UNDERSTANDING PROPORTIONAL REASONING
IN PRE-SERVICE TEACHERS

A Dissertation in
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by
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Abstract

The purpose of this study is to examine the proportional reasoning of pre-service teachers at the beginning of their teacher preparation program using the developmental shifts described by Lobato and Ellis (2010). They cast changes in proportional reasoning as transitions or “shifts” in students’ thinking and these shifts can serve as tools to “evaluate [a] student’s current thinking” (p. 61). Lobato and Ellis (2010) suggested that one needs to make the following transitions or shifts in his or her thinking in order to develop proportional reasoning: (1) from reasoning with one quantity to reasoning with two quantities, (2) from reasoning additively to reasoning multiplicatively, (3) from reasoning with composed-units to reasoning with multiplicative comparisons, and (4) from iterating a composed unit to reasoning with a set of infinitely many equivalent ratios.

Twenty-five pre-service teachers, who were either elementary education majors or secondary mathematics education majors, completed a questionnaire consisting of problems that were meant to elicit the four shifts identified in the model. Based on the questionnaire responses, these twenty-five were placed into four groups. Eleven of these pre-service teachers (2 from Group 1, 3 from Group 2, 3 from Group 3 and 3 from Group 3) were then interviewed to examine how their reasoning related to the shift model.

The analysis of the data suggests that model was hierarchical for those in Group 1 (participants who seemed to exhibit all four shifts) and Group 4 (participants who only provided evidence of shift 1). For those in Groups 2 and 3, the model did not appear to remain hierarchical given that the participants in these groups seem to inconsistently exhibit characteristics of the shifts. This means that when solving some proportional problems they would provide evidence of making all four shifts and on other problems they would revert back to evidence of making shifts 2,3, and 4. Groups 2 and 3’s reasoning was inconsistent with the shifts because they were in the process of making the transitions as they learned to be proportional reasoners.
The findings in this study provide teacher educators with knowledge about the nature of pre-service teachers’ proportional reasoning. In particular, this study highlights four assumptions and misconceptions about proportional reasoning seem necessary for pre-service teachers to transform. These four assumptions include: reasoning quantitatively, recognizing ratios as measurement, misconceptions about the concept of ratio and fraction, and the obstacle of linearity. Mezirow’s theory requires a disorienting dilemma in order to help individuals engage in rational discourse and critical reflection about previous assumptions. This study illustrates how four problems (Lemon/Lime problem, Dog/Cat problem, Housing problem and Track problem) were able to provide pre-service with a disorienting dilemma causing them to engage in rational discourse with the researcher and critically reflect on their previous assumptions in order to revise their strategies for solving these problems. By helping these pre-service teachers become aware of the assumptions they have about ratio, proportion and proportional reasoning through these disorienting dilemmas (i.e., thought-provoking problem solving tasks) they were able to think about proportional problems in new ways and make shifts in their proportional reasoning. This knowledge can be used to develop courses that can transform pre-service teachers’ understandings of ratio and proportion and enhance their proportional reasoning so that future teachers can ultimately improve their students’ learning.
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thankful for the encouragement and support he so freely offered me during this period of intensive study and research.
Chapter 1: Rationale

Teachers need a broad and deep understanding of the mathematics they teach (Ball & Cohen, 1999; Ma, 1999). Teachers’ content knowledge influences the mathematics that occurs in their classroom through the tasks they decide to use and the type of discussions they encourage. Teachers may use their content knowledge in ways that other professionals do not (e.g., using their mathematical knowledge when they ask students questions and provide feedback (National Research Council [NRC], 2001)). In general, “No one questions the idea that what a teacher knows is one of the most important influences on what is done in classrooms and ultimately on what students learn” (Fennema & Franke, 1992, p. 147).

If the United States wants to compete academically and economically with the rest of the world, it needs to provide its students with the best mathematics education possible (Organization for Economic Co-operation and Development [OECD], 2003; Stigler, Gonzales, Kawanaka, Knoll, & Serrano, 1999). This requires teachers who are mathematically proficient in all areas of mathematics. Teachers need, at a minimum, an understanding of the mathematical concepts that they are to teach (Simon & Blume, 1994; Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998; Thompson & Thompson, 1994). Teachers need to know how mathematics across grade levels is connected and consider how they might select and sequence tasks in ways that foster the development of key mathematical ideas (NRC, 2001; Stugler & Hiebert, 1999).

Limitations in Teacher Knowledge

In many states, both elementary certified and secondary certified teachers are permitted to teach middle school. Therefore, it makes sense to investigate both elementary and secondary teachers’ conceptions of the mathematical topics that are taught in middle school. Research has documented that pre-service elementary teachers often have procedural skill but limited conceptual knowledge of mathematics (Cramer & Lesh, 1988). Even though secondary mathematics majors take more advanced mathematics courses, research has indicated that the
knowledge they have of elementary mathematics topics is also limited (Ball, 1988; Even, 1993; Even & Tirosh, 1995; Wilson, 1994).

Some areas of mathematics merit particular attention by teachers, researchers, and teacher educators. The ability to reason multiplicatively is fundamental to students’ mathematical development (Harel & Confrey, 1994); therefore, teachers need an understanding of how to reason in multiplicative situations (Post, Harel, Behr, & Lesh, 1988; Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998). However, research has indicated that many teachers lack the ability to reason multiplicatively in ways that adequately advance their students’ mathematical proficiency (Graeber, Tirosh & Glover, 1989; Harel & Behr, 1995; Harel, Behr, Lesh, & Post, 1994; Post, Harel, Behr & Lesh, 1988; Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998; NRC, 2001). Teachers have a difficult time acquiring the in-depth understanding of multiplicative relationships required for them to implement adequate instruction (Simon & Blume, 1994; Sowder & Philipp, 1995; Sowder & Schappelle, 1995). Teacher education programs should include a focus on promoting development of teachers’ understanding of multiplicative situations (Harel & Behr, 1995; Post, Harel, Behr & Lesh, 1988; Sowder, 1995; Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998).

Ratio and proportion draw on multiplicative thinking and constitute an area of mathematics that appears to be problematic to teachers at all levels (elementary, middle and secondary) (Cramer, Post, & Currier, 1993; Heinz, 2000; Post, Harel, Behr, & Lesh, 1991; Simon & Blume, 1994; Smith, Silver, Leinhardt, & Hillen, 2003; Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998). Teachers who are unable to reason proportionally often exhibit the same characteristics and misconceptions (e.g., misusing additive reasoning) as students (Cramer, Post, & Currier, 1993; Simon & Blume, 1994). Even adults who can reason proportionally tend to have a procedural understanding of solving problems that involve ratio. The use of a cross-multiplication algorithm to solve proportions for a missing value often has little meaning attached to it. Lesh, Post, and Behr (1988) claim students poorly understand cross-multiplication because it
is not a naturally generated solution strategy and is taught in a way that avoids proportional reasoning instead of facilitating it. Research has shown that teachers who appear to reason proportionally in some situations exhibit limited understanding when they encounter more complex or unfamiliar proportionality problems (Post et al., 1991; Simon & Blume, 1994; Smith, Stein, Silver, Hillen, & Heffernan, 2001).

**Central Mathematical Idea: Proportional Reasoning**

In the past 5 decades there has been much discussion and research on ratio and proportion (Inhelder & Piaget, 1958; Karplus, Karplus & Wollman, 1974; Lamon, 1999; Lesh, Post & Behr, 1988; Noelting, 1980; Thompson, 1994; Vergnaud, 1983). The National Council of Teachers of Mathematics (NCTM, 1989, 2000) acknowledges that reasoning proportionally is “of great importance that merits whatever time and effort must be expanded to assure its careful development” (NCTM, 2000, p. 82). Proportional reasoning has been considered a cornerstone to achievement in upper mathematics and is the precursor to algebraic thinking (Lamon, 1999). It is also used in problem solving outside mathematics. Lesh, Post, and Behr (1988) used Piaget’s work to describe the importance of proportional reasoning. “In the psychology of human learning, proportional reasoning is widely recognized as a capability which ushers in a significant conceptual shift from concrete operational levels of thought to formal operational levels of thought” (Lesh, Post, & Behr, 1988, p. 101). This shift in understanding can lead to advanced mathematical thinking.

Proportional reasoning is important because it brings together the mathematics explored in the elementary grades and opens the door to high school mathematics and beyond. Lesh, Post, and Behr (1988) suggested that ratio and proportion are the “capstone of children’s elementary school arithmetic, and the cornerstone of all that is to follow” (p. 94-95). Proportional reasoning lays the groundwork for advanced mathematical topics including slope, probability, statistics, trigonometry and calculus. However, understanding of these advanced topics is difficult when curricula focus on procedural tasks (NRC, 2001) and “fail to develop children’s understanding of
ratio comparisons and move directly to formal procedures for solving missing-value proportion problems” (p. 417). Given the importance of proportional reasoning in the school curriculum, I turn now to a careful discussion of its meaning.

**Defining Proportional Reasoning**

Proportional reasoning merits a clear definition. I define *proportional reasoning* using a social-constructivist lens (Vygotsky, 1978) to be a cognitive process that is a mathematical form of reasoning about ratio and proportion based on experiences (Sowder, Phillipp, Armstrong, & Schappelle, 1998). For proportional reasoning, the mathematical entities about which one reasons are ratio and proportions. Lesh, Post and Behr (1988) characterize proportional reasoning as “the mental assimilation and synthesis of the various complements of [the ratios in a proportion]” (p. 93), including mathematical reasoning involving multiplicative comparisons, inferences, and prediction.

When researching a mental process it becomes necessary to identify aspects of the concept that can be observed by others. Others have used definitions that imply that in order to have proportional reasoning, one must be able not only to find the missing value or infer inequality between ratios but also to distinguish between problems that involve proportions and those that do not (Behr et al., 1992; Karplus, Pulos, & Stage, 1983). In particular, one must be able to determine if a situation is multiplicative or additive in nature. The Common Core State Standards for Mathematics recommended that in order for students to be proficient in ratio and proportion they need to be able to recognize the constant multiplicative relationship of a proportional situation in tables, graphs, equations, diagrams and verbal descriptions (The Common Core State Standards for Mathematics [CCSSI], 2010).

Evidence of proportional reasoning in previous studies was often determined by the ability to solve certain types of problems (e.g., comparison problems and missing value problems) (Fruedenthal, 1983; Lamon, 1993; Vergnaud, 1988). Additionally, Lamon (2007) stated that proportional reasoning involved:
Supplying reason in support of claims made about the structural relationships among four quantities in a context simultaneously involving covariance of quantities and invariance of ratios or products; this would consist of the ability to discern a multiplicative relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities. (p. 638)

The CCSSI (2010) has suggested the following as guidelines for multiplicative relationships: the ability to solve multistep problems, percent problems, problems that use the context of sale discounts, interest, taxes, tips, as well as scale drawings relating similar objects, graphing proportional relationships, understanding the unit rate as a measure of slope and the ability to distinguish proportional relationships from other relationships. I will provide more detail on previous research done on these areas in chapter 3.

Based on these characterizations, I argue that proportional reasoning requires the ability to solve problems that involve generating missing components, comparing different ratios, and distinguishing between additive and multiplicative relationships, as well as to provide reasons for strategies within the context of problems. If students are to be mathematically proficient in proportional reasoning, then teachers will need to have an understanding of how these contexts and situations are connected to other topics in mathematics. Teachers will need to understand how ratio and proportion are related to other mathematical topics such as slope and rate of change.

**Teachers Need Opportunities to Become Proportional Reasoners**

In order for students to develop ability to reason proportionally, teachers need to have an extensive knowledge of proportionality and be proportional reasoners themselves. By considering teachers’ opportunities to develop their capacity as proportional reasoners, we begin to understand why limitations in their knowledge exist. For example, the K-12 school experience that most pre-service teachers have with proportional reasoning may be limited to a few days of instruction. Sowder, Philipp, Armstrong and Schappelle (1998) characterized the treatment of
ratio in the curriculum as brief, impoverished and lacking in conceptual discussion of the topic. Successful proportional reasoning in school is typically defined as correctly solving missing value problems using taught procedures (e.g., cross multiplication) without necessarily indicating a student’s proportional reasoning. Curriculum materials usually focus on the procedural tasks of setting up proportions and then cross multiply. “Many school mathematics programs fail to develop children’s understanding of ratio comparisons and move directly to formal procedures for solving missing-value proportion problems” (NRC, 2001, p. 417). Teachers are not expected to make sense of or understand alternate strategies; but rather, to be able to use algorithms fluently (Conference Board of the Mathematical Sciences [CBMS], 2001). In addition, the topic of ratio is typically treated in isolation, not connected to other subjects or topics and lacking a variety of representations and contexts.

Teachers’ university experiences are often no different. Sowder, Bezuk and Sowder (1993) examined proportional reasoning in textbooks for teachers’ mathematics courses. Most of the textbooks contained a chapter on rational numbers that: defined proportion as ordered pairs \((a, b)\); presented discrete and continuous models of proportional situations, definitions, algorithms; and stated properties with exercises provided as opportunities to practice algorithms. The researchers claim the presentation of rational numbers is mechanical and done in a superficial manner lacking the depth needed for teachers to gain deeper conceptual understanding of the topic beyond procedural manipulation (Sowder, Bezuk, & Sowder, 1993). These textbooks leave pre-service teachers with the same proportional reasoning abilities they had when they entered college, resulting in many teachers (and other adults) who cannot reason proportionally (Lamon, 1999). How then can these teachers develop the variety of rich representations needed to offer instruction that helps students move from using informal strategies to expressing and understanding proportional relationships in algebraic terms?

Although secondary teachers complete numerous mathematics courses, few of those courses focus specifically on proportional reasoning. In addition, preparation for teachers of the
middle grades is often overlooked. The number of teacher preparation programs offering courses specifically targeted for middle-grade teachers is small (CBMS, 2001; 2012). If teachers are to meet the demands of curricula and become mathematically proficient, they need opportunities to develop their capacity as proportional reasoners. This capacity involves making appropriate transitions in their thinking in order to develop proportional reasoning.

**Developing Proportional Reasoning Using Social Constructivism**

This definition of proportional reasoning reflects a social cultural theory of learning that is the basis for this study. A key principle of constructivism is that meaning is actively constructed by learners (Airasian & Walsh, 1997; Piaget, 1926; Schunk, 2009; von Glaserfeld, 1995; Vygotsky, 1978). The theory of social constructivism suggests that knowledge is a construction by individuals in the context of social interaction (Sexton & Griffin, 1997). Schunk (2009) described social constructivism based on Vygotsky’s socio-cultural theory and Piaget’s epistemology of “constructivism” as a “constructivist theory that emphasizes the social environment as a facilitator of development and learning” (p. 242).

Social constructivism emphasizes the role of the other in the learning process. Concepts that might otherwise not be discovered on one’s own until later can be learned with assistance from another individual who already has developed those understandings. One of the most misinterpreted concepts of Vygotsky’s theory is the Zone of Proximal Development (ZPD), which is often viewed in a limited way that emphasizes the interpersonal at the expense of the individual. (Tudge & Scrimsher, 2003). Berk and Winsler’s text (1995) is an example of using ZPD as synonymous with “scaffolding,” focusing on role of the more competent other (teacher) to provide assistance to advance the child’s current thinking. This interpretation of ZPD misses what the child brings to the interaction (Griffin & Cole, 1999). Part of the confusion with Vygotsky’s concept of ZPD comes from the translation of the Russian word *obuchenie* as instruction, teaching or learning, whereas in fact the word connotes both teaching and learning (van der Veer & Valsiner, 1991; Wheeler & Unbegaun, 1984). This bi-directional translation of
obuchenie allows us to make better sense of Vygotsky’s position that “teaching/learning” occurs long before the child goes to school.

In this study, I will emphasize the proportional reasoning that the pre-service teachers bring to the classroom environment. By learning about their current understanding of proportional reasoning, I can better create an environment that supports their further development of the concept. Using both social constructivism and examples from empirical research that supports learning environments in which teachers supports the cognitive development of their students (e.g., Cognitively Guided Instruction, Carpenter, Fennema, Peterson, Chiang, & Loef, 1989), I will study the proportional reasoning schemes of pre-service teachers using a model of developmental transitions, as described below.

In Developing Essential Understanding of Ratio, Proportions and Proportional Reasoning, Lobato and Ellis (2010) described how learning to reason proportionally occurs “slowly, over time” (p. 61). They cast changes in proportional reasoning as transitions or “shifts” in students’ thinking and these shifts can serve as tools to “evaluate [a] student’s current thinking” (p. 61). Shifts in proportional reasoning can also be used as a guide to scaffolding. Gay (2000) defined scaffolding as a process of bridging, contextualizing, and using students’ prior knowledge base to connect to new experiences. By understanding the nature of a pre-service teacher’s current proportional reasoning, a teacher educator can scaffold their further development of proportional reasoning. The teacher educator should help these pre-service teachers understand the shifts that their future students need to make in order to become proficient proportional reasoners. The pre-service teachers need to develop their proportional reasoning skills so that they can scaffold their own students in their future classroom. Knowing pre-service teachers’ current proportional reasoning helps teacher educators scaffold their learning.
Shifts as a Model for Proportional Reasoning Development

Lobato and Ellis (2010) suggested that one needs to make the following transitions or shifts in his or her thinking in order to develop proportional reasoning: (1) from reasoning with one quantity to reasoning with two quantities, (2) from reasoning additively to reasoning multiplicatively, (3) from reasoning with composed-units to reasoning with multiplicative comparisons, and (4) from iterating a composed unit to reasoning with a set of infinitely many equivalent ratios. Lobato and Ellis presented them as a hierarchy in K-12 students’ development of proportional reasoning, and I want to determine whether they are a useful model for describing pre-service teachers’ proportional reasoning. The shifts might be the key to scaffolding pre-service teachers’ proportional reasoning.

Shift 1: From reasoning with one quantity to reasoning with two quantities.
Reasoning with two quantities involves making the transition from focusing on one quantity or unit in the problem (e.g., the size of the glass) to realizing that two quantities or units are involved in the relationship. For example, the orange juice mixture contained in a glass involves both orange concentrate and water, and it is the amount of each that determines the “orangey” taste (Harel et al., 1994). The attention to two quantities of a ratio is often an overlooked aspect of reasoning with ratio. Ratio reasoning must precede proportional reasoning given the fact that a proportion is an equivalence statement of two ratios. Coordinating two quantities presents a challenge after students have been focusing only on problems with one quantity. This change from univariate to multivariate reasoning involves a cognitive leap for students (Lamon, 1999). Making the shift from univariate reasoning to multivariate reasoning is imperative in developing proportional reasoning because, by the very definition, ratio involves reasoning with two or more variables that are dependent on each other.

Shift 2: From reasoning additively to reasoning multiplicatively. The second transition requires a multiplicative comparison of two quantities or the joining/composing of two quantities such that they maintain a multiplicative relationship. Multiplicative comparisons
answer the question “how many times larger or what fraction/part of one is the other?” (Lobato & Ellis, 2010, p. 18). In contrast, an additive comparison answers the question “how much larger or smaller is one than the other?” (Lobato & Ellis, 2010, p. 18). A ratio is a multiplicative comparison of two numbers, or it is sometimes defined as a comparison of two numbers using division and expressed as $a/b$. Lobato and Ellis (2010) claimed that many published curricula present ratio as a writing task, although it is really a cognitive task. For example, ratio is not about writing 3 boys to 4 girls in symbolic form (3:4), but it involves reasoning about two quantities in which “the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor” (Lobato & Ellis, 2010, p. 11). As students begin to recognize the multiplicative relationship between quantities, they begin to further their ratio reasoning on their way to formal proportional reasoning.

**Shift 3: From reasoning with composed units to reasoning with multiplicative comparisons.** The transition from composed unit strategies to multiplicative comparisons represents a step toward more abstract proportional reasoning. A composed unit is the joining of two quantities to create a new unit. Another way to form a ratio is to create a multiplicative comparison of two quantities in order to answer the question *how many times greater is one thing than another?* Pre-ratio reasoning involves creating equivalent ratios based on iterating and partitioning a composed unit in order to create a family of equivalent ratios (e.g., $2:3 \rightarrow 4:6 \rightarrow 8:12$) (Lesh, Post, & Behr, 1988). Those who do not have strong multiplicative reasoning skills may build up composed units to solve problems. This is often viewed as an extension of additive reasoning. However, when problems become more complicated, a student needs to realize that ratio represents a multiplicative comparison that can be used to solve problems more efficiently. In this case, the student’s reasoning moves from repeating a composed unit to anticipating the number of groups needed. The recognition that maintaining a proportional relationship involves multiplying or dividing each quantity by the same factor extends the student’s understanding of ratio. This shift represents the transition to a formal understanding of proportions as a student.
recognizes the invariance of ratio and can distinguish between situations that involve proportional reasoning and those that do not.

**Shift 4: From iterating a composed unit to reasoning with a set of infinitely many equivalent ratios.** The fourth shift occurs when a student makes the transition from iterating easily composed units (e.g., doubling and halving numbers) to creating infinite sets of equivalent ratios. Lobato and Ellis (2010) distinguished between the terms ratio and rate, not in terms of the context of the situation (some define rate as the comparison of two different units of measure and ratio as the comparison of two numbers with the same unit of measure) but in terms of how one conceives the concept. Thompson (1994) referred to a rate as a set of infinitely many equivalent ratios. This definition of rate expands understanding from a contextualized notion to a conceptual understanding of ratio, and helps connect ratio to bigger concepts in mathematics such as slope in a linear equation and rate of change in calculus. The use of rate by Thompson as a more sophisticated understanding of ratio leads students to reason and justify solutions, which allows students to make “sense” of mathematics (Lobato & Ellis, 2010). The final transition permits one to reason formally about proportions and apply their understanding to important concepts in more advanced mathematics including scalar problems, rate of change problems from Calculus, and everyday multiplicative reasoning situations.

**Why study proportional reasoning in terms of shifts?** The shifts might be the key to scaffolding pre-service teachers’ proportional reasoning. Lobato and Ellis (2010) presented the shifts as a hierarchy of transitions in thinking about proportion. If these transformations of thinking are necessary for students to reason formally about proportions, then we should make sure that prospective teachers have made these shifts. Previous research has suggested that pre-service teachers do not adequately reason proportionally (Post et al., 1991; Simon & Blume, 1994; Smith, et al., 2001), indicating that they might not have made these shifts in their thinking. This research indicates that we need to develop models to help teacher educators scaffold pre-service teachers’ understanding in order to become formal proportional reasoners.
The model of shifts presented by Lobato and Ellis (2010) may provide a useful tool for teacher educators. By learning more about prospective teachers’ current proportional reasoning, teacher educators can provide appropriate opportunities to learn formal proportional reasoning needed to become effective mathematics teachers. Hillen’s (2005) research indicated that pre-service teachers could reason formally about proportions when provided with experiences that elicit ideas similar to those expressed in the shifts. In her research, she compared two groups of pre-service teachers (treatment and non-treatment) using a pre/post test to determine if her practice-based mathematics methods course allowed teachers to construct their own understanding about proportional reasoning. The class focused on many of the same developmental transitions that exist within the shifts. For example, students were exposed to a variety of different strategies including unit strategies and multiplicative comparisons similar to Lobato and Ellis’s (2010) shift 3 in which one transitions from composed unit strategies to multiplicative comparisons. The success of her course suggests that such shifts need to occur in order for pre-service teachers to reason formally about proportions.

This study of prospective teachers’ proportional reasoning can be the building block for courses that will scaffold pre-service teachers’ understanding of an important concept in mathematics: proportional reasoning. If teacher educators are going to contribute to prospective teachers’ understanding, then they must first understand the proportional reasoning of pre-service teachers. This study is the first step in helping teacher educators in creating an environment in which prospective teachers can understand the nature of their existing proportional reasoning and make the needed shifts to become formal proportional reasoners. The model suggested by Lobato and Ellis (2010) allows us to determine where prospective teachers fit along a continuum of developmental transitions in proportional reasoning. We can then use this model as a way to determine which shifts we need to focus on in teacher preparation classes.

In this study I will describe a group of pre-service teachers’ proportional reasoning. As noted by Thompson and Thompson (1994):
There is little research on teachers’ schemes and on teachers’ abilities to reason conceptually. The little there is tends to emphasize the mathematics that teachers cannot do or cannot explain and not actual schemes by which they reason. It is essential that there be more research on the variety of teachers’ actual scheme in regard to central mathematical ideas if the long-term problem of preparing conceptually-oriented mathematics teachers is to be addressed productively. (p. 21-22)

This study seeks to understand pre-service elementary and secondary teachers’ proportional reasoning in terms of a developmental model of shifts.

The Study

The purpose of this study is to examine the proportional reasoning of pre-service teachers at the beginning of their teacher preparation program using the developmental shifts described by Lobato and Ellis (2010). I will determine how these potential teachers’ proportional reasoning is consistent with this hierarchy of shifts. In other words, are the shifts evident and hierarchical when one looks at adult proportional reasoning? This is the first step to determining whether the model proposed by Lobato and Ellis is useful as a scaffolding tool.

If future teachers, and subsequently their students, are to develop rich proportional reasoning abilities, pre-service teachers must be encouraged to make sense of proportional situations at both levels: informal and formal. They need to become mathematically proficient regarding this topic. In order to develop teacher education that enables pre-service teachers to be proficient proportional reasoners, we must first understand the nature of their existing knowledge of the topic. Learning more about pre-service teachers’ existing knowledge would be useful for improving courses that facilitate the development of proportional reasoning. This study wishes to answer the question:

To what extent is the nature of pre-service teachers’ proportional reasoning near the beginning of their formal teacher preparation [in a college-level program] consistent with the shifts described by Lobato and Ellis (2010)?
In addition, knowing the nature of pre-service teachers’ proportional reasoning would enable teacher educators to provide their students with opportunities to develop their proportional reasoning through a variety of rich representations that they will need in order to offer effective instruction to their future students. These prospective teachers’ ability to reason proportionally will help shift their students from using informal strategies to expressing proportional relationships in algebraic terms.
Chapter 2: A Framework for Studying Pre-Service Teachers’ Proportional Reasoning

In *Developing Essential Understanding of Ratio, Proportions and Proportional Reasoning*, Lobato and Ellis (2010) considered the question: “what is essential for teachers of mathematics in grades 6-8 to know about ratio, proportion and proportional reasoning to be effective in the classroom?” (p. 3). They drew on their personal experience, expertise of colleagues in mathematics and mathematics education, professional development, curricula materials, and research from mathematics teaching and learning to answer the question. They described how learning to reason proportionally occurs “slowly, over time” (p. 61). They cast changes in proportional reasoning as transitions or “shifts” in students’ thinking and a tool to “evaluate [a] student’s current thinking” (p. 61). Lobato and Ellis specified four such shifts:

“Shift 1: From one quantity to two” (p. 61)

“Shift 2: From additive to multiplicative comparisons” (p. 63)

“Shift 3: From composed-unit strategies to multiplicative comparisons” (p. 69)

“Shift 4: From iterating a composed unit to creating many equivalent ratios” (p. 71)

In addition to the four shifts, Lobato and Ellis (2010) described the ten Essential Understandings (EU), which describe what teachers need to know if they can expect their students to learn proportional reasoning:

1. Reasoning with ratio involves attending to and coordinating two quantities.

2. A ratio is a multiplicative comparison of two quantities or it is a joining of two quantities in a composed unit.

3. Forming a ratio as a measure of a real-world attribute involves isolating that attribute from other attributes and understanding the effect of changing each quantity on the attribute of interest.

4. A number of mathematical connections link ratio and fraction:
a. Ratios are often expressed in fraction notation, although ratios and fractions do not have identical meaning.

b. Ratios are often used to make “part-part” comparison, but fractions are not.

c. Ratios and fractions can be thought of as overlapping sets.

d. Ratios can often be meaningfully reinterpreted as fractions.

5. Ratios can be meaningfully reinterpreted as quotients.

6. A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change.

7. Proportional reasoning is complex and involves understanding that:
   a. Equivalent ratios can be created by iterating and/or partitioning a composed unit;
   b. If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and
   c. The two types of ratios – composed units and multiplicative comparisons – are related.

8. A rate is a set of infinitely many equivalent ratios.

9. Several ways of reasoning, all grounded in sense making, can be generalized into algorithms for solving proportion problems.

10. Superficial cues present in the context of a problem do not provide sufficient evidence of proportional relationships between quantities.

   The EUs are knowledge that teachers should have in order to help their students achieve the shifts. In particular, teachers should be able to use these EUs to see how their students’ reasoning fits into the shifts. In this spirit, we might be able to use these EUs to see how the pre-service teachers’ proportional reasoning fit into the shifts.

   Although Lobato and Ellis articulate both shifts and EUs, they did not explicitly link the two. However, they did have references to the EUs that are interspersed in the discussion of the
shifts. The following diagram illustrates how I believe the shifts and EUs fit along a continuum from pre-proportional reasoning to formal proportional reasoning. I intend to use the diagram as a way to analyze pre-service teachers’ responses to tasks designed to elicit evidence about the nature of their current understandings of proportional reasoning. I believe that the EUs may be able to provide explanations of why certain shifts in a pre-service teacher’s proportional reasoning have or have not occurred.

![Diagram of Continuum of Shifts and Essential Understandings](image)

**Figure 2.1: Continuum of Shifts and Essential Understandings fit together**

The diagram in Figure 2.1 represents a continuum of proportional reasoning. The left side represents a pre-proportional reasoning phase in which students do not recognize quantities as proportional to each other, and the right side represents the ability to use formal proportional reasoning. Lobato and Ellis (2010) suggested that one needs to make the following transitions or shifts in their thinking in order to develop proportional reasoning: (1) from reasoning with one
quantity to reasoning with two quantities, (2) from reasoning additively to reasoning multiplicatively, (3) from reasoning with composed-units to reasoning with multiplicative comparisons, (4) from reasoning with multiplicative comparisons to reasoning with a set of infinitely many equivalent ratios. In chapter 3, I will explore how these shifts are reflected in other research literature. In this chapter, I will explain how Figure 2.1 illustrates these shifts along a continuum; describe in more detail the transitions in thinking that each shift represents, and present research supporting these shifts in understanding as a basis for formal proportional reasoning. In addition, I will discuss how the EUs might help to further explain these transitions.

**Shift 1: From reasoning with one quantity to reasoning with two quantities**

Reasoning with two quantities involves making the transition from focusing on one quantity or unit in the problem to realizing that two quantities or units are involved in the relationship. This may include reasoning about whether orange juice poured from a container into two different sized glasses will have the same “taste of orange” (Harel, Behr, Lesh, & Post, 1994). Some may believe the larger glass of orange juice will have a stronger orange taste than the smaller glass because it is a bigger glass, others may believe the opposite is true. Shift 1 involves making the transition from focusing on one quantity or unit in the problem (i.e., the size of the glass) to realizing that two quantities or units are involved in the relationship. For example, the orange juice mixture contained in the glass involves both orange concentrate and water, and it is the amount of each that determines the orangey taste.

Looking along the continuum EU1 and shift 1 coincide. EU1 explains ratio reasoning as paying attention to two quantities. The attention to two quantities of a ratio is often an overlooked aspect of reasoning with ratio. Ratio reasoning must precede proportional reasoning given the fact that a proportion is a statement of equivalence of two ratios. Coordinating two quantities becomes important when students have been focusing only on problems with one quantity. Reasoning with a focus on one quantity usually precedes the ability to reason with two. This change from univariate to multivariate reasoning involves a cognitive leap for students
(Lamon, 1999). Harel and colleagues (1994) asked 6th graders about the constancy of taste in various sized samples of orange juice from the same container, exemplifying the conceptual challenges of this type of reasoning for students when first presented with proportional problems. Making the shift from univariate reasoning to multivariate reasoning is imperative in developing ratio reasoning, because by the very definition ratio involves reasoning with two or more variables that are dependent on each other.

Along the same transition as shift 1, we find EU3 in which real world ratio-as-measure involves two processes: isolating the attribute and understanding the effects that changing each quantity of that attribute has on the ratio. This seems to occur as part of shift one but before shift two for several reasons. First, isolating attributes is an important aspect of ratio-as-measure (Simon & Blume, 1994) as well as understanding the effects of change that occurs between these attributes. This means that students must recognize that there are two attributes in the problem, and they are dependent on each other. EU3 does not specify that this relationship is multiplicative; it does mention that one recognizes there are two attributes, and these can be isolated in order to reason about the effects. “Ratio-as-measure tasks involve aspects of modeling, like focusing on one quantitative relationship in a complex situation involving several relationships” (Lobato & Thanheiser, 2002, p. 164). This isolation and recognition of multiple units and their relationship is part of shift 1 but is a pre-cursor to shift 2, which recognizes the multiplicative relationship of ratio. An example of students’ recognition of two quantities involved in a ratio can be found in Lobato (2008). She observed that the majority of high school students had difficulty determining the steepness of a ramp when changing one quantity (i.e., the base or the height) at a time. Most students could reason correctly that an increase in height made the ramp steeper, but when there was a change in the base, students could not determine the effect on the steepness.
**Shift 2: From reasoning additively to reasoning multiplicatively**

The second shift involves the transition from additive reasoning to multiplicative reasoning. Ratio requires a multiplicative comparison of two quantities or the joining/composing of two quantities such that they maintain a multiplicative relationship. One must be able to distinguish between situations that involve additive reasoning or multiplicative reasoning.

In Figure 1, shift 2 and EU2 appear in the same location on the continuum, because both focus on multiplicative reasoning. EU2 states “a ratio is a multiplicative comparison of two quantities or the joining of two quantities in a composed unit” (Lobato & Ellis, 2010, p. 18). The transition from additive to multiplicative thinking depends on understanding the concept of ratio (Sowder, Armstrong, Lamon, Simon, Sowder & Thompson, 1998). However, authors of textbooks tend to focus only on descriptive ratio, and students reduce the concept of ratio to simply the comparison of two numbers (Clark, Berenson, & Cavey, 2003).

Multiplicative comparisons answer the question “how many times larger or what fraction/part of one is to the other?” (Lobato & Ellis, 2010, p. 18). In contrast, an additive comparison answers the question “how much larger or smaller is one than the other?” (Lobato & Ellis, 2010, p. 18). Ratio as a composed unit is the joining of quantities to form a new unit, evidenced by iterating or partitioning a composed unit (for example 10:4 → 30:12 → 2.5:1). A ratio is often defined as a multiplicative comparison of two numbers or sometimes as a comparison of two numbers using division and expressed as a fraction. Lobato and Ellis (2010) suggested that school curriculum presents ratio as a writing task rather than a cognitive task. If students are presented only with situations that involve writing a descriptive ratio, they will be left without the understanding that ratio involves multiplication. This incomplete recognition of the multiplicative nature of ratio may cause a stumbling block in achieving shift 2 and their development of mathematical proficiency in proportional reasoning.

Before individuals develop a formal understanding of proportions they need to recognize the invariance of ratio and know when to apply proportional reasoning to situations. In Figure 1,
EU6 is on the same step as shift 2. EU6 distinguishes proportion as an invariant multiplicative relationship of equality between two ratios. Individuals who have not yet mentally formed ratios as mathematical objects may interpret a template for proportions used in many textbooks as simply insertion of whole numbers, not as the equality of two ratios. If individuals come to understand a proportion only as a statement of simplified fractions in which only whole number understanding is involved, they will not proficiently reason proportionally. Using fractional quantities as part of the ratio will help determine whether pre-service teachers’ recognition of ratios are limited to whole number interpretations. For example 10 cm: 4 seconds can be interpreted as 10/4 cm: 1 second.

EU6 also includes conceptual understanding as students make connections between fractions, ratios and proportional reasoning. As students reinterpret problems in terms of composed units they employ different strategies to solve problems leading to flexibility in their understanding of ratio. Before they develop a formal understanding of proportions they need to recognize the invariance of ratio as well as when to apply proportional reasoning to situations. Lamon’s (1993, 1995) research suggested that students’ understanding of multiple interpretations of ratio allows them to make sense of proportions.

Shift 3: From reasoning with composed units to reasoning with a multiplicative comparison

Shift 3 is the transition from composed unit strategies to multiplicative comparisons. The transition represents a step toward more abstract proportional reasoning. Pre-ratio reasoning (Lesh, Post, & Behr, 1988) involves creating equivalent ratios based on iterating and partitioning a composed unit in order to create a family of equivalent ratios. One may use composed units to build up ratios to solve problems, some view this as only an extension of additive reasoning. It is a more intuitive way to work with ratios, especially for those who do not have strong multiplicative reasoning skills. Eventually one might realize that ratio represents a multiplicative comparison and use multiplication as a more efficient way to solve ratio problems. Their reasoning moves from repeating a composed unit to anticipating the number of groups needed.
This recognition involves maintaining the proportional relationship by multiplying or dividing each quantity by the same factor and includes extending this understanding to fractional factors. It is important that students can relate the two types of ratios (i.e., composed units and multiplicative comparisons) by asking questions like “what part of ___ is ___?” and “how many times greater is ___ than ___?” Questions such as these may help students make this distinction.

EU7 coincides with shift 3 since it directly involves proportional reasoning. EU7 states that proportional reasoning involves several types of understanding. First, “equivalent ratios can be created by iterating and/or partitioning a composed unit” (Lobato & Ellis, 2010, p. 36). Some view iterating composed units as a sophisticated form of additive reasoning, but it allows students to maintain the context of the problem in their solution (Clark et al., 2003). This might lead to a productive disposition in mathematics. Productive disposition is a belief that mathematics is sensible, useful and worthwhile (NRC, 2001) and is one of the strands of developing mathematical proficiency.

Second, the covariance of the proportional structure requires that “if one quantity of a ratio is multiplied or divided by a factor than the other quantity must also be multiplied or divided by the same factor to maintain proportional relationship” (Lobato & Ellis, 2010, p. 36). The use of multiplicative comparisons may lead to the development of procedural fluency when solving problems (NRC, 2001).

Third, there are two types of related ratios, composed units and multiplicative comparisons. Lobato and Ellis (2010) claimed, “using multiplicative comparisons is a powerful proportional reasoning strategy” (p. 40). Being able to connect composed units and multiplicative comparisons may also involve unit ratios. Steffe and Cobb (1988) found that unitizing or the ability to construct unit ratios to reinterpret a situation appeared to be critical in the development of more sophisticated proportional reasoning.

Figure 1 shows EU4 and EU5 on the same step as shift 3. Both deal with reinterpreting ratios and the use of alternative strategies to solve problems. EU4 discusses the reinterpretation
of ratio as fractions, and EU5 looks as ratios as quotients. EU4 discusses the important connections between ratios and fractions: ratios are often expressed \(a/b\) but do not have the same meaning, ratios can involve a part-part comparison, ratios and fractions are overlapping sets and ratios can be reinterpreted as fractions. The notation of \(a/b\) often confuses students’ understanding of the connection between ratio and proportion. Ratio can be part-part comparison whereas fraction is always a part-whole comparison. Fractions are also rational numbers, but ratios do not have to be rational, for example the ratio \(\pi/2\) is not rational. In addition, ratios can have a comparison to zero (i.e., 4:0) whereas this is not possible with fractions. Ratios can also involve more than 2 terms (i.e., 5: 3: 2). Therefore, ratios and fractions are intersecting sets of numbers (Clark et al., 2003). Despite the fact that ratios and fractions are not congruent, many ratios can be interpreted as a fraction. For example, the ratio of 2:5 can be interpreted as two-fifths as much \(____\) as \(____\) (e.g., a multiplicative comparison). A second way to interpret this ratio is by joining the two quantities and then partitioning or splitting them into equal parts (e.g., a composed unit) and saying “the amount of \(____\) is always \(2/5\) the amount of \(____\)” As students learn the distinction between ratio and fraction, multiplicative comparisons and composed units enable them to recognize the connection.

EU5 reinterprets ratios as quotients. Reinterpreting ratio as a quotient is useful for making sense of proportion. One interpretation of division is sharing, as in the amount of one quantity shared by the other quantity. The reinterpretation of ratio may be seen as a non-conventional strategy to solving problems with ratio. However, Clark and colleagues (2003) found that students who did not use fractional notation when solving missing value problems but used a quotient were more likely to maintain the context of the problem in their solution. The connections between quotients and ratio can help students develop their conceptual understanding of the strong multiplicative relationship that exists in advanced mathematics. The interpretation of ratio as a quotient may allow students to make sense of the ratios they are using to solve
problems. Both the fraction and quotient interpretations of ratio lead to reasoning with composed units and multiplicative comparisons needed for shift 3.

**Shift 4: From reasoning with a multiplicative comparison to reasoning with a set of infinitely many equivalent ratios**

The fourth shift occurs when one makes the transition from iterating easy composed units to creating infinite sets of equivalent ratios. Thompson (1994) distinguished between the terms ratio and rate, not in terms of the context that is being used (e.g., miles per hour) but in terms of how one conceives of the concept. Some define rate as the comparison of two different units of measure and ratio as the comparison of two numbers with the same unit of measure. A rate is used to determine a set of equivalent ratios. In this case the distinction between rate and ratio lies in a conceptual development of proportional reasoning in which rate is a more sophisticated understanding of the ratio concept. This definition of rate expands understanding from a contextualized notion to a conceptual understanding of ratio and helps connect ratio to big ideas such as slope in a linear equation and rate of change in calculus. The use of rate as a more sophisticated understanding of ratio leads students to reason and justify solutions, allowing them to make sense of mathematics. This final transition may permit one to reason more abstractly about proportions and apply his or her understanding to more advanced mathematical concepts, such as scalar problems, rate of change problems from Calculus and everyday multiplicative reasoning situations.

In figure 1, EU8 and shift 4 coincide because EU8 represents the final shift as the formation of “a set of infinitely many equivalent ratios” (Lobato & Ellis, 2010, p. 42). The treatment of proportion in many curricula, defined as a set of only two equivalent ratios, may leave students thinking that algorithmic procedures can be followed without involving reasoning. The conception of ratio as how one thinks about a ratio provides a language to distinguish between differences in levels of sophistication in pre-service teachers’ reasoning. For example, evidence that a pre-service teacher can use proportional reasoning to determine the amount of
blue and white paint needed for any quantity of a blue and white mixture of a certain ratio leads me to believe that they conceive a rate as a set of infinitely many equivalent ratios. This includes working with ratio that may involve more difficult numbers like fractions.

**Essential Understandings across the shifts**

In Figure 1, EU9 and EU10 appear outside the steps of the continuum. This is because these two EUs seem to be apparent throughout a student’s transition through the shifts. EU9 states that algorithms can be used to solve proportional problems. Algorithms are a general and efficient way to solve a class of problems, and cross-multiplication algorithm suspends sense making of problems as contexts. However, research has shown that algorithms such as cross-multiplication are frequently used without sense making, leading one to believe that a student has mastered proportional reasoning (Smith & Confrey, 1994). Alternative algorithms can also be generalized for any values such as the “unit ratio” method but can be used to develop associated understandings (Lobato & Ellis, 2010). The unit ratio method involves calculating the amount of a quantity per one of the second quantity, and it can generalize to other numbers.

EU10 involves the understanding that superficial cues from the context of a problem do not provide sufficient evidence of proportional relationships between quantities. Basically, EU10 illustrates that students look for cues in the 3 types of problems involving proportional reasoning. For example, a student reads the words “per”, “rate,” or “speed” in a problem and then assumes that he or she should apply the concept of proportions. Students often use what Schoenfeld (1985) called “cue-based” behavior as superficial cues from problems in order to determine what method to use to find a solution; this can lead to incorrect solutions and inappropriate applications of proportional reasoning. “Cue-based” behavior might be seen at any place along the continuum depending on the context of the problem. For this reason, EU9 and EU10 can help me explain pre-service teachers reasoning no matter where they might be along the continuum of the shifts. These two EUs provide reasons for the correct responses on questions that might be used to
determine a higher-level shift. More about this will be discussed in the data analysis section of Chapter 4.

The literature review will look at the research others have done on proportional reasoning and compare this work to the continuum proposed in this framework. Though others have also tried to explain the development of proportional reasoning in students, I will show how their work complements use of the shifts by Lobato and Ellis (2010) in explaining the development of proportional reasoning and how this framework may expand the understanding of previous research. I believe that this framework will be useful in characterizing pre-service teachers’ proportional reasoning because it will allow me to focus on certain aspects of their understanding. The continuum provides a basis for explaining their solutions to problems that involve ratio and proportion. For example, if a pre-service teacher uses additive reasoning to solve a ratio problem I might conclude that they have not made shift 2. This information can provide teacher educators with an idea of what they should focus on when preparing to teach pre-service teachers. If I know that many of my students do not understand that ratio is a multiplicative comparison of two quantities, I can make changes to the curriculum to provide them with the scaffolding they need to transition to formal proportional reasoning.
Chapter 3: Review of relevant literature

In this chapter I will review the literature on ratio and proportional reasoning and discuss how it relates to Lobato and Ellis’ (2010) shifts presented in the framework. I will begin by recognizing that much of the research on proportional reasoning occurred 20 or more years ago. A great deal of research on ratio and proportional reasoning was done in the eighties and nineties, but very little occurred in this century. In fact, the 2007 Second Handbook of Research on Mathematics Teaching and Learning chapter on rational numbers and proportional reasoning contained only 7 out of 140 references to work done in the 21st century. Lamon (2007) noted, “this crisis… stems from the complex nature of the mathematics, as well as from the difficulty of conducting such research” (p. 632). She also suggested that many adults, including middle school teachers and pre-service teachers struggle with the same misconceptions about ratio and proportional reasoning as do students. Lamon (2007) claimed these misconceptions “may be attributed to the fact that the mathematics curriculum has never appropriately addressed the multiplicative conceptual field involved in proportional reasoning, and teachers themselves have had the same school experiences as current students receive” (p. 633). Next, I will discuss what it means to reason proportionally.

Many scholars have studied what it means to reason proportionally and share the view that proportional reasoning is a long-term developmental process in which the understanding at one level forms a foundation for higher levels of understanding (Case, 1985, 1992; Hart, 1984, 1988; Inhelder & Piaget, 1958; Kieren, 1988, 1993; Lesh et al., 1988; Noelting, 1980). Lesh, Post and Behr (1988) described proportional reasoning as “the ability to generate successfully missing components [of proportional problems] regardless of the numerical aspects of the situation” (p. 93). As stated in Chapter 1, Lamon (2007) characterized proportional reasoning as: supplying reason in support of claims made about the structural relationships among four quantities in a context simultaneously involving covariance of quantities and invariance of ratios or products; this would consist of the ability to discern a multiplicative
relationship between two quantities as well as the ability to extend the same relationship to other pairs of quantities. (p. 638)

This requires them to be able to support their solutions by providing reasons for their answer in the context of the problem, not just as blind numbers. I believe that proportional reasoning requires not only the ability to generate missing components in solving proportional problems, comparing different ratios, and distinguishing between additive and multiplicative relationships, but also the ability to provide reasons for one’s strategies within the context of the problem.

Many researchers have tried to determine what abilities proportional reasoners should acquire. These include: solving a variety of problems (Carpenter, Gomez, Rousseau, Steinthorsdottir, Valentine, Wagner, 1999; Cramer, Post, & Currier, 1993; Heller, Ahlgren, Post, Behr, & Lesh, 1989; Karplus et al., 1983; Lamon, 1993; Noetling, 1980; Post, Behr, & Lesh, 1988; Vergnaud, 1988), discriminating proportional from non-proportional situations (CBMS, 2001; Cramer, Post, & Currier, 1993; Lamon, 1995; Sowder, Armstrong et al., 1998), and demonstrating an understanding of the mathematical relationships embedded in proportional situations (Cramer, Post, & Currier, 1993; Post, Behr, & Lesh, 1988). In the next section, I will discuss previous research on the various types of problems, the strategies that students used in solving these problems, and various learning trajectories for proportional reasoning. Additionally, I will discuss how this research is illustrated in Lobato and Ellis’ (2010) shifts and EUs. In the methodology, I will explain how I plan to address these skills in the problems I have chosen to use in the data collection instruments.

**Types of ratios involved in proportional reasoning**

Frueedenthal (1983) concluded that in any proportion between two measures there exist two multiplicative relationships: *within*-ratio and *between*-ratio. Measure spaces are interpreted as any two quantities related proportionally. For example, 4 cars can hold 12 kids. How many kids will 40 cars hold? The measure spaces are kids and cars. The multiplicative relationship *within* these measure spaces is the ratio of 12 kids: 4 cars or 3 kids per car; whereas the
multiplicative relationship *between* these measure spaces is 40 cars: 4 cars or 10 cars. Lamon (1993) used these terms in the opposite way. The *between* relationship is 4 cars: 12 kids or 3 and the *within* relationship is 4 cars: 40 cars or 10. Notice that in both of these situations the proportional relationship remains constant. In addition, the *within* relationship also defines a “functional ratio” and the *between* ratio is sometimes referred to as “scalar ratio” (Vergnaud, 1988).

A great deal of research has centered on *within* and *between* ratios used by students (Fruedenthal, 1983; Karplus et al., 1983; Lamon, 1993, 1999; Vergnaud, 1983). Most of these studies were inconclusive as to the preference of students in solving problems; neither *within* nor *between* appeared to be more natural. Students tended to use the method that produced integer ratios. Vergnaud (1983) concluded that *between* ratios were more frequent while Karplus and colleagues (1983) found that *within* ratios were preferred.

**Types of proportional reasoning problems**

Educational researchers have traditionally defined the domain of ratio and proportion in terms of two types of problems: *comparison* problems and *missing-value* problems. A *comparison* problem is one in which four values are given (*a*, *b*, *c*, and *d*) and the goal is to determine the order relation between the ratios *a/b* and *c/d*. For example, if at table A, 3 pizzas are to be shared with four friends and at table B, 4 pizzas are to be shared with seven friends, at which table does the person get more pizza? A *missing value* problem provides three of the four numbers in the proportion *a/b = c/d* and the goal is to find the missing value. For example, 3 balloons cost $2, how much would it cost for 24 balloons? I have used both of these types of problems in the data collection instruments that I will be using in this study (i.e., lemon/water problem and balloon problem in Appendix D).

As discussed in Chapter 1, CCSSI (2010) focused on different types of problems involving proportional reasoning, including many other situations and contexts of ratio and proportion. I have included these additional types of problems in my study of pre-service
teachers’ understanding of proportional reasoning. CCSSI (2010) recommended that proportional reasoning include multistep problems (I have purposely included problems that reflect this. For example, see Planet B problem in Appendix D), percent problems (basketball problem and sale problem in Appendix D), scale drawings (scalar problem and housing problem in Appendix D), recognition of proportional relationships and non-proportional relationships (track problem, and paint problem in Appendix D), understanding of unit rate (health food store problem and pizza problem in Appendix D) and use of multiple representations that identify a constant multiplicative relationship (health food store problem; Mr. Tall/Mr. Short problem in Appendix D). Depending on what classroom policy requires, the inclusion of these different aspects may contribute to a richer description of pre-service teacher proportional reasoning.

**Focus on student strategies**

Lamon (1994, 1996) focused on student strategies and used *unitizing* and *norming* in her studies about ratio and proportion. She defined *unitizing* as the construction of a reference unit or unit whole. I will call this a unit ratio. Lamon (1994) argued that the ability to unitize “appears to be critical to the development of increasingly sophisticated mathematical ideas” (p. 133). It makes sense that Lamon’s work is reflected in Lobato and Ellis’ (2010) EUs and shifts since her focus is on the development and understanding of proportional reasoning.

Lobato and Ellis (2010) used the idea of unit ratio in EU9 when they discussed ways to solve proportions so that they promote sense making for students as opposed to mechanical usage of an algorithm that often does not contain the context of the problem being solved. Lamon (1993) defined *norming* as the reinterpretation of a situation in terms of a composite unit. This can be used to determine the *scalar or functional* ratio in solving problems. Lobato and Ellis (2010) mentioned composed units as part of EU2 in which early ratio development involves forming composed units and iterating them to solve problems. Shift 3 later describes how this understanding is made more efficient with the use of multiplicative comparisons. Lamon (1993) described the use of composed units as a “building up” strategy that is less sophisticated than the
multiplicative comparison. Even less sophisticated is the use of the unit rate because it focuses only on single units or additive strategies. This seems to align with Lobato and Ellis’ (2010) shift 2: the transition from additive to multiplicative reasoning as well as shift 3 that recognizes that a transition from composed units to multiplicative comparisons is needed to achieve formal proportional reasoning. Therefore, Lamon’s work is reflected in the shifts of Lobato and Ellis (2010); however the shifts seem to include two more transitions in thinking, that of recognizing two quantities of a ratio and recognizing rate as a set of infinitely equivalent ratios. Lobato and Ellis’ (2010) ideas seem to expand Lamon’s work on unitizing and norming.

**Learning trajectories for proportional reasoning**

Others have also studied the learning trajectory of students studying proportional reasoning. Carpenter et al. (1999), Steinthorsdottir (2003) and Steinthorsdottir & Sriraman (2008) expanded Lamon’s work to develop a four-level “hypothetical learning trajectory” that attempts to account for the degree to which students conceptualize the ratio as a composite unit in order to develop more advanced levels of proportional reasoning. Their studies examined fourth and fifth graders’ solution strategies to missing value and comparison problems and determined four levels of understanding. The first level involved students’ incorrect solutions to proportionality problems using additive strategies (level 1, additive strategies). The second level involved students who could combine ratio units by adding or multiplying but were unable to solve ratio problems that must be partitioned (level 2, ratio as indivisible unit). In other words, these students could only enlarge or build up from the original ratio. The third level includes the view of ratio as a reducible unit (Level 3, ratio as reducible unit). Students could solve a broader range of problems including non-integers between ratios. For example, a student at this level might have reduced 3:2 to 1:0.67 and use it as a unit ratio. In addition, these students could build up and scale down from the original ratio when solving proportional problems. The fourth level comprised the ability to solve the largest class of problem because the student recognized both within and between ratios (Level 4: ratio as both within and between). They typically used the
most efficient strategy and had a flexible set of solution strategies. Since this work studies the development of proportional reasoning, it should reflect the work of Lobato and Ellis (2010).

Notice how these levels include similarities to Lobato and Ellis’ (2010) shifts. The first level (Level 1) in this learning trajectory involves the use of additive reasoning, and shift 2 discusses the transition from additive to multiplicative reasoning. The second and third levels (Level 2 and Level 3) include the use of composite units and unit ratios that are similar to shift 3, the transition from composite units to multiplicative comparisons. Finally, the fourth level (Level 4), which includes the ability to flexibly use different strategies efficiently, seems to be reflected in shift 3 in which the transition to multiplicative comparisons is determined by the efficiency of methods. Although this learning trajectory is more detailed than Lamon (1993), it still seems to lack elements of both shift 1 and shift 4 from Lobato and Ellis (2010), (e.g., the recognition that ratio involves two quantities and that rate is a set of infinitely many equivalent ratios).

Steinthorsdottir (2003) suggested that students’ transition from Level 1 to Level 2 quickly and from Level 2 to Level 3 somewhat easily; however, the transition from level 3 to level 4 was slow and infrequent. She found that asking students to find multiple ways to solve problems appeared to facilitate the transition from Level 3 to Level 4. This recognition and ability to solve problems in multiple ways is important in what Ball and colleagues (Ball, Bass, Delaney, Hill, Phelps, Lewis, Thames, & Zopf, 2005; Ball, Hill, & Bass, 2005; Ball & McDiarmid, 1990) call Mathematical Knowledge for Teaching (MKT) and seems relevant to the study of pre-service teachers’ knowledge of ratio and proportion. For example, I use the balloon problem in an interview setting (see Appendix D) to elicit multiple strategies from the pre-service teachers by asking them to solve the problem in another way.

Steinthorsdottir (2003) was able to use these levels to determine how students reason proportionally. She found that she needed to add intermediate levels in order to fully describe the strategies the fifth graders were using to solve missing value and comparison problems. I would
argue that these levels compare to Lobato and Ellis’ (2010) shifts but may be more detailed in developmental aspects of the transition that might already have occurred with pre-service teachers who have already made some of the transitions. Additionally, if CCSSI (2010) wants students to be able to expand beyond the missing value and comparison problems, it might be more difficult to use these levels to describe the nature of one’s understanding based on only these four levels. The use of Lobato and Ellis’ (2010) shifts and EU may offer a better way to describe this expanded problem set as well as shed light on the understandings that adults have when solving problems. Many of these studies on student strategies focused on elementary or middle school students. A recent study involving high school students was similar.

Parish (2010) looked at ninth and tenth grade students and used Lamon’s (1993) work to develop stages of development. The stages that were used reflect the work of Lobato and Ellis (2010). The first stage, not apparent, occurs when one cannot comprehend the required goal of the task. The next stage is visual/ignore, in which one either uses visual judgment or just ignores given the values. I might see this type of strategy in shift 1 in which the student does not recognize that there are two quantities involved with the ratio and therefore just guesses. An example of the additive stage is one in which a student attempts to quantify but uses additive rather than multiplicative strategies. This compares to shift 2 and is explained by EU2 when the student has not made the shift from additive reasoning to multiplicative reasoning. The pre-proportional reasoning stage includes the use of pattern recognition and replication but without recognizing the multiplicative structure. This again seems similar to shift 2 due to lack of recognition of the multiplicative reasoning involved with ratio.

The next stage Parish (2010) calls ratio-unit/build-up which includes a student’s recognition of the ratio as a unit and the ability to build up this unit maintaining the relative structure of the individual values. This is reflected in shift 2 in which one has made the shift from additive to multiplicative reasoning but uses primitive methods that mirror additive reasoning.
The sixth stage of Parish’s development involves functional and scalar reasoning in which the student can determine a unit value and use the multiplicative relationship between the ratios to determine the solution. This describes what shift 3 involves, the transition from composite units to multiplicative comparisons or the ability to use more sophisticated methods that can efficiently solve problems. Finally, Parish described what she calls quantitative proportional reasoning as the use of algebraic-type methods to represent and solve complex proportion problems. EU9 described the use of an algorithm to solve proportions but emphasized the fact that this should be done with meaning and not just mechanically. The cross-multiplication algorithm tends to ignore the context of the problem and this leads to incomplete proportional reasoning. The stages of development created by Parish (2010) compare to the shifts and EUs from Lobato and Ellis (2010), but Lobato and Ellis’ work breaks down the levels even further, making the EUs a useful tool for explaining and describing one’s strategies in terms of the shifts.

**Policy and proportional reasoning**

CCSSI’s (2010) goal is to have all students mathematically proficient in every area of mathematics including ratio and proportion. According to the National Research Council, (NRC, 2001) three skills are needed to be mathematically proficient on proportional reasoning tasks. The first skill states that students must learn to make comparisons based on multiplication rather than addition. The second is that students need to recognize which aspects of a proportion can change and which remain the same. And third, the NRC contends that students need to learn to build composite units in order to develop the idea of multiplicative comparisons. All of these aspects of proportional reasoning are reflected in Lobato and Ellis’ (2010) shifts.

The first skill of making multiplicative comparisons seems to directly align with Lobato and Ellis’ (2010) shift 2 in which one makes the transition to multiplicative reasoning from additive reasoning. The second skill of recognizing the various aspects of proportions seems similar to EU3 in which students must isolate the attributes and understand the effects that the
change in each quantity has on the other quantity. And the last skill, the use of both composite units and multiplicative comparisons, parallels shift 3: the transition from composed units to multiplicative comparisons. However, Lobato and Ellis (2010) go one step beyond the NRC recommendations by suggesting one make the shift to rate as a set of infinitely many ratios. Although the NRC focuses on skills that students should develop, it does not explain how or why those skills should appear in an assessment. I plan to use Lobato and Ellis’s 10 EUs as a way to explain the transitions that pre-service teachers demonstrate through both written tasks and individual interviews.

Throughout this review I have presented the work others have done with proportional reasoning involving students in grades K-12. I have focused on the types of ratios needed in proportional reasoning, the types of problems that can be solved, the different strategies that students use to solve these types of problems, and the possible learning trajectories students’ experience. I have compared these strategies, stages of development, and learning trajectories to the shifts in Lobato and Ellis (2010). I have made the argument that much of this research aligns with the shifts, but the EUs provide a way for me to explain pre-service teachers’ understanding using evidence from a questionnaire and individual interviews. I also make the argument that these stages have been used to describe students in elementary, middle school, and high school, but not college students. I believe that adults might have a slightly more sophisticated way of explaining their proportional reasoning, and the EUs might enable me to determine the nature of their proportional reasoning.
Chapter 4: Methodology

This research study wishes to answer the question: To what extent is the nature of pre-service teachers’ proportional reasoning near the beginning of their formal teacher preparation [in a college-level program] consistent with the shifts described by Lobato and Ellis (2010)? In order to answer to this question, I used qualitative research methods with a questionnaire as a tool for determining which participants I interviewed and then conducted individual interviews that delved more deeply into pre-service teachers’ proportional reasoning.

Participants

The participants selected for this study were pre-service teachers in both elementary education and secondary mathematics education majors at a large research university. I chose these groups because they were eligible to be middle grade teachers, and ratio and proportion represents a focus of middle school mathematics that middle grade teachers are required to teach.

Because my research question suggests looking at prospective teachers near the beginning of their teacher preparation, my participants were recruited and involved in the study before they were heavily engaged in their mathematics methods courses. I recruited secondary education students who were accepted into the secondary mathematics education major and were about to begin their secondary methods courses. Elementary education majors had completed their required mathematics and statistics courses but had not started their mathematics methods courses. Timing was important to gain understanding of the proportional reasoning that pre-service teachers bring with them to their methods courses with the long-term goal of developing these and similar methods courses to improve teacher knowledge; I will discuss implications in chapter 6.

In order to recruit participants, I contacted the instructors of relevant courses. The elementary education students were in language and literacy courses prior to their mathematics methods course. The secondary education students were at the start of their first of two semesters of mathematics methods courses. I contacted the language and literacy course instructors and the
secondary mathematics methods instructors and requested to speak with their class for 5-10 minutes to recruit volunteers for the study. I used the script submitted to IRB (approved by IRB and found in Appendix A) that included details about inclusion requirements for the participants (i.e., must be elementary education majors who had not taken a mathematics methods course or secondary education/mathematics option majors who had just begun their methods course). I also informed the methods students that all participants would be asked to complete a questionnaire of mathematics problems taking about 45 minutes and that a small number of participants would be interviewed. Additionally, I informed them that participants would receive a gift card for completing the questionnaire and a second gift card for completing the interviews. I explained that only a small number of people would be chosen because, in part, time does not allow everyone to be interviewed before the end of the semester. Methods students were informed that the interviews would be videotaped and involve mathematics problems and follow-up questions to help me better understand their answers on the questionnaire.

**Instruments**

Two instruments were used to determine the nature of pre-service teachers’ proportional reasoning. The first instrument was a questionnaire consisting of 9 problems that had been adapted from research literature on proportional reasoning (see Appendix B for the full questionnaire). The questionnaire was used as a selection tool for the interview. The second instrument was an individual interview schedule (see interview schedule in Appendix C) that helped me develop a rich description of some of the pre-service teachers’ proportional reasoning and the relationship of their proportional reasoning to the shifts proposed by Lobato and Ellis (2010). In this section, I elaborate on the construction and nature of each instrument.

**Questionnaire and Analysis of Problems**

The questionnaire requested that participants solve different kinds of proportional reasoning questions and explain their reasoning. To facilitate the analysis, the problems were ordered to align with Lobato and Ellis’ (2010) shifts and work in conjunction with the framework
of transformations previously discussed in chapter 2. The responses were used to determine the nature of the participants’ current proportional reasoning.

Both the questionnaire and interview questions used multiple contexts and numerical situations. Tourniare and Pulos’s (1985) review of literature discussed several variables that determine students’ success on proportional reasoning problems. First, the structure of the problem influences the students’ success. Problem structure includes the presence of integer ratios, order, and numerical complexity. Integer ratios (i.e., ratios that involve integers) make a problem easier (Heller et al., 1990; Kaput & West, 1994; Karplus et al., 1983). Hart (1981) and Noelting (1980) both discussed the difficulty students had with non-integer ratios, including ratios with fractions. Order refers to the place of the numbers with respect to the other numbers of the ratio, and numerical complexity refers to the size of the numbers used. All three of these factors were studied by Rupley (1981) and were found to impact a student’s success in solving proportional reasoning problems. Rupley (1981) found that the presence of integral ratios and small numerical values less than approximately 30 made problems considerably easier than those without integral ratios or larger numbers. The problems that I selected vary in their difficulty of integer ratios, order and numerical complexity.

Research has also found that the context of the problem influences a student’s success. One such context studied is mixture (in which elements of the ratio constitute a new object, i.e., yellow and blue paint make green paint) versus non-mixture problems (situations in which the objects are psychologically distinct, i.e., ounces of grain and cost per ounce). Quintero and Schwartz (1982) found that non-mixture problems yielded more quantitative solutions than mixture problems, and Van den Brink (1978) found that problems in which there were psychologically distinct objects were easier to solve. Similar to this context situation is the idea of continuous ratio versus discrete ratio. Horowitz (1981) found that participants saw problems that dealt with discrete ratio as easier than those that used continuous ratio. Tourniare (1986) found that familiarity with the context was also important in solving proportional reasoning
problems. Research has also found that students who maintained the context of the problem throughout their solution process were more successful in their proportional reasoning (Clark et al., 2003). Based on this research, I included various contexts that I believe were familiar to the pre-service teachers, problems that dealt with both mixture and non-mixture situations, some discrete problems and some that were continuous. Next I will discuss the problems I used to elicit each of the shifts on the questionnaire.

**Shift 1: From one quantity to two quantities.** Shift 1 involves distinction between two unrelated objects with a relationship wherein one is dependent on the other. This includes reasoning about whether orange juice poured from a container in two different sized glasses will have the same “taste of orange” (Harel et al., 1994). Some students may believe the larger glasses will have a stronger orange taste than the smaller glass because it is a bigger glass; others may believe the opposite is true. This is indicative of shift 1 involving the transition from focusing on one quantity or unit in the problem (i.e., the size of the glass) to realizing that two quantities or units are involved in the relationship (for example the orange juice mixture contained in the glass involves both orange concentrate and water and it is the amount of each that determine the orangey taste).

**Lemon/water mixture problem.** Shift 1 involves reasoning with two quantities. The Lemon/water mixture problem elicits pre-service teachers attention to two quantities in a mixture context. Harel and colleagues (1994) used an orange juice problem to elicit 6th graders’ understanding of the fact that ratios involve two quantities. I changed this problem to the context of lemonade since, based on trials of questionnaire items [by colleagues], many students today might not be familiar with orange juice from concentrate. I used the following question to help me determine if pre-service teachers have progressed past shift 1 is:

*Tracy and Kelly are each given 40 oz. of water. Tracy adds 20 oz. of lemon flavor to her water and Kelly adds 10 oz of lemon flavor to her water, how would the two mixtures of lemonade compare in terms of lemon taste? Explain your reasoning. (adapted from Harel et al.[1994, p. 331]).*
For adults with real-world experience, this problem might not pose the same difficulties as it did for the sixth grade student studied by Harel (1994). I would expect most pre-service teachers to have already acquired constancy of taste and in a qualitative sense respond that Tracy’s mixture has more lemon taste since you are adding more lemon flavor. If EU9 (proportional reasoning can be generalized into sense making algorithms) is taken into account, an adult who has a strong procedural understanding of ratio might try to use the numbers in this problem to find a numerical solution. A pre-service teacher might see three numbers and try to set up a proportion to find a missing value (for example, \[ \frac{20}{40} = \frac{10}{x} \] making the two mixtures equivalent). Also at play is the “uniformity” factor in which someone’s belief that the taste of a liquid is invariant under a change of volume only if viewed as ingredient free (i.e., if the mixture of lemon juice is changed to something like Coca-Cola). This problem consists of a mixture with two ingredients. The presence of three numbers in the problem statement may lead some pre-service teachers to try and find a missing number using a learned algorithm (\[ \frac{40}{20} = \frac{x}{10} \] or \[ \frac{x}{20} = \frac{40}{10} \]) and ignore the context of the question. Ultimately, this problem looks at the nature of the students’ ability to recognize the qualitative relationship that exists between two quantities of a ratio. In an adult’s eyes, the “Uniform Diffusion Principle” (Harel et al., 1994) explains the irrelevancy of the “uniformity” and “relative volume” variables. However, the “numerical data” refers to the adult’s belief that taste constancy holds only when the context is “number free.” In the case of a pre-service teacher this might elicit a solution based on the numerical data and not the qualitative data.

**Basketball Problem.** Along the continuum between shift 1 and shift 2 I have placed EU3, in which real world ratio-as-measure is understood as two processes: isolating an attribute and understanding the effects of changing each quantity of that attribute on the ratio. Lobato (2008) used a problem that had students compare steepness of various ramps by changing the width and length. This problem requires students to isolate attributes (i.e., the length of ramp)
from other attributes (i.e., the width of ramp) in order to understand the effects of changing various quantities. The problem reads:

A group of students were practicing basketball shooting. Here are the results:

<table>
<thead>
<tr>
<th>Name</th>
<th>Shots made</th>
<th>Shots Taken</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allen</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>Barbara</td>
<td>14</td>
<td>29</td>
</tr>
<tr>
<td>Charlie</td>
<td>9</td>
<td>19</td>
</tr>
<tr>
<td>Daisy</td>
<td>23</td>
<td>48</td>
</tr>
</tbody>
</table>

Who is the best player, based only on this information? Explain how you got your answer. (Adapted from Hines and McMahon [2005, p. 101])

Hines and McMahon (2005) used a similar problem that involves isolating the attributes of shots taken and shots made in order to determine “who played the best” to provide evidence that a student has not completely made shift 1. I decided to adapt the problem using larger numbers than Hines and McMahon since the smaller numbers might suggest that one needs only to pay attention to the “shots made” attribute. I changed my numbers similar to the way Bright, Joyner, and Wallis (2003) did when they used this question.

Someone who bases their answer only on largest number of shots made or on largest number of shots taken is not looking at the effects the quantities have on each other. They have not isolated the attributes in order to understand the effects one quantity has on the other. This problem (the basketball problem) is more concrete in nature then the lemon/water problem, since the two attributes are quantitative and discrete in nature. Hines and McMahon (2005) studied 6th grade student solutions and found that without a clear understanding of ratio students will not isolate the attribute being measured and not attend to how quantities affect each attribute, indicating a lack of making the first shift. Those students’ answers indicate that pre-service teachers might use percentage to describe the “best player” in which Allen’s 64% success rate is greater than Barbara’s 48%, Charlie’s 47%, and Daisy’s 48%, thus making Allen the best player even though he made the smallest number of shots. I changed the numbers in this problem to less recognizable percentages (before they were made 5 shots out of 10, now 7 out of 11) that do not
involve easy-to-solve integer relationships (Heller et al., 1990; Kaput & West, 1994; Karplus et al., 1983). I chose numbers that intentionally were close to one-half so that students would need to explore the relationship more closely. However, the problem can be solved without calculating any percentages, by recognizing that only Allen had made more than half of his shots and therefore was the “best player.” The fact that there are more than four quantities being compared might prevent some pre-service teachers from using ratios or algorithms to solve the problem. Some might respond that the best player is Daisy since she made the most shots. If students have not made shift 1 then they will have difficulty isolating and coordinating the variables to correctly solve the problem.

**Shift 2: From additive reasoning to multiplicative reasoning.** Shift 2 describes the transition from additive reasoning to multiplicative reasoning. Ratio requires a multiplicative comparison of two quantities or the joining/composing of two quantities such that they maintain a multiplicative relationship. As students begin to recognize the multiplicative relationship between quantities, they pass through shift 2 and begin to further their ratio reasoning along their way to proportional reasoning.

**Mr. Tall/ Mr. Short problem.** In order to elicit pre-service teachers’ shift 2 from additive reasoning to multiplicative reasoning, I utilized a problem used by Karplus and colleagues (1974). Misailidou and Williams (2003) developed a diagnostic test to determine children’s inappropriate use of different strategies when solving proportional problems; one such misconception that they focused on was the improper use of additive reasoning. One of the problems they used reads as follows:

>Mrs. Short has a friend Mr. Tall. The length of Mr. Short is 4 large buttons. The length of Mr. Tall is 6 large buttons. When paper clips are used to measure Mr. Short and Mr. Tall: The length of Mr. Short is 6 paper clips. What is the length of Mr. Tall in paper clips? (Problem adapted from Karplus, Karplus and Wollman [1974, p. 476]; Diagram adapted from Khoury [2002, p. 100])
Misailidou and Williams (2003) found in their study that the Mr. Short problem (originally used in Karplus, Karplus and Wollman, 1974) elicited additive strategies from 48% of the elementary students. This problem was used to determine if pre-service teachers use additive reasoning when confronted with a proportional reasoning problem. Karplus, Pulos, and Stage (1983) called this example a missing value problem: a problem in which three numbers are given and one is asked to find the fourth. Monteiro (2003) studied the difficulty prospective elementary teachers had with the Mr. Tall/Mr. Short problem. On a written assessment, 26% used additive reasoning. Despite the use of smaller numbers in this problem and the fact that the result is an integer, students struggled with the multiplicative nature of the relationship.

Khoury (2002) determined four levels of proportional thinking middle school students used when solving this problem. It was possible that I could observe these same levels with pre-service teachers. Level 1 (illogical) in which no explanation is given, for example “Mr. Tall is 10 paper clips tall, because 4+6=10”. Level 2 (additive) in which the student focuses on the difference between the 6 and 4 (2), so Mr. Tall should be 2 more paper clips taller than the number of buttons, making him 8 paper clips tall. Level 3 (transitional) includes using an additive approach that focuses on the fact that for every 2 buttons there is one more paperclip for Mr. Tall. So (2 + 1) + (2 + 1) + (2 + 1) = 9 paper clips for Mr. Tall. Level 4 (ratio) includes a multiplicative comparison for every 1 button for Mr. Short Mr. Tall is 1 ½ buttons leaving Mr. Tall 9 paperclips (6 x 1 ½) long. Additionally, I also expected some pre-service teachers to solve the problem by setting up a proportion of 4:6 = 6:x and solving for x. Although the proportion method arrives at a correct solution, it does not necessarily indicate that the individual
understands the underlying multiplicative properties of ratio, as EU9 suggests the use of algorithm without conceptual understanding leads to procedural knowledge without sense making. Additionally, if a pre-service teacher has not made shift 2 then they would most likely use an additive approach.

**Track problem.** In addition, shift 2 involves the recognition of situations that may involve rates (i.e., speed) but require additive reasoning in the solution process. EU6 describes proportions as an invariant multiplicative relationship of equality between two ratios. Individuals who have not yet mentally formed ratios as mathematical objects may interpret problems that involve three numbers in the context of speed as a missing value problem that uses a template for solving proportions. This next problem is one in which there is an additive relationship between the variables in the context of running. The pre-service teacher needs to recognize that this is not a multiplicative relationship despite the fact the problem’s context is speed. Cramer and colleagues (1993) used the following problem in their research:

*Sue and Julie were running equally fast around a track. Sue started first. When she had run 7 laps, Julie had run 3 laps. When Julie had completed 12 laps, how many laps had Sue run?* (Adapted from Cramer, Post and Currier [1993, p. 159])

They found that 32 out of 33 pre-service teachers solved this problem by setting up a proportion and using cross multiplication. This problem involves an additive relationship between the number of laps that Sue ran and the number of laps that Julie ran, as well as a multiplicative relationship (speed) as part of the context, it can be solved by recognizing that Sue is 4 laps ahead of Julie. Since they are running at the same speed, Sue maintains the lead so that when Julie completes 12 laps, Sue will have completed 12 + 4 or a total of 16 laps. However, Bright, Joyner and Wallis (2003) found that students who inappropriately applied proportional reasoning set up $3/7 = 12/x$ so $x = 28$ laps in which $x$ is the number of laps that Sue ran. A pre-service teacher who does not recognize the difference between multiplicative situations and
additive situations has not fully achieved shift 2 and may not be able to formally reason proportionally.

In my pilot study, I found that using numbers like 9 and 3 seemed to elicit a multiplicative response from the participants, possibly due to the multiplicative relationship between these two numbers. However, using numbers like 7 and 3 seemed to elicit an additive response. To avoid false positive instances of additive reasoning, I used 7 and 3 to help me determine if the pre-service teacher recognized the situation as additive and not just recognized numbers that have a multiplicative relationship and applying an algorithmic template to solve the problem.

**Shift 3: From composed units to multiplicative comparisons.** Shift 3 is the transition from composed unit strategies to multiplicative comparisons. Lesh, Post, and Behr (1988) suggested that pre-ratio reasoning involves creating equivalent ratios based on iterating and partitioning a composed unit in order to create a family of equivalent ratios. Students may use composed units to build up ratios (possibly only as an extension of their additive reasoning) to solve problems. This is a more intuitive way to work with ratios, especially for students who do not have strong multiplicative reasoning skills. This shift moves from repeating a composed unit to anticipating the number of groups needed and realizing that ratio represents a multiplicative comparison that is a more efficient way of solving problems with ratios.

**Health Food Store problem.** In order to elicit an understanding of the pre-service teachers’ shift 3, I used a problem that relates two types of ratios (i.e., composed units and multiplicative comparisons). Pre-service teachers should be able to answer questions like “what part of ___ is ___?” and “How many times greater is ___ than ___?” A missing value problem from Ercole, Frantz and Ashline (2011) asked pre-service teachers to answer both of these questions. The problem is:

*The Health Food Store sells granola by the ounce. The cost depends on the weight of the granola. Granola that weighs 8 ounces costs $1.50. Fill in the table below with cost or weight. (Adapted from Ercole, Frantz and Ashline [2011, p. 484])*
This problem required that students recognize the invariance of ratio and know when to apply proportional reasoning to each situation. It takes away the typical situation of missing value problems in which one is provided three numbers and are asked to find the fourth. In this situation, one is asked not only to find the cost of a particular weight, but also the weight for a particular cost. This requires that one recognizes the invariance of the situation and apply multiplicative comparisons, relating to EU7 in which one recognizes the covariance of the proportional structure. In addition, the chart may also cause individuals to perceive the situation as not proportional.

Various strategies could be applied in this scenario depending on which part of the chart a student is completing. The possibility of using both multiplicative strategies and unit strategies lends itself to determining whether a student has made shift 3 from only using unit ratios to recognizing the use of multiplicative comparisons in his or her thinking. One might just double 8 ounces to get 16 ounces so they double $1.50 to get $3.00; or one could even set up a proportion and cross multiply (i.e., \( \frac{6}{8} = \frac{x}{1.50}; \frac{8}{16} = \frac{1.50}{y}; \frac{8}{z} = \frac{1.50}{4.50} \)) and solving for x, y and z. Ercole, Frantz and Ashline (2011) suggested that students might use another strategy to find the unit ratio by dividing $1.50 and 8 ounces to find the cost per ounce. This is 0.1875; the person would then multiply 0.1875 by 6 to get $1.125 or $1.13 and 0.1875 by 16 to get $3.00. To find the weight for $4.50 you would divide by 0.1875 to get 24 ounces. Lamon (1993) found that unit rate strategies were common among 6th graders solving similar problems. However, those problems involved integers and this situation might make that strategy more difficult to use.
This problem allowed me to observe differences in the methods students use in various situations and determine their flexibility in proportional reasoning that is required for shift 3. If the pre-service teachers are only inefficiently using iterative methods with composed units then they have not completed shift 3. This problem required the use of a unit strategy or multiplicative comparison in order to find the cost of six ounces. The ability to transition to multiplicative comparisons in this problem provided evidence that a student had made shift 3.

Dog/Cat problem. On the same step as shift 3 along the continuum, EU4 distinguishes between fractions and ratio in order to determine whether pre-service teachers are operating at this point of the continuum. Specifically, EU4 suggests one must understand that ratio can represent a part-part comparison. I argue that to complete shift 3 one should have an understanding of ratio as a part-part comparison making it distinct from a fraction. Peled (2007) used a mathematically explicit problem that focuses on ratio as a part-part comparison. This problem reads:

When animal lover Mr. Henry died he left 240 thousand dollars to be divided amongst two animal shelters using a 2:3 ratio between the amounts that Cat Best Home and Dog Lover Home gets. How much money should each shelter get? (Adapted from Peled [2007, p. 2143])

This question is purposeful in seeking to understand if pre-service teachers interpret the ratio as part/whole. If they incorrectly apply a part/whole comparison it will lead to a whole number solution. The correct solution involves the ratio as a part-part ratio that also leads to whole number solution. Research indicates that students seem more accepting of whole number solutions (Heller et al., 1990; Kaput & West, 1994; Karplus et al., 1983) than non-whole number solutions. Since an incorrect solution method leads to whole number solutions, the pre-service teacher might not further explore the meaning of their answer in terms of the problem. This problem could reveal whether the student understands that ratios involve part-part relationships whereas fractions operate as a part-whole relationship. Although Peled did not use this problem as part of his research on students or prospective teachers, his argument was that this problem is
explicit in the knowledge needed to solve it (i.e., ratio, since it is mentioned in the question). The fact that this problem explicitly says “ratio” will prevent guessing if one understands what to do in a ratio situation. I would expect that those prospective teachers that understand ratio as a part-part comparison to solve the question by finding 2/5 of 240 thousand as $96,000 and 3/5 of 240 thousand as $144,000 in order to determine the amount of money each animal shelter should receive. Otherwise, I would expect some pre-service teachers to find 2/3 of 240 thousand as the amount of money that Cat Best home receives and 1/3 of 240 thousand as the amount that Dog Lover home receives. This second strategy assumes that the 2:3 ratio is a part/whole relationship or that the money is divided into three equal parts not five parts. This might indicate that the concept of ratio as a part-part comparison is not completely understood by the pre-service teacher. If a pre-service teacher does not correctly answer this question it raises doubt about whether he or she has completely made shift 3.

**Shift 4: From iterating composed units to creating equivalent ratios.** Shift 4 occurs when students make the transition from iterating “easy” composed units to creating infinite sets of equivalent ratios. Lobato and Ellis (2010) distinguished rate and ratio as changes in conceptual development of proportional reasoning in which rate is a more sophisticated understanding of the ratio concept. Their definition of rate expands understanding from a contextualized notion to a conceptual understanding of ratio. For example 3:4 could represent different things to a student, such as (1) not a ratio but simply 2 different quantities without a conceptual understanding of ratio, (2) as a ratio in which the composed unit 3:4 doubled would result in 6:8; however, the student is limited to simple halving and doubling scenarios, or (3) a rate in which the use of proportional reasoning is used to determine any equivalent ratio. This distinction helps connect ratio to concepts such as slope in a linear equation and rate of change in calculus. The use of rate as a more sophisticated understanding of ratio leads students to reason and justify solutions and allows students to make “sense” of mathematics.
**Planet problem.** Shift 4 allows students to reason formally about proportions and apply this understanding to important mathematical concepts in more advanced mathematics. These include scalar problems, rate of change problems from Calculus, and everyday multiplicative reasoning situations. When a student can recognize that there are infinite numbers of equivalent ratios that constitute a rate, they are able to formally reason proportionally. Lamon (1999) suggested using problems that do not involve whole number solutions in order to determine whether students are able to reason proportionally in unique situations. One problem that requires students to use more difficult reasoning skills such as generalizing is:

*NASA is building a space station on Planet B, given each pair of weights in the table, how can they determine how much an item on Earth will weigh on Planet B. Explain your reasoning.*

<table>
<thead>
<tr>
<th>Item</th>
<th>Earth (pounds)</th>
<th>Planet B (pounds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>120</td>
<td>45</td>
</tr>
<tr>
<td>2</td>
<td>27</td>
<td>10 1/8</td>
</tr>
<tr>
<td>3</td>
<td>42</td>
<td>15 ¾</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>1/10</td>
<td>9/240</td>
</tr>
</tbody>
</table>

a) *Determine the relationship between an item’s weight on Earth and Planet B.*
b) *Determine the weight of an item on Planet B if it weighs 200 lbs on Earth.*
c) *Determine the weight an item on Earth if it weighs 63 lbs on planet B.*
d) *Give an example of another item that would maintain the same relationship that exists between the weights on Earth and Planet B. (Adapted from Lobato & Ellis [2010, p. 72])**

Part “a” in this problem requires students to determine the ratio between the two planets and generalize a statement of how these weights are determined. Parts “b” and “c” require that the student applied this generalization to different situations. I would expect pre-service teachers to express the relationship of weight on Earth to weight on planet B as Earth weight = 8/3 (planet B weight) or Earth weight = 2.666 (planet B weight) or Planet B weight = 3/8 of Earth weight or planet B weight = .375 of Earth weight. Part “d” has the pre-service teacher apply the relationship to a situation of their choosing. This helps me determine if they understand that rates are a set of infinitely many equivalent ratios. In this problem, the equivalent ratio is 8/3. Pre-service teachers need to recognize this equivalence to determine the generalization necessary to
solve this problem. This rate is part of the final shift before formal proportional reasoning. In the first application, one would take \( \frac{3}{8} \times 200 = 75 \) pounds. In the second application, one would take \( \frac{8}{3} \times 63 = 168 \) pounds. The application of this rate leads to what Sfard (1991) referred to as a reification of proportion, in which ratio is more than just two numbers in a multiplicative relationship if it is also an object used to identify change in quantities. The reification in this problem occurs not only in the generalization of the weights but also in the ability to apply this generalization to multiple types of situations. The application extends the use of an infinite set of equivalent ratios leading to formal proportional reasoning. According to Lobato and Ellis (2010), this problem emphasizes quantitative relationships encouraging students to explain their reasoning, make generalizations and provide justifications.

**Scalar problem.** According to Lobato and Ellis (2010), students who made shift 4 reason formally about proportions and apply this to more advanced mathematics including scalar problems. I would ask pre-service teachers to demonstrate if they can successfully complete a scalar type problem. One problem comes from Miller and Fey (2000):

> A picture is 2 cm wide by 2.4 cm long. If Fran wants to enlarge the picture to be 5 cm wide, how long should it be? If Fran wants to enlarge the picture so that it is 7.2 cm long. How wide should it be?(Adapted from Miller & Fey [2000, p. 311])

This problem might indicate the application of formal proportional reasoning. Other researchers (Lamon, 1993; Lo & Watanabe, 1997; Vergnaud, 1988) have found that scalar problems involving ratios tend to indicate a higher level of proportional reasoning. This research found that students have difficulties due to growth problems containing continuous quantities as opposed to discrete quantities in other types of problems (Lamon, 1993). In addition, Lamon (1993) and others (Langrall & Swafford, 2000; Lo & Watanabe, 1997; Vergnaud, 1988) found that students did not recognize the multiplicative nature of the relationship between the length and width or the stretching and shrinking of a geometric object. This led to additive reasoning errors (Hart, 1981; Kaput and West, 1994). I would expect pre-service teachers who have not made
shift 2 to use additive reasoning and look at the difference in the widths as 3 cm and then add this difference to the length to arrive at a 5 cm x 5.4 cm picture in the first question. They would find the difference between the lengths of 7.2 and 2.4 to get 4.8 and add this to the 2 cm width to get a picture that is 6.8 cm x 7.2 cm. Miller and Fey (2000) found similar results in their study of proportional reasoning of middle school students.

If the pre-service teacher does recognize the multiplicative nature of the relationship (indicating that they have made shift 2), they might solve this problem using a proportion and then cross-multiplying (i.e., $2/5 = 2.4/x$ or $2/2.4 = 5/x$ depending on using a within or between strategy) this would yield a length of 6 cm. A similar process could be used to solve the second question and the proportion would look as follows: $2.4/7.2 = 2/y$ or $2.4/2 = 7.2/y$. This yields a width of 6 cm. The fact that both results are 6 cm may confuse some students and it would be interesting in an interview to ask them to further explain how this is possible.

In my pilot study, I found that pre-service teachers used a template of setting up proportions and solving them using cross-multiplication to find the missing value. It is not clear if this was done as EU9 suggests “without sense making” leading one to question if students have mastered proportional reasoning (Confrey & Smith, 1994). This would be something I would follow up on during the interview.

**Pizza Problem.** The final problem on the questionnaire involves the context of pizza and is what Vergnaud (1988) would consider a comparison problem. I used this problem to determine whether pre-service teachers made sense of the situation and not just blindly followed an algorithm (EU9) or take superficial cues from the context (EU10). The problem reads:

> When you join the party you could sit at a table where 7 people will be served 3 pizzas, or at a table where 4 pizzas will be served to 9 people. Given that you love pizza, where should you choose to sit in order to have the most pizza? Explain your reasoning. (Adapted from Peled [2007, p. 2143])

Notice how in the first sentence the ratio is people to pizza while in the second sentence it is pizza to people. I did this intentionally to determine if the pre-service teacher attends to this
nuance in their solution or follows an algorithm without sense making. It makes sense to compare the quantities only in terms of pizza per person. If one uses person per pizza the result will not make much sense in the context (i.e., you do not usually talk about a fraction of a person) and making it more difficult for them to determine which table to sit at in order to get more pizza. The context of this problem also required one to distinguish between the attributes and their effects on the situation. However, the correct solution method might also be due to “superficial cues from the context of the problem” as EU10 suggests, leaving one to question whether the pre-service teacher has mastered formal proportional reasoning.

The relationship of pizza per person is necessary to make a decision; however, one could simply choose to sit where there are four pizzas based on the number of pizzas, or one might make the decision based only on the number of people at the table. This problem will try to elicit one’s ability to apply proportional reasoning in a situation in which the unit ratio makes more sense in a particular way (i.e., pizza/person versus person/pizza).

**Interview and Analysis of Problems**

The second instrument used in this study is an individual interview conducted with participants selected from among those who completed the questionnaire. Their selection would be based on how their responses match the shifts. More specifically I based the selection on discrepancies between where they fall on the continuum of shifts and their methods of solving the problems on the questionnaire. For example, someone might use additive reasoning on the Mr. Tall/Mr. Short problem, but they solve the health food store problem using a unit ratio. The interview would delve more deeply into why they recognized one as a multiplicative relationship and another as additive.

The next section describes how the interview would provide me with a better understanding of the nature of the pre-service teachers’ proportional reasoning. The interview consisted of different problems that vary with respect to both the numbers and context of the situation to determine if those are factors in the pre-service teachers’ proportional reasoning
(Heller et al., 1990; Horowitz, 1981; Kaput & West, 1994; Karplus et al., 1983; Quintero and Schwartz, 1982; Van den Brink, 1978). The questions allowed me to determine what transitions the pre-service teachers had made along the continuum. I would not have time to present all questions during the individual interview; therefore, I chose the most appropriate questions based on the participants’ responses on the questionnaire.

For example, a pre-service teacher that used additive reasoning on the Mr. Tall/Mr. Short problem (which resulted in the wrong answer) might be asked to complete the Housing lot problem from the interview so I can observe if they consistently use the same type of additive reasoning. If this same pre-service teacher solved the Health food store problem correctly using cross multiplication, I would ask them to solve a similar balloon problem during the interview and to provide a second method of solving the problem without an algorithm. This will help me to determine if they reasoned proportionally or used an algorithm (i.e., cross multiplication) with little conceptual understanding of why or how that process works to solve the problem. In the next section, I will explain each question that might be used in the interview, why I chose those problems, and how I believe they will help me delve more deeply into the pre-service teachers’ proportional reasoning. In the data analysis section, I will explain how I chose problems to use for possible interviews. Additionally, I have created a table in Appendix D that provides each question from the questionnaire and interview and how they relate to the shifts enabling me to advance my understanding of the pre-service teachers’ proportional reasoning.

**Shift 1: From one quantity to two quantities.** The next two problems will be used to elicit proportional reasoning in terms of shift 1. The lemon/lime problem and the ski slope problem will be used during the interview to delve deeper into the participant’s transition from focusing on one quantity to recognizing the relationship of two quantities.

**Lemon/lime problem.** To gain deeper understanding of a pre-service teacher’s transition through shift 1, I would present a problem used by Heinz (2000) that uses manipulatives and stipulates that explanations cannot include computations. This would help me gain further insight
into the pre-service teachers’ conceptual understanding of ratio and their understanding of shift 1, reasoning with two quantities. The problem was:

*Jen and Alice are making lemonade. Jen mixes 3 cups of lemon juice with 2 cups of lime juice. Alice mixes 4 cups of lemon juice with 3 cups of lime juice. Whose mixture is more lemonier? Justify your solution without using calculations. You may rearrange the cubes in such a way that they demonstrate which one’s more lemony tasting or that they are the same. (Heinz [2000, p. 70])*

Heinz (2000) used this problem with pre-service elementary teachers in which they were asked to distinguish the effects of the lemon cubes and lime cubes used in a drink mixture without using computation. This problem differs from the questionnaire problem since it had students compare the mixtures’ taste based on the amount of lemon and lime cube in each mixture without using any numbers associated with the quantities. Due to the abstract nature of mixture problems, research has shown that the mixture context is often difficult for middle school students; however when the computation element is not allowed, Heinz (2000) found that pre-service teachers also struggled to explain their reasoning. Misailidou and Williams (2003) found that the use of two different colored cubes to represent the cups of lemon and lime juice made the mixture problem less abstract for students. Heinz (2000) found that pre-service teachers became frustrated when they were not allowed to quantify the result, but this led to discussions of their misinterpretations of mixtures. In addition, the use of the manipulative moves the problem from quantitative reasoning (used in the questionnaire lemon/water problem in which one can numerically compare the mixtures) to qualitative reasoning (in which one needs to rely on the manipulative and other reasoning to explain to compare the mixtures). This problem provided a different way for participants to explain their reasoning without using only numerical data. Heinz (2000) discussed how the stipulation that there are no calculations was non-trivial for students, and it was less likely that students would use recalled algorithms or procedures, as in EU9, in order to solve the problem. This caused the students to reason about the quantities according to their conceptions of how to compare the amount of lemon flavor.
Based on Heinz’ (2000) use of this problem, I would expect several possible solutions, although I believe that, for some pre-service teachers, the inability to use calculations to explain their reasoning may cause them insurmountable trouble. Heinz (2000) found that some students paired the one lemon with one lime for each mixture and realized that the volume of Alice’s mixture was larger but concluded that the mixtures would taste the same. Others argued with this solution that even though the volumes were different it did not mean that the taste was the same. Some explained that, even though there was one lemon cube left over in each mixture after the pairs were formed, it would be less diluted in the mixture when there were only two pairs of lemon-lime pairs. Although Heinz (2000) used this problem as part of a teaching experiment, I would expect to find similar solution processes during an interview setting even though there will not be group discussions on the topic to help persuade student thinking. I used this question even for pre-service teachers who answered the entire questionnaire correctly, especially those that provided little reasoning behind their solution methods. I anticipated that this problem would elicit explanations that prevented the use of cross multiplication or other algorithms to explain numeric results and helped me explore more deeply the pre-service teacher’s conceptual understanding of proportions.

During the pilot study interview, one of the participants believed that the mixtures would taste the same since there was only 1-cup difference between the two mixtures. However, when I told her she could use numbers she concluded that Jen’s mixture would have more lemon flavor since it had 3 lemons: 2 limes and Alice had 4 lemons: 3 limes. I might follow up by asking, how does the numeric reasoning differ from other types of reasoning? This might provide some insight into EU1 in which reasoning with ratios involves attending to and coordinating two quantities. Since this participant was using additive reasoning when they discussed the difference of one between the mixtures, I might also ask, what is a ratio? To delve deeper into EU2, a ratio is a multiplicaticative comparison of two quantities.
**Ski Slope problem.** This question is designed to delve deeper into EU3 and determine whether pre-service teachers can recognize that a ratio consists of two invariant attributes. In a follow-up interview, I might use the Simon and Blume (1994) problem. I chose to do this in an interview setting given the complexity of explaining one’s reasoning for the problem.

In Kansas, there are no mountains for skiing. An enterprising group built a series of ski ramps and covered them with a plastic fiber that permitted downhill skiing. It is your job to rate them in terms of most steep to least steep. You have available to you the following measurements for each hill: the length and width of the base and the height. How would you determine the relative steepness using the information you have? [Adapted from Simon & Blume [1994, p. 190] note: examples of multiple slopes would be provided and participants asked to rate them from steepest to least steep. To lessen the abstractness of the problem measurements for length, width and would be provided as illustrated by Lobato and Thanheiser (2002, p. 169) in their wheelchair activity.]

I would expect to find results similar to Simon and Blume (1994) (i.e., answers looking at steepness in two ways: ratio of height to length or difference between height and length). Despite the additive and multiplicative aspects required in this problem, I think the focus remains on whether students are able to isolate and recognize the attributes and the effects that they have on each other. Simon and Blume(1994) recognized that prospective elementary teachers in their study had no criteria for choosing between a ratio method of comparison and a “difference” method and that these two methods were inconsistent with each other.

I would expect that without a criteria method, pre-service teachers would continue to view these two comparison methods as equally viable solutions to this type of “steepness” problem. I would also expect to see some of the students respond in ways that were observed by Lobato and Thanheiser (2002) who used a similar problem involving steepness of ramps for wheelchairs. They found that students had difficulty isolating the characteristics of steepness in a ramp situation. They conjecture that the reason for this difficulty is that the attributes of height and length are not the most salient characteristics for those everyday situations involving slope. Once students had isolated the attributes, they need to determine how changes in the quantities affected the steepness. The distinction between additive methods of comparison and
multiplicative comparison methods indicates that the two attributes are related in terms of multiplication as in shift 2.

**Shift 2: from additive reasoning to multiplicative reasoning.** The next two problems were used to elicit proportional reasoning in terms of shift 2. The housing lot problem and the bike problem were used during the interview to delve deeper into the participant’s transition from utilizing additive reasoning to solve problems to solving problems with multiplicative reasoning.

**Housing Lot problem.** This question will be utilized to better understand if shift 2 has been fully completed and to make sure that the solution to question 3 on the questionnaire was not just a result of EU9 or a procedural understanding of ratio. During the interview I asked another question that exposed incorrect use of additive strategies. A problem used by Heinz (2000) found that pre-service teachers applied additive strategies when solving the following problem:

> A new housing subdivision offers lots of 3 different sizes: 185 feet by 245 feet; 75 feet by 114 feet; 455 feet by 508 feet. If you were to view these lots from above, which would appear most square? Which would be least square? Explain your answer. (Adapted from Heinz [2000, p. 150])

In this problem, one needs to determine which lot has a ratio closest to one. However, most pre-service teachers looked at the lot with the smallest difference between the two dimensions. Heinz (2000) found that 56% of the prospective elementary teachers had incorrect responses to this question, and 48% of those students calculated the difference between the length and width of the three rectangles to determine which was the most square. She also noted that 12% of the students, who eventually solved the problem correctly, began by looking at the difference. Only after looking at scale drawings of the rectangles did they determine that this method was incorrect. The use of additive strategies might imply that the student has not made shift 2 and needs to develop a deeper multiplicative understanding of ratio. The ability to make shift 2 requires one to recognize the multiplicative nature of the ratio, and this problem involves the recognition of “squareness” as a ratio of one to one. Pre-service teachers teaching ratio need
to understand when it is appropriate to use additive strategies versus multiplicative strategies. This problem provided me with an opportunity to see if they have made the needed transition in their thinking about ratio to proceed to shift 3.

During the interview I followed up this problem with a question to apply their method to two rectangles with dimensions 100 feet x 200 feet and 300 feet x 400 feet. If the pre-service teacher had used the difference of the side lengths to determine “squareness”, it may have given them pause when given this new situation. I was curious to see if they were willing to alter their method or give an explanation about whether or not it would work for this problem. I would also provide them with scale drawings of the two rectangles to see if that influences their reasoning about the problem.

During the pilot interview, one of the participants originally started comparing the squares using the difference between the sides of rectangles, explaining that a square has four equal sides and that if the difference is small than the rectangle is more “square”. After looking at the follow-up problem they decided that they should look at the ratio of the sides, but was not certain what should determine one rectangle being more “square” than the other. A case in which the participant brings up the idea of ratio, I might follow-up with the question, what is a ratio? How would a ratio of the sides of a square be related? This would help me to gain insight into EU2, especially if the participant responds that the ratio is a multiplicative relationship between two quantities.

**Bike problem.** Shift 2 also involves the recognition of situations that may involve rates (i.e., speed) but require additive reasoning in the solution process. Individuals who have not yet mentally formed ratios as mathematical objects may interpret problems that involve three numbers in the context of speed as a missing value problem that uses a template for solving proportions. This next problem is one in which there is an additive relationship between the variables in the context of bike riding. The pre-service teacher needed to recognize that in order to solve this problem one needed to focus on the additive relationship, despite the fact that the
problem’s context is speed. I have adapted this problem to be similar to the one Cramer and colleagues (1993) used in their research. It reads:

*Ben and John both ride their bikes to school at the same rate. Ben leaves his house first and meets John after riding 10 blocks at which time John has only rode 7 blocks. If John’s trip to school is 14 blocks, how many blocks does Ben ride his bike to school? (Adapted from Cramer, Post and Currier [1993, p. 159]*)

Since this problem involves an additive relationship between the number of blocks that Ben rode and the number of blocks that John rode there exists a multiplicative relationship (speed) as part of the context. The problem is solved by recognizing that Ben is 3 blocks ahead of John, and Ben will keep this lead since they are both riding at the same speed. When John completes 13 laps, Ben will have completed 13 + 3 or a total of 16 blocks. However, Bright, Joyner and Wallis (2003) found that students who inappropriately applied proportional reasoning set up \( \frac{10}{7} = \frac{x}{14} \) so \( x = 20 \) blocks where \( x \) is the number of blocks that Ben rode. A pre-service teacher who does not recognize the difference between multiplicative situations and additive situations has not fully achieved shift 2 and may not be able to formally reason proportionally.

During the pilot study, I found that the pre-service teachers focused more on the multiplicative nature of the numbers involved as opposed to the additive nature of the situation. I changed this problem to the context of blocks and bike riding to determine if the track problem was difficult due to the context. I also asked a follow up question:

*Sue runs twice as fast as Julie. Julie ran 3 laps before Sue started. If Julie completed 7 laps, how many laps did Sue complete? (Adapted from Cramer, Post and Currier [1993, p. 159]*)

This question focused not only on an additive nature of the laps but the multiplicative nature of different speeds. I had hoped that this would provide insight into the pre-service teachers’ proportional reasoning involving both additive and multiplicative contexts.

**Shift 3: From composed units to multiplicative comparisons.** The next two problems were used to elicit proportional reasoning in terms of shift 3. The balloon problem and the
boy/girl problem will be utilized during the interview to delve deeper into the participant’s use of composed units and multiplicative comparisons.

**Balloon problem.** This question seeks to observe transitions in thinking from composed unit strategies to multiplicative comparisons (shift 3). It attempts to determine if the pre-service teacher used efficient methods when solving the problem or relied on repeating composed units. In order to delve deeper into pre-service teachers’ use of multiplicative comparisons, I would use a problem by Lamon (1993) discussed in Langrall and Swafford (2000). The problem is:

_Ellen, Jim and Steve bought 3 helium-filled balloons and paid $2 for all three balloons. They decided to go back and buy enough for everyone in their class. How much did they pay for 24 balloons? Explain your reasoning._ (Lamon 1993, p. 53)

In this problem students can use whatever strategy they like; however, the problem lends itself nicely to both composed units and multiplicative comparisons. The difference is in the efficiency of these methods; the composed unit would be cumbersome and laborious while the multiplicative comparison requires simple calculations. Langrall and Swafford (2000) observed 5th-8th graders during interviews and found that they used various strategies to solve this problem. The less formal strategy included the composed unit (3 balloons: $2; 6 balloons: $4, … 24 balloons: $16), and the more formal strategy used multiplicative comparisons (24 balloons/3 = 8 balloon packs; 8 balloon packs x $2 = $16). This second strategy was more efficient but also more abstract in the fact that you must “create” balloon packs.

If a pre-service teacher has made shift 3 they should be able to understand this abstract use of multiplicative comparisons to solve this problem. If the pre-service teacher first solves the problem using a proportion and cross multiplying (i.e., \( \frac{3}{2} = \frac{24}{x} \)), I would ask them if there was another method they could use to solve the problem to further explore their understanding of multiplicative comparisons. This question delves more deeply into the part-whole relationship and explores whether pre-service teachers use composed units or multiplicative strategies to solve the question. The first strategy is the beginning of shift 3, whereas the use of multiplicative
comparison indicates a transition towards shift 4. If the participant states that they are using a proportion, I might ask them to tell me what they think a proportion is? If they tell me they are solving the proportion using cross-multiplication, I will ask them what they believe the relationship is between the cross-multiplication and the proportion? This will help me to gain insight into EU6 and EU9.

**Boy/girl class problem.** EU4 discusses the important connections between ratios and fractions: ratios are often expressed \( a/b \) but do not have the same meaning, ratios have a “part-part” comparison, ratios and fractions are overlapping sets, and ratios can be reinterpreted as fractions. The notation of \( a/b \) often confuses students’ understanding of the connection between ratio and proportion. Ratio can be part-part comparison, whereas fraction is always a part-whole comparison. During the interview, I asked students to discuss another question that elicits the use of multiplicative strategies as well as explores the part-part nature of ratio. This question had been modified from Langrall and Swafford (2000) who used it to explore proportional development of middle school students. It read:

*Mrs. Jones class has 30 students, and they are divided into groups using a 2:3 ratio of boys to girls. How many boys and how many girls does she have in her class? (Adapted from Langrall and Swafford [2000, p. 255]*)

This question delved more deeply into the part-part relationship and explored whether pre-service teachers understood that ratio is not only a multiplicative relationship like a fraction (part-whole) but also one that describes a relationship between two parts of a whole. Some pre-service teachers might set up a proportion of \( 3/2 = x/30 \); however, the \( 3/2 \) represents a part-part relationship whereas \( x/30 \) is a part-whole relationship. These two relationships are not equivalent. The pre-service teacher should realize that each group contains 5 students allowing the proportion to be \( 3/5= x/30 \) so that \( x=18 \) girls and \( 2/5 = y/30 \) or \( y=12 \) boys. The recognition that there is a total of five students in each group and a total of 30 students in the class is needed to explore the ratio in this problem. Lobato and Ellis (2010) suggested that understanding ratio involves linking
ratios to fractions and quotients. This question explores whether pre-service teachers use these
re-interpretations of ratio in order to solve this problem, indicating that they have made the
transition from shift 3 to shift 4.

During the pilot interview, I realized that the wording of this problem seemed to cause
difficulty in recognizing that this situation involved groups. I followed-up the problem by asking
a slightly different question with similar wording and different numbers:

*Mrs. Jones put her students into groups using a ratio of 3 girls to 2 boys. If she has 25
  students, how many girls and how many boys does she have in her class?*

This wording change seemed to help the participant recognize that there are 5 groups of 5
students. I would use this follow-up only if the participant is unable to make any progress with
the original problem. During the pilot study, one participant stated that they did not understand
the word ratio in the problem. To follow-up on such a statement, I might ask them; what they
think a ratio is? And how ratios might relate to fractions? Or how the ratio might relate to
division? This would help me delve deeper into EU2, EU4 and EU5.

**Shift 4: From iterating composed units to creating equivalent ratios.** The next three
problems will be used to elicit proportional reasoning in terms of shift 4. The pasta sauce
problem, sale problem, and the paint problem will be used during the interview to delve deeper
into the participant’s transition from iterating composed units to creating an equivalent set of
ratios.

**Pasta problem.** Shift 4 requires that students formally reason about proportions and
recognize that there are an infinite number of equivalent ratios that make up a set of rates. Lamon
(1999) suggested using problems that do not involve whole number solutions in order to
determine if students are able to reason proportionally in unique situations. One such problem is
from Lobato and Ellis (2010, p. 72) and says:

*Zandora sells pasta sauce and charges $3.00 for a 7-ounce jar or $16.00 for 2 larger jars
  that hold a total of 37 ½ ounces. Is buying a 7-ounce jar a better deal than buying the 2
Jars of 37 ½ ounces? How do you know? Explain. (Adapted from Lobato & Ellis [2010, p. 72])

Lobato and Ellis suggested that the “messy” numbers involved in this problem encourage the use of unit ratio strategies to solve this problem. One possible solution would be to compare \( \frac{3.00}{7 \text{ oz}} = \$0.429/\text{oz} \) to \( \frac{16.00}{37.5 \text{ oz}} = 0.426/\text{oz} \), making the 2 larger jars a slightly better deal. However, when the numbers are rounded to the nearest cent the two choices are essentially the same. According to Lobato and Ellis (2010), the use of multiplicative comparisons is necessary to efficiently solve this problem. To extend this problem I would ask participants to answer the following question:

*Can you give an example of another pricing structure that would give consumers the same deal as 2 jars of 37 ½ ounces for $16.00?*

This extension enabled me to determine the pre-service teachers’ understanding of rate as an infinite set of equivalent ratio that is necessary for shift 4. I might follow-up this question with, what is the pricing structure in this problem and how is it related to rate? This question might help me understand the participant’s proportional reasoning in terms of EU8 and shift 4.

**Sale problem.** Formal proportional reasoning involves situations in which one must distinguish between absolute versus relative comparison, and what that relationship represents for a given context ultimately determining whether a comparison is multiplicative or additive. Formal proportional reasoning involves carrying out the strategy in such a way that the original ratio remains invariant. The following problem is such an example (Baxter & Junker, 2001):

*Store A advertises a sale of $10.00 off any purchase. Store B advertises a sale of 10% off any purchase. If both stores sell the same things at the same prices, which store offers a better sale? (Adapted from Baxter & Junker, [2001, p. 11])*

In this case, I would expect students to conclude that the 10% discount from Store B is only better when the purchase is more than $100, but for sales less than $100, Store A’s sale of $10.00 off is better. If a student uses only one example to determine the better offer, e.g., a $50
purchase, they would conclude that the $10.00 off is a better deal. Whereas if they choose $150 to use as an example, they may conclude that the 10% off is the better deal.

**Paint problem.** Formal proportional reasoning studied by Lamon (1999) also includes the ability to make the distinction between problems that involve direct proportions and those that do not. Questions requiring this distinction involve an indirect or inverse proportion that is often confused with a direct proportion because it involves work and time. EU10 suggests looking for certain “cues” in problems. In this problem, three numbers are given, and another number is missing. This might suggest setting up a proportion that leads to an incorrect response. The problem reads:

> If one man can paint a room by himself in 3 hours, then how long will it take two men to paint the same room if they both paint at the same rate? (Adapted from Lobato & Ellis [2010, p. 46])

If a pre-service teacher only has an algorithmic approach to proportions, they might set this problem up as \( \frac{1}{3} = \frac{2}{x} \) and get a solution of 6 hours. However, common sense suggests that it should take less time for two people to paint the room, and this answer suggests that it will take more time. The pre-service teacher needs to recognize that the number of workers is indirectly proportional to the amount of time. Without considering the details of the situation, some pre-service teachers may take “cues” from the situation (EU10), mechanically set up this problem using an algorithm (EU9), and solve it without thinking about the answer in terms of the question being asked. If the participant uses cross multiplication, I might follow-up this question by asking what is the relationship between cross-multiplication and proportions? This would help me to delve deeper into EU9 and EU10.

All of these interview problems will enable me to delve deeper into pre-service teachers’ proportional reasoning in order to determine the nature of this knowledge. The individual interviews were used to help me explore the shifts and transitions in thinking that these students
have made in their proportional reasoning thus far. Ultimately, this analysis should allow me to answer my research question.

**Data Collection Plan**

Data collection occurred during two semesters, the spring and summer semesters of 2012. During the spring semester I focused on the secondary education majors. During the summer session, I focused on the elementary education majors taking their Language and Literacy Education (LLED) block. While collecting data for each major, there was a time selected (based on potential participants’ schedules) for the administration of the questionnaire, followed by time to analyze this data to help me determine whom I would choose to participate in a follow-up interview. In addition, there was time to design an interview schedule to be used for each interviewed participant, followed by an analysis of each participant’s individual data collected. Using this analysis, a description of each participant’s proportional reasoning in terms of the shifts would be written. A final analysis across all the participants would be done to look for patterns and help me describe pre-service teachers’ proportional reasoning in terms of the shifts. A plan for this analysis will be discussed in detail following the timeline.

**Timeline.** I planned to collect data during both spring semester 2012 and summer session 2012. I collected data from secondary education majors during the spring 2012 semester. This would be done at the very beginning of the semester because they would be currently taking their methods course when I recruited them, and I wanted to acquire data before they had much exposure in this course. The elementary education majors would be taking their Language and Literacy Education (LLED) block during the summer 2012 and would not have had their mathematics methods course until the following semester.
Table 4.1: Timeline of Data Collection 1

<table>
<thead>
<tr>
<th>Date</th>
<th>Secondary Education/Mathematics Option Majors</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week of 1/9</td>
<td>Contact instructors MTHED 411 (the first secondary mathematics methods course) to ask if I can recruit volunteers to complete questionnaire</td>
</tr>
<tr>
<td>Week of 1/16</td>
<td>Present recruitment flyer and script to MTHED 411 class with date, location and time of questionnaire administration</td>
</tr>
<tr>
<td>Week of 1/23</td>
<td>Remind MTHED 411 when and where questionnaire will be administered.</td>
</tr>
<tr>
<td>Week of 1/23</td>
<td>Administer questionnaire</td>
</tr>
<tr>
<td>Weeks of 1/30-2/3</td>
<td>Analyze questionnaires</td>
</tr>
<tr>
<td>Weeks of 2/2 and 2/6</td>
<td>Determine which participants to be interviewed and what the focus of their individual interview will consist of in terms of interview schedule, what questions to ask and why those questions</td>
</tr>
<tr>
<td>Weeks of 2/6 - 2/13</td>
<td>Contact participants to be interviewed to set up time and place of interview</td>
</tr>
<tr>
<td>Week of 2/20 and 2/27</td>
<td>Conduct interview (video and audio tape)</td>
</tr>
<tr>
<td>Week of 2/27</td>
<td>Transcribe and annotate interviews</td>
</tr>
<tr>
<td>3/5-3/31</td>
<td>Analyze interview and questionnaire</td>
</tr>
<tr>
<td>4/1-4/30</td>
<td>Look for patterns across and between questionnaires and interviews</td>
</tr>
</tbody>
</table>
Table 4.2: Timeline of Data Collection 2

<table>
<thead>
<tr>
<th>Date</th>
<th>Elementary Education Majors Data Collection 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week of 6/3</td>
<td>Contact instructors LLED block to ask if I can recruit volunteers to complete questionnaire</td>
</tr>
<tr>
<td>Week of 6/10</td>
<td>Present recruitment flyer and script to LLED classes with date, time and place of administration of questionnaire</td>
</tr>
<tr>
<td>Week of 6/17</td>
<td>Remind LLED classes of questionnaire administration</td>
</tr>
<tr>
<td>Week of 6/17</td>
<td>Administer questionnaire</td>
</tr>
<tr>
<td>Week of 6/24</td>
<td>Analyze questionnaire</td>
</tr>
<tr>
<td>Weeks of 7/1-7/7</td>
<td>Determine which participants to be interviewed and what the focus of their individual interview will consist of in terms of the interview schedule; questions to be asked and why those questions.</td>
</tr>
<tr>
<td>Week of 7/7</td>
<td>Contact participants to be interviewed to set up time and place of interview</td>
</tr>
<tr>
<td>Weeks of 7/14 and 8/9</td>
<td>Conduct interviews</td>
</tr>
<tr>
<td>Weeks of 7/14-8/9</td>
<td>Begin to transcribe and annotate interviews</td>
</tr>
<tr>
<td>7/21-8/9</td>
<td>Analyze interview and questionnaire</td>
</tr>
<tr>
<td>8/9-9/6</td>
<td>Look for patterns across and between questionnaires and interviews</td>
</tr>
</tbody>
</table>

As seen in the timeline shown in Tables 4.1 and 4.2, I planned to collect data during two semesters, the first in spring 2012 (from 10-15 secondary education majors). The data collection included administrating and analyzing the questionnaire prior to individual interviews. The analysis and data collection would occur in separate phases. First the participants would be asked to come at a specified time to a particular location to complete the questionnaire. Since the secondary pre-service teachers I recruited would be taking their methods course, it was important that they completed the questionnaire and even the interview as soon as possible after the start of the semester. For his or her participation in this part of the study, each volunteer would be compensated.

The questionnaire data would be analyzed to determine which participants were asked to return for a follow-up interview. I explained this process in phase two of the data collection and analysis section. During the interview, video and audio were used to record the participants’ work and discussion about each problem. All artifacts created during the interview were used as
data. These participants received additional compensation for completing the interview part of the study. The audio was transcribed by the researcher and then analyzed using the previously presented framework based on Lobato and Ellis (2010). This analysis provided a rich description of the nature of these pre-service teachers’ proportional reasoning.

**Data Collection and Analysis**

The data collection and analysis took place in five phases. The first phase was the administration and analysis of the questionnaire. This analysis helped me to conjecture what shifts the pre-service teachers had made and where they might fit along the continuum. Using the analysis of the questionnaire, the second phase involved the selection of pre-service teachers whom I interviewed. The presence or absence of a series of yes or no answers in a table (see Appendix E) to the question “Is the shift complete?” helped me identify participants whose reasoning appears to match or refute the shifts. I also identified participants whose responses on the questionnaire are incomplete or atypical. The third phase included the preparation and administration of the interview schedule for each participant’s individual interview. The fourth phase involved the analysis of each individual’s interview and questionnaire in order to develop a detailed description of the participants’ proportional reasoning in terms of the shifts and EUs. The fifth and final phase involved an analysis of similarities and dissimilarities in the descriptions of each participant’s proportional reasoning. This final phase helped me to articulate how the Lobato and Ellis (2010) shift model is or is not a viable model to describe the extent to which pre-service teachers have learned to reason proportionally.

**Phase 1: Analysis of questionnaire.** In order to manage the questionnaire data I created the table template seen in Appendix E. This table allowed me to look for patterns in terms of the shifts. A completed table for each participant let me organize the data and examine whether the pre-service teacher had made shifts based on the reasoning provided on their questionnaire. First, I took each questionnaire and attached a table to it, coding it with the person’s identification number. Then, I wrote in each row of the table whether the participant answered the question
correctly, and I coded the answer as “C” for correct or “I” for incorrect. Next, I summarized the reasoning that the participant provided (in column 3 of the table) and noted how this reasoning related to the shifts and EUs (in column 5 of the table). I realized that some participants may not provide detailed information about how they reasoned while solving the problems on the questionnaire. In the 3rd column of the table, I noted whether the description of the reasoning is complete, incomplete, unclear or atypical so that I can follow up on these questions during a possible interview.

For example, a participant in the summer 2011 sample data answered the Basketball problem incorrectly by selecting Barbara as the person who was the best player. I would put an “I” for incorrect in the second column. Their work showed that they made a computation error. I would make a detailed summary of the complete (I would write “complete” in column 4) data provided concerning this computation error and their reasoning in the third column. In this example the work also revealed that the volunteer compared the quantities of shots made to shots taken. This suggested that they had made shift 1 and recognized that proportional reasoning involves two quantities. I would note that I believed this participant has made shift 1 by writing “yes” in the fourth column and then explaining my reasoning in the fifth column under “why or why not?” The table helped me to quickly see how the provided reasoning and shifts fit together for each participant. It also let me look across the different questionnaires to identify patterns in reasoning.

In the sample data collected in summer 2011, I found that the volunteers appeared to fall into five groups. Group 1 provided evidence that they made all four shifts. They not only answered the questions correctly but also provided data that suggested that they recognize proportions as an infinite set of equivalent ratios and could distinguish between proportional situations and non-proportional situations. See appendix F for an example of a data analysis table of a volunteer who would be part of Group 1 (Volunteer #7). Group 2 seemed to use multiplicative comparisons (i.e., seemingly making shift 3) but did not distinguish between
proportional situations and non-proportional situations (i.e., Track problem). This makes me question whether they had completed shift 4. For an example of a volunteer who would fall into Group 2 see appendix F (Volunteer #4). Group 3 members used multiplicative reasoning to solve the problems (i.e., seemingly making shift 2) but were not able to work with a part-part ratio (i.e., Cat/Dog problem) nor recognize non-proportional situations (i.e., Track problem), making their reasoning appear to fall somewhere between shift 2 and shift 3. An example of a data analysis table from Group 3 can be found in appendix F (Volunteer #5). Group 4 used additive reasoning to solve problems (i.e., Mr. Tall/Mr. Short problem), suggesting that they had not made shift 2. An example of Group 4 can be found in Appendix F (Volunteer #9). Common across questionnaires for this group is an answer of “I don’t understand” or “I don’t know how to solve this problem” for many of the problems. Group 5 did not provide enough evidence of their reasoning to determine where they fell on the continuum. An example of Group 5 can be found in Appendix F (Volunteer #16).

I recognize that there are limitations to using the questionnaire as primary evidence of the participants’ proportional reasoning. The interview, described in phase 3, was used to gather more data on the incomplete or missing pieces of reasoning from the questionnaire and to help me describe in more detail the participants’ proportional reasoning in terms of the shifts. These five groups from the pilot data were the basis for the selection of participants I interviewed. I also noticed that these groups seemed to align with the four shifts as described above. I will describe the interview selection process in the next phase.

**Phase 2: Selection of interviewees.** The sample data collection from Summer 2011 suggests that the volunteers provided evidence that placed them into five possible groups. These seem to align with the shifts, for example Group 1 seems to have provided evidence of completing all four shifts. Group 2 seems to be somewhere between shift 3 and shift 4. Group 3 fell between shift 2 and shift 3. Group 4 fell between shift 1 and shift 2. Group 5 did not provide enough data to determine where they fall within the shifts. Considering most of the volunteers
who fell in Group 5 provided data that suggested that they do not understand how to solve proportional reasoning problems. I might claim that they are pre-shift 1; however, I recognize that lack of data does not mean lack of understanding only that they did not know how to do these problems.

The analysis of the questionnaire helped me determine which group the participant was a member of and where they might fall on the continuum in terms of those shifts. Based on the participant’s questionnaire solutions and explanations, I chose 1-2 members from each group to be interviewed so that I could delve further into the nature of their proportional reasoning. The interview helped me to gain a deeper understanding of how prospective teachers’ proportional reasoning might be described in terms of the four shifts. I realized that the volunteers from whom I collected data during summer 2011 were not from my sample population (i.e., some were still in high school and middle school). Therefore, I may not have had any pre-service teachers who fell into Group 5 (those who did not provide any data for proportional reasoning) making only four groups to study more in-depth. I anticipated interviewing 8-10 participants, 1-2 from each group; however, my priority was choosing participants who represented different combinations of the shifts.

For example, one volunteer whom I asked to complete the questionnaire correctly answered the first three questions (i.e., Orange Juice problem, Basketball problem, and Mr. Tall/Mr. Short problem) with explanations that provided evidence that they had made the first shift and the beginnings of shift 2. Their explanation for the problems provided evidence of their recognition that ratio consists of two quantities and is multiplicative in nature. They chose to use a multiplicative comparison to find the number of paperclips needed to find Mr. Tall’s height. This particular volunteer used the unit ratio to determine the cost of the granola and different weights in the Health Food Store problem; however, they did not identify ratio as a part–part comparison in the Cat–Dog problem nor correctly recognize the Track problem as requiring additive reasoning, making them a member of Group 3. The inconsistency in their answers on the
questionnaire led me to believe that one or more of the shifts had not been completed in this person’s proportional reasoning. I would select this volunteer to be interviewed as a typical member of Group 3.

**Phase 3: The individual interview.** Prior to the interview I developed individual interview schedules, selecting from questions discussed in the instruments section of this chapter. I planned to have the interview last 60 minutes, so I would need to use that time efficiently by focusing on those questions that seemed to provide me with the evidence needed to draw conclusions about the participant’s proportional reasoning. The questions I chose focused on areas of the participant’s questionnaire that seemed inconsistent with Lobato and Ellis’ (2010) shifts and on responses in which their reasoning was unclear, incomplete or atypical. I started by asking questions about the participant’s responses on the questionnaire. If they had solutions in which the reasoning was unclear, incomplete or atypical, I would ask them to explain to me how they solved the problem in more detail. This was used to later describe their proportional reasoning. I continued the interview by asking the participant to solve additional problems that helped me understand their proportional reasoning in terms of the shifts. The inconsistencies that I found in the analysis of their questionnaire would be the basis of the problems that I selected to focus on during the interview.

For example, the participant described in the interview selection phase was a member of Group 3. I used the interview to further determine the nature of the participant’s proportional reasoning. During the interview I would focus on questions that involved ratio as a part–part comparison as in the Boy–Girl class problem (p. 63) and follow-up by asking the participant to talk about the relationship between ratios and fractions. In order to gain more insight into shift 2, I would ask this particular volunteer to explain their reasoning for solving the Paint problem (p. 66) because it uses an inverse proportion, and the use of an algorithm may lead to a non-sensible result. The House problem (p. 58) would be asked because it tends to elicit additive reasoning, providing more evidence for shift 2. In addition, I would use the Balloon problem (p. 61) to
determine if a unit ratio is used to find the solution or if the volunteer iterates the ratio to find the solution, suggesting that they might not have made shift 3. If the participant indicated he or she used a proportion, I would follow-up this problem by asking them; what is a proportion? Similarly, if they described their work as “I used cross multiplication,” I would ask, how are cross-multiplication and proportions related? This would help me gain insights into EU6 and EU9. If there is extra time during the interview I would ask the volunteer to reason through the Lemon/Lime problem (p. 55) to gain understanding of their ability to explain their reasoning without using numbers. If the participant used the word ratio to describe their reasoning, I might ask them; what they think a ratio is and how it applies in this problem? This would help me delve deeper into their understanding of ratio in terms of proportional reasoning.

Phase 4: Combining interview and questionnaire data for each participant. I analyzed the interview data with the questionnaire data to provide me with a detailed description of each of the participants’ proportional reasoning in terms of the shifts. I transcribed and annotated the interview in order to analyze the data. A table similar to the one used with the questionnaire (without the question names in column 1) was created for each participant's individual interview transcript to help me organize the data and pinpoint evidence in terms of the shifts. I attached a table to the transcript and completed it similar to the way I described for the questionnaire. After completing a table for each participant interviewed, I first focused on the shifts that I had concrete evidence that the participant has completed and described their proportional reasoning in terms of those shifts. I would use the EUs and other research to help me explain why I believed the participant had provided evidence of making those shifts.

Next I would focus on questions that seemed to be inconsistent with the shifts and use the data from all of the questions (i.e., questionnaire and interview) to try and explain why the participant may or may not have made certain shifts. If it appeared that the data contradicts the shifts, I would use the EUs to help me explain why there might be conflicts with the participant’s proportional reasoning and the shifts. It was possible that the shift model would not be able to
explain these inconsistencies, in which case I would use other research to justify and describe the nature of the individual’s proportional reasoning. The purpose of this analysis was to create a detailed description of the participant’s proportional reasoning in terms of the shifts.

For example, if I interviewed Volunteer #4 and they were able to correctly apply a part–part comparison to the Boy–Girl problem (p. 63) as well as solve the inverse proportion Paint problem (p. 66) and the Bike problem (p. 60), I might conclude that they had indeed made all four shifts. It would be possible that the context of the Dog/Cat problem on the questionnaire may have caused them to reason incorrectly. I would use the responses to some of the open ended questions about ratio, proportions and their relationships to help me explain the participants’ proportional reasoning based on the EUs. Other research on problem context would be an important part of describing and explaining the evidence provided by the participant in their interview and questionnaire in terms of the shifts. A detailed description was created for each pre-service teacher interviewed and I used the EUs to help explain where each participant fit along the continuum of shifts. This description also included justification based on the evidence they provided, that supported whether some shifts had been made or not.

**Phase 5: Analysis across Participants.** Once I had a detailed description of each pre-service teacher’s proportional reasoning based on their questionnaire and interview, I looked across the data for patterns. I looked for patterns across the interviews and questionnaires for each individual and within the different groups (i.e., Groups 1-5) to help me describe the nature of pre-service teachers’ proportional reasoning. First, I looked within each group and would expect to find similarities in responses to certain questions. Next, I tried to explain the differences between the members of the same group based on the EUs and other research. I would analyze each group to develop a combined detailed description of its possible members. I would make recommendations, based on Lobato and Ellis (2010) and other research, on how to help pre-service teachers within each group transition through the shifts they had not made as they continued their pre-service education.
Across the groups, I would also look for similarities and differences among the participants’ proportional reasoning. I would look for explanations of these patterns in the EUs and other research. These patterns across groups would be based on the shifts and would help me to determine the extent to which Lobato and Ellis’ (2010) shifts is a useful model in describing the proportional reasoning of prospective middle school mathematics teachers. I hoped to find patterns of reasoning based on the shifts that could be supported by teacher educators to help those pre-service teachers make the shifts in the future.

A summary of the five phases. The five phases of data collection and analysis for this study provided me with a better understanding of the pre-service teachers’ proportional reasoning. The first phase involved a questionnaire in which the participant provided only written responses and explanations on paper. The questionnaire data used during phase 2 provided some evidence of the participant’s proportional reasoning. The second phase used the analysis of the questionnaire to select participants to be interviewed. The questionnaire helped me to determine what questions to ask during the individual interview with each participant. As illustrated in the example above, questions asked during the interview would be tailored to the individual’s proportional reasoning suggested by their questionnaire responses and corresponding to the shifts. During phase 3, I focused on those questions in which there was not sufficient evidence of the participant having made a particular shift. Phase 4 involved the interview transcript and questionnaire being analyzed to develop a detailed description of the nature of the pre-service teachers’ proportional reasoning. In phase 5, I looked for patterns across the data based on the shifts, using the EUs and other research to explain these shifts. Finally, using all of the analysis I determined the extent to which Lobato and Ellis’ (2010) shifts are useful in modeling the proportional reasoning of prospective middle school mathematics teachers. The final result of this study would be a summary of the findings based on the patterns found across the data and on the detailed description of each pre-service teachers’ proportional reasoning.
Potential Results

My results would begin with a detailed description of the participants’ proportional reasoning and how it is consistent or inconsistent with the shifts described by Lobato and Ellis (2010). I would describe the patterns found across the participants within a group and articulate how the shifts can explain these pre-service teachers’ proportional reasoning. Then I would look across the groups for patterns and use the EUs to help me explain why participants have or have not made a shift. For example, I might conclude that a participant who uses additive reasoning while solving a proportional reasoning problem does not understand that “a ratio is a multiplicative comparison of two quantities” (EU2). In addition, I would also include a description of what the participant does know about proportions. For example, the participant who uses additive reasoning might recognize that “ratio involves attending to and coordinating two quantities” (EU1) and that ratio “involves isolating that attribute from other attributes” (EU3), causing me to conclude that they have made shift 1.

Based on the volunteer questionnaires that I have analyzed and the previous literature on student proportional reasoning, I expected to find that pre-service teachers understand that ratio involves two quantities (shift 1), and that it is multiplicative in nature (shift 2). I expect that some may have difficulty distinguishing between proportional situations and non-proportional situations based on cues in the problems (EU10) or an algorithmic view of proportions (EU9). Among those who seemed to indicate evidence of shift 3, I think pre-service teachers might still demonstrate difficulty in understanding ratio as a part-part relationship. EU4 explains the link between ratios and fractions, and since most pre-service teachers have more experience with fractions, there is often a misconception that ratio and fraction are the same. I would explicitly ask these participants about their understanding of the relationship between ratio and fraction. A problem like the Dog/Cat problem may provide some difficulty to pre-service teachers if they do not completely understand the relationship between fractions and proportional reasoning. This allows EU4 to help me explain why the pre-service teacher may not have transitioned through
shift 3. I expected similar use of the other EU in helping me to explain pre-service teachers’ proportional reasoning with respect to the shifts. If the data supported these claims, I would conclude that the nature of pre-service teachers’ proportional reasoning is consistent with the shifts described by Lobato and Ellis (2010) and that the EU proposed in the model can be used to explain these shifts.

Using the descriptions of proportional reasoning created for each group, I would also provide suggestions on how to help pre-service teachers in each group make the shifts to develop their formal proportional reasoning. I would base these suggestions on the research of Lobato and Ellis (2010) and others. For example, I might suggest that a teacher educator present their prospective teachers with opportunities to represent composed units with a drawing in order to help them further develop their understanding and transition through shift 3. Understanding pre-service teachers’ proportional reasoning will enable teacher educators to create learning opportunities that further develop formal proportional reasoning for future middle school teachers. In this way, teacher educators can “scaffold” these pre-service teachers’ understanding of proportionality.

**Data Analysis Growth**

The data was analyzed keeping in mind the research question: To what extent is the nature of pre-service teachers’ proportional reasoning near the beginning of their formal teacher preparation [in a college-level program] consistent with the shifts described by Lobato and Ellis (2010)? It became apparent that in order to answer this question I needed to define what was meant by “consistent with the shifts.”

**Defining consistent.** The Oxford Learning Dictionary (2012) defines consistent as “always behaving in the same way or according to the same standards.” Given this definition, one would always have to behave in a way that provides evidence of having completed a shift or would provide evidence of having not completed a shift. One possibility is that the term consistent refers to whether the shifts are hierarchical. This means that once one completes the
first shift according to Lobato and Ellis’ (2010) definition, only then could one move to the next shift. For example, only after one recognizes a ratio as the joining or composing of two quantities such that they maintain a multiplicative relationship (shift 2) could they make the transition toward more abstract proportional reasoning with multiplicative comparisons (shift 3). Additionally, this means once one completely makes a shift in their reasoning they would not go backwards. For example, if one has completely made shift 2 they would not use additive reasoning when solving proportions because they would understand that ratios maintain a multiplicative relationship.

Not surprisingly a challenge arises when one is still in the process of making a shift. Since research suggests that shifts in an individual’s proportional reasoning takes a considerable amount of time (Lesh, Post & Behr, 1988), it is possible that during an interview they would still be in the process of making a shift and provide evidence inconsistent with the shift. For example, if one is in the process of making shift 2, they might provide evidence of using multiplicative reasoning when working with ratios on one particular problem but on another problem they might use additive reasoning with ratios.

Shift 3 and shift 4 provide their own unique challenges. Since shift 3 depends on using a particular strategy to solve a problem. It is possible that even if one has made this shift in their reasoning that the context of the question makes it more efficient to use an iterative strategy and not a multiplicative comparison. In this case the evidence provided might be considered inconsistent with the shifts, when in fact it is not. Additionally, shift 4 also presents a challenge because one can expand a set of ratios even to include fractions or decimals but not provide evidence or have an understanding that rates are a set of infinitely many equivalent ratios. All of these challenges were considered when analyzing the data to answer the research question.

**Addressing these challenges.** As discussed earlier in this chapter, specific questions were chosen to elicit evidence of each of the shifts. After the first round of data was collected, some tasks were modified or expanded in order to address the challenges of inconsistency that
seemed to exist within the shifts. Additionally, other tasks were added to the interview in order to focus on these same challenges. For example, a task involving sorting ratios into equivalent groups was added in order to address the participant’s understanding of expanding the set of ratios based on multiplicative reasoning and to elicit specifically the concept of rates as an infinite set of equivalent ratios. Furthermore, follow up questions were asked so I could determine if one was in the process of making a shift or if they had indeed made a shift.

For example, if I questioned whether a participant had made shift 2, I might ask if they believed 3:4 and 5:6 were equivalent because they had a relationship of +1. Someone who had made the shift would claim that they were not equivalent because equivalent ratios have to have multiplicative relationship. Someone who had not made shift 2 would believe that these ratios were equivalent. In the case of shift 3, questions might be asked to determine if iterative methods were being used because they found them to be more efficient or if they could not reason about the problem using a multiplicative comparison. In order to understand shift 3 better, the interview was also revised to include opportunities for participants to explain alternative methods to solve problems.

**Questionnaire Analysis and Placement in Groups.** After completing the questionnaire, participants were put into one of four groups based on their responses. Table 4.3 represents the number of participants that completed the questionnaire and the number placed in each group. The table also indicates the number in the first group of participants (Secondary education majors) and those in the second group of participants (Elementary education majors). These groups were discussed in detail earlier in this chapter under analysis of questionnaire. More detailed claims about the consistency of each group’s reasoning in terms of shifts will be discussed in subsequent sections and in Chapter 5.
Table 4.3: Percentage of participants placed in each group after analysis

<table>
<thead>
<tr>
<th>Group</th>
<th>Secondary</th>
<th>Elementary</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Group 1</td>
<td>4%(1/25)</td>
<td>8%(2/25)</td>
<td>12%(3/25)</td>
</tr>
<tr>
<td></td>
<td>52%(13/25)</td>
<td>48%(12/25)</td>
<td>100%</td>
</tr>
</tbody>
</table>

Group 1 was identified if they answered all the problems on the questionnaire correctly and provided complete reasoning. During the analysis of the interview, I found that those placed in this group had proportional reasoning consistent with the shifts. One participant was originally placed in this group, since she provided correct answers on the questionnaire but not the reasoning used to make the calculations. It became apparent during the analysis of the interview that this participant belonged in Group 2 because her reasoning was not consistent with the shifts. Group 2 was identified if they answered all the questions on the questionnaire correctly, except the Track problem. In this problem the participants placed in this group overextended the concept of proportion to use ratios in an additive scenario possibly due to the context of speed in the problem. This might be explained with EU9 and EU10. During analysis of the interview, I found that the reasoning presented by these participants was inconsistent with the shifts. Groups 3 consisted of participants that seemed to use multiplicative reasoning on proportional problems but were unable to solve the Dog/Cat problem and were unpredictable with their reasoning on the questionnaire. The analysis of the interview concluded that this group’s reasoning was also inconsistent with the shifts. This might be explained with EU4 and EU5. Group 4 seemed to use additive reasoning or iterative methods when solving proportional problems like the Health Food Store problem. Although this group’s solutions were incorrect, their reasoning appeared consistent with one that had not made shift 2. This might be explained with EU2 and EU6. There
was one participant placed in this group after the analysis of the questionnaire, whose reasoning during the interview seemed more consistent with someone trying to make shift 2; therefore the participants’ performance was inconsistent with the rest of group 4. I subsequently moved this person to Group 3 because their reasoning seemed to be inconsistent with the shifts.

Although the analysis questionnaire was used to place individual participants into four groups, it was analysis of the interview that determined the final placement of the participants into 4 groups. Two participants were placed in Group 1, in which their reasoning was consistent with the shift model. Seven participants were placed into Group 2 and Group 3, and their reasoning was inconsistent with the shift model, since they were in the process of making the shifts in understanding. Two participants were placed in Group 4; their reasoning was consistent with the shift model and confirmed that they had not made shift 2.

**Determining the consistency of groups.** How did I determine that some groups were consistent with the shift while others were inconsistent? I used two levels of evidence (macro and micro) to determine the consistency of someone’s reasoning. The macro or surface level of evidence would include just looking at the answer or the steps/calculations used to find the answers only on the questionnaire and not the reasoning involved in finding those responses. The micro level of evidence looks beyond the solution to the reasoning used to solve the problem. During the interview, I was able to ask probing questions that helped me determine the reasoning the participant used when answering problems providing a better understanding of their proportional reasoning in terms of the consistency with the shifts.

How did these different levels of looking at evidence allow me to determine if participants were placed into the correct group? Initially, participants were placed into four groups based only on their responses on the questionnaire. This was the macro or surface level analysis of their proportional reasoning and provided me with only a superficial understanding of the participant’s reasoning in terms of consistency with the shifts. For example, several participants were misplaced in groups based on the analysis of the questionnaire but upon a more
micro level examination of the data it was determined that this was due to the fact that parts of questions were left unanswered or only final answers with no explanation of their reasoning provided.

The interview allowed me to look at the participant’s reasoning at a micro level and ask follow up questions that might help me pinpoint their reasoning in terms of the shifts. For example, during the interview I could ask participants to clarify why they had not completed a problem or tell me what reasoning they used to arrive at a solution. These follow up questions allowed me to analyze the data at a micro level in terms of the consistency with the shifts.

**Triangulating analysis.** In addition to completing multiple passes through each participant’s data, I consulted my advisor and co-advisor about both the micro and macro level findings in terms of how the evidence aligned with the individual shifts. I also conferred with them about the patterns that I found within and between each group of participants. In addition, I trained a group of fellow graduate students and faculty in my analysis methods and asked them to code pieces of several different transcripts to determine if my methods were reliable. I found that there was considerable agreement between my initial coding and the coding decisions made by this group providing me with a degree of reliability with my procedures.

In the next chapter I will discuss the results of the macro level analysis of the questionnaire and the micro level analysis interviews and discuss the consistency of the shift model while answering the research question.
Chapter 5: Results

The data were analyzed in relation to the research question: To what extent is the nature of pre-service teachers’ proportional reasoning near the beginning of their formal teacher preparation [in a college-level program] consistent with the shifts described by Lobato and Ellis (2010)? In Developing Essential Understanding of Ratio, Proportions and Proportional Reasoning, Lobato and Ellis (2010) cast changes in proportional reasoning as transitions or “shifts” in students’ thinking and as a tool to “evaluate [a] student’s current thinking” (Lobato & Ellis, 2010, p. 61). Lobato and Ellis specified four such shifts:

Shift 1: From one quantity to two (Lobato & Ellis, 2010, p. 61)
Shift 2: From additive to multiplicative comparisons (Lobato & Ellis, 2010, p. 63)
Shift 3: From composed-unit strategies to multiplicative comparisons (Lobato & Ellis, 2010, p. 69)
Shift 4: From iterating a composed unit to creating many equivalent ratios (Lobato & Ellis, 2010, p. 71)

In addition to the four shifts, Lobato and Ellis (2010) described one big idea and ten EUs, which describe what teachers need to know if they can expect their students to learn proportional reasoning. The big idea is “when two quantities are related proportionally, the ratio of one quantity to the other is invariant as the numerical values of both quantities change by the same factor” (Lobato & Ellis, 2010, p. 11). The EUs are:

1. Reasoning with ratio involves attending to and coordinating two quantities.
2. A ratio is a multiplicative comparison of two quantities or it is a joining of two quantities in a composed unit.
3. Forming a ratio as a measure of a real-world attribute involves isolating that attribute from other attributes and understanding the effect of changing each quantity on the attribute of interest.
4. A number of mathematical connection link ratio and fraction:
a. Ratios are often expressed in fraction notation, although ratios and fractions do not have identical meaning.

b. Ratios are often used to make “part-part” comparison, but fractions are not.

c. Ratios and fractions can be thought of as overlapping sets.

d. Ratios can often be meaningfully reinterpreted as fractions.

5. Ratios can be meaningfully reinterpreted as quotients.

6. A proportion is a relationship of equality between two ratios. In a proportion, the ratio of two quantities remains constant as the corresponding values of the quantities change.

7. Proportional reasoning is complex and involves understanding that:

   a. Equivalent ratios can be created by iterating and/or partitioning a composed unit;

   b. If one quantity in a ratio is multiplied or divided by a particular factor, then the other quantity must be multiplied or divided by the same factor to maintain the proportional relationship; and

   c. The two types of ratios – composed units and multiplicative comparisons – are related.

8. A rate is a set of infinitely many equivalent ratios.

9. Several ways of reasoning, all grounded in sense making, can be generalized into algorithms for solving proportion problems.

10. Superficial cues present in the context of a problem do not provide sufficient evidence of proportional relationships between quantities. (p. 12-13)

    The EUs are understandings that teachers should have in order to help their students achieve the shifts. In particular, teachers should be able to use these EUs to identify how their students’ reasoning fits into the shifts. In this spirit, we might be able to use these EUs to explain how the pre-service teachers’ proportional reasoning corresponds with the shifts.
Consistent vs. Confirm

This research was to determine the extent to which a pre-service teacher’s proportional reasoning is “consistent with the shifts.” I reserve the words consistent or inconsistent to describe how a group of participants’ reasoning relates to the shift model. I use contradicts and confirms, when describing how a participant’s reasoning relates to a particular shift. As I refer to the confirmation of shifts parenthetically as shift \( x \), I refer to the contradiction of a shift parenthetically as not shift \( x \). For example, when one uses additive reasoning when reasoning proportionally this is evidence that contradicts shift 2 (i.e., not shift 2), however if one uses multiplicative reasoning when reasoning proportionally this is evidence that confirms shift 2.

If we assume that the shifts are a hierarchical model, then, in order to be consistent with the shifts, a shift must be completed or satisfied completely according to the Lobato and Ellis (2010) definition before one can move to the next shift. For example, only after one has made shift 2 can they make shift 3; additionally if one has made shift 3 they would not provide evidence that contradicts shift 2. This implies that in order for a pre-service teacher’s reasoning to be consistent with the shift model, they should always provide evidence of confirming a shift or always provide evidence of contradicting a shift. In general, I found that the 65% [see table 4.3] of pre-service teachers’ in this study had reasoning which was inconsistent with the shifts model.

Determining the Four Groups of Participants

As noted in chapter 4, the analysis of the questionnaire was used to place individual participants into four groups; however, it was the analysis of the interview that determined the final placement of the participants into these groups. Group 1 was identified if the participant answered all of the problems on the questionnaire correctly and provided complete and correct reasoning in the interview. Two participants were placed into Group 1 and their reasoning was consistent with the shifts model. Group 2 was identified if they answered all the questions on the questionnaire correctly, except the Track problem. In this problem these participants
overextended the concept of proportion to use ratios in an additive scenario due to the context of speed in the problem. Three participants were placed into Group 2. These participants seemed to be in the process of making one or more of the shifts and their reasoning was inconsistent with the shifts model, as will be discussed later in this chapter. Group 3 consisted of participants that seemed to use multiplicative reasoning on proportional problems and was unpredictable with their reasoning on the questionnaire. Four participants were placed into Group 3. The analysis of the interview revealed that this group’s reasoning was also inconsistent with the shifts. Group 4 seemed to use additive reasoning or iterative methods when solving proportional problems. Although this group’s solutions were incorrect, their reasoning confirmed that they had only completed shift 1. Two participants were placed into Group 4. In terms of the shift model, their reasoning appeared consistent with someone who had not made shift 2. Each participant was given a pseudonym and the following table lists their name and group affiliation.
Table 5.1: Pseudonym and Group Affiliation

<table>
<thead>
<tr>
<th>Pseudonym</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evelyn</td>
<td>Group 1</td>
</tr>
<tr>
<td>Sonny</td>
<td>Group 1</td>
</tr>
<tr>
<td>Elizabeth</td>
<td>Group 2</td>
</tr>
<tr>
<td>Susan</td>
<td>Group 2</td>
</tr>
<tr>
<td>Simon</td>
<td>Group 2</td>
</tr>
<tr>
<td>Seth</td>
<td>Group 3</td>
</tr>
<tr>
<td>Emma</td>
<td>Group 3</td>
</tr>
<tr>
<td>Eve</td>
<td>Group 3</td>
</tr>
<tr>
<td>Stephanie</td>
<td>Group 3</td>
</tr>
<tr>
<td>Steven</td>
<td>Group 4</td>
</tr>
<tr>
<td>Ellisa</td>
<td>Group 4</td>
</tr>
</tbody>
</table>

In summary, for those groups that have made all four shifts (Group 1) or who have not completed shift 2 (Group 4), consistency in terms of the shifts model seems to hold. Groups 2 and 3 seem to be in the process of making the shifts, and there was inconsistency in their reasoning in terms of the shifts model.

Consistency with Shifts

In the next section, I discuss my results in more detail. I look at the four groups and how their reasoning is consistent or inconsistent with the shifts model. I consider both the micro and macro levels of looking at the participants’ reasoning and describe the patterns that developed within a group after multiple passes through the analysis of the data. I make claims about their reasoning in terms of the shifts model and illustrate these claims with examples from the data.

Consistency of Group 1 with the hierarchy of shifts. Group 1 provided evidence of completing all four shifts and consistently answered proportional reasoning questions correctly.
Participants in this group would attend to and coordinate two quantities to solve proportional problems (shift 1). These pre-service teachers provided evidence that they understood ratios as a multiplicative relationship between two quantities (shift 2) and they could create a composed unit and use it as a multiplicative comparison (shift 3). Group 1 also provided evidence not only of being able to expand a set to include many equivalent ratios but also of recognizing that this set was infinite (shift 4). One example is the reasoning of Evelyn in response to the Health Food Store problem on the questionnaire, as seen in Figure 5.1.
From her work, we see a demonstration that Evelyn attended to the cost of the granola (1.5) and the weight (8), which is confirming evidence of shift 1. Evelyn coordinates these quantities using division to find the “cost per ounce,” indicating that she recognized there exists a multiplicative relationship between the two quantities providing confirming evidence of shift 2. However, it was not clear from her work shown on the questionnaire how she arrived at the missing costs and weights. The following is Evelyn’s response to a request for more detail about what she did to solve the Health Food Store problem on the questionnaire:

Evelyn: Well, it looks like you have some sort of ratio that you need to find, so I found the direct comparison. So we know that for 8 ounces of the, what is it? Granola it cost $1.50. So I tried to find the base of that so I want to find out like what one ounce would cost and so it looks like I divided 150 by eight to find that the cost per ounce was .1875 and then I just multiplied that by how many ounces there were to get the amount that would be the cost and then I did the opposite for this, so I would divide it, I think, I would assume by .1875 to get [3:54] the weight.

In this part of the interview, Evelyn provided evidence that she viewed this relationship as a ratio (line 73). She attended to and coordinated both weight and cost of the granola as quantities; this was confirming evidence of shift 1. She claimed that she “divided the 150 by 8 to find the cost per ounce” (lines 78-79), indicating that she recognized the multiplicative relationship that
existed between these two quantities and thus provided confirming evidence of shift 2. She then described how she used this unit ratio as a multiplicative comparison to find the missing weights and costs. She states how she multiplied the ounces by the unit price to find the missing cost or divided the cost by the unit price to find the missing weight, this was confirming evidence of shift 3. Similar to what occurred with most participants in Group 1, Evelyn’s reasoning about ratio as a set of infinitely many equivalent ratios is not readily evidenced in her solution to a problem. I followed up on this reasoning with Evelyn in her discussion of this problem. In the following interview excerpt regarding the Health Food Store (HFS) problem, Evelyn provided evidence that she could expand the set of equivalent ratios to include any size package:

I: So could you, basically could you find the cost of any size package?
Evelyn: If it was of granola by the ounce, yes.
I: Um, why does it make sense that you could find it for any size package?
Evelyn: Because I have a ratio that is consistent with everything.
I: Okay
Evelyn: So like it's not, you know like I have like ratios it’s the same throughout then you just have to manipulate the size in order to get it. I'm not sure if that makes sense but like you know.

Evelyn confirms that she understood there was “a ratio that is consistent” for different sizes of packages and that this set could be expanded to any size (shift 4). The other participant in Group 1 provided similar evidence of reasoning consistent with the shifts in all of the problems on the questionnaire and in the interview.

Furthermore, when Group 1 participants were presented with invalid ratio/proportional reasoning they were able to explain how this reasoning was faulty. Evelyn was asked “are 3:5 and 5:7 equivalent ratios since they have a relationship of +2.” She responded, “No, because … I increase by multiplication not addition in order for it to be a ratio.” Evelyn’s understanding that multiplication maintains a proportional relationship can be explained with EU7 because of her recognition of the invariance needed for proportional reasoning.
Additionally, participants in Group 1 successfully recognized questions that elicited an overextension of proportions to additive situations for participants from other groups. This recognition initially distinguished Group 1 members from those placed in Group 2. Group 1’s understanding of the Track problem can be explained using EU10, because they did not attend to speed as a superficial cue in the problem, but viewed speed as a necessary element of solving the problem. Both Evelyn and Sonny modeled the Track problem and Bike problem using a representation that illustrated the distance involved in finding the solution. For example, Sonny drew the following diagram in Figure 5.3 as he explained his reasoning on the Bike problem (Figure 5.2) during the interview.
Question #6: Ben and John both ride their bikes to school at the same speed. Ben leaves first, when Ben has rode 10 blocks, John has ridden 7 blocks. When John has completed 14 blocks, how many blocks has Ben rode his bike?

Figure 5.2: The Bike problem to which Sonny is responding

567    Sonny: So Ben might live here [SI draws circle in bottom
568            left of paper] and John might live here [SI draws
569            circle in right hand side of paper] and they are
570            going to meet here [SI make dot in middle of paper]
571            and right here is the 14 blocks [SI draws dot at top
572            of paper and writes 14 total blocks]...
Sonny: so, they were meeting here [Sonny draws the lines from two circles to the dot in the middle of the page], who did I say this was, who rides more, Ben, this is Ben, this is his 10 blocks [Sonny writes Ben on left line and 10 blocks], this is John’s 7 blocks [Sonny writes John and 7 blocks on right line] and so then they both have 7 blocks left to school [Sonny draws line from dot at top of page to dot in middle of page and writes 7 blocks]. Oh, this is John’s total. That’s just John’s total [Sonny crosses out 14 next to total blocks at top of page]. Ben’s total is 17. Ben rode 17 blocks. John rode 14 blocks. [Sonny writes Ben rode 17 blocks and John rode next to the 14].

Sonny used the drawing to explain the problem and to make sense of the distances John and Ben rode their bike. His use of representations in terms of sense making as the basis to solve problems can be explained using EU9 because Group 1 is able to use reasoning that is grounded in making sense of the problem. Initially, Sonny claimed that “Ben was 3 ahead and since they are riding at the same speed he is still going to be three ahead, no matter how many blocks they ride.” He tried to illustrate this claim in his representation as he indicated the distances that each boy rode his bike.

**Consistency of Group 4 with hierarchy of shifts.** Group 4’s reasoning was consistent with the hierarchy of shifts evidenced by their use of additive reasoning to solve proportional...
problems. The participants in this group were consistent in providing evidence of understanding ratios as only additive relationships, which contradicts shift 2. For an example of an additive relationship used in a proportional scenario see Steven’s work on the questionnaire (Figure 5.4).
Steven claimed in Mr. Tall/Mr. Short problem that since “the ratio from buttons to paperclips from Mr. Short is 6:4 or +2,” Mr. Tall would be 6+2 or 8 paper clips tall. Other researchers have found that a constant difference strategy, in which a difference between the two quantities of the ratio is used determine the missing term was common in childhood through adults (Karplus et al., 1974; Tourniaire & Pulos, 1985) this was similar to Steven’s strategy. Additionally, during the sorting activity Ellisa places ratios into equivalent groups based on a common difference (Figure 5.5).
Ellisa: Okay, um, so there. These are right.

[Ellisa points to the ratios sorted below].

Figure 5.5: Ellisa’s sorted ratios

I: So why are the ones that you have together why are they together?
Ellisa: Because they are all, they all have the difference of the same number. So 10 plus 5 is 15 and 15 + 5 is 20 [Ellisa points to 10:15 and 15:20 slips]. And these have a difference of 2 [Ellisa points to 4:6, 6:8 and 10:12], and these have a difference of 1 [Ellisa points to 7:8, 6:7, 5:6, 3:4, 2:3].

In this passage, Ellisa’s reasoning confirms that her understanding of ratio involves an additive comparison between quantities of a ratio. EU6 explains why Ellisa’s understanding of ratio involving the concept of invariance, however for her the invariance was based on an additive relationship. EU7 addressed that in order to maintain a proportional relationship both quantities must be multiplied or divided by the same quantity. Ellisa claimed that the ratios maintain their relationship if the same quantity is added to both quantities. Her reasoning contradicts shift 2 but is evidence that for Group 4 the shifts are a consistent model.
In terms of shift 1, Group 4 provided evidence of this shift by recognizing multiple quantities in the problem. For example, in the Scalar problem (Figure 5.6) Ellisa determined the difference between the width of the original rectangle (2.0) and the enlarged rectangle (5.0) and then added this difference to the original length (2.4) to find the enlarged length (5.4).
Ellisa’s statement during the interview that the rectangles were proportional and that “when the width increases by 3 the length increase by 3” (lines 315-316) indicated that she recognized the two quantities (shift 1) but was reasoning about them additively (not shift 2). Ellisa’s understanding of ratios as coordinating and attending to multiple quantities can be explained by EU1. The lack of recognition of the multiplicative relationship (shift 2) involved in working with proportions seemed to prevent Group 4 from providing evidence of shift 3 and shift 4. This might be why this group solved several problems using the iterative approach successfully. For example, Ellisa described in the interview how she used an iterative approach to solve the Health Food Store (HFS) problem on the questionnaire (Figure 5.7):
Figure 5.7: Ellisa’s work on HFS problem from the questionnaire

Ellisa: um, so there is a difference of 8 and 8 – 6 is a
difference of 2 [Ellisa points to 6 ounces and 8
ounces in the weight column] um, and then I figured
I think at first I started if you multiply that [Ellisa
points to 8 ounces in weight column and then 16
ounces] by 2 you get 16 so it’s asking what that [I
points to “?” under cost column] would be so 150
times 2 is $3.00 [Ellisa points to $1.50 and the $3.00]
um and then I was just trying to figure out the
difference between $1.50 and $2.25 is 75 cents;
$2.25 to $3.00 is 75 cents but then when I got here,
that [Ellisa points to $4.50] would have to be $3.75
o let’s see what I did [pause] um, so then I don’t
know what I did there.

This approach indicated an understanding of the covariance of two quantities involved,
but used additive relationship to arrive at the solution. In this example we saw Ellisa use the
difference between prices compared to the difference between weights and coordinate them to
find the new weight or price, keeping this change constant. Ellisa recognized that the ratio of two
quantities remained constant as corresponding quantities change, which enabled her to change
one and then the other, which can be explained by EU6.

One Group 4 member presented a temporary and interesting challenge to the consistency
of the model when she used a memorized procedure to a solve problem. However, she was
unable to explain how or why the procedure worked. Her responses gave the surface level
appearance that she had mastered proportional reasoning when she might have enacted only a
memorized set of rules when she saw the word ratio in a problem. For example, Ellisa explained how she used an algorithm to solve the Boy/Girl problem (Figure 5.8).
Question #8c: Mrs. Jones class has 40 students they are divided into groups using 3:5 ratio of boys to girls. How many girls and how many boys does Mrs. Jones have in her class?

Figure 5.8: The Boy/Girl problem given to Ellisa during the interview

Ellisa: So, I take 3 fifths, 3 plus 5 equals 8 [E12 writes 3:5]
8/40=5] and then I multiply 5 by 3 to get 15 and I multiply 5 by 5 to get 25 [E12 writes 5\times3=15 and 5\times5=25] and then when I add the two up just to check [E12 writes +15=40] I get 40 so there are 15 girls and 25 boys [E12 writes 15g and 25b] or vice versa 15 boys and 25 girls [E12 crosses out g and b and writes 15b and 25g].

Ellisa added 3 and 5 and divided this sum by 40 before multiplying the result by each of 3 and 5.

Ellisa’s use of this procedure on this problem and on the Dog/Cat problem gives the appearance that she had made all four shifts in reasoning. However, when asked what is the 3:5 ratio mean her response was “I don’t know how to explain it… I just know that there are more girls than boys” (lines 759-762). When asked how she knew to add the 3 and 5, her response was “that is what I learned in [a university Mathematics content course for pre-service teachers]” (line 773).

EU9 explains how Ellisa can have an algorithm to solve proportional problems, however in this case, the algorithm was not grounded in sense making. It was a memorized procedure recognized by superficial cues (i.e., the word ratio in the problem) that provoke a set of steps to be acted on without meaning (Smith & Stein, 1998). EU10 explains how Ellisa’s use of superficial cues present in problems did not provide evidence of proportional reasoning. Even
though Ellisa provided evidence in this problem of confirming all four shifts, in other problems she used only additive comparisons in her proportional reasoning indicating inconsistency with the shift model. Given the fact that she used an algorithm and admits she does not know how or why it works prevented me from using this as evidence of her proportional reasoning, and therefore I would not consider this as evidence of inconsistency with the shift model for Group 4.

**Inconsistency in Groups 2 and 3 with hierarchy of shifts.** Groups 2 and 3 differed in several ways. Group 2 answered more problems correctly on the questionnaire than Group 3. However, members of Group 3 were more willing to change their reasoning during the interview compared to Group 2 who seemed adamant with their reasoning on problems during the interview. Participants placed in Group 2 and 3 were found to be inconsistent with their reasoning in terms of the shifts model. They always provided confirming evidence of shift 1 but were inconsistent in their demonstration of reasoning for shifts 2–4. In particular there was inconsistency with the use of multiplicative reasoning (shift 2). On some problems these groups provided reasoning that confirmed all four shifts, however on other problems their solutions used additive reasoning and contradicted shift 2. For example, Emma’s (Group 3) questionnaire provides one example in Figure 5.9.
In this problem, Emma (Group 3) provided evidence of all four shifts. In part a), she coordinated the weight on Earth and the weight on planet B (shift 1) and used division to find the relationship (shift 2). In parts b) and c), Emma (group 3) used the relationship as a multiplicative comparison in order to find the missing weights (shift 3). Finally, she expanded the set of ratios to include 130 lbs and 48.75 lbs, indicating that she recognized that the set of ratios can be expanded to include more equivalent ratios (shift 4). Emma’s (Group 3) reasoning for this problem would lead me to believe that her proportional reasoning was consistent with all four shifts. However, on the Scalar problem on the questionnaire Emma used an additive operation in a proportional situation (Figure 5.10).
This seemed to be evidence that Emma (Group 3) had not made the transition in her understanding that proportions involve multiplicative relationships. The fact that she provided evidence of making all four shifts in the Planet problem and then evidence that contradicts shift 2 and shift 3 in the Scalar problem, seemed inconsistent with the shifts being a hierarchical model. This evidence led me to conclude that the proportional reasoning of the participants in groups 2 and 3 was inconsistent with the shifts model.

In addition to inconsistency with the shifts model with problems on the questionnaire, the data contains multiple examples of inconsistency with the shifts model while engaging during the interview. In particular, these seemed to provide evidence that contradicted shift 2. In the Lemon/Lime problem, most of the participants in these groups would first use additive reasoning to compare quantities (not shift 2) and then change their reasoning to use a multiplicative comparison to compare the quantities (shift 2). Initially the participants were asked to compare Jen’s mixture containing 3 cups lemon juice and 2 cups lime juice with Alice’s mixture containing 4 cups lemon juice and 3 cups of lime juice without using calculations. The participants’ claimed that the mixtures would taste the same because each mixture contained one more cup of lemon than it contained cups of lime, pointing to the difference between amount of lemon and lime. This attention to “one more” seemed to indicate that these participants viewed the relationship in the quantities in Lemon/Lime problem as additive. However, when the
participants were allowed to use calculations, all of them created a ratio and then used division or common denominators in order to compare the mixtures, suggesting a multiplicative relationship. Eve (Group 3) demonstrated this type of change in reasoning in the passage from her work during the interview on the Lemon/Lime problem. Initially she looked at the mixtures with a constant difference strategy based on an additive relationship (see Figure 5.11 and 5.12).

698 Eve: Jen has one extra of lemonade and so does Alice, so
699 it would be both equal at this point and they each
700 have an extra cup of lemonade or lime juice or
701 lemon juice I mean, so they have an extra cup of
702 lemonade in both of theirs which would make it the
703 same amount so although hers [places hand on
704 Alice’s representation] has a higher cup value the
705 lemonade or lemony taste would be the same [Eve
706 places the yellow cube back on the bottom of each
707 mixture representation].
I: Could you create another mixture that would taste the same as those?
Eve: So, just like a whole separate one?
I: Yeah
Eve: So you could do, five to four [Eve creates another set of cubes], which would be the same.
In this passage Eve (Group 3) provided evidence that she compares the mixtures based on the difference between the lemon and lime in the mixture (not shift 2). She even created another mixture with 4 lemons and 3 limes that also has a difference of 1 more lemon than lime and claimed that it would have the same taste (lines 716-717). This lead me to wonder if she viewed equivalent ratios based on an additive relationship (not shift 4). When I allowed her to use calculations her reasoning changed to multiplicative reasoning and she creates fractions as demonstrated in the following passage (Figure 5.13 and Figure 5.14).

Eve: So, if Jen has [Eve is writing 5 and 7] so I’d put it in a fraction I guess, so total cups for Jen is five [writes 5 above Jen’s name] and total cups for Alice is 7 [writes 7 above Alice’s name].
Eve: So, her lemon juice is, Jen’s lemon juice is 3 so she would be 3 out of her 5 [Eve writes \( \frac{3}{5} = \) uses calculator to compute], and let me find the percentage which would be 60% [Eve writes \( \frac{3}{5} \) then uses the calculator to compute 3 divided by 5 then writes 60%] and Alice was mixes 4 cups, so she is 4 out of 7 total which is less, 57% [E8 writes \( \frac{4}{7} \) then uses calculator to compute 4 divided by 7 then writes 57%].
Eve (Group 3) now compared the mixtures based on a quotient or percentage of lemon in the total mixture. She created a part/whole fraction and divided to find percentages that she can compare to each other. This indicated a multiplicative comparison of the two quantities and a change in her previous additive reasoning. Eve’s (Group 3) reasoning was similar to that of the other participants in groups 2 and 3. These participants discussed the proportional relationship in the same scenario using both an additive relationship (not shift 2) and a multiplicative relationship (shift 2). This change in reasoning seemed inconsistent with the shifts as a hierarchical model of reasoning because if they recognized ratio as a multiplicative comparison of two quantities they would not use an additive comparison when initially comparing the mixtures.

The Housing problem also seemed to demonstrate this same inconsistency with the participants of groups 2 and 3. In this problem, participants needed to utilize a ratio of length to width of a rectangular lot to determine which was the most square. Members of groups 2 and 3 initially claimed that the squarest lot would have the smallest difference between the length and width because a square has a difference of zero between the length and width. When Susan (Group 2) was presented with the Housing problem, she initially compared the lots based on the difference between length and width. I perturbed this reasoning by presenting a situation in which
the two lots have the same difference (100’x200’ and 300’x400’) between the length and width, as in lines 519-522 of the following passage:

```
519 Susan: It’s kind of like the proportion again, like 100 to
520   300 feet is like a 200 difference, which is the same
521   from 200 feet to 400 feet so they are like similar
522   even though they are like not the same size.
523   I: So the one rectangle is 300x400 and the other
524   rectangle is 100x200, so
525 Susan: Yeah, so
526   I: They are both the same as far as being square?
527 Susan: Yes
528   I: Because their difference is the same, the difference
529   between the sides is the same, the same reason that
530   it was over here [I points to the paper with 60, 39
531   and 53 on it]?
532 Susan: Not here, um, kind of like the picture problem that
533   was being enlarged, like these dimensions are just
534   being enlarged it’s not um, doing anything else so
535 they are similar and they are going to look the same
536   shape even though one of them is bigger.
```

In this passage, Susan (Group 2) identified the relationship as a proportion and compares the quantities using an additive relationship. She then concluded that the rectangles are similar figures but one has just been enlarged, so they have the same squareness. Research suggested that scalar reasoning, determining the multiplicative relationship between A and C in A/B=C/D, is more complex than other forms of proportional reasoning (Lamon, 1993; Langrall & Swafford, 2000; Lo & Watanabe, 1997; Vergnaud, 1988) which others might view as the cause of the difficulty this group had with this problem. However, several of the participants in groups 2 and 3 seemed to use a multiplicative comparison on the Scalar problem when they completed the questionnaire, which would suggest that scalar reasoning is not the issue. For instance, Susan (Group 2) used multiplicative reasoning on the Scalar problem on the questionnaire (Figure 5.15).
Susan (Group 2) used a proportion to find the missing dimensions in this problem indicating that she understood a procedure based on a multiplicative relationship. In the housing problem Susan (Group 2) used the difference between the length and width of the lot and claimed that the two lots are similar “like the picture problem [Scalar problem] that was being enlarged, like these dimensions are just being enlarged” (lines 532-534) and so “they are similar” (line 535). The contrast between her reasoning in the Housing problem and the Scalar problem provided more evidence that the shifts model is inconsistent for participants in Groups 2 and 3. One difference between the two groups is that when I challenged their thinking by pointing out that the two lots had the same difference between length and width, the participants in Group 3 changed their reasoning whereas the participants in Group 2 continued to compare the relationship additively. However, the participants in Group 3 changed their reasoning on the Housing problem when it was challenged by the fact that two lots had the same difference between the length and width, still the participants in Group 2 continued to compare the relationship additively.

8. A picture is 2 cm wide by 2.4 cm long. If Fran wants to enlarge the picture to be 5 cm wide, how long should it be? If Fran wants to enlarge the picture so that it is 7.2 cm long. How wide should it be?

\[
\begin{align*}
\text{Width:} & \quad x &= 2 \\
\text{Length:} & \quad x &= 2.4 \\
\text{New Width:} & \quad x &= 5 \\
\text{New Length:} & \quad x &= 7.2 \\
\end{align*}
\]

\[
\frac{5}{x} = \frac{x}{7.2} \Rightarrow 5 \cdot 7.2 = 6x \\
30 = 6x \\
x = 5
\]

Figure 5.15: Susan’s work on the Scalar problem from the questionnaire and interview
Susan’s (Group 2) use of an additive comparison in the Housing problem seemed to contrast her use of a multiplicative comparison in the Scalar problem. Herein lies the inconsistency with the shifts as a hierarchical model: it is not the fact that she used additive reasoning which contradicts shift 2, but that she compared the two quantities additively in the Housing problem and multiplicatively in the Scalar problem. In the Housing problem one needed to consider the ratio of length to width (multiplicative reasoning, shift 2) and not the difference between length and width (additive reasoning, not shift 2) in order to determine the squarest lot. The participants in groups 2 and 3 all initially used additive reasoning to compare the lots but in other problems provided evidence of using multiplicative reasoning with ratios. This contradiction of using additive reasoning and then changing one’s reasoning to multiplicative in the same problem seemed inconsistent with the shifts as a hierarchical model.

Although groups 2 and 3 often treated ratio as a multiplicative comparison, there were times when they used additive reasoning to compare ratios. This contradicts shift 2, and is inconsistent with the shifts being a hierarchical model. Additionally, the participants in these groups provided evidence of being able to expand a set of ratios to include many more equivalent ratios based on multiplicative reasoning.

To what extent are there inconsistencies with groups 2 and 3 as a hierarchical model? The examples of inconsistency with the model for Groups 2 and 3 seemed to suggest that these participants are “learners” or possibly in the process of making shifts since their reasoning in a questionnaire response often changed during the interview but only when it is perturbed. For example, Eve (Group 3) commented, “I am learning of things, too [referring to the interview]” (line 242). Other participants in groups 2 and 3 made similar comments. Additionally, there were many instances of these participants using incorrect reasoning on the questionnaire and when it was perturbed during the interview they would transition their reasoning to align with the shifts. This change in reasoning suggested that the participants might be learning or shifting their reasoning during the interview. This is often evidenced by changes in their reasoning from the
questionnaire to responses given in the interview. In several instances participants’ questionnaire responses provided conflicting evidence of shift 3; however, when their reasoning was perturbed during the interview, they provided confirming evidence of shift 3. For example Seth (Group 3) used a part/whole comparison in the Dog/Cat problem on the questionnaire (See Figure 5.16).
However, when asked to further explain his written statement about the 2:3 ratio “that the Cat home got 2 for every 3 the Dog home got,” he recognized that in his earlier solution he “thought of it as a straight fraction” and this was not correct. So I asked him how he could use this knowledge of two for every three if I gave him a twenty-dollar donation.

“I am trying to think of a number that would make it even. So let’s say the dogs got $18

Okay

So, just out of something, so 18/3 is 9. This isn’t going to be 20; by the way, I am trying to work with the ratio [Seth points to the 2:3 ratio from problem], so like 18 so that means that like the ratio occurred 9 times. Uh, I don’t know, sorry, I’m lost in my head right now, with what I am trying to figure out. Um, 2 to 3 ratio so, basically for every two dollars the cats get the dogs get three, so then again if the dog, cats get another 2 dollars.

So, they would have the 12 out of the 20 dollars [Seth uses calculator to compute 12/20], which was .6 and then the cats would have the other remaining 8 dollars, which is what I think we were at. I should have wrote it down, yeah 8 out of 20 and that would be 40%.

Uh, so if it was 240 and we knew the dogs were getting, what did we say, I think it was 60%, [Seth uses the calculator to compute .60 *240] they would get the 144. [33:56]

Okay

And the cats would get the remaining x amount.
In this passage Seth (Group 3) created a part/whole comparison 12/20 (shift 2), re-interpreted it as a quotient (60%) and applied it as a multiplicative comparison (shift 3) to determine how much money is donated to each shelter. EU4 explained the mathematical connection between ratio and fraction, and Seth (Group 3) seemed to now understand. This change in reasoning represented Seth’s (Group 3) shift in reasoning as he determined the meaning of the 2:3 ratio that presented difficulties to him on the questionnaire, but now viewed as a part/whole fractional relationship. Furthermore, notice the inconsistency with the shifts as a hierarchical model on the questionnaire in which there was evidence conflicting with shift 3 and in the interview when perturbed, Seth (Group 3) provided evidence confirming shift 3.

As part of the learning process these groups seemed to recognize their mistakes in reasoning during the interview. For example, during the interview Simon (Group 2) initially began the Bike problem by overextending the concept of proportions to the problem because the context of speed.
Question #6: Ben and John both ride their bikes to school at the same speed. Ben leaves first, when Ben has rode 10 blocks, John has ridden 7 blocks. When John has completed 14 blocks, how many blocks has Ben rode his bike?

Figure 5.17: Bike problem to which Simon is responding

427 Simon: [OC: S13 writes Ben 10, John 7 or
428 Ben/John=10/7=x/14, S13 cross-multiplies and
429 finds x = 20 blocks] so Ben would have rode 20
430 blocks. [32:00]
Simon (Group 2) set up a proportion and determined that “Ben would have rode 20 blocks” (lines 429-430). When I pressed him on the meaning of the 10/7, Simon (Group 2) recognized that there was a discrepancy in his reasoning. He then drew a diagram to illustrate what he referred to as the “head start” (line 461).

461 I: Ben had a head start. So, does that continue that head start?
462 Simon: Yes, because they are riding the same speed, so that John would never catch up to Ben.
464 I: So, what’s that mean. [OC: interviewer points to the word “meets” in the problem]
466 Simon: Oh, then I misread the question [OC: Simon crosses out what they have written on paper]. Okay. [pause]
468 I: So, what are you thinking?
471 Simon: I’m thinking that maybe the school is here, John’s house is here, Ben’s house is here [OC: Simon writes S*_____ J*__ B* in a line] since they are riding at the
Simon: same speed, this is a three block difference, they meet here, say they meet here [S13 makes vertical line | when he says “here”], well John has gone seven blocks, but Ben has gone 10 blocks, so this would be 14, this would not be 20 [S13 crosses out \( x=20 \) from previous work], it would only be three more. So. [OC: S13 writes Ben rides 17 blocks]
Additionally, Simon (Group 2) recognized that they were “riding at the same speed” (line 456) and “John would never catch up” (line 457). Simon (Group 2) also claimed that he “misread the question” (line 458). His reasoning changed as he learned to attend to different aspects of the question and not overextend the definition of proportion. Although this change is inconsistent with the shifts, Simon’s (Group 2) reasoning can be explained using EU10, because he did not attend to speed as a superficial cue anymore in the problem, but viewed speed as a necessary element of solving the problem. Simon’s (Group 2) change in reasoning on this problem suggested why I believe that the participants in these groups were “learning” to become proportional reasoners and why I claim that the shift model is an inconsistent hierarchical model when looking at pre-service teachers’ proportional reasoning. It is the change in reasoning that often occurs when one is learning that might make the shift model inconsistent.

Furthermore, the Housing problem was an instance in which Group 3 provided more evidence of being “learners” than Group 2 provides. Both Groups 2 and 3 initially compared the lots using the difference between the length and width. However, when presented with a problem in which the difference between length and width were the same, Group 3 recognized that their original method would not work and began to explore other options such as percentage, fractions with common denominators, and other multiplicative relationships. For example, Eve (Group 3) initially claimed that the lot that is the most square is the one with the smallest difference between length and width:

Eve: \[ E8 \text{ is draws 3 rectangles 185x245, 75x114 and 455x508, I will refer to these as the original rectangles} \text{ so 185 feet by 245 and then 75 by 114} \]
and then 455 by 508, so \textit{pause} I think my initial reaction to finding it would be to \textit{find the difference} between one length versus the other \textit{[points to 455 and 508 in drawing of rectangle]} to \textit{find out how much they are off by} so then I could say that 245 minus 185 so this one is 60 feet longer than this side, \textit{[E8 uses calculator then writes (60)]} this one would be 39 \textit{[E8 uses calculator then writes (39)]} and this one would be 53 \textit{[E8 uses calculator then writes (53)]}. So, \textit{pause} which goes against what I was first thinking, but I would say that this one \textit{[circles 75x114 drawing]} would be closer to a square because they are closer in distance.
Notice how Eve (Group 3) describes finding the difference between the length and width and used this difference to determine that the 75” x 114” lot “would be closer to a square because they are closer in distance” (lines 972-974). When I give Eve (Group 3) a prompt that asked whether a 100’ x 200’ lot or a 300’ x 400’ lot would be more square, Eve’s (Group 3) change in reasoning was closer to a multiplicative comparison between the length and the width than the previous reasoning with the difference, as illustrated in the passage below:

1043 Eve: So I guess this too, I just thought of this [traces 200 foot side] but this [traces 100 foot side] length has
1044 to be half of this width.
1046 I: Okay
1047 Eve: Where as this length [traces 400 foot side] is not
1048 half of the width [traces 300 foot side] it’s more
1049 than half, so it’s going to make it appear more
1050 square cause it has, there is more there.
1051 I: So would that, change or affect your answer to the first
1052 scenario or…? [I points to original rectangles]
1053 Eve: [long pause] yeah it would just change my thinking
1054 then ignoring the numbers in parentheses by saying
1055 if I were to relate these [points to scale drawings
1056 then to original rectangles] to this, this one [points
to 75x114] seems more like this one [points to
1058 100x200] because 75 is more like half of this
1059 [points to 114] versus 508 compared to 455 [points
to 508 then 455] is not half so it’s gonna look more
Notice how Eve (Group 3) claimed, “I just thought of this” (line 1043) and considered that the length was half the width (in the 100’x200’ lot) and in the other lot (300’x400’) was more than half the width. She is then able to apply this concept of comparing the length and width to half to the previous lots.

In contrast to Group 3 reasoning like Eve’s, Group 2 (as illustrated previously in the passage from Susan’s interview) remained firm in their belief that the difference between length and width was an adequate method to determine which lot was most square. This understanding remained even when presented with scale drawings to challenge their beliefs. The participants in Group 2 continued to claim that the lots would be the same because they had the same difference between length and width. For example, Susan (Group 2) claimed that they are the same because the “200 difference is the same” (line 520) making the rectangles “similar” (line 521). All of these examples provided evidence that participants in Group 2 and Group 3 provided reasoning that might be consistent with someone “learning” to reason proportionally but inconsistent with the shifts model being hierarchical in nature. The examples also suggested that Groups 2 and 3 differed in reasoning in that when Group 3’s reasoning was challenged they were more apt to change that reasoning, while group 2 was more likely to justify their reasoning as correct even if wrong.

Other findings or patterns found within groups. Other patterns that seemed to exist with the participants in Group 1 included the ability to interpret a ratio and its inverse in the context of the problem and apply it to solve new problems. For example, when Sonny (Group 1) was asked for another way to solve the Mr. Tall/Mr. Short problem he interprets 6/4 as “Mr. Tall
is 150% of Mr. Short” (line 345) and 4/6 as “Mr. Short is 66.7% of Mr. Tall” (line 335). This seemed to be an important part of their reasoning and is not observed in the reasoning of participants in the other groups. Participants in other groups were not able to interpret the inverse relationship of a ratio in the context of the problem. For example, Stephanie (Group 3) while working on the Lemon/Lime problem did not recognize that the comparison that she made was a ratio of the quantity of lime to the quantity of lemon, nor was she able to interpret this relationship for what it represented. She interpreted ratio as a percentage but then claimed that result was the same as the amount of lemon in a mixture.

424 Stephanie: This is two thirds this one is three fourths [OC: S12
425 writes 2/3 under Jen’s name and ¾ under Alice’s
426 name, then uses calculator to compute 2/3=.66 and
427 ¾=.75 and writes this on the paper] so, Alice does
428 have more of a taste than Jen. Alice's would be
429 more, would have a stronger lemon taste.
Figure 5.22: Stephanie’s comparison of the two mixtures as decimal

I: So, what does, what do these represent?

[Interviewer points to the $\frac{3}{4} = .75$ and the $\frac{2}{3} = .66$ written on the paper]

Stephanie: That would be like, um the amount of lemon taste in the mixture.

Even when pressed to explain the relationship that she made between lime and lemon, Stephanie (Group 3) still claimed that the calculations represent the amount of lemon taste to lime taste in the mixture. EU4 and EU5 explain the importance of reinterpreting ratios as both fractions and quotients. Additionally, participants in Group 1 seemed to represent the problems in context. For example they drew the percentage of pizza each person would receive in the Pizza problem or they drew the distance that Ben rode his bike in the Bike problem (illustrated previously with Sonny’s work from the Bike problem). These representations seemed to aid in their proportional reasoning. Furthermore, the use of unit ratio was important in their reasoning and was utilized especially in problems involving money and size. Evelyn used a unit ratio during the HFS problem that she referred to as the “base ratio” (line 76) and in the Pasta problem. She also referred to the “base ratio” during the sorting activity in which she claimed that the ratios were grouped because they have the same base.

Evelyn: And these are all base two to three [Evelyn points to $\frac{2}{3}$ with group $8:12$, $10:15$ and $4:6$], and I’ll say this is base 5 to 6 [E11 points to $5:6$ that is grouped with $10:12$], I’m just seeing that there are no interrelated bases, but I will say I am done.
Evelyn’s understanding of “interrelated bases” (line 981) can be explained by EU6 and EU7 since she recognized the invariance between proportions and the understanding that it is multiplication that determines the proportional relationship. Other researchers (i.e., Lamon, 1993) have found similar results in regards to unit ratio. Sonny and Evelyn both used the unit ratio to keep the context of the problem grounded in sense making with the solution method as suggested by EU9. For example in the Pasta problem, Sonny immediately considers the unit price as the way to compare the different sized jars; this maintained the unit ratios in the context of the problem.
Question #9: Zandora sells pasta sauce and charges $3.00 for a 7-ounce jar or $16.00 for 2 larger jars that hold a total of 37 ½ ounces. Is buying a 7-ounce jar a better deal than buying the 2 jars of 37 ½ ounces?

Figure 5.24: Prompt for Pasta problem given to Sonny

651 Sonny: So we want to find like the unit price, the price per ounce so, $3/7 ounces [Sonny writes $3/7 oz = .429 $/oz]

Finally, the use of proportions to solve scalar problems was evident in the Group 1 participants, as well as their acknowledgement that these relationships are ratios. As seen in this passage from the interview, Sonny referred to the relationship between Mr. Tall and Mr. Short as a ratio.
Sonny’s comment that “here we are doing ratios between paperclips and buttons” (lines 250-252) suggested that he indeed viewed these relationships as ratios. Participants in other groups seemed to provide correct multiplicative reasoning but never acknowledged that these relationships were ratios. With the exception of the Dog/Cat and Boy/Girl problem that used the word in the problem statement, Elizabeth (Group 2) did not use the word “ratio” at all during the interview. This might have indicated that she did not view these relationships as ratios. Elizabeth (Group 2) only used the word three times throughout her entire interview and all of those were prompted by questions from the interviewer regarding the Boy/Girl problem that contained the word ratio.

Other patterns found in Groups 2 and 3 involved the tendency to use learned procedures that involve setting up proportions and the cross-multiplying algorithm to solve many of the problems. It was not clear if this algorithm was done with or without meaning to the participant. For example, Susan (Group 2) used what she called a proportion to solve the Scalar problem.

Susan: Okay, so I’ll just use this one, um, this is [Susan is writing $\frac{5}{6} = \frac{x}{7.2} \rightarrow 5 \times 7.2 = 6x, \ 36x = 6, \ x = 6$].
Susan (Group 2) seemed to describe what “proportionate to each other” (lines 276-277) meant in terms of the pictures being similar or not similar if “if you are not making the width way longer then you would make the height” (lines 280-281). However, she did not describe what a proportion was and what it meant mathematically in the solution of this problem. This seemed to illustrate what happens when a student uses an algorithm without sense making. EU9 might provide an explanation of why this group was inconsistent with shift 2. It might also provide a reason why shift 3 became most evident when participants used a unit ratio or a proportion to solve a problem. EU9 claims that the unit ratio allows participants to keep the context of the problem to help them reason proportionally. Shift 3 was also evident with the part/part comparison, as the participants’ understanding of fractions seemed to play a part in interpreting the part/part comparison as two parts that make up a whole. EU4 explains how ratios and fractions are similar but not identical and how a fraction is a part/whole comparison, but a ratio
can be a part/part comparison. EU4 might explain the connection between ratios and fractions in these participants’ solutions. It might also explain why a strong understanding of fractions might help these students reason proportionally.

**Summary of Results**

The numbering of the four shifts in Lobato and Ellis (2010) implied that the shifts might be elements of a hierarchical model. Assuming a hierarchical model, if one provided confirming evidence of making a shift on one problem they would not provide conflicting evidence of that shift on another problem. When examining the data on both micro and macro levels I observed that the proportional reasoning of pre-service teachers is consistent with Lobato and Ellis’ (2010) shifts if the participant had successfully made all four shifts or if they had not made shift 2. The responses for Group 1, those who had completed four shifts, and for group 4, those who had not completed shift 2, were consistent with the model. Members of Groups 2 and 3 seemed to be in the process of transitioning through the shifts and their responses were inconsistent with the model.

Group 1 had provided confirming evidence of all four shifts and how reasoning is consistent with the shift model. Additionally, the participants in Group 1 provided evidence that they recognized invalid proportional reasoning and used representations to make sense of non-proportional situations involving the context of speed. Group 4’s consistent use of additive reasoning in proportional situations was inconsistent with shift 2. The one challenge to this consistency was a participant who used a memorized procedure to solve two problems. Because this procedure was used in the absence of additional reasoning, I do not consider it as an inconsistency with the model.

Groups 2 and 3’s reasoning was inconsistent with the shift model because on some problems they provided confirming evidence of all four shifts and on other problems they provided conflicting evidence of shifts 2, 3 and 4. In these cases, contrasts in reasoning occurred with two different problems from the questionnaire with the participant using multiplicative
reasoning on one problem and additive reasoning on another problem or took the form of change from additive reasoning to multiplicative reasoning on the same problem during the interview or from the questionnaire and the interview.

I contend groups 2 and 3’s reasoning was inconsistent with the shifts because they were in the process of making the transitions as they learned to be proportional reasoners. There was evidence of this learning in comments shared during the interview as they claimed that they were learning from this experience. Additionally, Group 3 provided multiple examples of changing their reasoning to align with the shifts model. It was through careful questioning and challenging of their reasoning that provided evidence that these participants were making transitions in their proportional reasoning. However, it was this change in reasoning from additive to multiplicative that made their reasoning inconsistent with the shifts as a model. It was clear from the evidence provided by groups 2 and 3 that some pre-service teachers are able to make the shifts but they may need to have their reasoning perturbed in ways that helps them develop the EUs of proportional reasoning necessary to teach in a classroom. In the next chapter, I discuss ways in which similar perturbation and learning might be accomplished in teacher education.
Chapter 6: Discussion

The purpose of this study was to investigate and characterize pre-service teachers’ reasoning of ratio and proportion with respect to the shifts model proposed by Lobato and Ellis (2010). As a result, I articulated aspects of pre-service teachers’ proportional reasoning in terms of these shifts; and postulated some reasons why certain groups of participants’ reasoning was inconsistent with the shift model and why other groups of participants’ reasoning seemed to be consistent. These findings could be used to extend the Lobato and Ellis model and to improve teacher education.

Discussion of major findings of the study

Teachers need a broad and deep understanding of the mathematics they teach (Ball & Cohen, 1999; Ma, 1999). A teacher’s knowledge of proportional reasoning influences the choices they will make in the classroom. If a teacher’s proportional reasoning skills need further development they may struggle with engaging students in tasks that involve ratios and with leading discussions about proportional reasoning. This study examines how we might better understand the nature of pre-service teachers’ proportional reasoning in terms of shifts. Lobato and Ellis’ (2010) shift model was examined in terms of consistency towards it being hierarchical. It was found that for those participants who showed evidence of completing all four shifts (Group 1) the model was consistent. Additionally, for those participants in Group 4 who had only completed shift 1 (from reasoning with one quantity to reasoning with two quantities) and had not shown evidence of making shift 2 (from reasoning additively to reasoning multiplicatively) the hierarchical aspects of the model also remained consistent. However, for those participants who were in Groups 2 and 3 there was inconsistency with the model as hierarchical. This was because the reasoning they provided confirmed that they were “learning” to reason proportionally. These groups provide us with a static snapshot of the developmental process of proportional reasoning.

I do not know how proportional reasoning developed in these pre-service teachers. However, the evidence from those who had made all of the shifts (Group 1) compared to those
who are still in the process of making the shifts (Group 2 and Group 3) may help us understand how proportional reasoning is developed. This study illustrates that the model is not hierarchical. It does not mean that the shift model is not useful as a taxonomic model. It can be useful as a tool to select, modify, and implement tasks and it provides us with language to discuss the development of proportional reasoning.

**Pre-service teachers’ proportional reasoning**

University experiences in mathematics courses do not always support the development of proportional reasoning (Sowder et al., 1993; CBMS, 2001, 2012). The responses of some of the secondary pre-service teachers in this study (52% of all participants), who take many more higher level mathematics courses than elementary pre-service teachers, were no more indicative of someone who had completed all four shifts than the responses of some of the elementary pre-service teachers (48%). Only 12% of all the participants (4% secondary pre-service teachers and 8% elementary pre-service teachers) provide evidence of making all four shifts. This observation suggests that college courses, either in mathematics or in mathematics education, need to challenge these pre-service teachers’ understandings of ratio and proportion in order for them to further their development of proportional reasoning.

Yetkiner and Capraro’s (2009) research summary for National Middle School Association stated that until teachers can develop specialized knowledge in multiplicative and proportional reasoning, they would struggle to provide students with multiple representations that can address the different learning styles found in their classroom. They also discuss the necessity for teachers to have the content and skills to be able to build upon students’ informal knowledge and to enhance students’ proportional thinking. However, many teacher education programs do not offer courses that target the content of middle grades mathematics (CBMS, 2001; CBMS, 2012). The evidence found in this study suggests that the mathematics education of the participating pre-service teachers may not have completely developed their specialized content knowledge of proportional reasoning, but that it is possible to develop this type of reasoning.
Pre-service teachers’ difficulties with proportional reasoning. Specifically, four distinct aspects of pre-service teachers’ difficulties with proportional reasoning surfaced from this study. These include the use of quantitative reasoning versus computational reasoning, ratio as measure, confusion between ratio and fraction, and the obstacle of linearity. For each difficulty, I will illustrate how each of the aspects arose in my data and how my observations compare to other research in the field. In addition, I will discuss how and why particular tasks might help to challenge pre-service teachers’ difficulties with the aspect of proportional reasoning. Finally, I will discuss how these findings might be used to further advance teacher education.

Quantitative reasoning versus computation. One of the difficulties pre-service teachers in this study had with proportional reasoning involved the distinction between quantitative reasoning and computation. Quantitative reasoning is making sense of the relationship among measureable attributes of objects in the situation (Thompson, 1994) while computation is the result of an arithmetic operation to evaluate quantities. An example of the difference between quantitative reasoning and computation is the Lemon/Lime problem. In this problem participants were asked to compare two different lemon/lime mixtures (3 lemon:2 lime and 4 lemon:3 lime) without calculations but by representing the mixtures with unifix cubes. The request to not use calculations posed a degree of difficulty for most of the pre-service teachers interviewed, because it forced them to reason quantitatively.

An example of the difficulty participants had with the request to not use calculations occurred, when Emma reasoned quantitatively about the mixture (without the use of calculations). She used an additive relationship claiming that there was one more cup of lemon in each mixture so the mixtures were the same. However, when she was allowed to utilize calculations she created a multiplicative relationship by dividing the quantities in order to compare the decimal representations of the mixtures. This interpretation of the relationship as a quotient allowed her to evaluate the mixtures and determine that the 3 lemon:2 lime mixture had more lemon taste than the 4 lemon:3 lime mixture. Emma exemplifies the difficulties pre-service teacher have when

only allowed to reason quantitatively about a situation. Additionally, her work illustrates how easily the use of calculations might be misinterpreted as having a complete understanding of proportional reasoning.

It is not surprising that Emma utilized an additive comparison when reasoning quantitatively about the situation. Kaput and West (1994) found in their research on proportional reasoning problems that the additive approach “appears as a default procedure when the student is confused – a way to do something in the face of confusion” (p. 252). The participants in this study (e.g., Emma) also seemed to default to additive reasoning when they were not allowed to use calculations to compare the mixtures. Similarly, Silvestre and Ponte (2011) found that students’ lack of understanding of multiplicative relationships made it difficult to compare quantities proportionally and often resulted in them using additive relationships when reasoning. The Lemon/Lime problem offered the participants a situation that required quantitative reasoning (e.g., which mixture has more lemon without using calculations) and encouraged the pre-service teachers to engage in proportional reasoning without manipulating numbers. The absence of calculations might allow them to distinguish between ratios as a manipulation of numbers and ratios as a multiplicative comparison of two quantities.

In general, reasoning about quantitative situations involves conceiving of circumstances in terms of quantities by constructing networks of quantitative relationships. EU2 addresses the importance of a ratio as both a multiplicative comparison of two quantities and the joining of two quantities as a composed unit. When participants were asked to compare the two mixtures in the Lemon/Lime problem without using calculations it led to “a more or less” comparison (additive) and not a “how many times more” comparison (multiplicative). This might be because pre-service teachers have come to view forming a ratio as “a writing task” [or computation task such as creating a proportion and cross multiplying] – not as a “cognitive task” (Lobato & Ellis, 2010, p. 22). The focus of ratio in many mathematics curricula is as a writing task, and this may not help students develop the concept of ratio as a multiplicative comparison of two quantities. It is
clear that researchers have found similar difficulties with students’ proportional reasoning (Kaput & West, 1994; Lobato & Seibert, 2002, Silvestre & Ponte, 2011), but researchers have also found that children are able to think multiplicatively without performing operations when they are provided opportunities to work with ratios quantitatively (Steffe, 1992; Harel, Behr, Lesh & Post, 1994; Lo & Wanatabe, 1997). In order for pre-service teachers to fully develop their proportional reasoning they need opportunities that allow them to reason quantitatively about ratio as a multiplicative comparison.

**Ratio as measurement (reasoning with intensive quantity).** The second difficulty in pre-service teachers’ proportional reasoning that this study highlights involves ratio as a measurement. Ratio as a measurement might be thought of in terms of intensive quantity. Schwartz (1988) describes quantities that can be counted or measured (e.g., distance or length) as extensive and quantities that form a composed unit between two quantities as a ratio (e.g., speed) as intensive. This distinction explains how ratios as measurements are intensive quantities. The Housing problem provides insight into the pre-service teachers’ reasoning with ratio as a measurement. In the problem, the participants are given the length and width of three different housing lots and are asked to determine which lot is most square. Many of the pre-service teachers used the difference between the length and the width of each lot to determine the squareness, and were did not reason about the relationship as multiplicative in nature or as a ratio of measure, such as a relationship of the form \( \text{width:length} \).

Simon and Blume (1994) confirm in their research that when pre-service teachers do not have a procedure for solving a multiplicative problem they often fall back on additive reasoning, such as use of a constant distance. Similarly, Lobato and Siebert (2002) found that almost all of the high school students that they interviewed about the steepness of a ramp did not recognize that the relationship should be multiplicative. Kaput and West (1994) also found that there was a strong tendency to adopt additive reasoning when problems involved geometric similarity and linear measurements. Its prevalence seemed to be tied to a lack of understanding of the
quantitative implications of similarity and the pull to compare linear measurements additively. This pull seems to be based on broad experience using additive differences in linear measurement comparisons. De Bock, Verschaffel and colleagues (2003) point out that students’ former real-life experiences with enlarging and reducing operations do not necessarily make them aware of the growth rates in lengths. They may see the covariance as an additive relationship (i.e., comparing the common difference between length and width) and not multiplicative (i.e., comparing length and width as a ratio). This emphasizes that the difficulty with intensive quantities might be due to the non-additive nature of the relationship. It also illustrates the importance of providing pre-service teachers with opportunities to further develop an understanding of the multiplicative relationship of intensive quantities.

Stavy and Tirosh (2000) found that the main challenge with intensive quantities was for students to overcome the “more A implies more B” comparison between the two extensive quantities involved in the relationship. EU3 draws attention to the fact that “forming a ratio as measure involves isolating the attributes and understanding the effect of changing each quantity on the attribute of interest” (Lobato & Ellis, 2010, p. 12). Nunes, Desli and Bell (2003) found that there are two challenges when reasoning with intensive quantities: 1) thinking in terms of proportional relationships (i.e., the effect of changing each quantity) and 2) understanding the connection between the intensive quantity and the two extensive quantities (i.e., the two attributes) of the ratio. Lobato and Ellis (2010) point to similar obstacles to proportional reasoning with EU3 and this study confirms that even pre-service teachers have difficulty with these challenges. Similarly, this study confirms the challenges pre-service teachers have when reasoning with intensive quantities or ratios as measures. The construct of intensive quantity could be added to EU3 to express the difficulty pre-service teachers have with ratio as measurement.

**Confusion between ratio and fraction.** A link between fractions and ratio is not often made explicit in many mathematics textbooks. In particular the link is superficial, “students are
shown how to represent information in proportion word problems as an equivalent fraction equation” (Karplus, et al., 1983, p. 79). The difficulty in proportional reasoning arises from a lack of opportunity for students to explore the differences that exist between ratio and fraction, leading them to believe that the two concepts have equivalent meanings. These difficulties surface if ratio and fraction are understood as equivalent mathematical terms when they are fundamentally different. Even though there is similarity in representations of ratio and fraction (i.e., fraction 2/3; ratio 2/3), the interpretations of those representations differ in important ways. In fractions, the numerator represents a part and the denominator always represents the whole (i.e., 2 parts out of 3 as the whole), while in the case of ratio both the numerator and the denominator can represent parts (i.e., 2 parts to 3 parts). Thus while the use of fraction notation in solving some proportion problems may seem expedient in setting up the equation $a/b=c/d$ in order to apply the cross-multiplying algorithm, it is likely to confuse students when it comes to problems that involve a part-part ratio.

The Dog/Cat problem was meant to elicit this type of difficulty in reasoning in pre-service teachers. The correct interpretation of the situation is a part-part relationship. The numbers were chosen so that regardless of whether the participant interprets the ratio as part-whole relationship or part-part relationship the ratio can be interpreted as an integer. For example, Seth’s reasoning on the Dog/Cat problem illustrates how he initially interpreted the ratio as a part-whole relationship and was comfortable with the result since it produced a result that was an integer. However, when his reasoning was challenged Seth recognized the part-part relationship and then applied it to solve the problem. Mathematics textbooks do not generally teach fractions and proportional reasoning in an integrated way (Sowder et al., 1993), and usually the distinction is not made explicit, making students’ confusion understandable. This study shows that when these misunderstandings are challenged with careful questioning, pre-service teachers such as Seth can begin to understand the subtle difference between ratio and fraction.
EU4 discusses the important connections between ratio and fraction, ratios are often expressed in the form \(a/b\), can involve a part-part comparison, and can be reinterpreted as fractions. However, ratios do not have the same meaning as fractions. The notation of \(a/b\) often challenges students’ understanding of the connection between ratio and proportion. Ratio can be a part-part comparison whereas fraction is always a part-whole comparison. Fractions are also rational numbers, but some ratios can be interpreted as irrational numbers, for example the numerical interpretation of the ratio \(\pi/2\) would not be a rational number. In addition, ratios can have a comparison to zero (e.g., 4:0) whereas this is not possible with fractions. Ratios can also involve more than 2 terms (e.g., 5: 3: 2). Therefore, ratio and fraction can be interpreted intersecting sets of numbers (Clark et al., 2003). The difficulty in reasoning arises from the fact that pre-service teachers have not had opportunities to explore the subtle differences and similarities between these two concepts. As illustrated in this study, the lack of explicit discussion about the link between ratios and fractions can lead to a misinterpretation of a ratio as part-whole relationship. This is often because students confuse fraction and ratio based on their similarities and treatment in the curriculum (Sowder et al., 1994).

The obstacle of linearity. Modestou and Gagatsis (2007) studied students’ improper proportional reasoning as an epistemological obstacle of linearity. An epistemological obstacle is not one in which there is a lack of knowledge, but one in which a piece of knowledge is appropriate only within particular contexts. The epistemological obstacle often generates false responses outside that context (Brousseau, 1997). These responses are recurrent, universal, and resistant to a variety of forms of support aimed at overcoming the problem (De Bock et al., 2003).

Problems involving proportionality are often characterized as an epistemological obstacle in linearity. Proportions include a basic structure of four quantities \((a, b, c, \text{ and } d)\) of which, in many cases, three are known and one is unknown. Many proportional problems involve the context of speed. The Bike problem is this type of scenario; it involves students riding their bikes to school at the same speed. It provides the participant with three numbers and asks them to find
the fourth. For example, Ben and John both ride their bikes to school at the same speed. Ben leaves first, when Ben has ridden 10 blocks, John has ridden 7 blocks. When John has completed 14 blocks, how many blocks has Ben ridden his bike? Despite the structure of a missing value problem and context of speed in this problem it does not involve a proportional relationship between the quantities. Research has found that this structure and context evoke a strong tendency of students to use direct proportions even if it does not fit the problem (DeBock, Verschaffel, Janssens, & Verschaffel 2002; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005).

Overextension of proportional reasoning in contexts that are non-proportional is the result of an epistemological obstacle of linearity. This aligns with the fact that many participants incorrectly answered the Bike problem, and implies that their responses are not due to their lack of knowledge but arise because they are required to expose the inaccuracy of their reasoning. The lack of awareness of their false assumptions would also imply that the participants’ reasoning needs to be perturbed in ways that allow them to overcome this obstacle. Other research has found that this phenomenon is partially caused by the characteristics of the problem formulation with which students have learned to associate proportional reasoning throughout their school life (DeBock, Verschaffel & Janssens, 2002; Van Dooren, De Bock, Hessels, Janssens, & Verschaffel, 2005; Modestou & Gagatsis, 2007). This study provides evidence that these obstacles can be overcome. For example, when Simon’s initial reasoning on the Bike problem was repeatedly challenged during the interview; he was able to recognize that the relationship was not proportional allowing him to overcome this obstacle.

Proportions appear to be deeply rooted in students’ intuitive knowledge and are used in a spontaneous and even unconscious ways—this makes the linear approach quite natural, unquestionable, and to a certain extent inaccessible for reflection (De Bock et al., 2002). EU10 illustrates how superficial cues presented in the context of a problem do not provide sufficient evidence of a proportional relationship between quantities. I would add that research also
suggests that the structure of the problem does not lead to a proportional relationship. Participants’ response to the Bike and Track problems demonstrate how pre-service teachers are drawn to the illusion of linearity in a problem and desire to solve it by setting up a proportion and cross multiplying even though there is not a proportional relationship. Verschaffel and colleagues (2000) claim it takes a radical conceptual shift to move from the uncritical application of this simple neat mathematical formula to the modeling perspective that takes into account the reality of the situation being described. Simon provides evidence that this conceptual shift is possible in pre-service teachers when their reasoning is repeatedly challenged. This study also suggests that certain tasks seem to elicit the challenges that face pre-service teachers with their proportional reasoning.

**Overcoming the four challenges**

**Theoretical tools.** All of the challenges in proportional reasoning raised by this study might be overcome if the learner is made aware of their prior assumptions. However, simply presenting pre-service teachers with information is not enough to help them transform their understanding of proportional reasoning. What is required is a transformation of their existing knowledge. Transformative learning theory attempts to establish and clarify a learner’s prior assumptions (Mezirow, 1991). Mezirow (1997, 1991) states that “we transform our frame of reference through critical reflection on assumptions” (p. 7) and that “rational discourse through communicative learning” matters (p. 78), and that this reflection often takes place within the context of problem solving (Mezirow, 1994). Transformative learning theory claims that only after learners are aware of their assumptions can they develop strategies to transform those assumptions. This suggests that in order for pre-service teachers to make shifts in their reasoning they need to have their reasoning perturbed in ways that encourage reflection on their prior assumptions about proportional reasoning. This study provides evidence that pre-service teachers frequently made shifts in their strategies to solve proportional problems after their reasoning was perturbed through discourse.
**Rational discourse: use of words.** “Rational discourse is a catalyst for transformation, as it induces the various participants to explore in depth the meaning of their various world-views and articulate those ideas to others” (Mezirow, 1991, p. 77). In this study the participants in Group 1 seemed to use the language of ratio and proportion. The participants in Group 1 evidenced this by calling relationships “ratio” and “proportion” during problem solving. They also recognized the properties of proportion and stated them when they applied them, which seemingly allowed these participants to articulate their ideas of ratio and proportion. For example, in the Housing problem Sonny refers to the multiplicative relationship between the length and width as a ratio or a “scale factor from one measurement to another.” Other participants did not recognize the multiplicative relationship nor did they refer to it as a ratio or scale factor. However, the participants in Group 2 and Group 3 were able to still participate in rational discourse with proportional problems. The way they use words in their reasoning provided cues for the researcher as to what concept might need to be perturbed. For example, if the participant referred to a ratio as a fraction, I would ask them to define ratio.

The lack of recognition of proportional relationships was evidenced by the fact that the pre-service teachers would solve proportional problems but never refer to the comparisons as ratios. Furthermore, they would often call the relationship “a fraction.” It is not surprising that many of them did not know how to define ratio. Although it was not necessary for the pre-service teachers to recognize the relationship in order to solve the problem correctly, it was surprising that they did not seem to view the relationship as proportional and they struggled to define the term ratio. Transformative learning theory would provide opportunities for pre-service teachers to engage in rational discourse, and critically reflecting on the proportional relationships in order to transform their frame of reference. This study indicates that pre-service teachers do not need to communicate about ratio and proportion using specific vocabulary in order to engage in rational discourse or transform their strategies for solving proportional problems.
Critical Reflection: Meaning schemes. The development of pre-service teachers’ language for proportional reasoning is important and part of that development depends on the choice of tasks that can lead to critical reflection. Existing research suggests that tasks and characteristics of questioning are useful in helping students’ reason proportionally. For example, Fantini and Gherpelli (2008) taught proportional reasoning through tasks consisting of the exploration of problem situations without first teaching the properties of proportions. They found that there was considerable improvement in the students’ proportional reasoning. Their research took place in secondary school mathematics but I believe their use of exploratory problems to develop proportional reasoning applied to teacher education.

Unlike the research of Fantini and Gherpelli (2008), pre-service teachers have already had experience with learning proportional reasoning as a set of properties. The fact that pre-service teachers have already formed meaning schemes (beliefs, values judgments, and feelings) for ratio and proportion requires that these schemes be challenged. Meaning schemes are frames of reference that are based on the totality of an individual’s experience over a lifetime of cultural assimilation (Mezirow, 1994). When one’s meaning scheme interprets and assimilates a new experience it may either reinforce the perspective or gradually stretch its boundaries (Mezirow, 2000). As a result, a novel experience is either flatly rejected or the experience is transformed in order to fit into existing meaning schemes. People may change meaning schemes as they add or assimilate new information into prior schemes and in fact this may occur through learning (Mezirow, 2003). Transformative learning theory describes the conditions and processes necessary for learners to make a shift or transformation in their meanings and perspectives, in this case transforming their understanding of proportional reasoning. It also suggests that critical reflection requires rational discourse with others in the context of problem solving. Critical reflection was exemplified in this study when participants in Group 2 and Group 3 solved thought-provoking problems, engaged in rational discourse, reflected on their assumptions through perturbations, and transformed their meaning schemes.
**Disorienting dilemmas: proportional tasks.** Transformative learning begins with what Mezirow (2000) refers to as the “disorienting dilemma” which sets the process in motion. I believe that a dilemma can come in the form of cognitively demanding problems that challenge the participant’s understanding of proportional reasoning. Four particular problems - the Lemon/Lime, Housing, Dog/Cat and Bike problems - addressed the challenges that pre-service teachers in this study faced when reasoning proportionally.

**Lemon/Lime problem: quantitative reasoning.** The Lemon/Lime problem challenged the quantitative aspects of proportional reasoning. The absence of calculations required pre-service teachers to explore the concept of ratio as a comparison. This problem also distinguishes proportion as a multiplicative relationship between quantities from proportion as a way to manipulate numbers. This distinction relates to EU2 by focusing pre-service teachers’ attention on ratio as a multiplicative comparison and EU6 on the invariance of quantities in a proportion. The construct of quantitative reasoning might be included with EU2 to expand its meaning since it emphasizes the difficulties with reasoning multiplicatively without calculations.

The difficulty in separating proportion from a way to manipulate numbers was elicited in the Lemon/Lime problem when pre-service teachers were not allowed to use calculation, and they were forced to reason quantitatively. This disorienting dilemma caused the participants to transform their thinking on how to compare the amount of lemon and lime in the mixture. Additionally, asking participants to connect their calculations to the claims they made while using the unifix cubes challenged their faulty assumption that calculations are the only way to reason proportionally. Removing calculation from the process led them to reflect in quantitative aspects of their proportional reasoning.

**Housing problem: ratio as measurement.** The second difficulty evidenced by pre-service teachers during this study was reasoning with ratio as measurement. The Housing problem elicits the use of ratio as measurement of squareness of non-square rectangular lots and intensive quantity that relates width and length. Additive relationships are often used with linear
measurements in mathematics; however, in the Housing problem, ratios are formed using linear measurements. This problem provides specific challenges to pre-service teachers’ initial inclination to use additive reasoning due to the presence of linear measurements given for the dimensions of the housing lots. Linear dimensions are appropriately compared additively but linear dimension are compared multiplicatively in an intensive quantity. Mezirow (1994) claims, “We resist anything that does not fit comfortably within our meaning structure (p. 223).” Research involving teachers indicates that they resist this kind of change of their meaning structure (Sengen, 1999; Simon, Tzur, Heinz, Kunzel, & Schwan Smith, 2000; Goos, 2004). Pre-service teachers in this study resisted changing their meaning of measurement that was limited to linear measurements and additive relationships to include ratio as measurement and intensive quantities. However, when their prior assumption to use a common difference was challenged by asking them to compare a 100'x200' lot to a 300'x400' lot, two lots that clearly did not have the same squareness. Many of pre-service teachers in this study were able to reflect on this dilemma and transform their reasoning about linear measurements to include ratio as measurements.

The construct of intensive quantity helps to explain the difficulties that pre-service teachers have with ratio as measurement and to explain the pull in their reasoning towards linear measurement when problems involve geometric similarities. EU3 relates to this difficulty by forming ratios as measures of real-world attributes and understanding the effects of changing each quantity and to EU7 by understanding the complexities of proportional reasoning. The Housing problem challenged the pre-service teachers’ limited assumptions that linear measurements should be compared additively and challenged them to include intensive quantities as valid measurements. The combination of a thought-provoking problem solving task, rational discourse with the researcher, and critical reflection caused them to transform their proportional reasoning to include a multiplicative comparison.

**Dog/Cat problem: confusion between ratios and fractions.** The Dog/Cat problem elicited a focus on the relationship between ratios and fractions addressing the third difficulty.
The choice of this problem was purposeful in seeking to understand if pre-service teachers interpret the ratio as a part-whole relationship. If they incorrectly apply a part-whole comparison it will lead to a whole number solution. The correct solution involves the ratio as a part-part relationship and also leads to whole number solution. Research indicates that students seem more accepting of whole number solutions than of non-whole number solutions (Heller et al., 1990; Kaput & West, 1994; Karplus et al., 1983). Since this problem involves whole number solutions even if the incorrect interpretation of ratio is applied, many pre-service teachers were satisfied with their strategy involving a part-whole interpretation of the ratio. The participants did not seem to recognize the distinction between ratio as part-part relationship or part-whole relationship and fraction as part-whole relationship. It was only after the participants were asked to explain what the 2:3 ratio meant that they seemed to recognize the subtle distinction between ratio (e.g., 2 parts are donated to the cat shelter and 3 parts are donated to the dog shelter) and fraction (e.g., 2 of the 5 total parts are donated to the cat shelter).

The distinction between ratios and fractions in proportional reasoning is expressed in EU4, in which a ratio can be interpreted as a fraction but the two underlying concepts have different meanings. A robust understanding of proportional reasoning involves understanding the distinction between ratio and fraction. When pre-service teachers are given an opportunity to explore and challenge their previous assumptions about the meaning of ratio and fraction, they more likely begin to recognize the distinction between the two concepts. As transformative learning theory suggests, the combination of disorienting dilemma, rational discourse, and critical reflection often leads to a transformation of misconceptions, such as the misconception that fraction and ratio are equivalent terms.

**Bike problem: obstacle of linearity.** The fourth difficulty evidenced in this study was the obstacle of linearity or the tendency to overextend proportions to non-proportional situations. The Bike and Track problems involve an additive relationship between variables in the context of speed and ask the participant to find one quantity when given three other quantities. Pre-service
teachers should recognize that this is not a multiplicative relationship. However, if they have not yet mentally formed ratios as mathematical objects they may interpret problems that involve three numbers in the context of speed as a missing value problem and use a template algorithm for solving proportions. In this study pre-service teachers’ reasoning was challenged by asking them to explain how the proportions they created in their solution related to the statement of the problem. This challenge led them to reflect critically on their solution and engage in rational discourse with the researcher in order to transform their strategy.

Mezirow (2000) describes three common themes of transformative learning these include the centrality of experience, critical reflection, and rational discourse. The starting point is the centrality of experience. People’s assumptions are generally constructed by their interpretations of experience. When participants were asked to interpret the proportions that they created, their previous assumptions about the problem became the center of the discussion. “We cannot critically reflect on an assumption until we are aware of it” (Cranton, 2002, p. 65). The obstacle of linearity could possibly be overcome if pre-service teachers are made aware of their assumptions, identify the difficulty through discourse, and reflect on the dilemma. The obstacle of linearity is related to EU10 because superficial cues in the context, such as a reference to speed, in a problem do not provide sufficient evidence that quantities are proportional. The construct of an obstacle of linearity could be included with EU10 to highlight the unconscious tendency to utilize proportional relationships when the situation involves non-proportional relationships. The Bike and Track problems challenge the pre-service teachers’ natural and unconscious inclination to use direct proportions even if it does not fit the problem.

Transformation learning is prompted when a learner faces a radically different and incongruent situation or information that cannot be assimilated (Mezirow, 1991). This study illustrates that pre-service teachers provided with opportunities to solve thought-provoking problems (e.g., Lemon/Lime, Housing, Dog/Cat and Bike and Track problems) and then carefully questioned about their previous assumptions can transform their understanding of their
proportional reasoning. During the interviews prior assumptions about the participants’ proportional reasoning were challenged by asking them to explain the strategies in the context of the problem. For example, participants were asked, “What does each ratio in the proportion they created mean in the context of the particular problem?” Through the use of innovative problems and rational discourse with the researcher, participants’ transformed their meaning of proportional reasoning.

**Extending Lobato and Ellis’ work**

This study provides a basis for extending the work of Lobato and Ellis (2010). Specifically, Group 1 exhibited reasoning consistent with the shift model but there were prominent aspects of their reasoning that were not evidenced by the other groups and not captured in Lobato and Ellis’ (2010) shift model. These differences in reasoning might be used to expand the list of essential understandings that can be used as tools to understand the shift model.

**Inverse ratios.** The term inverse ratios refers to the relationship a:b given b:a and the interpretation of this ratio in the context of the problem. Group 1 seemed to exhibit the ability to interpret an inverse ratio in the context of the problem. This was evident in Sonny’s work with the Planet problem when he used the relationship of Earth weight to planet B weight in one part of the question and then the relationship of planet B weight to Earth weight on the next part of the question. When he was asked about these two ratios, he was able to interpret the two relationships in the context of the problem and to explain why he chose to use to use a particular relationship in solving each part of the problem. In my review of literature, inverse ratios did not arise; however, Tall and Razali (1993) claim that professional mathematicians often misunderstand the notion of inverse function. Although inverse function and inverse ratio are different concepts, Tall and Razali’s claim suggests that the general notion of inverse might be challenging. If interpretation of inverse function is a source of confusion for mathematicians perhaps it is not surprising that pre-service teacher have difficulty with an interpretation of inverse ratios.
Lobato and Ellis (2010) do not discuss the concept of inverse ratio. It seems to be an important understanding for those in Group 1 and it allows them to think flexibly about solving proportional problems. Their knowledge of inverse ratio allowed them to recognize the difference between cost per ounce or ounces per cost. Both of these ratios can be used to make comparisons, but if one is asked to determine the “best deal” you need to recognize that the lowest cost per ounce or the greatest ounce per cost is the goal. Even when those in the Groups 2, 3, and 4 were asked to interpret the inverse ratio in the context of the problem they either claimed that they could not or that it would not make sense to interpret the inverse. Therefore, I believe that inverse ratio should be included in the essential understandings for ratio, proportion and proportional reasoning as a tool for communicating the Lobato and Ellis (2010) shift model. EU11 might read: ratios have inverses that can be meaningfully interpreted in the context of a problem.

**Representations of contexts and relationships.** Both Sonny and Evelyn used diagrams when solving problems that evoked the obstacle of linearity. Some participants who were also able to successfully solve this problem also used diagrams in their solution. However, many participants did not use diagrams. For example, many participants did not use representations to help them solve the Track or Bike problem and as a result overextended the idea of proportion to a non-proportional scenario. In contrast, Evelyn used the diagram to represent the situation described in the Bike problem, which seemed to identify the additive relationship between the quantities of the problem and not overextend the context to use proportions. In Figure 6.1 we notice that Evelyn illustrates the problem situation by indicating the number of blocks both John and Ben ride their bikes and then the number of blocks they ride their bikes together. The lines seem to indicate a linear measurement that utilizes an additive relationship. Her representation helps her illustrate the problem situation.
Diezmann and English (2001) noted that there are several difficulties that students have in using pictures or diagrams to solve problems. They note the need for teachers to actively facilitate the development of “diagram literacy” with their students. This need given the potential usefulness of diagrams suggests that college mathematics courses should provide pre-service teachers with opportunities to develop such literacy including knowing what types of drawings are appropriate for the context of the problem.

Several studies provide evidence of the benefit of including diagrams in pre-service education. Lim and Morera (2010) used diagrams as part of a course for pre-service teachers focused on overcoming overgeneralizations of proportionality. One such diagram, illustrated in Figure 6.2, contrasts the invariant ratio and invariant difference in a problem involving two candles. In one scenario the candles are lit at the same time and are burning at different but constant rates, in the second scenario the candles are lit at different times but burn at the same constant rate. The diagrams helped students differentiate between the different strategies needed to solve the two problems and allowed them to critically reflect on their tendency to overextend the proportional relationship to non-proportional situations. Lim and Morera’s findings seem to indicate that the use of diagrams is useful in overcoming the obstacle of linearity that challenged the proportional reasoning of many of the teachers in this study.
In addition to using diagrams to build concepts, pre-service education might include diagrams to understand relationships in a problem context. In the representation seen in Figure 6.3, Seth creates the representation while he solves the Dog/Cat problem. He first partitions the money into five groups of $48,000 (based on the 2:3 ratio of money), then divides the money giving two groups of $48,000 to the Cats shelter and three of the groups of $48,000 to the Dog shelter. Figure 6.3 explicitly captures the 2:3 part-part ratio that relates dogs to cats in the problem without using operations. It might be useful to share Seth’s representation of the Dog/Cat problem with Ellisa, who solved the Dog/Cat problem using a procedure but was not able to explain how her procedure related to the problem. Critical reflection regarding the representation might be productive in perturbing Ellisa’s use of operations on quantities without explanation. Seth’s representation might lead her to develop meaning schemes connecting her procedure within the context of the part-part ratio because he illustrates the solution not as just an operation on numbers but as a relationship between those two different objects.
The use of representations in the Housing problem helped some of the pre-service teachers construct new meanings of the relationship between linear measurements and ratio as measurements. Simon initially explores the difference between the length and width of the three lots to determine squareness, but he is not satisfied with this method. He then creates a lot with the dimensions 1 unit by 4 units and a difference of 3. Simon claims that this representation is much more rectangular than the other lots but has a smaller difference, so the difference between length and width is not a valid method to determine squareness. The representation in Figure 6.4 seemingly helped Simon to recognize the error of his previous assumption and allowed him to explore the option of ratio as measurement. The drawing provides him with a tool to engage in rational discourse about his previous strategy and reflect on alternative methods to find squareness such as intensive quantities.
The Lemon/Lime problem relies on the use of representations to determine which mixture has more lemon. It is interesting to note how those in Group 1 represent the mixtures in this problem and how they work with the unifix cubes to explore the context of the problem. Both participants whose physical representations of the situation are shown in Figure 6.5 separated the part of the representation representing lemon from the part of the representation representing lime during the comparison. This move seemed to help them recognize the multiplicative relationship between the two quantities and not focus on the fact that there is one-cup difference between the two mixtures. These representations are contrasted those used by the participants in other groups. As shown in the examples in Figure 6.6, the cubes remained attached and the focus remained on the fact that there was one more cup in each mixture. I believe there is a connection with this attached representation to linear measurement. The attached cubes resemble lines that would have length. Kaput and West (1994) found that there was a strong tendency to adopt additive reasoning when problems involved linear measurements. So by separating the cubes by color the participants in Group 1 are able to attend to both quantities in a multiplicative way because their prior assumptions about length are not brought forward by the representation. This attention to link linear representations and additive reasoning implies that different questioning is needed to perturb this use of linearity for those in Group 2 and 3. This
separation seems related to EU3, since the Group 1 isolates the attributes from each other in order to consider the effect each quantity has on the other attribute.
Figure 6.5: Evelyn’s (a) and Sonny’s (b) representations of the two mixtures in the Lemon/Lime problem
Lo (2004) suggests that providing pre-service teachers with mathematical tasks that are rich in context and encouraging them to develop drawings and representations that convey the meaning of their solution methods to other students appeared to have great potential to help them ground their mathematical reasoning. This might include proportional reasoning and the ability to distinguish between proportional and non-proportional situations. Although, Lobato and Ellis (2010) include the use of representations in the discussions of many of the essential understandings, the role of representation in developing and displaying is not given the priority it might warrant. This study suggests that greater importance should be given to representations that teachers produce while solving proportional problems. EU12 might read, representations allow individuals to attend to important aspects of their reasoning, including the two quantities involved in the ratios, the context of the problem, and the relationship between the quantities and the context.

Conclusion

This study provides further evidence that both secondary mathematics and elementary education pre-service teachers have underdeveloped proportional reasoning. Only 12% of participants provided reasoning on the questionnaire that indicated substantial understanding of proportional problems that was consistent with all four shifts (see Table 4.3). The lack of proportional reasoning evidenced in this study is not due to a deficiency in the number college
level mathematics courses experienced, since 92% of the secondary mathematics education majors also indicated incomplete proportional reasoning on the questionnaire. Additionally, this study suggests that strategic problem selection and careful questioning can help pre-service teachers make shifts in their proportional reasoning.

Lobato and Ellis’ (2010) shift model was used to discuss the proportional reasoning of pre-service teachers. The model was useful as a taxonomic model but it was found not to be hierarchical in nature. As an instrument for examining proportional reasoning, the model helps provide a language to discuss possible misconceptions. Lobato and Ellis’ essential understandings were tools that helped to apply the shift model. The data when analyzed in light of the shift model and essential understandings highlighted four difficulties in reasoning: obstacle of linearity, confusion between ratio and fraction, reasoning quantitatively and reasoning with ratios as measurement. When these misconceptions were challenged and pre-service teachers were encouraged to reflect on their own strategies and the strategies of others, they were often able to correctly solve problems.

Additionally, this study suggests that the list of essential understandings might be revised to incorporate other aspects of the development of pre-service teachers’ proportional reasoning. Two additions might help to better describe the proportional reasoning of pre-service teachers. These additions involve the interpretation of the inverse proportion in the context of a problem and the use of representations when solving proportional problems.

If teachers need to obtain a deeper understanding of ratios and proportions in order to provide their students with richer opportunities in their own classroom, then pre-service teachers need to have their prior assumptions about ratio and proportion challenged in their university experiences. The findings in this study provide teacher educators with knowledge about the nature of pre-service teachers’ proportional reasoning. In particular, this study highlights four assumptions and misconceptions about proportional reasoning seem necessary for pre-service teachers to transform. These four assumptions include: reasoning quantitatively, recognizing
ratios as measurement, misconceptions about the concept of ratio and fraction, and the obstacle of linearity. Mezirow’s theory requires a disorienting dilemma in order to help individuals engage in rational discourse and critical reflection about previous assumptions. This study illustrates how four problems (Lemon/Lime problem, Dog/Cat problem, Housing problem and Track problem) were able to provide pre-service with a disorienting dilemma causing them to engage in rational discourse with the researcher and critically reflect on their previous assumptions in order to revise their strategies for solving these problems. By helping these pre-service teachers become aware of the assumptions they have about ratio, proportion and proportional reasoning through these disorienting dilemmas (i.e., thought-provoking problem solving tasks) they were able to think about proportional problems in new ways and make shifts in their proportional reasoning. This knowledge can be used to develop courses that can transform pre-service teachers’ understandings of ratio and proportion and enhance their proportional reasoning so that future teachers can ultimately improve their students’ learning.
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Appendix A: IRB approved recruitment script

RECRUITMENT SCRIPT  (verbal, in person)

My name is Kim Johnson, a graduate student in the Department of Curriculum and Instruction, here at Pennsylvania State University. I would like to invite you to participate in my research study to understand the nature of pre-service teachers knowledge of ratio and proportion. You may participate if you are an Elementary and Kindergarten Education major or a Secondary Education major with a Mathematics options. Please do not participate if you have completed MTHED 420 or MTHED 411 or other optional MTHED courses.

As a participant, you will be asked to complete a questionnaire of mathematics problems that would take about 45 minutes. You might also be one of a small number participants asked to be interviewed that will be audio and videotaped. The interview will involve some more mathematics problems and follow up questions to better understand your reasoning.

Your participation in this research is confidential and voluntary. Participants will receive a $10 gift card for completing the questionnaire and another for completing the interview.

If you would like to participate in this research study, please contact me at (814) 574-0195 by either text or calling or by email at kxj5@psu.edu

Do you have any questions now? If you have questions later, please contact me at kxj5@psu.edu or you may contact my advisor, Dr. Rose Zbiek at rmz101@psu.edu.
Appendix B: Questionnaire

Code # __________

Please answer the following questions, include as much detail in your explanation as possible. Show all work used to find the solution. Thank you for participating in this study.

1. Tracy and Kelly are each given 40 oz. of water. Tracy adds 20 oz. of lemon flavor to her water and Kelly adds 10 oz of lemon flavor to her water, how would the two mixtures of lemonade compare in terms of lemon taste? Explain your reasoning.¹

2. A group of students were practicing basketball shooting. Here are the results:

<table>
<thead>
<tr>
<th>Name</th>
<th>Shots made</th>
<th>Shots Taken</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allen</td>
<td>7</td>
<td>11</td>
</tr>
<tr>
<td>Barbara</td>
<td>14</td>
<td>29</td>
</tr>
<tr>
<td>Charlie</td>
<td>9</td>
<td>19</td>
</tr>
<tr>
<td>Daisy</td>
<td>23</td>
<td>48</td>
</tr>
</tbody>
</table>

Who is the best player, based only on this information? Explain how you got your answer.

3. Mr. Short has a friend Mr. Tall. Mr. Short is 6 paper clips tall. When we measure their heights with buttons: Mr. Short’s height is four buttons and Mr. Tall’s height is six buttons. How many paper clips are needed for Mr. Tall’s height? Explain your reasoning.³

4. Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie had completed 15 laps, how many laps had Sue run? Explain your reasoning.⁴

5. The Health Food Store sells granola by the ounce. The cost depends on the weight of the granola. Granola that weighs 8 ounces costs $1.50. Fill in the table below with cost or weight. Explain your reasoning.⁵
<table>
<thead>
<tr>
<th>Weight</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 ounces</td>
<td>?</td>
</tr>
<tr>
<td>8 ounces</td>
<td>$1.50</td>
</tr>
<tr>
<td>?</td>
<td>$2.25</td>
</tr>
<tr>
<td>16 ounces</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>$4.50</td>
</tr>
</tbody>
</table>

6. When animal lover, Mr. Henry died he left 240 thousand dollars to be divided amongst two animal shelters using a 2:3 ratio between the amounts that Cat Best Home and Dog Lover Home gets. How much money should each chapter get? Explain your reasoning.

7. NASA is building a space station on Planet B, given each pair of weights in the table, how can they determine how much an item on Earth will weigh on Planet B. Explain your reasoning.

<table>
<thead>
<tr>
<th>Item</th>
<th>Earth (pounds)</th>
<th>Planet B (pounds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>120</td>
<td>45</td>
</tr>
<tr>
<td>2</td>
<td>27</td>
<td>10 1/8</td>
</tr>
<tr>
<td>3</td>
<td>42</td>
<td>15 3/4</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>1/10</td>
<td>9/240</td>
</tr>
</tbody>
</table>

e) Determine the relationship between an items weight on Earth and Planet B.
f) Determine the weight of an item on Planet B if it weighs 200 lbs on Earth.
g) Determine the weight an item on Earth if it weighs 63 lbs on planet B.
h) Give an example of another item that would maintain the same relationship that exists between the weights on Earth and Planet B.

8. A picture is 2 cm wide by 2.4 cm long. If Fran wants to enlarge the picture to be 5 cm wide, how long should it be? If Fran wants to enlarge the picture so that it is 7.2 cm long. How wide should it be?

9. When I joined the party I could sit at a table where 3 pizzas will serve 7 people, or at a table where 4 pizzas will be served to 9 people. Given that I love pizza, where should I have chosen to sit to have the most pizza? Explain your reasoning.
1 This problem adapted from Harel et al. (1994).
2 This problem adapted from Hines & McMahon (2005)
3 This problem adapted from Karplus, Karplus and Wollman (1974)
4 This problem adapted from Peled (2007)
5 This problem adapted from Ercole, Frantz and Ashline (2011, MTMS)
6 This problem adapted from Lobato and Ellis (2010, p. 72)
7 This problem adapted from Cramer and Post (1993)
8 This problem adapted Miller and Fey (2000)
9 This problem adapted Peled (2007)
Appendix C: Interview Questions

The following questions will be asked of participants who participate in the follow-up interviews. The responses will be written and orally explained to the principal investigator who might ask for further explanations and justifications of responses presented. The interview will be video and audio recorded.

1. Jen and Alice are making lemonade. Jen mixes 3 cups of lemon juice with 2 cups of water. Alice mixes 4 cups of lemon juice with 3 cups of water. Whose mixture is more lemony? Justify your solution. Create a diagram of the situation and use it to explain your answer. ¹

2. In Kansas, there are no mountains for skiing. An enterprising group built a series of ski ramps and covered them with a plastic fiber that permitted downhill skiing. It is your job to rate them in terms of most steep to least steep. You have available to you the following measurements for each hill: the length and width of the base and the height. How would you determine the relative steepness using the information you have? [note: examples of multiple slopes would be provided and participants asked to rate them from steepest to least steep, the measurements would be provided for length, width and height].²

3. A new housing subdivision offers lots of 3 different sizes: 185 feet by 245 feet; 75 feet by 114 feet; 455 feet by 508 feet. If you were to view these lots from above, which would appear most square? Which would be least square? Explain your reasoning.³

4. Mrs. Jones put her students into groups of 3 girls and 2 boys. If she has 25 students in her class, how many are girls and how many are boys?⁴

5. If one man can paint a room by himself in 3 hours. How long will it take two men to paint the same room if they both paint at the same rate?⁵

6. Zandora sells pasta sauce and charges $3.00 for a 7-ounce jar or $16.00 for 2 larger jars that hold a total of 37 ½ ounces. Is buying a 7-ounce jar a better deal than buying the 2 jars of 37 ½ ounces? How do you know? Explain.⁶

7. Ellen, Jim and Steve bought 3 helium-filled balloons and paid $2 for all three balloons. The decided to go back and buy enough for everyone in their class. How much did they pay for 24 balloons? Explain your reasoning.⁷

8. Store A advertises a sale of $10.00 off any purchase. Store B advertises a sale of 10% off any purchase. If both stores sell the same things at the same prices, which store offers a better sale? Explain.⁸

Examples of follow-up questions:
How did you arrive at that answer?
What information did you use to solve the problem?
Why did you solve the problem in that particular way?
Would you use that method to solve other problems?
Could you use another method to solve this problem?
## Appendix D: Questionnaire/Interview Question Comparison

<table>
<thead>
<tr>
<th>Shift</th>
<th>Questionnaire</th>
<th>Interview</th>
<th>How these questions provide evidence of the shifts?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Shift 1/EU1</strong></td>
<td>Tracy and Kelly are each given 40 oz. of water. Tracy adds 20 oz. of lemon flavor and Kelly adds 10 oz of lemon flavor, how would the two mixtures of lemonade compare in terms of lemon taste? Explain your reasoning. <em>(LEMON/WATER PROBLEM)</em></td>
<td>Jen and Alice are making lemonade. Jen mixes 3 cups of lemon flavor with 2 cups of lime flavor. Alice mixes 4 cups of lemon juice with 3 cups of lime flavor. Whose mixture is more lemonier? Justify your solution without using calculations. You may rearrange the cubes in such a way that they demonstrate which one’s more lemony tasting or that they are the same. <em>(LEMON/LIME problem)</em></td>
<td>These questions attempt to elicit a transition from focusing on only one quantity (i.e., the amount of water or amount of lemon flavor) to a realization that ratio involves a relationship between two quantities (i.e., a mixture of water and lemon or mixture of lemon and lime).</td>
</tr>
<tr>
<td><strong>Shift 1/ EU3</strong></td>
<td>A group of students were practicing basketball shooting. Here are the results: Name</td>
<td>Shots taken</td>
<td>Shots made</td>
</tr>
<tr>
<td><strong>Shift 2/ EU2</strong></td>
<td>Mr. Short has a friend Mr. Tall. The length of Mr. Short is 4 large buttons. The length of Mr. Tall is 6 large buttons. When paper clips are used to measure Mr. Short and Mr. Tall: The length of Mr. Short is 6 paper clips. What is the length of Mr. Tall in paper clips? <em>(MR. SHORT/MR. TALL problem)</em></td>
<td>A new housing subdivision offers lots of 3 different sizes: 185 feet by 245 feet; 75 feet by 114 feet; 455 feet by 508 feet. If you were to view these lots from about, which would appear most square? Which would be least square? Explain your reasoning. <em>(HOUSING LOT problem)</em></td>
<td>These two questions elicit the transition from additive reasoning to multiplicative reasoning. PST should recognize that the relationship between the two quantities in the ratio is multiplicative (i.e., buttons/paper clips or length/width) not additive (buttons - paper clips or length - width). Which attributes matter?</td>
</tr>
<tr>
<td>Shift 2/EU6 &amp; EU10</td>
<td>Sue and Julie were running equally fast around a track. Sue started first. When she had run 9 laps, Julie had run 3 laps. When Julie had completed 15 laps, how many laps had Sue run? (TRACK problem)</td>
<td>If one man can paint a room by himself in 3 hours. How long will it take two men to paint the same room if they both paint at the same rate? (PAINT problem)</td>
<td>Both these questions try to elicit if a PST can determine when to use proportional reasoning strategies to solve problems. In the track problem additive reasoning is needed to solve the problem and in the paint problem inverse proportions should be used to find the solution in the context of the situation.</td>
</tr>
<tr>
<td>Shift 3/ EU7</td>
<td>The Health Food Store sells granola by the ounce. The cost depends on the weight of the granola. Granola that weighs 8 ounces costs $1.50. Fill in the table below with cost or weight. Explain your reasoning. (HEALTH FOOD STORE problem)</td>
<td>Ellen, Jim and Steve bought 3 helium-filled balloons and paid $2 for all three balloons. The decided to go back and buy enough for everyone in their class. How much did they pay for 24 balloons? Explain your reasoning. (BALLOON problem)</td>
<td>These questions try to elicit the methods PST use when solving a proportion. The shift from repeating composed units to multiplicative strategies will be evidenced by how the PST solves these problems. If the PST repeats the composed unit (i.e., 3 balloons:$2; 6 balloons: $4) or use a more formal and efficient multiplicative comparison (i.e., 24/3=8 and 8x$2=16).</td>
</tr>
<tr>
<td>Shift 3/EU4 &amp; EU5</td>
<td>When animal lover, Mr. Henry died he left 240 thousand dollars to be divided amongst two animal shelters using a 2:3 ratio between the amounts that Cat Best Home and Dog Lover Home gets. How much money should each chapter get? (DOG/CAT problem)</td>
<td>Mrs. Jones put her students into groups of 3 girls and 2 boys. If she has 25 students in her class, how many are girls and how many are boys? (BOY/GIRL CLASS problem)</td>
<td>These questions underscore the ability to distinguish between fractions and ratio and especially the part-part relationship that exists with ratio. PST should recognize that the ratio in the problem is part/part (i.e., 2:3 is not two-thirds of the total) not part/whole.</td>
</tr>
<tr>
<td>Shift 4/EU8</td>
<td>Given each pair of weights in the table, how can you tell whether every weight B was determined on “another planet” was found on Planet B. (PLANET problem)</td>
<td>Zandora sells pasta sauce and charges $3.00 for a 7-ounce jar or $16.00 for 2 larger jars that hold a total of 37 ½ ounces. Is buying a 7-ounce jar a better deal than buying the 2 jars of 37 ½ ounces? How do you know? Explain. (PASTA SAUCE problem)</td>
<td>These two problems try to elicit students to recognize that there are an infinite number of equivalent ratios that make up a set of rates they are able to formally reason about proportions. Lamon (1999) suggests using problems that do not involve whole number solutions in order to determine if students are</td>
</tr>
</tbody>
</table>
able to reason proportionally in unique situations. Both of these problems use “not nice” numbers and require students to generalize beyond one ratio.

| Shift 4 | A picture is 2 cm wide by 2.4 cm long. If Fran wants to enlarge the picture to be 5 cm wide, how long should it be? If Fran wants to enlarge the picture so that it is 7.2 cm long. How wide should it be?  
*Scalar problem* | This problem uses the context of enlarging a picture to involve the PST understanding of scalar transformation. Some research suggests that this type of proportional reasoning is more advanced (Lamon, 1999). |
| Shift 4 | When I joined the party I could sit at a table where 3 pizzas will serve 7 people, or at a table where 4 pizzas will be served to 9 people. Given that I love pizza, where should I have chosen to sit to have the most pizza? Explain your reasoning.  
*PIZZA problem* | These two questions looks at situations where ratio varies depending on the situation. In the pizza problem one should look at the amount of pizza per person to make sense in the second problem the amount of the purchase plays a role. |
| Shift 4 | Store A advertises a sale of $10.00 off any purchase. Store B advertises a sale of 10% off any purchase. If both stores sell the same things at the same prices, which store offers a better sale?  
*SALE problem* | |
Appendix E: Data Analysis Worksheet

Code: _____________

Do I want to interview?

Why do I want to interview?

What do I focus on during the interview? What shifts do I want to observe?

Which interview questions?

Why those questions?

In what order do I present these questions?

Why this order?
### Appendix F: Data Analysis of Questionnaire

<table>
<thead>
<tr>
<th>Shift 1</th>
<th>Correct or Incorrect?</th>
<th>Explanation in terms of shifts</th>
<th>Is shift complete?</th>
<th>Why or why not?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1</td>
<td>Lemon/Water Problem</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q2</td>
<td>Basketball Problem</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shift 2</th>
<th>Correct or Incorrect?</th>
<th>Explanation in terms of shifts</th>
<th>Is shift complete?</th>
<th>Why or why not?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q3</td>
<td>Mr. Tall/ Mr. Short Problem</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q4</td>
<td>Track Problem</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shift 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>--------</td>
<td>----------</td>
<td>----------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q5</td>
<td>Health</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Food Store</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Problem</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q6</td>
<td>Dog/Cat</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Problem</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shift 4</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Q7</td>
<td>Planet</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Problem</td>
<td></td>
</tr>
<tr>
<td>Q8</td>
<td>Scalar</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Problem</td>
<td></td>
</tr>
<tr>
<td>Q9</td>
<td>Pizza</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Problem</td>
<td></td>
</tr>
</tbody>
</table>
## Appendix G: Example of Data Analysis from Group 1

### DATA ANALYSIS of QUESTIONNAIRE

<table>
<thead>
<tr>
<th>Shift</th>
<th>Question</th>
<th>Correct or Incorrect?</th>
<th>Explanation in terms of shifts</th>
<th>Is shift complete?</th>
<th>Why or why not?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shift 1</td>
<td>Q1 Lemon/Water Problem</td>
<td>C</td>
<td>I recognize that a mixture from the same container would have same taste.</td>
<td>Yes</td>
<td>The data does not provide evidence that I recognize that there are 2 quantities in a ratio only that they understand that mixture from same container have same taste.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Q2 Basketball Problem</td>
<td>C</td>
<td>I recognize that the best player is determined not only by shots made but also shots taken + use of % to compare the students</td>
<td>Yes</td>
<td>I reasoning suggests that they recognize that comparing ABC+D that one either has to have = shots or can use % to compare. This indicates that I recognize that ratio involves 2 quantities.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shift 2</td>
<td>Q3 Mr. Tall/ Mr. Short Problem</td>
<td>C</td>
<td>I not only sets up a template style proportion but uses a multi-comp 6/4x1.5x6=9 to solve the problem.</td>
<td>Yes</td>
<td>7 indicates that they understand ratio involves a multi relationship between 2 quantities.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Q4 Track Problem</td>
<td>C</td>
<td>I recognize that the situation involves additive reasoning.</td>
<td>Yes</td>
<td>This solution provides evidence that I has made shift 2 + recognizes situations that involve both additive reasoning + multi reasoning.</td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Shift 3</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>------------------------------------------</td>
<td>----------------------</td>
<td>-------</td>
<td>----------------------------------</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q5</td>
<td>7 correctly uses a</td>
<td>7</td>
<td>There is evidence that 7 is</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Health Problem</td>
<td>unit ratio</td>
<td></td>
<td>able to use multi-comp.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Food Problem</td>
<td>&amp; applies it to</td>
<td></td>
<td>to solve this problem.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>different situations</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q6</td>
<td>7 recognizes that 2:3</td>
<td>7</td>
<td>7 indicates they understand</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dog/Cat Problem</td>
<td>as a part-part</td>
<td></td>
<td>ratio as part of the problem.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>comparison &amp; is able</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>to interpret the</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>ratio as a</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>quotient</td>
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<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Shift 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q7</td>
<td>7 correctly</td>
<td>7</td>
<td>There is evidence that 7 has</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Planet Problem</td>
<td>generalizes an</td>
<td></td>
<td>made shift 4 &amp; 7, seems to</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>equation: B=2.661 x</td>
<td></td>
<td>recognize that</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>7 applies</td>
<td></td>
<td>Bell involves infinite</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>generalization</td>
<td></td>
<td>equivalent ratio despite</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>correctly but</td>
<td></td>
<td>incorrectly applying the</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>makes computional</td>
<td></td>
<td>generalization.</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>error</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q8</td>
<td>7 uses a multi-comp.</td>
<td>7</td>
<td>There seems to be evidence that</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Scalar Problem</td>
<td>to solve this</td>
<td></td>
<td>they have made shift 3 as</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>problem &amp; illustrates</td>
<td></td>
<td>the data in this question</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>further understanding of shift 3</td>
<td></td>
<td>suggests.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pizza Problem</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Page 2 Volunteer #7
Appendix H: Example of Data Analysis Group 2:

Evidence that Shifts 1 and 4 have been made and questionable evidence of Shifts 2 and Shift 3

There are several reasons why it would make for interesting interviewing. 4 did not recognize part-part ratios. Their reasoning on the track problem makes this participant's proportional reasoning in terms of the continuum questionable. An interview would provide more insight into their reasoning.

<table>
<thead>
<tr>
<th>DATA ANALYSIS of QUESTIONNAIRE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shift 1</td>
</tr>
<tr>
<td>Correct or Incorrect?</td>
</tr>
<tr>
<td>Q1 Lemon/Water Problem</td>
</tr>
<tr>
<td>Q2 Basketball Problem</td>
</tr>
</tbody>
</table>

<p>| Shift 2                        |
| Correct or Incorrect? | Explanation in terms of shifts | Is shift complete? | Why or why not? |
| Q3 Mr. Tall/ Mr. Short Problem | C | 4 divides 6 by 4 to determine how many ms per pc. Then multiply this quantity to determine the # of tps needed for Mr. Tall | Yes | 4 recognizes the multi. notation. The data involved in this problem 4 applies this multi comp. to this situation. |
| Q4 Track Problem | I | 4 does not recognize the additive nature needed for this problem. It uses a ratio of 3/4 to find the # of laps. This is an atypical answer and the reasoning was unclear | No | 4 is not able to distinguish between additive vs. multi. situations. |</p>
<table>
<thead>
<tr>
<th>Shift 3</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Q5 Health Food Store</td>
<td>C</td>
<td>4 divides 8 by 4; then multiply by 3 to get the price for dog. (1.25/4)(3) = .375 + uses</td>
</tr>
<tr>
<td>Problem</td>
<td></td>
<td>an iterative approach when... possibly the units.</td>
</tr>
<tr>
<td>Q6 Dog/Cat Problem</td>
<td>I</td>
<td>4 does not apply part-part comparison needed for this problem but attempts to apply a partwhole fraction.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>The inability to recognize the part part rate rapidly... requires this type of understanding.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shift 4</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Q7 Planet Problem</td>
<td>C</td>
<td>4 is able to generalize that E = 2.678; can apply this generalization correctly applied to several situations.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>This data provides evidence that 4 may have made shift 4.</td>
</tr>
<tr>
<td>Q8 Scalar Problem</td>
<td>C</td>
<td>4 was able to correctly solve the scalar problem. When multi-comp. in their reasoning found a rate of 1/10 of 1.2, then multi by 5. Also uses 7/12.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>4 is able to reason correctly about a problem involving scalar reasoning.</td>
</tr>
<tr>
<td>Q9 Pizza Problem</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Appendix I: Example of data analysis from Group 3
Evidence of shift 1 and possibly shift 2; no evidence of other shifts, response from Volunteer #5

<table>
<thead>
<tr>
<th></th>
<th>Correct or Incorrect?</th>
<th>Explanation in terms of shifts</th>
<th>Is shift complete?</th>
<th>Why or why not?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Shift 1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q1 Lemon/Water</td>
<td>C</td>
<td>5 claims that the 2 glasses would taste the same since it's all mixed together</td>
<td>Yes</td>
<td>5 does not indicate that the mixture contains 2 quantities</td>
</tr>
<tr>
<td>Problem</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Q2 Basketball</td>
<td>C</td>
<td>5 correctly realizes that with #4 photo may not have taken. The ratio in the comparison</td>
<td>Yes</td>
<td>5 has made shift 1 but provides evidence that a ratio consists of 2 quantities in a relationship</td>
</tr>
<tr>
<td>Problem</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Shift 2</strong></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Q3 Mr. Tall/</td>
<td>C</td>
<td>5 uses an iterative approach to solving this problem, drawing items for every 3 pc</td>
<td>Yes</td>
<td>5 recognizes that for every 3 ms there are 3 pc + ideas. This relationship to solve. 5 indicates understanding of the multiplicative nature of ratios</td>
</tr>
<tr>
<td>Mr. Short Problem</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Q4 Track Problem</td>
<td>I</td>
<td>5 does not recognize the additive nature of the problem. This is an apparent answer. It is unclear what reasoning is used</td>
<td>No</td>
<td>It might be that 5 does not add correctly. Saying “31” but 5 does not provide enough detail about his reasoning.</td>
</tr>
<tr>
<td>Shift 3</td>
<td></td>
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<td>---</td>
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<td>---</td>
<td></td>
</tr>
<tr>
<td><strong>Q5</strong></td>
<td><strong>5.</strong> Is able to determine</td>
<td><strong>No.</strong></td>
<td><strong>There is not enough evidence.</strong></td>
<td><strong>Food Store Problem</strong></td>
</tr>
<tr>
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<tr>
<td></td>
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<td></td>
<td></td>
<td><strong>That 5 has moved beyond 5</strong></td>
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<td><strong>Unlimited + may have allocated</strong></td>
</tr>
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<td></td>
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<td></td>
<td></td>
<td><strong>to solve this problem.</strong></td>
</tr>
<tr>
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<td></td>
<td><strong>Q6</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td><strong>No evidence; incomplete</strong></td>
<td></td>
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<tr>
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<td></td>
<td></td>
<td><strong>Dog/Cat Problem</strong></td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>No</strong></td>
</tr>
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<td></td>
<td><strong>Shift 4</strong></td>
</tr>
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<td></td>
<td></td>
<td><strong>Q7</strong></td>
</tr>
<tr>
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<td></td>
<td><strong>N/A</strong></td>
</tr>
<tr>
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<td></td>
<td></td>
<td><strong>Planet Problem</strong></td>
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<td></td>
<td></td>
<td><strong>Q8</strong></td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td><strong>N/A</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>Scalar Problem</strong></td>
</tr>
<tr>
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<td></td>
<td></td>
<td><strong>Q9</strong></td>
</tr>
<tr>
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<td></td>
<td></td>
<td><strong>Pizza Problem</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>No response</strong></td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td><strong>Partial data</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>5 is able to set up a proportion + cross multiply to solve the problem</strong></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>5 correctly used same algorithm</strong></td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td><strong>to solve this problem.</strong></td>
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<tr>
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<td></td>
<td></td>
<td><strong>No evidence that there were</strong></td>
</tr>
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<td></td>
<td><strong>Shift 4 made. EUR = EUR.</strong></td>
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<td></td>
<td><strong>might explain why they</strong></td>
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<td></td>
<td><strong>problem has been solved</strong></td>
</tr>
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<td></td>
<td><strong>correctly while others have</strong></td>
</tr>
<tr>
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<td></td>
<td></td>
<td><strong>not.</strong></td>
</tr>
</tbody>
</table>
Appendix J: Example of data analysis from Group 4
Evidence of shift 1; evidence of additive reasoning versus multiplicative reasoning indicating no shift 2; no evidence of other shifts.
Data analysis of Volunteer #9

<table>
<thead>
<tr>
<th>DATA ANALYSIS of QUESTIONNAIRE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Correct or Incorrect?</td>
</tr>
<tr>
<td>------------------------</td>
</tr>
<tr>
<td><strong>Shift 1</strong></td>
</tr>
<tr>
<td>Q1 Lemon/Water Problem</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td>Q2 Basketball Problem</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td><strong>Shift 2</strong></td>
</tr>
<tr>
<td>Q3 Mr. Tall/ Mr. Short Problem</td>
</tr>
<tr>
<td>Q4 Track Problem</td>
</tr>
<tr>
<td>Shift 3</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td><strong>Q5 Health Food Store Problem</strong></td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>9 uses a unit ratio to determine the cost per oz, then graph or draw as needed to solve other parts</td>
</tr>
<tr>
<td>9 is able to efficiently use the unit ratio to solve the problem, EU might provide some explanation as to how these problems are correct unless Shift 4 has not been reached.</td>
</tr>
<tr>
<td><strong>Q6 Dog/Cat Problem</strong></td>
</tr>
<tr>
<td>C</td>
</tr>
<tr>
<td>9 recognizes the part-part ratio and correctly reinterprets the ratio as a quotient 240/16 = 15</td>
</tr>
<tr>
<td>9 can re-interpret ratios as a quotient &amp; recognize the part-part concept of ratio. I questioned why the student had the correct idea but the book example used to determine an incorrect answer.</td>
</tr>
<tr>
<td>9 does not provide evidence that they can solve a problem that involves Shift 4.</td>
</tr>
</tbody>
</table>
Appendix K: Example of data analysis of Group 5

There is little evidence provided on the questionnaire for Volunteer #16

<table>
<thead>
<tr>
<th>Shift 1</th>
<th>Correct or Incorrect?</th>
<th>Explanation in terms of shifts</th>
<th>Is shift complete?</th>
<th>Why or why not?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q1 Lemon/Water Problem</td>
<td>C</td>
<td>Correct answer of “yes” but no explanation</td>
<td>?</td>
<td>no evidence</td>
</tr>
<tr>
<td>Q2 Basketball Problem</td>
<td>I</td>
<td>Incorrect / no explanation. Daisy might indicate only the use of shots made</td>
<td>?</td>
<td>no explanation</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Shift 2</th>
<th>Correct or Incorrect?</th>
<th>Explanation in terms of shifts</th>
<th>Is shift complete?</th>
<th>Why or why not?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q3 Mr. Tall/Mr. Short Problem</td>
<td>I</td>
<td>No indicates the use of additive reasoning stating “difference of 2”</td>
<td>no</td>
<td>evidence indicates use of additive reasoning w/ proper recognition of math</td>
</tr>
<tr>
<td>Q4 Track Problem</td>
<td>N/A</td>
<td>No data</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>Shift 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>---------</td>
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<td>---</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q5</td>
<td>I</td>
<td>no data</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Health</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Food Store</td>
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</tr>
<tr>
<td>Problem</td>
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<td></td>
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</tr>
<tr>
<td>Q6</td>
<td>N/A</td>
<td>no data</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dog/Cat</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Shift 4</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Q7</td>
<td>N/A</td>
<td>blank</td>
</tr>
<tr>
<td>Planet</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q8</td>
<td>N/A</td>
<td>no data</td>
</tr>
<tr>
<td>Scalar</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q9</td>
<td>N/A</td>
<td>no data</td>
</tr>
<tr>
<td>Pizza</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Problem</td>
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</tr>
</tbody>
</table>
VITA
Kim H. Johnson

Education
Ph.D. Curriculum and Instruction, The Pennsylvania State University, August 2013
M.S.ed. Mathematics Education, Millersville University of Pennsylvania, August 2006
B.S. Secondary Education: Mathematics, Millersville University of Pennsylvania, May 1991

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Instructor, MTHED 420: Teaching Elementary Mathematics Methods, 2009

Selected Publications

Selected Presentations


Fellowships and Awards
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Mid-Atlantic Center for Mathematics Teaching and Learning Fellowship