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OPTIMAL BIDDING STRATEGIES IN A LIMIT ORDER BOOK

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Abstract

We introduce a new model of the Limit Order Book, viewed as a noncooperative game, whose players are the agents submitting the Limit Orders. The incoming Market Order is modeled as a random variable.

If we allow prices to range in a compact interval, we can prove the existence or non-existence of Nash equilibria in the case of homogeneous players. When a Nash equilibrium exists, we prove that it is unique, and obtain explicit formulas for the optimal bidding strategies, by solving a system of Ordinary Differential Equations.

If prices take values in a finite set, we show the existence or non-existence of a Nash equilibrium in the more general case of heterogeneous players.

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Chapter 1

Introduction

1.1 What is the Limit Order Book

Consider a financial asset, for example a commodity or a stock. By *order* we mean an instruction that a client gives to a broker to buy or sell the asset on an exchange. There are two basic different types of orders: *market orders* and *limit orders*.

According to the Securities and Exchange Commission[1]

> A market order is an order to buy or sell a stock at the best available price. Generally, this type of order will be executed immediately. However, the price at which a market order will be executed is not guaranteed. It is important for investors to remember that the last-traded price is not necessarily the price at which a market order will be executed. In fast-moving markets, the price at which a market order will execute often deviates from the last-traded price or real time quote.

and[2]

> A limit order is an order to buy or sell a stock at a specific price or better. A buy limit order can only be executed at the limit price or lower, and a sell limit order can only be executed at the limit price or higher. A limit order is not guaranteed to execute. A limit order

[1] http://www.sec.gov/answers/mktord.htm
[2] http://www.sec.gov/answers/limit.htm

can only be filled if the stocks market price reaches the limit price. While limit orders do not guarantee execution, they help ensure that an investor does not pay more than a pre-determined price for a stock.

We can also make a distinction between two very different type of markets.

- *Quote-driven markets* are characterized by the presence of a market maker (or dealer, or specialist), who centralizes buy and sell orders and provides liquidity by setting bid and ask quotes.

- *Order-driven trading system* are electronic platforms that aggregate all outstanding limit orders in a limit order book that is available to market participants. Market orders are executed against the best available prices.

In the present dissertation, we will focus exclusively on the second type of markets, where no market maker is present, and the evolution of prices depends exclusively on the interaction of the buy and sell orders posted and executed by the various agents.

1.2 The Limit Order Book as a Noncooperative Game

This dissertation is concerned with a new mathematical model of the limit order book, viewed as a noncooperative game, whose players are the agents submitting the limit orders. In the following, we will use the terms *player, seller* and *agent* interchangeably. Our main goal is to study the existence and uniqueness of Nash equilibria, determining the optimal bidding strategies of the various agents.

We consider a one-sided limit order book: in our model the agents are exclusively the *sellers* who post limit orders. An external *buyer* will buy a positive random amount of X shares by executing a market order.

In other words, the external agent will buy the amount X at the lowest available price, while one or more sellers offer various quantities of this asset at different prices, competing to fulfill the random incoming order.

This modeling approach thus aims at describing the Limit Order Book as a static one shot game: the *players* are the selling agents, the *strategies* are the prices at which they submit the limit orders.

Each player (say, player i) is simply described by the following three quantities.

- The *reference price* $\overline{p}_i$. This represents the 'real' value of the asset, according to player i. This reference value could for example depend by the information that the player has about the asset and by his inventory. For example, an agent who owns a large quantity of the asset and wants to reduce his inventory risk might have a lower reference price than a different agent who has a smaller inventory.

- The assumed *distribution* of the incoming market order. Each agent has his own belief on what the probability distribution of the size of the incoming market order is. The tail distribution of the incoming order according to player i is denoted by ψ_i.

- The total amount put on sale by each agent, denoted by κ_i.

Having observed the prices asked by his competitors, each seller must determine an optimal strategy, maximizing his expected payoff, which in our model is simply his expected gain from the sale. More precisely, for player i the profit obtained by the sale of a particular share, which we label as β, is simply given by the difference between the price $\phi(\beta)$ at which the share is sold and his reference price:

$$\text{Unit payoff :} \quad \phi(\beta) - \bar{p}_i.$$

The *expected* profit will thus be given by the product of the above expression times the probability of actually selling share β.

$$\text{Expected unit payoff :} \quad (\phi(\beta) - \bar{p}_i) \times [\text{Probability of selling share } \beta]$$

This probability of selling share β will depend on the price $\phi(\beta)$, on the probability distribution ψ_i and on the strategies of the other players, who are selling at prices lower or equal than $\phi(\beta)$. A rigorous derivation of an expression for the payoff function will be given in Chapter 2.

We observe that in this model, player i will post limit order at prices strictly greater than $\bar{p}_i$: selling at prices lower than his reference price will yield a negative unit profit.

The output of the model is the *optimal* shape of the Limit Order book, which is determined by the collection of all the sell orders submitted by each agent. In particular the *ask price*, i.e. the smallest price at which somebody is willing to sell, will also be an outcome of the model. Another important quantity which can be computed is the *price impact function*, which quantifies the increase in the ask price, after a large trade is executed.

1.3 Literature Review

The recent literature on Mathematical Modeling order book is huge. The modeling approaches are very different, depending on what the main goal of the modeling is.

Some models of the Limit Order Book are concerned with the *Optimal Execution* problem: the main goals of these models is to optimize the way large trades are executed over time, through a sequence of Market Orders.

These models postulate a shape for the Limit Order Book, together with a random dynamics for the limit order prices. In some of these models ([10], [12], [1]), prices range in a continuum set of values, and the execution of a market order is modeled exactly as in our model: the external buyer buys the cheapest shares which are offered for sale and this moves the ask price .

The dual modeling approach is to take the perspective of the agents who post the Limit Orders. Several recent papers have been devoted to this modeling approach.

In the model by Avellaneda and Stoikov [2], the goal is to optimize the bid and ask prices, from the point of view of a Market Maker, who centralizes all outstanding orders, to provide liquidity to market participants. This model assumes stochastic dynamics for the asset price and derives optimal bid and ask quotes by solving an Hamilton-Jacobi-Bellmann PDE. This model is applicable to quote-driven markets.

In the model by Cont, Stoikov and Talreja [7], agents submit Limit Orders in an Order-driven trading system. Their modeling approach assumes stochastic dynamics of the asset price and is intrinsically discrete in the price variable: to each price there corresponds a queue of limit orders, which are executed according to a first-in-first-out schedule.

The main novelty of our model ([4], [5]) is that we consider the limit order book as the outcome of a noncooperative game, where several different players are trying to maximize their (possibly different) utility functions. In this aspect, our model is similar to the model by Roşu [14]. A big difference lies in the fact that in our model the utility functions of the players are determined by the probability distribution they assign to the size of the incoming market order, and the expected waiting time of order execution is not a factor.

In addition to the classical paper [9], for an introduction to non-cooperative games and Nash equilibria we refer to [3, 8, 15, 16].

1.4 Organization of this Dissertation

The dissertation is organized as follows. Chapters 2 to 4 are devoted to a model of the Limit Order Book, where prices vary in a continuum range of values $[0, \overline{P}]$. In this model, the "shape" of the Limit Order book is described in the most general way in terms of Radon measures supported on the interval $[0, \overline{P}]$.

In Chapter 2 we consider the optimization problem for a single agent, who observes the limit orders submitted by his competitors and wishes to optimally price his own assets. Under general regularity assumptions, the existence of an optimal pricing strategy is proved and a necessary condition for optimality is derived. This necessary condition motivates a classification of random variables, based on some properties of their distribution function. We call these two classes *type A* and *type B*. We prove that for random variables of type B, the optimal strategy always consists in putting all the assets for sale at the same price. We also prove some sufficient conditions for optimality, that are applicable in the case of random variables of type A.

Chapter 3 is devoted to the study of Nash equilibria in the particular case of homogeneous players. We consider a game for n players, putting on sale possibly different quantities $\kappa_1, \ldots, \kappa_n$ of the same asset. We consider the case when every player has the same reference price and assumes the same distribution for the incoming order size. When the random buying order X is a random variable of type A, we prove that this noncooperative game admits a unique Nash equilibrium, which is explicitly determined. On the other hand, if the random variable X is of type B, we show that no Nash equilibrium can exist.

In Chapter 4, we consider an asymptotic limit, where the total number of identical agents approaches infinity, while the amount of asset put on sale by each agent approaches zero. In this case, the limit order book approaches a well defined shape, determined by the probability distribution of the random variable X and by the reference price, common to all players.

In Chapter 5 we analyze the more realistic case of a Limit Order Book where prices vary in a discrete set. In this case, we consider the more general setting of heterogeneous players: each agent has a different belief on the probability distribution of the size of the incoming buy market order and a different reference price. We can prove the existence of Nash equilibria for a general class of random variables. The proof in this case does not rely on an explicit construction, but is obtained by a general topological argument. In particular, we cannot prove uniqueness of Nash equilibria in the discrete price model.

Chapter 2

The Optimization Problem for a Single Player

The present Chapter studies the optimization problem for a new agent who observes the Limit Order Book and needs to decide at what prices he is going to post his own sell Limit Orders. We denote the reference price of this new agent by p_0, the tail distribution that he associates to the incoming market order is denoted by ψ, and the total amount of shares that he puts on sale is κ.

2.1 The Optimization Problem

Let X be a non-negative random variable, with distribution function

$$Prob.\{X \leq s\} \;=\; 1 - \psi(s)\,. \tag{2.1}$$

The function $\psi(s)$ is sometimes called the *tail distribution* of X. Throughout the following we shall assume

(A1) *The map $s \mapsto \psi(s)$ is continuously differentiable and satisfies*

$$\psi(0) \;=\; 1\,, \qquad \psi(+\infty) \;=\; 0, \qquad \psi'(s) \;<\; 0 \qquad \textit{for all } s > 0\,. \tag{2.2}$$

We shall consider two main classes of random variables, depending on the decay properties of the function ψ.

Definition 1. *We say that a probability distribution is*

$$\text{of} \quad \textbf{type A} \quad \text{if} \quad (\ln\psi(s))'' \;>\; 0 \quad \text{for all } s>0\,; \tag{2.3}$$

$$\text{of} \quad \textbf{type B} \quad \text{if} \quad (\ln\psi(s))'' \;<\; 0 \quad \text{for all } s>0\,; \tag{2.4}$$

Moreover, we say that a probability distribution is

$$\text{of} \quad \textbf{type A}_+ \quad \text{if} \quad (\ln\psi(s))'' \;>\; 0 \quad \text{for all } s>0\,; \tag{2.5}$$

$$\text{of} \quad \textbf{type A}_\mathbf{0} \quad \text{if} \quad (\ln\psi(s))'' \;=\; 0 \quad \text{for all } s>0\,. \tag{2.6}$$

For example, the probability distributions determined by

$$\psi_1(s) \;=\; e^{-\lambda s} \qquad\qquad \lambda>0\,, \tag{2.7}$$

$$\psi_2(s) \;=\; \frac{1}{(1+s)^\alpha} \qquad\qquad \alpha>0\,, \tag{2.8}$$

are of type A_0 and A_+ respectively, while

$$\psi_3(s) \;=\; e^{-s^2}$$

yields a probability distribution of type B.

It is very easy to verify the following.

Remark 1. *A random variable is of type A_0 if and only if it is an exponential random variable, with tail distribution as in (2.7).*

Of course, we can consider more general probability distributions, where $(\ln\psi)''$ changes sign. For such random variables, the analysis will likely be more difficult.

We notice that is not necessary to assume that the second derivative exists. Indeed, we could simply define the different types of random variable based on the convexity of $\log\psi$. See Remark 8 for more details.

The next ingredient to formulate the optimization problem from the point of view of the new agent is the *observed* Limit Order Book. Let $\Phi_0 : [0,\overline{P}] \mapsto \mathbb{R}_+$ be a non-negative, nondecreasing function. For every p, we think of $\Phi_0(p)$ as the total amount of stock offered for sale at a price $\leq p$ by the other agents.

We now consider an agent who wishes to sell an amount κ of stock by posting Limit Orders.

Definition 2. *A* **pricing strategy** *for the new player is a nondecreasing, left continuous map* $\phi : [0,\kappa] \mapsto [p_0, \overline{P}]$.

Using the Lagrangian variable $\beta \in [0,\kappa]$ to label a particular share in possession of the new agent, by $\phi(\beta)$ we thus denote the price at which this particular share is put on sale. The total amount of shares that the new agent offers for sale at price $\leq p$ is thus computed by

$$\mu_1([0,p]) \;=\; meas\Big(\{\beta \in [0,\kappa]\,;\;\; \phi(\beta) \leq p\}\Big). \tag{2.9}$$

This is the push-forward of the Lebesgue measure on $[0,\kappa]$ w.r.t. the map ϕ.

Next, assume that the incoming order has size X. The total amount of stock sold by the new agent is

$$\beta(X) \;=\; \sup\;\Big\{\beta \in [0,\kappa]\,;\;\; \beta + \Phi_0(\phi(\beta)) \;\leq\; X\Big\}, \tag{2.10}$$

yielding the payoff

$$\int_0^{\beta(X)} (\phi(\beta) - p_0)\, d\beta\,.$$

Here $p_0 > 0$ is the value that the new player attaches to a unit amount of stock. For example, it could be the mean bid-ask price.

The optimization problem for the new seller can thus be formulated as

$$\text{Maximize:}\quad J(\phi) \;\doteq\; E\Big[\int_0^{\beta(X)} (\phi(\beta) - p_0)\, d\beta\Big] \tag{2.11}$$

among all pricing strategies $\phi : [0,\kappa] \mapsto [0,\overline{P}]$. Here $E[\cdot]$ denotes the expectation w.r.t. the probability distribution of the random variable X.
Observe that, by (2.1) and (2.10), we have the equivalent representation

$$J(\phi) \;=\; \int_0^{\kappa} (\phi(\beta) - p_0)\, \psi\Big(\beta + \Phi_0(\phi(\beta))\Big)\, d\beta\,. \tag{2.12}$$

Remark 2. *If Φ_0 has a jump at a point ξ, this means that a positive amount of stock is offered for sale by the other agents at the price ξ. Two main cases can arise.*

CASE 1: Φ_0 is left continuous, i.e. $\Phi_0(\xi) = \Phi_0(\xi-)$. This means that the new agent has selling priority. If he also puts on sale a positive amount of stock at the same price ξ, his stock will be the first to be sold.

CASE 2: Φ_0 is right continuous, i.e. $\Phi_0(\xi) = \Phi_0(\xi+)$. This means that the new agent does not have selling priority. If he also puts on sale a positive amount of stock at the same price ξ, his stock will be the last to be sold.

Notice that in Case 1 the function Φ_0 is lower semicontinuous. This property will play a key role in the proof of existence of an optimal strategy.

2.2 Existence of an Optimal Strategy

Our first result shows the existence of an optimal strategy for the new agent, assuming that he has selling priority.

Theorem 2.1 (existence)**.** *Let X be a random variable satisfying the assumptions (A1). Let $\Phi_0 : [0, \overline{P}] \mapsto \mathbb{R}_+$ be a left-continuous, nondecreasing function, and let $\kappa > 0$. Then there exists an optimal pricing strategy $\phi^* : [0, \kappa] \mapsto [p_0, \overline{P}]$ for the new agent, maximizing the expected payoff (2.11).*

Proof. Let $(\phi_\nu)_{\nu \geq 1}$ be a maximizing sequence of pricing strategies. Since all functions ϕ_ν are non-decreasing, using Helly's compactness theorem (see for example [13], p. 372), by extracting a subsequence and relabeling we can achieve the pointwise convergence

$$\phi_\nu(\beta) \ \to \ \phi^*(\beta) \qquad\qquad \text{for all } \beta \in [0, \kappa]\,.$$

We claim that the strategy ϕ^* is optimal.

Indeed, since Φ_0 is lower semicontinuous and ψ is strictly decreasing, the composite map $s \mapsto \psi(\Phi_0(s))$ is upper semicontinuous. Therefore, for every $\beta \in [0, \kappa]$ we have

$$\limsup_{\nu \to \infty} \ \psi\Big(\Phi_0(\phi_\nu(\beta))\Big) \ \leq \ \psi\Big(\Phi_0(\phi^*(\beta))\Big).$$

In turn, this yields

$$\begin{aligned}
\sup_\phi J(\phi) &= \lim_{\nu\to\infty} J(\phi_\nu) = \lim_{\nu\to\infty}\int_0^\kappa (\phi_\nu(\beta)-p_0)\,\psi\Big(\beta+\Phi_0(\phi_\nu(\beta))\Big)\,d\beta \\
&\leq \int_0^\kappa \limsup_{\nu\to\infty}\Big\{(\phi_\nu(\beta)-p_0)\,\psi\Big(\beta+\Phi_0(\phi_\nu(\beta))\Big)\Big\}\,d\beta \\
&\leq \int_0^\kappa (\phi^*(\beta)-p_0)\,\psi\Big(\beta+\Phi_0(\phi^*(\beta))\Big)\,d\beta = J(\phi^*)\,.
\end{aligned}$$

□

Example 1. *If the new player does not have priority, an optimal strategy may fail to exist. For example, assume that the other sellers offer a total amount of stock* κ_0*, all at the same price* $\overline{P}$*. This situation is described by the right continuous function*

$$\Phi_0(p) = \begin{cases} 0 & \textit{if} \quad p < \overline{P}\,, \\ \kappa_0 & \textit{if} \quad p = \overline{P}\,. \end{cases} \tag{2.13}$$

Assume that the new player has an amount κ *of stock to put on sale. For each* $\nu \geq 1$*, consider the pricing strategy* $\phi_\nu(\beta) \equiv \overline{P} - \nu^{-1}$*. Then* $(\phi_\nu)_{\nu\geq 1}$ *is a maximizing sequence. Writing* $a \wedge b \doteq \min\{a,b\}$*,* $a_+ \doteq \max\{a,0\}$*, the expected payoffs are*

$$J(\phi_\nu) = (\overline{P} - \nu^{-1} - p_0)\cdot E[X \wedge \kappa]\,.$$

However, the expected payoff $(\overline{P} - p_0)\cdot E[X\wedge\kappa]$ *could be achieved only if the new agent puts all his stock for sale at the maximum price* $\overline{P}$ *and has selling priority over the other agents (that would correspond to* Φ_0 *being left continuous). However, if* Φ_0 *is the function in (2.13), the new agent does not have priority. With the strategy* $\phi^*(\beta) \equiv \overline{P}$ *he only achieves*

$$J(\phi^*) = (\overline{P} - p_0)\cdot E\Big[(X-\kappa_0)_+ \wedge \kappa\Big].$$

2.3 Necessary conditions

In this section we seek necessary conditions for the optimality of a pricing strategy ϕ for the new agent. For this purpose given a non-negative, nondecreasing function $\Phi_0 : [0, \overline{P}] \mapsto \mathbb{R}_+$ as in (2.10), we introduce the functions

$$G^\beta(p) \doteq -\psi\Big(\beta + \Phi_0(p)\Big) \cdot \Big[(p - p_0)\psi'\Big(\beta + \Phi_0(p)\Big)\Big]^{-1}. \tag{2.14}$$

For $0 \le a < b \le \kappa$ we shall also consider the integrated function

$$G^{[a,b]}(p) \doteq -\int_a^b \psi\Big(\beta + \Phi_0(p)\Big) d\beta \cdot \Big[(p - p_0) \int_a^b \psi'\Big(\beta + \Phi_0(p)\Big) d\beta\Big]^{-1}.$$

Remark 3. *If the random variable X is of type A, then for every p the map $\beta \mapsto G^\beta(p)$ is non-decreasing. On the other hand, if X is of type B, then the maps $\beta \mapsto G^\beta(p)$ are strictly decreasing.*

In this section we do not make any assumption on the left or right continuity of Φ_0. It will thus be convenient to define the left continuous function

$$\Phi_0^\flat(p) \doteq \Phi_0(p-).$$

In other words, $\Phi_0^\flat$ is the unique left continuous function that coincides with Φ_0 everywhere with the possible exception of countably many points of jump. Call $J^\flat(\phi)$ the expected payoff achieved by a pricing strategy $\phi : [0, \kappa] \mapsto [0, \overline{P}]$ when Φ_0 is replaced by $\Phi_0^\flat$.

Lemma 2.2. *In the above setting, for every $\Phi_0 : [0, \overline{P}] \mapsto \mathbb{R}_+$ and $\kappa > 0$ one has*

$$\sup_\phi J(\phi) = \max_\phi J^\flat(\phi). \tag{2.15}$$

Proof. By Theorem 2.1, the maximum expected payoff on the right hand side of (2.15) is attained. Namely, there exists a pricing strategy ϕ^* such that

$$J^\flat(\phi^*) = \max_\phi J^\flat(\phi).$$

Consider the strategies

$$\phi_n(\beta) \;=\; \phi^*(\beta) - \frac{1}{n}\,. \tag{2.16}$$

The corresponding payoffs satisfy

$$\begin{aligned} J(\phi_n) \;&=\; \int_0^\kappa \left(\phi^*(\beta) - \frac{1}{n} - p_0\right)\psi\left(\beta + \Phi_0\left(\phi^*(\beta) - \frac{1}{n}\right)\right)\,d\beta \\ &\geq\; \int_0^\kappa \left(\phi^*(\beta) - p_0\right)\psi\left(\beta + \Phi_0\left(\phi^*(\beta) - \frac{1}{n}\right)\right)\,d\beta - \frac{\kappa}{n} \\ &\geq\; \int_0^\kappa \left(\phi^*(\beta) - p_0\right)\psi\left(\beta + \Phi_0^\flat\left(\phi^*(\beta)\right)\right)\,d\beta - \frac{\kappa}{n} \;=\; J^\flat(\phi^*) - \frac{\kappa}{n}. \end{aligned}$$

Therefore

$$\sup_\phi J(\phi) \;\geq\; \sup_n J(\phi_n) \;\geq\; \sup_n \left\{J^\flat(\phi^*) - \frac{\kappa}{n}\right\} \;=\; J^\flat(\phi^*) \;=\; \sup_\phi J^\flat(\phi).$$

The converse inequality is clear. Indeed, $\Phi_0^\flat(p) \leq \Phi_0(p)$ for every p. Hence $J^\flat(\phi) \geq J(\phi)$ for every admissible strategy $\phi : [0,\kappa] \mapsto [0,\overline{P}]$. □

Given a nondecreasing left continuous map $\phi : [0,\kappa] \mapsto [0,\overline{P}]$ one can isolate countably many disjoint intervals $S_j \doteq]a_j, b_j] \subseteq [0,\kappa]$ such that ϕ is constant on each S_j and strictly increasing elsewhere. Namely, defining $S \doteq \bigcup_j S_j$ one has

$$\beta_1 \;\notin\; S\,, \qquad \beta_1 \;<\; \beta_2 \qquad \Longrightarrow \qquad \phi(\beta_1) \;<\; \phi(\beta_2)\,.$$

In connection with the measure μ_1 introduced at (2.9), we observe that the atomic part of μ_1 is the measure μ_1^a concentrated on the points $\phi(a_j+) = \phi(b_j)$. Indeed, $\mu_1^a\Big(\{\phi(b_j)\}\Big) = b_j - a_j > 0$.

Theorem 2.3 (necessary conditions for optimality)**.** *Let the random variable X satisfy the assumptions (A1) and let $\Phi_0 : [0,\overline{P}] \mapsto \mathbb{R}_+$ be a nondecreasing map. If $\phi : [0,\kappa] \mapsto [p_0,\overline{P}]$ is an optimal pricing strategy, then the following holds.*

(i) For almost every $\beta \in [0,\kappa] \setminus S$, setting $x \doteq \phi(\beta)$ one has (see fig. 2.1)

$$\limsup_{\varepsilon\to 0-} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon} \;\leq\; G^\beta(x) \;\leq\; \liminf_{\varepsilon\to 0+} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon}\,. \tag{2.17}$$

(ii) If $\beta \in [a_i, b_i] \subset S$, with $\phi(\beta') = x < \overline{P}$ for all $\beta' \in [a_i, b_i]$, then

$$\begin{aligned} G^{a_i}(x) &\geq \limsup_{\varepsilon \to 0-} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon}\,, \\ G^{b_i}(x) &\leq \liminf_{\varepsilon \to 0+} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon}\,, \end{aligned} \tag{2.18}$$

$$\limsup_{\varepsilon \to 0-} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon} \;\leq\; G^{[a_i,b_i]}(x) \;\leq\; \liminf_{\varepsilon \to 0+} \frac{\Phi_0(x+\varepsilon) - \Phi_0(x)}{\varepsilon}\,. \tag{2.19}$$

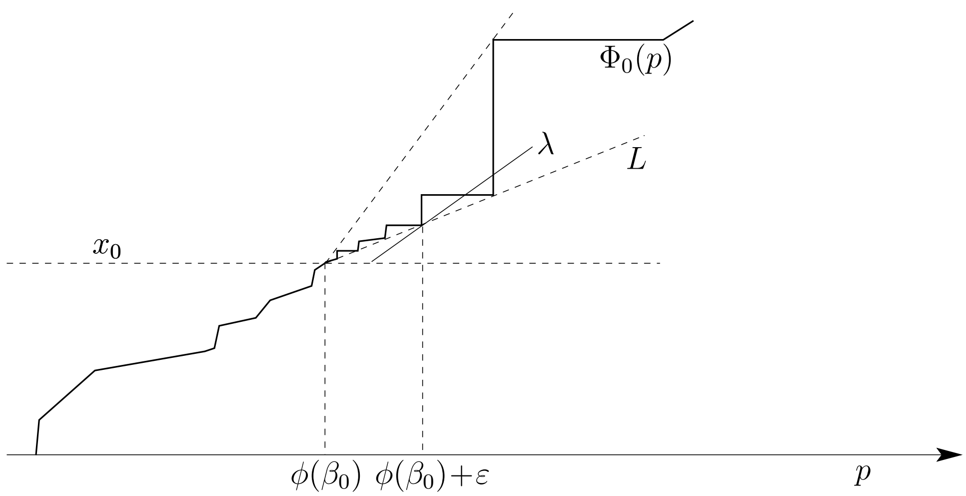

Figure 2.1. Deriving the necessary conditions for optimality. The solid line has slope λ and touches the graph of Φ_0 at the point $\phi(\beta_0) + \varepsilon$.

Proof. 1. Assume that the second inequality in (2.17) does not hold at some $\beta_0 \in [0, \kappa] \setminus S$. Setting $x_0 \doteq \phi(\beta_0)$, this clearly implies

$$L \;\doteq\; \liminf_{\varepsilon \to 0+} \frac{\Phi_0(x_0+\varepsilon) - \Phi_0(x_0)}{\varepsilon} \;<\; \infty\,. \tag{2.20}$$

Hence the nondecreasing function Φ_0 is right continuous at the point $x_0 = \phi(\beta_0)$. By continuity we can thus find λ and $\delta > 0$ such that

$$L \;<\; \lambda \;<\; G^{\beta}(p) \qquad \text{for all } \beta \in [\beta_0,\, \beta_0 + \delta], \quad p \in [x_0,\; x_0 + \delta]\,, \tag{2.21}$$

$$\psi(\zeta) + (p - p_0)\psi'(\zeta)\lambda \ > \ 0 \quad \text{for all } p \in [x_0,\ x_0 + \delta]\,, \qquad \text{for all } \zeta \in [\beta_0 + \Phi_0(x_0)\,,\ \beta_0 + \Phi_0(x_0) + \delta\lambda]\,. \tag{2.22}$$

2. We claim that there exists $\varepsilon \in]0, \delta]$ such that the following conditions hold (see Fig. 2.1).

$$\Phi_0(p) \ \geq \ \Phi_0(x_0 + \varepsilon) + \lambda\,(p - x_0 - \varepsilon) \qquad \text{for all } p \in [x_0,\, x_0 + \varepsilon], \tag{2.23}$$

$$\beta_1 \ \doteq \ \sup\{\beta\,;\ \ \phi(\beta) < x_0 + \varepsilon\} \ < \ \beta_0 + \delta\,. \tag{2.24}$$

Indeed, by definition of liminf there exists $\varepsilon_2 \in]0, \delta]$ such that

$$\Phi_0(x_0 + \varepsilon_2) \ < \ \Phi_0(x_0) + \lambda\varepsilon_2\,.$$

Consider the modified function

$$\Phi_0^\flat(p) \ \doteq \ \begin{cases} \Phi_0(p) & \text{if} \quad p \notin]x_0\,,\ x_0 + \varepsilon_2]\,, \\ \Phi_0(p-) & \text{if} \quad p \in]x_0\,,\ x_0 + \varepsilon_2]\,. \end{cases}$$

By lower semicontinuity, the function

$$\eta \ \mapsto \ \Phi_0^\flat(x_0 + \eta) - \lambda\eta \tag{2.25}$$

attains a strictly negative minimum on the interval $[0, \varepsilon_2]$. If

$$\varepsilon \in \operatorname*{argmin}_{\eta \in [0, \varepsilon_2]} \left\{ \Phi_0^\flat(x_0 + \eta) - \lambda\eta \right\} \tag{2.26}$$

is a point where this minimum is attained, then (2.23) holds.

3. Let $\phi^{\varepsilon+}$ be the perturbed strategy defined by

$$\phi^{\varepsilon+}(\beta) \ \doteq \ \begin{cases} \phi(\beta) & \text{if} \quad \beta \notin [\beta_0, \beta_1]\,, \\ x_0 + \varepsilon & \text{if} \quad \beta \in [\beta_0, \beta_1]\,. \end{cases}$$

Since $\psi' < 0$, using (2.23) and then (2.21)-(2.22), one obtains

$$
\begin{aligned}
& J^\flat(\phi^{\varepsilon+}) - J(\phi) \\
&= \int_{\beta_0}^{\beta_1} \Big[(x_0+\varepsilon-p_0)\cdot\psi\Big(\beta+\Phi_0^\flat(x_0+\varepsilon)\Big) \\
&\qquad\qquad -(\phi(\beta)-p_0)\cdot\psi\Big(\beta+\Phi_0(\phi(\beta))\Big)\Big]\, d\beta \\
&\geq \int_{\beta_0}^{\beta_1} \Big[(x_0+\varepsilon-p_0)\cdot\psi\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(x_0+\varepsilon-\phi(\beta))\Big) \\
&\qquad\qquad -(\phi(\beta)-p_0)\cdot\psi\big(\beta+\Phi_0(\phi(\beta))\big)\Big]\, d\beta \\
&= \int_{\beta_0}^{\beta_1}\int_{\phi(\beta)}^{x_0+\varepsilon} \frac{d}{dp}\Big[(p-p_0)\psi\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)\Big]\, dp\ d\beta \\
&= \int_{\beta_0}^{\beta_1}\int_{\phi(\beta)}^{x_0+\varepsilon} \Big[\psi\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big) \\
&\qquad\qquad +(p-p_0)\psi'\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)\lambda\Big]\, dp\, d\beta \\
&\geq\ \delta_0\ >\ 0\,,
\end{aligned}
\tag{2.27}
$$

for some positive constant δ_0. Using Lemma 2.2 we conclude

$$
J(\phi)\ =\ \sup_\varphi J(\varphi)\ =\ \sup_\varphi J^\flat(\varphi)\ \geq\ J^\flat(\phi^{\varepsilon+})\ \geq\ J(\phi)+\delta\,,
$$

reaching a contradiction.

The first inequality in (2.17) can be proved by an entirely similar argument.

4. The two statements (2.18)-(2.19) will be deduced as consequences of the more general necessary conditions

$$
\begin{aligned}
G^{[\xi,b_i]}(x) &\leq \liminf_{\varepsilon\to 0+} \frac{\Phi_0(x+\varepsilon)-\Phi_0(x)}{\varepsilon}, \text{ for all } \xi\in[a_i,b_i],\\
G^{[a_i,\xi]}(x) &\geq \limsup_{\varepsilon\to 0-} \frac{\Phi_0(x+\varepsilon)-\Phi_0(x)}{\varepsilon}, \text{ for all } \xi\in[a_i,b_i].
\end{aligned}
\tag{2.28}
$$

Indeed, the two inequalities in (2.18) are obtained by observing that

$$
\lim_{\xi\to b_i-} G^{[\xi,b_i]}(x) = G^{b_i}(x), \qquad \lim_{\xi\to a_i+} G^{[a_i,\xi]}(x) = G^{a_i}(x).
$$

Moreover, (2.19) follows from the two inequalities in (2.28), choosing $\xi = b_i$ and $\xi = a_i$, respectively.

5. It now remains to prove (2.28). Assume that the first inequality in (2.28) fails at $\beta_0\in[a_i,b_i]$, and call $x_0=\phi(\beta_0)$. Then by continuity we can find λ and $\delta>0$ such that

$$
\liminf_{\varepsilon\to 0+} \frac{\Phi_0(x_0+\varepsilon)-\Phi_0(x_0)}{\varepsilon} \;<\; \lambda \;<\; G^{[\xi,b_i]}(p) \qquad \text{for all } p\in[x,\ x+\delta],
$$

which implies that there exists $c_0>0$ such that

$$
\int_\xi^{b_i}\psi(\sigma)\,d\sigma+\lambda(p-p_0)\int_\xi^{b_i}\psi'(\sigma)\,d\sigma\geq c_0>0, \qquad \text{for all } p\in[x_0,\ x_0+\delta]. \tag{2.29}
$$

Choose $\varepsilon\in\,]0,\delta]$ such that the following conditions hold.

$$
\Phi_0(p) \;\geq\; \Phi_0(x+\varepsilon)+\lambda\cdot(p-x-\varepsilon) \qquad \text{for all } p\in[x_0,\ x_0+\varepsilon], \tag{2.30}
$$

$$
\beta_1 \;\doteq\; \sup\{\beta\,;\;\; \phi(\beta)<x+\varepsilon\} \;<\; b_i+\delta(\varepsilon)\,. \tag{2.31}
$$

where $\delta(\varepsilon)\downarrow 0$, as $\varepsilon\to 0$.

Let $\phi^{\xi,\varepsilon+}$ be the perturbed strategy defined by

$$
\phi^{\xi,\varepsilon+}(\beta) \;\doteq\; \begin{cases} x_0+\varepsilon & \text{if} \quad \beta\in\phi^{-1}([x_0,x_0+\varepsilon])\cap[\xi,\infty)\,,\\ \phi(\beta) & \text{otherwise.}\end{cases}
$$

One obtains

$$\begin{aligned}
&J^\flat(\phi^{\xi,\varepsilon+}) - J(\phi)\\
&\quad= \int_\xi^{\beta_1} \Big[(x_0+\varepsilon-p_0)\cdot\psi\Big(\beta+\Phi_0^\flat(x_0+\varepsilon)\Big)\\
&\qquad\qquad -(\phi(\beta)-p_0)\cdot\psi\Big(\beta+\Phi_0(\phi(\beta))\Big)\Big]\,d\beta\\
&\quad\geq \int_\xi^{\beta_1}\int_{\phi(\beta)}^{x_0+\varepsilon}\frac{d}{dp}\Big[(p-p_0)\psi\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)\Big]\,dp\,d\beta\\
&\quad= \int_{x_0}^{x_0+\varepsilon}\left(\int_\xi^{b_i}+\int_{b_i}^{\phi^{-1}(p)}\right)\Big[\psi\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)+\\
&\qquad\qquad +(p-p_0)\psi'\Big(\beta+\Phi_0^\flat(\phi(\beta))+\lambda(p-\phi(\beta))\Big)\lambda\Big]\,dp\,d\beta\\
&\quad\geq c_0\varepsilon+\varepsilon\delta(\varepsilon)=c_0\varepsilon+o(\varepsilon)>0
\end{aligned}$$

for $\varepsilon>0$ sufficiently small. Notice that the last inequality follows from (2.29) and (2.31). Using Lemma 2.2 we reach a contradiction. □

Corollary 2.4. *Assume that $\Phi_0(\cdot)$ is piecewise $\mathcal{C}^1$, and let $\phi(\cdot)$ be an optimal strategy. Then for almost every $\beta\in[0,\kappa]\setminus S$ one has*

$$\frac{d}{dp}\Phi_0(\phi(\beta)) \;=\; G^\beta(\phi(\beta)). \tag{2.32}$$

Proof. For a.e. $\beta\in[0,\kappa]\setminus S$ one has

$$\limsup_{\varepsilon\to0-}\frac{\Phi_0(\phi(\beta)+\varepsilon)-\Phi_0(\phi(\beta))}{\varepsilon}=\liminf_{\varepsilon\to0+}\frac{\Phi_0(\phi(\beta)+\varepsilon)-\Phi_0(\phi(\beta))}{\varepsilon}.$$

Hence $\frac{d}{dp}\Phi_0(\phi(\beta))$ exists for almost every $\beta\in[0,\kappa]\setminus S$ and (2.32) follows from (2.17). □

Example 2. *Assume that the random variable X has exponential distribution, so* *that*

$$\psi(s) \;=\; Prob.\{X < s\} \;=\; e^{-\lambda s}\,. \tag{2.33}$$

Let Φ_0 *be continuous, piecewise* $\mathcal{C}^1$*. If* $\phi : [0,\kappa] \mapsto [0,\overline{P}]$ *is an optimal pricing strategy, then the necessary conditions imply that the range of* ϕ *should by contained in the set*

$$\left\{p \in]p_0, \overline{P}[\,;\quad \Phi_0'(p) = \frac{1}{\lambda(p-p_0)}\right\} \cup \{\overline{P}\}\,.$$

2.4 Atomic optimal strategies

Our next goal is to prove that, if the random variable X is of type B, then any optimal pricing strategy for the new agent must be constant. Namely, all his assets should be put on sale sale at a single price.

Lemma 2.5. *Consider a random variable* X *satisfying the assumptions (A1) and (2.4). Let* $\phi : [0,\kappa] \mapsto [0,\overline{P}]$ *be a pricing strategy taking finitely many values* $p_1 < p_2 < \ldots < p_m$*. Call* ϕ^j *the constant strategy such that* $\phi^j(\beta) = p_j$ *for all* $\beta \in [0,\kappa]$*. Then*

$$J(\phi) \;\leq\; \max_{1\leq j\leq m} \left\{J(\phi^j)\right\}.$$

Proof.

1. Fix any $k \in \{1,\ldots,m-1\}$ and let $a,b,c \in [0,\kappa]$ be such that

$$\phi^{-1}(p_k) \;=\;]a,b], \qquad \phi^{-1}(p_{k+1}) \;=\;]b,c].$$

For $\theta \in [a,c]$, consider the interpolated strategy

$$\phi^\theta(\beta) \;\doteq\; \begin{cases} p_k & \text{if } \beta \in]a,\theta], \\ p_{k+1} & \text{if } \beta \in]\theta,c], \\ \phi(\beta) & \text{otherwise.} \end{cases} \tag{2.34}$$

The corresponding payoff is

$$J(\phi^\theta) \;=\; (p_k - p_0)\int_a^\theta \psi(\beta + \Phi_0(p_k))\,d\beta + (p_{k+1} - p_0)\int_\theta^c \psi(\beta + \Phi_0(p_{k+1}))\,d\beta \,+ C, \tag{2.35}$$

where C is constant with respect to θ.

2. We claim that the maximum of $J(\phi^\theta)$ in (2.35) can be attained only if $\theta = a$ or $\theta = c$. Assume, on the contrary, that there exists θ^*, such that

$$a \;<\; \theta^* \;<\; c\,, \qquad\qquad J(\phi^{\theta^*}) \;=\; \max_{\theta\in[0,\kappa]} J(\phi^\theta).$$

The optimality conditions yield

$$\frac{d}{d\theta}J(\phi^\theta)\bigg|_{\theta=\theta^*} \;=\; 0\,, \qquad\qquad \frac{d^2}{d\theta^2}J(\phi^\theta)\bigg|_{\theta=\theta^*} \;\leq\; 0\,.$$

In turn, these imply

$$\begin{cases} (p_k - p_0)\psi(\theta^* + \Phi_0(p_k)) \;=\; (p_{k+1} - p_0)\psi(\theta^* + \Phi_0(p_{k+1})), \\ \\ (p_k - p_0)\psi'(\theta^* + \Phi_0(p_k)) \;\leq\; (p_{k+1} - p_0)\psi'(\theta^* + \Phi_0(p_{k+1})). \end{cases} \tag{2.36}$$

We now recall that X is of type B, hence

$$s_1 < s_2 \quad\Longrightarrow\quad \frac{\psi'(s_1)}{\psi(s_1)} > \frac{\psi'(s_2)}{\psi(s_2)}. \tag{2.37}$$

From (2.36) we obtain

$$\frac{\psi'(\theta^* + \Phi_0(p_k))}{\psi(\theta^* + \Phi_0(p_k))} \;\leq\; \frac{\psi'(\theta^* + \Phi_0(p_{k+1}))}{\psi(\theta^* + \Phi_0(p_{k+1}))}\,. \tag{2.38}$$

Since $p_1 < p_2$, the first equality in (2.36) implies that $\psi(\theta^* + \Phi_0(p_1)) > \psi(\theta^* + \Phi_0(p_2))$, hence

$$s_1 \;\doteq\; \theta^* + \Phi_0(p_1) \;<\; \theta^* + \Phi_0(p_2) \;\doteq\; s_2\,.$$

The inequality (2.38) is thus in contradiction with (2.37). This proves our claim.

3. According to the previous step, given a pricing strategy ϕ taking m distinct values, one can construct a second strategy $\tilde{\phi}$ taking $m-1$ distinct values and yielding a payoff $J(\tilde{\phi}) \geq J(\phi)$. By induction on m, we conclude that there exists a constant pricing strategy $\phi^j(\beta) \equiv p_j$ with payoff $J(\phi^j) \geq J(\phi)$. □

Remark 4. *Consider the continuous function*

$$F(p_1,p_2,\theta,q_1,q_2) \doteq \max\left\{(p_1-p_0)\int_0^\kappa \psi(\beta+q_1)\,d\beta\,,\ (p_2-p_0)\int_0^\kappa \psi(\beta+q_2)\,d\beta\right\} -(p_1-p_0)\int_0^\theta \psi(\beta+q_1)\,d\beta-(p_2-p_0)\int_\theta^\kappa \psi(\beta+q_2)\,d\beta\,.$$

Let $\kappa_0 \doteq \Phi_0(\overline{P})$. *The proof of Lemma 2.5 shows that* $F>0$ *on the set*

$$\Omega \ \doteq\ \Big\{(p_1,p_2,\theta,q_1,q_2)\,;\quad 0\le p_1<p_2\le\overline{P}\,,\quad 0<\theta<\kappa\,,\quad 0\le q_1\le q_2\le\kappa_0\Big\}.$$

Given any $\varepsilon>0$, *consider the compact subset*

$$\Omega_\varepsilon \doteq \Big\{(p_1,p_2,\theta,q_1,q_2)\,;\ 0\le p_1\le p_2-\varepsilon\le p_2\le\overline{P}\,,\ \varepsilon<\theta<\kappa-\varepsilon\,,\ 0\le q_1\le q_2\le\kappa_0\Big\}.$$

Since F *is strictly positive on the compact set* Ω_ε, *it attains a strictly positive minimum* $\delta(\varepsilon)>0$ *on* Ω_ε. *In particular, this shows that given a positive* ε, *we can find* $\delta(\varepsilon)>0$ *such that the following holds. Assume that* $0\le p_1\le p_2-\varepsilon<p_2\le\overline{P}$ *and* $\theta\in[\varepsilon,\,\kappa-\varepsilon]$. *Then the pricing strategy* ϕ^θ *in (2.34) satisfies*

$$J(\phi^\theta)\ \le\ \max_{\alpha\in[0,\kappa]} J(\phi^\alpha)-\delta(\varepsilon). \tag{2.39}$$

Theorem 2.6. *Assume that the random variable* X *is of type B and satisfies the assumption (A1). Then, given any nondecreasing map* Φ_0, *any optimal solution* ϕ *of the problem (2.11) must be constant.*

Proof. Let ϕ be an optimal solution. Assuming that ϕ is not constant, we shall derive a contradiction.

1. Choose $\varepsilon>0$ and points $0<a<a+2\varepsilon<b<\overline{P}$ so that

$$meas\Big(\{\beta\in[0,\kappa]\,;\ \phi(\beta)<a\}\Big)\ >\ \varepsilon,\qquad meas\Big(\{\beta\in[0,\kappa]\,;\ \phi(\beta)>b+\varepsilon\}\Big)\ >\ \varepsilon\,.$$

Let $\delta(\varepsilon) > 0$ be the corresponding constant in (2.39), and choose an integer n large enough so that

$$\frac{\kappa}{n} < \min\{\varepsilon\,,\ \delta(\varepsilon)\}\,.$$

Introduce the points $p_j \doteq j/n$ and consider the approximate, piecewise constant strategy

$$\phi_n(\beta) = p_j \qquad \text{if} \quad p_j \leq \phi(\beta) < p_{j+1}\,.$$

This definition yields

$$J(\phi_n) \geq J(\phi) - \frac{\kappa}{n} > J(\phi) - \delta(\varepsilon)\,.$$

2. By construction, ϕ_n takes only finitely many values $p_0, \ldots, p_N$. Since the random variable X is of type B, by repeatedly using Lemma 2.5 we can replace the strategy ϕ_n with a strategy $\phi^\flat$ taking only three distinct values, P_1, P_2, P_3. More precisely, we can find three prices $P_1, P_2, P_3 \in \{p_1, \ldots, p_N\}$, with

$$0 < P_1 \leq a < P_2 < b - \varepsilon \leq P_3 \leq \overline{P},$$

such that the following holds. Defining the piecewise constant strategy

$$\phi^\flat(\beta) = \begin{cases} P_1 & \text{if} \quad \phi_n(\beta) \leq a\,, \\ P_2 & \text{if} \quad a < \phi_n(\beta) < b\,, \\ P_3 & \text{if} \quad \phi_n(\beta) \geq b\,, \end{cases}$$

one has $J(\phi^\flat) \geq J(\phi_n)$.

3. If now $P_2 - P_1 \leq P_3 - P_2$, we apply once again Lemma 2.5 and obtain a strategy $\phi^\sharp$ of the form

$$\phi^\sharp(\beta) = \begin{cases} Q_1 & \text{if} \quad \phi^\flat(\beta) \in \{P_1, P_2\}\,, \\ Q_2 & \text{if} \quad \phi^\flat(\beta) = P_3\,. \end{cases}$$

with

$$Q_1 \in \{P_1, P_2\}\,, \quad Q_2 = P_3\,, \qquad\qquad J(\phi^\sharp) \geq J(\phi^\flat)\,.$$

On the other hand, if now $P_2 - P_1 > P_3 - P_2$, we use Corollary 2.5 to obtain a strategy $\phi^\sharp$ of the form

$$\phi^\sharp(\beta) \;=\; \begin{cases} Q_1 & \text{if} \quad \phi^\flat(\beta) = P_1\,, \\ Q_2 & \text{if} \quad \phi^\flat(\beta) \in \{P_2, P_3\}\,. \end{cases}$$

with $Q_1 = P_1\,, Q_2 \in \{P_2, P_3\}\,, J(\phi^\sharp) \;\geq\; J(\phi^\flat)\,.$ In both cases we obtain a strategy $\phi^\sharp$ taking exactly two values Q_1, Q_2, with $Q_2 - Q_1 \geq \varepsilon$. Moreover

$$meas\Big(\{\beta \in [0,\kappa]\,;\;\; \phi^\sharp(\beta) = Q_1\}\Big) \;\geq\; \varepsilon\,, \quad meas\Big(\{\beta \in [0,\kappa]\,;\;\; \phi^\sharp(\beta) = Q_2\}\Big) \;\geq\; \varepsilon\,. \tag{2.40}$$

4. Finally, consider the two constant strategies

$$\phi_1^*(\beta) = Q_1\,, \qquad\qquad \phi_2^*(\beta) = Q_2\,.$$

By (2.40) and (2.39), we conclude

$$\max\Big\{J(\phi_1^*)\,,\; J(\phi_2^*)\Big\} \;\geq\; J(\phi^\sharp) + \delta(\varepsilon) \;\geq\; J(\phi_n) + \delta(\varepsilon) \;\geq\; J(\phi) - \frac{\kappa}{n} + \delta(\varepsilon) \;>\; J(\phi).$$

This contradicts the optimality of ϕ, proving the theorem. □

2.5 Sufficient conditions

We now consider a case where all strategies $\phi : [0, \beta] \mapsto [p_0, \overline{P}]$ which satisfy the necessary conditions stated in Theorem 2.3 are in fact optimal.

We make the following assumption on the regularity of Φ_0.

(A2) *The map $s \mapsto \Phi_0(s)$ is continuous on the half-open interval $[0, \overline{P}[$. Moreover, its derivative $\Phi_0'(p)$ is piecewise continuous.*

Theorem 2.7 (sufficient conditions)**.** *Let the assumptions (A1)-(A2) hold, and let X be a random variable of type A, so that (2.3) holds. Moreover, assume that one* *has*

$$G^\beta(p) \;\geq\; \Phi_0'(p) \quad \textit{for all} \;\; p \in [p_0, \phi(\beta)],$$
$$G^\beta(p) \;\leq\; \Phi_0'(p) \quad \textit{for all} \;\; p \in [\phi(\beta), \overline{P}]. \tag{2.41}$$

Then ϕ is optimal.

Proof. Assuming that the new agent has priority, by Theorem 2.1 an optimal strategy ϕ^* exists.

Let now ϕ be any admissible strategy which satisfies the conditions (2.41). Consider the interpolated strategy

$$\phi^\theta(\beta) \;\doteq\; \theta\phi(\beta) + (1-\theta)\phi^*(\beta). \tag{2.42}$$

Since ϕ^* is optimal, to prove that ϕ is also optimal it thus suffices to show that

$$\frac{d}{d\theta} J\Big(\theta\phi + (1-\theta)\phi^*(\beta)\Big) \;\geq\; 0\,. \tag{2.43}$$

We have

$$\frac{d}{d\theta} J(\phi^\theta) = \int_0^\kappa (\phi(\beta) - \phi^*(\beta))(\phi^\theta(\beta) - p_0)\psi'\left(\beta + \Phi_0(\phi^\theta(\beta))\right) \cdot$$

$$\left[\Phi_0'(\phi^\theta(\beta)) - G^\beta(\phi^\theta(\beta))\right] \, d\beta \;\geq\; 0.$$

Indeed, the inequality follows from the fact that $\psi'(s) < 0$ for every s, and

$$\begin{cases} \phi(\beta) \leq \phi^*(\beta) \quad \Longrightarrow \quad \phi^\theta(\beta) \geq \phi(\beta) \quad \Longrightarrow \quad \Phi_0'(\phi^\theta(\beta)) \geq G^\beta(\phi^\theta(\beta)), \\[2ex] \phi(\beta) \geq \phi^*(\beta) \quad \Longrightarrow \quad \phi^\theta(\beta) \leq \phi(\beta) \quad \Longrightarrow \quad \Phi_0'(\phi^\theta(\beta)) \leq G^\beta(\phi^\theta(\beta)). \end{cases}$$

Hence the integrand is nonnegative for every β. □

Example 3. *Assume that X is exponentially distributed, as in (2.33), and that* *there exists a subinterval* $[x_1, x_2] \subset [p_0, \overline{P}]$ *such that*

$$\Phi_0'(x) - \frac{1}{\lambda(x-p_0)} \quad \begin{cases} < \; 0 & \textit{if} \;\; x < x_1\,, \\ = \; 0 & \textit{if} \;\; x \in [x_1, x_2]\,, \\ > \; 0 & \textit{if} \;\; x > x_2\,. \end{cases}$$

Then a pricing strategy $\phi : [0, \kappa] \mapsto [0, \overline{P}]$ *is optimal if and only if it takes values inside the interval* $[x_1, x_2]$.

Indeed, in this particular case the function

$$G^{\beta}(p) \;=\; \frac{1}{\lambda(p-p_0)} \qquad \textit{for all } p \in [x_1, x_2], \quad \beta \in [0, \kappa]$$

does not depend on β. *The result follows directly from Theorem 2.7.*

Chapter 3

Nash Equilibria

This Chapter studies the existence and uniqueness of Nash equilibria for the bidding game with continuous prices for homogeneous players.

We assume that n agents compete, selling different amounts of the same asset. For $i = 1, \ldots, n$, let $\phi_i : [0, \kappa_i] \mapsto \mathbb{R}_+$ be the pricing strategy of agent i. We wish to study Nash non-cooperative equilibria, where the strategy of each player is an optimal reply to the strategies adopted by all the other players. We assume that all traders have the same payoff function: in particular, each player has the same reference price p_0 and assigns the same probability distribution to the random size X of the incoming order.

If ψ is of type A, we prove that a Nash equilibrium solution always exists, and explicitly determine the strategies of the various players. On the other hand, if ψ is of type B, we prove that no Nash equilibrium solution can exist.

3.1 Existence of a Nash Equilibrium

Definition 3. *Let $\phi_i^* : [0, \kappa_i] \mapsto [0, \overline{P}]$ be the pricing strategy of the i-th player. Define the right continuous, non-decreasing functions*

$$\Phi_i(p) \doteq \sum_{j \neq i} meas\Big(\{\beta \in [0, \kappa_j]\,; \quad \phi_j(\beta) \leq p\}\Big), \qquad i = 1, \ldots, n\,. \quad (3.1)$$

Then the n-tuple of strategies $(\phi_1^*, \ldots, \phi_n^*)$ is a **Nash equilibrium solution** *to the bidding game if each ϕ_i^* provides an optimal pricing strategy for the problem*

$$\text{Maximize:} \qquad J_i(\phi) \doteq \int_0^{\kappa_i} (\phi(\beta) - p_0)\, \psi\Big(\beta + \Phi_i(\phi(\beta))\Big)\, d\beta\,. \tag{3.2}$$

Remark 5. *The above definition does not mention the possible priority of one seller over another. Indeed, priority is irrelevant, because in any Nash equilibrium it is not possible that two sellers offer positive amounts of asset at the same price p^*. If this happens, the agent that does not have priority could offer his amount at price $p^* - \varepsilon$ with $\varepsilon > 0$ sufficiently small, and achieve a strictly larger expected payoff. This motivates our choice (3.1) of right-continuous functions Φ_i.*

As a preliminary example, given a random variable X of type A we construct the Nash equilibrium in the special case when all players have the exact same amount of shares to offer for sale.

Lemma 3.1 (Nash equilibrium for identical players)**.** *Assume that X is of type A and satisfies the assumptions (A1). Consider n players, each one putting on sale the same amount $\kappa = \kappa_1 = \cdots = \kappa_n$ of asset. Then the pricing strategies*

$$\phi_1^*(\beta) = \phi_2^*(\beta) = \cdots = \phi_n^*(\beta) = \phi(\beta), \tag{3.3}$$

with

$$\phi(\beta) \doteq p_0 + [\overline{P} - p_0] \left(\frac{\psi(n\beta)}{\psi(n\kappa)}\right)^{\frac{1-n}{n}}, \qquad \beta \in [0, \kappa], \tag{3.4}$$

provide a Nash equilibrium solution to the bidding game (3.2).

Proof. 1. Since $\psi' < 0$, the pricing strategies in (3.3)-(3.4) are strictly increasing. Moreover, for $i = 1, \ldots, n$, the functions $\Phi_1 = \cdots = \Phi_n = \Phi$ in (3.1) are all equal and satisfy

$$\Phi_i(\phi(\beta)) = \Phi(\phi(\beta)) = (n-1)\beta, \qquad \Phi'(\phi(\beta)) = \frac{n-1}{\phi'(\beta)}. \tag{3.5}$$

By (3.4)-(3.5), a direct computation shows that

$$\Phi(\overline{P}) \;=\; (n-1)\kappa\,, \qquad\qquad \Phi(p) \;=\; 0 \quad \text{for} \quad p \;\le\; p_A \;\doteq\; p_0+[\overline{P}-p_0]\,(\psi(n\kappa))^{\frac{n-1}{n}}\,,$$

$$\Phi(p) \;>\; 0, \qquad \Phi'(p) \;=\; \frac{-\psi\left(\frac{n}{n-1}\Phi(p)\right)}{(p-p_0)\psi'\left(\frac{n}{n-1}\Phi(p)\right)} \qquad\qquad \text{for} \;\; p_A < p < \overline{P}\,. \tag{3.6}$$

Here the ask price p_A is the minimum price at which some of the asset is offered for sale.

2. In order to check the necessary condition (2.32), we compute

$$G^\beta(p) \;=\; -\,\frac{\psi\,(\beta+\Phi(p)))}{(\phi(\beta)-p_0)\psi'\,(\beta+\Phi(p)))}\,.$$

By (3.5) and (3.6), this yields

$$G^\beta(\phi(\beta)) \;=\; -\,\frac{\psi\,(n\beta)}{(\phi(\beta)-p_0)\psi'\,(n\beta)} \;=\; \Phi'(\phi(\beta)), \tag{3.7}$$

showing that (2.32) holds.

3. To prove that the n-tuple of pricing strategies in (3.3)-(3.4) provides a Nash equilibrium, we need to show that each strategy satisfies the sufficient conditions for optimality (2.41).

Fix any value $\beta^* \in [0,\kappa]$ and call $p^* \doteq \phi(\beta^*)$.

Consider any two prices $p_1, p_2 \in [p_0, \overline{P}]$, with $p_1 < p^* < p_2$. As observed in Remark 2, since the random variable X is of type A, the map $\beta \mapsto G^\beta(p)$ is nondecreasing. Hence

$$G^{\beta^*}(p_2) \;\le\; G^{\beta_2}(p_2) \;=\; \Phi_0'(p_2)\,, \tag{3.8}$$

where $\beta_2 > \beta^*$ is such that $p_2 = \phi(\beta_2)$.

Next, if $p_1 > \phi(0)$, there exists $\beta_1 < \beta^*$ such that $\phi(\beta_1) = p_1$ and

$$G^{\beta^*}(p_1) \;\ge\; G^{\beta_1}(p_1) \;=\; \Phi_0'(p_1)\,. \tag{3.9}$$

On the other hand, if $p_1 \leq \phi(0)$, then $\Phi_0'(p_1) = 0$ and clearly

$$G^{\beta^*}(p_1) \;>\; \Phi_0'(p_1)\,. \tag{3.10}$$

The three inequalities (3.8), (3.9), (3.10) show that the sufficient conditions (2.41) are satisfied, and therefore $(\phi_1^*, \ldots, \phi_n^*)$ provides a Nash equilibrium. □

Remark 6. *In this Nash equilibrium the expected payoff of each agent is*

$$J(\phi) \;=\; \int_0^\kappa (\phi(\beta) - p_0)\cdot\psi(n\beta)\,d\beta \;=\; \frac{1}{n}\,(\psi(n\kappa))^{\frac{n-1}{n}}\cdot(\overline{P} - p_0)\cdot\int_0^{n\kappa}\psi(s)^{\frac{1}{n}}\,ds.$$

We now extend the previous result to an arbitrary number of players, putting on sale different amounts of the asset.

Theorem 3.2 (existence of a Nash equilibrium)**.** *Let X be a random variable of type A, satisfying the assumptions (A1). Given $n \geq 2$ players, putting on sale the amounts $\kappa_1, \ldots, \kappa_n > 0$ of the same asset, the bidding game (3.2) has a Nash equilibrium.*

Proof. 1. Without loss of generality, we can assume that

$$0 \;<\; \kappa_1 \;\leq\; \kappa_2 \;\leq\; \cdots \;\leq \kappa_n\,.$$

Define

$$h_1 \;\doteq\; \kappa_1\,, \qquad\qquad h_j \;\doteq\; \kappa_j - \kappa_{j-1} \qquad \text{if}\;\; 2 \leq j \leq n\,, \tag{3.11}$$

and, by backward induction,

$$\begin{cases} p_n \;\doteq\; \overline{P}, \\ p_j \;\doteq\; p_0 + \Big(\psi((n-j+1)h_j)\Big)^{\frac{n-j}{n-j+1}}\cdot[p_{j+1} - p_0] \;\text{ if }\; j = 1, \ldots, n-1. \end{cases} \tag{3.12}$$

We claim that a Nash equilibrium solution is provided by the following pricing strategies:

$$
\begin{aligned}
\phi_1(\beta) &\doteq p_0 + [p_2 - p_0]\left(\frac{\psi(n\kappa_1)}{\psi(n\beta)}\right)^{\frac{n-1}{n}} \qquad \beta\in[0,\kappa_1],\\
\phi_2(\beta) &\doteq \begin{cases} \phi_1(\beta) & \beta\in[0,\kappa_1],\\ p_0 + [p_3-p_0]\left(\dfrac{\psi((n-1)h_2)}{\psi((n-1)(\beta-\kappa_1))}\right)^{\frac{n-2}{n-1}} & \beta\in[\kappa_1,\kappa_2],\end{cases}\\
&\vdots\\
\phi_j(\beta) &\doteq \begin{cases} \phi_{j-1}(\beta) & \beta\in[0,\kappa_{j-1}],\\ p_0 + [p_{j+1}-p_0]\left(\dfrac{\psi((n-j+1)h_j)}{\psi((n-j+1)(\beta-\kappa_{j-1}))}\right)^{\frac{n-j}{n-j+1}} & \beta\in[\kappa_{j-1},\kappa_j],\end{cases}\\
&\vdots\\
\phi_n(\beta) &\doteq \begin{cases} \phi_{n-1}(\beta) & \text{if } \beta\in[0,\kappa_{j-1}],\\ \overline{P} & \text{if } \beta\in[\kappa_{n-1},\kappa_n].\end{cases}
\end{aligned}
\tag{3.13}
$$

2. Starting from the explicit formulas (3.13), a direct computation shows that the corresponding functions Φ_i in (3.1) satisfy

$$
0 \;\leq\; \Phi_n(p) \;\leq\; \Phi_{n-1}(p) \;\leq\; \cdots \;\leq\; \Phi_1(p), \qquad \text{for all } p\in[p_0,\overline{P}[\,. \tag{3.14}
$$

Moreover, for every $j=1,\ldots,n$ one has (see Fig. 3.1)

$$
\Phi_j(p) \;=\; \begin{cases} \Phi_n(p) & \text{for all } p\in[p_0,p_{j+1}[\,,\\ \dfrac{n+1-\ell}{n-\ell}\,\Phi_n(p) & \text{for all } p\in[p_\ell,p_{\ell+1}[\,, \quad \ell>j\,.\end{cases} \tag{3.15}
$$

To determine all functions Φ_j, it thus suffices to compute Φ_n. This is a continuous, nondecreasing, piecewise $\mathcal{C}^1$ function on $[0,\,\overline{P}[$, which satisfies

$$
\begin{aligned}
\Phi_n(p) \;&=\; 0 \quad \text{if} \quad p\leq p_1\,,\\
\Phi_n'(p) \;&=\; \frac{-\psi\left(\frac{n+1-j}{n-j}\Phi_n(p)\right)}{(p-p_0)\psi'\left(\frac{n+1-j}{n-j}\Phi_n(p)\right)} \quad \text{if} \quad p_j<p\leq p_{j+1}\,.
\end{aligned}
\tag{3.16}
$$

By (3.15) it follows

$$\Phi_j'(p) = \begin{cases} \frac{n+1-\ell}{n-\ell}\,\Phi_n(p) & p \in [p_\ell, p_{\ell+1}], \quad \ell > j\,, \\ \\ \Phi_n'(p) & p < p_{j+1}\,. \end{cases} \tag{3.17}$$

In particular, by (3.16)-(3.17) it follows that the necessary conditions $\Phi_i'(\phi_i(\beta)) = G^\beta(i(\phi_i(\beta)))$, stated in Corollary 2.4, are satisfied.

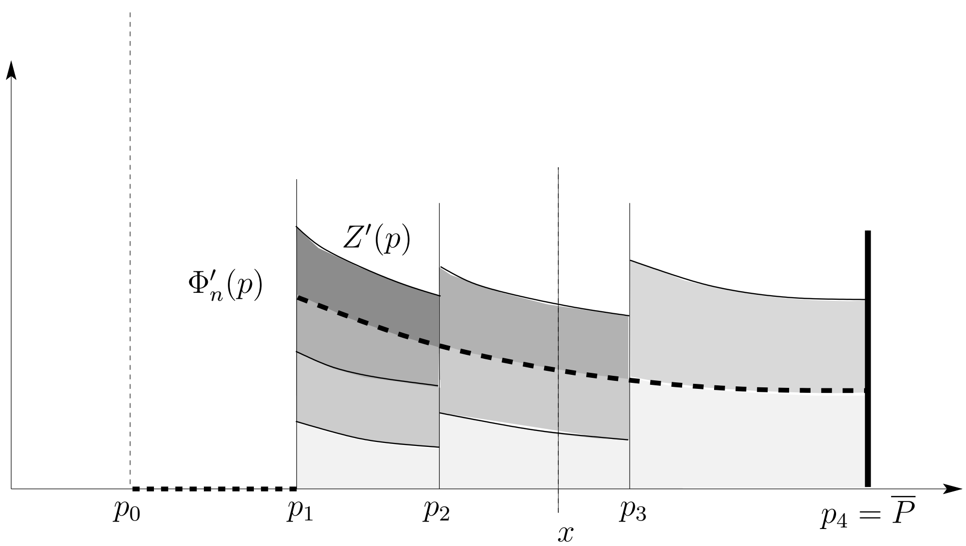

Figure 3.1. The Nash equilibrium solution in the case of 4 players. Player j prices his assets within the interval $[p_1, p_{j+1}]$. For any $x < \overline{P}$, the area of each colored region within the half-plane $\{p < x\}$ gives the amount of asset put on sale at price $\leq x$ by the corresponding player. In addition, Player 4 puts an amount $\kappa_4 - \kappa_3$ for sale at price $\overline{P}$.

3. In order to apply the sufficient condition for optimality stated in Theorem 2.7, given any $p^* = \phi_i(\beta^*)$, we need to check that

$$\begin{cases} \Phi_i'(p) \ \leq \ G_i^{\beta^*}(p) \quad \text{if} \quad p < p^*\,, \\ \\ \Phi_i'(p) \ \geq \ G_i^{\beta^*}(p) \quad \text{if} \quad p > p^*\,, \end{cases} \tag{3.18}$$

where $G_i^\beta(p)$ is defined as in (2.14), with Φ_0 replaced by Φ_i:

$$G_i^\beta(p) \ \doteq \ -\psi\Big(\beta + \Phi_i(p)\Big) \cdot \Big[(p-p_0)\psi'\Big(\beta + \Phi_i(p)\Big)\Big]^{-1}. \tag{3.19}$$

We observe that, since the random variable X is of type A, from (3.14) it follows

$$G_n^{\beta}(p) \;\leq\; G_{n-1}^{\beta}(p) \;\leq\; \cdots \;\leq\; G_1^{\beta}(p)\,. \tag{3.20}$$

To fix the ideas, assume $p^* \in [p_i, p_{i+1}]$. As in the proof of Lemma 3.1, we shall consider various cases.

CASE 1: $p < p_1$. In this case $\Phi_i'(p) = 0$ and the inequality $G_i^{\beta}(p) > \Phi_i'(p)$ is trivial.

CASE 2: $p_1 < p < p^*$. We can then find $\beta \in [0, \beta^*]$ such that $\phi_i(\beta) = p$. Since the random variable X is of type A, by Remark 2 this implies

$$G_i^{\beta^*}(p) \;\geq\; G_i^{\beta}(p) \;=\; \Phi_i'(p)\,.$$

CASE 3: $p^* < p < p_{i+1}$. Choose $\beta \in]\beta^*, \kappa_i]$ such that $\phi_i(\beta) = p$. Again by Remark 2, this implies

$$G_i^{\beta^*}(p) \;\leq\; G_i^{\beta}(p) \;=\; \Phi_i'(p)\,.$$

CASE 4: $p > p_{i+1}$. In this case, we can choose $\beta \in]\kappa_i, \kappa_n]$ such that $\phi_n(\beta) = p$. This yields

$$G_i^{\beta^*}(p) \;\leq\; G_i^{\kappa_i}(p) \;=\; G_n^{\beta}(p) \;=\; \Phi_n'(p) \;\leq\; \Phi_i'(p)\,.$$

□

Theorem 3.3 (nonexistence of a Nash equilibrium)**.** *Let X be a random variable of type B, satisfying the assumptions (A1). Then, for any number $n \geq 2$ of players offering amounts $\kappa_1, \ldots, \kappa_n > 0$ of the same asset for sale, a Nash equilibrium cannot exist (regardless of the selling priorities established among the players).*

Proof. 1. Assume, on the contrary, that a Nash equilibrium $(\phi_1^*, \ldots, \phi_n^*)$ exists. By Theorem 2.6, each pricing strategy ϕ_i^* must be constant, say

$$\phi_i^*(\beta) \;\equiv\; p_i, \qquad\qquad i = 1, \ldots, n.$$

We claim that

$$i \neq j \quad\Longrightarrow\quad p_i \neq p_j\,.$$

Otherwise, since one of the two players does not have the priority over the other, he could increase his expected payoff by pricing all his asset at $p_i - \varepsilon$, for some ε small enough.

2. Let

$$\varepsilon \doteq \min_{i \neq j} |p_i - p_j|.$$

Choose $k \in \{1, \ldots, n\}$ such that $p_k < \overline{P}$. Then the k-th player can unilaterally increase his payoff by using the strategy

$$\tilde{\phi}_k^*(\beta) = p_k + \frac{\varepsilon}{2}.$$

This contradiction shows that no Nash equilibrium can exist. □

3.2 Uniqueness of the Nash equilibrium

In this section we prove that, if the random variable X is of type A, then the Nash equilibrium constructed in Theorem 3.2 is unique.

In the following, given an n-tuple of pricing strategies $\phi : [0, \kappa_i] \mapsto [0, \overline{P}]$, we denote by

$$F_i(p) \doteq \sup \big\{ \beta \in [0, \kappa_i] \,; \quad \phi_i(\beta) \leq p \big\} \tag{3.21}$$

the amount of asset put on sale at price $\leq p$ by the i-th player. Moreover, we define

$$F(p) = \sum_{i=1}^{n} F_i(p).$$

Observe that, with these definitions, the functions Φ_i in (3.1) are expressed by

$$\Phi_i(p) = \sum_{j \neq i} F_j(p) = F(p) - F_i(p)\,.$$

Lemma 3.4. *Let the n-tuple $(\phi_1, \ldots, \phi_n)$ be a Nash equilibrium. Then the following holds.*

(i) There exists a Lipschitz constant C such that

$$F(p_2) - F(p_1) \;\leq\; C(p_2 - p_1) \qquad \textit{for all } p_0 < p_1 < p_2 < \overline{P}\,. \tag{3.22}$$

(ii) At most one of the functions F_i can have an upward jump at $p = \overline{P}$, while all the others are Lipschitz continuous on the entire interval $[0, \overline{P}]$.

(iii) There exists a minimum ask price p_A and a constant $\delta_0 > 0$ such that

$$F(p) \;=\; 0 \qquad \textit{for all } p \leq p_A\,, \qquad\qquad F'(p) \;\geq\; \delta_0 \qquad \textit{for a.e. } p \in [p_A, \overline{P}]. \tag{3.23}$$

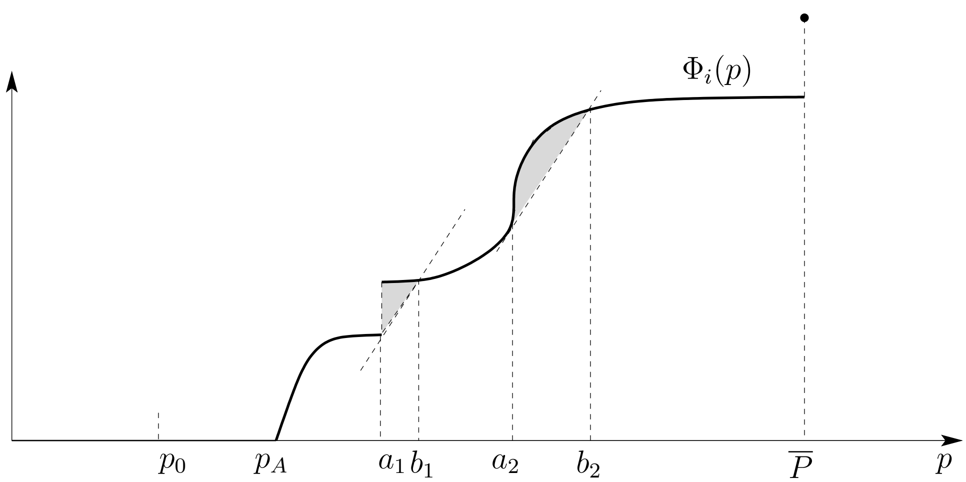

Figure 3.2. An illustration of the proof that $\Phi_i(p)$ is Lipschitz continuous in Lemma 3.4. In a Nash equilibrium, no other player can sell at a price $p \in]a_k, b_k]$.

Proof. 1. Let

$$p_* = p_0 + \psi(K)(\overline{P} - p_0) > p_0, \tag{3.24}$$

and

$$C \;\doteq\; \max\Big\{ G_i^\beta(p)\;;\; \beta \in [0, \kappa],\; p \in [p_*, \overline{P}],\; i \in \{1, 2, \ldots, n\} \Big\} + 1. \tag{3.25}$$

We claim that for every $i \in \{1, \ldots, n\}$, the set

$$S_i \;\doteq\; \Big\{ p \in [p_*, \overline{P}]\,;\quad \Phi_i(p) > \Phi_i(q) + C(p - q) \quad \text{for some} \quad q < p \Big\} \tag{3.26}$$

is empty.
Indeed, if $S_i \neq \emptyset$, we can write S_i as a union of intervals, say

$$S_i \;=\; \bigcup_k \,]a_k, b_k]\,.$$

Consider any other player, say the j-th player, with $j \neq i$. Then

$$meas\Big(\{\beta \in [0,\kappa_j]\,;\quad \phi_j(\beta) \in]a_k, b_k[\;\}\Big) \;=\; 0. \tag{3.27}$$

Otherwise, the j-th player could get a strictly higher expected payoff by using the strategy

$$\tilde{\phi}_j(\beta) \;\doteq\; \begin{cases} a_k & \text{if } \;\phi_j(\beta) \in]a_k, b_k[\,, \\ \phi_j(\beta) & \text{otherwise,} \end{cases}$$

as the following computation shows:

$$\begin{aligned} J(\tilde{\phi}_j) - J(\phi_j) \;\; &= \int_{\{\beta\;;\;\phi_j(\beta)\in[a_k,b_k]\}} (a_k - p_0)\cdot\psi(\beta + \Phi_j(a_k)) \\ &\qquad\qquad - (\phi_j(\beta) - p_0)\cdot\psi(\beta + \Phi_j(\phi_j(\beta))\,d\beta \\ &\geq \int_{\{\beta\;;\;\phi_j(\beta)\in[a_k,b_k]\}} \int_{\phi_j(\beta)}^{a_k} \frac{d}{dp}\Big((p-p_0)\psi\left(\beta + \Phi_j(a_k) - C(a_k - p)\right)\Big)\,dp\,d\beta \\ &\geq - \int_{\{\beta\;;\;\phi_j(\beta)\in[a_k,b_k]\}} \int_{a_k}^{\phi_j(\beta)} (p-p_0)\psi'\left(\beta + \Phi_j(a_k) - C(a_k - p)\right)\cdot \\ &\qquad\qquad \cdot\left(C - \frac{\psi\Big(\beta + \Phi_j(a_k) - C(a_k-p)\Big)}{(p-p_0)\psi'\Big(\beta + \Phi_j(a_k) - C(a_k-p)\Big)}\right)dp\,d\beta \\ &\geq - \int_{\{\beta\;;\;\phi_j(\beta)\in[a_k,b_k]\}} \int_{a_k}^{\phi_j(\beta)} (p-p_0)\psi'\left(\beta + \Phi_j(a_k) - C(a_k - p)\right)\cdot \\ &\qquad\qquad \cdot\left(C - G_j^{\beta}(a_k)\right)dp\,d\beta \;>\; 0 \end{aligned}$$

The first inequality follows from (3.26) and the Fundamental Theorem of Calculus, the third inequality follows from the fact that X is of Type A, and the strict inequality follows from the definition (3.25).

However, if (3.27) holds for every $j \neq i$, then the strategy ϕ_i for the i-th player is not optimal. Indeed, he could achieve a strictly higher payoff by setting

$$\tilde{\phi}_i(\beta) \doteq \begin{cases} b_k - \varepsilon & \text{if } \phi_i(\beta) \in [a_k, b_k[\,, \\ \phi_i(\beta) & \text{otherwise,} \end{cases}$$

for some $\varepsilon > 0$ sufficiently small. This proves that Φ_j is Lipschitz on the interval $[p_*, \overline{P}]$ for every $j \in \{1, \ldots, n\}$. Since

$$F = \frac{1}{n-1} \sum_{j=1}^{n} \Phi_j,$$

we conclude that F is Lipschitz continuous on $[p_*, \overline{P}[$.

Let p be a point such that $F'(p) > 0$. Then at least one agent is putting some shares on sale at the price p. From the necessary conditions (2.32) on the best reply of any of the n players, if $F' > 0$, then it satisfies the inequality

$$F'(p) \;\geq\; \Phi_j'(p) \;=\; \frac{-\psi(F)}{(p-p_0)\psi'(F)}\,, \qquad\qquad F(\overline{P}) \;\leq\; K \doteq \sum_{i=1}^{n} \kappa_i \quad p \in [p_*, \overline{P}[.$$

Denote by $Y(p)$ the solution to the terminal value problem

$$Y' \;=\; \frac{-\psi(Y)}{(p-p_0)\psi'(Y)}\,, \qquad\qquad Y(\overline{P}) \;=\; K\,.$$

By direct computation we see that

$$\psi(Y(p)) \;=\; \frac{\overline{P} - p_0}{p - p_0}\psi(K),$$

which implies that $Y(p_*) = 0$, where p_* is given by (3.24). By comparison, we see that $Y(p) \geq F(p)$ and therefore

$$p_A \;\doteq\; \inf\{p : F(p) > 0\} \geq p_* > p_0.$$

This proves the first assertion of the Lemma.

2. The second assertion is clear: if two players put a positive amount of asset for sale at the same price $\overline{P}$, the one that does not have priority can improve his expected payoff by selling the asset at price $\overline{P} - \varepsilon$.

3. Toward a proof of (iii), we show that there exists $\delta_0 > 0$ small enough so that, for any $p^* < \overline{P}$, the following implication holds:

$$F'(p^*) \leq \delta_0 \qquad \Longrightarrow \qquad F(p) = 0 \quad \text{for all} \quad p \in [0, p^*]\,. \tag{3.28}$$

Indeed, let

$$\delta_0 \doteq \frac{1}{2} \min \Big\{ G_i^\beta(p)\ ;\ \beta \in [0, \kappa],\ p \in [p_0, \overline{P}],\ i \in \{1, 2, \ldots, n\} \Big\},$$

and observe that $\delta_0 > 0$. By (i) it follows that F is differentiable at a.e. point $p \in [0, \overline{P}]$. Assume $F'(p^*) \leq \delta_0$ and consider the non-empty set

$$S^* \doteq \big\{ p < p^*\,;\quad F(p) > F(p^*) - 2\delta_0(p^* - p) \big\}\,.$$

If $F(p) = F(p^*)$ for all $p \in S^*$, recalling that F is Lipschitz continuous we conclude that $F(p) = F(p^*) = 0$ for all $p \leq p^*$, as claimed.

In the opposite case, there exist $p' < p^*$ such that

$$F(p') \ < \ F(p^*)\,, \qquad F(p) \ \geq \ F(p^*) - 2\delta_0(p^* - p) \qquad \text{for all } p \in [p', p^*]. \tag{3.29}$$

Clearly, at least one the the players is putting some assets for sale within the price interval $[p', p^*]$, say, the i-th player. This leads to a contradiction, because by (3.29)

$$\Phi_i(p) \ \geq \ \Phi_i(p^*) - 2\delta_0(p^* - p)\,,$$

Hence the strategy

$$\tilde{\phi}_i(\beta) \ = \ \begin{cases} p^* & \text{if} \quad \phi_i(\beta) \in [p', p^*]\,, \\ \phi_i(\beta) & \text{otherwise}\,, \end{cases}$$

yields a strictly higher expected payoff:

$$J(\tilde{\phi}_i) - J(\phi_i) = \int_{\{\beta\ ;\ \phi_i(\beta)\in[p',p^*]\}} \int_{\phi_i(\beta)}^{p^*} (p-p_0)\psi'\left(\beta+\Phi_i(p)\right) \cdot \Big(\Phi_i'(p) - G_i^\beta(p)\Big)\, dp\, d\beta$$

$$\geq \int_{\{\beta\ ;\ \phi_j(\beta)\in[p',p^*]\}} \int_{\phi_j(\beta)}^{p^*} (p-p_0)\psi'\left(\beta+\Phi_i(p)\right) \cdot \Big(2\delta_0 - G_i^\beta(p)\Big)\, dp\, d\beta \ > \ 0\,.$$

□

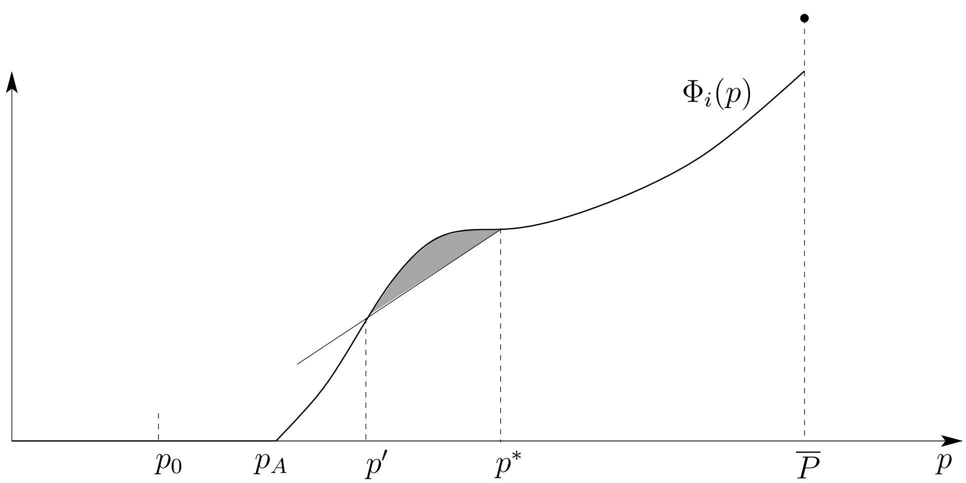

Figure 3.3. If $\Phi_i' \leq F'$ is small, then the i-th player can improve his expected payoff by asking the higher price p^* instead of a price $p \in [p', p^*]$.

Theorem 3.5. *In the same setting of Theorem 3.2, the Nash equilibrium is unique.*

Proof. 1. Let $(\phi_1, \ldots, \phi_n)$ be a Nash equilibrium. By Lemma 3.4, the corresponding functions F_i are Lipschitz continuous on $[0, \overline{P}[$, and all except at most one of them are Lipschitz continuous on the closed interval $[0, \overline{P}]$. Moreover, there exists a minimum ask price p_A such that (iii) in Lemma 3.4 holds.

2. By Rademacher's theorem, every function F_i is differentiable a.e. on $[0, \overline{P}[$.

For each p, consider the set of indices

$$\mathcal{I}(p) \doteq \{i\,;\quad F'(p) > 0\}$$

and call $N(p) \doteq \#\mathcal{I}(p)$ the cardinality of this set. By Lemma 3.4 the function $N(\cdot)$ is well defined and Lebesgue measurable. Moreover, $N(p) \geq 2$ for a.e. $p \in [p_A, \overline{P}]$.

For $p \in [p_A, \overline{P}[$, $i \in \mathcal{I}(p)$, let $\beta_i \in [0, \kappa_i]$ be such that $\phi_i(\beta_i) = p$. Recalling (2.14), from the necessary conditions (2.17) we deduce

$$\Phi_i'(p) = G_i^{\beta_i}(p) = \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))} \qquad\qquad i \in \mathcal{I}(p)\,.$$

Observing that

$$\Phi_i'(p) = \sum_{j\neq i} F_j'(p),$$

one obtains

$$\begin{cases} \Phi_i'(p) = \dfrac{N(p)-1}{N(p)} F'(p)\,, \quad F_i'(p) = \dfrac{F'(p)}{N(p)} & \text{for} \quad i \in \mathcal{I}(p)\,, \\[2ex] \Phi_i'(p) = F'(p), \qquad\qquad F_i'(p) = 0 & \text{for} \quad i \notin \mathcal{I}(p)\,. \end{cases}$$

The Lipschitz function F thus satisfies the ODE

$$F'(p) = \frac{N(p)}{N(p)-1} \cdot \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))} \tag{3.30}$$

at a.e. point $p \in [p_A, \overline{P}]$.

3. We claim that, for each $i \in \{1, \ldots, n\}$, the set of prices where the i-th player offers assets for sale is an interval $[p_A, p_{i+1}]$. Assume, on the contrary, that this is not the case. To derive a contradiction, call

$$S_i \doteq \{p \in [p_0, \overline{P}]\,;\quad F_i' > 0\,\}$$

and

$$\mathcal{L}_i \doteq \{p \in [p_0, \overline{P}]\,;\quad p \text{ is a Lebesgue point of } F_i'\,\}.$$

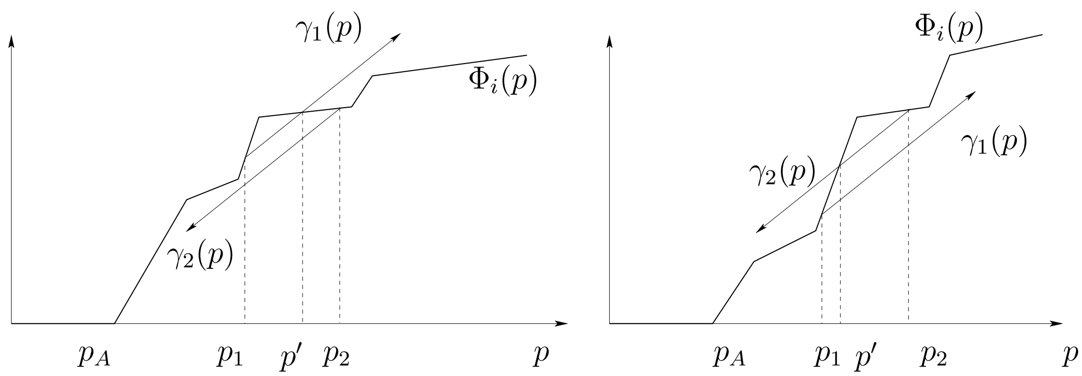

Figure 3.4. A graph of the function Φ_i. If player i sells something at price p_2 but nothing at price p_1, then his strategy is not optimal. Left: Case 1. Right: Case 2.

Let

$$q \in \big([p_A, \overline{P}] \cap \mathcal{L}_i\big) \setminus S_i$$

and assume that

$$[q, \overline{P}] \cap \mathcal{L}_i \cap S_i \neq \emptyset. \tag{3.31}$$

Let

$$q^* \doteq \inf[q, \overline{P}] \cap \mathcal{L}_i \cap S_i.$$

Then, for any $\delta_1 > 0$, the following two sets are non-empty:

$$A \doteq ([q^* - \delta_1, q^*] \cap \mathcal{L}_i) \setminus S_i \neq \emptyset, \qquad B \doteq [q^*, q^* + \delta_1] \cap \mathcal{L}_i \cap S_i \neq \emptyset.$$

Indeed, B is nonempty, by the definition of infimum. Moreover, if $q^* = q$ then $q^* \in A \neq \emptyset$, otherwise A is nonempty by the definition of infimum.

From the necessary conditions (2.32) we deduce

$$\Phi_i'(p) = F'(p) = \frac{N(p)}{N(p)-1} \cdot \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))} \geq \frac{n}{n-1} \cdot \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))}$$

for $p \notin S_i$, while

$$\Phi_i'(p) = \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))}$$

for $p \in S_i$.

Choose the intermediate slope

$$\lambda \doteq \left(\frac{2n-1}{2n-2}\right) \frac{-\psi(F(q^*))}{(q^*-p_0)\psi'(F(q^*))}\,. \tag{3.32}$$

By continuity we can choose $\delta_0 < \delta_1$ small enough so that

$$\lambda - \frac{-\psi(F(p))}{(p-p_0)\psi'(F(p))} > 0, \quad \text{for all } p \in [q^* - \delta_0, q^* + \delta_0].$$

Finally, let $p_1 \in A$ and $p_2 \in B$ be Lebesgue points of F_i' and consider the two lines

$$\gamma_1(p) = \Phi_i(p_1) + \lambda(p - p_1)\,, \qquad \gamma_2(p) = \Phi_i(p_2) + \lambda(p - p_2)\,.$$

We split the analysis into two cases (Fig. 3.4).

CASE 1: $\gamma_1 \geq \gamma_2$. We then consider the intermediate point

$$p' \doteq \min\left\{p > p_1\,; \quad \Phi_i(p) = \gamma_1(p)\right\}.$$

Observe that $p' > p_1$, because $p_1 \in \mathcal{L}_i \setminus \mathcal{S}_i$ and $\Phi_i'(p_1) > \lambda$.
Then the new pricing strategy

$$\tilde{\phi}_i(\beta) = \begin{cases} p_1 & \text{if} \quad \phi_i(\beta) \in [p_1, p']\,, \\ \phi_i(\beta) & \text{otherwise,} \end{cases}$$

yields a strictly better expected payoff:

$$\begin{aligned}
J(\tilde{\phi}_i) - J(\phi_i) &= \int_{\{\beta\ ;\ \phi_i(\beta)\in[p_1,p']\}} (p_1 - p_0)\cdot\psi(\beta + \Phi_i(p_1)) \\
&\qquad - (\phi_i(\beta) - p_0)\cdot\psi(\beta + \Phi_i(\phi_i(\beta)) \\
&\geq \int_{\{\beta\ ;\ \phi_i(\beta)\in[p_1,p']\}} \int_{\phi_i(\beta)}^{p_1} \frac{d}{dp}\Big((p-p_0)\psi\,(\beta + \gamma_1(p))\,\Big)dp\,d\beta \\
&= -\int_{\{\beta\ ;\ \phi_i(\beta)\in[p_1,p']\}} \int_{p_1}^{\phi_i(\beta)} (p-p_0)\psi'\,(\beta + \Phi_i(p))\,\cdot \\
&\qquad \cdot\left(\lambda - \frac{\psi\Big(\beta+\gamma_1(p)\Big)}{(p-p_0)\psi'\Big(\beta+\gamma_1(p)\Big)}\right)dp\,d\beta \ > \ 0.
\end{aligned}$$

CASE 2: $\gamma_1 < \gamma_2$. We then consider the intermediate point

$$p' \doteq \max\{p < p_2\,;\quad \Phi_i(p) = \gamma_2(p)\}\,.$$

An entirely similar argument now shows that the new pricing strategy

$$\tilde{\phi}_i(\beta) = \begin{cases} p' & \text{if} \quad \phi_i(\beta) \in [p', p_2]\,, \\ \phi_i(\beta) & \text{otherwise,} \end{cases}$$

yields a strictly better expected payoff.

In both cases we showed that ϕ_i is not optimal, thus reaching a contradiction.

4. From the previous step it follows

$$p_1 \ \leq\ p_2 \ \leq\ \cdots\ \leq\ p_n \ \leq\ p_{n+1} \ \leq\ \overline{P}.$$

We claim that $p_n = p_{n+1} = \overline{P}$.

Indeed, if $p_n < p_{n+1}$, this means that the n-th player is the only seller in the interval $[p_n, p_{n+1}]$. He could achieve a better expected payoff by taking all his assets originally on sale at a price $p \in [p_n, p_{n+1}]$ and offering them at the price p_{n+1} instead. This shows that $p_n = p_{n+1}$.

Finally we show that $p_n = \overline{P}$. Indeed, if this were not the case, we would have

$$F'(p) = 0, \quad \text{for all } p \in \,]p_n, \overline{P}] \, ,$$

contradicting the third statement in Lemma 3.4. □

Remark 7. *In the previous results, the requirement that the random variable X is of Type A can be weakened. Namely, instead of (2.3), it suffices to assume that the map $s \mapsto \log \psi(s)$ is convex for $0 < s < \sum_i \kappa_i$, where κ_i denotes the total amount of asset put on sale by the i-th player.*

Chapter 4

An infinite number of players

In this Chapter we study the limiting case where the number of sellers approaches infinity, but the total amount of asset offered for sale remains bounded. This model aims at representing a particular market where a very large number of homogeneous sellers are putting on sale very small quantities of the asset.

We show that the shape of the Limit Order Book converges to the solution of an ODE, which can be computed explicitly. In particular this allows us to derive a very simple, explicit formula for the price impact function.

We conclude the chapter with several examples, dealing with specific probability distributions.

4.1 An Infinite Number of Players

Example 4. *Consider the simple case of n players, each one selling the same amount K/n of asset. By (3.6) in the proof of Lemma 3.1, the total amount $Z_n(p) = \frac{n}{n-1}\Phi(p)$ of asset put on sale at price $\leq p$ is found by solving the ODE*

$$\frac{n-1}{n} Z_n' = \frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n)}, \qquad Z_n(\overline{P}) = K.$$

As $n \to \infty$, the limit distribution $Z(p) = \lim_{n\to\infty} Z_n(p)$ is clearly obtained by solving

$$Z' = \frac{-\psi(Z)}{(p-p_0)\psi'(Z)}, \qquad Z(\overline{P}) = K. \tag{4.1}$$

We wish to show that the same limit holds, without assuming that all players put on sale exactly the same amount of asset. Consider a sequence of bidding games, satisfying:

(G1) The n-th game involves n distinct players, selling the amounts $\kappa_{n,1}, \ldots, \kappa_{n,n}$ of the same asset.

(G2) The total amount of asset put on sale in the n-th game is $K_n \doteq \sum_{i=1}^{n} \kappa_{n,i}$, with $\lim_{n\to\infty} K_n = K$.

(G3) The largest amount of asset put on sale by any player in the n-th game approaches zero: $\lim_{n\to\infty} \left(\sup_{1\leq i\leq n} \kappa_{n,i}\right) = 0$.

The next result shows that, with the above assumptions, as $n \to \infty$ the limit order book approaches a well defined shape. In the following, we call $Z_n(p)$ the amount of asset offered for sale at price $< p$, in the Nash equilibrium solution (3.13) for the n-th game. Moreover, we let $Z(p)$ to be the solution to the Cauchy problem (4.1).

Observe that the right hand side of the ODE in (4.1) is well defined and uniformly positive as long as $Z \in [0, K]$. Indeed,

$$Z'(p) \;\geq\; \frac{C_0}{p - p_0}$$

for some constant $C_0 > 0$. By a comparison argument we conclude that there exists a value $p_A > p_0$ such that the solution of (4.1) satisfies

$$Z(p_A) \;=\; 0\,, \qquad\qquad Z(p) \;>\; 0 \quad \text{for} \quad p_A < p < \overline{P}\,. \tag{4.2}$$

We then extend the function Z to the entire interval $[0, \overline{P}]$ by setting

$$Z(p) \;\doteq\; 0 \qquad\qquad \text{for} \qquad p \in [0, p_A]\,. \tag{4.3}$$

Theorem 4.1. *Let X be a random variable of type A, satisfying the assumptions (A1). Consider a sequence of games for n players, satisfying (G1)–(G3).*

Then, for any $\varepsilon > 0$*, the following holds.*

$$\lim_{n\to\infty} Z_n(p) \;=\; Z(p) \qquad \textit{uniformly for all } p\in[0,\overline{P}]\,, \tag{4.4}$$

$$\lim_{n\to\infty} Z_n'(p) \;=\; Z'(p) \qquad \textit{uniformly for all } p\in[0,p_A-\varepsilon]\cup[p_A+\varepsilon\,,\;\overline{P}-\varepsilon]\,, \tag{4.5}$$

where Z *is defined by (4.1), (4.3), and* p_A *is determined by (4.2).*

Proof. 1. For a given $n \geq 1$, it is not restrictive to assume $\kappa_{n,1} \leq \kappa_{n,2} \leq \cdots \leq \kappa_{n,n}$. For $1 < i \leq n$ call $h_{n,i} \doteq \kappa_{n,i} - \kappa_{n,i-1}$. Moreover, set $h_{n,1} \doteq \kappa_{n,1}$. In the Nash equilibrium solution for the n-th game, the total amount $Z_n(p)$ put on sale at price $< p$ is characterized by the equations

$$Z_n(\overline{P}) \;=\; K_n - h_{n,n}\,, \qquad\qquad Z_n(p) \;=\; 0 \qquad \text{for}\;\; p\in[0,p_{n,1}]\,, \tag{4.6}$$

$$Z_n'(p) \;=\; \frac{n-i+1}{n-i}\cdot\frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))} \qquad \text{for}\;\; p_{n,i}<p<p_{n,i+1}\,,\quad 1\leq i<n\,. \tag{4.7}$$

Here the prices $p_{n,i}$ are determined by the inductive rule

$$p_{n,n} \;=\; \overline{P}, \qquad\qquad \int_{p_{n,i}}^{p_{n,i+1}} \frac{Z_n'(p)}{n-i+1}\,dp \;=\; h_{n,i-1} \qquad \text{for}\;\; i\geq 1\,. \tag{4.8}$$

Recalling that $\psi > 0$, $\psi' < 0$, from (4.7) we deduce

$$\frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))} \;\leq Z_n'(p) \leq\; \frac{-2\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))}\,, \qquad p_{n,1}<p<\overline{P}, \tag{4.9}$$

$$\frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))} \;\leq Z_n'(p) \leq\; \frac{-(m+1)}{m}\,\frac{\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))}, \qquad p_{n,n-m}<p<\overline{P}\,. \tag{4.10}$$

2. For any fixed $m \geq 1$, we claim that

$$p_{n,n-m} \;\to\; \overline{P}\,, \qquad\qquad Z_n(p_{n,n-m}) \;\to\; K \qquad\qquad \text{as}\quad n\to\infty. \tag{4.11}$$

Indeed, by (4.9) it follows that all maps $Z_n(\cdot)$ are increasing and uniformly Lipschitz continuous, say

$$Z_n(\overline{P})-C(\overline{P}-p) \ \le\ Z_n(p) \ \le\ Z_n(\overline{P}) \qquad\qquad \text{for all } \ p\in\left[\frac{p_A+\overline{P}}{2}\,,\ \overline{P}\right], \tag{4.12}$$

for some Lipschitz constant C. Since $Z_n(\overline{P}) = K_n - h_n \to K$ as $n\to\infty$, we can find $\delta > 0$ such that

$$\frac{K}{2} \ \le\ Z_n(p) \ \le\ 2K \qquad\qquad \text{for all } \ p\in[\overline{P}-\delta,\,\overline{P}] \tag{4.13}$$

and all n sufficiently large. By (4.8) one has

$$\begin{aligned}\int_{p_{n,n-m}}^{\overline{P}} Z_n'(p)\,dp \ &\le\ (m+1)\sum_{i=n-m}^{n-1}\int_{p_{n,i}}^{p_{n,i+1}}\frac{Z_n'(p)}{n-i+1}\,dp = (m+1)\sum_{i=n-m}^{n} h_{n,i-1}\\ &\le\ (m+1)\,(\kappa_{n,n-1}-\kappa_{n-m-1}) \ \le\ (m+1)\,\kappa_{n,n} \ \to\ 0\end{aligned} \tag{4.14}$$

as $n\to\infty$. Together, (4.13) and (4.14) imply (4.11). Indeed, using (4.10), (4.13) and the assumption (A1), it follows that, if $p_{n,n-m} < \overline{P}-\delta$, then

$$\int_{p_{n,n-m}}^{\overline{P}} Z_n'(p)\,dp \ \ge\ \int_{p_{n,n-m}}^{\overline{P}}\frac{-\psi(K/2)}{(\overline{P}-p_0)\,\psi'(Z_n(p))}\,dp \ \ge\ m_0\frac{\psi(K/2)}{(\overline{P}-p_0)}\delta\,,$$

where

$$m_0 \ \doteq\ \min_{s\in[K/2,\,K]}\ \frac{-1}{\psi'(s)} \ >\ 0.$$

By (4.14) we thus have $p_{n,n-m}\ge\overline{P}-\delta$ for all n sufficiently large. Therefore

$$\frac{K}{2}\cdot(\overline{P}-p_{n,n-m}) \ \le\ \int_{p_{n,n-m}}^{\overline{P}} Z_n'(p)\,dp \ \to\ 0,$$

showing that $p_{n,n-m}\to\overline{P}$ as $n\to\infty$. In turn, this implies

$$\begin{aligned}|Z_n(p_{n,n-m})-K| \ &\le\ |Z_n(p_{n,n-m})-Z_n(\overline{P})|+|Z_n(\overline{P})-K|\\ &\le\ C(\overline{P}-p_{n,n-m})+|K-K_n|+\kappa_{n,n} \ \to\ 0.\end{aligned}$$

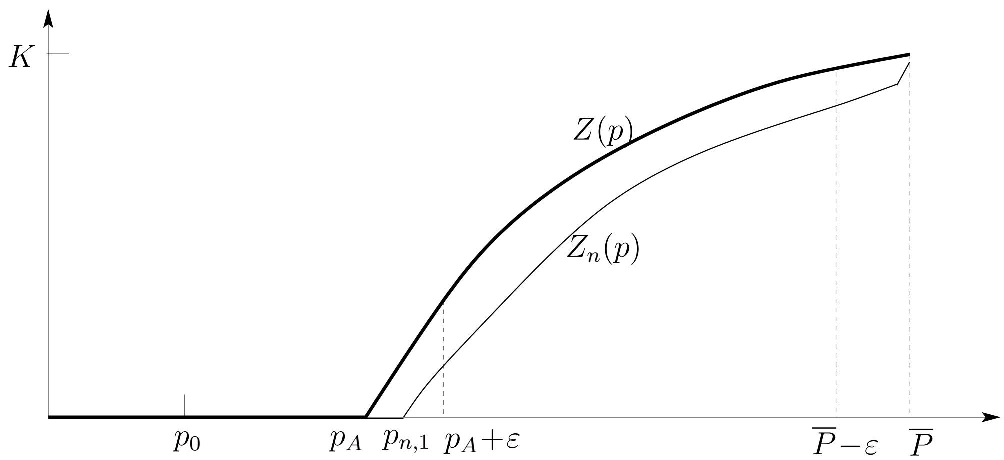

Figure 4.1. On any subinterval $[p_A+\varepsilon,\, \overline{P}-\varepsilon]$ we have the uniform convergence $Z_n(p) \to Z(p)$. Since each derivative Z_n' is uniformly positive on the region where $Z_n > 0$, this implies the convergence $p_n \to p_A$.

3. By the previous step, the function Z_n satisfies the differential inequalities

$$-\frac{m+1}{m}\frac{\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))} \;\leq\; Z_n'(p) \;\leq\; \frac{-\psi(Z_n(p))}{(p-p_0)\psi'(Z_n(p))}, \quad p_{n,1} < p < p_{n,n-m}, \tag{4.15}$$

with terminal conditions at $p = p_{n,n-m}$ satisfying (4.11). We now compare (4.15) and (4.11) with (4.1). By standard results on the continuous dependence of solutions to a Cauchy problem, for any $\varepsilon > 0$ we have the convergence (see Fig. 4.1)

$$Z_n(p) \;\to\; Z(p), \qquad\qquad Z_n'(p) \;\to\; Z'(p), \tag{4.16}$$

uniformly on the interval $[p_A+\varepsilon,\, \overline{P}-\varepsilon]$.

By (4.9), on the region where $Z_n > 0$ the derivative satisfies $Z_n'(p) \geq c_0$ for some constant $c_0 > 0$ and all $p > 0$, $n \geq 2$. Since in (4.16) we can choose $\varepsilon > 0$ arbitrarily small, we conclude that the value p_n in (4.6) satisfy

$$\lim_{n\to\infty} p_{n,1} \;=\; p_A \tag{4.17}$$

Observing that

$$\begin{aligned} Z_n(p) \;&=\; 0 \quad \text{for} \quad p \in [0, p_{n,1}]\,, \\ Z(p) \;&=\; 0 \quad \text{for} \quad p \in [0, p_A]\,, \end{aligned}$$

and that all functions Z, Z_n are uniformly Lipschitz continuous, from (4.16) and

(4.17) we deduce the convergence (4.4)-(4.5). □

4.2 Examples

In this section we consider in more detail the case when the probability distribution of size of incoming market order is given by (2.7) or (2.8).

Example 5. *Assume that the size of the incoming market order is exponentially distributed, with mean* λ^{-1}*. Two competing agents put on sale the amounts* $\kappa_1 < \kappa_2$ *of shares. The Nash equilibrium (3.13) is given by*

$$\phi_2^*(\beta) = \begin{cases} p_0 + e^{-\lambda\kappa_1+\lambda\beta} \cdot [\overline{P} - p_0], & \beta \in [0, \kappa_1]\,, \\ \overline{P} & \beta \in [\kappa_1, \kappa_2]\,, \end{cases} \tag{4.18}$$

$$\phi_1^*(\beta) = p_0 + e^{-\lambda\kappa_1+\lambda\beta} \cdot [\overline{P} - p_0], \qquad \beta \in [0, \kappa_1]\,.$$

The cumulative limit order book is thus given by

$$F(p) = \begin{cases} 0 & p \in [p_0, p_A[, \\ \dfrac{2}{\lambda} \ln \dfrac{(p - p_0)}{\overline{P} - p_0} & p \in [p_A, \overline{P}[, \\ \kappa_1 + \kappa_2 & p = \overline{P}. \end{cases}$$

This corresponds to a limit order book density

$$F'(p) = \frac{2}{\lambda(p - p_0)} \chi_{[p_A, \overline{P}]}(p) + (\kappa_2 - \kappa_1) \cdot \delta_{\overline{P}},$$

where $\delta_{\overline{P}}$ *denotes a unit Dirac mass located at* $p = \overline{P}$*, and the ask price* p_A *is given by*

$$p_A = p_0 + (\overline{P} - p_0) \cdot e^{-\lambda\kappa_1}.$$

The expected payoffs of the two agents in the Nash equilibrium configuration are *given by*

$$J_1 \;=\; \int_0^{\kappa_1} (\phi_1^*(\beta) - p_0)\cdot e^{-2\lambda\beta}\, d\beta \;=\; \frac{e^{-\lambda\kappa_1}(1-e^{-\lambda\kappa_1})(\overline{P}-p_0)}{\lambda}$$

$$J_2 \;=\; J_1 + E[(X-\kappa_1)_+ \wedge \kappa_2] \;=\; \frac{e^{-\lambda\kappa_1}(1-e^{-\lambda(\kappa_2+2\kappa_1)})(\overline{P}-p_0)}{\lambda}\,.$$

We observe that an increase in the total amount put on sale by the smaller player (hence by both players) lowers the ask price, and also decreases the expected payoff of both competitors.

$$J_1, J_2 \;\searrow\; 0, \quad \text{as } \kappa_1 \to \infty.$$

On the other hand, the larger player can increase his expected payoff by increasing the total amount of shares he puts on sale:

$$J_2 \;\nearrow\; \frac{e^{-\lambda\kappa_1}(\overline{P}-p_0)}{\lambda}, \quad \text{as } \kappa_2 \to \infty, \quad \kappa_1 \text{ fixed.}$$

Finally, using the explicit expression of the limit order book resulting from the Nash equilibrium, we can also derive an expression for the price impact function $\rho(X)$, which represents the increase in the ask price in response to a market order of size X. Indeed, $\rho(X)$ is defined by the following implicit equation

$$Z(p_A + \rho(X)) \;=\; X\,. \tag{4.19}$$

This yields

$$\rho(X) = \begin{cases} (e^{\frac{\lambda X}{2}} - 1)(\overline{P}-p_0)e^{-\lambda\kappa_1} & \text{if} \quad X \le 2\kappa_1\,, \\ \overline{P} - p_A & \text{if} \quad X > 2\kappa_1\,. \end{cases}$$

Example 6. *Consider the asymptotic limit of a large number of small agents, putting on sale a total amount of K shares. Assume that the size of the incoming market order is exponentially distributed, as in (2.7). In this case, the Cauchy problem (4.1) simplifies to*

$$Z'(p) \;=\; \frac{1}{\lambda(p-p_0)}\,, \qquad\qquad Z(\overline{P}) \;=\; K\,.$$

The expected payoff per unit amount of asset put on sale by any agent is given by

$$J_u \;=\; (\overline{P}-p_0)e^{-\lambda K}$$

The ask price is $p_A = p_0 + (\overline{P}-p_0)\cdot e^{-\lambda K}$*, while the price impact function is given by*

$$\rho(X) = \begin{cases} (e^{\lambda X}-1)(\overline{P}-p_0)e^{-\lambda K} & \text{if} \quad X \le K\,, \\ \\ \overline{P}-p_A & \text{if} \quad X > K\,. \end{cases}$$

Example 7. *Assume that the random size X of the incoming buying order is distributed according to the power law distribution (2.8). Consider n players, each one putting on sale the same amount κ of shares, for a total amount of $K = n\kappa$. The Nash equilibrium is thus given by (3.4):*

$$\phi_1^*(\beta) \;=\; \cdots \;=\; \phi_n^*(\beta) \;=\; \phi(\beta) \;\doteq\; p_0 + [\overline{P}-p_0]\cdot\left(\frac{1+n\beta}{1+n\kappa}\right)^{\frac{n-1}{n}\alpha},$$

and the corresponding ask price is

$$p_A^n \;=\; \phi(0) \;=\; p_0 + [\overline{P}-p_0]\cdot(1+n\kappa)^{\frac{1-n}{n}\alpha}.$$

The cumulative limit order book is thus given by

$$Z_n(p) \;=\; (1+n\kappa)\left(\overline{P}-p_0\right)^{-\frac{1}{\alpha}\frac{n}{n-1}}\cdot(p-p_0)^{\frac{1}{\alpha}\frac{n}{n-1}}-1, \qquad p\in[p_A^n,\overline{P}].$$

The corresponding order book density is then

$$Z_n'(p) \;=\; \frac{n}{\alpha(n-1)}\cdot(1+n\kappa)\left(\overline{P}-p_0\right)^{-\frac{1}{\alpha}\frac{n}{n-1}}\cdot(p-p_0)^{\frac{n(1-\alpha)+\alpha}{n\alpha-\alpha}}, \qquad p\in[p_A^n,\overline{P}].$$

From the above expressions we can easily compute the asymptotic limit as the number of players goes to infinity, for $K = n\kappa$ fixed. The ask price is $p_A = p_0 + [\overline{P}-p_0]\cdot(1+K)^{-\alpha}$ *and the shape of the limit order book is given by*

$$Z(p) \;=\; (1+K)\left(\overline{P}-p_0\right)^{-\frac{1}{\alpha}}\cdot(p-p_0)^{\frac{1}{\alpha}}-1, \qquad p\in[p_A,\overline{P}]\,,$$

$$Z'(p) \;=\; \frac{1}{\alpha}\cdot(1+K)\left(\overline{P}-p_0\right)^{-\frac{1}{\alpha}}\cdot(p-p_0)^{\frac{1-\alpha}{\alpha}}, \qquad\qquad p\in[p_A,\overline{P}].$$

In this case, the price impact function is given by

$$\rho(X)=\frac{\overline{P}-p_0}{(1+K)^{\alpha}}\cdot\left[(1+X)^{\alpha}-1\right], \quad X\le K.$$

Chapter 5

Discrete Prices

In this Chapter we study a model for the bidding game where there are only finitely many prices (say ν prices). This model is more realistic, since in real markets there is a positive *tick size*, and this, together with the assumption that there is an upper bound $\overline{P}$ on the set of feasible prices, implies that there is only a finite number of prices.

A pricing strategy can thus be identified with a vector taking positive values, representing how much is put on sale at each price. Since the total amount put on sale by each agent is fixed, it is clear that an optimal strategy for an agent always exists, being the maximum of a continuous payoff function over a compact subset of $\mathbb{R}^{\nu}_{+}$.

We consider here the more general model of heterogeneous players, with reference prices given by $\bar{p}_i$ and probability distributions for the random incoming order given by ψ_i. We show that if all the distributions are of type A_+ or A_0, then there exists a Nash equilibrium for the bidding game. The existence of a Nash equilibrium in the discrete model is based on the analysis of the best reply map. Under suitable assumptions, we prove that this map is upper semicontinuous with compact convex values, hence by Kakutani's theorem it has a fixed point.

We also show that, by possibly choosing a subsequence, this Nash equilibrium converges to an equilibrium for the model with continuous prices. In particular, this proves the existence of a Nash equilibrium for the continuum model also in the case of heterogeneous players.

Remark 8. *In Chapter 3, the existence of a Nash equilibrium was proved by solving a system of ODEs and explicitly constructing the optimal strategies for each player. That approach also yielded the uniqueness of the Nash equilibrium. Here we use a more standard topological technique, relying on Kakutani's fixed point theorem to a family of discrete approximation. By its nature, this approach is more general but does not yield information about the uniqueness of the Nash equilibrium.*

5.1 Discrete bidding strategies

Consider the finite set

$$\Omega_\nu \;=\; \Big\{p_k \doteq \frac{k}{\nu}\,\overline{P}\,;\quad k\in\{1,\ldots,\nu\}\Big\},$$

for some integer $\nu > 1$. We will sometimes refer to $\frac{\overline{P}}{\nu}$ as the *tick size.*

In this discrete model, a pricing strategy can thus be described by a vector

$$\mu_i \;=\; (\mu_{i1},\ldots,\mu_{i\nu}), \qquad\qquad \mu_{i\ell}\geq 0, \qquad \sum_{\ell=1}^{\nu}\mu_{i\ell} \;=\; \kappa_i\,, \tag{5.1}$$

where μ_{ik} denotes the amount offered for sale by the i-th player at the price p_k.

A major difference between the discrete and continuum case is how bids are prioritized. When prices range over all continuum values, as we showed in the previous Chapters, in a Nash equilibrium no two players offer a positive amount of asset exactly at the same price. Hence, if a buying order of size X arrives, the amounts sold by the various players are uniquely determined by the prices at which the sellers post their offers. However, in the discrete case, it may well happen that two or more players offer positive amounts of asset for sale at exactly the same price p_k. In this case, one needs to specify in which order the bids of the various players will be executed. As modeling assumption, we assume the following.

(H1) Players are given a fixed ranking. Hence, if they all put assets on sale at any given price p_ℓ, the buyer will start by taking assets from the first player, then from the second, and so on, until his order is fulfilled.

It will be convenient, as we did for the continuum model, to use the Lagrangian variable $\beta \in [0, \kappa_i]$ to describe a particular asset put on sale by the i-th agent. By a **pricing strategy** for the i-th player we mean any nondecreasing, left continuous function

$$\phi_i : [0, \kappa_i] \;\mapsto\; \Omega_\nu \;=\; \left\{ \frac{j\overline{P}}{\nu}\,, \quad j = 1, 2, \ldots, \nu \right\}. \tag{5.2}$$

Notice that ϕ_i is uniquely determined by the ν-tuple μ_i in (5.1). Indeed,

$$\phi_i(\beta) \;=\; p_k \qquad \text{if and only if} \qquad \sum_{j=1}^{k-1} \mu_{ij} \;<\; \beta \;\leq\; \sum_{j=1}^{k} \mu_{ij}\,, \tag{5.3}$$

$$\mu_{ij} \;=\; \text{meas}\Big(\{\beta \in [0, \kappa_i]\,; \quad \phi_i(\beta) = p_j\}\Big). \tag{5.4}$$

The expected payoff for the i-th player is then computed by

$$J_i(\phi_i, \Phi_i) \;\doteq\; \int_0^{\kappa_i} (\phi_i(\beta) - \overline{p}_i) \cdot \psi_i\Big(\beta + \Phi_i(\phi_i(\beta))\Big)\, d\beta\,. \tag{5.5}$$

We say that ϕ_i^* is an **optimal pricing strategy** if

$$J_i(\phi_i, \Phi_i) \;\leq\; J_i(\phi_i^*, \Phi_i)$$

for every other admissible strategy $\phi : [0, \kappa_i] \mapsto \Omega_\nu$.

Next, consider n agents offering for sale quantities $\kappa_1, \ldots, \kappa_n$ of the same asset. Let μ_{ik} be the amount put on sale at price p_k by the i-th agent. Recalling the prioritizing rule (H1), we define the functions

$$\Phi_i(p_\ell) \;\doteq\; \sum_{k<\ell,\, j\neq i} \mu_{jk} \;+\; \sum_{j<i} \mu_{j\ell}\,. \tag{5.6}$$

We say that an n-tuple of strategies $(\phi_1^*, \ldots, \phi_n^*)$ is a **Nash equilibrium** if, defining the corresponding functions Φ_i^* as in (5.6), one has

$$J_i(\phi_i, \Phi_i^*) \;\leq\; J_i(\phi_i^*, \Phi_i^*)$$

for every $i = 1, \ldots, n$ and any other pricing strategy $\phi_i : [0, \kappa_i] \mapsto \Omega_\nu$.

5.2 Necessary conditions for an optimal bidding strategy

Let a positive, nondecreasing function $\Phi_i : [0, \overline{P}] \mapsto \mathbb{R}_+$ be given, and consider the optimization problem for the i-th player, who wishes to maximize the expected payoff $J_i(\phi, \Phi_i)$ in (5.5) over all admissible strategies $\phi : [0, \kappa_i] \mapsto \Omega_\nu$. As remarked earlier, the set of admissible strategies can be identified with the set $\mathcal{S}_i$ of ν-tuples $\mu_i = (\mu_{i1}, \ldots, \mu_{i\nu})$ considered at (5.3)-(5.4).

In this section we seek necessary conditions for the optimality of μ_i. In the case of two similar players, these will provide an explicit formula describing the discrete Nash equilibrium.

Theorem 5.1. *Given a positive, nondecreasing function Φ_i let $\phi_i : [0, \kappa] \to \Omega_\nu$ be a best reply for the i-th player. If $\overline{\beta} \in]0, \kappa_i[$ is a point of jump, so that $\phi_i(\overline{\beta}+) > \phi_i(\overline{\beta}-)$, then*

$$(\phi_i(\overline{\beta}+) - \overline{p}_i) \cdot \psi_i \left(\overline{\beta} + \Phi_i(\phi_i(\overline{\beta}+))\right) \;=\; (\phi_i(\overline{\beta}-) - \overline{p}_i) \cdot \psi_i \left(\overline{\beta} + \Phi_i(\phi_i(\overline{\beta}-))\right). \tag{5.7}$$

Proof. Consider the following perturbation:

$$\phi^{\varepsilon-} \doteq \begin{cases} \phi_i(\overline{\beta}+) & \text{if} \quad \beta \in [\overline{\beta} - \varepsilon, \overline{\beta}], \\ \phi_i(\beta) & \text{otherwise}. \end{cases}$$

A direct computation yields

$$J_i(\phi^{\varepsilon-}) = \left(\int_0^{\overline{\beta}-\varepsilon} + \int_{\overline{\beta}}^{\kappa}\right) (\phi_i(\beta) - \overline{p}_i) \cdot \psi_i \left(\beta + \Phi_i(\phi_i(\beta))\right) d\beta$$

$$+ \int_{\overline{\beta}-\varepsilon}^{\overline{\beta}} (\phi_i(\overline{\beta}+) - \overline{p}_i) \cdot \psi_i \left(\beta + \Phi_i(\phi_i(\overline{\beta}+))\right) d\beta.$$

Since ϕ_i is optimal, we conclude

$$0 \;\geq\; \frac{d}{d\varepsilon} J_i(\phi^{\varepsilon -})\Big|_{\varepsilon=0} =$$

$$= -(\phi_i(\overline{\beta}-) - \overline{p}_i) \cdot \psi_i\left(\overline{\beta} + \Phi_i(\phi_i(\overline{\beta}-))\right) + (\phi_i(\beta+) - \overline{p}_i) \cdot \psi_i\left(\beta + \Phi_i(\phi_i(\beta+))\right).$$

Similarly, by considering the perturbation:

$$\phi^{\varepsilon +} \;\doteq\; \begin{cases} \phi_i(\overline{\beta}-) & \text{if} \quad \beta \in [\overline{\beta}, \overline{\beta}+\varepsilon], \\ \phi_i(\beta) & \text{otherwise}, \end{cases}$$

we obtain the converse inequality. Hence (5.7) holds. □

Remark 9. *The previous result can be restated in terms of the values μ_{ik} as follows:*

Given a positive, nondecreasing function Φ_i, let $\mu_i = (\mu_{i1}, \ldots, \mu_{i\nu})$ be a best reply for the i-th player.

If $\mu_{i\ell} > 0$ for some $\ell > 1$, then

$$(p_\ell - \overline{p}_i) \cdot \psi_i\left(\sum_{k=1}^{\ell-1} \mu_{ik} + \Phi_i(p_\ell)\right) \;\geq\; (p_{\ell-1} - \overline{p}_i) \cdot \psi_i\left(\sum_{k=1}^{\ell-1} \mu_{ik} + \Phi_i(p_{\ell-1})\right).$$

Moreover, if $\mu_{i\ell} > 0$ for some $\ell < \nu$, then

$$(p_\ell - \overline{p}_i) \cdot \psi_i\left(\sum_{k=1}^{\ell} \mu_{ik} + \Phi_i(p_\ell)\right) \;\geq\; (p_{\ell+1} - \overline{p}_i) \cdot \psi_i\left(\sum_{k=1}^{\ell} \mu_{ik} + \Phi_i(p_{\ell+1})\right).$$

In the particular case where the random incoming order is exponentially distributed, we obtain the following.

Corollary 5.2. *Assume that $\psi_i(s) = e^{-\lambda_i s}$, and let $(\mu_{i1}, \ldots, \mu_{i\nu})$ be an optimal pricing strategy for the i-th player. Then*

$$\mu_{i\ell} > 0 \quad \Longrightarrow \quad \Phi_i(p_\ell) - \Phi_i(p_{\ell-1}) \;\leq\; \frac{1}{\lambda_i} \ln\left(\frac{p_\ell - \overline{p}_i}{p_{\ell-1} - \overline{p}_i}\right),$$

$$\mu_{i,\ell-1} > 0 \quad \Longrightarrow \quad \Phi_i(p_\ell) - \Phi_i(p_{\ell-1}) \;\geq\; \frac{1}{\lambda_i} \ln\left(\frac{p_\ell - \overline{p}_i}{p_{\ell-1} - \overline{p}_i}\right).$$

5.3 Properties of the set of best replies

In this section, given a positive, nondecreasing function $\Phi_i : \Omega_\nu \mapsto \mathbb{R}$, we analyze the set of best replies for the i-th player. These are left-continuous, nondecreasing functions $\phi_i : [0, \kappa_i] \mapsto \Omega_\nu = \{p_1, p_2, \ldots, p_\nu = \overline{P}\}$ as in (5.2), which maximize the expected payoff (5.5).

To simplify our notation, given $0 < p_0 < \overline{P}$, $\kappa > 0$, and a positive nondecreasing function $\Phi_0 : \Omega_\nu \mapsto \mathbb{R}_+$, we consider the optimization problem

$$\text{maximize:} \quad J(\phi, \Phi_0) \doteq \int_0^\kappa (\phi(\beta) - p_0)\, \psi\Big(\beta + \Phi_0(\phi(\beta))\Big)\, d\beta\,. \tag{5.8}$$

The maximum is sought among all nondecreasing functions

$$\phi : \ [0, \kappa] \mapsto \Omega_\nu\,.$$

As in (5.3), setting

$$\mu_\ell \doteq \text{meas}\Big(\{\beta; \quad \phi(\beta) = p_\ell\}\Big),$$

the set of admissible strategies can be identified with the set of vectors

$$\mathcal{S} \doteq \left\{\mu = (\mu_1, \ldots, \mu_\nu)\,; \quad \mu_\ell \geq 0, \quad \sum_{\ell=1}^{\nu} \mu_\ell = \kappa\right\},$$

Clearly $\mathcal{S}$ is a compact, convex subset of $\mathbb{R}^\nu$. Setting $\mu_0 = 0$, a direct computation yields

$$J(\phi, \Phi_0) = J(\mu, \Phi_0) \doteq \sum_{k=1}^{\nu} \int_{\mu_1+\cdots+\mu_{k-1}}^{\mu_1+\cdots+\mu_k} (p_k - p_0) \cdot \psi\Big(\beta + \Phi_0(p_k)\Big)\, d\beta. \tag{5.9}$$

From (5.9) it is clear that the map $\mu \mapsto J(\mu, \Phi_0)$ is continuous. Hence it attains a maximum on the compact set $\mathcal{S}$. More precisely, the set $\mathcal{S}_{max}$ of vectors $(\mu_1, \ldots, \mu_\nu)$ where the maximum is attained is a nonempty, compact subset of $\mathcal{S}$. Aim of this section is to study the geometry of this set $\mathcal{S}_{max}$.

Lemma 5.3. *If the random variable X is of type A_+, then the optimization problem (5.8) admits a unique optimal solution.*

Proof. 1. Let $\phi_1, \phi_2 : [0, \kappa] \mapsto \{p_1, \ldots, p_\nu\}$ be two optimal strategies. If $\phi_1 \neq \phi_2$, then (by possibly permuting the indices 1,2) we can find an interval $[a, b] \subseteq [0, \kappa]$ such that

$$\begin{aligned} & [\phi_1(a-), \phi_1(a+)] \cap [\phi_2(a-), \phi_2(a+)] \;\neq\; \emptyset\,, \\ & [\phi_1(b-), \phi_1(b+)] \cap [\phi_2(b-), \phi_2(b+)] \;\neq\; \emptyset, \\ & \phi_1(\beta) \;\leq\; \phi_2(\beta) \qquad \text{for all} \;\; a < \beta < b. \end{aligned} \tag{5.10}$$

We claim that, for $\theta = a$ and for $\theta = b$, the above implies

$$\int_0^\theta (\phi_1(\beta) - p_0)\,\psi\Big(\beta + \Phi_0(\phi_1(\beta))\Big)\,d\beta \;=\; \int_0^\theta (\phi_2(\beta) - p_0)\,\psi\Big(\beta + \Phi_0(\phi_2(\beta))\Big)\,d\beta\,. \tag{5.11}$$

Indeed, if

$$\int_0^a (\phi_1(\beta) - p_0)\,\psi\Big(\beta + \Phi_0(\phi_1(\beta))\Big)\,d\beta \;>\; \int_0^a (\phi_2(\beta) - p_0)\,\psi\Big(\beta + \Phi_0(\phi_2(\beta))\Big)\,d\beta$$

then the admissible strategy

$$\tilde{\phi}(\beta) \;=\; \begin{cases} \phi_1(\beta) & \text{if} \quad \beta < a, \\ \phi_2(\beta) & \text{if} \quad \beta > a, \end{cases}$$

would yield an expected payoff strictly greater than ϕ_1 and ϕ_2. The other cases are ruled out by a similar argument.

2. If (5.10) holds, then for every $\theta \in [a, b]$ the interpolated strategy

$$\phi^\theta(\beta) \;=\; \begin{cases} \phi_1(\beta) & \text{if} \quad \beta < \theta, \\ \phi_2(\beta) & \text{if} \quad \beta > \theta, \end{cases}$$

is also admissible. We now define $J(\theta) \;\doteq\; J(\phi^\theta, \Phi_0)$. We claim that

$$J \;\in\; \mathcal{C}([a, b]) \cap \mathcal{C}^1(]a, b[). \tag{5.12}$$

Indeed,

$$J(\theta) = \int_0^\theta (\phi_1(\beta) - p_0) \cdot \psi\,(\beta + \Phi_0(\phi_1(\beta)))\,d\beta + \int_\theta^\kappa (\phi_2(\beta) - p_0) \cdot \psi\,(\beta + \Phi_0(\phi_2(\beta)))\,d\beta\,,$$

and the continuity of the map $\theta \mapsto J(\theta)$ is clear. To prove continuous differentiability, we compute

$$\frac{d}{d\theta} J(\theta) \;=\; (\phi_1(\theta) - p_0) \cdot \psi(\theta + \Phi_0(\phi_1(\theta))) - (\phi_2(\theta) - p_0) \cdot \psi(\theta + \Phi_0(\phi_2(\theta))). \tag{5.13}$$

At points where both ϕ_1 and ϕ_2 are constant, the continuity of the right hand side of (5.13) is trivial. Next, let θ be one of the finitely many points where ϕ_1 has an upward jump. Since ϕ_1 is optimal, the necessary conditions (5.7) yield

$$(\phi_1(\theta+) - p_0) \cdot \psi(\theta + \Phi_0(\phi_1(\theta+))) \;=\; (\phi_1(\theta-) - p_0) \cdot \psi(\theta + \Phi_0(\phi_1(\theta-))).$$

The same equality holds at points where ϕ_2 jumps. Hence the map $\theta \mapsto J(\theta)$ is continuously differentiable.

3. Next, consider an interior point $\theta^* \in]a, b[$ where $\frac{d}{d\theta} J(\theta) = 0$. We claim that J attains a strict local maximum at θ^*.

Indeed, choose $\delta > 0$ such that ϕ_1, ϕ_2 are both constant on the interval $]\theta^*, \theta^* + \delta]$. For $\theta \in]\theta^*, \theta^* + \delta]$ we then have

$$\frac{d^2}{d\theta^2} J(\theta) \;=$$

$$\frac{d}{d\theta}\Big[(\phi_1(\theta) - p_0) \cdot \psi(\theta + \Phi_0(\phi_1(\theta))) - (\phi_2(\theta) - p_0) \cdot \psi(\theta + \Phi_0(\phi_2(\theta)))\Big] \tag{5.14}$$

$$= \;(\phi_1(\theta) - p_0) \cdot \psi'(\theta + \Phi_0(\phi_1(\theta))) - (\phi_2(\theta) - p_0) \cdot \psi'(\theta + \Phi_0(\phi_2(\theta)))$$

$$= (\phi_1(\theta) - p_0) \cdot \psi(\theta + \Phi_0(\phi_1(\theta))) \left[\frac{\psi'(\theta + \Phi_0(\phi_1(\theta)))}{\psi(\theta + \Phi_0(\phi_1(\theta)))} - \frac{\psi'(\theta + \Phi_0(\phi_2(\theta)))}{\psi(\theta + \Phi_0(\phi_2(\theta)))} \right] +$$

$$+ \Big[(\phi_1(\theta) - p_0) \cdot \psi(\theta + \Phi_0(\phi_1(\theta))) - (\phi_2(\theta) - p_0) \cdot \psi(\theta + \Phi_0(\phi_2(\theta))) \Big] \cdot$$

$$\cdot \frac{\psi'(\theta + \Phi_0(\phi_2(\theta)))}{\psi(\theta + \Phi_0(\phi_2(\theta)))}$$

$$\doteq A(\theta) + B(\theta).$$

We now observe that

$$\lim_{\theta \to \theta^* +} A(\theta) < 0 \tag{5.15}$$

because X is of type A_+ and hence $(\ln \psi)'' > 0$. On the other hand, by (5.13)

$$\lim_{\theta \downarrow \theta^*} B(\theta) = \lim_{\theta \to \theta^* +} \Big[(\phi_1(\theta) - p_0) \cdot \psi(\theta + \Phi_0(\phi_1(\theta))) +$$

$$- (\phi_2(\theta) - p_0) \cdot \psi(\theta + \Phi_0(\phi_2(\theta))) \Big] \tag{5.16}$$

$$= \lim_{\theta \to \theta^* +} \frac{d}{d\theta} J(\theta) = 0.$$

From (5.15)-(5.16) it follows that $\frac{d^2}{d\theta^2} J(\theta) < 0$ for all $\theta \in]\theta^*, \theta^* + \varepsilon]$, with $\varepsilon > 0$ small enough. An entirely similar argument shows that $\frac{d^2}{d\theta^2} J(\theta) < 0$ also for $\theta \in [\theta^* - \varepsilon, \theta^*[$. Hence J attains a strict local maximum at θ^*.

4. As a consequence of (5.11), when $\theta = a$ and $\theta = b$ the maximum expected payoff is achieved:

$$J(\phi^a, \Phi_0) = J(\phi^b, \Phi_0) = J(\phi_1, \Phi_0) = J(\phi_2, \Phi_0).$$

Therefore, the function J in (5.12) must achieve its global minimum on $[a, b]$ at some interior point $\theta^* \in]a, b[$. This implies $\frac{dJ}{d\theta}(\theta^*) = 0.$ and hence, as proved in the previous step, J must attains a strict local maximum at θ^*. We thus reach a contradiction, proving the theorem. □

Lemma 5.4. *that the random variable X is of type A_0, with $\psi(s) = e^{-\lambda s}$. Then* *the problem (5.8) admits a compact, convex set of optimal solutions. Indeed, there is a subset $\Omega_{opt} = \{p_{i(1)}, p_{i(2)}, \ldots, p_{i(m)}\} \subseteq \Omega_\nu$ such that a strategy $(\mu_1, \ldots, \mu_\nu)$ is optimal if and only if $\mu_j = 0$ for all $j \notin \Omega^{opt}$.*

Proof. 1. Let $\mu = (\mu_1, \ldots, \mu_\nu)$ be an optimal strategy. We claim that there exists a constant C such that

$$(p_k - p_0)e^{-\lambda\Phi_0(p_k)} \;=\; C \tag{5.17}$$

for all $k \in \{1, \ldots, \nu\}$ such that $\mu_k > 0$.

Indeed, if there is only one price p_k such that $\mu_k > 0$, then the conclusion is trivial. Next, assume that μ is supported on the set $\Omega \doteq \{p_{j(1)}, p_{j(2)}, \ldots, p_{j(m)}\} \subseteq \Omega_\nu$, for some $j(1) < j(2) < \cdots < j(m)$. For $1 < \ell \leq m$, consider the perturbed strategy $\mu^\varepsilon \;\doteq\; (\mu_1^\varepsilon, \ldots \mu_\nu^\varepsilon)$ where

$$\mu_k^\varepsilon \;=\; \begin{cases} \mu_k & \text{if} \quad k \neq j(\ell-1), \quad k \neq j(\ell)\,, \\ \mu_k - \varepsilon & \text{if} \quad k = j(\ell-1), \\ \mu_k + \varepsilon & \text{if} \quad k = j(\ell). \end{cases}$$

Since μ^ε is admissible for $|\varepsilon|$ small enough, by optimality we must have

$$0 \;=\; \frac{d}{d\varepsilon}J(\mu^\varepsilon)\bigg|_{\varepsilon=0} \;=\; \lambda(p_{j(\ell)} - p_0)e^{-\lambda_i\Phi_0(p_{j(\ell)})} - \lambda(p_{j(\ell-1)} - p_0)e^{-\lambda\Phi_0(p_{j(\ell-1)})}.$$

By induction on $\ell = 2, 3, \ldots, m$, we conclude that all quantities in (5.17), for $\mu_k > 0$, coincide.

2. If $\tilde{\mu} \in \mathcal{S}$ is any strategy supported on the same set $\{p_{j(1)}, p_{j(2)}, \ldots, p_{j(m)}\}$ as μ, then $\tilde{\mu}$ is optimal as well. Indeed, thanks to (5.17) we obtain

$$J(\tilde{\mu}, \Phi_0) \;=\; C\int_0^\kappa e^{-\lambda\beta}\,d\beta \;=\; \sum_{k=1}^{\nu}\int_{\phi^{-1}(p_k)}(p_k - p_0)e^{-\lambda(\beta+\Phi_0(p_k))\,d\beta}$$

$$= C\int_0^\kappa e^{-\lambda\beta}\,d\beta \;=\; \frac{C}{\lambda}(1 - e^{\lambda\kappa}) \;=\; J(\mu, \Phi_0).$$

3. We now prove that the set of optimal strategies is convex. Let $\mu\,,\;\mu' \in \mathcal{S}$

be optimal. Denote by

$$\Omega \doteq \{p_{j(1)}, p_{j(2)}, \ldots, p_{j(m)}\}, \qquad \Omega' \doteq \{p_{j'(1)}, p_{j'(2)}, \ldots, p_{j'(m')}\},$$

the supports of μ and μ' respectively. Since μ is optimal, by (5.17) there exists a constant C such that

$$(p_{j(\ell)} - p_0)e^{-\lambda\Phi_0(p_{j(\ell)})} = C \qquad \ell = 1, \ldots, m.$$

Similarly, since μ' is optimal, there exists a constant C' such that

$$(p_{j'(\ell)} - p_0)e^{-\lambda\Phi_0(p_{j'(\ell)})} = C' \qquad \ell = 1, \ldots, m'.$$

Since μ and μ' are both optimal, they achieve the same payoff. Hence $C = C'$ and any strategy supported on $\Omega \cup \Omega'$ is also optimal. In particular, for any $\theta \in [0,1]$, the convex combination $\theta\mu + (1-\theta)\mu'$ is optimal as well. □

Lemma 5.5. *Let the random variable X be of type B, and let $\phi : [0,\kappa] \mapsto \Omega_\nu$ be an optimal strategy. Then ϕ is constant.*

Proof. By contradiction, assume that ϕ has a jump at $\beta^* \in]0,\kappa[$. Then the necessary conditions (5.7) hold. For $|\varepsilon|$ sufficiently small, define

$$\phi^\varepsilon(\beta) = \begin{cases} \phi(\beta^*+) & \text{if } \beta \in [\beta^* + \varepsilon, \beta^*] \\ \phi(\beta) & \text{otherwise} \end{cases} \qquad \text{for } \varepsilon < 0,$$

$$\phi^\varepsilon(\beta) = \begin{cases} \phi(\beta^*-) & \text{if } \beta \in [\beta^*,\, \beta^* + \varepsilon] \\ \phi(\beta) & \text{otherwise} \end{cases} \qquad \text{for } \varepsilon > 0.$$

By optimality we have

$$\frac{d}{d\varepsilon} J(\phi^\varepsilon, \Phi_0)\Big|_{\varepsilon=0} = 0. \tag{5.18}$$

For any $\varepsilon \in]0, \varepsilon_0[$ sufficiently small, a similar computation as in (5.14) now yields

$$
\begin{aligned}
&\frac{d^2}{d\varepsilon^2} J(\phi^\varepsilon, \Phi_0) = \\
&= \frac{d}{d\varepsilon}\Big[(\phi(\beta^*-) - p_0)\cdot \psi\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*-))\Big) + \\
&\qquad -(\phi(\beta^*+) - p_0)\cdot \psi\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*+))\Big)\Big] \\
&= (\phi(\beta^*-) - p_0)\cdot \psi'\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*-))\Big) + \\
&\qquad -(\phi(\beta^*+) - p_0)\cdot \psi'\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*+))\Big) \\
&= (\phi(\beta^*-) - p_0)\cdot \psi\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*-))\Big)\cdot \\
&\qquad \cdot\left[\frac{\psi'\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*-))\Big)}{\psi\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*-))\Big)} - \frac{\psi'\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*+))\Big)}{\psi\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*+))\Big)}\right] \\
&\qquad + \Big[(\phi(\beta^*-) - p_0)\cdot \psi\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*-))\Big) + \\
&-(\phi(\beta^*+) - p_0)\cdot \psi\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*+))\Big)\Big]\cdot \frac{\psi'\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*+))\Big)}{\psi\Big(\beta^* + \varepsilon + \Phi_0(\phi(\beta^*+))\Big)} \\
&\doteq A(\varepsilon) + B(\varepsilon).
\end{aligned}
$$

A similar argument as in (5.15)-(5.16), but using the fact that X is of type B and hence $(\ln \psi(s))'' < 0$ for all $s > 0$, we now obtain

$$
\lim_{\varepsilon \to 0+} A(\varepsilon) \; > \; 0, \qquad \lim_{\varepsilon \to 0+} B(\varepsilon) \; = \; 0.
$$

Therefore, for all $\varepsilon \in]0, \varepsilon_0]$ sufficiently small,

$$\frac{d^2}{d\varepsilon^2} J(\phi^\varepsilon, \Phi_0) \; > \; 0.$$

Together with (5.18), this proves that ϕ cannot be an optimal strategy. □

Assuming that the variable X is of type B, the following example shows that, for any $\kappa > 0$, one can construct a piecewise constant Φ_0 with exactly one jump such that the optimization problem (5.8) has exactly two solutions. In particular, the solution set is not convex.

Example 8. *Assume* $(\ln \psi(s))'' < 0$ *for all* $s > 0$, *and let*

$$\Phi_0(p) \;=\; \begin{cases} 0 & \text{if} \quad p < p_k\,, \\ \alpha & \text{if} \quad p \geq p_k\,. \end{cases} \tag{5.19}$$

Since by the previous Lemma any optimal pricing strategy must be constant, the only two optimal candidates are $\phi(\beta) \equiv p_{k-1}$, or $\phi(\beta) \equiv \overline{P}$. The corresponding payoffs are

$$J(p_{k-1}) = (p_{k-1} - p_0) \int_0^\kappa \psi(\beta)\, d\beta, \quad J(\overline{P}) = (\overline{P} - p_0) \int_\alpha^{\alpha+\kappa} \psi(\beta)\, d\beta.$$

Consider the function

$$f(\alpha) \;\doteq\; \int_\alpha^{\alpha+\kappa} \psi(\beta)\, d\beta.$$

It is easy to see that there exists a solution α^* to the equation $f(\alpha) = \frac{p_{k-1}-p_0}{\overline{P}-p_0} f(0)$. This follows from the fact that f is continuously differentiable, $f' < 0$, and $\lim_{\alpha\to\infty} f(\alpha) = 0$. Hence, by choosing $\alpha = \alpha^*$ in (5.19), we see that $\phi(\beta) \equiv p_{k-1}$ and $\phi(\beta) \equiv \overline{P}$ are both optimal, and this shows that the set of best replies is not connected, thus not convex.

5.4 Existence of a discrete Nash equilibrium

In this section we prove the existence of a Nash equilibrium, when each probability distribution ψ_i is either of type A_0 or of type A_+.

Theorem 5.6. *Consider a discrete pricing game for n players, with strategies given by (5.2) and payoffs as in (5.5), (5.6). Assume that the selling priorities are determined by (H1) and that each probability distribution ψ_i is either of type A_0 or A_+. Then the game admits a Nash equilibrium.*

Proof. The set of of admissible strategies for the i-th player can be identified with the compact convex set

$$\mathcal{S}_i \doteq \left\{ \mu_i = (\mu_{i1}, \ldots, \mu_{i\nu})\,; \quad \mu_{ij} \geq 0\,, \quad \sum_{j=1}^{\nu} \mu_{ij} = \kappa_i \right\}.$$

On the cartesian product $\mathcal{S} \doteq \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ consider the multifunction

$$R:\ (\mu_1, \mu_2, \ldots, \mu_n) \ \mapsto\ R_1 \times R_2 \times \cdots \times R_n\,,$$

where $R_i \subseteq \mathcal{S}_i$ is the set of all best replies for player i to the strategies adopted by the other $n-1$ players.

By the continuity of the maps J_i, which is clear from from (5.9), the multifunction R has closed graph. Moreover, by Lemmas 5.3-5.4, the sets R_i of best replies are compact, convex. Applying Kakutani's fixed point theorem (see for example [3]), we obtain an n-tuple of admissible strategies $(\mu_1^*, \ldots, \mu_n^*) \in \mathcal{S}_1 \times \cdots \times \mathcal{S}_n$ such that $\mu_i^* \in R_i(\mu_1^*, \ldots \mu_{i-1}^*, \mu_{i+1}^*, \mu_n^*)$ for every $i =, \ldots, n$. This provides a Nash equilibrium to the discrete pricing game. □

Example 9. *Consider a bidding game for two players with the same payoff functional. More precisely, assume that in (5.5) one has $\overline{p}_1 = \overline{p}_2 = \overline{p}$, $\psi_1(s) = \psi_2(s) = e^{-s}$. Assume that Player 1 has selling priority, in case both players ask the same price. For notational convenience, we write $v_\ell \doteq \mu_{1\ell}\,,\ u_\ell \doteq \mu_{2\ell}\,.$ Define the quantities*

$$\overline{v}_k \doteq \ln\frac{p_k - \overline{p}}{p_{k-1} - \overline{p}}\,, \qquad \overline{u}_k \doteq \ln\frac{p_{k+1} - \overline{p}}{p_k - \overline{p}}\,,$$

$$i^* \doteq \sup\left\{ i\ ;\ \sum_{k=i}^{n} \overline{v}_k \geq \kappa_2 \right\}, \qquad j^* \doteq \sup\left\{ j\ ;\ \sum_{k=j}^{n-1} \overline{u}_k \geq \kappa_1 \right\}.$$

Two cases can arise, depending on the total amounts κ_1, κ_2 of assets put on sale.

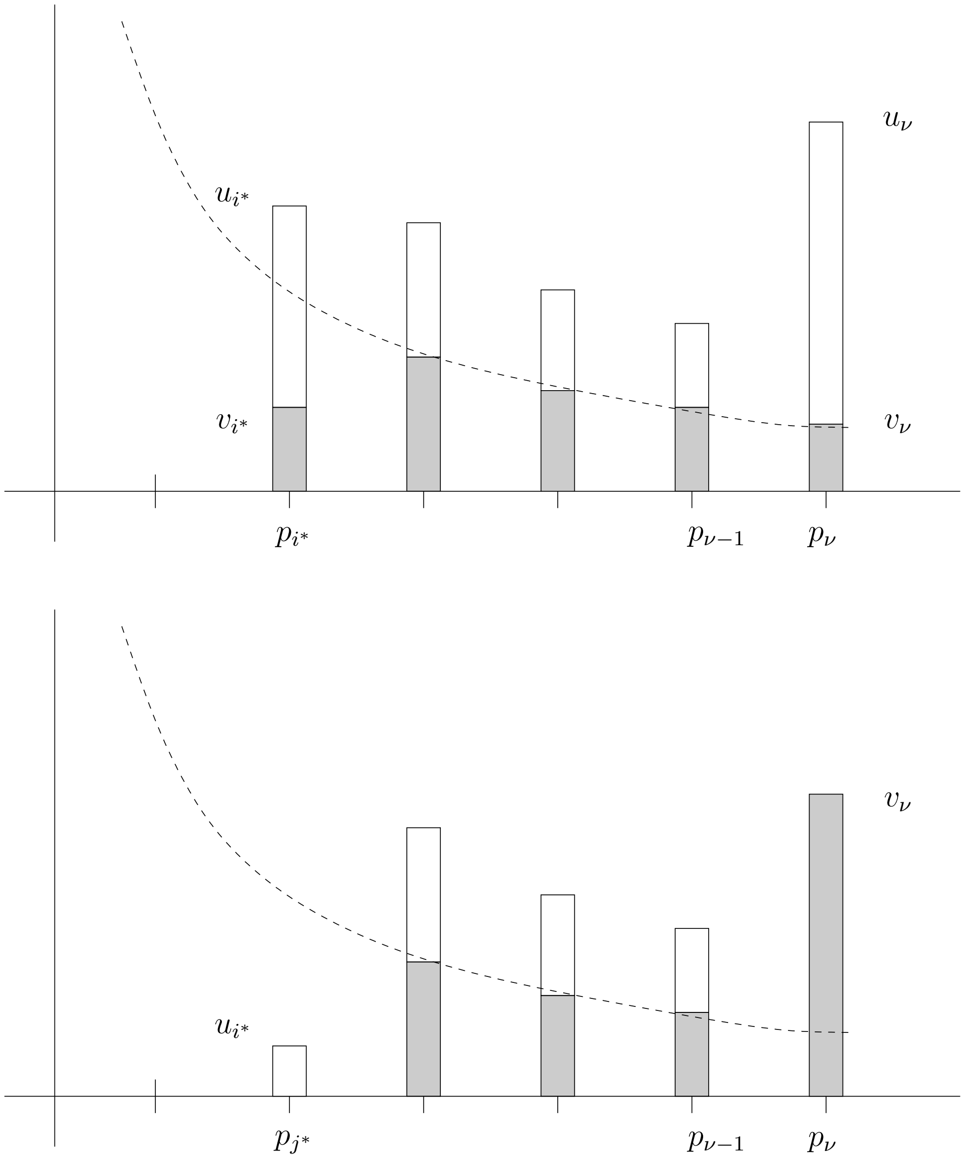

Figure 5.1. Two examples of Nash equilibria for two players with the same payoff function, selling different amounts of asset. Here $\overline{p}_1 = \overline{p}_2 = \overline{p}$, $\psi_1(s) = \psi_2(s) = e^{-s}$.

CASE 1: $\kappa_1 \geq \sum_{k=i^*}^{n-1} \overline{u}_k$. *Then a Nash equilibrium is given by:*

$$u_k = \begin{cases} 0 & k \leq i^* - 1 \\ \overline{u}_k & i^* \leq k \leq n-1 \\ \kappa_1 - \sum_{\nu=i^*}^{n-1} \overline{u}_\nu & k = n \end{cases}$$

$$v_k = \begin{cases} 0 & k \le i^* - 1 \\ \kappa_2 - \sum\limits_{\nu=i^*+1}^{n} \overline{v}_\nu & k = i^* \\ \overline{v}_k & i^* + 1 \le k \le n. \end{cases}$$

Indeed, the following optimality conditions for Player 2 are satisfied:

(i) Any pricing strategy ϕ_2 taking values within the set $\{p_{i^*}, \ldots, p_\nu\}$ yields the same payoff for Player 2;

(ii) Any strategy ϕ_2 taking values on a set which is not a subset of $\{p_{i^*}, \ldots, p_\nu\}$ yields a strictly lower payoff.

To verify (i) consider any pricing strategy $\phi_2 : [0, \kappa_2] \mapsto \{p_1, p_2, \ldots, p_\nu\}$, taking values inside the set $\{p_{i^*}, \ldots, p_\nu\}$. We have

$$\begin{aligned} J(\phi_2) &= \sum_{k=i^*}^{\nu} \int_{\phi_2^{-1}(p_k)} (p_k - \overline{p}) e^{-\beta - v_{i^*} - \ln \frac{p_k - \overline{p}}{p_{i^*} - \overline{p}}} \, d\beta \\ &= (p_{i^*} - \overline{p}) e^{-v_{i^*}} \sum_{k=i^*}^{\nu} \int_{\phi_2^{-1}(p_k)} e^{-\beta} \, d\beta \; = \; (p_{i^*} - \overline{p}) e^{-v_{i^*}} (1 - e^{-\kappa_2}). \end{aligned}$$

By the definition of i^* it follows

$$J(\phi_2) \; > \; (p_{i^*} - \overline{p}) e^{-\ln \frac{p_{i^*} - \overline{p}}{p_{i^*-1} - \overline{p}}} (1 - e^{-\kappa_2}) \; = \; (p_{i^*-1} - \overline{p})(1 - e^{-\kappa_2}),$$

which is the payoff corresponding to the constant pricing strategy $\phi_2(\beta) \equiv p_{i^*-1}$.

Consider now another strategy, $\phi_2^\varepsilon(\beta)$ where Player 2 sells an amount $\varepsilon > 0$ of shares at price p_{i^*-1}. The corresponding expected payoff is

$$\begin{aligned} J(\phi_2^\varepsilon) &= \int_0^\varepsilon (p_{i^*-1} - \overline{p}) e^{-\beta} \, d\beta + (p_{i^*} - \overline{p}) e^{-v_{i^*}} \int_\varepsilon^{\kappa_2} e^{-\beta} \, d\beta \\ &= (p_{i^*-1} - \overline{p})(1 - e^{-\varepsilon}) + (p_{i^*} - \overline{p}) e^{-v_{i^*}} (e^{-\varepsilon} - e^{-\kappa_2 + \varepsilon}). \end{aligned}$$

A simple computation shows that

$$\frac{d}{d\varepsilon}J(\phi_2^\varepsilon) \;=\; e^{-\varepsilon}[(p_{i^*-1}-\overline{p})-(p_{i^*}-\overline{p})e^{-v_{i^*}}(1+e^{-\kappa_2+2\varepsilon})]$$

$$< \;-\,(p_{i^*-1}-\overline{p})e^{-\kappa_2+\varepsilon} \;<\; 0 \;\; \textit{for all } \varepsilon > 0,$$

where we assume that ν *is large enough so that* $p_{i^*-1} > \overline{p}$ *and used the fact that* $v_{i^*} < \overline{v}_{i^*}$. *We conclude that*

$$J(\phi_2^\varepsilon) \;<\; J(\phi_2), \;\; \textit{for all } \varepsilon > 0$$

proving (ii).

With similar arguments, we can verify the same optimality conditions for Player 1. If $\phi_1(\beta)$ *is any pricing strategy taking values in* $\{p_{i^*}, \dots, p_\nu\}$, *then*

$$J(\phi_1) \;=\; \sum_{k=i^*}^{\nu}\int_{\phi_1^{-1}(p_k)}(p_k-\overline{p})e^{-\beta-\ln\frac{p_k-\overline{p}}{p_{i^*-1}-\overline{p}}}\,d\beta \;=\; (p_{i^*-1}-\overline{p})(1-e^{-\kappa_1})$$

and it is clearly not optimal to sell at lower prices, since Player 1 has the priority.

CASE 2: $\kappa_1 \le \sum_{k=i^*}^{n-1}\overline{u}_k$. *Then a Nash equilibrium is given by:*

$$u_k \;=\; \begin{cases} 0 & k \le j^*-1 \\ \kappa_1 - \sum_{\nu=j^*+1}^{n-1}\overline{u}_\nu & k = j^* \\ \overline{u}_k & j^*+1 \le k \le n-1 \\ 0 & k = n \end{cases}$$

$$v_k \;=\; \begin{cases} 0 & k \le j^* \\ \overline{v}_k & j^*+1 \le k \le n-1 \\ \kappa_2 - \sum_{\nu=j^*+1}^{n-1}\overline{v}_\nu & k = n. \end{cases}$$

Again,we can show that any strategy ϕ_1 *yields the same payoff to the first player if it takes values in the set* $\{p_{j^*+1}, \dots, p_\nu\}$ *and that the payoff for Player 2 is the* *same for any pricing strategy supported on* $\{p_{j^*},\ldots,p_{\nu-1}\}$. *Indeed, we have*

$$J(\phi_1) \;=\; \sum_{k=j^*+1}^{\nu}(p_k-\overline{p})\int_{\phi_2^{-1}(p_k)} e^{-\beta-u_{j^*}-\ln\frac{p_k-\overline{p}}{p_{j^*}-\overline{p}}} \;=\; (p_{j^*+1}-\overline{p})e^{-u_{j^*}}(1-e^{-\kappa_1}),$$

$$\begin{aligned} J(\phi_2) \;&=\; (p_{j^*}-\overline{p})\textstyle\int_0^{u_{j^*}} e^{-\beta}\,d\beta+\sum\limits_{k=j^*+1}^{\nu-1}(p_k-\overline{p})\int_{\phi_2^{-1}(p_k)} e^{-\beta-\ln\frac{p_k-\overline{p}}{p_{j^*}-\overline{p}}}\\ &=\; (p_{j^*}-\overline{p})(1-e^{-\kappa_2}). \end{aligned}$$

In Theorem 2, the assumption that every probability distribution is of type A_+ or A_0 was crucial. We now give another example showing that, if each probability distribution is of type B then a Nash equilibrium in general does not exist.

Example 10. *Consider a discrete bidding game for two players, putting on sale* $\kappa_1,\kappa_2>0$ *amounts of shares, and let* $\overline{p}_1\leq\overline{p}_2<\overline{P}$ *be given. Assume that* $(\ln\psi_i(s))''>0$ *for* $i=1,2$ *and all* $s>0$. *As usual, we assume that Player 1 has priority over Player 2. By Lemma 5.5, every optimal strategy* ϕ_i *is constant, hence any Nash equilibrium* (ϕ_1^*,ϕ_2^*) *has the form*

$$\phi_1^*(\beta)\equiv p_{j_1} \;=\; \frac{j_1\overline{P}}{\nu}\,, \qquad\qquad \phi_2^*(\beta)\equiv p_{j_2}=\frac{j_2\overline{P}}{\nu}\,,$$

for some $j_1,j_2\in\{1,\ldots,\nu\}$. *Clearly* $p_{j_1}>\overline{p}_1$ *and* $p_{j_2}>\overline{p}_2\geq\overline{p}_1$, *otherwise one of the payoffs would be negative.*

If ν *is large enough, then*

$$p_\nu\int_{\kappa_2}^{\kappa_2+\kappa_1}\psi_1(\beta)d\beta \;<\; p_{\nu-1}\int_0^{\kappa_1}\psi_1(\beta)d\beta\,, \tag{5.20}$$

$$p_\nu\int_{\kappa_1}^{\kappa_1+\kappa_2}\psi_2(\beta)d\beta \;<\; p_{\nu-1}\int_0^{\kappa_2}\psi_2(\beta)d\beta\,. \tag{5.21}$$

We now observe that

- *if* $j_1=j_2<\nu$, *then the strategy* $\phi_2\equiv p_\nu$ *yields a strictly higher expected payoff for Player 2;*
- *if* $j_1=j_2=\nu$, *then by (5.20) the strategy* $\phi_2\equiv p_{j_2-1}$ *yields a strictly higher expected payoff for Player 2;*

- *if* $j_1 < j_2$*, then the strategy* $\phi_1 \equiv p_{j_2}$ *yields a strictly higher expected payoff for Player 1;*

- *if* $j_2 < j_1 < \nu$*, then the strategy* $\phi_1 \equiv p_\nu$ *yields a strictly higher expected payoff for Player 1;*

- *if* $j_2 + 1 < j_1$*, then the strategy* $\phi_2 \equiv p_{j_2+1}$ *yields a strictly higher expected payoff for Player 2;*

- *if* $j_2 = \nu - 1$ *and* $j_1 = \nu$*, then by (5.20) the strategy* $\phi_2 \equiv p_{\nu-1}$ *yields a strictly higher expected payoff for Player 1.*

We conclude that a Nash equilibrium cannot exist.

5.5 Convergence of discrete approximations

In this section we let $\nu \to \infty$, so that the mesh size $\overline{P}/\nu$ approaches zero. We show that any weak limit of discrete Nash equilibria provides a Nash equilibrium for a bidding game where prices are allowed to range continuously over the reals.

Theorem 5.7. *Let* $\kappa_1, \ldots, \kappa_n > 0$ *be given. Assume that, for every* $\nu \geq 1$*, the* n*-tuple of strategies* $(\phi_1^\nu, \ldots, \phi_n^\nu)$ *provides a Nash equilibrium to the discrete bidding game in (5.5), (5.6). By selecting an infinite subset of indices* $I \subseteq \mathbb{N}$*, one can achieve the pointwise convergence*

$$\lim_{\nu \in I,\, \nu \to \infty} \phi_i^\nu(\beta) \;=\; \phi_i^*(\beta) \qquad \beta \in [0, \kappa_i]\,, \quad i = 1, \ldots, n\,, \tag{5.22}$$

for some nondecreasing functions $\phi_i^* : [0, \kappa_i] \mapsto [0, \overline{P}]$*. The* n*-tuple* $(\phi_1^*, \ldots, \phi_n^*)$ *provides a Nash equilibrium to the bidding game, with prices ranging continuously over the reals.*

Proof. 1. By possibly choosing a subsequence, since all functions ϕ_i^ν are nondecreasing, by Helly's compactness theorem we can assume the pointwise convergence in (5.22).

2. We claim that each limit strategy ϕ_i^* is optimal for the i-th player, in reply to the left continuous function

$$\Phi_i^-(p) \;=\; \sum_{j\neq i} \text{meas}\Big(\{\beta \in [0,\kappa_i]\,;\quad \phi_j^*(\beta) < p\}\Big). \tag{5.23}$$

Indeed, let

$$\Phi_i^\nu(p) \;=\; \sum_{j\neq i} \text{meas}\Big(\{\beta \in [0,\kappa_i]\,;\quad \phi_j^\nu(\beta) < p\}\Big).$$

Since $\Phi_i^-(p)$ is nondecreasing and left continuous, hence lower semicontinuous, for every $\beta \in [0,\kappa_i]$ the pointwise convergence $\phi_i^\nu(\beta) \to \phi^*(\beta)$ yields

$$\Phi_i^-(\phi_i^*(\beta)) \;\leq\; \liminf_{\nu\to\infty} \Phi_i^\nu(\phi_i^\nu(\beta)).$$

Therefore

$$\limsup_{\nu\to\infty}\; \psi_i(\beta + \Phi_i^\nu(\phi_i^\nu(\beta))) \;\leq\; \psi_i(\beta + \Phi_i^*(\phi_i^*(\beta))).$$

By Fatou's lemma, we conclude

$$J_i(\phi_i^*, \Phi_i^-) \;=\; \int_0^{\kappa_i} (\phi_i^*(\beta) - \overline{p}_i)\cdot \psi_i(\beta + \Phi_i^-(\phi_i^*(\beta)))\, d\beta$$

$$\geq\; \limsup_{\nu\to\infty} \int_0^{\kappa_i} (\phi_i^\nu(\beta) - \overline{p}_i)\cdot \psi_i(\beta + \Phi_i^\nu(\phi_i^\nu(\beta)))\, d\beta \;=\; \limsup_{\nu\to\infty} J_i(\phi_i^\nu, \Phi_i^\nu)\,. \tag{5.24}$$

Next, let $\phi_i : [0,\kappa_i] \mapsto [0,\overline{P}]$ be any admissible strategy for the i-th player and let $\varepsilon > 0$ be given. For each ν we can construct a (unique) discrete-valued strategy $\varphi_i^\nu : [0,\kappa_i] \mapsto \Omega_\nu$ such that

$$\phi_i(\beta) - \frac{\overline{P}}{\nu} \;<\; \varphi_i^\nu(\beta) \;\leq\; \phi_i(\beta) \qquad \text{for all} \quad \beta \in [0,\kappa_i]\,.$$

This strategy satisfies

$$J_i(\varphi_i^\nu, \Phi_i^-) \;\geq\; J_i(\phi_i, \Phi_i^-) - \kappa_i \frac{\overline{P}}{\nu}\,. \tag{5.25}$$

Indeed

$$\begin{aligned} J_i(\varphi_i^\nu, \Phi_i^-) &> \int_0^{\kappa_i} \left(\phi_i(\beta) - \frac{\overline{P}}{\nu} - \overline{p}_i\right) \psi_i(\beta + \Phi_i^-(\varphi_i^\nu(\beta)))d\beta \\ &= \int_0^{\kappa_i} \left(\phi_i(\beta) - \overline{p}_i\right) \psi_i(\beta + \Phi_i^-(\varphi_i^\nu(\beta)))d\beta - \kappa_i \frac{\overline{P}}{\nu} \\ &\geq J_i(\phi_i, \Phi_i^-) - \kappa_i \frac{\overline{P}}{\nu}\,. \end{aligned}$$

For all $\nu \geq 1$ sufficiently large we have

$$J_i(\phi_i, \Phi_i^-) \;\leq\; J_i(\varphi_i^\nu, \Phi_i^-) + \varepsilon \;\leq\; J_i(\varphi_i^\nu, \Phi_i^\nu) + 2\varepsilon \;\leq\; J_i(\phi_i^\nu, \Phi_i^\nu) + 2\varepsilon. \tag{5.26}$$

The first inequality in (5.26) is an immediate consequence of (5.25). The second follows from the lower semicontinuity of Φ_i^- and the pointwise relation $\Phi_i^-(p) \leq \liminf_{\nu\to\infty} \Phi_i^\nu(p)$. The third inequality follows from the optimality of ϕ_i^ν in response to Φ_i^ν. Together, (5.26) and (5.24) yield

$$J_i(\phi_i, \Phi_i^-) \;\leq\; \limsup_{\nu\to\infty}\; J_i(\phi_i^\nu, \Phi_i^\nu) + 2\varepsilon \;\leq\; J_i(\phi_i^*, \Phi_i^-) + 2\varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that ϕ_i^* is an optimal reply to Φ_i^-.

3. Our next goal is to prove that, for each $i = 1, \ldots, n$, the limit function Φ_i^- in (5.23) is Lipschitz continuous on $[0, \overline{P}]$. Define

$$\delta \;\doteq\; \min_j\; \psi_j(\kappa_1 + \cdots + \kappa_n) \cdot (\overline{P} - \overline{p}_j) \;>\; 0. \tag{5.27}$$

We claim that, for every $i \in \{1, \ldots, n\}$, it is never optimal for the i-th player to put anything on sale at price $p < \overline{p}_i + \delta$. Indeed, by putting the same asset for sale at price $\overline{P}$, his expected gain (per unit of asset) would be

$$\geq\; \psi_j(\kappa_1 + \cdots + \kappa_n) \cdot (\overline{P} - \overline{p}_i) \;\geq\; \delta \;>\; (p - \overline{p}_i).$$

This proves our claim.

4. Given an integer $\nu > 2\overline{P}/\delta$, let $(\mu_1^\nu, \ldots, \mu_n^\nu)$ be a discrete Nash equilibrium corresponding to the mesh size $\frac{\overline{P}}{\nu}$. Fix some $k < \nu$ and assume that some agent is selling at price p_k, i.e.

$$\sum_{i=1}^{n} \mu_{ik}^\nu \;>\; 0.$$

Let i^* be the player with lowest priority among those selling at price p_k. Then, by optimality (considering the first share he is selling at price p_k) we obtain

$$(p_k - \overline{p}_{i^*}) \cdot \psi\left(\sum_{\ell=1}^{k-1}\sum_{i=1}^{n} \mu_{i\ell}^\nu + \sum_{i<i^*} \mu_{ik}^\nu\right) > (p_{k-1} - \overline{p}_{i^*}) \cdot \psi\left(\sum_{\ell=1}^{k-2}\sum_{i=1}^{n} \mu_{i\ell}^\nu + \sum_{i\leq i^*} \mu_{i,k-1}^\nu\right)$$

By the mean value theorem, there exists some

$$\zeta \;\in\; \left[\sum_{\ell=1}^{k-2}\sum_{i=1}^{n} \mu_{i\ell}^\nu + \sum_{i\leq i^*} \mu_{i,k-1}^\nu \,,\; \sum_{\ell=1}^{k-1}\sum_{i=1}^{n} \mu_{i\ell}^\nu + \sum_{i<i^*} \mu_{i,k}^\nu\right] \;\subseteq\; \left[0, \sum_{i=1}^{n} \kappa_i\right]$$

such that

$$(p_k - \overline{p}_{i^*}) \cdot \left(\psi\left(\sum_{\ell=1}^{k-2}\sum_{i=1}^{n} \mu_{i\ell}^\nu + \sum_{i\leq i^*} \mu_{i,k-1}^\nu\right) + \psi'(\zeta) \cdot \left(\sum_{i>i^*} \mu_{i,k-1}^\nu + \sum_{i<i^*} \mu_{ik}^\nu\right)\right)$$

$$>\; (p_{k-1} - \overline{p}_{i^*}) \cdot \psi\left(\sum_{\ell=1}^{k-2}\sum_{i=1}^{n} \mu_{i\ell}^\nu + \sum_{i\leq i^*} \mu_{i,k-1}^\nu\right).$$

Hence

$$\left(\sum_{i>i^*} \mu_{i,k-1}^\nu + \sum_{i<i^*} \mu_{ik}^\nu\right) \;\leq\; \frac{\psi\left(\sum_{\ell=1}^{k-2}\sum_{i=1}^{n} \mu_{i\ell}^\nu + \sum_{i\leq i^*} \mu_{ik}^\nu\right)}{-\psi'(\zeta)} \cdot \frac{p_k - p_{k-1}}{p_k - \overline{p}_{i^*}} \;\leq\; \frac{1}{c_0} \cdot \frac{\overline{P}/\nu}{\delta}\,, \tag{5.28}$$

with $c_0 \doteq \inf\{-\psi'(\zeta)\,;\quad \zeta \in [0, \sum \kappa_i]\} > 0$, and δ as in (5.27).

In order to estimate $\mu_{i^*k}^\nu$, we observe that somebody with higher priority than i^* must be selling at price p_{k+1}, otherwise Player i^* would achieve a larger expected payoff by becoming the player with top priority at price p_{k+1}. Call $j^* < i^*$ the first seller at price p_{k+1}. Then by optimality, we have that

$$(p_{k+1} - \overline{p}_{j^*}) \cdot \psi \left(\sum_{\ell=1}^{k} \sum_{i=1}^{n} \mu_{i\ell}^{\nu} \right) > (p_k - \overline{p}_{j^*}) \cdot \psi \left(\sum_{\ell=1}^{k-1} \sum_{i=1}^{n} \mu_{i\ell}^{\nu} + \sum_{i \leq j^*} \mu_{ik}^{\nu} \right)$$

which, as before, yields the bound

$$\sum_{i > j^*} \mu_{ik}^{\nu} \leq \frac{\overline{P}}{\nu c_0 \delta}. \tag{5.29}$$

By combining (5.28) and (5.29), and recalling that $j^* < i^*$, we obtain

$$\sum_{i=1}^{n} \mu_{ik}^{\nu} \leq \frac{2\overline{P}}{\nu c_0 \delta}.$$

Finally, given any $0 < a < b < \overline{P}$, we have

$$\begin{aligned} \sum_{i=1}^{n} \mathrm{meas}\Big(\{\beta \; ; \; a < \phi_i^{\nu}(\beta) < b\} \Big) \; &= \; \sum_{a < p_k < b} \; \sum_{i=1}^{n} \mu_{ik}^{\nu} \\ &\leq \; \left(\frac{b-a}{\overline{P}/\nu} + 1 \right) \cdot \sup_{a < p_k < b} \sum_{i=1}^{n} \mu_{ik}^{\nu} \; \leq \; \frac{2}{c_0 \delta} \left(b - a + \frac{\overline{P}}{\nu} \right). \end{aligned} \tag{5.30}$$

Hence, by letting $\nu \to \infty$ we see that, for any i,

$$\mathrm{meas}\{\beta \; ; \; \phi_i^*(\beta) \in (a, b)\} \leq \frac{2}{c_0 \delta} (b - a)$$

which implies that Φ_i is Lipschitz continuous on $[0, \overline{P}[$. In particular, $\Phi_i(p) = \Phi_i^-(p)$ for $p < \overline{P}$.

5. We claim that, by possibly shrinking the countable set $I \in \mathbb{N}$, in the limit $(\phi_1^*, \ldots, \phi_n^*)$ at most one of the players puts a positive amount of assets for sale at price $\overline{P}$.

For each $\nu \geq 1$, let $(\mu_1^{\nu}, \ldots, \mu_n^{\nu})$ be a corresponding discrete Nash equilibrium. Let $\iota(\nu) \in \{1, \ldots, n\}$ denote the player with lowest priority, among those who are selling something at price $\overline{P}$:

$$\iota(\nu) \doteq \max\{i\ ;\ \mu^{\nu}_{i\nu} > 0\}.$$

By possibly choosing a further subsequence, we can assume that

$$\iota(\nu) = \iota^* \qquad \text{for all } \nu\,.$$

If $\iota^* = 1$, there is nothing to prove, since only one player is selling at the highest price. If $\iota^* > 1$, we use the optimality conditions (applied to the first share that player ι^* sells at price $\overline{P}$) to bound the amount of shares put on sale at $\overline{P}$ by the remaining players:

$$(\overline{P} - \overline{p}_{\iota^*}) \cdot \psi\left(\sum_{i=1}^{n} \kappa_i - \mu^{\nu}_{\iota^*\nu}\right) > (p_{\nu-1} - \overline{p}_{\iota^*}) \cdot \psi\left(\sum_{i=1}^{n} \kappa_i - \sum_{i=1}^{\iota^*} \mu^{\nu}_{i\nu} - \sum_{i>\iota^*} \mu^{\nu}_{i,\nu-1}\right)$$

which, as before, yields

$$\sum_{i=1}^{\iota^*-1} \mu^{\nu}_{i,\nu} \leq \frac{\overline{P}}{\nu c_0 \delta}\,. \tag{5.31}$$

As $\nu \to \infty$, the right hand side of (5.31) approaches zero. Combining (5.30) with (5.31), for every $\varepsilon > 0$ and $i \neq \iota^*$ we obtain

$$\begin{aligned}
&\text{meas}\Big(\{\beta \in [0,\kappa_i]\,;\ \phi^*_i(\beta) = \overline{P}\}\Big) \\
&\qquad \leq \limsup_{\nu\to\infty}\ \text{meas}\Big(\{\beta \in [0,\kappa_i]\,;\ \phi^{\nu}_i(\beta) \in [\overline{P} - \varepsilon,\ \overline{P}]\}\Big) < \frac{2\varepsilon}{c_0\delta}\,.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, for $i \neq \iota^*$ one has

$$\text{meas}\Big(\{\beta \in [0,\kappa_i]\,;\ \ \phi^*_i(\beta) = \overline{P}\}\Big) = 0,$$

proving our claim.

6. According to the previous analysis, for each $i \neq \iota^*$ the map

$$p \mapsto \text{meas}\Big(\{\beta \in [0,\kappa_i]\,;\ \ \phi^*_i(\beta) \leq p\}\Big)$$

is Lipschitz continuous. As a consequence,

$$J(\phi_i^*, \Phi_i^*) \ = \ J(\phi_i^*, \Phi_i^-) \qquad\qquad \text{for all} \quad i = 1, \ldots, n.$$

Recalling step 2 and observing that $\Phi_i^-(p) \leq \Phi_i^*(p)$, for every i and every admissible strategy $\phi_i : [0, \kappa_i]$ we have

$$J(\phi_i, \Phi_i^*) \ \leq \ J(\phi_i, \Phi_i^-) \ \leq \ J(\phi_i^*, \Phi_i^-) \ = \ J(\phi_i^*, \Phi_i^*).$$

proving that the n-tuple of pricing strategies $(\phi_1^*, \ldots, \phi_n^*)$ provides a Nash equilibrium. □

Remark 10. *In the case where all players have the same payoff function and assign the same probability distribution to the random incoming order X, it was proved in [4] that the Nash equilibrium in the continuum model is unique. In this case, the entire sequence of discrete Nash equilibria must converge to this unique limit.*

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