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CHARACTERIZING THE NATURE OF STUDENTS' FEATURE NOTICING-AND-USING WITH RESPECT TO MATHEMATICAL SYMBOLS ACROSS DIFFERENT LEVELS OF ALGEBRA EXPOSURE

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Patrick Sullivan

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The dissertation of Patrick Sullivan was reviewed and approved* by the following:

M. Kathleen Heid
Distinguished Professor of Education
Professor, Mathematics Education
Chair of Committee
Dissertation Advisor

Rose Mary Zbiek
Professor, Mathematics Education

Mark Levi
Professor, Mathematics

Glendon W. Blume
Professor, Mathematics Education
Graduate Director, Curriculum and Instruction

*Signatures are on file in the Graduate School.

ABSTRACT

SULLIVAN, PATRICK. Characterizing the Nature of Students' Feature Noticing-and-Using With Respect to Mathematical Symbols Across Different Levels of Algebra Exposure. (Under the direction of Dr. M. Kathleen Heid.)

The purpose of this study is to examine the nature of what students notice about symbols and use as they solve unfamiliar algebra problems based on familiar algebra concepts and involving symbolic inscriptions. The researcher conducted a study of students at three levels of algebra exposure: (a) students enrolled in a high school pre-calculus course, (b) college students enrolled in a second semester calculus course, and (c) prospective secondary mathematics teachers enrolled in a mathematics teaching methods course and who have completed three semesters of calculus, linear algebra, an introduction to proof and several upper level mathematics courses.

Six students from each level of algebra exposure were asked to reason about a series of novel algebra problems that involved symbolic inscriptions and content typical of a second-year algebra course. Data were analyzed for instances of recognizing, reasoning, and linking. One of the outcomes of the research was the development of a feature noticing-and-using taxonomy. The researcher found that students' feature noticing-and-using was characterized by three different strategies: manipulative, relational, and linking. Most students reasoned from a manipulative strategy, but it was found that students faced challenges reasoning from each of these strategies. Students at the highest level of algebra exposure were much more likely than the other two levels of algebra exposure to use multiple strategies in their reasoning.

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Chapter 1

STATEMENT OF THE PROBLEM

Many of the challenges of learning algebra are related to the symbols that give meaning to the subject. The activity of algebra involves generating, transforming, and utilizing strings of algebraic symbols, known commonly as symbolic representations. Being competent in algebra requires one to interact with symbolic representations in different ways. Not only must one be able to see symbolic representations as structured strings of symbols, objects in their own right, but they may also be seen as descriptors connecting some reality or situation familiar to the individual (Pimm, 1995). The complexity of these interpretations of symbolic representations is of great difficulty for those learning algebra (Rubenstein & Thompson, 2001). Choosing the appropriate interpretations or meanings of a symbolic representation that assist the student in solving a particular problem only adds to this complexity. This ability to discern different meanings and make interpretations of symbolic representations that are helpful in solving problems has been described in the research literature as *symbol sense*.

Background

What follows is an argument for the importance of a study that examined the nature of students' reasoning about symbols. This argument also involves making a case for the need for a taxonomy to classify the nature of this reasoning.

Importance of Symbol Sense

Symbol sense is considered by some the heart of algebraic competency (Arcavi, 1994). Symbol sense involves the core aspects of algebra, symbolizing generalizations and syntactically guided reasoning about generalizations expressed in conventional symbol systems (Kaput, Blanton, & Moreno, 2008). Picciotto and Wah (1993) suggested that symbol sense is the true prerequisite for further work in mathematics and science and should be the primary purpose of algebra.

Research on Symbol Sense

Over the past two decades headway has been made in better understanding some of the challenges students have in understanding the meaning of symbolic representations (MacGregor & Stacey, 1997). While much of the research focus has been on early- or beginning-algebra students' efforts to use symbolic representations to express relationships (Carraher & Schliemann, 2007; Kieran, 2007; Lee, 1996) there is a growing body of research focused on articulating and describing symbol sense.

Several have described symbol sense in general terms (Arcavi, 1994; Arzarello & Robutti, 2010; Keller, 1993; Kinzel, 2001; Zorn, 2002), whereas others have proposed a set of characteristics describing specific elements of symbol sense (Arcavi, 1994; Fey, 1990). Fey (1990), from a multiple representation perspective, characterized symbol sense as the ability to (a) scan an algebraic expression to make rough estimates of the patterns that emerge in numeric or graphic representations, (b) make informed comparisons of magnitudes of functions, (c) scan a table of function values or a graph to interpret verbally stated condition to identify the likely form of an algebraic rule that expresses the appropriate pattern, (d) inspect algebraic operations and predict the form of the result, or (e) determine which of several equivalent forms might be most appropriate.

Arcavi (1994) characterized symbol sense in ways that were helpful in thinking about aspects of this study. He stated symbol sense includes an (a) understanding of how and when symbols can be and should be used in order to display relationships, (b) ability to abandon symbols in favor of other approaches in order to make progress in solving a problem, (c) ability to manipulate and to "read" symbolic expression as complementary aspects of solving algebraic problems, (d) awareness that one can engineer symbolic relationships that express the verbal or graphical information needed to make progress in solving a problem, or (e) ability to select a possible symbolic representation of a problem.

Other researchers have focused on students' understanding of and difficulty with particular mathematical entities and ideas that are connected to aspects of students' symbol sense abilities. The current study examined students' reasoning about different elements of students' symbol sense in the context of problems that involved symbolic inscriptions. Elements of symbol sense discussed in this study include structure (Hoch & Dreyfus, 2004; Menghini, 1994; Pomerantsev & Korosteleva, 2003; Vaiyavutjamai, Ellerton, & Clements, 2005), linking representations (Knuth, 2000; Pierce, 2001; Sfard & Linchevski, 1994) , equivalence (Knuth, Stephens, McNeil, & Alibali, 2006; Linchevski & Herscovics, 1996) and meanings of letters (e.g., unknowns, variables, and parameters) (Bloedy-Vinner, 1994; Furinghetti & Paola, 1994).

Symbol Sense Frameworks

These studies shed light on the aspects of symbol sense that are challenging for students as they reason about symbols. These studies also suggest that there are many complexities as students reason about symbols. Krutetskii (1976) argued that any genuinely scientific approach to the study of a complex phenomenon requires an analysis of its structure and an isolation of its components. Understanding the nature of students' symbol sense, reasoning about symbols, is complex, and there is a need for organizational structures such as frameworks and taxonomies to examine the nature of students' reasoning about symbols and understand what this reasoning entails.

In more recent studies (Kenney, 2008; Pierce & Stacey, 2001), using elements of symbol sense as identified by Arcavi (1994), researchers have created frameworks that have been used to analyze the structure and isolate the components of students' symbol sense capabilities. Pierce and Stacey (Pierce, 2001; Pierce & Stacey, 2001) created a framework, Algebraic Insight, that characterizes elements of symbol sense in a computer algebra system (CAS) environment. Algebraic Insight, as described by Pierce and Stacey (2001), is the algebraic knowledge necessary for correctly entering expressions in a CAS, efficiently scanning for possible errors, and interpreting the output as conventional mathematics. The limitation of Pierce and Stacey's algebraic insight framework as a framework for describing students' reasoning about symbols is that it is designed to apply only to elements of symbol sense for which the CAS is helpful, the stage of solving a formulated problem. It does not provide a means to describe the activity in other stages of problem solving, such as formulating the problem and interpreting the solution.

Kenney (2008), in her dissertation study, expanded the work of Pierce and Stacey by incorporating their framework into a framework for identifying students' uses and understandings of symbolic structures in other stages of problem solving. While Kenney's framework does seem to provide an organization framework to study symbol sense, it has a few limitations. First, her framework does not address the back-and-forth movement between representations that seems to be characteristic of students' reasoning about symbols. Her framework categorizes reasoning from symbolic representations to graphical representations in broad terms, linking symbolic and graphical representations, without consideration for the direction of the link and the importance of this directionality. Second, findings from her study suggests that while her framework was helpful in categorizing aspects of symbol sense, it did not provide a lens to examine some of the challenges in students' reasoning about symbols that she saw in her data. Particular areas of difficulties that she mentioned include linking different representations, reasoning about symbol meaning in the context of the problem, and understanding the objects represented by the symbols (p. 302). The current body of research seems to need a tool to classify the nature of students' reasoning about symbols with an eye toward the difficulties that Kenney. Developing a tool that addresses this issue seems to be an important contribution to the body of research in the area of symbol sense.

Problem Statement

The problem(s) addressed in this study related to the nature of students' reasoning about symbolic representations in the context of solving problems presented using symbolic inscriptions. A *symbolic inscription* is defined as a symbol string. This study will account for several important aspects related to symbol sense that have not been addressed in prior studies. In particular, it will (a) account for the application of symbol sense to unfamiliar algebra problems based on familiar algebra concepts and involving symbolic inscriptions, (b) examine students' feature noticing-and-using about symbols across different levels of algebra exposure, and (c) describe the nature of students' feature noticing-and-using about symbols in a range of problem settings.

Symbol Familiarity

An assumption made in this research study is that feature noticing-and-using about symbols develops over time. It may take a significant amount of time for students to become comfortable enough with algebraic forms/notations to extract meaningful information from them (Arcavi, 1994). Likewise, Gray and Tall (1994) argue that it takes time working with new content for students to step back and reason about the symbolic representation in a conceptual manner. In the two prior studies (Kenney, 2008; Pierce & Stacey, 2001) that attempted to characterize symbol sense using a framework, the study participants were interviewed on content related to the class they were taking, calculus and precalculus, respectively.

Kenney's study involved students enrolled in a college precalculus class who were asked in an interview setting to reason about problems that involved precalculus content. This may have been too early to examine symbol sense capacities because the students may have been unfamiliar with what the symbolic representation revealed. For example, Kenney asked interviewees to reason about the following task, solve for *x:* $\frac{x-16}{x^2-3x-12}=0$. Results from her study suggest students had difficulty attending to the denominator of the rational expression to solve the problem (p. 71). One possible hypothesis regarding their difficulty is that solving rational equations was recently taught to students so it may have been too early for students to step back and engage the symbolic representation in a more conceptual manner, which would suggest a higher degree of symbol sense.

The current research will be able to contribute to the field by examining students' feature noticing-and-using, an aspect of symbol sense, on unfamiliar algebra problems involving symbolic inscriptions in which the algebra content is familiar to them. Students participating in this study will be given tasks involving content typical of a second-year high school algebra course, a level of content with which all students in the study are expected to be familiar, but the tasks themselves are designed to challenge students to reason beyond simply performing the steps to a procedure.

Levels of Algebra Exposure

Another issue to consider is the study population. There is evidence within the literature to suggest that students across a range of levels of algebra exposure face challenges when reasoning about symbols. Specific levels of algebra exposure described within the literature include advanced high school (Hoch & Dreyfus, 2004), undergraduate mathematics students (Crowley, 2000; Kenney, 2008; Pomerantsev & Korosteleva, 2003), and prospective mathematics teachers (Pomerantsev & Korosteleva, 2003; Vaiyavutjamai et al., 2005). It seems important to examine the nature of students' feature noticing-and-using from symbols across levels of algebra exposure in order to better understand the nature of and challenges in feature noticing-and-using both within a level of algebra exposure and across levels of algebra exposure.

Different levels of algebra exposure were chosen because one would expect that students' exposure to algebra would have some effect on their feature noticing-and-using. Precalculus students were chosen as the base level because it would be expected that these students would be comfortable reasoning about problems involving content typical of a second-year algebra course. Of all three levels of exposure their experience with second-year algebra course content would have been most recent. Calculus II students were chosen because of their assumed increased exposure working with algebraic symbols. One could argue that the Calculus II content involves more extensive symbol manipulating and graphical analysis (Stewart, 2008), and requires applications of symbolic manipulations in applied settings. The last level of exposure, prospective secondary mathematics teachers, were chosen not only because of their different exposure, but because they were prospective secondary mathematics teachers. These experiences include participating in at least 3 semesters of calculus, a linear algebra course, a proof course, two mathematics teaching methods courses, and possibly practice in teaching secondary mathematics classes. Each of these courses requires students to make mathematical arguments involving algebraic symbols.

Each level of algebra exposure represents an increased level of algebra exposure as well as a higher degree of mathematical importance with respect to career goals. The highest degree of importance lies with prospective secondary mathematics teachers. They have chosen a career that involves mathematics and are being trained to help others learn mathematics. It could be argued that this level of student represents those with a perceived high level of expertise who have at the very least chosen a career path that involves mathematics on a regular basis. In other words, mathematics is integral to their career choice. Prospective secondary mathematics teachers also, as part of their mathematics exposure, participated in methods courses in which there was presumed explicit attention given to multiple representations (including symbolic representations).

At the next level of importance, Calculus II students, there is a reasonable expectation that they have chosen a field of study that requires at least two semesters of calculus. It cannot be discerned that, unlike prospective mathematics teachers, their chosen career path will involve mathematics. The precalculus level represents the level where the importance of mathematics to the student could be the least. Many precalculus students take additional courses in mathematics while many others do not. Since the purpose of this study was to examine the nature of students' feature noticing-and-using, the goal was to examine the reasoning of students at levels that represent a range of exposures to algebraic symbols.

Purpose Statement

It is the intent of this study to examine the nature of students' reasoning about symbols, an aspect of symbol sense, as they solve problems involving symbolic inscriptions. Recall that a *symbolic inscription* is defined as a symbol string. The nature of students' reasoning was examined in fine-grained detail by identifying specific features of symbolic inscriptions noticed by students and how those features were used using a taxonomy designed by the researcher called *feature noticing-and-using* (this will be discussed in Chapter 2). *Feature noticing-and-using*, a term defined by the researcher, is the action triggered by noticing a feature of a symbolic inscription and using the feature to reason about a problem involving symbolic inscriptions. A *feature* is a form, characteristic, or structure of a symbolic inscription or an object or relationship represented by the symbolic inscription. The feature is said to be "noticed" when it is the focus of the student's attention, as inferred through verbal statements and/or written work, when he or she attends to the symbolic inscription during the solving process. In summary, this study not only examined the features noticed by students as they solve problems that involve symbolic inscriptions, but it also examined how they use these features in their reasoning and the meaning they attached to these features as they used them to solve problems.

This examination of feature noticing-and-using was conducted with students across different levels of algebra exposure. Specifically, the purpose of this dissertation was to answer questions—via analysis of semi-structured task-based interviews—about the nature of students' feature noticing-and using as students solve problems involving symbolic inscriptions. This analysis of students' feature noticing-and-using occurred across three levels of algebra exposure with the goal of determining the nature of similarities and differences in feature noticing-and-using across these levels of exposure. *Level of algebra exposure* is defined as the nature of students' mathematical experiences beyond a second- year course in algebra. Different levels of algebra exposure were attained through involvement of students from the following three groups: (a) students enrolled in a high school precalculus course, (b) college students enrolled in a second-semester calculus course, and (c) prospective secondary mathematics teachers who had taken several mathematics courses beyond three semesters of calculus and were enrolled in a secondary mathematics teaching methods course.

It was informative to the field of mathematics education to examine feature noticing-and-using across levels of algebra exposure because it was expected that this would look different as students gain more exposure. In addition to the fact that these three specific levels are described in the research as levels at which symbol sense difficulties have been identified, there are two other reasons it seemed important to examine the nature of feature noticing-and-using across levels of algebra exposure. First, to consider whether additional exposure to reasoning about symbolic inscriptions in the context of a calculus courses would impact the nature of students' feature noticing-and-using as compared to those students with less exposure. Second, to consider whether students' feature noticing-and-using at a particular level of exposure in which the career choice involves mathematics and the exposure to symbolic inscriptions is more extensive is different than those who have less exposure and who have not made the choice to teach secondary mathematics.

The current research contributes to the field by examining, in detail, the nature of students' feature noticing-and-using across different levels of algebra exposure. The questions addressed in this study are:

1. What is the nature of students' feature noticing-and-using as they solve unfamiliar algebra problems based on familiar algebra concepts and involving symbolic inscriptions?
2. Across levels of algebra exposure what is the nature of the similarities and differences in students' feature noticing-and-using as they solve unfamiliar algebra problems based on familiar algebra concepts involving symbolic entities?
3. What is a taxonomy that describes the nature of feature noticing-and-using as evidenced in students' reasoning about symbolically presented unfamiliar algebra problems that are based on familiar algebra concepts?

The analysis of students' feature noticing-and-using focused on what features of the symbolic inscription students notice as they solve problems, how they used those features, and the meaning they attach to the features.

Summary

It has been posited that students' feature noticing-and-using with respect to symbolic inscriptions, an aspect of symbol sense as claimed by the researcher, is an important issue and that there are many challenges students face in reasoning about features of symbolic inscriptions. Although the research community has identified specific features in which difficulties may lie, there have been few studies that have examined the nature of students' feature noticing-and-using and the meaning they attach to these features of symbolic inscriptions. Part of this difficulty has been due to the lack of an organizational structure, or taxonomy, that captures the different meanings students associate with features of symbolic inscriptions. This study addressed these issues by providing a taxonomy that captured the nature of students' feature noticing-and-using with respect to symbolic inscriptions as well as how this may look similar or different across levels of algebraic exposure.

Chapter 2

FEATURE NOTICING-AND-USING

It has been argued in Chapter 1 that there are many challenges students face in reasoning from features of symbolic inscriptions. The purpose of this research was to examine aspects of symbol sense in light of what noticing particular features of a symbolic inscription enabled the student to do. In other words, the researcher examined feature noticing-and-using with respect to what features were noticed and the meanings students attached to these features in solving problems that involved symbolic inscriptions. What is feature noticing-and-using? The descriptors of feature noticing-and-using used in this study are loosely based on Arcavi's (1994) descriptions of characteristics of symbol sense, Kenney's symbol sense framework (2008), and personal observations of students' reasoning about symbols. *Feature noticing-and-using* is the action triggered by recognizing a feature and reasoning from the feature in the context of solving problems involving symbolic inscriptions. In other words, feature noticing-and-using is about the features of a symbolic inscription that students notice, and the nature of students' activity after they notice that feature.

Key Terms

A definition of critical terms is in order. The researcher has defined *feature* as a form, characteristic, or structure of a symbolic inscription or an object or relationship represented by the symbolic inscription that is the focus of the student's attention, as inferred through verbal statements and/or written work, during the solving process. As previously stated, a *symbolic inscription* is defined as a symbol string. A symbolic inscription becomes a representation in a student's reasoning when the student attaches conceptual meaning, which can be evidenced by their verbal statements or written work, to the inscription. In other words, the symbolic inscription becomes a representation of something to the student. For example, some students may see the symbolic inscription $(x + 2)^2$ as something that can be manipulated without conceptual meaning, whereas other students may see the symbolic inscription as representing a variable expression of a number for values of *x*. In many instances determining whether a student views a particular symbolic inscription as a representation is difficult to discern. For the sake of flexibility, the term "inscription" or "symbolic inscription", as opposed to representation or symbolic representation, was used to allow for both possibilities.

Symbolic Capacity

Feature noticing-and-using is part of a taxonomy created by the researcher that was motivated by Kieran's (1996) model classifying the activities of school algebra into the three categories of generational, transformational, and global, meta-level. The researcher posited *symbolic capacity* as an overarching taxonomy for symbol sense in light of Kieran's different types of algebraic activity. The researcher defined symbolic capacity as the capacity that enables a person to (a) create symbolic representations of mathematical situations, (b) perform manipulations on symbolic representations, (c) reason from symbolic representations, and (d) utilize symbolic representations to solve problems and justify claims. Symbolic capacity involves knowing how to create symbolic representations to model mathematical situations and how to perform manipulations on symbolic representations. Symbolic capacity also involves noticing features of symbolic representation and using symbolic representations for mathematical purposes. There are four action-oriented dimensions of symbolic capacity: *generating*, *transforming*, *utilizing*, and *feature noticing-and-using*. The first three dimensions of symbolic capacity are similar to different aspects of algebraic activity as described by Kieran whereas the fourth dimension, feature noticing-and-using, extended Kieran's work.

Dimensions of Symbolic Capacity

While the focus of this study involves the dimension of feature noticing-and using, it is important to briefly describe the other dimensions of symbolic capacity. For the purpose of this study, a dimension is described as an aspect or facet of a situation. The first dimension of symbolic capacity is generating. *Generating* is the activity of creating symbolic representations to model relationships displayed in other representations, expressing rules governing numerical relationships, and forming equations containing unknowns that represent quantitative problem situations. Symbol generating is a prompted activity. That is, the mathematical activity of generating is implied within the task or requested by the teacher. For example, the task, "Write an algebraic expression to represent the sum of three consecutive integers." is a prompted activity. This activity is considered an example of generating.

Transforming is the activity of manipulating algebraic expressions and equations into equivalent forms. A student simplifying an expression such as $x + (x + 1) + (x + 2)$ or expanding the expression $(x + 2)^2$ is engaged in symbol transforming.

Another dimension of symbolic capacity is utilizing. *Utilizing* is the activity involved in reasoning about using symbols to solve problems, justifying claims, and establishing convincing arguments. Utilizing involves the purposeful use of symbolic representations in students' mathematical activity. Unlike generating, which is a prompted action, utilizing involves a conscious decision by the student to use a particular symbolic representation in his or her problem-solving endeavors. Utilizing is how one uses created symbolic representations in the context of solving a problem. For example, a student who, when asked to show that the sum of three consecutive integers is always divisible by 3, writes the expression $x + (x + 1) + (x + 2)$, transforms it into $3x + 3$ and into the expression $3(x + 1)$, reasons that the expression $3(x + 1)$ has a factor is 3, thus implying divisibility by 3, is exhibiting elements of utilizing. It is important to note that utilizing could involve feature noticing-and-using. For example, in the previous problem utilizing is the activity involved in producing the expression $x + (x + 1) + (x + 2)$, but noticing a feature (factor of 3) of the transformed expression $3(x + 1)$ and using the feature to reason about the expression being divisible by 3 represents an instance of feature noticing-and-using.

Feature Noticing-and-Using

The fourth dimension of symbolic capacity, feature noticing-and-using, is the dimension that is the focus of this study. *Feature noticing-and-using* is the action triggered by recognizing features of symbolic inscriptions and reasoning from those features in the context of solving problems that involve symbolic inscriptions. Feature noticing-and-using is operationalized by verbal statements expressed and written work shown during the problem-solving process. As students reason about a particular symbolic inscription in a problem context there are features of the inscription to which they attend. What is attended to plays a role in what is commented upon and what is written. For example, one algebra student, when asked to graph $y = 3x + 6$, may attend to the equal sign and see the symbolic inscription as representing an equation, whereas another algebra student may attend to the parameters in the equation and see the symbolic inscription as representing an equation for a line with a slope of 3 and a *y*-intercept of 6. Feature noticing-and-using may involve noticing particular properties and structures within the symbolic inscription and reasoning about those properties and structures. It also may involve linking features of other relational forms to features of the symbolic inscription.

The activities of generating, transforming, and utilizing symbolic inscriptions also involve feature noticing-and-using. Feature noticing-and-using occurs whenever symbols are present such as in transforming tasks (e.g., Let the lines $15x + 20y = -2$ and $x – y = -2$ intersect in point P. Find all values of k that ensure that the line $2x + 3y = k^2$ goes through point P) as well as in problem-solving tasks in which there is not an initial symbolic representation, but in which one is generated during the process. For example, in the aforementioned problem a student would need to notice features of the symbolic inscription(s) in order to perform an appropriate procedure to find point P and reason about the value of *k* to ensure the given line will go through point P. Feature noticing-and-using may occur in tasks in which there is a prompt for a symbolic representation, generating, or it may happen as part of a student's unprompted problem-solving activity, utilizing.

Components of Feature Noticing-and-Using

As a student interacts with a symbolic inscription he seems to have three options: (a) *recognizing* features of the symbolic inscription, (b) *reasoning* about the meaning of symbolic inscription, or (c) *linking* from features of the symbolic inscription to features of another representation of the symbolic inscription.

Recognizing

Recognizing is the activity of noticing features of a symbolic inscription. In many instances this activity may involve noticing features that cue symbol manipulating. For example, a student looking at the inscription $x^2 + x + 1 > 0$ in the context of solving for *x* may recognize that the inscription $x^2 + x + 1$ is of quadratic form, which may cue a procedure that involves applying the quadratic formula. Another instance of recognizing occurs when the student recognizes a mathematical object represented by the symbolic inscription. For example, in the context of solving $\frac{2x+3}{4x+6} = 2$ for *x* a student may recognize that the numerator and denominator of the rational term in the inscription $\frac{2x+3}{4x+6} = 2$ both have a factor of $2x + 3$. In other words, both terms represent algebraic expressions with the same factor. It is important to note that recognizing a mathematical object by naming the symbolic inscription does not mean the student has an understanding of the meaning of the object. This notion will be discussed later in Chapter 3 when a process-level understanding is distinguished from an object-level understanding.

Reasoning

Reasoning is the activity of describing the meaning of or reasoning about a symbolic inscription or the meaning of a part of the symbolic inscription within a particular problem context. For example, a student engaged in reasoning may reason that the inscription $x = \frac{a+b}{2}$ is the solution to the task, *solve the equation for x*, $|x - a| = |x - b|$ *where* $b > a$, because the inscription $x = \frac{a+b}{2}$ represents a point halfway between *a* and b on a number line and the problem involving the inscription $|x - a| = |x - b|$ is asking for the point that is the same distance from both a and b. In another case, a student may reason that the equation $\frac{2x+3}{4x+6} = 2$ has no solution because the equation, after mathematically valid symbol manipulating $\left(\frac{1}{2} = 2\right)$, is an untrue statement, meaning the original equation would also be an untrue statement, and that there is no value for x that would satisfy the equation because $\frac{1}{2} = 2$ and the original equation are equivalent equations.

Linking

Duval (2002) used the term *register* to describe types of representations that can be transformed. Representational forms that he distinguished include natural language, symbolic, and graphical representations. He used the term *conversion* to describe a transformation from a source register, the present representational form, to a target register, the new representational form that references the same object. For example, sketching the graph associated with the function rule $f(x) = x^2 + x + 1$ is a conversion in which both the symbolic inscription and the graphical representation of the symbolic inscription represent a quadratic function. The term *linking*, a specific type of conversion, is used to describe a link between features of a symbolic inscription and features of a nonsymbolic representation of the symbolic inscription. The conversion involves reasoning from features in one representational register to corresponding features in another representational register of the inscription. Specifically, conversion is reasoning that involves moving between different registers, symbolic to graphical, or vice-versa, as opposed to reasoning that involves staying within the same register, symbolic. For example, consider the problem, “For what values of a does the pair of equations, $x^2 - y^2 = 0$ and $(x - a)^2 + y^2 = 1$, have either 0, 1, 2, 3, 4, 5, 6, 7, or 8 solutions?” (Arcavi, 1994). A student who is able to analyze this symbolically stated task graphically by using the number of intersections between two diagonals, namely $y = \pm x$, and a family of circles with radius 1 is engaged in linking. The student who links from features of the function rule $f(x) = (x - 5)^2$ to features of the graphical representation of $f(x) = (x - 5)^2$ is engaged in linking. In each instance the student is reasoning from a feature of the source register, symbolic inscription, to a feature of target register, graphical representation of the symbolic inscription.

Another case occurs when the direction of the reasoning is changed. In other words, the student reasoned from the target register, graphical representation of symbolic inscription, to the source register, symbolic inscription. A student engaged in this type of linking, for example, might make statements connecting the solution back to the symbolic inscription in the original problem. For example, consider the earlier task, solve for *x*, $|x - a| = |x - b|$. A student engaged in linking may reason that a solution of $x = \frac{a+b}{2}$ makes sense because the graphical representations of $y = |x - a|$ and $y = |x - b|$ intersect at a point halfway between a and b and the original task, the solution of the original inscription $|x - a| = |x - b|$ is represented by a value of *x* that is the same distance from a and b.

Relationship Between Symbolic Capacity and Feature Noticing-and-Using

It is important to note that while generating, transforming, and utilizing may be thought of as separate dimensions, feature noticing-and-using is the activity that triggers recognizing, reasoning, and linking in these other dimensions. As a means to illustrate this relationship, consider a student who is asked to show that the sum of three consecutive integers is divisible by 3. Constructing the expression $x + (x + 1) + (x + 2)$ is indicative of the *utilizing* dimension of symbolic capacity. That is, the student has recognized that the sum of three consecutive integers can be represented by the symbolic inscriptions x, $x + 1$, and $x + 2$. *Features* of the inscription $x + (x + 1) + (x + 2)$ are *recognized*, which cue a procedure that *transforms* the inscription $x + (x + 1) + (x + 2)$ into another equivalent form $3(x + 1)$. *Recognizing* that 3 is a factor of the inscription $3(x + 1)$ it can be *reasoned* that the inscription $3(x + 1)$ representing the sum of three consecutive integers is divisible by 3, which means the sum of three consecutive integers is divisible by 3. It is the researcher's position that feature noticing-and-using precedes any generating, transforming, or utilizing activity. That is, a feature of the problem is attended to before activity happens. The relationship between feature noticing-and-using, other dimensions of symbolic capacity, and components of feature noticing-and-using are shown in Figure 2-1. The activities of generating, transforming, and utilizing involve feature noticing-and-using. Feature noticing-and-using has three different components. These components are manipulative, relational, and linking. It is important to note that after analysis of the data the components were changed from recognizing, reasoning, and linking to manipulative, relational, and linking. The rationale for this change is discussed in Chapter 5. The literature review focused on the initial three components (recognizing, reasoning, and linking) and the analysis (Chapter 5) and discussion of the results (Chapter 6) was organized around the three new components (manipulative, relational, and linking).

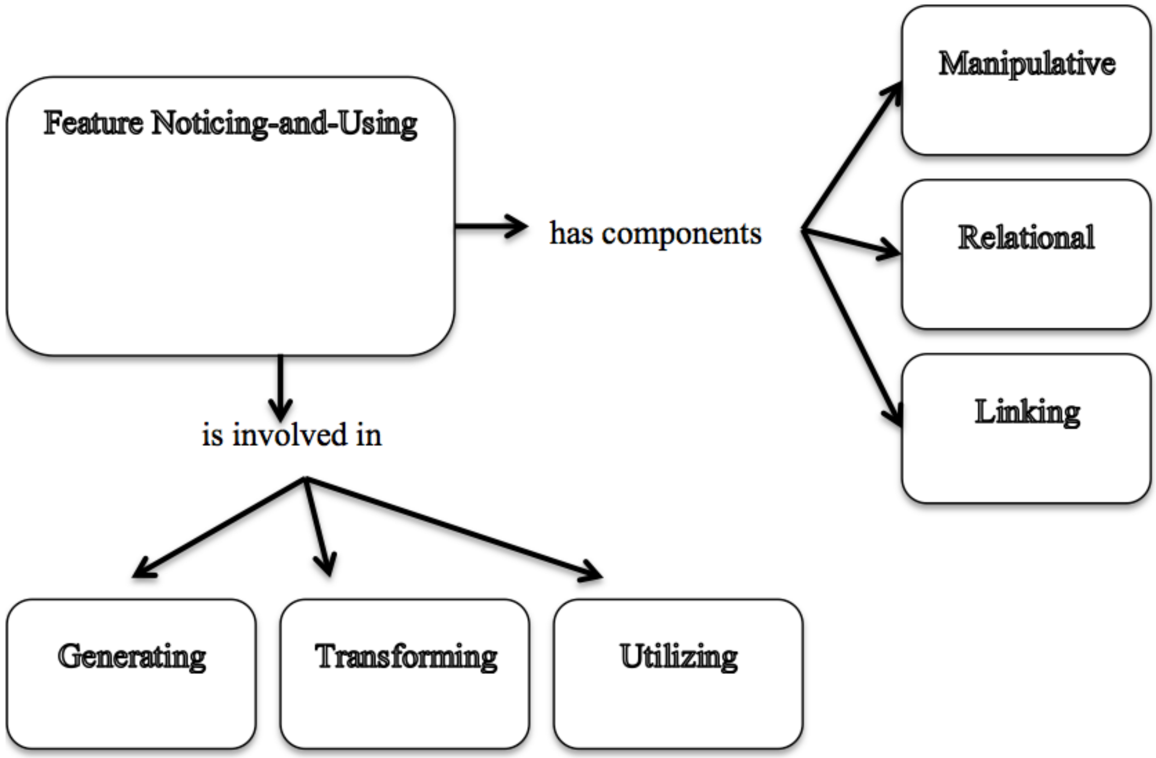

Figure 2-1. Dimensions of symbolic capacity and components of feature noticing-and-using.

It is the researcher's claim that feature noticing-and-using is at the heart of most algebraic activity. Feature noticing-and-using plays a role in the nature of the symbolic representations that are formed by students, how they are utilized in situations, and how they are transformed. Recall that the descriptors of feature noticing-and-using used in this study are loosely based on Arcavi's (1994) descriptions of characteristics of symbol sense, Kenney's symbol sense framework (2008), and personal observations of students' reasoning about symbols. Table 2-1 is a classification of these descriptors of feature noticing-and-using across the three components of feature noticing-and-using (recognizing, reasoning, and linking) that have been articulated up to this point. In other words, Table 2-1 represents a taxonomy for feature noticing-and-using. Prior to execution and analysis of the interviews the researcher anticipated that an analysis of data on students' work with symbolic inscriptions would reveal needed changes to the feature noticing-and-using taxonomy. These changes might include adjustments to instance codes and further development of the different components of feature noticing-and-using. The first column of Table 2-1 provides the components of feature noticing-and-using, while the second column provides a brief description of these components. The last column provides the specific instance codes associated with each component.

Table 2-1.
Feature noticing-and-using taxonomy.

Component	Description	Common Instance Codes
Recognizing	1. Activity of recognizing features of a symbolic inscription. This activity may include recognizing features that cue symbol-manipulating procedures.	1.1 Recognized a feature of the inscription that provided meaning.
		1.2 Recognized a feature during a symbol manipulating procedure.
		1.3 Recognized a feature that informed a symbol manipulating procedure.
		1.4 Recognized a feature that reduced written algebraic manipulations.
		1.5 Recognized a feature that cued a symbol manipulating procedure.
		1.6 Recognized a feature that resulted in abandoning one symbol manipulating procedure for another symbol manipulating procedure.

Reasoning	2. Activity of describing the meaning of or reasoning from a feature of the symbolic inscription.	2.1 Reasoning about the meaning of a symbolic solution. 2.2 Reasoning about meaning of the results of a symbol manipulating procedure. 2.3 Reasoning about the structure of a symbolic representation. 2.4 Reasoning about the meaning of symbols within a particular problem context.
Linking	3. Activity of linking features of the symbolic inscription with features in another representation of the symbolic inscription.	3.1 Linking features of an inscription expressed as a symbolic representation to features of another representational form. 3.2 Linking features of another representational form with features of an inscription expressed as a symbolic representation.

Summary

The purpose of this chapter is two-fold. First, to explain the dimensions of a taxonomy, symbolic capacity, developed by the researcher to characterize the different activities of algebra that involve symbolic inscriptions. A second purpose is to explain a specific dimension of the symbolic capacity framework, feature noticing-and-using. This explanation included describing the different components of feature noticing-and-using as well as describing a taxonomy that classifies the different instances of feature noticing-and using within each of these components.

Chapter 3

LITERATURE REVIEW

The purpose of this study was to investigate the nature of what students' feature noticing-and-using as they solve problems involving symbolic inscriptions. The nature of students' reasoning was examined in detail first by identifying specific aspects of symbol sense using a taxonomy designed by the researcher called *feature noticing-and-using.* This chapter provides a review of the research literature that underpins the characterization of symbol sense used in this study. The students in this study are users of algebra with two or more years of algebra exposure. Thus, the discussion of the literature review focuses on users of algebra with more exposure, not early-algebra students.

Reasoning About Symbols

One of the goals of this study was to characterize the nature of students' feature noticing-and-using as they solve problems involving symbolic inscriptions. A large part of what students do in algebra classrooms is to engage in activity involving symbolic inscriptions. Students' ability to interpret and make meaning of features of the symbolic inscriptions plays a role in their reasoning capabilities. Students at many different levels of algebra exposure have great difficulty reasoning about the meaning of features of symbolic inscriptions. The first part of the literature review provides an overview of how symbol sense is characterized in this study and how these characterizations form the basis for a construct that the researcher called feature noticing-and-using. The researcher examined the literature for descriptions and characterizations of noticing that were relevant to the current study. Also, the researcher examined the literature for the challenges students face with different aspects of feature noticing-and-using that were important to this study.

Defining Symbol Sense

Some have argued that symbol sense is too complicated to define because it interacts heavily with other senses such as numerical, graphical, and functional (Arcavi, 1994; Zehavi, 2004). Several have attempted to describe symbol sense. Some of these descriptions have been more global in nature: as a notational awareness (Kinzel, 2001), as a conceptual network that enables a person to relate symbolic expressions and operation properties (Keller, 1993), or as an informal skill to deal with symbolic representations (Fey, 1990).

Characterizing Symbol Sense

Despite the difficulty of defining symbol sense, several have described it in terms of specific abilities. Arzarello and Robutti (2010) suggested that students who are able to (a) call on symbols in the process of solving a problem, (b) abandon symbols for better tools, (c) recognize the meaning of a symbolic expression, and (d) have a sense of the different roles symbols can play in different contexts, are exhibiting aspects of symbol sense. Also, Zorn (2002) described symbol sense as an ability to extract mathematical meaning and structure from symbols, to encode meaning efficiently, to manipulate successfully, and to discover new mathematical meanings and structures.

Moving beyond descriptions, Arcavi (1994) characterized particular elements of symbol sense. Arcavi's description of these elements of symbol sense make up several of the common instances listed in the researcher-proposed taxonomy for feature noticing-and-using. These include, (a) a feeling of when to abandon symbols, (b) an ability to manipulate and read symbolic expressions, (c) an ability to purposefully and flexibly manipulate symbolic representations, (d) an understanding of the need to check symbol meanings while solving problems, and (e) a sense of the different roles symbols can play in different contexts.

Arcavi argued that even after years of study many students fail to make sense of algebraic symbols, often failing to use them as tools for understanding and communicating generalizations, revealing structure, or making connections between mathematical entities. He argued that symbol sense is at the heart of algebraic competence and is a necessary component of sense-making in mathematics.

Existing Symbol Sense Frameworks

As an understanding of symbol sense has progressed, researchers (Kenney, 2008; Pierce & Stacey, 2001) have developed frameworks to characterize the aspects of symbol sense occurring in the context of technological environments. Pierce and Stacey (2001) characterized calculus students' work using a computer algebra system (CAS) by identifying the specific aspects of symbol sense relating to what the CAS can provide: the mathematical solution to the problem. Their analysis led to a framework defined as *Algebraic Insight*, "…the algebraic knowledge and understanding which allows a student to correctly enter expressions into a CAS, efficiently scan the working and results for possible errors, and interpret the output as conventional mathematics" (p. 2).

Algebraic insight is divided into two parts by Pierce and Stacey (2001):

(a) *algebraic expectation* and (b) *linking representations*. Algebraic expectation is described as the thinking that takes place as one considers the possible outcomes or results of algebraic activity. The three elements of algebraic expectation include: recognition of conventions of basic properties, identification of structure, and identification of key features. The linking-representations component of their framework influenced the linking component of the feature noticing-and-using framework.

Kenney (2008) expanded Pierce and Stacey's (2001) algebraic insight framework to incorporate other aspects of symbol sense. Similar to Pierce (2001) who isolated aspects of Arcavi's symbol sense that apply only to the solving stage of problem solving, Kenney, using similar reasoning, categorized instances of symbol sense into the stages of problem solving as described by Polya. Kenney argued that the two components that make up algebraic insight, algebraic expectation and linking representations, could be identifiable whether students were working with or without technology. Table 3-1 displays Kenney's conceptual framework for symbol sense used to identify students' uses and understandings of symbolic representations at all stages of problem solving. Stages 2 and 3 represent the two components of algebraic expectation as described by Pierce and Stacey (2001) and stages 1 and 4 represent Kenney's extension of algebraic expectation into the other stages of problem solving. In both frameworks the checking stage of problem solving is not included.

Table 3-1

Kenney's Symbol Sense Framework (Kenney, 2008, p. 43)

Stage	**Elements**	**Common Instances**
1. Formulation	1.1 Linking verbal and algebraic	1.1.1 Know how and when to use symbols

	representation	1.1.2 Know when to abandon symbols for other approaches
		1.1.3 Ability to select possible symbolic representations
		1.1.4 Know that the chosen representation can be abandoned when it is not working.
2. Solving (Algebraic Expectation)	2.1 Recognition of conventions and basic properties	2.1.1 Know meaning of symbols
		2.1.2 Know order of operations
		2.1.3 Know properties of operations
	2.2 Identification of structure	2.2.2 Identify strategic groups of components
	2.3 Identification of key features	2.3.1 Identify form
		2.3.2 Identify dominant term
		2.3.3 Link form to solution type
3. Solving (Linking Representation)	3.1 Linking symbolic and graphic reps	3.1.1 Link form to shape
		3.1.2 Link key features to likely position
		3.1.3 Link key features to intercepts or asymptotes
	3.2 Linking symbolic and numeric reps	3.2.1 Link key features to critical intervals for a table
4. Interpretation	4.1 Recognition of meaning	4.1.1 Link symbol meaning to personal expectations

4.1.2 Use symbols to
communicate results

Kenney's framework does provide a lens to examine instances of symbol sense. This seems to be an important contribution to research on symbol sense since many have discussed challenges with aspects of symbol sense (Bloedy-Vinner, 2001; Hoch & Dreyfus, 2004; Knuth et al., 2006; Menghini, 1994; Sfard & Linchevski, 1994), but few (Pierce & Stacey, 2001) have provided a way to classify what students do with symbols. As described in detail in Chapter 2, the researcher's taxonomy, feature noticing-and-using, is an attempt to further the research community's understanding of the nature of students' noticing and using of features of symbolic inscriptions in their reasoning. Feature noticing-and-using is the researcher's attempt to characterize the nature of the meanings that underpins what students' identify, or describe, as they reason about problems that involve symbolic inscriptions.

Before discussing the different meanings that underpin what students notice about features of symbolic inscriptions it seems important to provide a description of noticing and the different distinctions researchers have made about different aspects of noticing. Over the next few pages the meaning of noticing will be discussed as well as some of the distinctions in the literature that advanced the researcher's thinking about feature noticing-and-using.

Noticing

Noticing is an attention to something (Mason, 2011) or an awareness that enables actions (Gattegno, 1987). Mason (2011) argued, "Attention is not a thing to be observed in others, but its influence can be inferred" (p. 46). He used the terms macro-structure, meso-structure, and micro-structure to describe the different ways people attend to things. From a *macro-structure* perspective, attention can vary a) in the focus, b) in the locus (source or basis of attention), c) in the strength or amplitude, and d) in the scope or breadth. From a *meso-structure* perspective, attention can be dominated by a particular collection of beliefs or perspectives and from a *micro-structure* perspective, attention is on discerning details, recognizing relationships, and perceiving properties.

LaGrange (2005) made three structural distinctions in which work is done in a mathematical domain. LaGrange (2005), using Chevallard's (1999) terminology, distinguished between three structural distinctions in which work in a mathematical domain is done. He described these three distinctions as tasks, techniques, and theories. LaGrange (2005) claimed that *tasks* are not just individual problems, but more general structures for problems and *techniques* are "a way of doing tasks." Techniques are the strategies or reasoning a student uses to complete a particular type of task. *Theories* are related to the consistency and effectiveness of the techniques. At this level of distinction the mathematical properties, concepts, and specific language appear.

In the next few paragraphs the relationship among Mason's different ways of attending, LaGrange's different structures, and the current study is discussed. Mason noted that the first way people attend to things is at a macro-structure level. In this study students were asked to reason about *tasks* that involved symbolic inscriptions so the focus of attention was somewhat controlled by the researcher. From a macro-structure perspective, the students in this study controlled the locus, strength, and scope of attention. That is, they noticed features of the symbolic inscription and used those features to reason about the task. The nature of their reasoning relied upon the meaning that the student had attached to the features of the symbolic inscription. This meaning, depending on students' understandings, had varying degrees of connectedness to other ideas. In other words, the strength and scope of attention was related to the meaning students attached to features of the symbolic inscription.

In thinking about the nature of students' reasoning about tasks that involve symbolic inscriptions there seems to be a connection between Mason's meso-structure way of attending and LaGrange's structural distinction of technique. Artigue (2002) argued that mathematical productions and thinking modes are dependent on the social and cultural contexts in which they develop. There is an enculturation that happens as students engage in learning in a particular domain. For example, it can be argued that when asked to complete a task that involves the phrase "solve for *x"* and in which *x* is part of symbolic inscription, students will be clued to perform a procedure related to the structure of the symbolic inscription because that is the norm in the social and cultural contexts in which they have experienced tasks of these types. In other words, from a meso-structure perspective, students' perspective or belief is that tasks that involve symbolic inscriptions that are algebraic in nature will require a procedure to be applied and that the results of that procedure are sufficient for answering the question.

Lagrange described technique as a "way of doing" a particular task. A student that has a particular way of doing a task has a perspective or belief about how tasks phrased in a particular way and of a particular symbolic form are done. As students reason about problems that involve symbolic inscriptions there are particular actions or *techniques* that are part of their routine for solving tasks that have particular features. Each domain of study, as is the case with algebra, has a way of doing tasks or solving problems that involve symbolic inscriptions. Learning these techniques requires attention to or noticing particular features of the symbolic inscriptions that are part of the particular domain. For example in learning to read, early readers develop an awareness of phonemes, a feature of the symbolic inscription that represents words. Young readers learn to hear, identify, and manipulate phonemes, features of the words, enabling them to differentiate meaning of different words, or symbolic inscriptions (Juel, 1988). Similarly, in algebra students learn to develop an awareness of what features of symbolic inscriptions to notice in order to perform procedures related to those features.

Hinkel (2004), in the context of language acquisition, stated two challenges related to noticing that seem relevant to noticing in other domains. He argued students needed to know what features of the symbolic inscription they should notice and what about those features requires attention. Similar to young readers, second language learners also become enculturated into a way of attending to and noticing particular features. Schmidt and Frota (1986) claimed there are two kinds of noticing that are necessary conditions for language acquisition: (a) the learner must attend to linguistic features of the input to which they are exposed, and (b) the learner must notice the gap between the current state of their developing linguistic system, realized in the output, and the target language system, available as input.

Similar to language acquisition, algebra also has aspects of noticing that are necessary conditions for learning algebra. That is, students need to attend to features of the symbolic inscription that cue particular procedures related to the form of the symbolic inscription. There is an awareness to the form of the symbolic inscription that cues a particular procedure. For example, a quadratic expression or equation may cue a factoring procedure or a procedure that involves applying the quadratic formula.

From this perspective it does not seem striking that experienced algebra students do not use other representations (Sfard & Linchevski, 1994) or that they do not seem to exhibit structure sense to solve problems involving symbolic inscriptions (Hoch & Dreyfus, 2004). The type of tasks and familiar techniques students associated with solving these types of tasks did not involve either exhibiting structure sense or other representations. The familiar technique associated with these types of tasks involved noticing features of the symbolic inscriptions and using those features to apply procedures related to those features.

The third perspective on attention as described by Mason (2011) is a *micro-structure* perspective. From this perspective there seems to be focus on the meaning of the symbolic inscription—LaGrange characterized this as a *theories*, a structural distinction. From this perspective, the focus of students' attention is on the meaning of features of the symbolic inscription. Students are able to able to discern details, recognize relationships, and perceive properties of symbolic inscriptions. Students' reasoning suggests that they see the symbolic inscriptions as representing mathematical objects and relationships. In other words, students' noticing of features, or attention, is from a micro-structure perspective and their using of features suggests a theories level of working.

Up to this point the literature review has focused on the noticing aspect of feature noticing-and-using. The ensuing literature review is focused more on the using aspect of feature noticing-and-using. A brief discussion of the noticing of those with more expertise will be followed by a discussion of the different meanings students attach to the symbolic inscriptions that were part of this study.

Expertise

The researcher has defined expertise as the amount of exposure to algebra. In this study those with the most expertise are prospective secondary mathematics teachers and those with the least expertise are high school precalculus students. Several researchers have made the distinction between expert and novice knowledge (National Research Council, 2001). These distinctions seem applicable to noticing features and the meaning students attach to the features of symbolic inscriptions they notice. For example, experts notice features and meaningful patterns of information that are not noticed by novices. Expert knowledge "is not simply a list of facts and formulas that are relevant to their domain; instead their knowledge is organized around core concepts or 'big ideas' that guide their thinking in their domain" (NRC, 2001, p. 36). Those with more expertise seem to possess an organization of knowledge with meaningful relationships among related elements that are governed by underlying concepts and principles. Expert knowledge means having more conceptual chunks, more connections between those chunks, and procedures for applying this information in problem-solving contexts (Chi, Feltovich, & Glaser, 1981). Specific to mathematics, experts are more likely than novices to first try to understand the problem, rather than to apply procedures (NRC, 2001). Over the next pages the focus of the literature review is on describing differences in the meaning students attach to features of the symbolic inscriptions that were part of this study as well as the challenges students face in reasoning from these features.

Meaning of Symbols

Kieran (2007) claimed that there are four sources of meaning related to symbolic inscriptions (a) algebraic structure (letter-symbol form), (b) other mathematical representations, (c) problem context, and (d) real-life applications. The ensuing discussion deals mainly with the first three sources identified by Kieran since the fourth is beyond the scope of this study because the tasks used in this study did not involve real-life applications.

Several researchers have made distinctions between the meanings attached to features of symbolic inscriptions. Yerushalmy and Gafni (1992) differentiated between two levels of meaning students attach to symbolic inscriptions. On the lower level is syntactic manipulation in which the student operates with basic algebraic rules from left to right by order of operations, using the same few algorithms constructed of sequences of steps such as: expand, collect, and factor. On the other hand, semantic interpretation involves analyzing top-level properties of the expression such as number of zeros, degree of variable, parameters, or constraints. In a similar manner, Skemp (1987) described two levels of structure related to features of symbolic inscriptions: surface structures and deep structures. Surface structures involve the written symbols, whereas the more difficult deep structures of language are those that involve the conceptual meanings of the symbolic inscriptions.

These two distinctions seem to describe the differences in meaning characterized by the recognizing and reasoning components of feature noticing-and-using. The recognizing component of feature noticing-and-using is characterized by recognizing features that cue particular procedures. In other words, the meaning is in the syntactic structure of the symbolic inscription. On the other hand, the reasoning component of feature noticing-and-using is characterized by the conceptual meaning students attach to the symbolic inscriptions.

Recognizing. A key aspect of feature noticing-and-using is recognizing the features that cue particular procedures. Kinzel (2001) argued that students' focus on manipulating may hinder their ability to evaluate their own work with respect to notational issues. Many students begin manipulating without noticing structural features of algebraic expressions (Hoch & Dreyfus, 2004; Sfard & Linchevski, 1994; Steinberg, Sleeman, & Ktorza, 1990). For example, Vaiyavutjamai, Ellerton, and Clements (2005) found that a third of experienced algebra students, when asked to find the zeros of the equation $(x-3)(x-5)=0$ transformed the equation into $x^2-8x+15=0$, without recognizing the zeros. In another study involving algebra students, Steinberg, Sleeman, and Ktorza (1990) found that while students knew how to use manipulations to solve simple equations, they did not use this knowledge to judge the equivalence of expressions. Menghini (1994) argued that "…many students seem to prefer monotonous, repetitive processes requiring very little concentration or reasoning… over brief concise processes requiring the active distinguishing of similarities and differences" (p. 12–13).

Brief and concise processes in manipulating symbols require recognition of features of symbolic entities that cue appropriate algebraic manipulations. Several authors have used the term *structure sense* to describe the nature of this ability. Linchevski and Livneh (1999) used the term to describe the ability to "…use equivalent structures of an expression flexibly and creatively" (p. 191). Hoch and Dreyfus (2004) elaborated on this definition of structure sense, describing it as, "…a collection of abilities, separate from manipulative ability, which enables students to make better use of previously learned algebraic techniques" (p. 49). More recently, Novotna and Hoch (2008) contended that students display structure sense if they can (a) recognize a familiar structure in its simplest form, (b) deal with a compound term as a single inscription and through a familiar structure in a more complex form, and (c) choose appropriate manipulations to make best use of the structure.

There are several other instances within the literature of students struggling with noticing specific structural features. In a study involving students with more advanced levels of high school, Hoch and Dreyfus (2004) found low levels of structure sense. None of the 92 students tested used structure sense to solve the equation, $1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$, for n. That is, they used a procedure such as multiplying both sides of the equation by $n + 3$ instead of recognizing the two sets of terms that were opposites. In the same study Hoch and Dreyfus found that students were more likely to use structure sense on tasks in which the structures required to be noticed involved parentheses or brackets. The parentheses seemed to assist students in seeing the necessary features needed to solve problems without engaging in symbol manipulating activity.

Reasoning. There are other challenges, aside from recognizing features, that students face as they solve problems involving symbolic inscriptions. These challenges involve the concepts, or objects represented by the symbolic inscriptions. Rubenstein and Thompson (2001) suggested these challenges include that (a) the same symbol may have different meanings, (b) multiple symbols may represent the same concept, (c) specific variables may be used in specific contexts, and (d) the family to which a function belongs is embedded in its symbolization (p. 268).

An example of struggles students face in reasoning about the meaning of features of symbolic inscriptions is revealed in a study conducted by Vaiyavutjamai, Ellerton, and Clements (2005). In a study of 29 second-year college students enrolled in a college algebra course who intended to become middle school mathematics specialists, the researchers found that almost a third of students were confused about the concept of variable in the context of questions of the form, solve for x in the equation $(x - a)(x - b) = 0$. Many assumed that different values or x could be used at the same time in the two sets of brackets.

As the study by Vaiyavutjamai and others suggested, reasoning about the meaning of symbolic inscriptions requires understanding the meaning of variables, parameters, and unknowns in a given problem context. For example, in the problem, solve for x in the equation $(x - a)(x - b) = 0$, the letter x denotes a variable and the letters a and b represent parameters. That is, for all a and b, there exists an x such that the equation $(x - a)(x - b) = 0$ is true. Namely, when x is equal to a or x is equal to b the equation is true. As the previous study alluded, many students do not understand the distinctions between parameters and unknowns or variables (Bloedy-Vinner, 2001; Furinghetti & Paola, 1994; Sfard & Linchevski, 1994). Many students recognized there are differences in the meanings of symbols, but they have a difficult time reasoning about these differences in the context of problems involving them.

Meaning of Equations/Inequalities

Feature noticing-and-using involves recognizing and reasoning about the meaning of features of symbolic inscriptions that they notice. The symbolic inscriptions students' reasoned about in this study represented relationships between mathematical expressions (e.g., $|x+1| = |x+2|$). Kieran (1981) commented on the distinction between the equal sign as a symbol to "do something signal" and as a symbol of equality. More recently, this difference has been characterized as the difference between operational and relational views of the equal sign (Knuth et al., 2006). In an operational view, students reason that the equal sign (the researcher would argue the inequality sign, as well) is a cue to perform a procedure related to the form of the expression or expressions on each side of the equal sign. For example, a student reasoning from an operational view may see the inscription $x^2 + x + 1 > 0$ and be cued to perform a procedure that involves applying the quadratic formula. On the other hand, a student reasoning from a relational view might see the inscription as a relationship between two algebraic expressions and describe values for x that are in the truth set of the relationship.

Herscovics and Linchevski (1994) argued that holding an operational view of the equal sign, instead of a relational view, made the transition from arithmetic to algebra difficult for many students. An operational view of the equal sign seems to be a view that even more experienced algebra students continue to hold. Clement (1982) suggested that college calculus students seem to be using the equal sign as merely a link between steps without reasoning about the meaning of the equivalence. More recently, McNeil and Alibali (2005) found that, as they expected, undergraduate and graduate students do hold a relational view of the equal sign, suggesting that a relational view does develop with more mathematics experience. It should be noted, however, that the equivalence context problem used in their study (i.e., 4 + 8 + 5 = 4 + ___) to make this claim is not typical of the types of problems in which more algebraically experienced students encounter the equal sign.

Process and Object

Another distinction that is helpful in thinking about the nature of students' reasoning about symbolic inscriptions and the meaning they attach to these inscriptions is the difference between process-level and object-level understanding. Sfard (1991) argued that a student considers a symbolic inscription an object if it is referred to as a real thing, static, and existing somewhere in space. Seeing a symbol as an object means being able to "…manipulate it as whole without going into details" (p. 4). To the student, on the other hand, seeing a symbol as a process means understanding is tied to a sequence of actions. A process becomes an object when the individual conveys an awareness of the totality of the process. This includes realizing the possibility of transformations that can be acted on it and the ability to construct those transformations (Cottrill et al., 1996).

Gray and Tall (1994) describe the transition from process to object as the movement from a procedure tied to a specific algorithm, to a process conceived as a whole, irrespective of algorithm. They define a procept as a symbol evoking both a process and a concept. Procept alludes to the fact that the symbol can serve jointly as a process or an object. The idea of procept is related to the notion of concept image, which consists of the mental pictures and associated properties and processes related to the concept in the mind of the individual.

While Sfard's theory parses development into an operational level (process) and structural level (object), Gray and Tall's theory is more focused on the relationship between mathematical processes, objects, and the symbols that dually evoke both. Gray and Tall argue that all symbols have process–object duality, allowing them to be classified as procepts. The conceptions of procepts change as one moves upward in mathematics. For many students this transition is the cause of substantial cognitive struggles.

There is great power is seeing a mathematical symbol as a procept. Speaking about the proceptual nature of numerical symbolism, Gray and others (1999) describe this power as being able to filter out information operating with the symbol as an object and being able to perform computation by connecting the symbol with some action schema usually tied to a mathematical procedure (p. 124). In algebra, as opposed to in arithmetic where the symbols are numbers and operations, the symbols are algebraic expressions that have the potential to be evaluated when the variables are given numerical values. Regardless, the symbols themselves still can be manipulated algebraically (Tall et al., 1999). While symbol manipulating ability can be developed by learning procedural techniques, it is Thomas and Tall's (2001) argument concerning viewing the symbolic representation in terms of (potential) procepts where the representation can be seen dually as a process of evaluation and a concept that gives students greater flexibility in their reasoning.

The symbolic inscriptions about which students are asked to reason in this study are equations, inequalities, and function rules. The process view, as described by Gray and Tall, seems to correspond in some ways with an operational view of the equal sign.

In other words, a feature of the inscription (e.g., equal or inequality sign) cues a related solving procedure. At a process level, students' reasoning is focused on the procedural steps. Students' attention is focused on the features that cue a particular procedure. For example, when given a problem such as solve $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$ for n, students reasoning at a process level notice the variable expressions in the denominator, which cues a procedure such as multiplying both sides of the equation by *n* + 3. Also, when given the problem, solve $x^2+x+1>0$ for x, students reasoning at a process level notice a feature—quadratic form—that cues a procedure that involves applying the quadratic formula. It is open to debate whether students actually notice the inequality or whether they merely recognize the quadratic expression x^2+x+1.

The challenge for students' reasoning at a process level occurs when their procedures require reasoning about the meaning of the symbolic inscription in the context of the problem, for example, when the solving procedure on $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$ results in the inscription $0=\frac{1}{72}$, or the quadratic formula applied to $x^2+x+1=0$ results in the inscription $x=\frac{-1\pm\sqrt{-3}}{2}$. In each case, successful reasoning requires an object-level understanding of the symbolic inscription. In the first instance students need to understand the meaning of the relationship between the two numbers and the what this relationship means in the context of solving the original equation $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$. That is, the inscription $0=\frac{1}{72}$ represents an untrue statement, which means there will be no value for *n* that will satisfy the equation

$1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$. In the second instance, students must not only understand that the inscription represents a complex number and that it represents the roots of the equations $x^2 + x + 1 = 0$ but also what complex roots mean in the context of solving the inequality $x^2 + x + 1 > 0$. That is, complex roots suggest either the algebraic expression $x^2 + x + 1$ is greater than zero for all values of *x* or $x^2 + x + 1$ is less than zero for all values of *x*.

An object level of understanding implies a relational view of equations and inequalities. In other words, the student views parts of the inscriptions and their totality as representing mathematical objects. For example, the entities $x^2 + x + 1 > 0$ and $|x + 1| = |x - 2|$ both represent relationships between mathematical expressions, and the mathematical expressions $x^2 + x + 1$, $|x + 1|$, and $|x - 2|$ represent numbers for values of *x*. Students reasoning at an object level may also be able to reason about the meaning of concepts such as squaring and absolute value in the context of the relationship between the algebraic expressions. In the absolute value example $|x + 1| = |x - 2|$, reasoning about the meaning of absolute value leads to viewing the value for *x* as the number that is equidistant from -1 and 2, greatly reducing the need for algebraic manipulations.

Tall and others (2000) argued that the nature of one's discourse, how one writes or talks about the inscription, serves as a significant marker as to whether one is reasoning in the process domain or in the object domain. The discourse of students reasoning at a process level is more narrative in nature, describing a procedure or series of steps. Conversely, the discourse of students reasoning at an object level is more descriptive in nature, describing properties of the object and its relationships to other objects.

It is important to note that much of the discussion and examples thus far have involved meaning as it relates to manipulations on symbolic inscriptions that result in other symbolic inscriptions. Duval (2002) characterized transformations of symbolic inscriptions that result in symbolic inscriptions as being in the same register. He used the term *treatment* to describe this type of transformation. For example, manipulating equations into equivalent forms is a treatment because the transformation involves staying in the same representational register. This is in contrast to another type of transformation that involves moving between registers. Duval used the term *conversions* to describe this type of transformation. The linking component of feature noticing-and-using is a specific type of conversion.

Meaning of Symbols Linked to Other Representations

Linking, the third component of feature noticing-and-using, involves reasoning between two representational registers. Duval (2002) used the term *conversion* to describe a transformation from a source register, the present representational form, to a target register, the new representational form that references the same object. In other words, transformations are a change in registers without changing the objects being denoted. There are two forms of linking in feature noticing-and-using. One instance is linking features from a symbolic inscription, source register, to features of another representational register, graphical representation of the symbolic inscription. The other instance is linking features from a representational register that is not a symbolic inscription, target register, to features of a symbolic inscription, source register.

Duval (2002) believed that there are two sources of incomprehension in mathematics in the conversion of representations. He argued that one of those sources is the variability in the congruent nature of the representational registers. He made the distinction between conversions of a representation that are either congruent, or transparent, and ones that are not congruent. For example, converting the statement, "The set of points whose ordinate is greater than the abscissa" is congruent to a symbolic representation, $y > x$, because there is term-by-term translation of the verbal description into algebraic notation. On the other hand, converting the statement, "The set of points whose abscissa and ordinate have the same sign" to a symbolic representation, $xy > 0$, is not congruent because the encoding or translation is not direct.

He argued that the second source of incomprehension is not necessarily the conversion itself but the direction of the conversion. In other words, there is a significant difference in a conversion when the source register is symbolic as compared to when the target register is symbolic.

Consider these two sources of incomprehension in the context of the symbolic inscription $y = x^2 + x + 1$ and the graphical representation of $y = x^2 + x + 1$. Asking students to reason about features of graphical representation of $y = x^2 + x + 1$ is much different from asking them to reason about features of the symbolic inscription represented by the graph of $y = x^2 + x + 1$. There are features of the symbolic inscription $y = x^2 + x + 1$ that reveal specific features of the graphical representation. These features include the y-intercept, direction the graph opens, and, possibly, location of the vertex. Other features of the graphical representation, specific coordinates, can be determined easily by substituting values for x to determine y. Reasoning from the graphical representation of $y = x^2 + x + 1$ seems a bit more challenging. While features of the symbolic inscription, constant term and sign of the coefficient of x^2, can be determined from the graphical representation, there is some challenge in determining the function rule that will represent the graphical representation. There is also a challenge in determining the symbolic form that best uses the information obtained from the graphical representation. Duval (2002) found different levels of success between students when they were asked to construct the graphical representation when given the symbolic representation and when they were asked to construct the symbolic representation given the graphical representation.

While there is evidence to suggest that more experienced algebra students are able to build tight links between symbolic and graphical representations (Crowley, 2000), there is evidence that linking is a challenging endeavor for many students (Kenney, 2008). Sfard and Linchevski (1994), citing Even's (1988) study with prospective mathematics teachers, argued that relating concepts in a symbolic representation to those corresponding concepts in a graphical representation is a difficult task for many learners. In their own study involving high school students, Sfard and Linchevski (1994) found few secondary school students using graphical representations to reason about tasks involving symbolic representations even when reasoning about the graphical representation would have led to a quick answer. Similarly, Even (1998) in a study of prospective mathematics teachers, found "…relating solutions of equations to values of corresponding functions in a graphical representation is a tough task with which not many learners can cope" (p. 110).

Driscoll (1999) argued that in the course of solving a problem students may fail to connect their graphical representations back to the original problem. Linking is challenging for many students because it requires evaluating their work in a way that involves coordinating representational forms that do not necessarily look the same, but have a connected meaning. For example, the equation $|x - a| = |x - b|$ where $b > a$ and the solution to the equation $x = \frac{a+b}{2}$ may not look alike, but there is implicit meaning (the absolute value of $x - a$ is the distance from x to a) that connects these two symbolic representations. That is, the original equation can be interpreted as the value of *x*, which is the same distance from both *a* and *b*—a value that would have to be halfway between *a* and *b*, and the solution $x = \frac{a+b}{2}$, an inscription that represents the point halfway between a and b.

Summary

As students work on algebraic problems they interact with symbolic inscriptions at some level. As the review of the literature suggests, the activity of solving problems and making meaning of the symbolic inscriptions within the problem situations is complex. It is Duval's (2002) position, and one with which the researcher agrees, that understanding of mathematical objects (or inscriptions) frequently passes through symbolic inscriptions that represent these objects. In a sense, the understanding of mathematical objects is tied to the features noticed as one engages in problem-solving activity that involve symbolic inscriptions. This study intends to add to the literature by analyzing and discussing the nature of feature noticing-and-using of students as they solve problems that involve recognizing features of symbolic inscriptions, reasoning about the meaning of these features, and linking features in one representational register of these inscriptions to features in another representational register of these inscriptions.

Chapter 4

METHOD

This chapter describes the method used to collect and analyze data related to the research questions:

1. What is the nature of students' feature noticing-and-using as they solve unfamiliar algebra problems based on familiar algebra concepts and involving symbolic inscriptions?
2. Across levels of algebra exposure what is the nature of the similarities and differences in students' feature noticing-and-using as they solve unfamiliar algebra problems based on familiar algebra concepts involving symbolic inscriptions?
3. What is a taxonomy that describes the nature of feature noticing-and-using as evidenced in students' reasoning about symbolically presented unfamiliar algebra problems that are based on familiar algebra concepts?

To answer these questions, a study was conducted that involved a task-based interview. Qualitative methods were used to collect data and analyze the results of the task-based interview.

Justification for the Method

It is the intent of this study to examine the nature of students' reasoning about symbols in the context of tasks involving symbolic inscriptions. The nature of students' understanding will be examined in detail first by identifying the nature of students' reasoning using a taxonomy designed by the researcher called *feature noticing-and-using* and, second, by viewing these understandings through the process/object lens of Gray and others (Gray, Pitta, Pinto, & Tall, 1999; Gray & Tall, 2001). Goldin (2000) argued that the task-based interview is appropriate when the "goal is to deepen our understanding of a particular phenomena…and to observe, record and interpret complex behavior" (p. 519). The nature of complex behavior, such as students' reasoning about symbols, is likely to be examined through students' verbal statements and written work as they solve problems. A task-based interview is appropriate because it provides the means to gather this type of data.

Choice of Levels of Algebra Exposure

MacGregor and Stacey (1997) argued that by dealing with students who have been successful in acquiring and using basic algebra, "Knowledge issues of cognitive level and different approaches to beginning algebra are avoided" (p. 49). From a cognitive perspective, Sfard and Linchevski (1994) suggest that examining the structure of symbolic representations, a necessary component of feature noticing-and-using, does not occur early in student's work with algebra. One of the goals of the study was to have the students solve unfamiliar algebra problems that involved familiar content and to examine the nature of their feature noticing-and-using across levels of algebra exposure. By involving students with at least 2 years of exposure to algebra, the challenges described by MacGregor and Stacey and Sfard and Linchevski were avoided.

Students who participated in this study were from three different levels of algebra exposure: (a) students enrolled in a high school precalculus course, (b) college students enrolled in a second-semester calculus course, and (c) prospective secondary mathematics teachers enrolled in a mathematics teaching methods course and who had completed 3 semesters of calculus, linear algebra, an introduction to proof, and several upper-level mathematics courses. The rationale for choosing these three levels of algebra exposure was discussed in Chapter 1.

Recruitment

Protocol approved by the institutional review board (# 37180) was followed for recruitment of students (see Appendix D and Appendix E). A script (see Appendix A) was read in high school precalculus courses, college-level Calculus II courses in an mathematics department of a large university, and college-level secondary mathematics methods course in an education department of the same university (see Appendix B) asking for volunteers to participate in the study. At the high school level those who volunteered were asked to read with their parents a recruitment letter (see Appendix C). Both the student and their parents were asked to sign a consent form to participate in the study (see Appendix D). Students at the college level were asked to give their consent to participate in the study (see Appendix E). To encourage students to participate in the study the researcher offered compensation of a $10 Starbucks card.

Selection of Students

Students at the first level of algebra exposure were chosen from a high school in a college town or high schools in the surrounding areas who were currently enrolled in a high school precalculus course. Volunteers were solicited from four precalculus classes. Ten students signed consent forms indicating their willingness to participate in the study. Of the 10 students who volunteered, 6 scheduled an interview. The 4 students who did not participate in the interview either were unable to fit the interview into their schedule or were ill the day of the interview and did not want to reschedule.

Students at the next level of algebra exposure were chosen from undergraduate students at a large university who were enrolled in a second-semester calculus course. Volunteers were solicited from four classes. Of the 120 students who were asked to participate, 18 signed consent form indicating their willingness to participate. Of the 18 who agreed to participate, 9 returned phone calls and emails indicating their willingness to schedule an interview. Of those 9, 7 scheduled and participated in the task interview. The 2 students who communicated with the researcher but did not participate in the interview cited scheduling conflicts as their reason for not participating.

Students at the highest level of algebra exposure were chosen from prospective secondary mathematics teachers near the completion of their undergraduate coursework who were enrolled in a mathematics teaching methods course at the same university. Of the 13 students enrolled in the class, 7 students signed consent forms indicating a willingness to participate. Of the 7 that signed consent forms, 6 scheduled and participated in the interview. The one student who agreed to participate who was not interviewed had a family emergency the day of the interview and stated he did not have time to reschedule. In total 18 students were interviewed—6 from each level.

Pilot Study

Before engaging in the full study, a pilot study was conducted. The purpose of the pilot study was two-fold. First, before engaging in a full-scale research study, the researcher believed that it was important to practice the interview protocol to further his thinking on possible task development and follow-up questions. Second, the pilot study was intended to provide insight into whether some forms of feature noticing-and-using were not addressed in the proposed taxonomy.

The pilot study informed the researcher in several ways. It suggested to the researcher that the tasks in the interview were appropriate for the different levels of algebra exposure that were studied. It was determined that the tasks were challenging, but reasonable for the levels of algebra exposure. Also, it provided an opportunity for the researcher to practice follow-up questions that were asked as the student worked on the interview problems. The researcher determined two very important questions to ask when the researcher perceived there was evidence of feature noticing-and-using. If the student recognized a feature that cued a procedure, the researcher would ask, "What made you think to do that?" Asking this question enabled the researcher to gain insight into the nature of the feature noticed by the student. If the student reasoned about the meaning of symbolic inscription, the researcher would ask, "What do you mean by what you said?" or "Can you explain what you mean again?" Asking these questions enabled the researcher to gain insight into the meaning students attached to a particular feature.

Choice of Interview Tasks

The rationale for the tasks that were used in the interview is presented in the next few paragraphs. The tasks are shared, along with an evaluation of the tasks by a panel of graduate students who had experience teaching at the three levels of algebra exposure from which students were selected for this study.

Task Novelty

One of the main principles in choosing tasks was to choose problems involving content that was accessible to all levels of algebra exposure (Goldin, 2000), but that students at all levels might see as being novel. In other words, the tasks needed to involve content that was typical of a high school advanced algebra course, such as equations and inequalities, but was also novel in nature. There were two ways in which the interview tasks were novel. First, many of the tasks involved familiar content, but the tasks were presented in unfamiliar ways. For example, while most students likely would have been exposed to the concept of absolute value, a problem such as "solve for *x*, $|x-1| = |x+2|$" was considered novel because it is not typically seen in an advanced algebra textbook section related to solving absolute-value equations. Second, several of the tasks involved symbolic inscriptions and were phrased in a manner that cued familiar procedures. In other words, the student could recognize features that cued routine procedures related to those symbolic inscriptions. The difficulty encountered by students was (a) that their routinized procedures would not necessarily work or (b) that reasoning about the meaning of the results of those procedures was challenging. For example, on Task 1 $\left(\text{Solve for } n\text{: } 1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\right)$, students might perform a procedure that resulted in the symbolic inscription $0 = \frac{1}{72}$—a result that challenged them to reason about its meaning in the context of the problem. These tasks not only enabled the researcher not only to examine the nature of students' feature noticing-and-using, but also to examine how students used what was noticed.

Interview Tasks

The interview tasks used are shared so that the ensuing discussion of the principles of task selection can be exemplified.

Task 1: Solve for n: $1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$.

Task 2a: Solve for x: $\left(\frac{1}{4} - \frac{x}{x-1}\right) - x = 6 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$.

Task 2b: Solve for x: $\frac{(2x+3)}{(4x+6)} = 2$.

Task 3: Solve for x: $x^2 + x + 1 > 0$

Task 4: Is it true that the following system of linear equations: $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ has a solution for every value of k?

Task 5: For what values of a does the pair of equations $\begin{cases} x^2 - y^2 = 0 \\ (x-a)^2 + y^2 = 1 \end{cases}$ have either 0, 1, 2, 3, 4, 5, 6, 7, or 8 solutions?

Task 6a: Solve for x: $|x + 1| = |x - 2|$.

Task 6b: Solve for x: $|x + 1| > |x + 2|$.

Task 6c: Solve for x: $|x - a| = |x - b|$.

Task 6d: Solve for x: $|x + 1| + |x + 2| = |x - 3|$

Task 6e: Solve for x: $|f(x)| = |g(x)|$.

Task 7a: Describe the graph of the function rule $f(x) = (x - 2)^2(x + 2)^2$.

Task 7b: Describe the graph of the function rule $g(x) = |(x - 4)^2(x + 2)^2|$.

Task 7d: Describe the graph of the function rule $j(x) = (x + 2)^2 + (x - 2)^2$

Task 8: Find the real values of x such that $4^{2x^2-7x+3} = 8^{x^2-x-6}$.

Feature Noticing-and-Using Potential

Another significant principle that guided task choice was the need for feature noticing-and-using. One of the goals of task selection was to find tasks that required limited symbol manipulating but also provided opportunities for students to reason about the structure of the symbolic inscriptions. It was realized that a certain level of symbol-manipulating activity can be expected on most algebraic problems, but the focus of the tasks presented in this study did not involve long and monotonous symbol-manipulating activity. Most of the tasks in the interview did not require a large amount of computational work. The goal was to focus on feature noticing-and-using and not on manipulations themselves. The only task that required multi-step manipulative work was Task 8. Most of the tasks were framed in such a way that there was potential for students to apply reasoning that would limit the amount of manipulative work required. For example, Task 1 and Task 2b could be done mentally by noticing the opposite terms and factors, respectively. Likewise, Task 6a and Task 6b could be done efficiently by reasoning about absolute value in terms of distances. In each case a high degree of feature noticing-and-using limited the need for algebraic manipulation. This was the intent of each of the tasks.

Another goal was to find tasks in which the student had to reason about the meaning of the results of the procedures with which they were likely to have had much experience. For example, in some instances applying standard solving procedures generated results that were considered atypical (e.g., typical solutions to Task 1 and Task

2b led to the equations $0 = \frac{1}{72}$ and $\frac{1}{2} = 2$, respectively). In Task 3, factoring over integers would not work and a procedure that involved applying the quadratic formula required students to reason about the meaning of nonreal numbers in the context of the problem. In Task 4 commonly understood procedures for solving systems worked, but students had to reason about the meaning of the parameter *k* in the context of the problem.

These tasks seemed to the researcher to be a good set of tasks to examine feature noticing-and-using because they were accessible to students across levels of algebra exposure, involved a range of content from a second-year algebra course, and challenged students to not only notice features but also to reason about the meaning of those features (feature-using) in the context of the problem. In summary, the tasks appeared adequate to address the research questions of this study.

Expert Panel

An expert panel was asked to determine whether the content of the tasks was typical of a high school student's advanced algebra exposure. The expert panel consisted of three graduate students enrolled in a doctoral mathematics education program who had taught at least at one of the three levels of algebra exposure considered in this study. They each agreed that the content of the tasks were typical of a high school student's advanced algebra experience.

The panel was also asked to evaluate each task with respect to the following questions: Thinking about the level of algebra exposure with which you are familiar, (a) would students be able to engage with the task, (b) would students view the task as being unfamiliar or novel, and (c) would students have an automatic process to solve the task?

Based on the feedback of the panel, the decision was made to not provide access to graphing calculators. It was the consensus of the panel that graphing calculators would make some of the tasks far less cognitively demanding and hinder the researcher's ability to answer the research questions. Also, they felt that the parentheses in Task 2a would focus students' attention on an automatic process to solve the task. Task 2a was removed from the protocol. They also felt Task 6d and Task 6e would be too difficult for students. Task 6d was removed from the protocol, but Task 6e was kept in the protocol based on a recommendation by a doctoral committee member.

In summary the tasks seemed appropriate for examining feature noticing-and-using across levels of algebra exposure. The tasks that were used in this study represented a range of mathematical content that was typical of a second-year algebra course. The content of these tasks included rational expressions, quadratic inequalities, systems of simultaneous linear equations, absolute value equations/inequalities, and exponential equations. The tasks did not involve long, tedious manipulative steps and could have been solved quickly by noticing and using features of the symbolic inscription. Also, a panel of graduate students deemed that the tasks were novel and accessible to the populations that were studied evaluated the tasks.

Task Decision-Making

Students were given tasks based on the researcher's decision-making protocol. The researcher's decision-making was based on the researcher's in-interview evaluation of a particular student's reasoning. If the researcher determined, based on a student's response, that a task was too cognitively demanding, a decision was made to give a less cognitively demanding task. Alternatively, if the researcher concluded that a student's reasoning was more advanced, a more cognitively demanding task was presented to the student. The deciding factor for the researcher in determining whether a task was too cognitively demanding for a student was whether the student had a procedure to solve the problem and/or whether they were able to reason about the meaning of the results of their procedure. In most cases the decision was made to give a less cognitively demanding task when the student did not show evidence of a procedure for solving the problem.

1. Tasks 1, 3, and 4 were given to all students in the study.
2. If the researcher sensed students were having difficulty in reasoning about Tasks 3 and 4, they were given Task 6A, a task that the researcher believed to be less cognitively demanding than Task 6B (equation instead of inequality).
3. If students reasoned algebraically and/or graphically about Task 6B, they were given Task 5 to further examine their algebraic/graphical reasoning.
4. If students were unable to reason about the meaning of absolute value in Task 6A, they were given Task 7A, a task that was believed to be less cognitively demanding (no absolute value); otherwise, they were given Task 7B.
5. If a student seemed to struggle reasoning about Task 7A or Task 7B, they were given Task 7D, a cognitively less demanding task.
6. Task 8 was given to all students who were interviewed.
7. If students struggled reasoning about Task 1 or reasoned from procedures, Task 2B was given as a follow-up later in the interview.
8. If students showed meaningful (algebraic or graphical) reasoning strategies on Tasks 6B they were given Task 6C.

9. If students showed meaningful (algebraic) reasoning strategies about Task 6C and time permitted they were given Task 6E.

10. After completing Task 7A or Task 7B the student was asked to reason graphically about Task 3 and Task 6A or Task 6B if they had not done so.

Data Collection

In order to investigate the nature of students' feature noticing-and-using it is important to examine their written and verbal statements as they solve problems involving symbolic inscriptions. As a means to facilitate this communication, the researcher conducted a semistructured interview that contained unfamiliar algebra problems that involved familiar symbolically stated algebraic content. The interview setting allowed the researcher to examine students' feature noticing-and-using and ask clarifying questions to better understand the nature of the student's feature noticing-and-using (Creswell, 1998).

An interview guide was prepared. The interview guide included the tasks to be used, identification of specific aspects of feature noticing-and-using that could arise, and possible follow-up questions (see Appendix G). Students participated in a 60- to 80-minute interview with the researcher. Students were provided with a ruler, protractor, graph paper, scratch paper, recordable pen and recordable paper. During the interview students were asked to think out loud and reason about several tasks that involved symbolic inscriptions. As the student was verbalizing his or her reasoning the researcher asked clarifying and probing questions (see Appendix G for examples) to better understand the student's verbal and written statements. The researcher did not attempt to direct the student's reasoning or teach the student how to solve the task. The researcher allowed the student to guide the interview, but asked follow-up questions such as “What made you think to do that?” and “Can you explain to me what you mean?” to clarify the researcher’s understanding of the student’s feature noticing-and-using. Also, the researcher did not provide the student with the answer to the interview tasks, and the student was not told whether his or her statements during the interview were correct or incorrect.

The interview session was recorded using digital audio and digital video. The researcher instructed the cameraperson to focus on written work. The camera was positioned to the left and behind the student so as to ensure as much anonymity in the videos as possible. Students were asked to use the recordable pen to write on the recordable paper. All written work was collected at the end of the interview. The recordable pen and paper provided a back-up audio file as well as a copy of the written work. Audio recordings were transcribed and annotated to reflect the actions that were visible in the video recording. Data was saved on a password-protected hard drive and all files were kept in a locked file cabinet. The only people that had access to the data were the researcher, his advisor, and the transcriptionist.

Human Subjects Compliance

An Institutional Review Board (IRB) proposal was submitted and approved before data collection (IRB approval #37180). All students who participated in the study were asked to do so voluntarily. All students were told that their participation had no effect on their class grades and that the information gained in the interviews would not be shared with their classroom teachers. All students were given a $10 Starbucks gift card as an incentive for participating. All interviews were audio recorded and video recorded using digital audio and video technology. The student and/or their guardian (Appendix D and Appendix E) gave consent as to the nature and circumstances of how the data could be shared with others. The researcher compiled the digital files for all sessions and stored them on a secure computer hard-drive as well as on DVDs stored in a locked file cabinet. All information about the students was kept confidential, including their names. Pseudonyms were used.

Data Analysis

After each interview was conducted the researcher made a backup DVD copy of the interview. He sent the backup copy of the interview to the transcriptionist to be transcribed. While the data were being transcribed, using the original data source, the researcher did an initial analysis of the data looking for observable instances of feature noticing-and-using. From the initial analysis of the data the researcher identified four interviews from each level of algebra exposure that seemed to provide the greatest opportunity to characterize the variability in students' feature noticing-and-using. The researcher examined each interview for those that seemed to have variations in feature noticing-and-using across tasks. This variation in feature noticing-and using took the form of either reasoning that was not apparent in other interviews or seemed to have reasoning on several tasks that went beyond recognizing features that cued procedures.

Using the feature noticing-and-using taxonomy described in Chapter 2 as a lens to analyze the reasoning of these 12 students, instances of feature noticing-and-using within students' work were identified. The researcher used the video recording and transcript to identify instances of recognizing, reasoning, and linking. The researcher's initial goal was to determine instances of recognizing, reasoning, and linking activity, and then code these instances using the instance codes that were part of the feature noticing-and-using taxonomy. However, after a partial analysis of the first interview the researcher decided, and his advisor agreed, that the instance codes were too ambiguous and did not accurately reflect the nature of students' feature noticing-and-using. Both agreed that it would be more productive to identify instances of recognizing, reasoning, and linking and describe these instances in detail with the goal of developing more descriptive instance codes.

The format of the analysis was changed to reflect this goal. Each interview was analyzed to identify instances of recognizing (R), reasoning/analyzing (A), and linking (L) activity. The researcher decided that any verbal statement made by the student or written work used by the student to reason about the problem, as perceived by the researcher, would be examined for feature noticing-and-using and coded either as recognizing (R), reasoning (A), or linking (L). Instances in which the student made an explicit statement about a feature of the symbolic inscription and used that feature to carry out a manipulation were coded as recognizing. For example, if a student mentioned that an expression was quadratic in form and followed with a procedure related to the quadratic form, the researcher claimed that student recognized a feature (quadratic form) that cued a procedure. Instances in which the student made claims with respect to solving the problem that seemed to follow from the noticed feature were coded as reasoning. In other words, the statements that followed seemed to suggest to the researcher that the student was using the feature to reason about the problem. Instances in which the student, during the solving process, made statements related to other nonsymbolic representations of the symbolic inscription were coded as linking. After the coding was completed, each student's verbal and written statements were examined, and the researcher wrote commentaries describing the nature of each student's feature noticing-and-using on every task. These commentaries focused on identifying the specific feature noticed and how the feature was used in students' reasoning. An example of this analysis is shown in Appendix I. It is important to note that after the analysis the feature noticing-and-using taxonomy was revised. Many of the instance codes of the feature noticing-and-using taxonomy were revised, but the meaning of the three components, recognizing, reasoning, and linking, corresponded with the meaning of the three components of the revised taxonomy, manipulative, relational, and linking.

Once the 12 interviews were analyzed, the researcher wrote summaries of all 12 students' feature noticing-and-using on each task. For each task, analyses of all students' reasoning on that task were compiled. In other words, the analysis was separated by task, not by student. From these compilations, task summaries were written and organized by level of algebra exposure. The researcher believed it would be much easier to examine the nature of students' feature noticing-and-using across levels of algebra exposure if the data were organized in this manner. These summaries were created from the earlier analysis of students' feature noticing-and-using. Each summary included a description of the nature of each student's feature noticing-and-using on a particular task. Students' solving strategies, in light of feature noticing-and-using, were described and key aspects of feature noticing-and-using were briefly summarized. The task summaries are shown in Appendix H. The task summaries were then used to describe the nature of students' feature noticing-and-using on each task.

Once the task summaries were completed, a narrative was written for each task that described similarities and differences in the nature of students' feature noticing-and-using both within a task and across levels of algebra exposure. The narratives for each task were organized by features that were noticed and by how those features were used in students' reasoning. Instances of feature noticing-and-using that occurred within the reasoning of multiple students on a task or added to the researcher's understanding of nature of feature noticing-and using were included in the task narrative.

A significant shift in the researcher's thinking occurred after sharing the Task 3 narrative with a mathematics education faculty member. Initially, the researcher attempted to characterize different levels of feature noticing-and-using using the terms *procedural*, *limited*, and *advanced*. Procedural was defined as students' reasoning from procedures. Limited was defined as students' reasoning about the meaning of the symbolic inscriptions and advanced was defined as students making links to other representation registers and reasoning about the meaning of the symbolic inscriptions. The faculty member suggested that these terms were not descriptive of what was being coded about students' reasoning. She suggested the term *strategy* to describe students' reasoning disposition and the categories *manipulative*, *relational*, and *representational* to reflect these different dispositions. The term *orientation* was changed to *strategy* to better reflect the nature of what students did as they reasoned about specific symbolic inscriptions. The researcher decided to use the terms *manipulative* and *relational* to describe two of the strategies and to keep the term *linking* from his original taxonomy, as opposed to *representational*, to describe the third reasoning strategy. The researcher believed the term *linking* better reflected the nature of students' feature noticing-and-using.

The task narratives for the other tasks were rewritten characterizing the nature of students' feature noticing-and-using with respect to the three strategies to reasoning (manipulative, relational, and linking). The narrative for each task was examined for similarities and differences to students' feature noticing-and-using within each strategy and across levels of algebra exposure. An example of the narrative for Task 3 is shown in Appendix K. Aspects of feature noticing-and-using within each of these strategies that seemed to be most prevalent in students' reasoning across tasks led to the claims that were made in Chapter 5. The narratives were rewritten and reorganized by each claim, instead of by task. The researcher then examined each narrative for the instances of students' feature noticing-and-using that would best exemplify each of these claims. These instances of students' feature noticing-and-using are shared in Chapter 5. After examining the claims and identifying, in broad terms, the nature of students' feature noticing-and-using, the researcher revised the taxonomy to better reflect the nature of students' reasoning. For example, instead of using an interview-specific description, "Reasoning about the relationship between variable expressions expressed in an equation," the researcher used a broader description, "Reasoning about a relationship between entities within a symbolic inscription" in the revised framework. The researcher believed that using broader descriptions would allow for the feature noticing-and-using taxonomy to encompass both symbolic inscriptions represented in the interview tasks and symbolic inscriptions that were not represented in those tasks.

It is important to note that after the analysis and discussion was completed and the research questions were addressed the researcher returned to the six interviews (2 from each level) that were not part of the original analysis. The researcher did this to determine whether the students had used any strategies that were not reflected in the original analysis. The researcher determined that there was one instance of students' feature noticing-and-using (Lucy) that added to the analysis. Reasoning about Task 2 $\left(\text{Solve for } x\text{: } \frac{(2x+3)}{(4x+6)} = 2\right)$ Lucy reasoned that the inscription $\frac{(2x+3)}{(4x+6)}$ represented an invariant relationship for any value of *x*, "Obviously, no matter what number you would substitute in for *x* you would always get one-half" (278-279). Also, there was one instance of a Calculus II student (Sandy) who recognized a feature (powers of 2) on Task 8 (Find the real values of x such that $4^{2x^2-7x+3} = 8^{x^2-x-6}$). This refuted the researcher's initial claim that all Calculus II students used logarithms to solve the problem.

Qualitative Research Standards

Important aspects of maintaining qualitative research standards are presented next. The aspects presented include quality and verification procedures relevant to this study as well as subjectivity and ethical concerns.

Quality and Verification

Creswell (1998) posited that there are certain techniques that make a qualitative study's results believable to a reader. He described eight procedures that make the results of a study believable: persistent observation, triangulation, debriefing, negative case analysis, clarifying research bias, member checks, rich, thick description; and external audits (p. 201). He recommended that at least two of these be engaged in any qualitative study. This study will incorporate two of these procedures: rich, thick descriptions, and clarifying research bias.

As described in the previous section, the researcher analyzed the data on multiple levels. From the initial analysis to the summaries to the narratives, the researcher attempted to accurately and succinctly describe the nature of students' feature noticing-and-using. The researcher met with his advisor on multiple occasions to clarify his thinking and ensure that the descriptions accurately reflected students' feature noticing-and-using. Also, the researcher presented a portion of his research to a group of mathematics education doctoral students with the purpose of evaluating whether the doctoral students viewed instances of feature noticing-and-using in the same manner as the researcher. This discussion assisted the researcher in clarifying the meaning of terms, such as *object*, that the researcher used to describe instances of feature noticing-and-using. Specifically, the doctoral students suggested that the use of the term *object* be clearly defined in the context of the study. Since the tasks in the study involved students' reasoning about mainly symbolic inscriptions representing equation and inequalities, the researcher analyzed the data with an eye toward making a distinction between a process-level understanding of the inscription, equations and inequalities, and an object-level understanding of these same inscriptions. In other words, the researcher looked for specific instances in students' reasoning that would describe characteristics of a process-level understanding and object-level understanding of equations and inequalities.

Subjectivity Statement

The researcher's own experience with learning algebra in high school was very rule-based. Most of the researcher's experience with mathematics, at least before entrance into the doctoral program, was focused on learning procedures and reasoning about the meaning of those procedures. His own learning experience and 10 years of teaching algebra (in which he concluded that students were leaving algebra without recognizing important features) were the impetus for the researcher's desire to better understand the nature of students' reasoning about algebraic symbols. These experiences may have influenced what the researcher did and did not see during the interview with students. To guard against this bias, the researcher constructed an interview protocol that was closely followed, and used a taxonomy to identify instances of feature noticing-and-using. This removed some of the possibility of the researcher's experience influencing what was or was not noticed.

There were limited opportunities for any preconceived notions by the researcher about what a student could or could not do influencing the study. Overall, the goal of the researcher in the interview setting was for the interview to be more of a conversation about mathematics during which the student's thoughts and opinions were heavily valued. During most interviews, the researcher's questions were typically, "What made you think to do that?" and "Can you tell me what you are thinking?" The only instance of the researcher prompting the student was near the end of the interview when he asked students if they could reason about tasks using a graph. The interviewer was merely there to present the tasks and ask clarifying questions related to student's verbal statements and written work.

Ethical Concerns

It is possible that the study may have had ethical issues. These ethical issues include the possibility that the interview would create anxiety for the student, the fact that the researcher knew a few of the students, and the fact that all students interviewed were compensated for their time.

Although there may be some anxiety associated with participating in an interview study, efforts were made by the researcher to alleviate this anxiety. At no time during the task-based interview was the correctness of the student's work evaluated. The interviewees were informed that the purpose of the study was to better understand their feature noticing-and-using, not to assess the correctness of their work. Clarifying questions were asked to better understand student feature noticing-and-using, but not to correct misunderstandings or mistakes. This hopefully minimized anxiety because students would have been less concerned with the correctness or incorrectness of their work. Also, at no point was information from the interviews shared with their classroom teachers/instructors.

The students could have also felt disappointment that their reasoning about problems would reflect poorly on their mathematics teachers. The researcher had limited knowledge of most of the students in the study. The researcher knew three of the students interviewed on a personal level. It should be noted that the researcher had been the mathematics teacher for two of these students, but not during the year in which the study took place. As previously stated, the researcher did not share the correctness of student's work with them and was careful not to give any verbal or nonverbal cues that would suggest correctness or incorrectness.

Also, although students were given compensation for their participation ($10 Starbucks gift card) the likelihood that this influenced their participation or performance appears low. The monetary value of the compensation did not seem to the researcher to be enough to significantly impact whether a student participated in the study. Each student was given the same compensation without regard to their performance on the interview tasks.

Chapter 5

RESULTS

This chapter presents the researcher's findings that resulted from a careful analysis of students' feature noticing-and-using on a series of interview tasks. The tasks that were used to analyze students' feature noticing-and-using in order to characterize the nature of their feature noticing-and-using are shown.

Task 1: Solve for n: $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$

Task 2b: Solve for *x*: $\frac{(2x+3)}{(4x+6)}=2$

Task 3: Solve for *x*: $x^2+x+1>0$

Task 4: Is it true that the following system of linear equations: $\begin{cases} k-y=2 \\ x+y=k \end{cases}$ has a solution for every value of k?

Task 6a: Solve for *x:* $|x+1|=|x-2|$

Task 6b: Solve for *x:* $|x+1|>|x+2|$

Task 6c: Solve for *x*: $|x-a|=|x-b|$

Task 6e: Solve for *x*: $|f(x)|=|g(x)|$

Task 7a: Describe the graph of $f(x)=(x-2)^2(x+2)^2$

Task 7b: Describe the graph of $g(x)=|(x-4)^2(x+2)^2|$

Task 8: Find the real values of *x* such that $4^{2x^2-7x+3}=8^{x^2-x-6}$

The analysis of the data was focused on addressing the following questions:

1. What is the nature of what students notice about symbols as they solve unfamiliar algebra problems based on familiar algebra concepts and involved symbolic inscriptions?
2. Across levels of algebra exposure what is the nature of the similarities and differences in what students notice about symbols as they solve unfamiliar algebra problems based on familiar algebra concepts involving symbolic inscriptions?
3. What is a taxonomy that describes the nature of feature noticing-and-using evidenced in students' reasoning about symbolically presented unfamiliar algebra problems that are based on familiar algebra concepts?

Findings with respect to each of these questions will be addressed separately, and students' feature noticing-and-using that exemplifies these findings will be described. Before presenting students' feature noticing-and-using, it seems important to describe the level of algebra exposure of each student whose work will be discussed. The level of algebra exposure of each student is shown in Table 5-1.

Table 5-1

Students' Level of Algebra Exposure

Precalculus (HS)	**Calculus II**	**Math Teaching Methods**
Betsy	Colin *	Ashley
Chris *	Jim	Becky *
Lucy *	Mandy *	Casey
Molly	Nadia	Dan
Roxie	Paul	Mandy *
Todd	Robin	Newt

*The feature noticing-and-using of the six students denoted was analyzed after the feature noticing-and-using of the other students was analyzed. The purpose was to cross-check the feature noticing-and-using of these six students against the claims that were already made and to look for instances of feature noticing-and-using that were not addressed in the analysis.

Nature of Feature Noticing-and-Using

Based on his analysis of the data the researcher made several claims about the nature of feature noticing-and-using. Each claim is described and evidence supporting the claim is provided. These claims relate to the different strategies students used to reason about problems involving symbolic inscriptions, the different meanings students attached to features of symbolic inscriptions, and the challenges they faced in reasoning about features of symbolic inscriptions.

Claim 1: Different Reasoning Strategies

> *In the context of solving problems presented using symbolic inscriptions, there seem to be differences in the nature of the features students notice and how these features are used in their reasoning. These differences are related to students' reasoning strategies.*

Based on the analysis of what students do as they solve problems presented using symbolic inscriptions there seems to be different strategies to students' reasoning. A strategy is defined as a characterization of what students do when they solve problems presented using symbolic inscriptions. The analysis of the data revealed three different strategies to reasoning: manipulative, relational, and linking.[1]

Underpinning students' reasoning in each of these strategies is the nature of feature noticing-and-using. A *manipulative strategy* is characterized by reasoning that involves noticing features of the symbolic inscription that cue a procedure. In a manipulative strategy a procedure is cued, but there is a lack of attention to the meaning of the mathematical objects represented by the symbolic inscription. In a manipulative strategy features are noticed and those features are used to apply a procedure. A *relational strategy* is characterized by noticing features that give meaning to the underlying objects represented symbolically and reasoning from the meaning of those objects. In a relational strategy features are noticed and used to reason about the objects represented by the feature of the symbolic inscription. Unlike a manipulative strategy, there is attentiveness to the meaning of the mathematical objects represented by the symbolic inscription. Feature noticing-and-using that involves recognizing features and reasoning about the meaning of what those features represent characterize a relational strategy. A *linking strategy* is characterized by feature noticing-and-using in which features of the symbolic inscription, one representational register, are linked to features of representations of the symbolic inscription in a different register. Linking also occurs

[1] In the initial analysis, the researcher began characterizing the data by level of feature-noticing (procedural, limited, and advanced). The researcher showed his initial of Task 3 to a faculty member. The faculty member suggested that the categories of strategies better reflected the nature of students'mathematical reasoning.

when students reason from features of the graphical representation of the symbolic inscription to features of the symbolic inscription. Over the next few pages each of these strategies is described in more detail.

Manipulative strategy. A student who recognized a feature that cued a procedure without attending sufficiently to the meaning of the objects represented by the symbolic inscription is exemplifying feature noticing-and-using from a manipulative strategy. These procedures typically involved factoring, simplifying, or solving. There are possible indicators of a manipulative strategy to reasoning. All of these indicators do not necessarily need to be present to qualify as reasoning from a manipulative strategy. The indicators serve as a guide for characterizing the nature of students' verbal statements and written work. Indicators of a manipulative strategy include evidence that (a) the student recognized features that cue the procedures, and (b) the student recognized conditions under which procedures can be applied.

Recognizing features that cue procedures or structural features. As previously stated, a manipulative strategy to reasoning involves noticing features and using those features to apply procedures. Most students' work started with a manipulative strategy. In many instances the feature recognized is a form or part of the original symbolic inscription that cued a particular procedure. For example, on Task 1, $\left(\text{Solve for } n\text{: } 1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}\right)$, Roxie recognized a variable expression in the denominator which cued a procedure that involved multiplying both sides of the equation by the common denominator, "First, I would need to find a common denominator" (31-33). In a similar manner on Task 3 (Solve for x: $x^2+x+1>0$), Newt recognized a form of the inscription that cued a procedure, “I see a quadratic inequality, and the first thing I think when solving it is just solve it like a quadratic equation I treat it like an equation, and try to factor it” (91-93). Similar to Newt, Jim, reasoning about Task 8 $\left(\text{Find the real values of } x \text{ such that } 4^{2x^2-7x+3} = 8^{x^2-x-6}\right)$ recognized a form that cued a procedure: “Whenever a number is raised to a power that has *x* in it, logarithms are usually involved” (355-369). In each instance a feature, usually a classification or particular form, of the symbolic inscription or part of the inscription cued a particular procedure.

Describing a procedure or stating a succession of steps characterizes the discourse of students reasoning from a manipulative strategy. Tall and others (2000) identify this type of discourse as narrative. Another indicator of a manipulative strategy is student feature noticing-and-using characterized by being satisfied with a procedure as long as the resulting inscription meets the student’s expectation. For example, students using a manipulative strategy, solving for a variable such as *n*, means that procedurally the last step should have *n* equaling a number. The last, but most important, indicator is a lack of attention to meaning. In other words, procedures are cued, but there is not much attention to the meaning of these procedures.

It is important to note that most interviewees began reasoning from a manipulative strategy when they solved problems presented as symbolic inscriptions. In some instances a manipulative strategy to reasoning led to a correct final answer to a problem. Newt’s feature noticing-and-using on Task 2 $\left(\text{Solve for } x\text{: } \frac{(2x+3)}{(4x+6)} = 2\right)$ exemplified a manipulative strategy to reasoning that led to a correct final answer.

Performing a series of procedural steps, he reasoned that the equation would have no real solution. He noticed a feature (variable term in denominator) of the inscription $\frac{(2x+3)}{(4x+6)} = 2$ and used the feature to perform a procedure (multiplying both sides by $4x + 6$) that led to a form with whose solution process he was familiar: "but I knew, [I] got rid of the denominator and multiplied that over [there], then eventually I just had a linear equation to solve" (708-709). After he completed the procedure that resulted in the symbolic inscription $x = -\frac{3}{2}$, he checked the domain restriction of the inscription $\frac{(2x+3)}{(4x+6)} = 2$ and reasoned that because the domain restriction and the answer generated by his procedure were the same, there would be no solution.

> Got *x* equals negative three halves, and I was about to say, hey, that works. But then I said, Oh, no, you can't divide by zero so I said, this denominator came to zero, and then I realized this is going to end up being the same thing that it was, so then I said, well the only solution I got doesn't actually work because then it gets me dividing by zero, so nothing works (711-717).

Newt's reasoning suggests that a feature of the original inscription, algebraic expression in denominator, cued a procedure for solving the equation from which he correctly reasoned about the answer. His reasoning was focused on the steps of the procedures. He attended to a procedural condition of implementing his procedure correctly, possible domain restrictions of *x*, as it related to the symbolic inscription. His comments suggest he has an operational view of the equation and a process-level understanding. That is, the equal sign seems to be a cue for him to perform a particular solving procedure. At no point in his procedure is there evidence that he faced a cognitive conflict. It can be argued that he did not need a relational view, or object-level understanding of equations, because his procedure worked for him.

It is important to note that there does seem to be a distinction within a manipulative strategy related to recognizing features but not describable as features that cue procedures. In their initial reasoning some students recognized features that reduced the number of equation-solving steps instead of recognizing features that cued particular procedures. On Task 1 $\left(\text{Solve for } n\text{: } 1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}\right)$ some students recognized the two sets of opposite terms while other students on Task 2b $\left(\text{Solve for } x\text{: } \frac{(2x+3)}{(4x+6)}=2\right)$ recognized the same factor in the numerator and denominator. Novotna and Hoch (2008) suggested that students choose appropriate manipulations to make best use of the structure of an inscription are exhibiting structure sense.

Newt's feature noticing-and-using on Task 1 $\left(\text{Solve for } n\text{: } 1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}\right)$ and Paul's feature noticing-and-using on Task 2b $\left(\text{Solve for x: } \frac{(2x+3)}{(4x+6)}=2\right)$ exemplify structure sense. Before any written work, Newt stated, "I am just gonna combine all the like terms without really thinking about anything else, and then I see the ones cancel right away, and so, oh wait, everything, so it's zero, one over seventy two" (79-83). In a similar manner on Task 2b Paul stated, "These, this is the same thing, I just think there's a factor of two on the bottom so I just take that out, and simplify… this goes away, this goes away, they cancel one another, on the top, and then you have one half equals two" (537-543). In each case students' feature noticing-and-using involved recognizing structural features of the inscriptions involving mathematical ideas (opposites and factors) leading to manipulations that used that structure.

Recognizing conditions under which procedures can be applied. Reasoning from a manipulative strategy involves recognizing features that cue procedures, but also involves recognizing the conditions under which those procedures can be applied. In most cases recognizing the conditions involves recognizing features of the original symbolic inscriptions. Molly's feature noticing-and-using on Task 8 (Find the real values of x such that $4^{2x^2-7x+3} = 8^{x^2-x-6}$) exemplified noticing the conditions under which a particular procedure can be applied. Specifically, she recognized a feature (same bases) of the inscription $2^{2(2x^2-7x+3)} = 2^{3(x^2-x-6)}$ that she used to peform procedure that involved setting the exponential terms on each side of the equation equal to each other. Reasoning from the inscription $2^{2(2x^2-7x+3)} = 2^{3(x^2-x-6)}$ (see Figure 5-1) she argued that she could "cross out" the bases and set the exponents equal to each other.

> I remember something that if you did that you could cross out like cancel out, if these numbers were the same, you could cancel them out. And then just have, solve for that equal to that. Top things, what are they called? Um, exponents, is that what it is? I can't remember the word for it. But… so then, that's what I did then, I crossed those out and then distributed the two and the three and got *x* squared minus eleven *x* plus twenty-three equals zero (385-386).

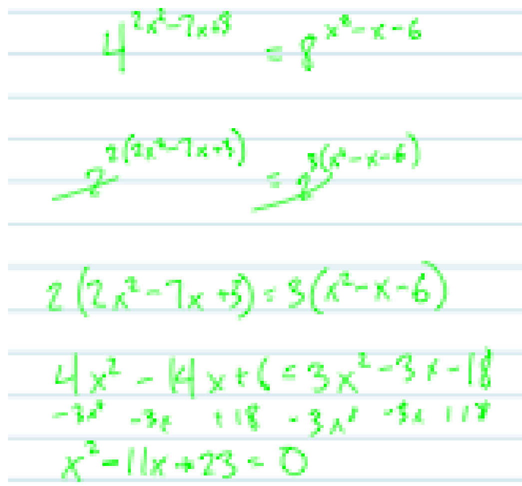

Figure 5-1. Molly's written work on Task 8.

Recognizing the conditions under which a certain procedure can be applied enabled the student to discern whether or not the procedure can be applied or whether the procedure is being applied in an appropriate situation. On Task 3 (Solve for x: $x^2 + x + 1 > 0$) many students recognized a feature (quadratic form) of the original inscription that cued a factoring procedure. They reasoned that the only possible integer factors of the constant term (i.e., 1) were -1 and 1 and that the condition required for factoring over the set of integers (linear term of x) would not be met. Ashley's feature noticing-and-using is typical of students' feature noticing-and-using.

> But I can't because if I factor it I'm just gonna break it up into two binomials where, um, factors of one, which are only one thing add up, and we need a two there [pointing to the linear term in the original inscription] to factor it" (Ashley, 92-95).

Ashley recognized a feature (linear term is $2x$) of the inscription resulting from her procedure (expanding $(x+1)(x+1)$ to x^2+2x+1) that did not match a feature (linear term of x) of the original inscription x^2+x+1.

Both Molly and Ashley recognized conditions underpinning their procedures. Casey recognized the bases had to be the same, which could, if she had not made a computation error, have led to a correct answer. Ashley, on the other hand, recognized a condition that would not be met, same middle term, which led her to abandon her factoring procedure for another procedure.

Recognizing features of the symbolic inscriptions that cue procedures and using those features to apply a procedure to the symbolic inscription guide the feature noticing-and-using of students who reason with a manipulative strategy. As a result, it is important that students recognize the features that enable the correct application of a procedure.

Relational strategy. The next type of strategy, relational, is characterized by a strategy to solving problems that involves noticing a feature of the symbolic inscription that represents an underlying mathematical object represented by the symbolic inscription. In a relational strategy there is attentiveness to the meaning of the object represented by the symbolic inscription.

Duval (2006) made a distinction that is helpful in describing other aspects of relational strategy. Duval defined *treatments* as transformations of representations that happen within the same register. In other words, a transformation from one form of a symbolic inscription to another form of the same symbolic inscription is a treatment. A relational strategy is characterized by reasoning between symbolic inscriptions or between symbolic and numerical inscriptions. These inscriptions are within the same register except when the conversion is from symbolic inscriptions to numerical expressions of those same inscriptions.

In some instances students' feature noticing-and-using in a relational strategy is characterized by reasoning about mathematical meaning of the treatments. This involves feature noticing-and-using that suggests there is an understanding of, for example, the meaning of the objects, and/or relationships represented by the symbolic inscription. Students who reason from a relational strategy seem to have a different view of the expressions, equations, and inequalities from those who reason from a manipulative strategy (Kieran, 1981; Knuth et al., 2006; McNeil & Alibali, 2005). Students reasoning from a relational strategy view the transformations of symbolic inscriptions represented by expressions, equations, and inequalities as representing a relationship between objects, and/or they view the symbolic inscriptions themselves as representing an object or a relationship between objects.

From a relational perspective, symbolic inscriptions of equations and inequalities represent relationships between algebraic expressions. Students reasoning from a relational strategy view the inscriptions or parts of the inscriptions as representing mathematical objects and relationships. Newt's feature noticing-and-using on Task 3 (Solve for *x*: $x^2 + x + 1 > 0$) exemplified a relational strategy. Mentally substituting numbers into the inscription $x^2 + x + 1$ he made claims about a set of numbers that satisfy the inequality $x^2 + x + 1 > 0$, "I can look and see that zero works, one works, anything greater than zero works right off the top of my head" (136–140). This led him to deductively make a generalization about a set of numbers satisfying the inequality, "*x* squared, if I plug in any positive real number uh, it's going to be positive, it's going to be positive, and one is greater than zero. So it's always going to be greater than zero" (142–144).

The descriptive nature of Newt's discourse is indicative of a relational strategy. As described by Tall and others (2001), descriptive discourse includes descriptions of properties, relationships with other inscriptions, and the ways in which they can be used. Unlike narrative discourse, descriptive discourse involves reasoning about the meanings of inscriptions resulting from procedures and relationships between inscriptions. Newt's reasoning suggests that he views the inscription $x^2 + x + 1$ as a variable expression representing a set of numbers, and that the inscription $x^2 + x + 1 > 0$ represents a relationship between numbers. Indicative of a relational strategy to reasoning Newt attended to the meaning of the symbolic inscription. To him the inscription $x^2 + x + 1 > 0$ represents a relationship between zero and a set of numbers of the variable expression $x^2 + x + 1$, which afforded him the understanding needed to reason about the solution set of the inequality.

Unlike a manipulative strategy, a relational strategy to solving problems involves recognizing features that represent the underlying mathematical objects represented by the symbolic inscription and reasoning about those features. In a relational strategy to reasoning there is attentiveness to the meaning of the inscription. While a manipulative strategy often involves recognizing features that cue procedures, a relational strategy is needed to reason about the symbolic inscriptions resulting from those procedures. Kinzel (2001) argued that students' manipulative abilities may hinder students' ability to evaluate the meaning of algebraic notation. That is, as long as students' manipulative strategies work, there is a disregard for reasoning about the meaning of the symbolic inscription in the context of the problem.

The challenge for students on most of the tasks in this study is that their usual manipulative strategies do not necessarily work in familiar ways for the given tasks, and, if they do work, their strategies result in atypical inscriptions $\left(\text{e.g.}\ \frac{1}{2} = 2\right)$ resulting in the need for students to reason about the meaning of the inscription. In this study—one that focused on symbolic equations, inequalities, and expressions—a relational strategy to reasoning suggested that students have at least one of the following understandings: (a) the truth set of the equation/inequality resulting from a procedure is the same as the truth set of the equation/inequality in the original task, (b) one form of an expression is equivalent to another form of the expression, or (c) algebraic, or variable, expressions represent numbers and the equations/inequalities that involve variable expressions represent relationships between numbers.

Same truth sets. Students' feature noticing-and-using in a relational strategy, unlike a manipulative strategy, includes an attention to the meaning of the inscription. These inscriptions could be the results of a procedure or they could be the original inscriptions themselves. For example, on Task 1 $\left(\text{Solve for } n\text{: } 1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\right)$ several students recognized the two sets of opposite terms leaving them with the inscription $0 = \frac{1}{72}$. "Newt stated, "That's not a true statement, so there is no value for *n* for which this is true…no solutions. It would be like five plus *n* equals seven plus *n*. There is no *n* for which that is true" (Newt, 84-89). Paul reasoned, "you get an untrue statement…you can plug any number in for *n* and it makes it equally untrue…there no like unique solution for *n*" (33-51).

Unlike a manipulative strategy, in a relational strategy students view equations and inequalities as expressing relationships rather than simply signifiers to perform procedures. Another distinction is students seeing the symbolic inscriptions as objects rather than processes. In the preceding examples that involve students' feature noticing-and-using about a feature of the inscription $0 = \frac{1}{72}$, it can be inferred that the inscription represented to them an equivalence relationship that is never true for any value of *n*. Students recognized a feature (untrue statement) of the inscription that provided meaning and they were able to use the feature to reason about a feature of the original problem. That is, the inscription $0 = \frac{1}{72}$ represented an untrue statement, which meant there would also be no values of *n* that would satisfy the equation $1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$, since the two were equivalent equations. In other words, the solution set of the equation $0 = \frac{1}{72}$ would be the same as the solution set of the equation $1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$.

Equivalent expressions. Students feature noticing-and-using from a relational strategy also understand that using a field property to re-express a symbolic expression results in another symbolic expression that is equivalent to the initial expression—has the same value when values are substituted for the variables. Lucy's and Molly's feature noticing-and-using on Task 2b $\left(\text{Solve for } x\text{: } \frac{(2x+3)}{(4x+6)} = 2\right)$ exemplify the difference between a student who has this understanding and one who does not. They both recognized a feature (numerator and denominator have the same factor of $2x + 3$) of the variable expression $\frac{(2x+3)}{(4x+6)}$ that resulted in the original symbolic inscription being rewritten as $\left(\frac{1}{2} = 2\right)$. Robin, not understanding the meaning of the resulting inscription, substituted numbers for *x* and stated the following.

> I was trying to think, do like a mental check , like putting random numbers in for *x*, and if it would ever equal two (196-197)… [substituting *x* = 1] it's five over the big number, which makes it, because this is a bigger number on the top, then it would always be a fraction, which would be less than one over two, so it would be less than two."(201-204)

In contrast, Lucy, understanding the meaning of the results, stated, "Obviously, no matter what number you would substitute in for *x* you would always get one-half" (278-279). Lucy's feature noticing-and-using suggests that she understood the equivalence relationship between the expressions $\frac{(2x+3)}{(4x+6)}$ and $\frac{1}{2}$ for $4x + 6 \neq 0$, whereas Robin's feature noticing-and-using suggests that she did understand this relationship. Specifically, Lucy understood that the variable expression $\frac{(2x+3)}{(4x+6)}$ will be equal to $\frac{1}{2}$ for any value of *x* that is substituted. It is unclear from her feature noticing-and-using whether she understood that this equivalence will hold for any value of *x* except for $x = -\frac{3}{2}$. Both Lucy and Robin recognized the same feature (factors) of the symbolic inscription and performed the same procedure that resulted in the same symbolic inscription. The difference was in how each used the feature in their reasoning.

Relationship between numbers. In a relational strategy, students' feature noticing-and-using is characterized by the recognition that the inscriptions express relationships and by reasoning about those relationships. In other words, features of the symbolic inscriptions are recognized and used to reason about the relationships between aspects of the symbolic inscriptions. Dan's and Ashley's feature noticing-and-using on Task 6b (Solve for x: $|x + 1| > |x + 2|$) demonstrated this aspect of a relational strategy. They both recognized that the inscriptions $|x + 1|$ and $|x + 2|$ were variable expressions representing numbers and that the inscription $|x + 1| > |x + 2|$ represented a relationship between those numbers. Substituting numbers into the inscription Ashley made claims about a set of numbers that would not be in the solution set of the inequality, "I'm going to just start to do values just to see … I'm thinking [if] they are positive numbers and zero. It wouldn't be true" (Ashley, 236-238) She made a deductive argument to support her claim, "and positive numbers and zeros you are adding one more over here [pointing to $|x + 2|$], so that would be greater than this guy [pointing to $|x + 1|$], not less than" (238-240). In a similar manner, Dan reasoned, "If two people start out with the same number of apples and I gave one, one apple and I gave one two apples, you know, it's back to like real basic stuff" (Dan, 302-303).

Casey's feature noticing-and-using on Task 3 (Solve for x: $x^2 + x + 1 > 0$) also exemplifies a relational strategy. Similar to Dan and Ashley, Casey recognized that the variable expression $x^2 + x + 1$ represented a set of numbers and the inscription $x^2 + x + 1 > 0$ represented a relationship between zero and numbers represented by the variable expression $x^2 + x + 1$ for values of x. The relational strategy afforded Casey the means to reason about a set of values in the solution set of the inequality.

> If you have any large negative number, this is going to be, then the x squared is going to be a large positive number, and then you're subtracting

> the negative number, so then it's still going to be greater than zero, and then adding one, you just kind of, um, not going to change it that much. If it's a large negative number, if it's a large positive number it's obviously going to be positive when you square and then add anything to it. And then when it's zero it also works, so, and even when it's a small negative number it works, you could say, and a small positive number works, so I guess you could say that *x* could be anything (Casey, 226-236).

As evidenced in their feature noticing-and-using, it is clear that all three students have a relational view of the symbolic inscription $|x + 1| > |x + 2|$. Their feature noticing-and-using suggests that the symbolic inscription represented a relationship between two variable expressions, and that the variable expressions represent numbers. In other words, they each recognized a feature of the object represented by the symbolic inscription and used the feature to reason about the solution set of the problem. Very few students at the lower levels of algebra exposure reasoned about Task 3 and Task 6 from a relational strategy.

Linking strategy. Similar to a relational strategy, a linking strategy is characterized by descriptive discourse. In a linking strategy these descriptions are characterized by links from features of an inscription in one representational register to features of that inscription in another representational register. A link, a specific type of conversion, is a transformation of representations from one register to another without changing the objects being denoted. The feature noticing-and-using of a student who is able to reason from a linking strategy is characterized as versatile reasoning, a type of reasoning involving the flexible use of a wide range of linkages between representations (Tall & Thomas, 1991).

A linking strategy is characterized by feature noticing-and-using in which features of the symbolic inscriptions are linked to features of representations of the symbolic inscription in nonsymbolic registers. There are three different types of links: (a) links from features of inscriptions resulting from procedures to features of graphical representations, (b) links from features of the original inscription and to features of graphical representations, and (c) links from features of the graphical representation to features of the original inscription. In the first two types of links the source register is the symbolic inscription while the target register is the graphical representation of the symbolic inscription. In the third type of link the source register and target registers are interchanged. The source register is the graphical representation of the symbolic inscription while the target register is the symbolic inscription. Duval (2006) argued that when the roles of the source and target registers are switched, the problem is radically changed for students. Each of these types of links will be exemplified next.

Links between features of inscriptions resulting from procedures and features of graphical representations of those inscriptions. Both Paul's and Todd's feature noticing-and-using on Task 3 (Solve for x: $x^2 + x + 1 > 0$) exemplify links between features of inscriptions resulting from procedures and features of graphical representations of related inscriptions. Paul performed a procedure that involved applying the quadratic formula. This procedure resulted in the inscription $x = \frac{-1 \pm \sqrt{-3}}{2}$. Paul made a link between a feature of the symbolic inscription $x = \frac{-1 \pm \sqrt{-3}}{2}$ (represents the roots of the equation) and a feature of the graphical representation (points where graph crosses *x*-axis).

> If you are dealing with real numbers, the roots of the equation would be like whether the equation is expressed as like a graph, so if I had something like I don't know, this would simply be *x* square root, like *x* squared, like a parabola type, the *x* axis, and those are like the roots of the equation. Where the equation equals zero. Yeah, so I guess if the equation is greater than zero, it's just everywhere. The entire graph is just greater than zero, or something like that. But, so basically I think it's just like this graph will never, this equation will never have a root of zero. So I guess, I guess *x* is zero everywhere. So *x* is greater than zero everywhere for all numbers (Paul, 121-132)

In a similar manner, Todd made a link between a feature of the inscription $x = \frac{-1 \pm \sqrt{-3}}{2}$ (not real) and a feature of the graphical representation (does not touch *x*-axis), "Well, it's not real but it would be where the, this graph of the quadratic equation here, the, would not touch the *x*-axis" (Todd, 117-119). In both instances there is an understanding that the resulting inscription $x = \frac{-1 \pm \sqrt{-3}}{2}$ implies the inscription $y = x^2 + x + 1$ has no real roots, which means the graphical representation of the inscription $y = x^2 + x + 1$ does not touch the x-axis.

A link from features of a symbolic inscription to features of a graphical representation of a symbolic inscription requires a coordination of meaning. The meaning of the same mathematical idea or object has to be understood in both representations. For example, in the previous instance Todd understood that a feature of the inscription $x = \frac{-1 \pm \sqrt{-\pm}}{2}$, no real roots, corresponded with a feature, does not cross the *x*-axis, of the graphical representation $y = x^2 + x + 1$.

In summary, in each instance students recognized features of symbolic inscriptions that resulted from procedures and used those features to describe or reason about features of the graphical representation of the symbolic inscription.

Links from features of the original inscription to features of the graphical representations of the symbolic inscription. In some instances the links are from features of the original inscription (rather than from features of the inscription resulting from a procedure) to features of a graphical representation of that inscription. For example, Newt, reasoning about the Task 6C (Solve for *x*: |*x* – a| = |*x* – b|), made links from features (coefficients of *x*) of the symbolic inscriptions $y = |x - a|$ and *y* = |*x* – *b*| to features (slopes are same so rays are parallel) of the graphical representations of $y = |x - a|$ and *y* = |*x* – *b*|. See Figure 5-2 for Newt's graphical representation of $= |x - a|$ and *y* = |*x* – *b*|. It is important to note that before Newt could make the links he had to recognize that the expressions on each side of the equal sign in the equation could each represent a function rule. In other words, he noticed features (variable expressions that represent function rules) and used those features to make links to graphical representations of functions represented by those rules.

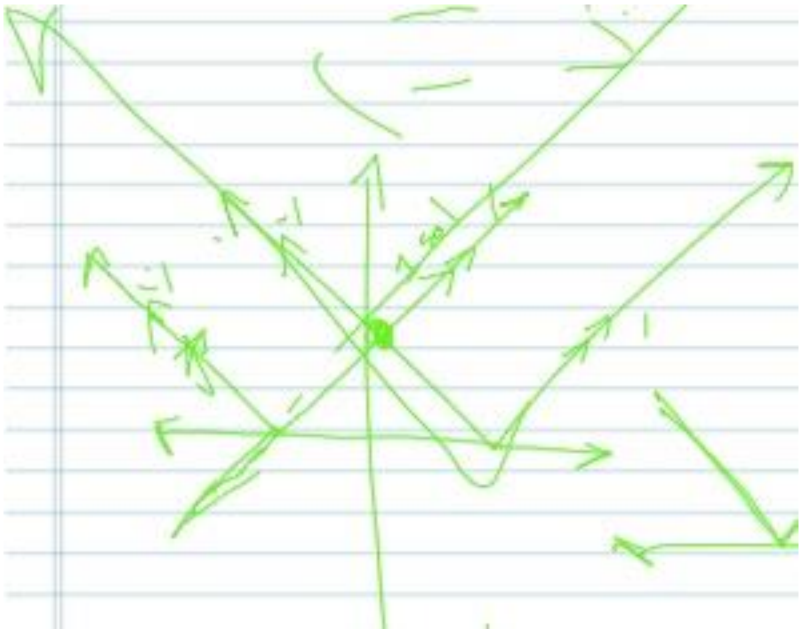

Figure 5-2. Newt's graph of $y = |x - a|$ and $y = |x - b|$.

> The inner arms are always gonna cross because, increasing in value that way [pointing to right branch of right "V" shape] and increasing in value that way [pointing to left branch of right "V" shape], and um, but there's no way to get two, because these are, these are parallel." [draws parallel markings on above left-side branches] (858-864). Because the slopes for absolute value function the way slopes are, is you know, there would just be a line otherwise, with a constant slope which would be the coefficient of the *x* term, and the only difference is, where there normally would have been a positive slope, but a negative *y* value, it's just the negative of the normal slope (870-875).

Newt's link is driven by his ability to coordinate meaning between features of

$y = |x - a|$ and $y = |x - b|$, coefficients of *x*, and corresponding features of their graphical representation, same slopes.

In a similar manner, Dan, reasoning about Task 7b (Describe the graph of $g(x) = |(x-4)^2(x+2)^2|$), made several links from the symbolic inscription to the graphical representation of the inscription. He made a link from a feature (zeros) of the inscription to a feature (*x*-intercepts) of the graphical representation of the inscription, "Four and negative two. I will get, it will intersect the *x*-axis" (Dan, 364-365). Also, he made a link from another feature (absolute value) of the inscription to a feature (graph will be above x-axis) of the graphical representation. From these links Dan was able to sketch a graphical representation of $g(x) = |(x-4)^2(x+2)^2|$ (see Figure 5-3) that seemed to portray features of $g(x) = |(x-4)^2(x+2)^2|$. It is interesting to note that both Dan and Newt attended to a feature (absolute value) of the function rule that did not affect the graph of $g(x) = |(x-4)^2(x+2)^2|$. In other words, $g(x) = |(x-4)^2(x+2)^2|$ and $h(x) = (x-4)^2(x+2)^2$ have the same graphical representation.

> But I am confident that negative two and negative four will be where the graph touches the x-axis, won't pass it and again, some sort of 'w' shape. Since given the absolute value being zero, no values on this side. All the outputs will be positive. (Dan, 395-401)

Figure 5-3. Dan's graph on Task 7b.

As was the case with Newt on Task 6c, Dan was able to coordinate meaning from features of the symbolic inscription, zeros and absolute value, to corresponding features of the graphical representation of the inscription, points that touch *x*-axis and no points below *x*-axis, respectively. Also, as with the earlier case, in each of these instances students noticed features of the symbolic inscription and used those features to describe or reason about features of the graphical representation of the symbolic inscription.

Links from features of the graphical representation to features of the original inscription. Of all the types of links, the link from features of the graphical representation to features of the original symbolic inscription may be the most important link. It seems important because, in most cases, it is the link from which student use to reason about the solution set of the original symbolic inscription. Driscoll (1999) argued that many students fail to make this link. Similarly, Duval (2002) found that a link from a graphical representation of the symbolic inscription to a symbolic inscription was a considerably more difficult link for students than links made in the other direction.

Ashley's feature noticing-and-using on several tasks is indicative of a link from features of the graphical representation to features of the original inscription. For example, on Task 3 (Solve for x: $x^2 + x + 1 > 0$) she made a link from a feature (always above *x*-axis) of the graphical representation of $y = x^2 + x + 1$ to a feature (all values of the expression $x^2 + x + 1$ are greater than zero) of the original inscription $x^2 + x + 1 > 0$ (see Figure 5-4).

> So this would be my *x*-axis, and I called this my *y*-axis, so any value of *x* that I substitute on this axis into this expression I'm getting out one of these values, a *y* value. So, by my graph right here it's always above zero on the *y*-axis, so I know this expression will always be positive, greater than zero (Ashley 147-152).

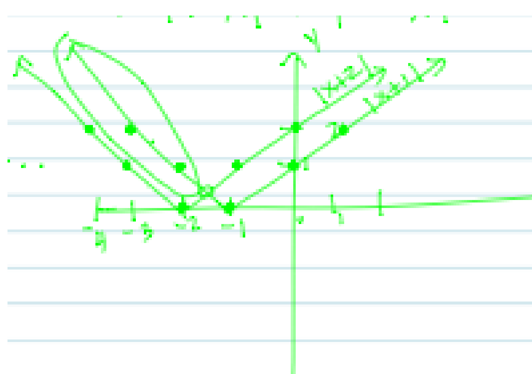

Figure 5-4. Ashley's graph on Task 3.

On Task 6b (Solve for x: $|x + 1| > |x + 2|$) Ashley made a link from a feature (interval of x for which the graph of y = |x + 1| is above the graph of y = |x + 2|) of the graphical representations of $y = |x + 1|$ and $y = |x + 2|$ to a feature (solution set) of initial inscription $|x + 1| > |x + 2|$. Ashley's reasoning about the solution set of the inequality from a linking strategy is shown in Figure 5-5.

> So I graphed the absolute value of *x* plus one and the absolute value of *x* plus two and I know from looking for where this side will be greater than this side, and need to look for where this graph is above this graph. Because when you substitute a value of *x* in I know it will be greater if its above because the values increase as you go up, so this part that I circled was the only part where the absolute value of *x* plus one graph was above the *x* plus two graph. Because from this point here over to the below the *x* plus two graph which was all those numbers that I said when I was substituting numbers didn't work.… so that matches the reasoning that I was just substituting things there as well (Ashley, 348-360).

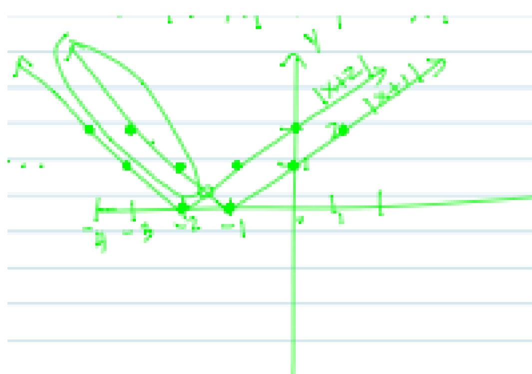

Figure 5-5. Ashley's graph on Task 6b.

In this instance Ashley has coordinated the meaning that the segment of the graph of $y = |x + 2|$ is above the graph of $y = |x + 1|$ with the values of x that generate the output values of $|x + 2|$ larger than the output values of $|x + 1|$. In other words, Ashley has coordinated the meaning of a part of the graph of $y = |x + 2|$ above $y = |x + 1|$ with values of *x* for which the inequality $|x + 2| > |x + 1|$ is true.

Other students' feature noticing-and-using was also characterized by links from features of the graphical representation to the solution set of the problem involving the original inscription. For example, Todd, reasoning from graphical representations of *y* = *k – x* and *y* = *-2* + *k* about Task 4 (Is it true that the following system of linear equations: $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ has a solution for every value of *k*?), made a link from a feature of the graph (horizontal line) of *y* = -2 + *k* to a feature of the original inscription (existence of a solution for all values of *k*). Having already determined that the inscription $y = -2 + k$ represented a horizontal line and the inscription $y = k - x$ represented a line with a slope of -1 and y-intercept of *k*, Todd stated the following.

> But they'll go on forever. So the statement is true, because there will be a corresponding *y* value for every *x* (187-188). It's coming out because this [pointing to $y = k - x$] is going to be the y-intercept. So that's not gonna change, um, whether the graph has a solution or not. It's just gonna change like where the solution falls? And so, like it could be any value and the line would just move up and down…Well, this one has no x value. No x variable, that is. And so it's just gonna be horizontal completely. And the horizontal line will just move up and down based on the value of

> k. And so then for every *x* value you just look right up there." (Todd, 204-235)

Todd's feature noticing-and-using suggests that he viewed k as a parameter and that this view is dynamic. He described the effect of changing k on the graphical representations of $y = -2 + k$ and $y = k - x$ as well as the impact of k on the solution set of the system of equations. Todd's feature noticing-and-using suggests that he has linked meaning from the intersection of the graphical representations $y = -2 + k$ and $y = k - x$ for any value k to a possible solution of the system of equations for any value of k.

Todd's and Ashley's reasoning exemplify links from features of the graphical representation of a symbolic inscription to features of the symbolic inscription in the context of the task. These links resulted from a coordination of meaning between the different representations. In each case students were able to form a link from the meaning of the mathematical idea in one representational register with the meaning of the same mathematical idea expressed in another representational register (symbolic). In each instance students recognized features of the graphical representation and used those features to reason about the solution set of the problem.

In summary, a linking strategy is characterized by three different forms of links that are directional in nature: (a) links from features of symbolic inscriptions resulting from procedures to features of graphical representations of those symbolic inscriptions, (b) links from features of the original symbolic inscription to features of graphical representations of the symbolic inscription, and (c) links from features of the graphical representation of a symbolic inscription to features of the symbolic inscription that was part of the original task.

Claim 2: Challenges in Reasoning From a Manipulative Strategy

> *Students' feature noticing-and-using frequently involved noticing a feature of the symbolic inscription and using the feature to apply a procedure related to the form of the symbolic inscription. The nature of this feature noticing-and-using was classified as a manipulative strategy—a strategy that seemed to limit one's ability to solve tasks involving symbolic inscriptions.*

It is claimed in this study that students who operated from a manipulative strategy have an operational view, not a relational view, of equations, inequalities and expressions. In other words, students recognized features of the symbolic inscriptions, representing equations, inequalities, and expressions, that cued particular procedures, but do not, or are unable to reason about the results of the procedure in the context of the problem. As a result, the operational view of the equations, inequalities and expressions can cause cognitive conflict for students as they reason from a manipulative strategy. There are several conditions, or features, related to the procedures and symbolic inscriptions that students enact that they need to attend to in order to successfully reason from a manipulative strategy. Students' difficulties in reasoning about these different types of conditions are discussed.

Subclaim 2A: Lack of Attention to Procedural Conditions

> *Students sometimes have difficulty reasoning using a manipulative strategy when features of a symbolic inscription do not meet the conditions under which a procedure can be applied.*

One aspect of a manipulative strategy to reasoning is noticing features that are used to perform procedures. In some cases there is a lack of attention to the conditions under which a procedure can be applied. In some instances this lack of attention to conditions is related to not understanding the meaning of a symbolic inscription, or a property of an object represented by the symbolic inscription. This is illustrated in Newt's reasoning on Task 3 (Solve for x: $x^2 + x + 1 > 0$) who, in the process of implementing a procedure for solving quadratic inequalities involving placement of roots on a number line, did not seem to know that the procedure was limited to roots that were real numbers. In other words, he did not seem to recognize a property of complex numbers—not ordered—making them impossible to position on a number line. His procedure resulted in the symbolic inscription $x = \frac{-1 \pm \sqrt{-3}}{2}$ which he rewrote as $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$. At no point does Newt mention these inscriptions represent the roots of the equation $x^2 + x + 1 = 0$.

The inscriptions are the result of a procedure for solving quadratic inequalities—procedure that involved applying the quadratic formula to the equation $x^2 + x + 1 = 0$, placing the resulting inscriptions on a number line and testing values between those inscriptions to determine whether certain intervals would be in the solution set of the inscription $x^2 + x + 1 > 0$ (see Figure 5-6). The problem is that this procedure only works when there are real roots. Reasoning about the inscriptions $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ in the context of the procedure created cognitive conflict for Newt. He needed to be able to quantify the complex numbers to perform the procedure because the procedure was dependent on placing those numbers on the number line. Although he did place the inscriptions on a line he was unable to reason about numbers between the inscriptions on the number line. While he did recognize the inscriptions represented complex numbers, he did not seem to know that complex numbers are not ordered, making it impossible to complete his procedure.

> I used the quadratic formula. And then I saw where they were going to be, and it was going to be a part of it. And so I tried to keep going and draw a number line, because at this point if these were real numbers, then I would test the point here and here and here… but since I don't have an understanding of what's bigger and smaller with complex numbers (196-202).

Figure 5-6. Newt's number line.

Newt's feature noticing-and-using is indicative of a manipulative strategy. Reasoning about the inscriptions resulting from the procedure is related to a role they play in a procedure for solving quadratic inequalities. There is no evidence in his reasoning to suggest he understood the meaning of the inscriptions $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ in the context of the problem. Although he enacted a procedure that required them, at no point in his reasoning is it suggested that Newt viewed the inscriptions $= -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ as representing the zeros of the equation $x^2 + x + 1 = 0$ and the meaning of this in the context of solving $x^2 + x + 1 > 0$. In summary, Newt recognized a feature (a complex number representation) of the symbolic inscription resulting from his procedure, but he was unable to use this understanding in his procedure for solving quadratic inequalities because he did not recognize a property of complex numbers.

Subclaim 2B: Results do not meet expectations

> *Students sometimes have difficulty reasoning from a manipulative strategy when the inscriptions resulting from procedures do not meet their expectations.*

Betsy and Roxie faced a similar struggle in reasoning about the inscription $0 = \frac{1}{72}$ which resulted from enacting different procedures to solve Task 1 $\left(\text{Solve for } n\text{: } 1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\right)$. In both Betsy's and Roxie's feature noticing-and-using the inscription $0 = \frac{1}{72}$ did not meet their expectations of the meaning of solving for *n*. Betsy stated, "you can't have zero equals one over seventy- two…Cause they're not equal to each other" (41-42). The focus of their reasoning was not on the meaning of the inscription $0 = \frac{1}{72}$, but on whether the inscription met their expectation of what it meant to solve—the expectation that to solve an inscription for *n* required a last step of *n* being equal to something.

One of the indicators of this operational view is the abandoning of a procedure or a changing of the direction of a procedure with the goal of meeting a particular expectation. Betsy's feature noticing-and-using on Task 1 is indicative of this view. She started another procedure (multiply both sides of the equation by $72(n + 3)$) that resulted in the inscription $0 = \frac{1}{72}$. She reasoned, "It [The equation in Task 1] would be unsolvable" (53) and to be solvable "you [would] end up with a variable on one side and a number on the other side" (58-59). While Betsy's feature noticing-and-using suggests a reasonable conclusion, she has given no indication that she knows why. As Betsy's reasoning implied, her meaning is tied to the expectations for the results of a procedure. In other words, to solve an equation for *n*, one needs a variable on one side and a number on the other side. Otherwise, it is not possible to solve the equation. There is no reference in Betsy's reasoning to what the inscription $0 = \frac{1}{72}$ described and its meaning as it related to the solution set of the original inscription. Her statements suggest that the meaning of the inscription is connected to whether or not a solving procedure can be applied in the situation as opposed to expressing an equivalence relationship between equations and/or its relationship to the original task $\left(\text{Solve for } n\text{: } 1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\right)$.

Similar to Betsy, Roxie reasoned that the result of her procedure $\left(0 = \frac{1}{72}\right)$ would mean it was not possible to solve for *n* since *n* did not exist in the results of her procedures.

> Well, if I do this, then that clears my *n'*s. So that obviously wouldn't work. Cause I want *n* all on one side and then to solve for *n*. I need *n* on one side. But from what I have it obviously can't be right because *n* plus three minus *n* plus three is zero… Because I get rid of *n*, so I can't solve for *n* if I don't have it. I'm gonna cross that off… Well, you can't solve for *n* if you don't have an *n* (Roxie, 49-58).

Betsy's feature noticing-and-using and Roxie's feature noticing-and-using were tied to a meaning of what it meant to solve an equation for *n*. That is, to solve for *n* meant that the enacted procedure must have *n* in its results. For both of them, recognizing a feature (the lack of a variable *n* in the equation that resulted from their procedure) of the resulting inscription $0 = \frac{1}{72}$ cued them to abandon one procedure for another procedure that they seemed to hope would have an *n* in its result. Betsy's and Roxie's additional procedures are not shown here, but in each instance the resulting inscription cued them to abandon one procedure for another procedure that would enable them to solve for *n*. Betsy's and Roxie's feature noticing-and-using related to the inscription $0 = \frac{1}{72}$ was typical of a manipulative strategy to reasoning.

This operational view of equations/inequalities was typical across levels of algebra exposure in a manipulative strategy to reasoning. For example, on Task 1 $\left(\text{Solve for } n\text{: } 1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\right)$ Jim, a Calculus II student, recognized a feature (common denominators of terms expressed as fractions) of a part of original inscription that cued a procedure (see *Figure* 5-7) that involved adding fractions, "well, to add fractions together you have to get a common denominator" (Jim, 33-34). At a point in his procedural working Jim reasoned about the inscription $\frac{0}{n+3} = \frac{1}{72}$ as follows, "I just added the numerators together. So, the n plus threes will cancel out. The plus one and the minus one will cancel out… I'm getting zero over *n* plus three. Which means that there isn't an answer, which I'm pretty sure is wrong" (49-53). Jim's feature noticing-and-using hints at a possible meaning of the inscription $\frac{0}{n+3} = \frac{1}{72}$, but his actions suggest otherwise. Instead of reasoning further about the meaning of the inscription he abandoned the cancelling of opposite terms step (see Figure 5-7) in his procedure for another procedural step that involved multiplying both sides of the inscription

$\frac{n+3-1-(n+3)+1}{n+3} = \frac{1}{72}$ by $72(n+3)$. This step led to the inscription $n+3=0$ which resulted in the inscription $n=-3$. By changing a step in his procedure Jim moved from an inscription $\frac{0}{n+3} = \frac{1}{72}$ that suggested to him that he had done something wrong to one about which he could reason ($n=-3$), "Yeah, that would be the solution that I got, but then plugging it back in would get you dividing by zero, which you can't do" (71-73).

It is interesting to note that all of Jim's algebraic work leading up to the inscription $\frac{0}{n+3} = \frac{1}{72}$ was algebraically correct, but similar to other students, his expectations of solving for *n* was not met, "Well it wasn't working out the way I had planned, so seeing the equal sign there and fractions, that would usually be my next step" (61-63). Instead of representing a relationship that was not true for any value of *n,* the symbolic inscription $\frac{0}{n+3} = \frac{1}{72}$ suggested to Jim that he had done something procedurally incorrect. Typical of a manipulative strategy, the cognitive conflict that Jim experienced led him to adjust his procedure to meet an expectation, solving for *n* and not to reason about the meaning the inscription $\frac{0}{n+3} = \frac{1}{72}$ in the context of the problem. He adjusted his procedure, which led to an inscription *n* = -3, about which he was comfortable reasoning in the context of the problem, "Yeah, that would be the solution that I got, but then plugging it back in would get you dividing by zero, which you can't do. Because dividing by zero can't happen" (71-74).

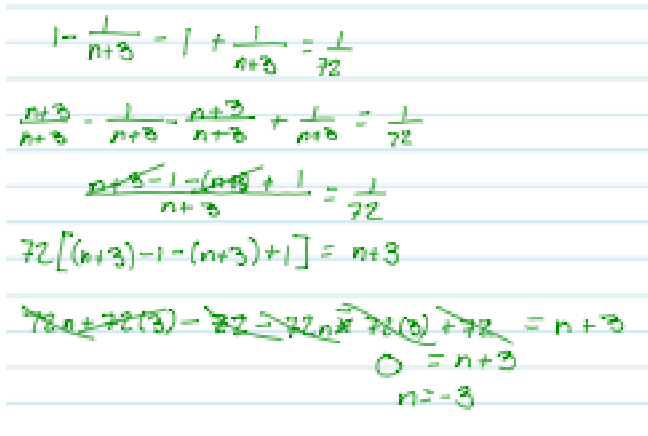

Figure 5-7. Jim's written work on Task 1.

Subclaim 2c: Lack of Attention to Features of the Structure of a Symbolic Inscription

> *Students reasoning from a manipulative strategy do not always attend to all the important features of the structure of symbolic inscription. The result is that procedures are cued but applied to the wrong situation.*

Knowing the conditions for using a particular procedure can also help students discern whether or not a particular procedure can be applied. A part of knowing the conditions is recognizing whether features of a particular inscription match the features of symbolic inscriptions in which a particular procedure can be applied. The consequence of not noticing a particular feature is exemplified in several student's feature noticing-and-using on Task 3 (Solve for x: $x^2 + x + 1 > 0$). Several students (Casey, Nadia, Roxie) rewrote the inscription $x^2 + x + 1 > 0$ as $x(x + 1) > -1$ and incorrectly applied a version of a property that mimicked the zero-product property, a procedure that resulted in the inequalities $x > -1$ and $x + 1 > -1$. Judging from student comments, the malformed procedure involved overgeneralizing the zero-product property from

$(ab = 0 => a = 0\ or\ b = 0)$ to $(ab > c => a > c$ or $b > c)$. For example, Casey reasoning about her procedure (see Figure 5-8) stated "if you factor in *x* is greater than negative one, and then you have *x* is greater than negative one, and then *x* plus one is greater than negative one." Similar to what a student would do when applying the zero-product property, Casey set each factor greater than [equal] to negative one [zero] and solved each inequality [equation] for *x*.

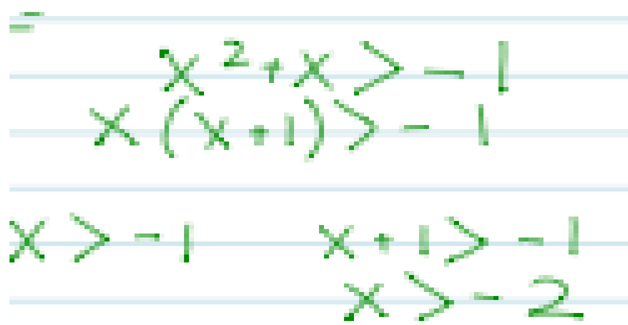

Figure 5-8. Casey's malformed procedure on Task 3.

Initially, Casey recognized a feature of the original inscription (product of factors) that met one of the conditions for applying the zero-product property. However, she did not recognize other features (-1 on right side of inequality or that it was inequality) that did not meet conditions for applying the zero-product property.

Unlike Casey, Nadia, after completing the procedure, recognized a desired feature (zero on right side of statement) that did not exist in the symbolic inscription $x^2 + x > -1$.

> But if I factor that out I won't be able to use the zero property because if I have *x* plus one minus one is greater than negative one, then I won't be

able to use the zero property on this because the zero property only works if you have zero on the right side (Nadia, 146-152).

A manipulative strategy to reasoning is predicated on recognizing features of symbolic inscriptions that cue procedures. Successful implementation of procedures requires recognition of all the necessary features of the structure of symbolic inscriptions. Unless all the relevant features of an inscription are recognized, there is the likelihood that procedures will be misapplied. This was the case for students who misapplied the zero-product property. Students recognized one of the features (product of factors) of the structure of the inscription $x(x+1) > -1$ that cued the zero-product property, but did not recognize other relevant features (relationship between the product of factors and zero) for applying the property.

Robin's feature noticing-and-using on Task 1 $\left(\text{Solve for } n\text{: } 1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\right)$ also exemplifies a less than complete attentiveness to relevant structures. She transformed the inscription $\frac{1}{n+3} + \frac{1}{n+3} = \frac{1}{72}$ into the inscription $72 = (n+3) + (n+3)$ arguing, "And then they were all over one, and so you could just flip them all, because it would be the same thing" (69-71). Although Robin is attending to a structure of the inscription, all fractions over one, she is not attending to a feature of the structure, multiplication of factors as opposed to addition of terms, that might allow for such a procedure. That is, "flipping" the symbolic inscriptions $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{c}$ or $\frac{1}{a} = \frac{1}{b}$ where $a, b, c \neq 0$ would result in equivalent equations $\left(\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{c} \Rightarrow ab = c\right)$, but $\frac{1}{a} \cdot \frac{1}{b} = \frac{1}{c}$ is not equivalent to $a + b + c$. Robin did not attend to the algebraic structural condition, multiplication, as opposed to addition, that would have made her procedure valid. As a result, her mathematical procedure led to an incorrect solution. It is argued that she is operating from an operational view of equations because the solution, though incorrect, does not lead to cognitive conflict. In other words, her procedure met her expectation, a number equal to variable *n*, so she did not feel a need to check the validity of her procedure.

Subclaim 2d: Lack of Attention to Mathematical Conditions Under Which a Selected Procedure can be Applied

> *Students reasoning from a manipulative strategy do not always attend to the features of a symbolic inscription that limit when a procedure can be applied. The result is that procedures are cued but mathematical conditions under which a procedure can be applied are not recognized.*

The previous claim was about not recognizing the mathematical structures under which a particular procedure could be applied. In most cases, not attending to the mathematical structures results in the misapplication of a procedure. In other instances, students performed appropriate algebraic procedures but did not check the mathematical conditions under which the procedure was valid. Depending on the focus of the student, this lack of attention could be related to the procedure, a structure of the inscription, or the mathematical concept related to the procedure. On Task 1 $\left(\text{Solve for } n\text{: } 1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}\right)$ and Task 2b $\left(\text{Solve for x: } \frac{(2x+3)}{(4x+6)}=2\right)$ students' reasoning was characterized by their recognizing features that cued procedures that included the step of multiplying each side of the inscription by the variable expression in the denominator. On Task 1 the step involved multiplying both sides of the equation by

n + 3, and on Task 2 the step involved multiplying both sides of the equation by 4x + 6. In each case multiplying by the inscription in the denominator is a valid as long as the condition that the multiplier is not equal to zero is fulfilled.

The mathematical issue is that when $n = -3$ or $x = -\frac{3}{2}$ multiplying by n + 3 or 4x + 6, respectively, is tantamount to multiplying by zero. Algebraic steps that involve multiplying both sides of the equation by the same variable expression are valid as long as the equivalence relationship between them is preserved. That is, the roots of the original equation are the same as the roots of the transformed equation. This does not happen when n = -3 on Task 1 or $x = -\frac{3}{2}$ on Task 2b. For example, on Task 1 $\left(\text{Solve for } n\text{: } 1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\right)$ multiplying both sides by $n + 3$ yielding the inscription $(n + 3) - 1 - (n + 3) = \frac{n+3}{72}$. If $n = -3$, then both sides of the equation have been multiplied by zero. Multiplying by zero does not preserve the equivalence of the equations. Multiplying by zero transforms the equation $1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$ into $(0)\left(1 - \frac{1}{n+3} - 1 + \frac{1}{n+3}\right) = \left(\frac{1}{72}\right)(0) \Rightarrow 0 = 0$. The inscription $0 = 0$ implies there is a solution for every n. Essentially, multiplying by zero does not preserve the equivalence relationship between the two equations. That is, the solution set of the original equation is not the same as the transformed equation.

The possibility exists that when implementing a procedure, such as the one just described, that an extraneous solution is introduced into the solution set of the original equation. Checking for an extraneous solution may or may not suggest a relational view of the equations. A relational view is suggested if there is evidence that the student viewed the original equation and the resulting equation as having the same roots or the student viewed the application of each property of equality or inequality as creating equivalent equations. That is, the roots are the same for the original equation and the transformed equation. An operational view, not a relational view, is suggested if checking the solution is part of the students' procedure for dealing with certain types of inscriptions or their reasoning does not involve an explanation that involves equivalence.

Nadia's feature noticing-and-using from a manipulative strategy on Task 2b $\left(\text{Solve for } x\text{: } \frac{(2x+3)}{(4x+6)} = 2\right)$ exemplified an operational view. Nadia described an otherwise valid procedure for solving Task 2b, except that she did not check to see whether the procedure introduced an extraneous solution. That is, the procedure resulted in the inscription $x = -\frac{3}{2}$ which was the value for *x* that would also make the inscription in the denominator (4*x* + 6) equal to zero. Thus, there would be no solution to the equation $\frac{(2x+3)}{(4x+6)} = 2$.

> So, I wanted *x* to be on the numerator so it would be easier to solve for *x*, so I just multiplied whatever was in the denominator on both sides and then that would just cross out the denominator, so then two *x* plus three would equal two times four *x* plus six. And then I multiplied that out so I could have it as a sum and I could move it to the other side and group like terms, so group, um, the *x*'s and the numbers, and then I subtracted this and I got negative six *x*, and then I subtracted, um, twelve, three from twelve, that's nine and nine divided by six is three halves with the negative still there. (Nadia, 195-2

In a similar manner on Task 2, Betsy and Ashley recognized a feature (variable in the denominator) that cued a procedure which also resulted in the symbolic inscription $x = -\frac{3}{2}$. Substituting $x = -\frac{3}{2}$ into $\frac{(2x+3)}{(4x+6)} = 2$ resulted in the statement $\frac{0}{0} = 2$. Both had difficulty reasoning about the meaning of the symbolic inscription. Ashley stated, "Again, dividing by zero…I think it was Calc Two where we did some kind of rule and some kind of problem with zero over zero (447-449). Betsy using correct terminology, but without any supporting reasoning to why it could be one stated, "But I guess it's just extraneous or something? Or maybe my multiplication is wrong…Yeah I have no idea what's going on" (267-270).

In each instance either students (Nadia) did not recognize a feature of the inscription (value for which it is undefined) that may have provided meaning or they had difficulty reasoning about the meaning of the resulting symbolic inscription in the context of the problem (Ashley, Betsy). It is claimed that students whose feature noticing-and-using is similar to Nadia's have an operational view of equations/inequalities. That is, as long as procedures provide a familiar form or what seems to be a reasonable solution, variable equal to a number, they seem not to feel a need to evaluate their work. If students' feature noticing-and-using is not characterized by a relational understanding of equations they will not see the need to determine whether the equivalence relationship holds for the operations involved in the procedure. Likewise, those who could not use the symbolic inscription $\frac{0}{0} = 2$ to reason about the problem did not view the inscription as expressing an equivalence relationship whose meaning was connected to the original equation. That is, the inscription $\frac{0}{0} = 2$ represents an untrue statement meaning the original equation would never be true for any value of *x*.

Students' feature noticing-and-using on Task 6b (Solve for *x:* $|x + 1| > |x + 2|$) also suggests a lack of attention to the mathematical conditions under which certain procedures could be performed**.** Several students' feature noticing-and-using was characterized by recognizing a feature of the inscription (absolute value) that cued a procedure (solving piecewise defined equations/inequalities). A step in the procedure involved re-writing the absolute value inequality $|x + 1| > |x + 2|$ as four inequalities. Newt's and Todd's algebraic work on Task 6b, are shown in Figure 5-9 and Figure 5-10, respectively.

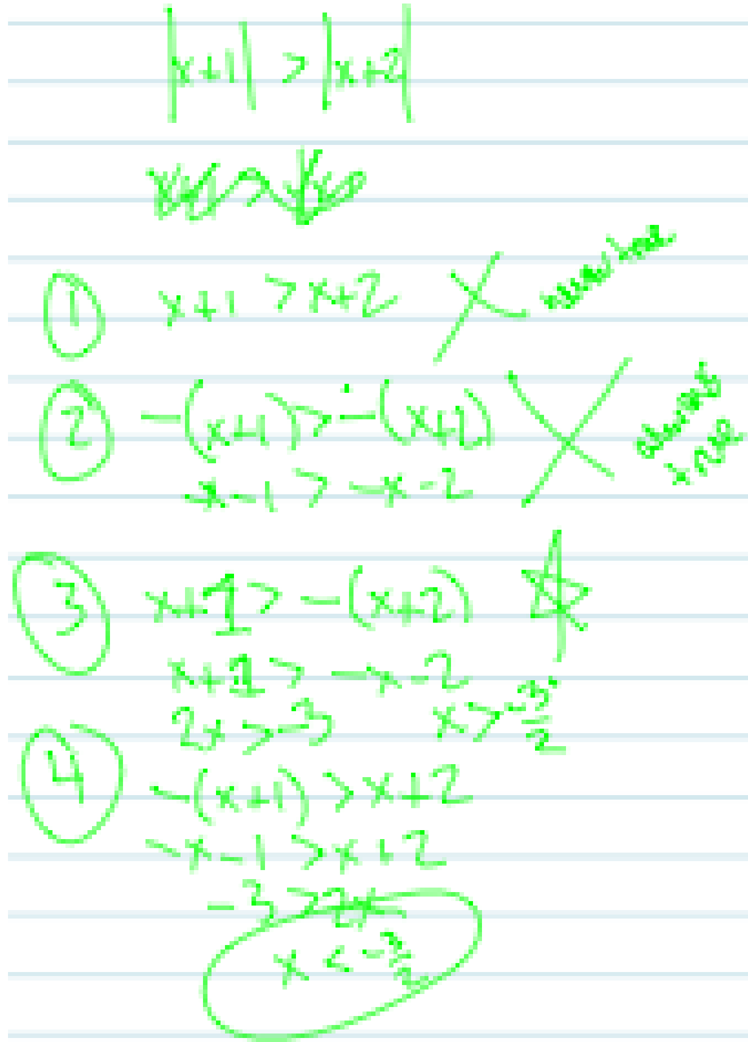

Figure 5-9. Newt's written work on Task 6b.

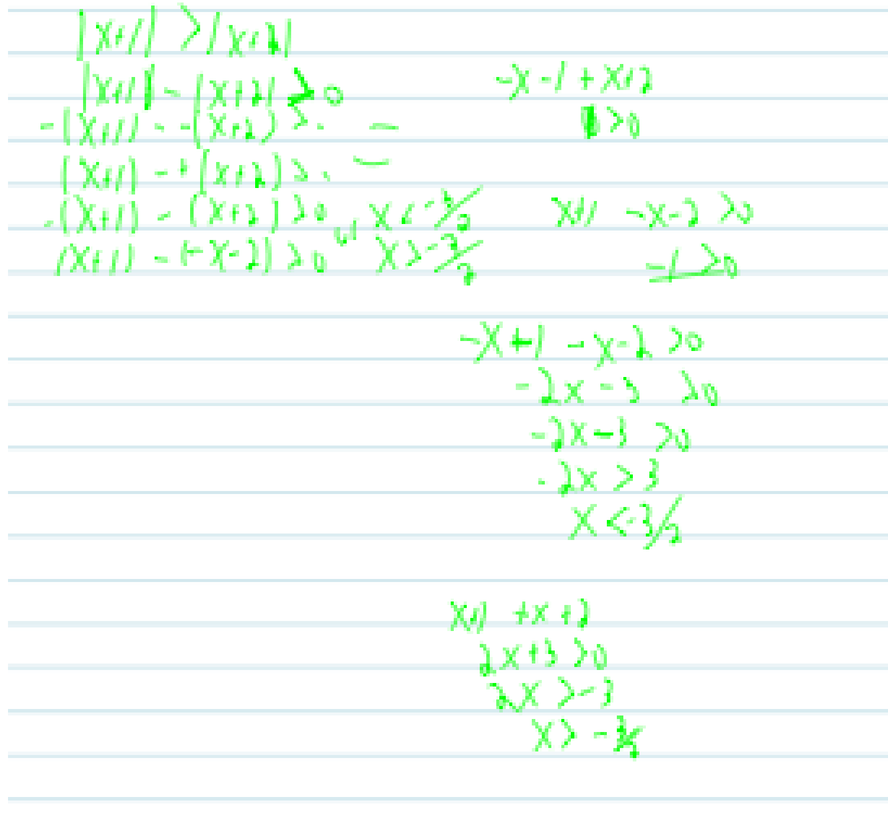

Figure 5-10. Todd's written work on Task 6b.

Their work suggests that neither Newt nor Todd attended to the mathematical conditions under which each of the piecewise functions is defined. Specifically, neither attended to the domain intervals under which each piecewise-defined inequality is defined. The absolute value functions $f(x) = |x + 1|$ and $g(x) = |x + 2|$ can be re-written as a piecewise defined functions with the following domain restrictions $f(x) = \begin{cases} x+1 & x \geq -1 \\ -(x+1) & x < -1 \end{cases}$ and $g(x) = \begin{cases} x+2 & x \geq -2 \\ -(x+2) & x < -2 \end{cases}$. Combining these domain restrictions in rewriting the inequality $|x + 1| > |x + 2|$ as piecewise defined functions results in three piecewise defined functions,

$$h(x) = \begin{cases} (x+1) > (x+2) & x \geq -1 \\ -(x+1) > (x+2) & -2 < x < -1. \\ -(x+1) > -(x+2) & x \leq -2 \end{cases}$$

Solving each of these equations yields the following results over the respective domains yields the following results.

Case 1: $(x \geq -1)$

$|x+1| > |x+2| \Rightarrow x+1 > x+2 \Rightarrow 1 > 2$ (true for no values of *x*)

Case 2: $(-2 < x < -1)$

$|x+1| > |x+2| \Rightarrow -(x+1) > x+2 \Rightarrow -2x > 3 \Rightarrow x < -\frac{3}{2}$ which implies the domain for Case 2 becomes $\left(-2 < x < -\frac{3}{2}\right)$

Case 3: $(x \leq -2)$

$|x+1| > |x+2| \Rightarrow -(x+1) > -(x+2) \Rightarrow -x-1 > -x-2 \Rightarrow -1 > -2$

(true for all $x < -2$)

Newt's and Todd's inattentiveness to the domain restrictions of the piecewise-defined functions led to an additional inequality $(x+1) > -(x+2)$ that when solved led to the inequality $x > -\frac{3}{2}$. That is, they both reasoned that an inequality with one absolute-value expression would result in two piecewise-defined (although they did not define the intervals) inequalities and so an inequality with two absolute-value expressions would result in four piecewise-defined inequalities. Their procedures resulted in an additional inequality $(x+1) > -(x+2)$. Todd's and Newt's procedure resulted in two inequalities, $x < -\frac{3}{2}$ and $x > -\frac{3}{2}$. Also, neither recognized that the inequality

$-(x+1) > -(x+2)$ was true $(-1 > -2)$ for all values of *x* in the domain for which it was defined $(x \leq -2)$.

Todd's written work suggests that he enacted a learned procedure without relating the solution set of his procedure to the solution set of the original inequality. His feature noticing-and-using suggests that he is working from an operational view of the inequality, "so I'm thinking this is, we've set it up to *x* is less than negative three halves, or *x* is greater than negative three halves, so, *x* is equal to all values except negative three halves" (613-615). He did not check to see whether the inequalities resulting from his procedure, $x < -\frac{3}{2}$ and $x > -\frac{3}{2}$, maintained the equivalence relationship with the original inequality, $|x+1| > |x+2|$. In other words, he did not check to see whether the truth values for both inequalities are the same. For Todd, the inscriptions resulting from his procedure apparently are merely the end result of a series of steps that were part of a learned procedure. There is no evidence in his feature noticing-and-using to suggest that he was treating the resulting inscriptions as representing a system of inequalities that have the same solution set as the original inequality, an indicator of a relational view of inequalities.

Newt's feature noticing-and-using, on the other hand, did seem to suggest a relational view of inequalities. Newt, using a similar procedure as Todd, reached the same solution as Todd, but was able to rule out one of the inequalities.

> *x* is greater than negative three halves and *x* is less than negative three halves. This is contradicting each other, *x* can't be greater than, this can't

> be true because zero doesn't work. And zero is bigger than negative three halves (Newt, 430-434).

Unlike Todd, Newt ruled out the inequality $x > -\frac{3}{2}$ as one of the solutions by relating the meaning of the results of his procedure to earlier reasoning in which he demonstrated that he had a relational understanding of the inequality $|x + 1| > |x + 2|$. His earlier feature noticing-and-using suggested he viewed the inscription $|x + 1| > |x + 2|$ as a relationship between mathematical expressions that represented numbers. Understanding the inscription as a relationship between numbers enabled him to make a deductive argument about a set of values of *x* that would not satisfy the relationship.

> Of the absolute value of something plus one, is it bigger than absolute value of something plus two. Well, for positive numbers this is never gonna be true, because the absolute value won't matter, because where, where, we'll be taking the absolute value of a positive number so for example, zero, one is not gonna be greater than two, two, one, like for zero, you have greater than two. It's just gonna keep happening. Those are true (315-322).

Unlike Todd, Newt knew, from his earlier reasoning, that $x = 0$ was not in the solution set of the inequality $|x + 1| > |x + 2|$. As a result, he noticed that since $x = 0$, a number not in the solution of $|x + 1| > |x + 2|$, satisfied the inequality, $x < -\frac{3}{2}$, but not the inequality $x > -\frac{3}{2}$, that the solution had to be $x > -\frac{3}{2}$. Newt's feature noticing-and-using suggests that he had a relational view of the original inequality and the system of transformed inequalities. That is, manipulations on the original inequality should have the same solution set as the inequalities that resulted from his procedure. In other words, Newt was able to reason from a relational strategy—a strategy that enabled him to determine that one of the inequalities resulting from his procedure was not indicative of the solution set.

Claim 3: Purposeful Movement Between Strategies

> *Those students whose feature noticing-and using seemed to be at a higher level were able to notice the objects and relationships represented by symbolic inscriptions.*

There were several instances of students' feature noticing-and-using being characterized by movement between different strategies. That is, features of symbolic inscriptions had several uses because of the different meanings students had attached to those features. For example, students may shift from a manipulative strategy to a relational or linking strategy in order to either reason about the meaning of an inscription resulting from a procedure or reason about a feature of the symbolic inscription in another representational register. Not all shifting between strategies led to correct answers, but those with correct answers had been able to shift purposefully between strategies. Moving effectively between strategies afforded the student the opportunity to use different meanings of features to (a) attend to student recognized gaps in reasoning, (b) confirm results of earlier reasoning, and (c) compensate for errors in reasoning that caused cognitive conflict. This movement was most prevalent on Task 3 (Solve for x: $\mathrm{x}^2 + x + 1 > 0$) and Task 6b (Solve for x: $|\mathrm{x} + 1| > |\mathrm{x} + 2|$). Reasoning effectively between strategies requires feature noticing-and-using that includes recognizing features that provide meaning and making links between different representations and inscriptions. The purpose of the next few sections is to exemplify the three different ways students' reasoning was advanced by moving between strategies.

Attending to gaps in reasoning. Reasoning from a manipulative strategy Newt enacted a procedure that involved applying the quadratic formula. The procedure resulted in the inscriptions $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$. As discussed earlier, to Newt the meanings of features of these inscriptions were linked to a procedure for solving quadratic inequalities. Because of his failure to reason about the magnitude of numbers between these complex numbers he abandoned the procedure and took a different strategy to solving the problem.

What enabled Newt to continue reasoning on the task is that he was able to view the inscription $x^2 + x + 1 > 0$ from a relational standpoint. In other words, he recognized the inscription as representing an inequality relationship between zero and a variable expression $x^2 + x + 1$ representing a set of numbers. Mentally substituting numbers in for *x*, he reasoned, "Well, kind of silly. Like, without even trying to deal with all of this complex stuff, I can look and see that zero works, one works, anything greater than zero works right off the top of my head"(136-140). This led to a deductive generalization about why all positive numbers would be in the solution set of the inequality $x^2 + x + 1 > 0$, "*x* squared, if I plug in any positive real number, it's going to be positive, it's [*x* term] going to be positive, and one is greater than zero. So it's always going to be greater than zero" (142-144).

Continuing from a relational strategy, Newt reasoned about a set of negative integers that would also be in the solution set. His reasoning led to a generalization about the solution set of the original inequality $x^2 + x + 1 > 0$ "Yeah, negative two works, negative four, yeah, every, all reals going to work here" (145-147). Adjusting his reasoning, he made a deductive generalization about negative numbers less than negative one being in the solution, "Any *x* less than negative one is going to work, because when you square it it's going to be positive and larger than just minus the constant value, and you have plus one is gonna be greater than zero" (Newt, 148-151).

Uncertain as to whether substituting numbers for *x* between *x* = -1 and *x* =0 would also yield positive numbers, " And, uh, so now It's just a matter of …between there and there"(153-154) he switched to a linking strategy. Preparing to reason about whether numbers for *x* between *x* = -1 and *x* =0 would also yield positive numbers he began reasoning about features of a graphical representation of $y = x^2 + x + 1$. Reasoning about features of the graphical representation signaled Newt's switch to a linking strategy. He recognized a feature (quadratic form) that cued a procedure $\left(\text{substituting coefficients a} = 1 \text{ and b} = 1 \text{ into the formula } x = \frac{-b}{2a}\right)$ for finding the vertex of a quadratic function. He made a link from the results of his procedure to the location of the vertex $\left(-\frac{1}{2}, \frac{3}{4}\right)$ on the graphical representation of $y = x^2 + x + 1$. Newt's graphical representation $y = x^2 + x + 1$ in Figure 5-11 seems to represent a quadratic function that does not pass through the x-axis.

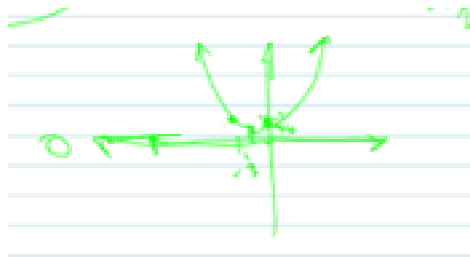

Figure 5-11. Newt's graph on Task 3.

Also, he made a link from a feature (positive coefficient of quadratic term) of the inscription $y = x^2 + x + 1$ to a feature (parabola opening upward) of the graphical representation of $y = x^2 + x + 1$. He used these links to reason about the solution set of the original inequality $x^2 + x + 1 > 0$. Making links from features of the graphical representation (vertex above *x*-axis and parabola opening upward) to the original symbolic inscription, $x^2 + x + 1 > 0$ (solution set is all real numbers) he reasoned about the solution set of the inequality.

> So if the vertex is there, it's going to make that I'd say all reals, just like that… this is like with zero, since it's an open up parabola, and the vertex is greater than zero, every other point is going to be greater than zero (Newt, 164-170).

Newt's shifts to other reasoning strategies was driven by his awareness of limitations in his feature noticing-and-using as well as understanding what other strategies could afford him. He shifted from a manipulative strategy to a relational strategy because he was unable to reason about the meaning of the inscription resulting from his procedure that involved applying the quadratic formula, "Since I don't have an understanding of what's bigger and smaller with complex numbers, I said well I'm just going to do something that's a lot easier than that which was just try values."(196-204). He shifted to a relational strategy because it afforded him the opportunity to reason more efficiently about a property (positive) of a set of values between $x = -1$ and $x = 0$ when substituted into the inequality $x^2 + x + 1 > 0$.

> I was testing values just by plugging them in. I was reasoning like zero, one, two, three, negative one, negative two, negative three… and I said, well rather than that, this is a parabola. I just thought hey, what can we plug in greater than zero. Then I don't have to worry about testing values. It's easier to think about. I like the picture (180-186).

Confirming reasoning. For some, the shifts in strategy are about supporting the reasoning in other strategies. Ashley's feature noticing-and-using on Task 6b (Solve for x: $|x + 1| > |x + 2|$) exemplifies switching to a linking strategy to support her reasoning in relational strategy. Ashley's feature noticing-and-using suggests that she has a relational understanding of the inscription $|x + 1| > |x + 2|$. Indicative of a relational strategy to reasoning, she noticed the symbolic inscription as representing a relationship between variable expressions of numbers. Specifically, the value of $|x + 1|$ must be greater than the value of $|x + 2|$ in order for x to be in the solution set of the inscription $|x + 1| > |x + 2|$. Using the feature of the original inscription in her reasoning she made a deductive generalization about a set of values (numbers greater than or equal to zero) not in the domain of the solution set, "I'm reasoning they are positive numbers and zero it wouldn't be true…and positive numbers and zeros you are adding one more over here [pointing to $|x + 2|$], so that would be greater than this guy not less than" (236-240). Substituting negative numbers in for x, she made an inductive generalization about a set of values (numbers less than or equal to -2) that would be in the domain of the solution set of the inscription $|x+1| > |x+2|$.

> Then if you substitute a negative number, I did negative one, I have zero is greater than one, negative two, one is greater than zero which could be true. I know negative two works. Negative three would be two is greater than, um, one which would work… three, I think… I think this is gonna keep going. Um…. y… when I did negative two I have one is greater than zero, and with negative three I had two is greater than one… negative four I had three is greater than two. So I feel like that pattern is gonna continue as well (240-249).

After she established that numbers less than negative two were in the solution set of the inscription $|x+1| > |x+2|$, she switched to a linking strategy to support her reasoning. She made links from features (absolute value, form) of the inscriptions $y = |x+1|$ and $y = |x+2|$ to features (shape, position of the vertex) of the graphical representation of those inscriptions, "And I know that this is the shape of an absolute value graph… If you have x plus one it's just shifted over to the left one" (301-304). She also recognized a feature (symmetry) of absolute value linear functions that aided her plotting procedure.

> I just tend to start with that one cause it's easy. Then I'll choose to do one to the right or to the left, and then from there see where that gets me. Um,

and then I can usually start making a shape, and I try to fill it out on the left and right with actual points" (319-326) (see Figure 5-12).

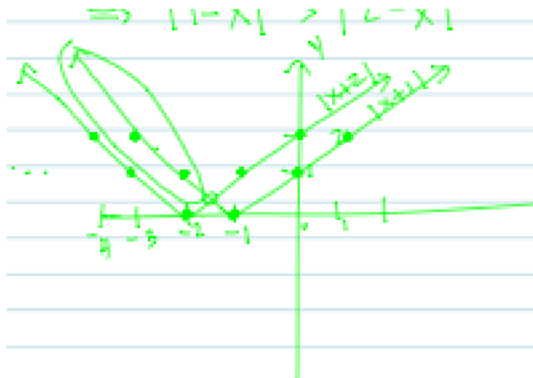

Figure 5-12. Ashley's graph on Task 6b.

She used links from features of the graphical representation of $y = |x + 1|$ and $y = |x + 2|$ to features of the original inscription $|\mathrm{x} + 1| > |\mathrm{x} + 2|$ to reason about the solution set of the original inscription. Specifically, she coordinated the meaning of the interval on the graphical representation where the graph of $y = |x + 1|$ was above the graph of $y = |x + 2|$ and the values of *x* for which the inequality $|\mathrm{x} + 1| > |\mathrm{x} + 2|$ is true. This link matched her earlier reasoning within a relational strategy.

> I know from looking for where this side will be greater than this side, and need to look for where this graph is above this graph. Because when you substitute a value of *x* in I know it will be greater if its above because the values increase as you go up, so this part that I circled was the only part where the absolute value of *x* plus one graph was above the *x* plus two graph. Because from this point here over to the below the *x* plus two graph which was all those numbers that I said when I was substituting

> numbers didn't work, so that matches the reasoning that I was just substituting things there as well. (Ashley, 348-360)

Ashley's feature noticing-and-using is not completely correct because the domain of the solution set of the inequality is $x < -1.5$, where *x* is a real number. She reasoned that the solution set of the inequality is $x < -2$. Although her reasoning lacked specificity in terms of the solution set, it does exemplify a shift between strategies to support reasoning. In this particular case, reasoning from the linking strategy confirmed her reasoning from a relational strategy. It is important to note a limitation in Ashley's feature noticing-and-using about absolute value graphs. It seems that her understanding of graphs of absolute value function is related to the absolute value linear functions. This is evidence by her reasoning on Task 7b ($g(x) = |(x-4)^2(x+2)^2|$) in which she sketched a graph that was "V-shaped" with vertices at the *x*-intercepts.

Compensating for errors in reasoning. As previously discussed in the analysis, Casey's feature noticing-and-using about Task 3 (Solve for *x*: $x^2 + x + 1 > 0$) involved a malformed procedure that mimicked the zero-product property (see Figure 5-12).

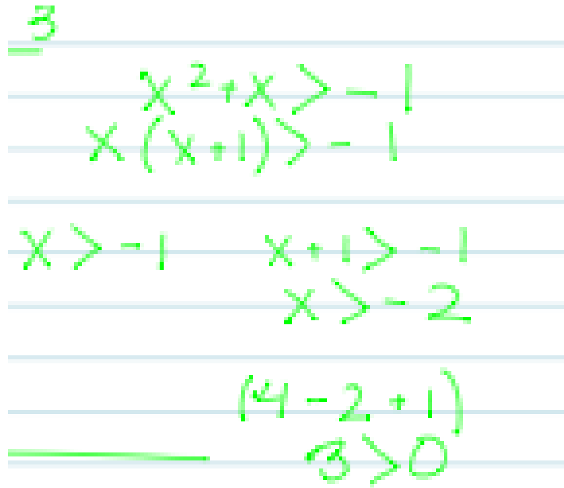

Figure 5-12. Casey's malformed procedure.

Holding a relational view of the inscriptions $x > -2$ and $x^2 + x + 1 > 0$ she reasoned that $x = -2$ would not satisfy either inequality. Reasoning that $x = -2$ would not be a truth value of the first inequality, she substituted the value into the second inequality expecting $x = -2$ to not be a truth value of the second inequality, "Because then this would be *x* is greater than negative two. And then, if you checked at negative two, you would have four minus two plus one. And then, I don't know. Now I'm confusing myself" (201-207). The fact that $x = -2$ did work caused Casey to have cognitive conflict. Casey likely is reasoning from a relational view because her substitution of $x = -2$ suggests that she understood that the truth values of the original inequality $x^2 + x + 1 > 0$ are the same as the truth values of the transformed inequality $x > -2$.

Casey's cognitive conflict accompanied with the substitution procedure seemed to have resulted in her viewing the inequality $x^2 + x + 1 > 0$ relationally. The relational view led to a generalization about a set of values of *x* that would be in the solution set of the inequality. In other words, substituting $x = -2$ seemed to have led her to call on her view of the inscription $x^2 + x + 1 > 0$ in terms of a relationship between the variable expression $x^2 + x + 1$ representing numbers and zero, a relationship that she was able to reason about over a set of numbers.

> I'm kinda reasoning that any, really like any number could work for *x*, because any number squared, even if it was a negative number, you're gonna, that's when it was a positive, so if it's gonna be greater than, like this *x* squared value is always gonna be greater than *x*, so it's still gonna be positive, and then just adding on, it's always gonna be greater than zero. Like if you had, even negative one, you have one plus one, or negative one, it's still greater than zero cause it's one negative one squared plus one, or minus one, plus one. So it's just one is greater than zero, which is true. (pause) Um, I would then say that really any number would work for *x*" (211-233).

Although Casey did make an error in her manipulation, her move to a relational strategy to reasoning contributed to her successfully dealing with the cognitive conflict that she faced. Her ability to notice the transformed inscriptions as representing a system of inequalities $(x > -1; x > -2)$ that should have the same truth values as the original inequality led to her cognitive conflict. It also led to her notice the original inscription $x^2 + x + 1 > 0$ represented a relationship between a variable expression of numbers $x^2 + x + 1$ and the number zero. Her ability to use the symbolic inscriptions to reason from a relational strategy compensated for her error in reasoning that she made when working from a manipulative strategy.

Claim 4: Linking Strategy and Productive Links

> *Students who were able to successfully use a linking strategy in their reasoning were able to notice features of a graphical representation of the symbolic inscription and use those features to reason about the original problem.*

In this study a *productive link* is defined as a link from features of a symbolic inscription to features of another representational form of that inscription back to features of original symbolic inscription. The link back to the original inscription involves using the features to reason about the solution of the original symbolically represented problem (see Figure 5-14). A productive link is not easy for students to make because it involves coordinating meaning across two changes in registers. The first coordination of meaning is from features of the source register, a symbolic inscription, to the features of the target register, a representation of the symbolic inscription. The second coordination of meaning is from different features of a source register, a representation of the symbolic inscription in another register, to features of a target register, the symbolic inscription represented in original problem.

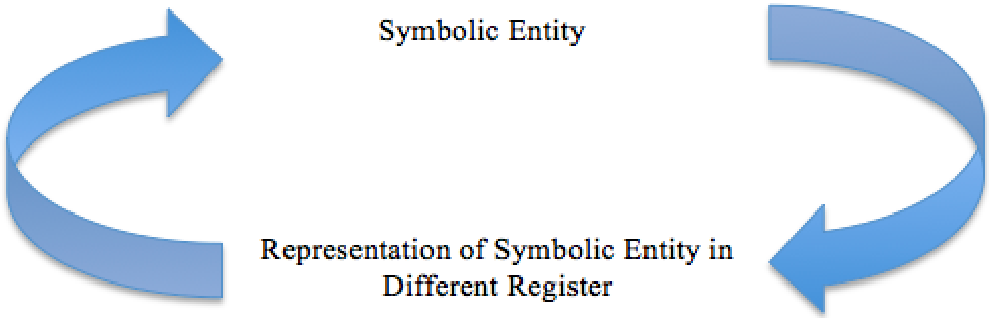

Figure 5-14. Dynamics of a productive link.

Instances of productive links. Todd's feature noticing-and-using on Task 4 (Is it true that the following system of linear equations $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ has a solution for every value of *k*?) exemplifies a productive link. He made a link between features (coefficient of linear term and value of constant term) of the inscription $y = k - x$ and features (slope and y-intercept) of the graphical representation of $y = k - x$. Also, he made a link between a feature ($-2 + k$ is a constant) of the inscription $y = -2 + k$ and a feature (horizontal line) of the graphical representation of the inscription $y = -2 + \mathrm{k}$. Without creating the graphical representations he made the following statements.

> So for this one [pointing to $y = k - x$], like *k* is just gonna be basically a *y*, a *y*- intercept. Again for both of these it's just going to change the *y*-intercept. It's still gonna be a linear, it's gonna be a line. This one will be horizontal [pointing to $y = 2 + k$] this one will have a slope of one in a negative direction [pointing to $y = 2 + k$] (Todd, 181-185).

Todd's feature noticing-and-using suggests that he coordinated meaning between features of the original inscriptions $y = k - x$ and $y = -2 + k$ and corresponding features of the graphical representations of those inscriptions. His feature noticing-and-using also suggests that he was able to coordinate the meaning of k in both inscriptions. Completing the cyclic form of a productive link, Todd used a link between a feature (horizontal line) of the graph of $y = -2 + k$ to reason about a feature of original inscription $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ (solution for all values of k).

> So that's not going to change, um, whether the graph has a solution or not. It's just going to change like where the solution falls? And so, like it could be any value and the line would just move up and down…Well, this one has no *x* value. No *x* variable, that is. And so it's just gonna be horizontal completely. And the horizontal line will just move up and down based on the value of *k*. And so then for every value you just look right up there (Todd, 204-235).

Todd's feature noticing-and-using suggests that he has a dynamic view of the parameter k. That is, there is a feature of the graphical representations (point of intersection) that holds for different values of k. The dynamic view of the parameter k seems to support his coordination of meaning between points of intersections for every real number k and the system of equations expressed in the problem having a solution for every value of k.

Paul's feature noticing-and-using (discussed earlier in the analysis) and Todd's feature noticing-and-using were the only instances of productive links evidenced in the work of students at the two lower levels of algebra exposure. Productive links were much more prominent at the highest level of algebra exposure. Newt's feature noticing-and-using on Task 3 (Solve for *x*: $x^2 + x + 1 > 0$) exemplifies a productive link of a student at the highest level of algebra exposure. He made links between features (opens upward, vertex above *x*-axis) of the graphical representation of $y = x^2 + x + 1$ and the solution set of the inequality $x^2 + x + 1 > 0$.

> So if the vertex is there, it's gonna make that I'd say all reals, just like that… this is like with zero, since it's an open up parabola, and the vertex is greater than zero, every other point is going to be greater than zero (Newt, 164-170).

On Task 6b (Solve for *x*: $|x + 1| > |x + 2|$) Dan made a productive link. Feature noticing-and-using from the graphical representation shown in Figure 5-15, Dan used a link from a feature of the graphical representations of $y = |x + 1|$ and $y = |x + 2|$ (point of intersection at $x = -1.5$) to reason about the solution set of the initial inscription $|x + 1| > |x + 2|$ (domain of solution set is $x < -1.5$).

> And we see right around somewhere between one and two, go ahead and stay one and a half, or negative one and a half, they intersect. And then everything before that point, whatever value we put in, we're gonna get a higher output for the *x* plus one (330-335). So when I'm looking at this and I'm thinking where would this one overtake the other, I'm not so much actually graphing in my head as I'm thinking of this concept of overtaking of being on top of something like that (344-347). When you plug in negative one and a half, you're gonna get one half is greater than

one half. That's not true. It's strictly greater it would say something more like, um, from negative infinity all the way up to negative one and one half not inclusive. That would be our solution (351-355).

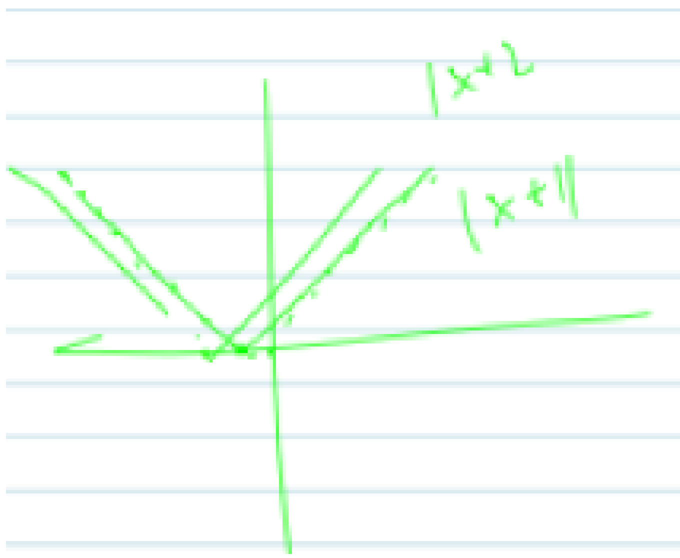

Figure 5-15. Dan's graph on Task 6b.

Newt's and Dan's feature noticing-and-using, similar to Todd's, is characterized by an ability to navigate the cyclic nature of a productive link. They were both able to coordinate meaning in two directions. First, they coordinated meaning from features of the original symbolic inscriptions to corresponding features of graphical representations of those symbolic inscriptions. Second, they used coordinated meaning from graphical representations of those inscriptions to reason about the solution of the problem involving the original symbolic inscriptions. Newt understood that the graph of $y = x^2 + x + 1$ lying above the *x*-axis corresponded with the all output values being greater than zero when numbers for *x* are substituted in the expression $x^2 + x + 1$. In a similar manner, Dan understood that the values for x where the graph of $y = |x + 1|$ was above the graph of $y = |x + 2|$ corresponded with the values of x for which $|x + 1| > |x + 2|$.

Lack of coordination of meaning. There seem to be two reasons why productive links did not happen in this study, and these reasons are related to a lack of coordination of meaning between the different registers. This lack of coordination of meaning can happen in two places, (a) from the source register to the target register, and (b) from the target register to the source register.

From the source register to the target register. Robin's feature noticing-and-using on Task 6a (Solve for x: $|x + 1| = |x – 2|$) exemplifies an incorrect coordination of meaning from what had originally served as the source register of the transformation, $y = |x + 1|$ and $y = |x - 2|$, to what had originally served as the target register, graphical representations of $y = |x + 1|$ and $y = |x - 2|$. It is important to note that in order to make this link Robin had to recognize that features (expressions) of original inscription $|x + 1| = |x – 2|$ represented function rules, $y = |x + 1|$ and $y = |x - 2|$. She coordinated meaning from a feature of the original inscription, absolute task value, to a feature of the graphical representations of the inscription, v-shaped form, but she was unable to coordinate the meaning of other important features. She did not recognize that the slopes of the piecewise-defined functions represented by $y = |x + 1|$ and $y = |x - 2|$ were 1 and -1 and that this feature corresponded with the branches of the graphical representation being parallel. Also, she did not correctly coordinate meaning from a feature, form of function, of the functions $y = |x + 1|$ and $y = |x - 2|$ to a feature of the graphical representations of these functions, a horizontal translation on the x-axis as opposed to a vertical translation on the y-axis (see Figure 5-16).

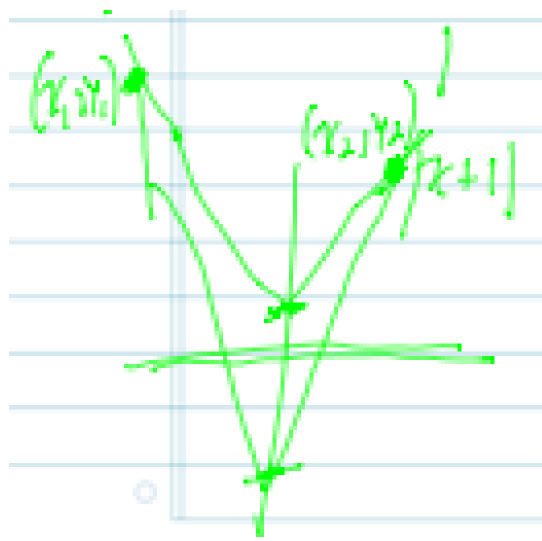

Figure 5-16. Robin's graph on Task 6a.

It is interesting to note that Robin's incorrect coordination of meaning did not create cognitive conflict for her because of a link she made from a feature of her graphical representation to the number of solutions in problems that she had done previously that involved absolute value. In other words, her graphical representation had two points of intersection that, to her, implied the equation would have two solutions. This matched her understanding of the number of solutions when solving problems that involved absolute value, "But I know that with absolute value you are supposed to get two answers, one where it would be negative and one where it wouldn't be negative. That's just to see what absolute value is; when you have two absolute values" (293-297).

Molly's feature noticing-and-using on Task 3 (Solve for *x*: $x^2 + x + 1 > 0$) also exemplifies an incorrect coordination of meaning from features of the source register, table of values, to features of the target register, graphical representation of table of values. She created a table of values and plotted points (see Figure 5-17), she reasoned the graph would be linear "because it goes up and over by the same amount every time…. For the most part… and it intersects at one on the *y* [axis]" (315-317).

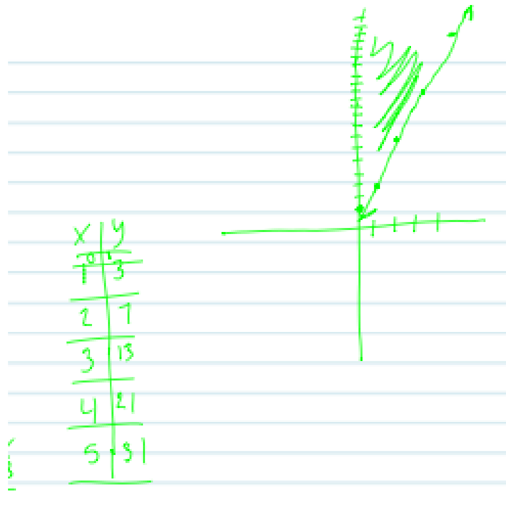

Figure 5-17. Molly's graph on Task 3.

In this instance the coordination of meaning is lacking in two different source registers. In the one case the incorrect coordination of meaning is from a feature of the table of values to a feature of the graphical representation. Her reasoning suggests she does not see the varying rate of change between points in the table of values corresponding with her graphical representation not being linear. In other words, Molly's table of values did not suggest linearity, but her graphical representation constructed from the table of values did suggest linearity. The other instance of a lack of coordination of meaning in Molly's reasoning is from a feature of the original inscription to a feature of the graphical representation, "I think with this kind don't you have to like shade certain parts [Molly shaded above line in Figure 5-16.]…all these answers or something?" (326-328). Her feature noticing-and-using suggests that the inequality symbol in the original inscription means to her that she needed to shade the region above the line and that the graph of the line would not extend beyond the y-axis. In both instances these statements suggest an incorrect coordination of meaning from a feature of a source register to a feature of a target register. Molly's ability to reason about the task from a graphical representation could have been hindered by her inability to correctly coordinate the meaning of the shape of the graphical representation represented by the original symbolic inscription as well as the meaning of the inequality symbol of the symbolic inscription in the context of the graphical representation of $y = x^2 + x + 1$. At no point in her reasoning does she make statements that would suggest links from the target register, graphical representation of the symbolic inscription, back to the source register, original symbolic inscription. In other words, she did not use the features of the graphical representation to reason about the problem involving the original symbolic inscription.

Target register to source register. Instances of Betsy's feature noticing-and-using on Task 6a (Solve for x: $|x + 1| = |x - 2|$) and on Task 4 (Is it true that the following system of linear equations $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ has a solution for every value of *k*?) exemplify the difficulties in creating productive links. In each instance, she did not definitively use a features of the graphical representation, the original target register, to reason about the solution set of task involving the initial symbolic inscription, the original source register.

For example, on Task 6a (Solve for *x:* $|x + 1| = |x - 2|$) Betsy was able to make links from features (absolute value, form) of the inscriptions $y = |x + 1|$ and $y = |x - 2|$ to features (shape of graph, location of vertex) of the graphical representations of $y = |x + 1|$ and $y = |x - 2|$.

Although she created an accurate graphical representations of $y = |x + 1|$ and $y = |x - 2|$ she did not reason from the graphical representations about the solution set of the original inscription $|x + 1| = |x - 2|$. Specifically, Betsy's feature noticing-and-using does not suggest that to her the *x*-value of the point of intersection of the graphical representation corresponded with the solution set of the task that involved the original inscription. When asked the meaning of the intersection of the two graphs, she stated, "I don't know. I don't think so because I don't think I've ever had to do that [Solve equations using graphical representations] " (433-434). In this particular instance Betsy was unable to use the meaning of the point of intersection of the graphical representations of $y = |x + 1|$ and $y = |x - 2|$ to reason about the solution set of the equation $|x + 1| = |x - 2|$. She reported that she had never solved equations using graphical representations.

In a similar manner, on Task 4 Betsy was able to make a series of links between the symbolic inscriptions $y = -2 + k$ and $y = -x + k$ and graphical representations of those inscriptions. Reasoning from the inscriptions $y = -2 + k$ and $y = -x + k$ she made links between features of both inscriptions ($-2 + k$ is a constant; $-x + k$ is not a constant) and features (horizontal line; nonhorizontal line) of the graphical representations of $y = -2 + k$ and $y = -x + k$. She also made links between a feature (point of intersection for any value of *k*) of the graphical representation of $y = -2 + k$ and $y = -x + k$ and a feature (solution for every value of k) of the original inscription $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$.

> If you have a horizontal line and a non-horizontal line, they are going to intersect at some point…Unless, no, even if it was just zero… then that would be a horizontal line and that would be like. So, I guess logically …

> a horizontal line and … any line not parallel to it would intersect. Any line that's not parallel to it would intersect. And they can't be parallel cause that one's horizontal and that one's not (Betsy, 208-219).

As was the case on Task 6a, Betsy did not coordinate meaning between the graphs always intersecting for any value of *k* and there being a solution for every *k*. Instead of reasoning further she stated, "I don't think that's what it's asking" (219-220) and abandoned her line of reasoning. Betsy's understanding of the graphical representations of $y = -2 + k$ and $y = -x + k$ seems to imply that *k* is a constant, but that she may not have seen *k* more dynamically as a parameter. Having a more dynamic view of *k* might have afforded her the means to form a productive link. That is for all *k* where *k* is any real number there exists non-horizontal line $y = -x + k$ and horizontal line $y = -2 + k$ that will intersect at a given point. In each instance her inability to coordinate meaning from the target register to the source register hindered her ability to use the links to reason successfully about the tasks.

Claim 5: Nature of Reasoning and Reasoning From Different Strategies

> *The ability of students to use a feature to successfully reason about a problem was related to the meaning students attached to the feature and whether the student viewed this meaning as being helpful in solving the problem.*

As prior examples have shown, students, reasoning from the same symbolic inscription, recognize different features of the inscription. In many instances students' feature noticing-and-using suggests that different students have different meanings associated with the same symbolic inscription. This represents one of the challenges of feature noticing-and-using from both a relational and a linking strategy. The same symbolic inscriptions may represent different ideas, depending on the meaning the student has associated with different features of the symbolic inscription.

This is exemplified in students' reasoning about the meaning of the system of equations in Task 4 (Is it true that the following system of linear equations $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ has a solution for every value of *k*?) and the meaning of absolute value on Task 7b (Describe the graph of the function rule $g(x) = |(x - 4)^2(x + 2)^2|$).

Different meanings within a relational strategy. On Task 4, the same inscription representing something different for different students characterized students' feature noticing-and-using. Several students recognized a feature (system of equations) of the inscription $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ that cued a procedure which led them to correctly reason that $x = 2$. After reasoning that $x = 2$, several students reasoned about the sameness of the two equations, "Since there were no restrictions on *x* and *y*, I said, um, let *x* equal two. And then which would give us the same equation" (Ashley, 225-226). Newt stated, "Well, so, I saw that these two were the same." (292)

It is important to note that the two equations in the system are not technically the same. It is a coincidence that the *x*-value in the solution set of the system of equations (*2, k – 2)* corresponded with the value for *x* that when substituted into the second equation of the system (*k – y = x)* made the two equations the same. Students reasoned about the sameness of the two equations in different ways.

For example, Newt, noticed a feature (sameness) of the two equations and used the feature to conclude that *k* could be any real number because the form of the equations in the system (2 + y = k and $k - y$ = 2) looked to him like the form of equations in other systems of linear equations that resulted in an infinite number of solutions.

> So it's just the same thing twice. Um, this is the same thing. And so there are infinite, there are infinite values for this, for k…Hey, whenever I have, you know, I'm solving for something in algebra and I have a equals a, I'm pretty sure that means infinite solutions, so I just wanted to confirm that so I did substitution method again, or instead of elimination of, the same thing happened and then I realized … and so for any given linear equation there are infinite values that work (Newt, 234-256).

For Newt, the inscription (two equations in the system after substituting $x = 2$ were both of the form $k - y = 2$) represented a system of equations that, based on his past experiences, resulted in an infinite number of solutions. He reasoned that k could be any value because the equations in the system would have an infinite number of solutions for any value of k. His reasoning about a prototype system of equations supported his claim. Writing $2x + 2y = 4$ and $x + y = 2$ two equations one of which is a linear combination of the other (i.e. $2x + 2y = 4$ is equivalent to $2(x + y = 2)$ which is a linear combination of 2 and $x + y = 2$), he stated the following:

> Except normally I'm used to just, you know, two y equals four, and then like x plus y equals two, and something like that there. I'm used to, you know, no . I'm used to something like that, having infinite solutions…Well, so, I was, I saw that these two were the same so I was like, is the reason there are infinite solutions because both equations in the

> system are the same. So I wrote two equations that are the same, and ah, so like one, one would work. And ah, zero and two would work, one half and three halves would be two plus, or one… yeah, so that is the rule (Newt, 262-298).

> I was confirming that I saw here, I was one hundred percent sure here that there are infinite solutions, but I was confirming that the reason for there being infinite solutions was because both equations in the system were the same. So, that's why it happens (306-310).

As discussed earlier, the flaw in Newt's noticing is that the two equations $k - y = x$ and $k - y = 2$ are not technically the same equation. It is a coincidence that the results of the procedures he has applied (substituting $x = 2$) is the value that when substituted for x in $k - y = x$ made the two equations the same. He did not understand that the substitution of $x = 2$ led to a new system of equations $(k - y = 2; k - y = 2)$ that was different from the original system $(k - y = 2; x + y = k)$, but had the same solution. His creation of the pair of equations that are linear combinations of each other $(2x + 2y = 4; x + y = 2)$ suggests that he is treating both y and k as variables as opposed to y being a variable and k a parameter. Newt's reasoning, unlike those who did successfully reason about the task, did not show that he understood that k was a parameter, not a variable.

Reasoning from the sameness of the two equations Ashley, similar to Newt, also concluded that k could be any number. However, her feature noticing-and-using that supported this conclusion was different (see Figure 5-18) than Newt's feature noticing-and-using of the same symbolic inscription.

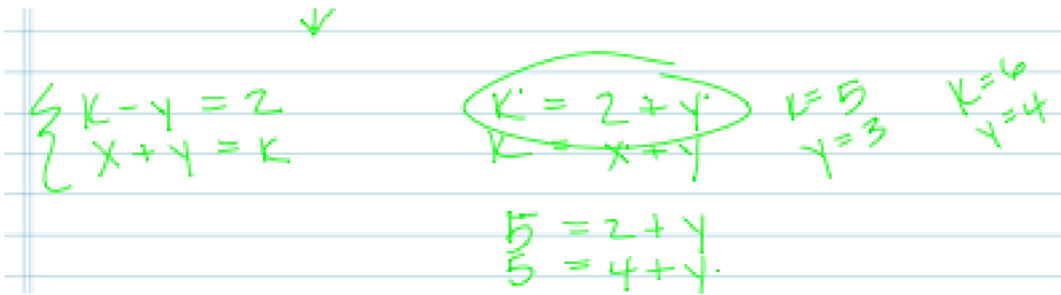

Figure 5-18. Ashley's written work on Task 4.

She recognized the same feature (sameness of the two equations), but used the feature to reason about a relationship between *k* and *y* that would hold for any number *k*.

> I would always say *x* equals two in this equation and then to do it might be the same, and then for any *k* since there is also no restriction on *y*, then I'd say I'd make *k* was five, then *y* would be three. Then if *k* was six, *y* would be four. And for whatever *k* is, I could then tell you what *y* is. (Ashley, 187-192)

Substituting numbers in for *k* and reasoning about the invariant relationship of *k* and *y* suggests that she is treating *k* as a parameter.

Unlike Newt's feature noticing-and-using where the sameness of the two equations represented a form of a system of equations with infinite solutions, Ashley's feature noticing-and-using seemed to suggest the sameness of the two equations represented an invariant relationship between *k* and *y* that held for any value of *k*. In other words, any value of *k* would maintain the sameness of the two equations since the *y* values, in each equation, would also have to be the same for the equations to represent true statements.

<u>Different meanings within a linking strategy.</u> On Task 7b (Describe the graph of the function rule $g(x) = |(x-4)^2(x+2)^2|$) several students' feature noticing-and-using was characterized by a link from the same feature (absolute value) of the inscription $g(x) = |(x-4)^2(x+2)^2|$ to different features of the graphical representation of that inscription. Although the links involved the same feature (absolute value) the meaning of the link varied. The link in some students' feature noticing-and-using (Ashley, Molly) involved connecting a feature of the graphical representations of linear absolute value functions to a feature of the graphical representation of the original inscription $g(x) = |(x-4)^2(x+2)^2|$. In Ashley's and Molly's feature noticing-and-using the meaning of absolute value was connected to the "V" shapeness of the graphical representation. For other students (Newt, Roxie, Todd) absolute value was connected to the meaning of the concept in the context of a graphical representation, that is, points below the *x*-axis on the graphical representation of the function whose rule did not have absolute value are reflected above the *x*-axis to represent the function whose rule did have absolute value. Thus, preserving the distance between the point and the *x*-axis.

In summary Ashley's and Molly's understanding of absolute value was limited to a feature (v-shaped) of graphical representations of linear absolute value functions. Creating the graph shown in Figure 5-18, Ashley reasoned, "So then … so it looks like the graph would be something like this which I expect that V-shape [pointing to tip of V emanating from (1,81) on graph shown in Figure 5-18] from the absolute value" (560-562). Molly, when asked to describe the shape of her graph stated, "It's like a really steep

V" (285). Molly uses her meaning of absolute value to create a graphical representation of the parent function represented by the rule $y = |x|$, whereas Ashley uses her meaning to create a feature (v-shapeness of minimums) of the graphical representation $g(x) = |(x-4)^2(x+2)^2|$ (see Figure 5-19).

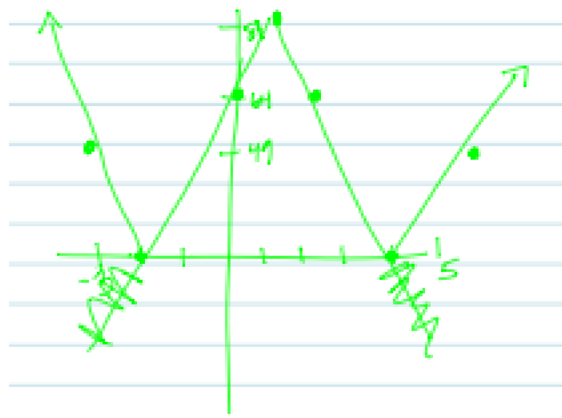

Figure 5-19. Ashley's graph on Task 7b.

Other students' feature noticing-and-using exemplifies a link based on the meaning of the concept of absolute value in a graphical context. For example, Roxie argued that absolute value meant that there would be no points on the graph below the *x*-axis. Her reasoning suggests that she understood absolute value in a graphical context in terms of an action, flipping negative-valued parts of a graph over the *x*-axis.

> Since it's absolute value, it would, nothing would be below the *x*-axis, because, well actually that's not true, well it is true… because in absolute value you flip anything that's down here to up there (Roxie, 326-330).

Newt's understanding of absolute value, unlike Roxie, is based on the definition of absolute value. The absolute value of a number represents its distance from zero. He is able to extend this definitional understanding to a graphical context. Newt created the graphical representations shown in Figure 5-20 and stated, "Absolute value…. So, the distance from zero for all these points [pointing to part of second graph below *x*-axis] would actually be positive" (Newt, 634-635). It is interesting to note that in the context of the graphical representation of the function rule $g(x) = |(x-4)^2(x+2)^2|$, the absolute value does not affect the shape of the graphical representation, but Newt chose to discuss the impact of the absolute value on a graphical representation that does extend below the *x*-axis.

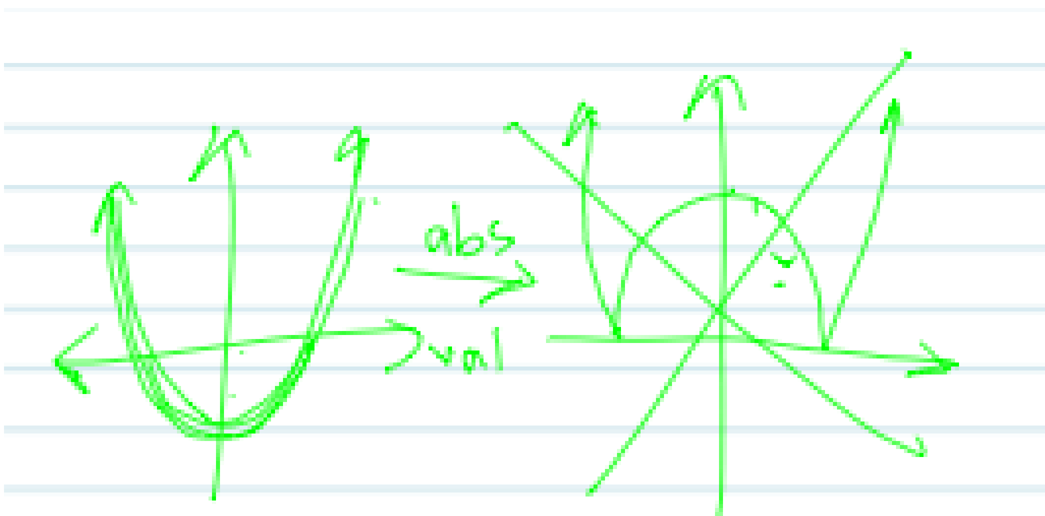

Figure 5-20. Newt's graphical representation to explain effect of absolute value.

Though incorrect, Betsy's feature noticing-and-using is also characterized by a link that involved the meaning of absolute value. Unlike Roxie's and Newt's feature noticing-and-using, Betsy's feature noticing-and-using suggests that she incorrectly linked the meaning of absolute value to the graphical representation of the inscription (see Figure 5-21), "I think it's one of the ones where you end up with half of a graph, because everything under the absolute value is positive. So like half of the parabola is negative, in here [shading the left side of Figure 5-21]" (339-341). Betsy's understanding of absolute value in a graphical context is that there cannot be any negative *x*-values represented on the graph. From a mathematical perspective, this would suggest it seems she did not notice, at least in the context of Task 7b, the fact that the absolute value function affects the output values, not the input values of the function rule $g(x) = |(x-4)^2(x+2)^2|$.

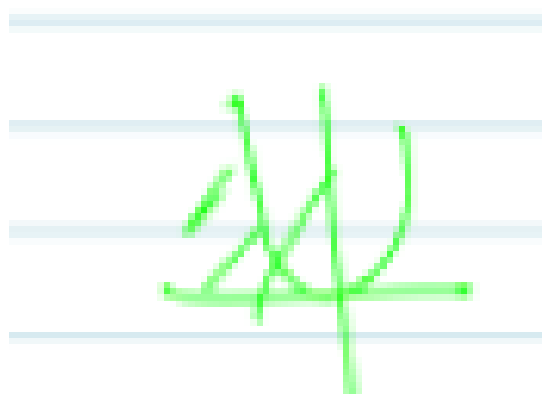

Figure 5-21. Betsy's graph of Task 7b.

In summary, Newt formed a link based on a concept (distance from zero) underpinning the definition of absolute value while Roxie's and Ashley's links were based on a procedural outcome of applying the absolute value function to a number (changes negative numbers to positive numbers) in a graphical context. In each instance the result was the same: Points below the *x*-axis were reflected over the *x*-axis. It is interesting to note that on this task none of the students in their reasoning needed to attend to the absolute value feature because the factors of the inscription

$|(x-4)^2(x+2)^2|$ were both squared and the product of these factors for any number *x* would always be nonnegative. In other words, the function rule $g(x) = |(x-4)^2(x+2)^2|$ and the function rule $h(x) = (x-4)^2(x+2)^2$ represent the same function.

Newt and Paul were the only students who recognized that the absolute value function would not have an impact on the graphical representation of $g(x) = |(x-4)^2(x+2)^2|$. Paul stated, "Well first of all dropped the absolute values cause you're squaring something, so the absolute value would always be positive" (368-370). Meanwhile, Newt, reasoned (erroneously) that inscription a^2b^2, which represented a form of the expression $(x-4)^2(x+2)^2$, would always be greater than zero for any values of *a* and *b*. [Note: This indicated to him that the absolute value sign in $y = |a^2b^2|$ would not have an effect on graphical representations of functions of the form $y = a^2b^2$.

> For this specific case, absolute value doesn't, absolute value signs don't matter cause I have two things that are being squared. I have *a* squared times *b* squared, and it's always gonna be greater than zero, so absolute value doesn't matter…So, it's just this specific case where you could factor it into binomial squared, but that, the absolute value doesn't matter cause you have two squared terms" (Newt, 673-687).

Unlike other students, and although it did not affect the graphical representation of $g(x) = |(x-4)^2(x+2)^2|$, Newt and Paul were able to reason about the meaning of absolute value in the context of the original inscription $g(x) = |(x-4)^2(x+2)^2|$ as well as in the context of the graphical representation of $g(x) = |(x-4)^2(x+2)^2|$.

These instances of students' feature noticing-and-using exemplify how recognizing the same feature of a symbolic inscription can be used differently in making links to graphical representations of the symbolic inscription. The same feature of a symbolic inscription can have different conceptual meanings.

Connected meaning to other links. Differences in students' feature noticing-and-using in reasoning about absolute value in a graphical context illustrate an important point. The nature of the coordinated meaning of the mathematical idea between the registers does play a role in students' ability to successfully reason using the link. This idea is exemplified in students' feature noticing-and-using of the quartic nature of the inscription $g(x) = |(x-4)^2(x+2)^2|$. Casey's meaning of quartic, for example, involved a link to a general shape, "two you're gonna have an *x* to the fourth which, I don't really, which might just be… the shape of a parabola maybe" (718-723). Newt's meaning also involved a link to a general shape as well, but he does make a distinction between the graphical representations of two function, $y = x^2$ and $y = x^4$.

> When I think of degree four I think of this shape, where between, if it was just, just *x* to the fourth, it's really flat between here because you're having decimal, decimals, fractions of numbers to the fourth, so it's gonna be smaller than the average. Smaller than a parabola, but then it shoots up, passing a parabola because each term is multiplying by itself (596-602).

Both Casey's feature noticing-and-using and Newt's feature noticing-and using are characterized by a meaning of quartic that is related to the general shape of the parent function of this type, $f(x) = x^4$. Neither is able to apply this meaning to successfully reason about the shape of the graph of $g(x) = |(x-4)^2(x+2)^2|$. In contrast, Todd and Dan were able to successfully reason about the task because their understanding of quartic meshed with other links in which they had correctly coordinated meaning of features across the different registers.

> I believe it's called; a quartic equation which is curved (384-386). Then since it touches, that solution happens twice, and there should be four solutions for *x* here. And so it touches twice…the exponents here, two and two, *x* to the fourth (Todd, 405-407).

> Again, I don't know where the concavity will switch, but I am confident that negative two and negative four will be where the graph touches the *x*-axis, won't pass it and again, some sort of "W" shape. Since given the absolute value being zero, no values of this side. All the outputs will be positive (Dan, 397-401).

Todd's coordinated meaning was between the multiplicity of the roots, two, in the original inscription and the graph "bouncing" at the *x*-intercepts. That is, since the same root happened twice, the graph would "bounce" at the *x*-intercepts as opposed to passing through the *x*-intercepts. In a similar manner, Dan's coordinated meaning was between the roots and absolute value that was part of the original symbolic inscription and the graphical representation of the symbolic inscription touching but not passing through the *x*-axis at the *x*-intercepts. That is, the absolute value sign in the original symbolic inscription suggested to Dan that the graphical representation of the symbolic inscription would not pass through the *x*-intercepts, but only touch them and extend upward.

Dan's and Todd's feature noticing-and-using about the meaning of *quartic* seemed to be related to other links that they had made. In other words, there was a connectedness between what they understood about the shape of quartic graphs and other features of the graphical representation of $g(x) = |(x-4)^2(x+2)^2|$. On the other hand, Newt's and Casey's reasoning about the meaning of quartic did not seem to be tied to other links that they made. For both of them the meaning of quartic was related to a general form of parent functions of this type. Newt's and Casey's reasoning about the meaning of quartic in a general sense without connection to other links seemed to hinder their ability to successfully reason about the task.

In summary, meaning plays an important role in students' ability to reason from both relational and linking strategies. It is apparent that students attribute different meanings to the same inscriptions and the nature and connectedness to other links can play a role in how successful students are in their reasoning.

Claim 6: Forms of Inscriptions and Revealing Features

> *Students seem more likely to notice features in some forms of symbolic inscriptions than other forms.*

This is best exemplified in students' feature noticing-and-using on Task 4, Is it true that the following system of linear equations: $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ has a solution for every value of *k?* Only three students' feature noticing-and-using (Todd, Casey, and Betsy) on Task 4 could be characterized as being from a linking strategy. These three students rewrote the original inscription $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ in terms of *y*, $\begin{cases} y = -2 + k \\ y = k - x \end{cases}$. Although both inscriptions are linear equations, the form of the second symbolic inscription seemed to have illuminated a link to features of graphical representations that the original symbolic inscription did not. Specifically, the form of the inscription, slope-intercept, seems to have revealed the features of slope and *y*-intercept. These features seemed not to be as noticeable in the original form of the inscription. Todd's feature noticing-and-using exemplifies the nature of the reasoning that followed after these features were revealed.

> So that's not gonna change, um, whether the graph has a solution or not. It's just gonna change like where the solution falls? And so, like it could be any value and the line would just move up and down…Well, this one has no *x* value. No *x* variable, that is. And so it's just gonna be horizontal completely. And the horizontal line will just move up and down based on the value of *k*. And so then for every *x* value you just look right up there" (Todd, 204-235).

It is interesting to note that Betsy created a graphical representation to reason about Task 4, but did not create a graphical representation to reason about Task 3 (Solve for x: $x^2 + x + 1 > 0$). When asked if she could reason about Task 3 using a graphical representation she stated, "if it was an equation it would have to have *y* in it" (399). This suggests that for Betsy the use of a graphical representation to reason about a task may be dependent on the symbolic inscription having an *x* and a *y*. Her reasoning on Task 3 supported this claim. The symbolic inscriptions in Task 4 seem to be forms, linear equations with variables represented by the common letters *y* and *x*, that are familiar to her and enable her to reason from graphical representations of those inscriptions. On the other hand, the symbolic inscription in Task 3 seems to be a form that is less familiar to her. Since the inscription is not an equation to her and does not have a *y*, symbols that are part of familiar forms of inscriptions for her, she appears unable to reason about the task from a linking strategy.

Claim 7: Prominent Features are Not Recognized

Students do not necessarily recognize prominent features of symbolic inscriptions.

The previous claim involves the notion that some forms of inscriptions are more apt to reveal features than other forms. In some instances, even when forms of inscriptions would seem to make certain features more prominent, students do not necessarily recognize those features. A *prominent feature* is defined as a feature that a student could recognize with an understanding of the form of the inscription and without any manipulative work. Prominent features are visible from the form of the symbolic inscription. Some forms of symbolic inscriptions reveal certain features more prominently than other equivalent forms of the same symbolic inscription. For example, in the function rule $f(x) = (x - 3)(x + 2)$ the zeros are prominent, whereas in the functional rule $g(x) = x^2 - x - 6$ the *y*-intercept and degree of the polynomial are prominent. Task 1 $\left(\text{Solve for } n\text{: } 1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\right)$ prominent features are the opposite terms—1 and 1 and the opposite terms $-\frac{1}{n+3}$ and $\frac{1}{n+3}$. Task 7b (Describe the graph of the function rule $g(x) = |(x - 4)^2(x + 2)^2|$) the prominent features are absolute value and the zeros, and Task 8 (Find the real values of x such that $4^{2x^2-7x+3} = 8^{x^2-x-6}$) the prominent feature is that the bases are powers of two. The claim is that students do not necessarily recognize prominent features. In other words, features that experienced algebra students are expected to attend to are not necessarily noticed. On Task 1 noticing the opposite terms and using those features to reason about the problem characterized the reasoning of only one third of the students. On Task 7b fewer than half of the students mentioned the zeros in their reasoning, and on Task 8 slightly over one half of the students who engaged in the task noticed the powers of two and used this feature in their reasoning. Only one of the six Calculus II students recognized the powers of two on Task 8.

In some instances students do a great deal of manipulative work before recognizing features that were prominent before their manipulative work. This is exemplified in Robin's work on Task 7a, a task in which she was asked to graph the function given by the rule $f(x) = (x-2)^2(x+2)^2$. Reasoning from a manipulative strategy, she enacted a procedure that involved expanding the inscription. After several algebraic steps she wrote $x^4 - 8x^2 + 16$. Rewriting the inscription as an equation $x^4 - 8x^2 + 16 = 0$ she started another procedure that resulted in the inscription $(x^2 - 4)(x^2 - 4) = 0$. At this point she noticed that the *x*-intercepts were $x = 2$ and $x = -2$, features that were prominent in the original symbolic inscription before she had done any manipulative work. Robin's feature noticing-and-using on this task is further evidence that some students begin manipulative work without a sense of what the symbolic inscriptions in their present form afford them.

Findings Across Levels of Algebra Exposure

Another research question to be addressed in this study involved the nature of similarities and differences in what students notice about symbolic inscriptions across different levels of algebra exposure. After analyzing the data, the researcher made the following claims about the nature of the similarities and differences in what students notice about symbolic inscriptions across levels of algebra exposure. These claims are.

- A manipulative strategy to reasoning was characteristic of the feature noticing-and-using of students with less algebra exposure.
- Students with the greatest exposure to algebra more frequently noticed features and used those features to reason in a manner that was indicative of a relational strategy.
- Students' current mathematical experiences seem to play a role in the nature of their feature noticing-and-using.
- The nature of the feature noticing-and using of those students with more algebra exposure more frequently enabled those students to reason from different strategies to confirm reasoning or fill in gaps in reasoning.

Each of these claims will be addressed individually with an explanation as to why this may have been the case.

Claim 1: Limitations in Feature noticing-and-Using of Students with Less Algebra Exposure

> *The feature noticing-and-using of students with less algebra exposure was characterized by a manipulative strategy to reasoning.*

Across interview tasks most precalculus and Calculus II students' feature noticing-and using was characterized by recognizing features and using those features to apply procedures. Their noticing suggests they had an operational view of equations (and inequalities). That is, the form of the symbolic inscriptions representing equations/inequalities cued particular procedures. It should be noted that only one student from the two lower levels, Paul (Calculus II) and Todd (Precalculus), was able to move consistently beyond a manipulative strategy in his reasoning.

For example, on Task 1 $\left(\text{Solve for } n\text{: } 1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}\right)$ and Task 2b $\left(\text{Solve for x: } \frac{(2x+3)}{(4x+6)}=2\right)$ most students at the precalculus and Calculus II levels did not recognize important features related to the structure of the inscriptions. On Task 1 they did not recognize the pairs of opposite terms, and on Task 2 they did not recognize the common factor between the terms $2x+3$ and $4x+6$. They also had difficulty reasoning about the meaning of resulting inscriptions such as $0=\frac{1}{72}$ and $\frac{1}{2}=2$. As previously stated, these difficulties are consistent with students holding an operational view, as opposed to a relational view, of the symbolic inscriptions. Holding an operational view implies the form of the inscriptions cue particular procedures related to those inscriptions. From an operational view students recognize structures (or features) of inscriptions that cue their procedures. For example, on Task 1 students with an operational view noticed the variable expression in the denominator that cues a related equation-solving procedure, multiplying both sides by n + 3. As long as cognitive conflict does not exist in the steps of the enacted procedure, an operational view is sufficient for solving equations/inequalities. The difficulty appears when students are faced with the need to reason about the meaning of unfamiliar results, $0=\frac{1}{72}$, that do not meet their expectations of solving for a variable. Unable to reason from a relational view about the meaning of an unfamiliar inscription $0=\frac{1}{72}$, most students abandoned their reasoning or resorted to other procedures with the goal of finding a familiar result such as $n=-3$.

For most students at the two lower levels of algebra exposure the operational view of equations/inequalities was evident across tasks. For example, on Task 3 (Solve for x: $x^2 + x + 1 > 0$) most students at the lower levels of algebra exposure recognized a feature (quadratic form) that cued a procedure that involved applying the quadratic formula. The procedure resulted in a form of the symbolic inscription, $x = \frac{-1 \pm \sqrt{-3}}{2}$, that most did not reason about the meaning of the symbolic inscription in the context of the problem. Betsy's feature noticing-and-using is typical of a student reasoning at a lower level of algebra exposure.

> Imaginary is always complex roots … cause it looks like a normal quadratic. Oh. It's an inequality. It's not an equation. But I don't actually remember what to do with that. So, I actually think I solved it wrong cause it's not an equation. I don't remember how to solve an inequality like that. (126-132)

Betsy was able to classify the symbolic inscription and provide a meaning as it related to the procedure of applying the quadratic formula, but she did not use the feature to reason about the problem.

On other tasks such as Task 4 (Is it true that the following system of linear equations $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ has a solution for every value of *k*?) and Task 6b (Solve for x: $|x + 1| > |x + 2|$ students' reasoning did not move beyond a manipulative strategy. In most cases, successfully reasoning about the tasks required students to move to either a relational or linking strategy of reasoning—a move that most students at the lower levels of algebra exposure did not make.

Claim 2: Reasoning of Students with Most Algebra Exposure

> *Students with the greatest exposure to algebra more frequently noticed features and used those features to reason in a manner that was indicative of a relational strategy.*

An aspect of a relational strategy to reasoning is feature noticing-and-using that involves the meaning of the objects and relationships represented by the symbolic inscription. This was characteristic of the feature noticing-and-using of students at the highest level of algebra exposure on tasks involving the symbolic inscriptions $x^2 + x + 1 > 0$ (Task 3) and $|x + 1| > |x + 2|$ (Task 6b). Those students recognized that the inscriptions represented relationships between the numbers represented by the variable expressions. Using these features led to generalized statements about the solution set of the inequalities. Casey's feature noticing-and-using on Task 3 and Dan's feature noticing-and-using on Task 6b exemplified the nature of this feature noticing-and-using.

> If you have any large negative number, this is gonna be, then the *x* squared is gonna be a large positive number, and then you're subtracting the negative number, so then it's still gonna be greater than zero, and then adding one, you just kinda, um, not gonna change it that much. If it's a large negative number, if it's a large positive number it's obviously gonna be positive when you square and then add anything to it. And then when it's zero it also works, so, and even when it's a small negative number it

> works, you could say, and a small positive number works, so I guess you could say that *x* could be anything (Casey, 226-236)

> If you consider zero and up, that statement will never be true. For if *x* is equal to any value zero and greater that statement can't be true…If two people start out with the same number of apples and I gave one, one apple and I gave one two apples, you know, it's back to like real basic stuff. No matter what, there's one more here than there is two there. (Dan, 300-307)

Casey and Dan both recognized that the symbolic inscriptions represented relationships between numbers represented by variables expressions. With an eye toward this relationship, their initial procedure of substituting numbers in for *x* led them to make generalizations about a set of values of *x* that would be in the solution set of the inequality.

It is interesting to note that students with less algebra exposure did substitute values for *x* into the variable expressions, but they did not reason about the relationships between the variable expressions. Students with the most algebra exposure viewed the substitution procedure as a means to better understand the relationships represented by the symbolic inscriptions. For example, Dan stated, "I'll just start putting some numbers in just to get a feel about what my range might be" (174-175). Beyond Paul's feature noticing-and-using (Calculus II) and Todd's feature noticing-and-using (Precalculus) there were very few instances of students' feature noticing-and-using at the lower levels of algebra exposure indicative of a relational strategy.

Claim 3: Impact of Current Mathematics Experience

A students' current mathematical experience seems to play a role in the nature of their feature noticing-and-using.

This claim is exemplified in students' feature noticing-and-using on Task 8 $\left(\text{Solve for } x\text{: } 4^{2x^2-7x+3} = 8^{x^2-x-6}\right)$. The feature noticing-and-using of most prospective mathematics teachers and precalculus students in the study involved recognizing features (bases are of the same power) that students used in procedures that involved rewriting the bases of the original inscription $4^{2x^2-7x+3} = 8^{x^2-x-6}$ as powers of 2 and applying properties of exponents. An example of the related procedure is shown in Ashley's written work (see Figure 5-22).

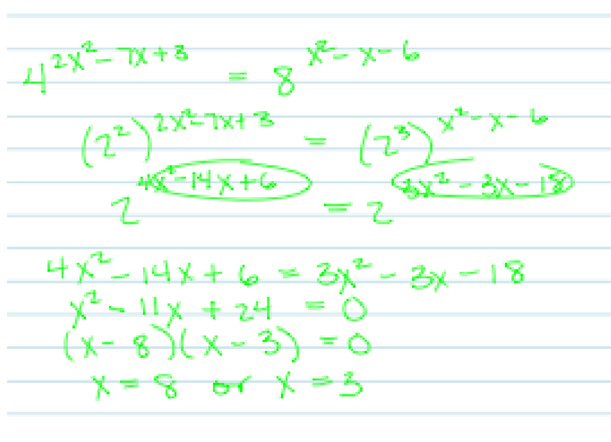

Figure 5-22. Ashley's written work on Task 8.

In contrast to student at the other two levels, the feature noticing-and-using of Calculus II students on Task 8 was characterized by noticing features (exponential expressions) that were used in procedures that involved applying properties of logarithms. For example, Jim recognized a feature of the original inscription (number raised to a variable) that cued a procedure (rewriting exponential function as a logarithmic function), “Whenever a number is raised to a power that has *x'*s in it, usually, logarithms are usually involved” (355-256) (see Figure 5-23). Jim actually abandoned the first line of reasoning and reasoned in the following manner.

> If you, any number can be written as *e* to the natural log of that number because *e* to the natural log of any number is just that number. So, four is also equal to *e* to the *ln* of four. So logarithmic rules say that if something, if the logarithm of something is raised to a power, you can bring that power to the front of the logarithm. So I just changed everything to *e* to the two *x* squared minus seven *x* *ln* of four, equals *e* to the *x* squared minus *x* minus six *ln* of eight. And then I had them both in the same base. So I can do this now. I am going to collect all the *x*s on one side (Jim, 355-369).

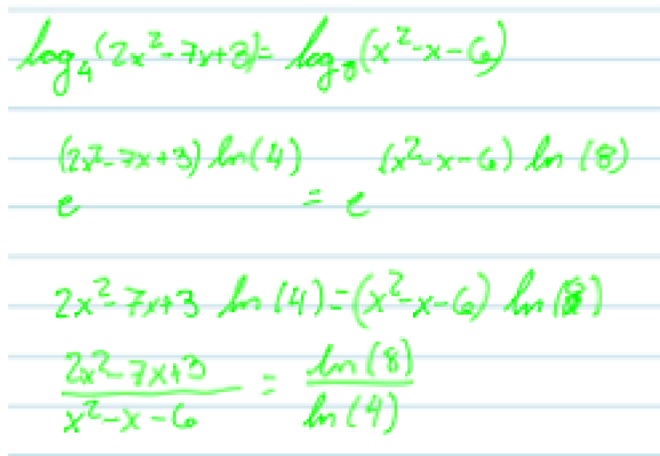

Figure 5-23. Jim's written work on Task 8.

It is interesting to note that Jim’s procedure beginning on the second line of Figure 5-23 would have worked, but he did not recognize the that the two polynomials in the rational expression were factorable, $\frac{2x^2-7x+3}{x^2-x-6}=\frac{(2x-1)(x-3)}{(x+2)(x-3)}$, and the expression $\frac{\ln 8}{\ln 4}$ can be simplified $\frac{\ln 8}{\ln 4}=\frac{\ln 4^{\frac{3}{2}}}{\ln 4}=\frac{\frac{3}{2}\ln 4}{\ln 4}=\frac{3}{2}$.

The feature noticing-and-using of the students who applied logarithmic properties to reason about Task 8 was somewhat similar. All recognized features that cued procedures that involved applying properties of logarithms and rewriting the inscription $4^{2x^2-7x+3}=8^{x^2-x-6}$ in terms of the natural log function. For example, Paul’s reasoning about his procedure stated, “You would have *e* to the natural log of this factor, is the same thing as just this number, cause they are inverse relationships to one another” (429-432). Jim reasoned, “If you, any number can be written as *e* to the natural log of that number because *e* to the natural log of any number is just that number” (358-359).

Most of the Calculus II students who reasoned about Task 8 seemed to have recognized features related to the content of the Calculus II class in which they were currently enrolled or the course in which they were enrolled the previous semester. The content of the calculus courses in the university at which these students were enrolled included solving exponential functions and calculating the derivatives of exponential functions (Stewart, 2008). Nadia in her feature noticing-and-using suggested that she was reasoning from her knowledge of calculus, "I just know that the derivative of *a* to the *x* is so… I just put the derivative on both sides… and … and I really don't know what to do" (385-387).

Although based only on students' feature noticing-and-using from one task the findings suggest that students' most recent mathematical background may play a role in the nature of the features of a symbolic inscription that are recognized. The other tasks in the study were more typical of a second year course in algebra and do not, unlike Task 8, seem as easily connected to the content of calculus courses.

Claim 4: Purposeful Movement and Students with Highest Exposure

> *The nature of the feature noticing-and using of those students with more algebra exposure frequently enabled those students to reason from different strategies to confirm reasoning or fill in gaps in reasoning.*

The fourth claim across levels of algebra exposure follows from the three previous claims. It has already been claimed that those algebra students' with the highest level of exposure more frequently have feature noticing-and-using capacities that enable them to reason with relational and linking strategies. The feature noticing-and-using of all four students at the highest level of exposure (Ashley, Casey, Dan, and Newt) exhibited multiple instances of reasoning from different strategies. Most of the students' feature noticing-and using in the two lower levels of algebra exposure enabled them only to reason with a manipulative strategy. Although there are instances of reasoning from both relational and linking strategies in the reasoning of those with less exposure, it is far less prevalent than at the highest level of algebra exposure. The feature noticing-and-using of students with the most exposure enabled them to move between the different strategies in their reasoning. As previously stated this purposeful movement is motivated by (a) reaching a dead end in reasoning and needing a different path, (b) filling a gap in reasoning, or (c) supporting earlier reasoning.

Students' feature noticing-and-using that exemplifies a purposeful movement between different strategies seems to be closely related to feature noticing-and-using that suggests a relational view of equations and inequalities and an ability to make productive links. In other words, students' feature noticing-and-using suggest they understood the relationships preserved between transformations of symbolic inscriptions into other forms of symbolic inscriptions and they understood that symbolic inscriptions expressing equations/inequalities represented relationships between numbers. They are also able to link features of inscriptions expressed in one representational register to features of inscriptions expressed in another representational register. Generally speaking, the nature of these students feature noticing-and-using seem to be connected to a deep understanding of the symbolic inscriptions, an understanding that Gray and others (2001) characterize as a *proceptual understanding*. In other words, purposeful movement between strategies suggests a flexibility of reasoning about symbolic inscriptions. That is, the purposeful movement between strategies suggests students have the understanding that these inscriptions represent both processes and objects, a proceptual understanding.

Adjustments to the Feature Noticing-and-Using Taxonomy

The development of the initial feature noticing-and-using taxonomy was a synthesis of others' descriptions of symbol sense activities (Arcavi, 1994; Driscoll, 1999; Fey, 1990; Kieran, 2007; Kinzel, 2001; Zorn, 2002) as well as the author's informal experiences with secondary mathematics students. After an analysis of the data from this study, it is evident that the initial feature noticing-and-using taxonomy (see Table 5-2) lacked the specificity and interconnectedness that seems to capture the nature of students' feature noticing-and-using as they reasoned about problems involving symbolic inscriptions. The following discussion will focus on the rationale for the changes to the feature noticing-and-using taxonomy.

Table 5-2

Initial Feature Noticing-and-Using Taxonomy

Component	**Description**	**Common Instance Codes**
Recognizing	1. Activity of recognizing features of a symbolic inscription. This activity may include recognizing features that cue symbol-manipulating procedures.	1.1 Recognized a feature of the inscription that provided meaning. 1.2 Recognized a feature during a symbol manipulating procedure. 1.3 Recognized a feature that informed a symbol manipulating procedure. 1.4 Recognized a feature that reduced written algebraic manipulations.

		1.5 Recognized a feature that cued a symbol manipulating procedure.
		1.6 Recognized a feature that resulted in abandoning one symbol manipulating procedure for another symbol manipulating procedure.
Reasoning	2. Activity of describing the meaning of or reasoning from a feature of the symbolic inscription.	2.1 Reasoning about the meaning of a symbolic solution.
		2.2 Reasoning about meaning of the results of a symbol manipulating procedure.
		2.3 Reasoning about the structure of a symbolic representation.
		2.4 Reasoning about the meaning of symbols within a particular problem context.
Linking	3. Activity of linking features of the symbolic inscription with features in another representation of the symbolic inscription.	3.1 Linking features of an inscription expressed as a symbolic representation to features of another representational form.
		3.2 Linking features of another representational form with features of an inscription expressed as a symbolic representation.

Rationale for Adjusting the Feature Noticing-and-Using Taxonomy

One of the outcomes of the study is a refinement of a taxonomy for feature noticing-and-using—a refinement that at its core has the three different strategies to reasoning that seem to be related to the nature of students' feature noticing-and-using. Before the study the three activities of feature noticing-and-using were identified as recognizing, reasoning, and linking. After a careful analysis of the data it was determined that the nature of students' feature noticing-and-using was related to the nature of their reasoning strategies. Specifically, recognizing features of symbolic inscriptions, symbol recognizing, was related to a manipulative strategy in which procedures are cued without attending to the meaning of the underlying mathematical objects. After analyzing the data the researcher determined that *reasoning* was too general of a term to enable the researcher to distinguish between feature noticing-and-using as it related to reasoning about steps of procedures and feature noticing-and-using about the meaning of symbolic inscriptions. The purpose of this study was to understand the nature of students' feature noticing-and-using as they reasoned about problems that involved symbolic inscriptions. As a result, the researcher made the decision to adjust the taxonomy to reflect the variations in students' feature noticing-and-using that were observed in the analysis of the data. The main categories were changed from *recognizing*, *reasoning*, and *linking* to *manipulative*, *relational*, and *linking* to reflect the relationship between students' feature noticing-and-using and their reasoning strategies.

Change in Instance Codes. In analyzing the data, the researcher determined that either some of the common instances codes (Table 5-2) were ambiguous or did not succinctly describe the nature of students' feature noticing-and-using on the interview tasks. For example, a few of the instance codes, recognizing feature during a symbol manipulative procedure (1.2) and recognizing a feature that informed a symbol manipulating procedure (1.3), were ambiguous. Other instance codes, were unhelpful in describing the nature of feature noticing-and-using. Recognizing a feature that resulted in abandoning one symbol manipulating procedure for another symbol manipulating procedure (1.6) is an example of a instance code that was not helpful in describing the nature of feature noticing-and-using. The analysis of the data suggests abandoning procedures is not necessarily about feature noticing-and-using, but about not noticing features.

In turn, instances of feature noticing-and-using that the analysis of the data revealed were added to the feature noticing-and-using taxonomy. These instances were changed to reflect the nature of students' feature noticing-and-using as revealed within the data . A reconceptualization of the feature noticing-and-using taxonomy is shown in Table 5-3. Instance codes that are part of the adjusted feature noticing-and-using taxonomy that were not part of the initial taxonomy are highlighted.

Table 5-3

Adjusted feature noticing-and-using taxonomy

Strategies	**Nature of Noticing-and-Using**	**Common Instances**
Manipulative	1. Noticing features of symbolic inscriptions and using those features to apply manipulative actions on the symbolic inscriptions.	1.1 Recognizing features of symbolic inscriptions that cue procedures. 1.2 Recognizing features of symbolic inscriptions that limit algebraic manipulations. 1.3 Recognizing features of symbolic inscriptions that involve attending to the mathematical conditions under which a particular procedure can be applied. 1.4 Recognizing features of symbolic inscriptions that involve the structural conditions under which a procedure can be applied.
Relational	2. Noticing features of symbolic inscriptions	2.1 Reasoning about a relationship between inscriptions within a symbolic

	and reasoning about the objects/concepts represented by those features. Involves reasoning within the same register except when the change in register is from symbolic to numerical.	inscription. 2.2 Reasoning about a relationship between a symbolic inscription and a transformation of the symbolic inscription. 2.3 Reasoning about the meaning of object represented by the symbolic inscription.
Linking	3. Noticing features in one representational register and using those features to reason about features in another representational register.	3.1 Links from features of the symbolic inscription to features of another representation of the symbolic inscription. 3.2 Links from features of another representation to features of the symbolic inscription.

Addition of Venn diagram structure to taxonomy. After analyzing the data and adjusting the feature noticing-and-using framework it became clear that the initial structure of the feature noticing-and-using taxonomy did not account for the dynamics of students' feature noticing-and-using and its relationship to different strategies used by students to reason about the same problem. It is proposed that a Venn diagram would be a better representation to capture the dynamic nature of students' feature noticing-and-using as they reason about a particular problem (see Figure 5-24). The feature noticing-and-using Venn diagram is meant to capture the nature of feature noticing-and-using and how those features are used to reason about a particular problem.

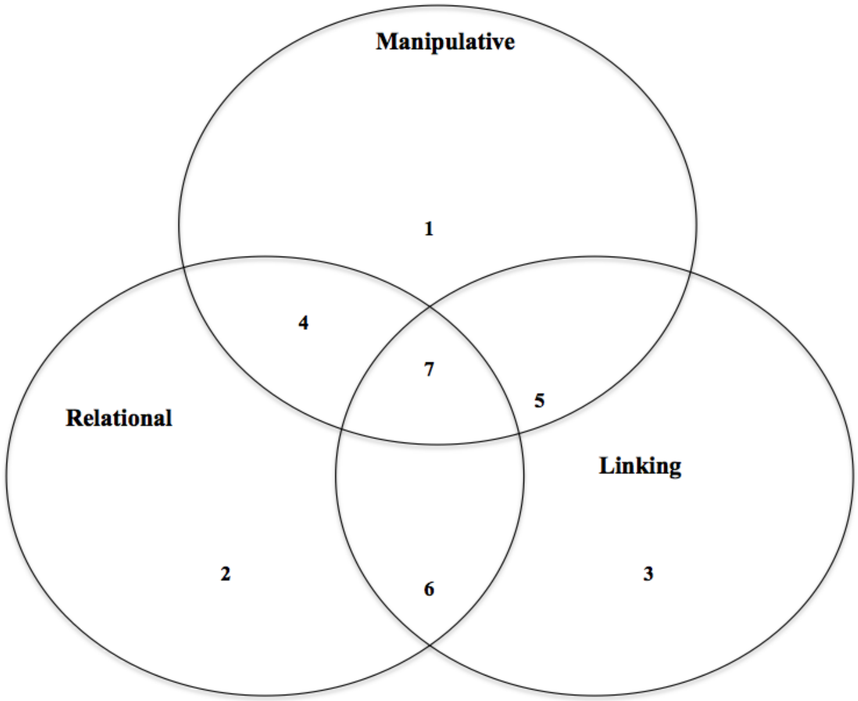

Figure 5-24. Feature noticing-and-using Venn diagram.

Students' feature noticing-and-using that draws on only one of the three strategies is represented by the isolated regions of the Venn diagram (Regions 1, 2, and 3). At other times students' feature noticing-and-using was characterized by reasoning that seemed to call upon multiple strategies. The regions between the strategies that do overlap represent this reasoning (Regions 4, 5, 6, 7). The overlapping regions represent instances of students' feature noticing-and-using that involved multiple strategies. In other words, there is a relationship in some manner between two strategies. For example, the student on Task 6b (Solve for x: $|x+1| > |x+2|$) who noticed the relationship between the two variable expressions and used the relationship in their reasoning is operating in Region 2. Region 6 represents using a linking strategy to fill a gap in reasoning related to a relational strategy. The gap in student's reasoning within a relational strategy motivated feature noticing-and-using that was related to a linking strategy. The overlapping regions signify a relationship between students' feature noticing-and-using and reasoning in one strategy with another strategy.

Newts' feature noticing-and-using on Task 3. In order to illustrate the feature noticing-and-using taxonomy (Table 5-3) and feature noticing-and-using Venn diagram (Figure 5-24) Newt's feature noticing-and-using on Task 3 (Solve for x: $x^2 + x + 1 > 0$) is discussed using both of these. The feature noticing-and-using taxonomy (Table 5-3) can be used to classify specific instances of feature noticing-and-using, and the Venn diagram (Figure 5-24) can be used to show into which strategy the specific feature noticing-and-using falls. In addition the Venn diagram can show the connectedness of the strategies.

Working from a manipulative strategy (Region 1) he noticed a feature that cued him to apply the quadratic formula [1.1]. His work resulted in the inscriptions $x = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$ and $x = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$ that represented a feature (complex numbers). He used the feature (complex numbers) in a procedure for solving quadratic inequalities. A step of the procedure required placing the inscriptions resulting from his procedure on a number line. Changing to a linking strategy he used the feature to place points on a line (see Figure 5-25) (Region 5).

Figure 5-25. Newt's representation of a number line.

> I used the quadratic formula. And then I saw there were going to be, and it was going to be a part of it. And so I tried to keep going and draw a number line, because at this point if these were real numbers, then I would test the point here and here and here… but since I don't have an understanding of what's bigger and smaller with complex numbers (196-202).

Unable to use the feature (complex numbers) in his procedure because his procedure required numbers that were ordered, and complex numbers are not ordered, he abandoned the procedure and began reasoning using a relational strategy (Region 2). [Note: There is no evidence to suggest that he recognized this property of complex numbers] He noticed a feature (relationship between variable expressions) and used the feature to reason about a set of numbers *x* that would satisfy relationship expressed by the symbolic inscription $x^2 + x + 1 > 0$ (Region 2) [2.1].

> I can look and see that zero works, one works, anything greater than zero works right off the top of my head *x* squared, if I plug in any positive real number uh, it's gonna be positive, it's gonna be positive, and one is greater than zero. So it's always gonna be greater than zero (136-144).

> Any *x* less than negative one is going to work because when you square it it's going to be positive and larger than just minus constant value, and you have plus one is gonna be greater than zero and also everything equal to zero is going to work because zero works, one works, and it's only gonna get bigger (148-153).

Uncertain of whether the numbers between x = -1 and $x = 0$ would satisfy the relationship, "And so now It's just a matter of between there and there" (154) his feature noticing-and-using switched to a linking strategy (Region 7) that involved a manipulative strategy.

He noticed a feature (quadratic form) that he used to apply a procedure for finding the vertex of the function $y = x^2 + x + 1$ (Region 6). Noticing that the symbolic inscription $y = x^2 + x + 1$ was of quadratic form he used to feature to graph a parabola that opened upward and had a vertex that was above the x-axis [3.1]. Newt noticed a feature of the graphical representation (all points lie above the x-axis) that he used to reason that all real numbers *x* were in the solution set of the inequality $x^2 + x + 1 > 0$ (Region 3) [3.2], "This is like with zero, since it's an open up parabola, and the vertex is greater than zero, every other point is going to be greater than zero"(168-169). The totality of his feature noticing-and-using on Task 3 is represented in the feature noticing-and-using Venn diagram shown in Figure 5-26.

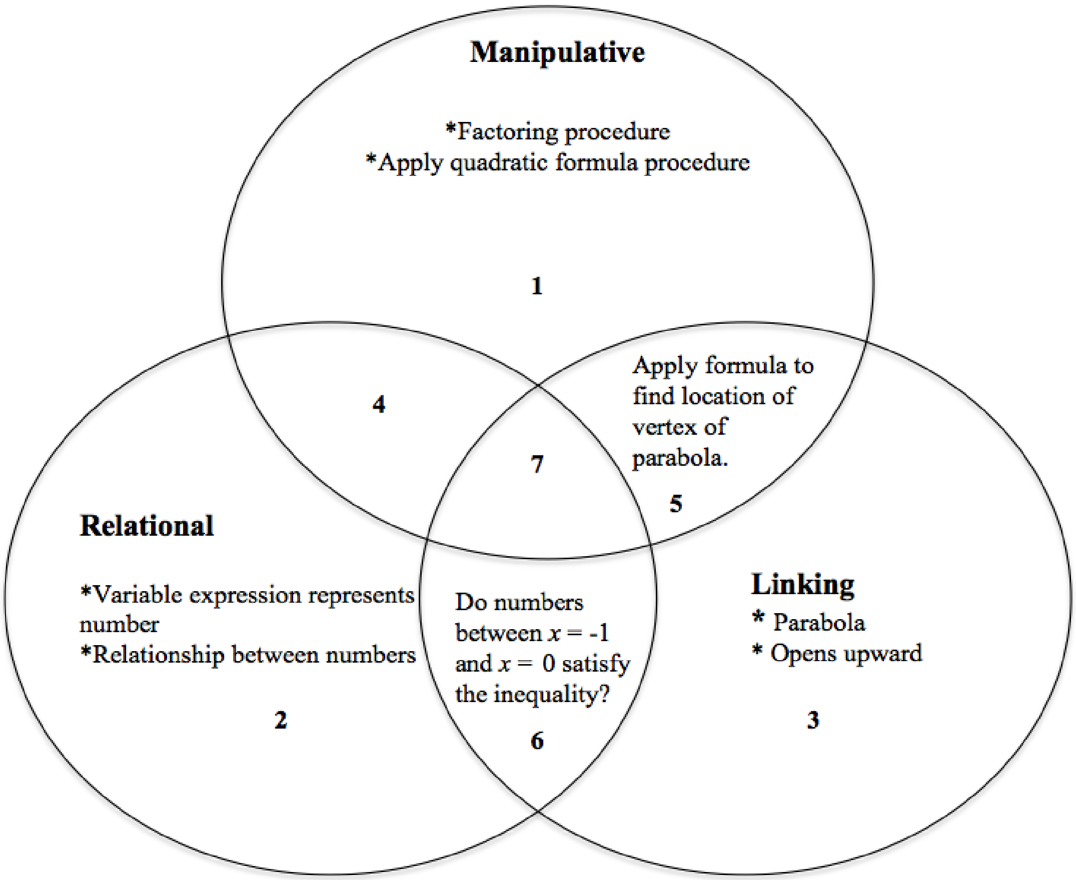

Figure 5-26. Newt's feature noticing-and-using on Task 3.

Purposeful shifts between different strategies characterized Newt's reasoning on Task 3. The isolated parts of each region (Regions 1, 2, and 3) represents Newt's feature noticing-and-using that was unrelated to other strategies. The overlapping parts (Regions 5 and 6) represent the regions in which Newt's feature noticing-and-using was related to other strategies. For example, Region 6 represented Newt's shift from a relational strategy to a linking strategy because he was uncertain whether numbers between $x = -1$ and $x = 0$ when substituted into the symbolic inscription $x^2 + x + 1$ would be greater than zero. He reasoned that the graphical representation of $y = x^2 + x + 1$ would tell him whether all values in this interval would be greater than zero. Region 5 represented Newt's use of a formula to determine the location of the vertex, manipulative strategy, which aided his reasoning from a linking strategy.

In summary, Newt's initial feature noticing-and-using was from a manipulative strategy (quadratic formula procedure, quadratic inequality procedure) that did not draw on other strategies. His quadratic inequality procedure, though a manipulative strategy in nature, required a linking strategy to use the feature (Region 5). After he abandoned the manipulative strategy he shifted to a relational strategy (Region 2). Uncertain whether a certain interval of numbers would satisfy the inequality represented in the original task he shifted to a linking strategy to reason (Region 6). Reasoning from a linking strategy he called on a formula to find the location of the vertex. This step required a manipulative strategy (Region 5) that was related to a linking strategy. Once he determined the location of the vertex he reasoned about the solution of the original task strictly from a linking strategy (Region 7).

Summary

The feature noticing-and-using taxonomy and Venn diagram as it relates to the different reasoning strategies provides a means to classify the nature of students' feature noticing-and-using. It provides a means with which teachers and researchers can identify the nature of students' feature noticing-and-using as they reason about tasks involving symbolic inscriptions. Researchers have suggested that symbol sense is at the heart of what it means to understand algebra. This researcher argues that students' symbol sense capabilities are related to the nature students' feature noticing-and-using. That is, symbolic sense capabilities are related to the features of symbolic inscriptions that students notice and how these features are used in their reasoning. Results from this study suggest that there is complexity to students' feature noticing-and-using as they reason about problems involving symbolic inscriptions. The two aspects of the feature noticing-and-using taxonomy provide an organizational structure with which to examine this complexity and identify the nature of students' feature noticing-and-using. Also, the feature noticing-and-using taxonomy provides the means to classify the nature of difficulties students have in reasoning about problems that involve symbolic inscriptions as well as the possible nature of students' advancements in these capabilities.

Chapter 6

DISCUSSION

There were three purposes to this study. The main purpose was to examine the nature of students' feature noticing-and-using as they solved unfamiliar algebraic problems based on familiar algebra concepts and involving symbolic inscriptions. Other purposes were to examine the similarities and differences in the nature of students' feature noticing-and-using across levels of algebra exposure and create a taxonomy that would account for the nature of feature noticing-and-using as evidenced in students' reasoning about problems involving symbolic inscriptions. Based on an analysis of the data, the researcher's feature noticing-and-using taxonomy was revised to better reflect the nature of features of symbolic inscriptions that were noticed and how students used these features to reason about problems that involved symbolic inscriptions.

Summary and Discussion of Research Findings

The purpose of this chapter is to provide a brief summary of the findings, a discussion of these findings as they relate to informing the research community, and their relationship to existing research in this area. This chapter also includes implications for teachers and researchers, suggestions for future research, and limitations of the study.

Nature of Feature noticing-and-using-and-Using

> *Research Question 1: What is the nature of students' feature noticing-and-using as they solve unfamiliar algebra problems based on familiar algebra concepts and involving symbolic inscriptions?*

The researcher made several claims about the nature of students' feature noticing-and-using. These claims are related to what features students noticed and how they used those features to reason about problems involving symbolic inscriptions. Kieran (2007) has argued that students attach different meanings to symbolic inscriptions and a great deal of this meaning is dependent on what students are asked to do with the symbolic inscriptions. In this study, the symbolic inscriptions were placed in the forefront. Students were asked to reason about a series of tasks that involved symbolic inscriptions and in which the phrasing of the tasks may suggested a familiar procedure related to the form of the symbolic inscriptions. Findings related to this research question are summarized:

- In the context of solving problems presented using symbolic inscriptions, there seem to be differences in the nature of the features students notice and how these features are used in their reasoning. These differences are related to students' reasoning strategies.
- Students' feature noticing-and-using frequently involved noticing a feature of the symbolic inscription and using the feature to apply a procedure related to the form of the symbolic inscription without attending to the mathematical meaning of the inscription. The nature of this feature noticing-and-using was classified as a manipulative strategy—a strategy that seemed to limit one's ability to solve tasks involving symbolic inscriptions.
- Those students whose feature noticing-and using seemed to be at a higher level were able to notice the objects and relationships represented by symbolic inscriptions.
- Several students' feature noticing-and-using was characterized by linking strategies. Those who were able to successfully use a linking strategy in their reasoning were

able to notice features of a graphical representation of the symbolic inscription and use those features to reason about the original problem.

- Some forms of symbolic inscriptions seem more amenable to noticing particular features than other forms.
- Students in their reasoning do not necessarily notice prominent features of symbolic inscriptions.

These points will be summarized in a discussion of how these points relate to existing research.

Discussion of Research Question 1

Many of the studies (Hoch & Dreyfus, 2004; Menghini, 1994; Sfard & Linchevski, 1994) that involve the features students notice in their reasoning about symbolic inscriptions have focused on deficiencies in students' noticing. The purpose of this research is not to focus on errors and limitations in students' noticing but to understand the nature of the features of symbolic inscriptions that students' notice and how they use these features in their reasoning as well as the challenges they face using features of symbolic inscriptions to reason about problems. The data from this study suggested three strategies for noticing and using features of symbolic inscriptions: manipulative strategies, relational strategies, and linking strategies. The discussion of the findings related to the first research question will be organized around the feature noticing-and-using within of these strategies.

Manipulative strategy. Before beginning a discussion of the findings related to the first research question it is important set the stage with respect to what students were asked to do and what would have been expected given their past experiences in similar situations. The nature of this experience will be discussed using LaGrange's (2005) terms *tasks* and *techniques.* Recall that *task* is "what is asked" and *technique* is "way of doing". It can be reasonably argued that the tasks that were used in this study($\mathrm{e.g.\,Solve\ for}\ x: x^2 + x + 1 > 0$) were of a type that would seem to suggest to the student, based on their experiences with algebra, to perform a procedure related to the form of the symbolic inscription. The familiar technique for students would be to notice a feature of the symbolic inscription and use that feature to implement a procedure. The challenge for students in their reasoning on these tasks was that their typical technique, while at times helpful, would not necessarily lead them to reason about the task. The nature of the interview tasks challenged students to reason about the meaning of the symbolic inscriptions resulting from their familiar techniques as well as use other techniques when their familiar techniques did not work.

From the familiar technique perspective it is not surprising that on most tasks across levels of algebra exposure students' initial strategy, described as manipulative, involved noticing features of the symbolic inscriptions and using the feature to apply procedures related to the form of the symbolic inscription. This seems to be consistent with what others have found (Even, 1998; Huntley & Davis, 2008). Of the 18 students interviewed, less than half (Newt, Dan, Ashley, Casey, Paul, Jim, Todd) noticed features and used those features in other capacities besides those related to manipulative strategies. It is almost as if students live in a manipulative strategy world and their experiences in algebra may be encouraging them to stay in this world.

Also, from this perspective it is not surprising that many students do not use structure sense to solve problems (Hoch & Dreyfus, 2004; Menghini, 1994) involving symbolic inscriptions or do not notice prominent features of symbolic inscriptions (Vaiyavutjamai et al., 2005). The dominance of a manipulative strategy to reason suggests that students are "doing before looking." In other words, the focus of students' attention is not on noticing structural features of symbolic inscriptions that provide meaning, but on noticing features that would cue procedures related to the form of the symbolic inscription. For example, noticing a quadratic form would suggest using factoring or applying the quadratic formula to reason and noticing an absolute value form would suggest using piecewise defined equations/inequalities to reason.

It is important to note that there were challenges in students' feature noticing-and-using at it related to a manipulative strategy. There was evidence to suggest that students did not attend to all the relevant features of a symbolic inscription before applying a procedure. In some cases this resulted in a malformed procedure being applied and in other cases it resulted in a procedure being applied in which there was a lack of attention to conditions of the procedure. For example, on Task 3 (Solve for x: $x^2 + x + 1 > 0$) several students incorrectly applied the zero-product property to $x(x + 1) > -1$ and on Task 6b (Solve for x: $|x + 1| > |x - 2|$) none of the students attended to the domain restrictions in their procedure that involved transforming the inequality into a set of piece-wise defined inequalities.

There were a few differences in students' feature noticing-and-using and the related strategies students used to reason about the tasks. Two students at the lower levels of algebra exposure (Todd, and Paul) and most of the students at the highest level of algebra exposure (Ashley, Casey, Dan, Newt) exhibited reasoning on multiple tasks that went beyond a manipulative strategy. For some (e.g. Dan and Todd) the technique for solving tasks involving symbolic inscriptions seemed to be different than others' technique. That is, Dan noticed features of symbolic inscriptions and used those features to make links to features of a graphical representation of the symbolic inscription. Dan's comment suggests that, unlike other students' initial technique, he has a proclivity to reason from graphical representations, when presented problems that involved symbolic inscriptions.

> I use graphs very often. I have to kind of, I don't have to, but I found out early on that it worked for me. I didn't find out maybe not early on in high school, but in my senior year in high school and then into college and everything. If I could make a graph of it, I could figure out how the numbers would behave, is really what I'm thinking about (338-343).

For other students (e.g. Ashley, Casey, Paul, and Newt) their initial technique may have involved a manipulative strategy, but they were able to reason from other strategies. Recall that on most of the interview tasks a manipulative strategy would not work to solve the problems—students had to call on other strategies or extend their feature noticing-and-using beyond manipulative strategies. Those who were able to move beyond a manipulative strategy were able to notice features of symbolic inscriptions related to the objects and relationships represented by the symbolic inscriptions, and they were able to use those features to reason about the problems they were asked to solve. These students seemed to have different meanings attached to the features of the symbolic inscription involved in the tasks than those who reasoned strictly from a manipulative strategy. In other words, these students had meanings attached to the symbolic inscriptions that suggested they were reasoning from relational and linking strategies, both strategies that suggest an object understanding of the symbolic inscriptions. Specific aspects of this object understanding, related to the findings of this study will be discussed in the context of the other two reasoning strategies, relational and linking.

Relational strategy. The feature noticing-and-using of students' relational strategies were characterized by noticing features of symbolic inscriptions that expressed the objects and relationships represented by the symbolic inscriptions and using the meaning of those features to reason about the presented problems. In other words, students seemed to have meanings attached to symbolic inscriptions that would suggest an object-level understanding. This type of strategy was most prevalent in the reasoning of those with the highest level of algebra exposure (Ashley, Casey, Dan, and Newt). Students using this strategy are attending to features at the micro-level structure (Mason, 2011) of the symbolic inscription. In other words, students are attending to the conceptual meaning of features of the symbolic inscription.

Students' feature noticing-and-using in a relational strategy is characterized by an attention to the features of the symbolic inscription that seem to provide meaning, as demonstrated by their verbal statements, before performing any procedures. Yerushalmy and Gafni (1992) characterizations of syntactic structures and semantic structures seem to distinguish the difference in students' feature noticing-and-using between a manipulative strategy and a relational strategy. Using terminology from this study in light of their descriptions they argued that, using a syntactic structure, students attend to the features of a symbolic inscription that are used to cue procedures (manipulative strategy) whereas at a semantic level students attend to the conceptual meaning relationships expressed by the features and use those features in their reasoning (relational strategy).

The feature noticing-and-using of those students who used relational strategies seemed to suggest they have an object-level understanding of root-preserving algebraic transformations on equations/inequalities (i.e. those transformations involving properties of equality) as well as of the equations themselves. For example, on Task 1 $\left(\text{Solve for } n\text{: } 1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}\right)$ and Task 2 $\left(\text{Solve for } x\text{: } \frac{(2x+3)}{(4x+6)}=2\right)$ students recognized features (structures) of the equations that provided meaning before performing any procedures and they recognized that transforming the equations resulted in new equations that preserved the equivalence relationship between the two equations. Specifically, they were able to reason about the meaning of the transformation of the equation in the context of the problem. That is, the resulting inscription represented an untrue statement (e.g. $\frac{1}{2}=2$) meaning there were no values for x that made the initial (equivalent) equation true.

The nature of these students' feature noticing-and-using suggest that they have an object-level understanding of these symbolic inscriptions representing equations as opposed to a process-level understanding (Gray & Tall, 2001; Tall et al., 2000). Specifically, students' reasoning suggested that they viewed the algebraic manipulations on the symbolic inscription $1-\frac{1}{n+3}-1+\frac{1}{n+3}=\frac{1}{72}$ which resulted in another symbolic inscription, such as $0=\frac{1}{72}$, as preserving an equivalence relationship between the two equations. That is, the roots of the original equation and those of the transformed equation are the same.

The RAND Mathematics Study Panel (2003) argued, "the notion of 'equal' is complex and difficult for students to comprehend" (p. 53). This complexity and difficulty can extend to students in high school and beyond (Kieran, 1981). Consistent with findings from this study, Kieran (1981) argued that "the procedures used by students to solve equations would seem to indicate that high school and college students interpret the equal sign as a signal rather than as a symbol for an equivalence relation" (p. 325). An operational view of symbolic entities representing equations (the researcher would include inequalities) is part of a manipulative strategy. From an operational perspective, similar to reasoning from a manipulative approach, the equal sign is a feature that is used to cue a procedure. The operational view is in contrast to a relational view in which the equal sign (or inequality sign) is seen as a feature that represents a relationship.

It is important to note that, counter to the findings of this study, McNeil and Alibali (2005) contended that more experienced algebra students (undergraduate and graduate students) interpreted the equal sign as a relational symbol of equivalence. Their conclusion, however, was based on results from students reasoning about tasks that, although algebraic in nature (e.g., $4 + 8 + 5 = 4 + \_\_\_$), did not involve feature noticing-and-using on symbolic inscriptions that would be more indicative of a high school classroom setting such as the tasks used in this study.

The current research extends the notion of an operational or relational view of equal sign to settings that involve students' reasoning about tasks in which the symbolic inscriptions represent algebraic equations (e.g. $|x - 2| = |x + 1|$) to the student. The findings from this study suggest that most students' feature noticing-and-using comes from an operational view of equations, but the feature noticing-and-using of students at the highest level of algebra exposure showed evidence of a relational view of equations.

Students' feature noticing-and-using in a relational strategy was also characterized by students noticing the feature that symbolic inscriptions represented relationships between algebraic expressions and using this feature to reason about the relationship expressed by the symbolic inscription (e.g. $x^2 + x + 1 > 0$). Dan's statement speaks to the nature of those who had more of a relational view of the relationship expressed by the symbolic inscription, "I'll just start putting some numbers in just to get a feel about what my range [Note: Dan was referring to values of *x* that would be in the truth set of the inequality] might be" (174-175).

It is interesting to note that students with less exposure to algebra also substituted numbers, which suggested that they noticed that the symbolic entities, $|x + 1|$ and $|x + 2|$, represented variable expressions for numbers. However, students with less exposure, unlike those students with the most exposure, did not use this knowledge to reason about the relationship between the variable expressions. In other words, students noticed a feature of the symbolic inscription (part of symbolic inscription represented a variable expression) but were unable to use that feature because they did not notice another feature of the symbolic inscription (the entire symbolic inscription represented a relationship between variable expressions).

A point that needs to be made is that because students notice a feature and are able to describe the conceptual meaning of the feature, suggestive of a relational strategy, does not mean students are able to use the feature to reason about the task. For example, on Task 6b (Solve for *x*: $|x + 1| > |x + 2|$) and Task 6a (Solve for *x*: $|x + 1| = |x - 2|$) most students recognized absolute value and stated that the concept of absolute value was "distance from zero", but only Newt (prospective secondary mathematics teacher) was able to use the feature to reason about interview tasks. Hinkel (1994), in a language acquisition context, argued that students need to know what features of the symbolic inscription they should notice and what about those features requires attention. As evidence from this study suggests in a domain that involves symbolic inscriptions that represent algebraic entities, knowing the features and what about those features require attention is not necessarily enough. There is also a need to know how the meaning of particular feature applies in the problem context. Although many students recognized absolute value and knew its conceptual meaning, they were unable to apply the meaning to a context that involved the symbolic inscription $|x + 1| = |x – 2|$. Specifically, they did not notice that the symbolic entity represented a relationship in which a number x was the same distance from -1 as it was from 2.

As previously stated, feature noticing-and-using indicative of a relational strategy was infrequent at the two lower levels of algebra exposure and most prevalent at the highest level of algebra exposure. A hypothesis for why this may have been the case will be discussed later in this manuscript.

Linking strategy. The feature noticing-and-using of students' linking strategies were characterized by noticing features of symbolic inscriptions and using those features to reason about features of the graphical representation of the symbolic inscription, and then using features of the graphical representation to reason about the solution set of the problem involving the original symbolic inscriptions. Feature noticing-and-using of this type was described by the researcher as a *productive link*. A few students' feature noticing-and-using (Newt, Casey, Ashley, Drew, Paul, and Todd) was characterized by productive links. Productive links require students to coordinate the meaning of objects represented in one register with the meaning of an object represented in another representational register. The challenge of a productive link is the coordination of meaning needed to carry out two conversions between registers. In other words, the first conversion is a coordination of meaning from the meaning of the object represented by the source register, represented as a symbolic inscription, to the same object represented by the target register, another representational register of the symbolic inscription. The second conversion is a coordination of meaning from a feature represented in the original target register back to the same feature represented in the initial source register, represented as a symbolic inscription.

Evidence of students' feature noticing-and-using on several tasks suggests there are challenges to making both conversions. As was the case in a relational strategy the noticed features in a linking strategy were objects represented by aspects of the symbolic inscription. One possible source of difficulty is related to the nature of the objects represented. Another source of difficulty is, in many instances, the features of the symbolic inscription that need attention in order to change between conversions. Using a linking strategy Task 3 (Solve for x: $x^2 + x + 1 > 0$) a possible feature noticing-and-using may involve a link in which features of the symbolic inscription are noticed and used to represent features of the graphical representation of the symbolic inscription. For example, this strategy would connect a parabola opening upward because of the positive value of the coefficient of x^2 with a vertex located at $\left(-\frac{1}{2}, \frac{3}{4}\right)$ based on applying the formula $x = -\frac{b}{2a}$ to $x^2 + x + 1$. These links represented one conversion. A possible second conversion, representing a change in direction of the link, is that since the entire graph lies above the *x*-axis, all real numbers *x* will satisfy the inequality $x^2 + x + 1 > 0$. Features of the graphical representation are used to reason about the solution set of the original problem. As this example illustrates, a productive link requires feature noticing-and-using that involves two conversions in which each conversion involves reasoning about different features represented by aspects of the symbolic inscription.

It is argued that those who are more successful in their reasoning about symbolic inscriptions are able to move flexibly between representations (Crowley, 2000; Kaput, 1989). As this study found, there are challenges students face in reasoning flexibly between representational registers. Findings from this study are consistent with what others have found. Students with algebra exposures ranging from precalculus (Dugdale, 1989; Kenney, 2008) to those of prospective secondary mathematics teachers (Even, 1998) have difficulty making links between symbolic entities and graphical representations of those entities. Similar to a specific context referenced by Even (1998) and Sfard and Linchevski (1994), few students in the study (Newt, Ashley, Dan, Paul), viewing a symbolic representation of a quadratic inequality $(x^2 + x + 1 > 0)$, were able to use a graphical representation of a quadratic function in their reasoning.

This study contributes to the existing research in this area by bringing specificity to the challenges students face in making links. Using Duval's (2006) characterization of conversions, the nature of links between features of the representation of an object in one register to a feature of the representation of the same object in another register was examined. The current research advances the field by adding specificity to the challenges students face in attempting to reason from graphical representations in tasks involving symbolic inscriptions. It is well-documented in the literature that students struggle reasoning from multiple representations (Crowley, 2000; Duval, 2006; Even, 1998; Kenney, 2008) to solve problems, but lacking has been a rich, detailed description, with examples, of the nature of these challenges. Results from this study suggest that there are two challenges students face as they move between representational registers to reason about symbolic inscriptions. The first challenge is students have to coordinate meaning between two conversions of registers. The second challenge is the focus of the feature noticing-and-using changes between the conversions. In other words, students link features of the symbolic inscription to another representational register of the symbolic inscription, one conversion, and then link from different features of the other representational register to features of the symbolic inscription, a second conversion.

This research, consistent with others' findings (Duval, 2006, citing research by Pavlopoulou, 2003); Driscoll (1997)), suggests that while some students are able to make the conversion from features of a symbolic inscription to features of the graphical representation of the symbolic inscription there is greater difficulty in making the link from features of the graphical representation of symbolic inscription back to features of the original symbolic representation. For example, Todd in reasoning about Task 3 (Solve for x: $x^2 + x + 1 > 0$) made links determining that the graphical representation of $y = x^2 + x + 1$ was above the x-axis for all values of x, but then reasoned in the following manner about the solution set of $x^2 + x + 1 > 0$, "x is not a real number (143)…because if there were any real answers in this graph here, then it would pass the x [axis] at a point, at a real point"(146-148). Todd's reasoning suggested that he attended to a feature of the graphical representation (not passing through x-axis) and related the feature to there being no solution to the original problem instead of the feature that all points being above the *x*-axis implied that there would be a solution for every value of *x*.

The researcher has also provided the field with the term *productive link* to describe students' feature noticing-and-using that characterizes successful coordination of meaning between two conversions that results in valid reasoning about the meaning of original symbolic inscription in the context of the task. The term productive link is helpful to the research community because it provides a lens through which to examine the nature of students' feature noticing-and-using as they use multiple representations in their reasoning and can assist in describing the specific challenge students may have in using multiple representations in their reasoning. The students in this study who made more than one productive link (Newt, Ashley, Dan, Paul, Todd) were prospective mathematics teachers who had higher levels of algebra exposure, and/or stated during the interview that they preferred a linking strategy to reason about problems involving symbolic inscriptions.

The last few pages have provided a summary and discussion related to the first research question. Important findings related to the research question and the relationship of these findings to existing research has been discussed. In a few instances contributions to the research community, especially with respect to a linking strategy, have been described.

Feature Noticing-and-Using Across Levels of Algebra Exposure

> *Research Question 2: Across levels of algebra exposure what is the nature of the similarities and differences in students' feature noticing-and-using as they solve unfamiliar algebra problems based on familiar algebra concepts involving symbolic inscriptions?*

The researcher made several claims about the nature of similarities and differences in students' feature notice-and-using across different levels of algebra exposure. It is important to address the differences in the algebra exposure of these three levels as well as the nature of the students at each level who participated in the study. The students that participated in this study were enrolled in one of the three courses: a precalculus class at a large rural high school, an undergraduate Calculus II course at a large university, and undergraduate mathematics teaching methods course at the same university. Precalculus students who participated in the study were enrolled in classes taught by two different teachers. There was no background information gathered to determine the nature of their prior mathematics experiences nor their future plans. Calculus II students who participated in the study were enrolled in classes taught by one teacher, and it was assumed that their content major required at least two semesters of calculus. Prospective mathematics teachers who participated in the study were enrolled in a class taught by the same teacher and it was assumed that they had each made a career choice that required extensive coursework in mathematics as well as training in teaching mathematics. The claims with respect to levels of algebra exposure are not about the development of feature noticing-and-using, but about the feature noticing-and-using of those with more exposure to algebra. In other words, the claims speak to the difference in feature noticing-and-using of those students with more exposure to algebra as compared to those students with less expertise. The claims suggest characteristics of feature noticing-and-using that could be possibly attained by those with less algebra exposure and not a development in feature noticing-and-using.

Findings related to this research question are summarized:

- The feature noticing-and-using of students with less exposure to algebra was related to a manipulative strategy to reasoning.
- Students with most exposure more frequently noticed features and used those features to reason in a manner that was indicative of a relational strategy.
- Students' current mathematical experiences seem to play a role in the nature of their feature noticing-and-using.
- The nature of the feature noticing-and-using of those students with more algebra exposure more frequently enabled those students to reason from different strategies to confirm reasoning or fill in gaps in reasoning.

In order to avoid redundancy, the discussion over the next few pages will focus only on aspects that were not addressed earlier in the discussion of the first research question.

There did not seem to be noticeable differences between the feature noticing-and-using of those at the two lower levels of algebra exposure. The most noticeable difference in the feature noticing-and-using of those at the two lower levels of algebra exposure involved the noticed features on Task 8 (Find the real values of x such that $4^{2x^2-7x+3} = 8^{x^2-x-6}$). Students enrolled in Calculus II, the middle level of algebra exposure, predominantly noticed a feature (exponential expressions on each side of symbolic inscription) that was used to reason about a procedure that involved applying properties of logarithms. Those students at the lowest level of algebra exposure, as well as the highest level of algebra exposure, noticed the bases of the two exponential expressions were the same and used the feature to reason about a procedure that involved rewriting the bases as powers of two. It does suggest that students' most recent mathematical experience may play a role in the features they notice.

The researcher's initial hypothesis was that exposure to calculus may impact the nature of students' feature noticing-and-using because the course content involved extensive work with symbolic inscriptions and analysis of graphical representations of those symbolic inscriptions. Evidence from this study suggests that Calculus II students' and precalculus students' feature noticing-and-using were similar in nature. It is the researcher's conjecture that for many students their experiences in calculus courses are extensions of tasks and techniques that were learned in earlier algebra courses. That is, in calculus courses the technique for students working on tasks involving symbolic inscriptions is to notice features of the symbolic inscriptions and to use those features to perform procedures related to the prompt and form of the symbolic inscriptions. Still living in a manipulative world the calculus students' task prompt has changed from "solve" to "differentiate" and "integrate", but the nature of the technique has not changed. Students are still being asked to perform procedures related to the form of the symbolic inscription.

The main difference in students' feature noticing-and-using occurred between those at the highest level of algebra exposure and most of the students at the other two levels of algebra exposure. Data from this study identified those with the most algebra exposure (Ashley, Casey, Dan, Newt) and a student at the next highest level of exposure

(Paul) had a feature noticing-and-using expertise that was not seen in other students. Consistent with how others have defined expertise (Chi, Feltovich, & Glaser, 1981; NRC, 2001) the feature noticing-and-using involved an understanding of the objects and relationships represented by the symbolic inscriptions and an ability to use those features to reason about tasks. The feature noticing-and-using of those at the highest level of algebra exposure suggested that they recognized that applications of properties of equality and properties of inequality on symbolic entities in the same register led to new symbolic inscriptions that preserved the equivalence relationship between the inscriptions. Students at the highest levels of algebra exposure (Ashley, Casey, Dan, Newt,) were more likely to reason from multiple strategies. As previously stated, the nature of this purposeful movement in students' reasoning was (a) to attend to recognized gaps in reasoning, (b) to confirm results of earlier reasoning, and (c) to compensate for errors in reasoning that caused cognitive conflicts. This suggested a robustness to their feature noticing-and-using that was not present in the feature noticing-and-using at the two lower levels of algebra exposure.

It is somewhat expected that students at the highest level of algebra exposure would have some expertise in feature noticing-and-using since they had chosen a career path that is focused on teaching others mathematics and have an extensive mathematics background. It is encouraging that those who have chosen this career path do have feature noticing-and-using expertise. Since it can be argued, being careful not to overgeneralize, from the results of this study that many students may leave high school with limited feature noticing-and-using capacities it is encouraging that there are prospective teachers who have potential to impart the same feature noticing-and-using as they have on their own students.

Results from this study suggest students' reasoning at the highest level of algebra exposure is different than that of most of the students at the two lower levels. Students at the highest level of algebra exposure are more likely to reason from both relational and linking strategies.

Adjustment to Feature Noticing-and-Using Taxonomy

> *Research Question 3: What is a taxonomy that describes the nature of feature noticing-and-using as evidenced in students' reasoning about symbolically presented unfamiliar algebra problems that are based on familiar algebra concepts?*

One of the outcomes of the study is a refinement of a taxonomy for feature noticing-and-using a refinement that is organized around the three different reasoning strategies. The development of the initial feature noticing-and-using taxonomy was a synthesis of others' descriptions of symbol sense activities (Arcavi, 1994; Driscoll, 1999; Fey, 1990; Kieran, 2007; Kinzel, 2001; Zorn, 2002) coupled with anecdotal evidence from the author's experiences with teaching secondary mathematics students. Rationale for adjustments to the taxonomy and the accompanying reconceptualization of the taxonomy are discussed in detail in Chapter 5. The purpose of the current discussion is to provide a rationale for why a taxonomy is needed.

Krutetskii (1976) argued that any genuinely scientific approach to the study of a complex phenomenon requires an analysis of its structure and an isolation of its components. As results from this study suggest, there is complexity in the nature of students' feature noticing-and-using in the context of solving problems involving symbolic inscriptions. The feature noticing-and-using taxonomy is the researcher's attempt, as Krutetskii recommends, to analyze the structure of students' feature noticing-and-using and to describe its different components. Existing frameworks (Kenney, 2008; Pierce & Stacey, 2001), or taxonomies, seem to capture the nature of features of symbolic inscriptions that are noticed by students, but they do not capture how these features are used in students' reasoning. The taxonomy arising from this study captures the complexities of the nature of the features of symbolic inscriptions noticed by students and how they use these features to reason about problems involving symbolic inscriptions.

Expanding Current Research

There are several ways this study expands current research. In the past, research has identified the difficulties and misconceptions have as they reason about symbolic inscriptions. This research can talk in broader (widely applicable) and more general terms (with examples) regarding the nature of how students notice and use features of symbolic inscriptions in their reasoning as opposed to focusing only on errors and limitations. Another contribution is that the current study offers a way to distinguish three types of symbolic related strategies in which features are noticed to reason procedurally, conceptually, and with multiple representations. This seems useful in thinking about the challenges students faced in high school algebra courses, college calculus courses, and other courses that involve reasoning about symbolic inscriptions. One could claim that a manipulative strategy only gives a choice, but a relational strategy and a linking strategy give an opportunity for something more.

The feature noticing-and-using taxonomy provides the research community with a way to classify the features of symbolic inscriptions that students notice and how they use these features to reason about problems involving symbolic inscriptions. The feature noticing-and-using taxonomy captures, in broad terms, the nature of students' feature noticing-and-using in relationship to three different strategies. Organized by the different strategies to students' reasoning, manipulative, relational, and linking, the instances of feature noticing-and-using, speak to the different meanings students attach to features of symbolic inscriptions and the actions they take to reason about problems involving symbolic inscriptions. The Venn diagram aspect of the feature noticing-and-using taxonomy captures the connectedness of the different strategies students use in their reasoning.

Conclusions

Over the next few pages the results of the study are examined in light of implications of this study for the learning and teaching of symbolic inscriptions. Suggestions for future research and limitations of the study are also discussed.

Implications for Teaching

Findings from this study seem to have three important implications for teaching—each of which is related to a specific strategy. These implications are discussed briefly and a set of questions that teachers should ask themselves as they teach content involving symbolic inscriptions is provided to assist them in examining students' feature noticing-and-using.

Manipulative strategy. Based on an analysis of students' feature noticing-and-using it is clear that even after years of algebra experience students have difficulties in reasoning about the procedures that they apply. Symbolic manipulation, the dominant technique, is an important aspect of students' feature noticing-and-using. Based on the results of this study, in spite of a great deal of attention in this area, students seem to have noticeable gaps in their understandings as it relates to their algebraic manipulations.

Students at a range of levels seem not to attend to the conditions under which a procedure can be applied. A hypothesis of the researcher is that the focus of students' attention is on performing the procedure and not necessarily on understanding the conditions under which the procedure could be applied. For example, students may very well have a procedure for factoring $x^2 + 5x + 6$, but may not notice feature(s) of related symbolic inscriptions that could be used to reason about factoring $x^6 + 5x^3 + 6$, or solving $x^2 + 5x + 6 = 0$, or determining whether the variable expression $x + 2$ is a factor of $x^2 + 5x + 6$. Each example suggests a deeper meaning of factor that extends beyond the quadratic expression. An analogy that seems to apply is that of an old television or computer monitor on which the same image appears again and again and is eventually burned into the screen. Related to the factoring example, students notice features of quadratic expressions that they use to reason about the factoring procedure. Eventually the meaning of the quadratic expression is attached to a factoring procedure resulting in the student being unable to "step back" and reason about the structure of the inscription or the meaning of the concepts related to the features of the symbolic inscriptions. The only "feature" of the symbolic inscription the student sees is related to a procedure that is devoid of greater procedural meaning or conceptual meaning.

The following questions could aid the teacher in examining the nature of student's feature noticing-and-using in a manipulative strategy and possibly in informing instructional practice.

- What features of the symbolic inscription did the student use in reasoning about a particular procedure?
- Did students notice the important mathematical conditions that underpin the procedure? If not, to which conditions did they not attend?
- Did students notice the important structure conditions of the symbolic inscription that underpin the procedure? If not, to which conditions did they not attend?

Relational strategy. Evidence from this study suggests that students with a deeper understanding of symbolic inscriptions representing equations and inequalities have a relational view of equations. Students with the highest level of algebra exposure seem to have a relational perspective to their feature noticing-and-using. The nature of their feature noticing-and-using has characteristics that could be the focus of attention in beginning-algebra experiences. Specifically, students' feature noticing-and-using in a relational strategy suggests they view certain kinds of algebraically valid transformations of symbolic inscriptions as preserving the equivalence relationships between the new symbolic inscription and the original symbolic inscription. Also, they notice that variable expressions represent numbers and the inequalities and equations and they are able to use this to reason about relationships between numbers (e.g. $x^2 + x + 1 > 0$). These are both understandings that could be addressed in a high school setting. Important questions that could be used by the teacher to shed light on the nature of students' feature noticing-and-using, and possibly inform instructional practice are provided.

- What is the nature of the meaning students have attached to a particular symbolic inscription? Does the student notice the objects (concepts) represented by the symbolic inscription? Does the student use these meanings to reason about the relationship expressed by the symbolic inscriptions?

- Does the student hold an operational and/or relational view of equations?
- What properties of the mathematical object (concept) represented by the symbolic entities does the student describe? Are these descriptions based on definitions or on a different meaning? Is the student able to use these descriptions in their reasoning?

Linking strategy. Evidence from this study suggests that students' feature noticing-and-using that is typical of a linking strategy are better able to address gaps in reasoning and confirm reasoning. Productive links are indicators of students who are more successful in reasoning about tasks involving symbolic inscriptions. The current research provides a lens to use in examining carefully the nature of students' feature noticing-and-using across different registers. Important questions that could be used by the teacher to shed light on the nature of students' feature noticing-and-using, and possibly guide instruction are provided.

- What is the nature of students' links? What is the nature of the meaning of the mathematical objects that underpin these links?
    - Are the links based on definitions of mathematical objects?
    - Are the links based on general forms (e.g., "U" shaped) of the representations of those objects or specific properties (e.g., end behavior) of representations of those objects?
- If a student is unable to make a productive link, what is the specific issue?
    - Is there a lack of or incorrect coordination of meaning from features of the source register to features of the target register?

- Is there a lack of or incorrect coordination of meaning from features of the target register to features of the source register?

As the results from the study suggest, there is complexity in understanding the nature of students' feature noticing-and-using as it relates to symbolic inscriptions. The questions are a means to examine the current nature of students' feature noticing-and-using and a means to think about planning instruction to advance students' feature noticing-and-using capabilities.

Suggestions for Future Research

The current research examined the nature of students' feature noticing-and-using by a small number of students on a small set of tasks involving symbolic inscriptions. A larger scale study examining this nature would possibly provide more information. Although the intent of study was to understand the nature of students' feature noticing-and-using, the possibility exists that there are instances of students' feature noticing-and-using about these same tasks that would shed more light on the feature noticing-and-using.

Gray and Tall (2001) suggest a more sophisticated understanding is an object-level understanding as opposed to a process-level understanding of a mathematical object. This more sophisticated strategy would suggest feature noticing-and-using that is typical of a relational strategy. Specific to this study, this idea relates to an operational versus relational view of equations. Most recent research in this area (Knuth et al., 2006; McNeil & Alibali, 2005) has either involved less experienced algebra students or symbolic inscriptions that did not involve tasks that were typical of those faced by more advanced algebra students. There does not seem to be much known about moving students from an operational to a relational view of equations in a high school algebra setting. A teaching experiment with the focus of understanding the nature of the students' development of a relational view in an algebraic setting over several years would seem to be of great importance to the field.

In a similar manner, Kieran (2007) argued there is a limited understanding in the research community of how students learn to make links between graphical and symbolic representations. Since this was a problematic area for many students across levels of algebra exposure it would be an interesting question to incorporate into future research. Findings from this research suggest there are several challenges in students' ability to make productive links. It would seem of interest to the research community to examine the developmental nature of a linking strategy.

In both relational and linking strategies the focus of attention is on the objects (concepts) represented by the symbolic inscriptions. Evidence from this study suggests that students, even those with calculus experience, have difficulty reasoning from these strategies. Studies that focused on developing students' feature noticing-and-using so that it would be more representative of a relational and linking strategy would seem to be a worthwhile endeavor.

Limitations of the Study

The limitations of this study are common to qualitative research studies (Creswell, 1998). The reasoning of fewer than 20 students was examined in this study. Little was known about their mathematics experience leading up to the study. Although preliminary questions were asked, it is not possible to determine whether the feature noticing-and-using of students interviewed at each level of algebra exposure was typical of students' feature noticing-and-using at each level of exposure. The researcher had difficulty soliciting volunteers to participate in the study at the two lower levels of algebra exposure so there was not an opportunity to be selective about those who agreed to participate. Also, in the interview setting students were asked to solve a small number of problems that were limited in scope. The students who were interviewed about experiences with reasoning about symbolic inscriptions were given tasks that were limited to equations and inequalities. It can be argued that tasks involving symbolic inscriptions that represent equations and inequalities are reasonable because these are the main symbolic inscriptions that students reason about in a second-year algebra course. Results from the study suggested that students' feature noticing-and-using at the highest level of algebra exposure was different than the feature noticing-and-using of those students at the two lower levels. The question remains whether this difference was the result of advancements in understandings or whether those students with more algebra exposure and choose a career path of becoming a secondary mathematics teacher have different characteristics than those at the two lower levels of algebra exposure. As a result, no claims were made about changes or development in students' feature noticing-and-using across levels of algebra exposure. Claims that were made across levels of algebra exposure were merely about the nature of students' feature noticing-and-using at one level of algebra exposure in comparison to another level.

Summary

Although the study involved a limited number of students, a limitation inherent in qualitative research, the purpose of the study was to characterize the nature of students' feature noticing-and-using in the context of problems involving symbolic inscriptions.

The current study provides the research community with a characterization of the nature of students' feature noticing-and-using with examples, a description of how students' feature noticing-and-using is different across levels of algebra exposure, and a taxonomy, in broad terms, that enables one to classify the different aspects of students' feature noticing-and-using

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Appendix A

High School Recruitment Script

I am seeking volunteers to participate in a research study. I am interested in learning more about what students reason about as they solve problems involving algebraic symbols. The research is part of my dissertation research work through Penn State. I am going to explain to you the study and try to answer any questions that you have. This study is meant to understand the nature of what students' reason about as they solve problems involving algebraic symbols.

If you agree to participate in this study you will be asked to participate in one-to-two 60-75 minute interview in which I will ask to you reason out loud as you solve problems involving algebraic symbols. Also, I would need you to allow me to audio and video record our conversations and interactions during the interview.

What I learn from these sessions will help me better understand the nature of how students reason about algebraic symbols. Since it is my desire to audiotape and videotape these interviews I will need your permission to do so. Please sign the form letting me know whether or not you are interested in participating in the study and whether or not you will you allow me to audiotape and videotape our interactions during the interview.

I will keep these recordings in a locked file cabinet. The only people who will have access to these recordings will be myself, my advisor--Dr. Kathleen Heid, and my transcriptionist. [I am sending home with you a form that requires both you and your parents to sign. I would like you to read through and sign in the presence of your parents.

This form asks for permission for you to participate in the study as well as permission to videotape and audiotape the classroom sessions. Please have them read the form and sign where they feel it would be most appropriate.]

It is important that you know that you do not have to participate in the study. The participation is voluntary. Also, if you do decide to participate you may change your mind at any time and remove yourself from the study. It is important for me to tell you that I will not necessarily interview everyone who volunteers. If you do volunteer and are interviewed you will be compensated for your time with a $10 Starbucks gift card. Do you have any questions?

One copy of the consent form is for your records. [Please take this form home and review it, along with the cover letter, with your parents. I am now going to hand out the consent forms. Once you and your parents have read and signed the forms, please give these forms to your teacher.]

Appendix B

College Recruitment Script

My name is Patrick Sullivan and I am a doctoral student in the department of Curriculum and Instruction at Penn State University. I am seeking research volunteers to participate in a research study titled, “Characterizing Understanding of Symbolic Feature noticing-and-using Across Different Levels of Algebra Experience.” The research is part of my dissertation study through the department of Curriculum and Instruction at Penn State. I am interested in learning more about the features students notice as they solve algebraic problems. The research is designed to understand the nature of how people with different levels of algebra exposure reason about algebraic symbols. The research could help those involved in the teaching and learning of algebra by contributing to the knowledge and understanding of how students’ think about algebraic symbols.

I am looking for volunteers who have taken beyond CALC 141 and/or are enrolled in MTHED 412 courses at Penn State. As a research volunteer, you will be asked to participate in one or two, 60-80 minute interviews where you would be asked to reason about a series of tasks involving symbolic representations. You must be 18 years or older to participate. If you are not 18, please do not agree to participate. Your decision to be in this research is voluntary. Please be assured you do not have to participate unless you wish to do so. As an incentive for your participation, those who complete the study will receive a $10 gift card to Starbucks. If you have any questions please feel free to reach me at (XXX) XXX-XXXX.

I am now going to hand out the consent forms. Once you have read and signed the forms, please place the forms in the provided envelope and return them to the folder on your teachers' desk. Do you have any questions?

Appendix C

Letter to Parents

May 1, 2012

Parents:

With your permission I would like to ask your son/daughter to participate in research that I am conducting as part of my dissertation research at Penn State through the department of Curriculum and Instruction. I am conducting a study that seeks to better understand the nature of students' understanding as they solve tasks involving algebraic symbols.

I am seeking students who are currently in high school precalculus. Students involved in the research study will be asked to participate in an approximately one-hour interview in which I will ask students to solve algebra problems. The type of tasks they will be asked to solve will involve content he/she has been exposed to at some point in their mathematics classes. During the interview process, students will be asked to communicate their explanations to me. My role during the interview will be to ask clarifying questions to make sure I understand what they are thinking. Again, with your permission, I would like to audiotape and videotape the interview process. These recordings would be used to carefully analyze the student's thinking as he/she worked the interview tasks

I have attached two copies of parental consent form asking for permission to conduct the research, as well as permission to audiotape and videotape the sessions. Please review the form with your child and sign where you feel appropriate. Keep one copy for your records and return a copy to your child's precalculus teacher. Your child's participation is completely voluntary and will have no effect on grades. Also, if your child chooses to participate s/he can remove themselves from the study at any point during the study.

Please sign and date the permission form if you and your child agree to participate in the study and have them return it to their teacher by **Tuesday, May 29**. If your child, with your permission, chooses to participate in the study he/she will receive a $10 Starbucks gift card or $10 cash. If you have any questions please do not hesitate to call, XXX-XXX-XXXX or email me.

Sincerely,

Patrick Sullivan

Appendix D

Informed Content (Under 18)

Informed Consent Form for Social Science Research

The Pennsylvania State University

Title of Project:	Characterizing Understanding of Symbol Feature-Noticing Across Different Levels of Algebra Experience
Principal Investigator:	*Patrick Sullivan* *235 Madison Street* *State College, PA 16801* *pls166@psu.edu* *(XXX)XXX-XXXX*
Advisor:	*Dr. Kathleen Heid* *270 Chambers* *University Park, PA 16803* *mkh2@psu.edu* *(814) 865-2430*

1. **Purpose of the Study:** The purpose of this research is to understand the nature of how people with different levels of algebra experience think and act upon algebraic symbols when solving problems. Between 20 and 30 students will be involved in this study.

2. **Procedures to be followed:** You will be asked to complete a series of algebra tasks in an interview setting in which our interactions will be recorded with audio and video devices.

3. **Discomforts and Risks:** Participants may feel performance anxiety, or general anxiety when participating in the research, especially some might feel uncomfortable when being audio, videotaped, or asked to explain how they arrived at the answer.

4. **Benefits:** You will have the opportunity spend time thinking out loud about mathematical tasks.

5. **Duration/Time:** A maximum of two interview sessions each of which will be between 60 and 80 minutes.

6. **Statement of Confidentiality:** Your participation in this research is confidential. The data will be stored and secured at the residence of Patrick Sullivan in a locked file cabinet. Code numbers and pseudonyms will be used when naming files and referring to data related to this study. The cameraperson during the interview and I will be the only people involved in the study who will have access to your interview. Transcribers and other research support person will only be given pseudonyms. The Pennsylvania State University's Office for Research Protections, the Institutional Review Board and the Office for Human Research Protections in the Department of Health and Human Services may review records related to this research study. In the event of a publication or presentation resulting from the research, no personally identifiable information will be shared. All records will be destroyed by June 15, 2021.

7. **Right to Ask Questions:** Please contact Patrick Sullivan at (XXX) XXX-XXXX with questions, complaints or concerns about this research. You can also call this number if you feel this study has harmed you. If you have any questions, concerns, problems about your rights as a research participant or would like to offer input, please contact The Pennsylvania State University's Office for Research Protections (ORP) at (814) 865-1775. The ORP cannot answer questions about research procedures. All questions about research procedures can only be answered by the research team

8. **Payment for participation:** The participant will receive a $10 Starbucks gift card for participating in the study.

9. **Voluntary Participation:** Your decision to be in this research is voluntary. You can stop at any time. You do not have to answer any questions you do not want to answer. Refusal to take part in or withdrawing from this study will involve no

penalty or loss of benefits you would receive otherwise.

Options for Use of Recording Devices

May the researcher use your audio records for future research? Circle two options:

1. I do not give permission for my recordings to be archived for future research projects. The records will be destroyed by June 15, 2021.

2. I do not give permission for my recordings to be archived for educational and training purposes. The records will be destroyed by June 15, 2021.

3. I give permission for my recordings to be archived for use in future research reports and publications.

4. I give permission for my recordings to be archived for educational and training purposes.

May the researcher use your video records for future research? Circle two options:

1. I do not give permission for my recordings to be archived for future research projects. The records will be destroyed by June 15, 2021.
2. I do not give permission for my recordings to be archived for educational and training purposes. The records will be destroyed by June 15, 2021.
3. I give permission for my recordings to be archived for use in future research reports and publications.
4. I give permission for my recordings to be archived for educational and training purposes.

If you agree to take part in this research study and the information outlined above, please sign your name, ask your child to sign; and return one countersigned copy to the yellow folder on your teacher's desk. Retain one countersigned copy for your future use. Please return the countersigned copy by **March 23, 2012**.

____ ____________________

PARENT CONSENT: Signature of Parent/Guardian Date

ASSENT: Teenagers age 13 and Older Signature Date

Signature of Principal Investigator/Person Obtaining Consent Date

Appendix E

Informed Content (Over 18)

Informed Consent Form for Social Science Research
Parent Permission and Child Assent Form
The Pennsylvania State University

Title of Project:	*Characterizing Understanding of Symbol Feature-Noticing Across Different Levels of Algebra Experience*
Principal Investigator:	*Patrick Sullivan* *235 Madison Street* *State College, PA 16801* *pls166@psu.edu* *(XXX)XXX-XXXX*
Advisor:	*Dr. Kathleen Heid* *270 Chambers* *University Park, PA 16803* *mkh2@psu.edu* *(814) 865-2430*

1. **Purpose of the Study:** The purpose of this research is to understand the nature of how people with different levels of algebra experience think and act upon algebraic symbols when solving problems. Between 20 and 30 students will be involved in this study.

2. **Procedures to be followed:** You will be asked to complete a series of algebra tasks in an interview setting in which our interactions will be recorded with audio and video devices.

3. **Discomforts and Risks:** Participants may feel performance anxiety, or general anxiety when participating in the research, especially some might feel uncomfortable when being audio, videotaped, or asked to explain how they arrived at the answer.

4. **Benefits:** You will have the opportunity spend time thinking out loud about mathematical tasks.

5. **Duration/Time:** A maximum of two interview sessions each of which will be between 60 and 80 minutes.

6. **Statement of Confidentiality:** Your participation in this research is confidential. The data will be stored and secured at the residence of Patrick Sullivan in a locked file cabinet. Code numbers and pseudonyms will be used when naming files and referring to data related to this study. The cameraperson during the interview and I will be the only people involved in the study who will have access to your interview. Transcribers and other research support person will only be given pseudonyms. The Pennsylvania State University's Office for Research Protections, the Institutional Review Board and the Office for Human Research Protections in the Department of Health and Human Services may review records related to this research study. In the event of a publication or presentation resulting from the research, no personally identifiable information will be shared. All records will be destroyed by June 15, 2021.

7. **Right to Ask Questions:** Please contact Patrick Sullivan at (XXX) XXX-XXXX with questions, complaints or concerns about this research. You can also call this number if you feel this study has harmed you. If you have any questions, concerns, problems about your rights as a research participant or would like to offer input, please contact The Pennsylvania State University's Office for Research Protections (ORP) at (814) 865-1775. The ORP cannot answer questions about research procedures. All questions about research procedures can only be answered by the research team.

8. **Payment for participation:** You will receive a $10 Starbucks gift card for participating in the study.

9. **Voluntary Participation:** Your decision to be in this research is voluntary. You can stop at any time. You do not have to answer any questions you do not want to answer.

Refusal to take part in or withdrawing from this study will involve no penalty or loss of benefits you would receive otherwise.

Options for Use of Recording Devices

May the researcher use your audio records for future research? Circle two options:

1. I do not give permission for my recordings to be archived for future research projects. The records will be destroyed by June 15, 2021.

2. I do not give permission for my recordings to be archived for educational and training purposes. The records will be destroyed by June 15, 2021.

3. I give permission for my recordings to be archived for use in future research reports and publications.

4. I give permission for my recordings to be archived for educational and training purposes.

May the researcher use your video records for future research? Circle two options:

1. I do not give permission for my recordings to be archived for future research projects. The records will be destroyed by June 15, 2021.

2. I do not give permission for my recordings to be archived for educational and training purposes. The records will be destroyed by June 15, 2021.

3. I give permission for my recordings to be archived for use in future research reports and publications.

4. I give permission for my recordings to be archived for educational and training purposes

You must be 18 years of age or older to consent to take part in this research study. If you agree to take part in this research study and the information outlined above, please sign your name and indicate the date below.

You will be given a copy of this consent form for your records.

____ _____________

Participant Signature Date

____ _____________

Person Obtaining Consent Date

Contact Information

____

Participant's Name

____

Participant's Email Address

____

Participant's Phone Number

Appendix F

Panel Task Evaluation

	HS Teaching	Collegiate Mathematics	Prospective Teachers
Task #1			
Task Engagement	Yes		Yes
Novelty	No	No, worked on problems solving single variable in a given equation	No
Automatic Process	Should have automatic process. Very few will recognize it is simpler than it is seen.	Isolate variable on one side of equation.	Apply algorithm without looking at structure
Task #2a			
Task Engagement	Yes	Yes	Yes

Novelty	No, exercise like task #1	Dependent on success with task #1	No, parentheses attract too much attention
Automatic Process	Automatic process prevents students from seeing that it is far simpler than it seems. Students will eliminate parentheses		I think that most preservice teachers would use structure sense (the parentheses attract too much attention). I know that many of my college algebra students would have approached this type of problem algorithmically.
Task #2b			
Task Engagement	Yes	Yes	Yes
Novelty	No	No	No
Automatic Process	Automatic cross-multiply processes. Multiply by denominator and solve for x.	Factor a 2 from denominator and cancel expression 2x + 3. Issue would be reconciling resulting equation.	Begin by multiplying both sides by (4x + 6). May noticed problems can be solved without manipulation.
Task #3			
Task Engagement	Yes	Yes	Yes
Novelty	No, depends on experience with graphing calculators	No, if graphing calculator is provided.	Depends on experience with graphing calculators

Automatic Process		Without graphing option may get bogged down in symbol manipulation	May use graphs to solve the problem.
Task #4			
Task Engagement	Yes	Yes	Not sure students will understand what question is asking. Challenging, but accessible.
Novelty	Yes	Yes	
Automatic Process	Guess-and-test, evaluating for collections of k-values.	Substitute values for k. May also attempt to solve first equation for y.	Not sure if problem accessible to high school and calculus students. Can think of a few prospective teachers who would not know how to approach problem.
Task #5			
Task Engagement	Yes	Yes	
Novelty	Yes	Yes	
Automatic Process	Choose value for A Graphing Calculuator	Is it seen as a quadratic of the form $2x^2$-$2ax$+$a^2 - 1 = 0$?	

Task #6A			
Task Engagement	Yes	Yes	
Novelty	Yes	Yes	
Automatic Process	Not routine Graphing Calculator	Square both sides and solve symbolically Graphical methods	
Task #6B			
Task Engagement	Yes	Yes	Yes
Novelty	Yes	Yes	Yes
Automatic Process	Not routine	Graphical approach	
Task #6C			
Task Engagement	Yes		Prospective teachers struggle with use of parameters Difficult for precalculus
Novelty	Yes		Yes

Automatic Process			
Task #7A			
Task Engagement	Yes	Yes	Yes
Novelty	Yes	Trivial with graphing calculator	Yes
Automatic Process	Graphing calculators would trivialize problem	Expanding expression by multiplying out terms	Flexibility with different representations depends on background
Task #7B			
Task Engagement	Yes		
Novelty	Yes		
Automatic Process	Begin w/ x-intercepts		
Task #8			
Task Engagement	Yes	Yes—only accessible to some	Yes

Novelty	Depends	Yes	Depends Prospective teachers have worked on similar problems, but some struggled
Automatic Process	Find common bases Possibly use logarithms	Depends on whether not student decides to write the bases as powers of 2.	

Appendix G

Interview Schedule

My name is Patrick Sullivan and I really appreciate your taking the time to meet with me today. I am interested in learning about how you reason as you solve problems in mathematics. I am going to start with a few conversational questions about your prior experiences in mathematics, and then I will ask you to work on a few mathematics problems with me. The tasks that I will give you in the interview will involve content that is typically covered in high school Algebra II classrooms, but may be problems with which you are not familiar. These tasks are not being graded so you do not have to worry whether you get them right or wrong. In fact, I will not tell you whether you are right or wrong. What you do here and say here will be held in the strictest of confidence. I will be analyzing the data and sharing some of this data with my advisor, Dr. Kathleen Heid. Also, the transcriber will hear and see the recordings. Unless you have consented no one else will see or hear your interview.

Since I am interested in learning about how you think about mathematics, it would really help me if you would describe as well as you can what you are thinking or doing in your head. While you are working I will probably ask a lot of questions to make sure that I am understanding what you are saying. I am not asking questions because what you are saying is correct or incorrect. I just want to make sure I understand what you are saying. Please let me know if you do not understand any question I am asking.

You are welcome to use any of the materials in front of you. I have provided a ruler, graph paper and notebook paper to write. I would like for you to use the recordable pen to write all your thoughts on the notebook paper.

Introductory Interview Question

1. What technology have you used in your mathematics classes?
2. Could you tell me a little bit about your experiences with algebra?

Problem Solving Tasks

Commentary: Tasks were chosen from a range of sources with an eye toward the different aspects of symbol-feature noticing. Unless cited, the researcher created the tasks. For each task a brief commentary citing pertinent research is provided along with an identification of possible feature noticing-and-using aspects. These are followed by possible follow-up questions that may glean a better picture of the student's understanding of feature noticing-and-using.

Task 1

Solve for n: $1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$.

Commentary: The problem is from Hoch & Dreyfus (2004). In a study involving 11th grade students not one of the 92 students used structure sense to solve.

Task 2a

Solve for x: $\left(\frac{1}{4} - \frac{x}{x-1}\right) - x = 6 + \left(\frac{1}{4} - \frac{x}{x-1}\right)$

Commentary: The problem is from *Hoch & Dreyfus (2004). In their study students had the most success (25%) on this problem. Students who have success with Task 1 may see this task as an exercise. I am interested in whether the brackets make a difference in symbol feature noticing-and-using. In other words, do the parentheses provide the learner with a appropriate "chunk" (Janson, Marriot, & Yelland, 2003) to notice like terms that was not obvious in Task 1? I will not probe thinking on Task 2 if I feel the task is an exercise.*

Task 2b

Solve for x: $\frac{(2x+3)}{(4x+6)} = 2$.

Commentary: Arcavi (1994)

Task 3

Solve for *x:* $x^2 + x + 1 > 0$.

Commentary: This problem is from Sfard & Linchevski (1994). Not one of the students in her study resorted to a graphical approach. The study was done almost 20 years ago. I am curious whether subjects' link representations at any level. Using Duval's (2006) language the conversion is from a symbolic representation to a graphical representation.

Task 4

Is it true that the following system of linear equations: $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ has a solution for every value of k?

Commentary: This problem is from Sfard & Linchevski (1994). Findings from their study found the problem very difficult for students to comprehend. The problem itself deals with the notion of variable and parameter and can be thought about either algebraically or geometrically.

Task 5

For what values of a does the pair of equations

$$x^2 - y^2 = 0$$
$$(x - a)^2 + y^2 = 1$$

have either 0, 1, 2, 3, 4, 5, 6, 7, or 8 solutions?

Commentary: Arcavi (1994)

Task 6A

Solve for x: $|x + 1| = |x - 2|$

Commentary: Task set 6 is a compilation of problems that I developed to push my own thinking. Using Brown and Walter's problem-posing ideas I created problems with tweaks in notation and structure to see what feature noticing-and-using ideas would carry over through a series of problems.

Task 6B

Solve for x: $|x + 1| > |x + 2|$

Task 6C

Solve for x: $|x - a| = |x - b|$ for any real numbers a and b.

Task 6D

Solve for x: $|x + 1| + |x + 2| = |x + 3|$

Task 6E

Solve for x: $|f(x)| = g(x)|$

Task 7A

Describe the graph of the function rule $f(x) = (x – 2)^2(x + 2)^2$?

Commentary: Task set 7 is a compilation of problems that I developed to use in this study. Using Brown and Walter's problem posing ideas I created problems with tweaks in notations and structure to examine whether feature noticing-and-using ideas would carry through a series of problems.

Task 7B

Describe the graph of the function rule $g(x) = |(x-4)^2(x+2)^2|$?

Task 7C

Describe the graph of the function rule $h(x) = (x – 3)^3(x + 2)^2$

Task 7D

Describe the graph of the function rule $j(x) = (x + 2)^2 + (x - 2)^2$

Task 8

Find the real values of x such that $4^{2x^2 - 7x+3} = 8^{x^2 - x - 6}$

Appendix H

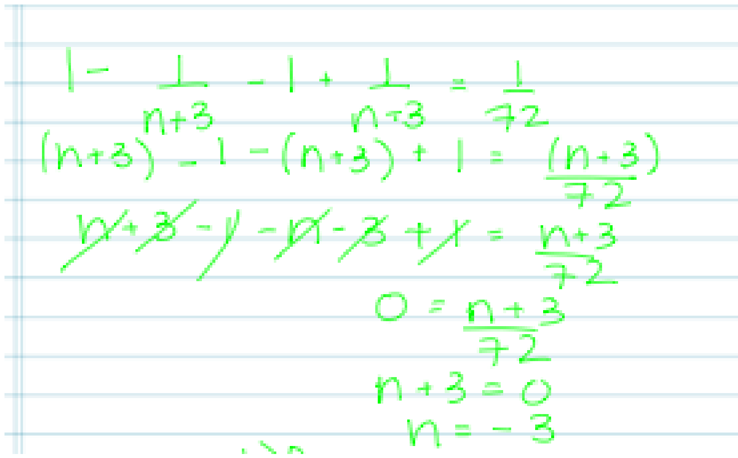
Task Summaries

Student	**Task 1**
Ashley	Ashley recognized which cued her to apply procedure to " get rid of denominators" (multiplied by 72(*n* + 3). Her procedure yielded answer of *n* = -3 which she substitutes back in and reasoned that "you can't divided by zero". Switched line of thinking and noticed two sets of additive inverse which results in $0 = \frac{1}{72}$. She interpreted this "as not true that the manipulations that you did on left will never give you the results on the right"

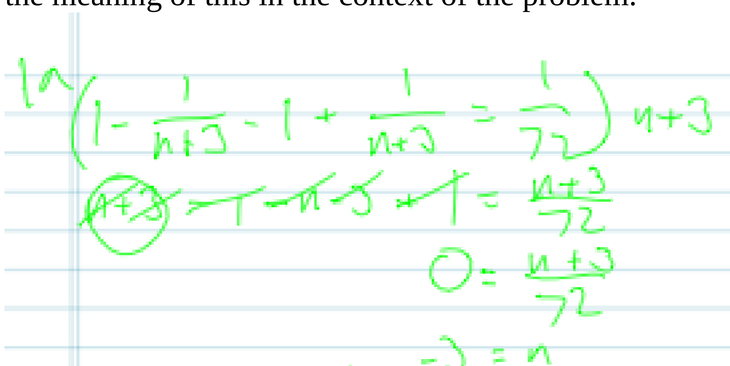
$$\left[1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\right] 72(n+3)$$
$$72(\cancel{n+3}) - \cancel{72} - 72(\cancel{n+3}) + \cancel{72} = n+3$$
$$0 = n+3$$
$$-3 = n$$

Casey	Casey began with a procedure (multiplied both sides by *n* + 3) but paused and noticed additive inverses. Her interpretation of $0 = \frac{1}{72}$ is" you wouldn't have any *n*'s left" which cued her to go through original procedure which results in *n* = -3. She plugged result into original equation and interpreted that term would be undefined—"zero in the denominator" which led her to interpret solution to problem as "no solution" because the one solution she found made denominator of a term undefined. Procedure confirmed her initial thinking.

Student **Task 1**



Dan — Dan recognized variable in denominator cuing multiplication of both sides by *n* + 3 which yielded answer of *n* = -3. He noticed substituting *n* = -3 into original equation will make terms one over zero, but he does not interpret the meaning of this in the context of the problem.



Newt — Newt, without writing anything down noticed common denominators and additive inverses. He Interpreted $0 = \frac{1}{72}$ as not a true statement and expounded on the meaning of not a true statement and interpreted this result in the context of the problem (no solution).

Jim — Jim began with a procedure (common denominator) that resulted in $\frac{0}{n+3} = \frac{1}{72}$. He interpreted this to mean, "there isn't an answer which I am pretty sure is wrong." He cross-multiplied at an intermediate step of first procedure which resulted in an answer of *n* = -3. He substituted the number into the original equation and reasoned that answer can't be *n* = -3 because "dividing by zero can't happen".

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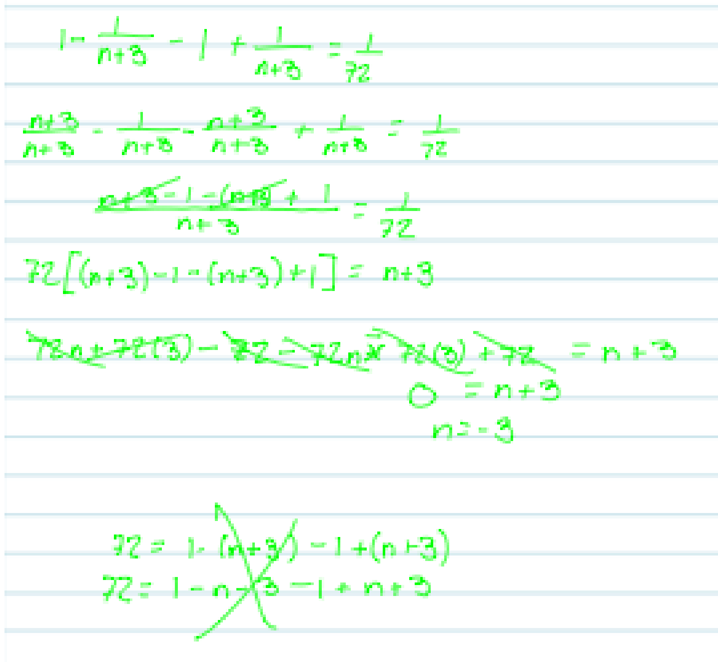
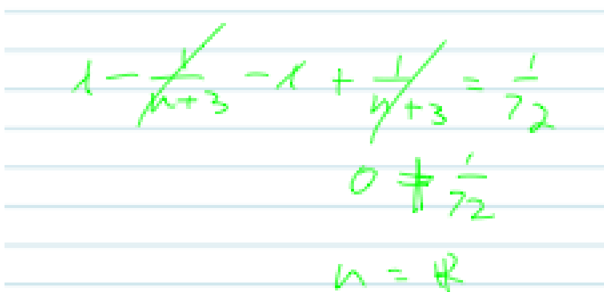
	
Paul	Paul, without written work, noticed the additive inverses—"noticed these two are opposites" and interpreted the meaning correctly, "you get an untrue statement, zero equals one over seventy two" and also interpreted in terms of *n*—"You can plug anything in for *n*…it still makes it equally untrue"
	
Nadia	Nadia, without written work, noticed the additive inverses, "these would cancel out" and interpreted results as "no solution". Not convinced she but invoked a procedure (adds -1 and 1 together and multiplies by n + 3). She stated her reason for the procedure as "I was thinking of like *n* have a value…realized I had to solve for *n*. She Substituted n = -3 back in and states that "one over zero is infinity I guess. …either I did it wrong or it doesn't have a solution"(98-102).

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$$\frac{-1}{n+3} = \frac{1}{72}$$

$$\left(\frac{1}{n+3} - \frac{1}{n+3}\right) = \left(\frac{1}{72}\right) n+3$$

$$1 - 1 = \frac{n+3}{72}$$

$$(0)\,72 = n+3$$

$$n = -3$$

Robin — Robin recognized equation not equal to zero (irrelevant structure)—“equal to one over seventy two…I would have thought I would have to solve for something, but then, like it would be equal to zero, because that’s how most problems are. She began a procedure that included an algebraic error (flipping over the sum of two rational expressions) that results in an incorrect answer. She did NOT interpret meaning of answer nor does she reason about its accuracy.

$$1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$$

$$\frac{1}{n+3} + \frac{1}{n+3} = \frac{1}{72} - \cancel{1+1} \rightarrow 0$$

$$72 = (n+3) + (n+3)$$

$$72 = 2n+6$$

$$\underline{-6} \qquad -6$$

$$\frac{66}{2} = \frac{2n}{2}$$

$$n = 33$$

Betsy — Betsy began with the additive inverse procedure which resulted in $0 = \frac{1}{72}$. She interpreted the result as something you can’t, which cued another procedure (multiply by common denominator)—a procedure that

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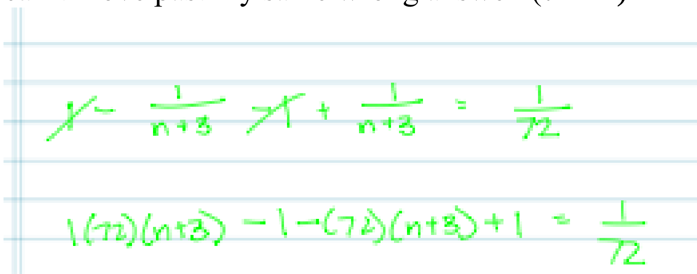
	yielded same results as the first procedure. Her interpretation of the results is that there is an invalid step in her procedures. She attempted a third procedure that led her to the same place. Her interpretation is about the accuracy of her answer, “I keep getting the same wrong answer, and I just can’t move past my same wrong answer”(72-74)



Molly .	Molly began with an additive inverse procedure but made a computational error that yielded a solution of $2 = \frac{1}{72}$. She interpreted the statement as the two numbers not being equivalent. She attempted another procedure that resulted in an answer of $\frac{0}{N+3} = -\frac{1}{72}$. She is not able to interpret the meaning of the result of the procedure or the meaning of this in the context of the problem.

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	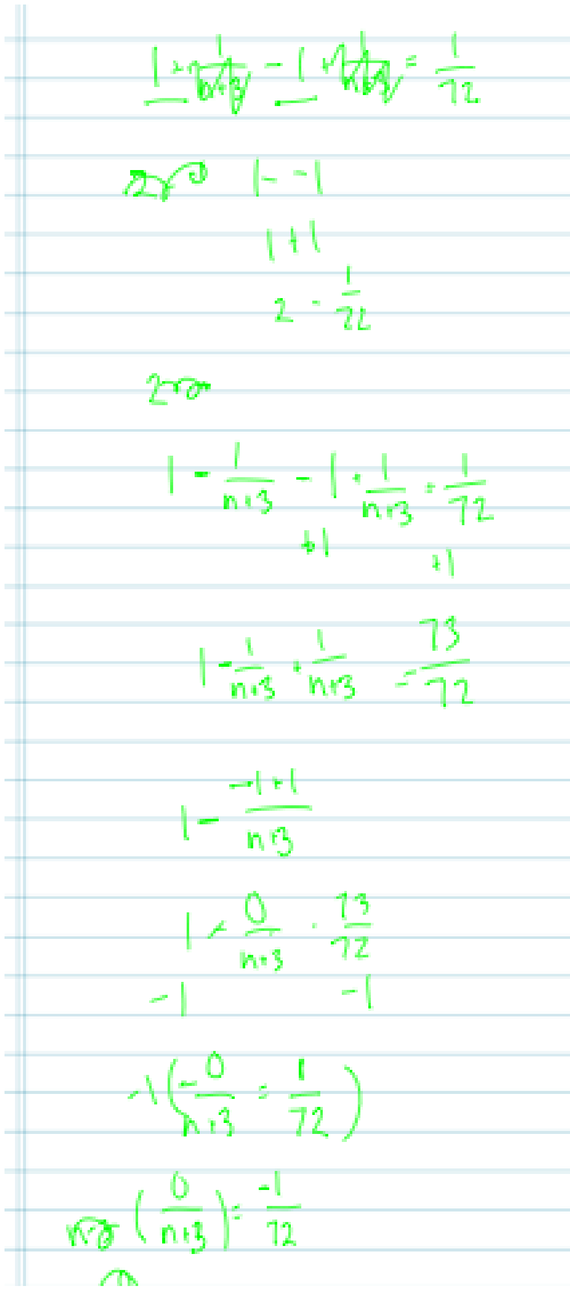
Roxie	Roxie began with a procedure (multiplying left side of equation by $n + 3$) She recognized left side would be 0 and her interpretation is that she can't solve for n if it doesn't exist. She attempted another procedure in the hopes of being able to solve for n. Her results are the same as the first procedure that she interpreted to mean she is unable to solve for n because it doesn't exist.

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$$1 - \frac{1}{(n+3)} - 1 + \frac{1}{(n+3)} = \frac{1}{72}$$

$$(n+3)\,1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$$

$$\cancel{(n+3) - 1 - (n+3) + 1} = \frac{1}{72}$$

$$(n+3)\cancel{-1} - (n+3)\cancel{+1} = \frac{1}{72}$$

$$0 = \frac{1}{72}$$

$$n = ?$$

$$\left(-\frac{1}{72}\right) 1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}\left(-\frac{1}{72}\right)$$

$$1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} - \frac{1}{72} = 0$$

$$\frac{\cancel{72(n+3)}\ \cancel{-72} - \cancel{72(n+3)}\ \cancel{+72} - (n+3)}{72(n+3)} = 0$$

$$-\frac{1}{72} = 0$$

Todd began with a procedure recognizing that -1 and 1 are additive inverses. His procedure resulted in an answer of *n* = -3 which led him to question the validity of a step in his procedure—"because the zero over N plus three, I think that should simplify to zero. I don't think you are allowed

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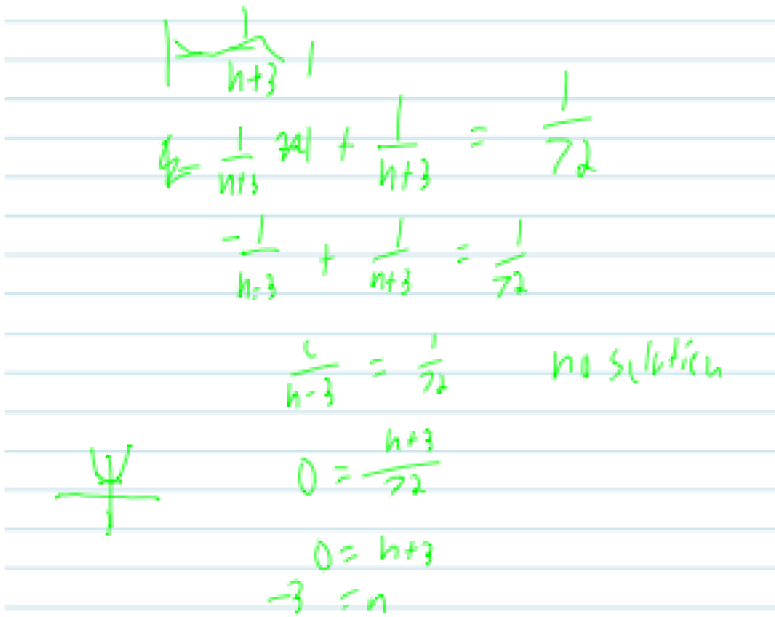
to multiply the more I think about it" (77-79).

He reasoned that there is no solution, but his interpretation is different than others, "if we were to graph this equation, where instead of N we used X is equal to Y, and with this it would not touch the X-axis"(56-58)



Student	**Task 2b**
Ashley	Ashley began with an algebraic procedure, multiplying both sides by 4x + 6. Her procedure resulted in the standard answer $x = -\frac{3}{2}$. She substituted the answer into original equation yielding an answer of $\frac{0}{0}$. She stated that she learned in Calculus 2 about some rule or some kind of problem with 0 over 0. She noticed that she can factor out the 2 which results in $\frac{1}{2} = 2$ and she stated this is not true. She stated it means you cannot find an *x* that makes the equation true.

$$\frac{(2x+3)}{(4x+6)} = 2 \qquad \frac{\cancel{(2x+3)}}{2\cancel{(2x+3)}} = 2$$

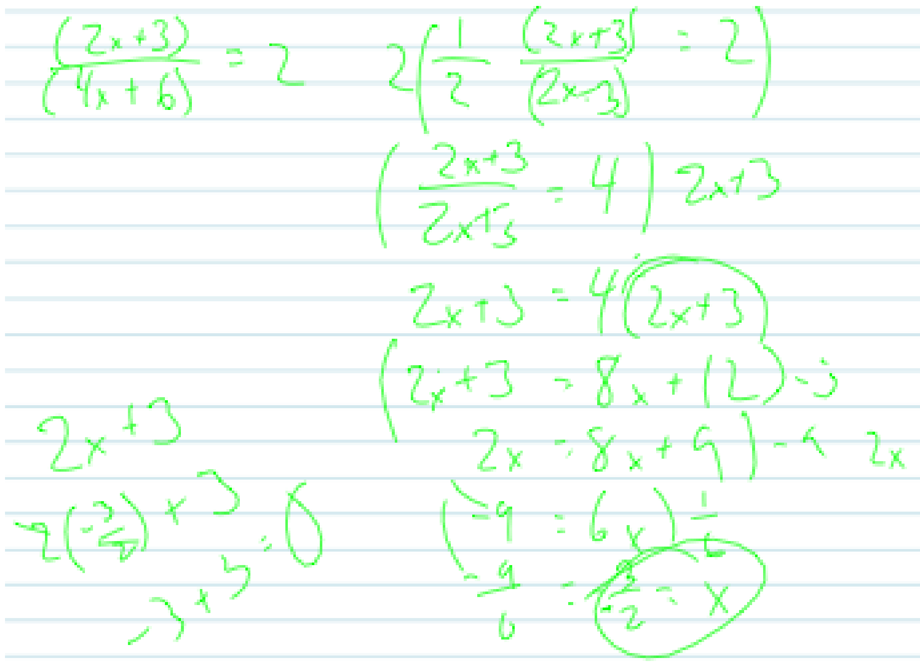
$$\frac{1}{2} = 2$$

$$\begin{aligned} (2x+3) &= 2(4x+6) \\ 2x+3 &= 8x+12 \\ -6x &= 9 \\ x &= -\frac{9}{6} \\ x &= -\frac{3}{2} \end{aligned}$$

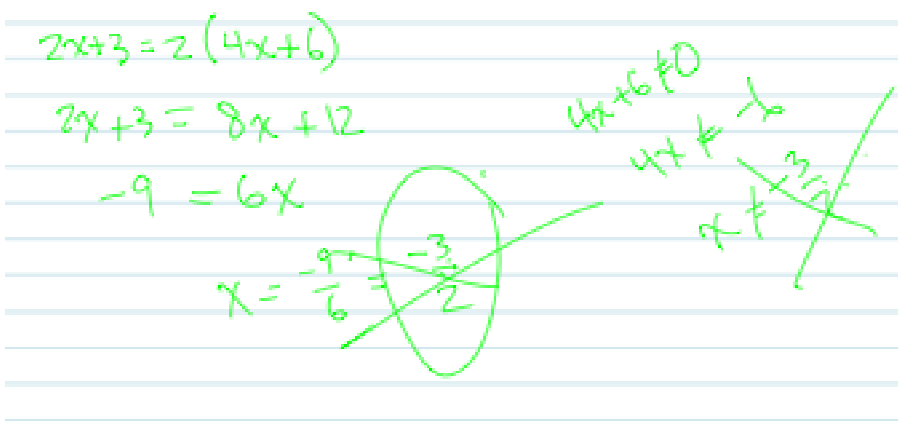
$$\frac{(-3+3)}{(-6+6)} = \frac{0}{0}$$

Casey	Did not ask Casey—ran out of time.
Dan	Dan began by factoring 2 out of the denominator. Unlike others he wrote the statement $\frac{1}{2}\frac{(2x+3)}{(2x+3)} = 2$. He multiplied both sides of the equation by 2. His procedure resulted in the standard answer. He stated that when he sees something like 2x + 3 = 4(2x + 3)—“something is the same as that same thing multiplied by four…only true as long as that something is zero”. He substituted his answer into the expression 2x + 3 = 0. He commented that same problem he had earlier comes up. That is, zero in the denominator. He stated I can understand two, but I cannot say that two equals a number that he does not understand.

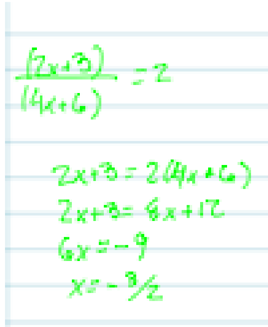
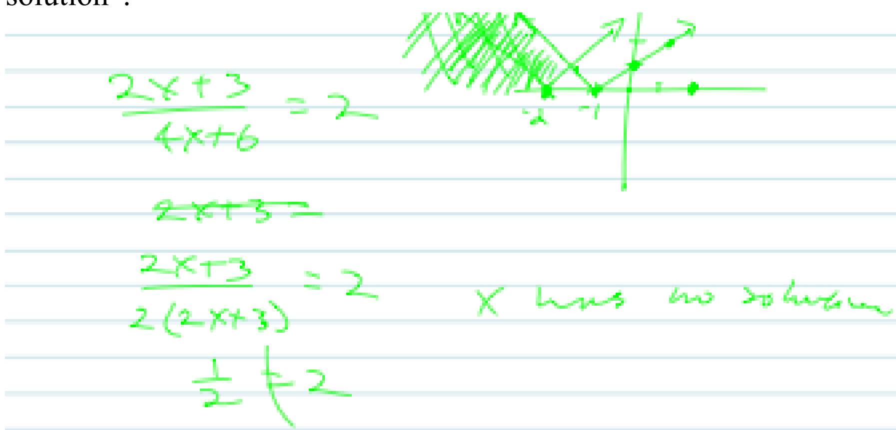
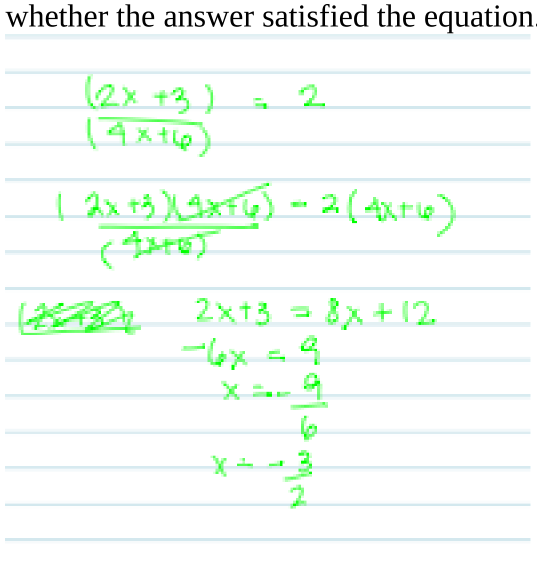
Student	**Task 2b**



Newt | Newt stated he has "linear looking stuff". He began with an algebraic procedure, multiplying both sides by 4x + 6. Her procedure resulted in the standard answer. He stated that 4x + 6 cannot be zero. He sets up the statement $4x + 6 \neq 0$ and solved for x, $x \neq -\frac{3}{2}$. He connected the two solutions and stated the answer is no solution noting that you cannot divide by zero which means nothing works.



Jim | Jim began with an algebraic procedure, multiplying both sides by 4x + 6. His procedure resulted in the standard answer, but he does not check to see the resulting answer is not in the solution set.

Student	**Task 2b**
	
Paul	Paul noticed the common factor in the denominator and factors. This yielded the statement $\frac{1}{2} \neq 2$ which he stated is "impossible" and "no solution".
	
Nadia	Nadia began with an algebraic procedure, multiplying by 4x + 6. She stated the procedure would just cross out the *x* in the denominator. Her procedure resulted in the standard answer. She does not check to see whether the answer satisfied the equation.
	
Robin	Robin began by classifying the two terms as binomials. She began

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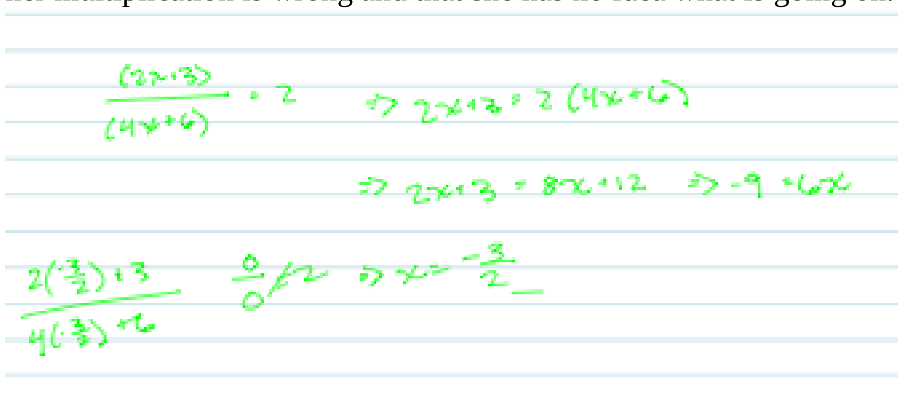
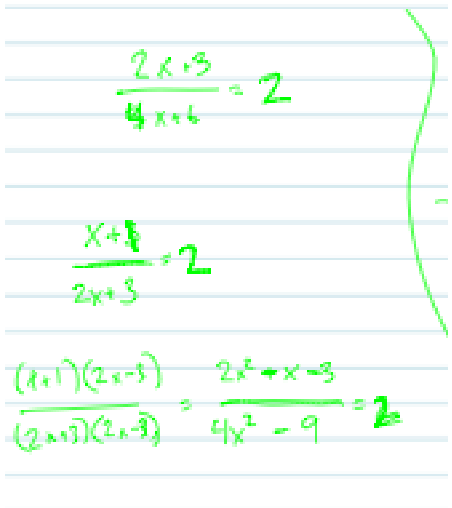
	with an algebraic procedure, factoring out the 2 and noticed they cancelled. She reasoned the resulting statement $\frac{1}{2} = 2$ is not true, not real, and a false equation. She started a substitution procedure (*x* = 1) to determine whether or not the equation will equal 2. She stated that it's "five over the big number, which makes, it because this is a bigger number on top, that it would always be a fraction, which would be less than one over two, so it would be less than two." Robin reasoned $\frac{1}{2} = 2$ is not true, but she did not see the inscription as representing a constant relationship for any value of *x*.
Betsy	Betsy began with an algebraic procedure, multiplying both sides by 4x + 6. Her procedure resulted in the standard answer. Betsy noticed the common factor in the denominator, but stated that if she cancelled it she would end up with 2 = 2 which is not true. She commented that the procedure would end up cancelling the *x*'s which meant she couldn't solve for *x* if she didn't have it. She also stated it would not get here anywhere. She substitutes the result of her answer to the first procedure into the original equation and while writing $\frac{0}{0} = 2$ she stated she gets "zero equals two" and stated it must be extraneous or maybe her multiplication is wrong and that she has no idea what is going on.



Molly	Molly begam with an invalid algebraic procedure, dividing first term of number and denominator by 2 and 2nd term of each by 3. She starts a second procedure, multiplying denominator by conjugate which she stated makes a bigger fraction. She stated she has no idea how to do the problem.

Student	**Task 2b**
	
Roxie	Roxie began with an algebraic procedure, multiplying both sides by 4x + 6 stating that she wanted to get rid of the denominators. She made a computational error which left her with the equation (4x + 6)(2x + 3)=2(4x + 6) as opposed to (2x + 3) = 2(4x + 6). This resulted in the equation $8x^2+16x+6=0$ which she factored into the resulting expression into 8x(x + 2) + 6 = 0. At this point she indicated she's not sure what to do next.

Student	**Task 2b**
	$\dfrac{(2x+3)}{(4x+6)} = 2$ $(4x+6)(2x+3) = 2(4x+6)$ $8x^2+12x+12x+18 = 8x+12$ $8x^2+12x+12x+18-8x-12=0$ $8x^2+16x+6=0$ $8x^2+16x = -6$ $8x(x+2) = -6$ $8x(x+2)+6=0$
Todd	Todd began with an algebraic procedure, multiplying both sides by 4x + 6. He stated the only way he knows how to solve an equation with *x* in the denominator is to move it from the denominator. His procedure resulted in the standard answer. After completing the next problem he asked to return to the problem and made a link between the initial equation (value for x that makes denominator equal to zero) and characteristic of the graph. He formed the statement $4x - 6 \neq 0$ and solved it. He connected the solution to both procedures and stated "no solution".

Student **Task 2b**

$$\frac{2x+3}{4x+6} = 2$$

$2x+3 = 2(4x+6)$

$2x+3 = 8x+12$

$3 = 6x+12$

$-9 = 6x$

$x = \frac{-9}{6}$

$x = -\frac{3}{2}$

$x \neq$

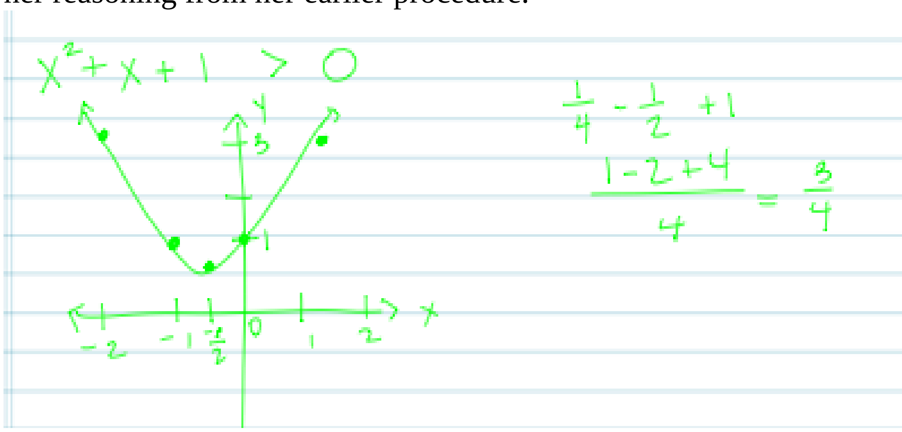
$4x+6 \neq 0$

$4x = -6$

$x \neq \frac{-6}{4}$

$x \neq \frac{-3}{2}$

no solution

Student	**Task 3**
Ashley	Ashley started with a factoring procedure and abandoned the strategy after realizing that factoring was not possible. She recognized the magnitude of $x^2 + x$ would always be positive and made a deductive generalization about positive numbers being part of the solution set. She substituted numbers [0, -1, and -2] for x in the expression $x^2 + x + 1$ and made an inductive generalization about negative numbers being part of the solution set. She began plotting points (0,1), (1,3), (-1, 3) found corresponding point to (1,3) via symmetry (-2,3) as well as the location of the vertex. She reasoned correctly from the shape of the graph about the solution set of the inequality. Her reasoning from her graphical procedure was consistent with her reasoning from her earlier procedure. 
Casey	Casey started with a factoring procedure followed by applying the quadratic formula to $x^2 + x + 1 = 0$. She did not discuss the meaning of the results of the quadratic formula procedure, but used the results of *x*, $x = \frac{-1\pm\sqrt{-3}}{2}$ to form two inequalities, $x > \frac{-1+\sqrt{-3}}{2}, x > \frac{-1-\sqrt{-3}}{2}$, and reasoned about solution set of the second inequality including the solution set of the first inequality. She explained what she meant with a number line. She started factoring II procedure, $x(x + 1) > -1$ and wrote two inequalities, $x > -1$ and $x + 1 > -1$ and solved each, $x > -1$ and $x > -2$ She substituted a number, $x = -2$, into the original inequality. The value, $x = -2$, fit the inequality, but it countered the results of her factoring II procedure $x > -1$ and $x > -2$. She noticed a feature, magnitude of $x^2 + x + 1$ that led to a deductive generalization about positive numbers being in the solution set and relied on one case, $x = -1$, to make an inductive generalization about negative numbers being in the solution set. In her explanation that all real numbers would be in the solution set she made distinctions in her statements between large and small negative numbers, zero, and large and small positive numbers. (Representational Orientation)

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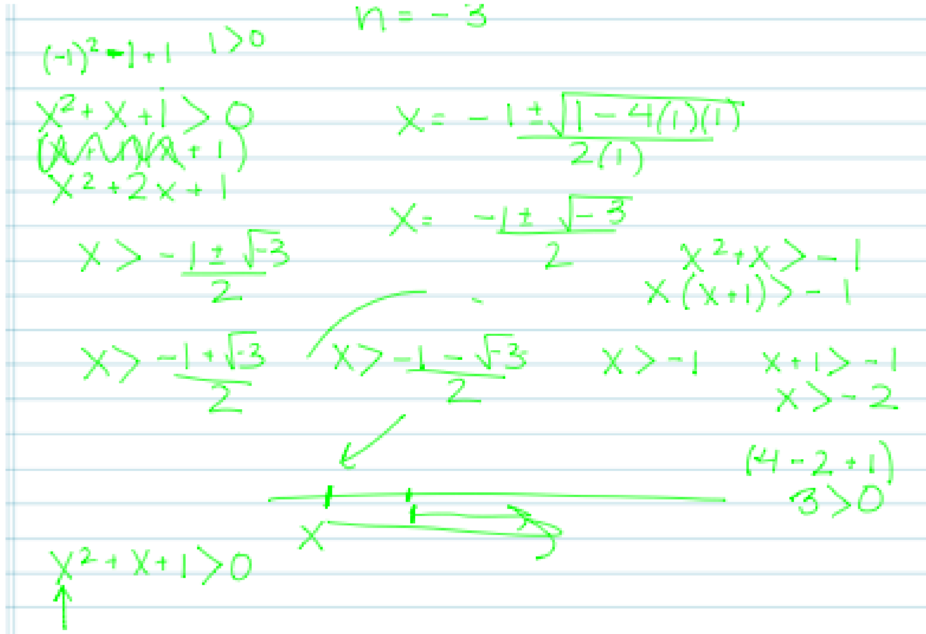
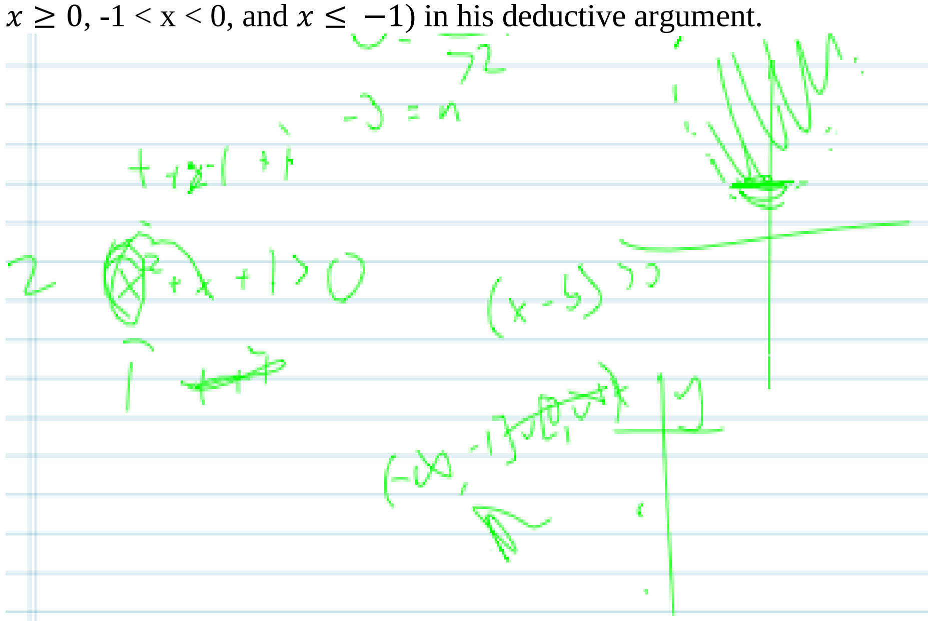


Dan — Dan recognized that the graph would be a parabola, but he constructed the parabola as a dotted curve because of the inequality symbol. Unlike many participants he did not apply the quadratic formula procedure. He began mentally substituting numbers into the expression (x = 1, 0, -1, 2). The procedure led to generalizations about the solution set of the inequality, all real numbers fit the inequality, differentiating between three intervals($x \geq 0$, -1 < x < 0, and $x \leq -1$) in his deductive argument.



Newt — Newt started with a factoring procedure followed applying the quadratic formula to $x^2 + x + 1 = 0$. He did not interpret the results of the quadratic formula procedure in the context of the problem. He attempted to place the x-values generated through the quadratic procedures on a number line, but

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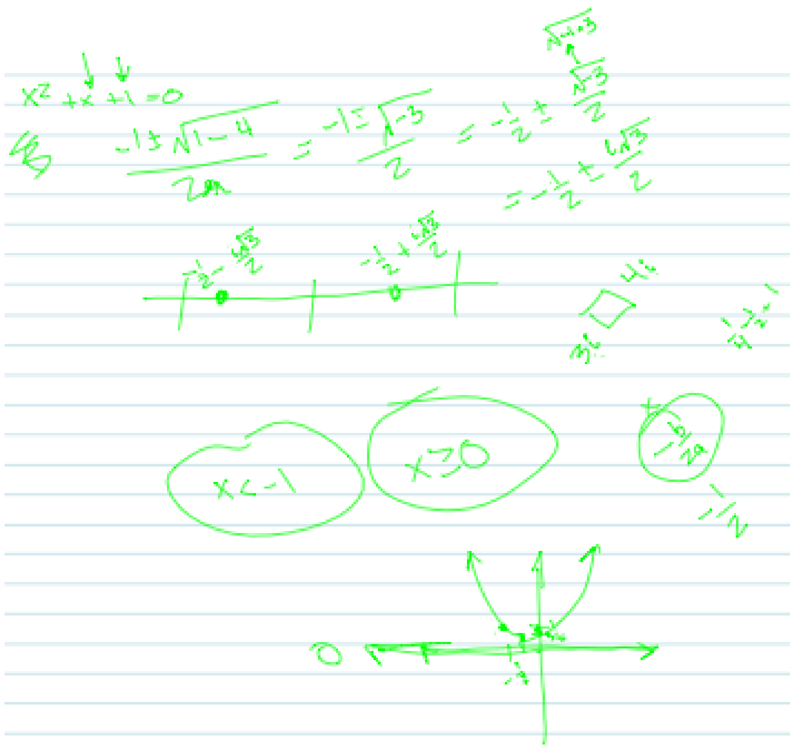
had difficulty trying to quantify complex numbers on a number line.(Manipulative-Representational-Manipulative-Relational) He started a substitution procedure [x = 0 and x = 1] into the expression $x^2 + x + 1$ and made a deductive statement about all positive numbers being in the solution set. In the middle of this strategy he started reasoning about the graph of $x^2 + x + 1 > 0$ determining the exact location of the vertex and the direction the parabola opened. He reasoned from these properties about the solution set of the inequality. (Relational) The substitution procedure seemed to be the springboard for the graphical procedure that seemed to refine his thinking about the solution set.



Jim

Jim started with a factoring procedure and explained why the procedure will not work. He followed with applying an incorrect quadratic formula, $\frac{-b\pm\sqrt{4ac-b^2}}{2ac}$ that resulted in the expression $\frac{-1\pm\sqrt{3}}{2}$. He placed the two expressions to a procedure into a factored form inequality, $\left[x + \left(\frac{-1+\sqrt{3}}{2}\right)\right]\left[x - \frac{1+\sqrt{3}}{2}\right] > 0$. He reasoned about the solution set of the inequality from the factored form stating the solution would be $x > \frac{1+\sqrt{3}}{2}$ because the solution set of the inequality $\left[x + \left(\frac{-1+\sqrt{3}}{2}\right)\right] > 0$ would be included in the inequality $\left[x + \left(\frac{-1+\sqrt{3}}{2}\right)\right] > 0$.

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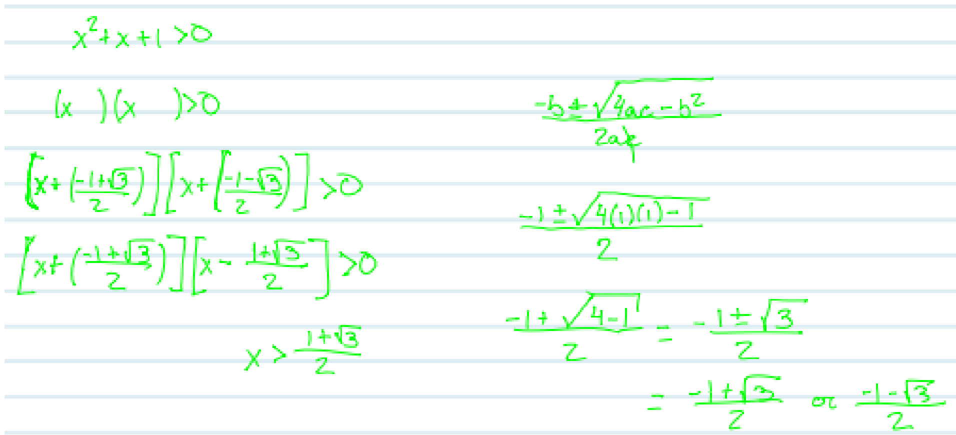
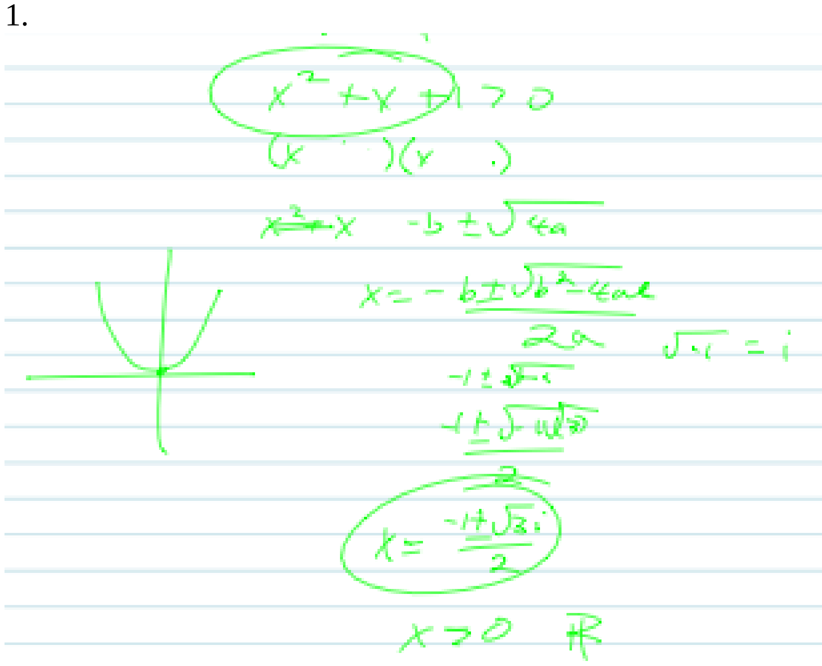
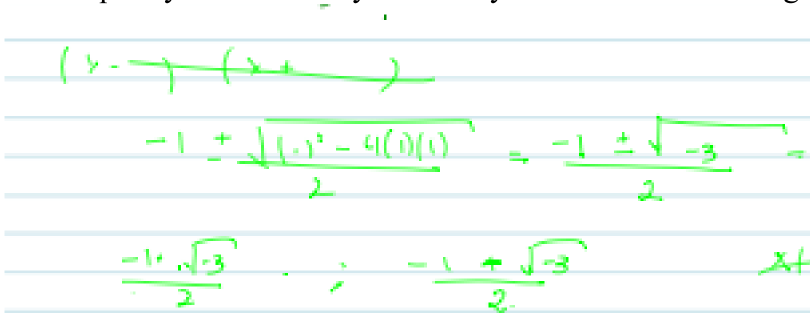


Paul

Paul began with a factoring procedure and explained why the procedure would not work. He performed a correct quadratic formula procedure (manipulative) and reasoned that since the roots are imaginary the graph would never touch the x-axis (relational) and the solution to the inequality would be all real numbers (representational) [Note: Roots that never touch the x-axis could also mean that all values of the expression could be below the x-axis which would mean there would be no solution.] He confirmed his reasoning about the solution set with a second procedure. He provided a deductive generalization about why both positive and negative numbers would be in the solution set. He does make the statement that "even with negative numbers, the x squared term is larger than the number itself", even if he meant that the squared numbers were greater than the absolute values of the number itself, the statement was not true for x values between 0 and 1.



Student	**Task 3**
Nadia	Nadia began with a factoring procedure and explained why it would not work. She followed applying the quadratic formula to $x^2 + x + 1 = 0$, but did not interpret the results this procedure in the context of the problem. She claimed that if the results were real numbers she would put them it into the equation to see if it would be greater than zero. She began factoring II procedure, *x(x + 1) > - 1* but stated the “zero property” would not work on the inequality because “only works if you have zero on the right side”. 
Robin	Robin began with a factoring procedure and explained that it will not work because it the expression is not a perfect square [Note: She explained what she means later and it is in line how others have explained why factoring procedure will not work]. She changed the inequality $x^2 + x + 1 > 0$ into $x^2 + x + 1 = 0$ and begins factoring II procedure, $x(x + 1) = -1$ that resulted in solutions of *x* = -1 and *x* = -2. She is prompted by researcher to think about the problem graphically. She plotted a few points (0,0), (1,3), (2,7) and traced a curved resembling the shape of a half-parabola. She interpreted the > symbol to mean the graph will not extend beyond y-axis.

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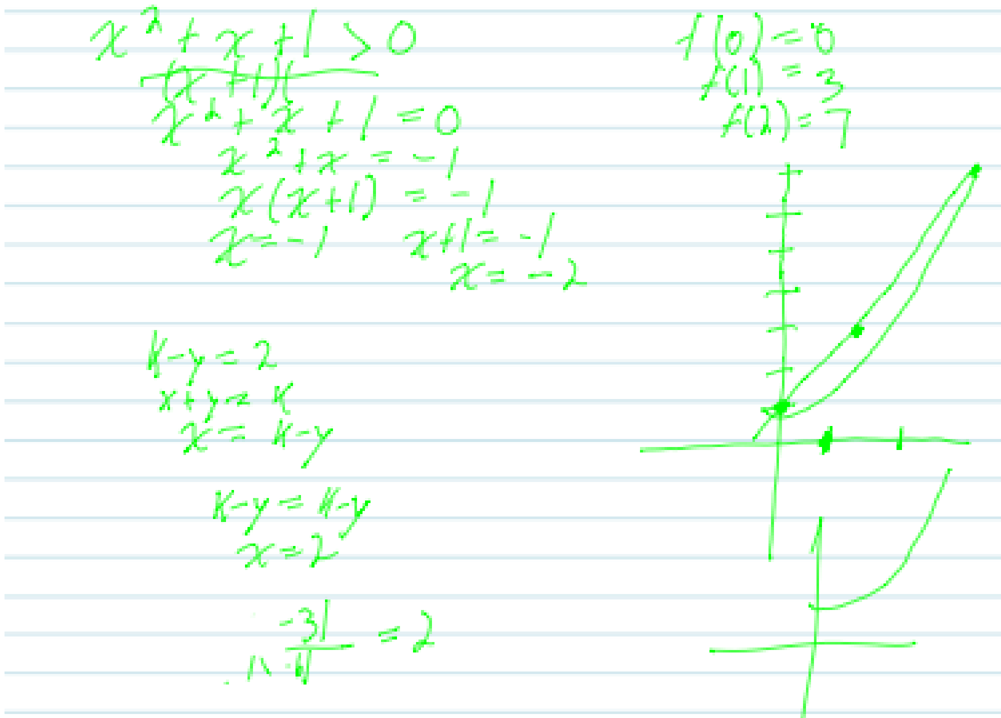
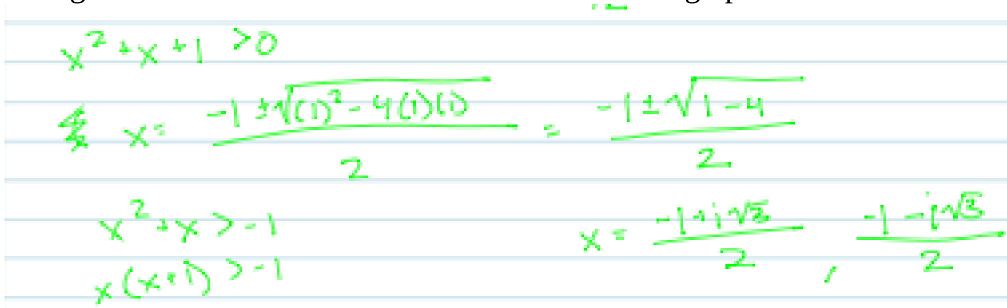


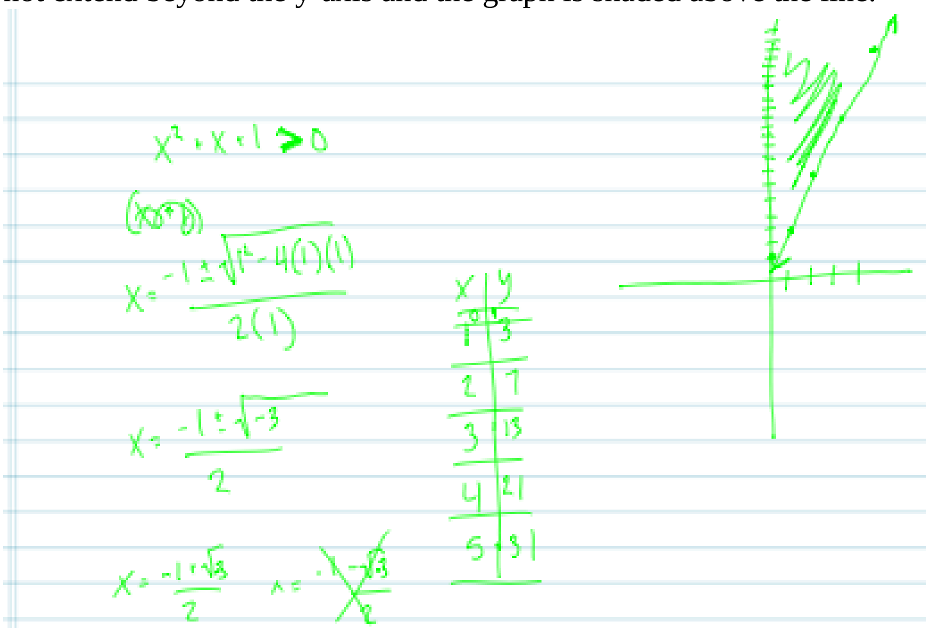
Betsy Betsy started with a factoring procedure and explained that is will not work because the requirements of the middle term cannot be met. She followed by applying the quadratic formula to $x^2 + x + 1 = 0$ and interpreted the result as "imaginary" which would mean the roots don't exist on number line. She later called the results of the quadratic procedure complex roots. She did not interpret the results of the quadratic formula procedure in the context of the problem. She abandoned the procedure and began factoring II procedure, $x(x + 1) > -1$, that she stated would not get her anywhere. She is prompted by the researcher to consider a graphical solution, but she is uncertain about doing it because $x^2 + x + 1 > 0$ does not have a Y in it. She explained that the graph would be a parabola because the expression is of degree two but she does not know how she would graph it.



Molly Molly started with a factoring procedure and stated that she is not sure if she can do it because she is not very good at it. She followed by applying

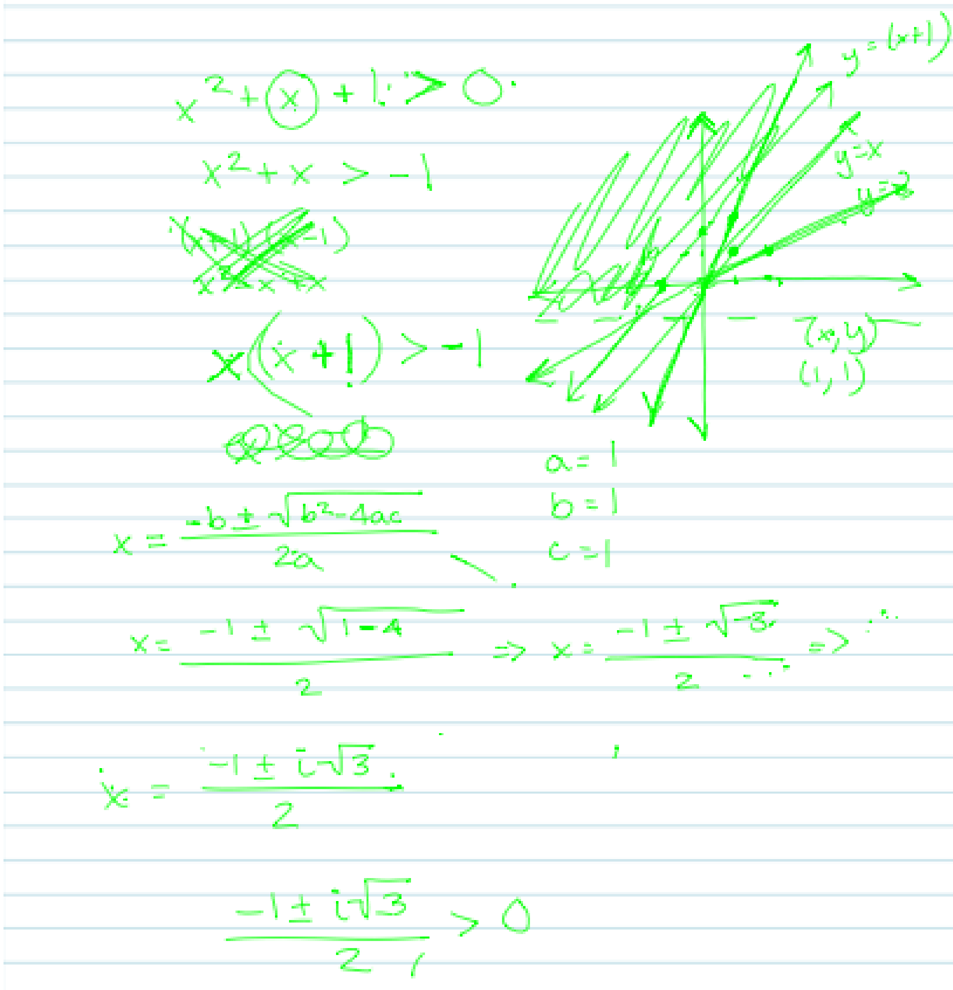
Student **Task 3**

the quadratic formula to
$x^2 + x + 1 = 0$. She did not correctly interpret the meaning of the results of the quadratic formula procedure [complex numbers]. She crossed out the solution $x = \frac{-1-\sqrt{-3}}{2}$ because "it would be negative" and the solution is supposed to be greater than zero (manipulative-representational). She is prompted by the researcher to consider a graphical strategy. She plotted several points [(1,3), (2,7), (3, 13), (4, 21)] and explained that the graph would be "linear". Her two interpretation of > symbol is the graph would not extend beyond the y-axis and the graph is shaded above the line.



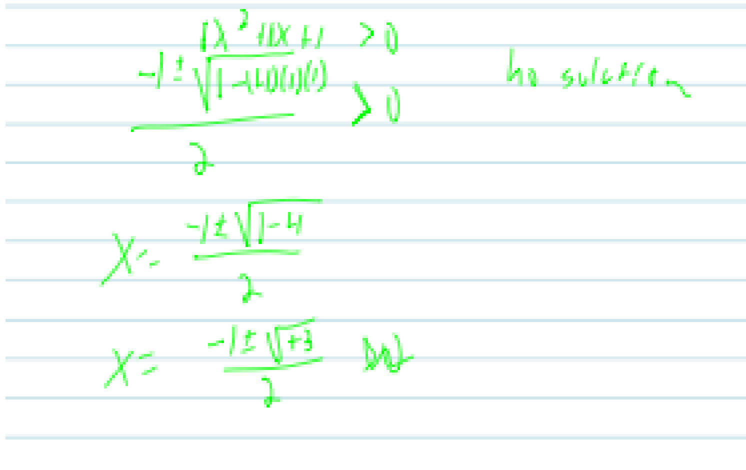
Roxie She started with a factoring procedure and explained for a specific example why it would not work. She followed with factoring II procedure, $x(x + 1) > -1$, but did not interpret the meaning of this in the results. She applied the quadratic formula to
$x^2 + x + 1 = 0$ but is unable to reason about the results of the procedure in the context of the problem (manipulative). She is prompted by the researcher to consider a graphical solution to the problem. From $x(x + 1) > -1$ she reasoned about the graphs of the two factors, x and x + 1, but she is unable to interpret the meaning of a factor of x and its impact on the graph of y = x + 1. She mentioned that none of her work with the graphical strategy has helped inform her about the solution to the problem.

Student **Task 3**

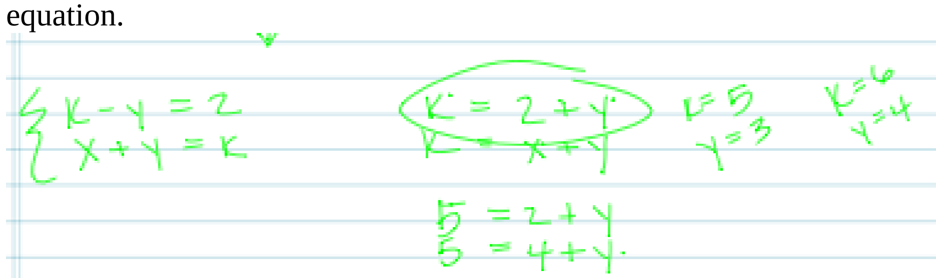


Todd Todd started with a factoring procedure and explained why it will not work. He followed by applying the quadratic formula to $x^2 + x + 1 = 0$ and linked the results of the procedure to a property of the graph, will not touch the x-axis (manipulative-representational-relational). He made a link between the original expression and another property of the graph, opens upward. From the two links and location of the y-intercept he reasoned that there would be no solution to the inequality because the graph never touches the x-axis. (relational)

Student **Task 3**

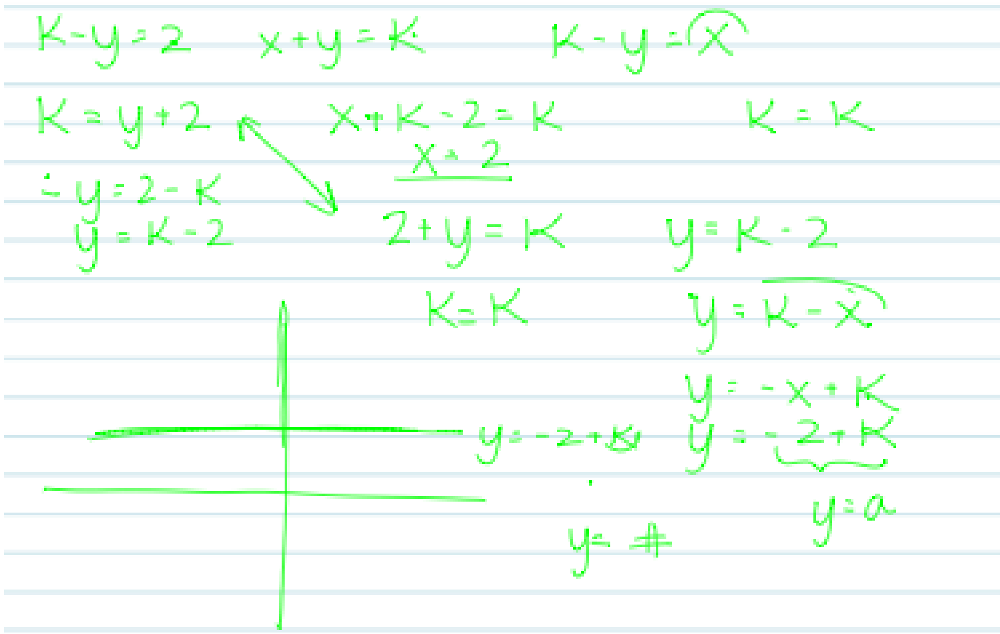


Student	**Task 4**
Ashley	Ashley solved both equations for *k* and in doing so she recognized that $x = 2$ made both equations the same. She seemed to have difficulty interpreting the question. She stated that she assumed $x = 2$ because it doesn't say that she can't and that if the question said for any *x* and *y* her thought process would be different. When questioned about whether *x* could be 4 she states it could not because doing so would not yield the same *y* value for each of the equations that she has solved for *k*. She reasoned that *x* has to be 2 and that no matter what *k* is the *y*'s would be same because they are the same equation.

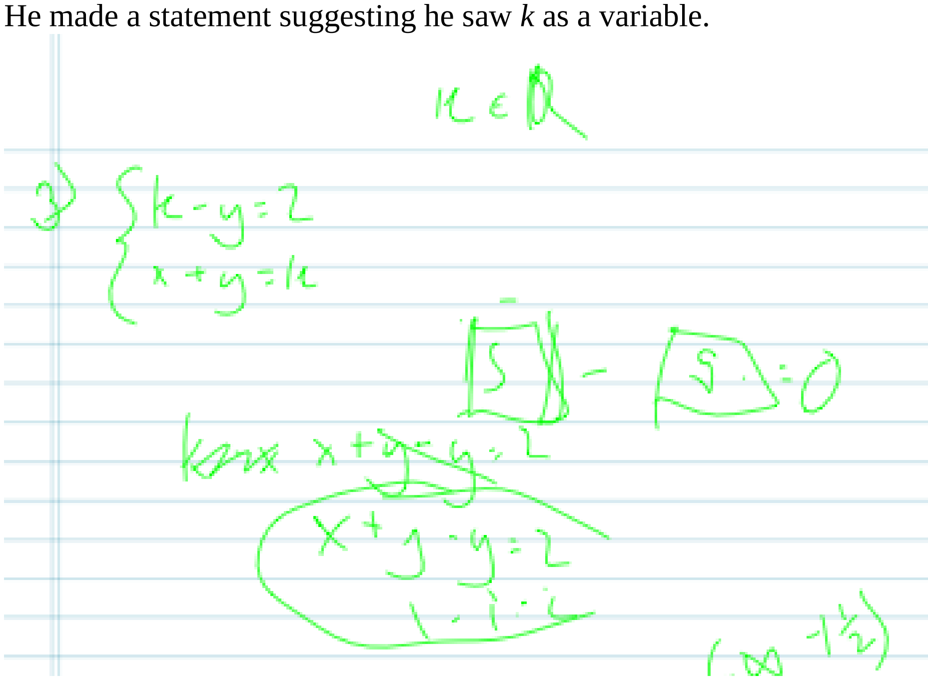


Casey	Casey solved the first equation for *k* and substituted for it in the second equation. This procedure yielded $x = 2$. She stated Her goal was to get $k = k$ which to her would mean infinite solutions. When she substituted 2 in for x she recognized the two equations are the same. She started another transforming procedure which resulted in the equation $k – y = x$ and reasoned that $x = 2$. She transformed $x + y = k$ into $y = -x + k$ and recognized the slope as -1 and the *y*-intercept as 2. She reasoned that since $x = 2$, $y = -2 + k$ would be a constant and linked to the graphical representation of a horizontal line. She reasoned that the system has a solution for every value of *k* because *y* is going to be some constant and you are going to get a horizontal line. She stated that $x = 2$ and *y* is whatever value you get after negative two plus *k*. Her response suggested that she is treating *x* and *y* as variables and *k* as a constant.

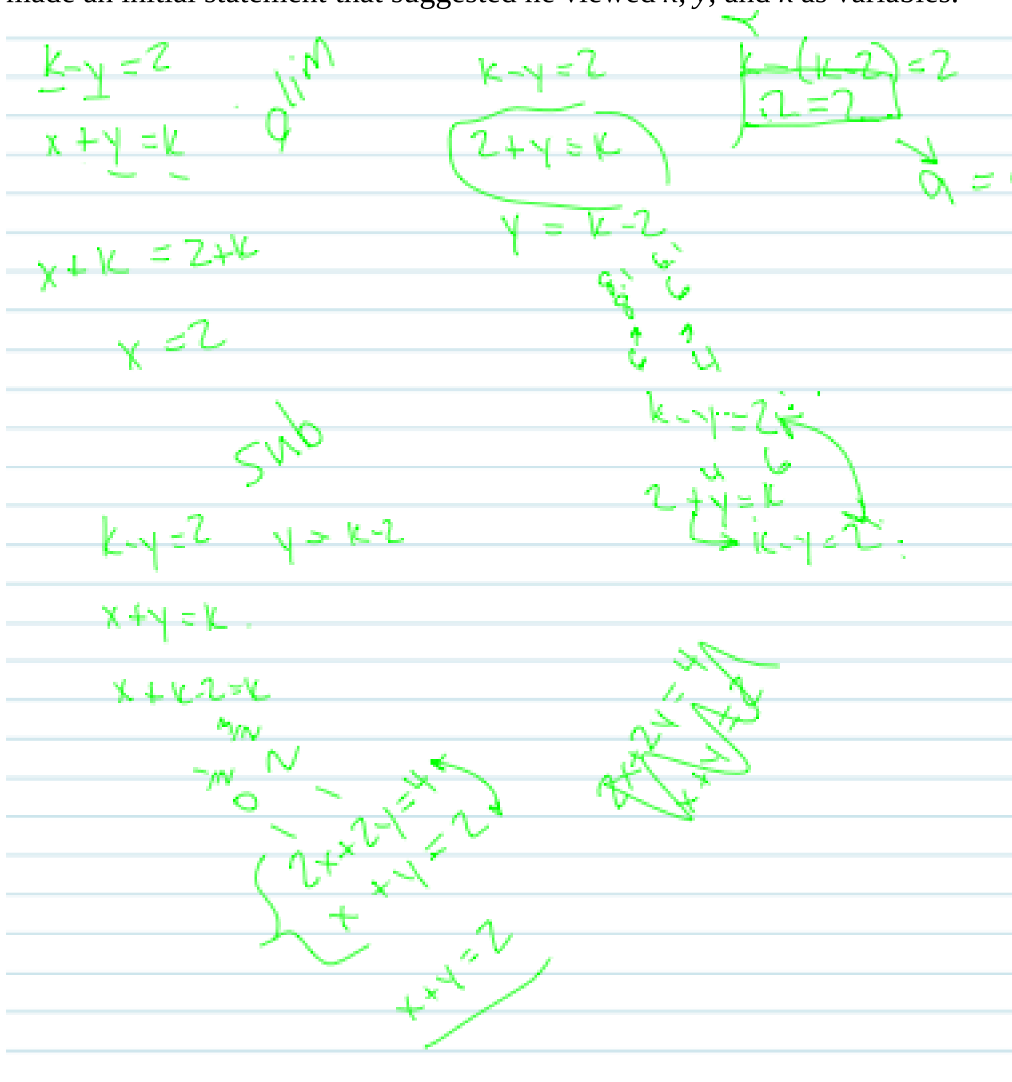
Student **Task 4**



Dan Dan applied a substitution procedure to the system of equations. It yielded an answer of $x = 2$. He reasons that k can be whatever reasoning from the intermediate statement $x + y - y = 2$ where he stated that y can be anything. He made a statement suggesting he saw k as a variable.



Newt Newt applied a elimination procedure on the system of equations which yields an answer x = 2. Substituting $x = 2$ into the equation $x + k = 2 + k$ resulted in 2 = 2. He indicated it told him nothing about y but it led him to believe that there will be a solution for every value of k. He applied a substitution procedure on the system of equations and reached the same

Student	**Task 4**
	result, *x* = 2. He reasoned since *y* is not in the results of any of his procedures that it doesn't matter what *y* is. He recognized that when 2 is substituted in for *x* that the two equations in the system are the same and that form suggested to him because whenever it happened before in algebra that the answer has been infinite solutions. Also, he stated the relationship between *y* and *k*, the difference between the two is always going to be 2. He provided an example of a system of equations with infinite solutions. He made an initial statement that suggested he viewed *x, y,* and *k* as variables. 
Jim Inductive	Jim mentally started with a substitution procedure that yielded a result of *x* = 2. He started another procedure solving both equations for *k* and reasoned that *x* = 2. He reasoned inductively from two cases (*k* = 0, *k* = 1) about the value that *k* would have. He does this for a couple more cases of k (*k*=10, *k*=1000) stating that this would mean there is always a solution. He stated that *x* must equal 2 and that for every value of *k*, *y* plus 2 would be that number. His response does not suggest that he is distinguishing between *x*

Student	**Task 4**
	and *y* being variables and *k* being a parameter.

$K = 2 + y$

$k = x + y$

$x = 2$

Paul Paul started with a substitution procedure which yielded a result of $x = 2$. He then used an elimination procedure on the system which also resulted in $x = 2$. He reasoned that no matter what value you put in for *k* that the *k*'s go away and so do the *y* values. His reasoned from the steps of his procedures that there is a solution for every value of *k.*

$k - y = 2 \qquad k = 2 + y$

$x + y = k$

$x + y = 2 + y$

$x - 2 = 0$

$x = 2$

$2 + y = 2 + y$

$K - \cancel{y} = 2$

$+\ x + \cancel{y} = K$

$K + x = k + 2$

$x = 2$

Nadia Nadia started with an elimination procedure which resulted in $x = 2$. She made a computational error in her procedure ($2 + k = 2k$) but it still yielded $x = 2$. She stated she really doesn't know how to do the problem.

$k - y = 2$

$x + y = k$

$k > \cdot 2k$

$x = 2$

Robin Robin solved the 2nd equation for *x* and reasons that $x = 2$ from the fact that $k - y = 2 - y$ and reasoned that it is true because it doesn't matter what you put in there but she is unsure that it proved that there is a solution for every value of *k*. She is unsure what else to do.

Student **Task 4**

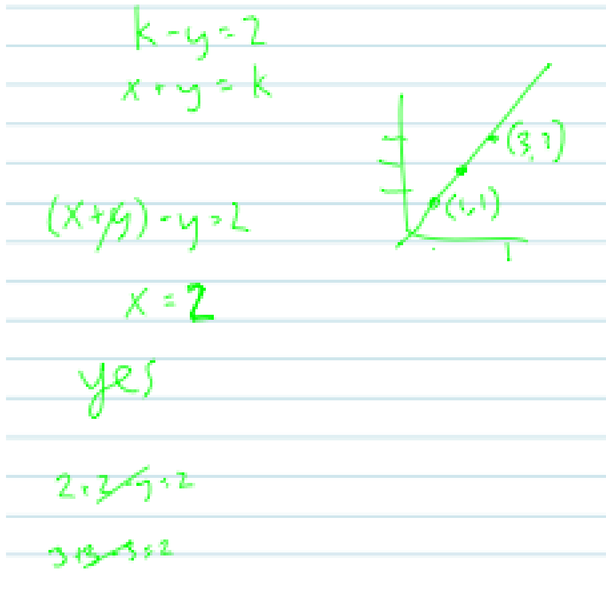
$$\begin{aligned} k - y &= 2 \\ x + y &= k \\ x &= k - y \\ k - y &= k - y \\ x &= 2 \end{aligned}$$

Betsy Betsy transformed the original equations into $k = y + 2$ and $k = y – x$. She stated she does not know what to solve for because there are three variables. She stated she is not sure if k is a constant. She transformed the equations solved for k into $y = k – 2$ and $y = k –x$ indicating that she is not sure what she doing. She reasoned that the sum of two equations has no solution which meant they would be parallel or one solution which means they intersect or else a million solutions because they are the same line. She stated that she doesn't think k is a variable because you can't have three variables in a system of two equations. She stated that if k is a constant the first equation $y = k – 2$ is a horizontal line because y would equal a number. She stated the second equation $y = k –$ x is non-horizontal because it has a number and two variables. She reasons that if you have a horizontal line and a non-horizontal line that they are going to intersect at some point. She reasoned that any horizontal line and any line not parallel would intersect and they can't be parallel because one's horizontal and one's not.

$$\begin{cases} k - y = 2 \\ x + y = k \end{cases} \Rightarrow \begin{matrix} k = y + 2 \\ k = y + x \end{matrix} \Rightarrow \begin{matrix} y = k - 2 \\ y = k - x \end{matrix}$$

Molly Molly applied a substitution procedure substituting $x + y$ into the first equation. This resulted in an answer of $x = 2$. In spite of the fact that she established x as a constant she stated that there will be different values for x and y each time. In explaining what she meant she substituted $x = 2$ and $y = 2$ in for x and y. The result was a true statement. She repeated the procedure for $x = 3$ and $y = 3$ which does not yield a true statement—substituting into (x + y) – y = 2 she gets 2 + 2 – 2 = 2 and then 3 + 3 – 3 = 2. She stated she does not know what it means for x to equal 2.

Student **Task 4**



Roxie

Roxie substituted k = 10 into both equations. She reasoned that y = 8 and x = 2. She reasoned that the equations would be true for any value of k because you don’t have an x variable in the equation. She pointed to $k - y = 2$ and stated that whatever you make k, you can make x whatever you need for it to fit in the equation $x + y = 2$ since there is not an x in the first equation. She substituted k = -8 into each equation: a procedure that resulted in y = -10 and x = 2. She stated again that x can be whatever because it is not in both equations and that it would have to work in both equations if it was, but since it is not it only has to work in one equation. This seemed to suggest that she is reasoning from a form that she is familiar (two equations and two unknowns) She made a statement suggesting that x, y, and k are variables and that x can be anything because it is not dependent on the equation $k - y = 2$.

Student **Task 4**

$10 - y = 2$ $\qquad -8 - (-10) = 2$

$x + y = 10$ $\qquad x + (-10) = -8$

$x = 2$

$k - y = 2$

$x + y = k$

Todd Todd solved both equations in the system for y. He stated that for every value of *k* if he added -2 he could determine the value of *y*. He reasoned that the graph of $y = -2 + k$ will have a horizontal slope and move up and down depending on *k* so *y* will have a solution every time. And he reasons that $y = k - x$ is a line that has a slope of -1 and that for both equations *k* is going to change the y-intercept. He reasoned that the solution could be found by looking at any value of *k* because for a give value of *x* you can solve for *y* and on the graph for every value of *x* you find the corresponding value of *y* that intersects the line. He placed a constraint on there being a solution on the graph—no breaks in the graph. He is prompted by the researcher to consider an algebraic approach, but stated the only way he can think of it is by looking at the graph.

$$\begin{cases} k - y = 2 \\ x + y = k \end{cases}$$

$$\begin{cases} -y = 2 - k \\ y = k - x \end{cases} \qquad |x| = 4$$

$$\begin{cases} y = -2 + k \\ y = k - x \end{cases}$$

$y = k - x$

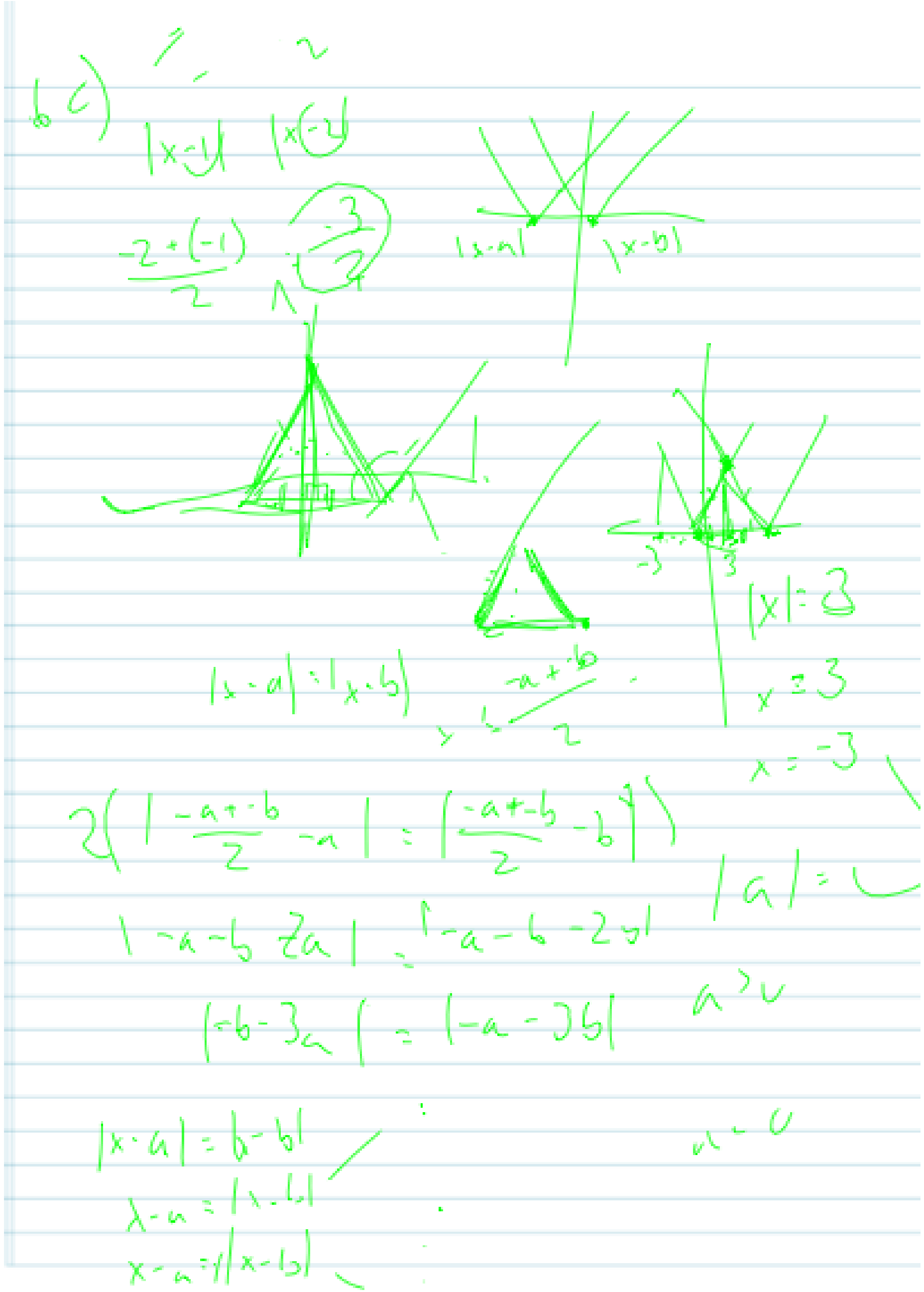
Student	**Task 6a**
Ashley	Given to Ashley after she was unable to engage in Task #6c. Ashley began with a substitution procedure (*x* = 0, 1, 2, 3, 4, 5, 6, -1, -2, -3). She stated she is looking to see whether the gap between the numbers closes. She stated they all differ by three. She does not recognize what is happening between *x* = 0 and *x* = 1—difference is not three, but it does not seem to lead to a refinement in her thinking. Did not pursue graphical strategy.

0	$1 \neq 2$	-1	$0 \neq 3$
1	$2 \neq 1$	-2	$1 \neq 4$
2	$3 \neq 0$	-3	$2 \neq 5$
3	$4 \neq 1$		
4	$5 \neq 2$		
5	$6 \neq 3$		

Casey	Task #6b given
Dan .	Dan was given Task 6b before he was given Task 6a and he mentioned that the two are similar in terms of how he made the graphs. He re-presented the equation and the graph of two functions that he has arbitrarily named |x – a| and |x – b|. He made a distinction between when *a* and *b* are the same and when they are different. He reasoned that if *a* = *b*, they would have every point in common and would be the same graph. He reasoned that the point of intersection will always be halfway between the two vertices, just like the case of two and one. At this point it is evident he has lost sight of the original problem (it would have been -1 and 2 if he would have been talking about the original problem). He provided a geometric justification as to why the solution would be halfway between the two vertices reasoning that an isosceles triangle will be formed between the point of intersection and the two vertices. He usesd a particular case with vertices at *x* = 0 and *x* = 6 and generalized about the forming of the isosceles triangle. He also established a requirement of the generalization stating that the would only be true with horizontal shifts because a horizontal shift would not change the shape or degree of the slope. He argued that once he proved that the triangle is isosceles he can say the distances are the same—he did not technically provide a proof of why the triangle would be isosceles—he assumed the distances would be the same from point of intersection to the vertices and only has the right angle by construction and the middle segment by transitive property of congruence. He computed that point of intersection of the graph of the functions y = |x – 1| and y = |x – 2 is at $x = -\frac{3}{2}$ reasoning from a formula $x = \frac{-2\pm-1}{2}$. This provided further evidence to suggest he is no longer reasoning from the original problem. His work also suggested a disconnect between the

Student	**Task 6a**
	actual location of the vertex as viewed through the function rule—should be at x = 1 and x = 2 and he is placing them at x =-1 and x = -2—this result corresponded with his earlier work where he determined it to be halfway between x = 1 and x = 2, but on the negative side. He created a formula for finding the point of intersection for any function rulewith vertices at $x = a$ and $x = b$. He represented the point of intersection as $x = \frac{-a \pm -b}{2}$, instead of $x = \frac{a+b}{2}$. Using the formula and points of vertex that he has labeled –a and –b he attempted to prove algebraically that the distances would be the same. He reasoned by substituting numbers into |-b-3a|=|-a – 3b| that there is something wrong with his solution. He returned to the malformed problem he created |x – 1|=|x - 2| and substituted x = -3/2. This does not result in a true statement that he stated made him begin to lose confidence in his earlier answer. He is prompted by the researcher for an algebraic solution. He changed |x – a| = |x – b| into |x – a| = |x – b] and |x – a| = -1|x – b| and reasoned that since he had two absolute values in the beginning problem that he would have four equations. He stopped the procedure arguing that he had not done it before and did not want to spend a whole lot of time doing the procedure if it was not going to be successful.

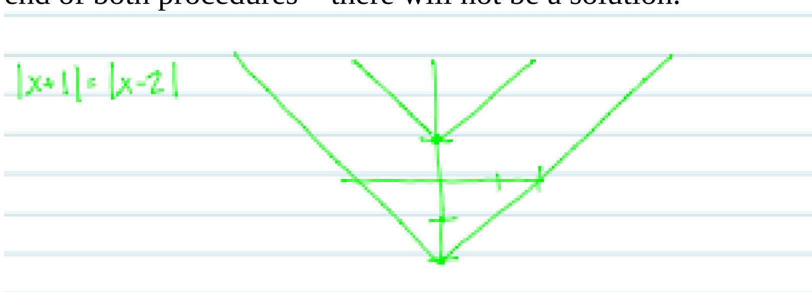
Student **Task 6a**



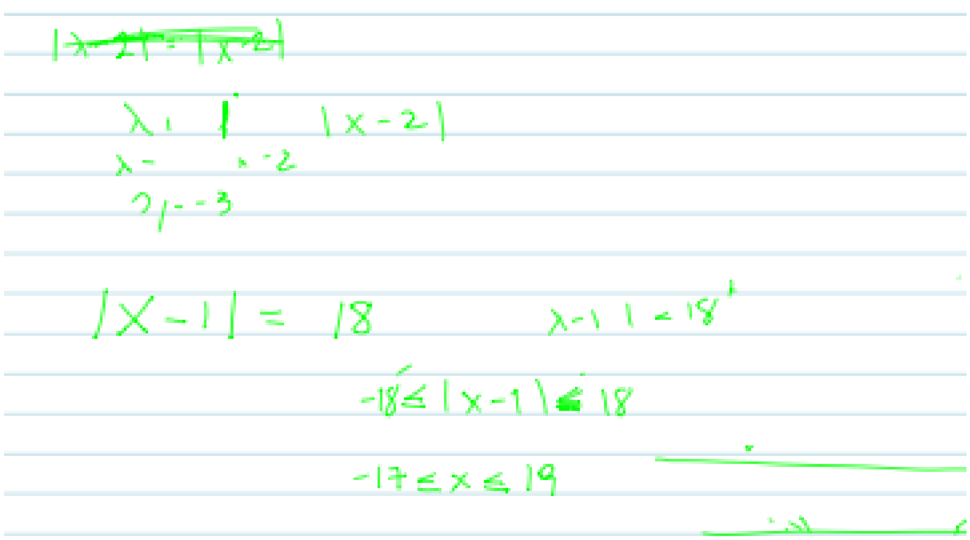
Newt Task #6b given

Jim Jim commented that he is trying to think of a scenario where a number plus one would equal a number minus two even with absolute value. He began with a substitution strategy where he substituted numbers with the hope of finding a number where the equation would be true (x = -5, -1, 0, 5) He generalized that there is no value that will satisfy the equation because he has both positive, negative numbers, x= 0 and x =1—numbers which hare sometimes exceptions to rules. He is prompted by

Student	**Task 6a**
	the researcher to reason about the problem graphically. He made an incorrect link between the function rules expression in the original problem and the location of the vertex—he sees the function rules in terms of vertical shifts instead of horizontal shifts. He reasoned since they will never intersect because they are straight lines with slopes of 1 and -1 [linked to function rules]. His reached the same conclusion at the end of both procedures—there will not be a solution.

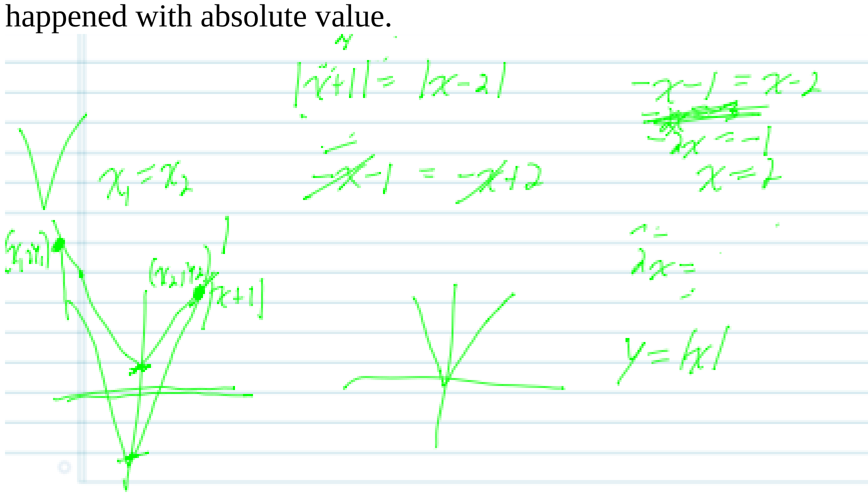


Paul	Task #6b given
Nadia	Nadia described absolute value as "whatever the answer, if you replace the value of *x*, whatever the answer, you get, you're gonna take the absolute value, so if it's negative, it will become a positive". Her first procedure is dropping the absolute value and solving. The solution of her work resulted in no variable. At this point the researcher recognized she was not able to reason about this problem so he gave her $|x – 1| = 18$ to reason about. She created a double-sided inequality and reasoned about a number line. She seemed to be relating pieces of procedures that she remembers without much coherence.



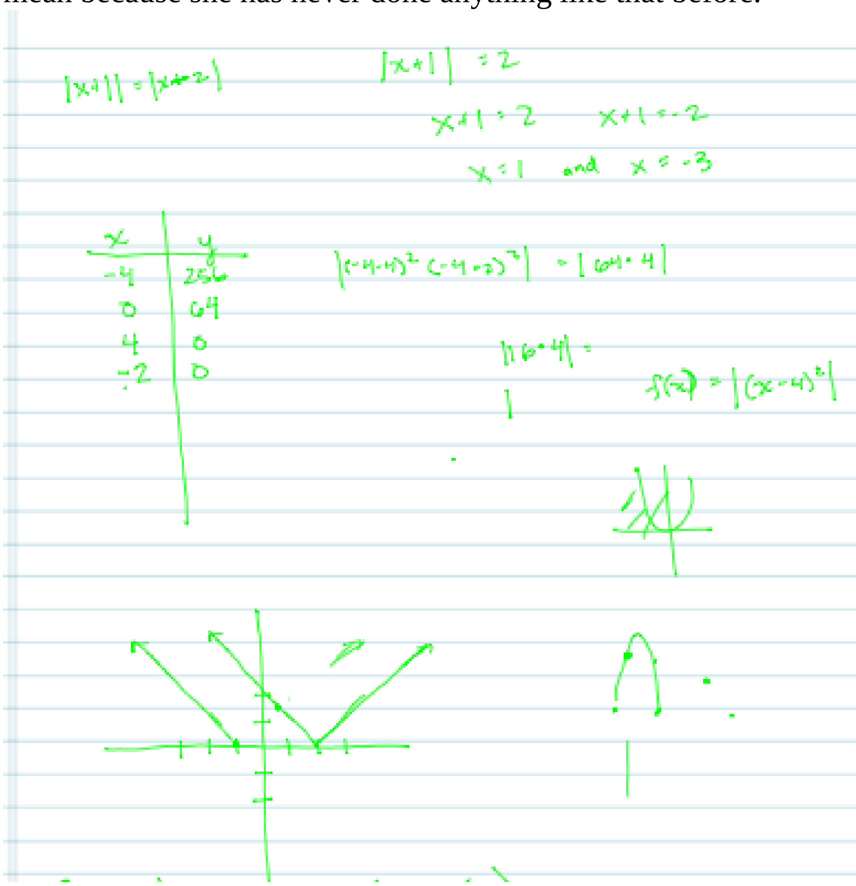
Robin	Robin defined absolute value as "that the answer is always going to be positive, the final answer, but what is inside could be negative. So if you're solving for *x* it could be a negative number if it is an absolute

Student	**Task 6a**
	value.” Her initial strategy is to move both absolute values to the same side, drop the absolute value symbols, and solve. Her result of -1 = 0 made her think she is doing the problem wrong. She started a substitution strategy substituting the numbers *x* =0, *x* = 1, and *x* = 2. She reasoned resulting statements are not true. She did not notice the changes in distance when entering *x* = 1 and *x* = 2. She began another procedure that involved solving piecewise defined equations. She stated she is making one negative and one positive but she is not sure she can do this. She wrote the equation –*x* – 1 = *x* – 2 and made a computational error that resulted in a solution of *x* = 2. She wrote the equation *x* + 1 = -*x* + 2 which resulted in an answer of *x* = ½. She does not check to see whether this is the correct answer and does not make any statements that suggested this is the correct answer. She stated that with absolute values you are supposed to get two answers, one where it would be negative and one where it would not be negative, but she is not sure. She started another procedure by rewriting the equation as |x + 1| - |x – 2| = 0 dropping the absolute value symbols and solving. She is prompted by the researcher to think about the problem graphically. She made a link between the equation (absolute value) and the shape of the graph (v-shaped) and stated incorrectly that y = |x +1| will be translated one unit upward as opposed to one unit to the left. Her sketch of the two function rules showed them intersecting in two points (shows a lack of specificity—not parallel) in graphing and that the two points of intersection correspond with two solutions which is what she thought happened with absolute value.

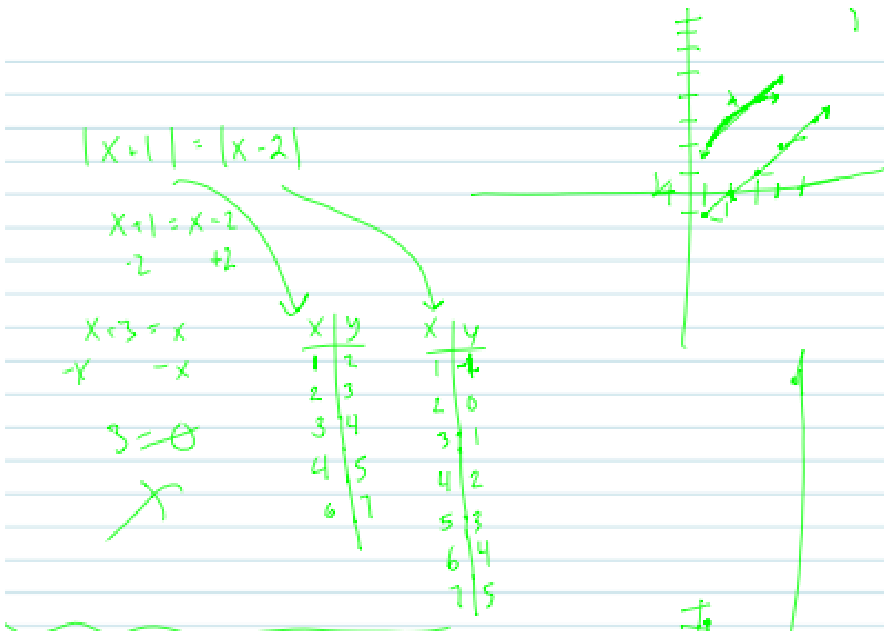


function rules and location of graph. Does not make a link	Betsy stated that she had never done double-sided ones in Algebra 2 and when prompted by the researcher to consider what she had done she wrote |x + 1| =2. She enacted a correct procedure to solve the problem. She stated she thought that you separate into two equations, but she does not know where to start with a double sided with an equation that has absolute value symbols on each side. She is prompted by the researcher

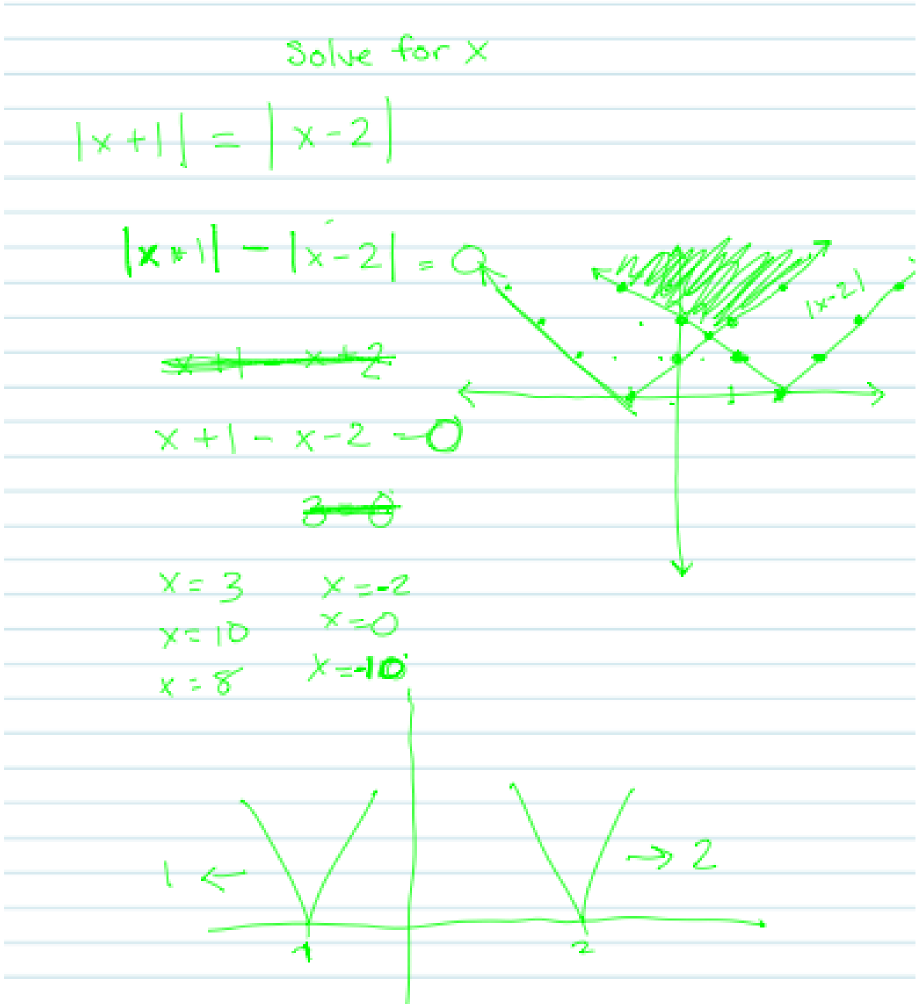
Student	**Task 6a**
between intersection of graph and solution of	to consider a graphical solution. She recognized that she can graph each side of the equation as a separate function rule and made an accurate link between the function rules and the location of vertex on the graph. She reasoned that she is looking for where they intersect but does not make any statements about the value of the solution nor does she make any statements connecting possible solution back to original problem. She stated that she does not know what the point of intersection would mean because she has never done anything like that before.



Molly	Molly began with a procedure that involved removing the absolute value symbols and solving the resulting equation. This resulted in the inscription a 0 = 3 that she reasoned would give her no solution. She described absolute value as “distance of a point of a point away from zero.” She is prompted by the researcher to consider a graphical strategy for solving the problem. She stated that she could graph both of these and find out where they intersect. She started an unrefined plotting strategy starting with $x = 1$ for each table. She used the points to create the graphs of each—lines with positive slopes and reasoned there is no solution because the lines will not intersect. She reasoned that each graph goes up one and over one. It’s interesting that on an earlier problem Molly recognized the graph of absolute value functions are v-shaped.

Student	**Task 6a**
	
Roxie	Roxie stated that absolute value is “well if it’s positive it becomes negative. Because you’re taking how far it is from zero instead of negative or positive-wise.” She began with a procedure that involved removing the absolute value and solving. This resulted in the inscription 3 = 0 which she crosses out. She followed with a substitution strategy looking for values for *x* that would yield a true statement. She tested *x* = 3, *x* = 10, *x* = -2, and **x** = 0. The choices of numbers in her substitution strategy does not afford for noticing of the narrowing of the outputs. She is prompted by the researcher to consider a graphical solution. She is able to think about the equation in terms of two function rules and is able to made a correct link, although not initially, between the function rule and the location of the vertex. She made the link between absolute value and shape (v-shaped) of the graph. She did not make the link between the intersection of the graphs and the solution of the equation. She linked what would solve it to be where the graphs share the same shaded region. She stated that she doesn’t think of solving problems in terms of graphs because they normally don’t help her.

Student **Task 6a**

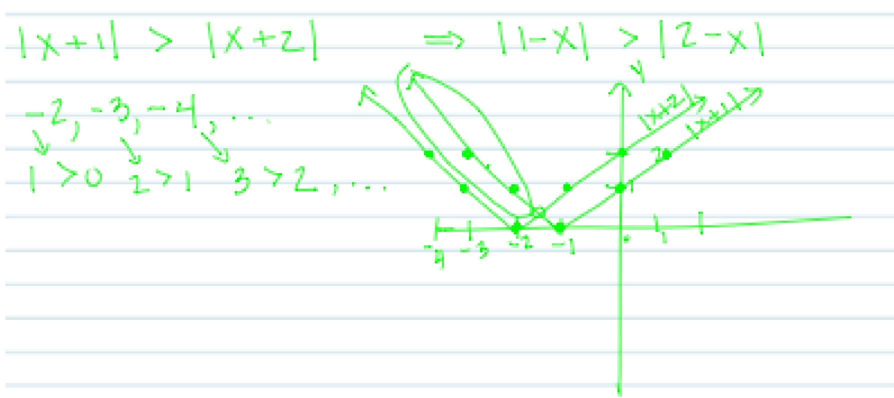


Todd Task #6b given

Student **Task 6b**

Ashley

Ashley began with a substitution procedure that led to a deductive generalization about zero and positive numbers not being in the solution set. She continued with the substitution procedure using negative numbers x = -2, -3, -4 which led to an inductive generalization about negative numbers less than -2 being in the solution set. She mentioned that she usually tried a procedure where she started substituting things in looking for patterns. She strategically plotted points to graph y = |x + 1|. She stated that she always starts with zero, the graph started to take shape and she chose points to the right and left once she figured out where the vertex is. She also stated she knows this is the shape of an absolute value graph and linked the equation form of the equation y = |x + 1| and the location of the vertex. In a similar manner she graphed y = | x + 2|. She linked her earlier solution to what she sees in the graphical representation. She reasoned that since the graph of y = |x + 1| is above y = |x + 2| it is where y = | x + 1| is greater than y = | x + 2| and stated where it would because everywhere else it is below. She states that what she saw on the graphical representation matched the reasoning when she substituted numbers. In other words, she formed a link between reasoning about the results of the substitution procedure and the graphical procedure.

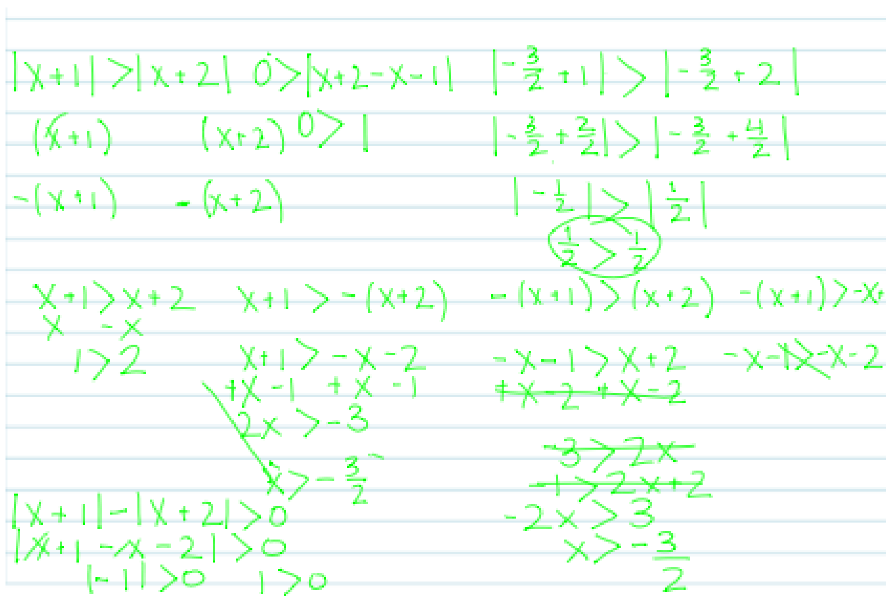


Casey

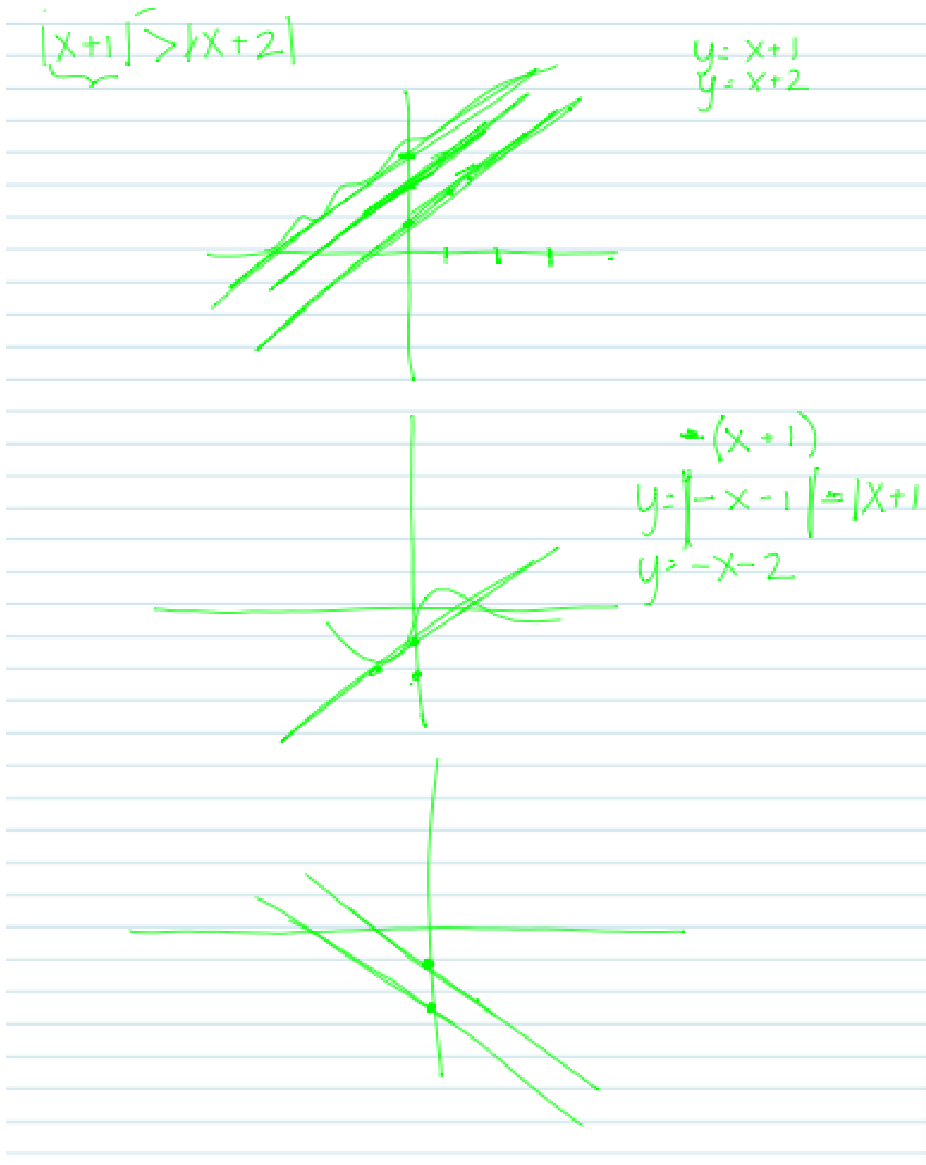
Casey began with a piecewise defined procedure solving the four inequalities. She made a computation error which resulted in her having two solutions $x > -\frac{3}{2}$. She substituted $x = -\frac{3}{2}$ into the initial inequality she seemed surprised that the result does not give her a true statement. She substituted *x* = 0 into the original inequality and is perplexed because it is in the solution set of the result of her procedure $x > -\frac{3}{2}$, but is not in the solution set of the original inequality. Using a procedure in which the initial inscription represented a relationship between numbers she made a deductive generalization about why positive numbers are not going to be in the solution set. She incorrectly expanded this statement into the set of negative numbers. She began another algebraic procedure writing $|x + 1| - |x + 2| > 0$ and incorrectly transformed the inequality into $|x + 1 – x – 2| > 0$ which resulted in 1 > 0. She repeated the same procedure moving both terms

Student **Task 6b**

to the left which yielded an answer of 0 > 1. This procedure, though flawed, reinforced her thinking that the inequality will never be true. She reiterated her reasoning in which the initial inscription represented a relationship between numbers. She is prompted by the researcher to consider thinking about the problem graphically. She created accurate representations of the graphs of $y = x + 1$ and $y = x + 2$. Instead of reasoning from the absolute value of each of these she reasoned from the graphs of $y = x + 1$ and $y = x + 2$ that the lines are parallel and that this supports her earlier thinking that the original inequality is never going to be true. In the end she has three strategies, though fraught with inaccurate steps, reinforced her thinking that there is no solution.



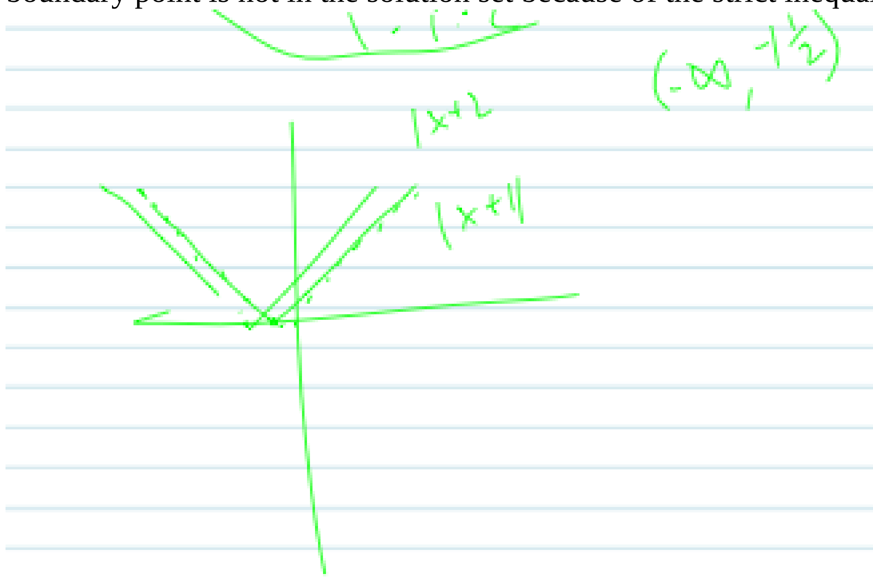
Student **Task 6b**



Dan

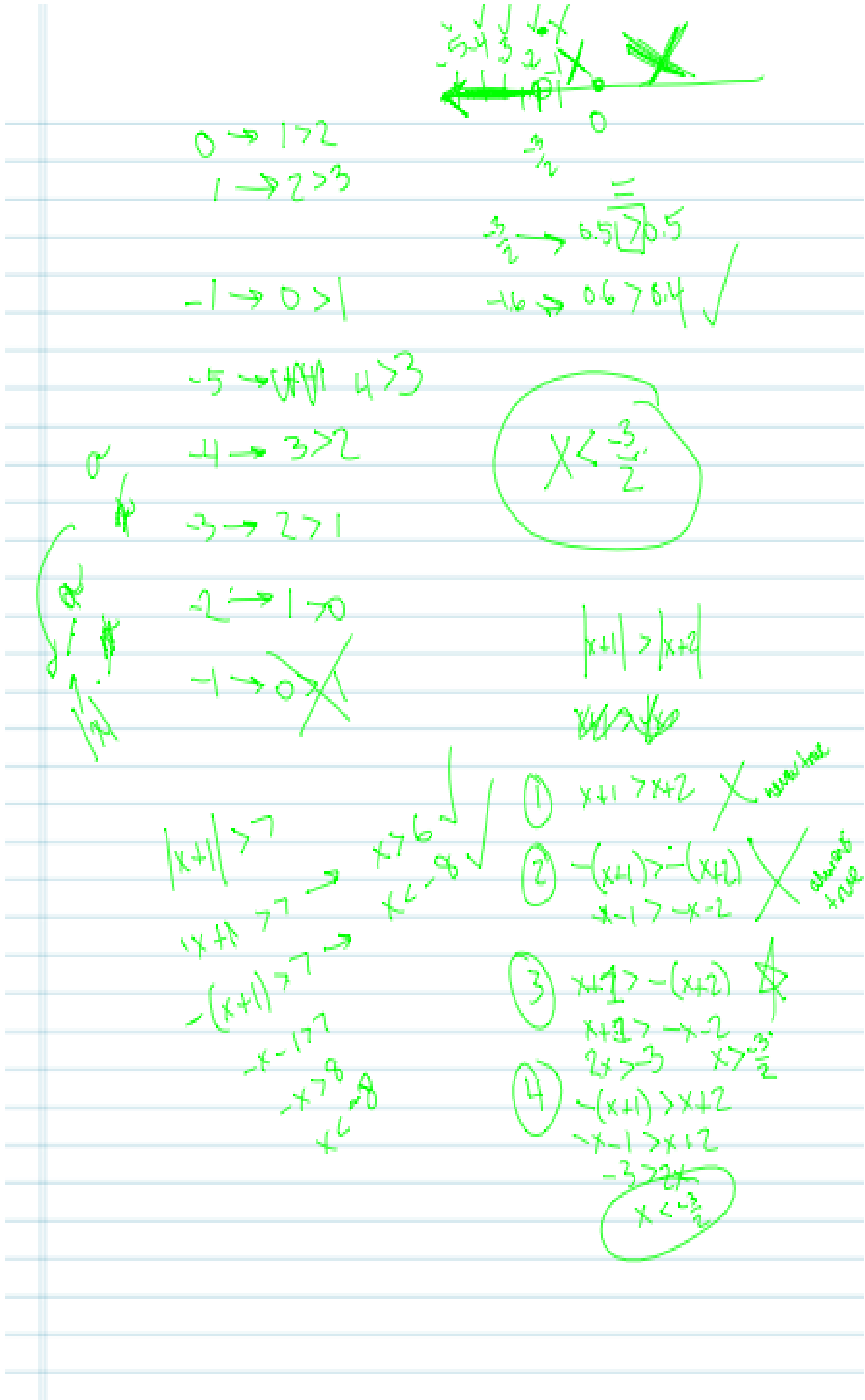
Dan began with a procedure where the original inscription represented a relationship between numbers making an analogy to apples. He used this to reason about nonnegative numbers being in the solution set. He followed up with a substitution procedure for *x* = -1 and *x* = -2. At this point he noticed there is a change in truth of the original inscription between -1 and -2. He stated, “I use graphs often, I don’t have to, but I found out early on that it worked for me…if I could make a graph of it I could figure out how the numbers would behave” (338-343). He rewrote the inequality as two function rules. He linked the absolute value and the shape of the graph, the function rule and the location of the vertex, the point of intersection and the boundary of the solution set and the interval where the graph of y = |x + 1| is greater than y = |x + 2|. He reasoned that the solution is where the one graph

Student	**Task 6b**
	overtakes the other. He tested and boundary point reasoning that the boundary point is not in the solution set because of the strict inequality.



Newt	Newt began with a procedure where the original inscription represented a relationship between numbers. He formed an inductive generalization about nonnegative numbers not being in the solution set. He explained his reasoning with two examples (*x* = 0 and *x* = – 1). He followed with a substitution procedure starting with *x* = -1 which is not true. He jumped to *x* = -5 which is true and worked down for *x* = -4, *x* = -3, *x* = -2 and reconfirmed that the inequality is not true for *x* = -1. As part of this procedure he created a link between the truth value of the outputs and a number line. This analysis afforded a refinement of the solution between x = -1 and x = -2. He substituted *x* = -1.5 into the initial inequality and interpreted this as the boundary point. He then tested *x* = -1.6 to confirm his thinking. Without being prompted he started i an algebraic procedure. He formed a prototype inequality |x + 1| > 7 and rewrote it as a piecewise defined inequality and solved both inequalities. He stated absolute value is “like saying the distance from x to zero so there’s two ways, there two distances since *x* can be positive or negative”. He stated that is a memorized procedure. His procedure resulted in solutions of x < -8 and x > 6 and confirmed that this is true by mentally substituting numbers into the original inequality. He goes back the original absolute value inequality creating four piecewise defined inequalities to solve. He solves all four and the two that have a variable when solved yield $x < -\frac{3}{2}$ and $x > -\frac{3}{2}$. He reasoned the second answer would not work because he knew that x = 0 is not in the solution set and that the other solution corresponded with what he found earlier.

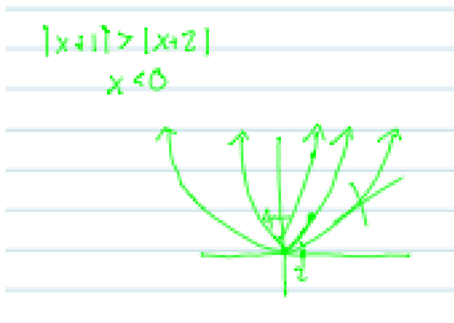
Student **Task 6b**



Jim Jim began by stating that the way it is written the absolute value of $x + 1$ will never be greater than the absolute value of $x + 2$. He supported his thinking substituting $x = 10$ into the inequality (result is a true statement),

Student **Task 6b**

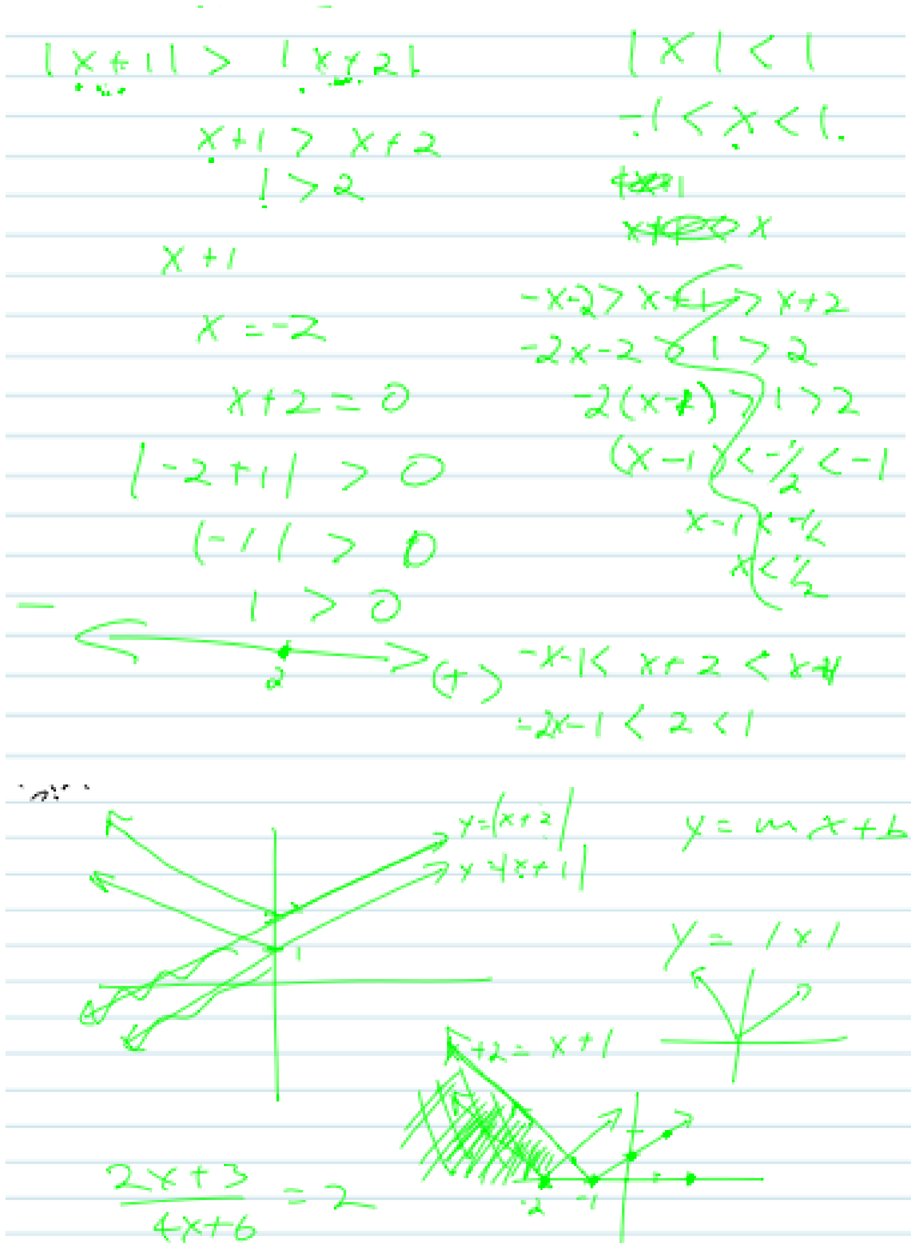
but adjusted his thinking substitution of x = -10 results in an untrue statement. He supported his inductive generalization that positive numbers would not be in the solution set substituting $x = 5$ and supported his inductive generalization that negative numbers would be in the solution set by substituting $x = -5$ into the solution set. Jim's analysis lacked specificity in that he does not hone in on the boundary of the solution set. Also, he does not reason graphically about this problem.



Paul

Paul began by dropping the absolute value and solving. This resulted in an untrue statement. Reasoning from a procedure where the initial inscription represented a relationship between numbers he made a deductive statement about why the solution does not include positive numbers. He reasoned about classes of numbers—very large, very small. He used a substitution procedure [x = -10, 0] to reason about negative numbers being in the solution set. He refined his thinking once he tested $x = -1$ and determined it is not in the solution set. While not precise he does reason from a substitution procedure that the solution included values less than -2. He states that absolute value is, "distance from the origin. So, on either side the distance is always a positive number. There is no such think as negative distance. So basically, you're just measuring the distance from zero." He is prompted by the researcher to think about an algebraic procedure for solving the problem. He created a double-sided inequality $-x - 2 > x + 1 > x + 2$ and solved. He made a computational error and ended with the inequality $x < \frac{1}{2}$. He explained the procedure is similar to when you have absolute value and rewrite $|x| < 1$ as $-1 < x < 1$. He repeated the procedure and wrote $-x - 1 < x + 2 < x + 1$ and gets $-2x - 1 < 2 < 1$. He seemed to have difficulty reasoning about the meaning of the results of his procedure. The researcher prompted him to consider a graphical procedure. He plotted y = x + 1 and y = x + 2 and then incorrectly demonstrates the effect of the absolute value on the graphs—bending the graph at the y-intercept of each as opposed to the x-intercepts. He reasoned from the slopes that they will never intersect. He begins to reason about the graph by strategically plotting points—starting where each output is zero and then for values where y = 1. At this point he recognized that he had not correctly graphed the two absolute value function the first time. His interpretation of greater seems to be about an area that is shaded reasoning that the region is where the $x + 1$ graph is above the $x + 2$.

Student	**Task 6b**
	In none of Paul's strategies does he pin down the boundary point of the interval.

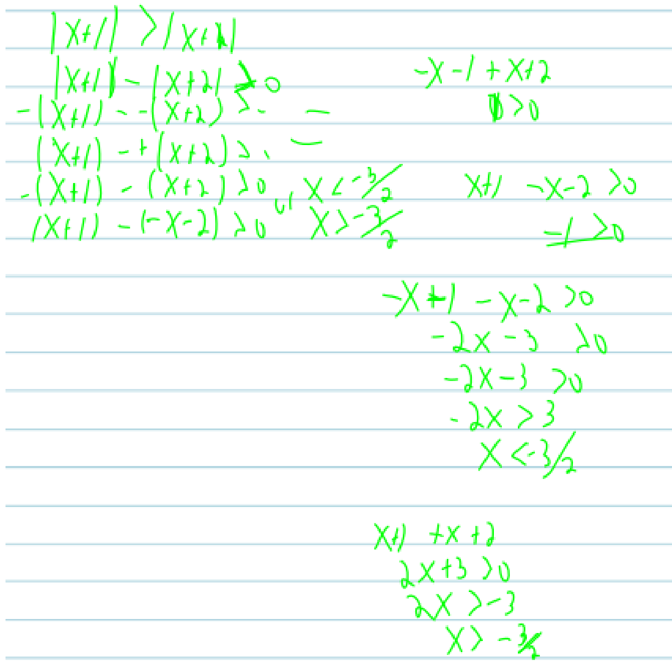


Nadia	N/A
Robin	N/A
Betsy	N/A
Molly	N/A
Roxie	N/A
Todd	Todd began with a piecewise defined inequality procedure writing four inequality statements. This procedure resulted in two solutions, $x < -\frac{3}{2}$ and

.

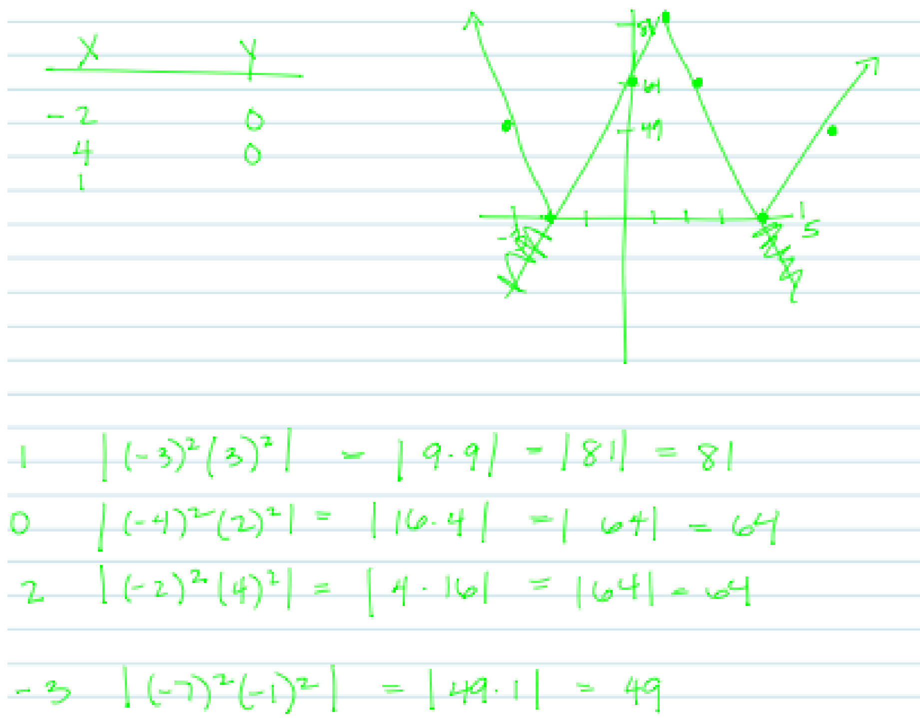
Student **Task 6b**

$x > -\frac{3}{2}$. He reasoned this to mean that *x* is equal to all values except negative three halves. He does not attend to the domain of the function.



Student **Task 7b**

Ashley began by finding the zeros, explaining why the are zeros, and then substituted a value, $x = 1$, into the function rule and followed by substituting $x = 0$ and $x = 2$ into the function rule She made an incorrect link between a characteristic of the function rule (absolute value) and the shape of the graph (v-shape emanating from (1,81) through (-2,0) and (4,0)). She stated that she will have an output for any value of x negative or positive. She made a link between a characteristic of the function rule (absolute value) and shape of the graph [scribbling out segments below x-axis and extending them upward]. She tested a point $x = -3$ to determine whether it matched her thinking about the impact of absolute value on the left-most branch of the graph. She makes a link between function rule (multiplying binomials to get x^4) and shape of the graph (W-shaped)



Casey Casey stated the graph is a quartic power and explained that if you expanded each binomial factor and multiplied them together the result would be a term to the 4th power. She made a link between the function rule (4th power) and shape of the graph (parabola)—drawing a sketch with vertex at (0,0).

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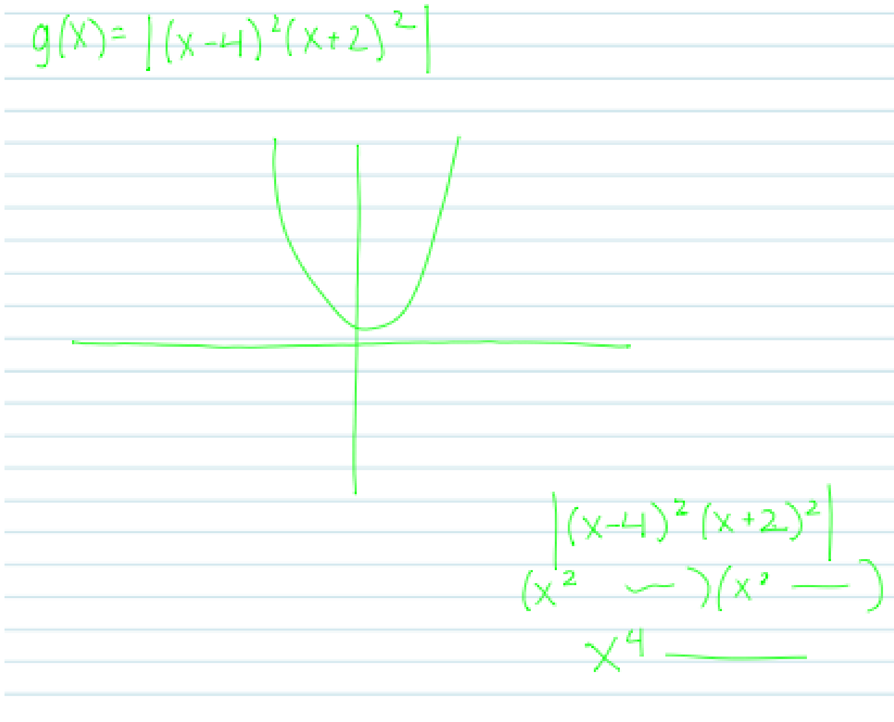
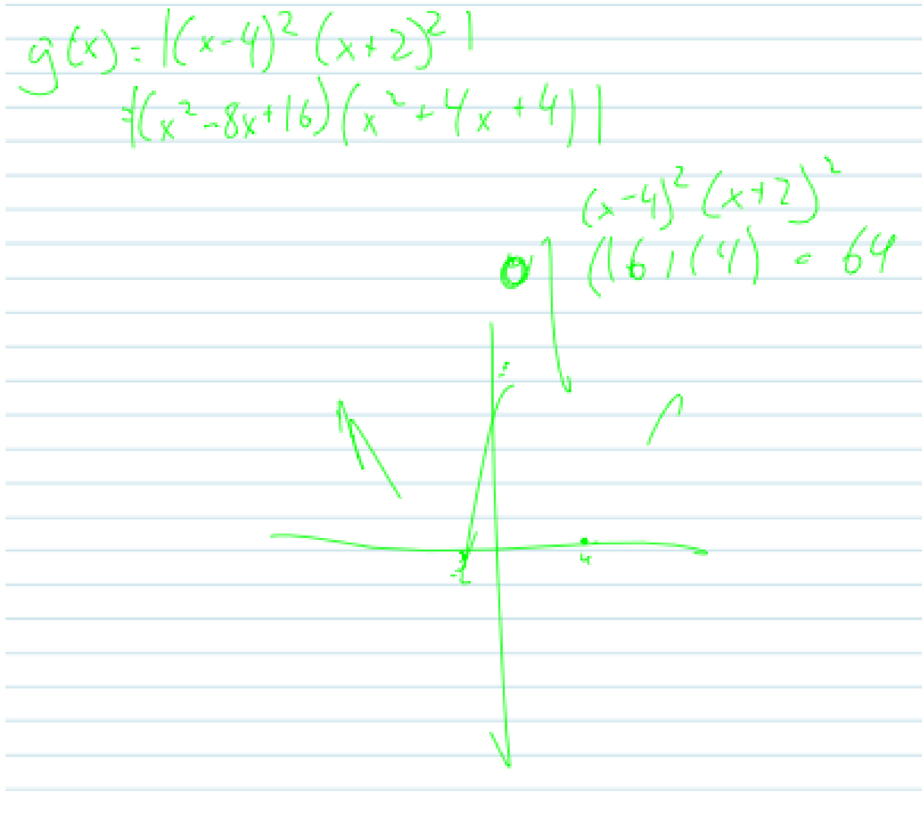


Dan	Dan began by expanding the two binomials in the function rule. He noticed the zeros in the function rule making a link between a characteristic of the function rule (zeros) and property of the graph (x-intercepts). He substituted $x = 0$ into the original function rule and plotted the point. He drew a curved, W-shaped graph and stated that he knew there are not other locations the graph crosses the x-intercepts reasoning from the original function rule. He reasoned from a specific example ($x = -100$) about the impact of absolute value on the graph. He linked a characteristic of the function rule (absolute rule) and a property of the graph (all outputs will be positive). He does not talk about the end behavior, but seemed to reason about this from the zeros and the absolute value along with location of the y-intercept. [Check paper sketch]

Student **Task 7b**



Newt

Newt began by stating the function rule is of degree four because of the product of the binomials. He made a link between a property of the original function rule (degree 4) and the shape of the graph (parabolic but flattened out) and showed more refined reasoning by talking about the “flattening out” between 0 and 1 and 0 and -1. He made another link between the original function rule (parent function of type $y = x^4$) and shape of graph (thinner than $y = x^2$) making the statement that the graph will “shoot up” after $x > 1$ and $x < -1$. Newt used a substitution procedure to reason about the shape of the graph and generalized about the impact of the interval of the domain on the shape of the graph. He made a link between a characteristic of the original function rule (absolute value) and a property of the graph (points below x-axis reflected over x-axis). He provided an example of a parabola opening up with vertex below x-axis and showed how it is reflected over *x*-axis. He made a link between the definition of absolute value (distance from zero) and its affect on the graph (vertical distance from x-axis changes from negative to positive). He expanded the original function rule and began a substitution procedure ($x = 0$) that he used to explain why the absolute value function will not impact the graph of the function rule. He made a generalization stating absolute value will not affect this specific case ($a^2b^2>0$) and gave the conditions as to why [binomials squared]. He did not discuss the

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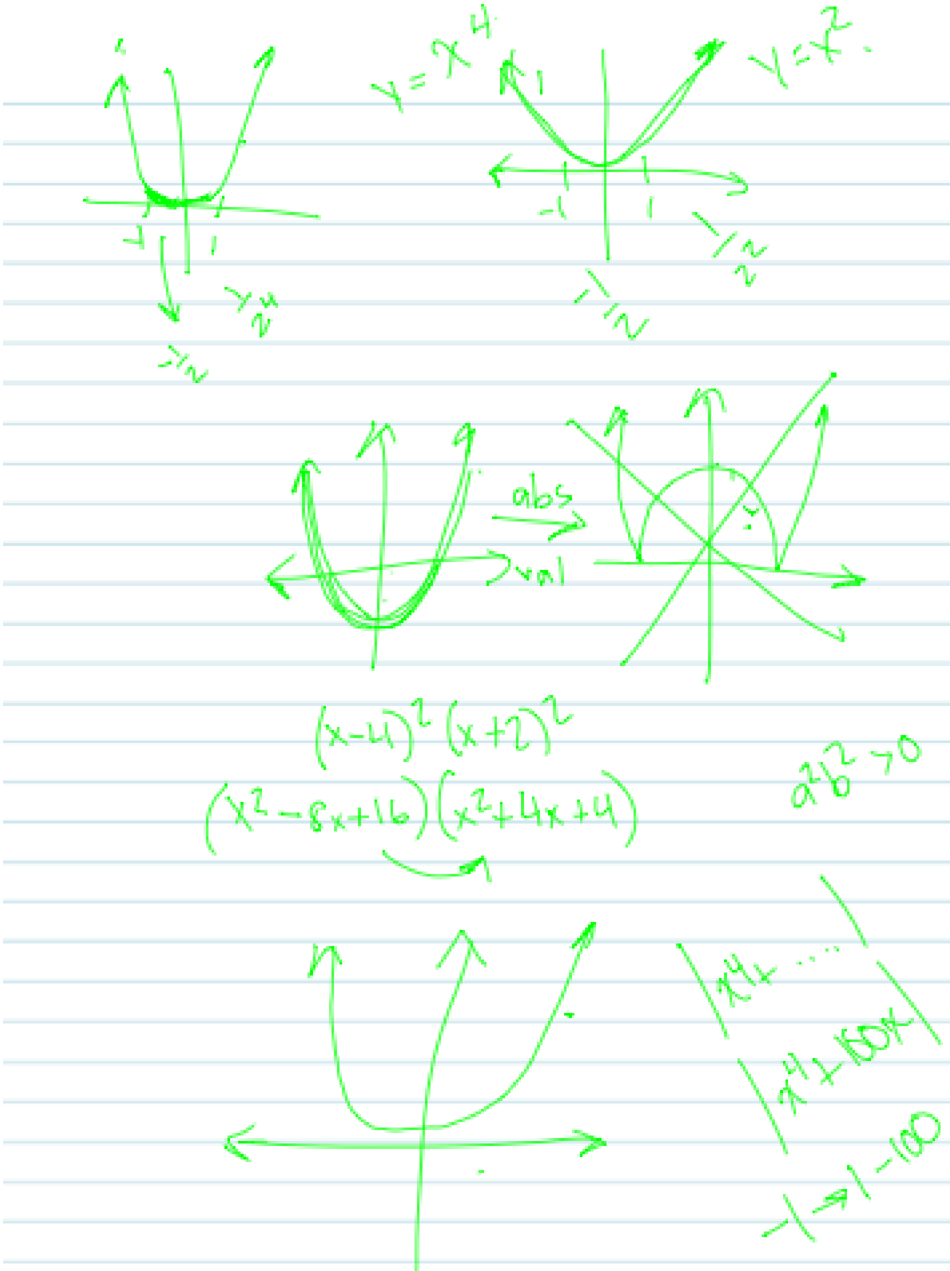
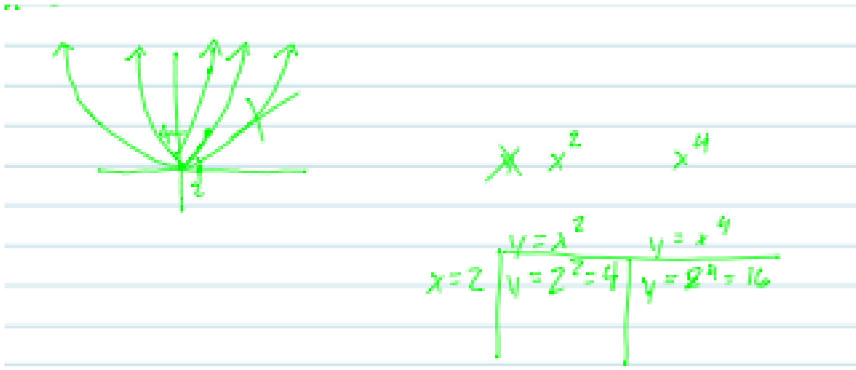
zeros or x-intercepts.



Jim   Jim reasoned from the initial function rule that the graph behaves like x^4, open upward (because of the absolute value even if the inside value was negative) and is wider than the graph of $y = x^2$ (incorrect). He reasoned that the graph would open upward because positive exponents mean the endpoints go in the same direction while negative exponents means it would be pointing down (researcher believes he is confusing even/odd with positive/negative). He begins a substitution procedure ($x = 2$) which

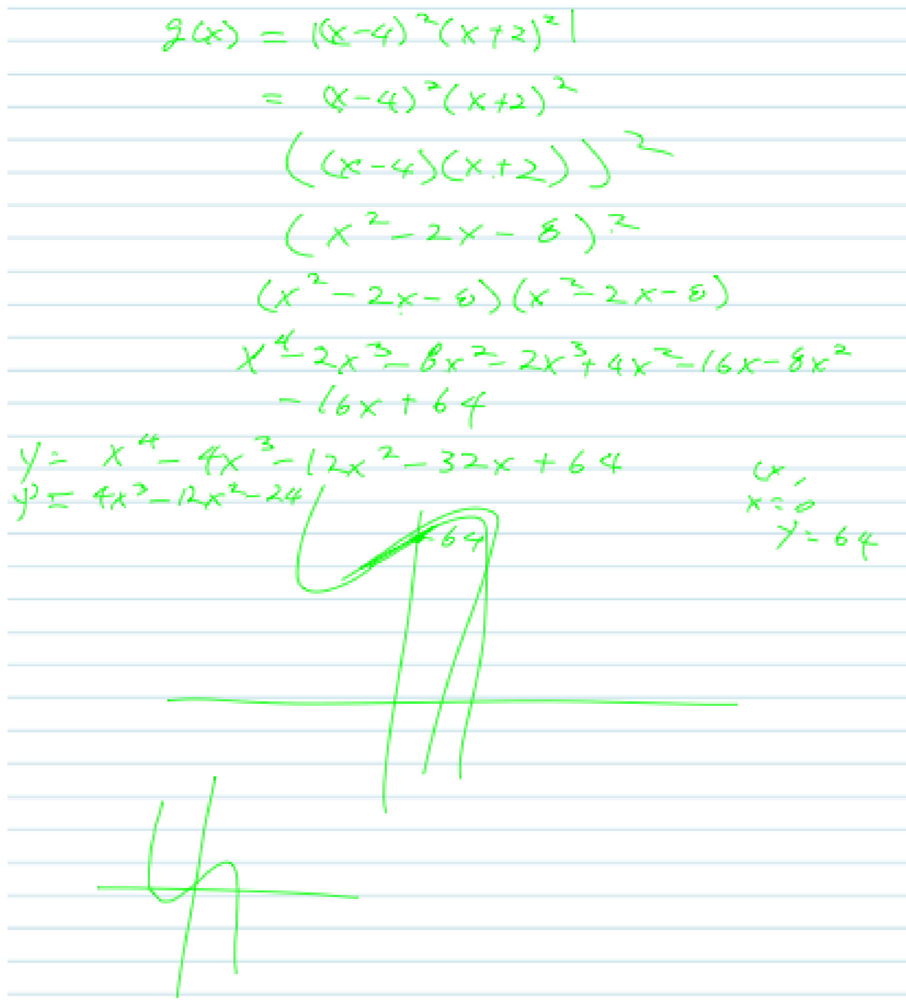
Student **Task 7b**

led him to refine his thinking stating that y = x^4 is "thinner" than y = x^2. He stated he cannot find the vertex without working out the problem which he does not do.



Paul — Paul reasoned that the graph will be a power of four because you multiply the squared terms together which leaves him with x^4. He dropped the absolute value because since you are squaring you will not need the absolute value (check his thinking on this). He followed with an expanding procedure on the initial function rule. He stated he could find the zeros, concavity, increasing, derivative is zero from this. Researcher asks him to use algebra. He noticed from the expanded form that the y-intercept is at 64. He placed this point on the graph. His sketch passes through the y-intercept of 64 but looks like the graph from the function family $y = -x^3$. He did not recognize the x-intercepts nor does he link what he knows about the function rule being of "power of 4". Also, he his sketch does not suggest a link between absolute value and the shape of the graph.

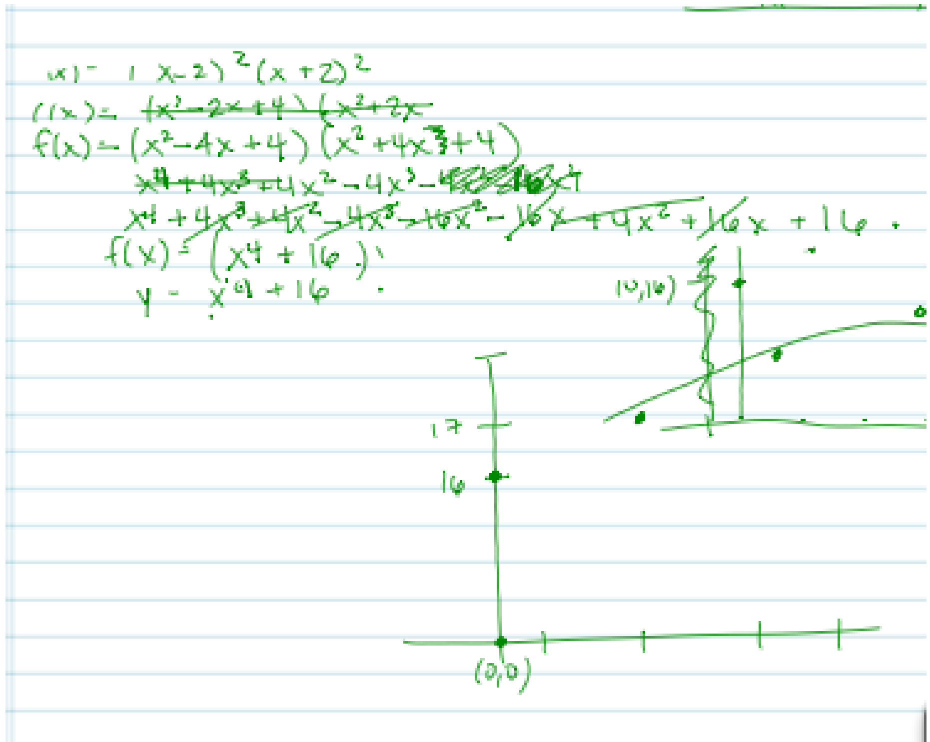
Student **Task 7b**



Nadia Nadia began by expanding the function rule. The result is the function rule y = x^4 – 16 [incorrect expansion]. She stated she does not know what the graph of x^4 would look like, but that she saw x^2 and knows that is a parabola. She made a link between the function rule and properties of the graph (increasing and having only positive values). Her reasoning is that even when you substituted negative values into the function rule you are going to get positive values. She substituted x = 1 and x = 2 into the function and begins plotting a graph.

She is asked by the researcher to reason about the graph y = |x – 2|. She drops the absolute value and made a link between the function rule y = x – 2 and the graph (connection to y = x, straight line, move it two to the left. She stated she does not remember the graph of $y = |x|$. She stated it might be like $y = x^3$ but she is not sure.

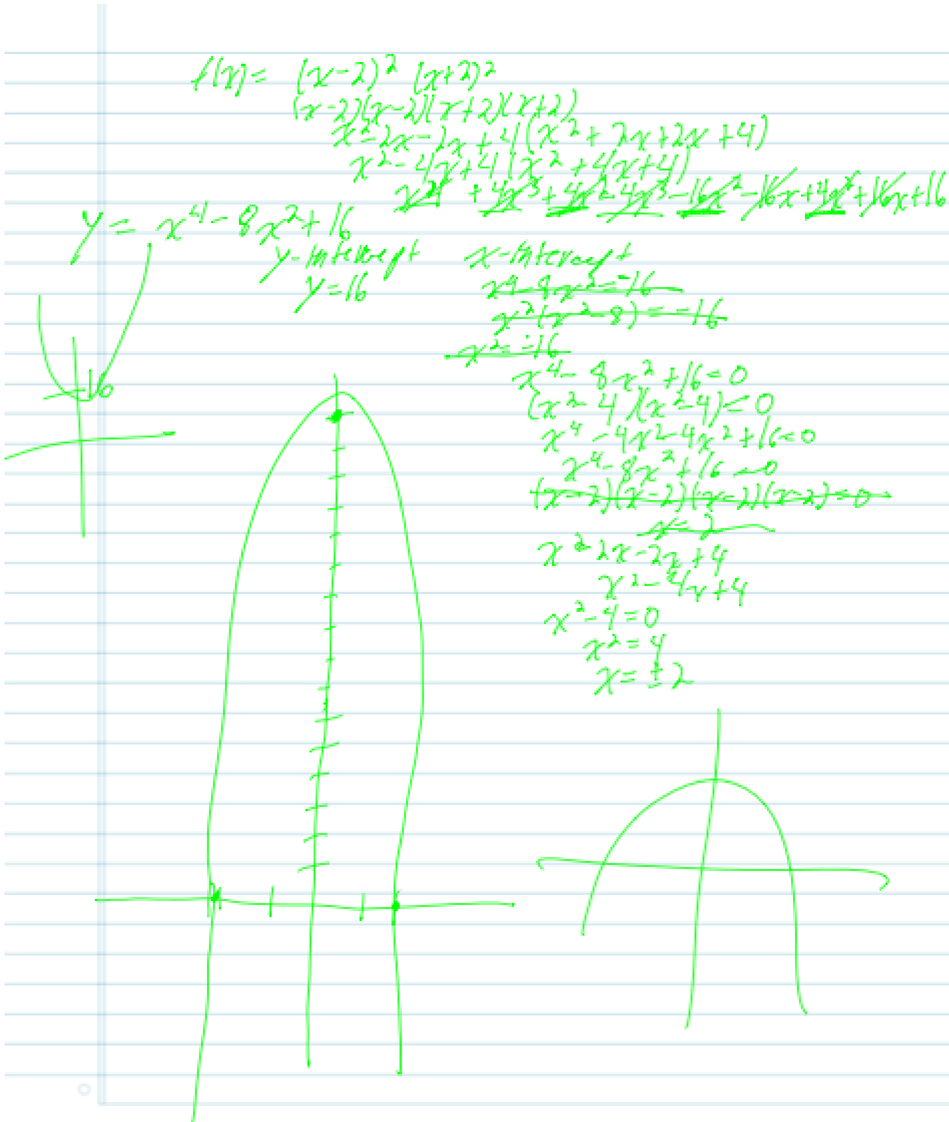
Student **Task 7b**



Robin

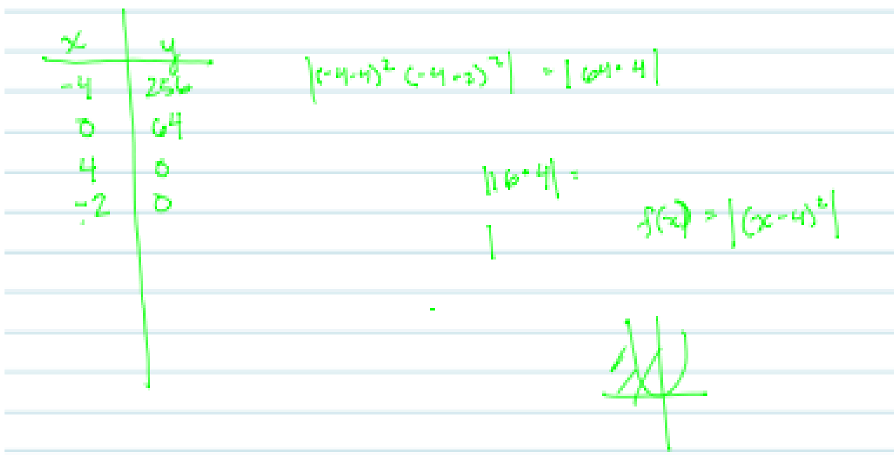
Robin expanded the product of the binomials. This resulted in the function rule y = $x^4 - 8x^2 + 16$. She substituted $x = 0$ into the function rule to determine the y-intercept and she began a procedure to find the x-intercepts. The result of this procedure is $x^2(x^2 - 8) = -16$. She started another procedure (zero-product property), but abandoned it once she realized it is not appropriate for the situation (not equal to zero). She continued with a factoring procedure on $x^4 - 8x^2 + 16 = 0$. The results of this procedure left her with $x = \pm 2$. She made links between the function rule and properties of the graph—x-intercepts, y-intercept, and shape of the graph. She sketched a parabola form opening downward. This caused her cognitive conflict because she believes it should not be an “upside down parabola” because the first term is not negative and because x to even powers are U-shaped.

Student **Task 7b**

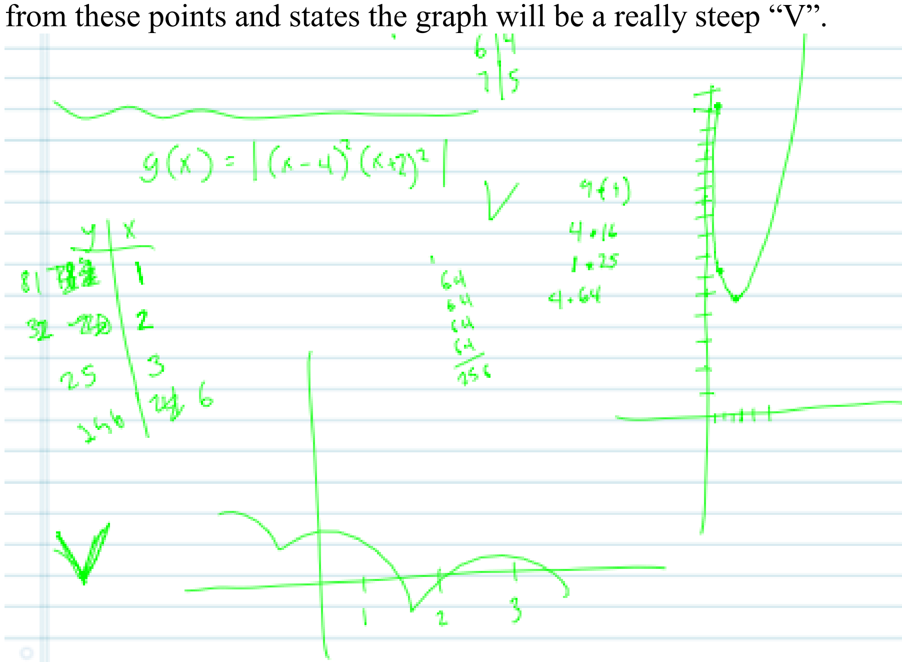


Betsy Betsy began with a table of values writing x = -4, but stopped, She created a prototype f(x) = |x – 4| and reasoned that it is one where you end up with half a parabola because everything under absolute value is positive so the negative part of the parabola (left of the y-axis) is not included. She made a statement about translating but provided no explanation to what she meant. She returned to her table of values strategy on the original function rule entering inputs of x = (-4, 0, 4, -2). She mentioned that she chose x=4 because that would make the output 0 and she liked the get the same number on each side—this point followed substituting input of 0 into the table of values—she does call these intercepts. She reasoned that the graph may be sideways and stated she is pretty sure it is only half a graph.

Student **Task 7b**



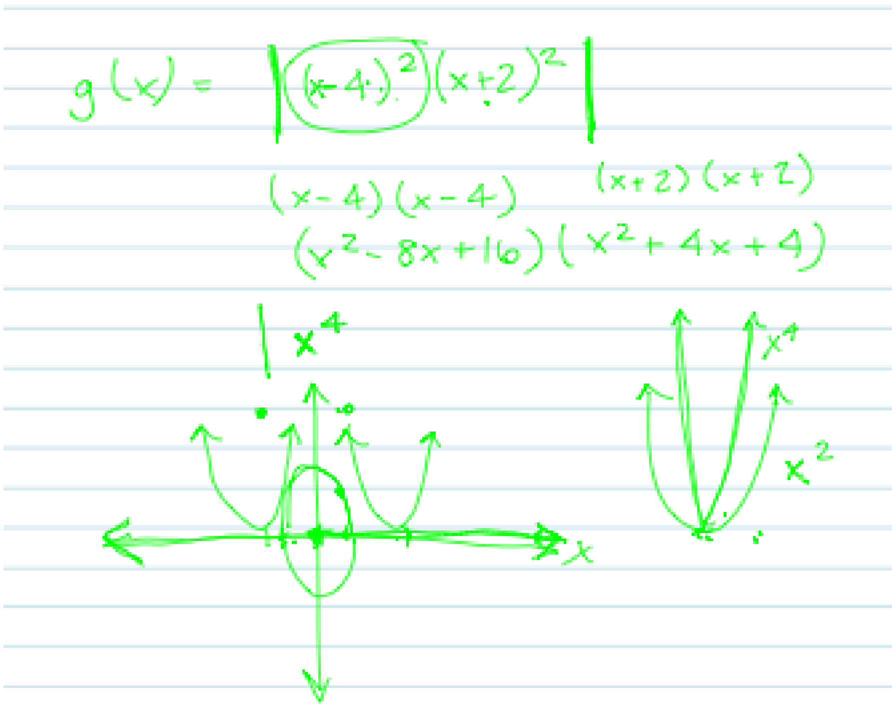
Molly Molly begins an unrefined table of values (x = 1, 2, 3, 6). Her table of values is labeled (y,x). She made a link between the function rule (absolute value) and shape of the graph (v-shaped). She made a sketch from these points and states the graph will be a really steep "V".



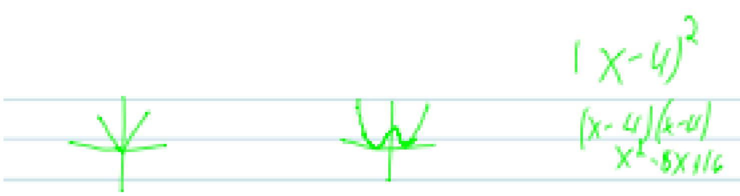
Roxie Roxie is uncertain of the parent function type. She debated between absolute value and a parabola. She begins with an expansion of each binomial factor. She stated the graph would be y = x^4, but does not provide meaning to this statement. She made a link between a characteristic of the function rule (absolute value) and a property of the graph (above x-axis) and provides an example of what she meant later in her reasoning process. She made a link between each factor in the

Student **Task 7b**

original function rule and the shape of the graph (parabolas with vertices on the x-axis). She mentioned that she does not know how to handle the task when the two factors are multiplied together. She stated that the graph of $y = x^4$ would be "horizontally compressed" or "stretched vertically" compared to $y = x^2$ making a general statement that "your y-values would be higher than they would be in the x^2 parabola."



Todd Todd reasoned that the graph will not go below the x-axis because of the absolute value function and explained why this is the case. He reasoned that it is a quartic function because it is a paired quadratic that he stated makes the graph curved. He made a refined link between the original function rule and the location of the *x*-intercepts (though he does not specifically state their location the researcher could infer he knew what he was saying by the relative placement on his sketch) and the behavior of the graph at those points—"bounces off" and interpreted the meaning of *x*- intercept—where $y = 0$. He stated the graph has four solutions. His sketch is W-shaped and seems to have all the relevant features.



Student	**Task 8**
Ashley	Ashley rewrote the bases as powers of 2 stating that when the bases are the same she can compare the exponents. She completed a valid algebraic procedure that resulted in a correct answer.

$$4^{2x^2-7x+3} = 8^{x^2-x-6}$$
$$(2^2)^{2x^2-7x+3} = (2^3)^{x^2-x-6}$$
$$2^{4x^2-14x+6} = 2^{3x^2-3x-18}$$
$$4x^2-14x+6 = 3x^2-3x-18$$
$$x^2-11x+24 = 0$$
$$(x-8)(x-3) = 0$$
$$x = 8 \text{ or } x = 3$$

Casey	Casey is not sure that it is true, but she rewrote the bases as powers of 2. She stated that if she has the same bases, then the powers are going to have to be equal. She completed a valid algebraic procedure that resulted in a correct answer.

Student **Task 8**

$4^{2x^2-7x+3} = 8^{x^2-x-6}$

$2^{\underbrace{2(2x^2-7x+3)}} = 2^{3(x^2-x-6)}$

$4x^2-14x+6 = 3x^2-3x-18$
$-3x^2+3x+18 \quad -3x^2+3x+18$

$x^2-11x+24=0$ 24

$(x-3)(x-8)=0$ 1 24; 2 12; 3 8

$x=3,\ x=8$

$4^{2(9)-21+3} \quad 8^{9-3-6}$

$4^{18-21+3} \quad 8^{9-9}$

$4^{-3+3} \quad 8^{0}$

$\begin{array}{r} 64 \\ \times 2 \\ \hline 128 \end{array}$ $1=1 \quad x=3$

$4^{2(64-56+3)}$

$\begin{array}{r} 128 \\ -53 \\ \hline 75 \end{array}$ $4^{2(64)-56+3}$

4^{128-53}

4^{75}

Dan Dan re-presented the equation as $4^a = 8^b$ and recognized the bases can be written in terms of powers of two. He performed an algebraic procedure that resulted in $2^{2a} = 2^{3b}$ and stated since he has the same bases, $2a = 3b$. He substituted in the known quantities in for a and b and performed an algebraic procedure that resulted in the correct answer.

Student	**Task 8**
	$4^a = 8^b$ $2^{2^a} = 2^{3^b}$ $2^{2a} = 2^{3b}$ $2a = 3b$ $2(2x^2 - 7x + 3) = 3(x^2 - x - 6)$ $(4x^2 - 14x + 6 = 3x^2 - 3x - 18) - 3x^2$ $(x^2 - 14x + 6 = -3x - 18) + 3x$ $(x^2 - 11x + 6 = -18) + 18$ $x^2 - 11x + 24 = 0$ $\quad x = 8, 3$ $(x - 8)(x - 3) = 0$ $x^2 - 3x - 8x + 24$
Newt	Ran out of time.
Jim Logarithms Natural logarithms and e	Jim began by stating the conditions under which he can perform a certain procedure (If the bases of two equal exponential expressions are the same, the bases can be dropped the exponential can be equated and solved for). He began an algebraic procedure that involved multiplying the left side of the equation by 2^{2x^2-7x+3} but quickly realized he cannot do this procedure. He began another procedure rewriting the exponential expressions as logarithmic expressions and then in terms of e and natural logarithms. He abandoned this procedure and is unsure what to do next.
	$4^{2x^2-7x+3} = 8^{x^2-x-6}$ $2^{2x^2-7x+3} \cdot 4^{2x^2-7x+3} = 8^{x^2-x-6}$ $8^{8x^2-14x+6} = 8^{x^2-x-6}$ $4x^2 - 14x + 6 = x^2 - x - 6$ $\log_4(2x^2 - 7x + 3) = \log_8(x^2 - x - 6)$ $e^{(2x^2-7x+3)\ln(4)} = e^{(x^2-x-6)\ln(8)}$ $2x^2 - 7x + 3 \ln(4) = (x^2 - x - 6)\ln(8)$ $\frac{2x^2-7x+3}{x^2-x-6} = \frac{\ln(8)}{\ln(4)}$
Paul	Paul recognized 4 is a factor of each base and rewrote the base of 8 as 4 * 2. He used a symbol to denote the function that is in the exponent of each side of the equation. Paul started another algebraic strategy (exponentiation) using properties of logarithms and exponents. When prompted to consider

Student	**Task 8**
	an easier way he stated that if both bases would have been to the same power it would have been easier or if both exponents would have been the same function.

$$4^{2x^2-7x+3} = 8^{x^2-x-6}$$

$$4^{2x^2-7x+3} = (4\cdot 2)^{x^2-x-6}$$

$$= 4$$

$$2x^2-7x+3 = X$$

$$x^2-x-6 = Y$$

$$4^{X} = (4\cdot 2)^{Y}$$

$$4^{X} = 4^{(X)}\cdot 2^{Y}$$

$$e^{2x^2-7x+3\ln 4} = e^{x^2-x-6\ln 8}$$

$$(2x^2-7x+3)\ln 4 = (x^2-x-6)\ln 8$$

$$\frac{2x^2-7x+3}{x^2-x-6} = \frac{\ln 8}{\ln 4} > \ln(8-4)$$

$$\frac{(2x-1)\cancel{(x-3)}}{\cancel{(x-3)}(x+2)} = \ln 4$$

$$\frac{2x-1}{x+2} = \ln 4$$

$$2x-1 = \ln 4\,(x+2)$$

$$2x-1 = \ln 4\, x + 2\ln 4$$

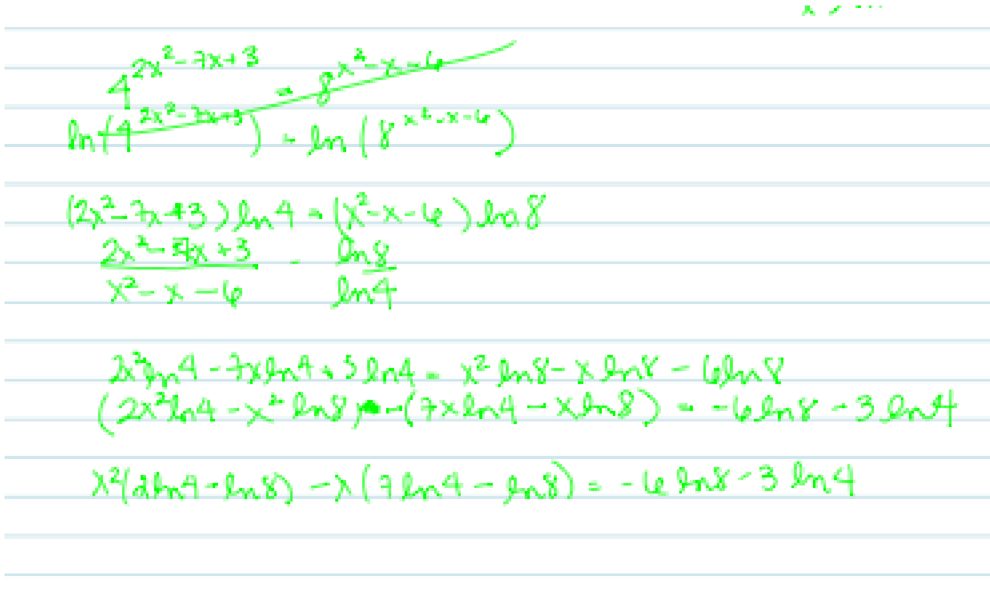
$$(2-\ln 4)x = 2\ln 4+1$$

$$x = \frac{2\ln 4+1}{2-\ln 4}$$

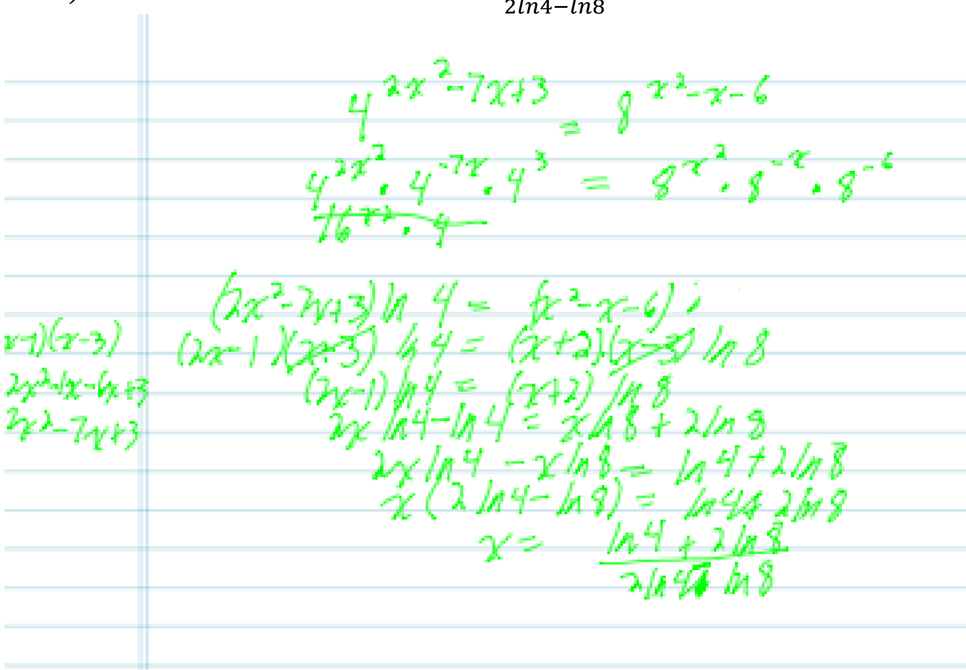
Nadia	Nadia began with an algebraic procedure (taking natural logarithm of both side) because she wanted to bring the x down so she can solve it. She

Student **Task 8**

mentioned that is taking the derivative of *a* to the *x*, but is unsure what to do.

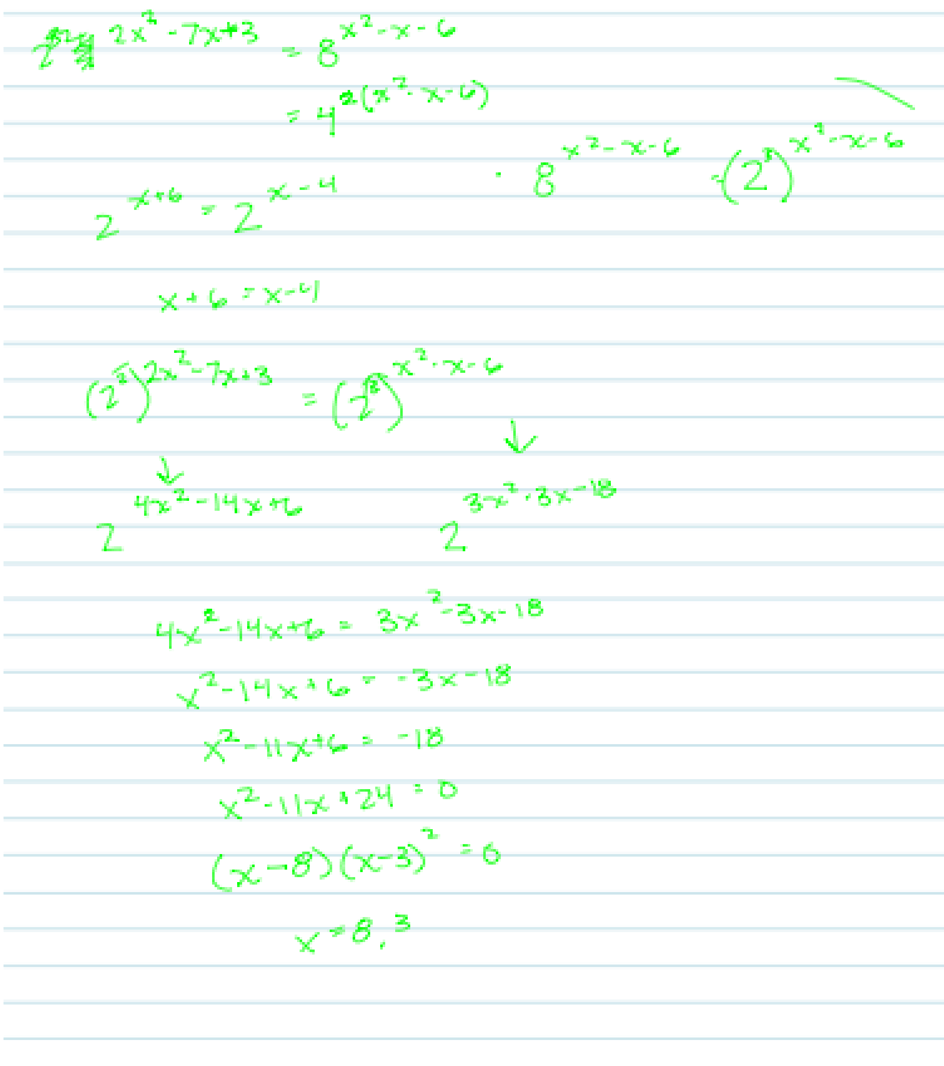


Robin Robin began an algebraic procedure rewriting the original equation as $4^{2x^2} * 4^{-7x} * 4^3 = 8^{x^2} * 8^{-x} * 8^{-6}$ with the goal of getting the fours to be the same base as eight, but she recognized her procedure would not work. She seemed to know that having the same base would help her solve the problem. She began another procedure (taking natural logarithm of both sides) that results in an answer of $x = \frac{\ln 4 + 2\ln 8}{2ln4 - ln8}$.



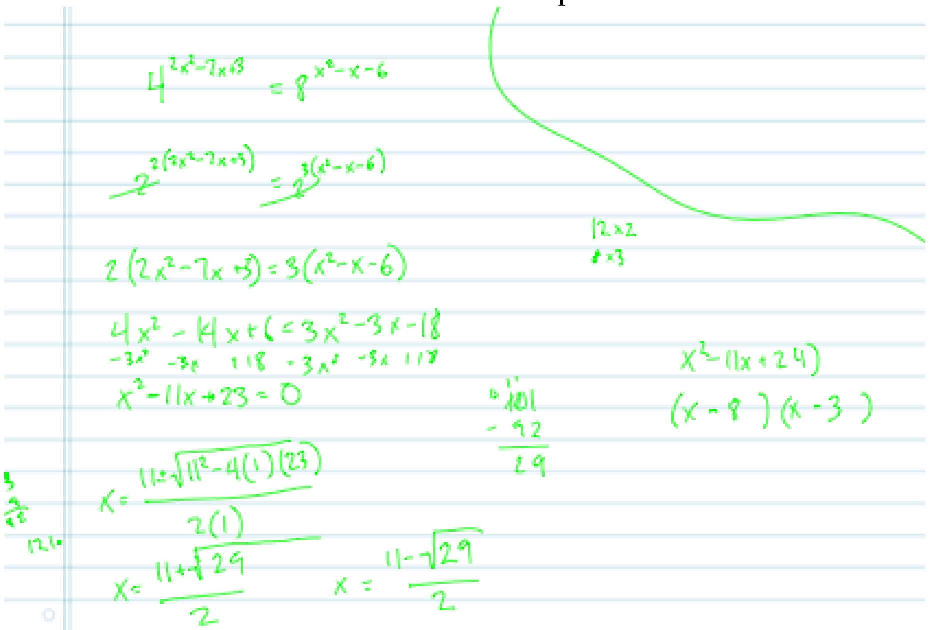
Student **Task 8**

Betsy Betsy created a prototype $2^{x+6} = 2^{x-4}$ and stated that if they have the same base that it can be solved and also stated it like a problem she did on the SAT. She began an algebraic strategy making the bases powers of 4. She realized that 4^2 is 16 not 8. She followed up with an algebraic strategy making the bases powers of 2. She stated she is not sure if she can do the step combining exponents, but performed all algebraically valid steps leading to the correct answer.

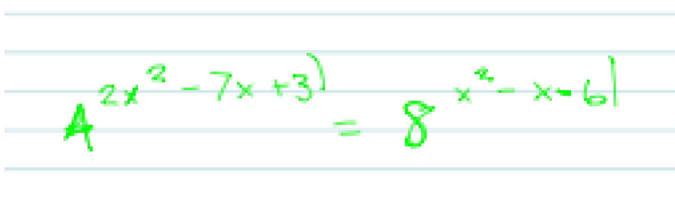


Molly Molly began by writing the bases as powers of 2. She stated that if the bases are the same you can cancel them out and solve for the exponents. She began with the common algebraic procedure, but instead of factoring she used the quadratic formula. She made a mistake in her computation. The researcher pointed out that she made a mistake and asked whether she could factor the $x^2 - 11x + 24$. She factored correctly, but is unable to interpret

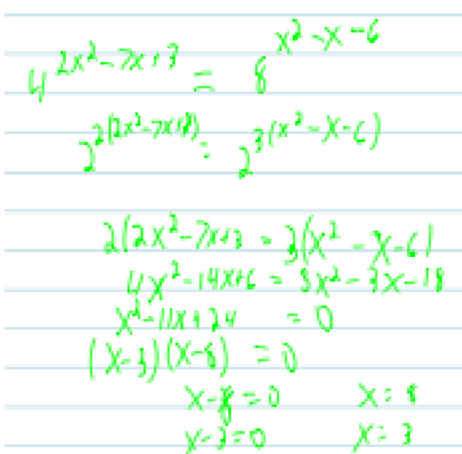
Student	**Task 8**
	what this would mean in the context of the problem.



Roxie Roxie is not able to engage with the problem.



Todd Todd began with an algebraic strategy making the bases powers of 2. He stated that if the bases are equal then the exponents are equal and set the exponents equal to each other. He performed all algebraically valid steps leading to the correct answer.



Appendix I

Student Task Completion

	1	2a	2b	3	4	5	6a	6b	6c	6e	7a	7b	7c	7d	7e	8
Methods																
Newt	x			x	x	x		x	x			x				x
Casey	x			x	x	x		x				x				x
Dan	x		x	x	x		x	x				x				x
Ashley	x		x	x	x	x	x	x	x			x				x
Becky																
Mandy																
Calc II																
Nadia	x		x	x	x		x				x					x
Sandy																
Robin	x			x	x		x				x					x
Jim	x		x	x	x		x	x				x				x
Paul	x		x	x	x			x				x				x
Chris																
PreCalc																
Todd	x	x	x	x	x	x	x	x		x		x		x		x
Lucy																
Molly	x	x	x	x	x		x					x				x
Colin																
Roxie	x	x	x	x	x		x					x				x
Betsy		x	x	x	x		x					x				x

Appendix J

Analysis of Newt's Reasoning

Background: Newt took Honors Algebra 2 in high school and stated he enjoyed teaching geometry more than algebra. He took Calc AB in high school.

Coding
R—Recognizing
A or I —Reasoning (possible difference between Analysis and Interpretation)
L—Linking
C—Connecting (How is it different from linking?)

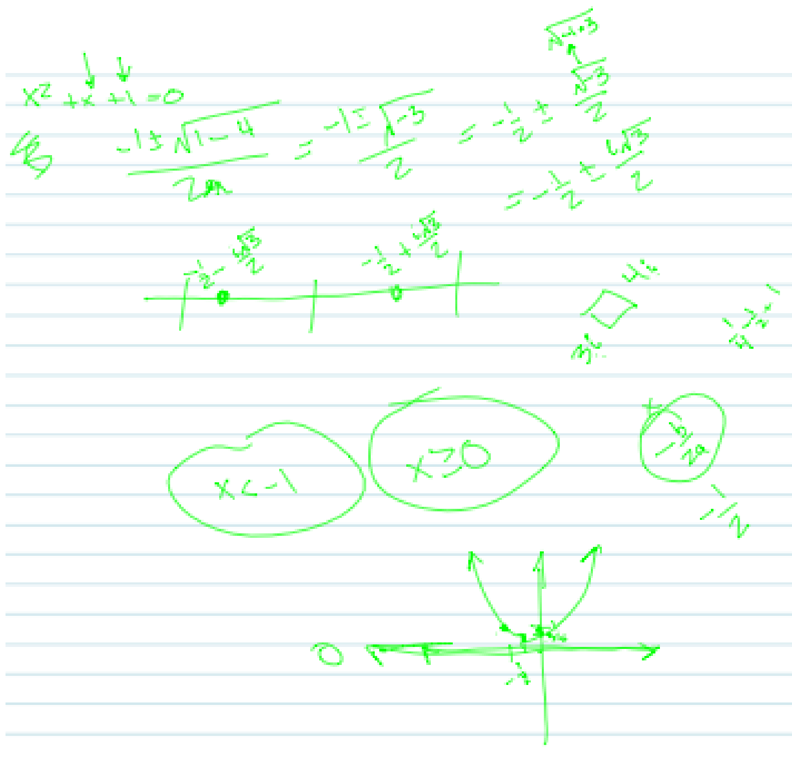
Task 1

Solve for n: $1 - \frac{1}{n+3} - 1 + \frac{1}{n+3} = \frac{1}{72}$

R1 E(original equation)—[Common Denominators]—T(Combining like terms)
R2 E(1 and -1 in original equation)—[Additive Inverses)—T(Cancelling)
"the first thing I see is common denominators. I am just gonna combine all the like terms without really thinking about anything else, and then I see the ones cancel right away, and so, oh wait, everything, so it's zero, one over seventy two."(79-83)
I E($0 = \frac{1}{72}$)—" that's not a true statement, so there is no value for *n* for which this is true."(84-85); "Ah, no solution. It's like, ah, it would be like five plus *n* equals seven plus *n*. There is no *n* for which that is true. "(88-89)
Commentary: Newt does not write anything down. He notices the feature, "everything cancels" that minimizes procedural working. Unlike other students, who interpret the solution as a cue to attempt another procedure, Newt interprets the solution in terms its meaning in the context of the problem.

Task 3

Solve for x: $x^2 + x + 1 > 0$.



C E(Original inequality)—[Form—Cuing]—Representation as an equation

R1 E(x^2+x+1=0)—[Form—cuing procedure]—T(factoring)

"I see a quadratic inequality, and uh, the first thing I think when solving it is just solve it like a quadratic equation and then I, you know, … number at this point, so … I treat it like an equation, and try to factor it. I got… I'm not gonna be able to factor it."(91-95)

I Describing why the expression x^2+x+1 cannot be factored.

"So after you multiply you are gonna get one, and add, or like the constants still have to multiply and get one, and add to get one, and the constants multiply to get one…and off the top of my head, one or like two and point five and they are never gonna add to get one."(98-102)

R2 E(x^2+x+1=0)—[Form—Cuing procedure]—T(Quadratic formula procedure)

"So, the next thing is the quadratic formula. So, negative B is minus the square root of B squared minus 4AC… so, the square root of a negative number, so the only solutions will be complex. And ah, um, let do that out…so its negative one plus or minus the square root of negative three over two. So it's negative one half… negative three over two… which … squared three… negative three times, it's the same thing as negative one times three. So it's ___ root three over two."(104-112)

I Describing the meaning of the results of the quadratic formula procedure
"So that would be a complex root if this was an equation. But since this is an inequality, so that would be two complex roots. But, ah, there would be like a negative one half minus then negative one half plus… but since it's complex I don't even know… how… an inequality would work, cuz I don't know how like … three I is greater than or equal to four I kinda of thing. Or, well, no, negative one half plus, so this would be negative one half minus the I root of three over two, this would be negative one half plus I root of three over two… um, but that's if I, assuming that if adding an imaginary number will get you a larger value, which I'm not even sure of. "(112-123)

L Results of quadratic formula procedure and location on number line
"I used the quadratic formula. And then I saw there were going to be, an I was going to be a part of it. And so I tried to keep going and draw a number line, because at this point if these were real numbers, then I would test the point here and here and here… but since I don't have an understanding of what's bigger and smaller with complex numbers, I said well I'm just gonna do something that's a lot easier than that which was just try values."(196-204)
Commentary: Newt seems to be inhibited by his inability to quantify the magnitude of the complex numbers that he has gotten. Newt's inability to *interpret the results of the quadratic formula procedure in the context of the problem seems to cause him to abandon this strategy.*

R3 E(original inequality)—[Form—cuing procedure]—T(Numerical-based analysis)
"Well, kinda silly. Like, without even trying to deal with all of this complex stuff, I can look and see that zero works, one works, anything greater than zero works right off the top of my head."(136-140)

A Reasoning from R3 about a set of numbers in the domain of the solution
"*x* squared, if I plug in any positive real number uh, it's gonna be positive, it's gonna\be positive, and one is greater than zero. So it's always gonna be greater than zero."(142-144)
Commentary: Inductive found, but deductive described generalization.

R4 E(original inequality)—[Form-cuing procedure]—T(Numerical-based analysis)
"Um, Yeah, negative two works, negative four, yeah, every, all reals gonna work here."(145-147)

A Reasoning from R4 about a set of numbers in the domain of the solution
"Any x less than negative one is going to work because when you square it it's going to be positive and larger than just minus constant value, and you have plus one is gonna be greater than zero and also everything equal to zero is going to work because zero works, one works, and it's only gonna get bigger."(148-153)

R5 E(x^2+x+1=0)—[Coefficients]—T(Procedure for finding location of vertex)

L E(Results of R5) and property of the graph(location of vertex)

- L Property of the graph(location of the vertex) and solution set of inequality
"But also another way I could think about it is just think about the graph. If I were to just graph the vertex is I think negative *b* over two *a*. It's the *x* value of the vertex, if I were to consider this to just be an equation. I think that's what the vertex is, so it would be negative one over two, so it would

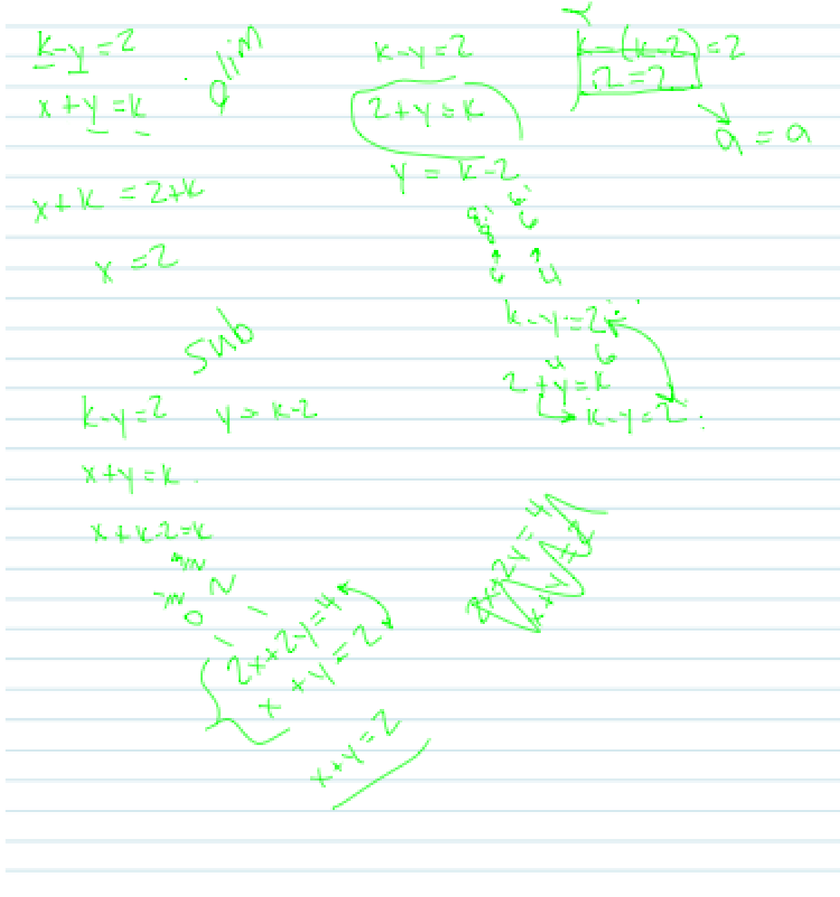
be negative one half. It would be like right there. And then *y* value of the vertex would be one fourth minus one half plus one. So that would be point five plus, so it would be point seven five. So this is one half.. one would be, three fourths, so if the vertex is there, it's gonna make that I'd say all reals, just like that."(155-165)

L E(x^2+x+1) and a property of the graph(opens upward)
"Ah, the coefficient of the *a* term is positive"(174-175);" The x squared term"(176)

A Reasoning from properties of the graph of $y=x^2+x+1$ about the solution
"Um, this is like with zero, since it's an open up parabola, and the vertex is greater than zero, every other point is going to be greater than zero."(168-169)

Task 4

Is it true that the following system of linear equations: $\begin{cases} k - y = 2 \\ x + y = k \end{cases}$ has a solution for every value of *k*?



R1 E(original system of equations)—[Form—cuing procedure]—T(elimination procedure)
"So if *x* equals two, use the elimination method, so I have, *x* equals two. Does that work nicely? And so if *x* equals two, how does that change things? *y*, two plus *y* equals *k*. So I have *y* equals *k* minus two, so then bring that into here, so *k* minus two, two, zero. So Two equals two."(210-215)

I Describing the meaning of the results of R1
"So Two equals two. So that tells me nothing about *y*. So, when I have something like this that is always true, I am starting to think that it's gonna work for every value of *k*."(214-215)

R2 E(original system of equations)—[Form—cuing procedure]—T(substitution procedure)
"So if I just use substitution to solve for *y*. I know based on that equation that *y* is *k* minus two, and so here I have *x* plus *k* minus two equals *k*… and again so I have *x* equals two."(223-225)

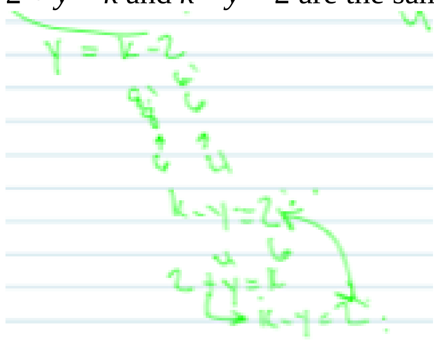
I Describing the meaning of the results of R1 and R2
"So I can get *x* equals two, and then, but I can never, any value of, I can't get a value of *y*. I just get two equals two. So that makes me think every *y*, it doesn't

even matter what *y* is. There, there are an infinite number of ways for this to work, I think."(225-230)

R3 E(*k*-*y* = 2)—[Form]—P(relationship between *k* and *y*)

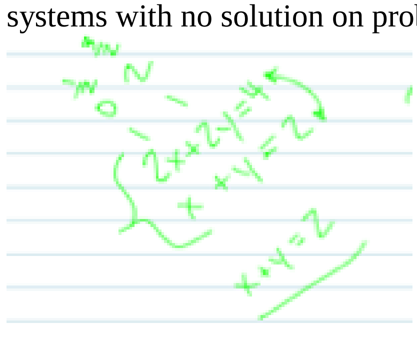
"If I have *k* plus [minus] *y* equals two. Two plus *y* equals *k*. Well, um, six minus four, oh yeah, I'm pretty sure.. cause these, the difference between them is always going to be two."(230-233)

C1 2 + *y* = *k* and *k* – *y* = 2 are the same equations



I Meaning of C1—" So it's just the same thing twice. Um, this is the same thing. And so there are infinite, there are infinite values for this, for *k*."(234-236);" Hey, whenever I have , you know, I'm solving for something in algebra and I have *a* equals *a*, I'm pretty sure that means infinite solutions, so I just wanted to confirm that so I did substitution method again, or instead of elimination of, the same thing happened and then I realized … and so for any given linear equation there are infinite values that work."(250-256)

C2 Between form of system with no solutions in this problem and form of systems with no solution on problems he has done in past.



"Except normally I'm used to just, you know, two *y* equals four, and then like *x* plus *y* equals two, and something like that there. I'm used to, you know, no *k*. I'm used to something like that, having infinite solutions."(262-265)

"Well, so, I was, I saw that these two were the same so I was like, is the reason there are infinite solutions because both equations in the system are the same. So I wrote two equations that are the same, and ah, so like one, one would work. And ah, zero and two would work, one half and three halves would be two plus, or one… yeah, so that is the rule."(292-298)

A Reasoning from two equations that are the same about the solution set

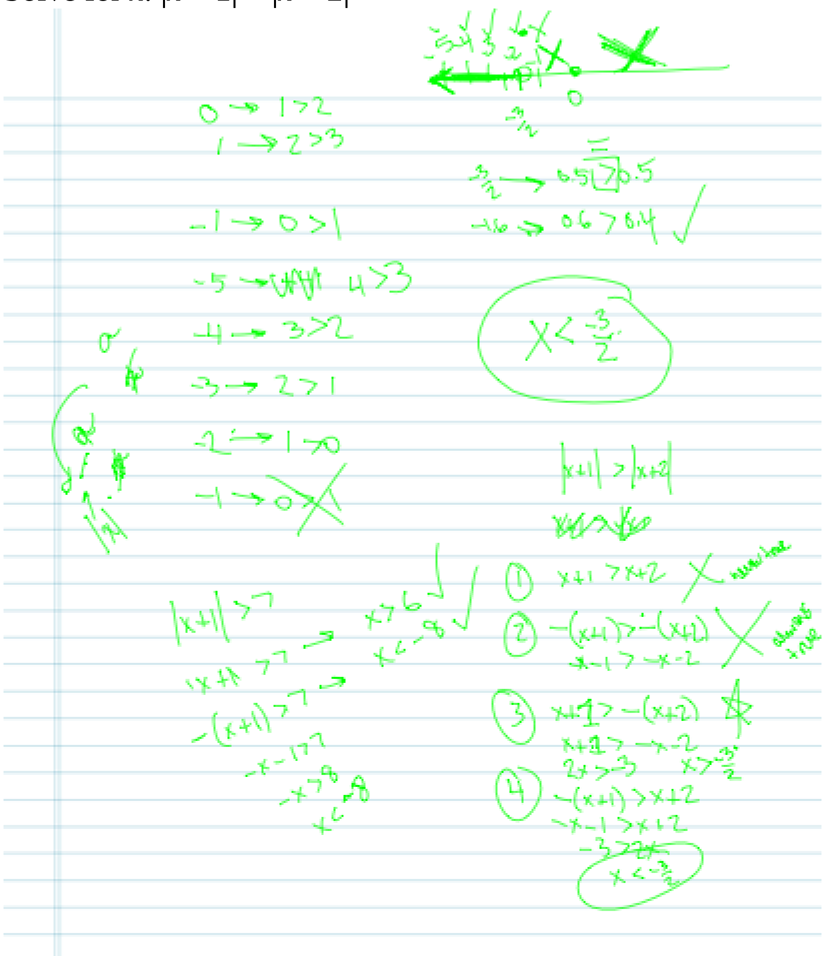
"Ah, if these are the same statement then there are infinite solutions… because it's just basically together these things are really just *x* plus *y* equals two. And there's an infinite number of real numbers that are two and two."(300-303)

"I was confirming that I saw here, I was one hundred percent sure here that there are infinite solutions, but I was confirming that the reason for there being infinite solutions was because both equations in the system were the same. So, that's why it happens."(306-310)

Commentary: The two equations are not technically the same. It happens that the solution set for the system is $x = 2$ and $y = k – 2$. The first equation equation $k – y = 2$ describes a relationship between the parameter k and the variable y and the second equation describes a relationship between k and the variables x and y. Newt does mention the relationship between y and k, but he seems to be treating x, y, and k as variables—" I think, maybe this does have a solution for every value of K because there's three very, oh wait."(206-208)

Task 6B

Solve for x: $|x + 1| > |x + 2|$



R1 E(original inequality)—[Form-Cuing Procedure]—T(Numerical-based analysis)

A Reasoning from the original inequality about a set of numbers [non-negative] in the solution set

"Of the absolute value of something plus one, is it bigger than absolute value of something plus two. Well, for positive numbers this is never gonna be true, because the absolute value won't matter, because where, where, we'll be taking the absolute value of a positive number so for example, zero, one is not gonna be greater than two, two, one, like for zero, you have greater than two. It's just gonna keep happening. Those are true."(315-322)

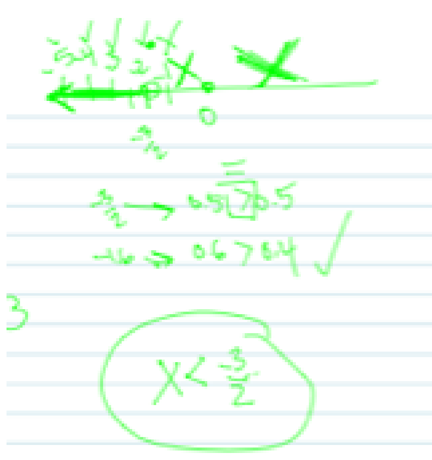
Commentary: Inductive generalization of the set of numbers in the solution set.

R2 E(original inequality)—[Form-Cuing Procedure]—T(Numerical-based analysis)

A Reasoning from the original inequality about a set of numbers (negative less than -1) in the solution set.

L Values in solution set of inequality and points on the number line
“And so then negative one, and even negative numbers, I just want to think, negative one is gonna get you zero is greater than one, and then you can look bigger negative numbers is gonna get you something like negative four … now it’s gonna be absolute value is gonna matter, so negative five plus one is negative four, is four is greater than negative five plus one, plus two. So now it’s gonna matter. So, It’s gonna be some here is a number line [creates number line shown below], so I know, this is one, this is zero. It’s all gonna be good over there [Places check mark on right hand side of number line]. It’s gonna be good up to negative one right now, too. But over at negative five, oh wait, what am I saying? It’s not gonna be true over here [Exchanges check mark for *x* on right-hand side of number line]. And it’s not gonna be true up to negative one, either. But I know at five it’s working. Right now I’m not even gonna bother doing algebra; I’m just gonna keep trying numbers. Let’s see how it works at negative four. Negative four would get me three, it’s greater than. Negative four plus two, it’s still working. Negative three would get me… negative three would get me two is greater than one, negative two… will get me one is greater than zero. And then negative one I just tried gave me zero is greater than one which is not correct. So, now it’s just a matter of what’s happening between negative one and negative two. If I keep going with this method… and… so let’s try to get three halves. Cause what happens is, whenever this is negative, oh, just let me think about that.

“Because everything was working, everything was working, I was going by integers. And one didn’t work. So I was like, ah, something special is between negative one and negative two, so I will try half way between them.”(363-366)



I (meaning of 0.5>0.5 in context of problem)—"Negative three halves would be point five is greater than, oh, point five, this looks like a border line. Now wait, no. Negative three halves is negative one point five, so it's one half negative one point five one half. So, it's gonna be an open circle here. So let's just confirm… negative one point six would give me point six is greater than …. Yeah. So I would say … (inaudible)…negative three halves"(322-355)

L (0.5>0.5) and number line

A Reasoning from a particular value substituted into the inequality about the solution set of the inequality.
"So let's just confirm… negative one point six would give me point six is greater than point four Yeah. So I would say x is less than negative three halves."(352-354)
"Yeah, because…just confirming… I knew things were working to the left of this [point to -1.5 on number line], like to the left of my border. I just wanted to confirm. Is it really working to the left of my border? Yes."(375-378)

Commentary: Unlike some participants, Newt is very intentional with his numerical reasoning strategy. He continues to refine his thinking until he reaches what he describes as the "border line". He jumps into thinking about the problem algebraically without being prompted. It seems like the numerical analysis enabled him to get a feel for the problem.

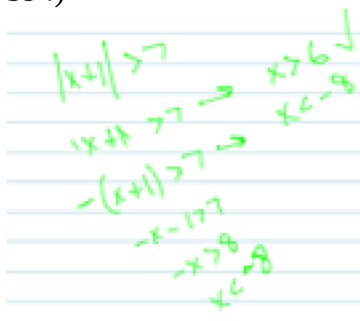
C (Form of original inequality, $|x + 1|>|x - 2|$) and (Form of prototype $|x + 1|>7$)—using an easier type of problem to evaluate the procedure for solving absolute value inequalities algebraically)

C E($|x + 1| > 7$)—[Form Re-presentation] into $(x + 1) > 7$ and $-(x + 1) > 7$
I(meaning of "absolute value")—" Well, absolute value of *x* is always going to be a positive… it's like saying the distance from *x* to zero, so there's two ways, there's two distances since *x* can be positive or negative, there's two distances from *x* to zero, there's you know, the distance from negative *x* to zero."(448-452)

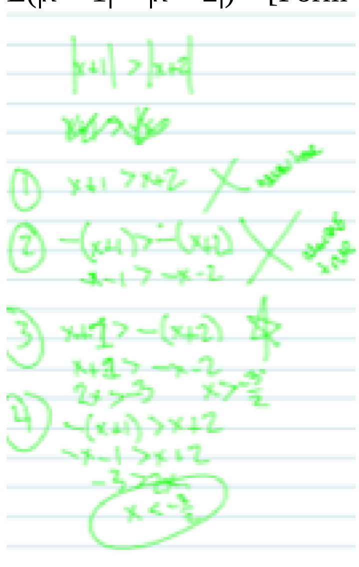
R3 E($(x + 1) > 7$ and $-(x + 1) > 7$)—[Form—cuing procedure]—T(Solving procedure)
"The *x* was seven … *x* could be positive, a positive or negative number. So if there was negative, it could be like negative number to zero or a positive number to zero. And both would be the same. So I need to do, you know, for positive and then negative. And it's also kinda just a memorized procedure I had."(453-457)

A Reasoning from results of R3 about the procedure for solving absolute value inequality problems of the form $|x +1| > |x + 2|$.
"I would get *x* is greater than six… and ah… eight….I want to see if that's true. And *x* is less than negative eight. That's true. So it's, checking here, thinking back, I was like hey, with absolute value, don't you just say, well, if it's absolute value the whole term was either, the terms were either all together positive or all together negative. And, cause if they were this

absolute value which is, yeah. So that's how I think about it. So in this case, I asked myself would I have to try positive and negative with that? And positive and negative with that, so I will have four different things to solve for, and then some would overlap and cancel each other out."(383-394)



C E(|x + 1| > |x + 2|) and (Piecewise defined inequalities)—part of procedure
R4 E(|x + 1| > |x + 2|)—[Form—cuing procedure]—T(Solving procedure)



"So, absolute value of *x* plus one, cause I'm not, I wouldn't do this because I'm not sure of it. But I'd say, just like here, I tried , you know, just removing the absolute value bars, or saying hey, it could all be negative … so … and I'm not even sure how I would, okay, number, the first one I'd try, make them both positive .. and actually that would never be true. So I could just say, one is greater than two, it's not gonna work. And then, the second one I'd try, all right, okay, let's make them both negative. Negative *x* plus one is greater than negative *x* plus two. In that case, it's negative *x* minus one is greater than, negative *x* minus two. So that would be negative one is greater than negative two, so it's always true. So that still doesn't tell me anything. So now I've tried. So I've made both positive and negative. Now I'll make the first one positive, the second one negative…. So it would be ah, *x* plus one, so it would just be *x* plus one is one, is greater than negative *x* minus two. So, the *x* terms are always cancelled. So, it's weird. So, I keep getting.. it's working for every *x* value, like here. I just, oh, no, what am I saying! Here I'd add *x* and they'd cancel, but here I'd add *x* and I'd have two *x* is greater than … ah, negative three. So here I'd get *x* is greater than … negative three halves. Now *x* is less than. But I just want to know, I made the first one positive, the second one negative. *x* plus one is greater than negative *x* minus two, two *x* is greater than negative Three. All right, so that's strange, but I'm gonna keep going. And negative *x* plus one is greater than, making the second term positive, the first one negative. So negative *x* minus one is greater than *x* plus two. I have … negative three is greater than two *x*. So here I have, here I have *x* is less than negative three halves. Right. *x* is less than negative three halves. Which is what my answer was. So, is this True? *x* is positive, *x* is greater than negative three halves and *x* is less than negative three halves. This is contradicting each other. What's going on. *x* can't be greater than, this can't be true because zero doesn't work. And zero is bigger than negative three halves. And so did I do something wrong here? I have positive first term, negative for the second term, negative *x* minus two, *x* plus one, add *x* both sides, I have two *x* is greater than negative three… *x* is greater than negative three halves. I don't know what's going awry here. But I mean, I got … down here and these I said forget about because it's, it will be, this is never true. So it doesn't tell me anything. And this is always true because it will just say negative one is greater than negative two. And this is telling me that this is true when it's actually not. And then this, is what I believe."(397-444)

A Reasoning from a particular value within the solution set as established in the numerical based strategy about the truth of the algebraic solutions $x < -\frac{3}{2}$ and $x > \frac{3}{2}$.

"So, is this True? *x* is positive, *x* is greater than negative three halves and *x* is less than negative three halves. This is contradicting each other. What's going on. *x* can't be greater than, this can't be true because zero doesn't work. And zero is bigger than negative three halves."(430-434)

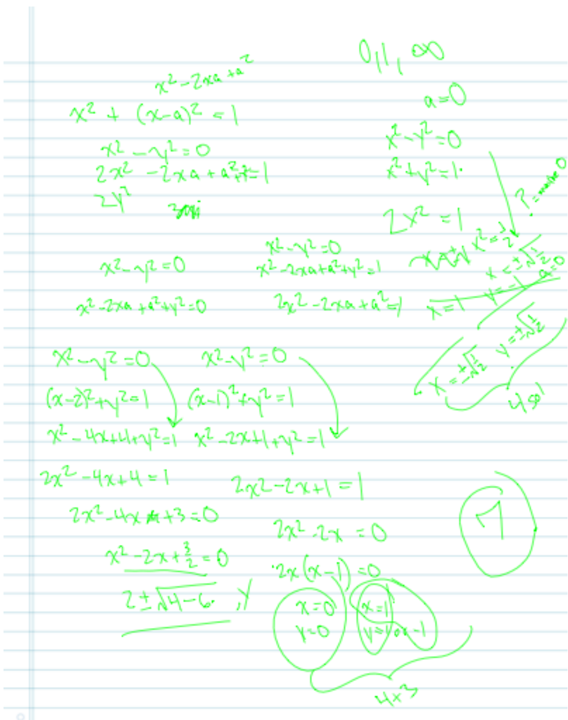
Commentary: Newt is like the other participants who have an algebraic procedure for solving the inequality in that their work is procedural in nature evidenced by the fact that they do not take domain restrictions into account when solving the equations. Not taking the domain restrictions into account creates some cognitive conflict for Newt when he solves the one inequality and gets and answer of $x > -\frac{3}{2}$*. [See underlined passage from above]*

Task 5

For what values of a does the pair of equations

$$x^2 - y^2 = 0$$
$$(x - a)^2 + y^2 = 1$$

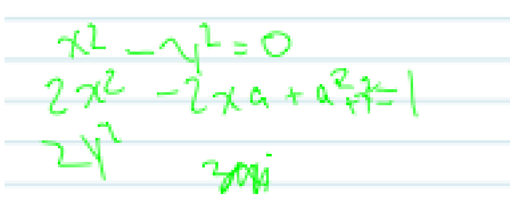
have either 0, 1, 2, 3, 4, 5, 6, 7, or 8 solutions?



R1 E(original system of equations)—[Form—cuing procedure]—T(elimination procedure)

R1A E($(x\text{-}a)^2$)—[Form—cuing procedure]—T(expanding procedure)

“, if we’re counting the number of solutions … the first thing I think of is, hey, is there either zero, one or infinite … but… but I’m not that sure of that, so I’m just gonna do elimination and see what kind of information I could gather. So, adding these two together I have *x* squared plus *x* minus *a* squared equals… so, *x* squared minus two *xa* plus *x* squared… So if I have two *x* squared minus two *xa* plus *a* squared equals … so I now I want to know…”(461-469)



Commentary: Abandons strategy: "so right now I'm just thinking about, okay, this is throwing me off. I don't like the A and X and the, I don't like the way this looks."(471-473)

R2 E($x^2 - y^2 = 1$)—[Form—Noticing Relationship—SI]—P(Relationship between x and y in solution)
"So, I was looking, just looking at these pairs of equations and trying things, cause okay, *x* has to be the same thing as *y*, or *x* has to be negative *y*."(473-476)
R3 E(original system of equations)—[Form—cuing representation of procedure]—T(substituting values for *a*)
C E(original system of equations) and $\begin{matrix} x^2 - y^2 = 0 \\ x^2 + y^2 = 1 \end{matrix}$.

I (*a* can be replaced with numbers)
R4 E(system of equations with $a = 0$)—[Form—cuing procedure]—T(elimination method)
Commentary: Instead of $2x^2 = 1$, Newt writes $x^2=1$
R5 E(x^2-y^2=1)—[Form—Noticing Relationship—SI]—P(-1 and 1 are in solution set)
A Reasoning from the system of equations about whether $x = 1$ and $x = -1$ are in solution set of the system.
"And ah, so I was like, all right, just make that constant zero. But then I'd have, that constant is zero, then I'd have *x* squared plus *y* squared and then I've have *x* squared plus *y* squared equals one. So then I'd have *x* squared equals one. I'd have *x* equals plus or minus one. And then *y*. So if I had *x* equals one and *y* equals negative one, does that work? If *a* is zero? I have one minus one is zero. One plus one doesn't work? Okay. That doesn't work. Okay. Based on elimination, if I say the constant is zero, then *x* does have to be plus or minus one, in which case it's either *x* equals one and *y* equals negative one, or *x* equals one and *y* equals one. Neither of which work. …. So I mean the fact that this fails is making me ask the question like maybe zero?"(479-493)
I (meaning of "*a*")—" But the constant definitely doesn't have to be zero. It can be anything."(493-494)

C E(original system of equations)—(Representation of expansion of 1st equation in system)
"So, maybe if from the start I do *x* minus *y* squared equals zero… and let's just multiply out that binomial squared. *x* squared minus two *xa* plus *a* squared plus *y* squared equals zero. Try to look at it like that and see if it makes, I can gather anything else from…"(494-499)
Commentary: At this point interviewer interrupts and asks what he was doing when he wrote $a = 0$. In the process Newt happens to find an error in his work in R3—"Over here I said now let a be zero. Let's just simplify things. Let's try a is zero. And see if I can. Cause if I made zero and I found at least one solution I

could say okay, at least it's not a zero. But then that didn't really get me anything. So"(501-505);" Yeah, so, when I did elimination I did, now wait! I had two x squared equals one. That was the problem.(508-509)

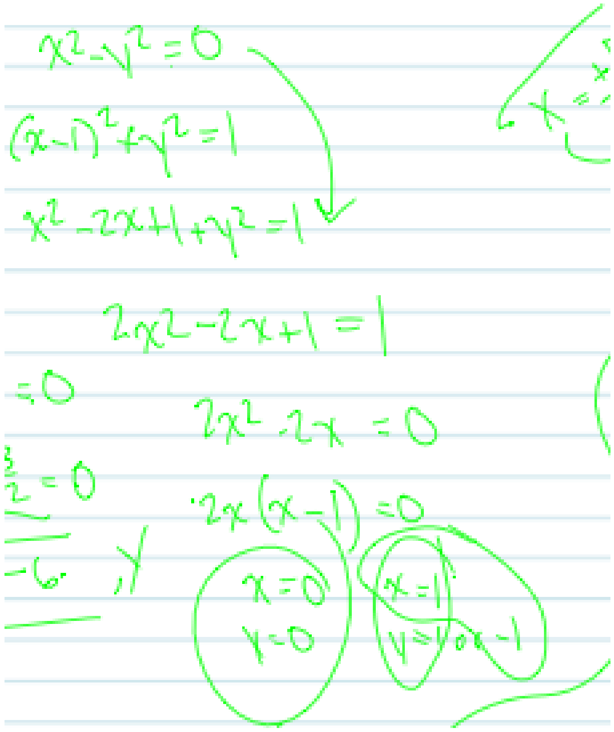
R4B E(system of equations with $a = 0$)—[Form—cuing procedure]—T(elimination method)

A From the results of R4B about the number of solutions of the system $a = 0$ "I had two *x* squared equals one. That was the problem. So *x* squared actually equals one half. So *x* equals plus or minus the square root of one half. So, we could have *x* equals … yeah, *x* equals plus or minus the square root of one half and *y*, now let's try that. Cause here I'd have one half minus one half is zero and then here I'd have one half plus one half is one. Oh! So there is at least one solution. So… Oh wait, there might be, just based on this there might be four because I could have positive, positive, positive, negative… negative positive, negative, negative, I think. So let's make sure of that. Yeah, because the squares are always gonna make it not matter whether it was positive or negative. So there, based on this, I'd say there is at least four solutions. Plus *a* is zero, I see four solutions."(508-523)

C Original system of equations and system of equations when $a = 1$

R4 E($\begin{matrix} x^2 - y^2 = 0 \\ (x-1)^2 + y^2 = 1 \end{matrix}$)—[Form—cuing procedure]—T(Elimination procedure)

"Because I'm not sure of any like specific procedure that's gonna give me a solution, so I'm just making this up as I go. *x* squared minus two *x* plus one plus *y* squared equals one. Add these two together *x* squared plus two *x* plus one … equals one. Two *x* squared minus two *x* equals zero. Zero works. …. *x* squared zero. If *x* is zero than Y must be zero. If *x* is one, *y* must be one or negative one…. So now I just, I plugged in *a* to *b*. I substituted one for *a* just to see what happens, and I got some things, so let's see if they actually work."(526-535)



A Reasoning from the results of the procedure (a = 1) about the solution set of the system of equations (solutions when a = 1)
"so let's see if they actually work. Zero, if x is zero and y is zero, it works. And if x is one and y is one, it doesn't work. If x is one and y is negative one … it works. So, that's three more solutions right here. I have x equals one, that and that. Right now I have seven; that makes seven. Three more."(534-539)

R5 E($\begin{matrix} x^2 - y^2 = 0 \\ (x-2)^2 + y^2 = 1 \end{matrix}$)—[Form—Cuing Procedure]—T(Elimination Procedure)

R5B E($x^2 + 2x - \frac{3}{2} = 0$)--[Form—Cuing Procedure)---T(Quadratic Formula Procedure)

I(Results of quadratic procedure)—" Ok, that's gonna get me complex answers"(552-558)

R E(original system of equations)—[Form-Cuing Procedure]—T(substation method)

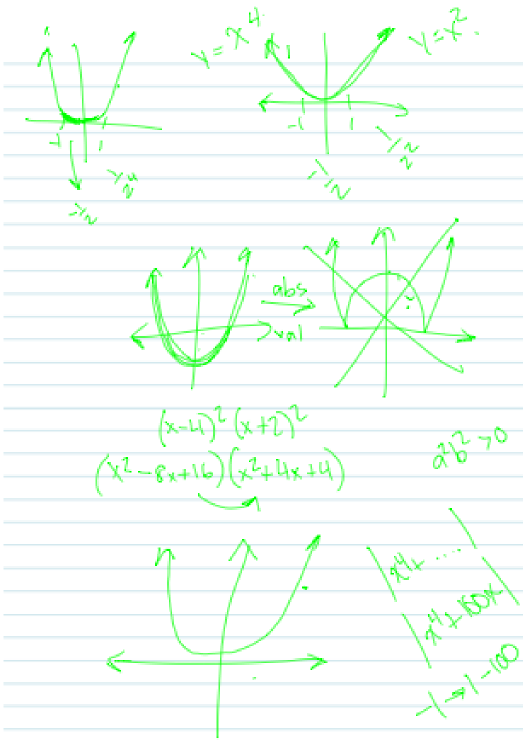
$$x^2 - y^2 = 0$$
$$x^2 - 2xa + a^2 + y^2 = 1$$
$$2x^2 - 2xa + a^2 = 1$$

"if I were to just do it from here in a general case, Ok, wait. If I were to do it in the general case… *x* squared minus *y* squared equals zero, and then *x* squared minus two *xa*… plus *a* squared plus *y* squared equals one… If I were to just do this in the general case, just do the substitution method, Two *x* squared minus two *xa*, plus *b* squared… equals one, and just say like do something from there… I mean, I tried *a* with zero… and that got me stuff, and then I tried *a* was one, that got me stuff. I tried *a* was two; that didn't get me anything."

Commentary: This seems to be another instance where Newt uses numerical-based reasoning to get a feel for the problem. The difficulty in this case is while specific cases of a give him possible solutions he is unable to make any generalized statements from the numerical analysis involving specific values of a.

Task 7B

Describe the graph of the function rule g(x) = |(*x*-4)^2(*x*+2)^2|.



R1 E(original function)—[Structure—SI]—P(Property of the function--Degree)

I (Describing why the function is of degree 4)

“Well, it’s gonna be, ah… degree four, because the largest, um, *x* squared, you’re gonna have *x* squared in here, so when you distribute, clean this up, there’s gonna be an *x* to the fourth term.”(588-591)

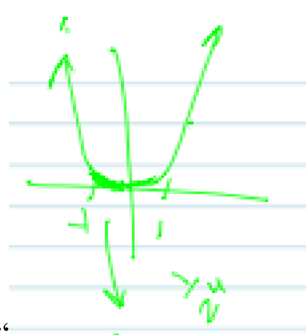
L E(Function of type y = x^4) and P(Shape of the graph)

L P(type of graph—parabolic) and P(property of graph of y = x^4)

“I know *x* to the fourth look like this; it looks parabolic but it’s not cause it flattens out. So I mean, oh, and it’s absolute value, which”(591-593)

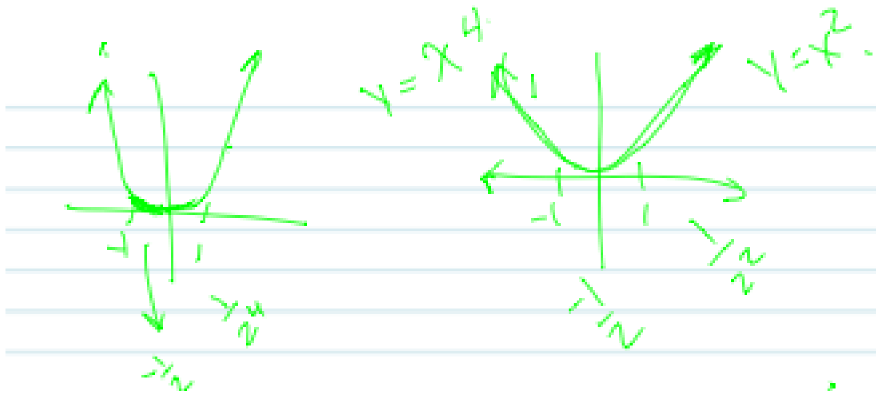
A Reasoning from function type y = x^4 about the shape of the graph

“



"When I think of degree four I think of this shape, where between, if it was just, just X to the fourth, it's really flat between here because you're having decimal, decimals, fractions of numbers to the fourth, so it's gonna be smaller than the average. Smaller than a parabola, but then it shoots up, passing a parabola because each term is multiplying by itself. (596-602)

"But I think this is, I think parabola, I think *x* squared and then I think *x* to the fourth except flatter… I think, you know, thinner for *x* to the fourth but I also remember that it's gonna be flatter at the bottom first because the numbers in here, like, negative one half, when you take them to the fourth, it's gonna be one over two the fourth, whereas here, *y* equals *x* squared is gonna be nearer between 8 point one. It's going to be… negative one would be one over two squared… bigger… so like here it's not gonna be going up very much between negative one and one. Because fraction results are only gonna get smaller. But here it's a little bit less small, but as soon as you get to one in each case it starts shooting up. In this case it's gonna start shooting up more because anything greater than the, anything greater than one or less than negative one to the fourth is gonna be a lot bigger than this."(605-621)



Commentary: Again Newt uses numerical based reasoning to talk about shape of the graph. He is able to generalize about the impact of intervals of the domain on the shape of the graph.

I (Meaning of absolute value on the shape of the graph)

"Well, as far as the absolute value goes, I don't… because say it was like this, and then you said, oh wait, but it's absolute value, just say this is like a quartic and it was not absolute value and then I saw absolute value and how do I think it would change? It would just be like … that… just make every negative positive, and so in this case all of this would be positive all the time. It just would be something like this, but if some values of *x* were to, if some values of *x* without the absolute value would be negative numbers, it would give you negative *y* values, and it would be, I expect there would be a bump like that[point to graph on right part reflected over x-axis."(621-631)

“Absolute value…. So, the distance from zero for all these points [point to part of LHS graph below x-axis] would actually be positive.”(634-635)

R2 E(Original function)—[Form—cuing procedure]—T(Expanding)
“if this was a test question and I actually wanted to get this right, I would start *x* squared minus eight *x* plus sixteen. I would start doing this, cause I always … wanna actually see it out.”(640-643)

“So I mean my intuition tells me this stuff, but I actually want to crunch the numbers. I actually want to distribute it and try some values of *x*. Yeah, I would want to distribute this completely, or at least try some values of *x* and see.”(645-649)

R3 E(Expanded form of function)—[Form—cuing procedure)—T(Numerical-based analysis)
“Or at least try some values of *x* and see … like for example, or even, it might be simpler if I just try some values of *x*. and leave it in this form… so if I try zero it’s gonna be positive”(648-651)

I Describing why absolute value won’t affect the graph
“the absolute value bars aren’t going to matter cause these are both terms are squared, so they’re always gonna be positive, so it doesn’t even matter. So this is not gonna happen… it’s never gonna happen with something to the fourth. Or it will, I don’t know about that, but it’s not gonna matter here because… if you have two squared terms, they’re automatically going to be positive. So absolute value doesn’t even matter” (652-659)

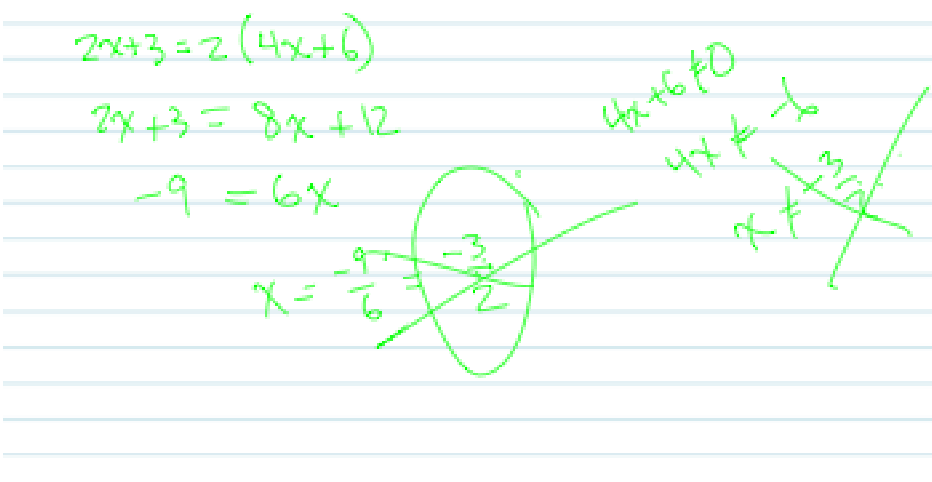
“, well, for this specific case, absolute value doesn’t, absolute value signs don’t matter cause I have two things that are being squared. I have A squared times B squared, and it’s always gonna be greater than zero, so absolute value doesn’t matter. But then I was wondering, would absolute value ever matter if I had *x* to the fourth plus stuff? Would it even matter because of *x* to the fourth, would this even happen, but this is *x* to the fourth plus, let’s find a number where this would be negative, like, ah, negative one plus hundred *x* here, negative one would get us one minus one hundred, so then the absolute value would actually be relevant there. So, <u>it’s just this specific case where you could factor it into binomial</u>

squared, but that, the absolute value doesn't matter cause you have two squared terms."(673-687)

Commentary: Newt generalizes from a particular case about why the absolute value will not affect the graph in this case. He limits his generalization ($a^2b^2>0$) to this particular problem which shows some refinement in thinking. It's interesting to note that he does not reason about the zeros (or x-intercepts) when they are visible in the form of the initial function.

Task 2b

Solve for x: $\dfrac{(2x+3)}{(4x+6)} = 2.$



R1 E(original equation)—[Form—cuing procedure]—T(Solving procedure)

"think here is I have linear looking stuff; I'm just gonna multiply… I expect it to work out because, *x* term, solve linear, eight *x* plus twelve, negative one equals Six *x*, and I have *x* equals negative nine sixths. Let's make sure that works,"(690-694)

"I can't do, I can't simplify this side any further, but I knew, ah, got rid of the denominator and multiplied that over, then eventually I just had a linear equation to solve, so I said let's skip that. … denominator"(705-709)

R2 E($4x + 6 = 0$)---[Form—SI]—P(Values not in the domain of equation)

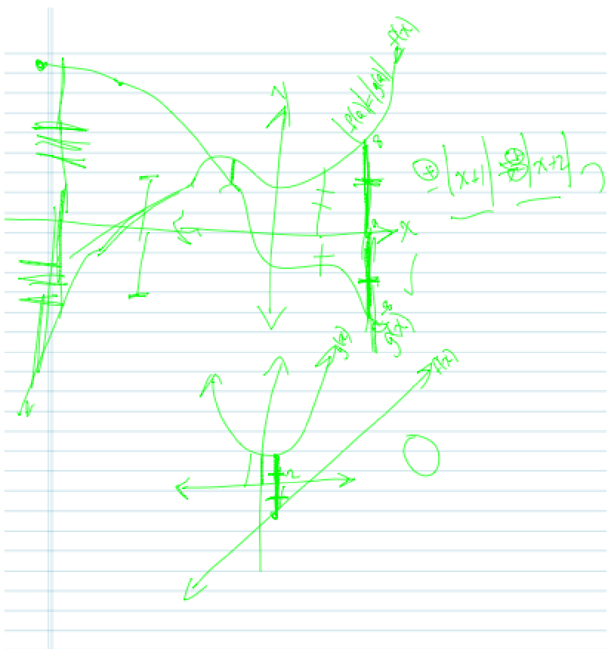
A Reasoning from R2 about the solution set of the equation.

"Let's make sure that works, cause you have a sum of four *x* minus six can't be zero. But that's probably what's gonna happen. But um, negative three halves would be negative three over two which , yeah, that will make it zero, so I would say no solutions. I am pretty sure I'd say no solutions. Let me confirm that. Yeah, cause four *x* cannot equal negative six. Cannot equal negative three *X*. And that's the answer, so. No solutions."(693-702)

> “got *x* equals negative three halves, and I was about to say, hey, that works. But then I said, Oh, no, you can’t divide by zero so I said, this denominator came to zero, and then I realized this is going to end up being the same thing that it was, so then I said, well the only solution I got doesn’t actually work because then it gets me dividing by zero, so nothing works”(711-717)

Task 6E

Solve for x: |f(*x*)| = |g(*x*)|.



C Re-presentation of original equation as two equivalent functions

L E(solution of original equation) and P(location on graph)
“The absolute value of a function equals the absolute value of another function… so you’re trying to find the intersection of two absolute value functions.”(721-723)

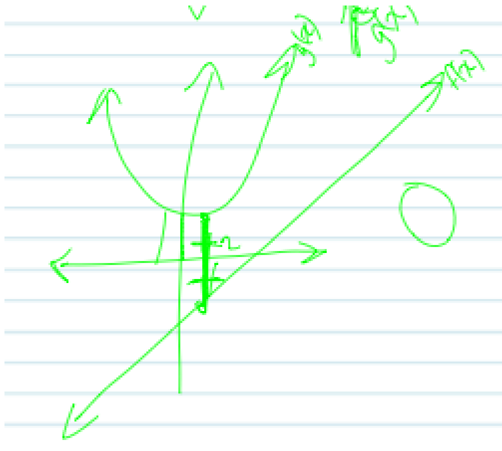
I (Describing the meaning of the locations of solutions on the graph)
“One thing I’m thinking about is you know, random function, random function, this is gonna be true when the distance from the *x* axis is the same. So that’s what I’m actually looking for. The concept I’m looking for is where are these the same

distance from? What value of *x* are they like equidistant from maybe here? It's what I'd actually be finding?"(723-729)

C Procedure for thinking about Task 6B and the procedure for this task.
"I think I would treat it about the same way that we did that one problem, which was *x* plus one is greater than the absolute value of *x* plus two, maybe. Except this would be an inequality, so I'd try this with positive and this was negative, and so I'd come up with four things, one with this positive, one with this negative, then positive positive, negative positive and negative, negative. Its something I'd try, and then, is there anything else I could do?"(730-738)
Commentary: Newt is posing "what if" scenarios at this point explaining that if he knew the functions this is how he would approach the problem and he follows up with an example showing how he would handle the problem if he knew the shape of the graphs.

L Type of functions and property(feature) of the solutions on the graph

I Describing finding the solution on a graphical representation of two functions
"I could know, if one was like quadratic and the other was linear, I would know, hey, it's only gonna work in like a certain area. It's only gonna work in a certain area, so…. Like if it's linear t here's only gonna be one point where it's a specific distance, and then quadratic is gonna be like two points a certain distance, and I'd have to start playing with things like that. But like, so are you saying how would <u>I solve something where I was given real</u>…"(739-746)



Commentary: Newt provides an example to explain his "what if" scenario—finding solutions if I were given two functions on a graph. While his idea of distance is accurate he does not take into account the absolute value when graphing his example of a linear function.

I Describing how he would go about the solving problems of this type "Things I would think about were… I would solve it by this method, where I set up four things and I'd also check that against what did I know the graphs to look like. And does my answer here make sense with what I see here? And also, based on the graph, and like just try some numbers."(748-752)
Commentary: This seems to be a general strategy that Newt uses to solve problems.

A Reasoning from a property of the graph about the solution of the problem.

"So this is just some random function. If *x* looked something like this, two *x* looked something like that, if the absolute values are equal, that means they are the same distance from the *x* axis. So, I would just look for *x* values for which they are equidistant from the *x*-axis. So if I drew, ah, a horizontal, a vertical line through the *x* value, the segment from the function to the *x*-axis would be the same on both sides. Maybe here it's not actually true. But here, maybe it would be"(755-763)

"Um, here it would be, if these actually were the same, like actually found, so this point is *a*, Actually found *f* of *a* equaled, the absolute value of *f* of *a* equaled the absolute value of *g* of *a*. So if this is like eight and this is like negative eight, and I'd say okay, that works. And then, I'd look around, it's not gonna work up here, up here. Not gonna work, not gonna work, not gonna work, not gonna work, gonna work. That's kinda just eyeballing it. Here, I'd say, well, since I got a constant, like the distance from, the change in the distance from the *x*-axis is constant, and here, there's gonna be two distances, except for here. There's always gonna be two. Oh, but I want an *x* value that works for both. So I would just look for, just look for a place like this. Where that was the same, and ah, but like I wouldn't be able to just do that based on the graph. I would figure out where that is based on what *f* of *x* is and what *g* of *x* is, and ah, figure out if this method is actually working based on, you know, I think something around two will work, and try that."(767-785)

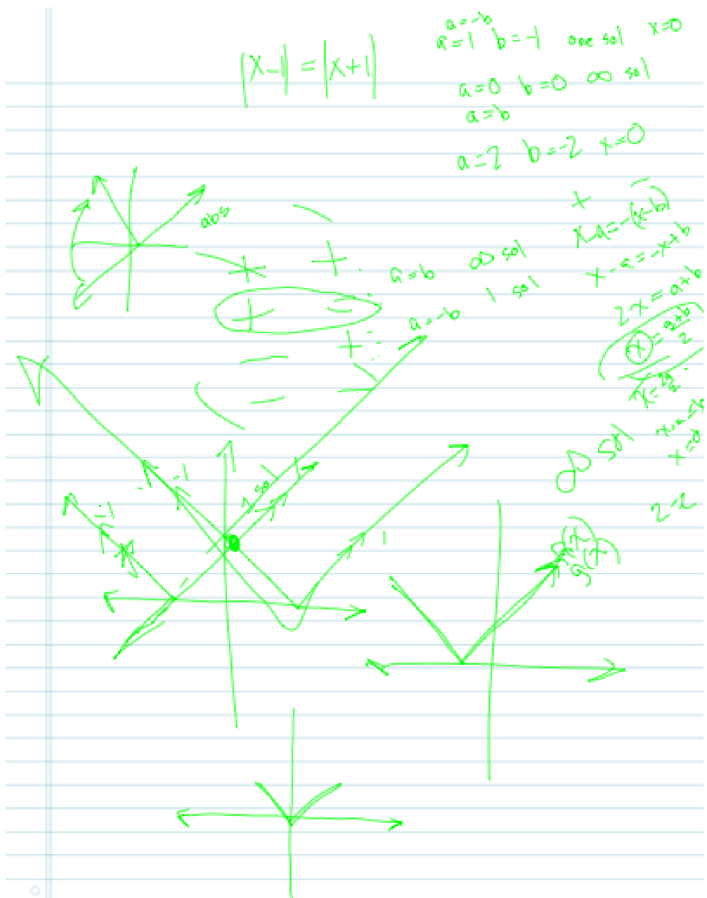
"Um, well, if the function ends here. But I mean, if it keeps going it could be infinite. Especially if, even if for like a, a millisecond it was like that, it would, parallel to the *x* axis, then an infinite number of points would be equidistant in that area."(798-802)

Commentary: Another example of Newt explaining a "What if" scenario.

Commentary: Newt reasons from the concept, absolute value, about the solution to the problem. Newt reasons about the location of the solutions, but does not recognize the solution where the two graphs intersect (would have same distance from x-intercept).

Task 6C

Solve for *x*: |*x* – *a*| = |*x* – *b*| for any real numbers *a* and *b*.



R (original equation)—[Form—noticing meaning]—P(*a* and *b* are parameters)

I Describing the number of solution for different values of *a* and *b*
“Yeah, um, if *a* and *b* can be anything, wait, so for any real numbers *a* and *b*, I would just say let *a* and *b* be zero, and then there’s infinite number solutions. And then, but if you let *a* be one and *b* be negative, *b* be two, then there’s no solutions.”(812-816)

A Reasoning from the initial equation about the solution set "the exact same [pointing to the initial equation] except for *a* and *b*. So, if you let *a* and *b* be the same thing, like one and one, zero and zero, you're gonna be good. There's gonna be infinite solutions."(821-824)

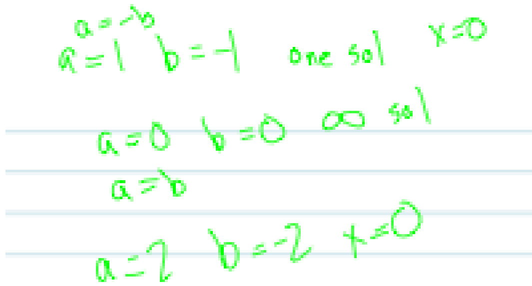
What distinguishes interpreting from analyzing

I Describing the number of solutions for different values of *a* and *b*

A Reasoning from the initial equation about the solution set

"But if you make *a* and *b* different, then this equal signs, these statements are not gonna be equal, unless what if *a* was one and *b* was negative one, let me think about this for a second. That would be true for *x* equals zero. So for *a* equals one and *b* equals negative one, you'd have one solution, so in some cases you'd have one solution. I know for, Here, I think you'd have one solution. For *a* equals zero and *b* equals zero, you'd have infinite solutions. And ah, I think as long as I keep making *a* and *b* like, you know negatives of each other, I keep getting one solution. It's gonna be zero. So I could either have one solution or infinite solutions. So this is any case where *a* equals *b*. And this is any case where *a* equals negative *b*. So yeah, *a* equals *b* I'd say infinite solutions. *A* equals negative *b* : one solution."(824-838)

Commentary: As he has done before Newt seems to refine his thinking from no solution, if a and b are different, to one solution by plugging number that are opposites for a and b. This is a strategy that he seems to rely upon. His reasoning about several cases seems to procedure a generalization about a specific instance, "this is any case where A equals negative B."(836-837). Also, another instance where Newt posed a "what if" scenario.



PROMPTED BY RESEARCHER TO THINK ABOUT PROBLEM GRAPHICALLY

L Form of the equation and the shape of the graph

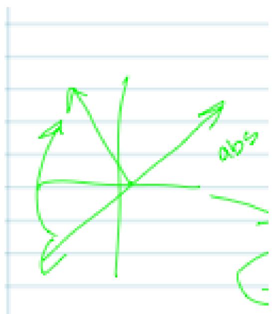
L E(initial equation) and property of the graph

"Absolute value of a linear function will always be like some sort of V with a slope of negative one and one."(843-845)

R1 E(Form of the original equation)—[Form—Structure]—P(function type)

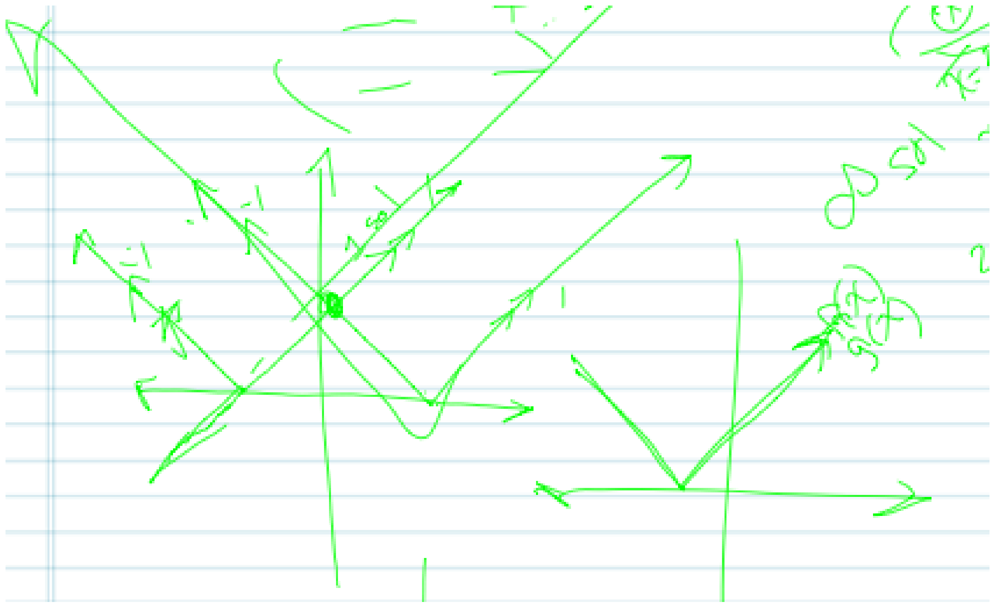
I Describing why the absolute value graph is "V-shaped"

"The reason I know that is because linear function is like this, but then when you make it absolute value this all gets reflected, like that. So this is how I know absolute value."(847-849)



L Number of solutions and shapes of graphs

"I have these both on the *x* axis but … that's not necessarily true. Oh yeah, is it? Yeah, the smallest it can be whenever *x* is *a* and *x* is *b*, it will always be on the *x* axis. So, this is the case where there's one solution, and then the case where there is an infinite solution [Draws graphs shown below] I when … this is like *f* of *x* and then… (inaudible) So the one solution case is like this, the two solution case or infinite solution case is when they are the same, and ah, there's no way to get, you know, anything between"(850-858)



A Reasoning from the above graph about why you will never get zero solutions

"there's no way to I get zero solutions… because this is always gonna happen [draws big dot on graph], because you're having something, yeah, yeah, the inner arms are always gonna cross because, increasing in value that way [pointing to right branch of right "V"] and increasing in value that way [pointing to left branch of right "V"], and um, but there's no way to get two, because these are, these are parallel." [draws parallel markings on above left-side graph] (858-864)

L Form of equation and property of the graph (slopes are -1 and 1)

"Because the slopes for absolute value function the way slopes are, is you know, there would just be a line otherwise, with a constant slope which would be the coefficient of the *x* term, and the only difference is, where there normally would have been a positive slope, but a negative *y* value, it's just the negative of the normal slope."(870-875)

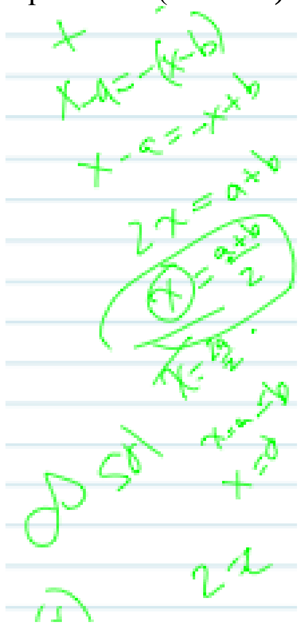
PROMPTED BY RESEARCHER TO REASON ABOUT THE PROBLEM ALGEBRAICALLY

"Well, because the only thing I can think of for dealing with these absolute values is where I make the first one positive, the second one positive, do that whole thing. First of all I don't ever feel like doing all that, and then also, like I started doing it, this is positive negative…"(886-890)

C E(original equation)—[Form]—(Representation as piecewise defined functions)

R2 E(piecewise defined equations)—[Form]—T(Solving procedure)

"I chose this one just because it's unusual. You know, but the same. So I decided to try that. And because I think that if I make both positive the *x* terms will just cancel, they will just cancel. So, these are ones that actually matter in this case. So, so I started doing it, and I got *x* equals *a* equals two"(893-898)



I E($\frac{a+b}{2}$)—" if *a* and *b* are the same, then I'd have like *x* equals two *a* over two. So *x* would equal *a*. So, hm…. I don't like the algebraically as much here."(880-882) "So, and I, I'm solving for *x*, and so this, so they will intersect at *a* plus *b* over two, so that will be either one intersection, if *a* and *b* are different, if *a* and *b* are the same it should be infinite, so it would then be *x* equals *a* which equals *b*, but that's not infinite. Which is startling to me because I, if *a* and *b* are the same, there is infinite solutions, so I'm wondering, okay, here if *a* and *b* are the same, there's not infinite solutions. So I have *x* equals *a* equals *b*, which seems like one solution. If *a* was four, *x* equals four would work. But there should be, or if like *a* and *b* were zero, it would be like this, so there should be infinite solutions. So if *a* and *b* were zero, this just says that *x* equals zero. Which… doesn't make sense to me. I'm expecting to get something like two equals two."(899-912)

Commentary: Newt's algebraic process leads to a result of $x = \frac{a+b}{2}$. *He interprets this result in the context of a and b being the same. This results in an outcome of x = a when plugged into formula. He has already reasoned that when a and b are the same there are infinite solutions. This result seems to cause cognitive conflict because he does not get a form for an answer that he is used to getting when the answer is infinite solutions—expectation is not met. The problem is functions chosen that resulted in* $x = \frac{a+b}{2}$,

$(x - a) = -x - b$ do not imply that a and b are the same. If they were same functions would be $|x - a| = |x - a|$ which would suggest infinite solutions. The correct interpretation of $x = \frac{a+b}{2}$ is that of a point half-way between a and b. In the case of a = b, it would represent the point halfway between a and a, or a. It's interesting that Newt interprets the meaning of $x = \frac{a+b}{2}$in an algebraic context, but does not link its meaning back to the graphical representation. As he has done many times before he tests a couple case— Unlike, other instances, this strategy does not seem to resolve the cognitive conflict.

Appendix K

Narrative of Task 3

The purpose of the next few pages is to provide describe incidents of students' thinking that characterize the different strategies of feature-noticing as they students reasoned about Task 3 (*Solve for x,* $x^2 + x + 1 > 0$*)*. These incidents exemplify the different characteristics that separate distinguish the three strategies. It is important to note that many students move between these strategies as they reason about a particular task while others are confined to one or two of these strategies.

Manipulative Strategy

As previously mentioned, a manipulative strategy to students' thinking is characterized by manipulative actions on the symbolic entity. Feature-noticing within a manipulative strategy is characterized by the following activities: 1) cuing a particular procedure, and 2) knowing when to abandon a procedure.

Cuing a procedure

In many instances students' feature-noticing is characterized by recognizing the tendency to recognize features of the entity that cue a particular procedure. For example, Betsy stated, "whenever I see a quadratic I automatically think to factor it" (89-90). Meanwhile Dan, also referring to a factoring procedure, exclaimed, "I just see a quadratic equation, or inequality…it's just one of those things it triggers" (167-180).

Recognizing features that cue procedures is an important aspect of students' thinking as they solve tasks involving symbolic representations. Successfully implementing a procedure requires recognizing the constraints conditions under which a procedure can be applied. For example, several students thinking of several students was characterized by their noticing a feature (product of factors) of the entity $x(x + 1) > -1$ that cued a procedure (applying zero-product property) which resulted in the inequalities $x > -1$ and $x + 1 > -1$. Judging from student comments, the malformed procedure involved overgeneralizing the zero-product property from $(ab = 0 => a = 0\ or\ b = 0)$ to $(ab > c => a > c\ or\ b > c)$. Most students who executed this malformed procedure, like Nadia, were able to recognize a feature (zero on the right) that convinced them that they had overgeneralized the procedure.

> But if I factor that out I won't be able to use the zero property, in which cause, because if I have *x* plus one minus one is greater than negative one, then I won't be able to use the zero property on this because, um, the zero property only works if you have zero on the right side.

As these incidents suggest, a manipulative strategy requires students to not only recognize features that cue particular procedures, but also to recognize features of the entity that indicate when a particular procedure can or cannot be applied.

Recognizing when to abandon a procedure

Another characteristic of a manipulative strategy is knowing the ability to know when to abandon a particular procedure. There are two reasons that why this happens. Either the student has reasoned that the procedure will not be productive or helpful to their problem- solving efforts or the student is not able to reason about the results of those procedures. The latter is exhibited in the thinking of most students in the study who reasoned that the factoring procedure could not be executed under over the set of integers. Specifically, students' recognized a feature of the symbolic entity (middle term of expression was 1*x*) during the procedural steps that informed their thinking. That is, they reasoned that the result of the factoring procedure would not yield a middle term of 1*x*.

> But I can't because if I factor it I'm just gonna break it up into two binomials where, um, factors of one, which are only one thing add up, and we need a two there [pointing to linear term] to factor it (Ashley, 92-95).

The second reason that a procedure is abandoned is because a student is unable to reason about the results of the procedure. In many instances reasoning from the results of a procedure requires a shift to a different strategy. This is exemplified in Betsy's thinking. She successfully performs a procedure (applying quadratic formula), but is unable to make the shift to either a relational or relational strategy to productively reason about the meaning of the results of the procedure.

> I think I multiplied wrong… but… I tried the quadratic formula… and under the radical I got a negative number, which would be imaginary. So it would be " I root three, but I don't think that's right…so… I would end up with an answer, but it's just imaginary (Betsy, 96-100).

In both instances students' thinking is characterized by an abandoning of a procedure. As previously stated, it is important to make a distinction between these two instances. In the first instance, abandoning the procedure was a result of recognizing a feature that suggested the procedure would not work. Meanwhile, in the second instance, abandoning the procedure was a result of not being able to reason about the results of the procedure.

Relational Strategy

Recall that a relational strategy is an ability to recognize features of the entity that provide meaning. What distinguishes a relational strategy from other strategys is that there is evidence in what is taken to be a student's reasoning that suggests that the symbolic entity represents to the student a mathematical object. This reasoning seems to occur in three ways: reasoning about the meaning of the object represented by the initial entity, reasoning about the meaning of the entity that expresses the results of a procedure performed on the initial entity, and reasoning about a property of the object represented by the initial entity.

Meaning of Object Underlying Entity

On Task 3, several students' thinking was characterized by reasoning about the meaning of the object (number) represented by the entity $x^2 + x + 1 > 0$. In this particular case, students recognized a feature ($x^2 + x + 1$ represented a number) that provided meaning. This is exemplified in Newt's and Casey's thinking.

> *x* squared, if I plug in any positive real number uh, it's gonna be positive, it's gonna be positive, and one is greater than zero. So it's always gonna be greater than zero (Newt, 142-144).
>
> If you have any large negative number, this is gonna be, then the *X x* squared is gonna be a large positive number, and then you're subtracting the negative number, so then it's still gonna be greater than zero, and then adding one, you just kinda, um, not gonna change it that much. If it's a large negative number, if it's a large positive number it's obviously gonna be positive when you square and then add anything to it. And then when it's zero it also works, so, and even when it's a

small negative number it works, you could say, and a small positive number works, so I guess you could say that *X* x could be anything (Casey, 226-236)

In both instances Newt and Casey recognized that the entity $x^2 + x + 1$ could be thought of as a number. Doing so led to both of them being able to reason about a set of values for x that would be in the solution set of the inequality. Moreover, their thinking was characterized by deductive generalizations about those sets of values. For Newt, the generalization was over the set of positive numbers while for Casey the generalization, though not completely accurate, was over the set of all real numbers.

Property of Underlying Object

Newt's and Casey's thinking was characterized by the ability to reasoning about the meaning of the object (number) represented by the entity $x^2 + x + 1 > 0.$ Dan's thinking was also characterized by this ability, but his thinking was also characterized by recognizingr a property, $x^2 + x$ is not always a positive number, of a part of the object. All three viewed the entity $x^2 + x + 1$ as a number, but only Dan recognized that $x^2 + x$ could be a negative number for values of x within a certain interval (between -1 and 0).

> And was talking to you about finding out those negatives, negative one half would still leave this with one half… one fourth, three fourths, that would work. No matter what, even anything between zero and negative one, this would turn to a positive, this become a positive plus something less than, or something … I'm trying to describe… something greater than negative one, and this would become, this would stay positive one. This will never be able to turn this guy negative, and this would be positive either way. (238-246)

Meaning of results of procedure

Students' thinking within a relational strategy requires an ability to describe the meaning of the entity that results from the application of a procedure. For example, several students' thinking was characterized by work included execution of a procedure that involved applying the quadratic formula to the equation $x^2 + x + 1 = 0$. The procedure resulted in the expression $x = \frac{-1 \pm \sqrt{-3}}{2}$. Several students notedIdentifying that the expression represented complex numbers; they expressed the desire to, but being were unable to quantify the complex numbers characterized several students' thinking. This is exemplified illustrated in Newt's thinking.

> I used the quadratic formula. And then I saw there were going to be, and I was going to be a part of it. And so I tried to keep going and draw a number line, because at this point if these were real numbers, then I would test the point here and here and here… but since I don't have an understanding of what's bigger and smaller with complex numbers (Newt, 196-202).

Newt's verbalized thinking suggested that he needed to be able to quantify the complex numbers so that he could perform a procedure that was dependent on placing those numbers on a number line. Unfortunately, complex numbers are not ordered so he would have been unable to complete this procedure.

Thinking within a relational strategy implies students gain meaning from the results of their procedures. For Newt, meaning was dependent on being able to quantify the results. It can be inferred from the procedure he wanted to enact that he knew the results were complex roots, but that he could not quantify place them on a number line. For Newt, the meaning of the results was tied to a procedure.

For others, the meaning of the entity expressing the results of a procedure is tied to a different strategy. This is exemplified in Paul's thinking. In reasoning about the results of a procedure that involved applying the quadratic formula he switched to a linking strategy making several links. Specifically, he made a link between a feature of the symbolic entity $x = \frac{-1 \pm \sqrt{-3}}{2}$ (represents the roots of the equation) and a feature of the graphical representation (points where graph crosses x-axis). Also, he made a link between a feature of the graphical representation (graph does not cross x-axis) and feature of the original symbolic entity $x^2 + x + 1 > 0$ (*x* is greater than zero for all numbers).

> If you are dealing with real numbers, the roots of the equation would be like whether the equation is expressed as like a graph, so if I had something like I don't know, this would simply be *x* square root, like *x* squared, like a parabola type, the *x* axis, and those are like the roots of the equation. Where the equation equals zero. Yeah, so I guess if the equation is greater than zero, it's just everywhere. The entire graph is just greater than zero, or something like that. But, so basically I think it's just like this graph will never, this equation will never have a root of zero. So I guess, I guess *x* is zero everywhere. So *x* is greater than zero everywhere for all numbers (Paul, 121-32)

In a similar manner, Todd switched to a linking strategy making a link between a feature of the entity (not real) and a feature of the graphical representation (does not touch x-axis), "Well, it's not real but it would be where the, this graph of the quadratic equation here, the, would not touch the *x* axis" (Todd, 117-119). In both instances, switching to a linking strategy afforded the student the means to reason about the results of their procedure.

In some cases, a movement between different stategies characterizes students' reasoning and explanations. Todd's thinking is an illustration of a movement from a relational strategy to a linking strategy. He is able to reason that the results of his procedure represented complex roots, reflecting a relational strategy, but he explained the meaning of complex roots making a link to a feature of a graphical representation, reflecting a linking strategy.

Linking Strategy

A linking strategy is characterized by noticing features that are related or linked to features of other representations. On Task 3 these links are distinguished as being between 1) the entity representing the results of a procedure and a graphical representation, 2) a part of original entity and a graphical representation, and 3) graphical representation of a part of original entity and original entity.

One instance of thinking within a linking strategy involves linking features of an entity representing that represents the results of a procedure and a graphical representation. For example, in the previous instance Todd's thinking about a feature (complex roots) of the entity $x = \frac{-1 \pm \sqrt{-3}}{2}$ was linked to a feature (will not touch x-axis) of the graphical representation of $y = x^2 + x + 1$.

Another instance of students' thinking within a linking strategy is linking features of the original entity with features of a graphical representation to of all or part of that entity. For example, Dan made a link between a feature (greater than) of the entity $x^2 + x + 1 > 0$ to and a feature (dotted line) of the graphical representation of $y = x^2 +$

$x + 1$, “that would have a dotted line since it’s strictly greater than” (Dan, 158-160). While the link was unproductive for Dan, other students’ thinking was characterized by productive links. For example, Todd made several links reasoning from a linking strategy. He made a link between a features (coefficient of x^2 term) of the entity $x^2 + x + 1$ and a features (parabola that opens upward) of the graphical representation of $y = x^2 + x + 1$, “And because, the *y*-intercept is gonna be one up here. And just, it’s above the *x*-axis… Because of the *y* intercept and *x* squared is positive so it opens up” (Todd, 119-134).

Another type of link shown in students’ thinking that characterized in students’ thinking within reflects a linking strategy is a link between a feature of a graphical representation of the original entity and a feature of the original entity. Ashley’s thinking exemplified illustrated a link between a feature (always above x-axis) of the graphical representation of $y = x^2 + x + 1$ and a feature (all values of *x* are greater than zero) of part of the original entity $x^2 + x + 1 > 0$.

> So this would be my *x*-axis, and I called this my *y*-axis, so any value of *x* that I substitute on this axis into this expression I’m getting out one of these values, a *y* value. So, by my graph right here it’s always above zero on the *y*-axis, so I know this expression will always be positive, greater than zero (Ashley 147-152).

Shifts Between Strategies

A characteristic of successful problem solving on Task 3 was an ability to move between the three different strategies. Moving between strategies requires thinking that includesd recognizing features that provided meaning and making links between different representations and entities. The purpose of the next few pages is to exemplify some of the more prominent shifts in strategy as revealed in students' thinking.

Newt's thinking

Newt's thinking represents the widest variety of shifts in strategy. Moving from a manipulative strategy to a relational strategy back to a manipulative strategy with an embedded linking strategy in a procedure, Newt unsuccessfully reasoned from the results of a procedure that involved applying the quadratic formula.

> I used the quadratic formula. And then I saw there were going to be, and I was going to be a part of it. And so I tried to keep going and draw a number line, because at this point if these were real numbers, then I would test the point here and here and here… but since I don't have an understanding of what's bigger and smaller with complex numbers (Newt, 196-202).

Newt working within a manipulative strategy recognized a feature (quadratic form) of the symbolic entity $x^2 + x + 1 > 0$ that cued a procedure (applying the quadratic formula). Shifting to a relational strategy he reasoned that the result $x = \frac{-1 \pm \sqrt{-3}}{2}$ represented complex numbers. Shifting back to a manipulative strategy he described a procedure for determining the intervals of the domain of the solution set of a quadratic inequality. That is, he indicated that he needed to place the roots on a number line and then test numbers within the three intervals separated by the roots to see if the resulting numbers would be positive or negative.

One of the steps in the procedure required a shift to a linking strategy. He needed to be able to place the results of the quadratic formula procedure on the number line. He attempted to make a link between a feature of the results of the procedure involving the quadratic formula (magnitude of the complex number) and a feature of the graphical representation (location on the number line). Unable to make the link he abandoned the procedure. Unlike real numbers, which are ordered, complex numbers are not ordered so he would not have been able to make the link between the magnitude of the complex numbers and locations on a number line.

After abandoning the procedure that involved applying the quadratic formula Newt successfully reasoned about the solution set shifting from a relational strategy to a linking strategy. Operating from a relational strategy he recognized a feature of the entity $x^2 + x + 1$ (it represented a number) that enabled him to reason about a set of values in the domain of the solution set of the inequality $x^2 + x + 1 > 0$.

> Any x less than negative one is going to work because when you square it it's going to be positive and larger than just minus constant value, and you have plus one is gonna be greater than zero and also everything equal to zero is going to work because zero works, one works, and it's only gonna get bigger. (Newt, 148-153)

Uncertain as to whether substituting numbers for *x* between *x* = -1 and $x = 0$ would also yield positive numbers, " And, uh, so now It's just a matter of …between there and there"(153-154) he switched to a manipulative strategy. He recognized a feature (quadratic form) that cued a procedure for finding the vertex of quadratic function. Switching to a relational strategy he reasoned about the solution set of the inequality making links between features of the graphical representation (vertex above x-axis and parabola opening upward) and the symbolic entity, $x^2 + x + 1 > 0$ (solution set is all real numbers).

> So if the vertex is there, it's gonna make that I'd say all reals, just like that… this is like with zero, since it's an open up parabola, and the vertex is greater than zero, every other point is going to be greater than zero (Newt, 164-170).

Paul's thinking exemplifies a different shift in strategy than different from Newt's shift. Unlike Newt, Paul was able to reason from the entity $x = \frac{-1 \pm \sqrt{-3}}{2}$ that resulted from the procedure that involved applying the quadratic formula to $x^2 + x + 1 = 0$. Reasoning from a linking strategy he made a link between a feature of the procedure (when equation equals 0) and a feature of the graphical representation (location where graph crosses x-axis).

> The root, well if you are dealing with real numbers, the roots of the equation would be like whether the equation is expressed as like a graph, so if I had something like I don't know, this would simply be X square root, like X squared, like a parabola type, the X axis, and those are like the roots of the equation. Where the equation equals zero (Paul, 121-126).

For both Newt and Paul the ability to move between different strategies suggests flexibility in their thought. In both cases this flexibility resulted in successful problem solving.

Differences Across Levels of Algebra Experience

There were two striking differences in the privileging of particular strategies privileging across the levels of algebra experience. The first difference involves a proclivity to a manipulative strategy and the other difference is about the ability to reason from a relational strategy.

While the thinking of all students in the study were characterized as operating from a manipulative strategy, most of the students at the two lower levels of algebra experience operated solely from a manipulative strategy on Task 3. Those who did operate strictly from a manipulative strategy were unable to reason about to a correct a solution to the problem.

This is in contrast to students who operated from a relational strategy in their thinking. All four prospective secondary mathematics teachers and one calculus student (Paul) were able to reason from a relational strategy about to a solution to the problem. The feature they all recognized was that the entity $x^2 + x + 1$ represented a number. Not only was their thinking characterized by recognizing, but also by reasoning about the object (number) in the context of the problem.

What is interesting about this is that the less experienced algebra students performed the same substitution procedure that helped the more experience algebra students to recognizing that the entity $x^2 + x + 1$ represented a number. For the students with less algebra experience the substitution procedure was used as a means to generate points to graph a graphical representation of $y = x^2 + x + 1$ but not as a means to reason about the solution set of $x^2 + x + 1 > 0$.

Another subtle difference between the levels of experience is the propensity to reason from a linking strategy. At the two lower levels of algebra experience, only one Pre-calculus student (Todd) and one Calculus student (Paul) were able to make productive links reasoning from a linking strategy. Todd moved fluently to a graphical representation in the process of building a deductive argument for his solution while Paul made a link between the results of the quadratic procedure and a feature of the graphical representation that supported his reasoning. No other students at the two lower levels of algebra experience reasoned from a linking strategy. Even when prompted to deductively reason deductively about the solution of the problem using graphical representations, students at these levels were seemed unable to make any productive links.

Of the four prospective secondary teachers, Ashley and Newt reasoned from a linking strategy. Newt made a productive link to support a gap, in his deductive argument about the domain of the solution. Meanwhile, Ashley made productive links to confirm the results of a correct argument about the domain of the solution set based on both inductive and deductive generalizations.

It is interesting to note that the two students at the lower levels of algebra experience (Paul and Todd) who reasoned from a linking strategy were the only students who, in their thinking, were able to productively reason productively from the results of the quadratic procedure. In each case they were able to switch strategies moving from a representation to a linking strategy to reason about the problem. In contrast, Newt, a prospective secondary mathematics teacher, was unable to productively reason about the entity that resulted from the procedure that involved applying the quadratic formula.

Vita

Patrick L. Sullivan

Education

Ph.D., Curriculum and Instruction (Mathematics Education), The Pennsylvania State University, 2013
Dissertation: Characterizing the Nature of Students' Feature Noticing-and-Using with Respect to Mathematical Symbols Across Different Levels of Algebra Exposure.

M.A., Teaching and Leadership, University of Kansas, 1997
Thesis: Effect of Real-World Experiences and Applications on the Attitudes and Beliefs of Students Toward Mathematics

B.S., Curriculum and Instruction (Mathematics), University of Kansas, 1992

Experience

Principal/Teacher, 2007-Present
Grace Prep High School

Academic Researcher, 2002-2007
The Pennsylvania State University
Courses: Mathematics content course (Data Analysis) and Teaching Secondary Mathematics I

Secondary Mathematics Teacher, 1992-2002
Department Chair, 1993-2002
USD 232 DeSoto School District (Kansas)
Courses: Algebra through Calculus

Presentations

Unpacking Secondary Teachers' Mathematical Knowledge, at Regional National Council of Teachers of Mathematics conference in Kansas City, MO (October, 2007)

Transforming your Teaching, at 2005 and 2006 National Council of Teachers of Mathematics Annual Conferences in Anaheim, CA and St. Louis, MO.

Grants and Fellowships

Mid-Atlantic Center for Learning and Teaching Fellowship, 2002-2007
Toyota TIME grant recipient, 2000

Honors

Presidential Award for Excellence in Science and Mathematics Teaching (Kansas, 1999)
American Teaching Award Nominee, 1997
Kansas Teacher of the Year Nominee, 1996