ON THE NUMBER OF $A \times B$ QUOTIENT DIAGRAMS OF INTEGER PARTITIONS

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Abstract

In the early 1980’s, James Propp queried about the enumeration of objects related to the theory of partitions which he called *quotient diagrams*; in the nearly 30 years since, this question has gone on largely unanswered. In this dissertation, we will formally introduce $a \times b$ quotient diagrams as objects of study and provide one theoretical framework in which they can be viewed. This analysis highlights the connection between quotient diagrams and the $t$-core partitions, a connection first hypothesized by Propp.

Using a computer algebra package, a conjecture for the generating functions for the number of $a \times b$ quotient diagrams was found by the author. The main goal of this work is to give proofs for four cases of the conjecture: the $2 \times 2$, $1 \times b$, $2 \times 3$, and $2 \times 4$ quotient diagrams. We will also prove ancillary facts about quotient diagrams along the way and provide formulas for calculating the quotient diagrams of 2- and 3-core partitions.
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Chapter 1

Integer Partitions and Quotient Diagrams

1.1 Introduction

The theory of integer partitions can be attributed back to Leibniz; he called these the "divulsions" of an integer. The first important result in the field, however, goes back to Leonard Euler when he proved that the number of partitions of an integer $n$ into odd parts is equal to the number of partitions of $n$ into distinct parts. It was in then that Euler first introduced the notion of a generating function. Over the years, generating functions and, since the 19th century, Ferrers diagrams have been used to prove both analytically and combinatorially a number of results involving partitions.

The quotient diagram of a partition is attributed to Jim Propp who asked George Andrews if he knew of an enumeration of these new combinatorial objects. In this chapter, we will begin with the important terminology from the theory of partitions and define the quotient diagram of a partition. At the end of the chapter, we will introduce the quotient diagram conjecture, which would answer Propp’s question.
1.2 Integer Partitions

In this section, we provide basic definitions and results from the theory of partitions [1].

**Definition 1.2.1.** A *partition* of a positive integer $n$ is any finite sequence 

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$$

of positive integers in non-increasing order whose sum is $n$. The number $n$ being partitioned is called the *weight* of the partition and is denoted $|\lambda|$. The $\lambda_i$ are called the *parts* of $\lambda$ and the integer $r$ is called the *length* of $\lambda$.

There are a wide variety of notations one can use for partitions. We will mostly use the notation above; however, we will sometimes switch to exponential notation $(\lambda_1^{m_1}, \ldots, \lambda_r^{m_r})$, meaning that the part $\lambda_i$ occurs $m_i$ times.

**Definition 1.2.2.** Let $\text{Par}$ denote the set of all partitions and let $\text{Par}(n)$ denote the set of partitions of $n$. Define the partition function 

$$p(n) = |\text{Par}(n)|.$$ 

The partition function generating function is well known to be 

$$\sum_{n=1}^{\infty} p(n)q^n = (1 + q + q^2 + \cdots)(1 + q^2 + q^4 + \cdots)(1 + q^3 + q^6 + \cdots) \cdots$$

$$= \prod_{i=1}^{\infty} \frac{1}{(1 - q^i)}.$$ 

**Remark 1.2.3.** Throughout this dissertation, we will be using the $q$-Pochhammer notation where 

$$(a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})$$

and 

$$(a; q)_{\infty} = \lim_{n \to \infty} (a; q)_n = \prod_{i=1}^{\infty} (1 - aq^{i-1}).$$
With this notation we have that
\[ \sum_{n=0}^\infty p(n)q^n = \frac{1}{(q;q)_\infty}. \]

An effective way to picture integer partitions is through the use of Ferrers diagrams. These diagrams, similar to Young diagrams, allow us to graphically manipulate partitions and provide combinatorial insight to many partition-theoretic results.

**Definition 1.2.4.** The **Ferrers diagram** of a partition \( \lambda = (\lambda_1, \ldots, \lambda_r) \), otherwise known as the *graphical representation*, is a left-justified array of dots which has \( \lambda_i \) dots in the \( i^{th} \) row.

**Definition 1.2.5.** The **conjugate** of a partition \( \lambda \), denoted \( \lambda' \), is the partition where \( \lambda'_i \) is the number of parts in \( \lambda \) which are at least \( i \). The Ferrers diagram of a partition is the transpose of the Ferrers diagram of its conjugate.

**EXAMPLE**

The following theorem, Euler’s Pentagonal Number Theorem, is seminal as it provides a simple recursive formula for the partition function and an example of the elegance of graphical proofs using Ferrers diagrams.

**Theorem 1.2.6** (Euler’s Pentagonal Number Theorem). Let \( p_e(D, n) \) and \( p_o(D, n) \) denote the number of partitions of \( n \) into an even number of distinct parts and an odd number of distinct parts, respectively. Then,

\[ p_e(D, n) - p_o(D, n) = \begin{cases} (-1)^m & n = \frac{1}{2} m(3m \pm 1) \\ 0 & \text{otherwise} \end{cases} \]

Equivalently,

\[ (q; q)_\infty = \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{1}{2}m(3m-1)}. \]

This theorem is proven graphically through an involution due to Franklin [6] which creates a correspondence between the sets enumerated by \( p_e(D, n) \) and
$p_o(D, n)$ which breaks down only when $n$ is a pentagonal number. As a consequence to Theorem 1.2.6, we have that

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + \cdots + (-1)^{n-1} p(n - \frac{1}{2} m(3m \pm 1)) + \cdots.$$  

This provides a relatively easy way to calculate the number of partitions of $n$ and was used by Percy MacMahon to calculate $p(n)$ for $n$ up to 200.

At this point, it is important to view partitions as Ferrers diagrams rather than sequences of integers. We will be discussing partitions and their Ferrers diagrams interchangeably.

### 1.3 $t$-Core Partitions

We will see that quotient diagrams have a strong connection to $t$-core partitions. The $t$-core partitions are perhaps most easily defined using Ferrers diagrams. Most of the information of this section comes from chapter 1 of [9].

**Definition 1.3.1.** The hook length of a dot in a Ferrers diagram at position $(i, j)$, denoted $h(i, j)$, is the number of dots to the right and below it, including the dot itself, so that

$$h(i, j) = \lambda_i + \lambda_j - i - j + 1.$$  

**Definition 1.3.2.** Let $t \geq 1$ be an integer. A partition of $n$ whose Ferrers diagram has no hook lengths divisible by $t$ is a $t$-core partition of $n$.

The $t$-core of a partition $\lambda$ can be obtained from the Ferrers diagram of $\lambda$ by removing all possible border strips of length $t$. What remains is the Ferrers diagram of the $t$-core of $\lambda$. This process does not depend on the order of border strips removed.

**EXAMPLE**

Another way to represent $t$-core partitions is with an abacus, a array of $t$ rows of beads. To the partition $\lambda$, we add the partition $\Delta := (k, k-1, \ldots, 2, 1)$, part-wise. It matters not what $k$ is chosen as long as $k + 1$ is at least the number of parts of
the purpose is to make distinct the parts of the partition. If the number \(rt + s\), with \(0 \leq s \leq t - 1\), is one of the parts of \(\lambda + \Delta\), then put a bead in the \(s^{th}\) row and \(r^{th}\) column. The act of removing a border strip of length \(t\) changes the abacus by sliding a bead one space to the left. A \(t\)-core partition corresponds to an abacus where all the beads are as far to the left as possible. The fact is that the choice of \(k\) is relatively unimportant; there are many abaci which represent the same \(t\)-core partition.

**Theorem 1.3.3** (Lemma 1 in [8]). The abaci whose row sums are \((r_1, r_2, \ldots, r_t)\) and \((r_t + 1, r_1, \ldots, r_{t-1})\) represent the same \(t\)-core partition.

If \(c_t(n)\) denotes the number of \(t\)-cores of \(n\), the generating function of \(c(t)\) is

\[
T_t(q) := \sum_{n \geq 0} c_t(n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{tn})^t}{(1 - q^n)}.
\]

The set of 2-core partitions are easy to describe; these are well-known to be the triangular partitions. The best known descriptions of the \(t\)-core partitions for \(t \geq 3\) is given in [13].

The \(t\)-core partitions first arose in the representation theory of the symmetric groups. The Murnahan-Nakayama rule [14] states that the characters of the symmetric group can be built up from border strips of a given length. By working backwards and removing border strips instead of adding them, we arrive at \(t\)-core partitions. The connection between these two objects comes originally from the fact that the conjugacy classes, and therefore the irreducible representations, of the symmetric group \(S_n\) are indexed by the partitions of \(n\). The Specht module is a representation of the symmetric groups whose basis elements are formal sums of Young tabloids, i.e. equivalence classes of Young diagrams filled in with the numbers 1, \ldots, \(n\); over a field of characteristic zero, the Specht modules form a complete set of irreducible representations.

The \(t\)-core partitions have also been studied in the field of modular forms. It is a well-known fact that the generating function for \(t\)-cores are themselves modular forms. Among other results, Ono and Granville [12] used this fact to prove that,
for $t \geq 4$, $c_t(n)$ is non-zero for all $n \geq 0$.

### 1.4 Quotient Diagrams

We are now poised to introduce quotient diagrams, which are to be our main objects of study. After covering some basic results, we will introduce the main theorems and conjectures that will be proved in this dissertation.

**Definition 1.4.1.** Consider the $a \times b$ rectangular lattice, that is, the free $\mathbb{Z}$-module, generated by $(0, a)$ and $(b, 0)$. This lattice divides $\mathbb{Z}^2$ into $ab$ equivalence classes in the obvious way. Define the $a \times b$ quotient diagram of a partition $\lambda$, denoted $Q := Q_{a,b}(\lambda)$, to be the $a \times b$ matrix where the entry $Q_{i,j}$ is equal to the number of points in the Ferrers diagram of $\lambda$ which fall in the equivalence class $(i, j)$.

**Definition 1.4.2.** Let $\text{Quo}_{a \times b}$ denote the set of $a \times b$ diagrams and let $\text{Quo}_{a \times b}(n)$ denote the set of $a \times b$ quotient diagrams of $n$. Let

$$d(a, b, n) := |\text{Quo}_{a \times b}(n)|$$

denote the number of distinct $a \times b$ quotient diagrams of partitions of $n$.

We will abuse language somewhat: when we discuss the weight of a quotient diagram we of course mean the weight of the partition. In this way, rather than constantly referring to quotient diagrams of partitions of $n$, we can simply talk about quotient diagrams of $n$.

**Example 1.4.3.** Consider the partition $\lambda = (8, 7, 6, 4, 2, 1, 1)$ and let $a = b = 2$. We can draw the $2 \times 2$ lattice around the Ferrers diagram of $\lambda$, as can be seen in the diagram below. The lattice is drawn offset by $(-\frac{1}{2}, -\frac{1}{2})$ so that the lines and dots do not intersect, however this does not interfere with any counting arguments.
Counting over all the $2 \times 2$ squares, there are a total 9 dots which occur in the top-left positions of square, 8 dots which occur in the top-right, 7 in the bottom-left, and 5 in the bottom-right. Therefore, the $2 \times 2$ quotient diagram corresponding to this partition $\lambda$ is

$$\begin{bmatrix} 9 & 8 \\ 7 & 5 \end{bmatrix}.$$ 

We can consider the same partition with the $3 \times 3$ lattice.

The $3 \times 3$ quotient diagram of $\lambda$ is

$$\begin{bmatrix} 6 & 4 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 2 \end{bmatrix}.$$ 

In discussing quotient diagrams, we will use adjectives such as “larger” and “smaller” to discuss the size of the quotient diagrams rather than their weight. 

**Definition 1.4.4.** The **preimage** of a quotient diagram $Q_0$ is the set of partitions

$$\{\lambda \in \text{Par} | Q(\lambda) = Q_0\}.$$
**Definition 1.4.5.** An $a \times b$ block, or block if the size is clear from context, is the Ferrers diagram of a non-zero partition that fits in an $a \times b$ rectangle. Blocks may be represented either as an array of nodes or 0-1 matrices.

There are $\binom{a+b}{a} - 1$ blocks of size $a \times b$; these blocks are in bijection with the partitions that fit inside an $a \times b$ rectangle which are enumerated by the Gaussian coefficients.

When we view the $a \times b$ lattice over the Ferrers diagram of a partition, we see that partition as made up of an arrangement of these blocks. If we view these blocks as 0-1 matrices, the $a \times b$ quotient diagram is the sum of these blocks.

**Example 1.4.6.** The set of $2 \times 2$ blocks is listed below in both representations:

- 
  \[
  \begin{bmatrix}
  1 & 1 \\
  1 & 1 \\
  \end{bmatrix}
  \]
  \[
  \begin{bmatrix}
  1 & 1 \\
  1 & 0 \\
  \end{bmatrix}
  \]
  \[
  \begin{bmatrix}
  1 & 1 \\
  0 & 0 \\
  \end{bmatrix}
  \]
  \[
  \begin{bmatrix}
  1 & 0 \\
  1 & 0 \\
  \end{bmatrix}
  \]
  \[
  \begin{bmatrix}
  1 & 0 \\
  0 & 0 \\
  \end{bmatrix}
  \]

The $2 \times 2$ quotient diagram in Example 1.4.3 can be seen as the sum of blocks

\[
5 \begin{bmatrix}
1 & 1 \\
1 & 1 \\
\end{bmatrix} + 2 \begin{bmatrix}
1 & 1 \\
1 & 0 \\
\end{bmatrix} + \begin{bmatrix}
1 & 1 \\
0 & 0 \\
\end{bmatrix} + \begin{bmatrix}
1 & 0 \\
1 & 0 \\
\end{bmatrix}.
\]

While enumerating quotient diagrams of partitions is not a simple task, it is not difficult to remark on some of their elementary properties. The theorems in these sections, while not terribly deep, will help provide insight into how to manipulate quotient diagrams and provide techniques which we will use in later sections. Of particular note is the shifting method of building quotient diagrams which we can use to study diagrams of families of partitions, such as $t$-core partitions.

Large quotient diagrams of small partitions are not very interesting. If the partition fits into an $a \times b$ box, like those enumerated by the Gaussian coefficients
[1], then the quotient diagram is a single block, with the 0-1 matrix representation. All of these quotient diagrams have only that single partition in their preimage. In fact, if \( n < a + b \), then all the \( a \times b \) quotient diagrams of \( n \) have a single partition in their preimage. This is summed up in the next theorem.

**Theorem 1.4.7.** For all \( a, b, n \in \mathbb{N} \),

\[
d(a, b, n) \leq p(n)
\]

and equality holds when \( n < a + b \).

**Proof.** The inequality is clear; there cannot be more quotient diagrams than partitions. A quotient diagram is guaranteed to have a single partition in its preimage if one of two criteria are met:

(a) if the partition fits into an \( a \times b \) box, as discussed above, or

(b) if there is an entire row (or column) of zeroes in the quotient diagram.

Indeed the second criterion guarantees uniqueness of the partition for if, without loss of generality, the last row of a quotient diagram is a row of zeroes, we then know that the quotient diagram has at most \( a - 1 \) parts and so the partition is determined by the sum of the rows of the quotient diagram.

The smallest \( n \) which has a partition that both does not fit inside an \( a \times b \) block and has at least \( a \) parts where the longest part is at least \( b \) is \( n = a + b \). The partition in question is \( \lambda = (b, 1^a) \), which has the same quotient diagram as the partition \( \lambda = (b + 1, 1^{a-1}) \).

Due to conjugation, there is symmetry between \( a \times b \) and \( b \times a \) quotient diagrams. This symmetry allows us to restrict our study of quotient diagrams to the cases where \( a \leq b \).

**Theorem 1.4.8.** For all \( a, b, n \in \mathbb{N} \),

\[
d(a, b, n) = d(b, a, n).
\]
Proof. The $b \times a$ quotient diagram of a partition $\lambda$ is the transpose of the $a \times b$ quotient diagram of the conjugate partition $\lambda'$.

Two corollaries follow immediately when $a = b$.

**Corollary 1.4.9.** The $a \times a$ quotient diagram of a partition is the transpose of the quotient diagram of the conjugate partition.

**Corollary 1.4.10.** The $a \times a$ quotient diagram of a self-conjugate partition is symmetric.

Note however that the converse to this last corollary is not true; for example, take the quotient diagram of the partition given at the end of the proof of 1.4.7.

In [9], Macdonald notes the relationship between the process of taking the $t$-core of partition and the Euclidean algorithm. Quotient diagrams share this relationship and are connected with the equivalence classes in which the parts of the partition lie.

**Definition 1.4.11.** The $(i,j) \pmod{(a,b)}$-subset of a Ferrers diagram, or just $(i,j)$-subset if $a$ and $b$ are understood from context, is the subset of dots whose positions $(i',j')$ are such that $i \equiv i' \pmod{a}$ and $j \equiv j' \pmod{b}$.

The $(i,j)$ entry in an $a \times b$ quotient diagram of a partition with Ferrers diagram $F$ is the number of elements in the $(i,j)$-subset of $F$.

**Theorem 1.4.12.** For $a, b \in \mathbb{N}$, if $1 \leq i < a$ and $1 \leq j < b$ and $Q(\lambda) = (d_{i,j})$ then the $d_{i,j} - d_{i,j+1}$ is the number of parts congruent to $j \pmod{b}$ whose index is $i \pmod{a}$.

**Proof.** The quantity $d_{i,j-1} - d_{i,j}$ is the number of dots in the $(i,j-1)$ subset of the Ferrers diagram of $\lambda$ which do not have a dot directly to the right of them. The only way this can happen is if this is the last dot in a row whose index is $i \pmod{a}$ and the part is congruent to $j-1 \pmod{b}$. 

**Remark 1.4.13.** A priori, there is no way to determine the number of parts congruent to $0 \pmod{b}$ since this depends on the number of parts in $\lambda$. If, however,
the number of parts is known to be \( r \), and \( r = ak + l \), with \( 0 \leq l \leq a - 1 \), then the number of parts in \( \lambda \) congruent to 0 (mod \( b \)) whose index is \( i \) (mod \( a \)) is

\[
d_{i,b} - d_{i,1} + k + \begin{cases} 1 & \text{if } i \leq l \\ 0 & \text{if } i > l \end{cases}
\]

Just as partitions can be built up from other partitions, so can quotient diagrams. There are two ways to build partitions which are immediately useful to the study of quotient diagrams. The first is to add 1 to one part of the partition, that is, adding one dot to the Ferrers diagram. This affects the quotient diagram by increasing one of the entries by one. The second way is to add a new row or column of dots to the Ferrers diagram.

If we consider a Ferrers diagram as just an array of dots in \( \mathbb{Z}^2 \), the action of moving the entire Ferrers diagrams down one cycles the rows of the quotient diagram down one, pushing the bottom row to the top. Likewise, moving all the diagram one to the right has the effect of cycling the columns of the quotient diagram right by one, where the rightmost column becomes the first column.

Now if, after we push a Ferrers diagram down by one, we insert a new swath of dots in the first row, we cycle the columns of the quotient diagram downwards and then add values only to the new first row of the quotient diagram.

To make this procedure concrete, let \( \lambda = (\lambda_1, \lambda_2, \ldots) \) be a partition and \( \lambda^* \) a new partition where \( \lambda_1^* = m \geq \lambda_1 \) and for \( i > 1 \), \( \lambda_i^* = \lambda_{i-1} \). If the \( a \times b \) quotient diagram of \( \lambda \) is the matrix whose rows are \( r_1, \ldots, r_a \), then the \( a \times b \) quotient diagram of \( \lambda^* \) is the matrix whose rows are \( r_a + M, r_1, \ldots, r_{a-1} \) where \( M \) is the \( a \times b \) quotient diagram of the partition \((m)\).

Symmetrically, let \( \lambda \) and \( \lambda^* \) be two partitions where the Ferrers diagram of \( \lambda^* \) is obtained by taking the Ferrers diagram of \( \lambda \) and adding a column of \( m \) dots to the left side, where \( m \) is greater than the number of parts of \( \lambda \). That is, \( \lambda^* \) has \( m \) parts and \( \lambda_i^* - \lambda_i = 1 \), for all the parts in \( \lambda^* \). If the \( a \times b \) quotient diagram of \( \lambda \) is the matrix whose columns are \( c_1, \ldots, c_b \), then the \( a \times b \) quotient diagram of \( \lambda^* \) is the matrix whose rows are \( c_b + M, c_1, \ldots, c_{b-1} \) where \( M \) is the \( a \times b \) quotient diagram of the partition \((1^m)\).
This procedure will be referred to as the *shifting method* of building quotient diagrams from previous quotient diagrams.

**Example 1.4.14.** Let $\lambda = (2, 2, 1)$ and $\lambda^* = (3, 3, 2, 1, 1)$; note that the Ferrers diagram of $\lambda^*$ is the Ferrers diagram of $\lambda$ with an additional column of 5 dots. We will use the shifting method to build $Q_{2 \times 2}(\lambda^*)$ from $Q_{2 \times 2}(\lambda)$. First, we must note that

\[ Q(\lambda) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \]

and that

\[ Q_{2 \times 2}((1^5)) = \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix}. \]

We can then use the shifting method to see that

\[ Q_{2 \times 2}(\lambda^*) = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}. \]

An important property of sequences of quotient diagrams obtained via successive shifts is that, if we know only a sparse number of quotient diagrams in the sequence, we can fill in the gaps and read off the rest of the quotient diagrams.

Consider a sequence of $b+1$ partitions $\lambda^{(1)}, \ldots, \lambda^{(b+1)}$ where the Ferrers diagram of the $i^{th}$ partition is the Ferrers diagram of the $(i-1)^{st}$ partition with an additional column of $f_i$ dots attached to the left side of the diagram, and let $Q_i$ denote the $a \times b$ quotient diagram of the $i^{th}$ partition in the sequence. If the quotient diagrams $Q_1$ and $Q_{b+1}$ are known, then the remaining $b-1$ quotient diagrams can be easily discerned.

In particular, if $Q_b$ has columns $c_1, c_2, \ldots, c_b$ and $Q_0$ has columns $c_{b+1}, c_{b+2}, \ldots, c_{2b}$, then the quotient diagram $Q_i$, where $1 \leq i \leq b$, has columns, $c_{b-i+1}, \ldots, c_{2b-i+1}$. We can show this by examining the relationship first between $Q_i$ and $Q_0$ and then between $Q_b$ and $Q_i$. By the shifting process, for $1 \leq j \leq b-i$, the $j^{th}$ column in $Q_1$ gets shifted to the right $i-1$ times and is now the $(j + i - 1)^{th}$ column in $Q_i$. On the other hand, for $1 \leq j \leq i$, the $j^{th}$ column in $Q_i$ is the $(b-j)^{th}$ column in $Q_b$.

In the following example, and for the rest of the dissertation, we let $t_k$ denote
the \(k^{th}\) triangular number.

**Example 1.4.15.** We can use the shifting method to exhibit the quotient diagrams of triangular partitions.

Consider the sequence of triangular partitions \(\Delta_k^2 := (k, k-1, \ldots, 2, 1)\); the Ferrers diagram of the \(k^{th}\) triangular partition is the Ferrers diagram of the \((k-1)^{st}\) triangular partition with an additional column of \(k\) dots attached to the left side.

Consider now the \(2 \times 2\) quotient diagrams of the the triangular partitions. If \(k = 2m - 1\), the \(2 \times 1\) quotient diagram of \((1^k)\) is \([m \ m - 1]\) and if \(k = 2m\), the \(2 \times 1\) quotient diagram of \((1^k)\) is \([m \ m]\). This means that

\[
Q(\Delta_{2m}^2) = Q(\Delta_{2m-2}^2) + \begin{bmatrix} m & m \\ m & m - 1 \end{bmatrix}.
\]

When \(m = 2\),

\[
Q(\Delta_2^2) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} t_1 & t_1 \\ t_1 & t_0 \end{bmatrix}
\]

and if

\[
Q(\Delta_{2m}^2) = \begin{bmatrix} t_m & t_m \\ t_m & t_{m-1} \end{bmatrix}
\]

then

\[
Q(\Delta_{2m+2}^2) = \begin{bmatrix} t_m & t_m \\ t_m & t_{m-1} \end{bmatrix} + \begin{bmatrix} m + 1 & m + 1 \\ m + 1 & m \end{bmatrix} = \begin{bmatrix} t_{m+1} & t_{m+1} \\ t_{m+1} & t_m \end{bmatrix}.
\]
Therefore, by induction,

\[ Q(\Delta^2_{2m}) = \begin{bmatrix} t_m & t_m \\ t_m & t_{m-1} \end{bmatrix}. \]

By the remark above, we can use the quotient diagrams for the \((2m - 2)^{th}\) and \(2m^{th}\) triangular partitions to get the quotient diagram for the \((2m - 1)^{st}\) triangular partition; specifically, since

\[ Q(\Delta^2_{2m}) = \begin{bmatrix} t_m & t_m \\ t_m & t_{m-1} \end{bmatrix} \text{ and } Q(\Delta^2_{2m-2}) = \begin{bmatrix} t_{m-1} & t_{m-1} \\ t_{m-1} & t_{m-2} \end{bmatrix}, \]

we can see that

\[ Q(\Delta^2_{2m-1}) = \begin{bmatrix} t_m & t_{m-1} \\ t_{m-1} & t_{m-1} \end{bmatrix}. \]

We can relate quotient diagrams of the same partition with different sizes if the length and width of one divides the other’s, respectively. The collection of dots in the \((1, 1) \mod (2, 2))\)-subset of the Ferrers diagram of a partition, \(\lambda\), is the union of the \((1, 1) \mod (2, 4))\) and \((1, 3) \mod (2, 4))\) subsets. This means that the \((1, 1)\) entry of the \(2 \times 2\) quotient diagram of \(\lambda\) depends only on the \((1, 1)\) and \((1, 3)\) entries of the \(2 \times 4\) quotient diagram of \(\lambda\). Graphically, this is because the \(2 \times 4\) is a sublattice of the \(2 \times 2\) lattice. We can extend this idea using the following maps.

**Definition 1.4.16.** If \(c \mid a\) and \(d \mid b\), then \(a \times b\) to \(c \times d\) **covering map**, denoted \(\text{Cov}_{c \times d}^{a \times b}\), is the function where, if \(Q\) is an \(a \times b\) quotient diagram, then \(\text{Cov}_{c \times d}^{a \times b}(Q)\) is the \(c \times d\) quotient diagram of the preimage of \(Q\).

**Theorem 1.4.17.** The covering maps are well-defined.

**Proof.** Let \(Q_{a \times b}\) and \(Q_{c \times d}\) be the \(a \times b\) and \(c \times d\) quotient diagrams of the same partition, respectively, and let \(\mathbb{Z}_{i,a}\) denote the set of integers congruent to \(i\) \((\text{mod } a)\). Since

\[ \mathbb{Z}_{i,c} = \bigcup_{k=0}^{2^2-1} \mathbb{Z}_{i+kc,a}. \]

the \(c \times d\) \((i, j)\)-subset of the Ferrers diagram of a partition is equal to the union of the \(a \times b\) \((i + kc, j + ld)\)-subsets of the Ferrers diagram as \(0 \leq k \leq \frac{a}{c} - 1\) and
0 ≤ l ≤ \frac{b}{d} - 1. Therefore, the (i, j) entry in \( Q_{c \times d} \) is equal to the sum of the (i + kc, j + ld) entries in \( Q_{a \times b} \) as 0 ≤ k ≤ \frac{a}{c} - 1 and 0 ≤ l ≤ \frac{b}{d} - 1. In particular, this means that the \( c \times d \) quotient diagram depends only on the \( a \times b \) diagram and therefore the covering map is well-defined.

Example 1.4.18. Let \( \lambda = (5, 4, 3, 2, 1) \) and consider the \( 2 \times 4 \) quotient diagram

\[
Q := Q_{2 \times 4}(\lambda) = \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}.
\]

The image of the \( 2 \times 4 \) to \( 2 \times 2 \) covering map is

\[
\text{Cov}_{c \times d}^{a \times b}(Q) = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 3 & 3 \end{bmatrix},
\]

which, from example 1.4.15, we know to be \( Q_{2 \times 2}(\lambda) \).

We will now present the conjecture for the generating function \( D(a, b, q) \).

Conjecture 1.4.19. The generating function for the number of \( a \times b \) quotient diagrams of \( n \), where \( a ≥ 1 \) and \( b ≥ 1 \) is

\[
D(a, b, q) := \sum_{n=0}^{\infty} d(a, b, n) q^n = \frac{1}{(q; q)_{\infty}} \prod_{i=1}^{a} (q^{a+b}; q^a)_{\infty}.
\]

This conjecture was derived by the author after calculating the values of \( d(a, b, n) \) for various small values of \( a, b, \) and \( n \) using a computer algebra package. In the second chapter of this dissertation, we prove this conjecture for \( a = b = 2 \). In the third chapter, we prove the conjecture for \( a = 1 \) and for the cases \( a = 2, b = 3 \) and \( a = 2, b = 4 \).
A Theory of Quotient Diagrams

2.1 Introduction

The first family of quotient diagrams studied by the author was the $2 \times 2$ quotient diagrams where the bottom-right entry is 0. The partitions in the preimages of these quotient diagrams all had to have exactly one rim hook and it was noted that, if the quotient diagram was

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix},$$

then either $a = b + c$ or $a - b - c = \pm 1$. Studying such quotient diagrams, and gradually increasing the possible values of the bottom-right entry gave rise to a signature map which corresponded to each $2 \times 2$ quotient diagram of an integer; these signatures were then extended to larger quotient diagrams by changing the target space from the set of integers to various matrix sets.

The kernel of the signature map provides a special set of quotient diagrams which can be effectively “modded out” by a process which will be known as reduction. This will allow us to reduce the problem of enumerating quotient diagrams to enumerating these remaining reduced quotient diagrams. In Section 2.4, we will visit the connection that quotient diagrams and $t$-core partitions share. We will further explore this connection in the last two sections of this chapter by explicitly describing the quotient diagrams of 2- and 3-core partitions.
2.2 Signatures and Crust Diagrams

Definition 2.2.1. If $\mathcal{Q} = [d_{i,j}]$ is an $a \times b$ quotient diagram, define the signature of $\mathcal{Q}$, denoted $\sigma(\mathcal{Q})$ as the $(a - 1) \times (b - 1)$ integer-valued matrix,

$$\sigma(\mathcal{Q})_{i,j} := d_{i,j} + d_{i+1,j+1} - d_{i+1,j} - d_{i,j+1}.$$  

We will refer to the set of integer-valued matrices $M_{a-1,b-1}(\mathbb{Z})$ as the $a \times b$ signature space.

Remark 2.2.2. To avoid confusion, quotient diagrams will be delimited with square brackets while signatures of quotient diagrams will be delimited with parentheses.

Theorem 2.2.3. The signature map

$$\sigma : M_{a,b}(\mathbb{N}) \rightarrow M_{a-1,b-1}(\mathbb{Z})$$

preserves matrix addition, that is, if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are two $a \times b$ quotient diagrams, then

$$\sigma(\mathcal{Q}_1 + \mathcal{Q}_2) = \sigma(\mathcal{Q}_1) + \sigma(\mathcal{Q}_2).$$

Proof. Let $\mathcal{Q}_1 = [d^1_{i,j}]$ and let $\mathcal{Q}_2 = [d^2_{i,j}]$.

$$\sigma(\mathcal{Q}_1 + \mathcal{Q}_2) = (d^1_{i,j} + d^2_{i,j}) + (d^1_{i+1,j+1} + d^2_{i+1,j+1}) - (d^1_{i+1,j} + d^2_{i+1,j}) - (d^1_{i,j+1} + d^2_{i,j+1})$$

$$= d^1_{i,j} + d^1_{i+1,j+1} - d^1_{i,j+1} + d^2_{i,j} + d^2_{i+1,j+1} - d^2_{i+1,j} - d^2_{i,j+1}$$

$$= \sigma(\mathcal{Q}_1) + \sigma(\mathcal{Q}_2).$$

Theorem 2.2.4. The signature map

$$\sigma : Q_{a,b} \rightarrow M_{a-1,b-1}(\mathbb{Z})$$

is a surjection. That is, for each $(a - 1) \times (b - 1)$ matrix with integer entries, there exists a partition whose quotient diagram has that matrix as its signature.
Proof. Define $U_{i,j}^+$ to be the $(a \times b)$ block given as the 0-1 $a \times b$ matrix $[u_{k,l}^+]$ where

$$u_{k,l}^+ = \begin{cases} 1 : & k \leq i \text{ and } l \leq j \\ 0 : & \text{otherwise} \end{cases}$$

and $U_{i,j}^-$ to be the block defined as the 0-1 $a \times b$ matrix $[u_{k,l}^-]$, where

$$u_{k,l}^- = \begin{cases} 1 : & k \leq i \text{ or } l \leq j \\ 0 : & \text{otherwise} \end{cases}.$$ 

The examples below should elucidate what these blocks look like in general.

$$U_{3,4}^+ = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad U_{3,4}^- = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Note that, if $i < a$ and $j < b$, the signature of $U_{i,j}^+$ (resp. $U_{i,j}^-$) is the $(a - 1) \times (b - 1)$ matrix whose entries are 1 (resp. -1) in position $(i,j)$ and 0 everywhere else.

Let $S \in M_{a-1,b-1}(\mathbb{Z})$. For all entries $S_{i,j}$ in $S$, consider the set which contains $S_{i,j}$ copies of the block $U_{i,j}^+$ if $S_{i,j} > 0$ and $S_{i,j}$ copies of $U_{i,j}^-$ if $S_{i,j} < 0$. Then, consider the matrix $V$, which is the sum of these blocks and $t_{r-1}$ copies of $U_{a,b} := U_{a,b}^+$, where

$$r := \sum_{i,j} |S_{i,j}|,$$

the total number of $U_{i,j}^{+/-}$ blocks. Because signatures preserve addition, the signature of $V$ is $S$. Therefore, it remains to show that $V$ is a quotient diagram by exhibiting a partition in its preimage.

Order the $U_{i,j}^{+/-}$ blocks with multiplicity, $U_1, U_2, \ldots, U_r$. Place $r - 1$ copies of $U_{a,b}$ and then $U_1$ in a row. Below them, place $r - 2$ copies of $U_{a,b}$ and then $U_2$, and
so on until you place $U_{a,b}$, $U_{r-1}$ to the right of it, and $U_r$ below it. The partition with this Ferrers diagram has $V$ as its quotient diagram.

Example 2.2.5. From the theorem, there is a $3 \times 3$ quotient diagram whose signature is

$$\begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}.$$ 

From the theorem, we need one copy of $U_{-1,1}$, two copies of $U_{2,1}^+$, and one copy of $U_{2,2}^-$. The matrix we then get is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{pmatrix} = \begin{pmatrix} 10 & 8 & 8 \\ 10 & 7 & 7 \\ 8 & 7 & 6 \end{pmatrix}$$

This is the quotient diagram of the partition $\lambda = (12, 10, 10, 7, 7, 6, 4, 4, 3, 3, 3, 2)$ as shown by the Ferrers diagram.

The fact that $U_{a,b}$ blocks have the zero signature is necessary for the construction in the previous theorem. All of the blocks $U_{i,b} := U_{i,b}^+$ and $U_{a,j} := U_{a,j}^+$, examples of which are shown below, have the zero signature and therefore contribute nothing to the signature of the partition to which these blocks belong.
In general, any quotient diagram with the zero signature is of special importance.

**Definition 2.2.6.** A crust diagram is a matrix whose signature is the zero matrix.

A priori, it is not assumed that all crust diagrams are quotient diagrams; this fact will be proved in the next theorem.

**Theorem 2.2.7.** If $C$ is a crust diagram, then it is a quotient diagram and there exist non-negative integers $u_i, v_j$ such that

$$C = \sum_{i=1}^{a} u_i U_{i,b} + \sum_{j=1}^{b-1} v_j U_{a,j}.$$  

**Proof.** Let $C = [c_{i,j}]$ be a matrix whose signature is the zero matrix. First we must have that the number of full blocks is $u_a = c_{a,b}$ since only full blocks contribute to the $(a,b)$-entry. Likewise, the $U_{a-1,b}$ are the only other blocks that contribute to the $(a - 1, b)$ entry and so $u_{a-1} = c_{a-1,b} - c_{a,b}$. Continuing in this fashion, we have that $u_i = c_{i,b} - c_{i+1,b}$.

Now let $C' = [c'_{i,j}]$ be the matrix you get after subtracting all the crust blocks found in the previous step;

$$C' = C - \sum_{i=1}^{a} u_i U_{i,b}^+.$$
Note that the last column of $C'$ is all zeroes. Also, it is important to note that since we have only subtracted crust diagrams, the signature of $C'$ is still the zero matrix. However this means that $c'_{i,b-1} - c'_{i+1,b-1} = 0$ for all $i < a$ and therefore the values in the $(b - 1)^{st}$ column of $C'$ are all equal to $c'_{a,b-1} = c_{a,b-1} - c_{a,b}$. So we have that $v_{b-1} = c_{a,b-1} - c_{a,b}$. After subtracting these blocks, we have that the $(b - 1)^{st}$ column is all zeroes and so we can repeat the process to get that $v_i = c_{a,i} - c_{a,i+1}$.

To get the partition whose quotient diagram is $C$, perform the following construction: Take the full blocks and line them up in a row. Then, for the next largest $i$, take the $u_i$ copies of $U_{i,b}$ and line them up to the right in the same row. Repeat this for all $i$. Then, take the $v_j$ copies of $U_{a,j}$ and line them up below the full blocks in a column.

**Example 2.2.8.** The quotient diagram

$$Q = \begin{bmatrix} 6 & 4 & 4 & 3 \\ 5 & 3 & 3 & 2 \end{bmatrix}$$

is a crust quotient diagram. From Theorem 2.2.7 we have that $u_{1,4} = 1$, $u_{2,4} = 2$, $v_{2,1} = 2$, $v_{2,2} = 0$, and $v_{2,3} = 1$. The partition whose quotient diagram is $C$ is then $\lambda = (12, 8, 3, 3, 1, 1, 1, 1)$ as can be seen by the Ferrers diagram below.

![Ferrers diagram](image)

In general, the sum of two quotient diagrams is not a quotient diagram. In the construction of the partition in the proof of Theorem 2.2.4, we had to include a number of copies of $U_{a,b}$ to act as a scaffolding for the other blocks. However, if one of the quotient diagrams is a crust diagram, then quotient diagram addition is well-defined.
**Theorem 2.2.9.** If $Q$ is a quotient diagram and $C$ is a crust diagram, then $Q + C$ is a quotient diagram. Moreover, the signatures of $Q$ and $Q + C$ are equal.

**Proof.** Decompose $C$ as in Theorem 2.2.7. Let $C_1$ be the quotient diagram made up of the collection of the $U_{i,b}$ blocks and let $C_2$ be the quotient diagram made up of the collection of the $U_{a,j}$ blocks. Using the construction in Theorem 2.2.7, let $\gamma^1 = (\gamma_1^1, \gamma_1^2, \ldots)$ and $\gamma^2 = (\gamma_2^1, \gamma_2^2, \ldots)$ be the partition in the preimage of $C_1$ and $C_2$, respectively. Note that all the parts in $\gamma^1$ are multiples of $b$ and all the parts in the conjugate of $\gamma^2$ are multiples of $a$.

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition in the preimage of $Q$. For all the parts in $\gamma^1$, attach $\gamma_i^1$ dots to the right of $\lambda_i$. The partition you get, $\lambda + \gamma^1$, has the quotient diagram $Q + C_1$. Now, for all the columns in $\gamma^2$, attach the $i$th column of $\gamma^2$ to the bottom of the $i$th column of $\lambda$. The resulting partition is $((\lambda + \gamma^1)' + \gamma^2)'$ and has the quotient diagram $Q + C_1 + C_2 = Q + C$. □

**Example 2.2.10.** Consider the sum of the quotient diagram

$$Q := \begin{bmatrix} 3 & 3 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

and the crust diagram

$$C := \begin{bmatrix} 4 & 4 & 2 \\ 3 & 3 & 1 \end{bmatrix}$$

which decomposes into

$$C_1 = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix}.$$  

The partition $\gamma^1 := (6, 3)$ is in the preimage of $C_1$ and the partition $\gamma^2 := (2, 2, 2, 2)$ is in the preimage of $C_2$ and the partition $\lambda := (5, 3, 2, 1)$ is in the preimage of $Q$.

As in the theorem, we attach $\gamma^1$ to the right of $\lambda$ and $\gamma^2$ to the bottom of $\lambda$; in the diagram, the dots in the Ferrers diagram of $\lambda$ are black, while those in the Ferrers diagrams of the $\gamma^i$ are white.
Note that the result is indeed

\[((\lambda + \gamma^1)' + \gamma^2)' = (11, 6, 2, 2, 2, 1)\]

and that the quotient diagram is

\[
\begin{bmatrix}
3 & 3 & 1 \\
2 & 1 & 1
\end{bmatrix} + \begin{bmatrix}
4 & 4 & 2 \\
3 & 3 & 1
\end{bmatrix} = \begin{bmatrix}
7 & 7 & 3 \\
5 & 4 & 2
\end{bmatrix}.
\]

The last two theorems together give us the picture that the set of \(a \times b\) crust diagrams forms a free \(N\)-module, where any number of crust diagrams can attached together and the quotient diagram only depends on the decomposition in Theorem 2.2.7. In fact, this is a graded \(N\)-module, where the grading is the weight of the quotient diagram. Since the weight of \(U_{i,b}\) is \(ib\) and the weight of \(U_{a,j}\) is \(aj\), we can get the generating function for the number of \(a \times b\) crust diagrams of weight \(n\).

**Corollary 2.2.11.** The generating function for the number of crust diagrams of partitions of \(n\) is

\[
D_C(a, b, q) = \frac{1}{(q^b; q^b)_a(q^a; q^a)_{b-1}}.
\]

### 2.3 Reduced Quotient Diagrams

We can abuse notation and say that, using the notation in Theorem 2.2.9, if \(Q' = Q + C\), then \(Q' - C = Q\). We cannot, however, use this notation in general; \(Q - C\) is not always a quotient diagram. In particular, there exist quotient diagrams such that \(Q - C\) is not a quotient diagram where \(C\) is any crust diagram. We will call such diagrams reduced and in this section show that every quotient diagram is
made up of a reduced quotient diagram and a crust quotient diagram in a unique manner.

**Definition 2.3.1.** A quotient diagram is **reduced** if it cannot be written as the sum of quotient diagram and a non-zero crust diagram.

We can imagine starting with a quotient diagram. One-by-one, we could try to subtract the various crust diagrams and see if the result is a quotient diagram. At some point, we would no longer be able to remove crust diagrams and the result would be a reduced quotient diagram. This process is called reduction and is formally explained in the following algorithm. Reduction can be seen as analogous to the process of removing border strips to create a \( t \)-core partition and it is through this analogy that many of the connections between quotient diagrams and \( t \)-cores arise.

**Definition 2.3.2.** Let \( Q \) be a quotient diagram. If all the values in the \( i^{th} \) row, for \( 1 \leq i < a \), are strictly greater that the values below them, then we say that \( Q \) has a **horizontal fault** between the \( i^{th} \) and \( (i+1)^{st} \) rows. If all the values in the \( j^{th} \) column, for \( 1 \leq j < b \), are strictly greater that the values to the right of them, then we say that \( Q \) has a **vertical fault** between the \( j^{th} \) and \( (j+1)^{st} \) columns.

**Algorithm 2.3.3.** (Reduction Algorithm) The reduction algorithm will occur in two steps. The first step will remove all possible non-full crust blocks and then the second removes the full blocks. For the first part, we will show that, if a quotient diagram has a horizontal fault, we can subtract a copy of \( U_{i,b} \) for some \( i < a \). By symmetry, we will also be able to subtract a copy \( U_{a,j} \) for some \( j < b \) if there is a vertical fault. Let \( Q = [d_{i,j}] \) be a quotient diagram with a horizontal fault. Assume that the fault occurs between the \( i^{th} \) and \( i + 1^{st} \) rows where \( i < a \), that is, \( d_{i,j} > d_{i+1,j} \) for \( 1 \leq j \leq b \).

Let \( \lambda \) be any partition in the preimage of \( Q \). Since \( d_{i,1} > d_{i+1,1} \), there is some dot in the \((i,1)\)-subset of \( \lambda \) which does not have a dot beneath it; circle this dot and the \( b-1 \) dots directly to the right of it or all dots to the right if there are only
Since $d_{i,k+1} > d_{i+1,k+1}$, there is some dot in the $(i,k+1)$-subset which does not have a dot beneath it; circle this dot and the $b-k-1$ to the right of it or all the dots to the right if there are only $k' < b-k-1$ dots. Continue with this process until you have circled $b$ dots. Removing these $b$ dots changes the quotient diagram by subtracting 1 from each of the entries of $Q$ in the $i^{th}$ row.

At this point, there is a horizontal fault between the $(i-1)^{st}$ and $i^{th}$ rows. We can repeat the process above to remove $b$ more dots, thus subtracting 1 from each of the entries of $Q$ in the $(i-1)^{st}$ row. We can again repeat until $i = 1$. The resulting quotient diagram is $Q - U_{i,b}$.

This procedure guarantees that, if a quotient diagram has a fault, it can be further reduced. However this part of the reduction algorithm only removes the blocks $U_{i,b}$ and $U_{a,j}$ for $i < a$ and $j < b$ but does not remove any copies of the full block $U_{a,b}$. Each family of quotient diagrams which are equal modulo a multiple of $U_{a,b}$ is linearly ordered according to their $d_{a,b}$ entries. In each of these families, the quotient diagram with minimal $d_{a,b}$ is the reduced quotient diagram.

**Theorem 2.3.4.** The result of the reduction algorithm on the quotient diagram $Q$, denoted red($Q$), is a reduced quotient diagram.

**Proof.** Let $Q$ be quotient diagram and consider $R := \text{red}(Q)$. The difference between $R$ and any non-full crust diagram is not a quotient diagram. If on the other hand, without loss of generality, $R - U_{i,b}$ was a quotient diagram for some $i$ less than $a$, then $R$ would have a horizontal fault between the $i^{th}$ and $(i+1)^{st}$ rows. However this contradicts the fact that all the faults of a quotient diagram are removed during the reduction algorithm.

It remains to argue that the difference between $R$ and a full crust diagram is not a quotient diagram. If this were true however, then the bottom-right entry in $R$ would not be minimal, which contradicts the algorithm.

We can refine the reduction algorithm if we know the largest part and number of parts of $\lambda$, one of the partitions in the preimage of a quotient diagram $Q = [d_{i,j}]$.

We perform a removal of dots, similar to the first part of the algorithm, which lie in the $b^{th}$ column. Specifically, we are looking for dots in the $(i,b)$ subset of the
Ferrers diagram that have no dots directly to the right. These dots to the right are in the \((i, 1)\) subset of the Ferrers diagram, along with the first column of dots. To account for this first column, we subtract the Ferrers diagram of the column alone,

\[
Q(1^r) = \begin{bmatrix} C_1 & \cdots & 0 \\ \vdots \\ C_a & \cdots & 0 \end{bmatrix},
\]

where \(r\) is the number of parts of \(\lambda\), from the original Ferrers diagram. Then compare the first and last columns. If \(d_{i,1} - C_i < d_{i,b}\) for all \(i\), then there are \(b\) dots which can be removed as before; this creates a fault between the second-to-last and last columns, and so we appeal to the case above and, in all, a full block can be removed. By symmetry, all the following remarks can be said about removing dots which lie in the \(a^{th}\) row. That is, if

\[
Q(\lambda_1) = \begin{bmatrix} R_1 & \cdots & R_a \\ \vdots \\ 0 & \cdots & 0 \end{bmatrix}
\]

then if \(d_{1,j} - R_j < d_{a,j}\) for all \(j\), then a full block can be removed.

If however, for all partitions in the preimage of \(Q\), \(d_{i,1} - C_i = d_{i,b}\) for some \(i\) and \(d_{i,j} - R_j = d_{a,j}\) for some \(j\), then no full blocks can be removed, i.e. the quotient diagram is reduced. This is true, because given the above equalities, there are no dots in the \((i, b)\) subset without a dot to the right of it and no dots in the \((a, j)\) subset without a dot below it and so those entries in the quotient diagram cannot be lessened.

**Theorem 2.3.5.** The following statements about reduction are true:

(a) if \(C\) is a crust diagram, then \(\text{red}(Q + C) = \text{red}(Q)\),

(b) if \(R\) is a reduced diagram, then \(\text{red}(R) = R\),

(c) and, in particular, \(\text{red}(R + C) = R\).

**Proof.** Parts (a) and (b) are obvious from how the reduction algorithm is defined. Part (c) follows immediately from (a) and (b). \(\square\)
**Theorem 2.3.6.** The signature map is a bijection between the set of reduced $a \times b$ quotient diagrams and $M_{a-1,b-1}(\mathbb{Z})$.

**Proof.** Assume, for the sake of contradiction, that there are two reduced quotient diagrams, $\mathcal{R}_1$ and $\mathcal{R}_2$, that have the same signature, $\sigma$. Since the signature map is addition-preserving, the signature of $\mathcal{R}_1 - \mathcal{R}_2$ is the zero matrix. Therefore, we get a matrix as in Theorem 2.2.7 with the caveat that the coefficients $u_i$ and $v_j$ can take positive and negative values in $\mathbb{Z}$. Using the same method as in the proof of Theorem 2.2.7, we can find crust diagrams $\mathcal{C}_1$ and $\mathcal{C}_2$ such that

$$\mathcal{R}_1 - \mathcal{R}_2 = \mathcal{C}_1 - \mathcal{C}_2,$$

where $\mathcal{C}_1$ adds over all the $u_i$ and $v_j$ which are positive and $\mathcal{C}_2$ adds over the absolute value of all those which are negative. By rearranging, we get that

$$\mathcal{R}_1 + \mathcal{C}_1 = \mathcal{R}_2 + \mathcal{C}_2$$

and so by reducing both sides and Theorem 2.3.5, we have that

$$\mathcal{R}_1 = \mathcal{R}_2;$$

this proves that the signature map is injective.

The fact that the signature is a surjection comes from applying reduction to the quotient diagrams that we get from Theorem 2.2.4.

\[ \square \]

**Theorem 2.3.7.** Every quotient diagram is uniquely the sum of a reduced diagram and a crust diagram. Consequently, if $D_C(a, b, q)$ and $D_R(a, b, q)$ are the generating functions which enumerate the crust diagrams and reduced diagrams, respectively, then

$$D(a, b, q) = D_C(a, b, q)D_R(a, b, q).$$

**Proof.** Let $\mathcal{Q}$ be a quotient diagram. From Theorem 2.3.6, the signature of $\mathcal{Q}$ uniquely corresponds to a reduced quotient diagram $\mathcal{R}$ of the same size. Since $\mathcal{Q}$
and \( R \) have the same signature, their difference is a matrix with the zero signature, hence a crust diagram.

\[ \square \]

**Conjecture 2.3.8.** (Quotient Diagram Conjecture B) The generating function for reduced quotient diagrams is

\[
D_R(a, b, q) = \frac{(q^b; q^b)_a(q^a; q^a)_{b-1}}{(q; q)_\infty} \prod_{i=1}^{a} (q^{a+ib}; q^a)_\infty.
\]

While this seems less intuitive than the original conjecture, this simplifies nicely in certain cases.

**Example 2.3.9.** If \( a = b = t \) for \( t \geq 2 \),

\[
\frac{(q^t; q^t)_t(q^t; q^t)_{t-1}}{(q; q)_\infty} \prod_{i=1}^{t} (q^{t+it}; q^t)_\infty
\]

\[= (q^t; q^t)_t(q^t; q^t)_{t-1} T_t(q) \prod_{i=1}^{t} \frac{1}{(q^t; q^t)_{i-1}} \]

\[= (q^t; q^t)_t(q^t; q^t)_{t-1} T_t(q) \prod_{i=1}^{t} \frac{1}{(1 - q^{it})^{t-i+1}} \]

\[= T_t(q) \prod_{i=1}^{t-1} \frac{1}{(1 - q^{it})^{t-i-1}} \quad (2.3.0.2) \]

If \( t = 2 \), the conjecture becomes

\[ D_R(2, 2, q) = T_2(q). \]

This is obvious; the process of \( 2 \times 2 \) reduction is identical to the process of taking 2-cores - both processes simply remove horizontal or vertical border strips of length 2. Therefore, we have the following theorem.

**Theorem 2.3.10.** The quotient diagram conjecture is true for \( a = b = 2 \); that is

\[ D(2, 2, q) = \frac{1}{(1 - q^2)^2(1 - q^4)} \frac{(q^2; q^2)_\infty^2}{(q; q)_\infty}. \]
Example 2.3.11. Another important case is when \(a\) and \(b\) are coprime. In this case, the product

\[
\frac{1}{(q; q)_\infty} \prod_{i=1}^{a} (q^{a+ib}; q^a)_\infty
\]

has only finitely many terms, a consequence of the Euclidean algorithm. These terms are all of the form

\[
\frac{1}{(1 - q^j)}
\]

where \(j\) runs over all positive integers which cannot be written as the sum \(ra + sb\) where both \(r\) and \(s\) are positive integers. Note that only the multiples of \(a\) or \(b\) less than or equal to \(ab\) remain as possible exponents. However these all cancel out from the product

\[
(q^b; q^b)_a (q^a; q^a)_{b-1}
\]

and so we have that

\[
D_R(a, b, q) = \prod_j \frac{1}{(1 - q^j)},
\]

where \(j\) runs over all the positive integers which cannot be written as the linear combination of \(a\) and \(b\) with non-negative integer coefficients. Integers of these forms have been studied in the Frobenius coin problem where it was proved that this set contains \(\frac{1}{2} (a - 1)(b - 1)\) integers and the largest such integer is equal to \(ab - a - b\). This fact is affectionately known as the “Chicken McNugget Theorem” [15]. This title came about in trying to ask what the largest number of chicken McNuggets one cannot order at McDonalds. Since McNuggets come in boxes of 9 or 20 pieces, it was determined that the largest impossible order is 151 McNuggets.

The following theorem is a shortcut to proving that a very specific type of quotient diagram is reduced and will be referenced several times during the third chapter. First, we must define the kind of quotient diagram this theorem affects.

**Definition 2.3.12.** A quotient diagram is **rigid** if any of the partitions in its preimage are made up of some triangular number, \(t_k\), of blocks where the \(k\) rightmost blocks have no dots in their last row or column and the \(k - 1\) second-to-rightmost blocks are all either the full block or the block with dots in all but the \((a, b)\)-position.
Example 2.3.13. An example of a rigid $2 \times 3$ quotient diagram is

\[
\begin{bmatrix}
6 & 3 & 3 \\
3 & 3 & 2
\end{bmatrix}
\]

since in its preimage is the partition

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

Theorem 2.3.14. For $a \geq 2$ and $b > 2$, let $\lambda$ be in the preimage of reduced $a \times b$ quotient diagram, $\mathcal{R}$. Let $\lambda^+$ be the partition you get from adding a dot to the $(i, j)$-subset of the Ferrers diagram of $\lambda$ and let $\mathcal{R}^+$ be this new partition’s quotient diagram.

(a) If $(i, j) \neq (a, b)$ and $\mathcal{R}^+$ has no faults then $\mathcal{R}^+$ is reduced.

(b) If $(i, j) = (a, b)$ and $\lambda$ is rigid, then $\mathcal{R}^+$ is reduced.

Proof. Part (a) would be true if being able to subtract a crust diagram not equal to a multiple of $U_{a,b}$ implied the existence of a fault; this was shown in the reduction algorithm. For part (b), notice that the only partitions in the preimage of $\mathcal{R}^+$ are $\lambda^+$ and the partitions you obtain from independently permuting the set of $k - 1$ second-to-rightmost blocks. A second thing to note is that the $(a, 1)$ entry and $(1, b)$ entry of $\mathcal{R}$ are both equal to $t_{k-1}$ whereas the $(1, 1)$ entry is equal to $t_k$. The same is true of $\mathcal{R}^+$. Third, for any $\lambda$ in the preimage of $\mathcal{R}$ and its corresponding $\lambda^+$ have both the same number of parts and same largest part. Therefore, for all partitions $\lambda^+$ in the preimage of $\mathcal{R}^+ := (d_{i,j})$, the equalities from the reduction algorithm, $d_{i,1} - C_1 = d_{1,b}$ and $d_{i,1} - R_1 = d_{a,1}$, hold; this proves that $\mathcal{R}^+$ is reduced.

We will end this section by seeing how we can compare reduced quotient diagrams of different sizes. To do this, we use the covering maps defined in Section 1.4.
Theorem 2.3.15. Let \( a, b, c, d \) be such that \( c \mid a \) and \( d \mid b \). For a partition \( \lambda \), if \( Q_{c \times d}(\lambda) \) is reduced, then so is \( Q_{a \times b}(\lambda) \).

Proof. Assume that \( Q := Q_{a \times b}(\lambda) \) is not reduced so that \( Q = R + C \) where \( C \) is a non-zero \( a \times b \) crust diagram. Then, by Theorem 1.4.17

\[
Q_{c \times d}(\lambda) = \text{Cov}_{c \times d}^{a \times b}(Q) = \text{Cov}_{c \times d}^{a \times b}(R) + \text{Cov}_{c \times d}^{a \times b}(C).
\]

We have that \( \text{Cov}_{c \times d}^{a \times b}(C) \) is a non-zero \( c \times d \) crust diagram since its signature is a sum of zeroes. This shows that \( Q_{c \times d}(\lambda) \) is not reduced. Geometrically, this can be seen as, if \( a = ic \), then removing a (possibly disjoint) border strip of length \( a \) is tantamount to removing \( i \) (possibly disjoint) border strips of length \( c \).

It is important to note that the converse of this theorem is false. For example, if \( \lambda = (2) \), the \( 2 \times 4 \) quotient diagram of \( \lambda \) is reduced whereas the \( 2 \times 2 \) quotient diagram is not.

2.4 Cores and Mantles

Equation 2.3.0.2 suggests a strong relationship between \( t \times t \) quotient diagrams and \( t \)-core partitions. This connection is supported by the fact that the process of reduction is similar to that of removing border strips of length \( t \). This connection is key to studying quotient diagrams; a better understanding of the geometry of \( t \)-cores will lead to a better understanding of quotient diagrams and vice versa.

Theorem 2.4.1. Partitions whose \( t \times t \) quotient diagrams are equal will also have the same \( t \)-core.

Proof. Recall that a \( t \)-core depends only on the number of beads in the rows of the partition’s abacus. If we let

\[
\lambda_{[i,j]}
\]

denote the number of parts of a partition \( \lambda \) whose index is congruent to \( i \) (mod \( t \)) and whose size is congruent to \( j \) (mod \( t \)) then the number of beads in the \( j^{th} \) row of the abacus is

\[
\alpha_j := \sum_{i=1}^{t} \mu_{[i,j]}.
\]
where \( \mu = \lambda + \Delta_{r-1} \) and \( r = kt \) is the smallest multiple of \( a \) larger than the number of parts of \( \lambda \).

Look at this sum of partitions as the triangular partition sliding from the left and forcing the parts in \( \lambda \) to the right. The \( i \)th part of \( \lambda \) is shifted \( r - i \) to the right. More importantly, parts whose indecies are congruent modulo \( t \) are shifted to the right an equal number of places modulo \( t \). Therefore,

\[
\lambda[i,j] = \mu[i,j+r-i] = \mu[i,j-i],
\]

or equivalently,

\[
\mu[i,j] = \lambda[i,j+i].
\]

Therefore,

\[
\sum_{i=1}^{t} \mu[i,j] = \sum_{i=1}^{t} \lambda[i,j+i] = \lambda[t-j,0] + \sum_{i+j \neq 0 \pmod{t}}^{1 \leq i \leq t} \lambda[i,j+i].
\]

From Theorem 1.4.12, we have that, if \( Q = [d_{i,j}] \) is the quotient diagram of \( \lambda \),

\[
\lambda[t-j,0] = d_{t-j,t} - d_{t-j,1} + k
\]

and

\[
\lambda[i,j+i] = d_{i,j+i} - d_{i,j+i+1}
\]

so that the total sum is

\[
\alpha_j = k + \sum_{i=1}^{t} (d_{i,j+i} - d_{i,j+i+1}).
\]

However, \( t \)-cores are not in a one-to-one correspondence with abaci; the abaci whose row sums are

\[(\alpha_0, \alpha_1, \ldots, \alpha_{t-1})\]

and the abaci whose row sums are

\[(\alpha_{t-1} + 1, \alpha_0, \ldots, \alpha_{t-2})\]

represent the same \( t \)-core partition ([8]). In particular, two row vectors of \( t \) non-
negative integers represent equivalent abaci if their difference is a multiple of the vector \([1 \ 1 \ \ldots \ 1]\). Therefore, all the partitions in the preimage of \(Q\) have equivalent abaci; the abaci of these partitions can only differ in their value of \(k\).

\[\square\]

**Definition 2.4.2.** We define the \(t\)-core of a quotient diagram, \(Q\), denoted \(C_t(Q)\) to be the quotient diagram of the \(t\)-core of a partition in the preimage of \(Q\).

Note that this operation is well-defined by the previous theorem.

**Theorem 2.4.3.** The \(t \times t\) quotient diagrams of \(t\)-core partitions are one-to-one and reduced. That is, the \(t\)-core of a quotient diagram is reduced and has a single partition in its preimage.

*Proof.* Let \(T\) be the quotient diagram, which we will assume to be non-reduced, of a \(t\)-core partition \(\lambda\) and consider the preimage of \(T\). Let \(\mu\) be a representative of this set, possibly different from \(\lambda\). However, since \(T\) is not reduced, \(\mu\) has a border strip of length \(t\), and is therefore not a \(t\)-core partition; note that the border strip in \(\mu\) is not necessarily connected. However we can rearrange the blocks of \(\mu\) so that this is the case. The \(t\)-core of \(\mu\) must have a smaller weight (by a multiple of \(t\)) than \(\lambda\) meaning the \(t\)-cores of these two partitions are different, which contradicts Theorem 2.4.1. Moreover, this also shows that \(\lambda\) must be the only partition in the preimage of \(T\).

\[\square\]

**Corollary 2.4.4.** If \(t \mid a\) and \(t \mid b\), the \(t\)-core of an \(a \times b\) quotient diagram is reduced and has a single partition in its preimage.

*Proof.* These facts follow from Theorems 2.3.15 and 2.4.3 and the fact that the preimage of an \(a \times b\) quotient diagram \(Q\) is a subset of the preimage of \(abtt(Q)\).

\[\square\]

Just as the crust diagrams are the difference between quotient diagrams and their reduced quotient diagrams, the mantle diagram is the difference between reduced quotient diagrams and their \(t\)-cores. Quotient diagrams can be thought of as the sum of three parts: a \(t\)-core diagram, a mantle diagram, and a crust diagram.
Definition 2.4.5. Let \( R \) be an \( a \times b \) reduced quotient diagram with \( t \)-core diagram \( T \), where \( t = \gcd(a, b) \). Define the \textbf{mantle diagram} of \( R \) to be the matrix \( R - T \).

Example 2.4.6. Consider the \( 2 \times 6 \) reduced quotient diagram

\[
R := \begin{bmatrix}
2 & 2 & 1 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 0
\end{bmatrix}.
\]

If \( \lambda \) is in the preimage of \( R \), then its \( 2 \times 2 \) quotient diagram is

\[
\text{Cov}_{2\times2}(R) = \begin{bmatrix}
4 & 4 \\
4 & 2
\end{bmatrix}.
\]

After reduction, this quotient diagram is

\[
\begin{bmatrix}
3 & 3 \\
3 & 1
\end{bmatrix}
\]

which has only the partition \((4, 3, 2, 1)\) in its preimage. Therefore, the \( 2 \)-core of \( R \) is

\[
\begin{bmatrix}
2 & 2 & 1 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 0 & 0
\end{bmatrix}
\]

and so its mantle is

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}.
\]

If the quotient diagram conjecture were true, then the generating function of reduced \( a \times b \) quotient diagrams would be the product of the generating function for \( t \)-cores and a rational function.

Definition 2.4.7. The quotient of \( D_R(a, b, q) \) and \( T_t(q) \) is called the \( a \times b \) \textbf{mantle generating function}.

It is conjectured that the mantle generating functions are all of the form

\[
\prod \frac{1}{(1 - q^{w_i})} \quad (2.4.0.3)
\]

for some set of positive integers \( \{w_i\} \). We could prove that this is the case by showing that the number of reduced quotient diagrams with any given \( t \)-core and
a mantle of weight \(n\) is the coefficient of \(q^n\) in (2.4.0.3). Note that this is similar to how the number of quotient diagrams with a given reduced quotient diagram and a crust diagram of weight \(n\) is the coefficient of \(q^n\) in Corollary 2.2.11.

However, we should be wary of thinking of mantle in the same way as crust diagrams. The same crust diagram can be added to any quotient diagram to get a new quotient diagram, however this is not true of mantles. The mantle of
\[
\begin{bmatrix}
1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
is
\[
\begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
(since the 2-core is
\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
), however the sum of
\[
\begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
and the 2-core
\[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]
is not a quotient diagram.

The trick then is to arrange, via algorithm or otherwise, an infinite digraph whose set of vertices is the set of \(a \times b\) reduced quotient diagrams with the following properties.

(i) If two reduced quotient diagrams are in the same connected component, they have the same \(t\)-core. Note that this implies that each connected component has exactly one \(t\)-core.

(ii) Each quotient diagram has exactly \(r\) edges coming out of it; these edges are labeled \(w_1, \ldots, w_r\). The quotient diagram at the end of the edge labeled \(w_i\) corresponds to a quotient diagram whose weight is \(w_i\) greater.

(iii) The paths commute, that is, going down a path \(w_i\) and then down a path \(w_j\) results in the same quotient diagram as going down \(w_j\) and then \(w_i\).

If for some \(a\) and \(b\) such a graph exists, then the mantle generating function is indeed (2.4.0.3) and the quotient diagram conjecture would be proven for the \(a \times b\) case.

There lie two challenges in building such a graph. The first is to ensure that all of your quotient diagrams are in fact reduced. This is done by building the quotient diagrams up one dot at a time and regularly appealing to Theorem 2.3.14. The second is proving that every reduced quotient diagram is present as a vertex. We can show this by considering the image of the vertex set of our graph under the signature map. If the image of the vertex set is onto the signature space, \(M_{a-1,b-1}(\mathbb{Z})\), then you are guaranteed to have a complete set of reduced quotient
diagrams. Injectivity comes from the fact that distinct reduced quotient diagrams have distinct signatures.

There is a caveat to the possible usefulness of this method: the only cases where this has been particularly effective are the cases $2 \times 3$, $2 \times 4$, and $3 \times 3$. The reason for the success of these cases comes from the fact that their mantle generating functions are $\frac{1}{1-q}$, $\frac{1}{1-q^2}$, and $\frac{1}{1-q^3}$, respectively. There is only one value of $m_i$ and therefore the digraph can be viewed as a family of sequences of quotient diagrams; each sequence starting at a $t$-core and then increasing equinumerously at each step.

2.5 Quotient Diagrams of 2-cores

In the previous section we remarked on the connection between the reduction algorithm and the conjectured generating function and $t$-cores, where $t = \gcd(a,b)$. This motivates taking an in-depth look into studying, and trying to fully classify, the quotient diagrams of $t$-cores. In this section we will explicitly derive the $a \times b$ quotient diagram of 2-cores for all $a$ and $b$.

From Example 1.4.15, we saw that the $2 \times 2$ quotient diagrams are

$$Q(\Delta^2_{2m-1}) = \begin{bmatrix} t_m & t_{m-1} \\ t_{m-1} & t_{m-1} \end{bmatrix} \text{ and } Q(\Delta^2_{2m}) = \begin{bmatrix} t_m & t_m \\ t_m & t_{m-1} \end{bmatrix}.$$  

A succinct way of stating this theorem is that

$$Q(\Delta^2_k) = \begin{bmatrix} t_{2,k} & t_{2,k-1} \\ t_{2,k-1} & t_{2,k-2} \end{bmatrix},$$  

where the sequence $t_{2,i}$ is

$$0, 0, 1, 1, 3, 3, 6, 6, \ldots,$$

that is, the sequence of triangular numbers where each term is repeated. This sequence has the generating function

$$\frac{1}{(1-q)(1-q^2)^2}.$$
**Theorem 2.5.1.** The $a \times b$ quotient diagram of $\Delta_k^2$ is

$$
\begin{bmatrix}
  x_{a,b,k} & x_{a,b,k-1} & \cdots \\
  x_{a,b,k-1} & x_{a,b,k-2} & \ddots \\
  \vdots & \ddots & \ddots \\
  x_{a,b,k-a-b+2} & & & & \ddots
\end{bmatrix},
$$

where $x_{a,b,e}$ is the sequence whose generating function is

$$
\frac{1}{(1 - q)(1 - q^a)(1 - q^b)}.
$$

In order to prove this theorem, we must define a family of sequences, the pseudo-triangular sequences, and discuss basic properties of these sequences.

**Definition 2.5.2.** Let $\{x_n\}$ be a sequence of integers with generating function $F(q)$.

(a) The **first difference** of $\{x_n\}$ is the sequence of integers $\{x_n - x_{n-1}\}$. The generating function for the first difference of $\{x_n\}$ is

$$
F'(q) = F(q)(1 - q).
$$

(b) The **$r$-repeater** of $\{x_n\}$ is the sequence where each term is repeated $r$ times. The generating function for the $r$-repeater of $\{x_n\}$ is

$$
F_{xr}(q) := F(q^r)(1 + q + \cdots + q^{r-1}) = \frac{F(q^r)(1 - q^r)}{(1 - q)}.
$$

(c) The **$r^{th}$ pseudo-triangular sequence**, denoted $\{s_r(n)\}$ (or $s(n)$, if $r = 2$) is defined to be the sequence whose first difference is the $r$-repeater of the sequence of counting numbers. The generating function of the $r^{th}$ pseudo-triangular sequence is

$$
S_r(q) := \frac{1}{(1 - q)^2(1 - q^r)}.
$$
The first few pseudo-triangular sequences are given below. When $r = 1$ we have the standard triangular numbers

$$0, 1, 3, 6, 10, 15, 21, \ldots$$

When $r = 2$, we have the square-rectangular numbers

$$0, 1, 2, 4, 6, 9, 12, 16, 20, \ldots$$

When $r = 3$, we have the sequence

$$0, 1, 2, 3, 5, 7, 9, 12, 15, 18, 22, \ldots$$

The fact that $S_r(q)(1 + q + \cdots + q^{r-1}) = S_1(q)$ corresponds to the fact that

$$\sum_{i=0}^{r-1} s_r(n + i) = t_{n+r-1}.$$

We will first prove that Theorem 2.5.1 is true for square quotient diagrams and then use that fact to prove the theorem in full.

**Lemma 2.5.3.** The $r \times r$ quotient diagrams of the triangular numbers are of the form given in Theorem 2.5.1 and the generating function for the sequence $x_{r,r,e}$ is

$$\frac{1}{(1-q)(1-q^r)^2}.$$

**Proof.** We will construct the sequence of triangular partitions using the shifting method by starting with the zero partition and then, on the $k^{th}$ step, adding 1 to the first $k$ parts, that is, adding a column of $k$ dots to the left of the Ferrers diagram.

During the $k^{th}$ term, where $k = mr + h$ and $0 \leq h < r$, the quotient diagram of $\Delta_{k-1}$ is cycled by one column to the right and then added to the matrix

$$\begin{bmatrix}
m + 1 & \cdots & 0 \\
\vdots \\
m & \cdots & 0
\end{bmatrix}.$$
where the first $h$ entries in the first column are equal to $m + 1$ and the rest are equal to $m$. Restricting this analysis to the first row shows that, between the $mr^{th}$ and $(m + 1)r^{th}$ terms, we cycle the entries right and add $m + 1$ to the first entry. Since we are constructing these quotient diagrams using the shifting method, we can restrict further since the first row is entirely dependent on how the top-left entry evolves. Since we started with the zero matrix, between terms $mr + 1$ and $(m + 1)r$, the $(1, 1)$ entry of the quotient diagram does not change; on term $mr + 1$, however, this entry increases by $m + 1$. Therefore, the sequence of values this entry takes has as its first difference the sequence

$$1, 0, \ldots, 0, 2, 0, \ldots, 0, 3, 0 \ldots,$$

where there are $r$ zeroes between each nonzero term. This first difference has the generating function

$$\frac{1}{(1 - q^r)^2},$$

and therefore, from the definitions above, the $(1, 1)$ entry has the sequence

$$\frac{1}{(1 - q)(1 - q^r)^2}.$$

The $h^{th}$ row, for $h > 1$ is identical to the first however shifted back $h$ terms thus proving the lemma. 

Proof of theorem 2.5.1. Let $r := \text{lcm}(a, b)$ so that $ua = vb = r$. Consider the covering map equality

$$\text{Cov}_{a \times r}^r(Q_{r \times r}({\Delta_k})) = Q_{a \times r}({\Delta_k}).$$

In $Q_{a \times r}({\Delta_k})$, the $(i, j)$ entry is equal to the sum of the $(i', j)$ entries in $Q_{r \times r}$ such that $i \equiv i'$ (mod $a$). This is equal to, for some $e$,

$$x_{r, r, e} + x_{r, r, e-a} + x_{r, r, e-2a} + \cdots + x_{r, r, e-(u-1)a}.$$
Therefore the generating function for the sequences of these sums is

\[
\frac{1}{(1-q)(1-q^a)^2(1+q^a + \cdots + q^{(u-1)a})} = \frac{1}{(1-q)(1-q^a)(1-q^r)}.
\]

We can use the same analysis for the covering map equality

\[
\text{Cov}^{a \times r}_{a \times b}(Q_{a \times r}^{\Delta_k}) = Q_{a \times b}^{\Delta_k}
\]

to show that the sequence is

\[
\frac{1}{(1-q)(1-q^a)(1-q^r)}(1 + q^b + \cdots + q^{(v-1)a}) = \frac{1}{(1-q)(1-q^a)(1-q^b)}.
\]

2.6 Quotient Diagrams of 3-cores

In the paper of Robbins [13], the 3-core partitions were fully classified.

**Theorem 2.6.1.** (Robbins) \(c_3(n)\) is the number of distinct ways that \(n\) can be represented in the form

\[r(r + m + k) + m(m + 1),\]

where \(r, m \geq 0\) and \(k = 0\) or \(1\). For each such representation, the corresponding 3-core partition of \(n\) is given by

\[
\Delta^3_{r,m,k} := (m + 2r + k - 1)(m + 2r + k - 3) \cdots (m + k + 1)m^2(m - 1)^2 \cdots 2^21^2.
\]

In this partition, \(r\) is the number of parts which occur once and \(m\) is the number of parts which occur twice. Note that from the construction of 3-core partitions due to Robbins, we can apply the shifting method of building quotient diagrams.

The easiest case to consider is when \(r = 0, m = 3i,\) and \(k = 0\). This is the 3-core partition \(\lambda = (m, m, \ldots, 2, 2, 1, 1)\). In going from \(\Delta^3_{0,m-3,0}\) to \(\Delta^3_{0,m-2,0}\), we add a
column of $2m - 4$ dots to the left hand side of the Ferrers diagram of $\Delta^3_{0,m-3,0}$. For the quotient diagram, this amounts to cycling the columns one to the right and then adding $\mathcal{V}_{2m-4}$ to the first column, where $\mathcal{V}_m$ is the $a \times 1$ quotient diagram of $(1^m)$. Going then to $\Delta^3_{0,m-1,0}$ takes the quotient diagram, cycles the columns again to the right and adds $\mathcal{V}_{2m-2}$ to the first column. Finally, going to $\Delta^3_{0,m,0}$ cycles the columns again to the right and adds $\mathcal{V}_{2m}$ to the first column. Explicitly, 

$$2m - 4 = 6i - 4 = 3(2i - 1) - 1$$

and so

$$\mathcal{V}_{2m-4} = \begin{bmatrix} 2i - 1 \\ 2i - 1 \\ 2i - 2 \end{bmatrix}.$$ 

We likewise have

$$\mathcal{V}_{2m-2} = \begin{bmatrix} 2i \\ 2i - 1 \\ 2i - 1 \end{bmatrix}, \mathcal{V}_{2m} = \begin{bmatrix} 2i \\ 2i \end{bmatrix},$$

and therefore

$$\mathcal{Q}(\Delta^3_{0,m,0}) = \mathcal{Q}(\Delta^3_{0,m-3,0}) + \begin{bmatrix} 2i & 2i & 2i - 1 \\ 2i & 2i - 1 & 2i - 1 \\ 2i & 2i - 2 & 2i - 2 \end{bmatrix}.$$ 

We can then add successively to get that

$$\mathcal{Q}(\Delta^3_{0,m,0}) = \begin{bmatrix} s(2i) & s(2i) & s(2i - 1) \\ s(2i) & s(2i - 1) & s(2i - 1) \\ s(2i) & s(2i - 2) & s(2i - 2) \end{bmatrix}.$$ 

The same reasoning can be used to increase $r$; assume now that $r = 3j, m = 3i$, and $k = 0$. In going from $\Delta^3_{r-3,m,0}$ to $\Delta^3_{r-2,m,0}$, we add a row of $m + 2r - 5$ dots to the top of the Ferrers diagram of $\Delta^3_{r-3,m,0}$. In the quotient diagram, we cycle the rows down one and add $\mathcal{H}_{m+2r-5}$ to the first row, where $\mathcal{H}_r$ is the $1 \times b$ quotient diagram of $(r)$, and so on. Since $m + 2r - 5 = 3(i + 2j - 1) - 2$,

$$\mathcal{H}_{m+2r-5} = \begin{bmatrix} i + 2j - 1 & 1 + 2j - 2 & i + 2j - 2 \end{bmatrix}.$$
and also that
\[ H_{m+2r-3} = \begin{bmatrix} i + 2j -1 & 1 + 2j - 1 & i + 2j - 1 \end{bmatrix}, \]
\[ H_{m+2r-1} = \begin{bmatrix} i + 2j & 1 + 2j & i + 2j - 1 \end{bmatrix}. \]
This means
\[
Q(\Delta_{r,m,0}^3) = \frac{Q(\Delta_{r,m,0}^3)}{Q(\Delta_{r-3,m,0}^3)} + \begin{bmatrix} i+2j & 1+2j & i+2j-1 \\ i+2j-1 & 1+2j-1 & i+2j-1 \\ i+2j-1 & 1+2j-2 & i+2j-2 \end{bmatrix},
\]
so that, by summing over the \( j \),
\[
Q(\Delta_{r,m,0}^3) = \frac{Q(\Delta_{r,m,0}^3)}{Q(\Delta_{0,m,0}^3)} + \begin{bmatrix} (i+2)j+2t_{j-1} & (i+2)j+2t_{j-1} & (i+2)j+2t_{j-1}-j \\ (i+2)j+2t_{j-1}-j & (i+2)j+2t_{j-1}-j & (i+2)j+2t_{j-1}-j \\ (i+2)j+2t_{j-1}-j & (i+2)j+2t_{j-1}-2j & (i+2)j+2t_{j-1}-2j \end{bmatrix}.
\]
Putting that together with the formula above gives
\[
Q(\Delta_{r,m,0}^3) = \begin{bmatrix} s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)-(i+2)j \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \end{bmatrix}.
\]

By the shifting method, the quotient diagrams of \( Q(\Delta_{3j}^{3j+i',3i' +i',0}) \), where \( 0 \leq i', j' \leq 2 \) are uniquely defined by \( Q(\Delta_{r,m,0}^3) \), \( Q(\Delta_{r+3,m,0}^3) \), \( Q(\Delta_{r,m+3,0}^3) \), \( Q(\Delta_{r+3,m+3,0}^3) \).

**Theorem 2.6.2.** The quotient diagrams for the 3-core partitions where \( k = 0 \), as in Theorem 2.6.1, are given below:
\[
Q(\Delta_{3j,3i,0}^3) = \begin{bmatrix} s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \end{bmatrix}.
\]
We can perform the same analysis for when \( r = 3j, m = 3i, \) and \( k = 1. \) Since the 3-cores given by \( r = 0 \) and \( k = 0 \) are the same as those given by \( r = 0 \) and \( k = 1 \) and so \( Q(\Delta^3_{3j+1,3i,0}) = Q(\Delta^3_{0,3i,0}). \) Moreover, since going from \( \Delta_{r-3,m,1} \) requires adding a part of size \( m + 2r - 4, \) a part of size \( m + 2r - 2, \) and one of size \( m + 2r, \)
we have that

\[ Q(\Delta_{r,m,1}^{3}) = Q(\Delta_{r-3,m,1}^{3}) + \begin{bmatrix} i+2j & 1+2j & i+2j \\ i+2j & 1+2j-1 & i+2j-1 \\ i+2j-1 & 1+2j-1 & i+2j-2 \end{bmatrix}, \]

so that, by summing over the \( j \),

\[ Q(\Delta_{r,m,1}^{3}) = Q(\Delta_{0,m,1}^{3}) + \begin{bmatrix} (i+2)j+2t_{j-1} & (i+2)j+2t_{j-1} & (i+2)j+2t_{j-1} \\ (i+2)j+2t_{j-1} & (i+2)j+2t_{j-1} & (i+2)j+2t_{j-1} \\ (i+2)j+2t_{j-1} & (i+2)j+2t_{j-1} & (i+2)j+2t_{j-1} \end{bmatrix}. \]

Putting that together with the formula above gives

\[ Q(\Delta_{r,m,1}^{3}) = \begin{bmatrix} s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ +2t_{j-1} & +2t_{j-1} & +2t_{j-1} \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ +2t_{j-1} & +2t_{j-1} & +2t_{j-1} \\ s(2i)+(i+2)j & s(2i)+(i+2)j & s(2i)+(i+2)j \\ +2t_{j-1} & +2t_{j-1} & +2t_{j-1} \end{bmatrix}. \]
Chapter 3

The Known Cases

3.1 Introduction

The first size of quotient diagram considered by Propp, and then again by the author, was the $2 \times 2$ quotient diagram. It is during the analysis of this case that we see noticeable restrictions on the set of matrices that can be quotient diagrams; if the bottom-right entry was not large enough compared to the other entries then the matrix could not be realized as the quotient diagram of a partition. We will see that the $1 \times b$ matrices do not have such problems; these quotient diagrams are “too small” for such restrictions to exist.

It seems however that the $2 \times 2$ case is too small itself. The proof methods needed to prove the conjecture for larger sized quotient diagrams cannot be gleaned from the $2 \times 2$ case. The simple reason for this is that the mantle, as defined in Section 2.4, of every $2 \times 2$ quotient diagram is the zero matrix.

In this chapter, we will give the proofs of all the cases where the quotient diagram conjecture is known to be true. These are the $2 \times 2$, $1 \times b$, $2 \times 3$, and $2 \times 4$ cases.

3.2 The $2 \times 2$ case

Although it would be desirable to get an exact formula for the number of $2 \times 2$ quotient diagrams of $n$, we can get a recursive formula akin to that of the number
of partitions of $n$ which we get from Euler’s Pentagonal Number Theorem. To get this formula, we use the $q$-partial fraction techniques developed by Munagi [11].

**Theorem 3.2.1.** If we define $d(2, 2, n) = 0$ if $n < 0$ and

$$f(n) = \begin{cases} 
(i + 1)^2 & \text{if } n = 4i \\
(i + 1)(i + 2) & \text{if } n = 4i + 2 \\
0 & \text{if } n \text{ is odd}
\end{cases}$$

then

$$d(2, 2, n) = f(n) + f(n - 1) + f(n - 3) + \cdots + f(n - \frac{1}{2}m(m + 1)) + \cdots$$

(3.2.0.1)

**Proof.** By Theorem 2.3.10 that

$$\sum_{n \geq 0} d(2, 2, n)q^n = \frac{1}{(1 - q^2)^2(1 - q^4)} \sum_{m \geq 0} q^{m(m+1)/2}.$$

So to prove formula (3.2.0.1), it suffices to show that

$$\frac{1}{(1 - q^2)^2(1 - q^4)} = \sum_{n \geq 0} f(n)q^n.$$

If we expand the left-hand side into its $q$-partial fraction, we get that

$$\frac{1}{(1 - q^2)^2(1 - q^4)} = \frac{7}{32(1-q)} + \frac{7}{32(1+q)} + \frac{5}{32(1-q)^2} + \frac{5}{32(1+q)^2} + \frac{1}{16(1-q)^2} + \frac{1}{16(1+q)^3} + \frac{1}{8(1+q^2)} + \frac{1}{8(1+q^2)^2} + \frac{1}{8(1+q^2)^3} + \frac{1}{8(1+q^2)^4} + \frac{1}{16} \sum_{j \geq 0} q^{2j} + \frac{5}{16} \sum_{j \geq 0} (2j + 1)q^{2j} + \frac{1}{8} \sum_{j \geq 0} (2j + 1)(2j + 2)q^{2j} + \frac{1}{8} \sum_{j \geq 0} (-1)^j q^{2j}.$$
The coefficient of $q^{2j}$ is then

$$\frac{7}{16} + \frac{5}{16}(2j + 1) + \frac{1}{8}(2j + 1)(j + 1) + \frac{1}{8}(-1)^j$$

$$= \begin{cases} 
\frac{1}{4}j^2 + j + 1 & \text{if } j \text{ is even} \\
\frac{1}{4}j^2 + j + \frac{3}{4} & \text{if } j \text{ is odd} 
\end{cases}$$

$$= \begin{cases} 
(i + 1)^2 & \text{if } j = 2i \\
(i + 1)(i + 2) & \text{if } j = 2i + 1 
\end{cases},$$

which proves the formula.

\[
\square
\]

So far, we have been concerned with enumerating the number of quotient diagrams of partitions of a number $n$. In other words, we are putting an equivalence on the set of partitions of $n$ and counting the number of equivalence classes. This naturally generates a dual question: can we enumerate the number of partitions that lie in each of these equivalence classes?

In the $2 \times 2$ case, this question was unknowingly answered by C. Boulet [5]. Boulet was trying to find a combinatorial proof of the following theorem of Andrews, found in [2].

$$\sum_{\lambda \in \text{Par}} r^{\theta(\lambda)} s^{\theta(\lambda')} q^{\mid \lambda \mid} = \prod_{i=1}^{\infty} \frac{1 + rsq^{2j-1}}{(1 - q^{4j})(1 - r^2q^{4j-2})(1 - s^2q^{4j-2})}$$

(3.2.0.2)

where $\theta(\lambda)$ is the number of odd parts in $\lambda$.

We use the notation that if $M = (m_i)$ is an $a \times b$ matrix (written as a $1 \times ab$ vector where the second row in $M$ is juxtaposed to the end of the first row, and so on) and $(x_1, \ldots, x_{ab})$ is a set of $ab$ variables, then $(x_1, \ldots, x_{ab})^M$ is the product $x_1^{m_1} \cdots x_{ab}^{m_{ab}}$. 
**Theorem 3.2.2.** (Theorem 1 in [5])

\[
\sum_{\lambda \in \text{Par}} (x_1, x_2, x_3, x_4)^{Q_{2 \times 2}(\lambda)} = \prod_{j=1}^{\infty} \frac{(1 + x_1^j x_2^{j-1} x_3^{j-1} x_4^{j-1}) (1 + x_1^j x_2^{j-1} x_3^{j-1} x_4^{j-1})}{(1 - x_1^j x_2^{j-1} x_3^{j-1} x_4^{j-1}) (1 - x_1^j x_2^{j-1} x_3^{j-1} x_4^{j-1})}.
\]

The transformation \(x_1 \mapsto rsq, \ x_2 \mapsto r^{-1}sq, \ x_3 \mapsto rs^{-1}q, \ x_4 \mapsto r^{-1}s^{-1}q\) in Theorem 3.2.2 gives us equation (3.2.0.2).

The coefficient of \((x_1, x_2, x_3, x_4)^Q\) in Theorem 3.2.2 is the number of partitions in the preimage of the quotient diagram \(Q\).

Boulet’s treatment of Theorem 3.2.2 is interesting because it highlights the connection between the quotient diagram of a partition and the number of parts having a certain parity in both the partition and its conjugate. This connection was noted earlier in Theorem 1.4.12. If we could find a generating function where the coefficient of \((x_1, \ldots, x_{ab})^{Q_{a \times b}} q^n\) is the number of partitions in the preimage of \(Q_{a \times b}\) we can use a transformation similar to that which Boulet uses to prove equations along the lines of (3.2.0.2) that count the number of parts of a partition and its conjugate which lie into equivalence classes with varying modulus. For example, a closed form for the sum

\[
\sum_{\lambda \in \text{Par}} (x_1, \ldots, x_9)^{Q_{3 \times 3}(\lambda)}
\]

would give us a generating function where the coefficients would tell us the number of partitions of \(n\) with a given number of parts congruent to 1 and 2 modulo 3 and whose conjugate also has a given number of parts congruent to 1 and 2 modulo 3.

### 3.3 The 1 \(\times\) b case

In this section, we will prove the 1 \(\times\) b cases for all \(b \geq 1\). These cases are special in that they cannot be handled using the theory developed in Chapter 2 because the signatures of these quotient diagrams are not defined. However, these cases are simple enough to be proven directly.

**Theorem 3.3.1.** The 1 \(\times\) b case of the quotient diagram conjecture is true; that
is,
\[ D(1, b, q) = \frac{1}{(q; q)_b}. \]

Proof. We will prove this result by showing that the $1 \times b$ quotient diagrams of $n$ are equinumerous with the partitions of $n$ having at most $b$ parts. If $\lambda = (\lambda_1, \ldots, \lambda_b)$ is such a partition (where some of the $\lambda_i$ may equal 0), then the $1 \times b$ quotient diagram of the conjugate of $\lambda$ is
\[ \begin{bmatrix} \lambda_1 & \cdots & \lambda_n \end{bmatrix} \]
and vice versa. \qed

**Theorem 3.3.2.** The generating function for the number of partitions with a given $1 \times b$ quotient diagram is
\[
\sum_{\lambda \in \text{Par}} (x_1, \ldots, x_b)^{Q_{1\times b}(\lambda)} = \prod_{j=1}^{\infty} \frac{1}{(1 - x_1^j x_2^j \cdots x_b^j)(1 - x_1^j x_2^j \cdots x_b^j)(1 - x_1^j x_2^j \cdots x_b^j)}.
\]

### 3.4 The $2 \times 3$ case

The $2 \times 3$ quotient diagrams present the first non-trivial example of quotient diagrams where $\gcd(a, b) = 1$. This means that the underlying $t$-core diagram of every reduced quotient diagram is the zero matrix. Reduced $2 \times 3$ quotient diagrams, and in general for $a \times b$ where $a$ and $b$ are coprime, are made up entirely by their mantle.

In this section we will prove the quotient diagram conjecture for the $2 \times 3$ case.

**Theorem 3.4.1.** The generating function for $2 \times 3$ quotient diagrams is
\[
D(2, 3, q) = \frac{1}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^4)(1 - q^6)};
\]
that is, Theorem 1.4.19 is true for the case $a = 2$ and $b = 3$.

By the Quotient Diagram Conjecture B, it suffices to show that the reduced $2 \times 3$ quotient diagrams have
\[
D_R(2, 3, q) = \frac{1}{1 - q}
\]
as their generating function. Of course, this would mean that there is exactly one reduced quotient diagram of weight \( n \) for each \( n \in \mathbb{N} \). We can prove this by constructing a sequence of reduced quotient diagrams where each term is obtained by adding 1 to an entry in the previous term. In actuality, we will construct a sequence of partitions where each partition’s Ferrers diagram, starting at the empty diagram, is obtained by adding 1 dot to the previous Ferrers diagram. We can then consider the quotient diagrams of the partitions in this sequence. The first task is then to prove that the quotient diagrams in this sequence are all reduced. Then finally, we must prove that the image of the signatures of these quotient diagrams is onto the signature space, \( \mathbb{Z}^2 \).

In this and future sections, we will be considering rather long sequences of matrices. For brevity, we will use the following notation:

**Definition 3.4.2.** If \( X \) and \( Y \) are matrices which are equal in all but a subset of entries and, if in that subset, the entries in \( X \) are equal to \( x_\alpha \) and the entries in \( Y \) are equal to \( x_\alpha + i \), respectively. Then the notation

\[
X \xrightarrow{i} Y
\]

denotes the sequence of matrices

\[
X = \begin{bmatrix}
\vdots \\
x_{\alpha} \\
\vdots
\end{bmatrix}, \begin{bmatrix}
\vdots \\
x_{\alpha} + 1 \\
\vdots
\end{bmatrix}, \ldots, \begin{bmatrix}
\vdots \\
x_{\alpha} + i \\
\vdots
\end{bmatrix} = Y.
\]

**Example 3.4.3.** For example,

\[
\begin{bmatrix}
5 & 1 \\
1 & 0
\end{bmatrix} \xrightarrow{3}
\begin{bmatrix}
5 & 4 \\
4 & 0
\end{bmatrix}
\]

represents the sequence of matrices

\[
\begin{bmatrix}
5 & 1 \\
1 & 0
\end{bmatrix}, \begin{bmatrix}
5 & 2 \\
2 & 0
\end{bmatrix}, \begin{bmatrix}
5 & 3 \\
3 & 0
\end{bmatrix}, \begin{bmatrix}
5 & 4 \\
4 & 0
\end{bmatrix}.
\]

The bold-facing of entries is to help the reader identify which values have been
increased.

**Algorithm 3.4.4.** We will now construct the sequence of partitions by induction. Start with the empty partition and add one to get the partition (1),

\[
\begin{array}{c}
\vdots
\end{array}
\]

Now assume, for some \( k \geq 0 \), that the partition in the sequence is

\[
(3k + 1, 3k, 3k - 2, 3k - 3, \ldots, 4, 3, 1),
\]

which has the Ferrers diagram

\[
\begin{array}{c}
\vdots
\end{array}
\]

The first \( k + 1 \) moves are to add a dot to the Ferrers diagram in the (1, 2) position of the rightmost box in each row. Then add \( k + 1 \) dots to position (2, 1), then \( k + 1 \) dots to position (1, 3), and then (2, 2). The next \( k + 2 \) dots get placed in the (1, 1) position of the next block to the right. Finally, place \( k + 1 \) dots in the (3, 3) position of the now second-to-rightmost block. This pattern changes the rightmost block of each row of the Ferrers diagram in the following way; keep in mind that these aren’t subsequent terms but rather they are \( k + 1 \) or \( k + 2 \) turns apart.

\[
\begin{array}{c}
\vdots
\end{array}
\]
The quotient diagrams of these partitions follow the sequence which starts as
\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
and then, for \( k \geq 0 \),
\[
\begin{bmatrix}
t_{k+1} & t_k & t_k \\
t_k & t_k & t_k
\end{bmatrix} \quad \overset{k+1}{\longrightarrow} \quad \begin{bmatrix}
t_{k+1} & t_{k+1} & t_{k+1} \\
t_k & t_k & t_k
\end{bmatrix} \quad \overset{k+1}{\longrightarrow} \quad \begin{bmatrix}
t_{k+1} & t_{k+1} & t_{k+1} \\
t_{k+1} & t_k & t_k
\end{bmatrix} \quad \overset{k+1}{\longrightarrow} \quad \begin{bmatrix}
t_{k+1} & t_{k+1} & t_{k+1} \\
t_{k+1} & t_{k+1} & t_k
\end{bmatrix}
\]

This sequence of \( 2 \times 3 \) quotient diagrams begins
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
2 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
3 & 1 & 1 \\
1 & 1 & 0
\end{bmatrix}, \quad \begin{bmatrix}
3 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}, \quad \ldots
\]

It now remains to be shown that the quotient diagrams in this sequence are all reduced and if the image of the signature map is the signature space, \( \mathbb{Z}^2 \).

**Theorem 3.4.5.** Every quotient diagram in the sequence of \( 2 \times 3 \) quotient diagrams obtained from Algorithm 3.4.4 is reduced and the set of signatures of these quotient diagrams is \( \mathbb{Z}^2 \).

**Proof.** We will prove this by induction. The first two quotient diagrams,
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
are both trivially reduced. Now for any \( k \geq 0 \), consider the subsequence \((3.4.0.3)\) and assume that the first term is reduced. The next \( 5k + 6 \) terms don’t have any vertical or horizontal faults and so by part \((a)\) of Theorem 2.3.14, all these quotient diagrams are reduced. The following \( k + 1 \) terms have their \((a,b)\) entry increased.
by one, however are also all rigid quotient diagrams and therefore are reduced by part (b) of Theorem 2.3.14. Therefore, by induction, all the quotient diagrams in the sequence are reduced.

Before continuing with the proof, consider the signatures of the quotient diagrams in the sequence as a path in $\mathbb{Z}^2$. Starting at $(0,0)$ the path follows the blackened edges in Figure 3.4.

Notice how the path is a tightly wound spiral. Since the segments in this path are parallel and exactly one unit (horizontally or vertically) apart at all time, it is clear that every point in $\mathbb{Z}^2$ will be traversed by the path. This finishes the proof of the $2 \times 3$ case of the quotient diagram conjecture.

\[\square\]

### 3.5 The $2 \times 4$ case

The $2 \times 4$ case is more akin to the $2 \times 2$ case than the $2 \times 3$ case. The first two share their relationship with 2-cores, a relationship the $2 \times 3$ quotient diagrams lack. In this section, we will prove the $2 \times 4$ case of the quotient diagram conjecture.
**Theorem 3.5.1.** The generating function for the $2 \times 4$ quotient diagrams is

$$D(2, 4, q) = D_R(2, 4, q) = \frac{(q^2; q^2)_\infty^2}{(1 - q^2)(q; q)_\infty^2}$$

From example 1.4.15, we know that Theorem 3.5.1 would be true if

$$D_M(2, 4, q) = \frac{1}{1 - q^2}.$$  

As per the method laid out in Section 2.4, this can be proven if we can construct a family of sequences of quotient diagrams with the following properties.

(i) All the quotient diagrams are reduced.

(ii) All the quotient diagrams in a single sequence have the same 2-core.

(iii) The successor of a quotient diagram has two more weight.

Then, after finding these sequences, we must prove that they form a complete set of reduced quotient diagrams. This will be done by showing that every signature in the signature space $\mathbb{Z}^3$ is represented. We can partition $\mathbb{Z}^3$ into $\mathbb{Z}$ copies of $\mathbb{Z}^2$, one plane for each 2-core,

$$\mathbb{Z}_g := \{ \mathcal{Q} \in \text{Quo}_{2 \times 4} \mid \sigma(\mathcal{Q}) = (a, b, c), (a + c) = g \}.$$  

Consider the planes

$$\mathbb{Z} \times \mathbb{Z}_g := (x, y, g - x) \mid x, y \in \mathbb{Z},$$

where $g \in \mathbb{Z}$. The reduced quotient diagrams whose signatures lie on the plane $\mathbb{Z} \times \mathbb{Z}_d$ have the same 2-core. Indeed, if a partition has $2 \times 4$ quotient diagram whose signature is $(a, b, c)$, then the signature of its $2 \times 2$ quotient diagram is $(a + c)$. If two $2 \times 4$ quotient diagrams have the same 2-core, then they have the same reduced $2 \times 2$ quotient diagram and so have the same $2 \times 2$ signature, and vice versa.

**Definition 3.5.2.** Define the truncated signature of a $2 \times 4$ quotient diagram, $\sigma^*(\mathcal{Q})$, to be the projection of the signature $\sigma(\mathcal{Q})$ onto its first two coordinates so...
that, if
\[ \sigma(Q) = (a, b, c), \]
then
\[ \sigma^*(Q) = (a, b). \]

If one knows the truncated signature and the 2-core of a quotient diagram, then its full signature is known. Specifically, if \( \sigma^*(Q) = (a, b) \) and the 2-core is the triangular partition \((k, k - 1, \ldots, 1)\), where

\[ g := \begin{cases} 
-k \over 2 & \text{if } k \text{ is even} \\
{k+1 \over 2} & \text{if } k \text{ is odd}
\end{cases}, \]

then \( \sigma(Q) = (a, b, g - a) \). Therefore, to prove Theorem 3.5.1 it suffices to show that the image of each sequence in our family is in bijection with \( \mathbb{Z}^2 \). For each \( g \in \mathbb{Z} \), or likewise, for each \( k \in \mathbb{N} \), we do this in two steps, the box phase and the spiral phase.

**Algorithm 3.5.3** (Box phase). Rather than working with quotient diagrams directly, we will first give a sequence of partitions and then calculate their \( 2 \times 4 \) quotient diagrams. This ensures that the matrices in these sequences are indeed quotient diagrams. Each sequence of quotient diagrams has to begin with its 2-core diagram, and therefore we must start our sequence of partitions with the \( k^{th} \) triangular partition, for some \( k \in \mathbb{N} \). These stages must be considering in four cases, depending on the value of \( k \) modulo 4.

For each \( i \) from 1 to \( (k \over 2 + 1)^{st} \), add 2 to the first \( i \) odd indexed parts, perform steps adding 2 to each of the first \( i \) even indexed parts, then stop when your two smallest parts are 5 and 2. Each addition of 2 counts as a term in the sequence.
If $k$ is even, then the partition you get from the algorithm is

$$(2k + 2, 2k + 1, 2k - 2, 2k - 3, \ldots, 6, 5, 2).$$

If $k$ is odd, then the partition you get is

$$(2k + 3, 2k, 2k - 1, 2k - 4, \ldots, 6, 5, 2).$$

In both cases, the length of the resulting partition is $k + 1$. Also note that immediately before the $i^{th}$ term, the first $i$ even indexed parts have the same parity modulo 4 and the same is true for the first $i$ odd indexed part.

**Example 3.5.4.** We will be discussing this algorithm along side two examples, the cases $k = 4$ and $k = 5$.

When $k = 4$, we start with the partition $(4, 3, 2, 1)$ and end with the partition $(10, 9, 6, 5, 2)$ in the following way:

$$(4, 3, 2, 1) \rightarrow (6, 3, 2, 1) \rightarrow (6, 5, 2, 1) \rightarrow (8, 5, 2, 1) \rightarrow (8, 5, 4, 1)$$

$$\rightarrow (8, 7, 4, 1) \rightarrow (8, 7, 4, 3) \rightarrow (10, 7, 4, 3) \rightarrow (10, 7, 6, 3) \rightarrow (10, 7, 6, 3, 2)$$

$$\rightarrow (10, 9, 6, 3, 2) \rightarrow (10, 9, 6, 5, 2).$$

**Example 3.5.5.** When $k = 5$, we start with the partition $(5, 4, 3, 2, 1)$ and end with the partition $(13, 10, 9, 6, 5, 2)$ in the following way:

$$(5, 4, 3, 2, 1) \rightarrow (7, 4, 3, 2, 1) \rightarrow (7, 6, 3, 2, 1) \rightarrow (9, 6, 3, 2, 1) \rightarrow (9, 6, 5, 2, 1)$$

$$\rightarrow (9, 8, 5, 2, 1) \rightarrow (9, 8, 5, 4, 1) \rightarrow (11, 8, 5, 4, 1) \rightarrow (11, 8, 7, 4, 1)$$

$$\rightarrow (11, 8, 7, 4, 3) \rightarrow (11, 10, 7, 4, 3) \rightarrow (11, 10, 7, 6, 3) \rightarrow (11, 10, 7, 6, 3, 2)$$

$$\rightarrow (13, 10, 7, 6, 3, 2) \rightarrow (13, 10, 9, 6, 3, 2) \rightarrow (13, 10, 9, 6, 5, 2).$$

---

This formula is only correct for $k > 0$; when $k = 0$, this algorithm ends after the first term. However, the spiral phase of the algorithm remains unchanged.
We will now compute the quotient diagrams of the sequence of partitions defined. Recall that the sequence \( s(n) \) is defined as

\[
s(n) := \begin{cases} 
m^2 & \text{if } n = 2m - 1 \\
m(m + 1) & \text{if } n = 2m
\end{cases}
\]

and that

\[
s(n) + s(n - 1) = t_n.
\]

In Section 2.5, we found that the \( 2 \times 4 \) quotient diagram of the triangular partition \( \Delta_k := (k, k - 1, \ldots, 1) \) is

\[
Q(\Delta_k) = \begin{cases} 
\begin{bmatrix} 
s(2m) & s(2m) & s(2m - 1) & s(2m - 1) \\
0 & s(2m) & s(2m - 1) & s(2m - 2)
\end{bmatrix} & \text{if } k = 4m \\
\begin{bmatrix} 
s(2m + 1) & s(2m) & s(2m) & s(2m - 1) \\
0 & s(2m) & s(2m - 1) & s(2m - 1)
\end{bmatrix} & \text{if } k = 4m + 1 \\
\begin{bmatrix} 
s(2m + 1) & s(2m + 1) & s(2m) & s(2m) \\
0 & s(2m + 1) & s(2m) & s(2m - 1)
\end{bmatrix} & \text{if } k = 4m + 2 \\
\begin{bmatrix} 
s(2m + 2) & s(2m + 1) & s(2m + 1) & s(2m) \\
0 & s(2m + 1) & s(2m) & s(2m)
\end{bmatrix} & \text{if } k = 4m + 3
\end{cases}
\]
which have signatures

\[
\sigma(Q(A_k)) = \begin{cases} 
(-m, m, -m) & \text{if } k = 4m \\
(m + 1, -m, m) & \text{if } k = 4m + 1 \\
(-(m + 1), m + 1, -m) & \text{if } k = 4m + 2 \\
(m + 1, -(m + 1), m + 1) & \text{if } k = 4m + 3 
\end{cases}
\]

\(k = 4m\): The first move changes the quotient diagram \(T_k\) by adding one to the (1, 1) and (1, 2) entries. The second move changes the new quotient diagram by adding one to the (2, 1) and (2, 4) entry. The third and fourth moves, in total, add two to the (1, 3) and (1, 4) entries, the fourth and fifth moves add two to the (2, 2) and (2, 3) entries, and so on. We change what two entries to which we are adding after each \(s(i)^{th}\) entry. Note that between the \(s(4i)^{th}\) and \(s(4i + 4)^{th}\) moves, the diagrams have the pattern:
The fact that
\[ s(4i) = 2i(2i + 1) + (8i + 6) = 4i^2 + 10i + 6 = (2i + 2)(2i + 3) = s(4i + 4) \]
means that the cycle above in fact ends on the \( s(4i + 4) \)th term.

In particular, the \( s(4m) \)th quotient diagram is
\[
\begin{bmatrix}
    s(2m) + s(2m - 1) & s(2m) + s(2m - 1) & s(2m - 1) + s(2m) & s(2m - 1) + s(2m) \\
    s(2m) + s(2m - 1) & s(2m - 1) + s(2m) & s(2m - 1) + s(2m) & s(2m - 2) + s(2m - 1) \\
    t_{2m} & t_{2m} & t_{2m} & t_{2m} \\
    t_{2m} & t_{2m} & t_{2m} & t_{2m-1}
\end{bmatrix}
\]

The last \( 4m + 1 \) quotient diagrams in the pattern we obtain by first adding one to entries \( (1,1) \) and \( (1,2) \) \( 2m + 1 \) times and then adding one to entries \( (2,1) \) and \( (2,4) \) \( m \) times. The final quotient diagram we get is
\[
\begin{bmatrix}
    t_{2m+1} & t_{2m+1} & t_{2m} & t_{2m} \\
    t_{2m+1} - 1 & t_{2m} & t_{2m} & t_{2m}
\end{bmatrix}
\]

Note that the total number of terms is \( s(k) + k + 1 = s(k + 2) - 1 \).

\( k = 4m + 2 \): The algorithm is nearly identical. The contour patterns between steps \( s(4i) \) and \( s(4i + 4) \) have the contour pattern below

\[
\begin{bmatrix}
    s(2m + 1) + s(2i - 2) & s(2m + 1) + s(2i - 2) & s(2m) + s(2i - 3) & s(2m) + s(2i - 3) \\
    s(2m + 1) + s(2i - 2) & s(2m) + s(2i - 3) & s(2m) + s(2i - 3) & s(2m) + s(2i - 2) \\
    s(2m + 1) + s(2i - 2) & s(2m) + s(2i - 3) & s(2m) + s(2i - 3) & s(2m - 1) + s(2i - 2) \\
    s(2m + 1) + s(2i - 2) & s(2m) + s(2i - 3) & s(2m) + s(2i - 3) & s(2m - 1) + s(2i - 2) \\
\end{bmatrix}
\]
$\rightarrow_{2i}^{\rightarrow_{2i}} \left[ \begin{array}{cccc}
 s(2m + 1) + s(2i) & s(2m + 1) + s(2i) & s(2m) + s(2i - 1) & s(2m) + s(2i - 1) \\
 s(2m) + s(2i - 1) & s(2m) + s(2i - 1) & s(2m - 1) + s(2i) & s(2m - 1) + s(2i) \\
 s(2m) + s(2i - 1) & s(2m) + s(2i - 1) & s(2m - 1) + s(2i) & s(2m - 1) + s(2i) \\
 s(2m) + s(2i - 1) & s(2m) + s(2i - 1) & s(2m - 1) + s(2i) & s(2m - 1) + s(2i) \\
 \end{array} \right]
$

and the final quotient diagram is

$\left[ \begin{array}{cccc}
 t_{2m+2} & t_{2m+2} & t_{2m+1} & t_{2m+1} \\
 t_{2m+2} & t_{2m+2} & t_{2m+1} & t_{2m+1} \\
 \end{array} \right].$

Again the total number of terms is $s(k + 2) - 1.$

$k = 4m + 1$:

The first move adds one to the $(1, 2)$ and $(1, 3)$ entries and the second move adds one to the $(2, 1)$ and $(2, 2)$ entries. The third and fourth move each add one to the $(1, 1)$ and $(1, 4)$ entries. The fifth and sixth move each add one to the $(2, 3)$ and $(2, 4)$ entries, and so on. Again, we change what two entries to which we are adding after each $s(i)^{th}$ entry; between the $s(4i)^{th}$ and $s(4i + 4)^{th}$ moves, the contour diagrams follow the sequence:

$\rightarrow_{2i-1}^{\rightarrow_{2i-1}} \left[ \begin{array}{cccc}
 s(2m + 1) + s(2i) & s(2m) + s(2i - 1) & s(2m) + s(2i - 1) & s(2m - 1) + s(2i) \\
 s(2m) + s(2i - 1) & s(2m) + s(2i - 1) & s(2m - 1) + s(2i) & s(2m - 1) + s(2i) \\
 s(2m) + s(2i - 1) & s(2m) + s(2i - 1) & s(2m - 1) + s(2i) & s(2m - 1) + s(2i) \\
 s(2m) + s(2i - 1) & s(2m) + s(2i - 1) & s(2m - 1) + s(2i) & s(2m - 1) + s(2i) \\
 \end{array} \right]$

$\rightarrow_{2i}^{\rightarrow_{2i}} \left[ \begin{array}{cccc}
 s(2m + 1) + s(2i + 2) & s(2m) + s(2i + 2) & s(2m) + s(2i + 2) & s(2m - 1) + s(2i + 2) \\
 s(2m) + s(2i + 2) & s(2m) + s(2i + 2) & s(2m) + s(2i + 2) & s(2m - 1) + s(2i + 2) \\
 s(2m) + s(2i + 2) & s(2m) + s(2i + 2) & s(2m) + s(2i + 2) & s(2m - 1) + s(2i + 2) \\
 s(2m) + s(2i + 2) & s(2m) + s(2i + 2) & s(2m) + s(2i + 2) & s(2m - 1) + s(2i + 2) \\
 \end{array} \right].$

In particular, the $s(4m + 2)^{th}$ quotient diagram in this sequence is

$\left[ \begin{array}{cccc}
 t_{2m+1} & t_{2m+1} & t_{2m+1} & t_{2m+1} \\
 t_{2m+1} & t_{2m+1} & t_{2m} & t_{2m} \\
 \end{array} \right].$
The last $2m+1$ quotient diagrams come from adding one to the $(1,1)$ and $(1,4)$ entries so that the sequence ends on the $s(4m+2) + 2m + 1 = s(k+2) - 1^{th}$ step with the quotient diagram
\[
\begin{bmatrix}
  t_{2m+2} - 1 & t_{2m+1} & t_{2m+1} & t_{2m+1} \\
  t_{2m+1} & t_{2m+1} & t_{2m} & t_{2m}
\end{bmatrix}.
\]

$k = 4m + 3$:

The first move adds one to the $(1,1)$ and $(1,4)$ entries. The second adds one to the $(2,3)$ and $(2,4)$ entries, and so on. Between the $s(4i)$ and the $s(4i + 4)$ moves, the quotient diagrams follow the sequence:

\[
\begin{bmatrix}
  s(2m + 2) + s(2i - 1) & s(2m + 1) + s(2i) & s(2m + 1) + s(2i) & s(2m) + s(2i - 1) \\
  s(2m + 1) + s(2i) & s(2m + 1) + s(2i) & s(2m) + s(2i - 1) & s(2m) + s(2i - 1)
\end{bmatrix}
\Rightarrow_{2i-1}^{2i-1}
\begin{bmatrix}
  s(2m + 2) + s(2i + 1) & s(2m + 1) + s(2i) & s(2m + 1) + s(2i) & s(2m) + s(2i + 1) \\
  s(2m + 1) + s(2i) & s(2m + 1) + s(2i) & s(2m) + s(2i - 1) & s(2m) + s(2i - 1)
\end{bmatrix}
\Rightarrow_{2i}^{2i}
\begin{bmatrix}
  s(2m + 2) + s(2i + 1) & s(2m + 1) + s(2i + 2) & s(2m + 1) + s(2i + 2) & s(2m) + s(2i + 1) \\
  s(2m + 1) + s(2i) & s(2m + 1) + s(2i) & s(2m) + s(2i + 1) & s(2m) + s(2i + 1)
\end{bmatrix}
\Rightarrow_{2i}
\begin{bmatrix}
  s(2m + 2) + s(2i + 1) & s(2m + 1) + s(2i + 2) & s(2m + 1) + s(2i + 2) & s(2m) + s(2i + 1) \\
  s(2m + 1) + s(2i) & s(2m + 1) + s(2i) & s(2m) + s(2i + 1) & s(2m) + s(2i + 1)
\end{bmatrix}
\]

In particular, the $s(4m + 4)^{th}$ quotient diagram is
\[
\begin{bmatrix}
  t_{2m+2} & t_{2m+2} & t_{2m+2} & t_{2m+1} \\
  t_{2m+2} & t_{2m+2} & t_{2m+1} & t_{2m+1}
\end{bmatrix}.
\]

The last $2m + 2$ quotient diagrams are
\[
\begin{bmatrix}
  t_{2m+2} & t_{2m+2} & t_{2m+2} & t_{2m+1} \\
  t_{2m+2} & t_{2m+2} & t_{2m+1} & t_{2m+1}
\end{bmatrix} \Rightarrow_{2m+2}^{2m+2}
\begin{bmatrix}
  t_{2m+3} - 1 & t_{2m+2} & t_{2m+2} & t_{2m+2} \\
  t_{2m+2} & t_{2m+2} & t_{2m+1} & t_{2m+1}
\end{bmatrix}.
\]
again, the sequence ends after the \( s(k + 2) - 1 \)th step.

**Example 3.5.6.** When \( k = 4 \), the box phase is the following sequence of \( s(k + 2) = 12 \) quotient diagrams:

\[
\begin{bmatrix}
2 & 2 & 1 & 1 \\
2 & 1 & 1 & 0
\end{bmatrix}, \begin{bmatrix}
3 & 3 & 1 & 1 \\
3 & 1 & 1 & 0
\end{bmatrix}, \begin{bmatrix}
3 & 3 & 1 & 1 \\
3 & 1 & 1 & 1
\end{bmatrix}, \begin{bmatrix}
3 & 3 & 2 & 2 \\
3 & 1 & 1 & 1
\end{bmatrix}, \begin{bmatrix}
3 & 3 & 3 & 3 \\
3 & 1 & 1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
3 & 3 & 3 & 3 \\
3 & 2 & 2 & 1
\end{bmatrix}, \begin{bmatrix}
3 & 3 & 3 & 3 \\
3 & 3 & 1 & 1
\end{bmatrix}, \begin{bmatrix}
4 & 4 & 3 & 3 \\
3 & 3 & 3 & 1
\end{bmatrix}, \begin{bmatrix}
5 & 5 & 3 & 3 \\
3 & 3 & 3 & 1
\end{bmatrix}, \begin{bmatrix}
6 & 6 & 3 & 3 \\
3 & 3 & 3 & 1
\end{bmatrix},
\]

\[
\begin{bmatrix}
6 & 6 & 3 & 3 \\
4 & 3 & 3 & 2
\end{bmatrix}, \begin{bmatrix}
6 & 6 & 3 & 3 \\
5 & 3 & 3 & 3
\end{bmatrix}.
\]

When \( k = 5 \), the box phase is the following sequence of \( s(k + 2) = 16 \) quotient diagrams:

\[
\begin{bmatrix}
4 & 2 & 2 & 1 \\
2 & 2 & 1 & 1
\end{bmatrix}, \begin{bmatrix}
4 & 3 & 3 & 1 \\
2 & 2 & 1 & 1
\end{bmatrix}, \begin{bmatrix}
4 & 3 & 3 & 1 \\
3 & 3 & 1 & 1
\end{bmatrix}, \begin{bmatrix}
5 & 3 & 3 & 2 \\
3 & 3 & 1 & 1
\end{bmatrix}, \begin{bmatrix}
6 & 3 & 3 & 3 \\
3 & 3 & 1 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
6 & 3 & 3 & 3 \\
3 & 3 & 2 & 2
\end{bmatrix}, \begin{bmatrix}
6 & 3 & 3 & 3 \\
3 & 3 & 3 & 3
\end{bmatrix}, \begin{bmatrix}
6 & 4 & 4 & 3 \\
3 & 3 & 3 & 3
\end{bmatrix}, \begin{bmatrix}
6 & 5 & 5 & 3 \\
3 & 3 & 3 & 3
\end{bmatrix}, \begin{bmatrix}
6 & 6 & 6 & 3 \\
3 & 3 & 3 & 3
\end{bmatrix},
\]

\[
\begin{bmatrix}
6 & 6 & 6 & 3 \\
4 & 4 & 3 & 3
\end{bmatrix}, \begin{bmatrix}
6 & 6 & 6 & 3 \\
5 & 5 & 3 & 3
\end{bmatrix}, \begin{bmatrix}
6 & 6 & 6 & 3 \\
6 & 6 & 3 & 3
\end{bmatrix}, \begin{bmatrix}
7 & 6 & 6 & 4 \\
6 & 6 & 3 & 3
\end{bmatrix}, \begin{bmatrix}
8 & 6 & 6 & 5 \\
6 & 6 & 3 & 3
\end{bmatrix},
\]

\[
\begin{bmatrix}
9 & 6 & 6 & 6 \\
6 & 6 & 3 & 3
\end{bmatrix}.
\]

**Algorithm 3.5.7 (Spiral Phase).** During this part of the algorithm, we begin with the partitions and quotient diagrams at the end of the box phase and continue to add dots two at a time.

\( k = 4m \),

\[
(8m + 2, 8m + 1, 8m - 2, 8m - 3, \ldots, 6, 5, 2) \rightarrow (8m + 2, 8m + 1, 8m - 2, 8m - 3, \ldots, 6, 5, 3, 1)
\]

\[
(8m + 4, 8m + 1, 8m - 2, 8m - 3, \ldots, 6, 5, 3, 1)
\]
\[ (8m + 4, 8m + 1, 8m, 8m - 3, \ldots, 6, 5, 3, 1) \rightarrow \cdots \]
\[ (8m + 4, 8m + 1, 8m, 8m - 3, \ldots, 8, 5, 3, 1) \]
\[ (8m + 4, 8m + 1, 8m, 8m - 3, \ldots, 8, 5, 4, 2) \]
\[ (8m + 4, 8m + 3, 8m, 8m - 3, \ldots, 8, 5, 4, 2) \rightarrow \cdots \]
\[ (8m + 4, 8m + 3, 8m, 8m - 1, \ldots, 8, 7, 4, 2) \]
\[ (8m + 4, 8m + 3, 8m, 8m - 1, \ldots, 8, 7, 5, 3) \]
\[ (8m + 6, 8m + 3, 8m + 2, 8m - 1, \ldots, 8, 7, 5, 3) \rightarrow \cdots \]
\[ (8m + 6, 8m + 3, 8m + 2, 8m - 1, \ldots, 10, 7, 5, 3) \]

and so on, in the obvious manner, switching regularly between increasing one part by 2 and increasing two parts by 1.

The \( i \)th section of the spiral phase, from \( i = 1 \), is

\[
\begin{align*}
\rightarrow_{2m} \begin{bmatrix}
t_{2m+i} & t_{2m+i} & t_{2m+i-1} & t_{2m+i-1} \\
t_{2m+i} & t_{2m+i-1} & t_{2m+i-1} & t_{2m+i-1} \\
t_{2m+i} & t_{2m+i-1} & t_{2m+i-1} & t_{2m+i-1} \\
t_{2m+i} & t_{2m+i-1} & t_{2m+i-1} & t_{2m+i-1}
\end{bmatrix} & \rightarrow_{i} \begin{bmatrix}
t_{2m+i} & t_{2m+i} & t_{2m+i-1} + i & t_{2m+i-1} \\
t_{2m+i} & t_{2m+i-1} & t_{2m+i-1} & t_{2m+i-1} \\
t_{2m+i} & t_{2m+i-1} & t_{2m+i-1} & t_{2m+i-1} \\
t_{2m+i} & t_{2m+i-1} & t_{2m+i-1} & t_{2m+i-1}
\end{bmatrix} \\
\rightarrow_{2m+1} \begin{bmatrix}
t_{2m+i+1} & t_{2m+i+1} - i & t_{2m+i} & t_{2m+i} \\
t_{2m+i} & t_{2m+i} & t_{2m+i} & t_{2m+i-1} \\
t_{2m+i} & t_{2m+i} & t_{2m+i} & t_{2m+i-1} \\
t_{2m+i} & t_{2m+i} & t_{2m+i} & t_{2m+i-1}
\end{bmatrix} & \rightarrow_{i} \begin{bmatrix}
t_{2m+i+1} & t_{2m+i+1} & t_{2m+i} & t_{2m+i} \\
t_{2m+i} & t_{2m+i} & t_{2m+i} & t_{2m+i} \\
t_{2m+i} & t_{2m+i} & t_{2m+i} & t_{2m+i} \\
t_{2m+i} & t_{2m+i} & t_{2m+i} & t_{2m+i}
\end{bmatrix}
\end{align*}
\]

\[ k = 4m + 1 \]
The \( i \)th section of the spiral phase, from \( i = 1 \), is

\[
\begin{align*}
\begin{bmatrix}
t_{2m+i+1} & t_{2m+i} & t_{2m+i} & t_{2m+i}
t_{2m+i+1} & t_{2m+i} & t_{2m+i} & t_{2m+i} - i
\end{bmatrix} & \mapsto \begin{bmatrix}
t_{2m+i+1} & t_{2m+i} & t_{2m+i} & t_{2m+i}
t_{2m+i+1} & t_{2m+i} & t_{2m+i} & t_{2m+i} - i
\end{bmatrix} \\
\begin{bmatrix}
t_{2m+i+1} & t_{2m+i} & t_{2m+i} & t_{2m+i}
t_{2m+i+1} & t_{2m+i} & t_{2m+i} & t_{2m+i} - i
\end{bmatrix} & \mapsto \begin{bmatrix}
t_{2m+i+1} & t_{2m+i} & t_{2m+i} & t_{2m+i} - i
\end{bmatrix}
\end{align*}
\]

\( k = 4m + 2 \)
signatures are one-to-one with the points in $\mathbb{Z}$

Algorithms 3.5.3 and 3.5.7, are all reduced quotient diagrams and their truncated
For any given
Lemma 3.5.8.

The crux is then to prove that the sequence of truncated signatures covers
subset, the partitions are rigid and therefore the quotient diagrams are reduced.

2.3.14; when each dot is added to a Ferrers diagram, if the dot is in the $(a,b)$-
Remark 3.5.9. The importance of this lemma is that it proves Theorem 3.5.1.

Proof. The fact that these quotient diagrams are all reduced comes from Theorem
2.3.14; when each dot is added to a Ferrers diagram, if the dot is in the $(a,b)$-
subset, the partitions are rigid and therefore the quotient diagrams are reduced. The crux is then to prove that the sequence of truncated signatures covers $\mathbb{Z}^2$.

In these sequences, in going from one reduced quotient diagram to another, you
increase the mantle by adding one of the following matrices.

$$
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
, \quad 
\begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
, \quad 
\begin{bmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
, \quad 
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}
, \quad 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
, \quad 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
, \quad 
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
, \quad 
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
, \quad 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
, \quad 
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}
$$

These mantles affect the truncated signature of the quotient diagram in the
follow manner:

Lemma 3.5.8. For any given $k$, the sequence of quotient diagrams to come out of Algorithms 3.5.3 and 3.5.7, are all reduced quotient diagrams and their truncated signatures are one-to-one with the points in $\mathbb{Z}^2$.  

Remark 3.5.9. The importance of this lemma is that it proves Theorem 3.5.1.
\[ \sigma^* \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sigma^* \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = (0, +1) \]
\[ \sigma^* \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sigma^* \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = (-1, 0) \]
\[ \sigma^* \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sigma^* \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} = (0, -1) \]
\[ \sigma^* \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \sigma^* \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} = (+1, 0) \]
\[ \sigma^* \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = (+1, +1) \]
\[ \sigma^* \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = (-1, -1) \]
\[ \sigma^* \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = (-1, +1) \]
\[ \sigma^* \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} = (+1, -1) \]

Graphically, we can imagine the sequence of truncated signatures as a spiral, as we did with the \(2 \times 3\) signatures in Theorem 3.4.5. We can see an example of this spiral when \(k = 4\) in figure 2. It is obvious from this example that the spiral intersects every point in \(\mathbb{Z}^2\) which proves the lemma.

\[ \square \]

### 3.6 Future Work

There are a few logical next steps in to trying to prove the Quotient Diagram Conjecture in full. The first would be to try to prove the \(3 \times 3\) case and try to extend the results to the \(t \times t\) cases. As noted before, this case is akin to the \(2 \times 3\) and \(2 \times 4\) cases in that its mantle quotient diagram is conjecture to have a single term. Furthermore, we already have a complete characterization of the
3 × 3 quotient diagrams of 3-core partitions. What would remain would be to organize these 3-core quotient diagrams so that their signatures can be truncated in a fashion similar to the 2 × 4 case.

It might be possible that, by using the covering maps, one could deduce the quotient diagram conjecture for any pair of $a$ and $b$ if the conjecture is known to be true for the $t \times t$ quotient diagrams where $t = \text{lcm}(a, b)$.

The other possible progression from this point would be to try to investigate
either the $2 \times 5$ or $2 \times 6$ case. Understanding these cases would possibly allow us to extend the known results to prove the $2 \times b$ cases for all $b$. Recall that these cases will be vastly different from one another; the $2 \times 5$ case will be similar to the $2 \times 3$ case and the $2 \times 6$ case will be more similar to the $2 \times 4$ case. The current problem with these cases is a matter of degrees of freedom which leads to ambiguity of how the case should be handled. For example, in the $2 \times 6$ case, the mantle generating function should turn out to be

$$\frac{1}{(1 - q^2)(1 - q^4)},$$

meaning that, to each reduced $2 \times 6$ quotient diagram, we can add either 2 dots or 4 dots. However, as of yet, there is no indication which of the two reduced $2 \times 6$ quotient diagrams of weight four is the result off adding 4 dots to the empty quotient diagram and which is the result of adding 2 dots to the quotient diagram of the partition (2).
Bibliography


Vita

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Matthew Katz was born in 1986 in New York City. He graduated from Pennsbury High School in Bucks County, PA, in 2004. Matt then went to Juniata College and graduated Magna Cum Laude, in 2008, with a major in mathematics and minors in physics, education, and Japanese studies. This dissertation concluded Matt’s tenure at Pennsylvania State University where, in 2013, he received his Ph.D in mathematics.