EFFECTIVE EQUATIONS OF COSMOLOGICAL MODELS IN
(LOOP) QUANTUM GRAVITY

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by
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Abstract

This dissertation focuses on the properties of several differing models within quantum cosmology. Specifically, by using the method of effective equations, we explore: a linear discrete Schrödinger model, a non-linear discrete Schrödinger model, factor ordering ambiguities in the Hamiltonian constraint (with a focus on large-volume behavior), and the use of the electric vector potential as deparameterized time. In the linear and non-linear Schrödinger models, we arrive at a new possibility for studying inhomogeneous quantum cosmology (where the non-linearities are interpreted as non-local deviations from the spatial average) that allows for a variety of dynamics and raises a number of questions for future research. We then turn our focus to the general effects of factor ordering ambiguities and their possible role in large-volume collapse of a $k = 0$ isotropic quantum cosmology with a free, massless scalar field. With the additional inclusion of holonomy and inverse-triad corrections, the choice in factor ordering of the Hamiltonian constraint is quite relevant; however, with our assumptions, we do not see any significant departure from classical large-volume behavior. The final model discussed is formulated with the electric vector potential as the global internal time in a Wheeler–DeWitt setting sourced by radiation. While further analysis is required to make a definitive statement on the impact that the choice of deparameterization makes, we find that the specific form of quantum state can affect early-universe dynamics and even lead to new possibilities.
Table of Contents

List of Figures vi
List of Tables ix
Acknowledgments x

Chapter 1
Introduction to Loop Quantum Cosmology 1
1.1 Canonical General Relativity . . . . . . . . . . . . . . . . . . . . . . 2
1.2 Canonical Effective Equations within Loop Quantum Cosmology . 4

Chapter 2
A Linear Discrete Schrödinger Model 8
2.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 8
2.2 Quantization . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
2.3 Effective Equations . . . . . . . . . . . . . . . . . . . . . . . . . . . 13
2.4 Numerical Calculations . . . . . . . . . . . . . . . . . . . . . . . . . 15
2.5 Appendix - Quantum Equations of Motion . . . . . . . . . . . . . . 22

Chapter 3
A Non-linear Extension in Quantum Cosmology 25
3.1 Introduction . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 25
3.2 Non-linear Dynamics in Quantum Cosmology . . . . . . . . . . . . 26
3.3 Non-linear Effective Equations . . . . . . . . . . . . . . . . . . . . 28
3.4 Conclusion . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 31
## List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>The normalized magnitude of the wavefunction as it changes in time. Originally it is Gaussian and it spreads symmetrically with time. With the addition of a ‘momentum’ term, the peak occurs at decreasing $n$ as time increases. <strong>a)</strong> A graph of $</td>
</tr>
<tr>
<td>2.2</td>
<td>The real and imaginary portions of $G^{Vh}(t)$ plotted with respect to $t$. Overlaid are the expected results from the equation of motion given by Eqn. (2.30). <strong>a)</strong> The real portion of the $G^{Vh}$ moment at various times shown is in good agreement with the effective equation. <strong>b)</strong> The imaginary portion of the $G^{Vh}$ moment is shown to be in good agreement as well.</td>
</tr>
<tr>
<td>2.3</td>
<td>Both $V(t)$ and $G^{VV}(t)$ are in good agreement with their theoretical functions (given by their commutators with the Hamiltonian). <strong>a)</strong> The expectation value of volume calculated at various times plotted with the equation of motion for the volume overlaid. <strong>b)</strong> $G^{VV}$ moment calculated by the expectation values at various times plotted with the expected function, Eqn. (2.35), from the equations of motion.</td>
</tr>
<tr>
<td>4.1</td>
<td>Effective equation solution for the expectation of volume $\nu(\phi)$ is plotted with a $-\phi$ axis. It is exponentially increasing $\propto -\phi$ over a large range of values until it begins a sharp descent when the squareroot term of Eqn. (4.58) approaches zero at $-\phi \approx 30.75$.</td>
</tr>
<tr>
<td>4.2</td>
<td>Relative fluctuation of volume $G^{\nu\nu}(\phi)/\nu^2(\phi)$ is plotted with a $-\phi$ axis. It is well behaved over a large range of values but does exponentially increase at $-\phi &gt; 30.75$ and is increasingly larger for $-\phi &lt; -20$ where $\nu(\phi)$ becomes exponentially smaller.</td>
</tr>
</tbody>
</table>
4.3 Relative covariance fluctuation of the volume and curvature expectation values $G^{bb}(\phi)/(b(\phi)\nu(\phi))$ is plotted with a $-\phi$ axis. It is reasonably well behaved over a large range of values, though it increases unboundedly at large $-\phi$, which is inconsistent with our semiclassical approximations in that region.

4.4 Solutions of effective equations for expectation values of volume $\nu(\phi)$ (top) and curvature $b(\phi)$ (bottom). Initial values have been set to $\nu_0 = 10, b_0 = .4, G^{bb}_0 = .2, G^{\nu\nu}_0 = 5$, and $G^{b\nu}_0 = .5$ such that our solutions are valid at time $\phi = 0$. In addition to the bounce at small $\nu(\phi)$, notice that there is some asymmetry in both graphs and that $b$ increases to where $(\sin b)/b \approx .9$, pressing the limits of our assumption.

4.5 Relative fluctuation of volume $G^{\nu\nu}(\phi)/\nu^2(\phi)$ is plotted and shown to be within the validity of the effective equations at inclusion of second order moments.

4.6 Relative fluctuation of curvature $G^{bb}(\phi)/b(\phi)$ is plotted and shown to increase beyond the validity of the model as $\phi$ increases through the bounce point.

5.1 Solutions of effective equations for expectation values, plotted as their ratios to the classical solutions $c_{\text{classical}}(A)$ (top) and $p_{\text{classical}}(A)$ (bottom). Initial fluctuations have been set to rather large values $-G^{pp}_0 = p_0 = G^{cc}_0 = c_0 = 1$ with $G^{cp}_0 = 0$ — to show the implications of quantum corrections more clearly. Nevertheless, the ratios to the classical solutions (with the same initial values $p_0 = c_0 = 1$) stay close to one.

5.2 Solution of effective equations for expectation value of $c(A)$ plotted as a ratio to the classical solutions $c_{\text{classical}}(A)$. Initial values are as follows $-G^{pp}_0 = p_0 = 1 , c_0 = .8$ with $G^{cp}_0 = .2$ — to show the implications of quantum corrections while forcing $c(A)|_{A=5000} = c_{\text{classical}}(A)|_{A=5000}$.

5.3 The $c$-variance $G^{cc}(A)$ (thin dashed) compared with the curvature expectation value $c(A)$ (thick), the covariance $G^{cp}(A)$ (thin) and $c_{\text{classical}}(A)$ (medium dashed) for large initial fluctuations $G^{pp}_0 = p_0 = G^{cc}_0 = c_0 = 1$ with $G^{cp}_0 = 0$. $c(A)$ matches well to $c_{\text{classical}}(A)$ (if we corrected the classical value by 87.5% they would be indistinguishable on this graph) and $G^{cc}$ decreases to zero quickly. $G^{cp}$ qualitatively agrees with the analytical solution $G^{cp}(A) \propto \text{const} - c(A)$ of Eqn. 5.11 with the appropriate constants.
5.4 The $p$-variance $G^{pp}(A)$ (dashed) compared with the expectation value $p(A)$ (solid). Both functions increase in nearly the same way even for large initial fluctuations as in Fig. 5.1, confirming our analytical solutions. We have $G^{pp} > p$, but relative fluctuations $G^{pp}/p^2$ become very small at large $A$. ................. 76

5.5 The expectation value $p(A)$(top) shows recollapse while the fluctuation $G^{pp}(A)$ as a ratio of $p(A)^2$ (bottom) surpasses $8p(A)^2$ corresponding to a maximum, with initial values $G_0^{pp} = 4p_0 = 4, c_0 = G_0^{cc} = 1$, and $G_0^{cp} = 0$. Important note: these demonstrate only interesting possibilities in a regime where our effective equations are no longer valid without consideration of higher order moments. 78
List of Tables

4.1 Sample values for $\alpha(n)$ and $\beta(n)$ ........................................ 51
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Dedication

To my parents and grandparents, for all of their love, support, and newspaper clippings.
Chapter 1

Introduction to Loop Quantum Cosmology

Classical general relativity holds quite well in regimes of large scale and small curvature, but when standard classical cosmological scenarios are evolved backwards in time, the equations break down and singularities arise. At those points of divergent energy density and small length scales, quantum effects become important and as such, we have need of a development of quantum gravity. Loop Quantum Gravity (LQG) is one possible formulation which is non-perturbative and background independent. In this introductory chapter, we will not describe the full theory of LQG, but rather focus on the main contributions and concepts used within Loop Quantum Cosmology (LQC): specifically on symmetry reduced cosmological models (minisuperspaces), the Dirac quantization of the classically reduced models, and the effective techniques which we predominantly utilize in this dissertation.

We will discuss linear and non-linear discrete Schrödinger equations in Chapters 2 and 3 respectively. The linear chapter will concentrate on the numerics of the solvable wave equations and show that they match the effective equation solutions, which are based entirely on expectation values and quantum moments – not any specific choice of wave function. We then motivate an extension to a non-linear term by analogy of Bose-Einstein condensate techniques and analyze the potential interpretation it may have for quantum cosmology. Chapter 4 will undertake an analysis on the possible effects of factor ordering quantization ambiguities, with
emphasis on the large volume regime and recollapse. In Chapter 5, we explore a new choice of deparameterized time variable in a regime where the usual choice is artificially included; we additionally check for sensitivity to the form of state in this regime. Now however, we give an overview of our methodology.

1.1 Canonical General Relativity

Classically, the gravitational phase space of full general relativity can be written in terms of the canonical pair \((A^i_a, E^a_i)\) on the spatial 3-manifold \(M\) where \(A^i_a\) is a SU(2) connection (referred to as the Ashtekar-Barbero connection) and \(E^a_i\) is its conjugate momenta, a densitized triad \([1,2,3]\). These are related to the spatial metric and extrinsic curvature by

\[
E^a_i E^b_i = q^{ab} \det q \tag{1.1}
\]

\[
A^i_a = \Gamma^i_a + \gamma K^i_a. \tag{1.2}
\]

where \(q^{ab}\) is the spatial metric, \(K^i_a\) is the extrinsic curvature, \(\Gamma^i_a\) is the spin connection, and \(\gamma > 0\) is the Barbero-Immirzi parameter \([4]\) (classically this parameter could be set to 1 by a canonical transformation, but plays a significant role in quantization \([5]\)). In this dissertation, we focus on the context of homogeneous and isotropic cosmology where our phase space is reduced, and we will instead use the canonical pair \((c, p)\). We can relate them to the above by fixing a flat fiducial metric \(q^{ab}\) with a volume \(V_o\) of the fiducial cell (over which integrations are done):

\[
A^i_a = c V_o^{-\frac{1}{3}} \delta^i_a \tag{1.3}
\]

\[
E^a_i = p V_o^{-\frac{2}{3}} \delta^a_i. \tag{1.4}
\]

The Poisson bracket of our canonical variables is given by

\[
\{c, p\} = \frac{8\pi G\gamma}{3} \tag{1.5}
\]

which we will use in our quantization procedure. Following Dirac quantization we arrive at a system governed by constraints; in the full theory, there are three constraints: the Gauss, Diffeomorphism, and Hamiltonian constraints. In homo-
geneous and isotropic cosmology, the first two are satisfied by the symmetry of our system and we are left with only the Hamiltonian constraint. For classical cosmology, space is represented by the Friedmann–Robertson–Lemaitre–Walker spatial metric, Eqn. (1.6), with dynamics governed by the Friedmann equation and the Friedmann acceleration equation, Eqns. (1.7) and (1.8) respectively (which are derived from the Einstein field equations of general relativity; here we have set $c$ the speed of light equal to 1 to remove any confusion with our canonical variable $c$).

\[ ds^2 = a(\tau)^2 \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \]  
\[ H^2 = \left( \frac{\dot{a}(\tau)}{a(\tau)} \right)^2 = \frac{8\pi G}{3} \rho - \frac{k}{a(\tau)^2} \]  
\[ \frac{\ddot{a}(\tau)}{a(\tau)} = -\frac{4\pi G}{3} (\rho + 3p) . \]

In the above, we have written the spatial metric using polar coordinates, $a(\tau)$ is the scale factor with $\tau$ proper time, $k$ is the intrinsic spatial curvature taking on values of zero or $\pm 1$, $H = \dot{a}/a$ is the Hubble parameter, `signifies here a derivative with respect to proper time $\tau$, $G$ is Newton’s gravitational constant, and $\rho$ is the energy density of the system with pressure $p$ (where we use bold font so the future use of $p$ as the canonical variable is clear, and $H$ as the Hamiltonian should be clear by statement and context). For the purposes of this dissertation, we further refine our cosmological setting to the $k = 0$ flat sector and simply write $a(\tau) \equiv a$ as the proper time scale factor unless otherwise noted.

Connecting together the various pieces, we see that the scale factor squared, $a^2$, defines our spatial metric, $q_{ab}$, and thus the densitized triad $E^a_i$ which in the symmetry reduced frame, is entirely encoded within our canonical variable $p$. Similarly for our curvature terms, which are related to the connection $A^i_a$ and thus to the canonical momentum $c$. For the purposes of this dissertation then, we will relate our canonical variables to the scale factor as

\[ |p| = a^2 \]  
\[ c = \gamma \dot{a} \]

where the absolute value comes about because triads are oriented, and so can take
positive or negative values, which we will without loss of generality choose to be positive unless noted otherwise. Hence, we can interpret our canonical pair \((c, p)\) as the curvature and area respectively. So far, we have only discussed classical general relativity, but going forward we expect any theory of quantum gravity should reproduce these dynamics – Eqns. (1.7) and (1.8) – in appropriately classical regimes, so it is perhaps not surprising that we then choose our Hamiltonian constraint such that it reduces to the Friedmann equation semiclassically, specifically in the low curvature regime. However, there are still many forms of ambiguity that can enter into the choice of Hamiltonian constraint, even between Hamiltonians that reduce to the same classical expression. Exploring the effects of such ambiguities and quantization choices is therefore an important endeavor, and one on which this dissertation embarks. Before doing so however, we review some methods of quantization.

1.2 Canonical Effective Equations within Loop Quantum Cosmology

Now that we have some background in the classical canonical system, we move towards a method of approximating complex quantum systems via effective equations [6, 7, 8, 9]. Typically quantum systems correspond to wave functions subject to partial differential equations, but effective equations allow for systems of ordinary differential equations that can be truncated to closed systems of finite order (truncated to the level of precision required). Thus quantum systems which are complicated conceptually and technically can be written as classical equations of motion with quantum corrections (though care must be taken for systems which are not exactly solvable where back reaction occurs, i.e. the quantum corrections influence the dynamics of the expectation values).

In quantum mechanics, a set of \(N\) basic operators \(\hat{J}_i\) with closed linear commutators

\[
[\hat{J}_i, \hat{J}_j] = \sum_k C_{ij}^k J_k
\]

(1.11)

(\text{perhaps including the identity operator if some commutators are constants}) pro-
vides a closed algebra for expectation values under Poisson brackets

\[ \{\langle \hat{J}_i \rangle, \langle \hat{J}_j \rangle \} = \frac{\langle [\hat{J}_i, \hat{J}_j] \rangle}{i \hbar}. \]  

(1.12)

If the operators are complete, any observable can be expressed as a function of the expectation values \( \langle \hat{J}_i \rangle \) and moments \( \Delta \left( \prod_i J_{a_i} \right) \equiv \langle \prod_i (\hat{J}_i - \langle \hat{J}_i \rangle)^{a_i} \rangle \) \[ \text{Symmetric} \]  

(1.13)

with operator products in totally symmetric ordering. Using linearity and the Leibniz rule for Poisson brackets, these expectation values and moments form a Poisson manifold. Their dynamics is then determined by the Hamiltonian flow generated by the expectation value \( H_Q \equiv \langle \hat{H} \rangle \) of the Hamiltonian constraint, another observable interpreted as a function of expectation values and moments. Hamiltonian equations of motion usually couple infinitely many moments to the expectation values, but a semiclassical expansion to some finite order in \( \hbar \) results in finitely coupled equations which can be solved at least numerically. Computer-algebra codes exist to automate the generation of equations to rather high orders \[ \text{[10]} \] (so far restricted to canonical commutators).

Writing \( \hat{J}_i = \langle \hat{J}_i \rangle + (\hat{J}_i - \langle \hat{J}_i \rangle) \) in the quantum Hamiltonian \( H_Q = \langle H(\hat{J}) \rangle \) and performing a formal expansion in \( (\hat{J}_i - \langle \hat{J}_i \rangle) \), the Hamiltonian flow is generated by

\[ H_Q = H(\langle \hat{J} \rangle) + \sum_{a_i} \frac{1}{a_1!} \cdots \frac{1}{a_N!} \partial^{a_1 + \cdots + a_N} H(\langle \hat{J} \rangle) \Delta \left( \prod_i J_{a_i} \right). \]  

(1.14)

The first term is the classical Hamiltonian evaluated in expectation values, and the series includes quantum corrections of progressing order \( \sum_i a_i \). Equations of motion for observables follow from Poisson brackets with the quantum Hamiltonian \( H_Q \)

\[ \frac{d\langle \hat{O} \rangle}{dt} = \{ \langle \hat{O} \rangle, H_Q \} = \frac{\langle [\hat{O}, \hat{H}] \rangle}{i \hbar}. \]  

(1.15)

At this point we have kept the notation very general, but in the following chapters, the notation will be explicitly defined and we will see that quantum moments used in the expansion at second order are the usual variance and co-variance of a
state, with the higher orders representing further description of the state such as skew. Also note that the time parameter $t$ can vary based on our constraint. It is typical in quantum cosmology that one uses a relational time variable through a deparameterization procedure: a popular choice is that of the free massless, scalar $\phi$. Proper time is then related to that choice by the Poisson bracket with the full Hamiltonian (using $\phi$ as a relational time for example)

$$\frac{d\phi}{d\tau} = \{\phi, H\} \text{ with some defined } \{\phi, p_\phi\} \quad (1.16)$$

but we will discuss further details of such a choice in subsections 4.2.4 and 5.3.1. Additionally, though our classical notation will be based on $(c, p)$ as defined previously, given the ambiguity of quantization scheme, the basic operators one uses can be different. One important difference in choice of operators is that of the Wheeler–DeWitt quantization compared to the loop quantization.

In a Wheeler–DeWitt quantization scheme, the classical variables $(c, p)$ with Poisson bracket $\{c, p\}$ are promoted directly to operators ($\hat{c}$ and $\hat{p}$) with commutator $[\hat{c}, \hat{p}] = i\hbar\{\langle \hat{c} \rangle, \langle \hat{p} \rangle\}$ as in Eqn. 1.12. One can then arrive at a Hamiltonian operator in terms of $\hat{c}$ and $\hat{p}$ which satisfies the Friedmann equation, but accordingly leads to a classical singularity at $a = 0$ (though there is some question on the genericness of that result which is beyond the scope of this dissertation).

In a loop quantization, one pulls guidance from the full theory of LQG; in the quantization of the full classical theory using $(A^i_a, E^a_i)$, the quantum operators are holonomies along edges of the fiducial cell and fluxes through the faces. The connection $A^i_a$ itself is not an operator, but rather one uses an exponentiated (and path ordered) function of the connection. Additionally, these holonomy operators act as shift operators on the wavefunctions of the full theory (an aspect which we see is important in the motivation of Chapter 2). In the homogeneous and isotropic sector, the basic operator related to curvature becomes $e^{i\delta(p)\hat{c}}$ where $\delta(p)$ is a function of $p$ (As a side note: $\delta(p)$ typically contains a constant which is derived from the spectrum of the area operator in LQG. The non-zero minimum of this operator gives the smallest quanta of area, referred to as the area gap $\Delta = 4\sqrt{3}\pi\gamma\ell^2_P$ with $\ell_P = \sqrt{G\hbar}$ the Planck length. We see this explicitly in Subsection 4.2.1.) Then, if one wishes to semiclassically arrive at $c$ in the classical Hamiltonian
constraint, holonomies are used

\[ \frac{e^{i\delta(p)c} - e^{-i\delta(p)c}}{2i\delta(p)} = \frac{\sin(\delta(p)c)}{\delta(p)} \text{ semiclassically} \rightarrow c. \]  

Thus, the use of such sine (or exponential) terms is referred to as holonomy corrections and can lead to generic resolution of the singularity, often referred to as bouncing models \[12, 13, 14, 15\]. We will discuss some of these differences and their effects on our dynamics in the chapters to follow.

Loop quantum gravity attempts to unify general relativity and quantum physics in a non-perturbative background-independent way. The full theory can be difficult to work with (there is no current method to derive cosmological models directly from the full theory), but symmetry reduced cosmological models allow us to explore fundamental properties with a simpler treatment, such as seen in [7]. Though quantitative effects depend on ambiguities due to how we quantize, we may still learn about the qualitative behavior given a particular choice of quantization scheme (some of which include aspects that come directly from the guidance of the full theory). Using the basis discussed in this chapter, we now focus on several specific models and perform an analysis on the effects possible due to varying quantization choices.
A Linear Discrete Schrödinger Model

We introduce a toy model which is governed by a specific linear discrete difference equation. This model is a stepping stone to Chapter 3 as it is exactly solvable in terms of effective equations as well as wavefunctions. As the importance for the model is that it adheres to a specific constraint formulacly, we must find a suitable model that yields such an equation and then identify a cosmological interpretation afterwards (using some foresight from the full theory of LQG). We then focus on the techniques of effective equations and numerics for the linear case in remaining sections of this chapter.

2.1 Introduction

This chapter describes work on the numerical analysis of solutions to the difference-differential equation

\[
i\partial_t \psi_n(t) = \Delta \psi_n(t) = \psi_{n+1}(t) - 2\psi_n(t) + \psi_{n-1}(t)
\]

which governs the basis functions of our model for a wavefunction of the universe in a mini-superspace model (\(\hbar\) is set to 1). We interpret the integer \(n\) as a label for the discrete eigenvalues of the volume for our system. This model, which is solvable, will be extended to the more complicated goal of incorporating a non-
linear term in Chapter 3. To motivate the addition of that term, we will look to the non-linearity of the Gross-Pitaevskii equation which describes a system of identical bosons, but for now we focus on this solvable model.

Recall that in loop quantum gravity there is no background geometry, and so constraint equations on the classical phase space generate the classical dynamics \([16]\). Similarly, the physical states of the quantum system are the solutions of the quantized constraints. Since we are interested in arriving at a particular quantized constraint, we must work backwards in some sense and find the classical constraint, and then interpret it as a cosmological model. In order to interpret our solutions, there must be some notion of time by which the equations evolve. We can consider the matter of the universe to be composed of dust, defined to have zero pressure \(p = 0\), to formulate a canonical time which can provide a Schrödinger equation where the dust time is separable \([17]\). We do not limit our matter to dust, but the formalism serves as motivation since we desire a separable Schrödinger difference equation.

From the cosmological perspective, let us now visit some of the concepts that will be pertinent here (and Chapter 3) for interpreting our intended difference equation. As discussed in Chapter 1.2, one may choose to follow Wheeler–DeWitt quantization or a loop quantization; in this and the preceding chapter we use the loop quantization, that is, we quantize the exponential of our momenta rather than the momenta itself. One reason we choose this is because within the full theory of loop quantum gravity, the holonomies are shift operators and thus one arrives at a difference equation for the Hamiltonian. Since a difference equation is our desired goal, it is logical that we should pursue a similar method. So if we wish to begin with

\[
\hat{H}\psi_n(t) = \psi_{n+1}(t) - 2\psi_n(t) + \psi_{n-1}(t)
\]

then we will need a Hamiltonian comprised of a positive plus a negative shift operator minus 2. Let us call these shift operators \(\hat{h}\) and \(\hat{h}^\dagger\) (due to their expected relation to holonomies), that is, we expect \(\hat{H} = \hat{h} + \hat{h}^\dagger - 2\). We will discuss the algebra and dynamics of \(\hat{h}\) later in the chapter, but for now let us focus on the connection to classical cosmology so that we might know how to interpret these shift operators. With that in mind, we first write a Hamiltonian constraint that classically compares to the Friedmann equation for spatially flat isotropic models.
in canonical variables $c = \gamma \dot{a}$ and $|p| = a^2$ (with a holonomy correction of sine due to our expectation of holonomies)

$$- \frac{3}{8\pi G \gamma^2} \frac{\sin(\delta(p)c)^2}{\delta(p)^2} \sqrt{|p|} + H_{\text{matter}} = 0.$$  \hspace{1cm} (2.3)

where $\delta(p)$ is some function of $p$ useful in defining the lattice refinement of our discrete geometry. As discussed previously for loop quantization, we do not use $c$ but rather some exponentiation of it, such as $e^{i\delta(p)c}$ which allowed us to write sine initially as

$$\sin(\delta(p)c)^2 = - \left( e^{i\delta(p)c} - e^{-i\delta(p)c} \right)^2 / 4 = .25 \left( e^{2i\delta(p)c} + e^{-2i\delta(p)c} - 2 \right).$$  \hspace{1cm} (2.4)

This is quite similar to the linear discrete Schrödinger difference equation we are interested in; if we neglect some numerical factors and concentrate on the fact that holonomies do act as shift operators, we can make the classical relation that $h = e^{ip^x c}$ where $p^x c$ is a canonical variable. For $x = 0$ this is simply $c$ and the conjugate partner is $p$. For $x = -1/2$ we get a different choice of canonical variables (which we use in Eqn. (4.5) ). For now though, let us just refer to it as some variable $P = p^x c$ with conjugate partner $V = p^{1-x} / (1-x)$ (which we identify as the volume of our system, $\hat{V}\psi_n = n\psi_n$) with constant $-1/2 < x < 0$. Returning to the Friedmann equation, let us simplify it as

$$- \sin^2(P) + p_t = 0$$  \hspace{1cm} (2.5)

such that we now have some relational time $t$ that was born of $H_{\text{matter}}$. Neglecting $\hbar$ we see this returns us to Eqn. (2.8) where the momentum of $t$ acts as the usual differential operator, $p_t = i\partial_t$, and our classical deparameterized Hamiltonian is (neglecting numerical coefficients)

$$H = \sin^2 P = e^{iP} + e^{-iP} - 2 = h + \bar{h} - 2.$$  \hspace{1cm} (2.6)

At small curvature $c \ll 1$ Eqn. (2.5) reduces to $p_t \sim p^{2x} c^2$ which we can then interpret as matter content of the full Hamiltonian constraint (using Eqn. (1.7),
\[ c = \gamma \dot{a}, \text{ and } |p| = a^2 \]

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{c^2}{p} = \frac{p_t}{p^{1+2x}} = \frac{p_t}{a^{2(1+2x)}} \sim \rho
\]

(2.7)

When considering matter as a perfect fluid \((p = w\rho)\) we can write \(\rho \propto a^{-3(1+w)}\) where \(w\) is the equation of state parameter. For \(-1/2 < x < 0\) we then have \(-1 < w < -1/3\) which corresponds to accelerated expansion of the Friedmann equation. Typically one would not expect any connection between our choice in lattice refinement (relating to \(x\)) and the parameter of state \(w\). However, as we are seeking a specific difference equation, we simplified our Hamiltonian constraint to \(\sin^2(P)\) rather than the usual holonomy correction, and so our interpretation varies accordingly. Also of note, due to this specification of the Hamiltonian, \(\sin(P)\) is constant (at least in the case of the linear discrete Hamiltonian of this chapter) and we see the consequences in our equations of motion (and numerics): i.e. this is not a bouncing model. Typically for holonomy corrections with a free, massless scalar, terms like \(\sin(P)\) grow as small volume is approached, and since \(\sin(P)\) is bounded, the scale factor must bounce when the limit is reached. That is not the case for this model; though our volume (which we discuss further in Section 2.3) is discrete, \(\sin(P)\) does not constrain it and there is no resolution of the singularity. However, the classical singularity occurs in proper time \(\tau\); with \(t\) as time, solutions \(V(t)\) are linear and at small curvature, we will see that the dynamics act as a free particle (that is, nothing in particular occurs as \(V(t) \to 0\)).

With those identifications, we arrive at a Poisson algebra of classical variables \((V, h, \bar{h})\) and a linear Hamiltonian, \(H = h + \bar{h} - 2\) which generates our evolution by Heisenberg-type equations of motion, i.e. \(dV/dt = \{V,H\}\). These equations of motion also hold for the expectation values; in our case, they form a closed system and we are able to solve them explicitly without information on the spread or deformation of the quantum state.

However, this is not generally the case: typically we must introduce time-dependent quantum variables which describe the wavefunction (generally written as Eqn. (1.13) but defined explicitly with our new variables in Eqn. (2.12)). The evolution of the moments is also governed by the Hamiltonian, but care must be taken as they are not just expectation values of the basic operators. Generally,
we would expect to find coupling between expectation values and quantum variables: that is, the expectation values could be influenced by the spread of the quantum state thus providing quantum corrections. Since this model is solvable, these quantum variables decouple from the expectation values and will not have quantum back-reaction.

By introducing the quantum variables though, we can obtain the expectation values and fluctuations by solving the system of ordinary differential equations (even without knowing the exact wavefunctions); though they are not necessary to solve our effective equations, we do so in order to compare them with the wavefunctions for which we numerically solve. The equations of motion for the quantum variables are confirmed and evolution of the wavefunction is graphed allowing us to visualize its spread and deformation.

2.2 Quantization

We now return to the difference-differential equation

\[ i\partial_t \psi_n(t) = \Delta_2 \psi_n(t) = \psi_{n+1}(t) - 2\psi_n(t) + \psi_{n-1}(t) \]  

(2.8)

which is motivated from a loop quantization of the Hamiltonian constraint of isotropic gravity with the proper identifications mentioned above. Eqn. (2.8) can easily be solved by separation of variables. With \( \exp(-i\omega t) \) for the time dependence, we must solve \( \omega \psi_n = \psi_{n+1} - 2\psi_n + \psi_{n-1} \). An ansatz \( \psi_n = \exp(i\lambda n) \) provides \( \cos \lambda = 1 + \omega/2 \). For fixed \( \omega \), the general solution is \( \psi_n^{(\omega)} = \alpha \exp(i\lambda n) \) with a complex constant \( \alpha \) and time dependence \( \exp(-i\omega t) = \exp(-i(2\cos(\lambda) - 2)t) \). Real and imaginary parts of the general solution are related to Tschebyscheff (Chebyshev) polynomials.

The time-independent solutions then form the basis functions, denoted \( E_n(\lambda) \equiv \exp(i\lambda n) \) with integer \( n \), such that the time-dependent general solution can be written as

\[ \psi_n(t) = \int_{-\pi}^{\pi} d\lambda c(\lambda) E_n(\lambda) \exp(-i(2\cos(\lambda) - 2)t). \]  

(2.9)
This provides all physical states, normalized by the inner product

\[
(\psi(t), \phi(t)) = \sum_{n=-\infty}^{\infty} \bar{\psi}_n(t_0) \phi_n(t_0)
\]

(2.10)

where the value chosen for \(t_0\) does not matter (since the Hamiltonian is self-adjoint) and the overbar denotes the complex conjugate. The coefficient function \(c(\lambda)\) can be determined from initial values for \(\psi_n(t)\), say at \(t = 0\):

\[
c(\lambda) = \frac{1}{2\pi} \sum_n \bar{E}_n(\lambda) \psi_n(0) = \frac{1}{2\pi} \sum_n \exp(-i\lambda n) \psi_n(0).
\]

(2.11)

Orthonormality used here follows from

\[
\int_{-\pi}^{\pi} \frac{dx}{2\pi} \bar{E}_m(x) E_n(x) = \int_{-\pi}^{\pi} \frac{dx}{2\pi} \exp(-ixm) \exp(ixn) = \delta_{mn}
\]

which is why we include the \(1/2\pi\) in \(c(\lambda)\).

The physical inner product will be used to compute expectation values and fluctuations of the volume and curvature as functions of time. Before showing examples, we now describe techniques of effective equations which provide analogous results, and then compare the solutions.

### 2.3 Effective Equations

Recall \(h \equiv \exp(iP)\), where \(P\) is related to the curvature term appearing in holonomies of loop quantum gravity and canonically conjugate to \(V\), which corresponds to the volume of the system. These variables allow us to formulate the difference equation as the Schrödinger equation for a Hamiltonian operator polynomial in \(\hat{h}\), since \(\hat{h}\) acts by shifting the eigenvalues of \(\hat{V}\) by unit steps. The relevant part of the spectrum of \(\hat{V}\) is thus discrete, and we identify its eigenvalues with the label \(n\) of the difference equation.

With these definitions, the system becomes exactly solvable (without quantum back-reaction) even with loop effects; we now define our Poisson algebra for the
classical variables

\[ \{V, h\} = ih \quad , \quad \{V, \bar{h}\} = -i\bar{h} \quad , \quad \{h, \bar{h}\} = 0 \]

with linear Hamiltonian \( H = h + \bar{h} - 2 \) generating evolution in \( t \). These linear structures can immediately be taken over to quantum theory, with the same relation for the commutator algebra of operators \( \hat{h} \) and \( \hat{V} \) and a Hamiltonian operator linear in \( \hat{h} \) and \( \hat{V} \). From the Ehrenfest theorem, we know the classical equations of motion agree exactly with the equations of motion \( \frac{d}{dt}\langle \hat{O} \rangle = \frac{\langle [\hat{O}, \hat{H}] \rangle}{i\hbar} \) for expectation values of the basic operators \( \hat{O} \in \{\hat{V}, \hat{h}, \hat{h}^\dagger\} \) in arbitrary quantum states. Classically, \( \frac{dV}{dt} = \{V, H\} = i(h - \bar{h}) \) with \( \frac{dh}{dt} = 0 = \frac{d\bar{h}}{dt} \) implies \( V(t) = 2t\text{Im}\bar{h} + V_0 \). The same behavior follows for expectation values. This linear time dependence is in agreement with the classical solutions expected of a free particle.

Similarly, equations of motion for quantum variables such as fluctuations can be solved. We present the solutions below, but more details and explicit calculations can be found in Appendix 2.5. We define the moments as

\[
G_{V \cdots V, h \cdots h, \bar{h} \cdots \bar{h}} = \langle (\hat{V} - \langle \hat{V} \rangle)^a(\hat{h} - \langle \hat{h} \rangle)^b(\hat{h}^\dagger - \langle \hat{h}^\dagger \rangle)^c \rangle_{\text{Symmetric}}
\]

where the subscript Symmetric denotes totally symmetric ordering of the operators (summing over all permutations, divided by \((a + b + c)!\)). Working up to second order \((a+b+c = 2)\), our moments are seen to be the usual variance and covariance

\[
G_{VV} = \langle \hat{V}^2 \rangle - \langle \hat{V} \rangle^2 \quad , \quad G_{Vh} = \frac{1}{2}\langle \hat{V}\hat{h} + \hat{h}\hat{V} \rangle - \langle \hat{V} \rangle \langle \hat{h} \rangle \quad , \quad \text{etc}
\]

and we find our equations of motion to be

\[
\frac{dG_{VV}}{dt} = 2i(G_{Vh} - G_{V\bar{h}}) \quad , \quad \frac{dG_{Vh}}{dt} = i(G_{hh} - G_{h\bar{h}}) \quad , \quad \frac{dG_{V\bar{h}}}{dt} = -i(G_{\bar{h}h} - G_{\bar{h}\bar{h}})
\]

as well as \( \frac{dG_{hh}}{dt} = \frac{dG_{h\bar{h}}}{dt} = \frac{dG_{\bar{h}h}}{dt} = 0 \). Solving the system of equations gives the solution as:

\[
\text{Re}G_{Vh}(t) = -t\text{Im}G_{hh} + \text{Re}G_{Vh}^{(0)}
\]
\[
\text{Im}G_{Vh}^V(t) = t(\text{Re}G_{hh}^h - G_{hh}^\bar{h}) + \text{Im}G_{Vh}^V_0 \tag{2.15}
\]
\[
G_{VV}^V(t) = -2t^2(\text{Re}G_{hh}^h - G_{hh}^\bar{h}) - 4t\text{Im}G_{Vh}^V_0 + G_{VV}^V_0. \tag{2.16}
\]

Reality conditions due to the complex \( h \) used can be imposed starting with \( \hat{h}\hat{h}^\dagger = \hat{h}^\dagger\hat{h} = 1 \) and taking expectation values: \( G_{hh}^{hh} = 1 - \langle \hat{h} \rangle \langle \bar{h} \rangle \).

The solutions for \( t \)-dependent moments directly show the amount of spreading out and squeezing. We will see direct examples of this from the numerical calculations in section 2.4. Even though we have been solving the equations for the loop quantized system, the behavior is in agreement with that of the free particle. In particular, nothing special happens at small \( V \) which, in the cosmology perspective, means the singularity is not resolved by a bounce.

### 2.4 Numerical Calculations

Now that we have found the general solutions to Eqn. (2.8) as well as the equations of motion, we would like to verify them and gain a better understanding of the moments by solving for an example numerically. We use the Poisson algebra to define our quantum operators \( \hat{V}, \hat{h} \) and \( \hat{\bar{h}} = \hat{h}^\dagger \) on a wavefunction \( \psi_n \) (for convenience, we drop the \(^\dagger\) notation) such that:

\[
V\psi_n = n\psi_n, \quad h\psi_n = \psi_{n+1}, \quad \bar{h}\psi_n = \psi_{n-1},
\]

which we can show to be consistent. For \( V \) and \( h \), simply note \([V, h] = i\{V, h\} = -h\) and check

\[
[V, h] \psi_n = (Vh - hV)\psi_n = [n - (n + 1)]\psi_{n+1} = -h\psi_n, \tag{2.17}
\]

where we used \( hV\psi_n = h(n\psi_n) = (n+1)\psi_{n+1} \). (More formally, we define the \( V \) and \( h \) operators on states \(|\Psi\rangle\) composed of eigenstates \(|n\rangle\) rather than wavefunctions such that

\[
V|\Psi\rangle = V \sum_n \psi_n|n\rangle = \sum_n n\psi_n|n\rangle \tag{2.18}
\]
\[
h|\Psi\rangle = h \sum_n \psi_n|n\rangle = \sum_n \psi_n|n - 1\rangle \tag{2.19}
\]
\[ [V, h] |\Psi\rangle = Vh \sum_n \psi_n |n\rangle - hV \sum_n \psi_n |n\rangle = \sum_n [(n - 1)\psi_n |n - 1\rangle - n\psi_n |n - 1\rangle] \]
\[ = \sum_n -\psi_n |n - 1\rangle = -h |\Psi\rangle \]  

(2.20)

The end result is the same, with \( h \) defined as decreasing the label \( n \) on the eigenstate \( |n\rangle \), in agreement with the increase of \( n \) for the wavefunction. We choose to use the wavefunction representation for convenience.

For our example, we must first choose an initial wavefunction, say at \( t = 0 \). Let it be a Gaussian centered around some \( n_0 \) with an initial ‘momentum’ such that the peak is moving leftward; we choose

\[ \psi_n = \exp(-(n - n_0)^2) \exp(ip(n - n_0)) \]

where \( n \) ranges over all integers and corresponds to the volume \( V \) (which acts like a position operator) as seen above, and \( p \) is a parameter which we relate to a classical momentum, but is not directly related to our basic variables used for the loop quantum cosmology analysis (its relationship with expectation values of \( h \) being more involved). For our calculations we set \( n_0 = 500 \) and \( p = 1 \) (see Figure 2.1 for a plot of the magnitude of the wavefunction at different times). This value of \( p \) is rather small, and so we expect wave functions to evolve mostly by spreading out. Nevertheless, we will see that expectation values in these spreading wave packets agree very well with the effective solutions.

We then solve for our coefficient \( c(\lambda) \) and the general time-dependent solution \( \psi_n(t) \)

\[ c(\lambda) = \sum_{n=-5+n_0}^{5+n_0} \frac{1}{2\pi} \exp(-i\lambda n - (n - n_0)^2 + ip(n - n_0)) \]  

(2.21)

\[ \psi_n(t) = \int_{-\pi}^{\pi} d\lambda c(\lambda) \exp(i\lambda n) \exp(-i(2\cos(\lambda) - 2)t) \]  

(2.22)

where numerically we approximate the infinite sum to only the 10 terms closest to the peak (values for \( |n - n_0| > 5 \) are of the order \( 10^{-16} \)) and the integration is done numerically in Mathematica 7. These wavefunctions are not normalized, but
Figure 2.1. The normalized magnitude of the wavefunction as it changes in time. Originally it is Gaussian and it spreads symmetrically with time. With the addition of a ‘momentum’ term, the peak occurs at decreasing $n$ as time increases. a) A graph of $|\psi_n(t)|$ vs $n$ for $n_0 = 500, p = 1$ at $t = 40$. b) The same wavefunction at $t = 80$.

by using Eqn. (2.10) we can find the coefficient which provides our normalization and include it in our calculations for the expectation values. Numerically we approximate the inner product as

$$
(\psi(t), \psi(t)) = \sum_{n=-2.5t-5+n_0}^{2.5t+5+n_0} .7866\bar{\psi}_n(t)\psi_n(t) = 1 \tag{2.23}
$$

where .7866 is our normalization constant and we choose to approximate our infinite sum as the $5t + 10$ closest $n$ values to our peak. From prior calculations, this was found to include all relevant terms as the wavefunction spreads in time and is approximately zero outside the range. Additionally, in the calculations we only use $t$ values which give integer $n$ in our summations. As usual, expectation values are taken as

$$
\langle A \rangle(t) = (\psi_n(t), A\psi_n(t)) = \sum .7866\bar{\psi}_n(t)A\psi_n(t)
$$

for an operator $A$, where the summation is that as defined above in Eqn. (2.23).

Now let us calculate the moments we expect to not change with time.

$$
G_{\hbar h} = \langle \hbar^2 \rangle - \langle \hbar \rangle^2 = \sum .7866\bar{\psi}_n(t)\psi_{n+2}(t) - \left[ \sum .7866\bar{\psi}_n(t)\psi_{n+1}(t) \right]^2 = .0882124 - 0.192748i \tag{2.24}
$$

$$
G_{\bar{\hbar}h} = \langle \bar{\hbar}^2 \rangle - \langle \bar{\hbar} \rangle^2 = \sum .7866\bar{\psi}_n(t)\psi_{n-2}(t) - \left[ \sum .7866\bar{\psi}_n(t)\psi_{n-1}(t) \right]^2 = .0882124 + 0.192748i \tag{2.25}
$$
\[ G^{hh} = \frac{1}{2}\langle h\bar{h} + \bar{h}h \rangle - \langle h \rangle \langle \bar{h} \rangle = 1 - \langle h \rangle \langle \bar{h} \rangle \]
\[ = 0.652691 - 5.93576 \times 10^{-12} i \]  
(2.26)

As anticipated, these values do not change with time and \( G^{hh*} = G^{\bar{h}h} \) where * denotes complex conjugation. We expected the latter since

\[ \langle h^2 \rangle = (\psi_n, \psi_{n+2}) = (\psi_{n+2}, \psi_n)^* = (\bar{h}^2)^* \]

and similarly \( \langle h \rangle = \langle \bar{h} \rangle^* \). Now let us calculate the other moments at a few sample times since they are time-dependent.

\[ G^{Vh}_{t=0} = 0.652691 - 5.93576 \times 10^{-12} i \]
\[ G^{Vh}_{t=50} = 9.63738 - 28.22391 i \]
\[ G^{Vh}_{t=100} = 19.27476 - 56.96850 i \]

Using the equations of motion from Eqn. (2.13) and the \( h\bar{h} \) moments found earlier, we can solve for \( G^{Vh}(t) \) and directly compare to the numerical results above.

\[ \frac{dG^{Vh}}{dt} = i(G^{hh} - G^{\bar{h}h}) = 0.192748 - 0.564478 i \]
\[ \Rightarrow G^{Vh}(t) = (0.192748 - 0.564478 i)t + G^{Vh}_0 \]  
(2.30)

Now we check whether \( G^{Vh}_t \), calculated from expectation values, is truly equal to \( G^{Vh}(t) \), the function from the theoretical equations of motion. Let \( G^{Vh}(0) \equiv 0 = G^{Vh}_0 \), then

\[ G^{Vh}(50) = 9.63738 - 28.22391 i, \]
\[ G^{Vh}(100) = 19.27476 - 56.44781 i \]

which gives us percent differences of .04% and 1.45% for the real and imaginary values (respectively) of \( t = 50 \), and .02% and .92% for \( t = 100 \). Though the er-
Figure 2.2. The real and imaginary portions of $G_{Vh}(t)$ plotted with respect to $t$. Overlaid are the expected results from the equation of motion given by Eqn. (2.30). a) The real portion of the $G_{Vh}$ moment at various times shown is in good agreement with the effective equation. b) The imaginary portion of the $G_{Vh}$ moment is shown to be in good agreement as well.

Error is small, it is unexpected. From the theory, we have made no approximations in solving for our equations of motion (see Appendix 2.5), and thus they should be exactly correct. This error is due to the approximations made in the finite summations which were chosen to decrease the computing time necessary for the calculations. Originally, these approximations left the calculated values invariant to $10^{-15}$. However, they do not seem to hold for the more involved calculations of the moments with time evolution. To confirm, these calculations were redone with an increased range of summation to verify that the error decreases. Specifically the sum was extended from the $5t + 10$ closest $n$ values to the peak to the $6t + 50$ closest. Doing so, we find the percent difference between numerical expectation values and our theoretical function of $G_{Vh}$ at $t = 50$ to be $4.36 \times 10^{-7}\%$ and $5.36 \times 10^{-7}\%$ for real and imaginary respectively. Though it is computationally prohibitive to solve to this accuracy, we see indeed that the error is drastically reduced. Thus, the difference in values comes from the numerical approximations, while our effective equations are exact solutions; by using greater numerical precision in our calculations, we can achieve a correspondingly small percent difference. However, for computational expediency, we continue with our original range of sums for the rest of this chapter.

Similarly, we can solve for $G_{V\bar{h}}$. However, since $G_{hh}^* = G_{\bar{h}\bar{h}}$ it is easy to see that $G_{V\bar{h}} = G_{Vh}^*$ as well. This is seen in the calculations, but since the absolute
values are the same, there is no difference in the analysis or the errors and so it is not shown here. Let us now calculate the only remaining moment $G^{VV}$ at varying times.

\[ G_{t=0}^{VV} = \langle V^2 \rangle(t) - \langle V \rangle(t)^2 = \sum \frac{1}{2} \psi_n(t) \frac{d}{dt} \psi_n(t) + \left[ \sum \frac{1}{2} \psi_n(t) \bar{\psi}_n(t) \right]^2 \]

\[ = 250000. - 500.2 = 0 \quad (2.31) \]

\[ G_{t=50}^{VV} = 2822.60584 + 3.49565 \times 10^{-12} \quad (2.32) \]

\[ G_{t=100}^{VV} = 11289.77781 - 2.89611 \times 10^{-13} \quad (2.33) \]

As before, we will now compare with the values given by the equations of motion we have derived, denoted $G^{VV}(t)$ (see Figure 2.3 for a graphical comparison).

\[ \frac{dG^{VV}(t)}{dt} = 2i(G^{Vh}(t) - G^{Vh}(t)) = 2i(G^{V} - G^{Vh}) = -4\text{Im}G^{Vh}(t) \]

\[ = -4(-.564478)t = 2.25791t \quad (2.34) \]

\[ \Rightarrow G^{VV}(t) = 1.12896t^2 + G_0^{VV} \quad (2.35) \]

With $G^{VV}(0) \equiv G_0^{VV} = 0$, we can use the function above to find

\[ G^{VV}(50) = 2822.60569 + 6.69741 \times 10^{-7}i \]

\[ G^{VV}(100) = 11289.77771 + 2.67878 \times 10^{-6}i \]

which gives us percent differences on the order of $5 \times 10^{-8}$ or better for the real values at both times, and though the percent difference for the imaginary portion is sizable, the actual values are close to zero for both. There are a couple of reasons why this error should be much lower than our previous: for one, the numbers used here are larger, so for the same absolute difference the percent difference is smaller. Secondly, in this calculation, we used the $G^{Vh}(t)$ functions to integrate rather than a contrived method using the original expectation values of $G^{Vh}$. This suggests that there may be some error only in the finite summations involving the time evolution of the $G^{Vh}$ whereas this function relies only on static values of $G^{hh}, G^{\bar{h}h},$ and $G^{\bar{h}\bar{h}}$.

Our final calculation will be to check the general solution from Eqn. (2.16) with the numerical results. In our example, the initial values for moments involving $V$
Both $V(t)$ and $G^{VV}(t)$ are in good agreement with their theoretical functions (given by their commutators with the Hamiltonian). a) The expectation value of volume calculated at various times plotted with the equation of motion for the volume overlaid. b) $G^{VV}$ moment calculated by the expectation values at various times plotted with the expected function, Eqn. (2.35), from the equations of motion.

were 0, which leaves our general solution as:

\[
\text{Re} G^{Vh}(t) = -t \text{Im} G^{hh} = t (1.192748) \quad (2.36)
\]

\[
\text{Im} G^{Vh}(t) = t (\text{Re} G^{hh} - G^{hh}) = t (-0.564478 - 2.62634 \times 10^{-13}i) \quad (2.37)
\]

\[
G^{VV}(t) = -2t^2 (\text{Re} G^{hh} - G^{hh}) = t^2 (1.12896 + 5.25268 \times 10^{-13}i) \quad (2.38)
\]

which is exactly what we found before for our functions from Eqns. (2.30) and (2.35) (where we neglect the $10^{-13}$ imaginary term). This should not be surprising as we found this solution using the equations of motion; since the values are the same, the errors too are the same as before and need no further analysis.

In the following chapter, we alter the original difference equation, Eqn. (2.8), by adding a non-linear term. The motivation is similar to the Gross-Pitaevskii equation, which describes interactions of weakly-interacting bosons; this then gives us some form of discrete non-linear Schrödinger equation\cite{18}, though ours will not describe bosonic point particles with delta-function interactions. With the addition of a non-linear term in the Hamiltonian, our equations of motion are changed and we can compare with the linear case.
2.5 Appendix - Quantum Equations of Motion

First, we derive the equations of motion. Here all classical values are treated as expectation values (for example \( h = \langle h \rangle \)) and we make use of

\[ \{ \langle A \rangle, \langle B \rangle \} = \frac{1}{i} \langle [A, B] \rangle \]  

(2.39)

where the commutator is given by \([ , ] = i \{ , \} \) (and we have set \( \hbar = 1 \)). Using this, the classical algebra, and the definition of the moments (Eqn. 2.12) we see:

\[
\frac{dG^{VV}}{dt} = \{ G^{VV}, H \} = \{ \langle V^2 \rangle - \langle V \rangle^2, h + \bar{h} - 2 \} = \frac{1}{i} \langle [V^2, h + \bar{h}] \rangle - \{ \langle V \rangle^2, \langle h + \bar{h} \rangle \} = \frac{1}{i} \langle V[V, h + \bar{h}] + [V, h + \bar{h}] V \rangle - (V \{ V, h + \bar{h} \} + \{ V, h + \bar{h} \} V) \\
= \frac{1}{i} \langle Vi[V, h + \bar{h}] + i[V, h + \bar{h}] V \rangle - 2i \left( \langle V \rangle \langle h \rangle - \langle V \rangle \langle \bar{h} \rangle \right) \\
= \frac{i}{2} \left( \langle Vh + \bar{h}V \rangle - \langle V\bar{h} + hV \rangle \right) - 2i \left( \langle V \rangle \langle h \rangle - \langle V \rangle \langle \bar{h} \rangle \right) = 2i(G^{Vh} - G^Vh)
\]

where on the last line we separate the \( Vh \) and \( V\bar{h} \) terms into the definitions of \( G^{Vh} \) and \( G^Vh \) respectively. Additionally, we see:

\[
\frac{dG^{Vh}}{dt} = \{ G^{Vh}, H \} = \{ \frac{1}{2} \langle Vh + hV \rangle - Vh, h + \bar{h} \} \\
= \frac{1}{2i} \langle [Vh + hV, h + \bar{h}] \rangle - \{ Vh, h + \bar{h} \} = \frac{1}{2i} \langle \{ V, h + \bar{h} \} h + h\{ V, h + \bar{h} \} - i(h - \bar{h})h \rangle \\
= \frac{1}{2} (i(h - \bar{h})h + ih(h - \bar{h}) - i\langle h \rangle^2 - \langle h \rangle \langle \bar{h} \rangle) \\
= i \left( \frac{1}{2} \langle \bar{h}^2 - 2h - h\bar{h} - \langle h \rangle^2 - \langle h \rangle \langle \bar{h} \rangle \rangle \right) \\
= i(\langle h^2 \rangle - \langle h \rangle^2 - \frac{1}{2} \langle \bar{h}^2 + h\bar{h} \rangle - \langle h \rangle \langle \bar{h} \rangle) = i(G^{hh} - G^{\bar{h}h})
\]

(2.40)

where in the third line we neglect the \( \{ h, h + \bar{h} \} \) terms as they are just zero, and on the final line we substitute using the definitions of the moments. Similarly, we find:

\[
\frac{dG^{V\bar{h}}}{dt} = \{ G^{V\bar{h}}, H \} = \{ \frac{1}{2} \langle V\bar{h} + \bar{h}V \rangle - V\bar{h}, h + \bar{h} \}
\]
\[ \frac{1}{2i} \langle [V\bar{h} + \bar{h}V, h + \bar{h}] \rangle - \{V\bar{h}, h + \bar{h}\} \]
\[ = \frac{1}{2} \langle \{V, h + \bar{h}\}\bar{h} + \bar{h}\{V, h + \bar{h}\} \rangle - i(h - \bar{h})\bar{h} \]
\[ = \frac{1}{2} \langle i(h - \bar{h})\bar{h} + i\bar{h}(h - \bar{h}) \rangle + i(\langle \bar{h}\rangle^2 - \langle h\rangle\langle \bar{h}\rangle) \]
\[ = -i\left(\frac{1}{2} \langle \bar{h}^2 - h\bar{h} - \bar{h}h \rangle - \langle \bar{h}^2 \rangle - \langle h\rangle\langle \bar{h}\rangle \right) \]
\[ = -i(\langle \bar{h}^2 \rangle - \langle h\rangle^2 - \frac{1}{2} \langle h\bar{h} + \bar{h}h \rangle - \langle h\rangle\langle \bar{h}\rangle) = -i(G^{hh} - G^{\bar{h}h}) \] (2.41)

Also, since \(G^{hh}, G^{\bar{h}h}\) and \(G^{\bar{h}h}\) moments only include \(h\) and \(\bar{h}\), their Poisson brackets with the Hamiltonian \(H\) are simply zero, and we have constant equations of motion for them as stated in Eqn. (2.13).

Finally, we will also derive the solution to the equations of motion. As mentioned above, the “\(\bar{h}\)” moments are constant, and thus would be the best choice to solve everything in terms of. From the equations of motion, we can solve for \(G^{Vh}(t)\) and \(G^{\bar{V}h}\) as such. Since they are complex conjugates of one another (as discussed in Section 2.4), we can use their sum and difference to find their real and imaginary parts. This gives:

\[ G^{Vh}(t) + G^{\bar{V}h}(t) = G^{Vh} + G^{Vh*} = 2\text{Re}G^{Vh}(t) \] (2.42)
\[ G^{Vh}(t) - G^{\bar{V}h}(t) = G^{Vh} - G^{Vh*} = 2i\text{Im}G^{Vh}(t). \] (2.43)

Rearranging and taking the derivatives we arrive at

\[ \frac{d}{dt} \text{Re}G^{Vh}(t) = \frac{d}{dt} i(G^{Vh}(t) + G^{Vh*}(t)) = \frac{i}{2} \left( G^{hh} - G^{\bar{h}h} - G^{\bar{h}h} + G^{\bar{h}h} \right) \]
\[ = \frac{1}{2} - 2\text{Im}G^{hh} = -\text{Im}G^{hh} \] (2.44)
\[ \Rightarrow \text{Re}G^{Vh}(t) = -t\text{Im}G^{hh} + \text{Re}G^{Vh}_0 \] (2.45)

and

\[ \frac{d}{dt} \text{Im}G^{Vh}(t) = \frac{d}{2it} (G^{Vh}(t) - G^{Vh*}(t)) = \frac{1}{2} \left( G^{hh} - G^{\bar{h}h} + G^{\bar{h}h} - G^{\bar{h}h} \right) \]
\[ = \frac{1}{2} (2\text{Re}G^{hh} - 2G^{\bar{h}h}) = \text{Re}G^{hh} - G^{\bar{h}h} \] (2.46)
\[ \Rightarrow \text{Im}G^{Vh}(t) = t(\text{Re}G^{hh} - G^{\bar{h}h}) + \text{Im}G^{Vh}_0 \] (2.47)
where on the second line of both calculations we used the fact that $G^{hh} = G^{hh*}$, and on the last line we show the integrations that give Eqns. (2.14) and (2.15). Additionally, $dG^{VV}(t)/dt$ depends on the difference of $G^{Vh}$ and $G^{Vh}$ which we just solved for. Thus, as shown before,

$$\frac{dG^{VV}}{dt} = 2i(G^{Vh} - G^{Vh}) = -4\text{Im}G^{Vh}(t) = -4t(\text{Re}G^{hh} - G^{hh}) - 4\text{Im}G^{Vh}_0$$

$$\Rightarrow G^{VV}(t) = -2t^2(\text{Re}G^{hh} - G^{hh}) - 4t\text{Im}G^{Vh}_0 + G^{VV}_0$$

which is exactly our solution found before.
Chapter 3

A Non-linear Extension in Quantum Cosmology

As hinted at previously, we will now extend our model from Chapter 2 to include a non-linear (and non-local) term. We use the same classical variables and Poisson algebra, but with the addition to the difference equation (and thus our Hamiltonian constraint) we find new dynamics and interpret the scheme as a new method for studying inhomogeneities in quantum cosmology.

3.1 Introduction

In this chapter, we look to the possibility of a non-linear term in our Hamiltonian constraint as an analog of the mathematical description of Bose-Einstein condensation. In such models, a many-body state $\Psi$ composes the dynamics of the 1-particle wave functions $\psi$ that all particles share identically. In simplest terms, we investigate the possibility that discrete geometric patches governed by individual isotropic patch wave functions (themselves single minisuperspace solutions of Eqn. (2.8)) can similarly be used to describe inhomogeneity in a product state wave function of a cosmological minisuperspace. This matches well with the common method for dealing with inhomogeneities by viewing quantum space at a moment in time as made up of small homogeneous patches. Each individual patch satisfies the quantum geometry of homogeneity and isotropy, but “interactions” between patches is governed by the quantized gravitational Hamiltonian (see [19].
We begin with the methods of Bose-Einstein condensation and write the many-body Hamiltonian of a condensate (assuming pointlike particle interactions with delta-function potentials of strength $\alpha$) as

$$\hat{H} = \sum_{i=1}^{n} \left( \frac{1}{2m} \hat{p}_i^2 + V(\hat{x}_i) \right) + \frac{1}{2} \alpha \sum_{i \neq j} \delta(\hat{x}_i - \hat{x}_j) \quad (3.1)$$

for $n$ particles of mass $m$ in individual potentials $V(x_i)$. With a product state $\Psi(x_1, x_2, \ldots) = \psi(x_1)\psi(x_2)\cdots$ for the condensate, we compute the expectation value of the Hamiltonian as

$$\langle \hat{H} \rangle_{\psi} = n\langle \hat{p}^2/2m + V(\hat{x}) \rangle_{\psi} + \frac{1}{2} n(n-1) \alpha \int d^3x |\psi(x)|^4. \quad (3.2)$$

The first term just adds up the 1-particle expectation values computed for the wave function $\psi$, but the second term is not equal to a 1-particle expectation value. We formally interpret it as the expectation value of a “potential” $|\psi(x)|^2$ depending on the wave function. Accordingly, the 1-particle dynamics and energy spectra are governed by a non-linear Schrödinger equation, the Gross–Pitaevskii equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V(x)\psi + \frac{1}{2} (n-1) \alpha |\psi(x)|^2 \psi. \quad (3.3)$$

(For a full and rigorous derivation, see [20, 21].) We therefore see that a mechanism exists to map interacting many-body dynamics of a condensate wave function to non-linear 1-particle dynamics.

### 3.2 Non-linear Dynamics in Quantum Cosmology

Including inhomogeneities in quantum cosmology is a difficult prospect, with varying open problems for the common approaches. As such, we do not claim here a detailed model which answers those problems, but rather a new option to study such deviations from homogeneity qualitatively (which perhaps raises further ques-
tions for future avenues of research). Given that we wish to follow the non-linear analogy of Eqn. (3.3) we still must make a few changes in keeping with loop quantum cosmology. Primarily, as we saw in the previous chapter, holonomies lead to shift operators and thus to a difference equation. As such, we would expect to deal with a discretized version of the dynamics (in addition to the discrete collection of patches in our model); one example is the discrete non-linear Schrödinger equation

\[ i\hbar \frac{\partial \psi_n}{\partial t} = \frac{1}{2} (\psi_{n+1} - 2|\psi_n|^2\psi_n + \psi_{n-1}). \]  \hspace{1cm} (3.4)

Additionally, we would not expect our interaction potential to follow the Bose-Einstein condensation potential; for particles in a condensate, a delta function of the distance between particles is a good approximation for nearly pointlike interactions (which can be smeared out to more-complicated functions for realistic systems). However, the interaction potential we obtain in cosmology, by expanding and discretizing the gravitational Hamiltonian constraint, is a quadratic polynomial in the distances in minisuperspace\[19\]. Thus while our resulting wave equation will be non-linear, as in the presence of any form of interaction, it will be more complicated than in the Gross–Pitaevskii equation.

Using the same starting point as in Bose–Einstein condensation, the key step is to evaluate the expectation value of the interaction Hamiltonian in a product state. To illustrate the main consequence, we consider just two variables \( V_1 \) and \( V_2 \) interacting with each other via a potential \( W_{int}(V_1, V_2) = \alpha(V_1 - V_2)^2/V^2 \). We divide by the total volume squared, treated as an external but time-dependent parameter, in order to have the correct scaling behavior of the Hamiltonian under a change of the spatial region. The expectation value of the quantized \( W_{int} \) then produces a term

\[
\langle \hat{W}_{int} \rangle_{\psi} = \frac{\alpha}{V^2} \int dV_1 dV_2 |\psi(V_1)|^2 |\psi(V_2)|^2 (V_1 - V_2)^2 \]  \hspace{1cm} (3.5)

\[
= \frac{\alpha}{V^2} \int dV_1 |\psi(V_1)|^2 \int d\delta V |\psi(V_1 + \delta V)|^2 (\delta V)^2 \]  \hspace{1cm} (3.6)

where we introduce \( \delta V \equiv V_2 - V_1 \). We can perform the second integration independently of the first over \( V_1 \). It depends on the wave function, but if we assume that \( \psi \) is sharply peaked around the expectation value \( \langle V_1 \rangle \), the dominant contribution
to \langle \hat{W}_{\text{int}} \rangle \psi \) comes from values of \( V_1 \) for which the second integration

\[
\int d\delta V \vert \psi (\langle V \rangle + \delta V) \vert^2 (\delta V)^2 = (\Delta V)^2
\]  

(3.7)
equals the quantum fluctuation of \( V \) in the state \( \psi (V) \). Instead of a non-linearity potential depending on \( \psi (V) \) or \( \psi_n \) as in Eqn. (3.4), we have a non-linearity potential that depends on the wave function via moments such as \( G^{VV} \equiv (\Delta V)^2 \). For instance, following the preceding arguments and noting that the minisuperspace \( V \) is quantized to a discrete parameter \( n \), we need to consider an equation of the form

\[
i\hbar \frac{\partial \psi_n}{\partial t} = \psi_{n+1} - 2 \left( 1 - \frac{1}{2} \alpha \frac{(G^{VV})_\psi}{V^2} \right) \psi_n + \psi_{n-1}.
\]  

(3.8)
We note that Eqn. (3.8) is not only non-linear but also non-local: the coefficient \( (G^{VV})_\psi = \sum_n (\hat{V} - \langle \hat{V} \rangle )^2 \vert \psi_n \vert^2 \) depends on all values of \( \psi_n \). However, as the non-local term is a second order moment of the wave function, we can use effective equations of the moments instead of \( \psi_n \) itself to reformulate the dynamics in variables that remove the issue of non-locality. Moreover, the equation as written is meaningful only for \( V \neq 0 \). At \( V = 0 \), the volume vanishes and we encounter a cosmological singularity. By inverse-triad corrections \([22]\), loop quantum cosmology resolves this singularity in such a way that \( 1/V \) is replaced by a bounded function. For simplicity, we will not discuss these terms here and instead focus on evolution at large \( V \).

### 3.3 Non-linear Effective Equations

Recall from Chapter \([2]\) that we have three basic operators, a multiplication operator \( \hat{V} \psi_n = n \psi_n \) (the volume operator) and two shift operators \( \hat{h} \psi_n = \psi_{n+1} \) and \( \hat{h}^\dagger \psi_n = \psi_{n-1} \). In terms of canonical variables \((V, P)\), we identify shift operators as quantizations of \( \hbar = \exp(iP) \). The commutators

\[
[\hat{V}, \hat{h}] = -\hbar \hat{h} \quad , \quad [\hat{V}, \hat{h}^\dagger] = \hbar \hat{h}^\dagger \quad , \quad [\hat{h}, \hat{h}^\dagger] = 0
\]  

(3.9)
then define the basic algebra of our loop-quantized theory, and correspondingly the Poisson brackets of expectation values and moments of \( V, h, \) and \( \bar{h} \)

\[
\{ V, h \} = \imath h \quad , \quad \{ V, \bar{h} \} = -\imath \bar{h} \quad , \quad \{ h, \bar{h} \} = 0.
\]

Moreover, since we introduced complex variables, the reality condition \( \hat{h} \hat{h}^\dagger = 1 \) as well as analogs for the moments (such as \( G^{hh} = 1 - \langle \hat{h} \rangle \langle \hat{h}^\dagger \rangle \)) must be satisfied. As before we also use the simplifying notation of \( \langle \hat{V} \rangle = V, \langle \hat{h} \rangle = h, \) and \( \langle \hat{h}^\dagger \rangle = \bar{h} \).

Following from Eqn. (3.8) we extend the quantum Hamiltonian (where we have expanded \( \langle \hat{V} - 2 \rangle \) in moments following the expansion procedure of Eqn. (1.14))

\[
H_Q = h + \bar{h} - (2 - \alpha A \langle \hat{V}^2 \rangle)
\]

\[
= h + \bar{h} - 2 + \alpha A \langle V^{-2} + 3V^{-4}G^{VV} - 4V^{-5}G^{VVV} + \cdots \rangle \quad (3.10)
\]

with the non-local coefficient \( A \) treated for now as an external parameter (we substitute \( A = G^{VV} \) in the following equations of motion after evolution by Poisson bracket with \( H_Q \)). Previously, our linear Hamiltonian did not contain factors of the inverse of \( V \) and as such nothing in particular occurred as \( V \to 0 \). In our non-linear Hamiltonian we do have powers of the inverse of \( V \), which is ill-defined at \( V = 0 \), and thus modifications due to inverse-triad corrections in loop quantum cosmology should be used at small \( V \). As mentioned above though, we neglect such corrections for simplicity and focus only on the large \( V \) regime. From Poisson brackets with \( H_Q \) we then arrive at our equations of motion:

\[
\dot{V} = i(h - \bar{h}) \quad (3.11)
\]

\[
\dot{h} = 2\alpha i h \frac{G^{VV}}{V^3} + 12\alpha i h \left( \frac{G^{VV}}{V^5} \right)^2 - 6\alpha i h \frac{G^{VV} G^{Vh}}{V^4} + \cdots \quad (3.12)
\]

\[
\dot{\bar{h}} = -2\alpha i \bar{h} \frac{G^{VV}}{V^3} - 12\alpha i \bar{h} \left( \frac{G^{VV}}{V^5} \right)^2 + 6\alpha i h \frac{G^{VV} G^{Vh}}{V^4} + \cdots \quad (3.13)
\]

\[
\frac{dG^{VV}}{dt} = 2i(G^{Vh} - G^{Vh}) \quad (3.14)
\]

\[
\frac{dG^{Vh}}{dt} = i(G^{hh} - G^{hh}) + 2\alpha i G^{Vh} \frac{G^{VV}}{V^3} + 12\alpha i \left( \frac{G^{VV}}{V^5} \right)^2 G^{Vh} +

6\alpha i \frac{G^{VV}}{V^4} \left( G^{Vh} + h G^{VV} - \frac{1}{6} \bar{h}^2 V \right) + \cdots \quad (3.15)
\]
\[ \frac{dG_{hh}^{\ddagger}}{dt} = -4\alpha i \frac{G_{hh}^{\ddagger}}{V^3} - 24\alpha i \frac{(G^{VV})^2 G_{hh}^{\ddagger}}{V^5} + 12\alpha i \frac{G^{VV} G_{Vh}^{\ddagger} + V G_{Vh}^{\ddagger}}{V^4} + \cdots \] (3.16)

Instead of \( \ddot{V} = 0 \) as in the linear case, we can combine the first two equations to obtain

\[
\ddot{V} = i(\dot{h} - \dot{h}) = -2\alpha (h + \bar{h}) \frac{G^{VV}}{V^3} - 12\alpha (h + \bar{h}) \frac{(G^{VV})^2}{V^5} + 6\alpha G^{VV} \frac{G_{Vh}^{\ddagger} + G_{Vh}^{\ddagger}}{V^4} + \cdots . \] (3.17)

Though these equations are rather complicated, we make a direct comparison to the linear version in the hope that some insight can be gleaned. Specifically, the major difference is that \( h \) and \( \bar{h} \) are no longer constants, and thus where before we had \( \dot{V} \) constant based on some "momentum" we chose numerically (corresponding to some value of \( P \)), that momentum can now change in time as seen in \( \ddot{V} \). However, it is not clear how \( P \) changes in relational time as \( h = e^{iP} \neq \langle \hat{h} \rangle \). For instance, in the numerics of Chapter 2, we had \( \langle \hat{h} \rangle = .32 + .5i \) which is clearly not of magnitude 1. Additionally, one might wish to make a simplification of the equations based on the inverse powers of \( V \) in the large \( V \) regime. However, using numerical values from the linear case (simply as initial conditions) with relative quantum fluctuations small, \( G^{VV}/V^2 \sim .01 \), shows that while terms with higher inverse orders of \( V \) are smaller, they are not negligible. For instance, in \( \dot{h} \) the real part of the leading term \( 2\alpha h G^{VV}/V^3 \) is only a factor of 2 larger than the real part of the two other terms shown. Using such numerics, we also see that such changes are small – on the order of \( \alpha 10^{-5} \), where \( \alpha = 8\pi G/3\gamma \sim 10^{-9} \) (using \( \gamma \approx .2375 \) as suggested by the black hole entropy calculations) 23 – but without solving the system of equations for the full dynamics, it is difficult to know whether such contributions could add up over long cosmological times (for instance, if \( \dot{h} \) is small, \( h \) may hold a particular value for long times which could play a role in Eqn. (3.19)).

Additionally, we note that this system of equations allows for many possibilities for the dynamics of \( V \). From reality conditions we can write

\[
\dot{V} = i(h - \bar{h}) = -2\text{Im}[h] \\
\ddot{V} = i(h - \bar{h}) = -2\text{Im}[h] \] (3.18)
\[
= 4\alpha \frac{G^{VV}}{V^3} \left( -\text{Re}[h] - 6\text{Re}[h] \frac{G^{VV}}{V^2} + 3 \frac{\text{Re}[G^{Vh}]}{V} \right). \tag{3.19}
\]

Though we do not have an exact understanding of the dynamics of \( h \), we do know that it can take positive and negative values for its real and imaginary components independently. Thus with the appropriate initial conditions, we have the choice to choose regimes in which \( \dot{V} \) and \( \ddot{V} \) are also independently positive and negative, leading to a variety of cosmological models (at least initially). Further analysis with regards to exploring the parameter space and methods for solving the system of effective equations (even approximately) is necessary before any definitive statement can be made, but the non-linear addition to our quantum Hamiltonian seems to allow for a number of possibilities, including a cosmological constant term.

### 3.4 Conclusion

We have introduced a new model for inhomogeneous quantum cosmology, aiming to capture essential features of the interacting dynamics of different parts of quantum space. The processes we describe therefore provide the dynamics of structure formation at a fundamental level. Using several approximations, justified when inhomogeneity is sufficiently small\[19\], and importing ideas of condensed-matter physics, we have been able to map the complicated many-body dynamics to a non-linear minisuperspace equation.\[1\]

In our cosmological interpretation, we propose “interactions” between discrete patches of our quantum geometry in the superspace of our model, not physical space. Rather than local interactions dependent on spatial distance as seen in the Bose-Einstein condensate, we instead have a non-local term dependent on deviations from the spatial average of all \( \psi_n \). Such non-linear wave equations can thus provide a new view of effects which occur from averaged inhomogeneity. Though further analysis is necessary, the possibility exists that small corrections can sum over large periods of time; that is, even with small quantum fluctuations and large volumes we may have relatively substantial effects. In that case, higher order mo-

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\[1\] Other versions of non-linear quantum cosmology have been proposed \[24, 25, 26, 27\], motivated by non-commutativity and information-theoretic arguments.
ments of Eqn. (2.12) may also become relevant, as their small contributions may similarly add (or subtract). Additionally, such a model could be further extended to include patches which contain anisotropies or other quantum corrections in the future.
Loop quantum cosmology implies quantum corrections to the classical equations of motion which can lead to significant departures from the classical trajectory, especially at high curvature near the big-bang singularity. Corrections could in principle be significant even in certain low-curvature regimes, provided that they add up during long cosmic evolution. The analysis of such terms is therefore an important problem to make sure that the theory shows acceptable semiclassical behavior. This chapter presents a general search for terms of this type as corrections in effective equations for a $k = 0$ isotropic quantum cosmological model with a free, massless scalar field. Specifically, the question of whether such models can show a collapse by quantum effects is studied.

### 4.1 Introduction

Effective equations are useful tools to analyze quantum effects by suitable correction terms to classical equations, for instance by using equations of motion for expectation values of operators of interest. Just like expectation values, effective equations that describe their evolution depend on the states or class of states whose
evolution is being approximated. The low-energy effective action often used in particle physics, as perhaps the best-known example, describes quantum corrections for states near the vacuum of the interacting theory. For other states, different quantum corrections arise. Effective equations are state-independent only in rare cases such as free theories or the harmonic oscillator, when no quantum back-reaction and therefore no dynamical quantum corrections occur. Such systems also exist in (loop) quantum cosmology, where they play a similar role in setting up perturbation theories in case interactions are present.

As an example for potentially large perturbative effects in quantum cosmology, it has been suggested that classically ever-expanding models could eventually collapse due to quantum corrections [28]. Such a feature is unexpected because the more the universe expands, the more classical it is supposed to become, with quantum corrections playing smaller and smaller roles. However, with long evolution involved, small quantum effects could potentially add up and eventually drive the universe into collapse even in classical regimes. As shown in [28], the outcome depends on what semiclassical quantum state is realized and how its fluctuations of different variables change in time. To test the proposed scenario, one needs information about dynamical semiclassical states over long evolution times, an issue that requires good control on quantum evolution and, when addressed with effective equations, a systematic scheme of going beyond the leading classical order.

The effective equations used in [28] were an extension of the classical limit obtained from methods derived in [29]. The scheme of going beyond classical order, however, is not entirely systematic because assumptions about the evolving state must be made; they cannot be self-consistently derived within the scheme. For this reason, the collapse suggested remained a possibility but could not be demonstrated firmly. (Note that going to the other extreme, the high densities of the big bang, requires even better control of evolving states no longer required to be semiclassical. For this reason, the high-curvature regime of quantum cosmology remains poorly understood.) In this article, we use a systematic perturbative framework to derive information about effective equations and dynamical semiclassical states and see whether the suggestions of [28] can be realized.

Although the low-curvature regime can be formulated in Wheeler–DeWitt
quantum cosmology \cite{30} and does not require additional corrections and modifications from loop quantum cosmology \cite{1 31}, we will use the latter for a general discussion. This choice also helps us keep our notation close to that of \cite{28}. In brief terms, loop quantum cosmology of spatially flat isotropic models proceeds by formulating the Friedmann equation in canonical variables $c = \gamma \dot{a}$ and $|p| = a^2$,

$$- \frac{3}{8\pi G \gamma^2} c^2 \sqrt{|p|} + H_{\text{matter}} = 0, \quad (4.1)$$

and modifying it by using a periodic function of $c$ (or, as seen in more detail below, $\delta(p)c$) instead of $c$. In this way, one mimics the fact that the full theory of loop quantum gravity provides operators only for holonomies of the connection corresponding to $c$, not for the connection itself. Such a modification to

$$- \frac{3}{8\pi G \gamma^2} \frac{\sin(\delta(p)c)^2}{\delta(p)^2} \sqrt{|p|} + H_{\text{matter}} = 0 \quad (4.2)$$

is most relevant in regimes where $\delta(p)c$ is large (at large curvature). If one expands the sine function, terms beyond the classical ones may be interpreted as contributions to higher-curvature terms. (They are not complete as higher-curvature corrections, however, because higher time derivatives have not been included. The latter would result from quantum back-reaction \cite{32}.) From equations of motion generated by Eqn. (4.2) used as a Hamiltonian constraint in proper-time gauge and assuming $\delta(p) = \delta/\sqrt{p}$, with $\delta$ a constant, one finds that the modified Friedmann equation in terms of $\ddot{a}$ or $H = \dot{a}/a$ takes the form

$$H^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{\rho}{\rho_{\text{crit}}}\right) \quad (4.3)$$

where $G$ is Newton’s constant, $\rho$ is the energy density, and $\rho_{\text{crit}} = 3/8\pi G \delta^2$ \cite{2, 33 34}. This form makes it especially clear that the modification is relevant only at large density.

Loop quantum cosmology represents the modified constraint as a difference equation \cite{35}, obtained from shift operators $\sin(\delta(p)c)$ acting on a wave function in the triad representation. From this difference equation or the underlying Hamiltonian-constraint operator, Eqn. (4.2) can be reproduced as the “clas-
sical” limit by computing the expectation value in Gaussian states and ignoring all fluctuation-dependent terms [29]. From the point of view of quantum dynamics, the result is the classical limit because fluctuations and their back-reaction on expectation values are ignored. Still, the result does not agree with the classical Eqn. (4.1) because loop quantum gravity implies quantum-geometry corrections in addition to the usual dynamical quantum corrections, which give rise to the modification of Eqn. (4.2) from (4.1). Only a combined classical and low-curvature limit brings loop quantum cosmology fully back to the classical Friedmann equation of Eqn. (4.1). (See also [36, 37].)

The modification in Eqn. (4.2) alone does not affect late-time evolution much. However, quantum evolution implies additional corrections of the usual type, which in canonical models can be computed by quantum back-reaction of fluctuations and higher moments of a state on expectation values. The leading-order expectation value that gives rise to the classical limit must be expanded systematically to include these terms. Such terms were computed in [28], with potentially significant low-curvature implications. In addition to the usual quantum corrections on small scales, higher-order corrections may allow for the possibility of an expanding universe to collapse at large scale when the energy density is suitably small, even in the absence of spatial curvature — a significant departure from classical theory. In that paper, lacking a derivation of properties of dynamical semiclassical states, a suitable coherent state for large volume and late times had to be chosen, as in most cases assumed to be Gaussian. With this choice, the classical-limit scheme of [29], developed further in [38], was used to calculate fluctuations and expectation values. The end result is a modified Friedmann equation

\[ H^2 = \frac{8\pi G}{3} \rho \left[ 1 - \frac{\rho}{\rho_{\text{crit}}} \left( 1 + \frac{\ell_P^6}{2\nu^2 \epsilon^2} \right) + \frac{\rho_P^6}{2\nu^2 \epsilon^2} - 2\epsilon^2 \frac{\rho_{\text{crit}}}{\rho} \right] \] (4.4)

where \( \ell_P = \sqrt{G\hbar} \) is the Planck length, \( \nu \sim a^{3/2} \) is the volume of the universe, \( \rho \) is the energy density, \( \rho_{\text{crit}} \) is the collection of constants mentioned above, and, most importantly, \( \epsilon \) is the quantum fluctuation of curvature. The equation has the correct classical limit when all fluctuations are set to zero. (As mentioned,
Eqn. (4.4) was derived with a Gaussian ansatz for the wave function, for which $1/\epsilon$, the inverse curvature fluctuation, is proportional to the volume fluctuation. The semiclassical behavior agrees with the classical one provided $\epsilon$ goes sufficiently fast to zero as the energy density decreases in an expanding universe. Indeed, in a semiclassical state peaked at smaller and smaller curvature, $\epsilon$ must decrease and cannot remain constant [12]. The question is whether it is possible for $\epsilon$ to decrease sufficiently slowly that the last term in Eqn. (4.4) can cancel the others and enable a recollapse.

In the present context, the important part to note from Eqn. (4.4) is that at late time and large volume, for an expanding universe with suitably low energy density ($\rho \sim 2\epsilon^2\rho_{\text{crit}}$), the universe could collapse. One way of looking at this is by distributing the $\rho$ in front to see that the final term is independent of the energy density and only depends on the quantum fluctuations. It functions as a negative cosmological-constant term if $\epsilon$ varies slowly. When the energy density is of the order of those fluctuations, the final term dominates and could lead to collapse—depending on the exact quantum dynamics.

From the perspective of effective equations, the classical model quantized in [28]—a spatially flat isotropic model sourced by a free, massless scalar—is identical to a harmonic one, without any quantum back-reaction [12]. The only corrections to classical evolution are due to quantum geometry; no quantum back-reaction should occur. However, compared to [12], [28] used a different factor ordering of the quantum constraint as well as quantum-geometry modifications of inverse-triad type, not just holonomy corrections as in Eqn. (4.2). These variations imply quantum back-reaction, which should not be large but could still be significant after long evolution. In this paper, we explore the possibility of such a term in the modified Friedmann equation found by embarking on a systematic method of deriving effective equations, including the underlying properties of dynamical semiclassical states [6, 32]. By way of the example of collapse solutions, we will therefore study several implications of terms in effective equations, including the role of factor ordering.
4.2 Loop Quantum Cosmology and Effective Equations

In loop quantum cosmology, the basic pair of canonical variables for spatially flat isotropic models is the extrinsic curvature \( c = \gamma da/d\tau \) and its conjugate \( |p| = a^2 \) where \( \gamma \) is the Barbero-Immirzi parameter, \( a \) is the scale factor, and \( \tau \) is proper time. The momentum \( p \) is derived from an isotropic densitized triad and so can take both signs due to the triad orientation. The fundamental Poisson bracket is \( \{c, p\} = 8\pi G \gamma /3 \) where \( G \) is Newton’s constant. Beginning with the Friedmann equation for an isotropic and homogeneous background, we can write the Hamiltonian constraint in these canonical variables as Eqn. (4.1) with \( H_{\text{matter}} = \frac{1}{2} |p|^{-3/2} \nu p_\phi^2 \) for a free, massless scalar \( \phi \) where \( p_\phi \) is the momentum conjugate to \( \phi \) such that \( \{\phi, p_\phi\} = 1 \).

4.2.1 The Model

As this work is motivated by [28], we will use the same notation for a direct comparison, and so by a canonical transformation we introduce new conjugate variables:

\[
b \equiv \delta \frac{c}{\sqrt{|p|}} , \quad \nu \equiv \frac{2}{3\delta} \text{sgn}(p)|p|^{3/2} \text{ such that } \{b, \nu\} = 8\pi G \gamma /3 ,
\]

where \( \delta \) is a length parameter that determines the size of holonomy corrections (specifically, it is defined in [28] as \( \sqrt{\Delta} \) with \( \Delta \equiv (2\sqrt{3}\pi \gamma) \ell_P^2 \)). Looking at large-scale evolution of large \( |p| \), we are free to fix the sign of \( p \) or \( \nu \), from now on assumed positive. (These variables are a specific case of the general parameterization of [39], providing different cases of lattice refinement. While the harmonic model has a dynamics independent of the specific choice of basic variables where \( \nu \) could be any power of \( p \), quantum corrections in models that differ for instance by factor ordering depend on such parameters. Here, however, we confine ourselves to the choice made in [28] because our aim is to explore the scenario suggested there.)
In the new variables, our Hamiltonian constraint is

\[ -\frac{9\nu b^2}{16\pi G\gamma^2\delta} + \frac{p_{\phi}^2}{3\delta\nu} = 0. \] (4.6)

We use the free scalar \( \phi \) as an internal time variable; the monotonicity of \( \phi(\tau) \) is guaranteed for the free case, and so evolution for \( b(\phi) \) and \( \nu(\phi) \) is governed by the Hamiltonian \( p_{\phi}(b,\nu) \) which is found as a phase-space function by solving for \( p_{\phi} \) in Eqn. (4.6). Doing this, we have \( p_{\phi} \sim |\nu b| \).

The momentum \( p_{\phi}(\nu, b) \) of internal time as a function of the remaining degrees of freedom functions as the Hamiltonian generating evolution. Since it is quadratic, the system is equivalent to a harmonic one, which, like the harmonic oscillator, can be quantized to a model without quantum back-reaction. However, in a model of quantum gravity there are quantum-geometry corrections as well, changing the Hamiltonian not just by quantum back-reaction terms but also by more-severe modifications. In loop quantum cosmology as the present setting, holonomy and inverse-triad corrections occur.

Holonomy corrections replace the \( b \) in \( p_{\phi}(\nu, b) \) by the non-linear \( \sin(b) \), making the Hamiltonian non-quadratic in canonical variables. Nevertheless, as shown in [12], surprisingly, they do not remove harmonicity even upon quantization provided a suitable factor ordering is chosen: If we define \( J \equiv \nu \exp(ib) \), the new set of (non-canonical) basic variables satisfies a linear algebra, in which the linear Hamiltonian \( p_{\phi} = |\text{Im}(J)| \) is just the right holonomy-modified version of the internal-time Hamiltonian \( p_{\phi} \). These properties are preserved when the model is quantized with Hamiltonian \( \hat{p}_{\phi} = -\frac{1}{2}i|\hat{J} - \hat{J}| \) in this specific ordering when seen as a function of \( b \) and \( \nu \). With a linear Hamiltonian in variables obeying a linear algebra, no back-reaction occurs. (Again, the absolute value turns out not to spoil harmonicity features.)

Unlike the harmonic oscillator in quantum mechanics, the quadratic Hamiltonian here is subject to factor ordering choices, and only one of them, as remarked in [12], results in an exactly solvable quantum system as just described. At the
quantum level, \( \hat{p}_\phi = |\text{Im} \hat{J}| = |\frac{1}{2}i(\hat{J} - \hat{J}^\dagger)| \) writes the modified \( \nu \sin(b) \) in the specific form \( \frac{1}{2}i(\nu \exp(ib) - \exp(-ib)\nu) \). Any other choice gives rise to quantum back-reaction, as spelled out in more detail below. Moreover, the second type of quantum-geometry modifications — inverse-triad corrections — implies quantum back-reaction even if the original ordering is chosen in which holonomy corrections alone would not spoil harmonicity. (In this case, the \( \nu \)- rather than \( b \)-dependence is modified.)

### 4.2.2 Effective Equations

To derive effective equations from our classical Hamiltonian above, we follow the general procedure of [6, 32]. For semiclassical (or even more general) states, the method of effective equations allows us to avoid the technical difficulties of working directly with wave functions and operators. Also the hard problem of deriving physical Hilbert spaces and explicit representations is avoided, and yet physical normalization is implemented by reality conditions. Large classes of states can be dealt with by suitable parameters, such as expectation values and fluctuations, of direct statistical significance for physical properties. Effective equations therefore allow much larger generality, and thereby more reliable conclusions, than traditional methods.

Instead of working with wave functions, we have a framework in which states are represented by the expectation values and moments they imply for basic operators. As quantum variables in addition to expectation values, we use the moments which describe a general quantum state:

\[
G_{\nu .. \nu, b .. b}^{m .. n} = \langle (\hat{\nu} - \langle \hat{\nu} \rangle)^m (\hat{b} - \langle \hat{b} \rangle)^n \rangle_{\text{symm}}
\]  

(4.7)

where \( m, n \) are positive integers such that \( m + n \geq 2 \) and the subscript denotes totally symmetric ordering of the operators. The moments and expectation values define a phase space with Poisson bracket \( \{\langle \hat{A} \rangle, \langle \hat{B} \rangle\} = \langle [\hat{A}, \hat{B}] \rangle / i\hbar \) in terms of the commutator, extended by the Leibniz rule to arbitrary polynomials of the expectation values as they occur in moments. Semiclassical regimes can be defined very generally to any integer order \( N \), by keeping moments only up to order \( 2N \). Indeed, in a Gaussian state, the prime example of a semiclassical one, the moments
behave as $G^{m,n} \sim \hbar^{(m+n)/2}$. However, our semiclassical regimes defined by the order of moments are much more general than the 1- or at most 2-parameter families of Gaussians, and thus avoid possible artifacts due to the selection of states.

The basic identity $d\langle \hat{A} \rangle/dt = \langle [\hat{A}, \hat{H}] \rangle / \hbar$ of quantum mechanics, used for instance to derive Ehrenfest’s equations, can then be written as a classical-type Hamiltonian flow generated by the quantum Hamiltonian $H_Q(\langle \hat{\nu} \rangle, \langle \hat{b} \rangle, G^{m,n}) = \langle \hat{H} \rangle$ where the expectation value is computed in a state specified by expectation values $\langle \hat{\nu} \rangle$ and $\langle \hat{b} \rangle$ and all of its moments $G^{m,n}$. Such expectation values of interacting Hamiltonians can be difficult to compute, but in semiclassical (or other) regimes in which only finitely many moments matter, they can be derived perturbatively by expansions such as

$$H_Q(\nu, b, G^{m,n}) \equiv \langle \hat{H} \rangle = \langle H(\hat{\nu}, \hat{b}) \rangle = \langle H(\langle \hat{\nu} \rangle + (\hat{\nu} - \langle \hat{\nu} \rangle), \langle \hat{b} \rangle + (\hat{b} - \langle \hat{b} \rangle)) \rangle$$
$$= H(\langle \hat{\nu} \rangle, \langle \hat{b} \rangle) + \sum_{m,n; m+n \geq 2} \frac{1}{m!n!} \frac{\partial^{m+n} H(\langle \hat{\nu} \rangle, \langle \hat{b} \rangle)}{\partial \langle \hat{\nu} \rangle^m \partial \langle \hat{b} \rangle^n} G^{m,n} \quad (4.8)$$

where we define $\nu \equiv \langle \hat{\nu} \rangle$ and $b \equiv \langle \hat{b} \rangle$ as a short cut used from now on. In this particular expression, we assume the Hamiltonian operator $\hat{H}$ to be totally symmetrically ordered just as the moments. If there are reasons for working with a different ordering, as explored below, it differs from the symmetric ordering by re-ordering terms which can be expressed in terms of the moments as a short cut used from now on. In this particular expression, we assume the Hamiltonian operator $\hat{H}$ to be totally symmetrically ordered just as the moments. If there are reasons for working with a different ordering, as explored below, it differs from the symmetric ordering by re-ordering terms which can be expressed in terms of the moments as well. Accordingly, there will be additional quantum corrections in Eqn. (4.8).

For instance, if $H(\nu, b) = \nu^2 b^2$, the symmetrically ordered quantization would be $\hat{H} = \frac{1}{6}(\hat{\nu}^2 \hat{b}^2 + \hat{\nu} \hat{b} \hat{\nu} + \hat{\nu} \hat{b}^2 \hat{\nu} + \hat{b} \nu^2 \hat{b} + \hat{b} \hat{\nu} \hat{b} \hat{\nu} + \hat{b}^2 \nu^2)$. Another symmetric ordering is $\hat{H}' = \frac{1}{2}(\hat{\nu}^2 \hat{b}^2 + \hat{b}^2 \nu^2)$. Using $\hat{\nu} \hat{b} \hat{\nu} + \hat{\nu} \hat{b} \hat{\nu} = \nu^2 b^2 + \hat{b}^2 \nu^2 - [\hat{b}, \nu]^2$ and $\hat{\nu} \hat{b}^2 \hat{\nu} + \hat{b} \hat{\nu}^2 \hat{b} = \nu^2 b^2 + \hat{b}^2 \nu^2 - 2[\hat{b}, \nu]^2$ for canonical commutators of $\hat{\nu}$ and $\hat{b}$, we have $\hat{H}' = \hat{H} + \frac{1}{2}[\hat{b}, \hat{\nu}]^2$. The effective Hamiltonians differ just by constants, but for higher polynomials moment-dependent factor-ordering terms can result. We will present a more detailed example below.

For the system at hand, quantum evolution — determined from commutators with the Hamiltonian operator $H_Q$ — generates our equations of motion for expectation values and quantum moments: $d\langle \hat{O} \rangle / d\phi = \{O, H_Q\}$. If the classical Hamiltonian is not quadratic in canonical variables (or non-linear in linear basic
variables) the last term in Eqn. (4.8) contains products of expectation values and moments. Their equations of motion are then coupled, with moments back-reacting on the dynamics of expectation values as a source of quantum corrections.

In this paper, we will work to leading order \( m + n = 2 \), and ignore higher moments as they are subdominant for semiclassical states. (For different analyses of cosmological quantum back-reaction using this scheme with moments of different orders, see [10, 14, 40].) Moreover, as we will see below, they appear with factors which are suppressed by inverse powers of \( \nu \) and become even more suppressed at large volume. Still, to leading order there is quantum back-reaction of fluctuations and correlations in expectation values. To compare evolution equations with a single Friedmann equation of the usual form, we should solve for all moments included and insert the solutions in equations of motion for expectation values. But as we are, for now, looking at particular terms in a general formulation, we do not have an explicit system of equations to solve.

As mentioned in the introduction, we are extending the typical analysis by including three additional forms of corrections — the holonomy replacement, factor ordering ambiguities, and inverse triad corrections — which we will now define and summarize. The holonomy replacement is motivated from the full theory of loop quantum gravity [23, 41, 42], as already discussed. Furthermore, holonomy corrections are of interest here because we are looking to reproduce the \( \epsilon^2 \) term of Eqn. (4.4). In our framework, \( \epsilon^2 \sim 2G^{0,2} \equiv 2G^{bb} \) which is the variance of \( b \). Thus, if we are looking for a \( G^{bb} \) dependence in our corrections to the Friedmann equation, we require \( H \) to have a non-zero third derivative with respect to \( b \). From Eqn. (4.8) we see that to obtain a \( G^{bb} \) term we need at least a non-zero second derivative, and as we will see, we need an additional derivative of \( b \) to get \( G^{bb} \) in the equations of motion from the Poisson bracket. The holonomy replacement conveniently satisfies this requirement.

Inverse-triad corrections modify the \( \nu \)-dependence of the Hamiltonian. The Hamiltonian constraint of Eqn. (4.1) to be quantized contains inverse powers of \( \nu \). Also here, as with holonomy corrections, the full theory suggests characteristic modifications by elementary properties of the form of quantum geometry realized. The variable \( \nu \), or rather \( p \) as the isotropic component of the densitized triad used as a basic field in the full theory, is quantized to flux operators with discrete spec-
tra containing zero. Such operators have no densely defined inverse, and therefore one must use more indirect quantization techniques, provided in \[43, 44\] to represent inverses of \( \nu \) or \( p \) in the Hamiltonian constraint. In isotropic models, the resulting correction functions can be computed explicitly \[22, 45, 46\]; they appear in quantized Hamiltonian constraints and determine, for instance, the coefficients of dynamical difference equations. For our present purposes, we take these effects into account by writing the \( \nu \)-factor of \( p_\phi \) as a general function \( f(\nu) \) which, in the large volume regime considered here, has the form \( f(\nu) \sim \nu(1 + O(1/\nu)) \).

A commonly used parameterization, for instance in phenomenological analysis, is \( \nu + c_\alpha (\ell_3^3/\nu)^{n_\alpha} \) with \( c_\alpha \) real valued and \( n_\alpha \) a positive integer.

Summarizing holonomy and inverse-triad corrections, we find our Hamiltonian with corrections to be

\[
H(b, \nu) \equiv p_\phi = \sqrt{\frac{27}{16\pi G}} f(\nu) \sin b. \tag{4.9}
\]

The next step is to use this example Hamiltonian to evaluate the dynamics of our system and arrive at an effective Friedmann equation.

### 4.2.3 Quantum Dynamics

Following the effective equation prescription just described, we find our quantum Hamiltonian, up to second order in moments, to be:

\[
H_Q = \frac{1}{\gamma} \sqrt{\frac{27}{16\pi G}} \left( f(\nu) \sin b - \frac{1}{2} G^{bb} f(\nu) \sin b + G^{bw} f'(\nu) \cos b + \frac{1}{2} G^{\nu\nu} f''(\nu) \sin b \right) \tag{4.10}
\]

where ' represents a derivative with respect to \( \nu \). As mentioned in the general discussion of effective equations, the specific terms written here assume that the Hamiltonian operator is symmetrically ordered. At this stage, factor-ordering choices may introduce additional quantum corrections which also depend on the moments, as discussed in more detail below.

From Eqn. (4.10), we can find the relational equations of motion for our expectation values via Poisson brackets. For instance, we have the expectation value of
the volume in terms of expectation values and quantum variables:

\[
\dot{\nu} \equiv \frac{d\langle \dot{\nu} \rangle}{d\phi} = \{\nu, H_Q\} \quad (4.11)
\]

\[
= -\sqrt{12\pi G} \left( f(\nu) \cos b - \frac{1}{2} G^{bb} f(\nu) \cos b - G^{bw} f'(\nu) \sin b + \frac{1}{2} G^{ww} f''(\nu) \cos b \right).\]

Here our quantum variables backreact on our classical trajectory. The moments are not constant but are subject to their own equations of motion; however we will not need them for the subsequent analysis of the Friedmann equation, but they are discussed in Appendix 4.5 at the end of this chapter.

### 4.2.4 Friedmann Equation

Now let us find the quantum corrections to the Friedmann equation for our example. From Eqn. (4.5) and the definition of \(|p| = a^2\), we have

\[
\frac{d\nu}{d\tau} = 2a^2 \frac{da}{d\tau} \quad \frac{d\phi}{d\tau} = \frac{2p_{\phi}}{3\delta \nu} \quad (4.12)
\]

where \(d\phi/d\tau\) is generated by the original Hamiltonian constraint\(^3\). Rearranging and substituting, we find

\[
\left( \frac{1}{a} \frac{da}{d\tau} \right)^2 = \frac{4}{81\delta^2} \left( \frac{\dot{\nu}}{\nu} \right)^2 \left( \frac{p_{\phi}}{\nu} \right)^2 \quad (4.14)
\]

\[
= \frac{4}{81\delta^2} 12\pi G \frac{f(\nu)^2}{\nu^2} \cos^2 b \left( 1 - \frac{(G^{bb})^2}{4} \right) \frac{27}{16\pi G} \frac{f(\nu)^2}{\nu^2} \sin^2 b
\]

where we used Eqns. (4.11) and (4.9) to arrive at the second line and dropped the \(f'\) and \(f''\) terms for simplicity. Note that these derivative terms are not neglected in the actual analysis, even though they will be small at large scales. In terms of

\(^3\)A more detailed discussion of this process can be found in subsection 5.3.1 where we comment on a model with a different choice of internal time variable.
energy density we have

\[ \rho = \frac{H_{\text{matter}}}{a^3} = \frac{2}{9\delta^2 \nu^2} \frac{p_\phi^2}{\nu^2} = \frac{3}{8\pi G \delta^2 \nu^2} \frac{f(\nu)^2}{\nu^2} \sin^2 b \]

(4.15)

where we used Eqn. (4.9) again. Rearranging to solve for \( \sin^2 b \) and using \( \cos^2 b = 1 - \sin^2 b \), we can substitute into Eqn. (4.14) to find

\[ \left( \frac{1}{a} \frac{da}{d\tau} \right)^2 = \frac{8\pi G}{3} \rho \left( \frac{f(\nu)}{\nu^2} - \frac{8\pi G}{3} \delta^2 \rho \right) \left( 1 - G^{bb} + \frac{(G^{bb})^2}{4} \right) \]

(4.16)

### 4.3 General factor ordering and collapse terms

In addition to the holonomy replacement and inverse triad corrections, we include the possibility of a factor ordering ambiguity as the main ingredient in this paper. None of these corrections are expected to be very large in low-curvature semiclassical regimes, but they could play a role in long-term evolution. To show that loop quantum cosmology has the correct large-scale behavior, all these terms must be studied carefully, but no such analysis has been completed yet. Here, we present systematic expansions by which the issue of long-term behavior can be addressed, focusing specifically on the question whether semiclassical effects can be so strong that they trigger a recollapse of a classically ever-expanding universe.

#### 4.3.1 Factor ordering

The Hamiltonian constraint of loop quantum cosmology (or loop quantum gravity in general) is far from being unique, owing for instance to choices in the representation of holonomies and factor orderings. Although uniqueness claims seem to exist in the literature, they are based on certain mathematical features posed ad-hoc, rather than physical considerations; they cannot be used when addressing the semiclassical limit, long-term evolution, or concrete physical effects.

In addition to ambiguities in the formulation of the Hamiltonian constraint operator, additional choices are usually made when one addresses the problem of time. One chooses or even introduces simple matter degrees of freedom, here and elsewhere assumed as a free and massless scalar field, whose momentum is exactly
constant and in particular never vanishing. The scalar degree of freedom therefore has no turning points when solved for as a function of proper time, and can be used as a mathematical evolution parameter itself.

Simple and manageable equations at the classical or quantum levels usually result. However, the question of whether different choices of internal times provide the same physics remains complicated to address; see e.g. [47]. (An exception is given by semiclassical regimes, where effective methods allow one to change time by gauge transformations of moments [48, 49, 50].) If independence of one’s choice of internal time can be achieved at all, as some kind of space-time anomaly problem, it likely requires detailed prescriptions of factor orderings and other quantization ambiguities. What one may choose in a deparameterizable version in rather simple terms may not be what is required for correct physics. Therefore, a general view on factor ordering and other choices is preferable for reliable statements, implemented here by general parameterizations.

Instead of directly quantizing $\nu \sin(b)$ with rather controlled factor ordering options, or in the harmonic formulation with the even more unique-looking $\text{Im}J$, we should start with the Hamiltonian constraint of Eqn. (4.2) and its terms $p_\phi^2/\nu$ and $\nu \sin^2(b)$. The types of factor-ordering ambiguities differ in those two cases, showing the special nature of deparameterized models regarding factor ordering.

In the former case, looking at possible orderings of the $\phi$-Hamiltonian $\nu \sin(b)$, we could take a symmetric ordering such as $\frac{1}{2}(\hat{\nu}\hat{\sin(b)} + \hat{\sin(b)}\hat{\nu})$ — the symmetric ordering of operators $\hat{\nu}$ and $\hat{\sin(b)}$ — but also $\sqrt{\nu}\hat{\sin(b)}\sqrt{\nu}$ or a more complicated re-ordering. The first two differ by $\frac{1}{2}(\sqrt{\nu}\hat{\sin(b)}, \sqrt{\nu}) + [\sqrt{\nu}, \hat{\sin(b)}]\sqrt{\nu} = \frac{1}{2} [\sqrt{\nu}, [\sin(b), \sqrt{\nu}]]$. Such terms, in a low-curvature expansion, just add positive powers of $b$ with ordering-dependent coefficients. The classical correspondence of the ordering term just derived, for instance, is

$$-\frac{1}{2} \hbar^2 \{\sqrt{\nu}, \{\sin(b), \sqrt{\nu}\}\} = -\frac{1}{9} \pi^2 \gamma^2 \ell_\text{p}^4 \nu^{-1} \sin(b). \quad (4.17)$$

As in this example, iterated commutators, or Poisson brackets in the classical correspondence, amount to terms of the order $\ell_\text{p}^4/\nu$ to some positive power, by which various orderings differ. (A single commutator contributes $\ell_\text{p}^2/\sqrt{\nu}$. However, there must always be an even number of commutators if the difference of two symmetric
orderings is considered to avoid factors of $i$.) Such terms can be interpreted as changing the form of inverse-triad corrections in $f(\nu)$ or the choice of holonomy corrections initially implemented by the sine function; therefore, they do not vary much from ambiguities of holonomy corrections.

Factor-ordering ambiguities can, however, be more radical if we start with the original Hamiltonian constraint rather than the deparameterized $\phi$-Hamiltonian. One could first take the square root of the quantized $p_\phi^2/\nu \propto \nu \sin^2(b)$, for instance of $\sin(b)\sqrt{\nu}\sin(b)$, and then multiply with $\sqrt{\nu}$. Factor-ordering choices now include expressions such as $\sqrt{\nu}^1/4\sqrt{\sin(b)\nu\sin(b)}\sqrt{\nu}^1/4$ which, compared with the previous ones, differ by terms that include quantizations of the inverse $(\sin(b)\nu\sin(b))^{-1/2}$.

It is now more difficult to compute commutators of square-root operators to see what factor-ordering terms arise. However, we can estimate contributions in an $\hbar$-expansion, suitable for effective equations, by noting that arguments of the square root such as $\sqrt{\sin(b)\nu\sin(b)}$ differ from squares of our previous Hamiltonians by double commutators involving a power of $\nu$ and $\sin^2(b)$, proportional to $\hbar^2\nu^{-1}(\sin^2(b) - \cos^2(b)) = -\hbar^2\nu^{-1}\cos(2b)$. In the semiclassical correspondence, such a term added to $\sin^2(b)$ under the square root implies a leading $\hbar$-correction of $\sqrt{\sin^2(b) - \hbar^2\nu^{-1}\cos(2b)} - \sin(b) = -\frac{1}{2}\hbar^2\nu^{-1}\cos(2b)/\sin(b) + O(\hbar^4/\sin^3(b))$, which is not analytic at $b = 0$.

The types of ordering just presented are more involved than simple ones when one directly quantizes $\nu \sin(b)$. However, they are no less natural; in fact, they are more natural from the point of view that the terms in the original Hamiltonian constraint, $1/\nu$ and $\nu \sin^2(b)$ should first be quantized, with a square root for $p_\phi$ taken afterwards at the quantum level. For full generality and independence of special features available only in deparameterized systems but not otherwise, one must take the properties of the orderings shown here into account.

In a low-curvature expansion, even inverse powers of $b$ can therefore appear due to factor ordering. This feature may sound surprising because such terms seem unduly large. However, they are always accompanied with factors of $\hbar$ and, in some cases, moments such as the curvature fluctuation which decrease as $b$ gets small. The fate of long-term evolution now crucially depends on properties of state parameters and thus on the detailed form of quantum corrections and effective equations; this requires a dedicated analysis on which we now embark.
4.3.2 Low-curvature expansion

So far, we have kept all terms expected from loop quantum cosmology, but they will not be fully required in the semiclassical regimes we are interested in. In the small-curvature regime, we can expand the Hamiltonian in powers of $b$:

$$H = H_1 b + H_3 b^3 + H_{-1} b^{-1}$$  \hspace{1cm} (4.18)

where the $H_n$ are functions of $\nu$ only. This analysis will include all possible terms expected for a minisuperspace quantization of the model without ad-hoc assumptions. (The analysis does not change with the inclusion of unexpected terms, such as from relaxing reality conditions, as we will see in the conclusion.)

In Eqn. (4.18), the linear term $H_1 b$ is what is expected classically. The cubic term is the first higher-order quantum correction due to the holonomy modification; higher orders are not relevant at low curvature. The origin of terms of less than the classical order in $b$ may be less obvious at first sight, but can arise depending on the factor ordering of $f(\nu)$ and $\sin(b)$ chosen for the square-root Hamiltonian $H = p_b$. Such Hamiltonians differ from the harmonic one by additive re-ordering terms depending on the commutator. Each commutator reduces the order of $b$ (and $\nu$) by one, and by reality there must be an even number of commutator factors in each term; thus only odd powers of $b$ result in the expanded Hamiltonian. We could expand to higher orders of positive and negative powers in $b$, but as we will discuss in the analysis, this does not lead to categorically different terms.

From Eqn. (4.18), we can then solve for the equation of motion for $\nu$

$$\dot{\nu} = \{\nu, H_Q\} = \{\nu, H\} + \frac{1}{2} G^{bb} \{\nu, \partial_b^2 H\} + G^{b\nu} \{\nu, \partial_b \partial_{\nu} H\} + \frac{1}{2} G^{\nu\nu} \{\nu, \partial_{\nu}^2 H\}$$  \hspace{1cm} (4.19)

which we expand term by term to find

$$\{\nu, H\} = \frac{8\pi G\gamma}{3} ( - H_1 - 3H_3 b^2 + H_{-1} b^{-2} ) \equiv A$$  \hspace{1cm} (4.20)

$$\frac{1}{2} G^{bb} \{\nu, \partial_b^2 H\} = \frac{1}{2} G^{bb} \partial_b^2 A = \frac{1}{2} G^{bb} \frac{8\pi G\gamma}{3} ( -6H_3 + 6H_{-1} b^{-4} )$$  \hspace{1cm} (4.21)

$$G^{b\nu} \{\nu, \partial_b \partial_{\nu} H\} = G^{b\nu} \partial_b \partial_{\nu} A = G^{b\nu} \frac{8\pi G\gamma}{3} \left( -6H_3 b - 2H_{-1} b^{-3} \right)$$  \hspace{1cm} (4.22)

$$\frac{1}{2} G^{\nu\nu} \{\nu, \partial_{\nu}^2 H\} = \frac{1}{2} G^{\nu\nu} \partial_{\nu}^2 A = \frac{1}{2} G^{\nu\nu} \frac{8\pi G\gamma}{3} \left( -H_1'' - 3H_3'' b^2 + H_{-1}'' b^{-2} \right)$$  \hspace{1cm} (4.23)
where we define $A$ in the first equation.

Now we have $\dot{\nu}$ expanded in powers of $b$. Recall however, that for the Friedmann equation, we are really interested in

$$\left(\frac{1}{a}\frac{da}{d\tau}\right)^2 = \frac{4}{81\delta^2} \left(\frac{\dot{\nu}}{\nu}\right)^2 \left(\frac{p_{\phi}}{\nu}\right)^2 = \frac{4}{81\delta^2} \left(\frac{\{\nu, H_Q\}}{\nu}\right)^2 \frac{H_Q^2}{\nu^2}$$

(4.24)

which requires us to look at the expansion of $\dot{\nu}^2 H_Q^2/\nu^4$. For terms we are interested in, those that depend on the second order moments but are independent of $\rho$, we can limit our search to terms with $b^0$ and $\nu^0$ which would resemble the term responsible for recollapse. (This term, specifically, contains $G^{bb}$ but no factor of $\nu$ or $b$; it turns out that the dependence on expectation values is already a restrictive condition, regardless of the dependence on moments.)

The simplest procedure is to look at all $b$-independent terms in $\dot{\nu}^2 H_Q^2$ and then restrict to all remaining ones proportional to $\nu^4$ to cancel the $\nu$-dependence. Since both $\dot{\nu} = \{\nu, H_Q\}$ and $H_Q$ itself are expanded as in Eqn. [4.18], all expansion terms in the product are quartic expressions in the $H_n$. Terms independent of $b$ but containing $G^{bb}$ include, for instance,

$$6 \left(\frac{8\pi G\gamma}{3}\right)^2 G^{bb} G^{\nu\nu} H_1'' H_3 H_1 H_{-1} \quad \text{or} \quad 18 \left(\frac{8\pi G\gamma}{3}\right)^2 G^{bb} G^{bb} H_3 H_3 H_1 H_{-1}. \quad (4.25)$$

Now we are interested in what is necessary such that the combination of $H(\nu)$ gives us terms with $\nu^0$, thus satisfying the $\rho$ independence, which could dominate the Friedmann equation in a regime with low energy density. With that in mind, let us categorize the $\nu$-dependence in the $H_n$ terms as

$$H_1 \sim f(\nu), \quad H_3 \sim f(\nu), \quad H_{-1} \sim \frac{f'(\nu)^2}{f(\nu)}, \quad H_{-3} \sim \frac{f'(\nu)^4}{f(\nu)^2}. \quad (4.26)$$

where these associations come from the power expansion of our original $H$: $H_1 \sim H_3$ are the coefficients of $\sin b$ (note that this extends to all higher-curvature terms since they are also obtained from expanding $\sin b$). $H_{-1}$ and $H_{-3}$ arise from the factor ordering implementation (which can be extremely ambiguous); for this analysis, we focus on factor ordering of the type demonstrated at the end of Section 4.3.1. Specifically, the $b^{-1}$ term is due to double commutators as described, but
higher orders of inverse powers are the result of the $\hbar$-expansion of the square root. Despite this specific choice in our definitions at this point, our final results hold for all choices of factor ordering implementations and power expansions (such as a factor ordering only approach, where each additional inverse power of $b$ requires another commutator of $\sin(b)$ with $\hat{f}(\nu)$).

Recall that $f(\nu)$ is due to the inverse triad corrections ($\hat{\nu}\nu^{-1} \approx 1 + 1/\nu$). Although we have an expectation for the form of $f(\nu)$ we do not presume it here; rather, we will solve for the necessary values that provide $\rho$ independence and compare for consistency.

With these classifications, we can then introduce a generalized expression:

$$H_n = f(\nu)^{\alpha(n)} f'(\nu)^{1-\beta(n)}$$

$$\alpha(n) \equiv \frac{\text{max}(\text{min}(n,1), 3 \text{ min}(n,1)) - 1}{2}$$

$$\beta(n) \equiv \text{min}(n,1)$$

which then allows us to write simple expressions for multiplicative combinations of $H_n$:

$$\mathbb{H}_{g,\bar{g}} \equiv H_i H_j H_k H_l \sim f(\nu)^S f'(\nu)^{4-S}$$

$$S \equiv \alpha(i) + \alpha(j) + \alpha(k) + \alpha(l)$$

$$\bar{S} \equiv \beta(i) + \beta(j) + \beta(k) + \beta(l)$$

These definitions may look complicated, but they encapsulate all orders into one equation and can be quickly calculated. One can easily check that $\bar{S}$ is the sum of $i, j, k,$ and $l$ with the exception that positive powers are all counted as 1, i.e. if $j \geq 1 \to \beta(j) = 1$; additionally $S$ is really the sum of $i, j, k,$ and $l$ with two exceptions: 1) higher powers follow the rules of $\bar{S}$ and 2) for inverse powers, the count is the negative of the order of the power expansion, i.e. if $j = -2x + 1 \to \alpha(j) = -x$ (see Table 4.1 for some specific values).

We are then interested in $\mathbb{H}_{g,\bar{g}}$ such that we have a $\nu^4$ dependence which cancels in Eqn. (4.24). Thus we can solve for $f(\nu)$ with respect to $\nu$ to see what is required for a $\nu$-independent term in the effective Friedmann equation. We need $f(\nu)^S f'(\nu)^{4-S} = \nu^4$, by separation of variables, we find $f(\nu) \sim \nu^{(8-S)/(4+S-S)}$ where
<table>
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<tr>
<th>$n$</th>
<th>$\alpha(n)$</th>
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<tr>
<td>5</td>
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Table 4.1. Sample values for $\alpha(n)$ and $\beta(n)$

$S$ and $\bar{S}$ are integers $\leq 4$. However, recall from Eqn. (4.25) that there can also be derivatives of $H_n$ in our combinations. We now look to the inclusion of derivative terms and their effects on our analysis:

\[
\partial_\nu H_n = nH_{n-1} + (1-n)H_n \frac{f''(\nu)}{f'(\nu)}
\]

\[
\partial_\nu^2 H_n = (1-n) \left( -nH_n \left( \frac{f''(\nu)}{f'(\nu)} \right)^2 + H_n \frac{f'''(\nu)}{f'(\nu)} \right) + n \left( 2-n \right) H_{n-1} \frac{f''(\nu)}{f'(\nu)} + (1-n)H_{n-1} \frac{f''(\nu)}{f'(\nu)} + n(n-1)H_{n-2}.
\]

With the benefit of foresight, it is now prudent to extend Eqn. (4.30) to allow for more than four terms by defining

\[
\mathbb{H}_{S,N,\bar{S}} = \underbrace{H_i H_j H_k \ldots}_{N \text{ terms}} f(\nu)^{S} f'(\nu)^{N-S}
\]

where $S$ and $\bar{S}$ are the sums of the labels as defined before and $N$ is how many $H_n$ terms are multiplied together. While we will continue to focus on the $N = 4$ case corresponding to our possible contributions to the effective Friedmann equation, this parameter will be important for categorizing derivatives. In fact, with this definition, we can now write terms which have derivatives as

\[
\mathbb{H}^{n'}_{S,N,\bar{S}} = \alpha(n) \mathbb{H}_{S-1,N,\bar{S}-1} + (1 - \beta(n))\mathbb{H}_{S,N-1,\bar{S}} f''(\nu)
\]

where the $n'$ superscript represents that there is a $H'_n$ term in the combination and we see that $N$ is altered in terms which have higher derivatives. We could also write out a general formula for combinations with more than one derivative term;
however, terms which include two derivative factors are included (up to numerical coefficients) in Eqn. (4.37) and combinations that contain more derivative terms lead to inconsistent conditions on $f(\nu)$, as we will see in Eqns. (4.38) and (4.39).

We can also express terms with second derivatives:

$$H_{S,N,\bar{S}}^{'''} = \alpha(n)(\alpha(n) - 1)H_{S-2,N,\bar{S}-2} + \alpha(n)(3 - 2\beta(n))H_{S-1,N-1,\bar{S}-1}f''(\nu) - \beta(n)(1 - \beta(n))H_{S,N-2,\bar{S}}f''(\nu)^2 + (1 - \beta(n))H_{S,N-1,\bar{S}}f'''(\nu). \quad (4.37)$$

Now that we have these expressions, we once again must solve for $f(\nu)$ such that we have a $\nu$-independent term in the effective Friedmann equation. These solutions are obtained from differential equations with non-linear products of varying orders of derivatives. We can generalize these differential equations from our terms as:

$$H_{S,N,\bar{S}}^{p}f''(\nu)^p f'''(\nu)^q \equiv f(\nu)^S f'(\nu)^{N-S} f''(\nu)^p f'''(\nu)^q = \nu^4 \quad (4.38)$$

assuming $p, q \geq 0$ (this describes all possible contribution terms). Assuming $f(\nu) \sim \nu^{a/b}$ for real numbers $a$ and $b$, we have solutions given by

$$f(\nu) \sim \nu^{\frac{4 + N + 2p + 3q - \bar{S}}{N + p + q - S - \bar{S}}} \quad (4.39)$$

with the added caveat that for $q \geq 0, f'''(\nu) \neq 0$. (We have made the ansatz $f(\nu) \sim \nu^{a/b}$ without loss of generality because the large-volume behavior is always a power law.)

All $\nu$-independent contributions to the corrected Friedmann equation can be written as this type of differential equation and thus can only allow solutions given by Eqn. (4.39). In fact, as hinted at previously, this includes expansions in higher positive and negative powers of $b$ in Eqn. (4.18); this is because the higher powers create more terms, but never categorically different results. For instance, higher positive powers in $b$ are due to the power expansion of $\sin b$ which will have the same $\nu$-dependence, that is $H_{2k+1} \sim H_5 \sim H_3 \sim H_1$ for non-negative integer $k$. Higher negative powers come from expansions in the factor ordering which leads to increasing powers of $f'(\nu)/\nu(\nu)$, but this is included in the analysis above with an appropriate choice of $S, \bar{S},$ and $N$. (In fact, with the appropriate choice
of $S$, $\bar{S}$, and $N$, one can include all general types of factor ordering ambiguities and expansions; though our $H_n$ are defined for a particular choice, Eqn. (4.39) is completely general.)

What then is the lowest possible exponent for $\nu$? For $f(\nu) \sim \nu^x$ where $x \leq 1$ we have:

$$4 + N + 2p + 3q - \bar{S} \leq N + p + q + S - \bar{S} \Rightarrow S \geq 4 + p + 2q.$$  \hspace{1cm} (4.40)

Thus for $p = q = 0$, $S = 4$ would give $f(\nu) \sim \nu$. Our expectation of $f(\nu)$ due to inverse triad corrections is only compatible with $f(\nu) \sim \nu$ in the case that there are no corrections or when $\nu$ is extremely large. However, the $S = 4$ case is never realized as it would require a four-term combination of only $H_1$ (or any $H_{2k+1}$ with $k \geq 0$) that still gives $b^0$ in the effective Friedmann equation. We do have terms with only these two factors in $H_Q^2$ but they correspond to positive, even powers of $b$ (in fact, $H_jH_k$ corresponds to terms with $b^{j+k}$). Similarly, terms with only these two factors from $\dot{\nu}^2$ also correspond to positive or zero powers of $b$ and thus the only way we could achieve a $\nu$-independent correction would require a $b^2$ or higher dependence.

Hence, for $b^0$, $S$ is an integer $\leq 3$. Additionally, $S \geq \bar{S}$ for all possible contributions while $p \geq 0$ and $q \geq 0$ since they are formed from our derivative terms which always lead to positive powers. Thus we see that all possible cases that could lead to a term independent of $\nu$ and $b$ in the effective Friedmann equation requires $f(\nu) \sim \nu^x$ with $x \geq 5/4$ which is incompatible with our expectation of $f(\nu)$ within our framework for an isotropic, homogeneous universe with a free, massless scalar field.

### 4.3.3 Factor Ordering Numerics

Though we did not find a term independent of $\nu$ and $b$ even in the very generalized case, we can still make some specific choices and numerically see how factor ordering might affect our equations of motion. For this example, we will include only $H_1$ and $H_{-1}$ terms and continue our focus on the regime of large volume and small curvature. We begin with our expanded Hamiltonian from Eqn. (4.18) and truncate accordingly:
\[ H = H_1 b + H_{-1} b^{-1}. \] (4.41)

This now encapsulates only inverse-triad corrections and factor ordering. Following the procedure outlined previously, we then arrive at our quantum Hamiltonian, up to second order in moments:

\[ H_Q = H_1 b + H_{-1} b^{-1} + \frac{1}{2} G^{\nu\nu}(H_1'' b + H_{-1}'' b^{-1}) + G^{bb}(H_1' - H_{-1}' b^{-2}) + G^{bb} H_{-1} b^{-3} \] (4.42)

We can then solve for the full system of equations of motion by Poisson brackets between our variables (classical and quantum) and the quantum Hamiltonian, where the \( \dot{\cdot} \) represents derivatives with respect to \( \phi \) as before.

\[
\begin{align*}
\dot{\nu} &= \frac{8 \pi G \gamma}{3} \left( -H_1 + H_{-1} b^{-2} + 3 H_{-1} G^{bb} b^{-4} + \right. \\
&\quad \left. \frac{G^{\nu\nu}}{2}(-H_1'' + H_{-1}'' b^{-2}) + 2 G^{bb} H_{-1}' b^{-3} \right) \\
\dot{b} &= \frac{8 \pi G \gamma}{3} \left( H_1' b + H_{-1}' b^{-1} + \frac{G^{\nu\nu}}{2} (H_1'' b + H_{-1}'' b^{-1}) + \\
&\quad G^{bb} (H_1' - H_{-1}' b^{-2}) + G^{bb} H_{-1}' b^{-3} \right) \\
\dot{G}^{bb} &= \frac{8 \pi G \gamma}{3} \left( 2 G^{bb} (H_1'' b + H_{-1}'' b^{-1}) + 2 G^{bb} (H_1' - H_{-1}' b^{-2}) \right) \\
\dot{G}^{\nu\nu} &= \frac{8 \pi G \gamma}{3} \left( -4 G^{bb} H_{-1} b^{-3} - 2 G^{\nu\nu} (H_1' - H_{-1}' b^{-1}) \right) \\
\dot{G}^{bb} &= \frac{8 \pi G \gamma}{3} \left( -2 G^{bb} H_{-1} b^{-3} + G^{\nu\nu} (H_1'' b + H_{-1}'' b^{-1}) \right).
\end{align*}
\] (4.43)

(4.44)

(4.45)

(4.46)

(4.47)

We can then simplify this system by truncating higher derivatives of \( H_n \) as those correspond to increasing inverse powers of \( \nu \). Keeping \( H_1 \sim f'(\nu) \sim 1 - \ell_0^b \nu^{-2} \) and \( H_{-1} \sim \pm f'(\nu)^2 \nu^{-1} \) to first order we are left with

\[
\begin{align*}
\dot{b} &= \frac{8 \pi G \gamma}{3} b \\
\dot{G}^{bb} &= \frac{8 \pi G \gamma}{3} 2 G^{bb} \\
\dot{G}^{bb} &= \frac{8 \pi G \gamma}{3} \left( \mp 2 G^{bb} \frac{1}{\nu b^3} \right).
\end{align*}
\] (4.48)

(4.49)

(4.50)
\[ \dot{G}^{\nu
u} = \frac{8\pi G\gamma}{3} \left( \mp 4G_{b\nu} - \frac{G_{\nu\nu}}{v^3b^3} - 2G^{\nu\nu} \right) \]  
\[ \dot{\nu} = \frac{8\pi G\gamma}{3} \left( -\nu \pm \frac{1}{\nu b^2} \pm \frac{3G_{bb}}{v^4b^4} \right) \]

where the ± comes from \( H_{-1} \) and the particular choice in factor ordering. Specifically the example we used in Section 4.3.1 had a negative sign (note that we did not need to take strict care of signs – or even numerical factors – in the previous section where the interest was for any terms which were independent of \( \nu \) and \( b \)). We can now solve this system of equations analytically; Eqns. (4.48) and (4.49) are independent of the other variables and greatly simplify the task.

\[ b = C_1 e^\phi \]  
\[ G_{bb} = C_2 e^{2\phi} \]  
\[ G^{\nu\nu}_+ = C_5 e^{-2\phi} - \frac{4e^{-2\phi}}{C_1^2(C_1^2 + 3C_2)} \left( C_1 C_4 \frac{\sqrt{C_1^2 C_3 + 2C_1^2 \phi + 6C_2 \phi}}{C_1^2 + 3C_2} + C_2 (C_1^4 C_3 + 2C_1^2 \phi + 6C_2 \phi) \right) \]  
\[ G^{\nu\nu}_- = C_5 e^{-2\phi} + \frac{4e^{-2\phi}}{C_1^2(C_1^2 + 3C_2)} \left( C_1 C_4 \frac{\sqrt{C_1^2 C_3 - 2C_1^2 \phi - 6C_2 \phi}}{C_1^2 + 3C_2} - C_2 (C_1^4 C_3 - 2C_1^2 \phi - 6C_2 \phi) \right) \]  
\[ G^{bb} = - \frac{C_2 \sqrt{\pm (2C_1^2 + 6C_2) \phi + C_1^4 C_3}}{C_1^2(C_1^2 + 3C_2)} + C_4 \]  
\[ \nu = \frac{e^{-\phi} \sqrt{\pm (2C_1^2 + 6C_2) \phi + C_1^4 C_3}}{C_1^2} \]

where our coefficients can be chosen to meet our initial conditions accordingly. Note that depending on the sign of \( H_{-1} \) we get ± in the equations for \( \nu(\phi) \) and \( G^{bb}(\phi) \) as well as slightly different entries for \( G^{\nu\nu}(\phi) \) labeled by the ± subscript. Looking at Eqn. (4.58), we see that there is a possibility of negative \( \phi \) within the squareroot, an indication that there could be turnover in the volume as \( \phi \to \pm \infty \) (depending on the sign of the factor ordering correction of \( H_{-1} \)). We also see that the constants under the squareroot will play an important role, with \( C_1 = b_0, C_2 = C_0^{bb}, \) and \( C_3 = \nu_0 \).
Let us then explore this system numerically for $H_{-1}$ positive, $\nu_0 = 10, b_0 = 1, G_{0}^{bb} = .2, G_{0}^{\nu\nu} = 2, G_{0}^{b\nu} = .25$, noting that our initial values satisfy the uncertainty relation (where we set $\hbar = 1$ for our numerics)

$$G^{\nu\nu}G^{bb} - (G^{b\nu})^2 \geq \frac{\hbar^2}{4}. \quad (4.59)$$

Using our truncated equations of motion, Eqns. (4.48) – (4.52), we can easily verify that the uncertainty relation is constant, that is:

$$\frac{d}{d\phi} \left( G^{\nu\nu}G^{bb} - (G^{b\nu})^2 \right) = 0 \quad (4.60)$$

and so will remain satisfied for all $\phi$. In Figure 4.1 we see that our values do indeed lead to a collapse at late negative time $\phi$ (plotted with $\phi$ increasingly negative to the right for convenience, as we will do with all of this section’s numerics). However, for this plot to be meaningful, we must verify the consistency of our assumptions within this region. In Figure 4.2 we see that the relative volume...
Figure 4.2. Relative fluctuation of volume $G^{\nu\nu}(\phi)/\nu^2(\phi)$ is plotted with a $-\phi$ axis. It is well behaved over a large range of values but does exponentially increase at $-\phi > 30.75$ and is increasingly larger for $-\phi < -20$ where $\nu(\phi)$ becomes exponentially smaller.

fluctuation $G^{\nu\nu}(\phi)/\nu^2(\phi)$ is well behaved for most of $\phi$, only increasing to the upper limits of our assumption of small relative fluctuations when $\nu(\phi)$ becomes exponentially small for $-\phi < -20$. In fact, from our analytic solutions given by Eqns. (4.55) and (4.58), we see that the relative volume fluctuations $G^{\nu\nu}/\nu^2$ asymptotically approaches a local maximum for $-\phi \to -\infty$:

$$\lim_{\phi \to -\infty} \frac{G^{\nu\nu}(\phi)}{\nu^2(\phi)} = \frac{4C_1^2C_2}{(C_1^2 + 3C_2)^2} = \frac{4b_0^2G_0^{bb}}{(b_0^2 + 3G_0^{bb})^2}$$

which is .3125 for the numerical example used for our graphs (however, graphing the large $\phi$ regime numerically would require a very large number of steps within Mathematica’s NDSolve function). However, near the turning point of $\nu$ at $-\phi > 30$, the relative volume fluctuation begins to exponentially increase, quickly surpassing our assumption of small relative values. From Eqns. (4.53) and (4.54) we see that relative curvature fluctuations $G^{bb}(\phi)/b^2(\phi)$ will remain constant according to where we set them by our constants $b_0$ and $G_0^{bb}$.

In Figure 4.3 we see that the relative covariance $G^{b\nu}/(b(\phi)\nu(\phi))$ also surpasses
Figure 4.3. Relative covariance fluctuation of the volume and curvature expectation values $G^{bo}(\phi)/(b(\phi)\nu(\phi))$ is plotted with a $-\phi$ axis. It is reasonably well behaved over a large range of values, though it increases unboundedly at large $-\phi$, which is inconsistent with our semiclassical approximations in that region.

our assumption of relatively small fluctuations as it approaches the turning point of $\nu$, becoming unboundedly large as $\nu$ collapses at large $-\phi > 30$. Thus, while we see collapse in Figure 4.1 it coincides with a breakdown of our assumptions in the regime where $b$ becomes exponentially small: a regime where higher inverse powers of curvature can become important, even when paired with inverse volume terms. We can also explore the opposite sign choice for $H_{-1}$ (negative) which is what our example from Subsection 4.3.1 had. In that case, the squareroot in $\nu(\phi)$ of Eqn. 4.1 goes to zero as $\phi \to \infty$. However, rather than insinuating a recollapse, this is the regime where $\nu(\phi) \to 0$ and $b(\phi)$ gets exponentially large. Regardless of sign choice, we would undoubtedly require a higher curvature expansion for our holonomy corrections before any physical meaning can be drawn from such analysis.
4.4 Conclusion

This chapter discusses the influence of factor-ordering choices on effective equations primarily in loop quantum cosmology. As we have emphasized, the quantization of the Hamiltonian constraint is far from being unique even in the most reduced models. Even though deparameterized models may offer simple quantization choices, they are not the most general or natural ones. Quantum corrections and an analysis of semiclassical physics must therefore take these ambiguities into account.

We have implemented such an analysis to analyze the question of whether quantum effects in long-term semiclassical evolution could lead to significant departures from classical behavior, for instance a recollapse of spatially flat isotropic models. The generality of effective equations indeed allows us to draw conclusions, indicating that too-drastic effects do not occur. Our results therefore provide support for the correct semiclassical behavior of loop quantum cosmology.

In particular, it does not appear possible to have quantum corrections of the modified Friedmann equation independent of the energy density in this formulation. Thus, quantum collapse at low energy densities or large scales does not seem feasible by this mechanism given our corrections and assumptions. The only possibility, according to our analysis, would be to have inverse-triad corrections with a function $f(\nu)$ increasing more strongly than linearly, but this is not acceptable by the classical limit.

Furthermore, we explored the effects of factor ordering choices at first order. Though the solutions to the effective equations hint at the possibility of a recollapse, our relative fluctuations are not small enough to satisfy our approximations for all $\phi$ of interest. Expanding such an analysis to higher quantum moments could be fruitful; for instance, the problem may be solved by corrections to $b$ such that it does not asymptotically approach zero, thus preventing factor ordering terms $1/b$ to approach infinity.
4.5 Appendix - Holonomy and Inverse Triad Numerics

In our example from Eqn. (4.11), we were only interested in the dynamics of $\nu$ as they related to the Friedmann equation of Eqn. (4.14). In this appendix though, we will explore the dynamics of the full system of equations of motion from that example. Recall that our quantum Hamiltonian for this example (up to second order in moments) is:

$$H_Q = \frac{1}{\gamma} \sqrt{\frac{27}{16\pi G}} \left( f(\nu) \sin b - \frac{1}{2} G^{bb} f(\nu) \sin b + G^{b\nu} f'(\nu) \cos b + \frac{1}{2} G^{\nu\nu} f''(\nu) \sin b \right)$$

which gives the following evolution equations:

$$\dot{\nu} \equiv \frac{d\langle \nu \rangle}{d\phi} = \{\nu, H_Q\}$$

$$\dot{b} = \sqrt{12\pi G} \left( f'(\nu) \sin b - \frac{1}{2} G^{bb} f'(\nu) \sin b + G^{b\nu} f''(\nu) \cos b + \frac{1}{2} G^{\nu\nu} f'''(\nu) \sin b \right)$$

$$\dot{G}^{bb} = 2G^{b\nu}(\nu) \sin b + 2G^{bb} f(\nu) \sin b$$

$$\dot{G}^{\nu\nu} = \sqrt{12\pi G} \left( -2G^{\nu\nu} f'(\nu) \cos b + 2G^{b\nu} f(\nu) \sin b \right)$$

$$\dot{G}^{b\nu} = \sqrt{12\pi G} \left( G^{\nu\nu} f''(\nu) \sin b + G^{bb} f(\nu) \sin b \right).$$

To solve the system, we drop the numerical factors and make a simplification for the small curvature regime such that $\cos b \rightarrow 1$ and $\sin b \rightarrow b$, we assume that $f(\nu) \sim \nu$ such that $f'(\nu) \sim 1$, and we focus on small relative quantum fluctuations to maintain the validity of our solutions. We then have the following much simpler system of equations:

$$\dot{\nu} = -\nu + \frac{1}{2} G^{bb} \nu + G^{b\nu} b$$ (4.62)

$$\dot{b} = b - \frac{1}{2} G^{bb} b$$ (4.63)

$$\dot{G}^{bb} = 2G^{bb}$$ (4.64)
Figure 4.4. Solutions of effective equations for expectation values of volume $\nu(\phi)$ (top) and curvature $b(\phi)$ (bottom). Initial values have been set to $\nu_0 = 10, b_0 = .4, G^\nu_0 = .2, G^{\nu\nu}_0 = 5, $ and $G^{b\nu}_0 = .5$ such that our solutions are valid at time $\phi = 0$. In addition to the bounce at small $\nu(\phi)$, notice that there is some asymmetry in both graphs and that $b$ increases to where $(\sin b)/b \approx .9$, pressing the limits of our assumption.

\[
\begin{align*}
\dot{G}^{\nu\nu} &= -2G^{\nu\nu} + 2G^{b\nu} \nu b \\
\dot{G}^{b\nu} &= G^{bb} \nu b .
\end{align*}
\] (4.65) (4.66)

Before we numerically solve this system, note that $b$ and $G^{bb}$ are separable. In fact,
Figure 4.5. Relative fluctuation of volume $G^{\nu \nu}(\phi)/\nu^2(\phi)$ is plotted and shown to be within the validity of the effective equations at inclusion of second order moments.

if we set $G^{b\nu} = 0$ for all times, we can solve this system explicitly.

\begin{align*}
\nu &= C[3] \exp \left[-1\phi + 0.25e^{2\phi}C[1]\right] \\
b &= C[2] \exp \left[1\phi - 0.25e^{2\phi}C[1]\right] \\
G^{bb} &= C[1]e^{2\phi} \\
G^{\nu \nu} &= C[4]e^{-2\phi}.
\end{align*}

We will find that these solutions match very well qualitatively with the numerical solutions we get even when $G^{b\nu} \neq 0$, and so they are useful to have an idea of the form of the solutions. For our initial values, we make sure to satisfy our assumptions above as well as the uncertainty relation

\begin{equation}
G^{\nu \nu}G^{bb} - (G^{b\nu})^2 \geq \frac{\hbar^2}{4}.
\end{equation}

In fact, from our equations above, it is easy to verify that if we satisfy the uncer-
Figure 4.6. Relative fluctuation of curvature $G^{bb}(\phi)/b(\phi)$ is plotted and shown to increase beyond the validity of the model as $\phi$ increases through the bounce point.

tainty condition at any time $\phi$, it remains at the same uncertainty for all time.

\[ \frac{d}{d\phi} \left( G^{\nu\nu} G^{bb} - (G^{b\nu})^2 \right) = 0 \]  

(4.72)

Continuing on in our analysis, we choose $\nu_0 = 10, b_0 = 4, G_0^{bb} = .2, G_0^{\nu\nu} = 5$, and $G_0^{b\nu} = .5$ for the following analysis. In Figure 4.4 we see the effective equation solutions for the expectation values of volume $\nu(\phi)$ and curvature $b(\phi)$. As one would expect from the inclusion of holonomies as well as the resulting Friedmann equation of this model seen in Eqn (4.14) there is a bounce as $\nu(\phi)$ becomes small (for our values, the bounce occurs near $\phi = .925$ with $\nu_{\text{min}} \approx 5.83$). Additionally, $b(\phi)$ reaches a maximum near $.8$ leading to a 10% variation from our assumption of $\sin b \approx b$.

Though our uncertainty relation is held constant, our fluctuations can still change: in Figure 4.5 we see the relative fluctuation of volume $G^{\nu\nu}(\phi)/\nu^2(\phi)$ plotted. As it nears the bounce point and quantum effects become large, it starts to increase but stays fairly well behaved throughout, returning to near zero as $\nu(\phi)$
increases exponentially for $\phi > 2$. However, in Figure 4.6 we see that the relative fluctuation of curvature $G^{bb}(\phi)/b(\phi)$ is not so well behaved as it approaches and continues past the bounce point at $\phi = .925$. Despite $b(\phi)$ increasing to a maximum within that range, $G^{bb}(\phi)$ grows exponentially, quickly surpassing our initial ratio for the relative fluctuations, and suggests that our simplifications and assumptions are not necessarily valid in this region without higher-curvature corrections and/or higher order moments.
Effective quantum cosmology is formulated with a realistic global internal time given by the electric vector potential. New possibilities for the quantum behavior of space-time are found using a Wheeler–DeWitt setting, and the high-density regime is shown to be very sensitive to the specific form of state realized.

5.1 Introduction

We now shift our focus to an investigation of the effects from two important notions in loop quantum cosmology: the choice of time and the sensitivity of the model to the state the universe is in. As discussed in Section 1.2 a common method to incorporate time is by deparameterization such as using a free, massless scalar or dust. However, if one wishes to consider the high curvature regime of the early universe, which is a radiation-dominated era, then those choices of global time are not realistic. Thus, in this model we will make a different choice and use electric fields as our time. Additionally, we will explore how the choice of state might affect the dynamics in this regime, where the more popular choices of Gaussian or semiclassical states are not necessarily apt. In this regime where quantum effects are expected to be strong, the departure from a semiclassical state can play an important role in the dynamics.
5.2 Radiation Hamiltonian

The Hamiltonian constraint for spatially flat isotropic Friedmann–Lemaître–Robertson–Walker models with radiation is

\[
H = -\frac{3}{8\pi G\gamma^2c^2}\sqrt{|p|} + \frac{E^2}{\sqrt{|p|}} = 0 \tag{5.1}
\]

with canonical gravitational variables \((c, p)\) with \(|p| = a^2\) and \(c = -\gamma \text{asgn}(p)\) (with the Barbero–Immirzi parameter \(\gamma\) relevant in loop quantizations [4, 51]) in terms of the scale factor \(a\), the derivative being by proper time. We have Poisson brackets \(\{c, p\} = 8\pi\gamma G/3\) while the momentum \(A\) of \(E\), \(\{A, E\} = 1\), does not appear in the Hamiltonian. The matter part is determined by the electric field \(E = |\vec{E}|\) assumed sufficiently small so as not to cause significant anisotropy. By writing the constraint in the form of a Friedmann equation, dividing \(H\) by \(|p|^{3/2}\), one can easily confirm that the matter term provides the correct behavior for radiation: the \(E\)-term amounts to an energy density \(\rho = E^2/p^2 = E^2/a^4\) where \(E\) is constant since \(H\) is independent of the conjugate momentum \(A\) of \(E\). (Alternatively, the Hamiltonian can be derived using the standard electric-field energy density: \(q_{ab}\vec{E}^a\vec{E}^b/\sqrt{\text{det}q}\) reduces to \(|p|E^2/|p|^{3/2}\) with an isotropic spatial metric \(q_{ab} = |p|\delta_{ab}\).)

The momentum of \(E\) is the electromagnetic vector potential and would contribute a non-zero term to \(H\) in the presence of a magnetic field. However, a magnetic field requires deviations from homogeneity for the rotation of \(\vec{A}\) to be non-zero. In the symmetric context used here, the restriction to pure electric fields is therefore meaningful. Since the electric field is canonically conjugate to the vector potential \(A\), which does not appear in the constraint, \(E\) is constant and \(A\) can be used as a global internal time. We will call this choice electric time. To realize \(A\)-evolution, we follow standard techniques of deparameterization and solve the constraint equation for the momentum

\[
p_A = E(c, p) = \pm \sqrt{\frac{3}{8\pi G\gamma^2}|c|\sqrt{|p|}}. \tag{5.2}
\]

\(^1\)The absolute value is taken with the flat Euclidean metric, \(|\vec{E}| = \delta_{ab}E^aE^b\), to keep the spatial metric \(q_{ab}\) as a physical degree of freedom independent of \(E\).
As a function on the gravitational phase space \((c, p)\), \(E(c, p)\) provides Hamiltonian equations of motion for the classical \(c(A)\) and \(p(A)\),

\[
\frac{dc}{dA} = \{c, E(c, p)\} \quad \text{and} \quad \frac{dp}{dA} = \{p, E(c, p)\},
\]
as well as the basis for the quantum Hamiltonian of effective equations with respect to \(A\). To transform equations or solutions to proper time \(\tau\), we can multiply all \(d/dA\) by \(dA/d\tau = \{A, H\} = 2E/\sqrt{|p|}\), using Eqn. (5.1). We confirm the correct classical equations of motion

\[
\frac{da}{d\tau} = \frac{\text{sgn}(p)}{2\sqrt{|p|}} \frac{dp}{d\tau} = \frac{E}{p} \frac{dp}{dA} = \frac{E}{p} \{p, E\} = \mp \sqrt{\frac{8\pi G}{3}} \frac{|E \text{sgn}(cp)|}{\sqrt{|p|}} = -\text{sgn}(p) \frac{c}{\gamma}
\]

substituting \(c\) for \(E\) in the last step, and

\[
\frac{1}{a} \frac{d^2a}{d\tau^2} = -\frac{\text{sgn}(p)}{\gamma a} \frac{dc}{d\tau} = -\frac{2E}{\gamma p} \frac{dp}{dA} = -\frac{2E}{\gamma p} \{c, E\} = \mp \frac{E}{\gamma p} \sqrt{\frac{8\pi G}{3}} \frac{|c| \text{sgn}(p)}{\sqrt{|p|}}
\]

where we have substituted \(E\) for \(c\) in the second line and used the electromagnetic expressions for energy density \(\rho\) and pressure \(p = \frac{1}{3} \rho\) to compare with the standard acceleration equation. (In what follows, we will set \(8\pi G/3 = 1\) and \(\gamma = 1\), so that \(\{c, p\} = 1\).)

### 5.3 Effective Dynamics

The sign in Eqn. (5.2) determines whether one considers solutions of positive or negative frequency with respect to time \(A\). Without loss of generality, we will use the negative choice, such that \(E(c, p) = -|c|\sqrt{|p|}\). Moreover, we can choose a definite sign of \(p\) (the orientation of space as measured by a triad) because we will consider only the approach to small \(p\), not a possible transition from positive to negative \(p\), or vice versa.\(^2\) Again without loss of generality, we will choose positive

\(^2\)If we wished to describe such a transition non-singularly, we would need to refer to wave functions which are solutions to a difference equation in loop quantum cosmology \([35, 52, 53]\).
Finally, since $E(c, p)$ is a conserved quantity, the sign of $c\sqrt{p}$ never changes dynamically and we can drop the absolute value, the two sign options here merging with the explicit $\pm$ in Eqn. (5.2). Even for quantum states, the fact that the Hamiltonian $E(c, p)$ and its quantization are conserved means that the absolute value can be dropped, provided that the expectation value $\langle c \sqrt{\hat{p}} \rangle$ is much larger than its quantum fluctuations.

We follow the procedure of canonical effective equations as outlined previously in Subsection 4.2.2 [6, 32]. That is, the quantum Hamiltonian $E_Q$ is a function on the quantum phase space with coordinates given by expectation values and moments

$$G_{c \cdots c, p \cdots p}^{m, n} = \langle (\hat{c} - \langle \hat{c} \rangle)^m (\hat{p} - \langle \hat{p} \rangle)^n \rangle_{\text{symm}} \quad (5.3)$$

where $m, n$ are positive integers such that $m + n \geq 2$ and the subscript denotes totally symmetric ordering of the operators. Recall that the moments and expectation values define a phase space with Poisson bracket $\{\langle \hat{A} \rangle, \langle \hat{B} \rangle\} = \langle [\hat{A}, \hat{B}] \rangle / i\hbar$ in terms of the commutator, extended by the Leibniz rule to arbitrary polynomials of the expectation values as they occur in moments. Recall that our semiclassical states can be expressed in terms of the moments which behave as $G_{m,n} \sim \hbar^{m+n}/2$ and that by the method of effective equations, we are allowed more generality than the 1- or at most 2-parameter families of Gaussians. Given a hierarchy, one can then truncate the infinitely many moments to a finite amount based on the desired level of approximation. This provides a practical method to study increasingly quantum regimes; higher moments, however, can lead to increasingly complex systems of equations which we do not study here, but efficient computational codes exist to derive and solve some systems to rather high orders [10]. Quantum corrections by higher moments are analogs of higher time derivatives in effective actions [54], amounting in quantum cosmology to important higher-curvature corrections.

The quantum Hamiltonian is a power series in the moments of $c$ and $p$, obtained by Taylor expanding the quantized $\langle \hat{E} \rangle = \langle E(\langle \hat{c} \rangle + (\hat{c} - \langle \hat{c} \rangle), \langle \hat{p} \rangle + (\hat{p} - \langle \hat{p} \rangle)) \rangle$ in $\hat{c} - \langle \hat{c} \rangle$ and $\hat{p} - \langle \hat{p} \rangle$:

$$E_Q := \langle \hat{E} \rangle = E(\langle \hat{c} \rangle, \langle \hat{p} \rangle) + \sum_{m, n} \frac{1}{m!n!} \frac{\partial^{m+n} E(\langle \hat{c} \rangle, \langle \hat{p} \rangle)}{\partial \langle \hat{c} \rangle^m \partial \langle \hat{p} \rangle^n} G_{m,n} \quad (5.4)$$
Since we assume an operator for $\hat{c}$ to exist, we will obtain the quantum Hamiltonian of a Wheeler–DeWitt quantization, as opposed to a loop quantization where only exponentials $\exp(i\delta \hat{c})$ exist, but no $\hat{c}$ \cite{1, 31}. Choosing totally symmetric ordering for $\hat{c} \sqrt{p}$ and expanding to quadratic terms with second-order moments (that is, $G^{cc}, G^{pp}$, and $G^{cp}$ which correspond to the variances and covariance respectively), we have

$$E_Q = -c \sqrt{p} - \frac{G^{cp}}{2 \sqrt{p}} + \frac{1}{8} \frac{G^{pp}}{p^{3/2}} c + \cdots,$$

(5.5)

Abbreviating $c = \langle \hat{c} \rangle$ and $p = \langle \hat{p} \rangle$ without risk of confusion. To this order, the Poisson structure provides effective equations

$$\frac{dp}{dA} = \sqrt{p} - \frac{1}{2} \frac{G^{pp}}{p^{3/2}},$$

(5.6)

$$\frac{dc}{dA} = -\frac{c}{2 \sqrt{p}} + \frac{1}{4} \frac{G^{cp}}{p^{3/2}} - \frac{3}{16} \frac{G^{pp}}{p^{5/2}} c,$$

(5.7)

$$\frac{dG^{pp}}{dA} = \frac{G^{pp}}{\sqrt{p}},$$

(5.8)

$$\frac{dG^{cp}}{dA} = \frac{1}{4} \frac{G^{pp}}{p^{3/2}} c,$$

(5.9)

$$\frac{dG^{cc}}{dA} = -\frac{G^{cc}}{\sqrt{p}} + \frac{1}{2} \frac{G^{cp}}{p^{3/2}} c,$$

(5.10)

derived from Hamiltonian equations of motion $df(c, p, \Delta(\cdot))/dA = \{f, E_Q(c, p, \Delta(\cdot))\}$.

We can then integrate all of Eqns. (5.6)–(5.10) perturbatively if we assume moments to be small throughout the whole evolution. We first solve the classical equations at zeroth order, ignoring all moments. We obtain

$$p_{\text{classical}}(A) = (\sqrt{p_0} + A/2)^2$$

and

$$c_{\text{classical}}(A) = \frac{c_0 \sqrt{p_0}}{\sqrt{p_0} + A/2},$$

with initial values $p_0$ and $c_0$ when $A = 0$. These solutions can then be assumed in the equations of motion for moments to find approximate solutions for the latter. We obtain

$$G^{pp}(A) \propto p(A), \quad G^{cp}(A) \propto -c(A) + \text{const} \quad \text{and} \quad G^{cc}(A) \propto c^4 + \text{const}' c^3 + \text{const}'' c^2.$$  

(5.11)
We then see relative fluctuations $G^{pp}/p^2 \propto p^{-1}$, $G^{cp}/(cp) \propto p^{-1}$ and $G^{cc}/c^2 \propto \text{const}$ remain small at small $c$. Moreover, the uncertainty product $G^{cc}G^{pp}$ is bounded from below by $\text{const}$, and the uncertainty relation will never be violated if we choose appropriate values for the constants. Thus solutions and their fluctuations remain valid (and increasingly more semiclassical) as we move towards large $p$, but conversely can have strong quantum effects and large fluctuations at small $p$. We will discuss the intrigues and limitations of these strong quantum effects in subsection 5.3.3 (which could depend on higher moment corrections).

5.3.1 Effective Equations in Proper Time

As we did in subsection 4.2.4 we can also discuss Eqns. (5.6)–(5.10) with regard to proper time. However we will see that, due to our new choice of internal time, the exact procedure to transform our equations of motion to proper time is slightly different. Classically, transforming from our deparameterized time to proper time involves evaluating $dA/d\tau = \{A, H[N]\}$ with a Hamiltonian constraint $H$ of lapse function $N = 1$ to find $dA/d\tau = 2E/\sqrt{p}$ with constant $E$. However, we are quantizing our deparameterized model: we solve the Hamiltonian constraint for $E(c,p)$ and then quantize, rather than quantizing $H$ directly. If we wish to return to a Hamiltonian constraint after our deparameterized quantization, we write $H_Q$ in terms of $E_Q$. This provides us with the corrected Hamiltonian which gives rise to the original deparameterized model when using $A$, the momentum of $E$, as time.

Having chosen $A$ as our time variable prior to quantizing means that $A$ remains classical within the quantum evolution equations, but by making that choice, $A$ is now conjugate to $E_Q$ at the quantum level, requiring one to use the electromagnetic Hamiltonian to compute the relation between $A$ and proper time: that is, $dA/d\tau = \{A, H_Q[E_Q]\} = 2E_Q(c,p,\Delta(\cdot))/\sqrt{p}$ (as we did for $d\phi/d\tau \sim p_\phi/\nu$ in Eqn. 4.24). These models are in differing variables (and regimes), further increasing the difficulty in directly comparing them, but as the relation of each time variable to proper time does have a dependence on quantum effects, which are themselves dependent on the choice of internal time, the resulting quantum theories are not likely to be equivalent. Furthermore, neither choice is equivalent to a model where the Hamiltonian constraint is quantized prior to deparameterization;
thus it is important to realize that deparameterized quantizations ignore some quantum corrections of the Hamiltonian constraint (in fact, differing corrections are ignored based on the chosen internal time). Nevertheless, a given choice in internal time is consistent and relevant so long as we do not attempt to relate it to other possible deparameterizations with other choices of internal time. This is a problem that lies with most quantum cosmological models which rely on quantization after parameterization. Despite this, we hold that our results, while not fully complete, give possible examples of the different features a choice in time might lead to.

Continuing on, we find our system of equations with respect to proper time is

\[
\frac{dp}{d\tau} = E_Q \left( 2 - \frac{1}{4} G^{pp} \right) = -2c\sqrt{p} - \frac{G^{cp}}{\sqrt{p}} + \frac{1}{2} G^{pp} \frac{c}{p^{3/2}} \tag{5.12}
\]

\[
\frac{dc}{d\tau} = E_Q \left( -\frac{c}{p} + \frac{G^{cp}}{2p^2} - \frac{3}{8} \frac{G^{pp}}{p^3} e \right) = \frac{c^2}{\sqrt{p}} + \frac{1}{4} G^{pp} \frac{c^2}{p^{5/2}} \tag{5.13}
\]

\[
\frac{dG^{pp}}{d\tau} = 2E_Q \frac{G^{pp}}{p} = -2G^{pp} \frac{c}{\sqrt{p}} \tag{5.14}
\]

\[
\frac{dG^{cp}}{d\tau} = \frac{1}{2} E_Q \frac{G^{pp}}{p^2} c = -\frac{1}{2} G^{pp} \frac{c^2}{p^{3/2}} \tag{5.15}
\]

\[
\frac{dG^{cc}}{d\tau} = E_Q \left( -2\frac{G^{cc}}{p} + \frac{cG^{cp}}{p^2} \right) = 2G^{cc} \frac{c}{\sqrt{p}} - G^{cp} \frac{c^2}{p^{3/2}} \tag{5.16}
\]

where, consistent with our approximation, we have ignored quadratic terms in the small second-order moments because they would compete with higher-order moments which are ignored here. As one can check explicitly, the moments satisfy

\[
\frac{d}{d\tau} \left( G^{pp} G^{cc} - (G^{cp})^2 \right) = 0 \tag{5.17}
\]

so that the uncertainty product is preserved. Any departure of a state from saturating the uncertainty relation

\[
G^{pp} G^{cc} - (G^{cp})^2 \geq \frac{\hbar^2}{4} \tag{5.18}
\]

remains constant. If an initial state saturates the uncertainty relation, it will keep saturating it in this regime, and we obtain a dynamical coherent state.
5.3.2 Numerics and Comparison to Classical Solutions

Now let us solve the full system of Eqns. (5.6)–(5.10) numerically so that we can test the validity of our assumptions with a comparison to the classical solutions. We assume some initial normalization such that $p(A)|_{A=A_0} = p_0 = 1$ and leave the other initial conditions as parameters ($G_{pp}^0, G_{cc}^0, c_0 \in \mathbb{R}^+$ and $G_{cp}^0 \in \mathbb{R}$) which satisfy the uncertainty relation of Eqn. 5.18. While these effective equations are valid for small relative fluctuations ($G_{pp}^0 \ll p^2$), we choose initial values of large relative fluctuations ($G_{pp}^0 \approx p_0^2$) to demonstrate the effects of the quantum corrections. Even with such pronounced fluctuations, our effective equations return near to classical solutions at late time.

In Figure 5.1 we see $c(A)$ and $p(A)$ plotted as a ratio to their respective classical solutions we found before. In the initial regime of large fluctuations, the ratios quickly depart from the value one, assumed by choosing as classical initial conditions the initial expectation values of the effective equations. After this initial dip, as $p(A)$ grows the ratios asymptote back to one over a rather large range of electric time $A$. Note that for $c(A)$ however, the asymptote is not 1 (even if we were to extend $A$ to much larger values). This difference cannot be explained from a renormalization of the scale factor since $|p| = a^2$ and $c \sim \dot{a}$, but rather because the large initial fluctuations have significantly altered $E_Q$ as compared to our classical solution. This can be seen in Eqn. 5.5, where $E_Q$ (a constant of motion) is reduced. The rate of proper time will then change relative to the rate of internal time, thus leading to a rescaling of $c(A)$ by a constant. This rescaling is what we see in the plot of $c(A)$ compared to the classical solution, which we can confirm quantitatively. For $G_{pp}^0 = 0$, as chosen here, we have $E_Q = -c_0\sqrt{p_0}(1 - \frac{1}{8}G_{pp}^0/p_0^2) + \cdots$. Our choice of initial fluctuations reduces $E_Q$ by 12.5% compared to the classical $E$. Accordingly, the effective solution for $c(A)$ is rescaled by $E_Q/E = 0.875$ (with the classical $E = -c_0\sqrt{p_0}$ as per our initial expectation values).

This observation raises an interesting conceptual question. For our Hamiltonian system, we use $E_Q$ as the momentum of time, with implications only for the rate of proper time. However, it has a physical meaning as the electric field and would therefore be measurable if our system were realistic. The choice of an initial quantum state in a regime of strong fluctuations (or correlations) could have sizable implications in semiclassical regimes. Or put differently, if one requires effective
Figure 5.1. Solutions of effective equations for expectation values, plotted as their ratios to the classical solutions $c_{\text{classical}}(A)$ (top) and $p_{\text{classical}}(A)$ (bottom). Initial fluctuations have been set to rather large values — $G_{0}^{pp} = p_{0} = G_{0}^{cc} = c_{0} = 1$ with $G^{cp} = 0$ — to show the implications of quantum corrections more clearly. Nevertheless, the ratios to the classical solutions (with the same initial values $p_{0} = c_{0} = 1$) stay close to one.

solutions to reproduce the values of all observables once a semiclassical regime is reached, possible initial states are restricted. With $E_{Q}$ in (5.5) being an observable, one cannot just use the equations of motion generated by the Hamiltonian $E_{Q} = E_{Q}(c, p, \ldots)$ but must also keep it as a constraint on the allowed initial values even if the system has been deparameterized. As an example, let us require a final condition, that $c(A)$ asymptotes to the classical solution at late times. This can be
Figure 5.2. Solution of effective equations for expectation value of $c(A)$ plotted as a ratio to the classical solutions $c_{\text{classical}}(A)$. Initial values are as follows — $G_0^{pp} = p_0 = 1$, $c_0 = .8$ with $G_0^{cp} = .2$ — to show the implications of quantum corrections while forcing $c(A)|_{A=5000} = c_{\text{classical}}(A)|_{A=5000}$.

done by a judicious choice of initial conditions (for instance, by iteratively solving for $c_0$ such that the solution for $c(A)$ from the system of differential equations converges to $c_{\text{classical}}$ at late times). In Figure 5.2, we see that the ratio of $c(A)$ to the classical solution now does asymptote to 1 at large values of time $A$. However, as noted in our previous discussion, this requires that $E_Q/E \approx 1$ which we can quantitatively confirm using the initial values $G_0^{pp} = p_0 = 1$, $c_0 = .8$ with $G_0^{cp} = .2$; thus the quantum corrections in $E_Q = -c_0\sqrt{p_0}(1 + G_0^{cp}/2c_0p_0 - \frac{1}{8}G_0^{pp}/p_0^2) + \cdots$ cancel and we are left with $E_Q/E = 1$ as expected. Thus, if we wish to constrain our system to return to the classical solution at late time, $G_0^{cp}$ is no longer a free parameter and our possible initial states are indeed restricted. It is also instructive to view plots of the moments as they change in time, as they are additional independent degrees of freedom compared to the classical model. For these purposes, we will return to the original initial values of $G_0^{pp} = p_0 = G_0^{cc} = c_0 = 1$ with $G_0^{cp} = 0$. In
Figure 5.3. The $c$-variance $G^{cc}(A)$ (thin dashed) compared with the curvature expectation value $c(A)$ (thick), the covariance $G^{cp}(A)$ (thin) and $c_{\text{classical}}(A)$ (medium dashed) for large initial fluctuations $G^{pp}_0 = p_0 = G^{cc}_0 = c_0 = 1$ with $G^{cp}_0 = 0$. $c(A)$ matches well to $c_{\text{classical}}(A)$ (if we corrected the classical value by 87.5% they would be indistinguishable on this graph) and $G^{cc}$ decreases to zero quickly. $G^{cp}$ qualitatively agrees with the analytical solution $G^{cp}(A) \propto \text{const} - c(A)$ of Eqn. 5.11 with the appropriate constants.

Figure 5.3 we see the slight difference expected from the $E_Q/E = 0.875$ rescaling as mentioned before. Additionally, the $c$-variance $G^{cc}$ is small and $G^{cp}$ asymptotes to a constant as $c(A)$ shrinks, matching the analytical solution of Eqn. 5.11 with the appropriate constant, that is $G^{cp}(A) \propto \text{const} - c(A)$. In Figure 5.4 we see that the $p$-variance $G^{pp}(A)$ increases as $p(A)$, also in accordance with Eqn. 5.11. Though $G^{pp}(A) > p(A)$, relative fluctuations $G^{pp}(A)/p(A)^2$ are small and decreasing, indicating an approach to semiclassical behavior.

5.3.3 Analysis of Very Large Relative Fluctuations

We now briefly delve into the realm where quantum moments are relatively quite large while maintaining our solutions from Eqs. (5.6)–(5.10) — it is important to note that in this regime, our effective equations would require the inclusion of
Figure 5.4. The $p$-variance $G^{pp}(A)$ (dashed) compared with the expectation value $p(A)$ (solid). Both functions increase in nearly the same way even for large initial fluctuations as in Fig. 5.1 confirming our analytical solutions. We have $G^{pp} > p$, but relative fluctuations $G^{pp}/p^2$ become very small at large $A$.

Higher order moments to remain valid, and though we have neglected them here, it is nevertheless interesting to see what our solutions may implicate. Thus, in this subsection we explore only interesting possibilities and make no claims for the non-semiclassical values where our original assumptions break down. First, we note the possibility of $dp/\,dA = 0$ in Eqn. (5.6) at $G^{pp} = 8p^2$. Hence, at very large fluctuations, we see there is a possibility of a turning point in $p(A)^2$. We can then check the second derivative of Eqn. (5.6) with respect to $A$, the electric time, to test whether our turning point corresponds to a minimum or a maximum (potentially indicating a bounce or a recollapse respectively):

$$\frac{d^2 p}{\,dA^2} = \frac{1}{2} - \frac{3}{128} \frac{(G^{pp})^2}{p^4}. \quad (5.19)$$

A more detailed analysis specifically near such turning points, as motivated by [55], could lead to parity violation and non-trivial chiral effects for the resulting wave function of the Universe, but this would require higher orders of effective equations and their numerics.
At $G^{pp} = 8p^2$ we then have $\frac{d^2p}{dt^2} = -1$ suggesting a recollapse as seen in Figure 5.5. As mentioned before, however, if $G^{pp}$ becomes this important in the equations of motion, higher moments might also become significant and alter the behavior. With that in mind, we see there are effects which can be quite sensitive to the precise form of the quantum state. This specific example with quantum fluctuations leading to recollapse is reminiscent of other models in the literature \cite{28, 56}.

5.4 Conclusion

In this chapter, we have focused on the basic description of deparameterized quantum cosmology with time provided by the electric field. The only matter source required to formulate this time evolution is radiation, which is expected to be significant in any early-universe model. We did not require any artificial matter sources such as dust or free massless scalars.

Unlike in Chapter 4.3.1, we did not include any modifications suggested by loop quantum cosmology, such as holonomy and inverse-triad corrections. This is because including further non-linearities such as replacing $c$ with $\sin(\delta c)/\delta$ (as seen in Eqn. (4.2) and (4.5)) would provide higher-curvature correction terms on the order of the quantum back-reaction by the moments. Thus, it would be impossible to isolate the effects of either holonomy corrections or quantum back-reaction by fluctuations, which is why we chose to evaluate within a Wheeler–DeWitt model. Alternatively, if one wished to study only holonomy corrections without effects of quantum back-reaction, the focus should be on models in which such back-reaction is weak or nonexistent (such as models of harmonic cosmology \cite{12, 57} or kinetic-dominated regimes \cite{14, 15, 40}).

Discussion of the quantum dynamics was done at the effective level of Wheeler–DeWitt quantum cosmology, allowing for an analysis of large classes of states without requiring any specific wave functions (such as Gaussians). Considering quantum corrections by fluctuations, we have found several new possibilities for early-universe dynamics, showing a high sensitivity to the specific forms of states. Differences with regard to choice of internal time are not clear cut; comparing electric time to a free, massless scalar for time differs even at the level of classical dynamics. However, as we saw in this chapter, there are additional differences in
Figure 5.5. The expectation value $p(A)$ (top) shows recollapse while the fluctuation $G^{pp}(A)$ as a ratio of $p(A)^2$ (bottom) surpasses $8p(A)^2$ corresponding to a maximum, with initial values $G_0^{pp} = 4p_0 = 4, c_0 = G_0^c = 1$, and $G_0^{cp} = 0$. Important note: these demonstrate only interesting possibilities in a regime where our effective equations are no longer valid without consideration of higher order moments.

timing due to very general effects of quantum back-reaction. Further analysis is required before a statement can be made on whether the choice of deparameterization does or does not affect generic results, but it is an important problem we face as current results in (loop) quantum cosmology are often obtained with the popular choice of scalar internal time. Research in effective-constraint methods
may shed some light on the matter, but is rather complicated to perform. As such, we conclude that further work is necessary before the high-curvature regime of quantum cosmology can be controlled, including a better understanding of the role of internal times.
Appendix

Linear Discrete Schrödinger
Equation Numerics

1 Introduction

Mathematica definitions of functions for the solvable wavefunctions and the computed expectation values. Note: ** denotes the complex conjugate of the term to the left multiplied by the term to the right.

2 Mathematica Functions and Values

c[j_, m0_, a_, P_]:= 
Sum[1/(2*Pi)Exp[-I*j*m - a*(m - m0)^2] * Exp[I*P*(m - m0)], 
{m, -5 + m0, 5 + m0}]

Psi[n_, m0_, a_, t_, P_]:= 
NIntegrate[c[l, m0, a, P] * Exp[I*l*n] * Exp[-I*t*(2*Cos[l] - 2)], 
{l, -Pi, Pi}]

Vol[m0_, t_, P_]:= 
Sum[Psi[n, m0, 1., t, P] * n * Psi[n, m0, 1., t, P]/1.2713415231539325, 
{n, -2.5t - 5 + m0, 2.5t + 5 + m0}]
\begin{align*}
V_{\text{olsq}}[\text{m0}_-, \text{t}_-, \text{P}_-] := \\
\text{Sum}[\Psi[n, \text{m0}, 1., t, P] * \Psi[n, \text{m0}, 1., t, P]] / 1.2713415231539325, \\
\{n, -2.5t - 5 + \text{m0}, 2.5t + 5 + \text{m0}\} \\
\text{h}[\text{m0}_-, \text{t}_-, \text{P}_-] := \\
\text{Sum}[\Psi[n, \text{m0}, 1., t, P] * \Psi[n + 1, \text{m0}, 1., t, P]] / 1.2713415231539325, \\
\{n, -2t - 25 + \text{m0}, 2t + 25 + \text{m0}\} \\
\text{h}_2[\text{m0}_-, \text{t}_-, \text{P}_-] := \\
\text{Sum}[\Psi[n, \text{m0}, 1., t, P] * \Psi[n + 2, \text{m0}, 1., t, P]] / 1.2713415231539325, \\
\{n, -2t - 25 + \text{m0}, 2t + 25 + \text{m0}\} \\
\text{hbar}[\text{m0}_-, \text{t}_-, \text{P}_-] := \\
\text{Sum}[\Psi[n, \text{m0}, 1., t, P] * \Psi[n - 1, \text{m0}, 1., t, P]] / 1.2713415231539325, \\
\{n, -2t - 25 + \text{m0}, 2t + 25 + \text{m0}\} \\
\text{hbar}_2[\text{m0}_-, \text{t}_-, \text{P}_-] := \\
\text{Sum}[\Psi[n, \text{m0}, 1., t, P] * \Psi[n - 2, \text{m0}, 1., t, P]] / 1.2713415231539325, \\
\{n, -2t - 25 + \text{m0}, 2t + 25 + \text{m0}\} \\
\text{Vh}[\text{m0}_-, \text{t}_-, \text{P}_-] := \\
\text{Sum}[\Psi[n, \text{m0}, 1., t, P] * \Psi[n + 1, \text{m0}, 1., t, P]] / 1.2713415231539325, \\
\{n, -2t - 25 + \text{m0}, 2t + 25 + \text{m0}\} \\
\text{hV}[\text{m0}_-, \text{t}_-, \text{P}_-] := \\
\text{Sum}[\Psi[n - 1, \text{m0}, 1., t, P] * \Psi[n, \text{m0}, 1., t, P]] / 1.2713415231539325, \\
\{n, -2t - 5 + \text{m0}, 2t + 5 + \text{m0}\} \\
\text{Vhbar}[\text{m0}_-, \text{t}_-, \text{P}_-] := \\
\text{Sum}[\Psi[n, \text{m0}, 1., t, P] * \Psi[n - 1, \text{m0}, 1., t, P]] / 1.2713415231539325, \\
\{n, -2t - 5 + \text{m0}, 2t + 5 + \text{m0}\} \\
\end{align*}
\[ h_{\text{bar}}V[m_0, t, P] := \]
\[ \text{Sum}\left[\Psi[n + 1, m_0, 1., t, P] \ast \ast n \ast \Psi[n, m_0, 1., t, P]/1.2713415231539325, \right. \]
\[ \left\{n, -2t - 5 + m_0, 2t + 5 + m_0\right\} \]
\[ nc[j, m_0, a, P] := \text{Sum}\left[1/(2 \ast \text{Pi})\text{Exp}[-I \ast j \ast m - a \ast (m - m_0)^2] \ast \text{Exp}[I \ast P \ast (m - m_0)], \right. \]
\[ \left\{m, -5 + m_0, 5 + m_0\right\} \]
\[ nPsi[n, m_0, a, t, P] := \]
\[ \text{NIntegrate}[nc[l, m_0, a, P] \ast \text{Exp}[I \ast l \ast n] \ast \text{Exp}[-I \ast t \ast (2 \ast \text{Cos}[l] - 2)], \{l, -\text{Pi}, \text{Pi}\}] \]

Increased range of sum and used coefficient from highest sum (10^-9 change)

\[ nIP[m_0, t] := \]
\[ \text{Sum}[nPsi[n, m_0, 1., t, 0] \ast \ast nPsi[n, m_0, 1., t, 0]/1.2713415241201664, \]
\[ \left\{n, -3t - 25 + m_0, 3t + 25 + m_0\right\} \]
\[ \text{Sum}[nPsi[n, 500, 5, 20, 1] \ast \ast nPsi[n, 500, 5, 20, 1]/1.2713415241201664, \]
\[ \left\{n, -30 - 25 + 500, 30 + 25 + 500\right\} \]

the normalization changes with a (something to keep in mind) also, for small a, probably need a large range on the sum for c(lambda)

\[ nh[m_0, t, P] := \]
\[ \text{Sum}[nPsi[n, m_0, 1., t, P] \ast \ast nPsi[n + 1, m_0, 1., t, P]/1.2713415241201664, \]
\[ \left\{n, -3t - 25 + m_0, 3t + 25 + m_0\right\} \]
\[ nh_{\text{bar}}[m_0, t, P] := \]
\[ \text{Sum}[nPsi[n, m_0, 1., t, P] \ast \ast nPsi[n - 1, m_0, 1., t, P]/1.2713415241201664, \]
\[ \left\{n, -3t - 25 + m_0, 3t + 25 + m_0\right\} \]
\[ nh2[m_0, t, P] := \]
\[ \text{Sum}[nPsi[n, m_0, 1., t, P] \ast \ast nPsi[n + 2, m_0, 1., t, P]/1.2713415241201664, \]
\[ \left\{n, -3t - 25 + m_0, 3t + 25 + m_0\right\} \]
\[\text{nVol}[m_0, t, P] := \sum[nPsi[n, m_0, 1, t, P] \cdot nPsi[n, m_0, 1, t, P]/1.2713415241201664,\]
\[\{n, -3t - 25 + m0, 3t + 25 + m0\}\]

\[\text{nVol2}[m_0, t, P] := \sum[nPsi[n, m_0, 1, t, P] \cdot nPsi[n, m_0, 1, t, P]/1.2713415241201664,\]
\[\{n, -3t - 25 + m0, 3t + 25 + m0\}\]

\[\text{nVh}[m_0, t, P] := \sum[nPsi[n, m_0, 1, t, P] \cdot nPsi[n + 1, m_0, 1, t, P]/1.2713415241201664,\]
\[\{n, -3t - 25 + m0, 3t + 25 + m0\}\]

\[\text{nhV}[m_0, t, P] := \sum[nPsi[n - 1, m_0, 1, t, P] \cdot nPsi[n, m_0, 1, t, P]/1.2713415241201664,\]
\[\{n, -3t - 25 + m0, 3t + 25 + m0\}\]

\[\text{nVhbar}[m_0, t, P] := \sum[nPsi[n, m_0, 1, t, P] \cdot nPsi[n - 1, m_0, 1, t, P]/1.2713415241201664,\]
\[\{n, -3t - 25 + m0, 3t + 25 + m0\}\]

\[\text{nhbarV}[m_0, t, P] := \sum[nPsi[n + 1, m_0, 1, t, P] \cdot nPsi[n, m_0, 1, t, P]/1.2713415241201664,\]
\[\{n, -3t - 25 + m0, 3t + 25 + m0\}\]

\[\text{nhbar2}[m_0, t, P] := \sum[nPsi[n, m_0, 1, t, P] \cdot nPsi[n - 2, m_0, 1, t, P]/1.2713415241201664,\]
\[\{n, -3t - 25 + m0, 3t + 25 + m0\}\]

\[\text{nhVhbar}[m_0, t, P] := \sum[nPsi[n - 1, m_0, 1, t, P] \cdot nPsi[n - 1, m_0, 1, t, P]/1.2713415241201664,\]
\[\{n, -3t - 25 + m0, 3t + 25 + m0\}\]
\[
\begin{align*}
nhbarVh[m0_-, t_-, P_] &:= \\
&\text{Sum}[n\Psi[n + 1, m0, 1, t, P] \ast n \ast n\Psi[n + 1, m0, 1, t, P] / 1.2713415241201664,} \\
&\{n, -3t - 25 + m0, 3t + 25 + m0\} \\
nVVh[m0_-, t_-, P_] &:= \\
&\text{Sum}[n\Psi[n, m0, 1, t, P] \ast n^2 \ast n\Psi[n + 1, m0, 1, t, P] / 1.2713415241201664,} \\
&\{n, -3t - 25 + m0, 3t + 25 + m0\} \\
nhVV[m0_-, t_-, P_] &:= \\
&\text{Sum}[n\Psi[n + 1, m0, 1, t, P] \ast n^2 \ast n\Psi[n, m0, 1, t, P] / 1.2713415241201664,} \\
&\{n, -3t - 25 + m0, 3t + 25 + m0\} \\
nhhV[m0_-, t_-, P_] &:= \\
&\text{Sum}[n\Psi[n - 1, m0, 1, t, P] \ast n^2 \ast n\Psi[n, m0, 1, t, P] / 1.2713415241201664,} \\
&\{n, -3t - 25 + m0, 3t + 25 + m0\} \\
nhVh[m0_-, t_-, P_] &:= \\
&\text{Sum}[n\Psi[n - 2, m0, 1, t, P] \ast n \ast n\Psi[n, m0, 1, t, P] / 1.2713415241201664,} \\
&\{n, -3t - 25 + m0, 3t + 25 + m0\} \\
nVhh[m0_-, t_-, P_] &:= \\
&\text{Sum}[n\Psi[n - 1, m0, 1, t, P] \ast n \ast n\Psi[n + 1, m0, 1, t, P] / 1.2713415241201664,} \\
&\{n, -3t - 25 + m0, 3t + 25 + m0\} \\
nVhV[m0_-, t_-, P_] &:= \\
&\text{Sum}[n\Psi[n, m0, 1, t, P] \ast n \ast n\Psi[n + 2, m0, 1, t, P] / 1.2713415241201664,} \\
&\{n, -3t - 25 + m0, 3t + 25 + m0\} \\
nVVhbar[m0_-, t_-, P_] &:= \\
&\text{Sum}[n\Psi[n, m0, 1, t, P] \ast n^2 \ast n\Psi[n - 1, m0, 1, t, P] / 1.2713415241201664,} \\
&\{n, -3t - 25 + m0, 3t + 25 + m0\}
\end{align*}
\]
\[
\begin{align*}
nhbarVV[m0\_, t\_, P\_]:= & \\
& \text{Sum}[n\Psi[n + 1, m0, 1., t, P] \ast n^2 \ast n\Psi[n, m0, 1., t, P]/1.2713415241201664,} \\
& \{n, -3t - 25 + m0, 3t + 25 + m0]\] \\
nVhbarV[m0\_, t\_, P\_]:= & \\
& \text{Sum}[n\Psi[n, m0, 1., t, P] \ast n \ast (n - 1) \ast n\Psi[n - 1, m0, 1., t, P]/1.2713415241201664,} \\
& \{n, -3t - 25 + m0, 3t + 25 + m0]\] \\
nhbarVh[m0\_, t\_, P\_]:= & \\
& \text{Sum}[n\Psi[n + 1, m0, 1., t, P] \ast n \ast n\Psi[n + 1, m0, 1., t, P]/1.2713415241201664,} \\
& \{n, -3t - 25 + m0, 3t + 25 + m0]\] \\
nhbarhbarV[m0\_, t\_, P\_]:= & \\
& \text{Sum}[n\Psi[n + 2, m0, 1., t, P] \ast n \ast n\Psi[n, m0, 1., t, P]/1.2713415241201664,} \\
& \{n, -3t - 25 + m0, 3t + 25 + m0]\] \\
nhbarVhbar[m0\_, t\_, P\_]:= & \\
& \text{Sum}[n\Psi[n + 1, m0, 1., t, P] \ast n \ast n\Psi[n - 1, m0, 1., t, P]/1.2713415241201664,} \\
& \{n, -3t - 25 + m0, 3t + 25 + m0]\] \\
nVhbarhbar[m0\_, t\_, P\_]:= & \\
& \text{Sum}[n\Psi[n, m0, 1., t, P] \ast n \ast n\Psi[n - 2, m0, 1., t, P]/1.2713415241201664,} \\
& \{n, -3t - 25 + m0, 3t + 25 + m0]\] \\
nVin[m0\_, t\_, P\_]:= & \\
& \text{Sum}[n\Psi[n, m0, 1., t, P] \ast n \ast n\Psi[n + 1, m0, 1., t, P]/1.2713415241201664,} \\
& \{n, -3t - 25 + m0, 3t + 25 + m0]\] \\
nhVin[m0\_, t\_, P\_]:= & \\
& \text{Sum}[n\Psi[n - 1, m0, 1., t, P] \ast n \ast n\Psi[n, m0, 1., t, P]/1.2713415241201664,} \\
& \{n, -3t - 25 + m0, 3t + 25 + m0]\]
nVinhbar[m0_, t_, P_] := 
Sum[nPsi[n, m0, 1, , t, P] / n * nPsi[n - 1, m0, 1, , t, P]/1.2713415241201664,
{n, -3t - 25 + m0, 3t + 25 + m0}]

nhbarVin[m0_, t_, P_] := 
Sum[nPsi[n + 1, m0, 1, , t, P] / n * nPsi[n, m0, 1, , t, P]/1.2713415241201664,
{n, -3t - 25 + m0, 3t + 25 + m0}]

nVVV[m0_, t_, P_] := 
Sum[nPsi[n, m0, 1, , t, P] * nPsi[n + 3, m0, 1, , t, P]/1.2713415241201664,
{n, -3t - 25 + m0, 3t + 25 + m0}]

code
\(-0.0563193 + 0.12306i\)

\[
G_{hh} = n_h^2 a - n_h a^2
\]

\(0.0882124 - 0.192748i\)

\[
G_{hbar} = 1. - n_h * n_{hbar}^{10}
\]

\(0.652691 + 4.1230612306^{*^-13}i\)

\[
G_{hbarhbar} = n_{hbar}^2 t^{10} - n_{hbar}^{10^2}
\]

\(0.0882124 + 0.192748i\)

\[
\text{Conjugate}[G_{hh}] - G_{hbarhbar}
\]

\(1.1366685370717278^{*^-11} + 2.497640982923599^{*^-11}i\)

\[
dG_{Vhdt} = I * (G_{hh} - G_{hbarhbar})
\]

\(0.192748 - 0.564478i\)

\[
G_{Vh0} = 0.5 * (n_{Vh0} + n_{hV0}) - n_{Vol0} * n_h
\]

\(1.9238211734773358^{*^-7} + 2.909345937496255^{*^-7}i\)

\[
G_{Vh50} = 0.5 * (n_{Vh50} + n_{hV50}) - n_{Vol50} * n_h
\]

\(9.63738 - 28.2239i\)

\[
G_{Vh100} = 0.5 * (n_{Vh100} + n_{hV100}) - n_{Vol100} * n_h
\]

\[
G_{Vht}[t_] = dG_{Vhdt} * t + G_{Vh0}
\]

\[(1.9238211734773358^{*^-7} + 2.909345937496255^{*^-7}i) + (0.192748 - 0.564478i) t\]

\[
G_{Vhbar0} = 0.5 * (n_{Vhbar0} + n_{hbarV0}) - n_{Vol0} * n_{hbar10}
\]

\(1.993009618292398^{*^-7} - 3.0106247095318395^{*^-7}i\)

\[
G_{Vhbar10} = 0.5 * (n_{Vhbar10} + n_{hbarV10}) - n_{Vol10} * n_{hbar10}
\]
1.92748 + 5.64478i

\[ GV_{\text{hbar}}[t_] = -I \ast (G_{\text{hbarhbar}} - G_{\text{hbarhbar}}) \ast t + GV_{\text{hbar}0} \]

\[ (1.9930095618292398 \ast -7 - 3.0106247095318395 \ast -7i) + (0.192748 + 0.564478i)t \]

\[ (GV_{\text{hbar}10} - GV_{\text{hbar}[10]})/GV_{\text{hbar}[10]} \]

\[-1.2072577033033693 \ast -8 - 1.1658068282709584 \ast -8i \]

\[ GV_{\text{ht}[50]} \]

"9.63738" − "28.2239"i

9.63738 − 28.2239i

\[ GV_{\text{ht}[100]} \]

"19.2748" − "56.4478"i

19.2748 − 56.4478i

\[ \text{Im}[GV_{\text{ht}[50]} - GV_{\text{h50}}]/\text{Im}[GV_{\text{h50}}] \]

1.56681398838015 \ast -8

\[ \text{Im}[GV_{\text{ht}[100]} - GV_{\text{h100}}]/\text{Im}[GV_{\text{h100}}] \]

9.228879795158633 \ast -9

\[ nV[t_] = 2 \ast t \ast \text{Im}[n_{\text{hbar10}}] + 500. \]

500. − 0.991808t

\[ \text{Re}[n_{\text{Vol10}} - nV[10]]/nV[10] \]

−1.549724276687308 \ast -9

\[ n_{\text{hVhbar[t]}} = -2 \ast \text{Im}[n_{\text{h}}] \ast t + n_{\text{Vhbar0}} \]

(501. − 1.404682334801567 \ast -15i) − 0.991808t
\[
\text{Re}[\text{nhVhbart}[50] - \text{nhVhbar50}]/\text{Re}[\text{nhVhbart}[50]]
\]

8.221675080464202*^-10

\[
\text{nbarVht}[t_] = -2 \times \text{Im}[\text{nha}] \ast t + \text{nbarVh0}
\]

\[
(499. + 1.2687265638358405*^15i) - 0.991808t
\]

\[
\text{Re}[\text{nbarVht}[50] - \text{nbarVh50}]/\text{Re}[\text{nbarVht}[50]]
\]

8.22511982354974*^-10

\[
\text{GVhhbar}[t_] = 2/3 \ast \text{nV}[t] + 1/6 \ast (\text{nhVhbart}[t] + \text{nbarVht}[t]) -
\]

\[
(\text{GVht}[t] \ast \text{nbar10} + \text{Ghhbar} \ast \text{nV}[t] + \text{GVhbar}[t] \ast \text{nha}) - \text{nV}[t] \ast \text{nha} \ast \text{nbar10}
\]

\[
\text{GVhbar1}[t_] = 2/3 \ast \text{nV}[t] + 1/6 \ast (\text{nhVhbart}[t] + \text{nbarVht}[t]) - \text{nV}[t] -
\]

\[
\text{nha} \ast \text{GVhbar}[t] - \text{nbar10} \ast \text{GVht}[t]
\]

\[
\text{GVhbar1}[100]
\]

43.7106 - 4.601759200469662*^10i

\[
\text{GVhbar}[100]
\]

43.7106 - 4.601759200469662*^10i

\[
\text{dGVhbardt} = I \ast \text{nha} \ast (\text{Gbarhbar} - \text{Ghbar}) - I \ast \text{nbar10} \ast (\text{Gh} - \text{Ghbar})
\]

0.437106 - 2.5398017022837394*^-12i

\[
\text{GVhbart}[t_] = \text{dGVhbardt} \ast t + \text{GVhbar}[0]
\]

\[
(-5.47254836827414*^12 - 2.061986532399971*^10i) + (0.437106 - 2.5398017022837394*^-12i) \ast t
\]

\[
(\text{GVhbart}[100] - \text{GVhbar}[100])/\text{GVhbar}[100]
\]

3.089413576167573*^11 - 6.642343799052856*^17i

\[
\text{nhhVt}[t_] = I \ast (\text{nhh10} - \text{nha}) \ast t + \text{nhhV0}
\]
\((-28.216 + 61.6531i) + (0.494381 - 0.329102i)t\)

\((\text{nhhV}30 - \text{nhhVt}[30])/\text{nhhVt}[30]\)

\(-1.6274102443291797^{*^\land}10 + 9.1677883381056^{*^\land}13i\)

\(\text{nhVht}[t_] = I \ast (\text{nhhh10} - \text{nha}) \ast t + \text{nhVh0}\)

\((-28.1597 + 61.53i) + (0.494381 - 0.329102i)t\)

\((\text{nhVh}30 - \text{nhVht}[30])/\text{nhVht}[30]\)

\(-1.628021892926963^{*^\land}10 + 8.635379308546834^{*^\land}13i\)

\(\text{nVhht}[t_] = I \ast (\text{nhhh10} - \text{nha}) \ast t + \text{nVhh0}\)

\((-28.1034 + 61.407i) + (0.494381 - 0.329102i)t\)

\((\text{nVhh}30 - \text{nVhht}[30])/\text{nVhht}[30]\)

\(-1.628630322844995^{*^\land}10 + 8.084818405376381^{*^\land}13i\)

\(\text{dnVVhdt}[t_] = I \ast (\text{nVhht}[t] + \text{nhVht}[t] - \text{nhbarVh}[t] - \text{nV}[t])\)

\(i((-1055.26 + 122.937i) + (2.97238 - 0.658204i)t)\)

\(\text{nVVht}[t_] = \text{Integrate}[\text{dnVVhdt}[t], t] + \text{nVVh0}\)

\((79445. + 123728.i) - (122.937 + 1055.26i)t + (0.329102 + 1.48619i)t^2\)

\((\text{nVVh100} - \text{nVVht}[100])/\text{nVVht}[100]\)

\(-9.20674868279385^{*^\land}11 + 9.029537575265571^{*^\land}10i\)

\(\text{dnVhVdt}[t_] = I \ast (\text{nVhht}[t] + \text{nhhVt}[t]) - 2 \ast I \ast \text{nV}[t]\)

\(-2i(500. - 0.991808t) + i((-56.3193 + 123.06i) + (0.988761 - 0.658204i)t)\)

\(\text{nVhVt}[t_] = \text{Integrate}[\text{dnVhVdt}[t], t] + \text{nVhV0}\)

\((79604.1 + 123976.i) - (123.06 + 1056.32i)t + (0.329102 + 1.48619i)t^2\)
\[
(nVhV100 - nVhVt[100])/nVhVt[100]
\]
\[
1.2010687812083489^{*^\wedge}-10 + 1.351959949782674^{*^\wedge}-9\imath
\]
\[
dnhVVdt[t_] = I \cdot (nhVht[t] + nhhVt[t] - nhVhbart[t]) - I \cdot nV[t]
\]
\[
-\imath(500. - 0.991808t) + \imath((-557.376 + 123.183\imath) + (1.98057 - 0.658204\imath)t)
\]
\[
nhVVt[t_] = \text{Integrate}[dhnVVdt[t], t] + nhVV0
\]
\[
(79763.4 + 124224.\imath) - (123.183 + 1057.38\imath)t + (0.329102 + 1.48619\imath)t^2
\]
\[
nVhbar[t_] = I \cdot (1 - nhbar2t10) \cdot t + nVhbar0
\]
\[
(159.367 - 248.2\imath) - (0.12306 - 1.05632\imath)t
\]
\[
(nVhbar[30] - nVhbar30)/nVhbar[30]
\]
\[
3.6032317952588624^{*^\wedge}-10 + 9.237033233914777^{*^\wedge}-11\imath
\]
\[
nhbarVt[t_] = I \cdot (1 - nhbar2t10) \cdot t + nhbarV0
\]
\[
(159.049 - 247.704\imath) - (0.12306 - 1.05632\imath)t
\]
\[
\]
\[
3.601313861609975^{*^\wedge}-10 + 9.254727890643492^{*^\wedge}-11\imath
\]
\[
dnV2[t_] = I \cdot (nVht[t] + nhVt[t] - nVhbart[t] - nhbarVt[t])
\]
\[
i((0. + 991.808\imath) - (0. + 4.22528\imath)t)
\]
\[
nV2t[t_] = \text{Integrate}[dnV2[t], t] + nVol2t0
\]
\[
(250000. + 5.892825493736356^{*^\wedge}-15\imath) - (991.808 + 0.\imath)t + (2.11264 + 0.\imath)t^2
\]
\[
(nV2t[30] - nVol2t30)/nV2t[30]
\]
\[
7.917411566855681^{*^\wedge}-10 + 2.339569115366053^{*^\wedge}-18\imath
\]
\[
nVhbarhbar[t_] = \text{Integrate}[I \cdot (nhbar10 - nhbar3t10), t] + nVhbarhbar0
\]
\[-28.216 - 61.6531i \pm (0.494381 + 0.329102i) t\]

\((nV_{hbar hbar})[20] - nV_{hbar hbar20})/nV_{hbar hbar}[20]\]

\[1.628724500081875*^9 - 10 + 2.91823901395245*^9 - 12i\]

\(\text{nhbarVhbar}[t.] = \text{Integrate}[I \ast (\text{nhbar10} - \text{nhbar3t10}), t] + \text{nhbarVhbar0}\]

\[-28.1597 - 61.53i \pm (0.494381 + 0.329102i) t\]

\((\text{nhbarVhbar}[20] - \text{nhbarVhbar20})/\text{nhbarVhbar}[20]\]

\[1.6292537793509515*^9 - 10 + 2.878429387242099*^9 - 12i\]

\(\text{nVVhbar}[t.] = \text{Integrate}[I \ast (nV[t] - nV_{hbar hbar}[t] + nV_{hbar hbar}[t] - \text{nhbarVhbar[t]}), t] + nV_{VVhbar0}\]

\[(79763.4 - 124224. i) \pm (123.183 - 1057.38i) t \pm (0.329102 - 1.48619i) t^2\]

\((\text{nVVhbar}[20] - \text{nVVhbar20})/\text{nVVhbar}[20]\]

\[3.725068185329578*^9 - 10 + 8.61191386270641*^9 - 11i\]

\(\text{nhbarhbarVt}[t.] = \text{Integrate}[I \ast (\text{nhbar10} - \text{nhbar3t10}), t] + \text{nhbarhbarV0}\]

\[-28.1034 - 61.407i \pm (0.494381 + 0.329102i) t\]

\((\text{nhbarhbarVt}[20] - \text{nhbarhbarV20})/\text{nhbarhbarVt}[20]\]

\[1.6298041566636404*^9 - 10 + 2.8354743705460815*^9 - 12i\]

\(\text{nVhbarVt}[t.] = \text{Integrate}[I \ast (2 \ast nV[t] - nV_{hbar hbar}[t] - \text{nhbarhbarVt[t]}), t] + nV_{hbarV0}\]

\[(79604.1 - 123976. i) \pm (123.06 - 1056.32i) t \pm (0.329102 - 1.48619i) t^2\]

\((\text{nVhbarVt}[20] - \text{nVhbarV20})/\text{nVhbarVt}[20]\]

\[3.243127446644767*^9 - 10 + 1.2214209881463168*^9 - 10i\]

\(\text{nhbarVVt}[t.] = \text{Integrate}[I \ast (nV[t] - \text{nhbarhbarVt[t]} + \text{nhbarVht[t]} - \text{nhbarVhbar[t]}), t]\)
\( \text{nhbarVV0} \)

\[
(79445. - 123728.i) - (122.937 - 1055.26i)t + (0.329102 - 1.48619i)t^2
\]

\[
(\text{nhbarVVt}[30] - \text{nhbarVV30})/\text{nhbarVVt}[30]
\]

\[
3.27083809016514*^-10 + 1.3409453507232057*^-10i
\]

\[\text{nVVVt}[t_\_] = \text{Integrate}[I * (\text{nVVht}[t] - \text{nVVhbart}[t] + \text{nVhVt}[t] - \text{nVhbarVt}[t] + \text{nhVVt}[t] - \text{nhbarVt}[t]), t] + \text{nVVV0} \]

\[
(1.25 \times 10^8 + 3.1139263242681203*^-10i) - (743857. - 2.9103830456476037*^-10i)t +
\]

\[
(3168.96 - 7.10542735511946*^-10i)t^2 - (2.97238 - 1.3837708756625489*^-10i)t^3
\]

\[
(\text{nVVVt}[30] - \text{nVV30})/\text{nVVVt}[30]
\]

\[
8.223518325909221*^-10 + 3.55624457970691*^-15i
\]

\[\text{GVVV[t]} = \text{nVVVt}[t] - 3 * \text{nV2t}[t] * \text{nV}[t] + 2 * \text{nV}[t] * \text{nV}[t] * \text{nV}[t] \]

\[
(1.25 \times 10^8 + 3.1139263242681203*^-10i) + 2(500. - 0.991808t)^3 - (743857. - 2.9103830456476037*^-10i)t +
\]

\[
(3168.96 - 7.10542735511946*^-15i)t^2 - (2.97238 - 1.3837708756625489*^-11i)t^3 -
\]

\[
3(500. - 0.991808t) ((250000. + 5.892825493736356*^-15i) - (991.808 + 0.i)t + (2.11264 + 0.i)t^2)
\]

\[\text{GVVVt}[t_\_] = \text{Integrate}[I * (\text{nVVht}[t] - \text{nVVhbart}[t] + \text{nVhVt}[t] - \text{nVhbarVt}[t] + \text{nhVVt}[t] - \text{nhbarVt}[t]) -
\]

\[3 * I * (\text{nVht}[t] - \text{nVhbart}[t] + \text{nVt}[t] - \text{nhbarVt}[t]) * \text{nV}[t] -
\]

\[3 * I * \text{nV2t}[t] * (\text{nha} - \text{nhbar10}) + 6 * I * \text{nV}[t] * \text{nV}[t] * (\text{nha} - \text{nhbar10}), t] + \text{GVVV}[0] \]

\[
(0.193442 + 0.025533941862075*^-10i) - (0.459457 - 0.0000103783i)t - (2.0264323108384*^-6 +
\]

\[
2.058652346087917*^-8i)t^2 + (1.36237 + 1.1827451628028073*^-11i)t^3
\]

\[\text{GVVht}[t_\_] = 1/3 * (\text{nVVht}[t] + \text{nVhVt}[t] + \text{nhVVt}[t]) - \text{nV}[t] * \text{nVht}[t] - \text{nV}[t] * \text{nhVt}[t] -
\]
\[
\text{dGVVh}[t] = \\
1/3 \cdot I \cdot (2 \cdot \text{nVht}[t] + 2 \cdot \text{nhVht}[t] + 2 \cdot \text{nhhVt}[t] - \text{nhVht}[t] - \text{nhVht}[t] - 4 \cdot \text{nV}[t]) - \\
I \cdot (2 \cdot \text{nV}[t] \cdot (\text{nh2t10} - 1) + (\text{nha} - \text{nhbar10}) \cdot (\text{nVht}[t] + \text{nhVt}[t])) - \\
I \cdot \text{nha} \cdot (\text{nVht}[t] + \text{nhVt}[t] - \text{nhVht}[t] - \text{nhVht}[t]) + \\
4 \cdot I \cdot (\text{nV}[t] \cdot \text{nha} \cdot \text{nha} - \text{nV}[t] \cdot \text{nhbar10} \cdot \text{nha}) \\
\]

\[
\text{GVVht}[t] = \text{Integrate}[\text{dGVVh}[t], t] + \text{GVVh}[0] \\
(0.0491679 + 0.0765788i) + (2.4500785400505265*^-7 + 6.597938408958726*^-7i)t + \\
(0.0387408 - 0.681185i)t^2 \\
\text{(GVVht}[20] - \text{GVVh}[20])/\text{GVVht}[20] \\
3.703263422889723*^-10 - 9.8474764406246*^-10i \\
\text{GVV}[t] = nV2t[t] - nV[t] \cdot nV[t] \\
(250000. + 5.892825493736356*^-15i) - (500. - 0.991808t)^2 - (991.808 + 0.\text{i}t) + \\
\]
$$(2.11264 + 0.i)t^2$$

$$\text{Ghh} = \text{nh}2t10 - \text{nha} * \text{nha}$$

$$0.0882124 - 0.192748i$$

$$\text{UP}[t_] = \text{Ghh} * \text{GVV}[t] - \text{GVht}[t]^2$$

$$-((1.9238211734773358*^\text{-7} + 2.909345937496255*^\text{-7}i) + (0.192748 - 0.564478i)t)^2 + (0.0882124 - 0.192748i)((250000. + 5.892825493736356*^\text{-15}i) - (500. - 0.991808t)^2 - (991.808 + 0.991808i)t)$$

$$\text{gvv}[t_] = ("250000." + 5.892825493736356*^\text{-15}i) - ("500." + "0.991808"t)^2 - ("991.808" + "0."i)t + ("2.11264" + "0."i)t^2$$

$$= (250000. + 5.892825493736356*^\text{-15}i) - (500. - 0.991808t)^2 - (991.808 + 0.i)t + (2.11264 + 0.i)t^2$$

$$\text{Expand}[\text{gvv}[t]]$$

$$= (0.214819 + 5.892825493736356*^\text{-15}i) - (4.368112058728002*^\text{-7} + 0.i)t + (1.12896 + 0.i)t^2$$

$$\text{nv}[t_] = "500." - "0.991808"t$$

$$500. - 0.991808t$$

$$\text{s}[t_] = \text{gvv}[t]^(1/2)/\text{nv}[t]$$

$$= \frac{1}{\text{500. - 0.991808t}} \sqrt{(250000. + 5.892825493736356*^\text{-15}i) - (500. - 0.991808t)^2 - (991.808 + 0.i)t + (2.11264 + 0.i)t^2}$$

$$\text{correlation}[t_] = \text{gvht}[t]/(\text{gvv}[t]*\text{ghh})^0.5$$

$$= ((1.9238211734773358*^\text{-7} + 2.909345937496255*^\text{-7}i) + (0.192748 - 0.564478i)t)/(((0.0882124 - 0.192748i) + (250000. + 5.892825493736356*^\text{-15}i) - (500. - 0.991808t)^2 - (991.808 + 0.i)t + (2.11264 + 0.i)t^2)$$

$$\text{GVh0} = 0.5 * (\text{nVh0} + \text{nhV0}) - \text{nVol0} * \text{nha}$$

$$= 1.9238211734773358*^\text{-7} + 2.909345937496255*^\text{-7}i$$
GVh10 = .5 * (nVh10 + nhV10) − nVol10 * nha
1.92748 − 5.64478i

GVh20 = .5 * (nVh20 + nhV20) − nVol20 * nha
3.85495 − 11.2896i

GVh30 = .5 * (nVh30 + nhV30) − nVol30 * nha
5.78243 − 16.9343i

GVh40 = .5 * (nVh40 + nhV40) − nVol40 * nha
7.7099 − 22.5791i

GVh50 = .5 * (nVh50 + nhV50) − nVol50 * nha
9.63738 − 28.2239i

GVh60 = .5 * (nVh60 + nhV60) − nVol60 * nha
11.5649 − 33.8687i

GVh70 = .5 * (nVh70 + nhV70) − nVol70 * nha
13.4923 − 39.5135i

GVh80 = .5 * (nVh80 + nhV80) − nVol80 * nha
15.4198 − 45.1583i

GVh90 = .5 * (nVh90 + nhV90) − nVol90 * nha
17.3473 − 50.803i

GVh100 = .5 * (nVh100 + nhV100) − nVol100 * nha
19.2748 − 56.4478i

Re[gvht[50] − GVh50]/Re[gvht[50]]
\[ -4.3576463605755 \times 10^{-9} \]

\[ \text{Vh}[500, 0, 0] \]

\[ "294.37" + 1.0973109027880899 \times 10^{-9}i \]

\[ \text{Vh}[500, 0, 1] \]

\[ "159.049" + "247.704"i \]

\[ \text{Vh}[500, 0, 2] \]

\[ "122.501" + "267.67"i \]

\[ \text{Vh}[500, 0, 3] \]

\[ "291.424" + "41.5415"i \]

\[ \text{Vh}[500, 0, 4] \]

\[ "192.413" - "222.78"i \]

\[ \text{Vh}[500, 0, 5] \]

\[ "83.5017" - "282.279"i \]

\[ \text{Vh}[500, 0, 6] \]

\[ "282.645" - "82.2516"i \]

\[ \text{Vh}[500, 0, 7] \]

\[ "221.926" + "193.397"i \]

\[ \text{Vh}[500, 0, 8] \]

\[ "42.8309" + "291.238"i \]

\[ \text{Vh}[500, 0, 9] \]

\[ "268.21" + "121.315"i \]
Vh[500, 0, 10]
− “246.998” − “160.144”i
Vh[500, 0, 10]
− “246.998” − “160.144”i
hV[500, 0, 0]
“294.96” + 1.0995077363489655*^-9i
hV[500, 0, 1]
“159.367” + “248.2”i
hV[500, 0, 2]
− “122.746” + “268.206”i
hV[500, 0, 3]
− “292.008” + “41.6247”i
hV[500, 0, 4]
− “192.798” − “223.226”i
hV[500, 0, 5]
“83.6689” − “282.844”i
hV[500, 0, 6]
“283.211” − “82.4163”i
hV[500, 0, 7]
“222.371” + “193.784”i
hV[500, 0, 8]
− “42.9166” + “291.821”\text{i}

hV[500, 0, 9]

− “268.747” + “121.558”\text{i}

hV[500, 0, 10]

− “247.492” − “160.464”\text{i}

HV = Array[hv, \{10, 2\}, \{1, 0\}]

\{\{hv[1, 0], hv[1, 1]\}, \{hv[2, 0], hv[2, 1]\}, \{hv[3, 0], hv[3, 1]\}, \{hv[4, 0], hv[4, 1]\}, \{hv[5, 0], hv[5, 1]\}\}

gVh = Array[gvh, \{11, 2\}, \{0, 0\}]

\{\{gvh[0, 0], gvh[0, 1]\}, \{gvh[1, 0], gvh[1, 1]\}, \{gvh[2, 0], gvh[2, 1]\}, \{gvh[3, 0], gvh[3, 1]\}, \{gvh[4, 0], gvh[4, 1]\}\}

ghV = Array[ghv, \{11, 2\}, \{0, 0\}]

\{\{ghv[0, 0], ghv[0, 1]\}, \{ghv[1, 0], ghv[1, 1]\}, \{ghv[2, 0], ghv[2, 1]\}, \{ghv[3, 0], ghv[3, 1]\}, \{ghv[4, 0], ghv[4, 1]\}\}

GVh[0, 0]

Symbol

Vh[500, 0, 1]

gVh[1, 1] = “159.049” + “247.704”\text{i}

159.049 + 247.704i

Vh[500, 10, 1]

gVh[2, 1] = “157.817” + “237.069”\text{i}

157.817 + 237.069i

Vh[500, 20, 1]

156.588 + 226.578i
gVh[[3, 1]] = "156.585" + "226.343"\ i
156.585 + 226.343i
Vh[500, 30, 1]
gVh[[4, 1]] = "155.353" + "215.63"\ i
155.353 + 215.63i
Vh[500, 40, 1]
gVh[[5, 1]] = "154.122" + "204.948"\ i
154.122 + 204.948i
Vh[500, 50, 1]
gVh[[6, 1]] = "152.891" + "194.297"\ i
152.891 + 194.297i
Vh[500, 60, 1]
gVh[[7, 1]] = "151.66" + "183.672"\ i
151.66 + 183.672i
Vh[500, 70, 1]
gVh[[8, 1]] = "150.43" + "173.068"\ i
150.43 + 173.068i
Vh[500, 80, 1]
gVh[[9, 1]] = "149.199" + "162.484"\ i
149.199 + 162.484i
Vh[500, 90, 1]
\[ g_{Vh}[10, 1] = "147.969" + "151.914" \cdot i \]

\[ 147.969 + 151.914i \]

\[ V_{h[500, 100, 1]} \]

\[ g_{Vh}[11, 1] = "146.739" + "141.358" \cdot i \]

\[ 146.739 + 141.358i \]
Bibliography


[25] T. P. Singh, Quantum mechanics without spacetime III: a proposal for a non-linear Schrödinger equation, [gr-qc/0306110]


[37] R. Helling, Higher curvature counter terms cause the bounce in loop cosmology, [arXiv:0912.3011]


Vita

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David Bryant Simpson was born in Hendersonville, Tennessee on April 22, 1985. He obtained his B.S. in Physics and Mathematics from the Honors College of East Tennessee State University in 2007. In the fall of that year, he attended the Pennsylvania State University where he later joined the research group of Professor Martin Bojowald. Among his honors are the National Science Foundation EAPSI Fellowship, multiple Student Leader Scholarships from Penn State Student Affairs, the Duncan Fellowship, and the Science Alliance Research Fellowship. His publications include:


