ESSAYS ON OPTIMAL FEASIBLE TAXATION

A Thesis in
Economics
by
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Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2004
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Abstract

In the first essay, we study the following question: If the provision of public good is financed by property (wealth or endowment) taxes, what is the optimal tax mechanism when agents have private information about his own endowment? Employing the state-dependent implementation model, under incentive compatibility and feasibility constraints, we provide the full characterization of an optimal tax mechanism with two agents and its properties. Some interesting comparative statics analyses are also provided. For the general $n$-agent case, some partial characterization results are obtained. In addition, we fully characterize the optimal tax mechanism for the corresponding infinitely large economy.

The main goal of the second essay is to extend the model of the first essay to a heterogeneous economy in which agents have some different wealth characteristics that are publicly observable, e.g., race. Using the same analysis as in the first essay where a homogeneous economy is studied, we provide the full characterization of an optimal tax mechanism for such a heterogeneous economy with two agents and its properties. In addition to the similar results in the first essay, we find that if the level of low endowment is low enough, only the incentive compatibility constraint of a minority agent is binding.

The third essay studies the optimal formation of local public good economies and the immigration incentive of an agent. Using the characterizations of optimal tax mechanisms in the first and second essays, we compare the social welfare between homogeneous formation for which each local community consists of agents with same observable characteristics, e.g., race, and heterogeneous formation for which each local community consists of agents with different observable characteristics. The comparison shows that if the expected endowment of the economy is low enough, homogeneous formation is optimal, while otherwise heterogeneous formation is optimal. We also study the immigration incentive and find that a minority (resp. majority) agent will choose a heterogeneous (resp. homogeneous) formation regardless of his endowment.
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Acknowledgements

First of all, I am very grateful to Professor Tomas Sjöström, my advisor, for his invaluable guidance, comments, and patience. I am indebted to my dissertation committee members, Professors Kalyan Chatterjee, Steven Huddart, and James Jordan. I also thank Neil Wallace for his encouragement. My special gratitude belongs to Professor Emeritus John Riew for his encouragement and guidance. Finally, I wish to express my deepest love to my wife, Sungran, and daughter, Jane.
Chapter I

A Characterization of Optimal Feasible Tax Mechanism

I.1 Introduction

This essay is motivated by a practical property (wealth or endowment) taxation problem: For an economy where the provision of public goods is to be financed by property taxes collected from agents, what is the optimal tax mechanism when a social planner does not know the agents’ property? What is important in this mechanism design problem is that we have to take into account not only the agents’ incentive to report their wealth truthfully, but also the feasibility of the designed tax schedule in the sense that each agent’s tax payment should be within their ability to pay. To see the problem of feasibility, consider the following simple example. Two thieves, Ali and Baba, want to build a door for their treasure cave. The quality of the door depends on the total contributions they make. Suppose that Ali is rich and has $200, and Baba is poor and has $100, but neither of them knows how much the other has. A social planner, who does not know how much Ali and Baba have, asks them to report their wealth in order to determine their contributions (taxes). What is the optimal tax mechanism that maximizes the expected sum of Ali and Baba’s utilities? If the social planner wants to collect $300 for the door-building, she cannot impose $150 to each of them because it is not (individually) feasible to Baba.

The theory of optimal taxation has a long history, but it was Mirrlees [1971] that started the modern optimal taxation theory that incorporates the He considered labor income taxation for an infinitely large economy and studied the optimality of redistributive taxation when each individual has private information about his own productivity. He showed that the redistributive

1 Throughout this dissertation, we use the terms wealth, endowment, or property of an agent interchangeably since they have the same meaning, the amount of resources the agent initially has.
2 This kind of problem is frequently observed in a small economy such as a club or village. Fund-raising is another example. See Andreoni [1998].
tax mechanism is subject to suboptimality due to the informational asymmetry between the public-
policy authority and the individuals. This situation is now well understood as a *second best taxation.* Following Mirrlees [1971], many authors have studied the characterization of optimal taxation and its properties, especially on the trade-off relationship between efficiency and equity.\(^3\) However, most of these works have assumed that there is a continuum of individuals and the tax schedule depends on an observable variable such as labor income so that there are no individual feasibility problems. Also, they have not provided full characterization results of optimal tax schedules.

The theory of implementation (or mechanism design), pioneered by Hurwicz [1972] and Maskin [1999], studies the implementability of various social choice rules and the characterization of the implementing mechanisms under different environments and informational assumptions.\(^4\) Most of results in this literature, however, assume that the set of feasible outcomes is fixed and common knowledge so that this set does not depend on the realization of the economic environment. This is a quite restrictive assumption, in particular, when agents have private information about their own endowments or production technologies. If a social planner is uninformed of the realization of agents’ endowments, she has to consider the feasibility problem when designing an implementing mechanism. The first study that tackles this type of state-dependent allocation problem was Hurwicz, Maskin and Postlewaite [1995].\(^5\) They studied the feasible implementation problem under complete information in which a social planner does not know the realization of agents’ endowments or production sets. Following them, there have been some extensions of their model to incomplete information cases.\(^6\) However, those results have mainly focused on the implementability of a general social choice rule, but not on the efficiency of the implementing mechanism, which is the main subject of this paper.

In this essay, we employ the so-called “endowment game” created by Hurwicz, Maskin and Postlewaite [1995] to model the optimal taxation problem of a public good economy with a finite number of agents. That is, under incentive compatibility and feasibility constraints, we set up the (Bayesian) mechanism design problem of a utilitarian social planner who is uninformed of the agents’ endowments. Since the number of agents in the economy is finite, each agent’s tax payment

---

\(^3\) For a survey of modern optimal taxation theory, see Stiglitz [1985]. Some authors have studied the equivalence of (optimal) taxation and mechanism design, called *taxation principle.* See Guesnerie [1995] and Rochet [1985].

\(^4\) For some recent surveys, see, e.g., Jackson [2000a, b], Palfrey [2002] and Maskin and Sjöström [2002].

\(^5\) The earlier version of this paper has been circulated since 1979.

\(^6\) See for example Hong [1996, 1998] and Tian [1999]. See also Dagan, Serreno, and Volij [1999], in which they study the feasible implementation of a given taxation method. However, their work considers the taxation problem from the *equity* point of view so that the total amount of taxes to be collected is exogenously given. In contrast, our model deals with the efficiency of a tax mechanism which endogenously determines the each agent’s tax payment as well as the total amount of taxes.
will depend not only on his own endowment but also on the other agents’ endowments.

We begin with the case of two agents and two potential types, and fully characterize its solution. Due to the low dimensionality of the problem, the solution can be illustrated graphically. The main implications of the characterization are as follows: First, if the expected total endowment of the economy is low enough or high enough, then incentive compatibility is no longer a matter so that first best feasible taxation can be obtained. This follows because a rich agent will not pretend to be poor for fear that too little of public good should be provided if the economy is expected to be poor enough. On other other hand, if the economy is rich enough, then the first best uniform taxation which does not involve any incentive problem can be implemented. Second, for the cases in which the incentive compatibility constraint does bind, the optimal tax mechanism imposes a high tax rate on a poor agent when his neighbor is rich. The intuition behind this regressive taxation is that levying a tax on the poor agent does not cause an incentive problem so that the social planner, who does not mind which agent pays what proportion of the total taxes, prefers to impose as much tax as possible on the poor agent rather than his rich neighbor who may request an informational rent as a reward for his revelation of true endowment. Third, the optimal tax schedule is increasing in the sense that the tax payment of an agent is increasing in his endowment.

Using the characterization results, we conduct some comparative statics analyses on how the optimal tax mechanism responds to a change in the initial parameter values. By these analyses, it will be shown explicitly how incentive compatibility and feasibility conditions are affected by a change in the primitives of an economy. In particular, one of the interesting implications is that if the possibility that the economy ends up with quite low provision of public good is not that high, then the tax burden of a rich agent is relatively light to that of a poor agent. This result reflects the intuition that the incentive compatibility problem is a more serious problem for an economy expected to be rich.

As a natural extension, we consider the case of more than two agents. Although it is impossible to fully describe the optimal tax mechanism for this case due to its high dimensionality and abundance of corner solutions, we obtain some partial results similar to those of the two-agent case under a mild assumption. Finally, we consider an infinitely large economy in which incentive compatibility is most extremely binding so that only uniform taxation is optimal.

One of the new features that distinguish our model from the previous literature on public goods is the continuous provision of public goods under incomplete information. Most of the previous models have dealt with the discrete (in fact, binary) provision of public goods. See, for example, D’Aspremont and Gérard-Varet [1979], Laffont and Maskin [1979], and Gradstein [1994].
discreteness makes the models mathematically simple and tractable,\(^8\) it is a restrictive assumption. In our model, the provision of public goods is continuous because it depends on the total amount of collected taxes.\(^9\)

The remainder of this essay is organized as follows. In Section 2, we present the model for a public good economy. In Section 3, we fully characterize the optimal tax schedule for the economy with two agents and two possible types. Using the characterization results, in Section 4, we discuss the properties of the optimal mechanism and provide some comparative statics analyses. As an extension, we consider the case of more than two agents in Section 5. In Section 6, we give concluding remarks and future research agenda.

I.2 The Model

I.2.1 The Economy

Consider a public good economy with \(n\) agents, \(2 \leq n < \infty\).\(^{10}\) Let \(N = \{1, \ldots, n\}\) denote the set of agents. There is one private good \(x \in \mathbb{R}_+\) and one pure public good \(y \in \mathbb{R}_+\), where the private good can be used to produce the public good according to a constant returns to scale technology. Without loss of generality, we normalize the production technology such that one unit of private good can be transformed into one unit of public good. Each agent \(i \in N\) has the same quasilinear von Neumann-Morgenstern utility function \(u\) on \(\mathbb{R}_+^2\),

\[
u(x_i, y) = \log y + x_i,
\]

where \(x_i\) is the consumption of private good by agent \(i\). Initially, each agent \(i\) is endowed with private good \(\omega_i \in \{\omega_L, \omega_H\}\) only, where \(0 \leq \omega_L < \omega_H < \infty\).\(^{11}\) Agent \(i\) is called poor when \(\omega_i = \omega_L\) and rich when \(\omega_i = \omega_H\). Let

\[
\Omega = \{ (\omega_L, \omega_H) \in \mathbb{R}_+^2 : \omega_L < \omega_H \}
\]

denote the set of all possible pairs of initial endowments.

---

\(^8\) One of many advantages that the discreteness assumption brings about is to make the individual utility depend linearly on the provision of public goods so that differential approach can be applied.

\(^9\) Exceptionally, Ledyard and Palfrey [1999] employ a model that allows continuous public goods provision. However, under their assumptions of linear production and risk-neutral preferences, it is equivalent to a discrete one. Also, Bergstrom, Blume, and Varian [1986] study a continuous case, but their model assumes complete information.

\(^{10}\) The case where there is only one agent in the economy is trivial. The case where \(n = \infty\) will be discussed in Section I.5.3.

\(^{11}\) In this essay, each agent’s initial endowment may be interpreted as a portion of his total wealth above the subsistence level. Thus, it can be called the agent’s taxable wealth for the provision of public goods. This interpretation will be made clear in Section I.4.1.
The information structure of this economy follows a standard incomplete information (Bayesian) model. The primitives of the economy are common knowledge, whereas each agent has private information about his own endowment. That is, agent $i$ knows the realization of his own endowment $\omega_i$ and the initial probability distribution of the other agents’ endowments, but does not know the realizations of the other agents’ endowments $\omega_{-i}$. Agents’ endowments are independently distributed according to

$$
\Pr(\omega_i = \omega_L) = p \in (0, 1) \quad \forall \ i \in N.
$$

Thus, an economic environment is equivalent to the realization of $\omega = (\omega_1, \ldots, \omega_n)$.

### I.2.2 The Tax Mechanism

A tax mechanism consists of message spaces $M_i$ for each agent $i \in N$, and an outcome function $f$ which maps each message profile $m \in M \equiv \prod_{i=1}^n M_i$ into agents’ tax burdens $t(m) = (t_1(m), \ldots, t_n(m)) \in \mathbb{R}_+^n$ and public good production $y$: $f : m \mapsto (t(m), y(m))$. The constant returns to scale technology implies that $y(m) \leq \sum_{i=1}^n t_i(m)$ for all $m \in M$, but without loss of generality, we can assume that the equality always holds since no taxes will be wasted. Hence, we have the following simple definition.

**Definition I.2.1 (Tax Mechanism and Schedule)** A tax mechanism $\Gamma$ is defined as $\Gamma = \langle M, t \rangle$, where $t : M \to \mathbb{R}_+^n$ is called a tax schedule.

Given a tax mechanism $\Gamma = \langle M, t \rangle$, let $s_i : \{\omega_L, \omega_H\} \to M_i$ denote the strategy (report) of agent $i$. By the Revelation Principle (see Myerson [1979]), we are able to restrict our attention to a direct incentive compatible tax mechanism. Thus, we assume that $M_i = \{\omega_L, \omega_H\}$ for each $i \in N$.

The expected utility of agent $i$ when his endowment is $\omega_i$ and he reports $s_i$, assuming the other agents are truthful, is

$$
U_i(s_i|\omega_i, t) = \mathbb{E}_{\omega_{-i}} \left[ u_i(\omega_i - t_i(s_i, \omega_{-i}), \sum_{j=1}^n t_j(s_i, \omega_{-i})) \bigg| \omega_i \right] \\
= \mathbb{E}_{\omega_{-i}} \left[ \log \left( \sum_{j=1}^n t_j(s_i, \omega_{-i}) \right) + (\omega_i - t_i(s_i, \omega_{-i})) \bigg| \omega_i \right].
$$

---

12 Notational convention applies here, that is, given a vector $a = (a_1, \ldots, a_n) \in A = \prod_{i=1}^n A_i$,

$$a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \in A_{-i} = \prod_{j \neq i} A_j, \text{ and } a = (a_i, a_{-i}).$$

13 This property may be viewed as a budget-balancedness. Compare with the Clarke-Groves mechanism where budget-balancedness is usually not satisfied, see Clarke [1971] and Groves [1973].
In this essay, we make two assumptions which a tax mechanism should satisfy. First, following Hurwicz, Maskin, and Postlewaite [1995], we employ the no exaggeration assumption.

**Assumption I.2.2 (No Exaggeration)** For each $i \in N$, $s_i(\omega_i) \leq \omega_i$.

That is, no agent is allowed to overstate his endowment when reporting.\(^{14}\) This assumption partially relieves the informational disadvantage of the social planner. Another assumption is the anonymity of a tax mechanism: A tax schedule should not be affected by the change of agents’ names.\(^{15}\) More specifically, this includes two conditions. First, an agent’s tax payment should not be affected by the change of order in the other agents’ reports as long as the distribution of their reports remains the same. Second, any two agents’ tax payments should be the same if they report the same endowment with other things being equal. Formally,

**Assumption I.2.3 (Anonymity)** For all $i, j \in N$,

i. $t_i(s_i, s_{i-}) = t_i(s_i, \sigma(s_{i-}))$,

ii. $s_i = s_j \implies t_i(s_i, s_i') = t_j(s_j, s_i') \ \forall s_i' \in \{\omega_L, \omega_H\}^{n-1}$,

where $\sigma(s_{i-})$ is a permutation of $s_{i-}$.

Under the anonymity assumption, let $t_{L,(n-k)L,(k-1)H}$ denote an agent’s tax payment when he and $(n - k)$ of the other agents report $\omega_L$ and the remaining $(k - 1)$ agents report $\omega_H$, $k = 1, \ldots, n$. Define $t_{H,(n-k)L,(k-1)H}$ similarly. Then, we can express a tax schedule as

$$t = \left( (t_{L,(n-k)L,(k-1)H})_{k=1}^n, (t_{H,(n-k)L,(k-1)H})_{k=1}^n \right).$$

Since we are considering a direct mechanism, we simply identify a (direct) tax mechanism $\Gamma = \langle M, t \rangle$ with a tax schedule $t$ in this essay.

To state the social planner’s problem, we need to look at three properties that a tax mechanism should satisfy: Feasibility, Incentive Compatibility, and Individual Rationality. First, feasibility, one of the most important concepts in this essay, implies that no tax mechanism should impose more than the announced endowment.\(^{16}\) That is,

**Definition I.2.4 (Feasibility)** A tax mechanism $t$ is feasible if for all $k = 1, \ldots, n$,

$$0 \leq t_{L,(n-k)L,(k-1)H} \leq \omega_L \text{ and } 0 \leq t_{H,(n-k)L,(k-1)H} \leq \omega_H.$$ 

\(^{14}\) It may be assumed that each agent is asked to put his report on the table.

\(^{15}\) This assumption must hold for every society where taxation is based on a democratic process.

\(^{16}\) In this sense, the feasibility can also be called no-bankruptcy.
Throughout this essay, we require all tax mechanisms considered to be feasible. Second, by the Revelation Principle, we consider an incentive compatible tax mechanism only. Note that due to the no exaggeration assumption, a poor agent cannot pretend to be rich, so the rich agent’s incentive compatibility is enough.

**Definition I.2.5 (Incentive Compatibility: IC)** A tax mechanism $t$ is (Bayesian) incentive compatible if for all $i \in N$, 

$$U_i(\omega_H|\omega_H, t) \geq U_i(\omega_L|\omega_H, t).$$  

(I.1)

Third, to make the agents participate in this public good economy, we need to make assumptions as to what will happen if an agent does not participate. Notice that we have to distinguish between the situations in which an agent does not want to report his endowment and in which an agent wants to leave the economy or refuses to pay the imposed tax. In the former case, we assume that the social planner can impose a tax on the agent as if he were to report $\omega_L$. Under this assumption, the expected utility of agent $i$ who did not report is $U_i(\omega_L|\omega_H, t)$ if his endowment is $\omega_H$, or $U_i(\omega_L|\omega_L, t)$ if his endowment is $\omega_L$. Since only incentive compatible tax mechanisms are considered, agent $i$ who reports his endowment will obtain $U_i(\omega_H|\omega_H, t)$ if his endowment is $\omega_H$, or $U_i(\omega_L|\omega_L, t)$ if his endowment is $\omega_L$. Thus, by inequality (I.1), every agent will report his endowment, which makes the individual rationality condition redundant in this model. In the latter case, we assume that the social planner can prevent the agent from enjoying the public good by, for example, expulsion from the economy. Under this assumption, the expected utility of the agent is $u(x_i, 0) = -\infty, \forall x_i \in \mathbb{R}_+$. Thus, the individual rationality constraint becomes redundant, too. As a result, we can ignore individual rationality by the above two assumptions.

Finally, we add one more definition for a tax mechanism.

**Definition I.2.6 (Increasingness)** A tax mechanism $t$ is increasing if for all $k = 1, \ldots, n$,

$$t_{L,(n-k)L,(k-1)H} \leq t_{H,(n-k)L,(k-1)H}.$$  

That is, a tax mechanism is increasing if an agent’s tax payment is increasing with his endowment.

**I.2.3 The Social Planner’s Problem**

The social planner (or tax authority), who does not know the true realization of the economic environment but knows its probability distribution, wants to find an incentive compatible and feasible
tax schedule \( t^* = \left( (t^*_{L,(n-k)L,(k-1)H})_{k=1}^n, (t^*_{H,(n-k)L,(k-1)H})_{k=1}^n \right) \) which maximizes the expected sum of agents’ utilities. Formally, given \((\omega_L, \omega_H) \in \Omega\) and \(p \in (0, 1)\), the social planner’s problem is

\[
\max_t W(t; p) = \mathbb{E} \left[ \sum_{i=1}^n U_i(\omega_i|\omega, t) \right]
\]

subject to

\[
\text{(IC)} \quad U_i(\omega_H|\omega_H, t) \geq U_i(\omega_L|\omega_H, t) \quad \forall \ i \in N,
\]

\[
\text{(Feasibility)} \quad t \in B(\omega_L, \omega_H) \equiv [0, \omega_L]^n \times [0, \omega_H]^n.
\]

Notice that only one (IC) constraint is binding. For notational simplicity, given \(p \in (0, 1)\), define a function \( \Delta : \mathbb{R}^{2n}_+ \rightarrow \mathbb{R} \equiv \mathbb{R} \cup \{-\infty, +\infty\} \) by

\[
\Delta(t; p) = U_i(\omega_H|\omega_H, t) - U_i(\omega_L|\omega_H, t).
\]

Then, a tax schedule \( t \) satisfies (IC) if \( \Delta(t; p) \geq 0 \).

### I.3 Optimal Tax Mechanism for \( n = 2 \)

In this section, we study the optimal tax mechanism for the case of two agents. For \( n = 2 \), a tax mechanism \( t \) can be written as

\[
t = (t_{LL}, t_{LH}, t_{HL}, t_{HH}),
\]

where, for example, \( t_{LH} \) is the tax payment of an agent when he reports \( \omega_L \) and the other agent reports \( \omega_H \). The social planner’s problem now becomes: Given \((\omega_L, \omega_H) \in \Omega\) and \(p \in (0, 1)\),

\[
\max_t W(t; p) = p^2 \left[ 2 \log(2t_{LL}) - 2t_{LL} \right] + 2p(1 - p) \left[ 2 \log(t_{LH} + t_{HL}) - (t_{LH} + t_{HL}) \right] + (1 - p)^2 \left[ 2 \log(2t_{HH}) - 2t_{HH} \right] + 2 \left( p\omega_L + (1 - p)\omega_H \right)
\]

subject to

\[
\text{(P}_2\text{)} \quad \begin{align*}
\log(t_{LH} + t_{HL}) - t_{HL} + (1 - p) \left[ \log(2t_{HH}) - t_{HH} \right] \\
\geq p \left[ \log(2t_{LL}) - t_{LL} \right] + (1 - p) \left[ \log(t_{LH} + t_{HL}) - t_{LH} \right],
\end{align*}
\]

\[
\text{(Feasibility)} \quad 0 \leq t_{LL} \leq \omega_L, \quad 0 \leq t_{LH} \leq \omega_L, \quad 0 \leq t_{HL} \leq \omega_H, \quad 0 \leq t_{HH} \leq \omega_H.
\]

Note that for \( n = 2 \),

\[
\Delta(t; p) = p \left[ \log \left( \frac{(t_{LH} + t_{HL})^2}{(2t_{LL})(2t_{HH})} \right) - (t_{LH} + t_{HL}) + (t_{LL} + t_{HH}) \right] - \left[ \log \left( \frac{t_{LH} + t_{HL}}{2t_{HH}} \right) + (t_{HH} - t_{LH}) \right].
\]
I.3.1 Possibility of First Best Taxation

To begin with, we examine the possibility of the first best tax schedule which is the solution to (P_2) without (IC) constraint. If the social planner knew the realization of each agent’s endowment, she could easily find the first best tax schedule. However, she lacks this information, so the question is when the (IC) constraint is not binding. First of all, to rule out the uninteresting cases, partition \( \Omega \) (see Figure I.1) into

\[
\Omega_1 = \{ (\omega_L, \omega_H) \in \Omega : \omega_L \in [0, 1) \}, \quad \text{and} \\
\Omega_2 = \{ (\omega_L, \omega_H) \in \Omega : \omega_L \in [1, \infty) \}.
\]

When \( (\omega_L, \omega_H) \in \Omega_2 \), the social planner can easily solve (P_2) by imposing a first best feasible tax schedule

\[
t^F \in \{ t \in B(\omega_L, \omega_H) : t_{LL} = t_{HH} = 1, \ t_{LH} + t_{HL} = 2, \ \text{and} \ 1 \leq t_{LH} \leq \omega_L \},
\]

since \( t^F \) satisfies the (IC) constraint; \( \Delta(t^F ; p) = - (1 - t^F_{LH}) \geq 0 \). If the social planner insists that the tax schedule be increasing, then the unique solution to (P_2) is \( t^F = (1, 1, 1, 1) \). Therefore, in the following we just focus on the case of \( (\omega_L, \omega_H) \in \Omega_1 \). According to the welfare function \( W(\cdot) \), it is easy to see that for \( (\omega_L, \omega_H) \in \Omega_1 \) the first best feasible tax schedule is

\[
t^F = (t^F_{LL}, t^F_{LH}, t^F_{HL}, t^F_{HH}) = (\omega_L, \omega_L, \min \{ \omega_L + \omega_H, 2 \} - \omega_L, \min \{ \omega_H, 1 \}) \text{.}^{19}
\]

To find the conditions under which \( t^F \) is the solution to (P_2), consider the (IC) constraint at \( t^F \):

\[
\Delta(t^F ; p) = p \left[ \log \left( \frac{\min \{ \omega_L + \omega_H, 2 \}}{2 \min \{ \omega_H, 1 \}} \right)^2 - \min \{ \omega_L + \omega_H, 2 \} + (\omega_L + \min \{ \omega_H, 1 \}) \right]
- \left[ \log \frac{\min \{ \omega_L + \omega_H, 2 \}}{2 \min \{ \omega_H, 1 \}} + (\min \{ \omega_H, 1 \} - \omega_L) \right].
\]

Lemma I.3.1 For \( (\omega_L, \omega_H) \in \Omega_1 \), \( \Delta(t^F ; p) \) is strictly increasing in \( p \).

**Proof:** Consider the two cases: (i) \( (\omega_L, \omega_H) \in [0, 1) \times (0, 1] \), and (ii) \( (\omega_L, \omega_H) \in [0, 1) \times (1, \infty] \).

Case (i) \( (\omega_L, \omega_H) \in [0, 1) \times (0, 1] \): In this case, it is clear that

\[
\frac{\partial \Delta(t^F ; p)}{\partial p} = \log \left( \frac{\omega_L + \omega_H}{2 \omega_L} \right) > 0.
\]

---

19 Strictly speaking, there is a continuum of first best feasible tax schedules if \( \omega_L + \omega_H > 2 \). However, given \( p \), \( t^F \) satisfies (IC) maximally in the sense that for any first best feasible schedule \( t^F', \Delta(t^F' ; p) \geq \Delta(t^F ; p) \), so we can assume without loss of generality that \( t^F \) is the unique first best tax schedule.
Case (ii) \((\omega_L, \omega_H) \in [0, 1) \times (1, \infty)\): If \(\omega_L + \omega_H < 2\),

\[
\frac{\partial \Delta(t^F; p)}{\partial p} = \log \left(\frac{(\omega_L + \omega_H)^2}{4\omega_L}\right) - (\omega_H - 1).
\]

Since \(\frac{\partial}{\partial \omega_H} \left(\frac{\partial \Delta(t^F; p)}{\partial p}\right) = \frac{2}{\omega_L + \omega_H} - 1 > 0\), it follows that

\[
\frac{\partial \Delta(t^F; p)}{\partial p} > \lim_{\omega_H \to 1} \frac{\partial \Delta(t^F; p)}{\partial p} = \log \left(\frac{(1 + \omega_L)^2}{4\omega_L}\right) > 0.
\]

If \(\omega_L + \omega_H \geq 2\), then

\[
\frac{\partial \Delta(t^F; p)}{\partial p} = \omega_L - \log \omega_L - 1 > 0.
\]

Therefore, we have the result.

For \((\omega_L, \omega_H) \in \Omega_1\), define \(\overline{p} \in \overline{\mathbb{R}}\) by \(\Delta(t^F; \overline{p}) = 0\), or equivalently,

\[
\overline{p} = \frac{\log \min\{\omega_L + \omega_H, 2\}}{\log \left(\frac{\min\{\omega_L + \omega_H, 2\}^2}{2\min\{\omega_H, 1\}}\right)} + \left(\min\{\omega_H, 1\} - \omega_L\right)
\]

and let \(\hat{\rho} = \min\{1, \overline{p}\}\). Define also (see Figure I.1)

\[
\Omega^F = \{\omega_H, \omega_L) \in \Omega_1 : \lim_{p \to 0} \Delta(t^F; p) \geq 0\}.
\]

**Proposition I.3.2** If \(p \geq \hat{\rho}\), then the first best feasible tax schedule \(t^F\) is the unique solution to \((P_2)\). In particular, if \((\omega_L, \omega_H) \in \Omega^F\), then \(t^F\) is the unique solution to \((P_2)\) for all \(p \in (0, 1)\).

**Proof:** By the definition of \(\hat{\rho}\) and Lemma I.3.1, if \(p \geq \hat{\rho}\), then \(\Delta(t^F; p) \geq 0\), which implies that \(t^F\) satisfies (IC). Since \(t^F\) is feasible, the fact that \(t^F\) is the unique first best feasible tax schedule proves the first result. It is obvious that \(\lim_{p \to 0} \Delta(t^F; p) \geq 0\) guarantees that \(\Delta(t^F; p) \geq 0\) for all \(p \in (0, 1)\).

Figure I.1 depicts the possibility of first best feasible taxation.

**I.3.2 Second Best Tax Schedule**

Assume that \(p < \hat{\rho}\). To characterize the second best feasible tax schedule, we begin with three lemmas. The main purpose of these lemmas is to lower the dimension of the social planner’s problem.

**Lemma I.3.3** Suppose \(t^*\) is a solution to \((P_2)\). Then,

\[
t^*_{HH} = \min\{\omega_H, 1\}.
\]
Proof: There are two cases: (i) $\omega_H \leq 1$, and (ii) $\omega_H > 1$.

Case (i) $\omega_H \leq 1$: Suppose by way of contradiction that $t_{HH}^* < \omega_H$. Choose $\varepsilon$ such that $0 < \varepsilon \leq \omega_H - t_{HH}^*$. Consider a new tax schedule $t' = (t_{LL}^*, t_{LH}^*, t_{HL}^*, t_{HH}^* + \varepsilon)$. Since $\log(2t_{HH}^*) - t_{HH}^*$ is strictly increasing in $t_{HH}^* \in (0, 1)$, it follows that

$$U_i(\omega_H|\omega_H, t') > U_i(\omega_H|\omega_H, t^*) \geq U_i(\omega_L|\omega_H, t^*) = U_i(\omega_L|\omega_H, t').$$

Hence, $t'$ satisfies (IC). Also, $t'$ satisfies (Feasibility) by the construction of $\varepsilon$. However, we have $W(t'; p) > W(t^*; p)$, a contradiction to the hypothesis that $t^*$ is a solution.

Case (ii) $\omega_H > 1$: Suppose by way of contradiction that $t_{HH}^* \neq 1$. If $t_{HH}^* < 1$, choose $\varepsilon$ such that $0 < \varepsilon \leq 1 - t_{HH}^*$. Then the same argument in Case (i) induces a contradiction. If $t_{HH}^* > 1$, choose $\varepsilon$ such that $0 < \varepsilon \leq t_{HH}^* - 1$, and consider a new tax schedule $t' = (t_{LL}^*, t_{LH}^*, t_{HL}^*, t_{HH}^* - \varepsilon)$. Then, the same argument in Case (i) also gives a contradiction.

Lemma I.3.4 Suppose $t^*$ is a solution to $(P_2)$. Then,

$$t_{LH}^* + t_{HL}^* \leq 2.$$

Proof: Suppose by way of contradiction that $t_{LH}^* + t_{HL}^* > 2$. Note that this case is possible only when $\omega_L + \omega_H > 2$. Choose $\varepsilon$ such that $0 < \log(t_{LH}^* + t_{HL}^*) - \log(t_{LH}^* + t_{HL}^* - \varepsilon) < \frac{\varepsilon}{2}$. Such an $\varepsilon$ is well defined since $\frac{d}{dy}(\log y) < \frac{1}{2}$ for $y > 2$. Consider a new tax schedule $t' = (t_{LL}^*, t_{LH}^*, t_{HL}^* - \varepsilon, t_{HH}^*)$. 


Then,

\[
U_i(\omega_H|\omega_H, t') = p \left[ \log(t_{LH}^* + t_{HL}^* - \varepsilon) - t_{HL}^* + \varepsilon \right] + (1 - p) \left[ \log(2t_{HH}^*) - t_{HH}^* \right] \\
> p \left[ \log(t_{LH}^* + t_{HL}^* - t_{HL}^* + \varepsilon) \right] + (1 - p) \left[ \log(2t_{HH}^*) - t_{HH}^* \right] \\
> p \left[ \log(t_{LH}^* + t_{HL}^*) - t_{HL}^* \right] + (1 - p) \left[ \log(2t_{HH}^*) - t_{HH}^* \right] \\
\geq p \left[ \log(2t_{LL}^*) - t_{LL}^* \right] + (1 - p) \left[ \log(t_{LH}^* + t_{HL}^*) - t_{LH}^* \right] \\
> p \left[ \log(2t_{LL}^*) - t_{LL}^* \right] + (1 - p) \left[ \log(t_{LH}^* + t_{HL}^* - \varepsilon) - t_{LH}^* \right]
\]

which implies that \( t' \) satisfies (IC). Also, \( t' \) satisfies (Feasibility) by construction. However, since

\[
2 \log(t_{LH}^* + t_{HL}^*) - (t'_{LH} + t'_{HL}) = 2 \log(t_{LH}^* + t_{HL}^* - \varepsilon) - (t_{LH}^* + t_{HL}^* - \varepsilon) \\
> 2 \log(t_{LH}^* + t_{HL}^*) - (t_{LH}^* + t_{HL}^*)
\]

we have \( W(t'; p) > W(t^*; p) \), a contradiction to the hypothesis that \( t^* \) is a solution.

\[ \square \]

**Lemma I.3.5** Suppose \( t^* \) is a solution to \((P_2)\). Then,

\[
t_{LH}^* = \omega_L, \quad \text{and} \quad t_{HL}^* \geq \omega_L.
\]

**Proof:** If \( \omega_L = 0 \), then the claim is trivial. Hence, consider the case of \( \omega_L > 0 \). Suppose by way of contradiction that (i) \( t_{HL}^* < \omega_L \); or (ii) \( t_{LH}^* < \omega_L \) and \( t_{HL}^* \geq \omega_L \).

Case (i) \( t_{HL}^* < \omega_L \): Consider a new tax schedule \( t' = (\omega_L, \omega_L, \omega_L, \min(\omega_H, 1)) \). Then,

\[
\Delta(t'; p) = -(1 - p) \left[ \log \frac{\omega_L}{\min(\omega_H, 1)} - \omega_L + \min(\omega_H, 1) \right] > 0, \tag{I.2}
\]

which implies that \( t' \) satisfies (IC). Also, \( t' \) satisfies (Feasibility). However, we have \( W(t'; p) > W(t^*; p) \), a contradiction to the hypothesis that \( t^* \) is a solution.

Case (ii) \( t_{LH}^* < \omega_L \) and \( t_{HL}^* \geq \omega_L \): According to Lemma I.3.4, we have two subcases: (a) \( t_{LH}^* + t_{HL}^* < 2 \); or (b) \( t_{LH}^* + t_{HL}^* = 2 \).

Subcase (a) \( t_{LH}^* + t_{HL}^* < 2 \): Choose \( \varepsilon \) such that \( 0 < \varepsilon \leq \min\{\omega_L - t_{HL}^*, 2 - (t_{LH}^* + t_{HL}^*)\} \). Consider a new tax schedule \( t' = (t_{LL}^*, t_{LH}^* + \varepsilon, t_{HL}^* - \varepsilon, t_{HH}^*) \). Then,

\[
U_i(\omega_H|\omega_H, t') = U_i(\omega_H|\omega_H, t^*) + p\varepsilon \\
> U_i(\omega_L|\omega_L, t^*) \\
= U_i(\omega_L|\omega_H, t') + (1 - p)\varepsilon \\
> U_i(\omega_L|\omega_H, t'), \tag{I.3}
\]

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which implies that $\Delta(t'; p) > 0$, that is, the (IC) constraint is not tightly binding. Since $\Delta(t; p)$ is continuous in $t_{HL}$, we can choose $\delta \in (0, \varepsilon)$ such that $t'' = t' + (0, \delta, 0, 0)$ still satisfies (IC) and (Feasibility). Notice that $t_{LL}^{*} + t_{HL}^{*} = t_{LL}' + t_{HL}' < t_{LL}'' + t_{HL}'' < 2$, and $t_{LL}'' = t_{LL}^{*}$, and $t_{HL}'' = t_{HL}^{*}$. Hence, we have $W(t''; p) > W(t'; p)$, a contradiction to the hypothesis that $t^{*}$ is a solution.

Subcase (b) $t_{LL}^{*} + t_{HL}^{*} = 2$: Notice that in this case $\omega > 1$ since $t_{LL}^{*} = 2 - t_{HL}^{*} > 2 - \omega > 1$. First, we want to show that $t_{LL}^{*} < \omega$. Suppose not, that is, $t_{LL}^{*} = \omega$. Since $t_{HL}^{*} = 1$ by Lemma I.3.3, it turns out that

$$\Delta(t^{*}; p) = p(\omega - \log \omega - 1) - (1 - \omega) < 0,$$

since $p < \hat{\rho} = \frac{1 - \omega}{\omega - \log \omega - 1}$. This is a contradiction to the hypothesis that $t^{*}$ satisfies (IC). So, $t_{LL}^{*} < \omega$.

Consider a new tax schedule $t' = (t_{LL}^{*}, t_{HL}^{*} + \varepsilon, t_{HL}^{*} - \varepsilon, t_{HL}^{*})$ where $\varepsilon$ is chosen such that $0 < \varepsilon \leq \omega - t_{LL}^{*}$. Then, by (I.3), we have $\Delta(t'; p) > 0$, that is, the (IC) constraint is not tightly binding. Since $\Delta(t; p)$ is continuous in $t_{LL}$ and $t_{LL}^{*} < \omega$, we can choose $\delta \in (0, \omega - t_{LL}^{*})$ such that $t'' = t' + (\delta, 0, 0, 0)$ still satisfies (IC) and (Feasibility). Since $\log(2t_{LL}^{*}) - 2t_{LL}^{*} > 2\log(2t_{LL}^{*}) - 2t_{LL}^{*}$, we have $W(t''; p) > W(t^{*}; p)$, a contradiction to the hypothesis that $t^{*}$ is a solution.

Therefore, we conclude that $t_{LL}^{*} = \omega$ and $t_{HL}^{*} \geq \omega$.

By Lemmas I.3.3–I.3.5, we can reduce the dimension of $(P_2)$ from four to two. Let $T = t_{LL} + t_{HL}$. Lemmas I.3.3–I.3.5 implies that we can restrict our attention to $(T, t_{LL}) \in [2\omega L, \min\{\omega L + \omega H, 2\}] \times [0, \omega L]$, which now can be called a tax schedule. Define (IC)-function $z(\cdot, \cdot; p) : [2\omega L, \min\{\omega L + \omega H, 2\}] \times [0, \omega L] \rightarrow \mathbb{R}$, by

$$z(T, t_{LL}; p) = \Delta(t_{LL}, t_{HL}, t_{HL}, t_{HL}; p)|_{t_{LL} = \omega L, t_{HL} = \min\{\omega H, 1\}} = p \left[ \frac{T^2}{(2t_{LL})(2\min\{\omega H, 1\})} - T + (t_{LL} + \min\{\omega H, 1\}) \right] - \left[ \frac{T}{2\min\{\omega H, 1\}} + (\min\{\omega H, 1\} - \omega) \right].$$

Thus, a tax schedule $(T, t_{LL})$ satisfies (IC) if $z(T, t_{LL}; p) \geq 0$.

Now, the social planner’s problem $(P_2)$ can be written as an equivalent but simplified version $(P'_2)$: Given $(\omega L, \omega H) \in \Omega_1$ and $p \in (0, 1)$,

$$\max_{(T, t_{LL})} W(T, t_{LL}; p) = p^2[2\log(2t_{LL}) - 2t_{LL}] + 2p(1 - p)[2\log T - T]$$

subject to

$$(P'_2) \quad (IC) \quad z(T, t_{LL}; p) \geq 0$$

and

$$(Feasibility) \quad (T, t_{LL}) \in [2\omega L, \min\{\omega L + \omega H, 2\}] \times [0, \omega L].$$

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To find the second best tax schedule, first consider the shape of the (IC)-curve $z(T, t_{LL}; p) = 0$. In fact, we can find a point that satisfies $z(T, t_{LL}; p) = 0$ for all $p \in (0, 1)$. For $(\omega_L, \omega_H) \in \Omega\setminus\Omega_F$, let

$$\widetilde{T} = 2 \min\{\omega_H, 1\} e^{-(\min\{\omega_H, 1\} - \omega_L)}, \quad \text{and}$$

$$\widetilde{t}_{LL} = -W_0 \left(-\exp \left\{ \log \frac{T}{2} - \widetilde{T} + \omega_L \right\} \right),$$

where $W_0$ is the principal branch of Lambert $W$ function.20 By the definition of $\widetilde{T}$, we can rewrite the (IC)-curve as

$$z(T, t_{LL}; p) = p \left[ \log \frac{T^2}{(2t_{LL})(\widetilde{T})} - T + (t_{LL} + \omega_L) \right] - \left[ \log \frac{T}{\widetilde{T}} \right],$$

so, it is clear that $z(\widetilde{T}, \widetilde{t}_{LL}; p) = 0$ for all $p \in (0, \tilde{p})$. That is, the (IC)-curve $z(T, t_{LL}; p) = 0$ always goes through the pivotal point $(\widetilde{T}, \widetilde{t}_{LL})$. Furthermore,

**Lemma I.3.6**

i. $(\widetilde{T}, \widetilde{t}_{LL}) \in (2\omega_L, \min\{\omega_L + \omega_H, 2\}) \times [0, 1)$.

ii. If $\widetilde{T} \leq 1$, then $\widetilde{t}_{LL} > \omega_L$.

**Proof:**

i. Since $\log(\min\{\omega_H, 1\}) - \min\{\omega_H, 1\} > \log(\omega_L) - \omega_L$, it is clear that $\widetilde{T} > 2\omega_L$. For $(\omega_L, \omega_H) \in \Omega_1 \setminus \Omega_F$, $\log \frac{\min\{\omega_L + \omega_H, 2\}}{2\min\{\omega_H, 1\}} + (\min\{\omega_H, 1\} - \omega_L) = -\lim_{p \to 0} \Delta(t^F; \tilde{p}) > 0$, so $\widetilde{T} < \min\{\omega_L + \omega_H, 2\}$. To see that $\widetilde{t}_{LL} \in [0, 1)$, by Lambert $W$ function, it suffices to show that $\varphi(\omega_L, \omega_H) \equiv \log \frac{T}{2} - \widetilde{T} + \omega_L < -1$. Note that since $\widetilde{T} > 2\omega_L$,

$$\varphi(\omega_L, \omega_H) = \log(\min\{\omega_H, 1\}) - \min\{\omega_H, 1\} - \widetilde{T} + 2\omega_L$$

$$< \log(\min\{\omega_H, 1\}) - \min\{\omega_H, 1\}$$

$$\leq -1.$$

ii. Since $0 \leq \widetilde{t}_{LL} < 1$, the result is equivalent to $\widetilde{t}_{LL} - \log \widetilde{t}_{LL} \leq \omega_L - \log \omega_L$, or $\log \frac{\widetilde{T}}{2\omega_L} - \widetilde{T} + 2\omega_L \leq 0$. By $2\omega_L < \widetilde{T} \leq 1$, we have the result.

This lemma tells that if $\widetilde{T} \leq 1$, the pivotal point $(\widetilde{T}, \widetilde{t}_{LL})$ is above the feasible set $[2\omega_L, \min\{\omega_L + \omega_H, 2\}] \times [0, \omega_L]$. Another property of the (IC)-curve is that it turns around the pivotal point $(\widetilde{T}, \widetilde{t}_{LL})$ counterclockwise as $p$ increases.

20 The Lambert $W$ function is defined to be the function satisfying $W(x)e^{W(x)} = x$. This function is defined on $[-e^{-1}, \infty)$, and has a single real value on $[0, \infty)$ and two real values on $[-e^{-1}, 0)$. $W_0$, called the principal branch, is the increasing part of $W$, $W_{-1}$, called the $(-1)$th branch, is the decreasing part of $W$. The solution of the equation $x b^x = a$ is $x = \frac{1}{\log b} W(a \log b)$. For more properties on the Lambert $W$ function, see Corless, et. al. [1996]. (We reluctantly employ the notational abuse, $W$, previously used for the welfare function. Hopefully, it may not cause any confusion in the following.)
**Lemma I.3.7** For all \( p, p' \in (0, 1) \) such that \( p < p' \), if \( z(T, t_{LL}; p) = 0 \), then

\[
z(T, t_{LL}; p') = \begin{cases} < 0 & \text{if } T < \tilde{T} \\ \geq 0 & \text{if } T \geq \tilde{T}. \end{cases}
\]

**Proof:** Since

\[
z(T, t_{LL}; p') = \left( \frac{p'}{p} \right) z(T, t_{LL}; p) + \left( \frac{p'}{p} - 1 \right) \left( \log \frac{T}{\tilde{T}} \right) = \left( \frac{p'}{p} - 1 \right) \left( \log \frac{T}{\tilde{T}} \right),
\]

we have the result easily.

---

Figure I.2 depicts the partition of \( \Omega_1 \setminus \Omega^F \) by \( (\tilde{T}, \tilde{t}_{LL}) \).

Now, consider the slope of \( (IC)\)-curve \( z(T, t_{LL}; p) = 0 \). Without loss of generality, we can restrict our attention to the domain of \([0, 2] \times [0, 1]\), which includes all of the possible \( (T, t_{LL}) \).

Using the Implicit Function Theorem,

\[
\frac{dt_{LL}}{dT} \bigg|_{z(T, t_{LL}; p) = 0} = -\frac{2p^{-1} - p}{p \left( 1 - \frac{1}{t_{LL}} \right)} \bigg|_{z(T, t_{LL}; p) = 0}.
\]

Since we restrict \( t_{LL} \) on \([0, 1]\), the denominator is negative. If \( p \leq \frac{1}{2} \), then the numerator is negative for all \( T \in [0, 2] \). If \( p \in \left( \frac{1}{2}, 1 \right) \), then the numerator is positive for \( T \in \left( 0, \frac{2p-1}{p} \right) \) and negative for \( T \in \left( \frac{2p-1}{p}, 2 \right) \). As a result, for \( (T, t_{LL}) \in [0, 2] \times [0, 1] \),

\[
\frac{dt_{LL}}{dT} \bigg|_{z(T, t_{LL}; p) = 0} = \begin{cases} > 0 & \text{if } p \in \left( \frac{1}{2}, 1 \right) \text{ and } T \in \left( 0, \frac{2p-1}{p} \right) \\ = 0 & \text{if } p \in \left( \frac{1}{2}, 1 \right) \text{ and } T = \frac{2p-1}{p} \\ < 0 & \text{otherwise} \end{cases}.
\]

\( ^{21} \) Note that \( z(T, t_{LL}; p) = 0 \) defines multiple curves on \( \mathbb{R}^2 \) while it defines a single curve on \([0, 2] \times [0, 1]\).


**Remark I.3.8** According to the inequality (I.2), it turns out that for \((\omega_L, \omega_H) \in \Omega_1\), the curve defined by \(z(T, t_{LL}; p) = 0\) on \((T, t_{LL}) \in [2\omega_L, \min\{\omega_L + \omega_H, 2\}] \times [0, \omega_L]\) has a negative slope because the point \((T, t_{LL})\) that has zero slope cannot be in \([2\omega_L, \min\{\omega_L + \omega_H, 2\}] \times [0, \omega_L]\).

The slope of welfare-curve \(\mathcal{W}(T, t_{LL}; p) = \overline{w}\), where \(\overline{w}\) is a constant, is

\[
\frac{dt_{LL}}{dT} \bigg|_{\mathcal{W}(T, t_{LL}; p) = \overline{w}} = -\frac{p(1-p)\left(\frac{\overline{w}}{T} - 1\right)}{p^2\left(\frac{1}{t_{LL}} - 1\right)} < 0, \tag{I.5}
\]

for \((T, t_{LL}) \in [0, 2] \times [0, 1]\). That is, the welfare-curve \(\mathcal{W}(T, t_{LL}; p) = \overline{w}\) has a negative slope on \([0, 2] \times [0, 1]\).

To describe the second best feasible tax schedule, we need some definitions. First, for \((\omega_L, \omega_H) \in \Omega_1\) such that \(\omega_L + \omega_H \leq 1\) and \(p \in (0, \rho]\), define \(t_{LL} \in (0, \omega_L]\) by \(z(\omega_L + \omega_H, 2, t_{LL}; p) = 0\). Second, for \(p \in (0, \rho]\), define \(T \in (2\omega_L, \min\{\omega_L + \omega_H, 2\})\) by \(z(T, \omega_L; p) = 0\). Third, define simply \(T^o = 1\).

Finally, for \(\bar{T} < 1\) and \(\bar{T} < 1\), define \(t_{LL}^o \in (0, \omega_L]\) by \(z(1, t_{LL}^o; p) = 0\).

Now, we can state the main result of this essay.

**Proposition I.3.9** For \(p < \rho\), the solution to \((P_2)\) is

\[
t^o = \begin{cases} 
(t_{LL}, \omega_L, \omega_H) & \text{if } \omega_L + \omega_H \leq T^o \\
(t_{LL}^o, \omega_L, T^o - \omega_L, \min\{\omega_H, 1\}) & \text{if } T \leq T^o \\
(\omega_L, t_{LL}, T - \omega_L, \min\{\omega_H, 1\}) & \text{if } T > T^o
\end{cases}
\]

**Proof:** Notice from (I.4) and (I.5) that the (IC)-curve \(z(T, t_{LL}; p) = 0\) is tangent to the welfare-curve \(\mathcal{W}(T, t_{LL}; p) = \overline{w}\) at \((T, t_{LL}) = (T^o, t_{LL}^o)\).

For the interior solution (the third case), we need to show that the tangent point \((T^o, t_{LL}^o)\) is maximizing the welfare function \(\mathcal{W}(\cdot)\) rather than minimizing. This can be done by showing that

\[
\frac{d^2t_{LL}}{dT^2} \bigg|_{\mathcal{W}(T^o, t_{LL}^o; p) = \overline{w}} > \frac{d^2t_{LL}}{dT^2} \bigg|_{z(T^o, t_{LL}^o; p) = 0}.
\]

---

\(22\) Using Lambert \(W\) function, we can express \(t_{LL}, T, \) and \(t_{LL}^o\) explicitly as

\[
t_{LL} = -W_0\left(\exp\left\{\log \frac{\omega_L + \omega_H}{2T} - \omega_H - \left(\frac{1}{\rho}\right) \log \frac{\omega_L + \omega_H}{T}\right\}\right);
\]

\[
T = -\left(\frac{2w-1}{p}\right)W\left(-\left(\frac{p}{2T+1}\right) \exp\left\{\left(\frac{1}{2T+1}\right) \log \left((2\omega_L)(T) - 2\omega_L\right) - \left(\frac{1}{2T+1}\right) \log(T)\right\}\right);
\]

\[
t_{LL}^o = -W_0\left(-\exp\left\{\log \frac{1}{2T} - 1 + \omega_L - \left(\frac{1}{p}\right) \log \frac{1}{T}\right\}\right).
\]

For \(T, W\) should be \(W_{-1}\) if \(p > \frac{1}{2}\) and \(W_0\) if \(p \leq \frac{1}{2}\).
Cases | \( t^*_LL \) | \( t^*_LH \) | \( t^*_HL \) | \( t^*_HH \)  \\ \hline \( p \geq \hat{\rho} \) (including \( \Omega^F \)) | \( \omega_L = \omega_L < \min \{ \omega_L + \omega_H, 2 \} - \omega_L \geq \min \{ \omega_H, 1 \} \) |  |  |  \\ \hline \( p < \hat{\rho} \) \( \omega_L + \omega_H \leq 1 \) | \( \omega_L \leq \omega_L < \omega_H = \omega_H \) |  |  |  \\ \hline \( \omega_L + \omega_H > 1 \) \( T \leq 1 \) | \( \omega_L \leq \omega_L < T^o - \omega_L < \min \{ \omega_H, 1 \} \) |  |  |  \\ \hline \( T > 1 \) | \( \omega_L = \omega_L < T - \omega_L \leq \min \{ \omega_H, 1 \} \) |  |  |  \\ \hline

Table I.1: Optimal Tax Schedules for \( n = 2 \).

Differentiating (I.4) and (I.5), it follows that

\[
\frac{d^2 t^*_{LL}}{dT^2} \bigg|_{W(T,t^*_{LL},\hat{\rho})=\bar{w}} - \frac{d^2 t^*_{LL}}{dT^2} \bigg|_{z(T^o,t^*_{LL},\hat{\rho})=0} = \frac{1}{pT(1-t^*_{LL})} \left[ \frac{t^*_{LL}}{T} - \left( \frac{1-T}{1-t^*_{LL}} \right) \frac{dt^*_{LL}}{dT} \right]_{T=T^o, t^*_{LL}=t^*_{LL}} = \frac{t^*_{LL}}{p(1-t^*_{LL})} > 0.
\]

For the corner solutions (the first and second cases), we need to show that

\[
T \leq 1 \implies \frac{dt^*_{LL}}{dT} \bigg|_{W(T,t^*_{LL},\hat{\rho})=\bar{w}} \leq \frac{dt^*_{LL}}{dT} \bigg|_{z(T,t^*_{LL},\hat{\rho})=0},
\]

respectively. From (I.4) and (I.5), we have

\[
\frac{dt^*_{LL}}{dT} \bigg|_{W(T,t^*_{LL},\hat{\rho})=\bar{w}} - \frac{dt^*_{LL}}{dT} \bigg|_{z(T,t^*_{LL},\hat{\rho})=0} = \frac{1}{p^2} \left( \frac{t^*_{LL}}{1-t^*_{LL}} \right) \left( \frac{T-1}{T} \right) \leq 0 \quad \text{for } T \leq 1,
\]

respectively, since \( t^*_{LL} \in [0,1) \). Therefore, we have \( t^* \) as stated.

Table I.1 summarizes the optimal tax schedules for each possible case.

### I.3.3 Simulated Examples

In this section, we illustrate some examples that show the specific optimal tax schedules for different parameter values. Due to the low dimensionality of the social planner’s problem, we can draw the results graphically.

**Example I.3.10**

i. Suppose first that \( (\omega_L, \omega_H) = (0.2, 0.5) \). In this case, \( \hat{\rho} \approx -0.28 \), so \( p \geq \hat{\rho} \) for all \( p \in (0,1) \).

In particular, this is the example of \( (\omega_L, \omega_H) \in \Omega^F \). Thus, the first best feasible tax schedule \( (t^*_{LL}, t^*_{LH}, t^*_{HL}, t^*_{HH}) = (0.2, 0.2, 0.5, 0.5) \) is obtained.

ii. Suppose that \( (\omega_L, \omega_H) = (0.1, 0.8) \). In this case, \( \hat{\rho} \approx 0.13 \).
(a) If \( p \geq \hat{\rho} \), the first best tax schedule \((t^*_LL, t^*_LH, t^*_HL, t^*_HH) = (0, 0.1, 0.8, 0.8)\) is obtained. Figure I.3(a) illustrates the case of \( p = 0.3 \).

(b) If \( p < \hat{\rho} \), by Proposition I.3.9, the second best tax schedule \( t^* = (t^*_LL, \omega_L, \omega_H) \) is obtained. Figure I.3(b) illustrates the case of \( p = 0.1 \) in which the optimal tax schedule is \((t^*_LL, t^*_LH, t^*_HL, t^*_HH) = (0.07, 0.1, 0.8, 0.8)\).

iii. Suppose that \((\omega_L, \omega_H) = (0.25, 0.8)\). In this case, \( \hat{\rho} \approx 0.4 \).

(c) If \( p = 0.35 \) as illustrated in Figure I.3(c), then \( T \approx 1.02 > 1 \), so the second best tax schedule \( t^* = (\omega_L, \omega_L, T - \omega_L, \omega_H) = (0.25, 0.25, 0.77, 0.8) \) is obtained.

(d) If \( p = 0.2 \) illustrated in Figure I.3(d), then \( T \approx 0.97 < 1 \), so the second best tax schedule \( t^* = (t^*_LL, \omega_L, T^o - \omega_L, \omega_H) = (0.21, 0.25, 0.75, 0.8) \) is obtained.

iv. Figure I.3(e)–(h) show some other cases that have the second best tax schedule for different parameter values. □

### I.4 Properties and Comparative Statics

#### I.4.1 Properties of Optimal Tax Schedules

First, consider the possibility of first best feasible taxation. For a first best feasible tax schedule to be a solution to the social planner’s problem, it should not give any incentive for an agent to misreport his endowment. Since the incentive compatibility constraint is unilateral in our model, this requirement says that a rich agent should have no incentive to lie. According to the characterization results in the previous section, when (i) \( p \geq \hat{\rho} \), or (ii) \((\omega_L, \omega_H) \in \Omega_2\), the corresponding first best feasible tax schedules can be a solution to the social planner’s problem (P2). For the case of (ii), the endowment of a poor agent is large enough that the first best feasible tax schedule could impose the same amount of tax on each agent for any case.\(^{23}\) Thus, a rich agent has no incentive to misreport his type. On the other hand, for the case of (i), since the overall endowment level of the economy is small enough (the case of \( \Omega^F \)) or the probability of low endowment is high enough, a rich agent worries mainly that too low an amount of public good would be provided if he misreports. Thus, he will not lie. Therefore, when the total endowment of the economy is low enough or high enough, first best feasible taxation satisfies the incentive compatibility constraint so that it can be the solution to (P2).

\(^{23}\) Note that it is assumed that an increasing tax schedule is used on \( \Omega_2 \).
\( t_{LL} = 0.1 \)

\( \omega_L = 0.1 \)

\( \omega_L + \omega_H = 0.9 \)

\( z() = 0 \)

\( W() = \bar{w} \)

\( 0 \quad 0.2 \quad 1 \)

\( 2\omega_L = 0.2 \quad \omega_L + \omega_H = 0.9 \)

(a) \( (\omega_L, \omega_H) = (0.1, 0.8), \ p = 0.3 \)

\( t_{LL} = 0.1 \)

\( \omega_L = 0.1 \)

\( \omega_L + \omega_H = 0.9 \)

\( z() = 0 \)

\( W() = \bar{w} \)

\( 0 \quad 0.2 \quad 1 \)

\( 2\omega_L = 0.2 \quad \omega_L + \omega_H = 0.9 \)

(b) \( (\omega_L, \omega_H) = (0.1, 0.8), \ p = 0.1 \)

\( t_{LL} = 0.25 \)

\( \omega_L = 0.25 \)

\( \omega_L + \omega_H = 1.05 \)

\( z() = 0 \)

\( W() = \bar{w} \)

\( 0 \quad 0.5 \quad 1 \)

\( 2\omega_L = 0.5 \quad \omega_L + \omega_H = 1.05 \)

(c) \( (\omega_L, \omega_H) = (0.25, 0.8), \ p = 0.35 \)

\( t_{LL} = 0.6 \)

\( \omega_L = 0.6 \)

\( \omega_L + \omega_H = 1.15 \)

\( z() = 0 \)

\( W() = \bar{w} \)

\( 0 \quad 0.7 \quad 1 \)

\( 2\omega_L = 0.7 \quad \omega_L + \omega_H = 1.15 \)

(d) \( (\omega_L, \omega_H) = (0.25, 0.8), \ p = 0.2 \)

\( t_{LL} = 0.6 \)

\( \omega_L = 0.6 \)

\( \omega_L + \omega_H = 1.4 \)

\( z() = 0 \)

\( W() = \bar{w} \)

\( 0 \quad 1 \quad 1.2 \quad 1.4 \)

\( 2\omega_L = 0.6 \quad \omega_L + \omega_H = 1.15 \)

(e) \( (\omega_L, \omega_H) = (0.35, 0.8), \ p = 0.5 \)

\( t_{LL} = 0.6 \)

\( \omega_L = 0.6 \)

\( \omega_L + \omega_H = 0.85 \)

\( z() = 0 \)

\( W() = \bar{w} \)

\( 0 \quad 0.4 \quad 1 \)

\( 2\omega_L = 0.4 \quad \omega_L + \omega_H = 0.85 \)

(f) \( (\omega_L, \omega_H) = (0.6, 0.8), \ p = 0.5 \)

\( t_{LL} = 0.6 \)

\( \omega_L = 0.6 \)

\( \omega_L + \omega_H = 1.9 \)

\( z() = 0 \)

\( W() = \bar{w} \)

\( 0 \quad 1 \quad 1.2 \quad 1.9 \)

\( 2\omega_L = 1.2 \quad \omega_L + \omega_H = 1.9 \)

(g) \( (\omega_L, \omega_H) = (0.2, 1.3), \ p = 0.15 \)

\( t_{LL} = 0.6 \)

\( \omega_L = 0.6 \)

\( \omega_L + \omega_H = 1.8 \)

\( z() = 0 \)

\( W() = \bar{w} \)

\( 0 \quad 0.4 \quad 1 \)

\( 2\omega_L = 0.4 \quad \omega_L + \omega_H = 1.8 \)

(h) \( (\omega_L, \omega_H) = (0.6, 1.3), \ p = 0.5 \)

Figure I.3: Examples of Optimal Tax Schedules
Second, the optimal tax schedule always imposes 100% tax rate on a poor agent when his neighbor is rich. That is, \( t_{LH}^* = \omega_L \) for all \((\omega_L, \omega_H) \in \Omega_1\) and all \(p \in (0, 1)\). This result reflects the effect of informational rent on the economy which pursues efficiency rather than equity as its objective. Due to the no exaggeration assumption, the incentive compatibility constraint in this model is unilateral so that levying a tax on a poor agent does not create any incentive problem as long as it is feasible. Thus, the social planner, who does not mind which agent pays how much proportion of the total taxes, prefers to impose as much tax as possible on the poor agent rather than his rich neighbor who may request informational rent. Of course, the absolute amount of tax payment of rich agent is strictly higher than that of poor agent. However, since there do exist some cases in which the tax rate imposed on the rich agent is strictly less than 100%, we can say that the optimal tax mechanism is regressive.

Third, the optimal tax schedule is increasing.

**Proposition I.4.1** For all \((\omega_L, \omega_H) \in \Omega_1\),

\[
\begin{align*}
    t_{LL}^* &\leq t_{HL}^* \\
    t_{LH}^* &\leq t_{HH}^*.
\end{align*}
\]

**Proof**: The first inequality is clear since Lemma I.3.5 implies that \( t_{LL}^* \leq \omega_L \leq t_{HL}^* \). The second inequality is also clear since \( t_{LH}^* = \omega_L < \min\{ \omega_H, 1 \} \).

Note, however, that this does not imply that marginal tax rate is increasing.

Finally, note that for all \((\omega_L, \omega_H) \in \Omega\), \( t_{LL}^* \leq t_{LH}^* \), with strict inequality for some cases, as can be seen in Table I.1. That is, the payment by a poor agent is greater when his neighbor is rich than when his neighbor is poor. This bears the same intuition given above; a poor agent should take extra informational burden caused by his rich neighbor, which would not have existed had his neighbor been poor. Of course, he is likely to enjoy more of the public good if his neighbor is rich. Thus, the natural question that arises is where the poor agent would move if he could choose his own neighbor. In Chapter III, we tackle this problem in detail by comparing the immigration incentives of an agent to communities with different expected endowments.

**I.4.2 Comparative Statics**

One of the interesting questions about the optimal tax schedule is how the optimal tax schedule \( t^* \) will respond to the change in the probability of low endowment \( p \). We are also interested in how \( t^* \)

\[\text{As mentioned in Section I.2.1, each agent’s initial endowment is considered as his taxable wealth. Thus, the 100% tax rate is acceptable in this sense.}\]
will change as \( \omega_L \) or \( \omega_H \) varies. In the following, we exclude the trivial case \( \Omega_2 \) in which first best taxation is always possible.

I.4.2.1 Responses of \( t^* \) to \( p \)

Since both \( t^*_LH \) and \( t^*_HH \) are independent of \( p \), it suffices to analyze the responses of \( t^*_LL \) and \( t^*_HL \).

Given \((\omega_L, \omega_H) \in \Omega_1\), if \( p \geq \hat{\rho} \), then \( t^* \) is independent of \( p \). Thus, suppose \( p < \hat{\rho} \). By Lemma I.3.7, we have the following subcases.

(1) \( \omega_L + \omega_H \leq 1 \)

In this case, only \( t^*_LL \) depends on \( p \). Since the principal branch \( W_0 \) of the Lambert \( W \) function is strictly increasing, it follows that \( \frac{dt^*_LL}{dp} > 0 \), that is, \( t^*_LL \) is strictly increasing.

(2) \( \omega_L + \omega_H > 1 \)

Case (i) \( \tilde{t}_LL \geq \omega_L \): If \( \frac{1}{2} \leq 1 \), then only \( t^*_LL = t^*_0L \) depends on \( p \). By the definition of \( t^*_0L \), it turns out that

\[
\frac{dt^*_LL}{dp} \Bigg|_{z(T^0,t^*_0L;p)=0} = -\left( \frac{1}{p} \right) \frac{\log T}{T} \left|_{z(T^0,t^*_0L;p)=0} \right. \geq 0,
\]

since the nominator is negative by \( t^*_0L < 1 \) and the denominator is negative by \( T < T^0 \leq 1 = T^0 \).

Hence, \( t^*_LL \) is strictly increasing. If \( \frac{1}{2} > 1 \), then only \( t^*_HL = T^0 - \omega_L \) depends on \( p \). Since \( \tilde{T} \leq T \) and \( \frac{T^0 - 1}{p} > 2p^{-1} \) by Remark I.3.8, it follows that

\[
\frac{dT^0}{dp} \Bigg|_{z(T^0,\omega_L;p)=0} = -\left( \frac{1}{p} \right) \frac{\log T^0}{T^0} \left|_{z(T^0,\omega_L;p)=0} \right. \geq 0,
\]

which implies that \( t^*_HL \) is increasing.\(^{25}\)

Case (ii) \( \tilde{t}_LL < \omega_L \): In this case, we claim that \( 1 < T \leq T^0 \). By Remark I.3.8, \( T \leq T^0 \) is clear. To see that \( 1 < T \), note that

\[
z(1,\omega_L;p) = p \left[ \log \frac{1}{(2\omega_L)(T)} - 1 + 2\omega_L \right] + \log \tilde{T} = p \left[ \log \frac{1}{2\omega_L} - 1 + 2\omega_L \right] + (1 - p) \log \tilde{T} > 0.
\]

That is, the (IC)-curve is always above the point \((1,\omega_L)\), which implies \( 1 < T \). Thus, only \( t^*_HL = T^0 - \omega_L \) depends on \( p \). By (I.6), \( \frac{dT^0}{dp} < 0 \), so \( t^*_HL \) is strictly decreasing.

(3) Interpretation

\(^{25}\) The equality holds only if \( \tilde{t}_LL = \omega_L \).
Table I.2 summarizes the responses of $t^*$ to $p$ for each possible case. Figure I.4 shows the examples for some different endowment values.\(^{26}\) Roughly speaking, $t^*_{LL}$ is (weakly) increasing as $p$ increases. However, $t^*_{HL}$ is (weakly) increasing for low $\omega_L$, but decreasing for large $\omega_L$. In fact, these results show how the (IC) constraint will change as the probability of low endowment $p$ varies.

Suppose first that the initial low endowment is small enough such that $\tilde{t}_{LL} \geq \omega_L$ (the areas of $\Omega_{\mathbf{L}_1}$ and $\Omega_{\mathbf{L}_2}$ in Figure I.2). In this case, the increase in $p$ makes the (IC) constraint less tight for both $t^*_{LL}$ and $t^*_{HL}$ in the sense that the set of incentive compatible and feasible tax schedules becomes larger.\(^{27}\) Thus, the social planner can increase $t^*_{LL}$ or $t^*_{HL}$ as long as the feasibility constraint is binding. That is, for $\mathcal{T} \leq 1$ (the interior solution case), $t^*_{LL}$ increases but $t^*_{HL}$ stays the same since the solution always occurs at $T^o = 1$, and for $\mathcal{T} > 1$ (the corner solution case), $t^*_{HL}$ increases but $t^*_{LL}$ is fixed at its feasible maximum $\omega_L$. The economic intuition behind this result is as follows: The social planner has to take into account the informational problem caused by a rich agent. When $\omega_L$ is relatively small such that $\tilde{t}_{LL} \geq \omega_L$, the social planner can design an incentive compatible tax mechanism without much worrying about such an informational problem because the rich agent is reluctant to lie to avoid too small provision of public good. Thus, as $p$ increases, the rich agent (IC) constraint becomes less tight (see Figure I.4(a),(b),(e), and (f)).

On the other hand, suppose that the initial low endowment $\omega_L$ is relatively large such that $\tilde{t}_{LL} < \omega_L$ (the area of $\Omega_{\mathbf{H}}$ in Figure I.2).\(^{28}\) In this case, the increase in $p$ makes the (IC) constraint less tight for $t^*_{LL}$ but tighter for $t^*_{HL}$, in the sense that the set of incentive compatible and feasible tax schedules becomes larger with respect to $t^*_{LL}$, but smaller with respect to $t^*_{HL}$. Thus, the social

<table>
<thead>
<tr>
<th>Cases</th>
<th>$\frac{dt^*_{LL}}{dp}$</th>
<th>$\frac{dt^*_{HL}}{dp}$</th>
<th>$\frac{dt^*_{LL}}{dp}$</th>
<th>$\frac{dt^*_{HL}}{dp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \geq \hat{p}$ (including $\Omega^F$)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$p &lt; \hat{p}$</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\omega_L + \omega_H \leq 1$</td>
<td>$\tilde{t}_{LL} \geq \omega_L$</td>
<td>$\mathcal{T} \leq 1$</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{T} &gt; 1$</td>
<td>0</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>$\tilde{t}_{LL} &lt; \omega_L$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

\(^{26}\) In Figure I.4, $\beta \in (0,1)$ is defined such that $z(T^o, \omega_L; \beta) = 0$.

\(^{27}\) Recall by Lemma I.3.7 that the (IC)-curve $z(T, t^*_{LL}; p) = 0$ turns counterclockwise around the pivotal point $(\bar{T}, \bar{t}_{LL})$ as $p$ increases.

\(^{28}\) Note that in this case, $\overline{T} > 1$ by Lemma I.3.6.
planner would like to decrease $t^*_HL$ and increase $t^*_LL$ as long as the feasibility constraint is binding. Since in this case $t^*_LL$ is already set at its maximum $\omega_L$ (the corner solution case), only $t^*_HL$ should be decreased. This result can be interpreted as follows: If $\omega_L$ is relatively large, then too small public good provision is no longer a big problem. Thus, as $p$ increases, the rich agent is more willing to lie, which implies that the social planner should decrease the rich agent’s tax payment $t^*_HL$ to make him honest (see Figure I.4(c),(d),(g), and (h)).

I.4.2.2 Responses of $t^*$ to $\omega_L$ or $\omega_H$

We have already studied rough responses of $t^*$ to $\omega_L$ or $\omega_H$ by its characterization for different cases on $\Omega$. Now, we provide some simulated examples for better understanding the optimal tax mechanism. Figure I.5(a)–(d) show the responses of $t^*$ to $\omega_L$ and Figure I.5(e)–(h) to $\omega_H$ when $p = 0.15$ or 0.5.

Consider first the responses to a change in $\omega_L$. As $\omega_L$ increases, we can see that $t^*_LL$ is increasing, but $t^*_HL$ is (weakly) decreasing for lower values of $\omega_L$ and then increasing for larger values. These results can be interpreted as follows. For lower $\omega_L$, a first best solution like Figure I.3(a) is possible so that $t^*_LL$ and $t^*_HL$ are set at their maximum. As $\omega_L$ increases more, a corner solution like Figure I.3(b) could occur depending on the values of $\omega_H$ and $p$. In this case, $t^*_HL$ stays the same at its maximal but $t^*_LL$ would be less than $\omega_L$. As $\omega_L$ increases furthermore, then an interior solution like Figure I.3(d) will happen. In this case, $t^*_LL$ will increase but $t^*_HL$ will decrease. As $\omega_L$ increases even further, a corner solution like Figure I.3(e) or (f) is obtained. In this case, $t^*_HL$ is decreasing initially and then increasing, and $t^*_LL$ is increasing with $\omega_L$. One of the interesting implications from these results is that an increase in $\omega_L$ can decrease the tax burden of a rich agent if $\omega_L$ is relatively small.

We can apply a similar interpretation to the change of $\omega_H$. That is, for lower $\omega_H$, a first best solution like Figure I.3(a) is obtained, and then a corner solution like Figure I.3(b) and/or an interior solution like Figure I.3(d) is obtained depending on the values of $\omega_L$ and $p$. Eventually, the optimal solution ends up an interior one like Figure I.3(d) or a corner solution like Figure I.3(e) or (f). Notice also that an increase in $\omega_H$ can decrease the tax burden of a poor agent if $\omega_L$ is relatively small and $\omega_H$ is relatively large (see Figure I.3(e)).
Figure I.4: Responses of $t_{LL}^*$ and $t_{HL}^*$ to $p$
Figure I.5: Responses of $t^*_{LL}$ and $t^*_{HL}$ to $\omega_L$ and $\omega_H$
Finally, we show how much public goods will be provided as \( p, \omega_L, \) or \( \omega_H \) varies. The expected total provision of public good is expressed as

\[
\mathbb{E}(y) = p^2 (2t_{LL}^*) + 2p(1-p) (t_{LH}^* + t_{HL}^*) + (1-p)^2 (2t_{HH}^*).
\]

The Figure I.6 illustrates some examples of those responses. Roughly speaking, \( \mathbb{E}(y) \) increases as \( \omega_L \) or \( \omega_H \) increase, and as \( p \) decreases.\(^\text{29}\) In particular, for large \( \omega_H \), the increase in \( \omega_L \) may reduce \( \mathbb{E}(y) \) (Figure I.6(d)). This fact reflects the observation that the increase of \( \omega_L \) may decrease \( t_{HL}^* \) so much. Thus, even if \( t_{LL}^* \) and \( t_{LH}^* \) increase with \( \omega_L \), the decrease of \( t_{HL}^* \) is still dominating, which

\(^{29}\) Although not provided here, the case in which \( \omega_L \) is quite small, say 0.05, shows the possibility that \( \mathbb{E}(y) \) increases as \( p \) increases. This is due to that fact that for a quite small \( \omega_L \) the increase of \( t_{LL}^* \) or \( t_{HL}^* \) as a small \( p \) increases (see Figure I.4(a)(b)(e) or (f)) could increase \( \mathbb{E}(y) \).
results in the smaller \( \mathbb{E}(y) \).

### I.5 Optimal Tax Mechanism for \( n > 2 \)

We now study the optimal tax schedule for \( 2 < n < \infty \), which is the solution to \((P_n)\).\(^{30}\) To begin with, consider the case of \((\omega_L, \omega_H) \in \Omega_2\). The social planner can easily solve \((P_n)\) by imposing a first best feasible tax schedule

\[
t^* \in \left\{ t \in [0, \omega_L]^n \times [0, \omega_H]^n : t_{L,(n-1)L} = 1, \ t_{H,0L,(n-1)H} = 1; \ (n-k)t_{L,(n-1-k)L,kH} + kt_{H,(n-k)L,(k-1)H} = n, \text{ and} \right. \\
1 \leq t_{L,(n-1-k)L,kH} \leq \omega_L, \text{ for } k = 1, \ldots, n-1 \right\}.
\]

since \( t^* \) satisfies the (IC) constraint;\(^{31}\)

\[
\Delta(t^F; p) = \sum_{j=1}^{n-1} n-j \left( (1-p)^{n-j} \left( t^F_{L,(j-1)L, n-jH} - 1 \right) \right) \geq 0. \tag{I.7}
\]

If the social planner insists that the tax schedule be increasing, then the unique solution to \((P_n)\) is \( t^* = (1, \ldots, 1; 1, \ldots, 1) \). Therefore, in the following we just assume that \((\omega_L, \omega_H) \in \Omega_1\).

#### I.5.1 Possibility of First Best Taxation

First of all, partition \( \Omega_1 \) into

\[
\Omega_{1A} = \{ (\omega_L, \omega_H) \in \Omega_1 : \omega_H \leq 1 \}, \\
\Omega_{1B_l} = \{ (\omega_L, \omega_H) \in \Omega_1 \setminus \Omega_{1A} : i \omega_L + (n-i) \omega_H < n \leq (i-1) \omega_L + (n-i+1) \omega_H \},
\]

\(^{30}\) The problem \((P_n)\) for the \( n \)-agent case is expressed explicitly as,

\[
\max_t W(t; p) = \sum_{j=0}^{n} nC_j p^j (1 - p)^{n-j} \left[ \log \left( j_{L,(j-1)L} n-jH + (n-j) t_{H,(j-1)L} n-jH \right) \right. \\
- \left. \left( j_{L,(j-1)L} n-jH + (n-j) t_{H,(j-1)L} n-jH \right) \right] + n \left( p \omega_L + (1-p) \omega_H \right)
\]

subject to

\[
(P_n) \quad \sum_{j=0}^{n-1} n-j \left( (1-p)^{n-j} \left( \log \left( j_{L,(j-1)L} n-jH \right) + (n-j) t_{H,jL} n-jH \right) \right. \\
- \left. \left( j_{L,(j-1)L} n-jH + (n-j) t_{H,(j-1)L} n-jH \right) \right] \right) \\
\text{(IC)} \quad \sum_{j=0}^{n-1} n-j \left( (1-p)^{n-j} \left[ \log \left( (j+1) t_{H,L} n-jH \right) \\
+ (n-1-j) t_{H,(j+1)L} n-jH \right] - t_{H,jL} n-jH \right) \right) \right),
\]

\[
\text{(Feasibility)} \quad 0 \leq t_{L,(n-k)L} \leq \omega_L, \quad \forall \ k = 1, \ldots, n, \\
0 \leq t_{H,(n-k)L} \leq \omega_H
\]

where \( nC_j \) is the number of ways of choosing \( j \) unordered outcomes from \( n \) possibilities. That is, \( nC_j = \binom{n}{j} = \frac{n!}{j!(n-j)!} \).

\(^{31}\) See Appendix A.1 for derivation.
for $i = 1, \ldots, n$. If $(\omega_L, \omega_H) \in \Omega_{1B}$, $i = 1, \ldots, n$, the first best feasible tax schedule is
\[
t^F \in \left\{ t \in [0, \omega_L]^n \times [0, \omega_H]^n : t_{L,(n-1)L,0H} = \omega_L, \ t_{H,0L,(n-1)H} = 1; \right.\]
\[
t_{L,(n-1-k)L,kH} = \omega_L, \ t_{H,(n-k)L,(k-1)H} = \omega_H, \ k = 1, \ldots, n - i; \]
\[
(n - k)t_{L,(n-1-k)L,kH} + kt_{H,(n-k)L,(k-1)H} = n, \ k = n - i + 1, \ldots, n - 1 \}.
\]
Since
\[
\lim_{p \to 0} \Delta(t^F; p) = \begin{cases} \log \frac{n}{\omega_L + (n-1)\omega_H} - (1 - \omega_L) & \text{if } i = 1 \\ -(1 - t^F_{L,0L,(n-1)\omega_H}) & \text{otherwise} \end{cases} < 0, \tag{I.8}
\]
there is no $(\omega_L, \omega_H) \in \Omega_{1B}$ for $i = 1, \ldots, n$ such that the first best feasible tax schedule $t^F$ can be the solution to $(P_n)$ for every $p \in (0, 1)$. Also, it turns out that the polynomial equation $\Delta(t^F; p) = 0$ of $p$ may have multiple roots so that it is impossible to define the unique $\bar{p}$ such that $\Delta(t^F; p) \geq 0$ for $p \geq \bar{p}$.

Now, suppose that $(\omega_L, \omega_H) \in \Omega_{1A}$. In this case, the unique first best feasible tax schedule is
\[
t^F = (\omega_L, \ldots, \omega_L; \omega_H, \ldots, \omega_H).
\]
To find the condition under which $t^F$ is the solution to $(P_n)$, consider the (IC) constraint at $t^*:\[33]\]
\[
\Delta(t^F; p) = \sum_{j=0}^{n-1} C_j p^{n-1-j} \left[ \sum_{k=0}^{n-j} (-1)^{k+\text{mod}(n-1-j,2)} n_j C_k \log (k\omega_L + (n-k)\omega_H) \right] - (\omega_H - \omega_L),
\]
where $\text{mod}(x, 2)$ is 0 if $x$ is even and 1 if $x$ is odd.

**Lemma I.5.1** For all $n \geq 2$ and all $j = 0, \ldots, n - 2$, $\Delta(t^F; p)$ is strictly increasing in $p^{n-1-j}$.

**Proof:** See Appendix A.3. \[\square\]

For $(\omega_L, \omega_H) \in \Omega_{1A}$, define $\bar{p} \in \mathbb{R}$ by $\Delta(t^F; \bar{p}) = 0$, and let $\bar{p} = \min\{1, \bar{p}\}$. Define also
\[
\Omega^F = \{(\omega_L, \omega_H) \in \Omega_{1A} : \lim_{p \to 0} \Delta(t^F; p) = \log \frac{n\omega_H}{\omega_L + (n-1)\omega_H} - (\omega_H - \omega_L) \geq 0 \}.
\]

**Proposition I.5.2** If $p \geq \bar{p}$, then the first best feasible tax schedule $t^F$ is the unique solution to $(P_n)$. In particular, if $(\omega_L, \omega_H) \in \Omega^F$, then $t^F$ is the unique solution to $(P_n)$ for all $p \in (0, 1)$.

**Proof:** Same as the proof of Proposition I.3.2. \[\square\]
Corollary I.5.3 \( \lim_{n \to \infty} \Omega^F = \emptyset. \)

Proof: Since \( \lim_{n \to \infty} \lim_{p \to 0} \Delta(t^F;p) = -(\omega_H - \omega_L) < 0, \) we have the result.

The intuition of this result is that as the number of agents increases, the incentive for a rich agent to misreport his type increases because the possibility that too little public good is provided decreases. Thus, it becomes more difficult to satisfy the (IC) constraint and finally the possibility of first best feasible tax schedule gets to disappear.

I.5.2 Second Best Feasible Tax Schedule

The same result as Lemma I.3.3 holds for \( n > 2. \)

Proposition I.5.4 Suppose that a tax schedule \( t^* \) is a solution to \((P_n)\). Then,

\[
t_{H,0L,(n-1)H}^* = \min\{\omega_H,1\}.
\]

Proof: Same as the proof of Lemma I.3.3.

On the contrary, the result like Lemmas I.3.5 does not hold for \( n > 2. \) Nonetheless, we can find a similar result with a mild assumption.

Proposition I.5.5 Suppose that a tax schedule \( t^* \) is a solution to \((P_n)\). For each \( j \in \{1, \ldots, n-1\}, \) if \( t_{H,jL,(n-1-j)H}^* > 0, \) then

\[
t_{L,(j-1)L,(n-j)H}^* = \omega_L.
\]

Proof: If \( \omega_L = 0, \) then the claim is obvious. Thus, assume that \( \omega_L > 0. \) Suppose by way of contradiction that \( t_{L,(j-1)L,(n-j)H}^* < \omega_L. \) Choose \( \varepsilon \) such that

\[
0 < \varepsilon \leq \min\{\omega_L - t_{L,(j-1)L,(n-j)H}^*, t_{H,jL,(n-1-j)H}^*\}.
\]

Consider another tax schedule \( t' \) which replaces \( t_{L,(j-1)L,(n-j)H}^* \) and \( t_{H,jL,(n-1-j)H}^* \) in \( t^* \) by

\[
t_{L,(j-1)L,(n-j)H}^* = t_{L,(j-1)L,(n-j)H}^* + \frac{\varepsilon}{j}, \quad \text{and} \quad t_{H,jL,(n-1-j)H}^* = t_{H,jL,(n-1-j)H}^* - \frac{\varepsilon}{n-j},
\]

respectively. Then, it follows that

\[
U_i(\omega_H|\omega_H, t') = U_i(\omega_H|\omega_H, t^*) + n-1C_j p^j (1 - p)^{n-1-j} \left( \frac{\varepsilon}{n-j} \right) \\
> U_i(\omega_L|\omega_H, t^*) \\
= U_i(\omega_W|\omega_W, t^*) + n-1C_j-1 p^{j-1} (1 - p)^{n-j} \left( \frac{\varepsilon}{j} \right) \\
> U_i(\omega_L|\omega_H, t'),
\]

29
which implies that \( \Delta(t'; p) > 0 \). Since \( \Delta(t; p) \) is continuous in \( t_{H,jL,(n-1-j)H} \) we can choose \( \delta \in (0, \varepsilon) \) such that a new tax schedule \( t'' \), which replaces \( t'_{H,jL,(n-1-j)H} \) in \( t' \) by \( t''_{H,jL,(n-1-j)H} = t'_{H,jL,(n-1-j)H} + \frac{\delta}{n-j} \), satisfies (IC). Note that \( t'' \) also satisfies (Feasibility) by its construction, and that

\[
j t''_{L,(j-1)L,(n-j)H} + (n-j)t''_{H,jL,(n-1-j)H} > j t^*_{L,(j-1)L,(n-j)H} + (n-j)t^*_{H,jL,(n-1-j)H}.
\]

Thus, we have \( W(t''; p) > W(t^*; p) \), a contradiction to the hypothesis that \( t^* \) is a solution.

This result implies that if the tax payment of rich agents is at least positive, poor agents have to pay 100\% tax in the optimal taxation scheme. Thus, the same interpretation as for the two-agent case applies; the optimal tax schedule imposes the burden caused by the rich agents’ informational rents on poor agents as much as possible. Also, it is possible that the tax payment of rich agents is \textit{absolutely} less than that of poor agents since the optimal tax schedule may not be increasing.\(^\text{34}\)

Therefore, as the number of agents increases, the optimal tax mechanism can be more regressive.

Furthermore, in addition to the above positivity assumption, if we assume that the tax schedule must be increasing and \( t_{H,(n-1)L,0H} \geq \omega_L \), we can show that the same result as Lemma I.3.5 for \( n > 2 \). However, even if we restrict the domain of tax mechanism by those assumptions, it is virtually impossible to describe the optimal tax schedule for \( n > 2 \) due to not only too many corner solutions but also the high dimensionality of social planner’s problem. In the next section, we study the case in which there is a continuum of agents in the economy.

**I.5.3 Infinitely Large Economy: \( n = \infty \)**

Suppose that there are infinitely many agents in the economy. With no loss of generality, we normalize the set of agents as \( N = [0, 1] \). In this case, a tax schedule can be expressed by \( t_\infty = (t_{L,\infty}, t_{H,\infty}) \) where for example \( t_{L,\infty} \) is the tax payment of agent \( i \in N \) when he reports \( \omega_L \). Since \( \Pr(\omega_j = \omega_L) = p \) for all \( j \in N \), the expected utility of agent \( i \) when his endowment is \( \omega_H \) is

\[
U_i(\omega_H|\omega_H, t_\infty) = \log(p t_{L,\infty} + (1-p) t_{H,\infty}) + \omega_H - t_{H,\infty}
\]

if he reports \( \omega_H \), and

\[
U_i(\omega_L|\omega_H, t_\infty) = \log(p t_{L,\infty} + (1-p) t_{H,\infty}) + \omega_H - t_{L,\infty}
\]

\(^\text{34}\) For the three-agent case, we can show that there exists a non-increasing optimal tax schedule under the assumption that a rich agent’s tax payment is positive.
if he reports $\omega_L$. Thus, given $(\omega_L, \omega_H) \in \Omega$ and $p \in (0, 1)$, the social planner’s problem is

$$
\max_{t_\infty} W(t_\infty; p) = \log(pt_{L,\infty} + (1-p)t_{H,\infty}) - (pt_{L,\infty} + (1-p)t_{H,\infty}) 
$$

subject to

(P$_\infty$) \hspace{1cm} (IC) \hspace{1cm} $t_{L,\infty} \geq t_{H,\infty}$

(Feasibility) \hspace{1cm} $(t_{L,\infty}, t_{H,\infty}) \in [0, \omega_L] \times [0, \omega_H]$. \hspace{1cm}

For $(\omega_L, \omega_H) \in \Omega_2$, the optimal tax schedule is

$$
t^*_\infty \in \{(t_{L,\infty}, t_{H,\infty}) \in [0, \omega_L] \times [0, \omega_H] : pt_{L,\infty} + (1-p)t_{H,\infty} = 1, t_{L,\infty} \geq t_{H,\infty}\}.
$$

If the social planner insists that the tax schedule be increasing, then the unique second best feasible tax schedule is

$$
t^*_\infty = (t^*_{L,\infty}, t^*_{H,\infty}) = (1, 1).
$$

**Proposition I.5.6** For $(\omega_L, \omega_H) \in \Omega_1$, the unique optimal tax schedule is

$$
t^*_\infty = (t^*_{L,\infty}, t^*_{H,\infty}) = (\omega_L, \omega_L).
$$

**Proof**: Straightforward. \hfill $\blacksquare$

This result implies that there is no way to prevent a rich agent from lying if $t_{H,\infty}$ is greater than $t_{L,\infty}$, so that imposing $\omega_L$ on every agent is optimal for $n = \infty$. Note that the poor agent’s tax rate is always 100%, but the rich agent’s is strictly less than 100%.

### I.6 Concluding Remarks

Most of the previous mechanism design models on a public good economy assume private information about the agent’s _valuation_ of the public good. In this essay, we introduced private information about the agent’s _wealth_, which is in some sense more realistic in practical taxation problem for a small economy. We modelled an optimal taxation problem using the Bayesian mechanism design approach, and fully characterized its solution for an economy with two agents. The comparative statics analyses provided some interesting implications that are helpful for better understanding the incentive compatibility and feasibility constraints. Also, we provided some partial characterization results for the case of more than two agents. In the following, we discuss some extensions of this study.

In this essay, we assumed that the social welfare is the sum of the agents’ utilities. However, if we assume another form of social welfare functions such as _weighted_ sum of the agents’ utilities
or Rawlsian welfare function, then finding an optimal tax mechanism would be a quite difficult problem because we are no longer able to reduce the dimension of the social planner’s problem. In fact, the feasibility constraint renders the optimal tax mechanism to have so many corner solutions that the high dimensionality of the problem will produce too many cases to handle.

Our optimal tax mechanism is not renegotiation-proof. Consider for example of Figure I.3(d). If two agents are both poor, then each one’s tax payment is $t_{LL}^* \approx 0.21 < 0.25 = \omega_L$. Thus, after the optimal taxation, they have an ex-post incentive to renegotiate for increasing the underprovided public good since the marginal benefit from the increase in public good is greater than the marginal cost, $-1$. In this case, we can easily make our optimal tax mechanism renegotiation-proof by imposing a constraint that $t_{LL}^* = \omega_L$ if $\omega_L \leq 1/2$ and $t_{LL}^* > 1/2$ if $\omega_L > 1/2$. Note that this renegotiation-proofness decreases $t_{HL}^*$ for some cases.

Finally, we may consider the model with more than two types. If a continuous type space is employed for each agent, we have to deal with the (IC) constraint which has a form of partial differential equation or inequality. Unfortunately, the standard differential approach used in mechanism design literature (e.g., Laffont and Maskin [1979]) is not applicable to this case. Thus, it is an open question in the future research how to transform such a partial differential equation suitable to be incorporated into the social planner’s objective function.
Chapter II

Optimal Feasible Tax Mechanism for a Heterogeneous Economy

II.1 Introduction

The main goal of this essay is to extend the model of optimal tax mechanism developed in the first essay to a heterogeneous economy in which agents can be distinguishable by some characteristics that are publicly observable. Each agent in an economy has many characteristics that determine his economic status. Most of the characteristics are observable only privately, but some of them are publicly observable. In particular, one can easily recognize that some publicly-observable characteristics of an agent are closely related to his or her wealth. Such characteristics include race (minority vs. majority), sex (female vs. male), class (blue-collar vs. white-collar) and so on. Thus, it is an important question how the optimal (property) tax mechanism will be affected by those observable characteristics that contain partial information about an agent’s wealth.

We already studied in the first essay the optimal tax mechanism and its properties for a homogeneous economy in which agents are indistinguishable. Using the same model as in the homogeneous case, we provide in this essay a full characterization of optimal tax mechanism for a heterogeneous economy with two agents, a minority agent with high probability to be poor and a majority agent with low probability to be poor. Although there are a bit more cases to be considered since there are two incentive compatibility constraints, finding the optimal tax schedule for a heterogeneous economy is virtually the same as for the homogeneous economy. As a matter of fact, the same results as in Chapter I are obtained; under incentive compatibility and feasibility, the optimal tax mechanism is (i) first best when the expected total endowment of the economy is low or high enough, (ii) regressive, and (iii) increasing. In addition, we find another interesting implication for

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1 Census Data indicate that there are notable income gaps by race and/or sex. In 2003, the average income is $33,185 for whites, $25,002 for blacks, and $41,057 for males, $23,619 for females. See U.S. Census Bureau [2004].
the heterogeneous case; (iv) if the level of low endowment is low, then only the incentive compatibility constraint of a rich minority agent will be binding, but otherwise, only that of a rich majority agent will be binding.\footnote{Strictly speaking, in the latter case, both the rich minority and rich majority agents’ incentive compatibility constraints are binding, but at an optimal tax mechanism, only the rich majority agent’s incentive compatibility is binding.} In other words, if a poor agent’s wealth is low, it is more likely for a rich minority to misreport his wealth than a rich majority, but otherwise \textit{vice versa}. This result follows because a rich majority worries more the too little provision of public goods than a rich minority if the level of low endowment is indeed low.

For a better understanding of the optimal tax mechanism, we conduct some comparative statics analyses when there is a change in the primitives of the economy. Although the there are many cases and corner solutions to be considered, the intuitions are similar to the corresponding homogeneous case.

The remainder of this essay is organized as follows.\footnote{We try to keep the same order of analyses as in Chapter I for the purpose of comparison, but omit many arguments to avoid repetition. Unless mentioned otherwise, the notations and arguments in Chapter I apply here after appropriately modified.} In Section 2, we present the model for a heterogeneous public good economy. In Section 3, we fully characterize the optimal tax schedule for the economy with two agents and two possible types. Using the characterization results, in Section 4, we discuss the properties of the optimal mechanism and provide some comparative statics analyses. In Section 5, we give concluding remarks.

II.2 The Model

Consider the same public good economy with two agents as given in Chapter I except the probability distribution of each agent’s endowment. The probabilities of agent 1 and 2 being poor are \( p \) and \( q \) respectively, where \( q > p \). That is,

\[
\Pr(\omega_1 = \omega_L) = p \in (0, 1),
\]

\[
\Pr(\omega_2 = \omega_L) = q \in (0, 1).
\]

We call agent 1 a \( p \)-type or \textit{majority} and agent 2 a \( q \)-type or \textit{minority}. This economy is called \textit{heterogeneous}.\footnote{Accordingly, the economy given in Chapter I is called \textit{homogeneous} since \( p = q \).}

Employing the same arguments and assumptions for a tax mechanism as in Chapter I, we can
express the social planner's problem as: Given \((\omega_L, \omega_H) \in \Omega\) and \(p, q \in (0, 1)\),

\[
\max_t \ W(t; p, q) = pq [2 \log(2t_{LL}) - 2t_{LL}]
\]

\[
+ [p(1-q) + (1-p)q] [2 \log(t_{LH} + t_{HL}) - (t_{LH} + t_{HL})]
\]

\[
+ (1-p)(1-q)[2 \log(2t_{HH}) - 2t_{HH}] + (p+q)\omega_L + (2-(p+q))\omega_H
\]

subject to

\[(P)\]

\[(IC_1)\]

\[
q[\log(t_{LH} + t_{HL}) - t_{HL}] + (1-q)[\log(2t_{HH}) - t_{HH}] \\geq q[\log(2t_{LL}) - t_{LL}] + (1-q)[\log(t_{LH} + t_{HL}) - t_{LH}];
\]

\[(IC_2)\]

\[
p[\log(t_{LH} + t_{HL}) - t_{HL}] + (1-p)[\log(2t_{HH}) - t_{HH}] \\geq p[\log(2t_{LL}) - t_{LL}] + (1-p)[\log(t_{LH} + t_{HL}) - t_{LH}],
\]

\[(Feasibility)\]

\[
0 \leq t_{LL} \leq \omega_L, \quad 0 \leq t_{LH} \leq \omega_L, \\
0 \leq t_{HL} \leq \omega_L, \quad 0 \leq t_{HH} \leq \omega_H.
\]

II.3 optimal Tax Mechanism

II.3.1 Possibility of First Best Taxation

To begin with, consider the possibility of first best taxation, which is the solution to \((P)\) without \((IC_1)\) and \((IC_2)\). By the same arguments for the homogeneous economy, we have the same result.

**Proposition II.3.1** If \(p \geq \hat{\rho}\), then the first best feasible tax schedule \(t^F\) is the unique solution to \((P)\). In particular, if \((\omega_L, \omega_H) \in \Omega^F\), then \(t^F\) is the unique solution to \((P)\) for all \(p, q \in (0, 1), q > p\).

II.3.2 Second Best Tax Schedule

To characterize the second best tax schedule, assume that that \(p < \hat{\rho}\). Relative to the homogeneous economy in Chapter I, the heterogeneous economy has more cases to be considered since we need to deal with two incentive compatibility constraints. However, the arguments and definitions are almost the same as before. Only the difference are the slope of the welfare curve \(W(T, t_{LL}; p, q) = \bar{w}_5\) and some more definitions. The slope of the welfare function is still negative since for \(T, t_{LL} \in [0, 2] \times [0, 1],\)

\[
\frac{dt_{LL}}{dT} \bigg|_{W(T, t_{LL}; p, q) = \bar{w}} = \frac{-[p(1-q) + (1-p)q] (\frac{T}{2} - 1)}{2pq \left( \frac{1}{t_{LL}} - 1 \right)} < 0. \quad (II.1)
\]

Note that \(W(\cdot)\) is now a function of \(p\) and \(q\).
Next, we need some more definitions. First, for \((\omega_L, \omega_H) \in \Omega_1\) such that \(\omega_L + \omega_H \leq 1\) and \(\rho \in (0, \bar{\rho})\), define \(t^p_{LL} \in (0, \omega_L)\) by \(z(\omega_L + \omega_H, 2, t^p_{LL}; \rho) = 0\). Second, for \(\rho \in (0, \bar{\rho})\), define \(T^p \in (2\omega_L, \min\{\omega_L + \omega_H, 2\})\) by \(z(T^p, \omega_L; \rho) = 0\). Third, define simply \(T^{po} = \frac{2\rho}{p + q}\). Finally, for \(\bar{T} < T^{po}\) and \(T^p < T^{po}\), define \(t^{po}_{LL} \in (0, \omega_L)\) by \(z(T^{po}, t^{po}_{LL}; \rho) = 0\).

Now, we state the main result of this essay.

**Proposition II.3.2** For \(p < \bar{\rho}\), the solution to (P) is: If \(\tilde{t}_{LL} \geq \omega_L\), then

\[
\tilde{t}^* = \begin{cases} 
\tilde{t}^p_{LL}, \omega_L, \omega_H, \omega_H & \text{if } \omega_L + \omega_H \leq T^{po} \\
t^{po}_{LL}, \omega_L, T^{po} - \omega_L, \min\{\omega_H, 1\} & \text{if } T^p \leq T^{po} < \omega_L + \omega_H \\
(\omega_L, \omega_L, T^p - \omega_L, \min\{\omega_H, 1\}) & \text{if } T^{po} < T^p 
\end{cases}
\]

If \(\tilde{t}_{LL} < \omega_L\), then

\[
\tilde{t}^* = \begin{cases} 
(\tilde{t}^{po}_{LL}, \omega_L, \bar{T} - \omega_L, \min\{\omega_H, 1\}) & \text{if } \bar{T} \leq T^{po} \\
t^{po}_{LL}, \omega_L, T^{po} - \omega_L, \min\{\omega_H, 1\} & \text{if } T^q \leq T^{po} < \bar{T} \\
(\omega_L, \omega_L, T^q - \omega_L, \min\{\omega_H, 1\}) & \text{if } T^{po} < T^q 
\end{cases}
\]

**Proof:** From (II.1), it follows that \(z(T, t_{LL}; q) = 0\) and \(z(T, t_{LL}; p) = 0\) are tangent to the welfare-curve \(W(T, t_{LL}; p, q) = \bar{w}\) at \((T^{po}, t^{po}_{LL})\) and \((T^{po}, t^{po}_{LL})\), respectively. The remaining part of proof is the same as for the homogeneous case *mutatis mutandis.*

Table II.1 summarize the optimal tax schedules and their relative size for each possible case.

### II.3.3 Simulated Examples

In this section, we illustrate some examples that show the specific optimal tax schedules for different parameter values. Due to the low dimensionality of the social planner’s problem, we can draw the

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6 To distinguish the definitions for (IC1) from those for (IC2), we add superscripts \(p\) and \(q\) to the old definitions. The explicit expressions for each variable can be easily deduced by the same calculations as in Chapter I and the Lambert \(W\) function.
Figure II.1: Examples of second best taxation
results graphically.

**Example II.3.3** Suppose that \((p,q) = (\frac{1}{3}, \frac{2}{3})\).

i. Suppose that \(\omega_H = 0.8\).

(a) If \(\omega_L = 0.15\), \(\hat{\rho} \approx 0.20\). Since \(p \geq \hat{\rho}\), the first best tax schedule \((t^*_{LL}, t^*_{HL}, t^*_{HH}, t^*_{SH}) = (0.15, 0.15, 0.8, 0.8)\) is obtained (Figure II.1(a)).

(b) If \(\omega_L = 0.25\), then \(\tilde{t}_{LL} \approx 0.33 > \omega_L\) and \(\hat{\rho} \approx 0.40\). Since \(T^{po} = \frac{2}{3} < 1.02 \approx T^p\), by Proposition II.3.2, the second best tax schedule \(t^* = (\omega_L, \omega_L, T^p - \omega_L, \omega_H)\) is obtained. Figure II.1(b) illustrates this case in which the optimal tax schedule is \((t^*_{LL}, t^*_{HL}, t^*_{HH}) = (0.25, 0.25, 0.77, 0.8)\).

(c) If \(\omega_L = 0.35\), then \(\tilde{t}_{LL} \approx 0.38 > \omega_L\), \(\hat{\rho} \approx 0.72\) and \(T^{po} = \frac{2}{3} < 1.05 \approx T^p\), thus the second best tax schedule \(t^* = (\omega_L, \omega_L, T^p - \omega_L, \omega_H)\) is obtained, too. Figure II.1(b) illustrates this case in which the optimal tax schedule is \((t^*_{LL}, t^*_{HL}, t^*_{HH}) = (0.35, 0.35, 0.70, 0.8)\).

(d) If \(\omega_L = 0.6\), then \(\tilde{t}_{LL} \approx 0.57 < \omega_L\) and \(\tilde{T} \approx 1.31 < \frac{4}{3} = T^{po}\). Thus, the second best tax schedule \(t^* = (\tilde{t}_{LL}, \omega_L, \tilde{T} - \omega_L, \omega_H)\) is obtained. Figure II.1(d) illustrates this case in which the optimal tax schedule is \((t^*_{LL}, t^*_{HL}, t^*_{HH}) = (0.57, 0.6, 0.71, 0.8)\).

ii. Suppose that \(\omega_H = 1.3\).

(e) If \(\omega_L = 0.15\), then \(\tilde{t}_{LL} \approx 0.28 > \omega_L\) and \(\hat{\rho} \approx 0.55\). Since \(T^{po} = \frac{2}{3} < 1.10 \approx T^p\), by Proposition II.3.2, the second best tax schedule \(t^* = (\omega_L, \omega_L, T^p - \omega_L, \omega_H)\) is obtained. Figure II.1(e) illustrates this case in which the optimal tax schedule is \((t^*_{LL}, t^*_{HL}, t^*_{HH}) = (0.15, 0.15, 0.95, 1)\).

(f) If \(\omega_L = 0.25\), then \(\tilde{t}_{LL} \approx 0.33 > \omega_L\) and \(\hat{\rho} \approx 0.86\). Since \(T^{po} = \frac{2}{3} < 1.04 \approx T^p\), by Proposition II.3.2, the second best tax schedule \(t^* = (\omega_L, \omega_L, T^p - \omega_L, \omega_H)\) is obtained. Figure II.1(f) illustrates this case in which the optimal tax schedule is \((t^*_{LL}, t^*_{HL}, t^*_{HH}) = (0.25, 0.25, 0.79, 1)\).

(g) If \(\omega_L = 0.35\), then \(\tilde{t}_{LL} \approx 0.38 > \omega_L\) and \(\hat{\rho} = 1\). Since \(T^{po} = \frac{2}{3} < 1.07 \approx T^p\), by Proposition II.3.2, the second best tax schedule \(t^* = (\omega_L, \omega_L, T^p - \omega_L, \omega_H)\) is obtained again. Figure II.1(g) illustrates this case in which the optimal tax schedule is \((t^*_{LL}, t^*_{HL}, t^*_{HH}) = (0.35, 0.35, 0.72, 1)\).

(h) If \(\omega_L = 0.6\), then \(\tilde{t}_{LL} \approx 0.56 < \omega_L\) and \(\tilde{T} \approx 1.34 > \frac{4}{3} = T^{po}\). Thus, by Proposition II.3.2, the second best tax schedule \(t^* = (t^{po}_{LL}, \omega_L, T^{po} - \omega_L, \omega_H)\) is obtained. Figure II.1(h) illustrates this case in which the optimal tax schedule is \((t^*_{LL}, t^*_{HL}, t^*_{HH}) = (0.56, 0.6, 0.73, 1)\) \(\square\)
II.4 Properties and Comparative Statics

II.4.1 Properties of optimal Tax Schedules

The results and intuitions in Chapter I are also valid to this heterogeneous economy. That is, (i) when the expected total endowments of the economy is relatively low or high enough, then first best taxation is possible; (ii) the second best feasible tax mechanism is regressive; and (iii) it is increasing. In addition, we have another interesting result about the incentive compatibility.

**Corollary II.4.1** *If the initial endowment \((\omega_L, \omega_H)\) satisfies \(\tilde{t}_{LL} \geq \omega_L\), then only the incentive compatibility constraint \((IC_2)\) is binding at \(t^*\). Otherwise, only \((IC_1)\) is binding.*

*Proof:* By Lemma I.3.4 and Proposition II.3.2, the result is straightforward.

According to Figure I.2, the subset of \(\Omega_1\) that satisfies \(\tilde{t}_{LL} \geq \omega_L\) (the areas of \(\Omega_{l_1}\) and \(\Omega_{l_2}\)) represents the case in which the amount of low endowment \(\omega_L\) is relatively low. Thus, this corollary can be interpreted as follows: If the endowment level of a poor agent is indeed low, a rich agent is reluctant to pretend to be poor since the size of public good provision could be too small provided the other agent was poor (and reported truthfully). Note that a rich majority agent is more reluctant than a rich minority agent because by definition a minority agent is more likely to be poor than a majority agent. Thus, the misreport incentive of a rich minority agent is greater than that of a rich majority. Combined with the no-exaggeration assumption, it follows that once the (IC) constraint of a rich minority agent has been satisfied, then that of a rich majority agent is obviously satisfied. On the other hand, if the level of low endowment is relatively high, the (IC) constraints of rich minority and majority are both binding, but in fact we find that only the rich majority’s (IC) constraint is binding at a solution.

II.4.2 Comparative Statics

In this section, we first study the responses of \(t^*\) to \(p\) and \(q\) analytically, and then show some examples of the responses of \(t^*\) to \(\omega_L\) and \(\omega_H\) by simulation approach. In the following, we exclude the trivial case \(\Omega_2\) in which first best taxation is always possible.

II.4.2.1 Responses of \(t^*\) to \(p\) and \(q\)

Since both \(t_{LH}^*\) and \(t_{HH}^*\) are independent of \(p\) and \(q\), it suffices to analyze the responses of \(t_{LL}^*\) and \(t_{HL}^*\). Given \((\omega_L, \omega_H) \in \Omega_1\), if \(p \geq \bar{p}\), then \(t^*\) is independent of the change in \(p\) and \(q\). Thus, suppose \(p < \bar{p}\). By applying a similar analysis in Rhee [2004a], we can obtain the responses of \(t^*\) to \(p\) and
Figure II.2: Responses of $t_{LL}^*$ and $t_{HL}^*$ to $p$ and $q$
(a) $\omega_H = 0.8, \ (p, q) = (0.1, 0.12)$

(b) $\omega_H = 0.8, \ (p, q) = (\frac{1}{3}, \frac{2}{3})$

(c) $\omega_H = 1.3, \ (p, q) = (\frac{1}{3}, \frac{2}{3})$

(d) $\omega_L = 0.15, \ (p, q) = (0.1, 0.12)$

(e) $\omega_L = 0.25, \ (p, q) = (\frac{1}{3}, \frac{2}{3})$

(f) $\omega_L = 0.6, \ (p, q) = (\frac{1}{3}, \frac{2}{3})$

Figure II.3: Responses of $t_{LL}^*$ and $t_{HL}^*$ to $\omega_L$ and $\omega_H$
\( \omega_L = 0.25 \) and \( (p, q) = \left( \frac{1}{3}, \frac{2}{3} \right) \)

\( \omega_H = 0.8 \) and \( (p, q) = \left( \frac{1}{3}, \frac{2}{3} \right) \)

\( \omega_L = 0.6 \) and \( (p, q) = \left( \frac{1}{3}, \frac{2}{3} \right) \)

Figure II.4: Expected Total Provision of Public Good: \( E(y) \)
Table II.2: Responses of $t^*$ to $p$ and $q$.

$q$. Table II.2 summarizes the responses of $t^*$ to $p$ and $q$ and Figure II.2 provides some examples for different parameter values.

The economic intuition is as follows. First, consider the change in $p$. Suppose first that the initial low endowment is small enough such that $\bar{t}_{LL} \geq \omega_L$ (the areas of $\varOmega_1$ and $\varOmega_2$ in Figure I.2). In this case, by Corollary II.4.1, only (IC$_2$) is binding. Thus, the increase in $p$ makes the set of incentive compatible and feasible tax schedules larger, which implies that the social planner can increase $t^*_{LL}$ or $t^*_{HL}$ as long as the feasibility constraint is binding. Figure II.2(a) and (c) depict this case. Given $q = \frac{2}{3}$, if $p < \hat{\rho}$, a corner solution like Figure II.1(b) is obtained, so $t^*_{LL}$ stays at its maximum $\omega_L$ but $t^*_{HL}$ increases as $p$ increases. Suppose instead that $(\omega_L, \omega_H)$ satisfies $\bar{t}_{LL} < \omega_L$. Then, by Corollary II.4.1, only (IC$_1$) is binding. Since $T^q = \frac{2q}{p+q}$ is increasing in $q$, the effects of $q$ on $t^*_{LL}$ and $t^*_{HL}$ are the inverse to those of $p$. That is, as $q$ increases, $t^*_{LL}$ stays initially at its maximum, and then decreases, and finally ends up at $\bar{t}_{LL}$ whereas $t^*_{HL}$ decreases initially, and then increases, and finally ends up at $\bar{T} - \omega_L$. Refer to
Figure II.2(f) and (h).

II.4.2.2 Responses of $t^*$ to $\omega_L$ or $\omega_H$

We provide some simulated examples for the responses of $t^*_{LL}$ and $t^*_{HL}$ to $\omega_L$ or $\omega_H$. The similar interpretations given in the homogeneous case apply here, so it is omitted. However, we want to emphasize the role of (IC$_1$) and (IC$_2$) depending on the relative size of $\omega_L$. Figure II.3(a)–(c) show the responses of $t^*$ to $\omega_L$ and Figure II.3(d)–(f) to $\omega_H$ when $(p, q) = (0.1, 0.12)$ and $(p, q) = (\frac{1}{3}, \frac{2}{3})$.

II.4.2.3 Expected Total Provision of Public Good

Finally, we show how much public goods will be provided as $p$, $q$, $\omega_L$, or $\omega_H$ varies (Figure II.4(a)–(h) respectively). The expected total provision of public good is expressed as

$$\mathbb{E}(y) = pq(2t^*_{LL}) + [p(1 - q) + (1 - p)q](t^*_{LH} + t^*_{HL}) + (1 - p)(1 - q)(2t^*_{HH}).$$

The quite similar interpretations to the homogeneous case apply here, too.

II.5 Concluding Remarks

In this essay, we generalize the homogeneous economy model studied in the first essay when agents have publicly-observable wealth characteristics. Using a Bayesian mechanism design approach, we fully characterized the optimal tax mechanism for a heterogeneous economy with two agents, and conducted some comparative statics analyses of the mechanism. These characterization results show how the optimal tax mechanism deals with the incentive problem and the (individual) feasibility simultaneously. In the next chapter, we study the efficiency aspect of different local community formations and the immigration incentives of each agent, which combine and compare the homogeneous and models.
Chapter III

Optimal Formation and Immigration of Local Public Goods Economies

III.1 Introduction

It is frequently observed that some communities are “poor” while others are “rich,” and interestingly, such communities are often adjacent. Sometimes, these communities have their own characteristics that are publicly observable: minority vs. majority, blue-collar vs. white-collar, and so on.\(^1\) This observation is even clearer if we look at the widespread existence of slums in metropolitan areas. The natural questions that arise are, the normative one, whether this kind of community formation is socially efficient, and, the positive one, how it has been formed. The former is on optimal formation of local communities, and the latter is on migration incentive of agents. In this essay, these two questions are studied using the models given in the previous two essays.

We start with the normative question. Consider a public good economy which consists of two local communities;\(^2\) where the provision of local public good is financed by the property taxes collected from its local agents. We can think of the two extreme local community formations. One is the *homogeneous* formation in which agents in a community are indistinguishable ex ante, and the other is the *heterogeneous* formation in which agents in a community can be distinguished by some publicly-observable characteristics, e.g., by race. If each agent in the economy has private information about his own wealth (property or endowment), a social planner, who wants to maximize the sum of all agents’ utilities, has to solve two problems: the design of an optimal (property) tax mechanism that is incentive compatible and feasible, and the choice of optimal local community formation. The solution of the first problem was presented in the previous two essays as the opti-

\(^1\) See, Footnote 1 in Chapter II.

\(^2\) In this essay, we choose the broad term *community* rather than *jurisdiction* since our model is only distantly related to the legal distinction of local public good economies.
mal tax mechanisms for homogeneous and heterogeneous economies. In this essay, using the same models, we tackle the second problem. In some sense, this kind of optimal community formation is indeed “optimal” since the “optimal” tax mechanism is being used in each community.

As the positive question, we analyze the immigration incentives of an agent. In other words, if an agent can move and choose one of the two community formations, which community formation will he prefer? For example, if a minority agent chooses to have minority neighbors, then the community formation becomes homogeneous.

Our findings are as follows. To the normative question, if the expected total endowment of the economy is low, then the homogeneous formation is optimal, and otherwise, heterogeneous formation is optimal. To see the intuition behind this result, we note a trade-off between two effects. First, the incentive compatibility this economy faces is tighter in the heterogeneous formation than in the homogeneous one. That is, the misreporting incentive of agents is greater in the heterogeneous formation. This implies that the social welfare loss caused by agents’ private information is greater in heterogeneous formation. However, we also note that the social welfare loss caused by low provision of public good is greater in homogeneous formation than in the heterogeneous formation. Therefore, if the economy is expected to be poor, the latter effect dominates the former, and otherwise vice versa.

To the positive question, we find that a minority agent always prefers the heterogeneous formation while a majority agent always prefers the homogeneous formation regardless of their endowment. In other words, both minority and majority agents always want to move to the community expected to be rich. To understand this result, note that an agent’s immigration incentive is determined by his expected tax payment and the expected amount of local public good. According to the characterization results in Chapters I and II, both an agent’s expected tax payment and the level of expected public good are higher if he immigrates into a community expected to be rich rather than into a community expected to be poor. Thus, the uniform incentive to move into a community expected to be rich implies that the expected benefit from higher public good always dominates the loss from higher tax payment. In fact, this result on immigration incentive is consistent with the historical experiences of the metropolitan areas in the U.S. as well as in some other countries. During the last three decades, for example, the U.S. metropolitan areas observe that a great deal of low-income minority immigrants settle down in the center of the cities while many high-income majority residents move into the suburban areas (see Jargowsky [2001]). This experience reflects

\[ \text{In particular, note that even a poor agent, whose tax rate is 100\% in our model if his neighbor is rich, is willing to move into a community expected to be rich.} \]
that minority agents (mostly blacks) immigrates seeking to enjoy the high level of public good the metropolitan ares usually provide, whereas majority agents (mostly whites) would like to make their own homogeneous community by forming a high-cost housing area.

Our model is new but the topics of optimal community (group, jurisdiction, or nation) formation and immigration have been studied widely and deeply. In his seminal paper, Tiebout [1956] conjectured that for a large economy with little transaction costs, individuals will choose the best community by voting with their feet, which induces a Pareto optimal formation. Following his conjecture, many authors (see Wooders [1978,1980], Guesnerie and Oddou [1981], Greenberg and Weber [1986], McGuire [1991], Benabou [1993], Scotchmer [1997], Glomm and Lagunoff [1999], Jehiel and Scotchmer [2001], Conley and Konishi [2002], among others) have proved or disproved Tiebout’s conjecture under various situation and solution concepts. However, most of those results are based on the models where the public good provision are exogenously given or determined by simple tax schemes like proportional tax or (lump sum) head taxes (see Konishi, Breton, and Weber [1998]) and/or where the economy considered are large enough so that incentive aspect, so called integer problem are avoided. In this sense, our model is indeed optimal formation since the tax mechanism implemented in a local community is also optimal.\footnote{Note that in our Bayesian model the optimality is subject to second best.} Also, for a finite economy mostly suboptimal results prevails. As an interesting approach, Dagan and Volij [2000] study the optimal formation of nations under some different redistributive policy. However, their model also ignores the optimality of tax mechanism implemented in a nation.

The remainder of this essay is organized as follows. In Section 2, the formal model and the optimal feasible tax mechanisms are provided. The welfare comparison between heterogeneous and homogeneous formations is given in Section 3, and the immigration incentives of each agent depending on the realization of his endowment are studied in Section 4. Section 5 provides some concluding remarks.

III.2 The Model

III.2.1 The Economy

Consider a public good economy with four agents, which consists of two local communities. Let $N = \{1, 2, 3, 4\}$ denote the set of agents. Suppose that each community can accommodate up to two agents, so always ends up a two-agent community.\footnote{In our model, a single-agent community is inferior, so not considered.} For each community $c$, there is one private good $x \in \mathbb{R}_+$ (money) and one local pure public good $y_c \in \mathbb{R}_+$, where the private good can be
used to produce the public good according to a constant returns to scale technology. Without loss of generality, we normalize the production technology such that one unit of private good can be transformed into one unit of public good. Each agent \( i \in N \) has the same quasilinear von Neumann-Morgenstern utility function \( u \) on \( \mathbb{R}^2_+ \),

\[
    u(x_i, y_c) = \log y_c + x_i,
\]

where \( x_i \) is the consumption of private good by agent \( i \) and \( y_c \) is the consumption of local public good in community \( c \). Initially, each agent \( i \in N \) is endowed with private good \( \omega_i \in \{ \omega_L, \omega_H \} \) only, where \( 0 \leq \omega_L < \omega_H < \infty \). Agent \( i \) is called poor when \( \omega_i = \omega_L \) and rich when \( \omega_i = \omega_H \). Let

\[
    \Omega = \{ (\omega_L, \omega_H) \in \mathbb{R}^2_+ : \omega_L < \omega_H \}
\]

denote the set of all possible pairs of initial endowments.

The information structure of this economy is as follows. The primitives of the economy are common knowledge, whereas each agent has private information about his own endowment. That is, agent \( i \) knows the realization of his own endowment \( \omega_i \) and the initial probability distribution of the other agents’ endowments, but does not know the realizations of the other agents’ endowments \( \omega_{-i} \). The probabilities of agent 1 and 2 being poor are \( p \) and those of agent 3 and 4 are \( q \), which are all independent. That is,

\[
    \Pr(\omega_i = \omega_L) = \begin{cases} 
    p \in (0, 1) & \text{for } i = 1, 2 \\
    q \in (0, 1) & \text{for } i = 3, 4
\end{cases}
\]

where \( q > p \). We call agent 1 and 2 a \( p \)-type or majority and agent 3 and 4 a \( q \)-type or minority. A local community is in the homogeneous formation if majority agents and minority agents forms their own communities \{\{1, 2\}, \{3, 4\}\} and in the heterogeneous formation if one majority and one minority forms a community, \{\{1, 3\}, \{2, 4\}\} or \{\{1, 4\}, \{2, 3\}\}.

### III.2.2 Optimal Feasible Tax Mechanisms

According to the results in Chapter I and II, once we employ the same assumptions, we have the optimal feasible tax mechanism for each community in the homogeneous or heterogeneous formation.\(^6\)

\(^6\) Unless mentioned otherwise, throughout this essay, we employ the same notations and analyses as in Chapter I and II. Therefore, the partition \( \{ \Omega_1, \Omega_2 \} \) of \( \Omega \), \( \bar{\rho}, (\bar{T}, \bar{t}_{LL}) \), the function \( \Delta(\cdot) \), and the (IC) function \( z(\cdot) \) have the same definitions.
III.2.2.1 Homogeneous Formation

Consider the optimal feasible tax schedule for a homogeneous community with \( \rho \)-type agents,

\[
(t^*_{LL}, t^*_{LH}, t^*_{HL}, t^*_{HH}),
\]

which is the solution to the social planner’s problem: Given \((\omega_L, \omega_H) \in \Omega\) and \(\rho \in (0, 1)\),

\[
\max_{t} W^\rho(t) = \rho^2[2 \log(2t_{LL}) - 2t_{LL}] + 2(1 - \rho)[2 \log(t_{LH} + t_{HL}) - (t_{LH} + t_{HL})] \\
+ (1 - \rho)^2[2 \log(2t_{HH}) - 2t_{HH}] + 2(\rho \omega_L + (1 - \rho)\omega_H)
\]

subject to

\[
\rho \log(t_{LH} + t_{HL}) - t_{HL} + (1 - \rho)[\log(2t_{HH}) - t_{HH}],
\]

\((\text{Feasibility})\)

\[
0 \leq t_{LL} \leq \omega_L, \quad 0 \leq t_{LH} \leq \omega_L, \\
0 \leq t_{HL} \leq \omega_H, \quad 0 \leq t_{HH} \leq \omega_H.
\]

First, for \((\omega_L, \omega_H) \in \Omega_1\) such that \(\omega_L + \omega_H \leq 1\) and \(\rho \in (0, \hat{\rho})\), define \(t^\rho_{LL} \in (0, \omega_L)\) by \(z(\omega_L + \omega_H, 2, t^\rho_{LL}; \rho) = 0\). Second, for \(\rho \in (0, \hat{\rho})\), define \(T^\rho \in (2\omega_L, \min(\omega_L + \omega_H, 2))\) by \(z(T^\rho, \omega_L; \rho) = 0\).

Third, define simply \(T_0 = 1\). Finally, for \(\bar{T} < 1\) and \(T^\rho < 1\), define \(t^\rho_{LL} \in (0, \omega_L)\) by \(z(T^\rho, t^\rho_{LL}; \rho) = 0\). Then, the optimal feasible tax schedule for the homogeneous community is: For \(\rho \geq \hat{\rho}\),

\[
t^\rho = (\omega_L, \omega_L, \min(\omega_L + \omega_H, 2) - \omega_L, \min(\omega_H, 1)).
\]

and for \(\rho < \hat{\rho}\),

\[
t^\rho = \begin{cases} 
(t^\rho_{LL}, \omega_L, \omega_H, \omega_H) & \text{if } \omega_L + \omega_H \leq T^\rho \\
(t^\rho_{LL}, \omega_L, T^\rho - \omega_L, \min(\omega_H, 1)) & \text{if } T^\rho \leq T^\rho \\
(\omega_L, \omega_L, T^\rho - \omega_L, \min(\omega_H, 1)) & \text{if } T^\rho > T^\rho 
\end{cases}.
\]

III.2.2.2 Heterogeneous Formation

Consider now the optimal feasible tax schedule for a heterogeneous community,

\[
t^{pq*} = (t^{pq*}_{LL}, t^{pq*}_{LH}, t^{pq*}_{HL}, t^{pq*}_{HH}),
\]
In addition to the definitions in the homogeneous case, define \( T_{\rho} \) and for heterogeneous community is: 

\[
W_{pq}(0) = \frac{2}{p^2 q}, \quad \text{and for } \tilde{T} < T_{pq}, \text{define } t_{pq}^{\tilde{T}} \in (0, \omega_L) \text{ by } z(T_{pq}^{\tilde{T}}, t_{pq}^{\tilde{T}}; \rho) = 0, \rho = p, q.\]

Then, the optimal feasible tax schedule for the heterogeneous community is: For \( p \geq \tilde{\rho} \),

\[ t_{pq}^{\tilde{T}} = \left( \omega_L, \omega_L, \min\{\omega_L + \omega_H, 2\} - \omega_L, \min\{\omega_H, 1\} \right). \]

and for \( \rho < \tilde{\rho} \), if \( \tilde{t}_{LL} \geq \omega_L \), then

\[
t_{pq}^{\tilde{T}} = \begin{cases} 
(t_{pq}^{\tilde{T}}, \omega_L, \omega_H, \omega_H) & \text{if } \omega_L + \omega_H \leq T_{pq} \\
(t_{pq}^{\tilde{T}}, \omega_L, T_{pq} - \omega_L, \min\{\omega_H, 1\}) & \text{if } T_p \leq T_{pq} < \omega_L + \omega_H, \\
(\omega_L, \omega_L, T_{pq} - \omega_L, \min\{\omega_H, 1\}) & \text{if } T_{pq} < T_p 
\end{cases}
\]

and if \( \tilde{t}_{LL} < \omega_L \), then

\[
t_{pq}^{\tilde{T}} = \begin{cases} 
(\tilde{t}_{LL}, \omega_L, \tilde{T} - \omega_L, \min\{\omega_H, 1\}) & \text{if } \tilde{T} \leq T_{pq} \\
(t_{pq}^{\tilde{T}}, \omega_L, T_{pq} - \omega_L, \min\{\omega_H, 1\}) & \text{if } T_q \leq T_{pq} < \tilde{T}, \\
(\omega_L, \omega_L, T_{pq} - \omega_L, \min\{\omega_H, 1\}) & \text{if } T_{pq} < T_q 
\end{cases}
\]

### III.3 Welfare Comparison

To find the optimal formation of local public good economies, consider the welfare difference between the homogeneous and heterogeneous formations. For given \((\omega_L, \omega_H) \in \Omega \) and \((p, q) \in (0, 1), q > p\), the difference can be expressed as

\[
\Delta W = 2W_{pq}(t_{pq}^{\tilde{T}}) - [W_p(t_{pq}^{\tilde{T}}) + W_q(t_{pq}^{\tilde{T}})].
\]

Note that \( t_{pq}^{\tilde{T}} \) is different from \( t_{pq}^{\tilde{T}} \).
Thus, if $\Delta W \leq 0$, the homogeneous formation is optimal, but if $\Delta W > 0$, the heterogeneous formation is optimal. In order to see the sign of $\Delta W$ for different parameter values, we need to consider all the possible cases in which $t^*, t^q*$, and $t^{pq*}$ are different so that $\Delta W$ has different expressions. Table III.1 summarizes the those cases that are distinguishable.

First, we can determine the sign of $\Delta W$ when first best taxation is possible.

**Proposition III.3.1** If $p \geq \hat{p}$, then $\Delta W > 0$.

**Proof:**

\[
\Delta W = 2(q - p)^2 \left[ \log \frac{\min \{\omega_L + \omega_H, 2\}^2}{(2\omega_L)(2\min \{\omega_L, 1\})} - \min \{\omega_L + \omega_H, 2\} + (\omega_L + \min \{\omega_H, 1\}) \right] > 0.
\]

For the other cases, due to the high dimensionality and complexity of $\Delta W$, unfortunately, it is almost impossible to determine the sign of $\Delta W$. Putting aside the detailed expression of $\Delta W$ in Appendix B.1, here we analyze the problem by simulations. In Figure III.1–III.3, we draw the welfare difference function $\Delta W = 0$ on $(p, q)$-space for some different parameter values of $(\omega_L, \omega_H)$.

---

8 Strictly speaking, there are some cases in which the sign $\Delta W$ is determined analytically. See Appendix B.1.

9 We used Mathematica as a simulation program in this essay. The program codes are available upon request.
Figure III.1: Welfare Comparisons: $\omega_H = 0.8$
Figure III.2: Welfare Comparisons: $\omega_H = 1.3$
Figure III.3: Welfare Comparisons: $\omega_H = 2.5$
From these graphs, two facts are observed. One is that given \((p, q)\), as \(\omega_L\) or \(\omega_H\) increases, the area of \(\Delta W \leq 0\) increases, that is, the possibility the the homogeneous formation is optimal increases. Another is that roughly speaking, the smaller are \(p\) and \(q\), the more likely \(\Delta W \leq 0\), that is, as both minority and majority agents are more likely to be rich, the homogeneous formation is more likely to be optimal. In summary, we can say that if the expected endowment of the economy is relatively low, then the homogeneous formation is optimal, and otherwise, the heterogeneous formation is optimal.

To see the intuition behind this result, we note first that the informational problem is more serious in heterogeneous formation than homogeneous one. That is, the incentive compatibility constraint of heterogeneous formation is tighter than that of homogeneous formation. Thus, the social welfare loss caused by private information is greater in heterogeneous formation. However, we also note another effect that the expected public good provision of heterogeneous formation is greater than that of the corresponding homogeneous formation. In some sense, the diversity of an economy increases the expected provision of public good. Thus, if the economy is expected to be poor enough, a social planner who maximizes sum of all agents’ utilities would like to have heterogeneous local community formation at the expense of some informational rent. If the economy is expected to be rich enough, however, homogeneous formation is more attractive from the efficiency point of view since informational problem is more important than the problem of small provision of public good. To this respect, we observe that for the case of \(\bar{t}_{LL} < \omega_L\) in which the level of low endowment \(\omega_L\) is relatively high (see Section II.4.1), the homogeneous formation is always better than the heterogeneous one.

**III.4 Immigration Incentives**

In this section, we study the immigration incentive of each agent. That is, if an agent can move and choose one of the two community formations, which community formation will he prefer? To answer this question, we slightly modify the model. Suppose that there are two communities, one of them consists of a minority agent and the other a majority agent. A potential immigrant has to choose one of them to form a two-agent community.

Denote the expected utility of a \(\rho\)-type agent with endowment \(\omega_k\) when he immigrates into a community under tax schedule \(t\) by \(U_\rho(\omega_k|t), \rho = p, q, k = L, H\). If he chooses to form a heterogeneous community, then \(t = \bar{t}_{pq}^{*}\) and if he chooses a homogeneous community, then \(t = \bar{t}_\rho^{*}\).
Define the difference of expected utilities between them as

\[ \Delta U_\rho(\omega_k) = U_\rho(\omega_k|t^q) - U_\rho(\omega_k|t^p). \]

Thus, if \( \Delta U_\rho(\omega_k) \geq 0 \), then the potential agent will choose a heterogeneous formation, but if \( \Delta U_\rho(\omega_k) < 0 \), then he will choose a homogeneous formation.

Table III.2 summarizes all the possible cases for a \( q \)-type (resp. \( p \)-type) agent. When first best taxation is possible, we can determine the sign of \( \Delta U_\rho(\omega_k) \).

**Proposition III.4.1** If \( p \geq \hat{\rho} \), then for all \( k = L, H \),

\[ \Delta U_\rho(\omega_k) \begin{cases} > 0 & \text{if } \rho = q \\ < 0 & \text{if } \rho = p \end{cases} \]
Proof: If $\rho = q$, then
\[
\Delta U_q(\omega_L) = (q - p) \left[ \log \frac{\min\{\omega_L + \omega_H, 2\}}{2\omega_L} \right] > 0,
\]
\[
\Delta U_q(\omega_H) = (q - p) \left[ \log \frac{2\min\{\omega_H, 1\}}{\min\{\omega_L + \omega_H, 2\}} + \left(\omega_L + \min\{\omega_H, 1\}\right) - \min\{\omega_L + \omega_H, 2\} \right] > 0.
\]
If $\rho = p$, then
\[
\Delta U_p(\omega_L) = (q - p) \left[ \log \frac{2\omega_L}{\min\{\omega_L + \omega_H, 2\}} \right] < 0
\]
\[
\Delta U_q(\omega_H) = (q - p) \left[ \log \frac{\min\{\omega_L + \omega_H, 2\}}{2\min\{\omega_H, 1\}} - \left(\omega_L + \min\{\omega_H, 1\}\right) + \min\{\omega_L + \omega_H, 2\} \right] < 0.
\]

For the other cases, due to the high dimensionality and complexity of $\Delta U_\rho(\omega_k)$, it is almost impossible to find the set of parameter values where the sign of $\Delta U_\rho(\omega_k)$ is uniquely determined.\(^{10}\)

Putting aside the detailed expression of $\Delta U_\rho(\omega_k)$ in Appendix B.2, here we analyze the problem by simulations. The result is in Figure III.4.\(^{11}\) Interestingly, the simulation results indicates that a q-type, namely a minority agent, always chooses the heterogeneous formation, and a p-type, namely a majority agent, always chooses the homogeneous formation. In other words, both the minority and majority wants to move to the community expected to be rich. To explain this result, we note the structure of optimal feasible tax schedule. According to the characterization results in

\(^{10}\) In fact, there are some cases in which the sign of $\Delta U_\rho(\omega_k)$ can be determined. See Appendix B.2.

\(^{11}\) We drew $\Delta U_\rho(\omega_k) = 0$, $\rho = p, q$, $k = L, H$, on $(p, q)$-space for $20 \times 20$ different endowment parameter values $(\omega_L, \omega_H)$ which were obtained by equally dividing $(0, 1) \times (0.5, 3)$. 

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the previous two chapters, the poor agent’s tax payment is relatively high (in fact, at 100% tax rate) when his neighbor is rich. Thus, the above result implies that the benefit from increasing the possibility of high provision of public good by moving to the community expected to be rich always dominates the benefit from decreasing the possibility of high tax payment by moving to the community expected to be poor. This result well explains why slums usually forms in a metropolitan area, a relatively rich area.

III.5 Concluding Remarks

In this essay, we study the formation of local public good communities and its immigration incentives. Although the model given in this paper is quite abstract, we can learn how the incentive problem is resolved under feasibility constraint. This study is a first attempt to explain what is the optimal local community formation and how incentive compatibility and feasibility problems are resolved from the mechanism design point of view. Also, the implications of this model well explains most experiences in a society with diversity.

As a future research, it is required to conduct some empirical research whether the results of this paper are consistent with the real data set. This kind of empirical work has to do with not only local public good community formation, but also club or world economy formation.
References


Appendix A

A.1 Derivation of Equation (I.7)

\[
\Delta(t^F; p) = \sum_{j=0}^{n-1} n_{-1}C_j p^j (1-p)^{n-1-j} \left( t^F_{L,j,L,(n-1-j)H} + t^F_{H,j,L,(n-1-j)H} \right) \\
= \sum_{j=1}^{n-1} n_{-1}C_{j-1} p^{j-1} (1-p)^{n-j} \left( t^F_{L,(j-1)L,(n-j)H} - 1 \right) + n_{-1}C_j p^j (1-p)^{n-1-j} \left( 1 - t^F_{H,j,L,(n-1-j)H} \right) \\
= \sum_{j=1}^{n-1} \left[ n_{-1}C_{j-1} p^{j-1} (1-p)^{n-j} \left( t^F_{L,(j-1)L,(n-j)H} - 1 \right) \\
+ n_{-1}C_j p^j (1-p)^{n-1-j} \left( \frac{j}{n-j} \right) (t^F_{L,(j-1)L,(n-j)H} - 1) \right] \\
= \sum_{j=1}^{n-1} n_{-1}C_{j-1} p^{j-1} (1-p)^{n-1-j} (t^F_{L,(j-1)L,(n-j)H} - 1).
\]

A.2 Derivation of Equation (I.8)

\[
\Delta(t^F; p) = \sum_{j=0}^{n-1} n_{-1}C_j p^j (1-p)^{n-1-j} \left[ \log \left( \frac{jω_L + (n-j)ω_H}{(j+1)ω_L + (n-1-j)ω_H} \right) - (ω_H - ω_L) \right] \\
= \sum_{j=0}^{n-1} n_{-1}C_j p^j (1-p)^{n-1-j} \log \left( \frac{jω_L + (n-j)ω_H}{(j+1)ω_L + (n-1-j)ω_H} \right) - (ω_H - ω_L) (p + (1-p))^{n-1} \\
= \sum_{j=0}^{n-1} n_{-1}C_j p^{n-1-j} \left[ \sum_{k=0}^{n-j} (-1)^{k+\text{mod}(n-1-j,2)} n_{-j}C_k \log(kω_L + (n-k)ω_H) \right] - (ω_H - ω_L),
\]

To see that the third equality holds, consider the coefficient of \(p^{n-1-j}, j = 0, \ldots, n-1\). Suppose \((n-1-j)\) is even. Then, the coefficient is

\[
\left( \begin{array}{c}
0_{n-1}C_0 \cdot n_{-1}C_{n-1-j} \\
0_{n-1}C_1 \cdot n_{-2}C_{n-2-j} \\
0_{n-1}C_2 \cdot n_{-3}C_{n-3-j} \\
\vdots \\
0_{n-1}C_{n-1-j} \cdot jC_0
\end{array} \right) \left[ \log(nω_H) - \log(ω_L + (n-1)ω_H) \right] \\
- \left( \begin{array}{c}
0_{n-1}C_1 \cdot n_{-2}C_{n-2-j} \\
0_{n-1}C_2 \cdot n_{-3}C_{n-3-j} \\
\vdots \\
0_{n-1}C_{n-1-j} \cdot jC_0
\end{array} \right) \left[ \log(ω_L + (n-1)ω_H) - \log(2ω_L + (n-2)ω_H) \right] \\
+ \left( \begin{array}{c}
0_{n-1}C_2 \cdot n_{-3}C_{n-3-j} \\
\vdots \\
0_{n-1}C_{n-1-j} \cdot jC_0
\end{array} \right) \left[ \log(2ω_L + (n-2)ω_H) - \log(3ω_L + (n-3)ω_H) \right] \\
\vdots \\
+ \left( \begin{array}{c}
0_{n-1}C_{n-1-j} \cdot jC_0
\end{array} \right) \left[ \log((n-1-j)ω_L + (j+1)ω_H) - \log((n-j)ω_L + jω_H) \right] \\
= \left( \begin{array}{c}
0_{n-1}C_0 \cdot n_{-1}C_j
\end{array} \right) \log(nω_H) - \left( \begin{array}{c}
0_{n-1}C_j \cdot jC_0
\end{array} \right) \log((n-j)ω_L + jω_H)
\]

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\[
\begin{align*}
&+ \sum_{k=1}^{n-j-1} (-1)^k \left( n-1 C_{k-1} \cdot n-k C_j + n-1 C_k \cdot n-k-1 C_j \right) \log(k \omega_L + (n-k) \omega_H) \\
&= \left( n-1 C_0 \cdot n-1 C_j \right) \log(n \omega_H) - \left( n-1 C_j \cdot n C_0 \right) \log((n-j) \omega_L + j \omega_H) \\
&+ \sum_{k=1}^{n-j-1} (-1)^{k+1} \left( n-1 C_j \cdot n-j C_k \right) \log(k \omega_L + (n-k) \omega_H) \\
&= n-1 C_j \sum_{k=0}^{n-j} (-1)^k n-j C_k \log(k \omega_L + (n-k) \omega_H).
\end{align*}
\]

A similar calculation shows that when \((n-1-j)\) is odd the coefficient is the same.

**A.3 Proof of Lemma I.5.1**

**Proof:** Let \(\xi_{n,j}(\omega_L, \omega_H) = \sum_{k=0}^{n-j} (-1)^{k+\text{mod}(n-1-j,2)} n-j C_k \log(k \omega_L + (n-k) \omega_H)\). First, we want to show that \(\xi_{n,j}(\omega_L, \omega_H)\) is strictly increasing in \(\omega_H\). For \(j = 0, \ldots, n-2\),

\[
\frac{\partial \xi_{n,j}(\omega_L, \omega_H)}{\partial \omega_H} = \frac{\partial}{\partial \omega_H} \left( \sum_{k=0}^{n-j} (-1)^{k+\text{mod}(n-1-j,2)} n-j C_k \log(k \omega_L + (n-k) \omega_H) \right)
\]

\[
= \sum_{k=0}^{n-j} (-1)^{k+\text{mod}(n-1-j,2)} \frac{(n-k)n-j C_k}{k \omega_L + (n-k) \omega_H}
\]

\[
= (n-1-j)! (\omega_H - \omega_L)^{n-1-j} \left[ \frac{n}{\prod_{k=0}^{n-j} (k \omega_L + (n-k) \omega_H)} - \frac{j}{\prod_{k=1}^{n-j} (k \omega_L + (n-k) \omega_H)} \right]
\]

\[
> 0.
\]

Thus, we have for all \(n \geq 2\) and all \(j = 0, \ldots, n-2\),

\[
\xi_{n,j}(\omega_L, \omega_H) > \xi_{n,j}(\omega_L, \omega_L) = \log(n \omega_L) \sum_{k=0}^{n-j} (-1)^{k+\text{mod}(n-1-j,2)} n-j C_k = 0
\]

by the Binomial Theorem. It follows that

\[
\frac{\partial \Delta(t^F;p)}{\partial p^{n-1-j}} = n-1 C_j \xi_{n,j}(\omega_L, \omega_H) > 0.
\]
Appendix B

B.1 Expressions of $\Delta W$

In this appendix, we provide the detailed expressions of $\Delta W$ for each possible cases given in Table III.1, and, if possible, show the sign of it. Suppose $p < \hat{\rho}$.

B.1.1 $[\hat{t}_{LL} \geq \omega_L]$ and $[\omega_L + \omega_H \leq 1]$

Case (1): $[\omega_L + \omega_H \leq \frac{2p}{p+q}]$ and $[q \geq \hat{\rho}]$

$$\Delta W = 2p \left[ \log \frac{\omega_L + \omega_H}{T} \right] + 4q \left[ \log \frac{T}{\omega_L + \omega_H} \right] + 2q^2 \left[ \log \frac{\omega_L + \omega_H}{4\omega_L \omega_H} \right].$$

Case (2): $[\omega_L + \omega_H \leq \frac{2p}{p+q}]$ and $[q < \hat{\rho}]$

$$\Delta W = 2p \left[ \log \frac{\omega_L + \omega_H}{T} \right] + 2q \left[ \log \frac{T}{\omega_L + \omega_H} \right] < 0.$$

Case (3): $[T^p \leq \frac{2p}{p+q} < \omega_L + \omega_H]$ and $[q \geq \hat{\rho}]$

$$\Delta W = 2p \left[ \log \frac{\left( \frac{2p}{p+q} \right)^2}{(\omega_L + \omega_H)(T)} - 2 + (\omega_L + \omega_H) \right] + 2q \left[ \log \frac{T}{\omega_L + \omega_H} + (\omega_L + \omega_H) \right] + 2q^2 \left[ \log \frac{\omega_L + \omega_H}{4\omega_L \omega_H} \right].$$

Case (4): $[T^p \leq \frac{2p}{p+q} < \omega_L + \omega_H \leq 1]$ and $[q < \hat{\rho}]$

$$\Delta W = 2p \left[ \log \frac{\left( \frac{2p}{p+q} \right)^2}{(\omega_L + \omega_H)(T)} - 2 + (\omega_L + \omega_H) \right] + 2q \left[ \log \frac{\omega_L + \omega_H}{\omega_L + \omega_H} + (\omega_L + \omega_H) \right].$$

Case (5): $[\frac{2p}{p+q} < T^p]$ and $[q \geq \hat{\rho}]$

$$\Delta W = 2p \left[ \log \frac{(T^p)^2}{(\omega_L + \omega_H)(T)} - T^p + (\omega_L + \omega_H) \right] + 2q \left[ \log \frac{T^{\omega_L + \omega_H}}{\omega_L + \omega_H} - T^p + (\omega_L + \omega_H) \right] + 2q^2 \left[ \log \frac{\omega_L + \omega_H}{4\omega_L \omega_H} \right].$$

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Case (6): \[ \frac{2p}{p+q} < T^p \] and \[ q < \bar{p} \]

\[ \Delta W = 2p \left( \log \frac{(T^p)^2}{(\omega_L + \omega_H)(T)} - T^p + (\omega_L + \omega_H) \right) + 2q \left( \log \frac{T}{\omega_L + \omega_H} - T^p + (\omega_L + \omega_H) \right). \]

B.1.2 \( [\bar{t}_{LL} \geq \omega_L] \) and \( [\omega_L + \omega_H > 1] \)

Case (7): \[ T^p \leq \frac{2p}{p+q} < 1 < \omega_L + \omega_H \] and \( T^q < 1 \)

\[ \Delta W = 2p \left( \log \frac{\left(\frac{2p}{p+q}\right)^2}{T} - 1 \right) + 2q \left( \log(\bar{T}) + 1 \right). \]

Case (8): \( \frac{2p}{p+q} < T^p \leq 1 \) and \( q \geq \bar{p} \)

\[ \Delta W = 2p \left( \log \frac{(T^p)^2}{T} - T^p + 1 \right) + 2q \left( 2 \log \frac{T}{\min\{\omega_L + \omega_H, 2\}} - \frac{T^p + \min\{\omega_L + \omega_H, 2\}}{\min\{\omega_L + \omega_H, 2\}} \right) \]
\[ + 2q^2 \left( \log \frac{\min\{\omega_L + \omega_H, 2\}^2}{(2\omega_L)(2\min\{\omega_H, 1\})} - \min\{\omega_L + \omega_H, 2\} + (\omega_L + \min\{\omega_H, 1\}) \right). \]

Case (9): \( \frac{2p}{p+q} < T^p \leq 1 \) and \( T^q \geq 1 \)

\[ \Delta W = 2p \left( \log \frac{(T^p)^2}{T} - T^p + 1 \right) + 2q \left( \log \frac{T}{T^q} - T^p + T^q \right). \]

Case (10): \( \frac{2p}{p+q} < T^p \leq 1 \) and \( T^q < 1 \)

\[ \Delta W = 2p \left( \log \frac{(T^p)^2}{T} - T^p + 1 \right) + 2q \left( \log(\bar{T}) - T^p + 1 \right). \]

Case (11): \( \frac{2p}{p+q} < 1 < T^p \) and \( q \geq \bar{p} \)

\[ \Delta W = 2p \left( \log \frac{T^p}{T} \right) + 2q \left( 2 \log \frac{T}{\min\{\omega_L + \omega_H, 2\}} - T^p + \min\{\omega_L + \omega_H, 2\} \right) \]
\[ + 2q^2 \left( \log \frac{\min\{\omega_L + \omega_H, 2\}^2}{(2\omega_L)(2\min\{\omega_H, 1\})} - \min\{\omega_L + \omega_H, 2\} + (\omega_L + \min\{\omega_H, 1\}) \right). \]

Case (12): \( \frac{2p}{p+q} < 1 < T^p \) and \( q < \bar{p} \)

\[ \Delta W = 2p \left( \log \frac{T^p}{T} \right) + 2q \left( \log \frac{T}{T^q} - T^p + T^q \right). \]

B.1.3 \( [\bar{t}_{LL} < \omega_L] \)

Case (13): \( \bar{T} \leq \frac{2q}{p+q} \)

\[ \Delta W = 2p \left[ \log \frac{\bar{T}}{T^p} - \bar{T} + T^p \right] + 2q \left[ \log \frac{\bar{T}}{T^q} - \bar{T} + T^q \right] < 0. \]
Case (14): \[ T^q \leq \frac{2q}{p+q} < \bar{T} \]
\[ \Delta W = 2p \left[ \log \frac{T}{T^p} + T^p \right] + 2q \left[ \log \left( \frac{2q}{(p+q)} \right)^2 - 2 + T^q \right]. \]

Case (15): \[ \frac{2q}{p+q} < T^q \]
\[ \Delta W = 2p \left[ \log \frac{T}{T^p} - T^q + T^p \right] + 2q \left[ \log \frac{T^q}{T} \right]. \]

B.2 Expressions of \( \Delta U_\rho(\omega_k) \) for \( \rho = p, q \) and \( k = L, H \)

B.2.1 Immigration Incentives of \( q \)-type Agent with \( \omega_L \)

B.2.1.1 \( [\tilde{t}_{LL} \geq \omega_L] \) and \( [\omega_L + \omega_H \leq 1] \)

case (1): \([\omega_L + \omega_H \leq \frac{2p}{p+q}] \) and \( [q \geq \hat{\rho}] \)
\[ \Delta U_q(\omega_L) = p \left[ \log \frac{\omega_L + \omega_H}{2\omega_H} \right] + q \left[ \log \frac{\omega_L + \omega_H}{2\omega_L} - \log \frac{\omega_L + \omega_H}{T} \right] > 0. \]

case (2): \([\omega_L + \omega_H \leq \frac{2p}{p+q}] \) and \( [q < \hat{\rho}] \)
\[ \Delta U_q(\omega_L) = (q - p) \left[ \log \frac{2\omega_H}{\omega_L + \omega_H} \right] > 0. \]

case (3): \([T^p \leq \frac{2p}{p+q} < \omega_L + \omega_H] \) and \( [q \geq \hat{\rho}] \)
\[ \Delta U_q(\omega_L) = p \left[ \log \frac{2\omega_H}{2\omega_H} - \left( \frac{2p}{p+q} \right) + (\omega_L + \omega_H) \right] - q \left[ \log \frac{2\omega_L}{\omega_L + \omega_H} \right] + \log \frac{\bar{T}}{\omega_L + \omega_H}. \]

case (4): \([\omega_L + \omega_H \leq \frac{2p}{p+q}] \) and \( [q < \hat{\rho}] \)
\[ \Delta U_q(\omega_L) = p \left[ \log \frac{2\omega_H}{2\omega_H} - \left( \frac{2p}{p+q} \right) + (\omega_L + \omega_H) \right] + q \left[ \log \frac{2\omega_H}{\omega_L + \omega_H} \right]. \]

case (5): \( [\frac{2p}{p+q} < T^p] \) and \( [q \geq \hat{\rho}] \)
\[ \Delta U_q(\omega_L) = p \left[ \log \frac{\omega_L}{T^p} \right] + q \left[ \log \frac{\omega_L + \omega_H}{2\omega_L} \right] + \log \frac{T^p}{\omega_L + \omega_H}. \]

case (6): \( [\frac{2p}{p+q} < T^p] \) and \( [q < \hat{\rho}] \)
\[ \Delta U_q(\omega_L) = p \left[ \log \frac{\omega_L}{T^p} \right] + q \left[ \log \frac{2\omega_H}{\omega_L + \omega_H} \right] + \log \frac{T^p}{T}. \]
**B.2.1.2** \( \bar{t}_{LL} \geq \omega_L \) and \( \omega_L + \omega_H > 1 \)

Case (7): \( \left[ T^p \leq \frac{2p}{p+q} \right] \)

\[
\Delta U_q(\omega_L) = p \left[ \log \frac{\left( \frac{2p}{p+q} \right)}{2 \min \{ \omega_H, 1 \}} - \left( \frac{2p}{p+q} \right) + (\omega_L + \min \{ \omega_H, 1 \}) \right] \\
- q \left[ \log \frac{1}{2 \min \{ \omega_H, 1 \}} - 1 + (\omega_L + \min \{ \omega_H, 1 \}) \right].
\]

Case (8): \( \left[ \frac{2p}{p+q} < T^p \right] \) and \([q \geq \hat{\rho}]\)

\[
\Delta U_q(\omega_L) = p \left[ \log \frac{2 \omega_L}{T^p} \right] - q \left[ \log \frac{2 \omega_L}{\min \{ \omega_L + \omega_H, 2 \}} \right] + \log \frac{T^p}{\min \{ \omega_L + \omega_H, 2 \}}.
\]

Case (9): \( \left[ \frac{2p}{p+q} < T^p \right] \) and \([T'^q \geq 1] \)

\[
\Delta U_q(\omega_L) = p \left[ \log \frac{2 \omega_L}{T^p} \right] - q \left[ \log \frac{2 \omega_L}{T'^q} \right] + \log \frac{T^p}{T'^q}.
\]

Case (10): \( \left[ \frac{2p}{p+q} < T^p \right] \) and \([T'^q < 1] \)

\[
\Delta U_q(\omega_L) = p \left[ \log \frac{2 \omega_L}{T^p} \right] - q \left[ \log \frac{1}{2 \min \{ \omega_H, 1 \}} - 1 + (\omega_L + \min \{ \omega_H, 1 \}) \right] + \log \frac{T^p}{T'^q}.
\]

**B.2.1.3** \( \bar{t}_{LL} < \omega_L \)

Case (11): \( \left[ \bar{T} < \frac{2q}{p+q} \right] \)

\[
\Delta U_q(\omega_L) = p \left[ \log \frac{T}{2 \min \{ \omega_H, 1 \}} - \bar{T} + (\omega_L + \min \{ \omega_H, 1 \}) \right] - q \left[ \log \frac{2 \omega_L}{T^p} \right] + \log \frac{T}{T'^q}.
\]

Case (12): \( \left[ T'^q \leq \frac{2q}{p+q} < \bar{T} \right] \)

\[
\Delta U_q(\omega_L) = p \left[ \log \frac{\left( \frac{2q}{p+q} \right)}{2 \min \{ \omega_H, 1 \}} - \left( \frac{2q}{p+q} \right) + (\omega_L + \min \{ \omega_H, 1 \}) \right] \\
- \frac{p}{q} \left[ \log \frac{\left( \frac{2q}{p+q} \right)}{T^p} \right] - q \left[ \log \frac{2 \omega_L}{T^p} \right] + \log \frac{\left( \frac{2q}{p+q} \right)}{T'^q}.
\]

Case (13): \( \left[ \frac{2q}{p+q} < T'^q \right] \)

\[
\Delta U_q(\omega_L) = (q - p) \left[ \log \frac{T'^q}{2 \omega_L} \right] > 0.
\]

**B.2.2** Immigration Incentives of \( q \)-type Agent with \( \omega_H \)

**B.2.2.1** \( \bar{t}_{LL} \geq \omega_L \) and \( \omega_L + \omega_H \leq 1 \)

Case (1): \( \left[ \omega_L + \omega_H \leq \frac{2p}{p+q} \right] \) and \([q \geq \hat{\rho}]\)

\[
\Delta U_q(\omega_H) = (q - p) \left[ \log \frac{2 \omega_H}{\omega_L + \omega_H} \right] > 0.
\]
Case (2): \( \omega_L + \omega_H \leq \frac{2p}{p+q} \) and \( [q < \hat{p}] \)

\[
\Delta U_q(\omega_H) = (q - p) \left[ \log \frac{2\omega_H}{\omega_L + \omega_H} \right] > 0.
\]

Case (3): \( T^p \leq \frac{2p}{p+q} < \omega_L + \omega_H \) and \( [q \geq \hat{p}] \)

\[
\Delta U_q(\omega_H) = p \left[ \log \left( \frac{2p}{p+q} \right) - \frac{2p}{p+q} (\omega_L + \omega_H) \right] - q \left[ \log \frac{\omega_L + \omega_H}{2\omega_H} \right].
\]

Case (4): \( \omega_L + \omega_H \leq \frac{2p}{p+q} \) and \( [q < \hat{p}] \)

\[
\Delta U_q(\omega_H) = p \left[ \log \left( \frac{2p}{p+q} \right) - \frac{2p}{p+q} (\omega_L + \omega_H) \right] - q \left[ \log \frac{\omega_L + \omega_H}{2\omega_H} \right].
\]

Case (5): \( \frac{2p}{p+q} < T^p \) and \( [q \geq \hat{p}] \)

\[
\Delta U_q(\omega_H) = p \left[ \log \frac{T^p}{2\omega_H} - T^p + \omega_L + \omega_H \right] - q \left[ \log \frac{\omega_L + \omega_H}{2\omega_H} \right].
\]

Case (6): \( \frac{2p}{p+q} < T^p \) and \( [q < \hat{p}] \)

\[
\Delta U_q(\omega_H) = p \left[ \log \frac{T^p}{2\omega_H} - T^p + \omega_L + \omega_H \right] - q \left[ \log \frac{\omega_L + \omega_H}{2\omega_H} \right].
\]

**B.2.2.2** \( I_{LL} \geq \omega_L \) and \( [\omega_L + \omega_H > 1] \)

Case (7): \( T^p \leq \frac{2p}{p+q} \)

\[
\Delta U_q(\omega_H) = p \left[ \log \frac{2p}{p+q} \frac{1}{2\min\{\omega_H, 1\}} - \frac{2p}{p+q} (\omega_L + \min\{\omega_H, 1\}) \right] \]

\[
- q \left[ \log \frac{1}{2\min\{\omega_H, 1\}} - 1 + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

Case (8): \( \frac{2p}{p+q} < T^p \) and \( [q \geq \hat{p}] \)

\[
\Delta U_q(\omega_H) = p \left[ \log \frac{T^p}{2\min\{\omega_H, 1\}} - T^p + \omega_L + \min\{\omega_H, 1\} \right] \]

\[
- q \left[ \log \frac{\min\{\omega_L + \omega_H, 2\}}{2\min\{\omega_H, 1\}} - (\omega_L + \min\{\omega_H, 1\}) + \min\{\omega_L + \omega_H, 2\} \right].
\]

Case (9): \( \frac{2p}{p+q} < T^p \) and \( [T^q \geq 1] \)

\[
\Delta U_q(\omega_H) = p \left[ \log \frac{T^p}{2\min\{\omega_H, 1\}} - T^p + \omega_L + \min\{\omega_H, 1\} \right] \]

\[
- q \left[ \log \frac{T^q}{2\min\{\omega_H, 1\}} - T^q + \omega_L + \min\{\omega_H, 1\} \right].
\]
Case (10): \[ \left[ \frac{2p}{p+q} < T^p \right] \text{ and } [T^q < 1] \]

\[
\Delta U_q(\omega_H) = p \left[ \log \frac{T^p}{2 \min\{\omega_H, 1\}} - T^p + (\omega_L + \min\{\omega_H, 1\}) \right] \\
- q \left[ \log \frac{1}{2 \min\{\omega_H, 1\}} - 1 + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

B.2.2.3 \[ \tilde{t}_{LL} < \omega_L \]

Case (11): \[ \tilde{T} \left[ \frac{2p}{p+q} \right] \]

\[
\Delta U_q(\omega_H) = p \left[ \log \frac{\tilde{T}}{2 \min\{\omega_H, 1\}} - \tilde{T} + (\omega_L + \min\{\omega_H, 1\}) \right] \\
- q \left[ \log \frac{T^q}{2 \min\{\omega_H, 1\}} - T^q + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

Case (12): \[ \left[ T^q \leq \frac{2q}{p+q} < \tilde{T} \right] \]

\[
\Delta U_q(\omega_H) = p \left[ \log \frac{\left( \frac{2q}{p+q} \right)}{2 \min\{\omega_H, 1\}} - \left( \frac{2q}{p+q} \right) + (\omega_L + \min\{\omega_H, 1\}) \right] \\
- q \left[ \log \frac{T^q}{2 \min\{\omega_H, 1\}} - T^q + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

Case (13): \[ \left[ \frac{2q}{p+q} < \tilde{T} \right] \]

\[
\Delta U_q(\omega_H) = (p - q) \left[ \log \frac{T^q}{2 \min\{\omega_H, 1\}} - T^q + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

B.2.3 Immigration Incentives of \( p \)-type Agent with \( \omega_L \)

B.2.3.1 \[ \tilde{t}_{LL} \geq \omega_L \] and \[ \omega_L + \omega_H \leq 1 \]

Case (1): \[ \omega_L + \omega_H \leq \frac{2p}{p+q} \]

\[
\Delta U_p(\omega_L) = (q - p) \left[ \log \frac{\omega_L + \omega_H}{2 \omega_H} \right] - \frac{q-p}{p} \left[ \log \frac{\omega_L + \omega_H}{T} \right] < 0.
\]

Case (2): \[ T^p \leq \frac{2p}{p+q} < \omega_L + \omega_H \]

\[
\Delta U_p(\omega_L) = q \left[ \log \left( \frac{\frac{2p}{p+q}}{2 \omega_H} \right) - \left( \frac{2p}{p+q} \right) + (\omega_L + \omega_H) \right] \\
- \frac{q}{p} \left[ \log \left( \frac{\frac{2p}{p+q}}{T} \right) \right] - p \left[ \log \frac{\frac{2\omega_L}{\omega_L + \omega_H}}{\omega_L + \omega_H} + \log \left( \frac{\frac{2p}{p+q}}{\omega_L + \omega_H} \right) \right].
\]

Case (3): \[ \frac{2p}{p+q} < T^p \]

\[
\Delta U_p(\omega_L) = q \left[ \log \frac{2 \omega_L}{T^p} \right] - p \left[ \log \frac{2 \omega_H}{\omega_L + \omega_H} \right] + \frac{T^p}{T}.
\]
B.2.3.2 \( [\bar{t}_{LL} \geq \omega_L] \) and \( [\omega_L + \omega_H > 1] \)

Case (4): \( [\bar{T}_p \leq \frac{2p}{p+q}] \)

\[
\Delta U_p(\omega_L) = q \left[ \log \frac{(2p)}{2\min\{\omega_H, 1\}} - \left( \frac{2p}{p+q} \right) + (\omega_L + \min\{\omega_H, 1\}) \right] \\
- p \left[ \log \frac{1}{2\min\{\omega_H, 1\}} - 1 + (\omega_L + \min\{\omega_H, 1\}) \right] - \frac{(q-p)}{p} \left[ \log \left( \frac{2p}{p+q} \right) \right].
\]

Case (5): \( [\frac{2p}{p+q} < \bar{T}_p \leq 1] \)

\[
\Delta U_p(\omega_L) = q \left[ \log \frac{2\omega_L}{\bar{T}_p} \right] - p \left[ \log \frac{1}{2\min\{\omega_H, 1\}} - 1 + (\omega_L + \min\{\omega_H, 1\}) \right] + \log \frac{T_p}{\bar{T}}.
\]

Case (6): \( [\frac{2p}{p+q} < 1 < \bar{T}_p] \)

\[
\Delta U_p(\omega_L) = (q-p) \left[ \log \frac{2\omega_L}{T_p} \right] < 0.
\]

B.2.3.3 \( [\bar{t}_{LL} < \omega_L] \)

Case (7): \( [\bar{T} < \frac{2q}{p+q}] \)

\[
\Delta U_p(\omega_L) = q \left[ \log \frac{\bar{T}}{2\min\{\omega_H, 1\}} - \bar{T} + (\omega_L + \min\{\omega_H, 1\}) \right] - p \left[ \log \frac{2\omega_L}{\bar{T}_p} \right] + \log \frac{\bar{T}}{T_p}.
\]

Case (8): \( [\frac{2q}{p+q} \leq \bar{T}_p < \bar{T}] \)

\[
\Delta U_p(\omega_L) = q \left[ \log \left( \frac{\bar{T}_p}{2\min\{\omega_H, 1\}} - \left( \frac{2q}{p+q} \right) + (\omega_L + \min\{\omega_H, 1\}) \right) \right] - p \left[ \log \frac{2\omega_L}{\bar{T}_p} \right] + \log \frac{\bar{T}_p}{T_p}.
\]

Case (9): \( [\frac{2q}{p+q} < \bar{T}_p] \)

\[
\Delta U_p(\omega_L) = q \left[ \log \frac{2\omega_L}{\bar{T}_p} \right] - p \left[ \log \frac{2\omega_L}{\bar{T}_p} \right] + \log \frac{T_p}{\bar{T}_p}.
\]

B.2.4 Immigration Incentives of \( p \)-type Agent with \( \omega_H \)

B.2.4.1 \( [\bar{t}_{LL} \geq \omega_L] \) and \( [\omega_L + \omega_H \leq 1] \)

Case (1): \( [\omega_L + \omega_H \leq \frac{2p}{p+q}] \)

\[
\Delta U_q(\omega_H) = (q-p) \left[ \log \frac{\omega_L + \omega_H}{2\omega_H} \right] < 0.
\]

Case (2): \( [\bar{T}_p \leq \frac{2p}{p+q} < \omega_L + \omega_H] \)

\[
\Delta U_q(\omega_H) = q \left[ \log \left( \frac{2p}{2\omega_H} \right) \right] - \left( \frac{2p}{p+q} \right) + (\omega_L + \omega_H) \right] - p \left[ \log \frac{\omega_L + \omega_H}{2\omega_H} \right].
\]
Case (3): \( \left[ \frac{2p}{p+q} < T^p \right] \)
\[
\Delta U_q(\omega_H) = q \left[ \log \frac{T^p}{2\omega_H} - T^p + (\omega_L + \omega_H) \right] - p \left[ \log \frac{\omega_L + \omega_H}{2\omega_H} \right].
\]

B.2.4.2 \( \tilde{t}_{LL} \geq \omega_L \) and \( [\omega_L + \omega_H > 1] \)

Case (4): \( \left[ T^p \leq \frac{2p}{p+q} \right] \)
\[
\Delta U_q(\omega_H) = q \left[ \log \frac{\omega_L}{2\min\{\omega_H, 1\}} - \left( \frac{2p}{p+q} \right) + (\omega_L + \min\{\omega_H, 1\}) \right]
- p \left[ \log \frac{1}{2\min\{\omega_H, 1\}} - 1 + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

Case (5): \( \left[ \frac{2p}{p+q} < T^p \leq 1 \right] \)
\[
\Delta U_q(\omega_H) = q \left[ \log \frac{T^p}{2\min\{\omega_H, 1\}} - T^p + (\omega_L + \min\{\omega_H, 1\}) \right]
- p \left[ \log \frac{1}{2\min\{\omega_H, 1\}} - 1 + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

Case (6): \( \left[ \frac{2p}{p+q} < 1 < T^p \right] \)
\[
\Delta U_q(\omega_H) = (q - p) \left[ \log \frac{T^p}{2\min\{\omega_H, 1\}} - T^p + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

B.2.4.3 \( \tilde{t}_{LL} < \omega_L \)

Case (7): \( \left[ \tilde{T} < \frac{2q}{p+q} \right] \)
\[
\Delta U_q(\omega_H) = q \left[ \log \frac{\tilde{T}}{2\min\{\omega_H, 1\}} - \tilde{T} + (\omega_L + \min\{\omega_H, 1\}) \right]
- p \left[ \log \frac{T^p}{2\min\{\omega_H, 1\}} - T^p + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

Case (8): \( \left[ T^q \leq \frac{2q}{p+q} < \tilde{T} \right] \)
\[
\Delta U_q(\omega_H) = q \left[ \log \frac{\omega_L}{2\min\{\omega_H, 1\}} - \left( \frac{2q}{p+q} \right) + (\omega_L + \min\{\omega_H, 1\}) \right]
- p \left[ \log \frac{T^p}{2\min\{\omega_H, 1\}} - T^p + (\omega_L + \min\{\omega_H, 1\}) \right].
\]

Case (9): \( \left[ \frac{2q}{p+q} < T^q \right] \)
\[
\Delta U_q(\omega_H) = q \left[ \log \frac{T^q}{2\min\{\omega_H, 1\}} - T^q + (\omega_L + \min\{\omega_H, 1\}) \right]
- p \left[ \log \frac{T^p}{2\min\{\omega_H, 1\}} - T^p + (\omega_L + \min\{\omega_H, 1\}) \right].
\]
Vita

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• B.A., International Economics, Seoul National University, Korea, 1994

Fields
• Microeconomics, Public Economics, Game Theory, IO

Presentation
• “A Characterization of Optimal Feasible Tax Mechanism,” at the 2004 Far Eastern Meeting of the Econometric Society

Experience
• Graduate Instructor: Intermediate Microeconomics, PSU, Summer 2003–Spring 2004
• Lecturer: Microeconomics (at Sookmyung Women’s Univ., Korea, Summer 1997), Microeconomics, Public Economics, Statistics (at Korea National Open Univ., Korea, Fall 1996–Spring 1998)
• Teaching Assistant: Gen-Ed, PSU, Fall 2001–Spring 2003
  Graduate and Intermediate Microeconomics, SNU, Spring 1997–Spring 1998
• Research Assistant to Prof. In-Koo Cho on “Refinement of Nash Equilibrium,” SNU, Spring 1998
• Research Assistant on “Structure of the Korean Telecommunication Market and Its Reform,” Institute of World Economy, SNU, Fall 1996

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• Graduate Assistantship, PSU, Fall 1998–Spring 2004
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