DEFORMED GENERAL RELATIVITY AND EFFECTIVE ACTIONS IN QUANTUM GRAVITY

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by
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Abstract

We will use canonical methods to construct effective actions from deformed covariance algebras, as implied by quantum-geometry corrections of loop quantum gravity. To this end, we extend classical constructions systematically to effective constraints of canonical quantum gravity and apply these constructions to model systems as well as general metrics, with the following conclusions: (i) Dispersion relations of matter and gravitational waves are deformed in related ways, ensuring a consistent realization of causality. (ii) Inverse-triad corrections modify the classical action in a way clearly distinguishable from curvature effects. In particular, these corrections can be significantly larger than often expected for standard quantum-gravity phenomena. (iii) Finally, holonomy corrections in high-curvature regimes do not signal the evolution from collapse to expansion in a "bounce," but rather the emergence of the universe from Euclidean space at high density. This new version of signature-change cosmology suggests a natural way of posing initial conditions, and a solution to the entropy problem.

The aforementioned corrections of canonical quantum gravity modify space-time structures, sometimes to the degree that no effective line elements exist to describe the geometry. An analysis of solutions, for instance in the context of black holes, then requires new insights. In this dissertation, standard definitions of horizons in spherical symmetry are first reformulated canonically, and then evaluated for solutions of equations and constraints modified by inverse-triad corrections of loop quantum gravity. For more general conclusions, canonical perturbation theory is developed to second order to include back-reaction from matter.

The work described in this dissertation regarding deformed algebras and their implications for space-time, matter, the universe, and black holes is based on previous publications by the author and his collaborators [102, 100, 101, 99], which may be consulted for further details and references.
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Dedication

To my friends and family,
who believed in me.
Chapter 1

Deconstructing the space-time manifold

The theory of General Relativity is often celebrated for the simplicity of its Lagrangian density, \( \mathcal{L} = \sqrt{-g} R \). In this form, as a scalar density, it is relatively straightforward to prove the space-time general covariance of the theory; that is, that the action and the equations of motion of the theory remain invariant under local diffeomorphisms \( x^\mu \rightarrow x^\mu + \xi^\mu(x) \). By contrast, neither simplicity nor general covariance seem obvious when General Relativity is presented in its Hamiltonian form. However, the simplicity of \( R \) lies in the fact that it has a simple physical interpretation. The full mathematical expression for the Ricci curvature scalar \( R \) in terms of the metric and its partial derivatives is long and complicated, and hardly seems intuitive. However, it takes on a new light once we realize that it is the simplest scalar (of no more than second order in derivatives of the metric) that can be constructed from the Reimann curvature tensor, itself a cumbersome expression that nevertheless arises naturally when looking at parallel transport and geodesic deviation on a curved space-time manifold. With this insight, the gravitational action becomes the space-time volume integral of the curvature scalar, and it seems elegant that the General Relativity is based on the idea of space-time trying to extremize this curvature based scalar action.

Having thus realized that beauty lies in the eye of the physically and mathematically well-informed observer, we shall first set up the necessary background to the Initial Value Formulation of General Relativity (the IVF of GR), and then
take another look at the Hamiltonian for GR and search for its significance and interpretation. More detailed discussions of the Initial Value Formulation may be found in [48, 85, 98, 106].

In the Lagrangian approach, the space-time is considered as a whole; the same applies to the traditional Einstein equations $G_{ab} = 8\pi GT_{ab}$, whose solutions describe the entire space-time. In the Hamiltonian approach, we shall assume that the space-time $M$ has the topology $M = \mathbb{R} \times \sigma$ where $\sigma$ is a compact manifold without boundary (throughout this dissertation we will ignore boundary terms; see [103, 104, 105] for a discussion of boundary terms and energy in General Relativity). We now use an embedding $X_t$ (parametrized by $t$) that maps $\sigma$ into $M$, given by $X: (\mathbb{R} \times \sigma) \to M; X_t : \sigma \to M; X_t : x^i \to X^\mu(t,x^i)$. We shall denote $X_t(\sigma)$ by $\Sigma_t$. We start with an initial space-like 3 dimensional manifold $\Sigma_{t_{in}}$, which we then evolve forward in time, under the action of a time-like vector field $t^a$. In this manner, the entire space-time is foliated into a series of space-like slices $\Sigma_{t_{in}}$; these slices are level sets of our time function $t$ that satisfies $t^a \nabla_a t = 1$. To enable an easier description of the evolution of a point on our initial slice, we construct a unit time-like normal ($n^\mu$) to the spatial slice. Using this, we may decompose the evolution of our initial point into two parts—a motion tangential to the spatial slice, described by a spatial vector field $N^\mu$ (called the shift vector), and a motion along the unit normal $n^\mu$, parametrized by a function $N$ (called the lapse function).

These are related to the time-like vector field by

$$t^\mu = \frac{\partial X^\mu(t,x)}{\partial t} |_{X=x(t)} = X^\mu_t = N n^\mu + N^\mu. \quad (1.1)$$

Our initial data is the three-metric of the spatial slice, variously denoted as $h_{\mu\nu}$, $q_{\mu\nu}$ or $g_{\mu\nu}$ (we shall favour the last to make for easier comparison with the results in [2, 1], and shall use $g_{\mu\nu}$ to distinguish the full space-time metric from the three-space metric $g_{\mu\nu}$), and its conjugate momentum $\pi^{\mu\nu}$. Unsurprisingly, the conjugate momentum is related to a kind of ‘velocity’ of the 3-metric: its Lie derivative along the normal, denoted as

$$K_{\mu\nu} = \frac{1}{2} \mathcal{L}_n g_{\mu\nu}, \quad (1.2)$$

better known as the extrinsic curvature. Thus our initial data contains information about both the intrinsic curvature of the spatial slice (which is encoded in $g_{\mu\nu}$), and
the extrinsic curvature \( K_{\mu \nu} \) of the slice (as encoded by \( \pi^{\mu \nu} \)), which describes how the slice is embedded in a larger space-time. They are related to the space-time quantities by \( g_{\mu \nu} = 4 g_{\mu \nu} + n_\mu n_\nu \) and \( K_{\mu \nu} = g^\alpha_\mu g^\beta_\nu \nabla_\alpha n_\beta \), and are known as the first and second fundamental forms of \( \Sigma_t \). Defining the vector fields \( X^\mu_{\ i} = \frac{\partial X^\mu}{\partial x^i} \) on \( \Sigma_t \), we can use them to pull back the relevant geometrical quantities from \( \Sigma_t \) to \( \sigma \).

The necessary relations we will use in the IV formulation are:

- regarding the initial value data:

\[
\begin{align*}
g_{ij} &= X^\mu_i X^\nu_j g_{\mu \nu} & K_{ij} &= X^\mu_i X^\nu_j K_{\mu \nu} \\
g &= \det[g_{ij}] & g^{ij} g_{jk} = \delta^i_k \\
K &= K^{ij} g_{ij} & \pi^{ij} &= \sqrt{g} (K^{ij} - K g^{ij})
\end{align*}
\]

- regarding the relations between the spatial and space-time metrics:

\[
\begin{align*}
N &= -4 g_{\mu \nu} t^\mu n^\nu \\
N^\nu &= g^\nu_\mu t^\mu \\
4 g_{\mu \nu} &= g_{\mu \nu} - n_\mu n_\nu \\
ds^2 &= 4 g_{\mu \nu} dX^\mu dX^\nu \\
&= 4 g_{\mu \nu} (X^\mu_1 dt + X^\mu_a dx^a)(X^\nu_1 dt + X^\nu_b dx^b) \\
&= 4 g_{\mu \nu} ((N n^\mu + N^\mu) dt + X^\mu_a dx^a)((N n^\nu + N^\nu) dt + X^\nu_b dx^b) \\
&= 4 g_{\mu \nu} (N n^\mu dt + X^\mu_a (dx^a + N^a dt))(N n^\nu dt + X^\nu_b (dx^b + N^b dt)) \\
&= -(Nd t)^2 + g_{ij} (N^i dt + dx^i)(N^j dt + dx^j)
\end{align*}
\]

Thus we have the matrix:

\[
4 g_{\mu \nu} = \begin{pmatrix}
-N^2 + N^i N_i & N_i \\
N_j & g_{ij}
\end{pmatrix}
\]

where \( \mu \) and \( \nu \) range from 0 to 3, and \( i \) and \( j \) range from 1 to 3.

Performing a Legendre transformation on the Lagrangian, and using the Dirac constraint procedure [86, 87, 48](or equivalently, noting as in [85] that the absence of \( \dot{N} \) and \( \dot{N} \) implies that they play the role of Lagrange multipliers), we finally
arrive at the Hamiltonian

$$H = \int \sqrt{g} \left[ -\frac{3}{2} R + g^{-1} \pi^{ij} \pi_{ij} - \frac{1}{2} g^{-1} \pi^2 \right] - 2N_j (g^{-\frac{3}{2}} \pi^i)_{|i} \right] \right)$$

$$= H[N] + D[N]$$

where \(|i| \sim D_{sp}^i\) is the spatial covariant derivative, adapted to the spatial metric \((D_{sp}^i (g_{jk}) = g_{jk|i} = 0)\).

As promised, the shift vector \(N^i\) and the lapse function \(N\) play the role of Lagrange multipliers, which means that the Hamiltonian of GR is simply the sum of two constraints

$$-\frac{3}{2} R + g^{-1} \pi^{ij} \pi_{ij} - \frac{1}{2} g^{-1} \pi^2 = 0 \quad (1.14)$$

$$\left(2g^{-\frac{3}{2}} \pi^{ij}\right)|_i = 0 \quad (1.15)$$

\(D[N]\) is variously known as the (spatial) diffeomorphism constraint or the super-momentum. \(H[N]\) goes by ‘Hamiltonian constraint’ or super-Hamiltonian; I shall favour the latter to avoid confusion with the total Hamiltonian. It is interesting to note that the super-Hamiltonian consists of two parts, \(-\frac{3}{2} R\) and \(g^{-1} \pi^{ij} \pi_{ij} - \frac{1}{2} g^{-1} \pi^2\), where \(-\frac{3}{2} R\) deals with the intrinsic curvature of the spatial slice, and \(g^{-1} \pi^{ij} \pi_{ij} - \frac{1}{2} g^{-1} \pi^2\) deals with its average extrinsic curvature (see [88] for an accessible account of how this relates to Gauss’ theorem egregium). However, to fully apprehend the physical significance of these expressions, we must move to the arena of the phase space. Constructing the Poisson brackets,

$$\{f(x), h(y)\} = \int d^3z \left[ \frac{\delta f(x)}{\delta g_{ij}(z)} \frac{\delta h(y)}{\delta \pi^{ij}(z)} - \frac{\delta f(x)}{\delta \pi^{ij}(z)} \frac{\delta h(y)}{\delta g_{ij}(z)} \right]$$

we note that just as how, in traditional mechanics, momentum generates translations, here the super-momentum generates spatial diffeomorphisms, described by the vector field \(N^a\). Thus the super-momentum generates deformations of the spatial slice that are tangential to the slice. Similarly, the super-Hamiltonian generates normal deformations of the spatial slice, moving it ‘forward’ as described by the lapse function \(N\). Together, they help describe the evolution of any physical quantity on the spatial slice. For an infinitesimal deformation of the spatial slice
by $\delta N$ and $\delta \vec{N}$, a phase space functional $F$ will change by an amount $\delta F$ given by:

$$\delta F = \{F, H[\delta N]\} + \{F, D[\delta \vec{N}]\}$$ (1.17)

Since $H[N]$ and $D[\vec{N}]$ are both constraints, they must be preserved as the system evolves. This means that we expect $H[N]$ and $D[\vec{N}]$ to form a first class set of constraints; that is, they will form a closed system of constraints under the action of the Poisson Bracket. We find that [2, 1, 1]:

$$\{D[N^i], D[M^j]\} = D[\mathcal{L}_M N^i]$$ (1.18a)

$$\{H[N], D[N^j]\} = H[\mathcal{L}_N N]$$ (1.18b)

$$\{H[N_1], H[N_2]\} = -D[g^{ij}(N_1 \partial_j N_2 - N_2 \partial_j N_1)]$$ (1.18c)

This set of relations among the constraints is variously known as the Dirac algebra, or the hypersurface deformation algebra, and encodes the general covariance of the theory. The vector fields generating the tangential and normal deformations of the spatial slice form an algebra (strictly speaking, a Lie algebroid) under the commutator, which finds a representation in the deformation algebra formed by the phase space quantities $H[N]$ and $D[\vec{N}]$ under the Poisson bracket. Significantly, Hojman et al [2, 1] showed that if we restrict our gravitational phase space to the variables $g_{ij}$ and $\pi^{ij}$, the expressions for $H[N]$ and $D[N]$ that satisfy (1.18) are uniquely given by

$$H[N] = \int N[G_{ijkl} \pi^{ij} \pi^{kl} - \frac{1}{2\kappa} \sqrt{g} (3R - 2\lambda)]$$ (1.19)

where $G_{ijkl} = \kappa(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl})$ is often referred to as the ‘supermetric’, and

$$D[\vec{N}] = \int N^i(-2\pi^i |_{\vec{N}})$$ (1.20)

which are the standard expressions in General Relativity, including the cosmological constant term $\lambda$. In this manner, they recovered the Hamiltonian for GR. Thus, under reasonable assumptions, General Relativity is the unique theory describing the motion of a three dimensional spatial slice (the hypersurface) embedded in a four dimensional space-time. The mantra of ‘Gravity is Geometry’ has once again
been validated.

Any change to the structure of the set of relations (1.18) will break general covariance, and change the properties of the space-time in which the theory takes place. To put it another way, the relations (1.18) teach us how to stitch together three dimensional spatial slices into a space-time. Whether this space-time is a regular four dimensional manifold or something more exotic depends on the relations 1.18. Here is one simple example of a change that can be made; if

\[ \{H[N_1], H[N_2]\} = -D[g^{ij}(N_1 \partial_j N_2 - N_2 \partial_j N_1)] \] (1.21)

is changed to

\[ \{H[N_1], H[N_2]\} = D[g^{ij}(N_1 \partial_j N_2 - N_2 \partial_j N_1)], \] (1.22)

we now have a description of the ‘motion’ of a three dimensional spatial slice (the hypersurface) embedded in a four dimensional Euclidean space (no time). Thus small changes in the Dirac algebra (1.18), lead to profound changes in the space-time structure. From this we can see that the proper arena of the theory is the infinite-dimensional space of all three dimensional spatial slices in a given higher dimensional space (or space-time). This arena was named hyperspace by Kuchař [5, 1].

Inspired by non-trivial modifications of the algebra [9, 8, 11] in models of Loop Quantum Gravity that still keep the algebra closed (that is, brackets of the super-Hamiltonian and super-momentum with one another do not lead to new constraints), we shall modify the algebra (1.18) to

\[ \{D[N^i], D[M^j]\} = D[\mathcal{L}_{M^j} N^i] \] (1.23a)
\[ \{H[N], D[N^j]\} = H[\mathcal{L}_{N^j} N] \] (1.23b)
\[ \{H[N_1], H[N_2]\} = -D[\beta g^{ij}(N_1 \partial_j N_2 - N_2 \partial_j N_1)] \] (1.23c)

where \( \beta \) is in general a functional of the phase space variables. We thus have a ‘deformed’ deformation algebra whose consequences we will examine.
1.1 Hypersurface deformations

The meaning and implications of the classical hypersurface-deformation algebra has been discussed in detail in the classic references [1, 2] and [5]. Nevertheless, it is useful to go through some of the arguments once again with a fresh perspective suggested by the deformed algebras found recently [9, 11], to see which assumptions and implications continue to hold, and which must be modified in light of the new algebras. We shall do so in this chapter and the next; the results in these chapters involving the correction function $\beta$ are mostly based on work done by this author and his co-authors in [99, 100, 101, 102].

1.1.1 Spatial diffeomorphisms

Most (but not all) of the proposed deformations of the constraint algebra found so far in loop quantum gravity leave the spatial part of the hypersurface-deformation algebra intact, which will also be one of our assumptions in this dissertation. There are several reasons for this assumption: First, spatial diffeomorphisms can be implemented directly in loop quantum gravity by moving graphs in the spatial manifold used to set up the canonical formulation. This action is the same as the one on classical fields, and so there are no corrections to the diffeomorphism constraint at an effective level. If one just assumes that the super-momentum associated with a vector field $\delta N^i$ generates relabellings $x^i \mapsto x^i + \delta N^i$ of points in the spatial manifold (that is, it generates a spatial diffeomorphism characterized by the vector field $\delta N^i$), any field on space must automatically change by the Lie derivative along $\delta N^i$. Spaces in a very general sense are described mathematically by labelling their elements in some way, but the physics of and in such spaces should be insensitive to how the labels are chosen. This is similar to how vector components may change under a passive change of axes, while the vector itself and the associated physics remain unchanged. Thus it is natural to expect a relabelling symmetry to be present at an effective level, even if the fundamental spatial structure may become discrete or non-commutative. From the relation $\int d^3y \{F(x), D_i(y)\} \delta N^i(y) = \mathcal{L}_{\delta N^i} F(x)$ and the usual expressions for Lie derivatives of the fundamental fields, one can then uniquely derive the phase-space expression that the diffeomorphism constraint must take [1]. In particular, it is
always linear in the momenta of the fields, a consequence which we will make use of later on. For example, for gravity in the ADM variables, we have:

\[ \int d^3y \{ g_{ij}(x), D_k(y) \} \delta N^k(y) = \mathcal{L}_{\delta g} g_{ij}(x) = \delta N_{j|i} + \delta N_{i|j}, \quad (1.24) \]

\[ \int d^3y \{ \pi^{ij}(x), D_k(y) \} \delta N^k(y) = \mathcal{L}_{\delta \pi} \pi^{ij}(x) = (\pi^{ij} \delta N^k)_{|k} + \delta N_{j|i} + \delta N_{i|j}, \quad (1.25) \]

keeping in mind that \( \pi^{ij} \) is a spatial tensor density of weight +1. However, this is equivalent to:

\[ \int d^3y \frac{\delta D_k(y)}{\delta \pi^{ij}(x)} \delta N^k(y) = \delta N_{j|i} + \delta N_{i|j}, \quad (1.26) \]

\[ \int d^3y \frac{\delta D_k(y)}{\delta g_{ij}(x)} \delta N^k(y) = (\pi^{ij} \delta N^k)_{|k} + \delta N_{j|i} + \delta N_{i|j}, \quad (1.27) \]

which are easily solved to find

\[ D_{i, \text{grav}}[N^i] = -2 \int d^3x N^i \pi^j \delta_{i|j}. \quad (1.28) \]

Once we have accepted the interpretation of the super-momentum as the generator of spatial diffeomorphisms, it automatically satisfies (1.18a), and the physical meaning of (1.18b) becomes clear. (1.18b) simply tells us of the transformation properties of the super-Hamiltonian under the action of a spatial diffeomorphism: the integrated super-Hamiltonian \( H[N] \) transforms as a spatial scalar. When dealing the deformed algebra (1.23), this tells us that the quantum corrected effective Hamiltonian \( H_{\beta}[N] \) is a spatial scalar, an assumption that seems safe because of the nature of effective constraints as integrated functionals on a spatial manifold. In what follows, we will make use not only of the assumption that the spatial part of the hypersurface-deformation algebra remains unmodified, but also of several further consequences regarding the form of the diffeomorphism constraint. Most importantly, the super-momentum appears on the right-hand side of (1.23c); thus, the expression it takes will influence the Hamiltonian constraint determined from the constraint algebra.
1.1.2 Normal deformations

The modification by $\beta$ in (1.23c) occurs for the commutator of two normal (transversal) deformations of spatial hypersurfaces along their normal vectors, by two different and position-dependent amounts $N_1$ and $N_2$. This part of the deformation algebra is distinguished from the spatial part in two ways: firstly, it is of dynamical content, owing to the presence of the Hamiltonian constraint and matter Hamiltonians, and secondly, the use of the normal vector to point the deformation normally implies a dependence on the space-time metric $\gamma_{\mu\nu}$, leading to phase-space degrees of freedom. Thus, though the spatial part (1.18a) of the constraint algebra closes in itself (the bracket of two supermomenta is another supermomentum) without the use of additional phase space functions, the total deformation algebra (1.18), has structure functions rather than just structure constants. As a result, (1.18) is not a true Lie algebra, but a Lie algebroid [91]. Implications of structure functions for canonical quantization, mainly negative ones owing to additional difficulties in commutator relationships, are well known [89]. In the present context they are however more interesting, as they serve as a general source of possible deformations by quantum corrections.

Unlike the spatial part of the deformation algebra, which directly shows its relation to infinitesimal deformations by the presence of the Lie derivative, relating the $\{H, H\}$ part of the algebra to transversal deformations is not so obvious. As indicated by the algebra, we consider two transversal deformations by lapse functions $N_1$ and $N_2$, done in a row but in the two different possible orders. Starting with an initial hypersurface $S_{in}$, we obtain two intermediate ones, $S_1$ by deforming $S_{in}$ by $N_1$ along the normal and $S_2$ by deforming $S_{in}$ by $N_2$ along the normal. From those two intermediate hypersurfaces, we obtain two final hypersurfaces, $S_{fin}^{(1)}$ by deforming $S_1$ by $N_2$ along the new normal of $S_1$ and $S_{fin}^{(2)}$ by deforming $S_2$ by $N_1$ along the new normal of $S_2$. Comparing the two final hypersurfaces should then yield a commutator of deformations according to (1.18c). In the process of computing the normals of $S_{in}$, $S_1$ and $S_2$, the metric tensor must be used. We will not fix the signature $s = \pm 1$ of the metric for our calculations in order to be able to incorporate possible sign changes due to quantum corrections, as suggested by holonomy corrections where $\beta$ can turn negative. (For Lorentzian signature with $s = -1$, we choose the time part of the metric to carry the minus sign.)
For simplicity, and without loss of generality, we choose space-time coordinates such that \( S_{\text{in}} \) is given by a constant-time slice, \( S_{\text{in}}: y^i \mapsto (t_{\text{in}}, y^i) \) with some spatial embedding coordinates \( y^i \). The general expression for the future-pointing unit normal to a hypersurface \( y^i \mapsto x^\mu(y^i) \),

\[
 n^\mu = s \frac{4 y^{\mu i} \epsilon_{\mu' \nu' \lambda' \kappa'} \partial_{y^i} x^{\mu'} \partial_{y^j} x^{\lambda'} \partial_{y^k} x^{\kappa'}}{|| \cdot ||}
\]

(with \( || \cdot || \) denoting the norm of the numerator) then simplifies to \( n^\mu_{\text{in}} = s 4 y^{\mu 0} / \sqrt{|4 y^{00}|} \).

The intermediate hypersurfaces, with infinitesimal transversal deformations, are obtained as

\[
 S_1: y^i \mapsto x^\mu(y^i) + N_1(y^i) n^\mu_{\text{in}} = (t_{\text{in}}, y^i) + \frac{s N_1(y^i) 4 y^{\mu 0}(y^i)}{\sqrt{|4 y^{00}|}}. 
\]

\[
 S_2: y^i \mapsto x^\mu(y^i) + N_2(y^i) n^\mu_{\text{in}} = (t_{\text{in}}, y^i) + \frac{s N_2(y^i) 4 y^{\mu 0}(y^i)}{\sqrt{|4 y^{00}|}}. 
\]

From these expressions, we obtain the new normals by the general formula (1.29), expanded to first order in the lapse functions for infinitesimal deformations:

\[
 n_1^\mu = s \frac{4 y^{\mu 0}}{\sqrt{|4 y^{00}|}} - \left(-s 4 y^{\mu i} + \frac{4 y^{\mu 0} 4 y^{i 0}}{|4 y^{00}|} \right) \partial_i N_1 + N_1 X + O(N_1^2)
\]

\[
 n_2^\mu = s \frac{4 y^{\mu 0}}{\sqrt{|4 y^{00}|}} - \left(-s 4 y^{\mu i} + \frac{4 y^{\mu 0} 4 y^{i 0}}{|4 y^{00}|} \right) \partial_i N_2 + N_2 X + O(N_2^2)
\]

with the spatial metric \( g^{\mu \nu} = 4 y^{\mu \nu} - s n^\mu_{\text{in}} n^\nu_{\text{in}} \) on the initial slice. The coefficient \( X \) denotes a combination of metric components and their derivatives; the precise form will not be important because these terms, depending on \( N_1 \) and \( N_2 \) but not on their derivatives, will drop out of the final commutator results. The two final
hypersurfaces are then parameterized as

\[ S^{(1)}_{\text{fin}}: y^i \mapsto x^\mu(y^i) + N_1(y^i) n^\mu_{\text{in}} + N_2(y^i) n^\mu_1 \]

\[ = (t_{\text{in}}, y^i) + s N_1(y^i) \frac{4g^{\mu0}}{\sqrt{|4g^{00}|}} \]

\[ + s N_2(y^i) \left( \frac{4g^{\mu0}}{\sqrt{|4g^{00}|}} - g^{\mu i} \partial_i N_1(y^i) \right) + N_1 N_2 X + O(N_1^2) \tag{1.36} \]

\[ S^{(2)}_{\text{fin}}: y^i \mapsto x^\mu(y^i) + N_2(y^i) n^\mu_{\text{in}} + N_1(y^i) n^\mu_2 \]

\[ = (t_{\text{in}}, y^i) + s N_2(y^i) \frac{4g^{\mu0}}{\sqrt{|4g^{00}|}} \]

\[ + s N_1(y^i) \left( \frac{4g^{\mu0}}{\sqrt{|4g^{00}|}} - g^{\mu i} \partial_i N_2(y^i) \right) + N_2 N_1 X + O(N_2^2) \tag{1.39} \]

With these expressions it is easy to notice that, writing \( S^{(1)}_{\text{fin}}: y^i \mapsto x^\mu_{\text{fin},1}(y^i) \), we have

\[ S^{(2)}_{\text{fin}}: y^i \mapsto x^\mu_{\text{fin},2}(y^i) = x^\mu_{\text{fin},1}(y^i) + \delta S^\mu(y^i) \tag{1.40} \]

with

\[ \delta S^\mu(y^i) = -sg^{\mu i}(N_1 \partial_i N_2 - N_2 \partial_i N_1) . \tag{1.41} \]

To leading order in the lapse functions, \( \delta S^\mu(y^i) \) (depending only on spatial metric components \( g^{\mu i} \)) is orthogonal to the normal vector and thus amounts to an infinitesimal spatial diffeomorphism along the hypersurface. The spatial deformation \( \delta S^\mu \) in (1.41) is obtained from the commutator of two normal deformations, and it reproduces the normal part of the algebra (1.18c) for \( s = -1 \). Thus the equation (1.18c) is equivalent to requiring ‘path-independence’—given the data on an initial spatial slice \( S_{\text{in}} \), the result on the final spatial slice \( S_{\text{fin}} \) must be independent (up to a relabelling via a spatial diffeomorphism) of the intermediate surfaces \( S_1 \) and \( S_2 \). It was also shown in [2] that this requirement of path-independence leads to the imposition of the constraints. Note also that a change of sign in the structure function is equivalent to signature change. (Formally, this implication of signature change can also be seen by replacing \( N \) with \( iN \).)
So far we have assumed the classical manifold structure and geometry in order to compute the normal vectors. The deformed algebra (1.23c) can be achieved formally by using $\beta g^{\mu\nu}$ instead of the inverse metric $g^{\mu\nu}$. For the inverse-triad corrections of Loop Quantum Gravity, such a modification would be expected because the inverse-triad corrections affect all inverse components of the metric in Hamiltonians. Nevertheless, the appearance of the correction function in the constraint algebra must have a more general origin than just modifying any appearance of the inverse metric because a deformation of the same form is obtained for some versions of LQG’s holonomy corrections. The latter do not affect inverse-metric components but rather appearances of extrinsic curvature or the Ashtekar–Barbero connection. However, only the spatial metric appears in the structure functions of the constraint algebra; deformations, therefore, cannot be reduced to simply applying the usual corrections of loop quantum gravity to the spatial metric-dependent structure functions. Such a procedure would be questionable, anyway, because the structure functions are not quantized but rather arise from the algebra satisfied by effective quantum constraints, with corrections following in a more indirect way.

\section{1.2 Constraints and space-time structure}

Having stated the Initial Value Formulation of General Relativity, and considered the physical meaning of the constraints, let us now consider how quantum mechanical corrections may modify these. Numerous approaches to quantum gravity have indicated that space-time, in the classical sense of a 4 dimensional Hausdorff manifold of Lorentzian signature, is an illusion. Whether it is by intimations of discreteness (Loop Quantum Gravity, Causal Dynamical Triangulations, Causal Sets) or of lower dimensionality at smaller length scales (Causal Dynamical Triangulations [93], Asymptotic Safety Gravity [94], possibly Loop Quantum Gravity [92]), there are signs that new physics takes place at a small length scale, and that this distorts the manifold structure implicit in classical space-time. Greenberger [95] gives a simple argument for the conflict between Quantum Mechanics and General Relativity, and the presence of a new length scale: if we accept the Equivalence Principle seriously, then the motion of a particle under (only) gravity is determined by the geometry of the space-time in which it moves, and should not depend on
the mass of the particle involved. That is, whether we are dealing with a cannon-ball, a feather or a neutron, as long as gravity is the only force acting on them, they should follow the same trajectory. However, a simple Bohr quantization of a ‘gravitational atom’ reveals that the Bohr radius of the orbit does in fact depend on the mass of the particle. Greenberger identifies the intrusion of the mass as coming from the Bohr-Sommerfeld quantization principle \( \oint p \, dx = mvr = n\hbar \) via the presence of mass in the momentum \( p \). He suggests that a modified quantization principle is needed, and puts forward \( \oint v \, dx = n\lambda_0 \), where \( \lambda_0 \) is a length scale indicative of new physics, though he declines to identify it with the Planck length. Note however, that the presence of a new length scale does not necessarily imply a new space-time structure, as seen in higher curvature theories that have the same constraint algebra as classical General Relativity.

We will take the viewpoint championed by Wheeler [96, 97], in which the dynamical object is space, not space-time. In classical General Relativity, the 3 dimensional spatial slices can be stitched together into a single space-time. However, as noted by Wheeler, once we make the transition to Quantum Geometrodynamics (QGMD), such a stitching into a classical space-time is no longer possible. We will go a step further—inspired by Loop Quantum Gravity, we will investigate deformations such as (1.23) that will allow us to stitch the slices into an effective, quantum corrected space-time. This space-time will no longer be a standard 4-dimensional Lorentzian signature manifold, but the similarities will allows us to see the changes that Quantum Effects bring about. In the next section, we will look at the quantum corrections in Loop Quantum Gravity that lead to these deformed algebras.

### 1.3 Overview of deformed constraint algebras in loop quantum gravity

Canonically, the quantum effects of interacting gravitational theories, often expressed by higher-curvature effective actions, are derived from quantum back-reaction [84]: While expectation values of semiclassical states follow nearly the classical trajectories, additional state parameters such as fluctuations and other
moments influence the quantum trajectory. Coupled equations of motion for expectation values and the moments can, in some regimes of adiabatic nature, be reformulated as the usual equations of low-energy effective actions [83].

Obviously, these effects should play a large role for quantum gravity and cosmology. But in addition to the ubiquitous quantum back-reaction (or corrections from loop diagrams in perturbative terms), there are characteristic quantum corrections expected for loop quantum gravity, providing two distinct quantum-geometry effects: (i) higher powers of spatial curvature components (intrinsic and extrinsic) stemming from the appearance of holonomies of the Ashtekar–Barbero connection instead of direct connection components in quantized constraints [15, 50], and (ii) natural cut-off functions of divergences of factors containing inverse components of the densitized triad, arising from spatial discreteness [50, 64]. The first type of quantum-geometry corrections is usually referred to as “holonomy corrections,” the second as “inverse-triad corrections” (or, in the context of nearly isotropic cosmology, “inverse-volume corrections”). Both can be expanded as series of corrections by components of spatial tensors in the constraints, not by scalar invariants of space-time tensors as one is used to from covariant effective actions. Neither the reconstruction of an action principle from the constraints nor properties of covariance are obvious in such a situation, and the only systematic way to determine such features is an analysis of the constraint algebra. As shown in several model systems so far, the hypersurface-deformation algebra is generically deformed by quantum-geometry. In particular, corrections cannot be written purely as higher-curvature terms added to the Einstein–Hilbert action, as often expected for quantum gravity. One of the main questions to be addressed in this article is what actions and covariance properties could be realized instead.

In this section we summarize the models investigated so far for their properties of deformations of the constraint algebra, split into the two types of quantum-geometry corrections. (Quantum back-reaction has not yet been analyzed to completion in this context, but the procedure would follow [83, 16, 17].) The set of models in which consistent deformations have been achieved is quite diverse, but the general form of $\beta$ appears to be insensitive to model specifications. The constraint algebra therefore displays universal implications for covariant space-time structure.
1.3.1 Inverse-triad corrections

In loop quantum gravity, space-time geometry is described by canonical fields $A_I^i$ and $E_I^i$, a connection related to curvature and the densitized triad, instead of the spatial metric $g_{ij}$ and its momentum $\pi^{ij}$. These fields have advantages for a background independent quantization because they can be smeared without reference to an auxiliary metric structure: The connection is integrated along curves $e$ in space to obtain holonomies $h_e(A) = \mathcal{P} \exp(\int_e \tau_I A_I^i \dot{e}^i d\lambda)$, and the densitized triad, dual to a 2-form, is integrated to fluxes $F_S(E) = \int_S \tau^I E_I^i n_i d^2y$ through surfaces $S$ in space. Here, $\tau_I = \frac{1}{2} i \sigma_I$ are generators of $\text{su}(2)$, related to the Pauli matrices. The canonical structure $\{ A_I^i(x), E_J^j(y) \} = 8\pi \gamma G \delta^i_j \delta_I^J \delta(x, y)$ with the Barbero–Immirzi parameter $\gamma$ [53, 52] provides a regular relation for $\{ h_e(A), F_S(E) \}$ free of delta functions $^1$.

Holonomies and fluxes are promoted to basic operators of the resulting quantum theory, and they represent the canonical fields in all composite operators such as Hamiltonians. Both types of basic operators imply some form of non-locality because they are integrated rather than point-like fields. Using holonomies for connection components, moreover, implies that there are higher-order corrections when the exponential is expanded, compared with classical expression which are usually polynomials of degree at most two in the connection. Fluxes also give rise to corrections in addition to their non-locality: They are quantized to operators with discrete spectra, containing zero as an eigenvalue. Such operators are not invertible, and yet an inverse of the densitized triad (or its determinant) is needed to quantize matter Hamiltonians (usually in the kinetic part) and the Hamiltonian constraint. Well-defined operators with inverse densitized triad components as their classical limit do exist [50], but they have strong quantum corrections for small values of the fluxes. Correction functions, obtained from expectation values of inverse-triad operators [18], then primarily depend on the fluxes, or on the

$^1$Triad variables are subject not only to the diffeomorphism and Hamiltonian constraints but also to the Gauss constraint. However, the latter does not appear in the space-time constraint algebra and plays no role for choosing the space-time gauge. Moreover, in loop quantum gravity it can easily be implemented and solved without quantum corrections, directly using the classical form of internal rotations of phase-space variables. Therefore, there is no harm in ignoring the Gauss constraint, as we will do in this paper. In some calculations of deformed algebras, the Gauss constraint had been fixed, but one can easily see that different gauge fixings all produce the same physical results. Therefore, no anomaly or deformation arises from the Gauss constraint.
densitized triad and the spatial metric.

### 1.3.1.1 Spherical Symmetry

A class of models in which inverse-triad corrections have been included consistently, non-perturbatively, is spherically symmetric models. Several different cases have been investigated: Poisson Sigma Models [10] (see [22, 23, 24, 25] for the classical models) and different versions of Lemaître–Tolman–Bondi models [67, 68]. In these models, it is noteworthy that non-trivial quantum corrections are possible even without any deformation of the constraint algebra, a property which we will discuss in more detail later.

We quote the corrected constraints in terms of triad variables rather than the metric $g_{ij}$ because one of the triad components is directly responsible for the corrections. (In the full theory, by comparison, it is primarily det $g$ which gives rise to inverse-triad corrections. Because det $g$ equals the squared determinant of the densitized triad, in the general case it will make no difference what variables we use.) As spherically symmetric phase space variables, with radial coordinate $x$ (not necessarily the area radius) and azimuth angle $\varphi$, we then have the radial component $E^x$ and angular component $E^\varphi$ of the densitized triad together with the radial component $K_x$ and angular component $K_\varphi$ of extrinsic curvature [60, 61]. The metric is related to $E^x$ and $E^\varphi$ by $g_{xx} = (E^\varphi)^2/|E^x|$ and $g_{x\varphi} = |E^x| \sin \vartheta$. Consistent deformations of the Hamiltonian constraint (with unmodified diffeomorphism constraint) have the form

$$H_\text{grav}^Q[N] = -\frac{1}{2G} \int dx \, N \left[ \alpha |E^x|^{-\frac{1}{2}} K_\varphi^2 E^\varphi + 2\bar{\alpha} K_\varphi K_x |E^x|^{\frac{1}{2}} 
+ \alpha |E^x|^{-\frac{1}{2}} E^\varphi - \alpha_\Gamma |E^x|^{-\frac{1}{2}} \Gamma_\varphi^2 E^\varphi + 2\bar{\alpha} \Gamma_\varphi |E^x|^{\frac{1}{2}} \Gamma_\varphi' \right]$$

(1.42)

with correction functions $\alpha, \bar{\alpha}, \alpha_\Gamma$ and $\bar{\alpha}_\Gamma$. In the second line, $\Gamma_\varphi = -(E^x)'/2E^\varphi$ is the angular component of the spin connection.

The four correction functions are not independent but must satisfy [68]

$$(\bar{\alpha}\alpha_\Gamma - \alpha\bar{\alpha}_\Gamma)(E^x)' + 2(\bar{\alpha}'\alpha_\Gamma - \bar{\alpha}\alpha_\Gamma')E^x = 0$$

(1.43)
for the Poisson bracket of two Hamiltonian constraints to be anomaly-free. If the two terms in this equation vanish separately, a case studied in [68], they imply $\alpha_\Gamma \propto \alpha$ and $\bar{\alpha}_\Gamma \propto \bar{\alpha}$ for a closed constraint algebra. For correction functions defined such that they approach the classical value one for large arguments, $\alpha_\Gamma = \alpha$ and $\bar{\alpha}_\Gamma = \bar{\alpha}$.

From the Poisson bracket $\{H[N], D[N^x]\}$, the only restriction is that both correction functions depend only on the radial triad component $E^x$, not on $E^\phi$. (This fact is easily understandable from transformation properties of the components: $E^x$ is a scalar in the radial manifold while $E^\phi$ is a scalar density [60].) Only the functions $\bar{\alpha}$ and $\bar{\alpha}_\Gamma$ appear in the deformed constraint algebra (1.45) via $\beta = \bar{\alpha}\bar{\alpha}_\Gamma$, not $\alpha$ or $\alpha_\Gamma$. By changing only $\alpha$ and $\alpha_\Gamma$, one can modify the spherically symmetric constraints while keeping their derivative order and the constraint algebra unchanged: In spherical symmetry, the classical dynamics does not follow uniquely from the hypersurface-deformation algebra.

1.3.2 Holonomy corrections

In spherically symmetric models also, a limited version of holonomy corrections has been implemented consistently: those that involve only the scalar component $K_\phi$ of extrinsic curvature but not the component $K_x$ of density weight one [11]. One can therefore consistently substitute $-i\delta^{-1}\exp(i\delta K_\phi)$ for $K_\phi$, but there is no known consistent way to use $\exp(i\int eK_xdx)$ for $K_x$. Accordingly, no spatial integration or discretization is required to ensure the existing forms of consistent correction functions to be scalar and to keep the $\{H, D\}$-part of the constraint algebra unmodified. This type of correction thus does not lead to non-locality, as holonomy corrections usually do owing to the spatial integrations involved in their definition. In this case, the form of the deformation is similar to the one obtained for inverse-triad corrections, with a Poisson bracket (1.23c) for a correction function now depending on extrinsic curvature instead of the densitized triad.

If we parameterize the Hamiltonian constraint as

$$H^Q_{\text{grav}}[N] = -\frac{1}{2G} \int dx \, N \left( \alpha |E^x|^{-\frac{1}{2}} E^\phi f_1(K_\phi, K_x) + 2\bar{\alpha} |E^x|^{\frac{1}{2}} f_2(K_\phi, K_x) \right)$$
\[
\beta(E^x)|E^x|^{-\frac{1}{2}}E^\varphi - \alpha_\Gamma|E^x|^{-\frac{1}{2}}\Gamma^\varphi_\varphi E^\varphi + 2\alpha_\Gamma\Gamma^\varphi_\varphi|E^x|^{\frac{1}{2}}
\]  
(1.44)

including inverse-triad corrections as well as holonomy corrections via two new functions \(f_1\) and \(f_2\), anomaly freedom can be realized if \(f_1 = F^2_1\) and \(f_2 = K_\varphi F_2\) provided that \(F_2 = F_1(\partial F_1/\partial K_\varphi)\alpha/\alpha_\Gamma\) [11]. If \(F_1\) is independent of \(E^x\), or at least depends on this triad variable in a way different from inverse-triad corrections, we obtain that \(\alpha_\Gamma = \alpha\) and also \(\bar{\alpha}_\Gamma = \bar{\alpha}\) must be realized, restricting the set of solutions of (1.43). Combinations of different corrections therefore can reduce the freedom of choices seen for just a single type. If we take \(F_1 = (\delta\gamma)^{-1}\sin(\gamma\delta K_\varphi)\), as often done for holonomy corrections, we see that \(F_2 = (2\gamma\delta)^{-1}\sin(2\gamma\delta K_\varphi)\). The algebraic deformation is then given by \(\beta(E^x,K_\varphi) = \bar{\alpha}\bar{\alpha}_\Gamma\partial F_2/\partial K_\varphi\). For the example provided, this means \(\beta(K_\varphi) = \cos(2\delta K_\varphi)\) if we include only holonomy corrections. Note that \(\beta\) can be negative for holonomy corrections, unlike for inverse-triad corrections.

Quantum-geometry corrections change the hypersurface-deformation algebra and accordingly the space-time structure: Normal deformations of spatial slices then behave differently from the classical case. Corresponding actions cannot be space-time covariant in the usual sense, but they are still covariant in a deformed sense, under an algebra of the type (1.23). We changed the space-time structure, and in the absence of a corresponding space-time tensor calculus, it is difficult to imagine the form of actions covariant with respect to the new space-time structures. Thus direct determination of an analogue of the Ricci curvature \(R\) is not possible, but fortunately, the constraint algebra allows such actions to be systematically derived or ‘regained’, in the language of [1, 2].

In this and the following section we will go in some detail through the steps outlined in [1], focusing our discussion on those steps that use assumptions that may no longer remain valid under our new space-time structure. According to the form of the deformed constraint algebra used here, and as a rather general consequence of canonical quantum gravity, the spatial structure on the one hand and the structure of hypersurface deformations within space-time, on the other, will play rather different roles. The algebraic effects considered here are thus truly dynamical and do not arise at the kinematical level of spatial manifolds.
1.4 Procedure for ‘regaining’ the Hamiltonian

The general procedure for the classical case is as follows: we start with the relation

\[ \{ H[N_1], H[N_2] \} = D[g^{ij}(N_1 \partial_j N_2 - N_2 \partial_j N_1)] \] (1.45)

Writing out the Poisson brackets explicitly, we have:

\[ \{ H[N_1], H[N_2] \} = \int \int d^3 z d^3 y d^3 x \left( \frac{\delta H(x)}{\delta g_{ij}(z)} \frac{\delta H(y)}{\delta \pi_{ij}(z)} - \frac{\delta H(y)}{\delta g_{ij}(z)} \frac{\delta H(x)}{\delta \pi_{ij}(z)} \right) N_1(x) N_2(y) \] (1.46)

Using the Legendre transform of the super-Hamiltonian, which we will call the super-Lagrangian, we see that:

\[ L(x) = \pi_{ij}(x)v_{ij}(x) - H(x) \text{ where} \] (1.47)

\[ v_{ij}(x) = (N(x))^{-1}\{g_{ij}(x), H[N]\} \] (1.48)

\[ \frac{\delta H(x)}{\delta g_{ij}(z)} \bigg|_{\pi_{ij} \text{ fixed}} = - \frac{\delta L(x)}{\delta g_{ij}(z)} \bigg|_{v_{ij} \text{ fixed}} \] (1.49)

\( v_{ij} \) is the ‘velocity’ of the metric \( g_{ij} \), and is classically related to the extrinsic curvature of the spatial slice by \( v_{ij} = 2K_{ij} \). Classically, we note that for the case of gravity (and similarly for the scalar field), that the normal change of the field at a point is related to the ‘velocity’/extrinsic curvature at that point. That is,

\[ \frac{\delta H[\delta N]}{\delta \pi_{ij}(x)} = \{g_{ij}(x), H[\delta N]\} = 2K_{ij}(x)\delta N(x), \] (1.50)

This implies that the super-Hamiltonian is local in \( \pi_{ij}(x) \), and thus that:

\[ \frac{\delta H(x)}{\delta \pi_{ij}(z)} = \frac{\partial H}{\partial \pi_{ij}}(x)\delta(x, z) = v_{ij}(x)\delta(x, z). \] (1.51)

This issue will be discussed in further detail below.

Using the relations (1.46), (1.49) and (1.51) in (1.45), we see that:

\[ \{ H(x), H(y) \} = \int d^3 z \left( \frac{\delta H(x)}{\delta g_{ij}(z)} \frac{\delta H(y)}{\delta \pi_{ij}(z)} - \frac{\delta H(y)}{\delta g_{ij}(z)} \frac{\delta H(x)}{\delta \pi_{ij}(z)} \right) \] (1.52)
\[
\int d^3z - \left( \frac{\delta L(x) \delta H(y)}{\delta g_{ij}(z) \delta \pi^{ij}(z)} - \frac{\delta L(y) \delta H(x)}{\delta g_{ij}(z) \delta \pi^{ij}(z)} \right) \quad (1.53)
\]

\[
\int d^3z - \left( \frac{\delta L(x)}{\delta g_{ij}(z)} v_{ij}(y) \delta(y, z) - \frac{\delta L(y)}{\delta g_{ij}(z)} v_{ij}(x) \delta(x, z) \right) \quad (1.54)
\]

\[
g^{ij} D_i(x) \delta_{ij}(x, y) - (x \leftrightarrow x') \quad (1.55)
\]

\[
\delta L(x) \frac{\delta}{\delta g_{ij}(y)} v_{ij}(y) + g^{ij} D_i(x) \delta_{ij}(x, y) - (x \leftrightarrow x') = 0 \quad (1.56)
\]

We then substitute in an explicit expression for \( D_i(x) \) in terms of \( g_{ij} \) and \( \pi^{ij} \) and replace \( \pi^{ij}(x) \) by \( \frac{\delta L(x)}{\delta v_{ij}} \). Using the locality of \( H(x) \) in \( \pi^{ij}(x) \), which implies the locality of \( L(x) \) in \( v_{ij}(x) \), we expand \( L \) as a power series in \( v_{ij} \). We now have an expression in terms of functions of the metric \( g_{ij} \) and powers of the velocity \( v_{ij} \), and can consider the equation order by order in powers of \( v_{ij} \).

This is the general outline of the procedure we will follow. However, the introduction of a deformed algebra (1.23) leads to a deformed space-time structure. We now consider some of the changes that may take place, and the failure of certain assumption that held good in the classical case.

### 1.4.1 Locality

As mentioned earlier, we will leave the spatial part (1.18a) of the deformation algebra unchanged, thus fixing the structure of the spatial slices. The next object to consider is the change of the spatial metric under a normal deformation of a spatial slice. Classically, given the embedding of the spatial slice in a space-time manifold, this deformation is given by the extrinsic-curvature tensor, \( \{ g_{ij}(x), H[\delta N] \} = 2K_{ij}(x)\delta N \), and it plays an important role in [1] in helping to show that the Hamiltonian constraint must be a local expression in the momentum:

Identifying

\[
\frac{\delta H[\delta N]}{\delta \pi^{ij}(x)} = \{ g_{ij}(x), H[\delta N] \} = 2K_{ij}(x)\delta N(x), \quad (1.57)
\]

the presence of the \( \delta N \) without derivatives on the right-hand side implies that \( H[\delta N] \) must be local in the momentum \( \pi^{ij}(x) \) without any dependence on \( \pi^{ij} \)-derivatives. The specific form of \( K_{ij} \) as extrinsic curvature does not matter for this conclusion, but it is important that it is a local function, and that no derivatives of \( \delta N \) appear on the right-hand side.
In the presence of deformed space-time structures, we cannot safely assume that transversal metric deformations are given in terms of extrinsic curvature. For the explicit examples of deformed constraint algebras, it is known that the relationship between momentum variables and extrinsic curvature deviates from the classical one; see e.g. the discussion in [68]. It should then be possible for the change of the metric under a transversal deformation to have a modified relationship with extrinsic curvature. In the absence of a geometrical interpretation of the change of the metric, that is, without knowing the exact nature of the spacetime in which it is embedded, one can compute it only by using the canonical formula \( \{ g_{ij}(x), H[\delta N] \} \); but then, one piece of independent information is lost and we cannot derive locality properties of the Hamiltonian constraint. If \( H[\delta N] \) is local in the momentum, \( \{ g_{ij}, H[\delta N] \} \) is local and vice versa, but there is no independent general statement that could determine whether this locality is realized.

Instead, we will make use of the following line of arguments: We know that the classical constraint must be local without spatial derivatives of \( \pi^{ij} \), and in most cases the form of corrections expected from loop quantum gravity tells us what locality properties new terms have. Most of the corrections are indeed non-local, for instance those arising from the use of holonomies as exponentiated line integrals of a connection related to extrinsic curvature, or inverse-triad corrections depending on fluxes through extended surfaces. In derivative expansions, whole series of spatial derivatives of \( \pi^{ij} \) or \( g_{ij} \) will result. The form of the corrections and their impact on effective constraints can thus be used to decide whether local or non-local constraints should be expected. The arguments put forward to regain the form of the constraint will then have to be adapted, depending on the locality properties realized. In most cases, effective equations include a derivative expansion, approximating non-local features locally. We can then assume a local Hamiltonian constraint, but, in contrast to the classical case, must take into account a possibly infinite number of additional spatial derivatives, for instance of \( K_{ij} \). The presence of the additional spatial derivative raises questions about the possible presence of additional time derivatives. These time derivatives may be introduced either via higher order derivatives, in which case the phase-space of the theory is enlarged, and care must be taken to avoid the Ostragradski instability [89, 90], or via higher powers of the ‘velocity’ of the spatial metric. Given that
we have abandoned the classical constraint algebra (and thus, by implication, the classical local Lorentz symmetry), it is not guaranteed that there should be equal amounts of space and time derivatives in the final action; but the need to reach the correct classical limit imposes interesting restrictions on the reconstructed action.

Similar considerations can be applied to the question of whether the matter Hamiltonian must be local in the matter momentum. Here, the assumptions made in [1], appear safer in the context of our deformed space-time for at least certain kinds of matter, compared to those for the corresponding gravitational terms. Instead of looking at transversal deformations of the spatial metric, we look at transversal deformations of a scalar matter field $\phi$ (not to be confused with the angular azimuthal variable $\varphi$, which will be used repeatedly later). The relation \[ \{\phi(x), H[\delta N]\} := V(x)\delta N \] then replaces the gravitational relation involving extrinsic curvature, with $V(x)$ interpreted as the velocity of the scalar field. In contrast to the gravitational part, there are some quantum corrections in matter Hamiltonians that, while changing the specific expression for $V(x)$, leave its local nature intact. Thus, in some cases we can assume the matter Hamiltonian to be local in the matter momentum even in the presence of corrections making the gravitational part non-local without a derivative expansion. This difference between gravitational and matter Hamiltonians may play an important role for the interplay of different contributions to the constraints ensuring that the algebra closes.

### 1.4.2 Gravity and matter

In [2], the closure of the matter constraints independently of the gravitational constraints is discussed in the classical case; we repeat the salient points here. Let us define:

\begin{align*}
H^T[N] &= H_{\text{grav}}[N] + H_{\text{matter}}[N] \\
D^T[\vec{N}] &= D_{\text{grav}}[\vec{N}] + D_{\text{matter}}[\vec{N}] \\
\{f, g\}_T &= \{f, g\}_{\text{grav}} + \{f, g\}_{\text{matter}}
\end{align*}

\[ (1.58, 1.59, 1.60) \]

In order for all the canonical variables (and their functionals) on one spatial slice
to vary smoothly and consistently to the next, we must have:

$$\delta F = \{F, H^T[\delta N]\}_T + \{F, D^T[\delta \vec{N}]\}_T.$$  \hfill (1.61)

Overall, the system must obey the constraints

$$H^T = 0 \quad D^T = 0,$$  \hfill (1.62)

and these constraints must be preserved under the evolution produced by the total Hamiltonian \((H^T[N] + D^T[\vec{N}])\), which means that the total super-momentum \(D^T[N]\) and the total super-Hamiltonian \(H^T[N]\) must obey the algebra (1.18). In the classical case, the notion of a vacuum is well defined, so we find that the gravitational super-momentum \(D_{grav}[\vec{N}]\) and super-Hamiltonian \(H_{grav}[N]\) also obey that algebra separately, by themselves (this can be verified explicitly). That leaves the matter terms. The matter super-momentum typically does not depend on the gravitational degrees of freedom at all (we shall verify this in section 2.1), so that:

$$\{D_{matter}[N^i], D_{matter}[M^j]\}_T = \{D_{matter}[N^i], D_{matter}[M^j]\}_{matt} = D_{matter}[\mathcal{L}_M N^i].$$  \hfill (1.63)

The matter super-Hamiltonian \(H_{matter}\) necessarily depends on the metric (it is required for the kinetic and spatial gradient terms, appearing via \(4g^{\mu\nu}\nabla_\mu \phi \nabla_\nu \phi\)), but if we restrict it to not depend on the gravitational momentum \(\pi^{ij}(x)\), and the gravitational super-Hamiltonian \(H_{grav}\) to not contain spatial derivatives of \(\pi^{ij}(x)\) (as is true already in the classical case), then it can be shown that:

$$\{H_{matter}[N], H_{matter}[M]\}_T = \{H_{matter}[N], H_{matter}[M]\}_{matt} = D_{matter}[g^{ij}(N \partial_j M - M \partial_j N)].$$  \hfill (1.64)

Thus, under these assumptions, for fields like the scalar field, we have

$$\{H^T[N_1], H^T[N_2]\} = \{H_{grav}[N_1], H_{grav}[N_2]\} + \{H_{matter}[N_1], H_{matter}[N_2]\}.$$  \hfill (1.65)
Thus the dependencies of the various super-Hamiltonians and super-momenta are:

\[
H_{\text{grav}}[N] = H_{\text{grav}}[N, g_{ij}, \pi^{kl}], \quad (1.66)
\]

\[
H_{\text{matter}}[N] = H_{\text{matter}}[N, g_{ij}, \phi, \phi_{|j}, V], \quad (1.67)
\]

\[
D_{\text{grav}}[\vec{N}] = D_{\text{grav}}[\vec{N}, g_{ij}, \pi^{kl}], \quad (1.68)
\]

\[
D_{\text{matter}}[\vec{N}] = D_{\text{matter}}[\vec{N}, \phi, \phi_{|j}, V], \quad (1.69)
\]

where \( \phi_{|i} \) is a spatial derivative of the scalar field \( \phi \). However, in the interests of concise notation, we will usually suppress the extended brackets, using the versions on the LHS.

Oddly enough, it is the remaining relation (1.18b) that cannot be copied exactly for the matter fields; however, we shall not require it for our ‘regaining’ of the classical scalar field Hamiltonian. With quantum corrections, however, the assumptions can be violated easily, depending on the type of the correction. Matter fields are usually introduced in loop quantum gravity via the values they take at the vertices of a spin network. Spatial derivatives as they occur in the Hamiltonians must be discretized and replaced by finite differences of the values at neighboring vertices before they can be quantized. Depending on how the differencing is done, one may have to refer to the gravitational connection, making the matter constraint dependent on the gravitational momentum. Another source of such a dependence may be counterterms as introduced in [7], required to close the constraint algebra. An extra momentum dependence can be avoided for a scalar field, but there may be reasons to prefer more complicated quantizations.

Coming back to the results found in the preceding subsection on locality, we can see a potential obstruction to the existence of consistent deformations of the classical constraint algebra. There are corrections expected from loop quantum gravity, most notably holonomy corrections, which are non-local in the connection and thus make the gravitational part of the Hamiltonian constraint non-local in the gravitational momentum. A scalar Hamiltonian in the presence of the same corrections, on the other hand, remains local in its momentum. If the gravitational part and the matter part are to satisfy the same deformed algebra for consistency, the mismatch of locality properties could be seen as an obstacle to the existence of a consistent deformation: The function \( \beta \) of (1.23c) would be non-local in one con-
tribution and local in another one, preventing one from adding up the constraint contributions to a consistent whole. However, the situation is not obviously inconsistent because the same property giving rise to the mismatch, non-locality and the presence of derivatives of $\pi^{ij}$, also violates the assumptions that led one to conclude that gravity and matter satisfy the hypersurface-deformation algebra independently. Non-locality, in a derivative expansion of holonomy corrections in effective constraints, makes the gravitational constraint depend on spatial derivatives of the momentum $\pi^{ij}(x)$, such that cross-terms between gravity and matter in (1.65) no longer cancel. It is reassuring that properties of non-locality thus restore the a-priori possibility of consistency, but the necessary appearance of gravity-matter cross-terms makes the explicit construction of consistent deformations for non-local momentum-dependent corrections more difficult than for local ones. As recalled in Sec. 1.3.2, results in spherical symmetry are indeed much easier to find in local versions of the corrections. Also for perturbative inhomogeneities as in [9] one so far assumes a local, pointwise form of holonomy corrections. The manipulations required for non-local modifications to be consistent appear to be rather complicated, a fact which may explain the difficulties found in constructing consistent deformations corresponding to the non-local holonomy or discreteness corrections. (On the other hand, tying matter terms more closely to gravitational ones rather than having them algebraically separated as in (1.65) may be of interest in the context of unification. It is worth noting that in the classical case the concept of a vacuum is well-defined, i.e, gravity, or rather, space may exist and ‘propagate’ by itself, whereas the matter fields cannot exist without space, and always produce gravitational effects. However, in the quantum theory, the vacuum is never truly empty, and thus it may not be surprising that ‘space’ and ‘matter’ never exist independently of each other, and thus require each other for a fully consistent formalism. See also [96, 97] for Wheeler’s thoughts on particles as distortions of space.)

1.4.3 Legendre transform

Instead of having to assume $\delta H/\delta \pi^{ij}$ (or $\delta H/\delta K_x$ and $\delta H/\delta K_\phi$ in spherical symmetry with triad variables) to be linear in the momenta, it is more general to treat
\( \frac{\delta H}{\delta \pi_{ij}}(x) =: v_{ij}(x) \) as a new independent variable in place of \( \pi_{ij} \), and then expand by this newly defined \( v_{ij} \). This change amounts to a Legendre transformation from \( (g_{ij}, \pi_{ij}) \) with Hamiltonian \( H \) to \( (g_{ij}, v_{ij}) \) with Lagrangian \( L = \pi_{ij} v_{ij} - H \), as proposed in [1]. We then have the equations

\[
\frac{\delta H}{\delta g_{ij}(x')} \bigg|_{\pi_{ij}(x)} = -\frac{\delta L}{\delta g_{ij}(x')} \bigg|_{v_{ij}(x)}.
\] (1.71)

There are now two differences to [1]. First, our \( v_{ij} \) here need not be geometrical extrinsic curvature because of modifications to space-time geometry. We simply define a new independent variable \( v_{ij} = (\delta N)^{-1}\{g_{ij}, H[\delta N]\} \), which we interpret as the rate of change of the metric, eventually providing time derivatives in an effective action. Secondly, we cannot always assume that the Hamiltonian is local and free of derivatives of \( \pi_{ij} \), which would imply that partial derivatives could be used to compute \( v_{ij} \).

Using \( v_{ij} \), we write the Poisson bracket of two smeared Hamiltonian constraints as

\[
\{H[N], H[M]\} = \int d^3x \frac{\delta H[N]}{\delta g_{ij}(x)} v_{ij}(x) M(x) - (N \leftrightarrow M)
\]

\[
= -\int d^3x \int d^3y \frac{\delta L(y)}{\delta g_{ij}(x)} v_{ij}(x) N(y) M(x)
\]

\[
- (N \leftrightarrow M)
\]

\[
= -\int d^3x \beta D^i(x)(NM_i - MN_i) \quad (1.72)
\]

with the local diffeomorphism constraint \( D^i \). Taking functional derivatives by \( N \) and \( M \), we arrive at the functional equation

\[
\frac{\delta L(x)}{\delta g_{ij}(x')} v_{ij}(x') + \beta D^i(x) \delta_{ij}(x, x') - (x \leftrightarrow x') = 0
\] (1.73)

for \( L(x) \), which can be solved once an expression for the diffeomorphism constraint \( D^i \) is inserted. With \( D^i \) linear in the momenta, a fact which remains true in the cases of deformed constraint algebras considered here, and momenta related to functional derivatives of \( L \) by \( v_{ij} \), a linear equation for \( L \) is obtained. The
importance of this consequence of the Legendre transform has been stressed in [1].

If gravity and matter split into independent constrained systems, as realized for matter constraints independent of the gravitational momentum and in the absence of derivatives of $\pi^{ij}(x)$ in $H_{\text{grav}}$, equation (1.73) can be derived in an analogous form for the matter part, just using canonical matter variables and the matter diffeomorphism constraint. Because the following calculations, integrating the functional differential equation, are easier for scalar matter, we will first consider this case as an illustration of the general procedure. As we will see, the Lagrangian viewpoint provides a new interpretation of conditions of anomaly freedom found earlier for inverse-triad corrections of a scalar field.
Chapter 2

Reconstructing Hamiltonians

2.1 Scalar matter

We will now illustrate the ‘reconstruction’ of a Hamiltonian from the Poisson bracket relations [1, 100]. As a simple introductory example of the general ‘reconstruction’ procedure we will follow, we first consider the case of the scalar field in standard gravity, under the assumptions that the matter Hamiltonian $H_{\text{matter}}[N] + D_{\text{matter}}[\vec{N}]$ does not depend on the gravitational momentum $\pi_{ij}$ or derivatives of the spatial metric, and that the gravitational Hamiltonian $H_{\text{grav}}[N] + D_{\text{grav}}[\vec{N}]$ is local in $\pi_{ij}$. As mentioned earlier, we can expect the matter super-Hamiltonian and super-momentum to satisfy the constraints (1.18a, 1.18c). We first reconstruct the super-momentum, following the procedure outlined for the classical case in [1]. The super-momentum for matter, must, like its gravitational counterpart, generate relabellings of the data on the initial value surface. Thus, we have:

$$\int d^3y \{ \phi(x), D_{k_{\text{matt}}}(y) \} \delta N^k(y) = \mathcal{L}_{\delta \vec{N}} \phi(x) = \phi_{|k}(x) \delta N^i(x), \quad (2.1)$$

$$\int d^3y \{ p_{\phi}(x), D_{k_{\text{matt}}}(y) \} \delta N^k(y) = \mathcal{L}_{\delta \vec{N}} p_{\phi}(x) = (p_{\phi}(x) \delta N^k(x))_{|k}, \quad (2.2)$$

keeping in mind that $p_{\phi}$ is a spatial tensor density of weight +1, and that for a scalar, a spatial covariant derivative is the same as a partial derivative. However,
this is equivalent to:

\[ \int d^3 y \frac{\delta D_{k \text{matt}}(y)}{\delta p_\phi(x)} \delta N^k(y) = \phi_{ik}(x) \delta N^i(x), \quad \text{and} \quad \int d^3 y \frac{\delta D_{k \text{matt}}(y)}{\delta \phi(x)} \delta N^k(y) = (p_\phi(x) \delta N^k(x))_{ik}, \]

which are easily solved to find

\[ D_{i \text{matt}}[N^i] = p_\phi(x) \phi_{ij}(x). \]

We now recover the super-Lagrangian, again inspired by the classical case considered in [1], but now taking into account the correction function \( \beta [100] \). With the classical spatial structure, the super-Lagrangian density of a scalar field \( \phi \) must be of the form \( \mathcal{L} = \sqrt{\det g} L(\phi, V, \psi) \) where \( V = (\delta N)^{-1}\{\phi, H[\delta N]\} \) is the normal scalar velocity introduced before and \( \psi = g^{ij} \phi_{ij} \) is the only remaining scalar that can be formed from \( \phi \) and its derivatives up to a total derivative order of at most two. Higher derivatives do not appear classically for equations of motion of second order, but they can easily be introduced by quantum effects. Higher spatial derivatives, in particular, are a natural consequence of discretization in loop quantum gravity, which in effective form combined with a derivative expansion will give rise to derivative terms of arbitrary orders. Higher time derivatives, on the other hand, follow from quantum back-reaction. The following considerations for matter assume the absence of higher-order derivatives, as realized for instance for inverse-triad corrections and some forms of holonomy corrections.

With the canonical variables of a scalar field and its diffeomorphism constraint \( D^i = p_\phi \phi_{ij} \), equation (1.73), adapted to a scalar field, assumes the form

\[ \frac{\delta L(x)}{\delta \phi(x')} V(x') + \beta \frac{\partial L(x)}{\partial V(x)} \phi_{ij}(x) \delta \phi_{ij}(x, x') - (x \leftrightarrow x') = 0. \]

As in [1], we write

\[ \frac{\delta L(x)}{\delta \phi(x')} = \frac{\partial L(x)}{\partial \phi(x)} \frac{\delta \phi(x)}{\delta \phi(x')} + 2 \frac{\partial L(x)}{\partial \psi(x)} \phi_{ij}(x) \delta \phi_{ij}(x, x') \]

(2.7)
and conclude, taking into account the additional factor of $\beta$, that

$$A^i := \phi^i \left( \beta \frac{\partial L}{\partial V} + 2V \frac{\partial L}{\partial \psi} \right)$$

(2.8)

satisfies the equation $A^i(x)\delta^i(x, x') - (x \leftrightarrow x') = 0$, shown in [1] to imply $A^i = 0$. Thus,

$$\beta \frac{\partial L}{\partial V} + 2V \frac{\partial L}{\partial \psi} = 0$$

(2.9)

and $L$ must be of the form $L(\phi, \psi - V^2/\beta)$.

This is a concrete indication that the deformed hypersurface-deformation algebra implies a modification of the usual covariance and of the dispersion relation of fields: The kinetic term of scalar Lagrangians does not depend on $\psi - V^2 = 4g^{\mu\nu} \phi_\mu \phi_\nu$ in space-time terms, but has its time derivatives in $\psi - V^2/\beta$ rescaled by the correction function $\beta$. Nevertheless, the system is covariant and consistent, albeit with a deformed notion of covariance as per the constraint algebra (1.23c). The dependence of the Lagrangian on the potential remains unrestricted, leaving the form of some counterterms as introduced in [7] more open.

It is illustrative to compare this form of the kinetic term with the one obtained for the matter Hamiltonian in a consistent deformation [7]. One begins with a matter Hamiltonian density of the form

$$H = \nu \frac{p_\phi^2}{2\sqrt{\det g}} + \frac{1}{2} \sigma \sqrt{\det g} \psi + \sqrt{\det g} W(\phi)$$

(2.10)

with metric factors corrected by inverse-triad corrections $\nu$ and $\sigma$, and some potential $W(\phi)$. The corresponding Lagrangian density, with $V = \nu p_\phi / \sqrt{\det g}$, takes the form

$$L = \sqrt{\det g} \left( \frac{V^2}{2\nu} - \frac{\sigma \psi}{2} - W(\phi) \right)$$

$$= -\sqrt{\det g} \frac{\sigma}{2} \left( \psi - \frac{V^2}{\beta} \right) - \sqrt{\det g} W(\phi)$$

(2.11)

with the kinetic dependence as derived above, provided that

$$\beta = \nu \sigma.$$

(2.12)
This condition, as derived in [7] for linear inhomogeneities around isotropic models, is exactly one of the requirements for anomaly freedom to ensure a closed constraint algebra of the form (1.23c) for inverse-triad corrections with $\beta = \bar{\alpha}^2$ from the gravitational constraint. The Lagrangian view clearly shows how this condition of anomaly cancellation is necessary to ensure a (deformed) covariant kinetic term in the action. With the same corrections in d’Alembertians, propagation speeds of massless matter and gravitational waves naturally agree, as explicitly shown for electromagnetic waves in [26].

From the new derivation of corrected scalar Lagrangians in this paper, we must expect corrections in matter terms also if $\beta$ results from holonomy corrections, provided they can be consistently implemented. Explicit examples for holonomy modifications required in matter terms have already been found in [27, 9]. However, in a scalar Hamiltonian quantized by the methods of loop quantum gravity [64] we do not expect holonomy corrections. Consistent formulations of holonomy corrections in the presence of matter therefore seem to encounter stronger difficulties than inverse-triad corrections. Another peculiar feature can be seen by recalling that $\beta$ for holonomy corrections can turn negative. The modified d’Alembertian $\psi - V^2/\beta$ then becomes one of Euclidean signature, or a 4-dimensional Laplacian, and fields no longer propagate. Also this property can explicitly be seen in the wave equations of [9]. We will discuss further consequences of this new form of signature change in Sec. 2.3.3.

2.2 Gravitational part

As in the case of scalar matter, we begin our discussion of the gravitational part by inserting the explicit expression of the diffeomorphism constraint in the general equation (1.73): In particular,

$$\beta D^i(x)\delta_{ij}(x,x') = -2\beta \pi^{ij}(x)\delta_{ij}(x,x').$$

(2.13)

We then proceed as in the example of spherical symmetry: We multiply this expression by two test functions $a(x)$ and $b(x')$ and integrate over $x$ and $x'$, observing that some terms symmetric in $a$ and $b$ cancel. After several steps, integrating by
parts, discarding total derivatives and using the symmetry of $\pi^{ij}$, we arrive at

$$\int dx \left[ 2\pi^{ij}(x)\beta_{ij}(x) \left( a(x)b_{ij}(x) - a_{ij}(x)b(x) \right) + 2\pi^{ij}(x)\beta \left( a(x)b_{ij}(x) - a_{ij}(x)b(x) \right) \right]$$

(2.14)

from the right-hand side of (2.13). Functional derivatives with respect to $a(y)$ and $b(z)$ give

$$\int dx \left[ 2\pi^{ij}(x)\beta_{ij}(x) \left( \delta(x,y)\delta_{ij}(x,z) - \delta_{ij}(x,y)\delta(x,z) \right) + 2\pi^{ij}(x)\beta \left( \delta(x,y)\delta_{ij}(x,z) - \delta_{ij}(x,y)\delta(x,z) \right) \right]$$

(2.15)

$$= 2\pi^{ij}(y)\beta_{ij}(y)\delta_{ij}(y,z) + 2\pi^{ij}(y)\beta(y)\delta_{ij}(y,z) - (y \leftrightarrow z)$$

(2.16)

$$= 2\frac{\partial L(y)}{\partial v_{ij}(y)}\beta_{ij}(y)\delta_{ij}(y,z) + 2\frac{\partial L(y)}{\partial v_{ij}(y)}\beta(y)\delta_{ij}(y,z) - (y \leftrightarrow z)$$

(2.17)

if no spatial derivatives of $v_{ij}$ appear in the corrections and the Lagrangian, such that $\pi^{ij}(y) = \delta L/\delta v_{ij}(y) = \partial L(y)/\partial v_{ij}(y)$. In combination with (1.73), we have

$$\frac{\delta L(x)}{\delta g_{ij}(x')}v_{ij}(x') + 2\beta_{ij}(x)\frac{\partial L(x)}{\partial v_{ij}(x)}\delta_{ij}(x,x')$$

$$+ \frac{\partial L(x)}{\partial v_{ij}(x)}\delta_{ij}(x,x') - (x \leftrightarrow x') = 0.$$  

(2.20)

In cases of derivative expansions of non-local terms in $v_{ij}$, we use

$$\frac{\delta L(x)}{\delta g_{ij}(x')}v_{ij}(x')\delta(x,x') + 2\beta_{ij}(x)\frac{\delta L(x)}{\delta v_{ij}(x')}\delta_{ij}(x,x')$$

$$+ 2\beta \frac{\delta L(x)}{\delta v_{ij}(x')}\delta_{ij}(x,x') - (x \leftrightarrow x') = 0$$

(2.21)

and write

$$\frac{\delta L(x)}{\delta v_{ij}(x')} = \frac{\partial L(x)}{\partial v_{ij}(x')}\delta(x,x') + \frac{\partial L(x)}{\partial v_{ij}(k)(x')}\delta_{ik}(x,x') + \cdots$$

(2.22)
2.2.1 Expansion

As the next crucial step in solving the functional equation (2.20), we expand both \( L \) and \( \beta \) as series in powers of the normal change of the metric, \( v_{ij} \):

\[
L = \sum_{n=0}^{\infty} L_{i_1j_1...i_nv_{n,j_n}}(x) L_{i_1,j_1}(x) \ldots v_{i_nv_{n,j_n}}(x) \tag{2.23}
\]

\[
\beta = \sum_{n=0}^{\infty} \beta_{i_1j_1...i_nv_{n,j_n}}(x) L_{i_1,j_1}(x) \ldots v_{i_nv_{n,j_n}}(x) \tag{2.24}
\]

assuming for now local functions without spatial derivatives. (See Sec. 2.2.4 for non-locality.) The expansion of \( \beta \) allows us to deal with inverse-triad corrections and local holonomy corrections at the same time. Holonomy corrections will then not appear as periodic functions such as \( \sin(\delta K_{\phi})/\delta K_{\phi} \) in spherical symmetry, but as perturbative terms of a power series in \( K_{\phi} \). Such an expansion is more consistent with the perturbative nature of these higher-order corrections, which are expected in a similar form from higher-curvature terms or quantum back-reaction. Including all terms in a power series of \( \sin(\delta K_{\phi}/\delta) \), even tiny ones at high orders, but ignoring quantum back-reaction would not be consistent. An expansion also makes it more clear how terms of higher order in \( v_{ij} \) can be combined with higher spatial derivatives of the metric.

We insert these expansions into (2.20) and first set \( v_{ij}(x) = 0 \) to obtain

\[
2L^{ij}(x)\beta^\theta_{ij}(x)\delta_{ij}(x,x') + 2L^{ij}(x)\beta^\theta(x)\delta_{ij}(x,x') - (x \leftrightarrow x') = 0. \tag{2.25}
\]

We multiply by test functions \( a(x) \) and \( b(x') \) and integrate over \( x \) and \( x' \), drop total divergences and terms that vanish due to the symmetry of indices of \( L^{ij}(x) \), cancel some other terms and are left with

\[
\int dx L_{i,j}^{ij}(x) \beta^\theta(a_{i}b - ab_{i}) = 0. \tag{2.26}
\]

Since \( a, b, a_{i}, \) and \( b_{i} \) can be chosen independently, we conclude that \( L_{i,j}^{ij}(x) \beta^\theta = 0. \)
Note that $\beta^0 \neq 0$ generically, so that we have

$$L_{ij}^0(x) = 0. \quad (2.27)$$

We return to equation (2.20), do a functional differentiation with respect to $v_{kl}(z)$ and then set $v_{ij}(x)$ to zero everywhere. With the notation

$$\delta_{ab}^i(x,z) = \frac{1}{2} (\delta^k_a \delta^l_b + \delta^l_a \delta^k_b) \delta(x,z) \quad (2.28)$$

we have

$$\frac{\delta L^0(x)}{\delta g_{kl}(x')} \delta(x',z) + 4L^{ijab}(x)\beta^0_{ij} \delta_{kl}(x,x') \delta_{ab}(x,z) + 4L^{ijab} \delta_{ab}(x,z) \delta_{lij}(x,x') + 2L^{ij}(x) \delta_{lij}(x,x') (\beta^a_{ij} \delta^{kl}_{ab}(x,z) + \beta^b_{ijkl}(x,z)) + 2L^{ij}(x) \beta^{a kl} \delta_{lij}(x,x') - (x \leftrightarrow x')$$

$$= - \frac{\delta L^0(x')}{\delta g_{kl}(x)} \delta(x,z) + \left(4L^{ijkl}(x)\beta^0_{ij} + 2L^{ij}(x)\beta^0_{lij}\right) \delta_{lij}(x,x') \delta(x,z) + 2L^{ij}(x) \beta^{a kl} \delta_{lij}(x,z) \delta_{li}(x,x')$$

$$+ (2L^{ij}(x)\beta^{a kl} + 4L^{ijkl}(x)\beta^0) \delta_{lij}(x,x') \delta(x,z) - (x \leftrightarrow x') = 0. \quad (2.29)$$

We use

$$2L^{ij}(x)\beta^{a kl} \delta_{lij}(x,z) \delta_{li}(x,x') = \left(2L^{ij}(x)\beta^{a kl} \delta(x,z) \delta_{lij}(x,x')\right)_{ij}$$

$$- 2L^{ij}(x)\beta^{a kl} \delta(x,z) \delta_{lij}(x,x') - 2L^{ij}(x)\beta^{a kl} \delta(x,z) \delta_{lij}(x,x')$$

$$- 2L^{ij}(x)\beta^{a kl} \delta(x,z) \delta_{lij}(x,x'), \quad (2.30)$$

drop the total divergence term in (2.31), and insert $L_{ij}^0(x) = 0$ from (2.27):

$$\left(- \frac{\delta L^0(x')}{\delta g_{kl}(x)} + 4L^{ijkl}(x)\beta^0_{ij} \delta_{lij}(x,x') + 4L^{ijkl} \beta^0_{lij}(x,x')\right) \delta(x,z) - (x \leftrightarrow x') = 0. \quad (2.32)$$

This equation can be solved as in [1] where $\beta^0 = 1$: define

$$A^{ij}(x,x') = \frac{\delta L^0(x)}{\delta g_{ij}(x')} - 4L^{ijkl}(x') \left(\beta^0_{ij}(x') \delta_{lk}(x', x) + \beta^0(x') \delta_{kl}(x', x)\right) \quad (2.33)$$
and rewrite (2.32) as

\[ A^{ij}(x, x') \delta(x', z) - A^{ij}(x', x) \delta(x, z) = 0. \]  

(2.34)

Integrating over \( x' \), we find

\[ A^{ij}(x, x'') = F^{ij}(x) \delta(x, x'') \]  

with

\[ F^{ij}(x) = \int d^3 x' A^{ij}(x', x), \]

a function of only one variable, and thus

\[ \frac{\delta L^0(x)}{\delta g_{ij}(x')} = F^{ij}(x) \delta(x, x') \]  

\[ + 4 L^{ijkl}(x') \left( \beta^{\theta}_{ij}(x') \delta(x', x) + \beta^{\theta}(x') \delta_{ijkl}(x', x) \right). \]

(2.35)

### 2.2.2 Coefficients

As a spatial scalar density, \( L^0 \) can depend on the metric and its spatial derivatives only via the metric itself and suitable contractions of products of the spatial Riemann tensor. To second order in spatial derivatives,

\[ L^0(x) = L^0(g_{ij}(x), (3) R_{ij}(x)), \]  

(2.36)

a fact, used in [1], that remains true in the deformed case with our assumption that the spatial part of the algebra stays classical. Define

\[ \varphi^{ij} = \frac{\partial L^0(g_{kl}, (3) R_{kl})}{\partial g_{ij}} \quad \Phi^{ij} = \frac{\partial L^0(g_{kl}, (3) R_{kl})}{\partial (3) R_{ij}} \]  

(2.37)

and write

\[ \delta L^0 = \left( \varphi^{ij} + \frac{1}{2} (3) R^{i}_{kl} \delta^{j} + \frac{1}{4} (3) R_{k}^{i} \Phi^{k} + \frac{1}{4} (3) R_{k}^{i} \Phi^{k} \right) \delta g_{ij} \]  

\[ + \frac{1}{4} \left( \Phi^{kl} g^{jl} + \Phi^{jl} g^{kl} + \Phi^{jk} g^{il} + \Phi^{il} g^{jk} \right) \delta g_{ij|kl}. \]

(2.38)

From (2.35), we also have

\[ \delta L^0 = \delta g_{ij} \left( F^{ij} + 4 L^{ijkl}_{lk} \beta^{\theta} + 4 L^{ijkl}_{l} \beta^{\theta}_{k} \right) \]  

\[ + \delta g_{ij|kl} \left( 8 L^{ijkl}_{l} \beta^{\theta} + 4 L^{ijkl}_{l} \beta^{\theta}_{k} \right) + \delta g_{ij|kl} \left( 4 L^{ijkl}_{l} \beta^{\theta} \right). \]

(2.39)
Comparing the various coefficients, we get

$$L^{ijkl}_{\beta^0} = \frac{1}{16} (\Phi^{ik} g^{jl} + \Phi^{il} g^{jk} + \Phi^{jk} g^{il} + \Phi^{jl} g^{ik} - 2\Phi^{ij} g^{kl} - 2\Phi^{kl} g^{ij}) \tag{2.40}$$

$$2L^{ijkl}_{\beta^0} + \Phi^{ijkl}_{\beta^0} = 0 \tag{2.41}$$

$$F^{ij} + 4L^{ijkl}_{\beta^0} \beta^0 + 4L^{ijkl}_{\beta^0} \beta^0 = \varphi^{ij} + \frac{1}{2} (3) R_{kl}^{ij} \beta^0 + \frac{1}{4} (3) R_{kl}^{ij} \Phi^{kl} + \frac{1}{4} (3) R_{kl}^{ij} \Phi, \tag{2.42}$$

Thus, $0 = 2L^{ijkl}_{\beta^0} + \Phi^{ijkl}_{\beta^0} = -\beta^0 \Phi^{ijkl} + 2(L^{ijkl}_{\beta^0} \beta^0).$ We compute each term using (2.40), and write

$$0 = -\frac{\beta^0}{16\beta^0} \left( \Phi^{ik} g^{jl} + \Phi^{il} g^{jk} + \Phi^{jk} g^{il} + \Phi^{jl} g^{ik} - 2\Phi^{ij} g^{kl} - 2\Phi^{kl} g^{ij} \right)$$

$$+ \frac{1}{8} \left( \Phi^{ikj} + g^{jk} \Phi^{il} + \Phi^{ikl} + g^{ij} \Phi^{kl} - 2\Phi^{ijl} - 2\Phi^{klj} \right). \tag{2.43}$$

We contract this with $g_{ij},$ use $\delta^i_i = 3,$ and denote $\Phi^i$ as $\Phi:

$$\frac{\beta^0}{8\beta^0} \left( \Phi^{kl} + \Phi g^{kl} \right) - \frac{1}{4} \left( \Phi^{kl} + \Phi^{ijkl} g_{ij} \right) = 0. \tag{2.44}$$

Note that $\Phi^{ijkl} g_{ij} = \Phi^{ik} = (\Phi g^{kl})_{ij}.$ With $\Phi^{kl} + \Phi g^{kl}$ denoted as $\bar{\Phi}^{kl},$

$$0 = \frac{\beta^0}{8\beta^0} \bar{\Phi}^{kl} - \frac{1}{4} \bar{\Phi}^{kl}$$

$$= \frac{1}{4} \sqrt{|\beta^0|} \left( \beta^0 \Phi^{kl} - \frac{1}{2} |\beta^0| \frac{1}{2} \bar{\Phi}^{kl} - |\beta^0| \frac{1}{2} \bar{\Phi}^{kl} \right)$$

$$= \frac{1}{4} \sqrt{|\beta^0|} \left( |\beta^0| - \frac{1}{2} \bar{\Phi}^{kl} \right) \tag{2.45}$$

Again maintaining our assumption of an unmodified spatial structure, the only covariantly constant 2-tensors constructed from the metric and its derivatives up to second order are the metric itself and the spatial Einstein tensor. Noting the density weight one of $\bar{\Phi}^{kl},$ inherited from $L^0,$ we conclude that

$$\frac{\bar{\Phi}^{kl}}{\sqrt{|\beta^0|}} = A \sqrt{\det g} \left( (3) R^{kl} - \frac{1}{2} (3) R g^{kl} \right) + B \sqrt{\det g} g^{kl} \tag{2.46}$$
where $A$ and $B$ are constants. This gives

$$\Phi^{kl} = A \sqrt{|\beta^0| \det g} \left( (3) R^{kl} - \frac{3}{8} (3) R g^{kl} \right) + \frac{B}{4} \sqrt{|\beta^0| \det gg^{kl}}. \tag{2.47}$$

Inserting this into (2.43), we find, after cancelling terms, that

$$A \sqrt{|\beta^0| \det g} \left[ (3) R^{ik} g^{jl} + (3) R^{il} g^{jk} + (3) R^{jk} g^{li} + (3) R^{jl} g^{ik} - 2 (3) R^{ij} g^{kl} \right] = 0. \tag{2.48}$$

For this to be satisfied for general metrics, we must set $A = 0$. Writing $B = \frac{1}{16\pi G}$,

$$\Phi^{kl} = \frac{1}{16\pi G} \sqrt{|\beta^0| \det gg^{kl}}. \tag{2.51}$$

Then, from (2.40)

$$L^{ijkl} = \frac{1}{16\pi G \beta^0} \left( \sqrt{|\beta^0| \det g} (g^{ik} g^{jl} + g^{il} g^{jk} + g^{jk} g^{li} + g^{jl} g^{ik}) \right) + \sqrt{|\beta^0| \det gg^{kl} g^{ij} - 2 \sqrt{|\beta^0| \det gg^{ij} g^{kl} - 2 \sqrt{|\beta^0| \det gg^{kl} g^{ij}}} \tag{2.52}$$

We also have

$$\frac{\partial L^0(g_{kl}, (3) R_{kl})}{\partial (3) R_{ij}} = \Phi^{ij} = \frac{1}{16\pi G} \sqrt{|\beta^0| \det gg^{ij}} \tag{2.53}$$

from the definition (2.37). We integrate this to get

$$L^0 = \frac{1}{16\pi G} \sqrt{\det g} \left( \sqrt{|\beta^0|} (3) R + f(g) \right) \tag{2.54}$$

where, for a scalar density, $f(g) = -2\lambda$ must be a constant, the cosmological constant. (The previous equations do not determine $f(g)$ because it would follow from $\varphi^{ij}$ according to (2.37), which by (2.42) is related to the free function $F^{ij}$.)
Combining (2.52) and (2.54), the regained Lagrangian up to second order is

\[ L = \frac{\sqrt{\text{det} g}}{16\pi G} \left( \frac{\text{sgn} \beta^\theta v^i v^j - v_i^i v_j^j}{4} + \sqrt{|\beta^\theta|} (3) R - 2\lambda \right). \]  

(2.55)

For \( \beta^\theta = 1 \), the classical Lagrangian is recovered with \( v_{ij} = 2K_{ij} \) related to extrinsic curvature. But already to second order in derivatives, loop quantum gravity implies corrections to the Lagrangian from inverse-triad corrections with \( \beta^\theta \neq 1 \), a property that cannot be mimicked by any form of higher-curvature effective actions. Also holonomy corrections cannot provide a similar modification because they always come with higher powers of \( v_{ij} \). Inverse-triad corrections can thus easily be distinguished from other quantum effects. (Holonomy corrections can provide similar modifications if the \( v_{ij} \) expansion is resummed; see Sec. 2.3.3.1.)

The correction function \( \beta^\theta[g_{ij}] \) relevant for these corrections must be scalar, which is not possible classically if only the metric can be used. For this reason, the full dynamics is more unique than the spherically symmetric one, where \( E^x \) is a scalar metric component without a density weight in the reduced model. In an effective formulation of quantum gravity, additional quantities become available that explicitly refer to properties of an underlying state, such as the discreteness scale in loop quantum gravity. It is then possible to construct non-trivial scalars of density weight zero by referring to the metric and state parameters, such as elementary fluxes [7].

Compared with the results in spherical symmetry, the full effective action is more unique, as already discussed. Other properties of the corrections are, however, very similar: The correction function \( \beta^\theta \) features in the same way in the curvature potential. Also the kinetic term is corrected in the same way, if we only note that a factor of \( \sqrt{|\beta^\theta|} \) was obtained in spherical symmetry, where we used momenta \( K_x \) and \( K_\phi \) instead of the normal change \( v_{ij} \) of the metric. If we substitute the normal changes \( \delta H/\delta K_x \) and \( \delta H/\delta K_\phi \) for \( K_x \) and \( K_\phi \) in spherical symmetry, we also obtain a kinetic term divided by \( \sqrt{|\beta^\theta|} \). The sign of \( \beta^\theta \) appears in different places in our expressions for spherical symmetry and the full theory, but the relative sign between the curvature and the kinetic terms is the same. The absolute placement of the sign is ambiguous because in the derivations it first appears in derivatives, for instance when we introduce \( B_1 \) after (3.25), or in (2.45).
2.2.3 Higher orders

To second order, $\Phi^{kl}$ determines both $L^{ijkl}$ from (2.40) and $\partial L^0/\partial (\kappa R_{ij})$ from (2.37), ensuring that time derivatives of $g_{ij}$ and spatial Ricci contributions are combined to space-time covariant curvature terms. The same interplay is repeated for higher orders in the $v$-expansion, although with an increasing number of terms.

For the next order, as an example, we start again from (2.20) and gather all terms which are quadratic in $v_{ij}(x)$ and its derivatives.

\[
\frac{\delta L^{ab}(x)}{\delta g_{ij}(x')} v_{ab}(x) v_{ij}(x') + 6 L^{abcdij} v_{ab}(x) v_{cd}(x) \beta^a_{ij}(x, x') \\
+ 4 L^{abij} v_{ab}(x) \left( \beta^{ef}_{ij} v_{ef}(x) \right)_{ij} \delta^a_i(x, x') \\
+ 2 L^{ij}(x) \left( \beta^{cdef} v_{cd}(x) v_{ef}(x) \right)_{ij} \delta^a_i(x, x') \\
+ 6 L^{abcdij} v_{ab}(x) v_{cd}(x) \beta^a_{ij}(x, x') + 4 L^{abij} v_{ab}(x) \beta^{cd} v_{cd}(x) \delta^a_{ij}(x, x') \\
+ 2 L^{ij}(x) \beta^{cdef} v_{cd}(x) v_{ef}(x) \delta^a_{ij}(x, x') - (x \leftrightarrow x').
\] (2.56)

We multiply this by two test functions, $a(x)$ and $b(x')$ and integrate over $x$ and $x'$. After integrating by parts, discarding total divergences, removing terms that disappear due to the symmetry and anti-symmetry of various indices, and using (2.27), we arrive at

\[
\int dx dx' \left( \frac{\delta L^{ab}(x)}{\delta g_{ij}(x')} - \frac{\delta L^{ij}(x')}{\delta g_{ab}(x)} \right) v_{ab}(x) v_{ij}(x') a(x)b(x') \\
- \int dx \left( 6 L^{abcdij} v_{ab}(x) v_{cd}(x) \right)_{ij} \beta^a_{ij}(a(x)b_i(x) - a_i(x)b(x)) \\
- \int dx \left( 4 L^{abij} v_{ab}(x) v_{cd}(x) \right) \beta^{cd} v_{cd}(x) (a(x)b_i(x) - a_i(x)b(x)) = 0. \] (2.57)

Since $v_{ab}(x)$, $v_{abij}(x)$, $a(x)$, $b(x)$, $a_i(x)$ and $b_i(x)$ can all be varied independently, we arrive at the following three conditions:

First, setting $(a(x)b_i(x) - a_i(x)b(x)) = 0$, we get

\[
\left( \frac{\delta L^{ab}(x)}{\delta g_{ij}(x')} - \frac{\delta L^{ij}(x')}{\delta g_{ab}(x)} \right) v_{ab}(x) v_{ij}(x') a(x)b(x') = 0.
\] (2.58)
Following the arguments in [1], we see that this eventually implies

$$\frac{\delta L^{ab}(x)}{\delta g_{ij}(x')} - \frac{\delta L^{ij}(x')}{\delta g_{ab}(x)} = 0.$$  \hfill (2.59)

This equation restricts the form of terms linear in $v_{ij}$ in the action, which are absent anyway if the theory is time-reversal invariant. Then setting $v_{abij}(x) = 0$,

$$6L_{|ij}^{abcdij} \beta^\emptyset + 4L_{|ij}^{ijab} \beta^{cd} = 0.$$  \hfill (2.60)

And finally:

$$12L_{|ij}^{abcdij} \beta^\emptyset + 4L_{|ij}^{abij} \beta^{cd} = 0.$$  \hfill (2.61)

We relabel indices, multiply (2.61) with $\beta^\emptyset_{|j}$ and use (2.41) to rewrite it.

$$12L_{|ijklmn}^{ijklmn} \beta^\emptyset_{|j} + 4L_{|ijkl}^{ijkl} \beta^\emptyset_{|l} = 0.$$  \hfill (2.62)

(We use $L^{ijklmn} = L^{ijmnkl}$, referring to the definition in (2.23).) Using (2.60), we can write $24L_{|ijklmn}^{ijklmn}(\beta^\emptyset)^2 + 24L_{|ijkl}^{ijkl} \beta^\emptyset \beta^\emptyset_{|l} = 0$. Generically, $\beta^\emptyset \neq 0$, and so we have $(L_{|ijklmn}^{ijklmn} \beta^\emptyset)_{|l} = 0$ solved by the classical covariantly constant quantities with the corresponding index structure, divided by $\beta^\emptyset$.

The third order in $v_{ij}$ will therefore have terms with a factor of $1/\beta^\emptyset$ times corresponding orders possible for higher-curvature actions, while the quadratic order had a factor of $1/\sqrt{|\beta^\emptyset|}$, and the zeroth order a factor of $\sqrt{|\beta^\emptyset|}$. The same pattern is repeated at higher orders in the $v$-expansion: To order $n$ in $v_{ij}$, we have terms as in higher-curvature actions but multiplied with $|\beta^\emptyset|^{(1-n)/2}$. To see this, we notice that Eq. (2.20), when expanded by powers of $v_{ij}$, has a first term which contains expansion coefficients of $L^{-1}$ two orders lower than the rest, which are all multiplied with $\beta^\emptyset$. If we use the equation to derive the $L$-coefficients by recurrence, we solve for a coefficient two orders higher by dividing by $\beta^\emptyset$. Starting with zeroth order in $v_{ij}$ of magnitude $\sqrt{|\beta^\emptyset|}$ in the prefactor, the quoted orders follow. (If the corrected theory is not time-reversal invariant and odd orders appear in the $v$-expansion, the same powers of $|\beta^\emptyset|$ per order are obtained.)
2.2.4 Non-locality

So far, we have assumed only a local dependence on $v_{ij}$, with no spatial derivatives of $v_{ij}$ that would otherwise be implied by a derivative expansion. In the classical case, locality follows from the relation of $v_{ij}$ to extrinsic curvature, but it can easily be violated by some of the correction functions in quantum gravity.

In an effective action, non-locality usually makes itself noticeable in a derivative expansion of the fields. The basic equation (1.73) is valid also for non-local theories, without explicit terms with spatial derivatives of $v_{ij}$. However, (2.20) must be replaced by (2.21), and the general expansions (2.23) and (2.24) must also include terms with spatial derivatives of $v_{ij}$. We now define

$$L(x) = \sum_{n=0}^{\infty} \sum_{N_1,\ldots,N_n=0}^{\infty} L^{(i_1,j_1,k_1^{(1)},\ldots,k_1^{(N_1)})\ldots(i_n,j_n,k_n^{(1)},\ldots,k_n^{(N_n)})} [g_{ij}] v_{i_1,j_1,k_1^{(1)}\ldots,k_1^{(N_1)}\ldots,k_n^{(1)}\ldots,k_n^{(N_n)}](2.63)$$

$$\beta = \sum_{n=0}^{\infty} \sum_{N_1,\ldots,N_n=0}^{\infty} \beta^{(i_1,j_1,k_1^{(1)},\ldots,k_1^{(N_1)})\ldots(i_n,j_n,k_n^{(1)},\ldots,k_n^{(N_n)})} [g_{ij}] v_{i_1,j_1,k_1^{(1)}\ldots,k_1^{(N_1)}\ldots,k_n^{(1)}\ldots,k_n^{(N_n)}].(2.64)$$

Derivative terms in the expansion of $\beta$ then require new terms in the Lagrangian that contain spatial derivatives. Going through the recurrence, an order $n$ in the $v$-expansion again receives a coefficient of $|\beta^0|^{(1-n)/2}$.

In this context, we can distinguish between two expansions of the action, one by powers of $v_{ij}$ and its spatial derivatives as in (2.63), and one by the total order of derivatives. The total order of derivatives is the crucial one for a comparison with higher-curvature terms in an effective action, which come arranged by the order of time and space derivatives. With $v_{ij}$ related to the normal change of the metric, it counts as a derivative (by time) of order one. A term of $v_{i_1,j_1,k_1^{(1)}\ldots,k_1^{(N_1)}}$ counts as a derivative of order $1 + N_1$, and therefore a general expansion term in (2.63) with coefficient $L^{(i_1,j_1,k_1^{(1)},\ldots,k_1^{(N_1)})\ldots(i_n,j_n,k_n^{(1)},\ldots,k_n^{(N_n)})}$ counts as a derivative of order $\sum_{i=1}^{n}(1 + N_i) = n + \sum_{i=1}^{n} N_i$. Terms of the same $v$-order $n$, that is with the same number of factors of $v_{ij}$ or its spatial derivatives, have different derivative orders of at least $n$. If we reorganize the expansion by derivative orders $N$, keeping track of $\beta^0$-factors that depend only on the $v$-order, we obtain effective-action terms of
the schematic form

\[ |\beta^\emptyset|^{(1-N)/2} v^N + |\beta^\emptyset|^{(2-N)/2} (v^{N-1} g' + v^{N-2} v') \]
\[ + |\beta^\emptyset|^{(3-N)/2} (v^{N-2} (g'' + (g')^2) + v^{N-3} (v'' + v' g') + v^{N-4} (v')^2) + \ldots . \]

The highest power of \(1/\sqrt{|\beta^\emptyset|}\) for a given derivative order is always obtained for the term \(v^N\) free of spatial derivatives. For small \(\beta^\emptyset\), time derivatives in a derivative or curvature expansion are dominant.

### 2.3 Applications and conclusions

One of the main results of this paper, of general importance for loop quantum gravity, follows from the effective action (2.55), valid to second order in extrinsic curvature. Although we did allow for holonomy and higher-curvature corrections as well, only inverse-triad corrections are active at this order. This result is an independent confirmation, in addition to [13, 3, 4], that inverse-triad corrections can be much more significant than higher-curvature and holonomy corrections, both of which occur only at higher orders in \(v_{ij}\) and are of the tiny size \(\ell_P^2/\ell_H^2\) throughout most of nearly isotropic cosmology with the Hubble distance \(\ell_H\). Our calculations show, for the first time, how different quantum effects in loop quantum gravity without any symmetry assumptions can be included all at once, but still show their own characteristic consequences. The complete correction function \(\beta\) in the constraint algebra may contain contributions from both inverse-triad and holonomy corrections, including non-local effects, but it is only the \(v\)-independent part \(\beta^\emptyset\) which appears at second order of the effective action. This coefficient is affected by inverse-triad corrections, which therefore present the most important modification of the classical dynamics unless curvature is extremely large. Holonomy corrections, on the other hand, modify terms of higher order in \(v_{ij}\); they mix with higher-curvature terms and can rarely be used in isolation. Moreover, \(U(1)\) calculations of inverse-triad correction functions are reliable because non-Abelian features would change merely the higher-\(v\) behavior.
The clear separation of some of the corrections allows us to discuss their cosmological consequences in very general terms.

### 2.3.1 Enhanced BKL scenario and the absence of singularities in consistent loop quantum gravity

All $v_{ij}$-terms in the effective action (2.55), to all orders, have at least one additional factor of $1/\beta^0$ compared with the spatial curvature term at zeroth order (or a factor of $1/\sqrt{|\beta^0|}$ if there are linear terms in $v_{ij}$, breaking time-reversal invariance). At higher orders, as shown in Secs. 2.2.3 and 2.2.4, $v_{ij}$-terms free of spatial derivatives have at least an additional factor of $1/\sqrt{|\beta^0|}$ compared to spatial-derivative terms of the same derivative order. When $\beta^0$ is very small, all spatial derivatives and curvature potentials are suppressed compared with the normal change of the metric in $v_{ij} = N^{-1}\{g_{ij}, H[N]\}$. Inverse-triad corrections, computed in Abelian models [19], imply that $\beta^0$ approaches zero for vanishing components of the densitized triad, right at classical singularities. As we approach such a singularity, quantum corrections become stronger, which could altogether stop the evolution down to smaller volumes. If this is the case, the singularity is resolved. However, such “bounces” have been difficult to generalize beyond the simple models in which they can be realized explicitly (see also Sec. 2.3.3 below), and therefore it is not guaranteed that vanishing components of the densitized triad can always be avoided. However, if such small values are approached, inverse-triad corrections become significant and suppress spatial derivatives. The evolution then follows a nearly homogeneous behavior of Bianchi-I type, for which singularity resolution in loop quantum cosmology can be shown in general terms by quantum hyperbolicity [28, 29, 30, 31], based on properties of difference equations for wave functions. Even without symmetry assumptions and without restricting the class of quantum corrections included, the dynamics of loop quantum gravity is singularity-free. The same mechanism is hereby shown to apply in symmetric models [28, 30, 32, 33] and the full theory.

The concrete mechanism is reminiscent of the BKL scenario [34] in that spatial derivatives are suppressed and the dynamics becomes almost homogeneous near singularities. The present scenario, however, is much more general. We need not
rely on details of the evolution because it is terms in the effective action itself that show the suppression. Moreover, the arguments are easily seen to be independent on what gauge, or spatial slicing in the classical setting, is chosen, because they make use of a consistent and anomaly-free theory exhibiting general covariance (in a deformed sense). Spatial terms are suppressed even in the \( \{H, H\} \)-algebra itself. This feature is also responsible for the covariance of the mechanism: if \( \beta \) is very small, normal deformations of hypersurfaces, governed by \( \{H, H\} \) as in (1.23c), do not generate spatial displacements from \( D \). With the suppression by small \( \beta \), normal deformations form a subalgebra of the full hypersurface-deformation algebra and can be considered in separation, eliminating the need of homogeneity assumptions.

### 2.3.2 Dispersion relations and causality

Our results show how consistent deformations of the type (1.23c), for which several examples have been found in models of loop quantum gravity as recalled in Section 1.3, affect the form of action principles reconstructed from them. From this perspective, the universal modification — irrespective of the precise form of the correction function \( \beta \) — is that a new coefficient \( \beta \) rescales time derivatives relative to spatial derivatives in matter terms as well as gravitational ones. The usual d’Alembertian \( \Box = -\partial_t^2 + g^{ij} \partial_i \partial_j \) is replaced by \( \Box_\beta := -\beta^{-1} \partial_t^2 + g^{ij} \partial_i \partial_j \). Dispersion relations and propagation speeds are then modified in a compatible way for matter and gravity, as shown explicitly in the special cases considered in [26]. (Counterterms in perturbative realizations of consistency lead to interesting new effects for non-propagating modes [12, 13].) In particular, while \( \beta \neq 1 \) implies that speeds of massless modes differ from the classical speed of light, they all propagate at the same speed as light in space-time according to deformed relativity. All massless excitations propagate with the velocity \( \sqrt{\beta} \) times the classical speed of light for \( \beta > 0 \). If \( \beta < 0 \), which is possible for holonomy corrections, the d’Alembertian changes to a Euclidean-signature Laplacian, and all propagation ceases.
2.3.3 Signature change

Holonomy corrections cannot easily be analyzed in general terms because their mixing with higher-curvature corrections requires the latter to be derived in detail, too. In loop quantum gravity, however, the derivation of higher-curvature terms or their analog in quantum back-reaction remains incomplete. But there is one general property of holonomy corrections realized when they are large and near their maximum value. When this is the case, we must be careful with the $v$-expansions used. One consequence, fortunately, can be seen very generally.

2.3.3.1 The high-density regime in models of loop quantum gravity

In existing consistent examples, holonomy corrections always have the following form: A connection or extrinsic-curvature component in the classical Hamiltonian constraint is replaced by a bounded and periodic function of the same component (possibly depending also on the triad). For instance, in spherical symmetry we can consistently replace $K_\phi$ by $\delta^{-1}\sin(\delta K_\phi)$ with some parameter $\delta$ [11], and in isotropic models we can replace the isotropic connection component $c$ by $\delta^{-1}\sin(\delta c)$ [9]. The parameter $\delta$ may depend on triad components $E^x$ or $a$. When these bounded functions take their maximum value, at $\delta K_\phi = \pi/2$ or $\delta c = \pi/2$, holonomy corrections are large and the Hamiltonian constraint ensures that we are at high energy densities if matter is present. As recalled in Sec. 1.3, in the constraint algebra we obtain a deformation with correction function $\beta(K_\phi) = \cos(2\delta K_\phi)$ and $\beta(c) = \cos(2\delta c)$, respectively. These functions are negative, and in fact are equal to $-1$, when $\delta K_\phi = \pi/2$ or $\delta c = \pi/2$. This change in the sign of $\beta$ (from $\beta = 1$ in the classical regime to $\beta = -1$ when holonomy corrections are significant) effectively changes our space-time from Lorentzian signature to Euclidean. There is no evolution in Euclidean space, so how does the ‘Big Bounce’ of LQG occur?

2.3.3.2 Euclidean space at the Bounce

Negative $\beta^\theta$, in all models studied consistently so far, are a necessary consequence of holonomy modifications in the high-density phases in which they may resolve singularities. With negative $\beta^\theta$, however, the dispersion relation is positive definite and the hypersurface-deformation algebra is of Euclidean signature, as seen in
Sec. 1.1.2. These consequences are consistent with a formal transformation from positive to negative $\beta$ in (1.23c) by the replacements of $N$ or $t$ by $iN$ or $it$, respectively. With a Euclidean action, the initial/boundary-value problem changes its form significantly and propagation in time no longer exists. Loop quantum gravity, in this way, provides a concrete mechanism for signature change.

In loop quantum cosmology, going through the Planck regime near the big bang does therefore not at all correspond to a bounce, as minisuperspace models are sometimes interpreted as suggesting [36]. The big bang is rather a transition from Euclidean 4-dimensional space to Lorentzian space-time which only appears dynamical in the homogeneous background. This observation shows some of the pitfalls and unexpected subtleties of minisuperspace models. We are also reminded that we have to be careful with gauge-fixings or deparameterization, which do not determine the constraint algebra and cannot show the consequences seen here (see e.g. [27]).

In addition to these cautionary remarks for some scenarios in loop quantum cosmology, the new picture of signature change also provides larger unity among the different scenarios for singularity resolution. The main mechanism [28] is based on properties of the underlying difference equations that appear with a loop quantization [37], with difference operators on minisuperspace. The resulting recurrence scheme of the wave function depending on an integer geometrical quantity, taking both signs thanks to orientation, allows one to evolve uniquely from one side of the classical singularity in minisuperspace to the other. With unique evolution, the singularity is resolved making use of geometrical internal time. A scenario of less generality is realized for deparameterizable models sourced by a scalar field when its energy is almost all kinetic. Here, using the scalar as internal time, the minisuperspace evolution is non-singular with a minimum volume achieved at high density.

These pictures look inconsistent at first sight, with the oriented volume used as unbounded recurrence variable in the first one, but bouncing back from a small value in the second one. With the results of this paper we see that what is inconsistent is not the role of volume in the recurrence, but rather the interpretation of evolution as a smooth bounce. In both cases, a collapsing branch of shrinking volume is connected to an expanding branch of growing volume by a non-classical
space-time region. In the first picture, based on a recurrence analysis of discrete wave equations, the non-classical part is modeled as a tunneling process of the wave function through small volume, while it becomes a Euclidean chunk of 4-dimensional space in the second picture. We may draw an analogy with the way the wave-function of a particle tunnels through a classically disallowed region of ‘negative kinetic energy’ to reach another allowed state. This scenario not only unifies different mechanisms of singularity resolution in loop quantum cosmology, it also shows an interesting and unexpected overlap with the tunneling aspects of [38] and the postulated signature change of [39].

2.3.3.3 The question of cosmological initial values

We arrive at several new possibilities for cosmological model building: Initial values can be posed only in the Lorentzian regime. Holonomy-induced super-inflation, as it appears in the background evolution in loop quantum cosmology at high density, is not realized; the corresponding background piece is not part of space-time but rather corresponds to a Euclidean chunk of 4-dimensional space. (Super-inflation from inverse-triad corrections [40, 41] has a positive $\beta$ and could happen in the space-time part.) While the background equations, taken on their own, might be interpreted as implying super-inflationary evolution, they fail to provide any insight into the correct initial/boundary-value problem. Only an extension at least to perturbative inhomogeneity, without gauge fixing or deparameterization so as to have access to the off-shell constraint algebra, can provide this important input, and it shows the Euclidean nature. With the corresponding boundary-value instead of initial-value problem, even the background equations can no longer be interpreted as evolution equations in time.

The Euclidean nature of high-density regimes with holonomy corrections have several unanticipated consequences for initial values in cosmology. One cannot use this phase to evolve or generate structure, or to pose initial conditions within it, such as at the bounce of maximum density. Models making use of the super-inflationary phase to supply initial values, even if only for the background equations as suggested for instance in [42], are not consistent with quantum geometry. It becomes, however, very natural to pose initial values right at the boundary of Euclidean space, cutting off super-inflation. This procedure would be similar to
the usual choice of initial values or an initial vacuum state before slow-roll inflation, but providing stronger justification of the choice.

There are several advantages. First, we can pose well-defined initial values in a non-singular regime. Classically, if we go back as far as possible to pose initial conditions close to what can be considered the beginning, we end up at the big-bang singularity. If there is a bounce [43], we end up far back at large volume in the preceding collapse branch. In the deformed solutions with holonomy corrections of loop quantum gravity, we end up at the non-singular beginning of the Lorentzian branch, a clearly distinguished and non-singular moment in time. Secondly, methods of Euclidean quantum gravity may be used to shed light on what initial conditions one should expect. These initial conditions would not be transferred from the collapse phase bordering the Euclidean chunk at its other end: In Euclidean 4-space we must choose boundary conditions for a well-posed formulation of partial differential equations for inhomogeneity. This boundary includes the initial-value slice of the expanding branch of the universe model and the final-value surface of the collapsing branch. Field values on these surfaces can be specified independently and freely for a complete set of Euclidean boundary conditions. We could, for instance, evolve the collapsing branch from its initial data to obtain field values at one piece of the Euclidean boundary. Boundary conditions will then be completed by choosing values on the rest, including the initial-data surface of the expanding branch. Therefore, the final values of the collapse do not determine initial values for expansion. There is no deterministic evolution across the Euclidean high-density phase. Rather, the scenario describes a beginningless beginning, with a concrete physical realization of a distinguished initial-value surface. Although our scenario has cyclic features in that it combines collapsing and expanding branches, connected by Euclidean space not causally but at least as manifolds, we do not encounter the entropy problem. Entropy, like anything else, will simply not be transmitted through the Euclidean piece.

\[1\] Some indications of non-deterministic evolution through bounces of loop quantum cosmology can be found already at the level of background evolution, where cosmic forgetfulness implies that not all moments of a pre-big bang state can be recovered after the big bang [45, 46].
2.3.4 Additional modifications

Non-local corrections are possible in our formalism, extended from [1], but have not yet been realized explicitly in effective actions. We have identified additional difficulties which may prevent simple realizations of consistent deformations: gravity and matter terms in the constraints can no longer satisfy the hypersurface-deformation algebra independently. Instead, there must be delicate cancellations between matter and gravity Poisson brackets so as to ensure that the total constraints satisfy a consistently deformed algebra.

In addition to non-locality, modifications to the spatial part of the constraint algebra would prevent the steps followed here from going through. From the perspective of effective constraints, modifications to the spatial part may not seem likely because these constraints are formulated for fields on some manifold, which may not obey the classical geometry but nevertheless is a collection of points labeled, for the formulation of physical theories, by coordinates. The choice of coordinates cannot matter for the physics, and so there must be relabelling invariance. Such an invariance, in turn, leads very generally to the spatial part of the constraint algebra just based on properties of the Lie derivative [1].

Also from the point of view of full loop quantum gravity, modifications to the part of the constraint algebra involving the diffeomorphism constraint may not be called for. This constraint, unlike the Hamiltonian constraint, is implemented directly by its action on subsets in space (points or graphs) without any regularization or modification required to quantize it consistently. The final verdict on this question has not arrived, however, as shown by recent attempts to construct diffeomorphism constraint operators amenable to a closed operator algebras for the constraints [44].

The constraint algebra opens the way to specific results for space-time geometry in loop quantum gravity, extending some minisuperspace results to more general situations. A crucial open issue remains: deriving consistent deformations in more general terms than available now. Our results here do not provide new cases of consistent deformations, because we must assume consistency in order to employ our algebra. But the new methods do show how different terms in a consistently modified Hamiltonian constraint must be related to one another, as seen in conditions for dispersion relations and in the relations of \( v^n \)-terms to spatial metric
derivatives. Thus, our methods help in finding new consistent models. But even for existing ones, the effective actions obtained provide new insights and several unexpected cosmological consequences.
Chapter 3

Modified gravity in Spherically Symmetric systems

3.1 Spherical symmetry

For the sake of easier comparison with calculations of modified constraints motivated by loop quantum gravity, we will present equations in this subsection for triad variables. A spherically symmetric spatial densitized triad has two components $E^x$ and $E^\varphi$, for the radial coordinate $x$ and one angular coordinate $\varphi$, which determine the spatial metric by $g_{xx} = (E^\varphi)^2 / |E^x|$ and $g_{\varphi\varphi} = \sin^2 \vartheta |E^x|$. We will assume $E^x > 0$ to avoid dealing with sign factors.

The spherically symmetric metric in terms of these variables is

$$ds^2 = -N^2dt^2 + \frac{E^\varphi^2}{|E^x|}((dx + N^x dt)^2 + |E^x|d\Omega^2) \quad (3.1)$$

where $N(t, x)$ is the lapse function and $N^x(t, x)$ the only nonzero component of the shift vector.

Instead of working with spatial curvature tensors, in this context it turns out to be useful to refer to the angular spin-connection component and its spatial and functional derivatives,

$$\Gamma_\varphi = -\frac{(E^x)^\prime}{2E^\varphi} \quad (3.2)$$
\[ \Gamma' = -\frac{(E^x)''}{2E^\varphi} + \frac{(E^x)'(E^\varphi)'}{(2E^\varphi)^2} \]  \hspace{1cm} (3.3)

\[ \frac{\delta \Gamma_\varphi(y)}{\delta E^x(x)} = -\frac{1}{2E^\varphi(y)} \delta'(y, x) \] \hspace{1cm} (3.4)

\[ \frac{\delta \Gamma_\varphi(y)}{\delta E^\varphi(x)} = \frac{(E^\varphi)'(y)}{2(E^\varphi(y))^2} \delta(y, x) \] \hspace{1cm} (3.5)

\[ \frac{\delta \Gamma_\varphi(y)}{\delta E^x(x)} = -\frac{1}{2E^\varphi(y)} \delta''(y, x) + \frac{(E^\varphi)'(y)}{2(E^\varphi(y))^2} \delta'(y, x) \] \hspace{1cm} (3.6)

\[ \frac{\delta \Gamma_\varphi(y)}{\delta E^\varphi(x)} = \frac{(E^\varphi)''(y)}{2(E^\varphi(y))^2} \delta(y, x) + \frac{(E^\varphi)'(y)}{2(E^\varphi(y))^2} \delta'(y, x) \] \hspace{1cm} (3.7)

\[ \frac{\delta \Gamma_\varphi(y)}{\delta E^\varphi(x)} = \frac{(E^\varphi)'(y)(E^\varphi)'(y)}{(E^\varphi(y))^3} \delta(y, x). \] \hspace{1cm} (3.8)

(The radial component of the spin connection does not have any gauge-invariant contribution [60].)

Moments of the densitized triad are classically given by extrinsic-curvature components \( K_x \) and \( K_\varphi \) with

\[ \{K_x(x), E^x(y)\} = 2G\delta(x, y) \] \hspace{1cm} (3.9a)

and

\[ \{K_\varphi(x), E^\varphi(y)\} = G\delta(x, y). \] \hspace{1cm} (3.9b)

With these properties, the commutator relationship (1.45) to exploit here reads

\[ \{H(x), H(y)\} = G\int d^3 z \left( 2 \frac{\delta H(x)}{\delta K_x(z)} \frac{\delta H(y)}{\delta E^x(z)} - 2 \frac{\delta H(y)}{\delta K_x(z)} \frac{\delta H(x)}{\delta E^x(z)} \right. \]

\[ + \frac{\delta H(x)}{\delta K_\varphi(z)} \frac{\delta H(y)}{\delta E^\varphi(z)} \frac{\delta H(y)}{\delta K_\varphi(z)} \frac{\delta H(x)}{\delta E^\varphi(z)} \right) \]

\[ = \beta \frac{E^x(x)}{(E^\varphi(x))^2} D(x) \delta'(x, y) - (x \leftrightarrow y) \] \hspace{1cm} (3.10)

with the local diffeomorphism constraint

\[ D(x) = \frac{1}{2G} \left( 2E^\varphi K_\varphi' - K_x(E^x)' \right). \] \hspace{1cm} (3.11)
With a modified Hamiltonian, $K_x$ and $K_\varphi$ may no longer be components of extrinsic curvature. However, they are still canonically conjugate to $E^x$ and $E^\varphi$, and we continue to use the same letters for momentum variables.

For now, we will be looking only for constraints with quadratic "kinetic" term in momenta and no non-locality or spatial derivatives of $K$,

$$H = {}^{00}H + {}^{11}HK_xK_\varphi + {}^{20}HK_xK_x + {}^{02}HK_\varphi K_\varphi$$

(3.12)

(without linear terms, assuming time reversal symmetry), and have linear functional derivatives

$$G \frac{\delta H(x)}{\delta K_x(z)} = (A_1(x)K_x(x) + B_1(x)K_\varphi(x)) \delta(x, z)$$

(3.13)

$$G \frac{\delta H(x)}{\delta K_\varphi(z)} = (A_2(x)K_x(x) + B_2(x)K_\varphi(x)) \delta(x, z).$$

(3.14)

We then identify $^{11}HG = A_2 = B_1$, $^{02}HG = B_2/2$, $^{20}HG = A_1/2$, which may all depend on the triad components. The Poisson bracket of two Hamiltonian constraints becomes

$$\{H(x), H(y)\} = \frac{\delta H(y)}{\delta E^x(x)} (2A_1K_x(x) + 2B_1K_\varphi(x))$$

$$+ \frac{\delta H(y)}{\delta E^\varphi(x)} (A_2K_x(x) + B_2K_\varphi(x)) - (x \leftrightarrow y)$$

$$= \beta \frac{E^x(x)}{G(E^\varphi(x))^2} \left( E^\varphi(x)K'_\varphi(x) - \frac{1}{2}K_x(x)(E^x)'(x) \right) \delta'(x, y) - (x \leftrightarrow y).$$

(3.15)

We continue by comparing coefficients of $K_x$ and $K_\varphi$. In this section, we will assume that $\beta$ does not depend on $K_x$ or $K_\varphi$, thus considering the case of inverse-triad corrections.

For $K_x = 0$, $K_\varphi = 0$, the equation is automatically satisfied. For the first-order coefficients in $K_x$, we operate with the functional derivative $\delta/\delta K_x$ and then set
\( K_x = 0, K_\varphi = 0: \)

\[
\left( 2 \frac{\delta^{00}H(y)}{\delta E^x(x)} A_1(x) + \frac{\delta^{00}H(y)}{\delta E^\varphi(x)} A_2(x) \right) \delta(x, z) - (x \leftrightarrow y)
= -\frac{\beta E^x(x)(E^\varphi(x))(x)}{2G(E^\varphi(x))^2} \delta'(x, y) \delta(x, z) - (x \leftrightarrow y). \tag{3.16}
\]

For functional derivatives of \( 00 H \) by \( E^x \) and \( E^\varphi \), we must know the general triad-dependent terms possible. In addition to a direct dependence on the fields, \( 00 H \) can depend on the triad via spatial curvature which, in turn, depends on the spin connection and its derivatives. We thus have to expect a dependence on \( E^x, E^\varphi, \Gamma_\varphi \) and \( \Gamma'_\varphi \). Higher derivatives are not included because here, as in (3.12), we expand only to second order in momenta and derivatives.

We then have the chain rule

\[
\frac{\delta^{00}H(y)}{\delta E^x(x)} = \frac{\partial^{00}H(y)}{\partial \Gamma_\varphi(y)} \frac{\delta \Gamma_\varphi(y)}{\delta E^x(x)} + \frac{\partial^{00}H(y)}{\partial \Gamma'_\varphi(y)} \frac{\delta \Gamma'_\varphi(y)}{\delta E^x(x)} + \frac{\partial^{00}H(y)}{\partial E^\varphi(x)} \frac{\delta E^\varphi(x)}{\delta E^x(x)} \tag{3.17}
\]

and a similar relation for \( \delta^{00}H(y)/\delta E^\varphi(x) \) to rewrite (3.16). We substitute our expressions (3.5)–(3.8) for \( \delta \Gamma_\varphi(y)/\delta E^x(x) \) and so on, multiply with test functions \( a(x), b(y), \) and \( c(z) \), and integrate over \( x, y, \) and \( z \). We state the result obtained after several integrations by parts (surface terms are discarded):

\[
\int \mathrm{d}y \left[ -(a'cA_1) \frac{b}{E^\varphi} \frac{\partial^{00}H}{\partial \Gamma_\varphi} + (a'cA_1) \frac{b(E^\varphi)'}{(E^\varphi)^2} \frac{\partial^{00}H}{\partial \Gamma'_\varphi} \\
+ (a'cA_2) \frac{b(E^\varphi)'}{2(E^\varphi)^2} \frac{\partial^{00}H}{\partial \Gamma'_\varphi} + 2(a'cA_1)b \left( \frac{1}{E^\varphi} \frac{\partial^{00}H}{\partial \Gamma'_\varphi} \right)' \\
+ (a''cA_1) \frac{b}{E^\varphi} \frac{\partial^{00}H}{\partial \Gamma'_\varphi} - \frac{\beta E^x(E^\varphi)'}{2G(E^\varphi)^2} a'c \right] - (a \leftrightarrow b) = 0. \tag{3.18}
\]

(Several terms that cancel in the antisymmetrization with respect to \( a \) and \( b \) have not been written explicitly.) Collecting the coefficients of \( c(a''b - b''a) \) and \( c(a'b - b'a) \), respectively, we get

\[
\frac{A_1}{E^\varphi} \frac{\partial^{00}H}{\partial \Gamma'_\varphi} = 0, \tag{3.19}
\]
\[-A_1 \frac{\partial^{00} H}{E^\varphi} \frac{\partial}{\partial \Gamma'_\varphi} + \frac{A_1 (E^\varphi)'}{2(E^\varphi)^2} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} + \frac{A_2 (E^x)'}{2(E^x)^2} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} + \left( \frac{1}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} \right)' 2A_1 - \frac{\beta E^x (E^x)'}{2G(E^\varphi)^2} = 0. \] 

(3.20)

Going back to (3.15) to look at the first order in \(K_\varphi\) (and zeroth in \(K_x\)), and performing similar operations, we get

\[
\int dy \left[ - (a'cB_1) \frac{b}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} + (a'cB_1) \frac{b(E^\varphi)'}{(E^\varphi)^2} + (a'cB_2) \frac{b(E^x)'}{2(E^x)^2} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} + 2(a'cB_1) \left( \frac{1}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} \right)' + (a''cB_1) \frac{b}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} - a''bc \frac{\alpha^2 E^x}{E^\varphi}
\right.
\]

\[
\left. - a'bc \left( \frac{\beta E^x}{G E^\varphi} \right)' \right] - (a \leftrightarrow b) = 0.
\] 

(3.21)

Collecting the coefficients of \(c(a''b - b''a)\) and \(c(a'b - b'a)\), respectively, results in

\[
\frac{B_1}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} - \beta E^x \frac{\partial^{00} H}{G E^\varphi} = 0
\] 

(3.22)

\[
- \frac{B_1}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} + \frac{B_1 (E^\varphi)'}{(E^\varphi)^2} + \frac{B_2 (E^x)'}{2(E^x)^2} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} + \left( \frac{1}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} \right)' 2B_1 - \left( \frac{\beta E^x}{G E^\varphi} \right)' = 0.
\] 

(3.23)

Equation (3.22) implies that \(\delta^{00} H / \delta \Gamma'_\varphi\) cannot be zero. With this condition, we find \(A_1 = 0\) from (3.19),

\[
A_2 = \frac{\beta E^x}{G} \left( \frac{\partial^{00} H}{\partial \Gamma'_\varphi} \right)^{-1} = B_1
\] 

(3.24)

from (3.20) and (3.22), and

\[
- \frac{B_1}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} + \frac{\partial^{00} H}{\partial \Gamma'_\varphi} \left( \frac{B_1 (E^\varphi)'}{(E^\varphi)^2} + \frac{B_2 (E^x)'}{2(E^x)^2} \right) + 2B_1 \left( \frac{1}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma'_\varphi} \right)' = \frac{1}{G} \left( \frac{\beta E^x}{E^\varphi} \right)'.
\] 

(3.25)
This tells us that

\[
G \frac{B_1}{E^\varphi} \frac{\partial^{00} H}{\partial \Gamma_\varphi} = \left( \frac{B_2}{B_1} \frac{E^x}{2E^\varphi} + 1 \right) \frac{\beta(E^x)'}{E^\varphi} + \frac{E^x}{E^\varphi} \left( \beta' - 2 \frac{B'_1}{B_1} \beta \right). \tag{3.26}
\]

To solve these equations, we introduce a function \( b_1 \) such that \( B_1 = -\sqrt{|\beta|} b_1 \sqrt{E^x} = A_2 \). The factors are chosen so as to cancel several terms in (3.26):

\[
\frac{\beta(E^x)'}{E^\varphi} + \frac{E^x}{E^\varphi} \left( \beta' - 2 \beta B'_1 / B_1 \right) = -2 \beta \frac{E^x b'_1}{E^\varphi b_1}.
\]

For the correct density weights in the first term in (3.26), \( B_2 \) must be proportional to \( E^\varphi \). (The other factors \( B_1 \) and \( E^x \) are scalar and cannot change the density weight.) With another free function \( b_2 \), we write \( B_2 = -b_1 b_2 \sqrt{|\beta|} E^\varphi / \sqrt{E^x} \), with factors other than \( E^\varphi \) chosen for later convenience. The coefficients \( A_1, A_2, B_1 \) and \( B_2 \) determine the form of momentum contributions to the Hamiltonian constraint:

\[
\begin{align*}
1^{11} H &= \frac{B_1}{G} = -\sqrt{|\beta| E^x b_1} / G, \\
2^0 H &= \frac{A_1}{2G} = 0, \\
0^2 H &= \frac{B_2}{2G} = -b_1 b_2 \sqrt{|\beta|} E^\varphi / 2G \sqrt{E^x}. \tag{3.29}
\end{align*}
\]

With these solutions, we obtain \( \partial^{00} H / \partial \Gamma_\varphi = -\text{sgn}(\beta) \sqrt{|\beta|} E^x / G b_1 \) from (3.26) and \( \partial^{00} H / \partial \Gamma_\varphi = \text{sgn}(\beta) \sqrt{|\beta|} (b_2 - 4(d b_1 / d E^x)(E^x / b_1)) E^x \Gamma_\varphi / (G b_1 \sqrt{E^x}) \) from (3.24), or integrated,

\[
0^0 H = -\frac{\text{sgn}(\beta) \sqrt{|\beta|}}{G} \left( \frac{\sqrt{E^x}}{b_1} \Gamma_\varphi - \frac{1}{b_1} \left( \frac{b_2}{2} - \frac{2E^x}{b_1} \frac{d b_1}{d E^x} \right) \frac{E^\varphi}{\sqrt{E^x} \Gamma_\varphi} \right) + f(E^x) E^\varphi. \tag{3.30}
\]

Comparing with the general form (3.31), we read off

\[
\bar{\alpha} = \sqrt{|\beta|} b_1, \quad \alpha = \sqrt{|\beta|} b_1 b_2, \\
\bar{\alpha}_\Gamma = \text{sgn}(\beta) \frac{\sqrt{|\beta|}}{b_1}, \quad \alpha_\Gamma = \text{sgn}(\beta) \frac{\sqrt{|\beta|}}{b_1} \left( b_2 - 4 \frac{d \log b_1}{d \log E^x} \right).
\]
With these relationships, the correction functions can easily be seen to satisfy the condition (1.43) as well as $\beta = \bar{\alpha} \bar{\alpha}_\Gamma$.

Modifications to the spherically symmetric dynamics are not entirely determined by the constraint algebra, consistent with the results of [67, 68]. The function $b_1$ is related to the ratio of $\bar{\alpha}$ to $\bar{\alpha}_\Gamma$, and $b_2$ determines how $\alpha$ differs from $\bar{\alpha}$. The $E^x$-dependence of $^{00}H$ in (3.30) (which may include a cosmological constant term) is not fully determined because $E^x$ is a scalar with no density weight and can, for the purpose of the constraint algebra, be inserted rather freely in the constraints. In this feature we can see why the full dynamics is more unique than the spherically symmetric one: Without symmetry, there is less freedom in the choice of spatial tensors with the correct transformation properties. Indeed, as we will see later, spatial transformation properties play an important role for the regaining procedure. Without spherical symmetry $\Gamma'_\varphi$ and $\Gamma^2_\varphi$ would be part of the same contribution $^{(3)}R$, which cannot be split apart by different correction functions if the spatial structure of geometry remains unmodified. The case of $\alpha = \bar{\alpha}$ ($b_2 = 1$) and $\alpha_\Gamma = \bar{\alpha}_\Gamma$ ($b_1$ constant and therefore $b_1 = 1$ for it to approach one at large fluxes) is then preferred, with all corrections determined by the algebraic deformation $\beta$.

Now that we have determined the general form of the corrected Hamiltonian for the deformed algebra, we shall investigate its consequences for black hole geometry. The discussion below follows that in [11]; thus we will assume the special case of $\alpha = \alpha_\Gamma$, and $\bar{\alpha} = \bar{\alpha}_\Gamma$. We will introduce as matter a scalar field whose modified action will also be subject to constraints arising from the deformed algebra. To distinguish between the matter and gravitational parts of the Hamiltonian, and also between the classical and the deformed Hamiltonian constraint, we will use the notation $H^Q_{\text{grav}}$.

The modified gravitational part of the Hamiltonian constraint we consider here is

$$H^Q_{\text{grav}}[N] = -\frac{1}{2G} \int dx \, N \left[ \alpha |E^x|^{-\frac{1}{2}} K^2 E^\varphi + 2\bar{\alpha} K_\varphi K_x |E^x|^{\frac{1}{2}} + \alpha |E^x|^{-\frac{1}{2}} (1 - \Gamma^2_\varphi) E^\varphi + 2\bar{\alpha} \Gamma'_\varphi |E^x|^{\frac{1}{2}} \right]$$

(3.31)

There is no inverse triad component in the diffeomorphism constraint, and
its action is directly represented on graph states by the spatial deformations it generates. Thus, we will keep the diffeomorphism constraint unmodified.

The Poisson algebra of modified constraints closes:

\[
\{ H^Q_{\text{grav}}[N], D_{\text{grav}}[N^x] \} = -H^Q_{\text{grav}}[N^x N'], \tag{3.32}
\]

\[
\{ H^Q_{\text{grav}}[N], H^Q_{\text{grav}}[M] \} = D_{\text{grav}}[\bar{\alpha}^2 E^x] (E^x)^{-2} (NM' - MN'). \tag{3.33}
\]

The Poisson bracket relations (3.33) with \( \bar{\alpha} = 1 \) express the fact that dynamics takes place on space-like hypersurfaces embedded in a pseudo-Riemannian space-time [2]. However, for generic \( \bar{\alpha} \neq 1 \), this algebra no longer coincides with the algebra of spherically symmetric hypersurface deformations of general relativity and generally space-time covariant systems. (Defining \( \bar{\alpha}N = \bar{N} \), we have \( \bar{\alpha}^2 (NM' - MN') = (\bar{N}M' - \bar{M}N') \) and the algebra can formally be written in classical form \( \{ H^Q_{\text{grav}}[\bar{N}/\bar{\alpha}], H^Q_{\text{grav}}[\bar{M}/\bar{\alpha}] \} = D_{\text{grav}}[E^x] (E^x)^{-2} (\bar{N}M' - \bar{M}N') \). However, using \( \bar{N} \) in the full algebra will then modify the Poisson bracket (3.32) of the Hamiltonian with the diffeomorphism constraint. The deformation we consider is thus a non-trivial one.)

The algebra is a fundamental object, encoding not only gauge properties of gravity but the structure of space-time as well. Thus, for \( \bar{\alpha} \neq 1 \) not only the dynamics (reflected by the modified Hamiltonian constraint (3.31)) but also the structure or symmetries of the space-time manifold are changed by these quantum corrections. Gauge symmetries will, in general, no longer coincide with coordinate transformations in our models and we cannot interpret the dynamical fields \( E^x \) and \( E^\varphi \) as components of a pseudo-Riemannian metric as in (3.1): modified gauge transformations of \( E^x \) and \( E^\varphi \) no longer match with coordinate transformations of \( dx^\mu \) to form an invariant \( ds^2 \). (Possible candidates for space-time models corresponding to the corrected gauge transformations are non-commutative manifolds or Finsler geometries.)

For the correction functions, we generically have \( 1 \neq \alpha \neq \bar{\alpha} \neq 1 \). However, solutions with \( \bar{\alpha} \neq 1 \) can formally be related to the solutions with \( \bar{\alpha} = 1 \) by means of the substitutions

\[
N \rightarrow \frac{N}{\bar{\alpha}}; \quad \alpha \rightarrow \frac{\alpha}{\bar{\alpha}}, \tag{3.34}
\]

where \( N_\alpha \) refers to the value of \( N \) in the \( \bar{\alpha} = 1 \) case. (The substitutions are
well-defined because $\bar{\alpha}$, by virtue of having to approach one in the classical limit, cannot identically vanish.) We will, when possible, keep the matter terms general, referring only to the energy density and energy flow without specific matter models. Thus, the substitutions will lead to different matter terms compared to the case with $\bar{\alpha} = 1$, but will not change the analysis of equations. Note, moreover, that since $\bar{\alpha} \neq 1$ in the original constraints, the system is not generally covariant in the classical sense and coordinate transformations between the different gauges do not exist.

It will then suffice to consider in detail the two cases $\alpha \neq \bar{\alpha} = 1$ (Sections 3.2 and 3.4.1) and $\alpha = \bar{\alpha} \neq 1$ (Section 3.4.2), the latter giving rise to a genuinely modified constraint algebra. Using the substitutions, we will then briefly deal with the generic case in Section 3.4.3.

We end this general exposition with a remark on the for the first choice of corrected constraints that leaves the constraint algebra unmodified. The Hamiltonian constraint is replaced by its modified counterpart (3.31) with $\bar{\alpha} = 1$, while the diffeomorphism constraint (3.11) remains unchanged, as does the overall form of the constraint algebra. How does this first type of modification square with the result [2, 1] that given the constraint algebra of classical general relativity, the Hamiltonian that depends only on the 3-metric and the extrinsic curvature is uniquely the classical Hamiltonian of general relativity? First note that while the relations (3.9) still hold, we can no longer interpret $K_a$ as extrinsic curvatures of the metric. This can be easily seen from the equation for $K_x$ derived from the corrected evolution equations, (B.6) and (B.7) below. Classically, we have:

$$K_x = \frac{1}{N}\left(\frac{\dot{E}^\phi}{\sqrt{|E^x|}} - \frac{E^\phi \dot{E}^x}{2|E^x|^{3/2}} + \frac{N^x E^\phi (E^x)'}{2|E^x|^{3/2}} - \frac{(N^x E^\phi)'}{\sqrt{|E^x|}}\right)$$  \hspace{1cm} (3.35)

but with the modified equations we get:

$$K_x = \frac{1}{N}\left(\frac{\dot{E}^\phi}{\sqrt{|E^x|}} - \frac{\alpha E^\phi \dot{E}^x}{2|E^x|^{3/2}} + \frac{N^x \alpha E^\phi (E^x)'}{2|E^x|^{3/2}} - \frac{(N^x E^\phi)'}{\sqrt{|E^x|}}\right)$$  \hspace{1cm} (3.36)

which cannot be derived from relation (1.2). The component $K_\phi = (\dot{E}^x - N^x (E^x)')/2N \sqrt{|E^x|}$ is not modified if $\bar{\alpha} = 1$. We can relate the $\alpha$-modified $K_a$, which are conjugate to the densitized triad, to the geometric extrinsic curvature components from (1.2),
denoted here by $\bar{K}_a$:

$$K_\phi = \bar{K}_\phi, \quad K_x = \bar{K}_x - \frac{(\alpha - 1)E^x}{E^x} \bar{K}_\phi. \quad (3.37)$$

Substituting this into (3.31), we get the following Hamiltonian in terms of the densitized triad and the extrinsic curvatures:

$$H_{\text{grav}}^Q[N] = -\frac{1}{2G} \int dx \, N \left[ (2 - \alpha)|E^x|^{-\frac{1}{2}} K_\phi E^\rho + 2\bar{K}_\phi \bar{K}_x |E^x|^\frac{1}{2} + 
\alpha |E^x|^{-\frac{1}{2}} (1 - \Gamma^2_\phi) E^\rho + 2\Gamma'_\phi |E^x|^\frac{1}{2} \right] \quad (3.38)$$

The Hamiltonian constraint is thus modified also in the geometrical variables $\bar{K}_a$ which correspond to the classical form of extrinsic curvature but are no longer canonically conjugate to the densitized triads. Compared with the full situation [2, 1], the constraint algebra in spherical symmetry is thus not as restrictive, and different sets of constraints can give rise to the same algebra.

### 3.2 Background solutions for undeformed constraint algebra

For $\bar{\alpha} = 1$ the constraints obey the classical algebra, and thus generate coordinate changes as gauge transformations and allow the existence of effective line elements to describe the modified geometries. Thus, $\alpha$ is the only non-trivial correction function in this case. In this subsection, we analyze its implications for effective vacuum line elements. The usual properties of black holes can then be studied by standard means; just corrections in metric coefficients appear.

#### 3.2.1 Vacuum line elements

For comparison and later reference, we derive vacuum solutions in two commonly used space-time gauges, producing line elements in the Schwarzschild and Painlevé–Gullstrand form.
3.2.1.1 Modified Schwarzschild metric

In vacuum, we produce a Schwarzschild-type line element by imposing the static gauge $K_x = K_\phi = N^x = 0$, and the diffeomorphism constraint is automatically satisfied. The Hamiltonian constraint requires

$$(1 - \Gamma^2_x) \frac{\alpha E^\varphi}{\sqrt{E^x}} + 2\Gamma^x \sqrt{E^x} = 0.$$  \hspace{1cm} (3.39)$$

With the vanishing $K_\phi$ obeying (B.10), we have the further equation

$$N' = \frac{N\alpha (E^\varphi)^2}{E^x E^x'} + \frac{N\alpha E^x'}{4E^x} - \frac{NE^x'}{2E^x}.$$  \hspace{1cm} (3.40)$$

With these two differential equations for $E^\varphi$ and $N$ one can check that (B.11) is identically satisfied.

Next we specify the coordinate gauge $E^x = x^2$ so as to refer by $x$ to the area radius. Eq. (3.39) then becomes

$$\alpha (E^\varphi)^3 - 2x^2 E^\varphi + 2x^3 E^\varphi' - \alpha x^2 E^\varphi = 0.$$  \hspace{1cm} (3.41)$$

With the classically motivated ansatz $E^\varphi = x / \sqrt{1 - 2M f_\alpha(x)/x}$ we obtain the equation

$$f_\alpha'(x) = \frac{1 - \alpha}{x}.$$  \hspace{1cm} (3.42)$$

The behavior of solutions to this equation for different refinement schemes will be shown below. The functional form of the resulting line element, which we first continue to derive, is largely independent of the refinement scheme.

Using the solution for $E^\varphi$ along with the choice $E^x = x^2$ in (3.40) gives,

$$\frac{2N'x}{N} = \frac{\alpha}{1 - 2M f_\alpha(x)/x} + \alpha - 2.$$  \hspace{1cm} (3.43)$$

Again we use a classically motivated ansatz $N = g_\alpha(x) \sqrt{1 - 2M f_\alpha(x)/x}$ where $f_\alpha(x)$ is the function found above, and obtain

$$\frac{g_\alpha'}{g_\alpha} = \frac{\alpha - 1}{x}.$$  \hspace{1cm} (3.44)$$
Comparing this with (3.42) we see that the solution for \( g_\alpha(x) \) is the inverse of the solution for \( f_\alpha(x) \). In what follows we will interchangeably use \( g_\alpha(x) \) and \( 1/f_\alpha(x) \).

We thus see that both \( E^\phi \) and \( N \) pick up corrections due to the inclusion of quantum effects. And since we already verified that the condition \( \dot{K}_x = 0 \) is satisfied assuming that the solution is static, we have a valid solution for the modified Schwarzschild line element:\(^1\)

\[
\text{d}s^2 = -g_\alpha^2 \left( 1 - \frac{2Mf_\alpha}{x} \right) \text{d}t^2 + \left( 1 - \frac{2Mf_\alpha}{x} \right)^{-1} \text{d}x^2 + x^2 \text{d}\Omega^2.
\]

(3.45)

Provided that \( f_\alpha \) and \( g_\alpha \) approach one in the asymptotic region of large \( x \), the classical Schwarzschild space-time is recovered. If one were to use this solution all the way down to \( x = 0 \), there is a strongly modified region at small \( x \), but the singularity at \( x = 0 \) would not be resolved. (Notice that the Ricci term does not necessarily vanish even in vacuum if quantum corrections are present.)

Between the asymptotic regime and the strongly modified one, we encounter the possibility of horizon formation. The equation for a horizon is given by \( 2Mf_\alpha(x) = x \) or solving for \( M \) we have that \( M = x/2f_\alpha(x) \) as the value of mass for which we have a horizon, implicitly defined as a function of the horizon radius \( x \).

\[3.2.1.2\] Modified Painlevé–Gullstrand metric

With the classical constraint algebra satisfied for the corrections with \( \bar{\alpha} = 1 \), we can look for a coordinate transformation to produce the Painlevé–Gullstrand form of the corrected metric (3.45). This coordinate system has as its time variable the proper time measured by a freely falling observer in the Schwarzschild space-time (starting at rest from infinity and moving radially; see e.g. [72]). To determine this proper time we proceed as follows. The corrected Schwarzschild metric is independent of time and therefore \( \xi_t \) is a Killing vector. Now consider the geodesic of a (radially) freely falling observer, with the tangent to the geodesic denoted by \( u^a \). Then we have \( u^a\xi_t^a = E \) constant. If we parameterize the geodesic

\(^1\)The modifications show some similarity with space-times found from deformed dispersion relations [71] rather than general relativistic constraints. Differences are that the correction functions in the latter case are energy dependent, and that the radial position of the horizon is not corrected but the horizon area is rescaled.
by its proper time $T$ and choose $E = -1$, we have
\[ g_{ab}u^au^b = \frac{1}{f_α^2} \left( 1 - \frac{2Mf_α}{x} \right) \frac{dt}{dT} = -1. \] (3.46)

In addition, $g_{ab}u^au^b = -1$, or
\[ -\frac{1}{f_α^2} \left( 1 - \frac{2Mf_α}{x} \right) \left( \frac{dt}{dT} \right)^2 + \left( 1 - \frac{2Mf_α}{x} \right)^{-1} \left( \frac{dx}{dT} \right)^2 = -1 \] (3.47)
and with (3.46) we obtain
\[ \frac{dx}{dT} = -\sqrt{f_α^2 - 1 + \frac{2Mf_α}{x}}, \] (3.48)
where the negative sign for the square root corresponds to an infalling observer. Thus, $u_a = (-1, -(1 - 2Mf_α/x)^{-1} \sqrt{f_α^2 - 1 + 2Mf_α/x}, 0, 0)$ and
\[ dT = -u_a dx^a = dt + \left( 1 - \frac{2Mf_α}{x} \right)^{-1} \sqrt{f_α^2 - 1 + \frac{2Mf_α}{x}} dx. \] (3.49)

Inserting for $dt$ from above in the Schwarzschild metric and simplifying we arrive at the metric in Painleve-Gullstrand-like coordinates
\[ ds^2 = -\frac{1}{f_α^2} \left( 1 - \frac{2Mf_α}{x} \right) dT^2 + f_α^{-2} dx^2 + 2f_α^{-2} \sqrt{f_α^2 - 1 + \frac{2Mf_α}{x}} dT dx + x^2 dΩ^2. \] (3.50)

(Notice that $T = \text{const}$-slices, which are classically flat, are no longer so.)

In this derivation, we have made use of the fact that coordinate changes are gauge transformations for this type of corrections, and have used the usual geodesic properties in space-time. We can explicitly verify the first property by checking that the constraints are satisfied for the new form of the line element as well. By comparison with (3.1) we obtain $E^x = x^2$ and $E^r = x/f_α$, as well as $N = 1$ and $N^x = (f_α^2 - 1 + 2Mf_α/x)^{1/2}$. These when used in (3.36) and the analogous equation for $K_ϕ$, obtained from (B.6), give
\[ K_ϕ = -\sqrt{f_α^2 - 1 + \frac{2Mf_α}{x}} , \quad K_x = \frac{αf_α + Mα/x - f_α}{x\sqrt{f_α^2 - 1 + 2Mf_α/x}}. \] (3.51)
The diffeomorphism constraint amounts to $2E^\phi K'_\phi = K_x E^x$ which is satisfied, as is the Hamiltonian constraint. For later use, we note that $N = 1$ appears to be a suitable way to specify the Painlevé–Gullstrand form without directly referring to space-time properties (while spatial flatness, as seen, can be violated by quantum corrections).

### 3.3 Second-order perturbations

We now perform perturbative calculations for the classical vacuum constraints, to be used in the context of matter back-reaction. Although the classical vacuum constraints can be solved exactly, the perturbative procedure as well as some of the equations will be useful later. We will also take this opportunity to state our background gauge conditions for the two versions of the space-time, Schwarzschild and Painlevé–Gullstrand. For each of the backgrounds considered, we will perform the following steps:

**Step 1** 1st-order perturbation of the Hamiltonian and diffeomorphism constraints.

**Step 2** 1st-order perturbation of the equations of motion.

**Step 3** 2nd-order perturbation of constraints including matter fields.

**Step 4** 2nd-order perturbation of the equations of motion, as necessary.

**Step 5** Calculation of the perturbed form of the metric and evaluation of horizon conditions to find area-mass relationships.

In addition, the following features are common to all the calculations:

- The perturbations of the fundamental variables ($E^x, E^\phi, K_x, K_\phi$) will be denoted as:

  $$ E^\phi \to E^\phi + \Delta E^\phi = E^\phi + \delta E^\phi + \delta_2 E^\phi $$

  (3.52)

and so on. In every case, a $\delta$ without a subscript is to be taken to refer to a first-order perturbation of the relevant quantity. Fields without any kind of delta refer to the background values.
Since we are interested in possible changes to the area of various surfaces, we will, for simplicity of calculation, make the gauge choice

\[ E^x = x^2 \]  

(3.53)

to fix the diffeomorphism constraint. In particular, we set \( \Delta E^x = 0 \), and the perturbation of \( E^x \) at every order is set to zero. We will not be fixing the gauge completely. Rather, the presence of gauge-dependent terms (under transformations generated by the perturbed Hamiltonian constraint) will be taken as one of the criteria to distinguish between the horizon conditions used in various models with different types of inverse-triad corrections. A key consistency requirement will be that horizon conditions be gauge invariant.

We consider the matter field and its corresponding conjugate momentum to be first order perturbations (the background space-time is vacuum), which implies that the energy density \( \rho \) and the energy-momentum flux \( J^x \) are to be included only in the second order and higher perturbations of the constraints.

The different slicings (Schwarzschild and Painlevé–Gullstrand) are implemented by specifying the background fields. We will carry out the steps of the calculation in detail for the Painlevé–Gullstrand metric for an uncharged non-rotating black hole. For subsequent calculations we will only list the relevant changes. First, we provide two canonical versions of horizon conditions to be used.

### 3.3.1 Horizon conditions

We define horizons in canonical variables in order to be able to apply them to equations corrected by effects from canonical quantum gravity. For comparison, we provide two versions which would classically be equivalent in the context of spherically symmetric geometries. In doing so, we must use space-time notions to capture the meaning of a horizon, and it is not guaranteed that such definitions are reasonable for models with a deformed constraint algebra and their new versions of space-time structures. The motivation for providing two versions of horizon conditions is that we can test whether they remain equivalent in the deformed context and then have a chance of capturing the same effects. For cases with
an uncorrected constraint algebra, we will furthermore compare with the direct space-time analysis.

3.3.1.1 Trapping horizon

Horizons of our perturbative solutions can be analyzed by an expansion of the usual conditions, for instance of [73]. In spherical symmetry, the cross-section of a spatial slice with a spherical trapping horizon as the boundary of spherical marginally trapped surfaces, can be defined simply as a sphere at radius $x$ whose co-normal $dx_a$ is null. This condition may be written as $g^{xx} = 0$; one can verify that zero expansion of null geodesics is then implied. In triad variables with line element (3.1) one obtains the condition

$$\frac{E^x}{(E^\phi)^2} - \left(\frac{N^x}{N}\right)^2 = 0 \quad (3.54)$$

which can easily be analyzed perturbatively. To second order in the perturbations, it expands to:

$$\left[ \frac{E^x}{(E^\phi)^2} - \left(\frac{N^x}{N}\right)^2 \right]_0 + \left[ -2 \frac{E^x \delta E^\phi}{(E^\phi)^3} - 2 \frac{N^x \delta N^x}{N^2} + 2 \frac{(N^x)^2 \delta N}{N^3} \right]_1 + \left[ -2 \frac{E^x \delta E^\phi}{(E^\phi)^3} - 2 \frac{N^x \delta_N^x}{N^2} + 2 \frac{(N^x)^2 \delta_N}{N^3} + 3 \frac{E^x (\delta E^\phi)^2}{(E^\phi)^4} - \frac{(\delta N^x)^2}{N^2} - 3 \frac{(N^x \delta N)^2}{N^4} + 4 \frac{N^x \delta N^x \delta N}{N^3} \right]_2 = 0 \quad (3.55)$$

3.3.1.2 Isolated horizon

Alternatively, for the Schwarzschild slicing we can define a spherical horizon by using the specialization of isolated horizon conditions [74] to spherical symmetry. Since matter is still allowed outside the horizon, a situation comparable to the previous definition is obtained, but the condition is more restrictive because no matter is allowed at the horizon.
We are now dealing with the condition [75]

\[ A_\varphi = \sqrt{K_\varphi^2 + \Gamma_\varphi^2} = 0. \]  

(3.56)

In the Schwarzschild metric this gives us two conditions:

\[ K_\varphi^2 = 0 \quad \text{and thus} \quad \delta K_\varphi^2 = 0 \]  

(3.57)

and

\[ \Gamma_\varphi^2 = 0 \quad \text{or} \quad \Gamma_\varphi^2 + 2\Gamma_\varphi \delta \Gamma_\varphi + 2\Gamma_\varphi \delta_2 \Gamma_\varphi + (\delta \Gamma_\varphi)^2 = 0. \]  

(3.58)

The fact that we have two conditions instead of just one as in (3.55) demonstrates the more restrictive notion. In spherical symmetry, it turns out that the difference does not matter much classically, but it will become important with space-time-deforming quantum corrections.

3.3.1.3 Comparison and gauge

The origin of the additional condition arising for isolated horizons can be seen in the fact that isolated horizons, defined as boundaries of space-time, freeze gauge transformations generated by the Hamiltonian constraint on the horizon by boundary conditions. The additional condition on \( K_\varphi \), then formally replaces a possible gauge-fixing condition one might choose in a treatment where the horizon is not a boundary. Classically, the trapping-horizon condition (3.55) is gauge invariant, and its evaluation does not depend on which gauge fixing is used. It thus implies results equivalent to those produced by the isolated-horizon condition.

However, it turns out that the condition (3.55) is no longer gauge invariant for some versions of quantum corrected constraints. The horizon condition itself will then have to be corrected so as to cancel the gauge dependence, thereby shedding some light on what quantum horizon conditions could look like. For an isolated horizon, on the other hand, having the Hamiltonian gauge fixed by boundary conditions eliminates the important option of seeing how horizon conditions must be corrected in addition to the dynamics of quantum gravity. We will address these questions in detail by the examples provided in the rest of this article.
3.3.2 Painlevé–Gullstrand

The Painlevé–Gullstrand form of the Schwarzschild space-time is

\[ ds^2 = - \left( 1 - \frac{2M}{x} \right) dt^2 + dx^2 + 2\sqrt{\frac{2M}{x}} dtdx + x^2d\Omega^2. \] (3.59)

It is characterized by several interesting properties, such as having flat spatial slices of constant \( t \). In what follows, the background solutions will appear as coefficients of perturbation equations, partially identifying the gauge in which perturbations are analyzed. For the Painlevé–Gullstrand background,

\[ E^{\varphi} = x, \quad N = 1, \quad N^x = \sqrt{\frac{2M}{x}}, \quad K_x = \sqrt{\frac{M}{2x^3}}, \quad \text{and} \quad K_\varphi = -\sqrt{\frac{2M}{x}} \]

in addition to (3.53).

3.3.2.1 First order perturbation of the constraints

We expand the Hamiltonian and diffeomorphism constraint equations \( \delta H[N]/\delta N = 0 \) and \( \delta D[N^x]/\delta N^x = 0 \) to first order in metric perturbations, obtaining the general forms

\[ 2(K_\varphi E^{\varphi} + K_x E^x) \delta K_\varphi + 2K_\varphi E^x \delta K_x + (K_\varphi^2 - \Gamma_\varphi^2 + 1) \delta E^{\varphi} \]
\[ + 2(K_\varphi K_x + \Gamma_\varphi') \delta E^x - 2\Gamma_\varphi E^\varphi \delta \Gamma_\varphi + 2E^x \delta \Gamma_\varphi' = 0 \] (3.60)

and

\[ 2E^{\varphi} \delta K'_\varphi + 2K'_\varphi \delta E^\varphi - K_x \delta E^{x'} - E^{x'} \delta K_x = 0 \] (3.61)

Inserting the unperturbed form of the densitized triad and extrinsic curvature corresponding to (3.59), and applying the gauge condition \( \delta E^{x} = 0 \), with the additional corollaries that \( \delta E^{x'} = 0 \) and \( \delta \Gamma_\varphi = \frac{(E^{x'})'}{2E^{x}} \delta E^\varphi \), we have:

\[ -\sqrt{2Mx} \delta K_\varphi - 2\sqrt{2Mx^3} \delta K_x + \frac{2M}{x} \delta E^\varphi + 2x \delta E^{x'} = 0 \] (3.62)

and

\[ 2\sqrt{2Mx} \delta K'_\varphi + \frac{2M}{x^2} \delta E^\varphi - 2\sqrt{2Mx} \delta K_x = 0. \] (3.63)
To proceed solving the equations as far as possible, we subtract (3.62) and (3.63) to obtain

\[ \delta E' = \frac{1}{2} \sqrt{\frac{2M}{x}} \delta K' + \sqrt{2Mx} \delta K' = (\sqrt{2Mx} \delta K')' , \quad (3.64) \]

which can immediately be integrated. If we impose the boundary conditions that all the perturbations fall off to zero at infinity, and in particular, that

\[ \sqrt{x} \delta K \rightarrow 0 \quad \text{as} \quad x \rightarrow \infty , \quad (3.65) \]

this equation can be solved by

\[ \delta E = \sqrt{2Mx} \delta K , \quad (3.66) \]

and, substituting this back in (3.63)

\[ \delta K_x = \frac{M}{x^2} \delta K + \delta K' . \quad (3.67) \]

### 3.3.2.2 Perturbation of the Equations of motion

We obtain the linear equations of motion by expanding the general spherically symmetric equations (B.6)–(B.11) with \( \bar{\alpha} = \alpha = 1 \). Equation (B.6) gives to first order

\[ \delta \dot{E} = 2|E^x| \frac{1}{2} (K \delta N + N \delta K) + NK \delta |E^x| - \frac{1}{2} \delta E^x + Nx \delta E^x + E^x \delta N^x \quad (3.68) \]

or

\[ \delta \dot{E} = -2x \sqrt{\frac{2M}{x}} \delta N + 2x \delta K - \frac{1}{x} \sqrt{\frac{2M}{x}} \delta E^x + \sqrt{\frac{2M}{x}} \delta E^x + 2x \delta N^x \quad (3.69) \]

with the background solution (3.59) inserted for the unperturbed variables. We have \( \delta \dot{E} = \sqrt{2Mx} \delta K' \) from (3.66). Using the equations of motion, this provides a second relation between \( \delta N, \delta K', \) and \( \delta N^x \) which turns out to be identically satisfied.

To implement the gauge for the perturbations, we set \( \delta E^x \) and all its derivatives
to zero, to give:

\[-\sqrt{\frac{2M}{x}} \delta N + \delta K_\varphi + \delta N^x = 0.\] (3.70)

If we make the further choice that \(\delta N = 0\), we can use the relations derived above to arrive at simplified equations for the other perturbations; in particular \(\delta N^x = -\delta K_\varphi\) and:

\[\delta \dot{K}_\varphi = \sqrt{\frac{2M}{x}} \delta K'_\varphi - \frac{1}{2x} \sqrt{\frac{2M}{x}} \delta K_\varphi.\] (3.71)

The first order set of equations is solved by the general solution to (3.71):

\[\delta K_\varphi = \sqrt{x} F \left(2x^{3/2}/3 + \sqrt{2M} t\right)\]

for an arbitrary function \(F\) of one variable as indicated, satisfying the asymptotic condition (3.65). However, this extra gauge condition \(\delta N = 0\) is not necessary for our later results. The expressions for \(\delta K_x\), \(\delta E^\varphi\) and \(\delta N^x\) in terms of \(\delta K_\varphi\) are consistent, and satisfy equations (B.10) and (B.11) for \(\delta \dot{K}_\varphi\) and \(\delta \dot{K}_x\).

### 3.3.2.3 Second order perturbation of the constraints including matter

The second-order diffeomorphism constraint including matter is

\[
2\delta E^\varphi \delta K'_\varphi + 2E^\varphi \delta_2 K'_\varphi + 2K'_\varphi \delta_2 E^\varphi - (E^x)'\delta_2 K_x - 8\pi E^\varphi |E^x|^\frac{1}{2} J_x = 0, \tag{3.72}
\]

so in our coordinates and using first order results we have

\[
4M \delta K_\varphi \delta K'_\varphi + 2\sqrt{2Mx} \delta_2 K'_\varphi + \frac{2M}{x^2} \delta_2 E^\varphi - 2\sqrt{2Mx} \delta_2 K_x - 8\pi x^2 \sqrt{\frac{2M}{x}} J_x = 0. \tag{3.73}
\]

The second-order Hamiltonian constraint

\[
(K^2_\varphi - \Gamma^2_\varphi + 1)\delta_2 E^\varphi + 2K_\varphi E^x \delta_2 K_x + 2(K_\varphi E^\varphi + K_x E^x)\delta_2 K_\varphi
- 2\Gamma_\varphi E^\varphi \delta_2 \Gamma_\varphi + 2E^x \delta_2 \Gamma_\varphi + 2K_\varphi \delta K_\varphi \delta E^\varphi + E^\varphi (\delta K_\varphi)^2
+ 2E^x \delta K_\varphi \delta K_x - 2\Gamma_\varphi \delta \Gamma_\varphi \delta E^\varphi - E^\varphi (\delta \Gamma_\varphi)^2 - 8\pi E^\varphi |E^x| \rho = 0 \tag{3.74}
\]
with \( \delta_2 \Gamma_\phi = \frac{(E_\rho')'}{2E_\rho}(\delta E_\rho)^2 \), requires a little more work and gives

\[
- \sqrt{\frac{2M}{x}} \delta_2 K_\phi - \sqrt{8Mx} \delta_2 K_x + \frac{2M}{x^2} \delta_2 E_\phi + 2\delta_2 E_\phi' + \left( (x - 4M)(\delta K_\phi)^2 \right)' - 8\pi x^2 \rho = 0, \tag{3.75}
\]

Subtracting these constraints,

\[
2\delta_2 E_\phi' = \sqrt{\frac{2M}{x}} \delta_2 K_\phi + 2\sqrt{2Mx} \delta_2 K'_\phi - ((x - 6M)(\delta K_\phi)^2)' + 8\pi x^2 H \tag{3.76}
\]

and integrating gives

\[
\delta_2 E_\phi = \sqrt{2Mx} \delta_2 K_\phi - \frac{1}{2} (x - 6M)(\delta K_\phi)^2 - \frac{1}{2} \int_x^\infty dz 8\pi z^2 H \tag{3.77}
\]

where we have used

\[
H := N \rho - N^x J_x. \tag{3.78}
\]

### 3.3.2.4 Second order perturbation of the equations of motion

We may proceed putting (3.77) back into the diffeomorphism constraint (3.73) to get an equation for \( K_x \) in terms of \( K_\phi \). Equation (B.6) gives

\[
- \sqrt{\frac{2M}{x}} \delta_2 N + \delta_2 K_\phi + \delta_2 N^x + \delta N \delta K_\phi = 0 \tag{3.79}
\]

and (B.7), upon using these and the first order equations, results in an evolution equation for \( K_\phi \) consistent with equation (B.10). Since we will not use these equations for the horizon conditions we will not write them here.

### 3.3.2.5 Perturbation of the metric and horizon

After inserting the relevant expressions into (3.55), we find that the zeroth order terms are naturally the same as for the background, the first order terms vanish — which is to be expected since the matter terms have not yet played a part — and the second order terms include an influence from the matter fields. The condition
on the horizon becomes:

\[ 1 - \frac{2M}{x} + \frac{2}{x} \int_{x}^{\infty} dz \, 4\pi z^2 \mathcal{H} = 0 \]  

(3.80)

which tells us that

\[ R_{\text{hor}} = 2 \left( M - \int_{R_{\text{hor}}}^{\infty} dz \, 4\pi z^2 \mathcal{H} \right) . \]  

(3.81)

The horizon radius is simply shifted from the vacuum value \(2M\) in terms of the asymptotic mass by the amount of energy contributed by matter between the horizon and spatial infinity. The dependence on \(\Delta K_\varphi\) in some solutions, for instance in (3.77), automatically cancels when they are combined to the horizon condition: the resulting condition is gauge invariant.

### 3.3.3 Schwarzchild

We proceed with the calculations in the Schwarzschild metric in a manner analogous to the Painlevé–Gullstrand case.

#### 3.3.3.1 First-order constraints

In the Schwarzschild metric, assuming \(\delta E^x = 0\), the first order Hamiltonian constraint (3.60) can be simplified to

\[ \left( 2 \left( 1 - \frac{2M}{x} \right)^{3/2} \delta E^\varphi \right)' = 0 . \]  

(3.82)

The simplest solution to satisfy this constraint is to have \(\delta E^\varphi = c(1 - 2M/x)^{-3/2}\). However this choice blows up near the horizon faster than \(E^\varphi = x(1 - 2M/x)^{-1/2}\), so we make the choice \(\delta E^\varphi = 0\).

In the Schwarzschild gauge, the first order diffeomorphism constraint (3.61) becomes

\[ \delta K'_\varphi = \sqrt{1 - \frac{2M}{x}} \delta K_x . \]  

(3.83)

This relation will be used repeatedly to simplify the second order constraints and equations of motion.
3.3.3.2 First-order equations of motion

From the first order perturbation of equation (B.6) for $E^x$ we derive

$$\delta N^x = -\sqrt{1 - \frac{2M}{x}} \delta K_\varphi. \tag{3.84}$$

Considering (B.7), the equation of motion for $E^\varphi$, we find:

$$\delta \dot{E}^\varphi = \delta K_\varphi + x \sqrt{1 - \frac{2M}{x}} \delta K_x + \left( x \left( 1 - \frac{2M}{x} \right)^{-1/2} \delta N^x \right)'$$

which, using (3.83) and (3.84), simplifies to

$$\delta \dot{E}^\varphi = \delta K_\varphi + x \delta K'_\varphi + (-x \delta K_\varphi)' = 0 \tag{3.85}$$

and ensures that $\delta E^\varphi$ remains zero.

Finally, equation (B.10) gives the additional relation

$$\delta \dot{K}_\varphi = \left( 1 - \frac{2M}{x} \right) \delta N' - \frac{M}{x^2} \delta N \tag{3.86}$$

3.3.3.3 Second-order constraints

The second order Hamiltonian constraint (3.74), after simplification and discarding terms containing $\delta E^\varphi$, becomes

$$\left( 2 \left( 1 - \frac{2M}{x} \right)^{3/2} \delta_2 E^\varphi \right)' + \left( x (\delta K_\varphi)^2 \right)' - 8\pi x^2 \rho = 0 \tag{3.87}$$

and implies

$$2 \left( 1 - \frac{2M}{x} \right)^{3/2} \delta_2 E^\varphi + x (\delta K_\varphi)^2 = - \int_x^\infty dz \, 8\pi z^2 \rho. \tag{3.88}$$

The relation provided by the diffeomorphism constraint (3.72) and the second order equations of motion are not needed here to derive the horizon condition, so we may proceed directly to step 5 as listed at the beginning of this section.
3.3.3.4 Perturbed metric and horizon conditions

The condition on the horizon is:

\[ 1 - \frac{2M}{x} + \frac{2}{x} \int_x^\infty dz \, 4\pi z^2 \mathcal{H} = 0 \]  

(3.89)

where in the Schwarzschild slicing \( \mathcal{H} = \rho \). This agrees with our Painlevé–Gullstrand result.

Additionally, we can use the isolated horizon conditions, and we find that (3.58) gives, after setting \( \delta E^\varphi = 0 \):

\[ 1 - \frac{2M}{x} + \frac{2}{x} \left( 1 - \frac{2M}{x} \right)^{3/2} \delta^2 E^\varphi = 0. \]  

(3.90)

But since \( \delta K_{\varphi}^2 = 0 \) from (3.57) at the isolated horizon, we once again have:

\[ 1 - \frac{2M}{x} + \frac{2}{x} \int_x^\infty dz \, 4\pi z^2 \rho = 0. \]  

(3.91)

showing that we get equivalent results for the two methods of deriving the position of the horizon.

3.4 Inverse-triad corrections

We are especially interested in horizon conditions in the presence of back-reaction and quantum corrections. For \( \bar{\alpha} = 1 \) the constraints satisfy the classical hypersurface-deformation algebra despite the presence of corrections. Effective line elements can thus be used to describe the space-time geometry and standard horizon definitions are applicable. We will first evaluate these definitions in the presence of corrections, which still provide equivalent results. This outcome is non-trivial since the modified dynamics could have led to stronger changes of the horizon behavior, rendering different definitions inequivalent. Moreover, the results of horizon conditions will be gauge invariant.

For \( \alpha = \bar{\alpha} \neq 1 \) we have a modified constraint algebra but can obtain horizon formulas simply by substitution after absorbing \( \alpha \) in the lapse function as far as the gravitational part of the Hamiltonian constraint is concerned.
The most interesting case is thus that of \( 1 \neq \bar{\alpha} \neq \alpha \), which as stated previously can be related to these two special cases. Here, the standard horizon conditions will no longer be gauge invariant, but we present a modification leading to satisfactory results. We will come back to conclusions drawn from this case in the discussions.

### 3.4.1 Classical algebra

Modified dynamics in the presence of ordinary space-time structure can directly be evaluated by the canonical horizon definitions.

#### 3.4.1.1 Modified Painlevé–Gullstrand gauge

We consider the modified Painlevé–Gullstrand metric (3.50) as our background. The correction function \( \alpha \) depends only on \( E^x \), so by assuming \( \delta E^x = 0 \), we also have \( \delta \alpha = 0 \). We will use the short hand notation:

\[
h := f_\alpha^2 - 1 + \frac{2Mf_\alpha}{x}.
\]  

(3.92)

#### 3.4.1.1.1 First-order constraints

##### Modified Hamiltonian constraint \( H[N] \):

To first order, assuming \( \delta E^x = 0 \), the modified Hamiltonian constraint reads

\[
2(\alpha K_\varphi E^\varphi + K_x E^x) \delta K_\varphi + 2K_\varphi E^x \delta K_x + \alpha(K_\varphi^2 - \Gamma_\varphi^2 + 1) \delta E^\varphi \\
- 2\alpha\Gamma_\varphi E^\varphi \delta \Gamma_\varphi + 2E^x \delta \Gamma_\varphi' = 0.
\]  

(3.93)

For the modified Painlevé–Gullstrand metric, using the relation between \( \alpha \) and \( f_\alpha \), this simplifies to:

\[
\left(\frac{x\sqrt{h}}{f_\alpha}\right)' \delta K_\varphi + x\sqrt{h} \delta K_x - \frac{f_\alpha^2}{x}(x \delta E^\varphi)' + \frac{f_\alpha}{x} \left( x \frac{Mf_\alpha}{x} + f_\alpha^2 \right) \delta E^\varphi = 0.
\]  

(3.94)

##### Diffeomorphism constraint \( D[N^x] \):

Equation (3.61) becomes:

\[
\frac{x\sqrt{h}}{f_\alpha} \delta K_\varphi' - \sqrt{h}(\sqrt{h})' \delta E^\varphi - x\sqrt{h} \delta K_x = 0.
\]  

(3.95)
Adding these first order equations, we get an expression that simplifies to:

$$\left( \frac{x \sqrt{\hbar}}{f_\alpha} \delta K_\varphi \right)' - (f_\alpha^2 \delta E_\varphi)' = 0$$

(3.96)

which implies, with the appropriate fall off conditions at infinity, that

$$\delta E_\varphi = \frac{x \sqrt{\hbar}}{f_\alpha^3} \delta K_\varphi .$$

(3.97)

### 3.4.1.1.2 Equations of motion

The equation of motion (B.6) for $E^x$ to first order gives us the relation

$$-\sqrt{\hbar} \delta N + \delta K_\varphi + \delta N^x = 0$$

(3.98)

in this modified Painlevé–Gullstrand metric.

Similarly for the second order perturbation of the same equation, we derive

$$\delta N \delta K_\varphi - \sqrt{\hbar} \delta_2 N + \delta_2 K_\varphi + \delta_2 N^x = 0 .$$

(3.99)

### 3.4.1.1.3 Second-order constraints

Adding the second order constraints, integrating and rearranging, we get

$$\frac{2x \sqrt{\hbar}}{f_\alpha} \delta_2 K_\varphi - \left( \frac{x}{f_\alpha} - \frac{3x \hbar}{f_\alpha^3} \right) (\delta K_\varphi)^2 - 2f_\alpha^2 \delta_2 E_\varphi = \int_x^\infty dz \, 8\pi z^2 \frac{\mathcal{H}}{f_\alpha}$$

(3.100)

where

$$\mathcal{H} = N \rho - N^x J_x .$$

(3.101)

### 3.4.1.1.4 Horizon conditions

The condition on the horizon in the modified metric is:

$$1 - \frac{2M f_\alpha}{x} + \frac{2f_\alpha}{x} \int_x^\infty dz \, 4\pi z^2 \frac{\mathcal{H}}{f_\alpha} = 0$$

(3.102)

which agrees with the classical Painlevé–Gullstrand result in the limit that $f_\alpha \to 1$. Gauge-dependent terms such as $\delta K_\varphi$ drop out and there is no need to fix the Hamiltonian gauge.
3.4.1.2 Modified Schwarzschild gauge

3.4.1.2.1 First-order constraints

3.4.1.2.1.1 Modified Hamiltonian constraint $H[N]$:

Using the relation between $\alpha$ and $f_\alpha$, equation (3.93) simplifies to:

$$\left( \frac{2}{f_\alpha} \left( 1 - \frac{2Mf_\alpha}{x} \right)^{3/2} \delta E^\varphi \right)' = 0.$$ \hspace{1cm} (3.103)

The simplest solution to satisfy this constraint is to have $\delta E^\varphi = 0$.

3.4.1.2.1.2 Diffeomorphism constraint $D[N^x]$:

Equation (3.61) for this metric gives the relation:

$$\delta K'_\varphi = \sqrt{1 - \frac{2Mf_\alpha}{x}} \delta K_x.$$ \hspace{1cm} (3.104)

3.4.1.2.2 First-order equations of motion

From the first order perturbation of the equation of motion for $E^x$, we have:

$$\delta N^x = -\frac{\delta K_\varphi}{f_\alpha} \sqrt{1 - \frac{2Mf_\alpha}{x}}.$$ \hspace{1cm} (3.105)

For the equation of motion of $\delta E^\varphi$, we find

$$\delta \dot{E}^\varphi = \left( -\frac{x}{f_\alpha} \delta K_\varphi \right)' + \frac{x}{f_\alpha} \left( \delta K_\varphi \right)' + \left( \frac{1}{f_\alpha} - \frac{x f'_\alpha}{f_\alpha^2} \right) \delta K_\varphi = 0$$ \hspace{1cm} (3.106)

which, once again, ensures that $\delta E^\varphi$ remains zero.

3.4.1.2.3 Second-order constraints

The second order Hamiltonian constraint, after simplification and discarding terms which contain $\delta E^\varphi$, becomes:

$$\left( \frac{2}{f_\alpha} \left( 1 - \frac{2Mf_\alpha}{x} \right)^{3/2} \delta_2 E^\varphi \right)' + \left( \frac{x (\delta K_\varphi)^2}{f_\alpha} \right)' - \frac{8\pi x^2 \rho}{f_\alpha} = 0.$$ \hspace{1cm} (3.107)

As in the classical Schwarzschild case, the relations from the second order diffeomorphism constraint (3.72) and equations of motion are not needed to derive
the expression for the horizon condition.

### 3.4.1.2.4 Horizon condition

For the horizon condition, we arrive at

\[
1 - \frac{2Mf_\alpha}{x} + \frac{2f_\alpha}{x} \int_x^\infty dz \frac{dz}{z^2} \frac{\rho}{f_\alpha} = 0 \quad (3.108)
\]

which agrees with the classical Schwarzschild result in the limit that \( f_\alpha \to 1 \).

Equivalent results, (3.102) and (3.108) are obtained with both slicings and, in the Schwarzschild slicing, with both definitions of horizons. Moreover, for vacuum the result is in agreement with the direct space-time analysis performed in Sec. 3.2, which applies in this subsection where the classical algebra is assumed in the presence of corrections. In both cases, \( \delta K_\varphi \)-terms automatically cancel in the horizon equation.

### 3.4.2 Modified algebra, absorbable

Before we evaluate horizon conditions in the case of a modified constraint algebra, first restricting attention to the case \( \alpha = \bar{\alpha} \), we present calculations that show the overall consistency of the equations of motion and constraints. We will perform some of the calculations explicitly for the choice \( N_\alpha = 1 \) with a scalar matter field, illustrating how the anomaly-freedom condition is necessary to obtain consistent equations. (See the Appendix for an illustration of the inconsistency of line elements in this case with modified space-time structures.)

#### 3.4.2.1 Dynamical consistency

First-order equations and results for this case are identical to those in sections 3.3.2.1 and 3.3.2.2. The second-order diffeomorphism constraint is the same as (3.73), and in the second order Hamiltonian constraint (3.75) the matter term is replaced by \(-2x\alpha^{-1}(\nu \mathcal{H}_\pi + \sigma \mathcal{H}_\varphi + \mathcal{H}_U)\). Again, combining these equations and integrating gives

\[
\delta_2 E^\varphi = \sqrt{2Mx} \delta_2 K_\varphi - \frac{1}{2} (x - 6M)(\delta K_\varphi)^2 - \mathcal{E} \quad (3.109)
\]
where now we use the short hand notation

\[
\mathcal{E} := \int_x^\infty dz \ 4\pi z^2 (N\rho_{mod} - N^x J_x)
\]

\[
= 4\pi \int_x^\infty dz \left[ \frac{1}{\alpha} \left( \frac{\nu}{2 x^2} p_\chi^2 + \sigma \frac{z^2}{2} \chi'^2 + \frac{z^2}{2} U \right) + \sqrt{\frac{2M}{z}} p_\chi \chi' \right].
\]  

(3.110)

Putting (3.109) back into the diffeomorphism constraint (3.73)

\[
\delta_2 K = \frac{M}{x^2} \delta_2 K_\varphi + \delta_2 K'_x - \frac{1}{4x^2} \sqrt{\frac{2M}{x}} (x - 6M) (\delta K_\varphi)^2
\]

\[
+ \sqrt{\frac{2M}{x}} \delta K_\varphi \overline{\delta K_\varphi} - \frac{1}{2x^2} \sqrt{\frac{2M}{x}} \mathcal{E} + \frac{4\pi}{x} p_\chi \chi'.
\]  

(3.111)

Equation (B.6) gives again

\[
\delta_2 N^x = -\delta_2 K_\varphi
\]  

(3.112)

and (B.7), upon using (3.109), (3.111), (3.112) and the first-order equations,

\[
\sqrt{2Mx} \delta _2 K_\varphi = -\frac{M}{x} \delta_2 K_\varphi + 2M \delta_2 K'_x - \sqrt{2Mx} \delta K_\varphi \delta K'_\varphi
\]

\[
+ \frac{1}{x} \sqrt{\frac{2M}{x}} \mathcal{E} - \sqrt{\frac{2M}{x}} \mathcal{E}' + 4\pi p_\chi \chi' + \dot{\mathcal{E}}.
\]  

(3.113)

On the other hand, equation (B.10) for the time evolution of the extrinsic curvature \(K_\varphi\) gives

\[
\delta_2 K_\varphi = -\frac{1}{2x} \sqrt{\frac{2M}{x}} \delta_2 K_\varphi + \sqrt{\frac{2M}{x}} \delta_2 K'_x - \delta K_\varphi \delta K'_\varphi
\]

\[
+ \frac{1}{x^2} \mathcal{E} - \frac{2\pi}{\alpha} \left( \frac{\nu}{x^3} p_\chi^2 + \sigma x \chi'^2 - xU \right)
\]  

(3.114)

comparing each term of this equation with (3.113) we must, for consistency, have the identity

\[
\dot{\mathcal{E}} = \sqrt{\frac{2M}{x}} \mathcal{E}' - 4\pi p_\chi \chi' - 2\pi \sqrt{\frac{2M}{x}} \left( \frac{\nu}{x^3} p_\chi^2 + \sigma x \chi'^2 - xU \right)
\]
or, simplifying the RHS using (3.110),

\[
\dot{\mathcal{E}} = -4\pi \left[ \left( 1 - \frac{2M}{x} \right) p_x \chi' + \frac{\sqrt{2M x}}{\alpha} \left( \frac{\nu}{x^3} p_x^2 + \sigma x \chi'' \right) \right] + \text{surface term}.
\]

That this is indeed the case can be readily verified using the (first order) equations of motion for the matter field, (B.8) and (B.9) or

\[
\dot{\chi} = \frac{\nu}{\alpha x^2} p_x + \sqrt{\frac{2M}{x}} \chi', \quad \dot{p}_x = \left( \frac{\sigma x^2}{\alpha^2} \chi' \right)' - \frac{x^2}{2\alpha} dU + \left( \frac{\sqrt{2M x} p_x}{x} \right)'.
\]

in the present gauge, to compute the time derivative of \( \mathcal{E} \) from its definition (3.110):

\[
\dot{\mathcal{E}} = 4\pi \int_x^\infty dz \left[ \left( \frac{\nu}{\alpha^2} - \frac{2M}{z} \right) p_x \chi' + \frac{\sqrt{2M z}}{\alpha} \left( \frac{\nu}{z^3} p_x^2 + \sigma z \chi'' \right) \right]' .
\]

Comparing (3.115) and (3.117), we see here how the anomaly-freedom condition (2.12) is required for consistency (remember that here, \( \beta = \alpha^2 \)).

Once anomaly freedom is implemented, equations of motion can be consistently used to evaluate the dynamics even in the absence of a classical space-time structure. We will now turn to the issue of horizons, whose primary motivation and definition is closely tied to classical space-time intuition.

### 3.4.2.2 Classical horizon conditions

We introduce inverse-triad corrections in the Hamiltonian constraint by replacing \( N/\sqrt{E^x} \) by \( N\alpha/\sqrt{E^x} \). For the Schwarzschild gauge this can be accounted for most simply by setting:

\[
N\alpha = \sqrt{1 - \frac{2M}{x}}
\]

and replacing \( \rho \) by \( \rho_{\text{mod}}/\alpha \) where \( \rho_{\text{mod}} \) contains further corrections such as \( \nu \) and \( \sigma \) used above for a scalar field. By following the procedure in Sec. 3.3.3 and simple substitution in (3.55) we have

\[
1 - \frac{2M}{x} + \frac{2}{x} \int_x^\infty dz 4\pi \frac{\rho_{\text{mod}}}{\alpha} z^2 - (\alpha^2 - 1)(\delta K^\phi)^2 = 0 .
\]
Now, $\delta K_\varphi$ no longer cancels because different powers of $\alpha$ appear in the terms of (3.55) with different powers of $N$ in the denominators. The isolated horizon condition gives the results from (3.90) and (3.91), with $\rho$ replaced by $\rho_{\text{mod}}/\alpha$, and $\delta K_\varphi$ vanishes by definition. Thus the two horizon conditions give different results, becoming equivalent only in the case when $\delta K_\varphi = 0$. One may choose this value to fix the Hamiltonian gauge, but the more general condition of trapping horizons remains gauge dependent.

For the Painlevé-Gullstrand gauge, we have, again up to second order, the horizon condition (3.55) as

$$1 - \alpha^2 \frac{2M}{x} + (\alpha^2 - 1) \left(2 \sqrt{\frac{2M}{x}} \delta K_\varphi + 2 \frac{2M}{x} \delta_2 K_\varphi - (\delta K_\varphi)^2\right) + \frac{2}{x} \mathcal{E} = 0.$$  

(3.120)

In contrast to the Schwarzschild case with the same correction in the Hamiltonian constraint, even the background terms are modified as a consequence of the term $(N^x/N)^2$ in (3.55), now with a non-vanishing shift vector. Different slicings do not give rise to the same area-mass relationship of horizons, further illustrating the gauge dependence of the original horizon condition.

3.4.2.3 Horizon conditions for modified space-time structures

The case of a modified, yet consistent constraint algebra provides several interesting lessons. Not only do the horizon conditions we use lead to different results (3.119) and (3.120) for different choices of slicings, for each slicing they depend on the gauge-dependent quantity $\delta K_\varphi$. With this dependence, the horizon conditions are no longer meaningful. The application of conventional space-time intuition to quantum gravity, embodied here by some of its effects on modified constraints, is thus highly non-trivial. In Section 3.5 we will discuss this set of problems and its ramifications further.

We recall that the modified equations are fully consistent dynamically; it is only the horizon conditions which must be adapted as well by using as yet unknown notions of quantum horizons. To provide an idea of the required modifications of
horizon conditions, it turns out that the modified trapping-horizon condition

\[ \frac{E_x}{(E^x)^2} - \left( \frac{N^x}{\bar{\alpha} N} \right)^2 = 0 \]  

(3.121)

when evaluated for all cases considered here produces satisfactory results: there is no gauge dependence in the area-mass relationships, and they all agree for the different slicings, correcting the classical relationship by

\[ 1 - \frac{2M}{x} + \frac{2}{x} \int x \, d\bar{z} \, 4\pi \frac{\rho_{\text{mod}}}{\alpha} \bar{z}^2 = 0. \]  

(3.122)

Moreover, the corrections differ from those found in the non-absorbable case with classical constraint algebra, where we have (3.108).

The combination of fields appearing in the modified horizon condition may be interpreted as the inverse-metric component \( g^{xx} \) for a metric with rescaled lapse function \( \bar{\alpha} N \), but in the case of a modified constraint algebra the notion of line elements or metrics is not applicable. Instead, the modification can be read off from the dynamical equations used here, ensuring that evaluations for horizons are gauge invariant. The isolated-horizon condition fixes the Hamiltonian gauge before quantization or putting in corrections, and thus removes the gauge-dependent term by fiat. This form of gauge fixing before quantization, or before including corrections, eliminates important consistency conditions, and thus, if it is used as the sole means to determine horizons, further necessary conditions to the horizon condition such as (3.121) would be overlooked.

### 3.4.3 Modified algebra, non-absorbable

The equations for \( 1 \neq \alpha \neq \bar{\alpha} \neq 1 \) can be mapped to those analyzed in Section 3.4.1 by absorbing \( \bar{\alpha} \) in the lapse function. We can thus skip analyzing this general case anew and simply cite the conclusions drawn earlier: Corrections to the area-mass relation do arise, even in vacuum space-times. However, as in Section 3.4.2, absorbing a correction function in the lapse function makes the horizon conditions differ in the two definitions used here, and gauge-dependent terms no longer drop out, unless the horizon condition is corrected to (3.121). Combining the previous
area-mass relationships, we arrive at

\[ 1 - \frac{2M f_{\alpha/\bar{\alpha}}}{x} + \frac{2f_{\alpha/\bar{\alpha}}}{x} \int_{x}^{\infty} d\bar{z} \frac{4\pi \rho}{f_{\alpha/\bar{\alpha}} \bar{\alpha}} \bar{z}^2 = 0 \]  

(3.123)

where \( f_{\alpha/\bar{\alpha}} \) is computed as in the case of \( \bar{\alpha} = 1 \), but replacing \( \alpha \) with \( \alpha/\bar{\alpha} \).

3.5 Discussion

When quantum gravity changes the structure of space and time, as expected in many different ways at a fundamental level, the usual notions of geometry and physical implications for instance in the behavior of black holes must be reanalyzed. In particular, one cannot always make use of definitions that refer directly or indirectly to space-time manifolds or even coordinates. The line element, one of the basic concepts often used in classical general relativity, is the main example for this; and constructions based on its properties such as some notions of horizons cannot always be applied in the presence of quantum-gravity corrections. But even if one does not rely on line elements or metric components, the concept of a horizon crucially refers to test-particle propagation in space-time (e.g. for trapping surfaces or causal properties). The notion of test particles does not exist in fundamental quantum-gravity theories, and even at effective levels this notion can lead to additional difficulties if space-time structures change.\(^2\)

In this article, we have illustrated some of these features by different examples of inverse-triad corrections in spherically symmetric models of loop quantum gravity, showing the various ways in which the area-mass relationship of horizons is modified by inverse-triad corrections. While our calculations of the dynamics are not at the full quantum level of the theory, which is still too difficult to handle explicitly, several features such as modified space-time structures as evidenced by non-classical constraint algebras, can be highlighted. This led us to stress the importance of rethinking definitions of horizons suitable for quantum gravity.

In order to probe properties of black-hole horizons in a more general context, allowing for corrections to the constraint algebra, we have developed a canonical

\(^2\)For instance, in [76] apparent superluminal effects arise, but only because the space-time notion used for null lines is not applicable for the deformed constraint algebra.
version of spherically symmetric perturbation theory in connection variables. Several perturbation equations can be solved completely in the presence of matter, providing general formulas for the dynamics of trapping horizons. In the classical case, these formulas are not new, but their new derivation allows an easy extension to geometries arising from canonical quantizations and the related modified space-time structures.

Quantum-gravity corrections, from this perspective, can be split into two classes: those that modify the dynamics of general relativity but not its space-time structure, leaving the classical constraint algebra unchanged; and those that modify both the dynamics and the space-time structure. We have presented a detailed analysis of a model falling in the former class, where a standard space-time analysis is available in the presence of inverse-triad corrections, used for the results presented in Sec. 3.2. As seen there, the horizon behavior is affected by the corrections, for instance regarding the relationship between mass and size, or Hawking radiation. But the classical notion of a horizon is still valid, illustrated by the result of Section 3.4.1 that different horizon conditions agree with each other and are gauge independent. Moreover, in this case ($\bar{\alpha} = 1$) the canonical horizon conditions produce the same result as a direct space-time analysis.

We have not attempted to address the question of how in general to define horizons in modified space-time, but we have provided an example where direct extensions of classical conditions fail when quantum gravity modifies space-time structures. Properties of horizons according to the classical definitions then depend on the slicing chosen, and are gauge dependent. In the examples considered here, a simple modification of the classical horizon conditions (3.121) by the correction function that also changes the dynamics leads to satisfactory results. In particular, the area-mass relationship is corrected to the implicit condition

$$R_{\text{hor}} = 2 f_{\alpha/\bar{\alpha}} (R_{\text{hor}}) \left( M - \int_{R_{\text{hor}}}^{\infty} dx 4\pi x^2 \frac{\rho(x)}{f_{\alpha/\bar{\alpha}}(x)\bar{\alpha}(x)} \right)$$

(3.124)

for the area radius $R_{\text{hor}}$ of the horizon, with $f_{\alpha/\bar{\alpha}}$ related to the primary correction functions $\alpha$ and $\bar{\alpha}$ by $f'_{\alpha/\bar{\alpha}}/f_{\alpha/\bar{\alpha}} = (1 - \alpha/\bar{\alpha})/x$.

No gauge-dependence appears in the condition for the horizon radius, and the different slicings lead to equivalent results. But the modified horizon condition was
not obtained by quantum space-time intuition; rather, we looked for a modification that served to eliminate gauge-dependent terms. Our results especially in the case of modified yet first-class constraint algebras, the general case expected for loop quantum gravity, thus show the need to develop appropriate horizon definitions for quantum space-times without referring to the usual classical notions such as the expansion of light rays which are no longer available. Some steps in this direction have already been undertaken, for instance in [77, 78, 79, 80] and recently in [81], but most of them remain tied to the classical notion of expansion and they are difficult to evaluate in a dynamical context. Our results also show that the more restrictive notion of isolated horizons, based on an additional gauge fixing compared with trapping horizons, does not seem sufficient to derive corrected horizon conditions.

Our considerations provide a cautious note regarding the reliability of black-hole entropy calculations in loop quantum gravity, which are based on a classical implementation of isolated horizons treated as boundaries of space-time [54]. The properties of horizon definitions found here indicate that the implementation of isolated horizons via boundary conditions derived before quantization may not include all possible quantum features relevant for horizons. Even though quantum-gravity corrections are expected to be small for realistic black holes, the value of the Barbero–Immirzi parameter derived from entropy countings could change. (These quantum corrections are distinct from those discussed in [56, 57], which arise from the quantized boundary field theory obtained after imposing classical boundary conditions. Conceptually, the implications of quantum corrections for conclusions about the Barbero–Immirzi parameter nevertheless seem comparable at a qualitative level.) In this way, new interesting and non-trivial tests of the quantization may be possible. On the other hand, as a supportive statement for some of the assumptions behind the current counting procedures, our results for the case of quantum effects leaving the classical constraint algebra intact also show that corrections to the area and temperature laws arise from modifications in the dynamics even if classically motivated horizon conditions are used. The fact that, at least in some cases, classical definitions can consistently be used even for the quantum-modified dynamics shows, among other things, that a possible renormalization of Newton’s constant, as sometimes suggested [82], need not necessarily
be taken into account for the horizon condition itself (or for countings of entropy based on it);³ it will in any case arise once horizon conditions are evaluated for a dynamical solution, producing the area-mass relationship. Inverse-triad corrections, considered here as an important contribution from quantum geometry, do not constitute the usual source of renormalization. But the canonical methods developed and applied here can also be used for quantum back-reaction, which in its canonical form formulated in [83, 84] corresponds to the familiar quantum-dynamical corrections of interacting quantum theories. Our results thus provide a first step toward possible implications of renormalization in dynamical solutions of loop quantum gravity.

³There may be other motivations to introduce renormalization at the level of horizon conditions independent of the present context.
Appendix A

Space-time transformations with modified constraint algebra

In this appendix, we compare different coordinate representations of solutions in the case of a modified constraint algebra, showing that they are not related by coordinate transformations [99]. To be specific, we choose the absorbable case $\alpha = \bar{\alpha} \neq 1$.

A.1 Schwarzschild-like

A Schwarzschild-like solution can be obtained by assuming $K_\phi = K_x = N_x = 0$. Since the vacuum Hamiltonian-constraint equation is the same as in the classical case we have the Schwarzschild solution for $E^\phi$ if we assume the gauge $E^x = x^2$. Only the form of the lapse function changes and using (B.10) is found to be $N = \alpha^{-1}(1 - 2M/x)^{1/2}$, as already suggested by the absorbable nature of the inverse-triad correction in the case under consideration. If we were to assume that even with the modified algebra there is a spacetime interpretation, we would write the solution as the corresponding Schwarzschild-like line element

$$\delta s^2 = -\alpha^{-2}\left(1 - \frac{2M}{x}\right) \delta t^2 + \left(1 - \frac{2M}{x}\right)^{-1} \delta x^2 + x^2 \delta \Omega^2. \quad (A.1)$$

(The slashed $\delta$s indicate that the line element in the present context is a purely formal construction, with $\delta x^a$ not subject to the usual coordinate transformations.)
A.2 Painlevé–Gullstrand-like

Following the analysis of section 3.1.2 we now consider the transformation to a Painlevé–Gullstrand like metric. Since (A.1) is time independent, there is a time-like Killing vector \( \xi(t) = \partial_t \). If \( u^a \) is the tangent to a radial freely falling geodesic (parameterized by \( T \)) then \( u_a \xi^a(t) = E \), where \( E \) is a constant which we choose to be equal to one. This implies

\[
g_{ab} u^a \xi^b(t) = -\alpha^{-2} \left( 1 - \frac{2M}{x} \right) \frac{dt}{dT} = -1
\]

or, with \( g_{ab} u^a u^b = -1 \),

\[
\frac{dx}{dT} = -\sqrt{\alpha^2 - 1 + \frac{2M}{x}}.
\]

The time differential \( dT = -u^a dx^a \) with \( u_a = (-1, -(1-2M/x)^{-1}\sqrt{\alpha^2 - 1 + 2M/x}, 0, 0) \) reads

\[
dt = dT - \left( 1 - \frac{2M}{x} \right)^{-1} \sqrt{\alpha^2 - 1 + \frac{2M}{x}} \ dx.
\]

Substituting this back in the Schwarzschild metric we obtain

\[
\eta^2 = -\alpha^{-2} \left( 1 - \frac{2M}{x} \right) dT^2 + \alpha^{-2} dx^2 + 2\alpha^{-2} \sqrt{\alpha^2 - 1 + \frac{2M}{x}} \ dx \ dT + x^2 \ d\Omega^2
\]

which can be considered as the Painlevé–Gullstrand version of (A.1).

For our phase-space functions, (A.2) implies \( E^x = x^2 \), \( E^\phi = x/\alpha \), \( N^x = \sqrt{\alpha^2 - 1 + 2M/x} \), \( N = 1 \), \( K^\phi = -\sqrt{\alpha^2 - 1 + 2M/x}/\alpha \), \( K_x = (2M\alpha' + M\alpha/x - \alpha'/x) \alpha^3 x \sqrt{\alpha^2 - 1 + 2M/x} \). However, substituting this form of the metric back in the constraints we find that the diffeomorphism constraint satisfied, but not the Hamiltonian constraint. This is an illustration of the fact that the modified form of the constraint algebra prevents coordinate transformations from being gauge transformations: they do not map solutions of the constraints to other solutions.

With a version of inverse-traid corrections not modifying the constraint algebra, on the other hand, the analysis of Section 3.2.1.2 showed that the metric in the new coordinates did satisfy all constraints and was a solution representing the same spacetime.

Earlier, we have seen that \( N\alpha = 1 \) solves the Hamiltonian constraint, but it
does not correspond to the Painlevé–Gullstrand form obtained by following the spacetime procedure to transform from the Schwarzschild metric. As discussed in Section 3.1, absorbing the correction function in the lapse function does not amount to reducing the constraint algebra to classical form. Conversely to the transformation attempted here, one may start with the Painlevé–Gullstrand-like solution solving the constraints and transform to some Schwarzschild form. For the static form of the Schwarzschild line elements combined with our usual gauge fixing of $E^x$, two coefficients, $g_{tt}$ and $g_{xx}$, have to be determined. If the Painlevé–Gullstrand form is given, one may follow the procedure used above backwards, asking what Schwarzschild-like coefficient would provide the desired Painlevé–Gullstrand form in this way. With three non-trivial coefficients to be produced for the Painlevé–Gullstrand form, but only two free coefficients for a Schwarzschild-like form, three equations for two unknowns must be solved. Classically, there is a consistent solution, but there is none when the constraints of a modified algebra are used.
Appendix B

Equations of Motion in Spherical Symmetry

The modified gravitational part of the Hamiltonian constraint we consider here is

\[
H_{\text{grav}}^Q[N] = -\frac{1}{2G} \int dx \: N \left[ \alpha |E^x|^{-\frac{1}{2}} K_\varphi^2 E^\varphi + 2\bar{\alpha} K_\varphi K_x |E^x|^{\frac{1}{2}} + \\
+ \alpha |E^x|^{-\frac{1}{2}} (1 - \Gamma_\varphi^2) E^\varphi + 2\bar{\alpha} \Gamma'_\varphi |E^x|^{\frac{1}{2}} \right]
\] (B.1)

For concreteness, we consider as a matter source a scalar field \( \chi \) with general potential \( U(\chi) \) [99, 11]. The matter contribution to the diffeomorphism constraint reads:

\[
D_{\text{matter}}[N^x] = 4\pi \int dx \: N^x p_\chi \chi'
\]

and to the Hamiltonian constraint it is

\[
H_{\text{matter}}[N] = \int dx \: N (\tilde{\mathcal{H}}_\pi + \tilde{\mathcal{H}}_\nabla + \tilde{\mathcal{H}}_U)
\]

where the kinetic, gradient and potential terms are, respectively,

\[
\tilde{\mathcal{H}}_\pi = 4\pi \frac{p_\chi^2}{2|E^x|^{\frac{3}{2}} E^\varphi} \quad , \quad \tilde{\mathcal{H}}_\nabla = 4\pi \frac{|E^x|^{\frac{3}{2}} \chi'^2}{2 E^\varphi} \quad , \quad \tilde{\mathcal{H}}_U = 4\pi \frac{|E^x|^{\frac{3}{2}} E^\varphi U[\chi]}{2}.
\]

Since \( \tilde{\mathcal{H}}_{\text{matter}} = \sqrt{\det q} \rho \) and \( \tilde{D}_{\text{matter}} = -\sqrt{\det q} J_x \), the energy density \( \rho = \)
$T_{abc}n^a n^b$ and energy flux $J_a = q^b_a T_{bc} n^c$ are, respectively,

$$
\rho = \frac{p^2_\chi}{2|E^x|E^{\phi^2}} + \frac{|E^x|\chi'^2}{2E^{\phi^2}} + \frac{U}{2}
$$

and

$$
J_x = -\frac{1}{|E^x|\frac{1}{2}E^\phi p_\chi \chi'}. \tag{B.2}
$$

Again, we introduce general quantum correction functions $\nu$ and $\sigma$ into the matter part of the Hamiltonian constraint to account for the quantization of inverse-triad operators as:

$$
H^Q_{\text{matter}}[N] = \int dx \, N (\nu \tilde{\mathcal{H}}_x + \sigma \tilde{\mathcal{H}}_\nabla + \tilde{\mathcal{H}}_U). \tag{B.2}
$$

As before, only a dependence of the correction functions on $E^x$ is possible for consistency with the unmodified diffeomorphism constraint. The potential term is not expected to acquire quantum corrections because it does not contain an inverse of the triad. There is no inverse of $E^x$ in the gradient term, either, which may thus be expected to be unmodified by $E^\phi$-independent corrections. For generality, we nevertheless insert a second correction function $\sigma(E^x)$ for this term (in contrast to the potential term) because without spherical symmetry there is an inverse-triad component in the gradient term and it would be corrected. For all our subsequent calculations, it will nevertheless be consistent to assume $\sigma = 1$.

The presence of matter makes the constraint algebra more non-trivial. In the gravitational part, one can sometimes absorb correction functions in the lapse function if $\alpha = \bar{\alpha}$ at least as far as the dynamics is concerned. With a matter potential, even if $\nu$ and $\sigma$ would equal $\alpha$, the correction does not simply amount to a rescaling of the lapse function and the closure of the constraint algebra becomes a nontrivial requirement that restricts the form of the correction functions. The total Hamiltonian $H^Q[N] = H^Q_{\text{grav}}[N] + H_{\text{matter}}[N]$ and diffeomorphism constraint $D[N^x] = D_{\text{grav}}[N^x] + D_{\text{matter}}[N^x]$ satisfy the algebra:

$$
\{H^Q[N], D[N^x]\} = -H^Q[N^x N'], \tag{B.3}
$$

$$
\{H^Q[N], H^Q[M]\} = D_{\text{grav}}[\bar{\alpha}^2 |E^x|(E^\phi)^{-2}(NM' - MN')].
$$
\[ + D_{\text{matter}}[\nu \sigma |E^x|(E^x)^{-2}(NM' - MN')] \]  \hspace{1cm} (B.4)

The requirement of anomaly-freedom thus imposes the condition

\[ \bar{\alpha}^2 = \nu \sigma \]  \hspace{1cm} (B.5)

(see (2.12)) and quantization ambiguities are somewhat reduced by relating correction functions.

The canonical equations of motion obtained from the corrected Hamiltonian are

\[ \dot{E}^x = 2N\bar{\alpha}K_\varphi |E^x|^2 + N^x E^x' \]  \hspace{1cm} (B.6)

\[ \dot{E}^\varphi = N(\bar{\alpha}K_\varphi |E^x|^2 + \alpha K_\varphi^2 |E^x|^2 - \frac{1}{2}) + (N^x E^x)' \]  \hspace{1cm} (B.7)

\[ \dot{\chi} = \frac{N\nu}{|E^x|^2 E^\varphi} p_\chi + N^x \chi' \]  \hspace{1cm} (B.8)

\[ \dot{p}_\chi = \left(\frac{N\sigma |E^x|^2 K_\varphi}{E^\varphi}\right)' - \frac{1}{2} N |E^x|^2 E^\varphi \frac{\partial U}{\partial \chi} + (N^x p_\chi)' \]  \hspace{1cm} (B.9)

\[ \dot{K}_\varphi = \frac{N}{2} |E^x|^{-\frac{1}{2}} \left[ -\alpha K_\varphi^2 + (2\bar{\alpha} - \alpha) \frac{E^x^2}{4E^\varphi^2} - \alpha \right] + N^x K_\varphi' + (N\bar{\alpha})' \frac{|E^x|^2 E^x'}{2E^\varphi^2} \]

\[- 2\pi G N \left[ \nu \frac{p_\chi^2}{|E^x|^2 E^\varphi^2} + \sigma \frac{|E^x|^2 K_\varphi^2 E^\varphi^2}{E^\varphi^2} - |E^x|^2 U[\chi] \right] \]  \hspace{1cm} (B.10)

\[ \dot{K}_x = -N\bar{\alpha}|E^x|^{-\frac{1}{2}} K_x K_\varphi + N\alpha \frac{|E^x|^2 E^\varphi}{2} \left( K_\varphi^2 + 1 - \frac{E^x^2}{4E^\varphi^2} \right) \]

\[ + N\bar{\alpha}|E^x|^{-\frac{1}{2}} \left( \frac{E^x^n}{2E^\varphi} - \frac{E^{x'} E^\varphi'}{2E^\varphi^2} \right) + N(\bar{\alpha} - \alpha) \left( |E^x|^2 \frac{E^x'}{2E^\varphi} \right)' \]

\[ + [2(N\bar{\alpha})' - (N\alpha)'] \frac{|E^x|^2 E^x'}{2E^\varphi} - (N\alpha)' \frac{|E^x|^2 E^\varphi'}{2E^\varphi} + (N\bar{\alpha})' \frac{|E^x|^2}{E^\varphi} \]

\[ + (N^x K_\varphi)' - N \frac{\partial \alpha}{\partial E^x} |E^x|^{-\frac{1}{2}} (K_\varphi^2 E^\varphi + E^\varphi (1 - \Gamma^2)) \]

\[ - 2N \frac{\partial \bar{\alpha}}{\partial E^x} |E^x|^2 (K_\varphi K_\varphi + \Gamma^2) + 2GN \left( \frac{\partial \nu}{\partial E^x} \hat{H}_\pi + \frac{\partial \sigma}{\partial E^x} \hat{H}_\varphi \right) \]
\[ + 2\pi GN \left( -\nu \frac{p^2}{|E^x|^2 E^\varphi} + \sigma \frac{3|E^x|^{\frac{1}{2}} \chi^2}{E^\varphi} + \frac{E^\varphi U[\chi]}{|E^x|^{\frac{1}{2}}} \right) \]. \quad (B.11) \]
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Vita

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