_{jk}\cdot\chi(\delta_j\wedge\delta_k)$$

as operators on $HP^{\bullet}_{\mathfrak{g}}(A)$. The result is immediate from Proposition 4.4.7 and Corollary 4.4.5. □

Corollary 4.4.10. *For any idempotent $P \in M_N(A)$ and $\omega \in \Lambda^{2m}\mathfrak{g}$,*

$$\begin{aligned}\frac{d}{dt}&\gamma(\omega)(P,\ldots,P)\\ &= -\frac{(4m+2)}{2\pi i}\sum_{j>k}\theta_{jk}\cdot\gamma(\delta_j\wedge\delta_k\wedge\omega)(P,\ldots,P).\end{aligned}$$

Corollary 4.4.11. *For any invertible $U \in M_N(A)$ and $\omega \in \Lambda^{2m+1}\mathfrak{g}$,*

$$\begin{aligned}\frac{d}{dt}&\gamma(\omega)(U^{-1},U,\ldots,U^{-1},U)\\ &= -\frac{m}{2\pi i}\sum_{j>k}\theta_{jk}\cdot\gamma(\delta_j\wedge\delta_k\wedge\omega)(U^{-1},U,\ldots,U^{-1},U).\end{aligned}$$

Proof. Use Corollary 4.3.3 and the explicit form of Chern characters. □

Let us specialize to the case $n = 2$. Here, the noncommutative torus is determined by a single real parameter $\theta := \theta_{21}$, and we shall denote the algebra by $\mathcal{A}_\theta$. We shall consider $\{\mathcal{A}_\theta\}_{\theta\in J}$ as a smooth one-parameter deformation, where $J \subset \mathbb{R}$ is an open interval containing 0. Let A be the algebra of sections and let τ_2 be the cyclic 2-cocycle $\tau_2 = \frac{1}{\pi i}\cdot\gamma(\delta_1\wedge\delta_2)$, which is given explicitly by

$$\tau_2(a_0,a_1,a_2) = \frac{1}{2\pi i}\tau(a_0\delta_1(a_1)\delta_2(a_2) - a_0\delta_2(a_1)\delta_1(a_2)).$$

Corollary 4.4.10 says that for any idempotent $P \in M_N(A)$,

$$\frac{d}{d\theta}\tau(P) = \tau_2(P,P,P),$$

and

$$\frac{d^2}{d\theta^2}\tau(P) = \frac{d}{d\theta}\tau_2(P,P,P) = 0$$

because $\gamma(\delta_2 \wedge \delta_1 \wedge \delta_2 \wedge \delta_1) = 0$. Thus we have shown that for any idempotent $P \in M_N(A)$,

$$\tau(P) = \tau(P)(0) + \tau_2(P, P, P)(0) \cdot \theta.$$

Now the idempotent $P(0) \in M_N(\mathcal{A}_0) \cong M_N(C^\infty(\mathbb{T}^2))$ corresponds to a smooth vector bundle over $\mathbb{T}^2$ and the value $\tau(P)(0)$ is the dimension of this bundle. The number $\tau_2(P, P, P)(0)$ is the first Chern number of the bundle, which is an integer. So P satisfies

$$\tau(P) = C + D\theta$$

for integers C and D.

This result suggests that $\mathcal{A}_\theta$ may contain an idempotent of trace θ, a fact which is now well-known and was first shown in [29]. Let $P_\theta \in \mathcal{A}_\theta$ be such an idempotent. One could ask about the possibility of extending P_θ to an idempotent $P \in A$. This is not possible, because such an idempotent would necessarily satisfy $\tau_2(P, P, P) \neq 0$, and the only idempotents in $\mathcal{A}_0$, namely 0 and 1, do not. However, this can be done in the situation where the parameter space J doesn't contain any integers.

One can show that there exists an idempotent $P \in M_2(A)$ of trace $1 + \theta$ in the case where J is a small enough interval containing 0. However, this type of phenomenon cannot happen in the following two situations.

1. If the parameter space $J = \mathbb{R}$, then for any idempotent $P \in M_N(A)$, we necessarily have $\tau(P)$ is a constant integer-valued function. If not,

$$\tau(P) = C + D\theta$$

for some nonzero D. This contradicts the fact that $\tau(P)(k) \geq 0$ for all integers k because $\mathcal{A}_k \cong C^\infty(\mathbb{T}^2)$.

2. Since the deformation is periodic with period 1, we can consider the parameter space to be $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. In this case, the algebra of sections has underlying space $C^\infty(\mathbb{T}, \mathcal{S}(\mathbb{Z}^2))$, and the multiplication is defined in the usual way. The trace is now a map $\tau : A \to C^\infty(\mathbb{T})$. In this case, every idempotent $P \in M_N(A)$ has constant integer-valued trace. Indeed, by the above results,

$\tau(P)$ must be smooth and locally a linear polynomial in θ. The only such functions in $C^\infty(\mathbb{T})$ are constant.

In a similar fashion, one can consider the cyclic 1-cocycles

$$\tau_1^1 = \gamma(\delta_1), \qquad \tau_1^2 = \gamma(\delta_2).$$

For any invertible $U \in M_N(A)$,

$$\frac{d}{d\theta}\tau_1^j(U^{-1}, U) = 0, \qquad j = 1, 2$$

by Corollary 4.4.11, and one can show that $\tau_1^j(U^{-1}, U)(0)$ is integer-valued. Thus, $\tau_1^j(U^{-1}, U)$ is a constant integer-valued function.

Chapter 5

Rigidity of A_∞-algebras

We return to the general type of deformation considered in Chapter 4. Suppose $\{A_t\}_{t\in J}$ is a $\bar{\otimes}$-smooth one parameter deformation of algebras with algebra of sections A, and

$$\frac{d}{dt}(a_1a_2) = \frac{da_1}{dt}a_2 + a_1\frac{da_2}{dt} + \sum_{i=1}^{N} X_i(a_1)Y_i(a_2), \qquad \forall a_1, a_2 \in A$$

for some commuting family $\{X_1, \ldots, X_N, Y_1, \ldots, Y_N\}$ of operators on the underlying space that are derivations on each A_t. The methods of the last chapter, as applied to noncommutative tori, relied on this property and the fact that the action of $\mathfrak{g} = \mathrm{Span}\{X_1, \ldots, X_N, Y_1, \ldots, Y_N\}$ was the infinitesimal of a torus action. The torus action provided a "Fourier series" decomposition that was used in the proofs of Theorem 4.2.2 and Theorem 4.2.4.

In this chapter, we prove that the periodic cyclic homology of any such deformation $\{A_t\}_{t\in J}$ is preserved, where we need not assume that the action of $\mathfrak{g}$ integrates to a torus action. The proof appeals to the machinery of A_∞-algebras developed in sections 2.7 and 3.4.

5.1 The Chevalley-Eilenberg differential graded algebra

Suppose a finite dimensional abelian Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ acts on a locally convex $\bar{\otimes}$-algebra A by continuous derivations. Let

$$\Omega^\bullet(A) = A\bar{\otimes}\Lambda^\bullet\mathfrak{g}^*$$

denote the Chevalley-Eilenberg cochain complex that computes the Lie algebra cohomology of $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module A, see [1] or [36]. Notice that since $\Lambda^\bullet\mathfrak{g}^*$ is finite dimensional,

$$A\bar{\otimes}\Lambda^\bullet\mathfrak{g}^* = A \otimes \Lambda^\bullet\mathfrak{g}^*.$$

Then $\Omega^\bullet$ is a locally convex graded $\bar{\otimes}$-algebra, as the tensor product of two algebras. Explicitly,

$$(a_1 \otimes \omega_1)(a_2 \otimes \omega_2) = (a_1a_2) \otimes (\omega_1 \wedge \omega_2), \qquad \forall a_1, a_2 \in A, \omega_1, \omega_2 \in \Lambda^\bullet\mathfrak{g}^*.$$

To describe the differential d, choose a basis $\{X_1, \dots, X_n\}$ of $\mathfrak{g}$ and denote the corresponding dual basis of $\mathfrak{g}^*$ by $\{dX_1, \dots, dX_n\}$. Define

$$d : A \to \Omega^1(A), \qquad da = \sum_{i=1}^n X_i(a) \otimes dX_i.$$

This expression is independent of the choice of basis. Then

$$d : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$$

is given by

$$d(a \otimes \omega) = (da)(1 \otimes \omega), \qquad \forall a \in A, \omega \in \Lambda^\bullet\mathfrak{g}^*.$$

It is routine to verify that d is a degree $+1$ derivation on $\Omega^\bullet(A)$. To see that $d^2 = 0$, notice that

$$d^2 = \frac{1}{2}[d, d]$$

is a degree $+2$ derivation on $\Omega^\bullet(A)$, so it suffices to prove that $d^2 = 0$ on elements of the form $a \otimes 1$ and $1 \otimes \omega$, which generate $\Omega^\bullet(A)$ as an algebra. But

$$d^2(a \otimes 1) = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i(X_j(a)) \otimes dX_i \wedge dX_j = \sum_{i<j} [X_i, X_j](a) \otimes dX_i \wedge dX_j = 0$$

and

$$d^2(1 \otimes \omega) = d(0) = 0.$$

Thus we have shown the following.

Proposition 5.1.1. *If the abelian Lie algebra $\mathfrak{g}$ acts on a locally convex $\bar{\otimes}$-algebra A by continuous derivations, then the Chevalley-Eilenberg cochain complex $\Omega^\bullet(A)$ is a locally convex differential graded $\bar{\otimes}$-algebra.*

Given $X \in \mathfrak{g}$, define the *contraction by X*

$$\iota_X : \Lambda^\bullet \mathfrak{g}^* \to \Lambda^{\bullet-1}\mathfrak{g}^*$$

to be the operator

$$\iota_X(\alpha_1 \wedge \ldots \wedge \alpha_k) = \sum_{i=1}^{k} (-1)^{i+1} \alpha_i(X) \alpha_1 \wedge \ldots \wedge \widehat{\alpha_i} \wedge \ldots \wedge \alpha_k,$$

where $\widehat{\alpha_i}$ indicates that α_i is omitted from the wedge product. For each $X \in \mathfrak{g}$, the contraction ι_X is a degree -1 derivation. We shall also denote by ι_X the operator

$$1 \otimes \iota_X : \Omega^\bullet(A) \to \Omega^{\bullet-1}(A),$$

which is also a degree -1 derivation. The *Lie derivative along $X \in \mathfrak{g}$* is the operator

$$L_X = X \otimes 1 : \Omega^\bullet(A) \to \Omega^\bullet(A).$$

It is a degree 0 derivation because X is a derivation. Then we have the following classical Cartan Homotopy Formula.

Theorem 5.1.2 (Cartan Homotopy Formula)**.** *For any $X \in \mathfrak{g}$,*

$$[d, \iota_X] = L_X.$$

Proof. The commutator $[d, \iota_X]$ is a degree 0 derivation, so it suffices to prove $[d, \iota_X] = L_X$ on generators of the form $a \otimes 1, 1 \otimes \omega \in \Omega^\bullet(A)$. It is clear that

$$[d, \iota_X](1 \otimes \omega) = 0 = L_X(1 \otimes \omega), \qquad \forall \omega \in \Lambda^\bullet \mathfrak{g}^*.$$

It is also clear if $X = 0$. Given a nonzero $X \in \mathfrak{g}$, extend to a basis $\{X, Y_1, \ldots, Y_{n-1}\}$ of $\mathfrak{g}$. Then

$$\begin{aligned}
[d, \iota_X](a \otimes 1) &= \iota_X(d(a \otimes 1)) \\
&= \iota_X \left(X(a) \otimes dX + \sum_{i=1}^{n-1} Y_i(a) \otimes dY_i \right) \\
&= X(a) \otimes 1 \\
&= L_X(a \otimes 1).
\end{aligned}$$

□

In particular, it follows that $[d, L_X] = 0$ for all $X \in \mathfrak{g}$.

5.2 A_∞-deformations and cyclic homology

Let $\{A_t\}_{t \in J}$ is a $\bar{\otimes}$-smooth deformation of algebras equipped with commuting continuous linear maps $X_1, \ldots, X_N, Y_1, \ldots Y_N$ on the underlying space that are derivations with respect to each product in the deformation. Suppose additionally that the algebra of sections satisfies

$$\frac{d}{dt}(a_1 a_2) = \frac{da_1}{dt} a_2 + a_1 \frac{da_2}{dt} + \sum_{i=1}^{N} X_i(a_1) Y_i(a_2)$$

Let $\mathfrak{g}$ denote the abelian Lie algebra spanned by $X_1, \ldots, X_N, Y_1, \ldots, Y_N$. Then for each $t \in J$, we can form the Chevalley-Eilenberg differential graded algebra $\Omega^\bullet(A_t)$. In this way we obtain a deformation $\{\Omega^\bullet(A_t)\}_{t \in J}$ of differential graded algebras, which we view as a deformation of A_∞-algebras.

Proposition 5.2.1. *The family $\{\Omega^\bullet(A_t)\}_{t \in J}$ form a smooth deformation of locally convex A_∞-algebras.*

Proof. By Proposition 3.4.2, we need only check separately that $\{\Omega^\bullet(A_t)\}_{t\in J}$ is a $\bar{\otimes}$-smooth deformation of complexes and a $\bar{\otimes}$-smooth deformation of algebras. It is constant as a deformation of complexes, because the family of derivations $X_1,\ldots,X_N,Y_1,\ldots Y_n$ do not depend on t. That $\{\Omega^\bullet(A_t)\}_{t\in J}$ is a smooth deformation of algebras follows at once from the fact that $\{A_t\}_{t\in J}$ is a $\bar{\otimes}$-smooth deformation of algebras. □

Theorem 5.2.2. *The deformation $\{\Omega^\bullet(A_t)\}_{t\in J}$ is trivial as a deformation of A_∞-algebras. Consequently, for each $s,t\in J$ there is an A_∞-isomorphism*

$$\Omega^\bullet(A_s)\cong\Omega^\bullet(A_t).$$

Proof. We shall construct an explicit integrable connection on the A_∞-algebra $\Omega^\bullet(A)$ of sections that is an A_∞-derivation. In terms of the component maps

$$\nabla_n : (s\Omega(A))^{\bar{\otimes}n}\to s\Omega(A)$$

as described in section 3.4.1, our connection is

$$\nabla=\nabla_1+\nabla_2,$$

where

$$\nabla_1=\frac{d}{dt},\qquad \nabla_2=-\sum_{i=1}^{N}\iota_{X_i}\smile L_{Y_i}.$$

Recall that the Hochschild coboundary is given by

$$\delta D=[m_1,D]+[m_2,D],$$

where $m_1=d$ and $m_2(\omega_1,\omega_2)=(-1)^{\deg\omega_1}\omega_1\omega_2$, see section 2.6. On $\Omega^\bullet(A)$, we have

$$\frac{d}{dt}(\omega_1\omega_2)=\frac{d\omega_1}{dt}\omega_2+\omega_1\frac{d\omega_2}{dt}+\sum_{i=1}^{N}L_{X_i}(\omega_1)L_{Y_i}(a_2),$$

which follows naturally from the corresponding formula on A. This shows that

$$[m_2, \frac{d}{dt}] = \sum_{i=1}^{N} L_{X_i} \smile L_{Y_i}.$$

On the other hand, the differential in $\Omega^\bullet(A_t)$ does not depend on t, so

$$[m_1, \frac{d}{dt}] = 0.$$

As in Example 2.6.1, we see that

$$[m_2, \iota_{X_i}] = 0, \qquad [m_2, L_{Y_i}] = 0$$

because contraction and Lie derivative operators are graded derivations. Thus,

$$\begin{aligned}\delta(\iota_{X_i} \smile L_{Y_i}) &= (\delta\iota_{X_i}) \smile L_{Y_i} \;+\; \iota_{X_i} \smile (\delta L_{Y_i}) \\ &= [m_1, \iota_{X_i}] \smile L_{Y_i} \;+\; \iota_{X_i} \smile [m_1, L_{Y_i}] \\ &= L_{X_i} \smile L_{Y_i}\end{aligned}$$

by the Cartan Homotopy Formula. We conclude that

$$\delta\nabla = \delta\nabla_1 + \delta\nabla_2 = \sum_{i=1}^{N} L_{X_i} \smile L_{Y_i} - \sum_{i=1}^{N} L_{X_i} \smile L_{Y_i} = 0.$$

Thus, ∇ is an A_∞-derivation. By Proposition 3.4.6, ∇ is integrable because $\nabla_1 = \frac{d}{dt}$ is integrable. By applying Proposition 3.4.5, we conclude that $\{\Omega^\bullet(A_t)\}_{t\in J}$ is a trivial deformation of A_∞-algebras. $\square$

By applying Theorem 2.7.8, we obtain that periodic cyclic homology of the deformation $\{A_t\}_{t\in J}$ is preserved. For all $s, t \in J$, there are isomorphisms

$$HP_\bullet(A_s) \cong HP_\bullet(A_t).$$

Applying this to the noncommutative tori deformation $\{\mathcal{A}_{t\Theta}\}_{t\in\mathbb{R}}$ proves that

$$HP_\bullet(\mathcal{A}_\Theta) \cong HP_\bullet(C^\infty(\mathbb{T}^n)),$$

so that $HP_\bullet(\mathcal{A}_\Theta)$ does not depend on Θ. Applying our result to the crossed product deformation $\{B \rtimes_{\alpha^t} \mathbb{R}\}_{t\in\mathbb{R}}$, see section 3.1.2, proves that

$$HP_\bullet(B \rtimes_\alpha \mathbb{R}) \cong HP_\bullet(\mathcal{S}(\mathbb{R})\widehat{\otimes}B).$$

As $HP_\bullet(\mathcal{S}(\mathbb{R})\widehat{\otimes}B) \cong HP_{\bullet+1}(B)$, we obtain the Thom isomorphism

$$HP_\bullet(B \rtimes_\alpha \mathbb{R}) \cong HP_{\bullet+1}(B).$$

Appendix A

Single variable calculus in locally convex vector spaces

A.1 Differentiation

In this appendix, let X be a complete, locally convex, Hausdorff vector space and let $J \subseteq \mathbb{R}$ be an open interval. A function $f : J \to X$ is *differentiable at* $t \in J$ if

$$f'(t) := \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

exists in X. We say f is *differentiable* if $f'(t)$ exists for all $t \in J$. So we have a new function $f' : J \to X$ and we can ask if it is differentiable. The n-th derivative of f, if it exists, will be denoted $f^{(n)}$. The function f is *smooth* if $f^{(n)}$ exists for every n. As it follows from the definition that differentiability implies continuity, the n-th derivatives of a smooth function are, in particular, continuous. Let $C^\infty(J, X)$ denote the vector space of all smooth functions from J into X.

Proposition A.1.1. *If $f : J \to X$ satisfies $f' = 0$, then f is constant.*

Proof. For any continuous linear functional $\varphi \in X^*$, we have that $\varphi \circ f : J \to \mathbb{C}$ is differentiable with derivative $\varphi \circ f' = 0$. Thus, by calculus in finite dimensional Euclidean space, $\varphi(f(t)) = \varphi(f(s))$ for all $t, s \in J$. The Hahn-Banach theorem implies that X^* separates points in X, and so $f(t) = f(s)$ for all $t, s \in J$. $\square$

A.2 Integration

We shall now develop the theory of the Riemann integral in X. We shall consider various spaces of functions (not necessarily continuous) valued in X. Let K be a compact interval and let $\mathcal{F}(K, X)$ be the vector space of all functions $f : K \to X$ such that for each continuous seminorm p on X,

$$p^{\infty}(f) := \sup_{t \in K} p(f(t)) < \infty.$$

We give $\mathcal{F}(K, X)$ the locally convex topology generated by the collection $\{p^{\infty}\}$ of seminorms as p varies through a generating family of seminorms for X. This is the topology of uniform convergence on K. The completeness of X implies that $\mathcal{F}(K, X)$ is complete. The space of all continuous functions $C(K, X)$ is a closed subspace of $\mathcal{F}(K, X)$.

Given an interval $L \subset K$, let $1_L : K \to \mathbb{C}$ be the characteristic function of L, so that

$$1_L(t) = \begin{cases} 1, & t \in L \\ 0, & t \notin L. \end{cases}$$

Here, L can be open, closed, or half-open. A *step function* $g : K \to X$ is a finite sum

$$g = \sum_{i=1}^{N} x_i \cdot 1_{L_i},$$

for subintervals $L_1, \ldots, L_N$ and elements $x_1, \ldots, x_N \in X$. The vector space of all step functions will be denoted $\mathcal{S}(K, X)$. It is a linear subspace of $\mathcal{F}(K, X)$. Let $\mathcal{R}(K, X)$ denote its closure in $\mathcal{F}(K, X)$. For our purposes, the following proposition is the most important fact we need to know about $\mathcal{R}(K, X)$.

Proposition A.2.1. *The space of continuous functions $C(K, X)$ is a closed subspace of $\mathcal{R}(K, X)$.*

Proof. Given $f \in C(K, X)$ and a continuous seminorm p on X, we must show that for every $\epsilon > 0$, there is a step function $g \in \mathcal{S}(K, X)$ such that

$$p^{\infty}(f - g) < \epsilon.$$

As K is compact, f is uniformly continuous with respect to the seminorm p. That is, there is $\delta > 0$ such that if $|s-t| < \delta$, then $p(f(s)-f(t)) < \epsilon$. Choose an integer N large enough so that $\Delta t := |b-a|/N < \delta$, where $K = [a,b]$. Let $t_0 = a$ and $t_i = a + i\Delta t$ for $i = 1, \ldots, N$. For each i, let $L_i = [t_i, t_{i+1})$, and let

$$g = \sum_{i=0}^{N-1} f(t_i) \cdot 1_{L_i}.$$

Then if $t \in L_i$, we have $|t - t_i| < \delta$, and so

$$p(f(t) - g(t)) = p(f(t) - f(t_i)) < \epsilon.$$

Since the L_i cover K, $p^\infty(f-g) < \epsilon$. This shows that the continuous functions are contained in the closure of the step functions. □

Now, on to the integral. Given $g \in \mathcal{S}([a,b], X)$, $g = \sum_{i=1}^{N} x_i \cdot 1_{[a_i,b_i]}$, we define

$$\int g = \sum_{i=0}^{N} |b_i - a_i| x_i.$$

It is routine to check that this expression is independent of the presentation of g as a step function. Thus we have a well-defined linear map

$$\int : \mathcal{S}([a,b], X) \to X.$$

Rewriting the step function so that the intervals $[a_i, b_i]$ are mutually disjoint, we see that for any continuous seminorm p on X,

$$p(\int g) = \sum_{i=0}^{N} |b_i - a_i| p(x_i) \leq |b-a| p^\infty(g).$$

So $\int$ is continuous and therefore extends to a continuous linear map

$$\int : \mathcal{R}([a,b], X) \to X$$

which satisfies

$$p(\int f) \leq |b-a| p^{\infty}(f)$$

for all $f \in \mathcal{R}([a,b], X)$. In particular, we have defined the integral of a continuous function $f \in C([a,b], X)$. Now, for any $f \in C(J, X)$ and $[a,b] \subset J$, we define

$$\int_a^b : C(J,X) \to X$$

as the composition

$$C(J,X) \longrightarrow C([a,b],X) \xrightarrow{\int} X$$

where the first map is the restriction map. Thus, $\int_a^b$ is a continuous linear map, and we have the estimate

$$p(\int_a^b f) \leq |b-a| \sup_{t \in [a,b]} p(f(t)).$$

Theorem A.2.2 (Fundamental theorem of calculus)**.** *Given $f \in C(J,X)$ and $t_0 \in J$, the function $F : J \to X$ given by*

$$F(t) = \int_{t_0}^t f$$

is differentiable and $F'(t) = f(t)$.

Proof. For any continuous seminorm p on X, we have

$$\begin{aligned} p(\frac{1}{h}(F(t+h) - F(t)) - f(t)) &= p(\frac{1}{h}(\int_{t_0}^{t+h} f - \int_{t_0}^{t} f) - f(t)) \\ &= p(\frac{1}{h}\int_t^{t+h} f - \frac{1}{h} f(t) \int_t^{t+h} 1) \\ &= p(\frac{1}{h}\int_t^{t+h} (f(s) - f(t)) ds) \\ &\leq \sup_{s \in [t, t+h]} p(f(s) - f(t)), \end{aligned}$$

which goes to 0 as $h \to 0$ because f is continuous at t. As this holds for all continuous seminorms p on X, we have $F'(t) = f(t)$ as desired. □

Corollary A.2.3. *If f' is continuous, then*

$$\int_s^t f'(u)du = f(t) - f(s).$$

Proof. By Theorem A.2.2,

$$\frac{d}{dt}\left[\int_s^t f'(u)du - f(t)\right] = 0,$$

so for every t, $\int_s^t f'(u)du - f(t) = x$ for some fixed $x \in X$ by Proposition A.1.1. Plugging in $t = s$ shows that $x = -f(s)$, which gives the result. □

Corollary A.2.4. *If $f_n \in C(J, X)$ are a sequence of continuously differentiable functions that converge pointwise to f, and if f_n' converge uniformly on compact sets to a function g, then $f' = g$. In particular, if the series $h = \sum h_n$ is absolutely convergent and $\sum h_n'$ converges absolutely and uniformly on compact sets, then $h' = \sum h_n'$.*

Proof. We will show that $f' = g$ on an arbitrary subinterval $[a, b] \subset J$. Indeed,

$$\begin{aligned}\int_s^t g &= \int_s^t \lim_{n\to\infty} f_n' \\ &= \lim_{n\to\infty} \int_s^t f_n' \\ &= \lim_{n\to\infty} f_n(t) - f_n(s) \\ &= f(t) - f(s),\end{aligned}$$

where we have used the continuity of the integral as an operator with domain $C([s,t], X)$. Differentiating with respect to t gives the result. □

Appendix B

Noncommutative calculus proofs

This appendix contains the algebraic proofs of identities involving operators on the cyclic complex.

B.1 Lie derivatives

Suppose $\mathfrak{g}$ is a graded Lie algebra that decomposes as a product

$$\mathfrak{g} = \prod_{k=0}^{\infty} C^k,$$

where each C^k is a graded subspace. We emphasize that each C^k is graded in its own right, and so the integer k does not give the grading of $\mathfrak{g}$. Additionally, we assume,

$$[C^k, C^l] \subset C^{k+l-1}$$

for all k, l.

Now suppose we have a collection of graded vector spaces $\{X_n\}_{n=0}^{\infty}$, and suppose the cyclic group $\mathbb{Z}_{n+1}$ acts on X_n by a degree 0 operator, which we shall denote τ. Suppose $\mathfrak{g}$ acts on this data in the following sense. For each $D \in C^k$ of homogeneous degree $|D|$, there are linear maps

$$D_i : X_n \to X_{n-k+1}, \qquad i = 0, \ldots, n-k+1$$

of degree $|D|$ such that

$$[D,E]_i = \sum_{j=i}^{i+k-1} D_i E_j - (-1)^{|D||E|} \sum_{j=i}^{i+l-1} E_i D_j$$

and

$$\begin{aligned} D_i E_j &= (-1)^{|D||E|} E_{j-k+1} D_i && \text{if } i < j-k+1 \\ \tau D_i &= D_{i+1}\tau && \text{if } i < n-k+1 \\ \tau D_{n-k+1} &= D_0 \tau^k. \end{aligned}$$

for all $D \in C^k, E \in C^l$.

Definition B.1.1. We shall refer to the spaces $\{X_n\}_{n=0}^{\infty}$ with this structure as a ∞-*precyclic* $\mathfrak{g}$-*module.*

Example B.1.2. A *cyclic module* consists of a collection $\{X_n\}_{n=0}^{\infty}$ of vector spaces equipped with a collection of face maps

$$d_i : X_n \to X_{n-1}, \qquad i = 0, \dots, n,$$

degeneracies

$$s_i : X_n \to X_{n+1}, \qquad i = 0, \dots, n,$$

and a cyclic map

$$t : X_n \to X_n$$

satisfying the simplicial module relations

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j \\ s_i s_j &= s_{j+1} s_i && \text{if } i \le j \\ d_i s_j &= s_{j-1} d_i && \text{if } i < j \\ d_i s_j &= \mathrm{id} && \text{if } i = j, j+1 \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j+1. \end{aligned}$$

and the additional relations

$$
\begin{aligned}
d_i t &= t d_{i-1} && \text{if } 1 \le i \le n \\
d_0 t &= d_n \\
s_i t &= t s_{i-1} && \text{if } 1 \le i \le n \\
s_0 t &= t^2 s_n \\
t^{n+1} &= \mathrm{id}\,.
\end{aligned}
$$

Given any cyclic module, we can define an ∞-precyclic $\mathfrak{g}$-module as follows. Define a graded Lie algebra $\mathfrak{g} = C^0 \oplus C^2$, where C^0 is spanned by a single degree -1 element σ and C^2 is spanned by a single degree 1 element m. All brackets in $\mathfrak{g}$ are zero. We define a grading on X_n by declaring it to be concentrated in degree $-n$. The action of $\mathfrak{g}$ is given by

$$
\begin{aligned}
m_i &= (-1)^i d_i && \text{if } i = 0, \dots n-1 \\
\sigma_0 &= t s_n \\
\sigma_i &= (-1)^i s_{i-1} && \text{if } i = 1, \dots, n+1.
\end{aligned}
$$

One can verify that the axioms of a cyclic module imply that this is a ∞-precyclic $\mathfrak{g}$-module. The converse nearly holds in that given any ∞-precyclic $\mathfrak{g}$-module of the graded Lie algebra $\mathfrak{g}$ defined above, if one defines operators

$$
\begin{aligned}
d_i &= (-1)^i m_i && \text{if } i = 0, \dots n-1 \\
d_n &= m_0 t \\
s_i &= (-1)^{i+1} \sigma_{i+1} && \text{if } i = 0, \dots, n,
\end{aligned}
$$

then all the axioms of a cyclic module are satisfied except for the condition

$$
d_i s_j = \mathrm{id} \qquad \text{if } i = j, j+1.
$$

Instead we have the weaker condition

$$
d_i s_{i-1} = d_i s_i.
$$

We shall return to this deficiency later.

Example B.1.3. Let V be a graded vector space and let $X_n = V^{\otimes(n+1)}$. The cyclic action on X_n is given by the signed cyclic permutation

$$\tau(v_0 \otimes \ldots \otimes v_n) = (-1)^{|v_n|(|v_0|+\ldots+|v_{n-1}|)} v_n \otimes v_0 \otimes \ldots v_{n-1}.$$

For the graded Lie algebra, we shall take

$$\mathfrak{g} = \mathrm{Coder}(T^c V) \cong \mathrm{Hom}(T^c V, V).$$

We have

$$\mathfrak{g} = \prod_{k=0}^{\infty} C^k, \qquad C^k = \mathrm{Hom}(V^{\otimes k}, V).$$

Given $D \in C^k$, we define

$$D_i = 1^{\otimes i} \otimes D \otimes 1^{\otimes(n-i-k+1)} : X_n \to X_{n-k+1}.$$

It is straightforward to verify that this action gives an ∞-precyclic $\mathfrak{g}$-module.

Given an ∞-precyclic $\mathfrak{g}$-module, and an element $D \in C^k$, define the *Lie derivative along D* to be the operator

$$L_D : X_n \to X_{n-k+1}$$

given by

$$L_D = \sum_{i=0}^{n-k+1} D_i + \sum_{i=1}^{k-1} D_0 \tau^i.$$

Theorem B.1.4. *For any $D, E \in \mathfrak{g}$,*

$$[L_D, L_E] = L_{[D,E]}.$$

Thus $L : \mathfrak{g} \to \mathrm{End}(X_\bullet)$ is a homomorphism of graded Lie algebras.

Proof. Assume $D \in C^k$ and $E \in C^l$. We shall write

$$L'_D = \sum_{i=0}^{n-k+1} D_i, \qquad L''_D = \sum_{i=1}^{k-1} D_0 \tau^i$$

and similarly for E. The n refers to the index of the space X_n on which L_D acts. We shall write

$$(D \circ E)_i = \sum_{j=i}^{i-k+1} D_i E_j,$$

so that

$$[D, E]_i = (D \circ E)_i - (-1)^{|D||E|}(E \circ D)_i,$$

though we remark that we do not assume that $D \circ E$ makes sense as an element of $\mathfrak{g}$. We shall first verify that

$$[L'_D, L'_E] = L'_{[D,E]}.$$

We calculate directly

$$\begin{aligned}
L'_D L'_E &= \sum_{i=0}^{n-k-l+2} \sum_{j=0}^{n-l+1} D_i E_j \\
&= \sum_{i=0}^{n-k-l+2} \left(\sum_{j=0}^{i-1} D_i E_j + \sum_{j=i}^{i+k-1} D_i E_j + \sum_{j=i+k}^{n-l+1} D_i E_j \right) \\
&= \sum_{i=0}^{n-k-l+2} \left(\sum_{j=0}^{i-1} (-1)^{|D||E|} E_j D_{i+l-1} + (D \circ E)_i + \sum_{j=i+k}^{n-l+1} (-1)^{|D||E|} E_{j-k+1} D_i \right).
\end{aligned}$$

We can reindex these sums, so that

$$\begin{aligned}
\sum_{i=0}^{n-k-l+2} \sum_{j=0}^{i-1} (-1)^{|D||E|} E_j D_{i+l-1} &= \sum_{j=0}^{n-l-k+1} \sum_{i=j+1}^{n-l-k+2} (-1)^{|D||E|} E_j D_{i+l-1} \\
&= \sum_{j=0}^{n-l-k+1} \sum_{i=j+l}^{n-k+1} (-1)^{|D||E|} E_j D_i
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=0}^{n-k-l+2} \sum_{j=i+k}^{n-l+1} (-1)^{|D||E|} E_{j-k+1} D_i &= \sum_{i=0}^{n-k-l+2} \sum_{j=i+1}^{n-k-l+2} (-1)^{|D||E|} E_j D_i \\
&= \sum_{j=1}^{n-k-l+1} \sum_{i=0}^{j-1} (-1)^{|D||E|} E_j D_i.
\end{aligned}$$

Putting it together,

$$\begin{aligned}
L'_D L'_E - L'_{[D,E]} &= \sum_{j=0}^{n-k-l+2} (-1)^{|D||E|}(E \circ D)_j + \sum_{j=0}^{n-l-k+1} \sum_{i=j+l}^{n-k+1} (-1)^{|D||E|} E_j D_i \\
&\quad + \sum_{j=1}^{n-k-l+1} \sum_{i=0}^{j-1} (-1)^{|D||E|} E_j D_i \\
&= \sum_{j=0}^{n-k-l+2} \sum_{i=j}^{j+l-1} (-1)^{|D||E|} E_j D_i + \sum_{j=0}^{n-l-k+1} \sum_{i=j+l}^{n-k+1} (-1)^{|D||E|} E_j D_i \\
&\quad + \sum_{j=1}^{n-k-l+1} \sum_{i=0}^{j-1} (-1)^{|D||E|} E_j D_i \\
&= \sum_{j=0}^{n-k-l+2} \sum_{i=0}^{n-k+1} (-1)^{|D||E|} E_j D_i \\
&= (-1)^{|D||E|} L'_E L'_D.
\end{aligned}$$

To finish, we must show

$$[L'_D, L''_E] + [L''_D, L'_E] + [L''_D, L''_E] = L''_{[D,E]}.$$

We calculate

$$\begin{aligned}
L'_D L''_E &= \sum_{i=0}^{n-k-l+2} \sum_{j=1}^{l-1} D_i E_0 \tau^j \\
&= \sum_{j=1}^{l-1} D_0 E_0 \tau^j + \sum_{i=1}^{n-k-l+2} \sum_{j=1}^{l-1} (-1)^{|D||E|} E_0 D_{i+l-1} \tau^j \\
&= \sum_{j=1}^{l-1} D_0 E_0 \tau^j + \sum_{i=l}^{n-k+1} \sum_{j=1}^{l-1} (-1)^{|D||E|} E_0 D_i \tau^j.
\end{aligned}$$

Notice that the ∞-precyclic $\mathfrak{g}$-module relations imply

$$\tau^j D_i = \begin{cases} D_{i+j} \tau^j & \text{if } i \leq n-j-k+1, \\ D_{i+j-n+k-2} \tau^{j+k-1} & \text{if } i \geq n-j-k+2. \end{cases}$$

Using this, we see

$$
\begin{aligned}
L''_E L'_D &= \sum_{j=1}^{l-1} \sum_{i=0}^{n-k+1} E_0 \tau^j D_i \\
&= \sum_{j=1}^{l-1} \left(\sum_{i=0}^{n-j-k+1} E_0 \tau^j D_i + \sum_{i=n-j-k+2}^{n-k+1} E_0 \tau^j D_i \right) \\
&= \sum_{j=1}^{l-1} \left(\sum_{i=0}^{n-j-k+1} E_0 D_{i+j} \tau^j + \sum_{i=n-j-k+2}^{n-k+1} E_0 D_{i+j-n+k-2} \tau^{j+k-1} \right) \\
&= \sum_{j=1}^{l-1} \sum_{i=j}^{n-k+1} E_0 D_i \tau^j + \sum_{j=k}^{k+l-2} \sum_{i=0}^{j-k} E_0 D_i \tau^j \\
&= \sum_{i=1}^{l-1} \sum_{j=1}^{i} E_0 D_i \tau^j + \sum_{i=l}^{n-k+1} \sum_{j=1}^{l-1} E_0 D_i \tau^j + \sum_{i=0}^{l-2} \sum_{j=k+i}^{k+l-2} E_0 D_i \tau^j.
\end{aligned}
$$

after reindexing and changing the order of summation. Next,

$$
\begin{aligned}
L''_E L''_D &= \sum_{i=1}^{l-1} \sum_{j=1}^{k-1} E_0 \tau^i D_0 \tau^j \\
&= \sum_{i=1}^{l-1} \sum_{j=1}^{k-1} E_0 D_i \tau^{i+j} \\
&= \sum_{i=1}^{l-1} \sum_{j=i+1}^{i+k-1} E_0 D_i \tau^j.
\end{aligned}
$$

Putting it all together, we have

$$
\begin{aligned}
&L'_D L''_E - (-1)^{|D||E|} L''_E L'_D - (-1)^{|D||E|} L''_E L''_D \\
&\quad = \sum_{j=1}^{l-1} D_0 E_0 \tau^j - (-1)^{|D||E|} \sum_{i=1}^{l-1} \sum_{j=1}^{i} E_0 D_i \tau^j - (-1)^{|D||E|} \sum_{i=0}^{l-2} \sum_{j=k+i}^{k+l-2} E_0 D_i \tau^j \\
&\qquad - (-1)^{|D||E|} \sum_{i=1}^{l-1} \sum_{j=i+1}^{i+k-1} E_0 D_i \tau^j \\
&\quad = \sum_{j=1}^{l-1} D_0 E_0 \tau^j - (-1)^{|D||E|} \sum_{i=0}^{l-1} \sum_{j=1}^{k+l-2} E_0 D_i \tau^j + (-1)^{|D||E|} \sum_{j=1}^{k-1} E_0 D_0 \tau^j
\end{aligned}
$$

$$= \sum_{j=1}^{l-1} D_0 E_0 \tau^j - (-1)^{|D||E|} \sum_{j=1}^{k+l-2} (E \circ D)_0 \tau^j + (-1)^{|D||E|} \sum_{j=1}^{k-1} E_0 D_0 \tau^j.$$

Interchanging the roles of D and E, the same calculation gives

$$\begin{aligned} &-(-1)^{|D||E|} L'_E L''_D + L''_D L'_E + L''_D L''_E \\ &\quad = -(-1)^{|D||E|} \sum_{j=1}^{k-1} E_0 D_0 \tau^j + \sum_{j=1}^{k+l-2} (D \circ E)_0 \tau^j - \sum_{j=1}^{l-1} D_0 E_0 \tau^j. \end{aligned}$$

Adding the two equations gives

$$[L'_D, L''_E] + [L''_D, L'_E] + [L''_D, L''_E] = L''_{[D,E]}$$

as desired. □

Next, we shall consider ∞-precyclic $\mathfrak{g}$-modules that have additional structure. Suppose that the Lie algebra $\mathfrak{g}$ has a distinguished element m of degree 1 such that $[m, m] = 0$. Then $\mathfrak{g}$ is a DGLA with coboundary map

$$\delta D = [m, D].$$

Let m^k denote the component of m in C^k. Suppose also that there is another distinguished element $\sigma \in C^0$ of degree -1 with $\delta\sigma = 0$ such that

$$(m^k)_i \sigma_j = \begin{cases} \mathrm{id}, & k = 2, \qquad j = i \\ -\mathrm{id}, & k = 2, \qquad j = i+1 \\ 0, & k \neq 2, \qquad j = i, \dots, i+k-1 \end{cases}$$

for all $D \in C^k$.

Definition B.1.5. An ∞-*cyclic* $\mathfrak{g}$-*module* is an ∞-precyclic $\mathfrak{g}$-module with an m and σ as above.

Example B.1.6. Let $\mathfrak{g}$ be the two-dimensional differential graded Lie algebra spanned by $m \in C^2$ in degree 1 and $\sigma \in C^0$ in degree -1. Then an ∞-cyclic $\mathfrak{g}$-module is the same thing as a cyclic module.

Example B.1.7. If A is a unital A_∞-algebra and

$$\mathfrak{g} = \mathrm{Coder}(B(A)) \cong \mathrm{Hom}(B(A), sA)$$

is the Hochschild DGLA of A, then we can define an ∞-precyclic $\mathfrak{g}$-module where $D \in C^k = \mathrm{Hom}((sA)^k, sA)$ acts on $X_n = (sA)^{n+1}$ via the operators

$$D_i = 1^{\otimes i} \otimes D \otimes 1^{\otimes n-i-k+1}$$

as in Example B.1.3. The element $m \in \mathfrak{g}$ is the degree 1 coderivation that gives the A_∞-structure, and $\sigma \in \mathrm{Hom}(\mathbb{C}, sA) \cong sA$ is the unit of the A_∞-algebra. It follows that this data gives an ∞-cyclic $\mathfrak{g}$-module.

Given an ∞-cyclic $\mathfrak{g}$-module, we define the *Hochschild boundary* to be the map $b = L_m$ on $X_\bullet$. Then it follows that

$$b^2 = \frac{1}{2}[L_m, L_m] = \frac{1}{2}L_{[m,m]} = 0$$

by Theorem B.1.4. We also have

$$[b, L_D] = [L_m, L_D] = L_{[m,D]} = L_{\delta D}.$$

Define the *normalized subcomplex* $\overline{X}_\bullet$ by

$$\overline{X}_n = X_n / \left(\sum_{i=1}^{n} \sigma_i(X_{n-1}) \right).$$

We shall call $D \in C^k$ *normalized* if

$$D_i \sigma_j = 0, \qquad j = i, \ldots, i+k-1.$$

Define $\overline{\mathfrak{g}}$ to be the set of all elements of $\mathfrak{g}$ whose components are normalized.

Proposition B.1.8. *Given an ∞-cyclic $\mathfrak{g}$-module,*

1. *the subspace $\overline{\mathfrak{g}}$ is a sub-DGLA of $\mathfrak{g}$.*

2. *the operator L_D descends to an operator on $\overline{X}_\bullet$ for all $D \in \overline{\mathfrak{g}}$.*

3. the Hochschild boundary b descends to an operator on $\overline{X}_\bullet$.

Proof. We first show that $\bar{\mathfrak{g}}$ is closed under brackets. So for normalized $D \in C^k$ and $E \in C^l$, we must show that

$$[D, E]_i \sigma_j = 0, \qquad j = i, \ldots, i + k + l - 2.$$

It suffices to show

$$D_i E_p \sigma_j = 0,$$

where

$$i \leq j \leq i + k + l - 2, \qquad i \leq p \leq i + k - 1.$$

There are three cases, depending on the value of j. If $i \leq j \leq p - 1$, then

$$D_i E_p \sigma_j = (-1)^{|E|} D_i \sigma_j E_{p-1} = 0$$

because $i \leq j \leq i + k - 2$. If $p \leq j \leq p + l - 1$, then $E_p \sigma_j = 0$, which completes that case. Thirdly, if $p + l \leq j \leq i + k + l - 2$, then

$$D_i E_p \sigma_j = (-1)^{|E|} D_i \sigma_{j-l+1} E_p = 0$$

because $i + 1 \leq j - l + 1 \leq i + k - 1$.

Next, we shall show that $\delta D = [m, D] \in \bar{\mathfrak{g}}$ for all $D \in C^k$. By the previous part, we can assume without loss of generality that $m \in C^2$. We must show

$$[m, D]_i \sigma_j = 0, \qquad j = i, \ldots, i + k.$$

Now

$$(m \circ D)_i \sigma_j = m_i D_i \sigma_j + m_i D_{i+1} \sigma_j = \begin{cases} (-1)^{|D|} D_i, & j = i \\ 0, & i + 1 \leq j \leq i + k - 1 \\ -(-1)^{|D|} D_i, & j = i + k. \end{cases}$$

On the other hand,

$$\begin{aligned}(D\circ m)_i\sigma_i &= \sum_{p=i}^{i+k-1} D_i m_p \sigma_i \\ &= D_i m_i \sigma_i + \sum_{p=i+1}^{i+k-1} D_i m_p \sigma_i \\ &= D_i - \sum_{p=i+1}^{i+k-1} D_i \sigma_i m_{p-1} \\ &= D_i,\end{aligned}$$

$$\begin{aligned}(D\circ m)_i\sigma_{i+k} &= \sum_{p=i}^{i+k-1} D_i m_p \sigma_{i+k} \\ &= \sum_{p=i}^{i+k-2} D_i m_p \sigma_{i+k} + D_i m_{i+k-1}\sigma_{i+k} \\ &= -\sum_{p=i}^{i+k-2} D_i \sigma_{i+k-1} m_p - D_i \\ &= -D_i,\end{aligned}$$

and if $i+1 \le j \le i+k-1$,

$$\begin{aligned}(D\circ m)_i\sigma_j &= \sum_{p=i}^{i+k-1} D_i m_p \sigma_j \\ &= \sum_{p=i}^{j-2} D_i m_p \sigma_j + D_i m_{j-1}\sigma_j + D_i m_j \sigma_j + \sum_{p=j+1}^{i+k-1} D_i m_p \sigma_j \\ &= -\sum_{p=i}^{j-2} D_i \sigma_{j-1} m_p - D_i + D_i - \sum_{p=j+1}^{i+k-1} D_i \sigma_j m_{p-1} \\ &= 0.\end{aligned}$$

Thus,

$$(\delta D)_i\sigma_j = (m\circ D)_i\sigma_j - (-1)^{|D|}(D\circ m)_i\sigma_j = 0$$

for $j = i, \dots, i+k$, as required. So $\overline{\mathfrak{g}}$ is a sub-DGLA of $\mathfrak{g}$.

To show that L_D descends to $\overline{X}_\bullet$, we first introduce some notation. Let

$$Y_n = \sum_{i=1}^{n} \sigma_i(X_{n-1}), \qquad Z_n = \sum_{i=0}^{n} \sigma_i(X_{n-1}).$$

Each of the following is straightforward to verify for $D \in C^k$.

- $D_0(Z_n) \subset Y_{n-k+1}$,
- $D_i(Y_n) \subset Y_{n-k+1}, \qquad i = 1, \dots, n-k+1,$
- $\tau(Y_n) \subset Z_n$.

From these, it follows that

$$L_D = \sum_{i=0}^{n-k+1} D_i + \sum_{i=1}^{k-1} D_0\tau^i$$

maps Y_n to Y_{n-k+1}. Thus, L_D descends to an operator on $\overline{X}_\bullet = X_\bullet/Y_\bullet$.

To show that $b = L_m$ descends to $\overline{X}$, we may assume without loss of generality that $m \in C^2$ by the last part. Consider the operators $\sigma_j : X_{n-1} \to X_n$ for $j = 1, \dots, n$. If $j = 1, \dots, n-1$, then

$$\begin{aligned}
b\sigma_j &= \sum_{i=0}^{n-1} m_i\sigma_j + m_0\tau\sigma_j \\
&= \sum_{i=0}^{j-2} m_i\sigma_j + m_{j-1}\sigma_j + m_j\sigma_j + \sum_{i=j+1}^{n-1} m_i\sigma_j + m_0\sigma_{j+1}\tau \\
&= -\sum_{i=0}^{j-2} \sigma_{j-1}m_i - \mathrm{id} + \mathrm{id} - \sum_{i=j+1}^{n-1} \sigma_j m_{i-1} + \sigma_j m_0\tau \\
&= -\sum_{i=0}^{j-2} \sigma_{j-1}m_i - \sum_{i=j+1}^{n-1} \sigma_j m_{i-1} + \sigma_j m_0\tau,
\end{aligned}$$

and

$$b\sigma_n = \sum_{i=0}^{n-1} m_i\sigma_n + m_0\tau\sigma_n$$

$$
\begin{aligned}
&= \sum_{i=0}^{n-2} m_i\sigma_n + m_{n-1}\sigma_n + m_0\sigma_0 \\
&= -\sum_{i=0}^{n-2} \sigma_{n-1}m_i - \mathrm{id} + \mathrm{id} \\
&= -\sum_{i=0}^{n-2} \sigma_{n-1}m_i.
\end{aligned}
$$

This shows $b(Y_n) \subset Y_{n-1}$, and we are done. □

Next, we define the operator $B : \overline{X}_n \to \overline{X}_{n+1}$ by

$$B = \sum_{i=0}^{n} \sigma_0\tau^i.$$

Since,

$$\tau\sigma_0 = \sigma_1\tau = 0$$

on the normalized complex, it follows at once that $B^2 = 0$.

Theorem B.1.9. *1. For any $D \in \overline{\mathfrak{g}}$, $[B, L_D] = 0$.*

2. $[b, B] = bB + Bb = 0$.

Proof. We have that

$$
\begin{aligned}
L_D B &= \sum_{i=0}^{n-k+2}\sum_{j=0}^{n} D_i\sigma_0\tau^j + \sum_{i=1}^{k-1}\sum_{j=0}^{n} D_0\tau^i\sigma_0\tau^j \\
&= \sum_{i=1}^{n-k+2}\sum_{j=0}^{n} D_i\sigma_0\tau^j
\end{aligned}
$$

because $D_0\tau^i\sigma_0 = D_0\sigma_i\tau^i = 0$ for $0 \le j \le k-1$ and $D_0\sigma_0 = 0$. Hence,

$$L_D B = (-1)^{|D|}\sum_{i=0}^{n-k+1}\sum_{j=0}^{n} \sigma_0 D_i\tau^j$$

We also have

$$BL_D = \sum_{j=0}^{n-k+1} \left(\sum_{i=0}^{n-k+1} \sigma_0 \tau^j D_i + \sum_{i=1}^{k-1} \sigma_0 \tau^j D_0 \tau^i \right).$$

Now

$$\begin{aligned}
&\sum_{j=0}^{n-k+1} \sum_{i=0}^{n-k+1} \sigma_0 \tau^j D_i \\
&\quad = \sum_{j=0}^{n-k+1} \left(\sum_{i=0}^{n-j-k+1} \sigma_0 \tau^j D_i + \sum_{i=n-j-k+2}^{n-k+1} \sigma_0 \tau^j D_i \right) \\
&\quad = \sum_{j=0}^{n-k+1} \left(\sum_{i=0}^{n-j-k+1} \sigma_0 D_{i+j} \tau^j + \sum_{i=n-j-k+2}^{n-k+1} \sigma_0 D_{i+j-n+k-2} \tau^{j+k-1} \right) \\
&\quad = \sum_{j=0}^{n-k+1} \sum_{i=j}^{n-k+1} \sigma_0 D_i \tau^j + \sum_{j=k-1}^{n} \sum_{i=n-j+1}^{n-k+1} \sigma_0 D_{i+j-n-1} \tau^j \\
&\quad = \sum_{j=0}^{n-k+1} \sum_{i=j}^{n-k+1} \sigma_0 D_i \tau^j + \sum_{j=k-1}^{n} \sum_{i=0}^{j-k} \sigma_0 D_i \tau^j \\
&\quad = \sum_{i=0}^{n-k+1} \sum_{j=0}^{i} \sigma_0 D_i \tau^j + \sum_{i=0}^{n-k} \sum_{j=i+k}^{n} \sigma_0 D_i \tau^j
\end{aligned}$$

after changing the order of summation. Also,

$$\begin{aligned}
\sum_{j=0}^{n-k+1} \sum_{i=1}^{k-1} \sigma_0 \tau^j D_0 \tau^i &= \sum_{j=0}^{n-k+1} \sum_{i=1}^{k-1} \sigma_0 D_j \tau^{i+j} \\
&= \sum_{j=0}^{n-k+1} \sum_{i=j+1}^{j+k-1} \sigma_0 D_j \tau^i \\
&= \sum_{i=0}^{n-k+1} \sum_{j=i+1}^{i+k-1} \sigma_0 D_i \tau^j
\end{aligned}$$

by relabeling i and j. Putting it together, we see that

$$BL_D = \sum_{i=0}^{n-k+1} \sum_{j=0}^{n} \sigma_0 D_i \tau^j = (-1)^{|D|} L_D B$$

as desired.

To show $[b, B] = [L_m, B] = 0$, we can assume $m \in C^2$ without loss of generality by the last part. The exact same computation as above when $D = m$ gives

$$BL_m = \sum_{i=0}^{n-1} \sum_{j=0}^{n} \sigma_0 m_i \tau^j.$$

On the other hand,

$$\begin{aligned} L_m B &= \sum_{i=0}^{n} \sum_{j=0}^{n} m_i \sigma_0 \tau^j + \sum_{j=0}^{n} m_0 \tau \sigma_0 \tau^j \\ &= \sum_{j=0}^{n} m_0 \sigma_0 \tau^j + \sum_{i=1}^{n} \sum_{j=0}^{n} m_i \sigma_0 \tau^j + \sum_{j=0}^{n} m_0 \sigma_1 \tau^{j+1} \\ &= \sum_{j=0}^{n} \tau^j - \sum_{i=1}^{n} \sum_{j=0}^{n} \sigma_0 m_{i-1} \tau^j - \sum_{j=0}^{n} \tau^{j+1} \\ &= -\sum_{i=0}^{n-1} \sum_{j=0}^{n} \sigma_0 m_i \tau^j \\ &= -BL_m. \end{aligned}$$

□

Given an ∞-cyclic $\mathfrak{g}$-module $\{X_n\}_{n \in \mathbb{Z}}$, let $X_{n,k}$ denote the degree k component of X_n. We define the *Hochschild complex* to be

$$C_\bullet(X) = \bigoplus_{n \in \mathbb{Z}} X_{n,-(\bullet+1)}$$

with differential b. The homology of $(C_\bullet(X), b)$ will be called the *Hochschild homology* and will be denoted $HH_\bullet(X)$. For any homogeneous $D \in \mathfrak{g}$, we have shown that the Lie derivative is a chain map

$$L_D : C_\bullet(X) \to C_{\bullet-|D|}(X),$$

and so induces a map on homology

$$L_D : HH_\bullet(X) \to HH_{\bullet-|D|}(X).$$

Similarly, we define the *normalized Hochschild complex* to be

$$\overline{C}_\bullet(X) = \bigoplus_{n\in\mathbb{Z}} \overline{X}_{n,-(\bullet+1)}$$

with differential b. Its homology is the *normalized Hochschild homology* $\overline{HH}_\bullet(X)$, and any for any $D \in \overline{\mathfrak{g}}$, there are Lie derivative operators

$$L_D : \overline{C}_\bullet(X) \to \overline{C}_{\bullet-|D|}(X), \qquad L_D : \overline{HH}_\bullet(X) \to \overline{HH}_{\bullet-|D|}(X).$$

The *periodic cyclic chain complex* is $C_{\mathrm{per}}(X) = C_{\mathrm{ev}}(X) \bigoplus C_{\mathrm{odd}}(X)$, where

$$C_{\mathrm{ev}}(X) = \prod_{n\in\mathbb{Z}} C_{2n}(X), \qquad C_{\mathrm{odd}}(X) = \prod_{n\in\mathbb{Z}} C_{2n+1}(X).$$

Then $C_{\mathrm{per}}(X)$ is a $\mathbb{Z}/2$-graded complex with differential $b + B$. It's homology is the *periodic cyclic homology* $HP_\bullet(X)$. Similarly, we can define the *normalized periodic cyclic chain complex* $\overline{C}_{\mathrm{per}}(X)$ and the *normalized periodic cyclic homology* $\overline{HP}_\bullet(X)$. As above, there are Lie derivative chain maps

$$L_D : C_{\mathrm{per}}(X) \to C_{\mathrm{per}}(X), \qquad L_D : \overline{C}_{\mathrm{per}}(X) \to \overline{C}_{\mathrm{per}}(X).$$

Example B.1.10 (Hochschild and cyclic homology of an A_∞-algebra)**.** Given an A_∞-algebra A, let X be the ∞-cyclic $\mathfrak{g}$-module, as in Example B.1.7, associated to the unitization $\widetilde{A} \cong A \bigoplus \mathbb{C}e$. Then we define

$$C_0(A) := \overline{C}_0(X)/\mathbb{C}e, \qquad C_n(A) := \overline{C}_n(X) \qquad (n \neq 0)$$

and

$$C_{\mathrm{ev}}(A) = \prod_{n\in\mathbb{Z}} C_{2n}(A), \qquad C_{\mathrm{odd}}(A) = \prod_{n\in\mathbb{Z}} C_{2n+1}(A)$$

to be the Hochschild and periodic cyclic chain complexes of A. We identify the

DGLA $\mathfrak{g}(A) := \mathrm{Coder}(B(A))$ as a sub-DGLA of $\overline{\mathfrak{g}}$. Thus, $\mathfrak{g}(A)$ acts on the complexes $C_\bullet(A)$ and $C_{\mathrm{per}}(A)$ by Lie derivatives.

B.2 Cartan homotopy formula

For our discussion of contractions and the Cartan homotopy formula, we shall restrict to the case where $m = m^{(1)} + m^{(2)}$ for $m^{(1)} \in C^1$ and $m^{(2)} \in C^2$. In terms of Example B.1.7, this corresponds to restricting from arbitrary A_∞-algebras to differential graded algebras. This will simplify some of what follows, but it is not a necessary restriction [12].

Definition B.2.1. Given a homogeneous $D \in C^k$, we define the *contraction by* D to be the operator

$$\iota_D : X_n \to X_{n-k}$$

of degree $|D| + 1$ given by $\iota_D = m_0^{(2)} D_1$.

For a homogeneous $D \in C^k$, define the operator

$$S_D : X_n \to X_{n-k+2}$$

of degree $|D| - 1$ by

$$S_D = \sum_{i=1}^{n-k+1} \sum_{j=0}^{n-k+1-i} \sigma_0 \tau^j D_i.$$

Proposition B.2.2. *If $D \in \overline{\mathfrak{g}}$, then ι_D and S_D descend to operators on $\overline{X}_\bullet$.*

Proof. We assume $D \in \overline{C}^k$. If $i = 1, \ldots, k$, then

$$\iota_D \sigma_i = m_0^{(2)} D_1 \sigma_i = 0.$$

If $i = k+1, \ldots, n+1$, then

$$\iota_D \sigma_i = m_0^{(2)} D_1 \sigma_i = (-1)^{|D|} m_0^{(2)} \sigma_{i-k+1} D_1 = -(-1)^{|D|} \sigma_{i-k} m_0^{(2)} D_1.$$

This shows that ι_D descends to $\overline{X}_\bullet$.

In terms of the notation from the proof of Proposition B.1.8,

- $D_i(Y_n) \subset Y_{n-k+1}, \qquad i = 1, \dots, n-k+1,$
- $\tau(Y_n) \subset Z_n,$
- $\sigma_0(Z_n) \subset Y_{n+1}.$

Thus we see that $S_D(Y_n) \subset Y_{n-k+2}$, which gives the result. □

Definition B.2.3. Given $D \in \overline{\mathfrak{g}}$, the *cyclic contraction by* D is the operator

$$I_D = \iota_D + S_D.$$

Lemma B.2.4. *If* $D \in \overline{C}^1 \subset \overline{\mathfrak{g}}$ *satisfies* $[m^{(2)}, D] = 0$, *then*

$$[L_D, I_E] = (-1)^{|D|} I_{[D,E]}$$

for $E \in \overline{\mathfrak{g}}$.

Proof. We claim that the following identities hold

(i) $[L_D, \sigma_0] = 0.$

(ii) $[L_D, \tau] = 0.$

(iii) $[L_D, E_j] = [D, E]_j$, for $E \in \mathfrak{g}$

The identities

$$[L_D, \sigma] = 0, \qquad [L_D, \tau] = 0$$

are easy to see directly. For $E \in C^l$,

$$\begin{aligned}
L_D E_j &= \sum_{i=0}^{n-l+1} D_i E_j \\
&= \sum_{i=0}^{j-1} D_i E_j + D_j E_j + \sum_{i=j+1}^{n-l+1} D_i E_j \\
&= \sum_{i=0}^{j-1} (-1)^{|D||E|} E_j D_i + (D \circ E)_j + \sum_{i=j+1}^{n-l+1} (-1)^{|D||E|} E_j D_{i+l-1} \\
&= \sum_{i=0}^{j-1} (-1)^{|D||E|} E_j D_i + [D, E]_j + (-1)^{|D||E|} (E \circ D)_j
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=j+l}^{n} (-1)^{|D||E|} E_j D_i \\
&= [D, E]_j + \sum_{i=0}^{n} (-1)^{|D||E|} E_j D_i \\
&= [D, E]_j + (-1)^{|D||E|} E_j L_D.
\end{aligned}
$$

So we have

$$
\begin{aligned}
[L_D, \iota_E] &= [L_D, m_0^{(2)} E_1] \\
&= [L_D, m^{(2)}] E_1 + (-1)^{|D|} m_0^{(2)} [L_D, E_1] \\
&= (-1)^{|D|} m_0^{(2)} [D, E]_1 \\
&= (-1)^{|D|} \iota_{[D,E]}
\end{aligned}
$$

and

$$
\begin{aligned}
[L_D, S_E] &= \sum_{i=1}^{n-l+1} \sum_{j=0}^{n-l+1-i} [L_D, \sigma_0 \tau^j E_i] \\
&= \sum_{i=1}^{n-l+1} \sum_{j=0}^{n-l+1-i} \Big([L_D, \sigma_0] \tau^j E_i + (-1)^{|D|} \sigma_0 [L_D, \tau^j] E_i \\
&\qquad + (-1)^{|D|} \sigma_0 \tau^j [L_D, E_i] \Big) \\
&= (-1)^{|D|} \sum_{i=1}^{n-l+1} \sum_{j=0}^{n-l+1-i} \sigma_0 \tau^j [D, E]_i \\
&= (-1)^{|D|} S_{[D,E]}.
\end{aligned}
$$

Thus,

$$
[L_D, I_E] = (-1)^{|D|} I_{[D,E]}.
$$

□

Proposition B.2.5. *1. For any $D \in \mathfrak{g}$, $[b, \iota_D] = -\iota_{\delta D}$.*

2. For any $D \in \overline{\mathfrak{g}}$, $[B, \iota_D] + [b, S_D] = L_D - S_{\delta D}$.

Proof. By the proof of the previous lemma, we have

$$[L_{m^{(1)}}, \iota_D] = -\iota_{[m^{(1)},D]}.$$

Next, we compute

$$\begin{aligned}
\iota_D L_{m^{(2)}} &= \sum_{i=0}^{n-1} m_0^{(2)} D_1 m_i^{(2)} + m_0^{(2)} D_1 m_0^{(2)} \tau \\
&= m_0^{(2)} D_1 m_0^{(2)} + \sum_{i=1}^{k} m_0^{(2)} D_1 m_i^{(2)} + \sum_{i=k+1}^{n-1} m_0^{(2)} D_1 m_i^{(2)} \\
&\qquad + m_0^{(2)} D_1 m_0^{(2)} \tau \\
&= -(-1)^{|D|} m_0^{(2)} m_1^{(2)} D_2 + m_0^{(2)} (D \circ m^{(2)})_1 + (-1)^{|D|} \sum_{i=k+1}^{n-1} m_0^{(2)} m_{i-k+1}^{(2)} D_1 \\
&\qquad - (-1)^{|D|} m_0^{(2)} m_1^{(2)} D_2 \tau \\
&= -(-1)^{|D|} m_0^{(2)} [m^{(2)}, D]_1 + (-1)^{|D|} m_0^{(2)} m_1^{(2)} D_1 + (-1)^{|D|} \sum_{i=2}^{n-k} m_0^{(2)} m_i^{(2)} D_1 \\
&\qquad - (-1)^{|D|} m_0^{(2)} \tau m_0^{(2)} D_1 \\
&= -(-1)^{|D|} \iota_{[m^{(2)},D]} + (-1)^{|D|} \sum_{i=1}^{n-k} m_0^{(2)} m_i^{(2)} D_1 - (-1)^{|D|} m_0^{(2)} \tau \iota_D \\
&= -(-1)^{|D|} \iota_{[m^{(2)},D]} - (-1)^{|D|} \sum_{i=0}^{n-k-1} m_i^{(2)} m_0^{(2)} D_1 - (-1)^{|D|} m_0^{(2)} \tau \iota_D \\
&= -(-1)^{|D|} \iota_{[m^{(2)},D]} - (-1)^{|D|} L_{m^{(2)}} \iota_D.
\end{aligned}$$

So $[L_{m^{(k)}}, \iota_D] = -\iota_{[m^{(k)},D]}$ for $k = 1, 2$. Thus,

$$[b, \iota_D] = [L_m, \iota_D] = -\iota_{[m,D]} = -\iota_{\delta D}$$

because $m = m^{(1)} + m^{(2)}$.

For the next part, we have

$$[L_{m^{(1)}}, S_D] = -S_{[m^{(1)},D]}$$

from the previous lemma. So it suffices to show

$$[B, \iota_D] + [L_{m^{(2)}}, S_D] = L_D - S_{[m^{(2)},D]}.$$

We first calculate

$$\begin{aligned}
(-1)^{|D|}\iota_D B &= (-1)^{|D|}\sum_{i=0}^{n} m_0^{(2)} D_1 \sigma_0 \tau^i \\
&= \sum_{i=0}^{n} m_0^{(2)} \sigma_0 D_0 \tau^i \\
&= \sum_{i=0}^{n} D_0 \tau^i \\
&= \sum_{i=0}^{k-1} D_0 \tau^i + \sum_{i=k}^{n} \tau^{i-k+1} D_{n-i+1} \\
&= \sum_{i=0}^{k-1} D_0 \tau^i + \sum_{i=1}^{n-k+1} \tau^{n-i-k+2} D_i
\end{aligned}$$

and so

$$[B, \iota_D] = \sum_{i=0}^{n-k} \sigma_0 \tau^i m_0^{(2)} D_1 + \sum_{i=0}^{k-1} D_0 \tau^i + \sum_{i=1}^{n-k+1} \tau^{n-i-k+2} D_i.$$

Next, we shall calculate $L_{m^{(2)}} S_D$. First, we have that

$$m_0^{(2)} S_D = \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} m_0^{(2)} \sigma_0 \tau^i D_j = \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} \tau^i D_j$$

and

$$\begin{aligned}
m_0^{(2)} \tau S_D &= \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} m_0^{(2)} \tau \sigma_0 \tau^i D_j \\
&= \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} m_0^{(2)} \sigma_1 \tau^{i+1} D_j \\
&= -\sum_{j=1}^{n-k+1} \sum_{i=1}^{n-k+2-j} \tau^i D_j
\end{aligned}$$

so that

$$m_0^{(2)} S_D + m^{(2)} \tau S_D = \sum_{j=1}^{n-k+1} D_j - \sum_{j=1}^{n-k+1} \tau^{n-k+2-j} D_j.$$

Now

$$\begin{aligned}
&\sum_{l=1}^{n-k+1} m_l^{(2)} S_D \\
&\qquad = \sum_{l=1}^{n-k+1} \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} m_l^{(2)} \sigma_0 \tau^i D_j \\
&\qquad = -\sum_{l=0}^{n-k} \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} \sigma_0 m_l^{(2)} \tau^i D_j \\
&\qquad = -\sum_{j=1}^{n-k-1} \sum_{i=2}^{n-k+1-j} \sum_{l=0}^{i-2} \sigma_0 m_l^{(2)} \tau^i D_j - \sum_{j=1}^{n-k} \sum_{i=1}^{n-k+1-j} \sigma_0 m_{i-1}^{(2)} \tau^i D_j \\
&\qquad\qquad - \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} \sum_{l=i}^{n-k} \sigma_0 m_l^{(2)} \tau^i D_j \\
&\qquad = -\sum_{j=1}^{n-k-1} \sum_{i=2}^{n-k+1-j} \sum_{l=0}^{i-2} \sigma_0 \tau^{i-1} m_{n-k+2+l-i}^{(2)} D_j - \sum_{j=1}^{n-k} \sum_{i=1}^{n-k+1-j} \sigma_0 \tau^{i-1} m_0^{(2)} \tau D_j \\
&\qquad\qquad - \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} \sum_{l=i}^{n-k} \sigma_0 \tau^i m_{l-i}^{(2)} D_j \\
&\qquad = -\sum_{j=1}^{n-k-1} \sum_{i=1}^{n-k-j} \sum_{l=n-i-k+1}^{n-k} \sigma_0 \tau^i m_l^{(2)} D_j - \sum_{j=1}^{n-k} \sum_{i=0}^{n-k-j} \sigma_0 \tau^i m_0^{(2)} \tau D_j \\
&\qquad\qquad - \sum_{j=1}^{n-k+1} \sum_{i=0}^{n-k+1-j} \sum_{l=0}^{n-i-k} \sigma_0 \tau^i m_l^{(2)} D_j.
\end{aligned}$$

Thus

$$\begin{aligned}
[B, \iota_D] + L_{m^{(2)}} S_D - L_D = &-\sum_{j=1}^{n-k-1} \sum_{i=1}^{n-k-j} \sum_{l=n-i-k+1}^{n-k} \sigma_0 \tau^i m_l^{(2)} D_j \\
&- \sum_{j=2}^{n-k+1} \sum_{i=0}^{n-k+1-j} \sum_{l=0}^{n-i-k} \sigma_0 \tau^i m_l^{(2)} D_j
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=0}^{n-k-1} \sum_{l=1}^{n-i-k} \sigma_0 \tau^i m_l^{(2)} D_1 \\
&- \sum_{j=1}^{n-k} \sum_{i=0}^{n-k-j} \sigma_0 \tau^i m_0^{(2)} \tau D_j \\
= &-(F + G + H + J).
\end{aligned}
$$

Notice that in the triple sum for F, we always have $l > j$, and so

$$
\begin{aligned}
F &= \sum_{j=1}^{n-k-1} \sum_{i=1}^{n-k-j} \sum_{l=n-i-k+1}^{n-k} \sigma_0 \tau^i m_l^{(2)} D_j \\
&= (-1)^{|D|} \sum_{j=1}^{n-k-1} \sum_{i=1}^{n-k-j} \sum_{l=n-i-k+1}^{n-k} \sigma \tau^i D_j m_{l+k-1}^{(2)} \\
&= (-1)^{|D|} \sum_{j=1}^{n-k-1} \sum_{i=1}^{n-k-j} \sum_{l=n-i}^{n-1} \sigma_0 \tau^i D_j m_l^{(2)}.
\end{aligned}
$$

Also,

$$
\begin{aligned}
G = &\sum_{j=2}^{n-k+1} \sum_{i=0}^{n-k+1-j} \left(\sum_{l=0}^{j-2} \sigma_0 \tau^i m_l^{(2)} D_j + \sum_{l=j+1}^{n-i-k} \sigma_0 \tau^i m_l^{(2)} D_j \right. \\
&\qquad \left. + \sigma_0 \tau^i m_{j-1}^{(2)} D_j + \sigma_0 \tau^i m_j^{(2)} D_j \right) \\
= &\sum_{j=2}^{n-k+1} \sum_{i=0}^{n-k+1-j} \sum_{l=0}^{j-2} (-1)^{|D|} \sigma_0 \tau^i D_{j-1} m_l^{(2)} \\
&+ \sum_{j=2}^{n-k+1} \sum_{i=0}^{n-k-1-j} \sum_{l=j+1}^{n-i-k} (-1)^{|D|} \sigma_0 \tau^i D_j m_{l+k-1}^{(2)} + \sum_{j=1}^{n-k} \sum_{i=0}^{n-k-j} \sigma_0 \tau^i m_j^{(2)} D_{j+1} \\
&+ \sum_{j=2}^{n-k} \sum_{i=0}^{n-k-j} \sigma_0 \tau^i m_j^{(2)} D_j \\
= &(-1)^{|D|} \sum_{j=1}^{n-k} \sum_{i=0}^{n-k-j} \sum_{l=0}^{j-1} \sigma_0 \tau^i D_j m_l^{(2)} + (-1)^{|D|} \sum_{j=2}^{n-k-1} \sum_{i=0}^{n-k-1-j} \sum_{l=j+k}^{n-i-1} \sigma_0 \tau^i D_j m_l^{(2)} \\
&+ \sum_{j=1}^{n-k} \sum_{i=0}^{n-k-j} \sigma_0 \tau^i (m^{(2)} \circ D)_j - \sum_{i=0}^{n-k-1} \sigma_0 \tau^i m_1^{(2)} D_1.
\end{aligned}
$$

Next,

$$\begin{aligned}
H &= \sum_{i=0}^{n-k-1} \sigma_0\tau^i m_1^{(2)} D_1 + \sum_{i=0}^{n-k-2}\sum_{l=2}^{n-i-k} \sigma_0\tau^i m_l^{(2)} D_1 \\
&= \sum_{i=0}^{n-k-1} \sigma_0\tau^i m_1^{(2)} D_1 + (-1)^{|D|}\sum_{i=0}^{n-k-2}\sum_{l=2}^{n-i-k} \sigma_0\tau^i D_1 m_{l+k-1}^{(2)} \\
&= \sum_{i=0}^{n-k-1} \sigma_0\tau^i m_1^{(2)} D_1 + (-1)^{|D|}\sum_{i=0}^{n-k-2}\sum_{l=k+1}^{n-i-1} \sigma_0\tau^i D_1 m_l^{(2)},
\end{aligned}$$

so that

$$\begin{aligned}
G + H = (-1)^{|D|}&\sum_{j=1}^{n-k}\sum_{i=0}^{n-k-j}\sum_{l=0}^{j-1} \sigma_0\tau^i D_j m_l^{(2)} \\
&+ (-1)^{|D|}\sum_{j=1}^{n-k-1}\sum_{i=0}^{n-k-1-j}\sum_{l=j+k}^{n-i-1} \sigma_0\tau^i D_j m_l^{(2)} + \sum_{j=1}^{n-k}\sum_{i=0}^{n-k-j} \sigma_0\tau^i (m^{(2)}\circ D)_j \\
= (-1)^{|D|}&\sum_{j=1}^{n-k}\sum_{i=0}^{n-k-j}\sum_{l=0}^{j-1} \sigma_0\tau^i D_j m_l^{(2)} \\
&+ (-1)^{|D|}\sum_{j=1}^{n-k-1}\sum_{i=0}^{n-k-1-j}\sum_{l=j+k}^{n-i-1} \sigma_0\tau^i D_j m_l^{(2)} + \sum_{j=1}^{n-k}\sum_{i=0}^{n-k-j} \sigma_0\tau^i [m^{(2)}, D]_j \\
&+ (-1)^{|D|}\sum_{j=1}^{n-k}\sum_{i=0}^{n-k-j}\sum_{l=j}^{j+k-1} \sigma_0\tau^i D_j m_l^{(2)} \\
= (-1)^{|D|}&\sum_{j=1}^{n-k}\sum_{i=0}^{n-k-j}\sum_{l=0}^{n-i-1} \sigma_0\tau^i D_j m_l^{(2)} + S_{[m^{(2)},D]}.
\end{aligned}$$

and

$$\begin{aligned}
F + G + H &= (-1)^{|D|}\sum_{j=1}^{n-k}\sum_{i=0}^{n-k-j}\sum_{l=0}^{n-1} \sigma_0\tau^i D_j m_l^{(2)} + S_{[m^{(2)},D]} \\
&= (-1)^{|D|}\sum_{l=0}^{n-1} S_D m_l^{(2)} + S_{[m^{(2)},D]}.
\end{aligned}$$

Also,

$$
\begin{aligned}
J &= \sum_{j=1}^{n-k} \sum_{i=0}^{n-k-j} \sigma_0 \tau^i m_0^{(2)} \tau D_j \\
&= \sum_{j=1}^{n-k} \sum_{i=0}^{n-k-j} \sigma_0 \tau^i m_0^{(2)} D_{j+1} \tau \\
&= (-1)^{|D|} \sum_{j=1}^{n-k} \sum_{i=0}^{n-k-j} \sigma_0 \tau^i D_j m_0^{(2)} \tau \\
&= (-1)^{|D|} S_D m_0 \tau.
\end{aligned}
$$

Putting it together gives

$$
\begin{aligned}
[B, \iota_D] + L_{m^{(2)}} S_D - L_D &= -(F + G + H + J) \\
&= -(-1)^{|D|} S_D L_{m^{(2)}} - S_{[m^{(2)}, D]},
\end{aligned}
$$

which finishes the proof. □

Theorem B.2.6. *(Cartan homotopy formula) For any $D \in \overline{\mathfrak{g}}$,*

$$
[b + B, I_D] = L_D - I_{\delta D}.
$$

Proof. We compute

$$
\begin{aligned}
[b + B, I_D] &= [b, \iota_D] + [B, \iota_D] + [b, S_D] + [B, S_D] \\
&= -\iota_{\delta D} + L_D - S_{\delta D} \\
&= L_D - I_{\delta D},
\end{aligned}
$$

using the fact that $BS_D = S_D B = 0$ on $\overline{X}_\bullet$. □

B.3 Some higher operations

Here, we specialize to the ∞-cyclic $\mathfrak{g}$-module associated to an ungraded unital associative algebra A, viewed as an A_∞-algebra concentrated in degree zero, to prove the formulas asserted in section 2.5.2. Here $m \in C^2$ is the multiplication map. We shall work with cochains $D, E \in C^1$ satisfying

$$[m, D] = 0, \qquad [m, E] = 0.$$

In other words, D and E are derivations on A. Since the grading of A is trivial, we have $|D| = |E| = 0$. So we have

$$D_i E_j = E_j D_i, \qquad i \neq j.$$

The cup product $D \smile E \in C^2$ satisfies $|D \smile E| = 1$ and

$$(D \smile E)_i = -m_i D_i E_{i+1}.$$

Define the operators

$$I\{D, E\} : \overline{X}_n \to \overline{X}_{n+1}, \qquad L\{D, E\} : \overline{X}_n \to \overline{X}_n$$

by

$$I\{D, E\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \sigma_0 \tau^r D_i E_j, \qquad L\{D, E\} = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} D_i E_j + \sum_{i=1}^{n} E_0 D_i.$$

Theorem B.3.1. *In the above situation,*

$$[b + B, I\{D, E\}] = L\{D, E\} + I_{D\smile E} - I_E I_D$$

Proof. First note that we trivially have

$$BI\{X, Y\} = I\{X, Y\}B = S_Y S_X = 0$$

because we are working on the normalized complex $\overline{X}$. It is straightforward to verify that

$$\iota_{D\smile E} = \iota_E \iota_D.$$

Therefore, it suffices to prove

$$[b, I\{D,E\}] = L\{D,E\} + S_{D\smile E} - \iota_E S_D - S_E \iota_D.$$

To show this, we first see

$$\begin{aligned} m_0 I\{D,E\} &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} m_0 \sigma_0 \tau^r D_i E_j \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \tau^r D_i E_j \end{aligned}$$

and

$$\begin{aligned} m_0 \tau I\{D,E\} &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} m_0 \tau \sigma_0 \tau^r D_i E_j \\ &= -\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=1}^{n-j+1} \tau^r D_i E_j \end{aligned}$$

since $m_0 \tau \sigma_0 = -\tau$. Combining these,

$$\begin{aligned} (m_0 + m_0\tau) I\{D,E\} &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left(D_i E_j - \tau^{n-j+1} D_i E_j \right) \\ &= L\{D,E\} - \sum_{i=1}^{n} E_0 D_i - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \tau^{n-j+1} D_i E_j. \end{aligned}$$

Next, notice that

$$\begin{aligned} \iota_E S_D &= \sum_{i=1}^{n} \sum_{j=0}^{n-i} m_0 E_1 \sigma_0 \tau^j D_i \\ &= \sum_{i=1}^{n} \sum_{j=0}^{n-i} m_0 \sigma_0 E_0 \tau^j D_i \\ &= \sum_{i=1}^{n} \sum_{j=0}^{n-i} E_0 \tau^j D_i \end{aligned}$$

$$= \sum_{i=1}^{n} E_0 D_i + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \tau E_n \tau^{j-1} D_i$$
$$= \sum_{i=1}^{n} E_0 D_i + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \tau^j E_{n-j+1} D_i$$
$$= \sum_{i=1}^{n} E_0 D_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \tau^{n-j+1} E_j D_i$$

and consequently

$$(m_0 + m_0\tau) I\{D, E\} + \iota_E S_D = L\{D, E\}$$

because $j > i$ implies $D_i E_j = E_j D_i$. Now

$$\begin{aligned}
\sum_{l=1}^{n} \partial_l I\{D, E\} &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \sum_{l=1}^{n} m_l \sigma_0 \tau^r D_i E_j \\
&= -\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \sum_{l=0}^{n-1} \sigma_0 m_l \tau^r D_i E_j \\
&= -\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \Bigg(\sum_{l=0}^{r-2} \sigma_0 m_l \tau^r D_i E_j + \sigma_0 m_{r-1} \tau^r D_i E_j \\
&\qquad\qquad + \sum_{l=r}^{n-1} \sigma_0 m_l \tau^r D_i E_j \Bigg) \\
&= -\sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{r=2}^{n-j} \sum_{l=0}^{r-2} \sigma_0 \tau^{r-1} m_{n+1+l-r} D_i E_j \\
&\qquad - \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=1}^{n-j} \sigma_0 \tau^{r-1} m_0 \tau D_i E_j \\
&\qquad - \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \sum_{l=r}^{n-1} \sigma_0 \tau^r m_{l-r} D_i E_j \\
&= -\sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{r=1}^{n-j-1} \sum_{l=n-r}^{n-1} \sigma_0 \tau^r m_l D_i E_j \\
&\qquad - \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=0}^{n-j-1} \sigma_0 \tau^r m_0 \tau D_i E_j
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \sum_{l=0}^{n-r-1} \sigma_0 \tau^r m_l D_i E_j \\
&= -J - K - L.
\end{aligned}
$$

Notice that in summation J, we always have $l > j > i$, and so

$$
J = \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{r=1}^{n-j-1} \sum_{l=n-r}^{n-1} \sigma_0 \tau^r X_i Y_j m_l.
$$

Now,

$$
\begin{aligned}
K &= \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=0}^{n-j-1} \sigma_0 \tau^r m_0 \tau D_i E_j \\
&= \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=0}^{n-j-1} \sigma_0 \tau^r m_0 D_{i+1} E_{j+1} \tau \\
&= \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=0}^{n-j-1} \sigma_0 \tau^r D_i E_j m_0 \tau,
\end{aligned}
$$

so

$$
J + K = \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=0}^{n-j-1} \sigma_0 \tau^r D_i E_j \left(\sum_{l=n-r}^{n-1} m_l + m_0 \tau \right).
$$

Next, we decompose L depending on how l compares to j:

$$
\begin{aligned}
L &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \sum_{l=0}^{n-r-1} \sigma_0 \tau^r m_l E_j D_i \\
&= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \sum_{l=0}^{j-2} \sigma_0 \tau^r m_l E_j D_i \\
&\quad + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \sum_{r=0}^{n-j} \sigma_0 \tau^r m_{j-1} E_j D_i + \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=0}^{n-j-1} \sigma_0 \tau^r m_j E_j D_i \\
&\quad + \sum_{i=1}^{n-3} \sum_{j=i+1}^{n-2} \sum_{r=0}^{n-j-2} \sum_{l=j+1}^{n-r-1} \sigma_0 \tau^r m_l E_j D_i
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=0}^{j-1}\sigma_0\tau^r E_j m_l D_i \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r m_j E_{j+1} D_i + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r m_j E_j D_i \\
&\quad + \sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{r=0}^{n-j-2}\sum_{l=j+1}^{n-r-1}\sigma_0\tau^r E_j m_l D_i \\
&= P + Q + R + T.
\end{aligned}
$$

In the summation for T, we always have $l > i$, and so

$$
T = \sum_{i=1}^{n-3}\sum_{j=i+1}^{n-2}\sum_{r=0}^{n-j-2}\sum_{l=j+1}^{n-r-1}\sigma_0\tau^m D_i E_j m_l.
$$

Notice that

$$
\begin{aligned}
Q &= \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r m_j E_{j+1} D_i + \sum_{i=1}^{n-1}\sum_{r=0}^{n-i-1}\sigma_0\tau^r m_i E_{i+1} D_i \\
&= \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r m_j E_{j+1} D_i - S_{D\smile E}.
\end{aligned}
$$

and so

$$
\begin{aligned}
Q + R &= \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r m_j (E_j + E_{j+1}) D_i - S_{D\smile E} \\
&= \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r E_j m_j D_i - S_{D\smile E} \\
&= \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r D_i E_j m_j - S_{D\smile E}
\end{aligned}
$$

using the fact that $[m, E] = 0$ and that $j > i$. Combining these together gives

$$
Q + R + T = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=j}^{n-r-1}\sigma_0\tau^r D_i E_j m_l - S_{D\smile E}
$$

and further

$$J+K+Q+R+T = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r D_i E_j\left(\sum_{l=j}^{n-1} m_l + m_0\tau\right) - S_{D\smile E}$$
$$= I\{D,E\}b - S_{D\smile E} - \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=0}^{j-1}\sigma_0\tau^r D_i E_j m_l.$$

So to finish the proof, it suffices to show that

$$P = S_E\iota_D + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=0}^{j-1}\sigma_0\tau^r D_i E_j m_l.$$

To show this, we decompose P depending on how l compares to i:

$$\begin{aligned}
P &= \sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=0}^{j-1}\sigma_0\tau^r E_j m_l D_i \\
&= \sum_{i=2}^{n-1}\sum_{j=i}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=0}^{i-2}\sigma_0\tau^r E_j m_l D_i \\
&\quad + \sum_{i=1}^{n-1}\sum_{j=i}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r E_j m_{i-1} D_i + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r E_j m_i D_i \\
&\quad + \sum_{i=1}^{n-3}\sum_{j=i+2}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=i+1}^{j-1}\sigma_0\tau^r E_j m_l D_i \\
&= \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=0}^{i-1}\sigma_0\tau^r D_i E_j m_l \\
&\quad + \sum_{i=0}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r E_j m_i D_{i+1} + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sigma_0\tau^r E_j m_i D_i \\
&\quad + \sum_{i=1}^{n-3}\sum_{j=i+2}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=i+1}^{j-1}\sigma_0\tau^r D_i E_j m_l \\
&= P_1 + P_2 + P_3 + P_4.
\end{aligned}$$

We have

$$\begin{aligned}P_2 + P_3 &= \sum_{j=1}^{n-1}\sum_{r=0}^{n-j-1} \sigma_0\tau^r E_j m_0 D_1 + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1} \sigma_0\tau^r E_j m_i(D_i + D_{i+1})\\ &= S_E\iota_D + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1} \sigma_0\tau^r E_j D_i m_i,\end{aligned}$$

using the fact that $[m, D] = 0$. Thus,

$$P = P_1 + P_2 + P_3 + P_4 = S_E\iota_D + \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{n-j-1}\sum_{l=0}^{j-1} \sigma_0\tau^r D_i E_j m_l$$

as desired. ◻

Corollary B.3.2. *For D, E as above,*

$$[b + B, L\{D, E\}] = -L_{D\smile E} + L_E I_D - I_E L_D$$

Proof. We apply the commutator with $b + B$ to Theorem B.3.1 to see

$$\begin{aligned}0 &= [b + B, [b + B, I\{D, E\}]]\\ &= [b + B, L\{D, E\}] + [b + B, I_{D\smile E}] - [b + B, I_E I_D]\\ &= [b + B, L\{D, E\}] + L_{D\smile E} - [b + B, I_E]I_D + I_E[b + B, I_D]\\ &= [b + B, L\{D, E\}] + L_{D\smile E} - L_E I_D + I_E L_D,\end{aligned}$$

which gives the result. ◻

B.4 Commuting connections

The purpose of this section is to prove the assertion of Lemma 4.1.7 that the two connections

$$\nabla_{GM} = L_\nabla - \sum_{i=1}^{N} L_{X_i\smile Y_i}, \qquad \widetilde{\nabla} = L_\nabla + \sum_{i=1}^{N} L\{X_i, Y_i\}$$

commute on the invariant complex $C^{\mathfrak{g}}_\bullet(A)$.

The notation from Chapter 4 conflicts with the notation of this appendix. The family of derivations $\{X_1, \ldots, X_N, Y_1, \ldots, Y_N\}$ shall be written with superscripts as $\{X^1, \ldots, X^N, Y^1, \ldots, Y^N\}$, whereas a subscript shall refer to the operator on the ∞-cyclic module, as in the notation of section B.1. For example,

$$L\{X^i, Y^i\} = \sum_{j=0}^{n-1} \sum_{k=i+1}^{n} X^i_j Y^i_k$$

on $C^{\mathfrak{g}}_n(A)$, using Proposition 4.1.4.

Now, because ∇ commutes with the X^i and Y^i,

$$[L_\nabla, L\{X^i, Y^i\}] = 0.$$

Consequently,

$$[\widetilde{\nabla}, \nabla_{GM}] = -[L_\nabla, \sum_{i=1}^{N} I_{X^i \smile Y^i}] - [\sum_{i=1}^{N} L\{X^i, Y^i\}, \sum_{i=1}^{N} I_{X^i \smile Y^i}],$$

therefore it suffices to show

$$[L_\nabla, \sum_{i=1}^{N} \iota_{X^i \smile Y^i}] + [\sum_{i=1}^{N} L\{X^i, Y^i\}, \sum_{i=1}^{N} \iota_{X^i \smile Y^i}] = 0$$

and

$$[L_\nabla, \sum_{i=1}^{N} S_{X^i \smile Y^i}] + [\sum_{i=1}^{N} L\{X^i, Y^i\}, \sum_{i=1}^{N} S_{X^i \smile Y^i}] = 0.$$

Now $[L_\nabla, \cdot]$ and $[\sum_{i=1}^{N} L\{X^i, Y^i\}, \cdot]$ are both derivations on the algebra of endormorphisms of $C^{\mathfrak{g}}_\bullet(A)$. Since

$$\iota_{X^i \smile Y^i} = -m_0 m_1 X^i_1 Y^i_2, \qquad S_{X^i \smile Y^i} = -\sum_{j=1}^{n-1} \sum_{k=0}^{n-j-1} -\sigma_0 \tau^k m_j X^i_j Y^i_{j+1},$$

we consider these derivations applied to the elementary operators $\sigma_0, \tau, m_j, X^i_j, Y^i_j$.

As in the proof of Lemma B.2.4, we have

$$[L_\nabla, m_j] = [\nabla, m]_j = -(\delta\nabla)_j = -\sum_{i=1}^{N}(X^i \smile Y^i)_j,$$

and

$$[L_\nabla, \tau] = 0, \qquad [L_\nabla, \sigma_0] = 0.$$

Since ∇ commutes with X^i and Y^i, we also have that for any j, k,

$$[L_\nabla, X^j_k] = 0, \qquad [L_\nabla, Y^j_k] = 0.$$

We shall establish the identities

$$[\sum_{i=1}^{N} L\{X^i, Y^i\}, m_j] = \sum_{i=1}^{N}(X^i \smile Y^i)_j$$

$$[\sum_{i=1}^{N} L\{X^i, Y^i\}, \tau] = [\sum_{i=1}^{N} L\{X^i, Y^i\}, \sigma_0] = 0,$$

$$[\sum_{i=1}^{N} L\{X^i, Y^i\}, X^j_k] = [\sum_{i=1}^{N} L\{X^i, Y^i\}, Y^j_k] = 0.$$

From these relations, the desired result follows because the derivations $[L_\nabla, \cdot]$ and $[\sum_{i=1}^{N} L\{X^i, Y^i\}, \cdot]$ are negatives of each other on a subalgebra of $\mathrm{End}(C^{\mathfrak{g}}_\bullet(A))$ containing $\sum_{i=1}^{N} I_{X^i \smile Y^i}$.

The identities $[\sum_{i=1}^{N} L\{X^i, Y^i\}, X^j_k] = [\sum_{i=1}^{N} L\{X^i, Y^i\}, Y^j_k] = 0$ follow at once from the fact that the X^i and Y^i generate an abelian Lie algebra. Since $X^i_0\sigma_0 = 0$, it follows that $[\sum_{i=1}^{N} L\{X^i, Y^i\}, \sigma_0] = 0$. We shall now show $[\sum_{i=1}^{N} L\{X^i, Y^i\}, \tau] = 0$, which is an identity that only holds on the invariant complex $C^{\mathfrak{g}}_\bullet(A)$. We calculate

$$\begin{aligned}\tau L\{X^i, Y^i\} &= \sum_{j=0}^{n-1}\sum_{k=j+1}^{n} \tau X^i_j Y^i_k \\ &= \sum_{j=1}^{n}\sum_{k=j}^{n} X^i_j \tau Y^i_k\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \left(\sum_{k=j}^{n-1} X^i_j \tau Y^i_k + X^i_j \tau Y^i_n \right) \\
&= \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} X^i_j Y^i_k \tau + \sum_{j=1}^{n} X^i_j Y^i_0 \tau \\
&= L\{X^i, Y^i\}\tau - \sum_{k=1}^{n} X^i_0 Y^i_k \tau + \sum_{j=1}^{n} X^i_j Y^i_0 \tau \\
&= L\{X^i, Y^i\}\tau + X^i_0 Y^i_0 - X^i_0 Y^i_0 \\
&= L\{X^i, Y^i\}\tau,
\end{aligned}
$$

where the second to last equality holds because be are working on the $\mathfrak{g}$-invariant complex, and $L_{X^i} = L_{Y^i} = 0$.

Lastly, we calculate

$$
\begin{aligned}
&m_l L\{X^i, Y^i\} + (X^i \smile Y^i)_l \\
&\quad = \sum_{j=0}^{n-1} \sum_{k=j+1}^{n} m_l X^i_j Y^i_k - m_l X^i_l Y^i_{l+1} \\
&\quad = \sum_{j=0}^{l-1} \sum_{k=j+1}^{n} m_l X^i_j Y^i_k + \sum_{k=l+1}^{n} m_l X^i_l Y^i_k + \sum_{k=l+2}^{n} m_l X^i_{l+1} Y^i_k \\
&\qquad\qquad + \sum_{j=l+2}^{n-1} \sum_{k=j+1}^{n} m_l X^i_j Y^i_k - m_l X^i_l Y^i_{l+1} \\
&\quad = \sum_{j=0}^{l-1} \sum_{k=j+1}^{n} m_l X^i_j Y^i_k + \sum_{k=l+2}^{n} m_l (X^i_l + X^i_{l+1}) Y^i_k + \sum_{j=l+2}^{n-1} \sum_{k=j+1}^{n} m_l X^i_j Y^i_k \\
&\quad = \sum_{j=0}^{l-1} \sum_{k=j+1}^{n} X^i_j m_l Y^i_k + \sum_{k=l+2}^{n} X^i_l m_l Y^i_k + \sum_{j=l+1}^{n-2} \sum_{k=j+2}^{n} X^i_j m_l Y^i_k \\
&\quad = \sum_{j=0}^{l-1} \left(\sum_{k=j+1}^{l-1} X^i_j m_l Y^i_k + X^i_j m_l (Y^i_l + Y^i_{l+1}) + \sum_{k=l+2}^{n} X^i_j m_l Y^i_k \right) \\
&\qquad\qquad + \sum_{j=l}^{n-2} \sum_{k=j+2}^{n} X^i_j m_l Y^i_k \\
&\quad = \sum_{j=0}^{l-1} \left(\sum_{k=j+1}^{l-1} X^i_j Y^i_k m_l + X^i_j Y^i_l m_l + \sum_{k=l+1}^{n-1} X^i_j Y^i_k m_l \right) + \sum_{j=l}^{n-2} \sum_{k=j+1}^{n-1} X^i_j Y^i_k m_l
\end{aligned}
$$

$$= \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} X_j^i Y_k^i m_l$$
$$= L\{X^i, Y^i\} m_l,$$

where we have used the fact that $[m, X^i] = [m, Y^i] = 0$. This completes the proof that

$$[\widetilde{\nabla}, \nabla_{GM}] = 0.$$

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