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PARTIAL RANDOMNESS AND KOLMOGOROV COMPLEXITY

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Mathematics

by

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Abstract

Algorithmic randomness and Kolmogorov complexity provide a computational framework for the study of probability theory and information theory. In this dissertation we prove the following theorems about partial randomness and complexity.

Fourteen notions of partial randomness are considered: three using generalized Martin-Löf tests, six using generalized Solovay tests, one in terms of martingales, and four using either a priori (KA) or prefix-free (KP) complexity. Under suitable hypotheses, every test or martingale variant of partial randomness can be characterized using either KA or KP, via generalizations of celebrated theorems of Schnorr and Levin. New partial randomness notions are introduced, based on the first Borel-Cantelli lemma; these notions are equivalent to other notions under suitable conditions on f. It is also shown that while effectivizations of the first Borel-Cantelli lemma can be used to characterize Martin-Löf randomness, effectivizations of the second Borel-Cantelli lemma instead characterize Kurtz randomness.

Using complexity arguments we weakly separate notions of partial randomness. Under suitable hypotheses on the function f, there exists an X such that $\mathrm{KP}(X \upharpoonright n) \geq f(n) + O(1)$ for all n but $\mathrm{KP}(X \upharpoonright n) \leq f(n) + O(1)$ infinitely often. A similar result holds with KP replaced by KA. Here X is an infinite sequence of 0's and 1's and $X \upharpoonright n$ is the length n initial segment of X.

A major new theorem is that under suitable hypotheses on f, there is an X such that $\mathrm{KP}(X \upharpoonright n) \geq f(n)$ for all n, but no Y recursive in X satisfies $\mathrm{KP}(Y \upharpoonright n) \geq f(n) + 2\log_2 f(n) + O(1)$ for all n (also the analogous result for KA). This theorem, which will be published in The Journal of Symbolic Logic, generalizes the theorem of Miller that there exists a Turing degree of effective Hausdorff dimension 1/2. The new theorem also implies that there exists an X of effective Hausdorff dimension 1 which does not compute a Martin-Löf random, a result originally due to Greenberg and Miller.

X is K-trivial if its initial segment complexity is always minimal, that is $\mathrm{KP}(X \upharpoonright n) = \mathrm{KP}(n) + O(1)$ for all n. We give a new and simpler proof that no K-trivial can be random relative to a continuous measure. We also show that if X is K-trivial and Y is computable from the halting problem $\varnothing'$ then there is no measure μ such that X and Y are mutually relatively μ-random and $\mu(\{X, Y\}) = 0$.

Contents

List of Figures **vi**

Acknowledgments **vii**

1 Introduction **1**

1.1 Background 1

1.1.1 Infinite and finite sequences 1

1.1.2 Recursion theory 2

1.2 Algorithmic randomness 4

1.2.1 Kolmogorov complexity 5

1.3 Partial randomness 8

1.4 Summary of results 10

2 The Borel-Cantelli lemmas **18**

2.1 Classical Borel-Cantelli lemmas 18

2.1.1 The first Borel-Cantelli lemma 18

2.1.2 The second Borel-Cantelli lemma 19

2.2 The Borel-Cantelli lemmas for Δ_1 and Σ_1 19

2.2.1 When the sums of measures are computable 24

2.3 Borel-Cantelli lemmas for more general classes 25

3 Analogues of Martin-Löf randomness **28**

3.1 Functions as scales 28

3.1.1 Types of functions 28

3.1.2 Defining “measures” using arbitrary functions 30

3.2 Analogues of randomness which scale according to an arbitrary function 31

3.2.1 Analogues of Martin-Löf randomness 31

3.2.2 Analogues of Solovay randomness 35

3.3 Implications among partial randomness notions 38

3.3.1 Weak separation of pwt-f-random and vwt-f-random by example 43
3.3.2 "Upstream" implications 46
3.4 Optimal covers 49
3.4.1 Optimal and approximate f-covers 50
3.4.2 Boundedly optimal and approximate f-covers 51
3.4.3 Convex and strongly convex functions 56
3.4.4 Convex functions and the existence of optimal f-covers 60
3.4.5 Properties of approximate and optimal f-covers 62

4 Partial randomness and Kolmogorov complexity 64
4.1 Defining f-complexity 64
4.1.1 Connecting f-randomness and f-complexity 66
4.2 Martingales and betting strategies 70
4.3 Weak separations 71

5 Strong separations and extracting randomness 76
5.1 Forcing with optimal f-covers 78
5.1.1 Optimal f-covers in the length-invariant case 78
5.1.2 Defining f-conditions 81
5.1.3 Properties of f-conditions 81
5.2 Extracting additional complexity is difficult 85
5.3 The non-length-invariant case 88

6 Lower bounds on Kolmogorov complexity 90
6.1 K-triviality, low for K, and NCR 90
6.2 Oracles which compute lower bounds for KP 92
6.3 Applications to NCR and K-trivials 94

7 Conclusion 96
7.1 Hausdorff dimension and partial randomness 96
7.2 Partial randomness and DNR 97
7.3 Open problems 100

References 103

List of Figures

1.1 Partial randomness for recursive $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ 13
1.2 Implications for recursive $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ 14
1.3 Implications for recursive convex $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ 15
1.4 Implications for test notions of f-random, general case 16
1.5 Connecting f-random and f-complex, general case 16
1.6 Implications for test notions of f-random, convex case 17
1.7 Connecting f-random and f-complex, convex case 17

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Chapter 1

Introduction

1.1 Background

1.1.1 Infinite and finite sequences

Definition 1.1.1. $\mathbb{N}$ is the set of natural numbers, which includes 0. ***The Cantor set*** is the set $\{0,1\}^{\mathbb{N}} = \{X : \mathbb{N} \to \{0,1\}\}$, the set of all infinite sequences of 0's and 1's. The set of all finite sequences of 0's and 1's is $\{0,1\}^{<\mathbb{N}} = \{\sigma : \{0,\dots,k-1\} \to \{0,1\} \mid k \in \mathbb{N}\}$. Elements of $\{0,1\}^{\mathbb{N}} \cup \{0,1\}^{<\mathbb{N}}$ are called strings or sequences.

For $\sigma \in \{0,1\}^{<\mathbb{N}}$ the ***length*** of σ is the cardinality of the domain of σ, written $|\sigma|$. λ denotes the empty sequence, the unique string such that $|\lambda| = 0$. For each n we let $\{0,1\}^n = \{\sigma \in \{0,1\}^{<\mathbb{N}} \mid |\sigma| = n\}$ and $\{0,1\}^{\le n} = \bigcup_{k\le n}\{0,1\}^k$; the sets $\{0,1\}^{<n}$, $\{0,1\}^{\ge n}$, and $\{0,1\}^{>n}$ are defined in an analogous fashion.

For $\sigma,\tau \in \{0,1\}^{<\mathbb{N}}$ say σ is a ***prefix*** of τ, written $\sigma \subseteq \tau$, when $\sigma(n) = \tau(n)$ for all $n < |\sigma|$. Equivalently we say that τ ***extends*** σ in this case. σ is a ***proper prefix*** of τ, denoted $\sigma \subset \tau$, when $\sigma \subseteq \tau$ and $\sigma \ne \tau$. $\sigma,\tau \in \{0,1\}^{<\mathbb{N}}$ are said to be ***comparable***, written $\sigma \asymp \tau$, if either $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$. Write $[\sigma]^{<\mathbb{N}}$ to denote the set of all finite strings extending σ, $[\sigma]^{<\mathbb{N}} = \{\tau \in \{0,1\}^{<\mathbb{N}} \mid \sigma \subseteq \tau\}$. For $A \subseteq \{0,1\}^{<\mathbb{N}}$ define $[A]^{<\mathbb{N}} = \bigcup_{\sigma\in A}[\sigma]^{<\mathbb{N}}$. The notation $\widehat{A}$ means the set of minimal elements of A, that is $\widehat{A} = \{\sigma \in A \mid \neg(\exists\tau \subset \sigma)(\tau \in A)\}$.

For $\sigma \in \{0,1\}^{<\mathbb{N}}$ and $X \in \{0,1\}^{\mathbb{N}}$ say that σ is a ***prefix*** of X, or $\sigma \subset X$, if $\sigma(n) = X(n)$ for all $n < |\sigma|$. In this case we also say that X ***extends*** σ. For $X \in \{0,1\}^{\mathbb{N}}$ and $k \in \mathbb{N}$ the length k ***initial segment*** of X is the unique $\sigma \subset X$ such that $|\sigma| = k$; this element of $\{0,1\}^{<\mathbb{N}}$ is denoted $X \upharpoonright k$. For $\sigma \in \{0,1\}^{<\mathbb{N}}$ define $[\sigma] = \{X \in \{0,1\}^{\mathbb{N}} \mid \sigma \subset X\}$ and similarly for $A \subseteq \{0,1\}^{<\mathbb{N}}$ define $[A] = \bigcup_{\sigma\in A}[\sigma]$. Finally for $X \in \{0,1\}^{\mathbb{N}}$ define $X^{<\mathbb{N}} = \{\sigma \in \{0,1\}^{<\mathbb{N}} \mid \sigma \subset X\}$.

Informally, $\sigma^\frown\tau$ means the ***concatenation*** of strings σ and τ, that is the string which has all bits of σ followed by all bits of τ. More formally, for $\sigma, \tau \in \{0,1\}^{<\mathbb{N}}$ we define $\sigma^\frown\tau$ to be the unique string of length $|\sigma^\frown\tau| = |\sigma| + |\tau|$ such that $\sigma \subseteq \sigma^\frown\tau$ and $(\sigma^\frown\tau)(|\sigma| + n) = \tau(n)$ for each $0 \le n < |\tau|$. Similarly for $\sigma \in \{0,1\}^{<\mathbb{N}}$ and $X \in \{0,1\}^{\mathbb{N}}$ we define $\sigma^\frown X$ to be the element of $\{0,1\}^{\mathbb{N}}$ such that $\sigma \subset \sigma^\frown X$ and $(\sigma^\frown X)(|\sigma| + n) = X(n)$ for all $n \in \mathbb{N}$.

For $X, Y \in \{0,1\}^{\mathbb{N}}$ the ***join*** of X and Y is $X \oplus Y$, the element of $\{0,1\}^{\mathbb{N}}$ defined by $(X \oplus Y)(2n) = X(n)$ and $(X \oplus Y)(2n+1) = Y(n)$ for all n. Similarly for $\sigma, \tau \in \{0,1\}^{\mathbb{N}}$ such that $|\sigma| - 1 \le |\tau| \le |\sigma|$ the ***join*** of σ and τ is the string $\sigma \oplus \tau \in \{0,1\}^{<\mathbb{N}}$ defined by $(\sigma \oplus \tau)(2n) = \sigma(n)$ for all $n < |\sigma|$ and $(\sigma \oplus \tau)(2n+1) = \tau(n)$ for all $n < |\tau|$.

1.1.2 Recursion theory

For a complete introduction to recursion theory see for example Rogers [37] or Cooper [9].

Definition 1.1.2. The notation $f :\subseteq D \to R$ means that f is a function whose domain is a subset of D and whose range is a subset of R. A function $f :\subseteq \mathbb{N} \to \mathbb{N}$ is ***partial recursive*** if there is a Turing machine M which when run with n on the input tape halts with m on the output tape if and only if $f(n) = m$. We write $f(n)\downarrow$ to mean n is in the domain of definition of f or equivalently that M halts on input n; we write $f(n)\uparrow$ otherwise. $\mathbb{N}^k$, $\mathbb{Q}$, and $\{0,1\}^{<\mathbb{N}}$ are all countable sets which can be coded as subsets of $\mathbb{N}$. Then there is a natural sense in which this definition can be extended to functions on these sets: $f :\subseteq D \to R$ is ***partial recursive*** if $D, R \subseteq \mathbb{N}$ and there is a partial recursive $g :\subseteq \mathbb{N} \to \mathbb{N}$ such that $g|_D = f$. Kleene's enumeration theorem provides an effective enumeration $\{\varphi_e\}$ of all partial recursive functions; moreover there is a partial recursive function $U :\subseteq \mathbb{N}^2 \to \mathbb{N}$ such that $U(e,n) = \varphi_e(n)$ for all e and n. A function f on D is ***(total) recursive*** if it is partial recursive and total, that is $f(x)\downarrow$ for all $x \in D$. We can also extend computability to real-valued functions. For $D \subseteq \mathbb{N}$, a function $f : D \to \mathbb{R}$ is ***recursive*** if there is a recursive $g : D \times \mathbb{N} \to \mathbb{Q}$ such that $|g(x,n) - f(x)| < 2^{-n}$ for all n and x.

There is a natural sense in which a set may be recursive as well. A set $A \subseteq \mathbb{N}$ is ***recursive*** if its characteristic function

$$\chi_A(n) = \begin{cases} 1 & \text{if} \quad n \in A \\ 0 & \text{if} \quad n \notin A \end{cases}$$

is recursive. Similarly a subset of $\mathbb{N}^k$, $\mathbb{Q}$, or $\{0,1\}^{<\mathbb{N}}$ is recursive if its characteristic function is. A is ***recursively enumerable (r.e.)*** if it is either empty or there is a recursive $f : \mathbb{N} \to \mathbb{N}$ such that $A = \{n \mid (\exists m)(f(m) = n)\}$; say that f ***witnesses*** that A is r.e. If A is nonempty, define for each k the set $A^k = \{n \mid (\exists m < k)(f(m) = n)\}$; the cardinality of such a set is clearly at most k. The sets $\{A^k\}$ are an an ***enumeration*** of A. While a given r.e. set may have many enumerations, we will always assume some particular enumeration is fixed. A sequence $\langle A_n \subseteq \mathbb{N} \mid n \in \mathbb{N}\rangle$ is ***uniformly recursively enumerable*** if the set $\{(n,m) \mid n \in A_m\}$ is an r.e. subset of $\mathbb{N}^2$. This is a stronger property than just having each A_n be r.e.; the enumerability is uniform in n.

In a similar fashion we can define r.e. and uniformly r.e. for sequences $\langle A_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$. Since the collection of cylinders $\{[\sigma] \mid \sigma \in \{0,1\}^{<\mathbb{N}}\}$ is a basis for the usual toplogy on $\{0,1\}^{\mathbb{N}}$, every open set $A \subseteq \{0,1\}^{\mathbb{N}}$ has the form $A = [B]$ for some $B \subseteq \{0,1\}^{<\mathbb{N}}$. This allows us to make related definitions for $\{0,1\}^{\mathbb{N}}$. $A \subseteq \{0,1\}^{\mathbb{N}}$ is ***effectively open*** if there is an r.e. set $B \subseteq \{0,1\}^{<\mathbb{N}}$ such that $A = [B]$. $\langle A_n \subseteq \{0,1\}^{\mathbb{N}} \mid n \in \mathbb{N}\rangle$ is ***uniformly effectively open*** if there is a uniformly r.e. $\langle B_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ such that $A_n = [B_n]$ for all n. A sequence $\langle A_n\rangle$ is said to be ***uniformly effectively clopen*** if both $\langle A_n\rangle$ and $\langle \{0,1\}^{\mathbb{N}} \setminus A_n\rangle$ are uniformly effectively open.

We can use recursively enumerable sets to define the meaning of recursively enumerable for a function as well. A function $f : \mathbb{N} \to \mathbb{R}$ is ***(left) recursively enumerable*** if the set $\{(q,n) \in \mathbb{Q} \times \mathbb{N} \mid q < f(n)\}$ is recursively enumerable. Another name for this is recursively enumerable ***from below***. An equivalent formulation of recursively enumerable for a function $f : \mathbb{N} \to \mathbb{R}$ is the existence of a recursive function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that $F(n,t+1) \geq F(n,t)$ for all n and t and $f(n) = \lim_{t\to\infty} F(n,t)$ for all n. A function $f : \mathbb{N} \to \mathbb{R}$ is ***right recursively enumerable*** if the set $\{(q,n) \in \mathbb{Q}\times\mathbb{N} \mid q > f(n)\}$ is recursively enumerable, or equivalently if $g = -f$ is recursively enumerable. Another name for this is recursively enumerable ***from above***. Similarly a function with domain $\mathbb{N}^k$ or $\{0,1\}^{<\mathbb{N}}$ can be left or right recursively enumerable.

Relativization

Definition 1.1.3. An ***oracle machine*** with oracle $X \in \{0,1\}^{\mathbb{N}}$ is a Turing machine which additionally is able to query values $X(n)$ for any $n \in \mathbb{N}$. Then for any recursion-theoretic definition above, we can append the term ***relative to X*** by replacing Turing machines with oracle machines with oracle X. For instance, a function $f :\subseteq \mathbb{N} \to \mathbb{N}$ is partial recursive relative to X if there is an oracle machine with oracle X such that $f(n) = m$ if and only if the machine run with input n halts with output m.

The enumeration theorem extends to allow for an enumeration of all partial recursive relative to X functions, $\{\varphi_e^X\}$. Then we say that a function f is ***recursive in X***, written $f \leq_T X$, if $\varphi_e^X = f$ for some f. Similarly for a set A, we write $A \leq_T X$ if $\chi_A \leq_T X$. Write $X =_T Y$ to mean that $X \leq_T Y$ and $Y \leq_T X$.

For any $X \in \{0,1\}^{\mathbb{N}}$ the ***Turing jump*** of X, written X' is the element of $\{0,1\}^{\mathbb{N}}$ such that $X'(n) = 1$ if and only if $\varphi_n^X(n)\downarrow$. We can also iterate Turing jumps: for every X we write $X^{(1)}$ to mean X', and write $X^{(n+1)}$ to mean $X^{(n)'}$ for each $n > 1$. We write $\varnothing'$ to denote the Turing jump of the recursive function $f(n) = 0$ for all n. It is a fact that for any X, X' is not recursive in X.

1.2 Algorithmic randomness

While many different recursive measures can be placed on $\{0,1\}^{\mathbb{N}}$, we will use the ***fair-coin measure***, the unique measure μ on $\{0,1\}^{\mathbb{N}}$ such that $\mu([\sigma]) = 2^{-|\sigma|}$ for all $\sigma \in \{0,1\}^{<\mathbb{N}}$. With the exception of Chapter 6 we will always use μ to mean the fair-coin measure. One interpretation of algorithmically random for $X \in \{0,1\}^{\mathbb{N}}$ is that X should not have any properties that occur with effective measure 0. The most widely used measure theoretic notion of algorithmic randomness is the following.

Definition 1.2.1 (Martin-Löf [29]). A ***Martin-Löf test*** is a uniformly effectively open sequence $\langle A_n \subseteq \{0,1\}^{\mathbb{N}} \mid n \in \mathbb{N}\rangle$ such that $\mu(A_n) \leq 2^{-n}$ for all n. $X \in \{0,1\}^{\mathbb{N}}$ is ***Martin-Löf random*** if there is no Martin-Löf test $\langle A_n\rangle$ such that $X \in A_n$ for all n.

The following closely related definition is due to Solovay as communicated to Chaitin [8].

Definition 1.2.2 (Solovay). A ***Solovay test*** is a uniformly effectively open sequence $\langle A_n \subseteq \{0,1\}^{\mathbb{N}} \mid n \in \mathbb{N}\rangle$ such that $\sum_n \mu(A_n) < \infty$. $X \in \{0,1\}^{\mathbb{N}}$ is ***Solovay random*** if for any Solovay test $\langle A_n\rangle$ the set $\{n \mid X \in A_n\}$ is finite.

Theorem 1.2.3 (Solovay, Chaitin [8]). $X \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random if and only if it is Solovay random.

While the above definitions can be thought of as falling under the *measure-theoretic paradigm* of randomness, algorithmic randomness can be formalized alternatively in terms of betting strategies or martingales.

Definition 1.2.4 (Schnorr). A function $\delta : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is a ***super-***

martingale if $\delta(\sigma) \geq \frac{1}{2}\delta(\sigma\hat{\ }0) + \frac{1}{2}\delta(\sigma\hat{\ }1)$ for all σ. A supermartingale δ ***succeeds on X*** if $\limsup_{n\to\infty} \delta(X \upharpoonright n) = \infty$.

A gambler should not be able to win unlimited capital in a casino if a game is truly random; so a sequence X is algorithmically random if no appropriately picked supermartingale can succeed on X. Then when *appropriately picked* means recursively enumerable, the algorithmic randomness notion is still Martin-Löf randomness.

Theorem 1.2.5 (Schnorr [40])**.** $X \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random if and only if there is no r.e. supermartingale which succeeds on X.

There is a third paradigm we use to discuss randomness, that of compressibility or complexity.

1.2.1 Kolmogorov complexity

Several inequalities in this section hold only up to a constant. For this reason we use the following notation: a statement of the form "$A(x) \leq^+ B(x)$ for all $x \in C$" means that there exists a constant k such that $A(x) \leq B(x) + k$ for all $x \in C$. Similarly "$A(x) =^+ B(x)$ for all $x \in C$" means that $A(x) \leq^+ B(x) \leq^+ A(x)$ for all $x \in C$.

Roughly speaking, the complexity of an object is the length of its shortest description. The original idea of complexity is due independently to Solomonoff [43] and Kolmogorov [21]. We formalize the concept in a manner similar to that of Li and Vitanyi [26].

Prefix-free complexity

Definition 1.2.6. Let $\langle\ ,\ \rangle : \{0,1\}^{<\mathbb{N}} \times \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ be the pairing function defined by $\langle\sigma,\tau\rangle = 0^{|\sigma|}\hat{\ }1\hat{\ }\sigma\hat{\ }\tau$ for all σ and τ.

Definition 1.2.7. For a partial recursive function $f :\subseteq \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ we define a function $\mathrm{K}_f(\ |\ ) : \{0,1\}^{<\mathbb{N}} \times \{0,1\}^{<\mathbb{N}} \to \mathbb{N} \cup \{\infty\}$ by

$$\mathrm{K}_f(\sigma \mid \tau) = \inf\{|\rho| \mid f(\langle\tau,\rho\rangle) = \sigma\}$$

for all σ and τ. Note that $\mathrm{K}_f(\sigma \mid \tau) = \infty$ if no such ρ exists. $\mathrm{K}_f(\sigma \mid \tau)$ is called the ***conditional complexity of σ given τ according to f***.

There are many partial recursive functions and thus many different possible ways of defining conditional complexity; for example one could use a universal partial recursive function. There are natural objections to this approach to complexity (see Li and Vitanyi [26] for an explanation of these objections). For this reason we consider prefix-free functions, also known as self-delimiting codes.

Definition 1.2.8. $A \subseteq \{0,1\}^{<\mathbb{N}}$ is ***prefix-free*** if there are no $\sigma \asymp \tau$ such that both σ and τ are in A. $f :\subseteq \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ is ***prefix-free*** if its domain is prefix-free; in other words, if $f(\sigma)\downarrow$ and $\sigma \asymp \tau$ then $f(\tau)\uparrow$.

Definition 1.2.9 (Levin [25], Chaitin [6])**.** A partial recursive prefix-free function $f :\subseteq \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ is said to be ***additively optimal*** for defining complexity if for any other partial recursive prefix-free g, we have $\mathrm{K}_f(\sigma \mid \tau) \leq^+ \mathrm{K}_g(\sigma \mid \tau)$ for all σ and τ. Many such functions must exist.

Fixing such an f, we define for every σ and τ the ***(conditional) prefix-free complexity of $\boldsymbol{\sigma}$ given $\boldsymbol{\tau}$*** to be $\mathrm{KP}(\sigma \mid \tau) = \mathrm{K}_f(\sigma \mid \tau)$. For any σ the ***(unconditional) prefix-free complexity of $\boldsymbol{\sigma}$*** is $\mathrm{KP}(\sigma) = \mathrm{KP}(\sigma \mid \lambda)$.

Definition 1.2.10. For convenience, we also use n to mean the string 0^n. Then for example we write $\mathrm{KP}(\sigma \mid n)$ to mean $\mathrm{KP}(\sigma \mid 0^n)$, $\mathrm{KP}(m \mid n)$ to mean $\mathrm{KP}(0^m \mid 0^n)$, and $\mathrm{KP}(n)$ to mean $\mathrm{KP}(0^n)$.

Facts about KP

The results stated here are due to a great number of authors, and some results were obtained independently by several parties. Some have trivial proofs which are immediately evident; for the rest Li and Vitanyi [26] give full details of the proofs.

Consider the function $f :\subseteq \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ defined by $f(\langle \sigma, \tau \rangle) = \tau$ for all σ and τ. While it is clear that $\mathrm{K}_f(\sigma) \leq |\sigma|$ for all σ, unfortunately f is not prefix-free. A similar bound is not possible for KP, but nevertheless we have an upper bound which is not far off.

Lemma 1.2.11. $\mathrm{KP}(\sigma) \leq^+ |\sigma| + 2\log_2 |\sigma|$ for all σ.

Closely related to this lemma we have:

Lemma 1.2.12. $\mathrm{KP}(n) \leq^+ 2\log_2 n$ for all n.

We can also find strings with very high conditional complexity.

Lemma 1.2.13. For any $n \in \mathbb{N}$ and $\tau \in \{0,1\}^{<\mathbb{N}}$, there is a σ of length n such that $\mathrm{KP}(\sigma \mid \tau) \geq |\sigma| = n$.

The following is a very useful result about prefix-free complexity.

Lemma 1.2.14. $\sum_{\sigma \in \{0,1\}^{<\mathbb{N}}} 2^{-\mathrm{KP}(\sigma)} < 1$.

Moreover, KP can be characterized as the minimal (up to a constant) right recursively enumerable function with this property. Another useful result tells us about when a machine can be built to fulfill a set of requests or requirements which are given in an r.e. way.

Lemma 1.2.15. Let $A \subseteq \{0,1\}^{<\mathbb{N}} \times \mathbb{N}$ be a ***Kraft-Chaitin set***, that is an r.e. set such that $\sum_{(\sigma,n)\in A} 2^{-n} < \infty$. Then $\mathrm{KP}(\sigma) <^{+} n$ for all $(\sigma, n) \in A$.

The final result we will use has to do with the complexity of pairs.

Theorem 1.2.16 (Gács [14], Levin)**.** For σ and τ define the complexity of a pair by $\mathrm{KP}(\sigma,\tau) = \mathrm{KP}(\langle\sigma,\tau\rangle)$. Then for every σ and τ we have

$$\mathrm{KP}(\sigma,\tau) =^{+} \mathrm{KP}(\sigma) + \mathrm{KP}(\tau | \langle\sigma, \mathrm{KP}(\sigma)\rangle).$$

A priori complexity

Prefix-free complexity is defined in terms of machines and minimal description length. Other variants of Kolmogorov complexity that are definable in this way include plain complexity, monotone complexity, process complexity, and decision complexity. The following complexity notion cannot be characterized in terms of miminum description length, but is closely related to monotone complexity.

Definition 1.2.17 (Zvonkin and Levin [52])**.** A function $\delta : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is a ***continuous semimeasure*** if $\delta(\lambda) \leq 1$ and $\delta(\sigma) \geq \delta(\sigma^\frown 0) + \delta(\sigma^\frown 1)$ for all σ. An r.e. continuous semimeasure δ is said to be ***multiplicatively maximal*** if for any other r.e. continuous semimeasure γ there is constant k such that $\gamma(\sigma) \leq k \cdot \delta(\sigma)$ for all σ. Such semimeasures must exist.

Letting δ be a fixed multiplicatively maximal r.e. continuous semimeasure, for each σ the ***a priori complexity of $\boldsymbol{\sigma}$*** is $\mathrm{KA}(\sigma) = -\log_2 \delta(\sigma)$. Since the negative logarithm of a multiplicatively maximal function is additively optimal in the earlier sense, we have $\mathrm{KA}(\sigma) \leq^{+} -\log_2 \delta(\sigma)$ for all σ and any fixed r.e. continuous semimeasure δ.

The following result is closely related to Lemma 1.2.14.

Lemma 1.2.18. If $P \subseteq \{0,1\}^{<\mathbb{N}}$ is a prefix-free set, then $\sum_{\sigma\in P} 2^{-\mathrm{KA}(\sigma)} \leq 1$.

For a discussion of KP, KA, and the other varieties of Kolmogorov complexity mentioned above, see for example Uspensky and Shen [49] or Downey and Hirschfeldt [13].

Kolmogorov complexity and Martin-Löf randomness

As mentioned before, complexity provides a third paradigm for algorithmic randomness. A random sequence $X \in \{0,1\}^{\mathbb{N}}$ should not have any patterns that allow for significant compression of its initial segments. Formally, this can be seen in the following theorem.

Theorem 1.2.19 (Schnorr [42] and Levin [24])**.** $X \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random if and only if $\mathrm{KP}(X \upharpoonright n) \geq^{+} n$ for all n if and only if $\mathrm{KA}(X \upharpoonright n) \geq^{+} n$ for all n.

Prefix-free complexity relative to an oracle

The definitions of both conditional and unconditional prefix-free complexity can be relativized to an oracle X by replacing prefix-free partial recursive functions with prefix-free and partial recursive relative to X functions. The existence of an additively optimal version of K_f relative to X still holds. Then we write $\mathrm{KP}^X(\sigma \mid \tau)$ and $\mathrm{KP}^X(\sigma)$ to mean respectively conditional prefix-free complexity relative to X and (unconditional) prefix-free complexity relative to X. The proofs of both Lemma 1.2.14 and Lemma 1.2.15 relativize to KP^X as well.

1.3 Partial randomness

The variants of partial randomness studied in this dissertation are largely extensions of definitions given throughout the literature. The following is not meant as an exhaustive timeline of the history of partial randomness, but rather to give an idea of the origins of several key concepts.

Tadaki [48] gave the first partial randomness definition that was a direct generalization of Martin-Löf randomness. Tadaki can be credited with introducing, in the linear case, the notions of dwt-f-random, Borel-Cantelli dwt-f-random, KP-f-complex, and strongly KP-f-complex. Significantly, Tadaki also noticed that several key results about Martin-Löf randomness translated well into the case of partial randomness. In the linear case, Tadaki established the equivalence of dwt-f-random and KP-f-complex as well as that of Solovay dwt-f-random and strongly KP-f-complex. Using a generalized construction of Chaitin's Ω, Tadaki also showed that natural examples of dwt-f-randoms existed in the linear case.

While Tadaki was the first to formalize partial randomness in this way, as early as two decades prior others had been investigating partial randomness from the point of view of Kolmogorov complexity. Where $C(\sigma \mid n)$ is conditional plain complexity, Staiger [44] studied growth rates of $\mathrm{C}(X \upharpoonright n \mid n)$ for X belonging to effectively closed sets $F \subseteq \{0,1\}^{\mathbb{N}}$. In particular the upper bound $\mathrm{C}(X \upharpoonright n \mid n) \leq^{+} H_F \cdot n$ always holds, where H_F is the entropy of F as defined by Kuich [22]. Staiger also provided what is essentially the first weak separation, by showing that an $X \in F$ can be found satisfying $C(X \upharpoonright n \mid n) \geq^{+} (H_F - \epsilon) \cdot n$ for all n, where $\epsilon > 0$ is an arbitrary fixed number. In the same vein, Staiger [45] expanded on these results, defining the quantities

$$\underline{\kappa}(X) = \liminf_{n\to\infty} \frac{\mathrm{C}(X \upharpoonright n \mid n)}{n} \text{ and } \kappa(X) = \limsup_{n\to\infty} \frac{\mathrm{C}(X \upharpoonright n \mid n)}{n}$$

for an arbitrary $X \in \{0,1\}^{\mathbb{N}}$. In addition to the connection between these concepts and entropy, they are related to Hausdorff dimension. Of course if X is Martin-Löf random, then $\underline{\kappa}(X) = \kappa(X) = 1$, and so in a sense these quantities are measures of the degree of randomness of a given X. Ryabko [38, 39] and Staiger [46] also used supermartingales to further this line of investigation. In particular, they noted that there exists a relationship between $\underline{\kappa}(X)$ and the speed with which a recursive martingale can succeed on X. This can perhaps be viewed as a precursor to the supermartingale characterizations of partial randomness notions that were later made explicit.

Lutz [27, 28] and Mayordomo [30] investigated the quantities $\underline{\kappa}$ and κ, finding similar results and characterizations in terms of effective Hausdorff dimension. In various papers they use linear versions of f-supermartingales to study constructive dimension, an effective form of Hausdorff dimension.

Calude, Staiger, and Terwijn [4] first introduced the notion of pwt-f-random for linear functions f. For all linear f, they also observed the fundamental link between pwt-f-randomness and f-supermartingales: X is pwt-f-random if and only if no r.e. f-supermartingale succeeds on X. Because of the connection between f-supermartingales and KA, this implies as a corollary that X is pwt-f-random if and only if it is KA-f-complex. These authors also gave the first definition of Solovay dwt-f-random in the linear case, proving that it was equivalent to strongly KP-f-complex and therefore to Borel-Cantelli dwt-f-random as well. At this point historically, it was still unclear which notions of partial randomness could be separated.

Reimann [33, 34] greatly expanded on what was known about partial randomness by considering for the first time functions which were not length-invariant. He showed that pwt-f-random and KA-f-complex are equivalent not just for linear functions f, but for any f belonging to a special class of length-invariant functions known as dimension functions. Another result of Reimann combined with a result attributed to Kjos-Hanssen proved the equivalence of pwt-f-random and vwt-f-random, a notion introduced earlier by Kjos-Hanssen, in the linear case.

Reimann and Stephan [36] endeavored to provide a much more complete understanding of variants of partial randomness. Compared to the previous literature, they studied dwt-f-randomness, pwt-f-randomness, and Solovay dwt-f-randomness for a much broader class of functions f. They extended known implications to greater generality as well as proving some previously unknown implications. They also provided the first known method of constructing an X which is dwt-f-random but not Solovay dwt-f-random, where f is a particular type of length-invariant convex function; since they also established that pwt-f-random implies Solovay dwt-f-random, this finally separated dwt-f-random and pwt-f-random as notions of partial randomness. They also separated Solovay dwt-f-randomness from pwt-f-randomness for the same class of functions.

The definitions and results of Chapters 3 and 4 are largely a continuation of this trend to consider notions of partial randomness in even greater generality, and to study what known equivalences extend to these generalizations and attempt to separate notions where an equivalence does not hold.

1.4 Summary of results

In Chapter 2, the connection between Martin-Löf and Solovay randomness is further explored. The definition of Solovay randomness and its connection to Martin-Löf randomness can be seen as an effectivization of the first Borel-Cantelli lemma. The measure-theoretic paradigm for algorithmic randomness sets out to classify those sequences which do not belong to any effectively measure 0 set. The first Borel-Cantelli lemma says that for any sequence $\langle A_n \subseteq \{0,1\}^{\mathbb{N}} \mid n \in \mathbb{N}\rangle$ such that $\sum_n \mu(A_n) < \infty$, there is a measure 0 set $\{X \mid X \in A_n \text{ for infinitely many } n\}$. Then if we let the sequence considered be uniformly effectively open as well, the measure 0 set has measure 0 effectively. This is precisely the content of the theorem that Martin-Löf random implies Solovay random. Along these same lines, then, we look at other effectivizations of the first Borel-Cantelli lemma. Theorem 2.2.4 shows that Martin-Löf randomness can also be characterized in terms of the first Borel-Cantelli lemma for effectively clopen sequences.

In addition to the first Borel-Cantelli lemma, the second Borel-Cantelli lemma and its effectivizations are investigated as well. Instead of providing a similar characterization of Martin-Löf randomness, Theorem 2.2.7 shows that both the effectivization of the second Borel-Cantelli lemma in terms of uniformly effectively open sequences and the one in terms of uniformly effectively clopen sequences characterize Kurtz randomness. We also look at the second Borel-Cantelli and its effectivization to Σ_n, $\Delta_1^{\varnothing^{(m)}}$ and $\Sigma_n^{\varnothing^{(m)}}$ sequences, and show that no such effectivization can be used to characterize Martin-Löf randomness. While this is of interest on its own, the motivation for this investigation was to assess the feasibility of using the second Borel-Cantelli lemma to define a notion of partial randomness. The fact that no characterization of Martin-Löf randomness can be done means that a notion of partial randomness stemming from the second Borel-Cantelli lemma would likely not provide interesting results.

In Chapters 3 and 4 we introduce all of the definitions of partial randomness studied in this dissertation. The emphasis of these two chapters is placed on proving every possible implication in the most general setting, as well as attempting to separate two notions of partial randomness when this is not possible. The definitions and results of these two chapters are summarized by Figures 1.1, 1.2, and 1.3. Figures 1.4, 1.5, 1.6, and 1.7 present this same information in a different way, so as to emphasize the more important results.

In Chapter 5 we look at the question of strong separation of partial randomness notions. Miller [31] proved that there is an X which has Turing degree of effective Hausdorff dimension 1/2. An analysis of that proof shows that in fact the constructed X satisfies $\mathrm{KP}(X \upharpoonright n) \geq^{+} n/2$ for all n, and that no $Y \leq_T X$ satisfies $\mathrm{KP}(X \upharpoonright n) \geq^{+} (1/2 + \epsilon)n$ for all n, for any $\epsilon > 0$. Thus we can strongly separate dwt-f-randomness from dwt-g-randomness for different linear length-invariant functions f and g. We prove, using forcing conditions inspired by Miller's, that a much greater strong separation exists. Theorem 5.0.8 proves that for any length-invariant convex f such that KA-f-complex is not equivalent to Martin-Löf random, there is a KA-f-complex X which does not Turing compute any Y which is KP-g-complex for g recursively strongly dominating f. Theorem 5.3.2 shows moreover that the hypothesis cannot be weakened by removing the length-invariant condition, and that in fact there is a strongly convex but non-length-invariant f such that every KA-f-complex computes a KA-$(2f)$-complex Y.

In Chapter 6 we look at points of Cantor space which are either K-trivial or low for K. In a certain sense, something which is K-trivial or low for K is as far as possible from being partially random, as they have minimal initial segment complexity (as low as a recursive point in fact). The key new result of this chapter is Theorem 6.2.2, which states that any X powerful enough to compute a monotonic unbounded lower bound on $\mathrm{KP}(n)$ must also be powerful enough to compute the halting problem $\varnothing'$. This can be used to give a new and much simpler proof of the fact that every K-trivial is never continuously random (NCR). Additionally we can apply Theorem 6.2.2 to show that a K-trivial X and a $Y \leq_T \varnothing'$ cannot be mutually relatively random for any measure such that neither X nor Y is an atom. This is the content of Theorem 6.3.3.

Finally in Chapter 7 we relate the results of this dissertation to other research in mathematical logic. In particular both Chapters 5 and 6 are connected to the notion *complex*. Additionally, the results of Chapter 5 can be reframed in terms of *mass problems*.

Figure 1.1: Partial randomness for recursive $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$

Test-based variants of f-randomness

Where wt is dwt, pwt, or vwt:

***wt*-*f*-random.** X is wt-f-random if there is no uniformly r.e. sequence $\langle A_n \rangle$ such that $wt_f(A_n) \leq 2^{-n}$ for all n and $X \in \bigcap_n [A_n]$.

Solovay *wt*-*f*-random. X is Solovay wt-f-random if there is no r.e. A such that $wt_f(A) < \infty$ and $A \cap X^{<\mathbb{N}}$ is infinite.

Borel-Cantelli *wt*-*f*-random. X is Borel-Cantelli wt-f-random if there is no uniformly r.e. sequence $\langle A_n \rangle$ such that $\sum_n wt_f(A_n) < \infty$ and $\{n \mid X \in [A_n]\}$ is infinite.

Martingale-based variant of f-randomness

Supermartingale *f*-random. X is supermartingale f-random if there is no r.e. function $\delta : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ such that

$$2^{f(\sigma)}\delta(\sigma) \geq 2^{f(\sigma\frown 0)}\delta(\sigma\frown 0) + 2^{f(\sigma\frown 1)}\delta(\sigma\frown 1)$$

for all σ and $\limsup_{n\to\infty} \delta(X \upharpoonright n) = \infty$.

Variants of f-complexity:

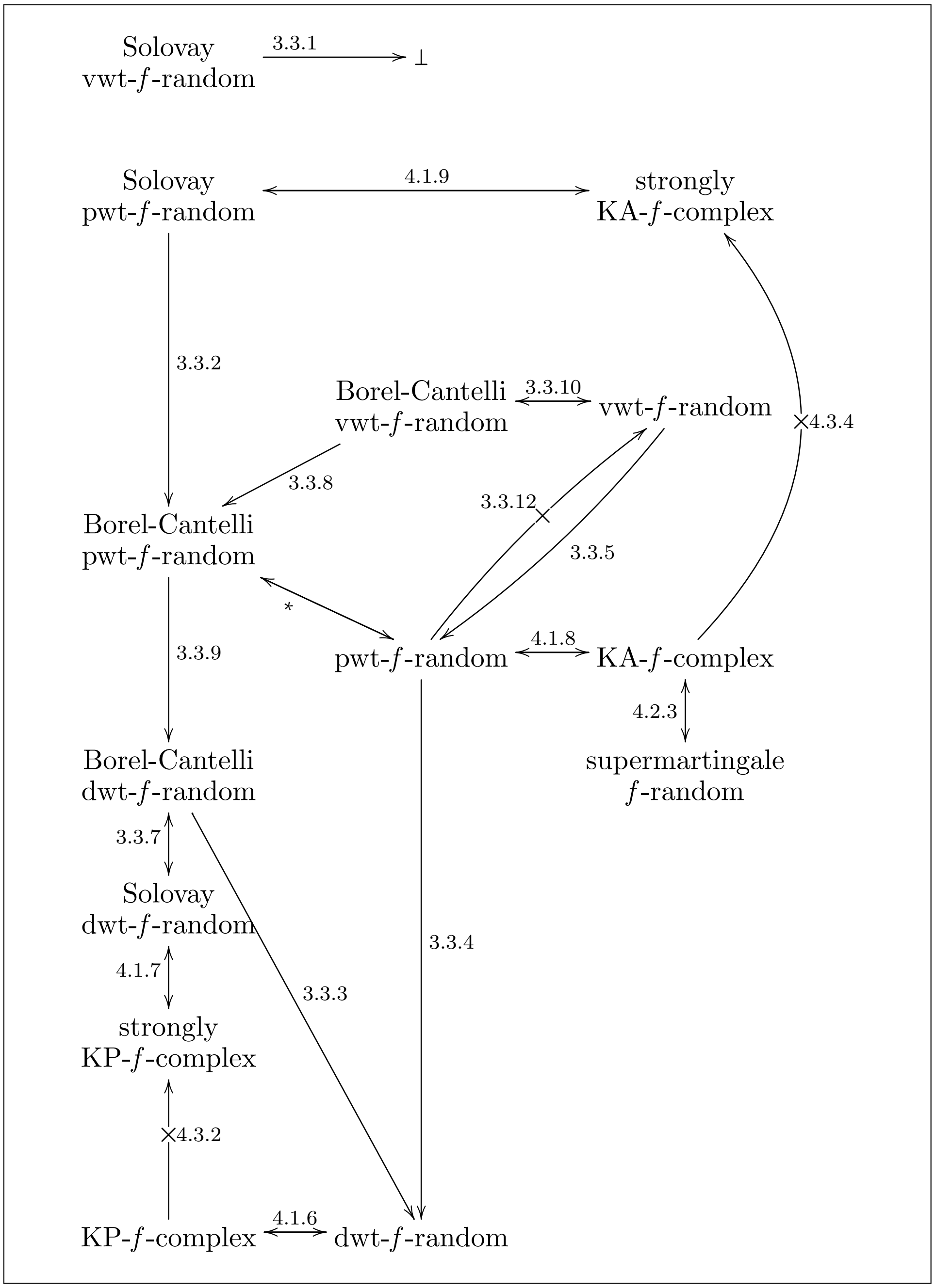
Where K is KP or KA:

***K*-*f*-complex.** X is K-f-complex if $K(X \upharpoonright n) \geq^+ f(X \upharpoonright n)$ for all n.

Strongly *K*-*f*-complex. X is strongly K-f-complex if

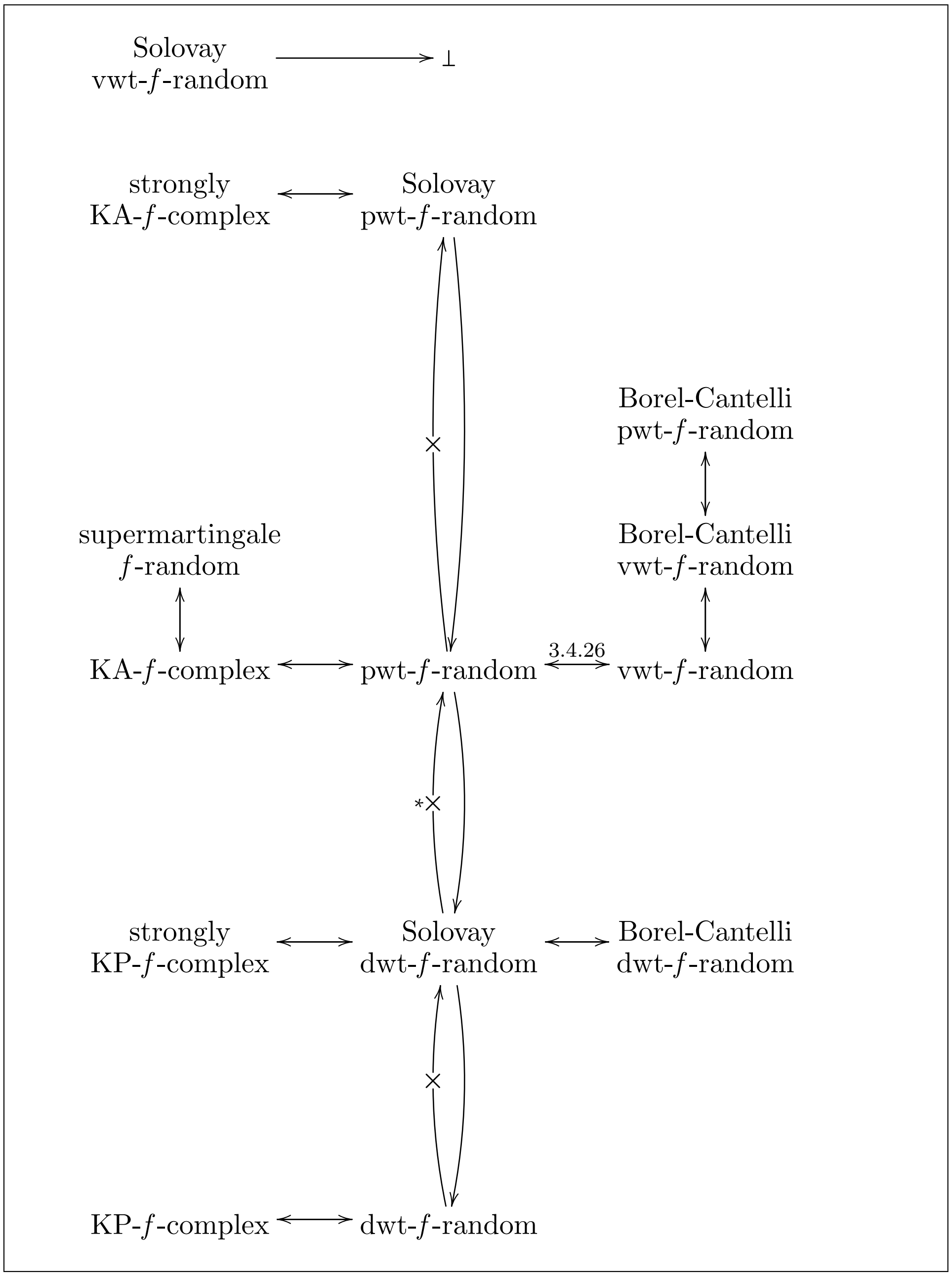
$$\lim_{n\to\infty} K(X \upharpoonright n) - f(X \upharpoonright n) = \infty.$$

Figure 1.2: Implications for recursive $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$



* Due to Higuchi, Hudelson, Simpson, and Yokoyama [18].

Figure 1.3: Implications for recursive convex $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$



* Due to Reimann and Stephan [36].

Figure 1.4: Implications for test notions of f-random, general case

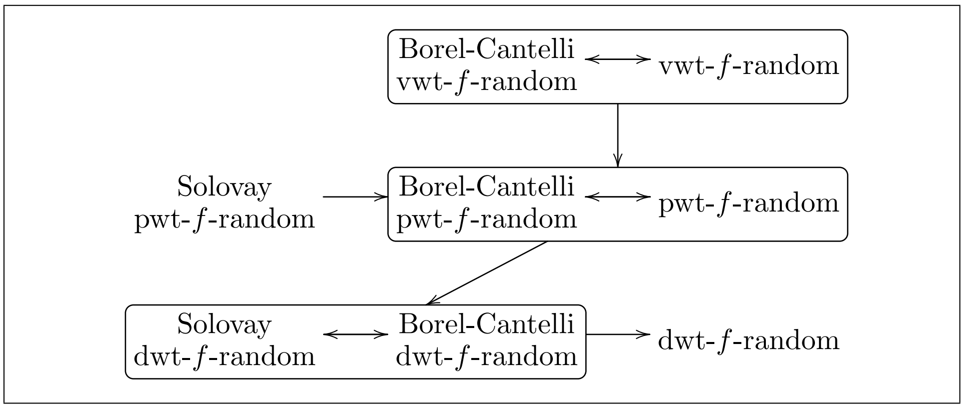


Figure 1.5: Connecting f-random and f-complex, general case

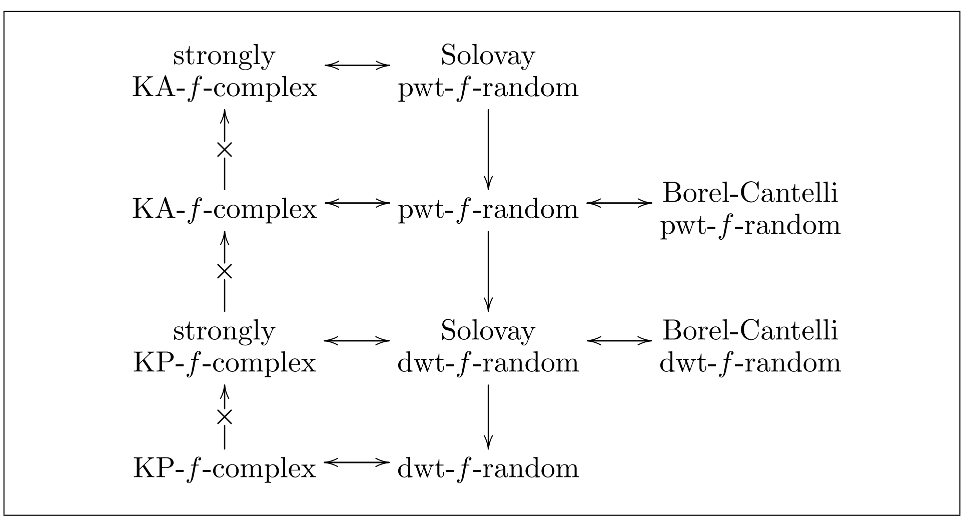


Figure 1.6: Implications for test notions of f-random, convex case

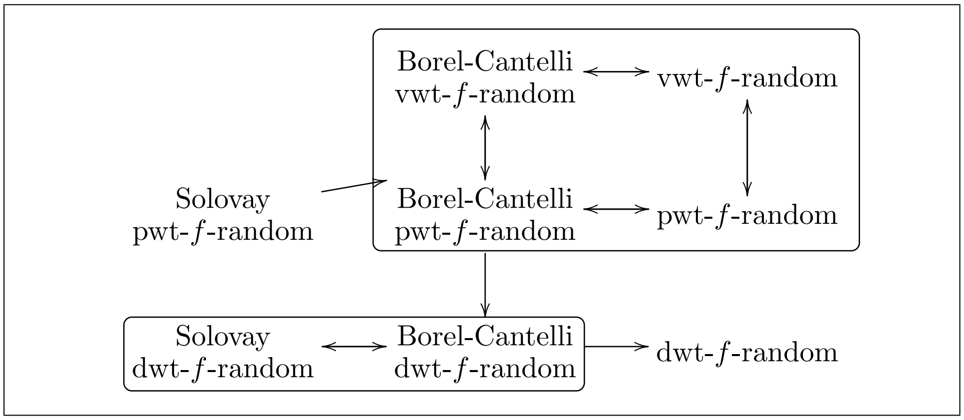
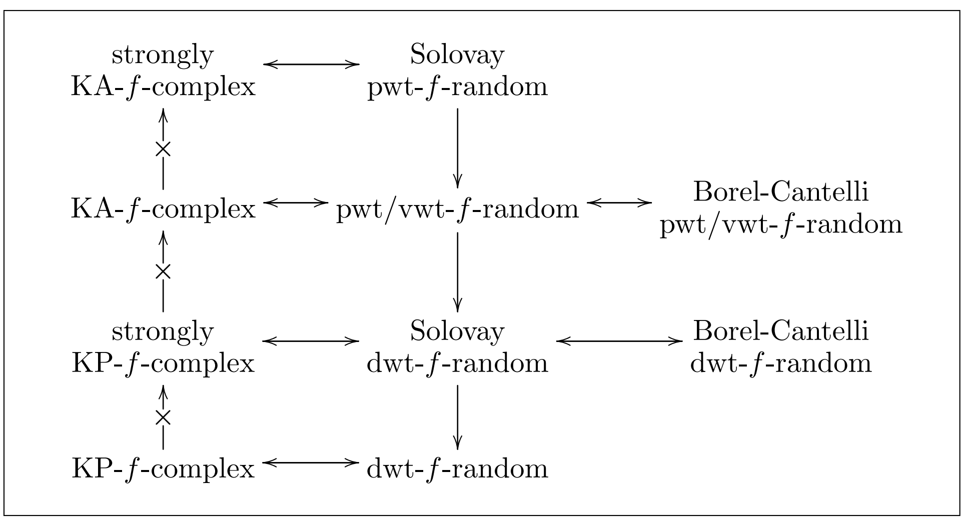


Figure 1.7: Connecting f-random and f-complex, convex case



Chapter 2

The Borel-Cantelli lemmas

The motivation for this chapter is the relationship between Solovay randomness and Martin-Löf randomness, which we restate below.

Theorem 2.0.1. $X \in \{0,1\}^{\mathbb{N}}$ is Martin-Löf random if and only if it is Solovay random.

There are numerous other notions of random one could consider; some will turn out to be equivalent to Martin-Löf random, while others will not. Our interest in Solovay randomness is because of the remarkable similarity its definition shares with the classical result known as the first Borel-Cantelli lemma. In this chapter we will further explore the connections between the Borel-Cantelli lemmas and algorithmic randomness. We will make formal the connection between Martin-Löf randomness and the first Borel-Cantelli lemma, and show that a similar connection does not hold for the second Borel-Cantelli lemma.

2.1 Classical Borel-Cantelli lemmas

2.1.1 The first Borel-Cantelli lemma

Known as the first Borel-Cantelli lemma or simply the Borel-Cantelli lemma, the following result is due independently to Borel [3] and Cantelli [5].

Lemma 2.1.1. Let $\langle A_n \subseteq \{0,1\}^{\mathbb{N}} \mid n \in \mathbb{N}\rangle$ be any sequence such that $\sum_n \mu(A_n) < \infty$. With probability 1, X belongs to only finitely many A_n.

This relates to Theorem 2.0.1 in the following way: given a Solovay test $\langle A_n\rangle$ the sequence necessarily satisfies $\sum_n \mu(A_n) < \infty$, while a Martin-Löf test $\langle B_n\rangle$ corresponds to a measure 1 set $\bigcup_n(\{0,1\}^{\mathbb{N}} \setminus B_n)$; Theorem 2.0.1 allows us to find a specific measure 1 set witnessing Lemma 2.1.1 for $\langle A_n\rangle$.

2.1.2 The second Borel-Cantelli lemma

Definition 2.1.2. A finite set of events $\{A_n \subseteq \{0,1\}^{\mathbb{N}} \mid n < k\}$ is called ***mutually independent*** if $\prod_{n<k}\mu(A_n) = \mu(\bigcap_{n<k} A_n)$; an infinite set of events $\{A_n \mid n \in \mathbb{N}\}$ is called ***mutually independent*** if every finite subset of A is mutually independent.

The following result is a partial inverse to the first Borel-Cantelli lemma, and hence referred to as the second.

Lemma 2.1.3. Let $\langle A_n \subseteq \{0,1\}^{\mathbb{N}} \mid n \in \mathbb{N}\rangle$ be a sequence of events such that $\sum_n \mu(A_n) = \infty$ and $\{A_n \mid n \in \mathbb{N}\}$ is mutually independent. Then with probability 1, X belongs to infinitely many of the A_n.

Given the connection between the first Borel-Cantelli lemma and randomness, we will also explore what connections exist between the second lemma and randomness.

2.2 The Borel-Cantelli lemmas for Δ_1 and Σ_1

Theorem 2.0.1 can be thought of as an effectivization of Lemma 2.1.1. But what exactly does effectivization mean in this context? In particular, which types of sets should we allow?

Definition 2.2.1. $\mathcal{P}(\{0,1\}^{\mathbb{N}} \times \mathbb{N})$ denotes the set of all sequences of the form $\langle A_n \subseteq \{0,1\}^{\mathbb{N}} \mid n \in \mathbb{N}\rangle$. For any $\mathcal{A} \subseteq \mathcal{P}(\{0,1\}^{\mathbb{N}} \times \mathbb{N})$ let $\mathrm{FBC}(\mathcal{A})$ denote the set of X such that the first Borel-Cantelli lemma holds for $\mathcal{A}$-sequences:

$$\mathrm{FBC}(\mathcal{A}) = \left\{X \mid (\forall\langle A_n\rangle \in \mathcal{A})\left[\sum \mu(A_n) < \infty \to (\forall^\infty n)(X \notin A_n)\right]\right\}.$$

Similarly let $\mathrm{SBC}(\mathcal{A})$ mean the set of all X such that the second Borel-Cantelli lemma holds for $\mathcal{A}$-sequences:

$$\mathrm{SBC}(\mathcal{A}) = \left\{X \mid (\forall\langle A_n\rangle \in \mathcal{A}^*)\left[\sum \mu(A_n) = \infty \to (\exists^\infty n)(X \in A_n)\right]\right\}.$$

Here $\mathcal{A}^*$ is the subset of $\mathcal{A}$ consisting of exactly the mutually independent sequences.

These definitions now allow us to relativize the first and second Borel-Cantelli lemmas to specific classes of sets, and see in what ways the resulting relativizations provide characterizations of known randomness concepts.

Natural candidates for sets $\mathcal{A}$ include the arithmetical hierarchy, two pieces of which we define below.

Definition 2.2.2.

$$\Delta_1 = \{\langle A_n \rangle \in \mathcal{P}(\{0,1\}^{\mathbb{N}} \times \mathbb{N}) \mid \langle A_n \rangle \text{ is uniformly effectively clopen}\}.$$

$$\Sigma_1 = \{\langle A_n \rangle \in \mathcal{P}(\{0,1\}^{\mathbb{N}} \times \mathbb{N}) \mid \langle A_n \rangle \text{ is uniformly effectively open}\}.$$

Δ_1 is characterized by the fact that $\langle A_n \rangle$ belongs to Δ_1 if and only if both $\langle A_n \rangle \in \Sigma_1$ and $\langle \{0,1\}^{<\mathbb{N}} \setminus A_n \rangle \in \Sigma_1$. Alternatively, $\langle A_n \rangle \in \Delta_1$ if and only if each A_n is clopen and there is a recursive function f such that $f(n)$ is a code for A_n for each n. Note that a code for A_n might mean one of two things. First, if we take a recursive enumeration $\{x_n\}$ of all finite subsets of $\{0,1\}^{<\mathbb{N}}$, then there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $[x_{f(n)}] = A_n$ for each n. Second, if we take an enumeration $\{B_n\}$ of all effectively open subsets of $\{0,1\}^{\mathbb{N}}$, then there is a recursive function $f : \mathbb{N} \to \mathbb{N}^2$ such that if $f(n) = (p,q)$ then $A_n = B_p$ and $\{0,1\}^{<\mathbb{N}} \setminus A_n = B_q$ for all n. Using the compactness of $\{0,1\}^{\mathbb{N}}$, one can prove that these two methods of coding are equivalent.

It is clear that $\Delta_1 \subset \Sigma_1$ and that this inclusion is strict. A special subset of Δ_1 is the set of sequences of basic clopen sets:

$$\{\langle A_n \rangle \in \mathcal{P}(\{0,1\}^{\mathbb{N}} \times \mathbb{N}) \mid \exists \text{ rec. } f : \mathbb{N} \to \{0,1\}^{<\mathbb{N}} \text{ s.t. } (\forall n)(A_n = [f(n)])\}.$$

The first Borel-Cantelli lemma

If $\langle A_n \rangle \in \Sigma_1$ and $\sum \mu(A_n) < \infty$, then $\langle A_n \rangle$ is a Solovay test. Then not surprisingly, given Theorem 2.0.1, the Martin-Löf randoms are exactly the points at which the first Borel-Cantelli lemma holds for all Σ_1 sequences. Actually though, the same is true for Δ_1 sequences as well. First we need the following lemma.

Lemma 2.2.3. If $\langle A_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N} \rangle$ is a uniformly r.e. sequence then there is a uniformly r.e. sequence $\langle B_n \subseteq \{0,1\}^{<\mathbb{N}} \rangle$ such that $[A_n] = [B_n]$ and B_n is prefix-free for each n.

Proof. Fix n, and suppose that there is some t such that $\tau \in A_n^{t+1} \setminus A_n^t$ has a prefix $\sigma \in A_n^t$. If $[\tau] \subseteq [A_n^t]$ then define $B_n^{t+1} = B_n^t$. Otherwise, $[\tau] \setminus [A_n^t]$ is clopen and there exist strings $\rho_1, \dots, \rho_k$ such that $[\tau] \setminus [A_n^t] = [\{\rho_1, \dots, \rho_k\}]$. Moreover these can be found effectively and such that the set $\{\rho_1, \dots, \rho_k\}$ is itself prefix-free. Then define $B_n^{t+1} = B_n^t \cup \{\rho_1, \dots, \rho_k\}$. Then putting $B_n =$

$\bigcup_t B_n^t$ for each n, this defines $\langle B_n \rangle$ as a uniformly r.e. sequence. Moreover it is clear that each B_n is prefix-free and that $[B_n] = [A_n]$ for all n. □

Theorem 2.2.4. For each $X \in \{0,1\}^{\mathbb{N}}$, the following are equivalent:

1. $X \in \mathrm{FBC}(\Sigma_1)$,
2. $X \in \mathrm{FBC}(\Delta_1)$, and
3. X is Martin-Löf random.

Proof. $(1) \Rightarrow (2)$: Since $\Sigma_1 \supset \Delta_1$, this is immediate.

$(2) \Rightarrow (3)$: Let $X \in \mathrm{FBC}(\Delta_1)$ and let $\langle A_n \rangle$ be a Martin-Löf test. Since $\langle A_n \rangle$ is uniformly effectively open, there is a uniformly r.e. sequence $\langle B_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N} \rangle$ such that $A_n = [B_n]$ for each n; we can assume by Lemma 2.2.3 that each B_n is prefix-free. The set $B = \bigcup_n B_n$ is recursively enumerable as well, and infinite so long as the original Martin-Löf test has infinitely many non-empty levels of the test.

Now let $f : \mathbb{N} \to \{0,1\}^{<\mathbb{N}}$ be a function that enumerates B without repetition. Defining $C_n = [f(n)]$ for all n, the sequence $\langle C_n \rangle$ clearly belongs to Δ_1. Moreover

$$\sum_n \mu(C_n) = \sum_n \mu([f(n)]) \leq \sum_m \mu([B_m]) = \sum_m \mu(A_m) \leq 2.$$

Then since $X \in \mathrm{FBC}(\Delta_1)$, the set $\{n \mid X \in C_n\}$ is finite. In other words, $f(n) \subset X$ for only finitely many n. But then if we define the number $m = \max\{|f(n)| \mid f(n) \subset X\}$ it is clear that $\{f(n) \mid f(n) \subset X\} \cap B_{m+1}$ is empty and so $X \notin A_{m+1}$. Since $\langle A_n \rangle$ was an arbitrary Martin-Löf test, X is Martin-Löf random.

$(3) \Rightarrow (1)$: Let X be Martin-Löf random and let $\langle A_n \rangle$ be a uniformly effectively open sequence such that $\sum_n \mu(A_n) < \infty$. As noted above, $\langle A_n \rangle$ is a Solovay test and so by Theorem 2.0.1 the set $\{n \mid X \in A_n\}$ must be finite. Therefore $X \in \mathrm{FBC}(\Sigma_1)$. □

In other words, the first Borel-Cantelli lemma provides us with two ways of characterizing Martin-Löf randomness.

The second Borel-Cantelli lemma

At this point it is natural to ask whether the second Borel-Cantelli lemma, effectivized to either Δ_1 or Σ_1, will provide another characterization of Martin-Löf randomness. While this will not be the case, we will end up with a characterization of a slightly different type of randomness. The following definition is due to Kurtz [23].

Definition 2.2.5. A ***Kurtz test*** is an effectively open set $A \subseteq \{0,1\}^{\mathbb{N}}$ such that $\mu(A) = 1$. X is ***Kurtz random*** if $X \in A$ for any Kurtz test A.

Note that the definition of a Kurtz test is not quite analogous with that of a Martin-Löf test; a random sequence should avoid belonging to a Martin-Löf test, while only a non-random sequence can avoid belonging to a Kurtz test. They are similar, though, in the sense that effective sets of measure 0 should not contain random points. Only the interpretation of "effective sets of measure 0" changes between the two test types. Wang [51] showed that Kurtz randomness can be characterized as follows.

Theorem 2.2.6. $X \in \{0,1\}^{\mathbb{N}}$ is Kurtz random if and only if there is no uniformly effectively clopen sequence $\langle A_n \rangle$ such that $\mu(A_n) = 2^{-n}$ for each n and $X \in A_n$ for all n as well.

This is somehow related to Theorem 2.0.1: both theorems show that a certain measure 1 set relates to sets whose measures decay to 0 effectively. The second Borel-Cantelli lemma provides the following characterizations of Kurtz randomness.

Theorem 2.2.7. For each $X \in \{0,1\}^{\mathbb{N}}$, the following are equivalent:

1. $X \in \mathrm{SBC}(\Sigma_1)$,
2. $X \in \mathrm{SBC}(\Delta_1)$, and
3. X is Kurtz random.

Proof. (1) $\Rightarrow$ (2): Since $\Delta_1 \subset \Sigma_1$ this is immediate.

(2) $\Rightarrow$ (3) : Suppose that X is not Kurtz random. Then there exists a uniformly effectively clopen sequence $\langle A_n \rangle$ such that $X \in A_n$ and $\mu(A_n) = 2^{-n}$ for each n. Now define $A_0^* = A_0$ and for each n define A_{n+1} by first taking $A_n^* \cap A_{n+1}$ and then padding the resulting set to get a clopen subset of A_n^* of measure 2^{n-1}. Then $\langle A_n^* \rangle$ is also an effectively clopen sequence. Moreover, if $X \in \bigcap_n A_n$ then $X \in \bigcap_n A_n^*$. Therefore we can assume without loss of generality that our original effectively clopen sequence had the nesting property: $A_{n+1} \subset A_n$ for all n.

We will now describe how to define a particular sequence in Δ_1. Define $E_0 = \{0,1\}^{\mathbb{N}} \setminus A_1$. Note that this set is clopen and $\mu(E_0) = \mu(\{0,1\}^{\mathbb{N}} \setminus E_0) = 1/2$. Also by assumption we have $X \notin E_0$.

Now assume by induction that we have defined E_n such that

- $\mu(E_n) = \mu(\{0,1\}^{\mathbb{N}} \setminus E_n) = 1/2$,
- $A_{n+1} = (\{0,1\}^{\mathbb{N}} \setminus E_0) \cap \cdots \cap (\{0,1\}^{\mathbb{N}} \setminus E_n)$ and so $X \notin E_n$, and
- $\mu(C_0 \cap \cdots \cap C_n) = 2^{-n-1}$ when each C_i is either E_i or $\{0,1\}^{\mathbb{N}} \setminus E_i$.

We will say how to define E_{n+1}. Consider the set A_{n+2}. By assumption $A_{n+2} \subset A_{n+1}$ and $\mu(A_{n+2}) = \frac{1}{2}\mu(A_{n+1})$. So put A_{n+2} into $\{0,1\}^{\mathbb{N}} \setminus E_{n+1}$ and put $A_{n+1} \setminus A_{n+2}$ into E_{n+1}. Doing so ensures that

$$A_{n+2} = (\{0,1\}^{\mathbb{N}} \setminus E_0) \cap \cdots \cap (\{0,1\}^{\mathbb{N}} \setminus E_{n+1})$$

and so $X \notin E_{n+1}$ as required.

Next, consider each other intersection $D = C_0 \cap \cdots \cap C_n$ where every C_i is either E_i or $\{0,1\}^{\mathbb{N}} \setminus E_i$ (we have already dealt with the case of the intersection $(\{0,1\}^{\mathbb{N}} \setminus E_0) \cap \cdots \cap (\{0,1\}^{\mathbb{N}} \setminus E_n)$, so look at the other such intersections). D is clopen and $\mu(D) = 2^{-n-1}$. We can effectively break D in half, that is find a clopen set $D_1 \subset D$ such that $\mu(D_1) = \mu(D \setminus D_1) = \mu(D)/2$. Doing so, put all of D_1 into E_{n+1} and all of $\{0,1\}^{\mathbb{N}} \setminus D_1$ into $\{0,1\}^{\mathbb{N}} \setminus E_{n+1}$. Then we have ensured that $\mu(C_0 \cap \cdots \cap C_{n+1}) = 2^{-n-2}$ whenever each C_i is either E_i or $\{0,1\}^{\mathbb{N}} \setminus E_i$. Finally note that there are exactly 2^{n+1} sets of the form $C_0 \cap \cdots \cap C_n$, each is disjoint of measure 2^{-n-2}, and we have constructed E_{n+1} to be clopen and satisfy $\mu(E_{n+1}) = \mu(\{0,1\}^{\mathbb{N}} \setminus E_{n+1}) = 2^{n+1} \cdot 2^{-n-2} = 1/2$.

The sequence $\langle E_n \rangle$ is uniformly effectively clopen by construction, and so belongs to Δ_1. By induction it is clearly mutually independent, as each E_i cuts any other E_j exactly in half. Therefore $\langle E_n \rangle \in \Delta_1^*$. Since $\sum_n \mu(E_n) = \infty$ and $X \notin E_n$ for all n, this implies $X \notin \mathrm{SBC}(\Delta_1)$.

(3) ⇒ (1): Suppose that X is Kurtz random and let $\langle A_n \rangle$ be a sequence in Σ_1^* such that $\sum_n \mu(A_n) = \infty$. For arbitrary m, the classical version of the second Borel-Cantelli lemma implies that $\mu(\bigcup_{n>m} A_n) = 1$. Moreover, since $\langle A_n \rangle$ is uniformly effectively open, it is clear that $\bigcup_{n>m} A_n$ is effectively open. Then since X is Kurtz random, $X \in \bigcup_{n>m} A_n$. In other words there is an $n > m$ such that $X \in A_n$. Since m was arbitrary, there must then exist infinitely many n for which $X \in A_n$. This proves that $X \in \mathrm{SBC}(\Sigma_1)$. □

So unlike the first Borel-Cantelli lemma, the second does not provide a characterization of Martin-Löf randomness when relativized to Δ_1 or Σ_1. As we have seen though, Kurtz randomness is closely related to Martin-Löf randomness. In fact, Wang [51] showed that martingales can be used to characterize Kurtz randomness, while Downey, Griffiths, and Reid [12] showed that Kurtz randomness can be characterized using Kolmogorov complexity.

2.2.1 When the sums of measures are computable

In our effectivization of the first Borel-Cantelli lemma to Σ_1, we are requiring that the sets be uniformly effectively open and that the sum of the measures be finite. We could further effectivize the result by forcing the sum of the measures to converge to a recursive finite number as well.

Definition 2.2.8. For $\mathcal{A} \subseteq \mathcal{P}(\{0,1\}^{\mathbb{N}} \times \mathbb{N})$ let $\mathcal{A}^r$ be the set of $\mathcal{A}$-sequences $\langle A_n \rangle$ such that $\sum_n \mu(A_n) < \infty$ is a recursive real.

This further effectivization was investigated by Davie [10], who proved the following result.

Theorem 2.2.9. For any $\langle A_n \rangle \in \Sigma_1^r$, there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that if $\mathrm{KP}(X \upharpoonright n) \geq n - k$ for all n then $X \notin A_m$ for every $m > f(k)$.

Of course, $\mathrm{KP}(X \upharpoonright n) \geq^+ n$ if and only if X is Martin-Löf random. So not only does a Martin-Löf random only belong to finitely many of the A_n, but we can tell in a certain sense how long it takes a Martin-Löf random to leave the A_n. Then this result contains a degree of uniformity not found in Theorem 2.2.4.

On the other hand, as an effectivization of the first Borel-Cantelli lemma Theorem 2.2.9 is slightly unsatisfactory. In particular Galatolo, Hoyrup, and Rojas [16] proved the following about these sequences with recursive sums of measures.

Theorem 2.2.10. If $\langle A_n \rangle \in \Sigma_1^r$ then there is a recursive $X \in \{0,1\}^{\mathbb{N}}$ such that $\{n \mid X \in A_n\}$ is finite.

Furthermore, the result of Davie cannot be extended to provide another characterization of Martin-Löf randomness, as we show below. First we introduce a related randomness notion due to Schnorr [41].

Definition 2.2.11. A ***Schnorr test*** is a uniformly effectively open sequence $\langle A_n \rangle$ such that $\mu(A_n) = 2^{-n}$ for all n. X is ***Schnorr random*** if there is no Schnorr test $\langle A_n \rangle$ such that $X \in A_n$ for all n.

Then we have the following characterization of Schnorr randomness.

Theorem 2.2.12. For each $X \in \{0,1\}^{\mathbb{N}}$ the following are equivalent:

1. $X \in \mathrm{FBC}(\Sigma_1^r)$,
2. $X \in \mathrm{FBC}(\Delta_1^r)$, and
3. X is Schnorr random.

Proof. $(1) \Rightarrow (2)$: Again this is trivial since $\Sigma_1^r \supset \Delta_1^r$.

$(2) \Rightarrow (3)$: Suppose that $X \in \mathrm{FBC}(\Delta_1^r)$ and let $\langle A_n \rangle$ be a Schnorr test. As in the proof of Theorem 2.2.4, there is a uniformly r.e. sequence $\langle B_n \rangle$ such that $A_n = [B_n]$ for each n and each B_n is prefix-free. Define a set $B = \bigcup_n B_n$, and let $\{\sigma_n\}$ be an enumeration without repetition of B. Note that $\mu([B])$ is recursive since each $\mu(A_n)$ is.

Define a sequence $\langle C_n \rangle = \langle [\sigma_n] \rangle$. This sequence belongs to Δ_1^r since $\sum_n \mu(C_n) = \mu([B])$. Then by assumption $\{n \mid X \in C_n\}$ must be a finite set. Calculating $m = \max\{|\sigma_n| \mid \sigma_n \subset X\}$ it is clear that $X \notin A_{m+1}$. Since $\langle A_n \rangle$ was an arbitrary Schnorr test, X is Schnorr random.

$(3) \Rightarrow (1)$: Let X be Schnorr random and let $\langle A_n \rangle$ be a uniformly effectively open sequence such that $\sum_n \mu(A_n)$ is a recursive number. Then there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $\sum_{k > f(n)} \mu(A_k) < 2^{-n}$ for each n. Then the sets $B_n = \bigcup_{k > f(n)} A_k$ also have recursive measures; we can pad these sets until their measure is exactly 2^{-n}, forming new sets $C_n \supseteq B_n$. The resulting sequence $\langle C_n \rangle$ is a Schnorr test, and so there is some n such that $X \notin C_n \supseteq \bigcup_{k > f(n)} A_k$. Then $\{n \mid X \in A_n\}$ is finite and so $X \in \mathrm{FBC}(\Sigma_1^r)$. □

In a certain sense then, Theorem 2.2.10 can be seen as a corollary of the above result, since a recursive X can easily be constructed which avoids the intersection of any given Schnorr test.

Recursive sum of measures and the second Borel-Cantelli lemma

How does adding a recursive condition to the sum of measures impact effectivizations of the second Borel-Cantelli lemma? In fact it has no impact at all. This is due to the following: if $\sum_n \mu(A_n) = \infty$ for a sequence in either Δ_1 or Σ_1, then the sum goes recursively to infinity. That is, there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $\sum_{k < f(n)} \mu(A_k) > 2^n$ for all n.

2.3 Borel-Cantelli lemmas for more general classes

Chapter 3 introduces partial randomness notions based on both the definition of Martin-Löf randomness and on characterizations of Martin-Löf randomness. The first Borel-Cantelli lemma for Σ_1 characterizes the Martin-Löf randoms and so generalizations of the first Borel-Cantelli lemma give rise to meaningful definitions of partial randomness. In this section we try to answer the question: Could the second Borel-Cantelli lemma also provide a useful partial randomness notion? If so, then there should be an effectivization of the second Borel-Cantelli lemma that characterizes the Martin-Löf randoms.

The additional classes we consider are the following.

Definition 2.3.1. For $k > 1$ a set $A \subseteq \{0,1\}^{\mathbb{N}}$ is Σ_k if there is a recursive set $R \subseteq \mathbb{N}^k \times \{0,1\}^{<\mathbb{N}}$ such that $A_n = [\{\sigma \mid \exists n_1 \forall n_2 \exists n_3 \cdots (n_1, \ldots, n_k, \sigma) \in R\}]$. A sequence $\langle A_n \rangle$ is in Σ_k if there is a recursive set $R \subseteq \mathbb{N}^{k+1} \times \{0,1\}^{<\mathbb{N}}$ such that $A_n = [\{\sigma \mid \exists n_1 \forall n_2 \exists n_3 \cdots (n, n_1, \ldots, n_k, \sigma) \in R\}]$ for each n. Essentially, a sequence $\langle A_n \rangle$ belongs to Σ_n if and only if there is an effective sequence of Σ_{n-1} sequences $\langle A_{n,k} \mid k \in \mathbb{N} \rangle$ such that $A_n = \bigcup_k (\{0,1\}^{\mathbb{N}} \setminus A_{n,k})$ for all n.

While Σ_1 was not strong enough to capture Martin-Löf randomness, it is clear that Σ_2 is sufficiently strong.

Lemma 2.3.2. If $X \in \mathrm{SBC}(\Sigma_2)$ then X is Martin-Löf random.

Proof. Let $X \in \mathrm{SBC}(\Sigma_2)$ and let $\langle A_n \rangle$ be a Martin-Löf test. The set $\bigcap_n A_n$ has measure 0. If we define $B = \{0,1\}^{\mathbb{N}} \setminus \bigcap_n A_n = \bigcup_n (\{0,1\}^{\mathbb{N}} \setminus A_n)$ then it is clear that B is the effective intersection of the complements of a Σ_1 sequence. Therefore the sequence $\langle B_n \rangle$ where $B_n = B$ for all n belongs to Σ_2. Moreover $\mu(B_n) = \mu(B) = 1$ for all n and so $\langle B_n \rangle$ is mutually independent. Then X must be in B and so $X \notin A_n$ for some n. Since $\langle A_n \rangle$ is an arbitrary Martin-Löf test, X is Martin-Löf random. □

However the second Borel-Cantelli lemma for Σ_2 actually characterizes something much stronger than Martin-Löf randomness. Consider the following definitions.

Definition 2.3.3. $X \in \{0,1\}^{\mathbb{N}}$ is ***Kurtz n-random*** if it belongs to every Σ_n set $A \subseteq \{0,1\}^{\mathbb{N}}$ of measure 1. For $n > 0$, an ***n-Martin-Löf test*** is a uniformly effectively open relative to $\varnothing^{(n-1)}$ sequence $\langle A_n \rangle$ such that $\mu(A_n) \leq 2^{-n}$ for all n. X is ***n-random*** if there is no n-Martin-Löf test $\langle A_n \rangle$ such that $X \in A_n$ for all n.

Theorem 2.3.4. For each $n > 1$ and every X, X is Kurtz n-random if and only if $X \in \mathrm{SBC}(\Sigma_n)$.

Proof. First assume that $X \in \mathrm{SBC}(\Sigma_n)$ and let A be a Σ_n set of measure 1. Then define $\langle A_k \rangle$ by $A_k = A$ for all k. $\langle A_k \rangle$ is in Σ_n, and since $\mu(A_k) = 1$ for all k it is also mutually independent. Then since $\sum_k \mu(A_k) = \infty$ it must hold that $X \in A_k$ for some k, and therefore $X \in A$. Therefore X is Kurtz n-random.

Now assume that X is Kurtz n-random and let $\langle A_k \rangle \in \Sigma_n^*$ be such that $\sum_k \mu(A_k) = \infty$. For every fixed m the set $B_m = \bigcup_{k>m} A_k$ has measure 1 by the classical second Borel-Cantelli lemma. Of course the set B_m is also Σ_n and so $X \in B_m$. Therefore $X \in A_k$ for some $k > m$. Since this is true for arbitrary m, there are infinitely many k such that $X \in A_k$, and so $X \in \mathrm{SBC}(\Sigma_n)$. $\square$

It is additionally known, due to Kurtz [23], that n-randoms and Kurtz n-randoms are related as follows: If X is n-random then it is Kurtz n-random as well. If X is Kurtz $(n+1)$-random then it is n-random as well. In addition, neither reverse implication holds. Then since $\mathrm{FBC}(\Sigma_2)$ characterizes Kurtz 2-randomness, it cannot also characterize Martin-Löf randomness.

Another possibility is that we could characterize Martin-Löf randomness using a weaker generalization of Σ_1 sets.

Definition 2.3.5. A sequence $\langle A_n \rangle$ is in $\Delta_1^{\varnothing^{(n)}}$ if it is uniformly effectively clopen relative to the oracle $\varnothing^{(n)}$. Similarly, a sequence $\langle A_n \rangle$ is in $\Sigma_1^{\varnothing^{(n)}}$ if it is is uniformly effectively open relative to $\varnothing^{(n)}$.

Then it is clear that Theorem 2.2.4 relativizes to the following.

Theorem 2.3.6. For $X \in \{0,1\}^{\mathbb{N}}$ and $n > 0$, the following are equivalent:

1. $X \in \mathrm{FBC}(\Sigma_1^{\varnothing^{(n-1)}})$,
2. $X \in \mathrm{FBC}(\Delta_1^{\varnothing^{(n-1)}})$, and
3. X is n-random.

So could Martin-Löf randomness possibly be characterized by an effectivization of the second Borel-Cantelli lemma to some $\Sigma_1^{\varnothing^{(n)}}$? Theorem 2.2.7 relativizes to the following.

Theorem 2.3.7. For each $X \in \{0,1\}^{\mathbb{N}}$ and n, the following are equivalent:

1. $X \in \mathrm{SBC}(\Sigma_1^{\varnothing^{(n)}})$,
2. $X \in \mathrm{SBC}(\Delta_1^{\varnothing^{(n)}})$, and
3. X is Kurtz random relative to $\varnothing^{(n)}$.

However, it is also well known that Kurtz random relative to $\varnothing^{(n)}$ is not equivalent to Martin-Löf random for any n (using n-generics). Then it appears that no effectivization of the second Borel-Cantelli lemma can be used to characterize Martin-Löf randomness.

Chapter 3

Analogues of Martin-Löf randomness

In this chapter we introduce the ten notions of partial randomness which arise from generalizations of the definitions of Martin-Löf and Solovay randomness and the characterization of Martin-Löf randomness in terms of martingales. While Tadaki [48] and others considered partial randomness scaled according to a fixed constant, we consider partial randomness which scales according to a function $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$.

3.1 Functions as scales

Since we strive to prove results in the greatest generality possible, we will occasionally state results for functions $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ and use key lemmas to reduce the result to one for functions $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$. In fact, the distinction between these two types of functions rarely matters.

3.1.1 Types of functions

Definition 3.1.1. A function $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is ***integer-valued*** if $f(\sigma) \in \mathbb{N}$ for every σ; in other words the function actually has the form $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$. f is ***length-invariant*** if $f(\sigma) = f(\tau)$ whenever $|\sigma| = |\tau|$. Two functions $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ are ***equivalent***, denoted $f \approx g$, if $|f(\sigma) - g(\sigma)| \leq^+ 0$ for all σ.

Equivalence as above is a true equivalence relation on functions, and will be useful when discussing the relationship between partial randomness and complexity more fully. In particular, we will ensure the following each time we introduce a new notion of f-randomness or f-complexity.

Claim 3.1.2. For any notion of *f-random* which scales according to a given function f, if $f \approx g$ then any X is f-random if and only if it is g-random.

This fact allows us to work with easier to present functions when possible. For example if a function is equivalent to either a length-invariant or an integer-valued function, working with the equivalent function may be preferable. In fact any recursive function is equivalent to an integer-valued one.

Lemma 3.1.3. If $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is recursive then there is a recursive $g : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ such that $f \approx g$.

Proof. This is a simple consequence of the fact that f is recursive. There is a recursive function $\Phi : \{0,1\}^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}^{\geq 0}$ such that $|\Phi(\sigma, n) - f(\sigma)| \leq 2^{-n}$ for all σ and n. So define $g(\sigma) = \lfloor \Phi(\sigma, 0) \rfloor \in \mathbb{N}$ for all σ. This defines g recursively, and in such a way that $|g(\sigma) - f(\sigma)| \leq 2$ for all σ. Thus $f \approx g$ as required. □

The increasing set and weakly equivalent functions

Definition 3.1.4. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. The ***increasing set of f*** is the set $\mathcal{I}(f) = \{\sigma \mid (\forall \tau \subset \sigma)(f(\tau) < f(\sigma))\}$.

For recursive f it is not hard to see that $\mathcal{I}(f)$ is an r.e. set. If f is integer-valued, then $\mathcal{I}(f)$ is actually recursive. However, $\mathcal{I}(f)$ is not necessarily recursive in general because of the following: a machine cannot tell deterministically whether $f(\sigma)$ and $f(\tau)$ are equal as real numbers or just very close but ultimately different. To deal with this we introduce the following definition.

Definition 3.1.5. A function $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is ***strongly recursive*** if it is recursive and the function $\Phi : \{0,1\}^{<\mathbb{N}} \times \{0,1\}^{<\mathbb{N}} \to \{0,1\}$ such that $\Phi(\sigma, \tau) = 0$ if and only if $f(\sigma) < f(\tau)$ is recursive.

When looking at the increasing set of a strongly recursive function, we do not encounter the same problem that might arise for a recursive function. If f is strongly recursive $\mathcal{I}(f)$ must be recursive. We now define a weak form of equivalence using the increasing set.

Definition 3.1.6. $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ are ***weakly equivalent***, written $f \sim g$, if $f(\sigma) \leq^+ g(\sigma)$ for all $\sigma \in \mathcal{I}(f)$ and $g(\sigma) \leq^+ f(\sigma)$ for all $\sigma \in \mathcal{I}(g)$.

In practice, the most common case of weak equivalence is when two functions satisfy $\mathcal{I}(f) = \mathcal{I}(g)$ and $f|_{\mathcal{I}(f)} =^+ g|_{\mathcal{I}(g)}$. Weak equivalence is a weaker notion than equivalence: if $f \approx g$ then $f \sim g$ as well. But it is sufficient to ensure the following strengthening of Claim 3.1.2.

Claim 3.1.7. For any notion of *f-random* which scales according to a given function f, if $f \sim g$ then any X is f-random if and only if it is g-random.

Since this implies Claim 3.1.2, we will instead verify that this new condition is met every time we introduce a new randomness notion.

3.1.2 Defining "measures" using arbitrary functions

Consider the case that $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ is defined by $f(\sigma) = |\sigma|$ for all σ. Then for any prefix-free set $A \subset \{0,1\}^{<\mathbb{N}}$, we can calculate the Lebesgue measure of $[A] \subseteq \{0,1\}^{\mathbb{N}}$ by

$$\mu([A]) = \sum_{\sigma\in A} 2^{-|\sigma|} = \sum_{\sigma\in A} 2^{-f(\sigma)}.$$

This equation no longer holds if the set A is not required to be prefix-free. However if A is not prefix-free there is a prefix-free set $P \subset A$ such that $[P] = [A]$, and so $\mu([A]) = \mu([P])$. In fact, we cannot find a prefix-free subset $P \subset A$ such that $\mu([P]) > \mu([A])$, and so for any A we can write

$$\begin{aligned}\mu([A]) &= \sup\{\mu([P]) \mid P \subseteq A \text{ is prefix-free}\} \\ &= \sup\left\{\sum_{\sigma\in P} 2^{-f(\sigma)} \mid P \subseteq A \text{ is prefix-free}\right\}.\end{aligned}$$

Finally, any set B such that $[B] \supseteq [A]$ clearly has $\mu([B]) \geq \mu([A])$. Since we can also pass to a prefix-free subset of such a B we have

$$\mu([A]) = \inf\{\mu([B]) \mid [B] \supseteq [A]\} = \inf\left\{\sum_{\sigma\in B} 2^{-f(\sigma)} \mid [B] \supseteq [A]\right\}.$$

This gives us three different calculations using the function f which all yield the Lebesgue measure of a set. These three different calculations motivate the notions of direct, prefix, and vehement f-weight respectively.

Definition 3.1.8. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and $A \subseteq \{0,1\}^{<\mathbb{N}}$. ***The direct f-weight of A*** is the quantity

$$\mathrm{dwt}_f(A) = \sum_{\sigma \in A} 2^{-f(\sigma)}.$$

The prefix f-weight of A is the quantity

$$\mathrm{pwt}_f(A) = \sup\{\mathrm{dwt}_f(P) \mid P \subseteq A \text{ is prefix-free}\}.$$

The vehement f-weight of A is the quantity

$$\mathrm{vwt}_f(A) = \inf\{\mathrm{dwt}_f(B) \mid [B] \supseteq [A]\}.$$

These three quantities are always related in the following way.

Lemma 3.1.9. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and let $A \subseteq \{0,1\}^{<\mathbb{N}}$. Then $\mathrm{vwt}_f(A) \leq \mathrm{pwt}_f(A) \leq \mathrm{dwt}_f(A)$.

Proof. First, observe that $\widehat{A} \subseteq A$ is prefix-free and $[\widehat{A}] = [A]$. Then by definition of vehement f-weight and prefix f-weight

$$\mathrm{vwt}_f(A) \leq \mathrm{dwt}_f(\widehat{A}) \leq \mathrm{pwt}_f(A)$$

which establishes the first inequality. For any $P \subseteq A$, and therefore for prefix-free P in particular, we have $\mathrm{dwt}_f(P) \leq \mathrm{dwt}_f(A)$. This establishes the second inequality. □

3.2 Analogues of randomness which scale according to an arbitrary function

3.2.1 Analogues of Martin-Löf randomness

Recall that a Martin-Löf test is a uniformly effectively open sequence $\langle A_n \subseteq \{0,1\}^{\mathbb{N}} \mid n \in \mathbb{N}\rangle$ such that $\mu(A_n) \leq 2^{-n}$ for all n. If we let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be defined by $f(\sigma) = |\sigma|$ for all σ, we can think about Lebesgue measure in terms of f in three different ways. Then the following are equivalent for a uniformly effectively open sequence $\langle A_n \subseteq \{0,1\}^{\mathbb{N}}\rangle$:

- $\langle A_n\rangle$ is a Martin-Löf test,
- there is a uniformly r.e. sequence $\langle B_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ such that $A_n = [B_n]$ and $\mathrm{dwt}_f(A_n) \leq 2^{-n}$ for each n,

- there is a uniformly r.e. sequence $\langle B_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ such that $A_n = [B_n]$ and $\mathrm{pwt}_f(A_n) \leq 2^{-n}$ for each n, and

- there is a uniformly r.e. sequence $\langle B_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ such that $A_n = [B_n]$ and $\mathrm{vwt}_f(A_n) \leq 2^{-n}$ for each n.

In other words Martin-Löf tests can be characterized in terms of uniformly r.e. sequences $\langle B_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ satisfying some property about either direct, prefix, or vehement f-weight. This motivates the following definitions.

Definition 3.2.1. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and let $\langle A_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ be uniformly recursively enumerable. Call $\langle A_n\rangle$ a ***dwt-f-test*** if $\mathrm{dwt}_f(A_n) \leq 2^{-n}$ for each n. Similarly we call $\langle A_n\rangle$ a ***pwt-f-test*** if $\mathrm{pwt}_f(A_n) \leq 2^{-n}$ for each n, and a ***vwt-f-test*** if $\mathrm{vwt}_f(A_n) \leq 2^{-n}$ for each n.

Each type of test corresponds to a randomness notion as well.

Definition 3.2.2. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. We say that $X \in \{0,1\}^{\mathbb{N}}$ is ***dwt-f-random*** if there is no dwt-f-test $\langle A_n\rangle$ such that $X \in [A_n]$ for all n. Similarly X is called ***pwt-f-random*** if there is no pwt-f-test $\langle A_n\rangle$ such that $X \in [A_n]$ for all n, and ***vwt-f-random*** if there is no vwt-f-test $\langle A_n\rangle$ such that $X \in [A_n]$ for all n.

As promised, we will look at each of these three notions with respect to weakly equivalent functions (and therefore equivalent functions as well). But first, a key lemma we will use about weakly equivalent functions is the following.

Lemma 3.2.3. Let $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive such that $f \sim g$. Then there is a constant c with the following property. For any set A there is a set $A^* \subseteq \mathcal{I}(f)$ such that

1. $[A^*] \supseteq [A]$,

2. $\mathrm{dwt}_g(A^*) \leq 2^c \cdot \mathrm{dwt}_f(A)$,

3. if $\mathrm{dwt}_f(A) < \infty$ and $A \cap X^{<\mathbb{N}}$ is infinite then $A^* \cap X^{<\mathbb{N}}$ is infinite as well, for any X, and

4. if A is r.e., then so is A^* and a code for A^* can be found uniformly in a code for A.

Proof. Let $k \in \mathbb{N}$ be a constant such that $f(\sigma) \leq g(\sigma) + k$ for all $\sigma \in \mathcal{I}(f)$. Let A have a fixed enumeration, not necessarily recursive. For each $\sigma \in A$ pick a string σ^* with the following properties:

- $\sigma^* \subseteq \sigma$,
- $\sigma^* \in \mathcal{I}(f)$, and
- $f(\sigma^*) > f(\sigma) - 1$.

Since $\mathcal{I}(f)$ is a recursively enumerable set, we will eventually find out if some $\tau \subseteq \sigma$ belongs to $\mathcal{I}(f)$. If it does, checking if $f(\tau) > f(\sigma) - 1$ is also an r.e. process, in that we will find out eventually if that is the case. So for the first τ that satisfies these conditions, define $\sigma^* = \tau$. At the same time, we check if $f(\tau) < f(\sigma) - 1/2$ for all $\tau \subset \sigma$. This is also an r.e. condition to check, but if this holds then $\sigma \in \mathcal{I}(f)$. In this case define $\sigma^* = \sigma$. One of these two processes must halt eventually, so whichever halts first defines σ^*.

Now we simply define $A^* = \{\sigma^* \mid \sigma \in A\}$. It is clear that this process defines an enumeration of A^* uniformly from the enumeration of A; then if A is r.e. so is A^*, satisfying (4). Since $\sigma^* \subseteq \sigma$ for every σ, (1) holds. Noting that $A^* \subseteq \mathcal{I}(f)$ and that every $\sigma \in A^*$ is equal to τ^* for some $\tau \in A$

$$\mathrm{dwt}_g(A^*) = \sum_{\sigma \in A^*} 2^{-g(\sigma)} \leq \sum_{\sigma \in A^*} 2^{-f(\sigma)+k} < \sum_{\tau \in A} 2^{-f(\tau)+1+k} = 2^{k+1}\mathrm{dwt}_f(A).$$

This proves (2) with constant $c = k + 1$. Finally if $A \cap X^{<\mathbb{N}}$ is infinite for some X but $\mathrm{dwt}_f(A) < \infty$ then for any m there must be a $\sigma \in A$ such that $\sigma \subset X$ and $f(\sigma) > m$. Then however it is defined, $f(\sigma^*) > m - 1$. Since $f(\sigma^*)$ must still be finite, it is clear that there will then be infinitely many distinct σ^*, each of which is still a prefix of X. Thus $A^* \cap X^{<\mathbb{N}}$ is infinite and so (3) is satisfied. □

This implies something similar for prefix and vehement f-weight as well.

Corollary 3.2.4. Let $f, g : \{0,1\}^{<\mathbb{N}} \to [0, \infty)$ be recursive such that $f \sim g$. Then there is a constant c such that $\mathrm{pwt}_g(A^*) \leq 2^c \mathrm{pwt}_f(A)$ and $\mathrm{vwt}_g(A) \leq 2^c \mathrm{vwt}_f(A)$ for any $A \subseteq \{0,1\}^{<\mathbb{N}}$.

Proof. Let c be the constant from Lemma 3.2.3.

Note that it is possible that $\sigma^* = \tau^*$ for some $\sigma \neq \tau$, and so there could be some B which is not prefix-free such that B^* is. However, if B^* is prefix-free then there is a prefix-free $P \subseteq B$ such that $P^* = B^*$. If two elements $\sigma, \tau \in B$ have $\sigma^* = \tau^*$ then only one of them needs to belong to P. Moreover any set

$P \subseteq A^*$ has $P = B^*$ for some B. So for any set A we have

$$\begin{aligned}\mathrm{pwt}_g(A^*) &= \sup\{\mathrm{dwt}_g(P) \mid P \subseteq A^* \text{ is prefix-free}\}\\ &= \sup\{\mathrm{dwt}_g(P^*) \mid P^* \subseteq A^* \text{ is prefix-free}\}\\ &= \sup\{\mathrm{dwt}_g(P^*) \mid P \subseteq A \text{ is prefix-free}\}\\ &\leq \sup\{2^c \mathrm{dwt}_f(P) \mid P \subseteq A \text{ is prefix-free}\}\\ &= 2^c \mathrm{pwt}_f(A).\end{aligned}$$

Also since $[B^*] \supseteq [B]$ for any B, for any A we have

$$\begin{aligned}\mathrm{vwt}_g(A) &= \inf\{\mathrm{dwt}_g(B) \mid [B] \supseteq [A]\}\\ &\leq \inf\{\mathrm{dwt}_g(B^*) \mid [B^*] \supseteq [A]\}\\ &\leq \inf\{\mathrm{dwt}_g(B^*) \mid [B] \supseteq [A]\}\\ &\leq \inf\{2^c \mathrm{dwt}_f(B) \mid [B] \supseteq [A]\}\\ &= 2^c \mathrm{vwt}_f(A).\end{aligned}$$

This completes the proof. □

Now we return to verifying Claim 3.1.7.

Lemma 3.2.5. Let $f, g : \{0,1\}^{\mathbb{N}} \to [0,\infty)$ be recursive functions such that $f \sim g$. Then for any $X \in \{0,1\}^{\mathbb{N}}$

1. X is dwt-f-random if and only if it is dwt-g-random,
2. X is pwt-f-random if and only if it is pwt-g-random, and
3. X is vwt-f-random if and only if it is vwt-g-random.

Proof. For each claim it is sufficient to prove only one direction. The other direction follows by reversing the roles of f and g. Let $c \in \mathbb{N}$ and A^* for any A be as guaranteed by Lemma 3.2.3.

(1) Let X be dwt-g-random. For any dwt-f-test $\langle A_n \rangle$ we have

$$\mathrm{dwt}_g(A^*_{n+c}) \leq 2^c \mathrm{dwt}_f(A_{n+c}) \leq 2^{c-n-c} = 2^{-n}$$

for all n. Since codes for A^*_n can be uniformly found from the codes for the original sequence, $\langle A^*_{n+c} \rangle$ is uniformly r.e. and therefore a dwt-g-test. So $X \notin [A^*_{n+c}] \supseteq [A_{n+c}]$ for some n. This proves that X is dwt-f-random.

(2) Let X be pwt-g-random. Then for any pwt-f-test $\langle A_n \rangle$ we have

$$\mathrm{pwt}_g(A^*_{n+c}) \leq 2^c \mathrm{pwt}_f(A_{n+c}) \leq 2^{c-n-c} = 2^n$$

for all n. Similar to before, $\langle A^*_{n+c}\rangle$ is therefore a pwt-g-test and so $X \notin [A^*_{n+c}] \supset [A_{n+c}]$ for some n. Thus X is pwt-f-random as well.

(3) Let X be vwt-g-random. For any vwt-f-test $\langle A_n\rangle$ we have

$$\mathrm{vwt}_g(A_{n+c}) \leq 2^c \mathrm{vwt}_f(A_{n+c}) \leq 2^{c-n-c} = 2^{-n}$$

for all n. Then $\langle A_{n+c}\rangle$ is a vwt-g-test and $X \notin [A_{n+c}]$ for some n. Therefore X is vwt-f-random. $\square$

With each of the above notions of f-randomness, in some sense it is not A itself that matters but only $[A]$. As a consequence of this, we can easily trim A to get rid of extraneous strings. This will be useful later.

Lemma 3.2.6. Let $\langle A_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ be a uniformly r.e. sequence. There is a uniformly r.e. sequence $\langle B_n\rangle$ such that

1. $B_n \subseteq A_n$ for all n,
2. $[B_n] = [A_n]$ for all n, and
3. if $\sigma \in B^t_n$ and $\tau \in B^{t+1}_n \setminus B^t_n$ then $\tau \not\supset \sigma$.

It follows immediately that:

1. $\langle B_n\rangle$ is a dwt-f-test if $\langle A_n\rangle$ is,
2. $\langle B_n\rangle$ is a pwt-f-test if $\langle A_n\rangle$ is, and
3. $\langle B_n\rangle$ is a vwt-f-test if $\langle A_n\rangle$ is.

Proof. This is a straightforward modification to A_n. If $\sigma \in A^t_n$, $\tau \in A^{t+1}_n \setminus A^t_n$ and $\tau \supset \sigma$ then we simply omit τ from B_n. Otherwise, B_n is identical to A_n. Then $A_n \supseteq B_n \supseteq \widehat{A}_n$ at the very least and so the other claims all hold. $\square$

3.2.2 Analogues of Solovay randomness

In Chapter 2 we looked at the relationships among Martin-Löf randomness, Solovay randomness, and the first Borel-Cantelli lemma. Just as we did with Martin-Löf randomness, we would like to introduce variants of Solovay randomness that scale according to a function f and use one of direct, prefix, or vehement f-weight.

Definition 3.2.7. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and let $A \subseteq \{0,1\}^{<\mathbb{N}}$ be an r.e. set. We say that A is a ***Solovay dwt-f-test*** if $\mathrm{dwt}_f(A) < \infty$. Analogously we say that A is a ***Solovay pwt-f-test*** if $\mathrm{pwt}_f(A) < \infty$ and a ***Solovay vwt-f-test*** if $\mathrm{vwt}_f(A) < \infty$.

Each type of Solovay test defines a new randomness notion.

Definition 3.2.8. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. $X \in \{0,1\}^{\mathbb{N}}$ is ***Solovay dwt-f-random*** if $A \cap X^{<\mathbb{N}}$ is finite for any Solovay dwt-f-test A. Similarly, X is called ***Solovay pwt-f-random*** if $A \cap X^{<\mathbb{N}}$ is finite for any Solovay pwt-f-test A, and ***Solovay vwt-f-random*** if $A \cap X^{<\mathbb{N}}$ is finite for any Solovay vwt-f-test A.

In our definition X fails to be Solovay dwt-f-random if there is an r.e. set A such that $\mathrm{dwt}_f(A) < \infty$ and $A \cap X^{<\mathbb{N}}$ is infinite. It is clear then that removing finitely many elements of A would not affect whether or not this happens for a particular X. So we can delete enough elements from A that $\mathrm{dwt}_f(A) \leq 1$ without impacting the test. Note that the same argument does not work for Solovay pwt-f-randomness or Solovay vwt-f-randomness.

Generalizing from the notion of a Solovay test in a slightly different way, we get the following notions. We use the term Borel-Cantelli here because of the similarity to the first Borel-Cantelli lemma.

Definition 3.2.9. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and let $\langle A_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ be uniformly recursively enumerable. We call $\langle A_n\rangle$ a ***Borel-Cantelli (BC) dwt-f-test*** if $\sum_n \mathrm{dwt}_f(A_n) < \infty$. Similarly we call $\langle A_n\rangle$ a ***Borel-Cantelli (BC) pwt-f-test*** if $\sum_n \mathrm{pwt}_f(A_n) < \infty$, and a ***Borel-Cantelli (BC) vwt-f-test*** if $\sum_n \mathrm{vwt}_f(A_n) < \infty$.

Each type of test corresponds to a randomness notion.

Definition 3.2.10. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. We say that $X \in \{0,1\}^{\mathbb{N}}$ is ***Borel-Cantelli (BC) dwt-f-random*** if there is no BC dwt-f-test $\langle A_n\rangle$ such that $X \in [A_n]$ for infinitely many n. Similarly X is called ***Borel-Cantelli (BC) pwt-f-random*** if there is no BC pwt-f-test $\langle A_n\rangle$ such that $X \in [A_n]$ for infinitely many n, and ***Borel-Cantelli (BC) vwt-f-random*** if there is no BC vwt-f-test $\langle A_n\rangle$ such that $X \in [A_n]$ for infinitely many n.

If $\sum_n \mathrm{dwt}_f(A_n) < \infty$ then there is a k such that $\sum_n \mathrm{dwt}_f(A_{n+k}) \leq 1$; by defining $B_n = A_{n+k}$ for each n, it is clear that there exist infinitely many n such that $X \in [A_n]$ if and only if there exist infinitely many n such that $X \in [B_n]$, for any X. In other words we can assume without loss of generality that a BC dwt-f-test not only has a finite condition on the sum of the direct f-weights, but that the sum is bounded by 1. Similarly we can assume without loss of generality that $\sum_n \mathrm{pwt}_f(A_n) \leq 1$ for any BC pwt-f-test, and that $\sum_n \mathrm{vwt}_f(A_n) \leq 1$ for any BC vwt-f-test.

Now let us check that each new notion satisfies Claim 3.1.7.

Lemma 3.2.11. Let $f, g : \{0,1\}^{\mathbb{N}} \to [0,\infty)$ be recursive functions such that $f \sim g$. For any $X \in \{0,1\}^{\mathbb{N}}$:

1. X is Solovay dwt-f-random if and only if it is Solovay dwt-g-random,
2. X is Solovay pwt-f-random if and only if it is Solovay pwt-g-random,
3. X is Solovay vwt-f-random if and only if it is Solovay vwt-g-random,
4. X is BC dwt-f-random if and only if it is BC dwt-g-random,
5. X is BC pwt-f-random if and only if it is BC pwt-g-random, and
6. X is BC vwt-f-random if and only if it is BC vwt-g-random.

Proof. Again, it is sufficient to prove only one direction of each claim. Let $c \in \mathbb{N}$ and A^* be as in the proof of Lemma 3.2.3.

(1) Let X be Solovay dwt-g-random. For any Solovay dwt-f-test A

$$\mathrm{dwt}_g(A^*) \le 2^c \mathrm{dwt}_f(A) < \infty.$$

Therefore A^* is a Solovay dwt-g-test and so $A^* \cap X^{<\mathbb{N}}$ is finite. Then $A \cap X^{<\mathbb{N}}$ is finite as well, and X is Solovay dwt-f-random.

(2) Let X be Solovay pwt-g-random. Then for a Solovay pwt-f-test A

$$\mathrm{pwt}_g(A^*) \le 2^c \mathrm{pwt}_f(A) < \infty.$$

Then A^* is a pwt-g-test and $A^* \cap X^{<\mathbb{N}}$ must be finite. So $A \cap X^{<\mathbb{N}}$ is finite as well. Thus X is Solovay pwt-f-random.

(3) Let X be Solovay vwt-g-random. For any Solovay vwt-f-test A

$$\mathrm{vwt}_g(A) \le 2^c \mathrm{vwt}_f(A) < \infty.$$

Then A is a Solovay vwt-g-test and $A \cap X^{<\mathbb{N}}$ is finite. Therefore X is Solovay vwt-f-random.

(4) Let X be BC dwt-g-random. For any BC dwt-f-test $\langle A_n \rangle$ we have

$$\sum_n \mathrm{dwt}_g(A_n^*) \le \sum_n 2^c \mathrm{dwt}_f(A_n) = 2^c \sum_n \mathrm{dwt}_f(A_n) < \infty.$$

Therefore $\langle A_n^* \rangle$ is a BC dwt-g-test and so $X \in [A_n^*] \supseteq [A_n]$ for only finitely many n. This proves that X is BC dwt-f-random.

(5) Let X be BC pwt-g-random. For any BC pwt-f-test $\langle A_n \rangle$ we have

$$\sum_n \mathrm{pwt}_g(A_n^*) \le \sum_n 2^c \mathrm{pwt}_f(A_n) = 2^c \sum_n \mathrm{pwt}_f(A_n) < \infty.$$

So $\langle A_n^* \rangle$ is a pwt-g-test and so $X \in [A_n^*] \supseteq [A_n]$ for only finitely many n. Thus X is BC pwt-f-random as well.

(6) Let X be BC vwt-g-random. For any BC vwt-f-test $\langle A_n \rangle$ we have

$$\sum_n \mathrm{vwt}_g(A_n) \leq \sum_n 2^c \mathrm{vwt}_f(A_n) = 2^c \sum_n \mathrm{vwt}_f(A_n) < \infty.$$

Then $\langle A_n \rangle$ is a BC vwt-g-test and $X \in [A_n]$ for only finitely many n. Therefore X is BC vwt-f-random. □

We will spend the rest of this chapter discussing some of the connections and differences between the randomness notions introduced so far. In particular we are interested in any implications or non-implications that occur between two notions, since all of these notions stem from the definition of Martin-Löf randomness or properties characterizing Martin-Löf randomness.

3.3 Implications among partial randomness notions

Let us first look at the two types of randomness which come out of generalizations of Solovay randomness.

Lemma 3.3.1. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. No $X \in \{0,1\}^{\mathbb{N}}$ is Solovay vwt-f-random.

Proof. Define $A = \{0,1\}^{<\mathbb{N}}$ which is clearly an r.e. set. Since $[\lambda] = [A]$, we have $\mathrm{vwt}_f(A) \leq 2^{-f(\lambda)} < \infty$ and so A is a Solovay vwt-f-test. But $X \upharpoonright n \in A$ for every n, and so X is not Solovay vwt-f-random. □

Lemma 3.3.2. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. Then for any X:

1. if X is Solovay dwt-f-random it is also BC dwt-f-random;
2. if X is Solovay pwt-f-random it is also BC pwt-f-random; and
3. if X is Solovay vwt-f-random it is also BC vwt-f-random.

Proof. (1) Let X be Solovay dwt-f-random and let $\langle A_n \rangle$ be a BC dwt-f-test. Then defining $A = \bigcup_n A_n$, A is an r.e. set satisfying $\mathrm{dwt}_f(A) \leq \sum_n \mathrm{dwt}_f(A_n) < \infty$. So A is a Solovay dwt-f-test, and the set $A \cap X^{<\mathbb{N}}$ must be finite. Observe that $\lim_{k\to\infty} \sum_{n>k} \mathrm{dwt}_f(A_n) = 0$. Then since $\mathrm{dwt}_f(\sigma) > 0$ for any σ, we can always find a k_σ such that $\sigma \notin \bigcup_{n>k_\sigma} A_n$. As a corollary of this, there is a k large enough that $A \cap X^{<\mathbb{N}} \cap \bigcup_{n>k} A_n = \varnothing$. Then $X^{<\mathbb{N}} \cap \bigcup_{n>k} A_n = \varnothing$ as well; therefore $X \in [A_n]$ for only finitely many n and so X is BC dwt-f-random.

(2) Let X be Solovay pwt-f-random and let $\langle A_n \rangle$ be a BC pwt-f-test. Again defining $A = \bigcup_n A_n$, A is now an r.e. set satisfying

$$\begin{aligned}
\mathrm{pwt}_f(A) &= \sup\{\mathrm{dwt}_f(P) \mid P \subseteq A \text{ is prefix-free}\} \\
&\le \sup\left\{\sum_n \mathrm{dwt}_f(P \cap A_n) \mid P \subseteq A \text{ is prefix-free}\right\} \\
&\le \sum_n \sup\{\mathrm{dwt}_f(P \cap A_n) \mid P \subseteq A \text{ is prefix-free}\} \\
&= \sum_n \mathrm{pwt}_f(A_n) < \infty.
\end{aligned}$$

Then A is a Solovay pwt-f-test and so $A \cap X^{<\mathbb{N}}$ must be finite. We use a similar argument to that used in part (1). Note $\lim_{k\to\infty} \sum_{n>k} \mathrm{pwt}_f(A_n) = 0$ and $\mathrm{dwt}_f(\sigma) > 0$ for any σ. Since $\mathrm{pwt}_f(B) \ge \mathrm{dwt}_f(\sigma)$ if $\sigma \in B$, we can always find a k_σ such that $\sigma \notin \bigcup_{n>k_\sigma} A_n$. So there is a k large enough that $A \cap X^{<\mathbb{N}} \cap \bigcup_{n>k} A_n = \varnothing$. It follows that $X^{<\mathbb{N}} \cap \bigcup_{n>k} A_n = \varnothing$ as well and so $X \in [A_n]$ for only finitely many n. Then X is BC pwt-f-random as required.

(3) This is a trivial corollary of Lemma 3.3.1. □

Next we look at the relationships between BC randomness and notions which arise from generalizations of the definition of Martin-Löf random.

Lemma 3.3.3. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and let $X \in \{0,1\}^{<\mathbb{N}}$. Then

1. if X is BC dwt-f-random, it is also dwt-f-random,
2. if X is BC pwt-f-random, it is also pwt-f-random, and
3. if X is BC vwt-f-random, it is also vwt-f-random.

Proof. (1) Let X be BC dwt-f-random and let $\langle A_n \rangle$ be a dwt-f-test; then it is also a BC dwt-f-test. Then there are finitely many n such that $X \in [A_n]$. In particular, there exists at least one n such that $X \notin [A_n]$. Therefore X is dwt-f-random.

(2) Let X be BC pwt-f-random. If $\langle A_n \rangle$ is a pwt-f-test, then it is also a BC pwt-f-test. So there are only finitely many n such that $X \in [A_n]$. It follows that $X \notin [A_n]$ for some n and so X is pwt-f-random as well.

(3) Let X be BC vwt-f-random and let $\langle A_n \rangle$ be a vwt-f-test. Then it is also a BC vwt-f-test, and so there are at most finitely many n such that $X \in [A_n]$; $X \notin [A_n]$ for some n as a result. So X is vwt-f-random. □

Next, we will look at some implications among the types of weights. The first two are straightforward.

Lemma 3.3.4. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If $X \in \{0,1\}^{\mathbb{N}}$ is pwt-f-random then it is also dwt-f-random.

Proof. Let $X \in \{0,1\}^{\mathbb{N}}$ be pwt-f-random, and let $\langle A_n \rangle$ be a dwt-f-test. Then by Lemma 3.1.9, $\mathrm{pwt}_f(A_n) \le \mathrm{dwt}_f(A_n) \le 2^{-n}$ for each n and so $\langle A_n \rangle$ is also a pwt-f-test. Therefore $X \notin [A_n]$ for some n. Since $\langle A_n \rangle$ is arbitrary X is dwt-f-random. □

Lemma 3.3.5. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If $X \in \{0,1\}^{\mathbb{N}}$ is vwt-f-random then it is also pwt-f-random.

Proof. Let $X \in \{0,1\}^{\mathbb{N}}$ be vwt-f-random, and let $\langle A_n \rangle$ be a pwt-f-test. Then $\langle A_n \rangle$ is also a vwt-f-test, by Lemma 3.1.9; therefore $X \notin [A_n]$ for some n. Since $\langle A_n \rangle$ is arbitrary X is pwt-f-random. □

It follows that if X is vwt-f-random it is also dwt-f-random, but in fact something stronger holds as well.

Lemma 3.3.6. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If $X \in \{0,1\}^{\mathbb{N}}$ is vwt-f-random then it is Solovay dwt-f-random as well.

Proof. Let X be vwt-f-random and let A be a Solovay dwt-f-test; we may assume without loss of generality that $\mathrm{dwt}_f(A) \le 1$. For each $n \in \mathbb{N}$ define $B_n = \{\sigma \in A \mid \exists \tau_0, \dots, \tau_n \in A \text{ s.t. } \tau_0 \subset \cdots \subset \tau_n = \sigma\}$, the set of elements of A which have at least n proper prefixes in A. Since A is an r.e. set, the B_n are uniformly r.e. in n. Define $C_0 = \widehat{A}$ and for each $n > 0$ define $D_n = A \setminus \bigcup_{k<n} C_k$ and $C_n = \widehat{D_n} = \{\sigma \in A \mid \text{ exactly } n \text{ proper prefixes of } \sigma \text{ are in } A\}$. For any $k < n$ we have $[C_k] \supseteq [B_n]$; so $\mathrm{vwt}_f(B_n) \le \mathrm{dwt}_f(C_k)$. Also the C_k are all disjoint subsets of A and so

$$n \cdot \mathrm{vwt}_f(B_n) \le n \cdot \inf\{\mathrm{dwt}_f(C_k) \mid k < n\} \le \sum_{k<n} \mathrm{dwt}_f(C_k) \le \mathrm{dwt}_f(A) \le 1.$$

Therefore $\mathrm{vwt}_f(B_n) \le 1/n$ for each n.

So we define a new sequence by $A_n = B_{2^n}$ for each n. $\langle A_n \rangle$ is a vwt-f-test and so there must be some n such that $X \notin [A_n] = [B_{2^n}]$. Then X has fewer than $2^n + 1$ initial segments occurring in A. This proves that X is Solovay dwt-f-random. □

Next we have the first equivalence of randomness notions.

Theorem 3.3.7. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. $X \in \{0,1\}^{\mathbb{N}}$ is BC dwt-f-random if and only if it is Solovay dwt-f-random.

Proof. If X is Solovay dwt-f-random, then it is BC dwt-f-random by Lemma 3.3.2. So suppose that X is BC dwt-f-random and let A be a Solovay dwt-f-test. If A is finite then $A \cap X^{<\mathbb{N}}$ is trivially finite. If A is infinite then let $\langle \sigma_n \rangle$ be a recursive enumeration of the elements of A without repetition. Defining $A_n = \{\sigma_n\}$ for each n, $\langle A_n \rangle$ is a uniformly r.e. sequence satisfying $\sum_n \mathrm{dwt}_f(A_n) = \mathrm{dwt}_f(A) < \infty$. Then $\langle A_n \rangle$ is a BC dwt-f-test, and therefore $X \in [A_n]$ for only finitely many n. This is equivalent to saying $\sigma_n \subset X$ for only finitely many n. In particular there are only finitely many $\sigma \in A$ such that $\sigma \subset X$. We conclude that X is Solovay dwt-f-random. $\square$

Lemma 3.3.8. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If X is BC vwt-f-random, then X is BC pwt-f-random as well.

Proof. Recall by Lemma 3.1.9 that $\mathrm{vwt}_f(A) \leq \mathrm{pwt}_f(A)$ for any set A. So let X be BC vwt-f-random and $\langle A_n \rangle$ be a BC pwt-f-test. Then

$$\sum_n \mathrm{vwt}_f(A_n) \leq \sum_n \mathrm{pwt}_f(A_n) < \infty$$

and so $\langle A_n \rangle$ is a BC vwt-f-test as well. Then there are finitely many n such that $X \in [A_n]$; therefore X is BC pwt-f-random as well. $\square$

Lemma 3.3.9. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If X is BC pwt-f-random, then it is BC dwt-f-random as well.

Proof. By Lemma 3.1.9 $\mathrm{pwt}_f(A) \leq \mathrm{dwt}_f(A)$ for any A. So let X be BC pwt-f-random and $\langle A_n \rangle$ be a BC dwt-f-test. Then

$$\sum_n \mathrm{pwt}_f(A_n) \leq \sum_n \mathrm{dwt}_f(A_n) < \infty$$

and so $\langle A_n \rangle$ is a BC pwt-f-test as well. Then there are finitely many n such that $X \in [A_n]$; therefore X is BC dwt-f-random as well. $\square$

Next we have another equivalence.

Theorem 3.3.10. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. Then X is BC vwt-f-random if and only if X is vwt-f-random.

Proof. If X is BC vwt-f-random then it is vwt-f-random by Lemma 3.3.3. So suppose that X is vwt-f-random, and let $\langle A_n \rangle$ be a BC vwt-f-test. Without loss of generality assume $\sum_n \mathrm{vwt}_f(A_n) \leq 1/2$ (for any k the sequence $\langle A_{n+k} \rangle$ is also a BC vwt-f-test, and $\{n \mid X \in [A_{n+k}]\}$ is finite if and only if $\{n \mid X \in [A_n]\}$ is finite).

Define a function $s : \{0,1\}^{<\mathbb{N}} \to \mathbb{N} \cup \infty$ by $s(\sigma) = \#\{n \mid [\sigma] \subseteq [A_n]\}$ for each σ. Since each $[A_n]$ is an r.e. subset of $\{0,1\}^{\mathbb{N}}$ this function is r.e. Then for each n define $B_n = \{\sigma \mid s(\sigma) \geq 2^n\}$. This defines $\langle B_n \rangle$ in a uniformly r.e. way as well. Note that $X \in [B_n]$ if and only if there are at least 2^n choices of k for which $X \in [A_k]$.

We would ultimately like to show that $\langle B_n \rangle$ is a vwt-f-test. To do this we need to construct a cover of each B_n with sufficiently small direct f-weight. For each i choose C_i such that $[C_i] \supseteq [A_i]$ and $\mathrm{dwt}_f(C_i) \leq \mathrm{vwt}_f(A_i) + 2^{-i-2}$ (which is possible since the vehement f-weight of A_i is the infimum of the direct f-weights of its covers). While there may not be a recursive or uniform way to find such C_i, such sets must exist. Then

$$\sum_i \mathrm{dwt}_f(C_i) \leq \sum_i (\mathrm{vwt}_f(A_i) + 2^{-i-2}) = \sum_i \mathrm{vwt}_f(A_i) + 1/2 \leq 1.$$

The intuition of what comes next is as follows. If we shuffle the elements of the C_i in some way, the sums of the direct f-weights will not get any larger. So we will shuffle them in such a way that we end up with several sets that completely cover a given B_n. If we end up with many sets covering B_n then the direct f-weight of at least one of them must be very small. We formalize this idea now.

Define $S_i, T_i \subseteq \{0,1\}^{<\mathbb{N}} \times \mathbb{N}$ for each i as follows. Let $S_0 = \bigcup_{j \in \mathbb{N}} C_j \times \{j\}$. For each $i \in \mathbb{N}$ let

$$T_i = \{(\sigma, j) \in S_i \mid (\forall (\tau, k) \in S_i)[(\tau \subseteq \sigma \wedge k \leq j) \to (\tau, k) = (\sigma, j)]\}$$

and let $S_{i+1} = S_i \setminus T_i$. Essentially we are finding and collecting all of the shortest strings present in $\bigcup C_i$, then setting them aside; we then continue by finding the shortest remaining strings, over and over.

For any i the set $\pi(T_i) = \{\sigma \mid (\exists n)((\sigma, n) \in T_i)\}$ is prefix-free. Also, whenever $k < 2^n$ we have $[\pi(T_k)] \supseteq [B_n]$. To see this fix $Y \in [B_n]$. There exist indices $i_1, \ldots, i_{2^n}$ and strings $\sigma_j \in C_{i_j}$ such that $\sigma_j \subset Y$ for all j; since all are prefixes of Y, any two σ_j are compatible. For each $1 \leq j \leq 2^n$ the pair $\langle \sigma_j, i_j \rangle$ is in C, but at most k of these pairs can be in the set $T_1 \cup T_2 \cup \cdots \cup T_{k-1}$. Therefore $Y \in [\pi(S_k)]$ and so $Y \in [\pi(T_k)]$ as well.

Then since $[\pi(T_j)] \supseteq [B_n]$ for each $j < 2^n$ we have

$$1 \geq \sum_i \mathrm{dwt}_f(C_i) \geq \sum_{j < 2^n} \mathrm{dwt}_f(\pi(T_j)) \geq 2^n \cdot \mathrm{vwt}_f(B_n).$$

Therefore $\mathrm{vwt}_f(B_n) \leq 2^{-n}$ and $\langle B_n \rangle$ is a vwt-f-test. Thus $X \notin [B_n]$ for some n and so there are fewer than 2^n values of i for which $X \in [A_i]$. So X is BC vwt-f-random, as required. $\square$

3.3.1 Weak separation of pwt-f-random and vwt-f-random

Later we will see that pwt-f-randomness and vwt-f-randomness are equivalent for convex f; it is natural to ask if this result extends to all f. The next example will illustrate that this equivalence does not hold in general, and that for at least one f we can weakly separate the notions.

First we need the following standard result.

Lemma 3.3.11. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. Then there is universal pwt-f-test, that is a pwt-f-test $\langle A_n \rangle$ such that for any other pwt-f-test $\langle B_n \rangle$ we have $\bigcap_n [B_n] \subseteq \bigcap_n [A_n]$.

Proof. Assume by Lemma 3.1.3 and Claim 3.1.2 that f is integer-valued. Let $\langle B_{(s,t)} \rangle$ be a uniformly r.e. sequence indexed by both $s, t \in \mathbb{N}$ such that if $\langle B_n \rangle$ is a uniformly r.e. sequence then $B_n = B_{(s,n)}$ for some s and all n. Then define a uniformly r.e. sequence $\langle A_{(s,t)} \rangle$ as follows. $A_{(s,t)}$ enumerates the same set as $B_{(s,t)}$ unless $\mathrm{pwt}_f(B_{(s,t)}) > 2^{-t}$. In this case, the enumeration of $A_{(s,t)}$ stops immediately before this happens; this ensures that $\mathrm{pwt}_f(A_{(s,t)}) \leq 2^{-t}$ for all s and t. And of course if $\mathrm{pwt}_f(B_{(s,t)}) \leq 2^{-t}$ then $A_{(s,t)} = B_{(s,t)}$. In particular if $\langle B_n \rangle$ is a pwt-f-test then $B_n = B_{(s,n)} = A_{(s,n)}$ for fixed s and all n.

Now define for each n the set $A_n = \bigcup_k A_{(k,n+k+1)}$. This defines $\langle A_n \rangle$ as a uniformly r.e. sequence such that

$$\mathrm{pwt}_f(A_n) \leq \sum_k \mathrm{pwt}_f(A_{(k,n+k+1)}) \leq \sum_k 2^{-n-k-1} = 2^{-n}$$

for all n. Then $\langle A_n \rangle$ is a pwt-f-test. So let $\langle B_n \rangle$ be an arbitrary pwt-f-test. Since $B_n = A_{(s,n)}$ for all n and some fixed s, $A_n \supseteq A_{(s,n+s+1)} = B_{n+s+1}$ for all n and so $\bigcap_n [B_n] \subseteq \bigcap_n [A_n]$. This confirms that $\langle A_n \rangle$ is a universal pwt-f-test. $\square$

Now we are ready to construct an f for which vwt-f-randomness and pwt-f-randomness can be separated.

Example 3.3.12. There is a recursive $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ and an $X \in \{0,1\}^{\mathbb{N}}$ such that X is pwt-f-random but not vwt-f-random.

Proof. The idea of the proof is as follows. If we know that $\langle A_n \rangle$ is a universal pwt-f-test, it is sufficient to show that there is a vwt-f-test $\langle B_n \rangle$ and some $X \in \bigcap_n [B_n] \setminus \bigcap_n [A_n]$. In our proof we will need to alter f based on what is enumerated into each A_n and so it is not feasible to get an index for a universal pwt-f-test beforehand. So instead we will construct for any uniformly r.e. sequence $\langle A_n \rangle$ a uniformly r.e. sequence $\langle B_n \rangle$ with the following properties:

1. $\langle B_n \rangle$ is a vwt-f-test and
2. if $\langle A_n \rangle$ is a pwt-f-test then there is an $X \in \bigcap_n [B_n] \setminus \bigcap_n [A_n]$.

Then no matter the index of the universal pwt-f-test, it cannot cover all the same points as some vwt-f-test.

Setting up the construction. For each k define the set

$$T_k = \left\{ 0^k{}^\frown 1^\frown \sigma \mid \sigma \in \{0,1\}^{<\mathbb{N}} \right\}$$

and let $\langle A_{(k,n)} \mid n \in \mathbb{N} \rangle$ be the kth uniformly r.e. sequence in some fixed enumeration of all such sequences.

To define f we will define $f|_{T_k}$ for each k. If σ does not belong to T_k for any k then define $f(\sigma) = |\sigma|$. Simultaneously to defining $f|_{T_k}$, we build for each k uniformly r.e. sets $B_{(k,n)}$ as the union of recursive sets $B^s_{(k,n)}$.

Defining f on T_k. For convenience write $A = A_{(k,k+2)} \cap T_k$ and recall that A^s is the set of elements of A enumerated within s steps. Let $\sigma \in T_k$ be of length $|\sigma| = n$. If no proper prefix $\tau \subset \sigma$ belongs to A^n then define $f(\sigma) = |\sigma|$. On the other hand if some $\tau \subset \sigma$ is in A^n then define $f(\sigma) = 2|\sigma|$. This defines f recursively on T_k.

Let us make some important observations based on the definition f:

1. For any $\sigma \subseteq 0^k{}^\frown 1$ we have $f(\sigma) = |\sigma| < k+2$. So if $\langle A_{(k,n)} \rangle$ is a pwt-f-test then no $\sigma \subseteq 0^k{}^\frown 1$ can belong to $A = A_{(k,k+2)}$. So let us assume that this is the case, since we ultimately only care when it is a pwt-f-test.

2. If $\sigma \in T_k$ then $f(\sigma) \geq |\sigma|$ and so $\mathrm{dwt}_f(\sigma) \leq \mu([\sigma])$. Then for prefix-free $P \subseteq A$ we have $\mathrm{dwt}_f(P) \leq \mu([P]) \leq \mu([A])$ and so $\mathrm{pwt}_f(A) \leq \mu([A])$. On the other hand, if $\sigma \in \widehat{A}$ then $f(\sigma) = |\sigma|$ and so $\mathrm{dwt}_f(\sigma) = \mu([\sigma])$. Therefore $\mathrm{pwt}_f(A) \geq \mathrm{dwt}_f(\widehat{A}) = \mu([\widehat{A}]) = \mu([A])$ and so $\mathrm{pwt}_f(A) = \mu([A])$. Then by our most assumption in (1),
$$\mu([A]) \leq \mathrm{pwt}_f(A_{(k,k+2)}) \leq 2^{-k-2}.$$

3. If $\sigma \in A$ then $\sigma \in A^n$ for some $n > |\sigma|$. Then any $\rho \supset \sigma$ such that $|\rho| = n$ will satisfy $f(\rho) = 2|\rho| = 2n$. In particular $C = \{0,1\}^n \cap [\sigma]^{<\mathbb{N}}$ covers $[\sigma]$ and has $\mathrm{dwt}_f(C) = 2^{n-|\sigma|}2^{-2n} \leq 2^{-n}$. Since n is arbitrary $\mathrm{vwt}_f(\sigma) = 0$. Therefore $\mathrm{vwt}_f(A) = 0$.

Defining $B^s_{(k,n)}$. Fix n and s. Still operating under the assumption that $\mu([A]) \le 2^{-k-2}$, there is a unique set $C^{(k,s)}_n \subseteq [T_k]$ with the following properties:

1. $X \in C^{(k,s)}_n$ if and only if $X <_{\text{lex}} \sigma_{(n,k,s)}$ for some fixed $\sigma_{(n,k,s)} \in T_k$, and
2. $\mu(C^{(k,s)}_n \setminus [A^s]) = 2^{-n-k-3}$.

To see that this holds, note that $[A^s] \subset [T_k]$ is a clopen set of measure no greater than 2^{-k-2}; therefore $[T_k]$ has measure at least $2^{-n-k-3} < 2^{-k-2}$ left for $C^{(k,s)}_n$ to occupy. The choice of $\sigma_{(n,k,s)}$ is not unique, so we just pick some suitable one with the property that $|\sigma_{(n,k,s)}| > s$. Then define $B^s_{(k,n)}$ so that $[B^s_{(k,n)}] = C^{(k,s)}_n$. Since A^s is finite, for convenience we pick $B^s_{(k,n)}$ such that $|\sigma| > |\tau|$ for every $\sigma \in B^s_{(k,n)}$ and $\tau \in A^s$.

Now we make several observations about each $B^s_{(k,n)}$ and $B_{(k,n)} = \bigcup_s B^s_{(k,n)}$:

1. As s grows, so does $C^{(k,s)}_n$. That is, for any n and s we have $C^{(k,s)}_n \subseteq C^{(k,s+1)}_n$. So $[B_{(k,n)}] = \bigcup_s C^{(k,s)}_n$ has the form $X \in [B_n]$ if and only if $X <_{\text{lex}} Y_{(n,k)}$ for some fixed $Y_{(n,k)} \in [T_k]$. Specifically $Y_{(n,k)}$ is the lexicographically least Y such that $\sigma_{(n,k,s)} <_{\text{lex}} Y$ for each s.
2. There must be infinitely many s such that $\sigma_{(n,k,s)} \subset Y_{(n,k)}$; otherwise $Y_{(n,k)}$ would not be the lexicographically least such point in $[T_k]$.
3. Since $f(\sigma) \ge |\sigma|$ for all σ it holds that $\text{vwt}_f(P) \le \mu([P])$ for any $P \subseteq [T_k]$. So for each n and s we have
$$\begin{aligned}\text{vwt}_f(B^s_{(k,n)}) &\le \text{vwt}_f(A^s) + \text{vwt}_f(B^s_{(k,n)} \setminus [A^s]^{<\mathbb{N}})\\ &\le 0 + \mu(C^{(k,s)}_n \setminus [A^s]) = 2^{-n-k-3}.\end{aligned}$$
For any s there is a $t > s$ such that $\sigma_{(n,k,t)} \subset Y_{(n,k)}$ and therefore $[B_{(k,n)}] \subseteq [B^t_{(k,n)} \cup \{\sigma_{(n,k,t)}\}]$. So
$$\text{vwt}_f(B_{(k,n)}) \le \text{vwt}_f(B^t_{(k,n)}) + \text{vwt}_f(\sigma_{(n,k,t)}) < 2^{-n-k-3} + 2^{-t}$$
from which it follows that $\text{vwt}_f(B_{(k,n)}) \le 2^{-n-k-3}$ for any n. So in particular $\langle B_{(k,n)} \rangle$ is a vwt-f-test. This will also hold if $\langle A_{(k,n)} \mid n \in \mathbb{N} \rangle$ fails to be a pwt-f-test, which is important.

Verifying the construction for fixed k. Assuming $\langle A_{(k,n)} \mid n \in \mathbb{N}\rangle$ is a pwt-f-test, then let X be the lexicographically least element of $[T_k]$ not belonging to $[A]$; such an X must exist since $\mu([A]) < \mu([T_k])$. Then of course $\{Y \mid Y <_{\mathrm{lex}} X\} \subset [A]$, and so for any n there is an s such that

$$\mu(\{Y \mid Y <_{\mathrm{lex}} X\} \smallsetminus [A^s]) < 2^{-n-k-3}.$$

Then $X <_{\mathrm{lex}} \sigma_{(n,k,s)}$ and so $X \in [B^s_{(k,n)}] \subseteq [B_{(k,n)}]$. Therefore $X \in \bigcap_n [B_{(k,n)}]$ while $X \notin [A_{(k,k+2)}] \subseteq \bigcap_n [A_{(k,n)}]$ by assumption.

Putting everything together. By Lemma 3.3.11, the constructed f allows for a universal pwt-f-test. Then some $\langle A_{(k,n)} \mid n \in \mathbb{N}\rangle$ will be a universal pwt-f-test. Then the constructed $\langle B_{(k,n)} \mid n \in \mathbb{N}\rangle$ shows that there is some $X \in \bigcap_n [B_{(k,n)}] \smallsetminus \bigcap_n [A_{(k,n)}]$ and so this X is pwt-f-random but not vwt-f-random. □

3.3.2 "Upstream" implications

Later we will use Theorem 4.3.2 to show that dwt-f-random does not imply Solovay dwt-f-random and Theorem 4.3.4 to show that pwt-f-random does not imply Solovay pwt-f-random. Additionally, Reimann and Stephan [36] have shown that Solovay dwt-f-random does not imply pwt-f-random. In other words, unless otherwise indicated implications do not go upwards in the diagram of Figures 1.2 and 1.3. However, a similar upstream implication holds by weakening the function. Reimann and Stephan [36] first proved a version of part (1) of the following, for geometric premeasures.

Lemma 3.3.13. Let $X \in \{0,1\}^{\mathbb{N}}$ and let $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive functions such that $\lim_{n\to\infty} f(X \upharpoonright n) - g(X \upharpoonright n) = \infty$.

1. If X is dwt-f-random, it is also Solovay dwt-g-random.

2. If X is pwt-f-random, it is also Solovay pwt-g-random.

Proof. (1). Let X be dwt-f-random and let A be a Solovay dwt-g-test; we may assume without loss of generality that $\mathrm{dwt}_g(A) \le 1$. Define for each n the set $A_n = \{\sigma \in A \mid f(\sigma) > g(\sigma) + n\}$. Then $\langle A_n\rangle$ is a uniformly r.e. sequence and

$$\mathrm{dwt}_f(A_n) = \sum_{\sigma\in A_n} 2^{-f(\sigma)} < \sum_{\sigma\in A_n} 2^{-g(\sigma)-n} \le 2^{-n}\mathrm{dwt}_g(A) \le 2^{-n}$$

for each n. Therefore $\langle A_n\rangle$ is a dwt-f-test and so $X \notin [A_n]$ for some n. Then since $\lim_{k\to\infty}(f(X \upharpoonright k) - g(X \upharpoonright k)) > n$ there must be some N such that $X \upharpoonright k \notin A$ for any $k > N$. Therefore X is Solovay dwt-g-random.

(2). Now assume that X is pwt-f-random and let A be a Solovay pwt-g-test. Define for each n the set $A_n = \{\sigma \in A \mid f(\sigma) > g(\sigma) + n\}$. $\langle A_n \rangle$ is uniformly r.e. and for any prefix-free $P \subseteq A_n$

$$\mathrm{dwt}_f(P) = \sum_{\sigma \in P} 2^{-f(\sigma)} < \sum_{\sigma \in P} 2^{-g(\sigma)-n} \leq 2^{-n}\mathrm{dwt}_g(P) \leq 2^{-n}\mathrm{pwt}_g(A).$$

Therefore $\mathrm{pwt}_f(A_n) \leq 2^{-n}\mathrm{pwt}_g(A)$ for every n. If we fix k large enough that $\mathrm{pwt}_f(A) < 2^k$ then $\langle A_{n+k} \mid n \in \mathbb{N} \rangle$ is a pwt-f-test. Then $X \notin [A_n]$ for some n, and therefore $X \upharpoonright m \notin A$ for fixed N and all $m > N$. Therefore X is Solovay pwt-g-random. $\square$

We could ask if Solovay dwt-f-random implies pwt-g-random for any X such that $\lim_{n\to\infty} f(X \upharpoonright n) - g(X \upharpoonright n) = \infty$; it seems unlikely that this would hold, due to known relationships between KP and KA and the relationships that will be established in Chapter 4. To get an upstream implication between these two notions, we first introduce a stronger type of divergence between two functions.

Definition 3.3.14. Let $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ and $g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. We say that ***f* is strongly dominated by *g***, denoted $f \ll g$, if there is a function $h : \mathbb{N} \to \mathbb{Z}$ such that

1. $f(\sigma) + h(f(\sigma)) \leq g(\sigma)$ for all $\sigma \in \mathcal{I}(f)$, and
2. $\sum_{n\in\mathbb{N}} 2^{-h(n)} < \infty$.

More generally, if $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ are recursive, we write $f \ll g$ if $f' \ll g$, where $f' : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ is the function constructed to satisfy $f' \approx f$ in Lemma 3.1.3.

Remark 3.3.15. As it is defined, strong domination is not transitive. That is, we can easily construct recursive functions $f, g, h : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ such that $f \ll g$ and $g \ll h$ but $f \not\ll h$. However, this is not possible if we restrict to non-decreasing functions: a function f is ***non-decreasing*** if $f(\sigma) \leq f(\tau)$ for all $\sigma \subset \tau$. Then if f, g, h are non-decreasing functions such that $f \ll g$ and $g \ll h$ it follows that $f \ll h$. To see this, note that for non-decreasing f and g we can replace part (1) in Definition 3.3.14 with

1. $f(\sigma) + h(f(\sigma)) \leq g(\sigma)$ for all σ

without changing the content of the definition. Of course, every recursive $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ is also equivalent to a non-decreasing one.

Strong domination gives us the power to convert known bounds on prefix f-weight into bounds on direct g-weight.

Lemma 3.3.16. Let $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ and $g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and such that $f \ll g$. There is a constant c such that if $A \subseteq \mathcal{I}(f)$ then $\mathrm{dwt}_g(A) \leq c \cdot \mathrm{pwt}_f(A)$.

Proof. Let $h : \mathbb{N} \to \mathbb{N}$ be a function witnessing the fact that $f \ll g$ and define $c = \sum_n 2^{-h(n)} < \infty$. Note that every set of the form $A_n = \{\sigma \in A \mid f(\sigma) = n\}$ is prefix-free since $A \subseteq \mathcal{I}(f)$. In addition, $A = \bigcup_n A_n$. Then

$$\begin{aligned}
\mathrm{dwt}_g(A) &= \sum_n \mathrm{dwt}_g(A_n) \\
&= \sum_n \sum_{\sigma \in A_n} 2^{-g(\sigma)} \\
&\leq \sum_n \sum_{\sigma \in A_n} 2^{-f(\sigma) - h(f(\sigma))} \\
&= \sum_n \sum_{\sigma \in A_n} 2^{-f(\sigma) - h(n)} \\
&= \sum_n 2^{-h(n)} \mathrm{dwt}_f(A_n) \leq \sum_n 2^{-h(n)} \mathrm{pwt}_f(A) = c \cdot \mathrm{pwt}_f(A).
\end{aligned}$$

This completes the proof. □

This implies two new upstream implications. In fact, both of these imply the previously hinted at upstream implication from dwt-g-random to pwt-f-random. Part (1) of this result is also given by Higuchi, Hudelson, Simpson, and Yokoyama [18].

Corollary 3.3.17. Let $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and such that $f \ll g$. For any $X \in \{0,1\}^{\mathbb{N}}$:

1. if X is dwt-g-random then it is pwt-f-random, and
2. if X is Solovay dwt-g-random then it is Solovay pwt-f-random.

Proof. It is sufficient to prove this for integer-valued f. Let c be the constant from Lemma 3.3.16. Let m be such that $c < 2^m$.

(1): Let X be dwt-g-random and let $\langle A_n \rangle$ be a pwt-f-test. One implication of Lemma 3.2.3 is that we may assume without loss of generality that $A_n \subseteq \mathcal{I}(f)$ for all n. Then $\mathrm{dwt}_g(A_n) \leq 2^m \cdot \mathrm{pwt}_f(A_n) \leq 2^{m-n}$ for all n. Therefore $\langle A_{m+n} \mid n \in \mathbb{N} \rangle$ forms a dwt-g-test, and so $X \notin [A_{m+n}]$ for some n; since $\langle A_n \rangle$ was an arbitrary pwt-f-test, X is pwt-f-random.

(2): Now assume that X is Solovay dwt-g-random ant let A be a Solovay pwt-f-test. Again we may assume without loss generality that $A \subseteq \mathcal{I}(f)$, thanks to Lemma 3.2.3. Then $\mathrm{dwt}_g(A) \leq c \cdot \mathrm{pwt}_f(A) < \infty$ and so A is a Solovay dwt-g-test. Therefore $A \cap X^{<\mathbb{N}}$ is finite, and X is Solovay pwt-f-random as well. □

In Chapter 5 we will also look at a notion related to strong domination for length-invariant functions. Note the following.

Lemma 3.3.18. If $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ and $g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ are recursive and length-invariant, then the following are equivalent:

1. $f \ll g$, and
2. $\sum_{0^n \in \mathcal{I}(f)} 2^{f(0^n)-g(0^n)} < \infty$.

Proof. If $f \ll g$ pick h such that $f(\sigma) + h(f(\sigma)) \leq g(\sigma)$ for all $\sigma \in \mathcal{I}(f)$ and $\sum_n 2^{-h(n)} < \infty$. The restriction of f to the set $\mathcal{I}(f) \cap \{0^n \mid n \in \mathbb{N}\}$ is one-to-one, and so

$$\sum_{0^n \in \mathcal{I}(f)} 2^{f(0^n)-g(0^n)} \leq \sum_{0^n \in \mathcal{I}(f)} 2^{-h(f(0^n))} \leq \sum_{m \in \mathbb{N}} 2^{-h(m)} < \infty.$$

On the other hand, if $\sum_{n \in \mathcal{I}(f)} 2^{f(0^n)-g(0^n)} < \infty$, define a function h by $h(k) = \lfloor g(0^n) - f(0^n) \rfloor$ if $f(0^n) = k$; if there is no n such that $f(0^n) = k$, then define $h(k) = k$. Then it follows that

$$\begin{aligned} g(\sigma) &= f(\sigma) + (g(\sigma) - f(\sigma)) \\ &= f(\sigma) + (g(0^{|\sigma|}) - f(0^{|\sigma|})) \geq f(\sigma) + h(f(0^{|\sigma|})) = f(\sigma) + h(f(\sigma)) \end{aligned}$$

for all $\sigma \in \mathcal{I}(f)$ and

$$\sum_{n \in \mathbb{N}} 2^{-h(n)} \leq \sum_{0^m \in \mathcal{I}(f)} 2^{f(0^m)-g(0^m)+1} + \sum_{n \in \mathbb{N}} 2^{-n} < \infty.$$

This proves that $f \ll g$. □

3.4 Optimal covers

The notion of an optimal cover was first developed by Miller [31] in a very specific setting. Here we develop our own notion of optimal cover, which can be thought of as a generalization or extension of the original one. In particular, the optimal covers we develop have many desirable properties.

The original idea of optimal covers was to pick a canonical set that is best — in a precise sense — at covering a given set. Actually though, there could be multiple sets tied for best; so we broaden the concept to allow for many optimal covers of a given set. We also introduce the related notion of an approximate cover, as well as the variant boundedly optimal and boundedly approximate covers.

One application of optimal f-covers is that it allows us to show the equivalence of pwt-f-random and vwt-f-random for a large class of functions, the convex functions. Of course this equivalence does not hold in general, as seen in Example 3.3.12.

3.4.1 Optimal and approximate f-covers

The essence of an optimal f-cover is that it is the cheapest way to cover a given set A: it is the set B of smallest direct f-weight such that $[B] \supseteq [A]$.

For any $\sigma \notin \mathcal{I}(f)$ there is a $\tau \subset \sigma$ such that $\mathrm{dwt}_f(\tau) \leq \mathrm{dwt}_f(\sigma)$ and $\tau \in \mathcal{I}(f)$. So for at most the cost of covering $[\sigma]$, we can cover the larger set $[\tau] \supset [\sigma]$. We will specify optimal f-covers to exclude such obviously bad choices, then.

Definition 3.4.1. Let $f : \{0,1\}^{<\mathbb{N}} \to [0, \infty)$ be recursive. $B \subseteq \{0,1\}^{<\mathbb{N}}$ is an ***optimal f-cover*** of $A \subseteq \{0,1\}^{<\mathbb{N}}$ if

1. $[B] \supseteq [A]$,
2. $\mathrm{dwt}_f(B) = \mathrm{vwt}_f(A)$, and
3. $B \subseteq \mathcal{I}(f)$.

An optimal f-cover of A has minimal direct f-weight among all covers of A. It is possible that a set might have multiple optimal covers, or none. But even if A is strongly recursive and an optimal cover exists, it might be computationally hard to specify it. Then consider the following related definition.

Definition 3.4.2. Let $f : \{0,1\}^{<\mathbb{N}} \to [0, \infty)$ be recursive. $B \subseteq \{0,1\}^{<\mathbb{N}}$ is an ***approximate f-cover*** of $A \subseteq \{0,1\}^{<\mathbb{N}}$ if

1. A and B are recursively enumerable and
2. $\widehat{B^n}$ is an optimal f-cover of A^n for every n.

An approximate f-cover of an r.e. set A can be thought of as constantly revised guesses as to the optimal f-cover of A; the guesses need to be revised as more information is revealed about A. Note the following: If $A \subseteq \{0,1\}^{<\mathbb{N}}$ is an r.e. set with approximate f-cover B, then $\widehat{B}$ is the point-wise limit of $\widehat{B^k}$ as k approaches infinity. Moreover $\mathrm{dwt}_f(\widehat{B}) = \lim_{k\to\infty} \mathrm{dwt}_f(\widehat{B^k})$ and this limit is the limit of a monotonically increasing sequence.

Optimal and approximate covers are related as follows.

Lemma 3.4.3. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If B is an approximate f-cover of A then $\widehat{B}$ is an optimal f-cover of A.

Proof. It is clear that $[B] \supseteq [A]$ since $[B^k] \supseteq [A^k]$ for each k. It is also immediate that $\widehat{B} \subseteq \mathcal{I}(f)$ since $\widehat{B^k} \subseteq \mathcal{I}(f)$ for each k.

So suppose that $\widehat{B}$ is not an optimal f-cover of A. Then there must be a set C proving this: that is $[C] \supseteq [A]$ and $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(\widehat{B})$. But $\mathrm{dwt}_f(\widehat{B}) = \lim_{k\to\infty} \mathrm{dwt}_f(\widehat{B^k})$, and so there is some $k \in \mathbb{N}$ such that $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(\widehat{B^k})$. Since $[C] \supseteq [A^k]$ and $\mathrm{dwt}_f(\widehat{B^k}) > \mathrm{dwt}_f(C)$, $\widehat{B^k}$ is not an optimal f-cover of A^k. This is a contradiction. □

Additionally, optimal f-covers are optimal everywhere: if we start removing pieces from both the cover and the covered set, what is left should still be an optimal f-cover.

Lemma 3.4.4. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If B is an optimal f-cover of A and no proper prefix of σ is in B, then $B \smallsetminus [\sigma]^{<\mathbb{N}}$ is an optimal f-cover of $A \smallsetminus [\sigma]^{<\mathbb{N}}$.

Proof. Suppose this is not the case. Then there is a $C \subseteq \{0,1\}^{<\mathbb{N}}$ such that $[C] \supseteq [A] \smallsetminus [\sigma]$ but $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(B \smallsetminus [\sigma]^{<\mathbb{N}})$. So define $D = C \cup (B \cap [\sigma]^{<\mathbb{N}})$ and note that $\mathrm{dwt}_f(D) < \mathrm{dwt}_f(B)$ and $[D] \supseteq [A]$, which is a contradiction. □

3.4.2 Boundedly optimal and approximate f-covers

The main drawback of optimal and approximate f-covers is that they may fail to exist. For example, if f is defined by $f(\sigma) = 2|\sigma|$ for all σ, then $\mathrm{vwt}_f(A) = 0$ for every $A \subseteq \{0,1\}^{<\mathbb{N}}$, while no set has direct f-weight of 0. Then no non-empty set has an optimal f-cover. Similarly for an arbitrary recursive f, we do not necessarily know what f will do on all very long strings, and how that might affect what we currently believe to be an optimal cover.

To address these issues we define the following concepts.

Definition 3.4.5. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. For $A \subseteq \{0,1\}^{<\mathbb{N}}$ the ***boundedly vehement f-weight of*** A is

$$\mathrm{bwt}_f(A) = \inf\{\mathrm{dwt}_f(C) \mid [C] \supseteq [A] \text{ and } (\forall\sigma \in C)(\exists\tau \in A)(\sigma \subseteq \tau)\}.$$

Boundedly vehement f-weight is defined similarly to vehement f-weight. The difference is that we only look at covers that come from prefixes of the given strings. This is only "boundedly vehement" as it ignores the possibility that A might be covered cheaply by some set containing very long extensions of the elements of A; it is basically vehement, but modified to look at only a bounded collection of covering sets. A useful fact about boundedly vehement f-weight is as follows.

Lemma 3.4.6. For any $\sigma \in \mathcal{I}(f)$, $\mathrm{dwt}_f(\sigma) = \mathrm{bwt}_f(\sigma)$.

Proof. Obviously $\mathrm{bwt}_f(\sigma) \le \mathrm{dwt}_f(\sigma)$. On the other hand, if $\tau \subset \sigma$ then $f(\sigma) > f(\tau)$ since $\sigma \in \mathcal{I}(f)$. So any set A containing some $\tau \subset \sigma$ will satisfy $\mathrm{dwt}_f(A) > \mathrm{dwt}_f(\sigma)$. Therefore $\mathrm{bwt}_f(\sigma) = \mathrm{dwt}_f(\sigma)$. ☐

With boundedly vehement f-weight in place of vehement f-weight we get the following modified definitions.

Definition 3.4.7. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. $B \subseteq \{0,1\}^{<\mathbb{N}}$ is a ***boundedly optimal f-cover*** of $A \subseteq \{0,1\}^{<\mathbb{N}}$ if

1. $[B] \supseteq [A]$,
2. $(\forall\sigma \in B)(\exists\tau \in A)(\sigma \subseteq \tau)$,
3. $\mathrm{dwt}_f(B) = \mathrm{bwt}_f(A)$, and
4. $B \subseteq \mathcal{I}(f)$.

In addition to changing prefix f-weight to boundedly vehement f-weight, we have added the condition that all elements of B be prefixes of elements of A. In other words, A not only achieves but also witnesses the minimum direct f-weight in the definition of boundedly vehement f-weight.

Definition 3.4.8. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. $B \subseteq \{0,1\}^{<\mathbb{N}}$ is a ***boundedly approximate f-cover*** of $A \subseteq \{0,1\}^{<\mathbb{N}}$ if

1. A and B are r.e. sets and
2. $\widehat{B^k}$ is a boundedly optimal f-cover of A^k for each k.

Just as with approximate f-covers, if B is a boundedly approximate f-cover then $\mathrm{dwt}_f(\widehat{B}) = \lim_{k\to\infty} \mathrm{dwt}_f(\widehat{B^k})$ and this limit is of a monotonically increasing sequence of reals. Analogous to Lemma 3.4.3 we have the following.

Lemma 3.4.9. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If B is a boundedly approximate f-cover of A then $\widehat{B}$ is a boundedly optimal f-cover of A.

Proof. First, it is clear that $[B] \supseteq [A]$, $(\forall\sigma \in B)(\exists\tau \in A)(\sigma \subseteq \tau)$, and $B \subseteq \mathcal{I}(f)$ since each $\widehat{B^k}$ is itself a boundedly optimal f-cover of A^k.

So suppose $\widehat{B}$ is not a boundedly optimal f-cover of A. Then there is a set C such that $[C] \supseteq [A]$, $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(\widehat{B})$, and $(\forall\sigma \in C)(\exists\tau \in A)(\sigma \subseteq \tau)$. But $\mathrm{dwt}_f(\widehat{B}) = \lim_{k\to\infty} \mathrm{dwt}_f(\widehat{B^k})$ and so there is some $k \in \mathbb{N}$ such that $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(\widehat{B^k})$. Define $C' = \{\sigma \in C \mid (\exists\tau \in A^k)(\sigma \subseteq \tau)\}$. Note that $\mathrm{dwt}_f(C') < \mathrm{dwt}_f(\widehat{B}^k)$ and $[C'] \supseteq [A^k]$. Then C' shows that $\widehat{B^k}$ is not a boundedly optimal f-cover of A^k, which is a contradiction. □

And analogous to Lemma 3.4.4 we have:

Lemma 3.4.10. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If B is a boundedly optimal f-cover of A and no proper prefix of σ is in B, then $B \setminus [\sigma]^{<\mathbb{N}}$ is a boundedly optimal f-cover of $A \setminus [\sigma]^{<\mathbb{N}}$.

Proof. Suppose this is not the case. Then there is a set C such that $(\forall\sigma \in C)(\exists\tau \in A \setminus [\sigma]^{<\mathbb{N}})(\sigma \subseteq \tau)$, $[C] \supseteq [A] \setminus [\sigma]$, and $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(B \setminus [\sigma]^{<\mathbb{N}})$. Then define $D = C \cup (B \cap [\sigma]^{<\mathbb{N}})$, noting that $(\forall\sigma \in D)(\exists\tau \in A)(\sigma \subseteq \tau)$, $\mathrm{dwt}_f(D) < \mathrm{dwt}_f(B)$, and $[D] \supseteq [A]$. But then D shows that B is not a boundedly optimal f-cover of A, which is a contradiction. □

So far, boundedly optimal f-covers and optimal f-covers appear to be very similar concepts. The difference is that for a simple enough set, such as a finite set, a boundedly optimal f-cover must exist.

Lemma 3.4.11. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. If $A \subseteq \{0,1\}^{<\mathbb{N}}$ is finite then some B is a boundedly optimal f-cover of A.

Proof. Since A is finite the infimum used to define $\mathrm{bwt}_f(A)$ becomes a minimum. So let $C \subseteq \{0,1\}^{<\mathbb{N}}$ be such that $[C] \supseteq [A]$, $(\forall\sigma \in C)(\exists\tau \in A)(\sigma \subseteq \tau)$, and $\mathrm{dwt}_f(C) = \mathrm{bwt}_f(A)$. Note that we have already met all necessary criteria, except it is not necessarily true that $C \subseteq \mathcal{I}(f)$.

Define a new set by $B = \{\tau \in \mathcal{I}(f) \mid (\exists \sigma \in C)(\tau \subseteq \sigma \wedge f(\tau) = f(\sigma))\}$. In other words, B is what we get when we shorten each member of C just enough so that it belongs to $\mathcal{I}(f)$. Obviously then $B \subseteq \mathcal{I}(f)$.

Of course, we would like to be sure that every $\sigma \in C$ can be shortened in this way. Obviously there is a $\tau \in \mathcal{I}(f)$ such that $\tau \subseteq \sigma$ and $f(\tau) \geq f(\sigma)$ (otherwise σ itself would be in $\mathcal{I}(f)$ and have this property). But if there were some $\tau \subset \sigma$ such that $f(\tau) > f(\sigma)$, then this would show that $\mathrm{bwt}_f(A) < \mathrm{dwt}_f(C)$, which is not true. So defining B in this way, each element of C gets shortened to something in B.

This shows that $[B] \supseteq [C] \supseteq [A]$. Finally, B inherits several properties from C: $\mathrm{dwt}_f(B) \leq \mathrm{dwt}_f(C) = \mathrm{bwt}_f(A)$ and also $(\forall \sigma \in B)(\exists \tau \in A)(\sigma \subseteq \tau)$. The first property is true since if $\sigma \in B$ there is some corresponding $\tau \in C$ such that $f(\sigma) > f(\tau)$. The second property is trivial, since we only ever shortened elements of C. Then B is a boundedly optimal f-cover of A. □

It is possible for a set to have multiple boundedly optimal f-covers. For example, let f be given by $f(\sigma) = |\sigma|$ for all σ, and consider a set as simple as $A = \{00, 01\}$. Then it is not hard to show that both A itself and the set $B = \{0\}$ are boundedly optimal f-covers (in fact optimal f-covers) for A.

The important thing is that we now know that finite sets have finite boundedly optimal f-covers. More importantly, though, we can find these covers effectively. This effectivity allows us to build boundedly approximate f-covers of infinite sets, assuming the infinite sets are recursively enumerable and never enumerate extensions of elements they already contain.

Theorem 3.4.12. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be strongly recursive and let $A \subseteq \{0,1\}^{<\mathbb{N}}$ be an r.e. set such that for any t if $\sigma \in A^t$ and $\tau \in A^{t+1} \setminus A^t$ then $\tau \not\supseteq \sigma$. Then some r.e. set B is a boundedly approximate f-cover of A.

Proof. We may assume without loss of generality that A is infinite and enumerated without repetition. We build B by defining sets B^k recursively in k and letting $B = \bigcup_k B^k$.

Step 1*:* If $A^1 = \{\sigma_1\}$, then we define $B^1 = \{\tau_1\}$ where $\tau_1 \in \mathcal{I}(f)$ is such that $\tau_1 \subseteq \sigma_1$ and $f(\tau_1) = \max\{f(\rho) \mid \rho \subseteq \sigma_1\}$. The choice of τ_1 is unique since we are requiring it to be in $\mathcal{I}(f)$.

Step $k+1$*:* Let $\sigma_{k+1} \in A^{k+1} \setminus A^k$, noting that such σ_{k+1} is unique. If $[A^{k+1}] \subseteq [B^k]$ then define $B^{k+1} = B^k$. Otherwise let $\tau_{k+1} \subseteq \sigma_{k+1}$ be the longest such prefix for which τ_{k+1} belongs to a boundedly optimal f-cover of A^{k+1}. We can find τ_{k+1} effectively, since there are only finitely many sets which could possibly be boundedly optimal f-covers of A^{k+1}. For later reference, write C_{k+1} to mean the boundedly optimal f-cover of A^{k+1} which contained τ_{k+1}. Then define $B^{k+1} = B^k \cup \{\tau_{k+1}\}$.

Verifying the construction at step 1. We wish to show that $\widehat{B^1}$ is a boundedly approximate f-cover of A^1. It is clear that $\widehat{B^1} \subseteq \mathcal{I}(f)$ and $\tau_1 \subseteq \sigma_1 \in A^1$. We picked τ_1 such that $f(\tau_1) \geq f(\rho)$ for all $\rho \subseteq \sigma_1$ and so $2^{-f(\tau_1)} \leq 2^{-f(\rho)}$ for all such ρ as well; it follows that $\mathrm{bwt}_f(\sigma_1) = \mathrm{dwt}_f(\tau_1)$. This confirms that $\widehat{B^1}$ is a boundedly optimal f-cover of A^1.

Verifying the construction at step $k+1$ when $B^{k+1} = B^k$. If $B^{k+1} = B^k$ then by construction it must have been the case that $[\sigma_{k+1}] \subseteq [B^k]$. If $\widehat{B^k}$ is not a boundedly optimal f-cover for A^{k+1}, let C be such that $\mathrm{dwt}_f(C) < \mathrm{dwt}_f(\widehat{B^k})$, $[C] \supseteq [A^{k+1}]$ and $(\forall \sigma \in C)(\exists \tau \in A^{k+1})(\sigma \subseteq \tau)$. Define a new set $C' = \{\sigma \in C \mid (\exists \tau \in A^k)(\sigma \subseteq \tau)\}$. It is clear that $\mathrm{dwt}_f(C') < \mathrm{dwt}_f(\widehat{B^k})$ and $[C'] \supseteq [A^k]$ and so C' shows that $\widehat{B^k}$ is not a boundedly optimal f-cover of A^k; this is a contradiction.

Verifying the construction at step $k+1$ when $B^{k+1} \neq B^k$. The string τ_{k+1} is necessarily in $\widehat{B^{k+1}}$, and so no prefix of τ_{k+1} is in B^k. Applying Lemma 3.4.10, we get that $\widehat{B^k} \smallsetminus [\tau_{k+1}]^{<\mathbb{N}} = \widehat{B^{k+1}} \smallsetminus \{\tau_{k+1}\}$ is a boundedly optimal f-cover for $A^k \smallsetminus [\tau_{k+1}]^{<\mathbb{N}} = A^{k+1} \smallsetminus [\tau_{k+1}]^{<\mathbb{N}}$. By the same lemma $C_{k+1} \smallsetminus \{\tau_{k+1}\}$ is a boundedly optimal f-cover for $A^{k+1} \smallsetminus [\tau_{k+1}]^{<\mathbb{N}}$. Since two boundedly optimal f-covers of the same set have the same direct f-weight we have $\mathrm{dwt}_f(\widehat{B^{k+1}} \smallsetminus \{\tau_{k+1}\}) = \mathrm{dwt}_f(C_{k+1} \smallsetminus \{\tau_{k+1}\})$ and so $\mathrm{dwt}_f(\widehat{B^{k+1}}) = \mathrm{dwt}_f(C_{k+1}) = \mathrm{bwt}_f(A^{k+1})$. It is clear from the construction that $[\widehat{B^{k+1}}] \supseteq [A^{k+1}]$, $\widehat{B^{k+1}} \subseteq \mathcal{I}(f)$ and that $(\forall \sigma \in \widehat{B^{k+1}})(\exists \tau \in A^{k+1})(\sigma \subseteq \tau)$. Therefore $\widehat{B^{k+1}}$ is a boundedly optimal f-cover for A^{k+1} as required.

Then since B is r.e., B is a boundedly approximate f-cover for A. □

Note that this construction gives preferential treatment to longer strings: if both τ and ρ belong to boundedly optimal f-covers of some set and $\tau \subset \rho$, then the construction will always pick ρ to be in B. This provides a uniformity to the construction, which we will use in Lemma 3.4.24.

Corollary 3.4.13. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be strongly recursive and let $A \subseteq \{0,1\}^{<\mathbb{N}}$ be an r.e. set such that for any t if $\sigma \in A^t$, $\tau \supset \sigma$, and $\tau \notin A^t$ then $\tau \notin A$. Then some set B is a boundedly optimal f-cover of A.

Proof. Apply Lemma 3.4.9 to the previous theorem. □

Not only does Theorem 3.4.12 prove the existence of boundedly approximate and boundedly optimal f-covers, but it shows how to construct one uniformly from an index for an r.e. set with the correct properties.

3.4.3 Convex and strongly convex functions

Now we introduce a condition on functions which will be sufficient to guarantee the existence of optimal f-covers as well. This type of function will also be necessary when we discuss weak separations in Chapter 4.

Definition 3.4.14. A function $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is said to be ***convex*** if $\mathrm{dwt}_f(\sigma) \le \mathrm{dwt}_f(\sigma^\frown 0) + \mathrm{dwt}_f(\sigma^\frown 1)$ for every σ and ***strongly convex*** if $f(\sigma^\frown i) \le f(\sigma) + 1$ for any σ and i.

It is not hard to see that any strongly convex function must also be convex. A simple example demonstrating that the converse implication fails is the function f defined by $f(\sigma) = \sum_{\sigma(n)=0} 2$ for all σ. For any σ we have

$$\mathrm{dwt}_f(\sigma) = \mathrm{dwt}_f(\sigma^\frown 1) < \mathrm{dwt}_f(\sigma^\frown 0) + \mathrm{dwt}_f(\sigma^\frown 1)$$

and so f is convex; however $f(\sigma^\frown 0) = f(\sigma) + 2$ for any σ as well, and so f is not strongly convex. In the case of length-invariant functions, however, this kind of example is not possible.

Lemma 3.4.15. If $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is recursive and length-invariant, then it is convex if and only if it is strongly convex.

Proof. It is trivial that strongly convex implies convex, so we assume that f is convex. Then for any σ we have $f(\sigma^\frown 0) = f(\sigma^\frown 1)$ and therefore

$$2^{-f(\sigma)} \le 2^{-f(\sigma^\frown 0)} + 2^{-f(\sigma^\frown 1)} = 2 \cdot 2^{-f(\sigma^\frown 0)} = 2^{-f(\sigma^\frown 0)+1}.$$

From this it follows that $f(\sigma^\frown 0) \le f(\sigma)+1$. Then $f(\sigma^\frown 1) = f(\sigma^\frown 0) \le f(\sigma)+1$ as well, and f is strongly convex. $\square$

Lemma 3.1.3 shows that any function is equivalent to an integer-valued one. If we start with a convex function, does the process described in the proof of that lemma produce an equivalent function which is also convex? If not, can we modify the process to achieve this? Unfortunately both of these questions can be answered negatively.

Example 3.4.16. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be defined by

$$f(\sigma) = \sum_{n<|\sigma|} [2(1-\sigma(n)) + \sigma(n)/3] = \sum_{\sigma(n)=0} 2 + \sum_{\sigma(n)=1} 1/3$$

for all σ. Then f is recursive and convex, yet there is no recursive convex $g : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ such that $f \approx g$.

Proof. It is clear that f is recursive. To see that f is convex, observe that $2^{-2} + 2^{-1/3} > .25 + .79 > 1$ and so for all σ

$$\mathrm{dwt}_f(\sigma) < (2^{-2} + 2^{-1/3})\mathrm{dwt}_f(\sigma) = \mathrm{dwt}_f(\sigma\widehat{\ }0) + \mathrm{dwt}_f(\sigma\widehat{\ }1).$$

Now let $g : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ be recursive and convex. We will construct an $X \in \{0,1\}^{\mathbb{N}}$ such that $\lim_{n\to\infty}(f(X \upharpoonright n) - g(X \upharpoonright n)) = \infty$, ensuring $f \not\approx g$.

First note that there are only a few possibilities for $g(\sigma\widehat{\ }0)$ and $g(\sigma\widehat{\ }1)$ once we know $g(\sigma)$. Since g is convex one of the following cases holds for a fixed σ:

0. $g(\sigma\widehat{\ }0) \le g(\sigma) + 1$, or
1. $g(\sigma\widehat{\ }0) > g(\sigma) + 1$ and $g(\sigma\widehat{\ }1) \le g(\sigma)$.

Then let $h : \{0,1\}^{<\mathbb{N}} \to \{0,1\}$ be the function such that $h(\sigma) = 0$ if σ satisfies case (0) and $h(\sigma) = 1$ if σ satisfies case (1).

Define X as follows. $X(0) = h(\lambda)$ and $X(n) = h(X \upharpoonright n)$ for all $n > 0$. In other words, if case (0) holds when setting $\sigma = X \upharpoonright n$ then define $X(n) = 0$; otherwise define $X(n) = 1$. For each n one of the following occurs:

- $X \upharpoonright (n+1) = (X \upharpoonright n)\widehat{\ }0$ and $g(X \upharpoonright (n+1)) \le g(X \upharpoonright n) + 1$. Then since $f(X \upharpoonright (n+1)) = f(X \upharpoonright n) + 2$ we have
$$f(X \upharpoonright (n+1)) - g(X \upharpoonright (n+1)) \ge f(X \upharpoonright n) - g(X \upharpoonright n) + 1.$$
- $X \upharpoonright (n+1) = (X \upharpoonright n)\widehat{\ }1$ and $g(X \upharpoonright (n+1)) \le g(X \upharpoonright n)$. Then since $f(X \upharpoonright (n+1)) = f(X \upharpoonright n) + 1/3$ we have
$$f(X \upharpoonright (n+1)) - g(X \upharpoonright (n+1)) \ge f(X \upharpoonright n) - g(X \upharpoonright n) + 1/3.$$

Then we have $\lim_{n\to\infty}(f(X \upharpoonright n) - g(X \upharpoonright n)) = \infty$ and so $f \not\approx g$. $\square$

The problem here is that f grows too fast, and no integer-valued convex g can keep up. For any σ, g will always fall farther behind f on either $\sigma\widehat{\ }0$ or $\sigma\widehat{\ }1$. By just following a path on which g always falls farther behind f, we see that $f \not\approx g$. In this example we have $\mathcal{I}(f) = \{0,1\}^{<\mathbb{N}}$ and so it follows that there is also no convex integer-valued g such that $f \sim g$.

The above construction relies on the fact that f can grow very quickly along some paths. If we further restrict how fast f can grow, this type of construction is not possible.

Lemma 3.4.17. If $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is recursive and strongly convex, there is a recursive strongly convex $g : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ such that $f \approx g$.

Proof. Here g will be a modified version of the function defined in the proof of Lemma 3.1.3. There is a recursive function $\Phi : 2^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}^{\geq 0}$ such that $|\Phi(\sigma, n) - f(\sigma)| \leq 2^{-n}$ for all σ and n. Define $g(\lambda) = 0$. For each σ and i let $g(\sigma\frown i) = \min\{g(\sigma) + 1, \lfloor \Phi(\sigma\frown i, 0) \rfloor\}$. It is clear that g is recursive and $g(\sigma\frown i) \leq g(\sigma) + 1$ for all σ and i. So we need only show that $f \approx g$.

In the definition we see that $g(\sigma) \leq \lfloor \Phi(\sigma, 0) \rfloor < \Phi(\sigma, 0) + 1 \leq f(\sigma) + 2$ for any σ. The last thing we need is $f(\sigma) \leq^+ g(\sigma)$ or $f(\sigma) - g(\sigma) \leq^+ 0$ for all σ. For each σ and i we have either

- $g(\sigma\frown i) = g(\sigma) + 1$ in which case

$$f(\sigma\frown i) - g(\sigma\frown i) \leq f(\sigma) + 1 - g(\sigma) - 1 = f(\sigma) - g(\sigma),$$

- or $g(\sigma\frown i) = \lfloor \Phi(\sigma\frown i, 0) \rfloor$ in which case

$$f(\sigma\frown i) - g(\sigma\frown i) \leq f(\sigma\frown i) - \Phi(\sigma\frown i, 0) + 1 \leq 2.$$

This shows that $f(\sigma) - g(\sigma) \leq^+ 0$ as required, and completes the proof. □

Lastly, recall that $\mathcal{I}(f)$ will be recursive so long as f is strongly recursive. Then the following modification allows us to replace any recursive function with an equivalent strongly recursive one, without impacting whether or not it is convex or strongly convex.

Lemma 3.4.18. Let $f : \{0,1\}^{<\mathbb{N}} \to [0, \infty)$ be recursive. Then there is a strongly recursive $g : \{0,1\}^{<\mathbb{N}} \to [0, \infty)$ such that $f \approx g$. Moreover if f is (strongly) convex then so is g.

Proof. For convenience, assume that $f(\sigma) \geq 1$ for all σ. If this is not the case, then the function $\tilde{f}(\sigma) = f(\sigma) + 1$ has this property, is equivalent to f, and is (strongly) convex if and only if f is. Let $\Phi(\sigma, n)$ be the recursive function which gives the approximate rational values of $f(\sigma)$, with accuracy at worst 2^{-n}.

We define g by $g(\sigma) = f(\sigma) - h(\sigma)$ where $h : \{0,1\}^{<\mathbb{N}} \to [0, 1) \cap \mathbb{Q}$ is a recursive function such that $h(\sigma\frown i) \geq h(\sigma)$ for all σ and i. We will postpone defining h for last. It is clear that since h is a bounded non-negative function, $f \approx g$. Moreover if f is convex then

$$\begin{aligned}
\mathrm{dwt}_g(\sigma) = 2^{-f(\sigma)+h(\sigma)} &\leq 2^{-f(\sigma\frown 0)} 2^{h(\sigma)} + 2^{-f(\sigma\frown 1)} 2^{h(\sigma)} \\
&\leq 2^{-f(\sigma\frown 0)} 2^{h(\sigma\frown 0)} + 2^{-f(\sigma\frown 1)} 2^{h(\sigma\frown 1)} \\
&= \mathrm{dwt}_g(\sigma\frown 0) + \mathrm{dwt}_g(\sigma\frown 1)
\end{aligned}$$

and so g is convex as well. More simply, if f is strongly convex then

$$\begin{aligned} g(\sigma\widehat{\ }i) &= f(\sigma\widehat{\ }i) - h(\sigma\widehat{\ }i) \\ &\leq f(\sigma) + 1 - h(\sigma\widehat{\ }i) \leq f(\sigma) - h(\sigma) + 1 = g(\sigma) \end{aligned}$$

and so g is strongly convex.

Finally we must define h such that g becomes strongly recursive. Let $\{\sigma_n\}$ be an enumeration without repetition of $\{0,1\}^{<\mathbb{N}}$ such that if $|\sigma_i| < |\sigma_j|$ then $i < j$. Note in our enumeration that $\sigma_0 = \lambda$. In addition to h we will define a recursive function $t : \mathbb{N} \to \mathbb{N}$. We begin by defining $h(\lambda) = 0$ and $t(0) = 0$.

Assume inductively that we have defined $h(\sigma_i)$ and $t(i)$ for all $i < n$ in such a way that for all $i < j < n$

$$|(\Phi(\sigma_i, t(n-1)) - h(\sigma_i)) - (\Phi(\sigma_j, t(n-1)) - h(\sigma_j))| > 3 \cdot 2^{-t(n-1)}.$$

Let $i < n$ be such that σ_i is the longest proper prefix of σ_n. Then let $t > n+3$ be the least number such that for all $j < n$ either

$$|(\Phi(\sigma_j, t) - h(\sigma_j)) - (\Phi(\sigma_n, t) - h(\sigma_i))| > 8 \cdot 2^{-t}$$

or

$$|(\Phi(\sigma_j, t) - h(\sigma_j)) - (\Phi(\sigma_n, t) - h(\sigma_i))| < 2^{-t}.$$

It is clear that such a number must exist since there are only finitely many $j < n$ and either these values are close together or there is space between them. Then define $h(\sigma_n) = h(\sigma_i) + 4 \cdot 2^{-t}$. Letting $t(n) = t$ this ensures that

$$|(\Phi(\sigma_i, t(n)) - h(\sigma_i)) - (\Phi(\sigma_j, t(n)) - h(\sigma_j))| > 3 \cdot 2^{-t(n)}$$

for all $i, j \leq n$. Note that $\Psi(\sigma, n) = \Phi(\sigma, n) - h(\sigma)$ is recursive and approximates $g(\sigma) = f(\sigma) - h(\sigma)$ within 2^{-n}. Then for any σ, τ there are $i, j < N$ for some N such that $\sigma = \sigma_i$ and $\tau = \sigma_j$. Then we have

$$\begin{aligned} |g(\sigma) - g(\tau)| &\geq |\Psi(\sigma_i, t(N)) - \Psi(\sigma_j, t(N))| - 2 \cdot 2^{-t(N)} \\ &\geq 3 \cdot 2^{-t(N)} - 2 \cdot 2^{-t(N)} = 2^{-t(N)}. \end{aligned}$$

In other words, $g : \{0,1\}^{<\mathbb{N}} \to [0, \infty)$ is an injection. Then since g is recursive, we will eventually find out whether $g(\sigma) < g(\tau)$ or $g(\tau) < g(\sigma)$ for any $\sigma \neq \tau$. Thus g is strongly recursive, completing the proof. □

Note in the above that if f is length-invariant then g will not be. However, a simple modification of the above argument shows that for a length-invariant function a similar modification can be made to add strong recursiveness without impacting the length-invariance.

3.4.4 Convex functions and the existence of optimal f-covers

For a large class of r.e. sets, boundedly optimal f-covers always exist. What does this say about the existence of optimal f-covers? We will show that for convex functions, this implies the existence of optimal f-covers as well, for any r.e. set.

Lemma 3.4.19. A function $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is convex if and only if $\mathrm{dwt}_f(\sigma) \le \mathrm{dwt}_f(A)$ for any σ and all $A \subseteq [\sigma]^{<\mathbb{N}}$ s.t. $[A] = [\sigma]$.

Proof. Assume that the latter condition holds and let σ be arbitrary. Letting $A = \{\sigma\frown 0, \sigma\frown 1\}$, we have $A \subseteq [\sigma]^{<\mathbb{N}}$ and $[A] = [\sigma]$ from which it follows that f is convex.

On the other hand, assume that f is convex. Let σ be fixed, and let $\prec$ be the partial order on finite subsets of $\{0,1\}^{<\mathbb{N}}$ defined as follows:

$$A \prec B \iff (A \ne B) \wedge (\forall \tau \in B)(\exists \rho \in A)(\rho \subseteq \tau).$$

Clearly a finite set A has at most finitely many $\prec$-predecessors. By compactness, if there is an $A \subseteq [\sigma]^{<\mathbb{N}}$ such that $[A] = [\sigma]$ and $\mathrm{dwt}_f(\sigma) > \mathrm{dwt}_f(A)$, then there is a finite prefix-free such A. So let A be a $\prec$-minimal such set with these properties.

Let $\rho\frown 0, \rho\frown 1 \in A$ be of maximum length among all strings in A (such a pair must exist, since otherwise A could be trimmed still). Then since f is convex, we have $\mathrm{dwt}_f(\rho) \le \mathrm{dwt}_f(\{\rho\frown 0, \rho\frown 1\})$. So define $A' = (A \setminus \{\rho\frown 0, \rho\frown 1\}) \cup \{\rho\}$. Clearly $[A'] = [A] = [\sigma]$, $\mathrm{dwt}_f(A') \le \mathrm{dwt}_f(A) < \mathrm{dwt}_f(\sigma)$ and A' is finite and prefix-free. But $A' \prec A$ which contradicts the assumption that A was a $\prec$-minimal such set. Then no such set exists, proving the claim. □

The implication of this lemma is that for convex f and a string τ, the set $[\tau]$ cannot be covered using only extensions of τ for cheaper than the cost of τ itself.

Corollary 3.4.20. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be convex. Then for any set $A \subseteq \{0,1\}^{<\mathbb{N}}$ we have $\mathrm{bwt}_f(A) = \mathrm{vwt}_f(A)$.

Proof. Of course it is clear that $\mathrm{vwt}_f(A) \le \mathrm{bwt}_f(A)$ for any A, and so we need only look at the reversed inequality.

First, assume that $A = \{\sigma\}$ for some σ. For any $\epsilon > 0$ there exists a prefix-free set A_ϵ such that $[A_\epsilon] \supseteq [\sigma]$ and $\mathrm{dwt}_f(A_\epsilon) < \mathrm{vwt}_f(\sigma) + \epsilon$. We may assume without loss of generality that $\tau \in A_\epsilon$ implies $\tau \asymp \sigma$, and so either $A_\epsilon \subseteq \{\tau \mid \tau \subseteq \sigma\}$ or $A_\epsilon \subset [\sigma]^{<\mathbb{N}}$. If the former holds, then $\mathrm{bwt}_f(\sigma) \le \mathrm{dwt}_f(A_\epsilon) < \mathrm{vwt}_f(\sigma) + \epsilon$ by definition of boundedly vehement f-weight. If the latter holds, then $\mathrm{bwt}_f(\sigma) \leq \mathrm{dwt}_f(\sigma) \leq \mathrm{dwt}_f(A_\epsilon) < \mathrm{vwt}_f(\sigma) + \epsilon$ by Lemma 3.4.19. Since this is true for any $\epsilon > 0$ we have $\mathrm{bwt}_f(\sigma) \leq \mathrm{vwt}_f(\sigma)$ which proves the claim in this special case.

Now we consider the case of more general A. Actually, $\mathrm{vwt}_f(A) = \mathrm{vwt}_f(\widehat{A})$ and $\mathrm{bwt}_f(A) \leq \mathrm{bwt}_f(\widehat{A})$ and so we need only consider the case that A is prefix-free. For fixed $\epsilon > 0$ there is a prefix-free set A_ϵ such that $\mathrm{dwt}_f(A_\epsilon) < \mathrm{vwt}_f(A) + \epsilon$ and $[A_\epsilon] \supseteq [A]$. For each $\sigma \in A$ define $B_\sigma = [\sigma]^{<\mathbb{N}} \cap A_\epsilon$. Also define the sets $B = A_\epsilon \setminus (\cup_\sigma B_\sigma)$ and

$$A' = \{\sigma \in A \mid [\sigma] \nsubseteq [B]\} = \{\sigma \in A \mid B_\sigma \neq \varnothing\}.$$

Then applying the simple case we have

$$\begin{aligned}\mathrm{dwt}_f(A_\epsilon) &= \mathrm{dwt}_f(B) + \sum_{\sigma \in A'} \mathrm{dwt}_f(B_\sigma) \\ &\geq \mathrm{dwt}_f(B) + \sum_{\sigma \in A'} \mathrm{dwt}_f(\sigma) \geq \mathrm{dwt}_f(A' \cup B) \geq \mathrm{bwt}_f(A),\end{aligned}$$

since $(\forall \sigma \in A' \cup B)(\exists \tau \in A)(\sigma \subseteq \tau)$ and $[A' \cup B] \supseteq [A]$. Since $\epsilon > 0$ was arbitrary, this proves that $\mathrm{vwt}_f(A) = \mathrm{bwt}_f(A)$ as required. □

A further corollary of this is the following lemma.

Lemma 3.4.21. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and convex, and let $A \subseteq \{0,1\}^{<\mathbb{N}}$. If B is an optimal f-cover of A, $\sigma \in \mathcal{I}(f)$ and $\mathrm{dwt}_f(\sigma) > \mathrm{vwt}_f(A)$ then $[\sigma] \nsubseteq [B]$.

Proof. Combining Theorem 3.4.20 with Lemma 3.4.6, it holds that $\mathrm{dwt}_f(\sigma) = \mathrm{bwt}_f(\sigma) = \mathrm{vwt}_f(\sigma)$. If $[A] \supseteq [\sigma]$ then by definition of vehement f-weight it would follow that $\mathrm{vwt}_f(\sigma) \leq \mathrm{vwt}_f(A)$ which is a contradiction. □

Another immediate consequence of Corollary 3.4.20 is that every boundedly optimal f-cover constructed so far is also an optimal f-cover when f is convex. But actually, we can infer from this that every r.e. set has an optimal f-cover.

Theorem 3.4.22. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and convex. For any r.e. $A \subseteq \{0,1\}^{<\mathbb{N}}$ there is an r.e. B which is an approximate f-cover for A. It follows that $\widehat{B}$ is an optimal f-cover for A.

Proof. Let $\tilde{A}$ be the r.e. set $\tilde{A} = \bigcup_k \widehat{A^k}$. In other words, remove any elements of A which are enumerated after some proper prefix has already been enumerated. Then by Theorem 3.4.12 some B is a boundedly approximate f-cover for $\tilde{A}$, and so by Corollary 3.4.20 it must also be an approximate

f-cover of $\tilde{A}$. Since $[A] = [\tilde{A}]$ we have $\mathrm{vwt}_f(A) = \mathrm{vwt}_f(\tilde{A})$ and so B is an approximate f-cover for A as well. It follows that $\widehat{B}$ is an optimal f-cover for A. $\square$

Based on these facts and the uniformity of the construction in the proof of Theorem 3.4.12 we make the following definitions.

Definition 3.4.23. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and convex and let $A \subseteq \{0,1\}^{<\mathbb{N}}$ be an r.e. set. Let $\tilde{A} = \bigcup_k \widehat{A^k}$. Define ***the approximate f-cover of A***, denoted $\mathbb{A}_f(A)$, to be the specific approximate f-cover of $\tilde{A}$ constructed in the proof of Theorem 3.4.12. Define ***the optimal f-cover of A***, to be $\mathbb{O}_f(A) = \widehat{\mathbb{A}_f(A)}$.

3.4.5 Properties of approximate and optimal f-covers

First, note that our choice of the optimal f-cover respects intersections.

Lemma 3.4.24. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and convex. If A is an r.e. set and $[\sigma] \nsubseteq [\mathbb{O}_f(A)]$ then $\mathbb{O}_f(A \cap [\sigma]^{<\mathbb{N}}) = \mathbb{O}_f(A) \cap [\sigma]^{<\mathbb{N}}$, where the enumeration of $A \cap [\sigma]^{<\mathbb{N}}$ is the one inherited from A.

Proof. Define $A_\sigma = A \cap [\sigma]^{<\mathbb{N}}$ and $O_\sigma = \mathbb{O}_f(A) \cap [\sigma]^{<\mathbb{N}}$. We first show O_σ is an optimal f-cover of A_σ. $[\mathbb{O}_f(A)] \supseteq [A_\sigma]$ is true, but no prefix of σ can belong to $\mathbb{O}_f(A)$, and so $[O_\sigma] \supseteq [A_\sigma]$ as well. So either O_σ is an optimal f-cover for A_σ or else there is some other set $[B_\sigma] \supseteq [A_\sigma]$ such that $\mathrm{dwt}_f(O_\sigma) > \mathrm{vwt}_f(A_\sigma) = \mathrm{dwt}_f(B_\sigma)$. Defining $B = (\mathbb{O}_f(A) \setminus O_\sigma) \cup B_\sigma$ this would imply that $[B] \supseteq [A]$ and $\mathrm{dwt}_f(B) < \mathrm{dwt}_f(\mathbb{O}_f(A))$, which is a contradiction to the fact that $\mathbb{O}_f(A)$ is an optimal f-cover of A. Then O_σ must be an optimal f-cover for A_σ.

One question remains: is O_σ *the* optimal f-cover of A_σ (rather than just some optimal f-cover)? Here we appeal to the deterministic nature of the construction in Theorem 3.4.12. Any set C is an optimal f-cover of A_σ if and only if $(\mathbb{O}_f(A) \setminus O_\sigma) \cup C$ is an optimal f-cover of A. Then since A_σ has inherited its enumeration from A, any choices made about strings belonging to $\mathbb{O}_f(A)$ and extending σ will be the same choices made about strings belonging to $\mathbb{O}_f(A_\sigma)$. $\square$

An optimal f-cover of A is a set which achieves the minimum possible direct f-weight of a cover of A. The following lemma shows that approximate f-covers are optimal in a weak sense: an approximate f-cover of A achieves the minimum possible prefix f-weight of a cover of A.

Lemma 3.4.25. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and convex and let A be an r.e., set. If B is an approximate f-cover of A such that $B = \bigcup_k \widehat{B^k}$ then $\mathrm{vwt}_f(A) = \mathrm{bwt}_f(A) = \mathrm{pwt}_f(B)$. In particular this holds for $B = \mathbb{A}_f(A)$.

Proof. By Corollary 3.4.20 $\mathrm{vwt}_f(A) = \mathrm{bwt}_f(A)$. The inequality $\mathrm{vwt}_f(A) \leq \mathrm{dwt}_f(\widehat{B}) \leq \mathrm{pwt}_f(B)$ is clear. So we need only show that $\mathrm{pwt}_f(B) \leq \mathrm{vwt}_f(A)$.

Suppose that this property does not hold for some A and B as in the statement. Then there is a $k \in \mathbb{N}$ such that $\mathrm{pwt}_f(B^k) \leq \mathrm{vwt}_f(A^k)$, but $\mathrm{pwt}_f(B^{k+1}) > \mathrm{vwt}_f(A^{k+1})$. So let $P \subseteq B^{k+1}$ be such that $\mathrm{dwt}_f(P) > \mathrm{vwt}_f(A^{k+1})$ and let $\sigma \in B^{k+1} \setminus B^k$. By assumption about B it holds that $\sigma \in \widehat{B_k}$. It must also hold that $\sigma \in P$ since otherwise we would have $P \subseteq B^k$ as well, implying that $\mathrm{pwt}_f(B^k) > \mathrm{vwt}_f(A^k)$. Now define $P' = (P \setminus \{\sigma\}) \cup \widehat{C}$ where $C = \{\tau \in B^k \mid \sigma \subseteq \tau\}$. Clearly $P' \subseteq B^k$ and P' is prefix-free. Note also that $\widehat{C} \subseteq \widehat{B^k}$ since $\sigma \in B^{k+1} \setminus B^k$ and $\sigma \in \widehat{B^{k+1}}$.

Since the $\widehat{B^i}$ are optimal f-covers for the A^i, we have

$$\mathrm{vwt}_f(B^{k+1}) - \mathrm{vwt}_f(B^k) = \mathrm{dwt}_f(\widehat{B^{k+1}}) - \mathrm{dwt}_f(\widehat{B^k}) = \mathrm{dwt}_f(\sigma) - \mathrm{dwt}_f(\widehat{C}),$$

and so we have

$$\begin{aligned} \mathrm{dwt}_f(P') &= \mathrm{dwt}_f(P) - \mathrm{dwt}_f(\sigma) + \mathrm{dwt}_f(\widehat{C}) \\ &= \mathrm{dwt}_f(P) - \mathrm{vwt}_f(B^{k+1}) + \mathrm{vwt}_f(B^k) > \mathrm{vwt}_f(B^k). \end{aligned}$$

But then $\mathrm{pwt}_f(B^k) > \mathrm{vwt}_f(A^k)$, which is a contradiction. □

We will ultimately use optimal f-covers to construct forcing conditions in Chapter 5, but for now the most relevant corollary of these results on optimal f-covers is the following.

Theorem 3.4.26. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and convex. Then $X \in \{0,1\}^{<\mathbb{N}}$ is pwt-f-random if and only if it is vwt-f-random.

Proof. If X is vwt-f-random then it is pwt-f-random by Lemma 3.3.5. So let X be pwt-f-random and let $\langle A_n \rangle$ be a vwt-f-test. Then defining $B_n = \mathbb{A}_f(A_n)$ for all n, it is clear that $\langle B_n \rangle$ is a pwt-f-test since it is uniformly r.e. and $\mathrm{pwt}_f(\mathbb{A}_f(A_n)) = \mathrm{vwt}_f(A_n) \leq 2^{-n}$ for all n. Then there is an n such that $X \notin [B_n] = [\mathbb{A}_f(A_n)] \supset [A_n]$. Since $\langle A_n \rangle$ was arbitrary, we conclude that X is vwt-f-random. □

Chapter 4

Partial randomness and Kolmogorov complexity

In this chapter we study partial randomness notions that arise from Kolmogorov complexity, and their relation to those partial randomness notions introduced in Chapter 3. We also consider a formulation of f-random based on supermartingales. We will see that each new notion coincides with an old one, and additionally that Kolmogorov complexity is a very powerful tool for weakly separating notions of f-randomness.

4.1 Defining f-complexity

The main definitions we will consider are:

Definition 4.1.1. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. We say that X is ***KP-f-complex*** if $\mathrm{KP}(X \upharpoonright n) \geq^+ f(X \upharpoonright n)$ for all n and ***KA-f-complex*** if $\mathrm{KA}(X \upharpoonright n) \geq^+ f(X \upharpoonright n) + k$ for all n.

These definitions will allow us to study partial analogues of Schnorr's and Levin's theorems characterizing the Martin-Löf randoms as exactly the X such that $\mathrm{KP}(X \upharpoonright n) \geq^+ n$ or $\mathrm{KA}(X \upharpoonright n) \geq^+ n$ for all n. In fact X is Martin-Löf random if and only if $\lim_{n\to\infty} \mathrm{KP}(X \upharpoonright n) - n = \infty$ and so we consider the following related definitions as well.

Definition 4.1.2. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. We say that X is ***strongly KP-f-complex*** if $\lim_{n\to\infty} \mathrm{KP}(X \upharpoonright n) - f(X \upharpoonright n) = \infty$ and ***strongly KA-f-complex*** if $\lim_{n\to\infty} \mathrm{KA}(X \upharpoonright n) - f(X \upharpoonright n) = \infty$.

While there is no X such that $\lim_{n\to\infty} \mathrm{KA}(X \upharpoonright n) - n = \infty$, strong KA-

f-complexity is still studied for the sake of completeness. Just as with our earlier notions of partial randomness, notions of f-complex respect both equivalence and weak equivalence of functions. For equivalence this is trivial.

Lemma 4.1.3. Let $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be such that $f \approx g$. Then:

1. X is KP-f-complex if and only if X is KP-g-complex.
2. X is KA-f-complex if and only if X is KA-g-complex.
3. X is strongly KP-f-complex if and only if X is strongly KP-g-complex.
4. X is strongly KA-f-complex if and only if X is strongly KA-g-complex.

Proof. Each property is a trivial consequence of the definitions, since f and g are within a constant. □

To show that notions of f-complex are stable under weak equivalence is more difficult, but fortunately we can take advantage of the work that we have done for notions of f-random.

Lemma 4.1.4. Let $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be such that $f \sim g$. Then:

1. X is KP-f-complex if and only if X is KP-g-complex.
2. X is KA-f-complex if and only if X is KA-g-complex.
3. X is strongly KP-f-complex if and only if X is strongly KP-g-complex.
4. X is strongly KA-f-complex if and only if X is strongly KA-g-complex.

Proof. Corollaries of Theorems 4.1.6, 4.1.8, 4.1.7, and 4.1.9 respectively □

It is well known that X is Martin-Löf random if and only if there is a computable infinite set $A \subseteq \mathbb{N}$ such that $\mathrm{KP}(X \restriction n) \geq^{+} n$ for all $n \in A$. While a similar statement does not necessarily hold for f-complexity, it is true for at least one infinite set, namely $\mathcal{I}(f)$. It is natural to ask which other recursive sets have this same property, and in particular which recursive subsets of $\mathcal{I}(f)$ would suffice for the following result.

Corollary 4.1.5. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. Then:

- X is KP-f-complex if and only if $\mathrm{KP}(X \restriction n) \geq^{+} f(X \restriction n)$ for all n such that $X \restriction n \in \mathcal{I}(f)$.
- X is KA-f-complex if and only if $\mathrm{KA}(X \restriction n) \geq^{+} f(X \restriction n)$ for all n such that $X \restriction n \in \mathcal{I}(f)$.

Proof. Assume without loss of generality that f is strongly recursive, by Theorem 3.4.18 and Lemma 4.1.4. Let g be defined by $g(\sigma) = f(\sigma)$ if $\sigma \in \mathcal{I}(f)$ and $g(\sigma) = 0$ otherwise. Then $\mathcal{I}(g) = \mathcal{I}(f)$ and $f \sim g$. Applying Lemma 4.1.4 again, each of the results holds. □

4.1.1 Connecting f-randomness and f-complexity

Just as Martin-Löf randomness is intimately linked to Kolmogorov complexity, each of the four notions of f-complex is tied to a notion of f-random.

Theorem 4.1.6. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. Then $X \in \{0,1\}^{\mathbb{N}}$ is dwt-f-random if and only if X is KP-f-complex.

Proof. By Lemma 3.1.3 and Claim 3.1.2 it suffices to prove the result for recursive $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ only; so assume that f is integer-valued.

First suppose that X is dwt-f-random. For each $k \in \mathbb{N}$ define

$$A_k = \{\sigma \mid \mathrm{KP}(\sigma) < f(\sigma) - k\}.$$

Since f is recursive and KP is right r.e. the sequence $\langle A_k \rangle$ is uniformly r.e. as well. For each k we have

$$\mathrm{dwt}_f(A_k) = \sum_{\sigma \in A_k} 2^{-f(\sigma)} < \sum_{\sigma \in A_k} 2^{-\mathrm{KP}(\sigma)-k} = 2^{-k} \sum_{\sigma \in A_k} 2^{-\mathrm{KP}(\sigma)} \leq 2^{-k}.$$

This last inequality holds due to Lemma 1.2.14. This shows that $\langle A_k \rangle$ is a dwt-f-test; since X is dwt-f-random there must be a k such that $X \notin [A_k]$. Then for this fixed k and for all n, we have $\mathrm{KP}(X \upharpoonright n) \geq f(X \upharpoonright n) - k$. So X is KP-f-complex.

Now suppose that X is KP-f-complex. Let $\langle A_k \rangle$ be an arbitrary dwt-f-test. Define $A \subseteq \{0,1\}^{<\mathbb{N}} \times \mathbb{N}$ by $A = \{(\sigma, f(\sigma) - k) \mid \sigma \in A_{2k}\}$. Since $\langle A_k \rangle$ is uniformly r.e. and f is recursive A is r.e. as well. Moreover

$$\begin{aligned}\sum_{(\sigma,n)\in A} 2^{-n} &= \sum_k \sum_{\sigma \in A_{2k}} 2^{-f(\sigma)+k} \\ &= \sum_k 2^k \mathrm{dwt}_f(A_{2k}) \leq \sum_k 2^{-2k+k} = \sum_k 2^{-k} < \infty.\end{aligned}$$

Then A is a Kraft-Chaitin set and so by Lemma 1.2.15 $\mathrm{KP}(\sigma) \leq^+ n$ for all $(\sigma, n) \in A$. Therefore $\mathrm{KP}(\sigma) \leq^+ f(\sigma) - k$ for all $\sigma \in A_{2k}$. Since X is KP-f-complex, there is a k such that $X \notin [A_{2k}]$. $\langle A_k \rangle$ was an arbitrary dwt-f-test, and so X is dwt-f-random. □

Theorem 4.1.7. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and let $X \in \{0,1\}^{\mathbb{N}}$. Then X is strongly KP-f-complex if and only if X is Solovay dwt-f-random.

Proof. By Lemma 3.1.3 and Claim 3.1.2 we may assume without loss of generality that f is integer-valued.

Let X be Solovay dwt-f-random and fix $k \in \mathbb{N}$. Define

$$A = \{\sigma \mid \mathrm{KP}(\sigma) < f(\sigma) + k\}.$$

A is r.e. since f is recursive and KP is right recursively enumerable. Also

$$\mathrm{dwt}_f(A) = \sum_{\sigma \in A} 2^{-f(\sigma)} < \sum_{\sigma \in A} 2^{-\mathrm{KP}(\sigma)+k} = 2^k \sum_{\sigma \in A} 2^{-\mathrm{KP}(\sigma)} \leq 2^k.$$

The last inequality follows from Lemma 1.2.14. This shows that A is a Solovay dwt-f-test, and so there is some N such that $X \upharpoonright n \notin A$ for all $n > N$. So $\mathrm{KP}(X \upharpoonright n) \geq f(X \upharpoonright n) + k$ for all $n > N$. Then

$$\liminf_{n \to \infty} (\mathrm{KP}(X \upharpoonright n) - f(X \upharpoonright n)) \geq k$$

for every k, and so X is strongly KP-f-complex.

Now suppose that X is strongly KP-f-complex. Let A be a Solovay dwt-f-test; without loss of generality we can assume that $\mathrm{dwt}_f(A) < 1$. Define a set $B = \{(\sigma, f(\sigma)) \mid \sigma \in A\}$. B is an r.e. subset of $\{0,1\}^{<\mathbb{N}} \times \mathbb{N}$ since A is r.e. and f is recursive. Additionally

$$\sum_{(\sigma,n) \in B} 2^{-n} = \sum_{\sigma \in A} 2^{-f(\sigma)} = \mathrm{dwt}_f(A) < 1$$

and so B is a Kraft-Chaitin set. By Lemma 1.2.15 we have $\mathrm{KP}(\sigma) \leq^+ n$ for all $(\sigma, n) \in B$. Therefore $\mathrm{KP}(\sigma) \leq^+ f(\sigma)$ or equivalently $\mathrm{KP}(\sigma) - f(\sigma) \leq^+ 0$ whenever $\sigma \in A$. Since X is strongly KP-f-complex

$$\liminf_{n \to \infty} (\mathrm{KP}(X \upharpoonright n) - f(X \upharpoonright n)) = \infty$$

and so there can be at most finitely many n such that $X \upharpoonright n \in A$. Since A was arbitrary, this proves that X is Solovay dwt-f-random. □

Theorem 4.1.8. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and $X \in \{0,1\}^{\mathbb{N}}$. Then X is pwt-f-random if and only if X is KA-f-complex.

Proof. Again, by Lemma 3.1.3 and Claim 3.1.2 it suffices to prove the theorem only for recursive $f:\{0,1\}^{<\mathbb{N}}\to\mathbb{N}$; so we assume that f is integer-valued. Suppose that X is pwt-f-random. For each $k\in\mathbb{N}$ define a set

$$A_k=\{\sigma \mid \mathrm{KA}(\sigma)<f(\sigma)-k\}\,.$$

Since f is recursive and KA is right r.e., the sequence $\langle A_k\rangle$ is uniformly r.e. For each k and prefix-free $P\subseteq A_k$ we have

$$\mathrm{dwt}_f(P)=\sum_{\sigma\in P}2^{-f(\sigma)}<\sum_{\sigma\in P}2^{-\mathrm{KA}(\sigma)-k}=2^{-k}\sum_{\sigma\in P}2^{-\mathrm{KA}(\sigma)}<2^{-k}.$$

The last inequality holds by Lemma 1.2.18. So $\mathrm{pwt}_f(A_k)\le 2^{-k}$ for each k, and thus $\langle A_k\rangle$ is a pwt-f-test. By assumption X is pwt-f-random and so there must be a $k\in\mathbb{N}$ such that $X\notin[A_k]$. For this k and for all n, $\mathrm{KA}(X\upharpoonright n)\ge f(X\upharpoonright n)-k$; therefore X is pwt-f-complex.

Now suppose that X is KA-f-complex, and let $\langle A_n\rangle$ be a pwt-f-test. Define a function $\delta:\{0,1\}^{<\mathbb{N}}\to[0,\infty)$ as follows:

$$\delta(\sigma)=\sum_k 2^{k-1}\mathrm{pwt}_f(A_{2k}\cap[\sigma]^{<\mathbb{N}})$$

for each σ. Then we have

$$\delta(\lambda)=\sum_k 2^{k-1}\mathrm{pwt}_f(A_{2k})\le\sum_k 2^{k-2k-1}=\sum_k 2^{-k-1}=1$$

and for all σ

$$\begin{aligned}\delta(\sigma)&=\sum_k 2^{k-1}\mathrm{pwt}_f(A_{2k}\cap[\sigma]^{<\mathbb{N}})\\ &\ge\sum_k 2^{k-1}\left[\mathrm{pwt}_f(A_{2k}\cap[\sigma^\frown 0]^{<\mathbb{N}})+\mathrm{pwt}_f(A_{2k}\cap[\sigma^\frown 1]^{<\mathbb{N}})\right]\\ &=\delta(\sigma^\frown 0)+\delta(\sigma^\frown 1).\end{aligned}$$

Observe that for any r.e. set B

$$\mathrm{pwt}_f(B)=\lim_{n\to\infty}\mathrm{pwt}_f(B^n)$$

and that $\mathrm{pwt}_f(B^n)$ is recursive uniformly in n, and so $\mathrm{pwt}_f(B)$ is r.e., uniformly in a representation of B. This implies that δ is an r.e. function, and an an r.e. continuous semimeasure in particular.

By Definition 1.2.17 we have $\mathrm{KA}(\sigma)\le^{+}-\log_2\delta(\sigma)$ for all σ. If $\sigma\in A_{2k}$ it is clear that $\mathrm{pwt}_f(A_{2k}\cap[\sigma]^{<\mathbb{N}})\ge\mathrm{dwt}_f(\sigma)=2^{-f(\sigma)}$ and so

$$\mathrm{KA}(\sigma)\le^{+}-\log_2\delta(\sigma)=-\log_2\mathrm{pwt}_f(A_{2k}\cap[\sigma]^{<\mathbb{N}})-k-1\le f(\sigma)-k-1.$$

Then since $\mathrm{KA}(X \upharpoonright n) \geq^+ f(X \upharpoonright n)$ there must be some k such that $X \notin [A_{2k}]$. It follows that X is pwt-f-random as required. $\square$

Theorem 4.1.9. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. Then $X \in \{0,1\}^{\mathbb{N}}$ is strongly KA-f-complex if and only if X is Solovay pwt-f-random.

Proof. By Lemma 3.1.3 and Claim 3.1.2, we can again assume that f is integer-valued. Suppose that X is Solovay pwt-f-random, and let $k \in \mathbb{N}$ be fixed. Define $A \subseteq \{0,1\}^{<\mathbb{N}}$ by

$$A = \{\sigma \mid \mathrm{KA}(\sigma) < f(\sigma) + k\}.$$

A is an r.e. set and if $P \subseteq A$ is prefix-free then

$$\mathrm{dwt}_f(P) = \sum_{\sigma \in P} 2^{-f(\sigma)} < \sum_{\sigma \in P} 2^{k - \mathrm{KA}(\sigma)} < 2^k$$

where the last inequality is due to Lemma 1.2.18. Therefore $\mathrm{pwt}_f(A) \leq 2^k$ and so A is a Solovay pwt-f-test. Since X is Solovay pwt-f-random there is an N such that $X \upharpoonright n \notin A$ when $n > N$. That is $\mathrm{KA}(X \upharpoonright n) \geq f(X \upharpoonright n) + k$ for all $n > N$, and therefore

$$\liminf_{n\to\infty} (\mathrm{KA}(X \upharpoonright n) - f(X \upharpoonright n)) \geq k.$$

Since k was arbitrary, X is strongly KA-f-complex.

Now suppose that X is strongly KA-f-complex and let A be a Solovay pwt-f-test such that $\mathrm{pwt}_f(A) < 2^k$. Define $\delta : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ by

$$\delta(\sigma) = 2^{-k}\mathrm{pwt}_f(A \cap [\sigma]^{<\mathbb{N}})$$

for all σ. Observe that $\delta(\lambda) = 2^{-k}\mathrm{pwt}_f(A) < 1$ and for any σ

$$\begin{aligned}\delta(\sigma) &= 2^{-k}\mathrm{pwt}_f(A \cap [\sigma]^{<\mathbb{N}}) \\ &\geq 2^{-k}\left[\mathrm{pwt}_f(A \cap [\sigma^\frown 0]^{<\mathbb{N}}) + \mathrm{pwt}_f(A \cap [\sigma^\frown 1]^{<\mathbb{N}})\right] = \delta(\sigma^\frown 0) + \delta(\sigma^\frown 1).\end{aligned}$$

Additionally, since A is r.e. each $A \cap [\sigma]^{<\mathbb{N}}$ is r.e. in σ. Noting that

$$\mathrm{pwt}_f(B) = \lim_{n\to\infty} \mathrm{pwt}_f(B^n)$$

and that $\mathrm{pwt}_f(B^n)$ is recursive uniformly in n for any r.e. set B, the function δ is recursively enumerable. In other words δ is an r.e. continuous semimeasure.

By Definition 1.2.17 we have $\mathrm{KA}(\sigma) \leq^{+} -\log_2 \delta(\sigma)$ for all σ. If $\sigma \in A$ it is clear that $\mathrm{pwt}_f(A \cap [\sigma]^{<\mathbb{N}}) \geq \mathrm{dwt}_f(\sigma) = 2^{-f(\sigma)}$ from which it follows that

$$\mathrm{KA}(\sigma) \leq^{+} -\log_2 \delta(\sigma) = -\log_2 \mathrm{pwt}_f(A \cap [\sigma]^{<\mathbb{N}}) + k \leq f(\sigma) + k.$$

Then if $A \cap X^{<\mathbb{N}}$ is infinite we have

$$\liminf_{n\to\infty} (\mathrm{KA}(X \upharpoonright n) - f(X \upharpoonright n)) \leq^{+} k$$

which is not the case since X is KA-f-complex. So $A \cap X^{<\mathbb{N}}$ is finite, and X is Solovay pwt-f-random. □

4.2 Martingales and betting strategies

Recall that the third paradigm for Martin-Löf randomness was in terms of r.e. betting strategies or martingales. We can define partial randomness in an analogous way.

Definition 4.2.1. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. A function $\delta : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is an ***f-supermartingale*** if

$$2^{-f(\sigma)}\delta(\sigma) \geq 2^{-f(\sigma^\frown 0)}\delta(\sigma^\frown 0) + 2^{-f(\sigma^\frown 1)}\delta(\sigma^\frown 1)$$

for all σ. For an f-supermartingale δ and $X \in \{0,1\}^{\mathbb{N}}$ say that ***δ succeeds on X*** if $\limsup_{n\to\infty} \delta(X \upharpoonright n) = \infty$. We say that $X \in \{0,1\}^{\mathbb{N}}$ is ***supermartingale f-random*** if there is no recursively enumerable f-supermartingale δ which succeeds on X.

Note that f-supermartingales are related to continuous semimeasures in the following way.

Lemma 4.2.2. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and let $\delta : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be such that $\delta(\lambda) \leq 1$. Then δ is an f-supermartingale if and only if the function $\psi : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ defined by $\psi(\sigma) = 2^{-f(\sigma)}\delta(\sigma)$ for all σ is a continuous semimeasure.

Proof. This is clear from the definitions of the two concepts. □

It follows that there is a close connection between a priori complexity and f-supermartingales as well.

Theorem 4.2.3. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. Then $X \in \{0,1\}^{\mathbb{N}}$ is KA-f-complex if and only if it is supermartingale f-random.

Proof. Let X be KA-f-complex and let δ be an r.e. f-supermartingale. Picking an integer $k > \delta(\lambda)$, it is clear that the function $\delta'(\sigma) = \delta(\sigma)/k$ is also an f-supermartingale and that δ' succeeds on X if and only if δ succeeds on X. So we can assume without loss of generality that δ satisfies $\delta(\lambda) \leq 1$. Then the function $\psi(\sigma) = 2^{-f(\sigma)}\delta(\sigma)$ is an r.e. continuous semimeasure. By Definition 1.2.17 we have

$$\mathrm{KA}(\sigma) \leq^{+} -\log_2 \psi(\sigma) = f(\sigma) - \log_2 \delta(\sigma)$$

for all σ. But by assumption $\mathrm{KA}(X \upharpoonright n) \geq^{+} f(X \upharpoonright n)$ for all n and so we must have $\log_2 \delta(X \upharpoonright n) \leq^{+} 0$ for all n. Therefore $\delta(X \upharpoonright n) \leq^{+} 0$ for all n as well. Then δ does not succeed on X and X is supermartingale f-random.

Now suppose that X is supermartingale f-random. By Lemma 4.2.2 and the definition of KA the function δ defined by $\delta(\sigma) = 2^{f(\sigma)-\mathrm{KA}(\sigma)}$ is an r.e. f-supermartingale. Then δ does not succeed on X and so

$$\delta(X \upharpoonright n) = 2^{f(X\upharpoonright n)-\mathrm{KA}(X\upharpoonright n)} \leq^{+} 0$$

for all n. Therefore $f(X \upharpoonright n) - \mathrm{KA}(X \upharpoonright n) \leq^{+} 0$ for all n as well. Equivalently $\mathrm{KA}(X \upharpoonright n) \geq^{+} f(X \upharpoonright n)$ and so X is KA-f-complex. □

Claim 3.1.7 for supermartingale f-randomness is a corollary of this.

Corollary 4.2.4. Let $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive such that $f \sim g$. Then any $X \in \{0,1\}^{\mathbb{N}}$ is f-supermartingale random if and only if it is g-supermartingale random.

Proof. A corollary of the previous result and Lemma 4.1.4. □

4.3 Weak separations

Up till now we have focused mainly on proving implications between randomness notions and a few very basic non-implications. In this section, we will weakly separate various partial randomness concepts. By weak separation, we mean find a single X which is f-random in one sense but not in another.

First we need a definition.

Definition 4.3.1. For $X \in \{0,1\}^{\mathbb{N}}$, a function $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is ***unbounded along X*** if the sequence $\{f(X \upharpoonright n) \mid n \in \mathbb{N}\}$ is unbounded. f is said to be ***strongly unbounded*** if it is unbounded along X for every $X \in \{0,1\}^{\mathbb{N}}$.

Now we are ready to show that dwt-f-randomness and Solovay dwt-f-randomness do not always coincide (unlike the case of Martin-Löf randomness and Solovay randomness).

Theorem 4.3.2. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive, strongly unbounded, and such that there exists a constant $t < 1$ for which $f(\sigma^\frown i) < f(\sigma) + t$ for all σ and i. Then there is an X which is KP-f-complex and not strongly KP-f-complex. Equivalently X is dwt-f-random and not Solovay dwt-f-random.

Proof. Note that any such f is strongly convex, but not every strongly convex f satisfies the necessary conditions. Fix $0 < t < 1$ as in the statement of the lemma.

Handling constants. Letting φ be the partial recursive function used to define KP, define a partial recursive $\psi : \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ as follows. Recall that we may use n to mean the string 0^n for convenience. $\psi(\langle\lambda, \langle\beta,\alpha\rangle\rangle) = \langle\sigma,\tau\rangle$ when $\varphi(\langle\lambda,\alpha\rangle) = \sigma^\frown\tau$ and $|\varphi(\langle\lambda,\beta\rangle)| = |\tau|$; otherwise ψ is undefined. In particular, if $\varphi(\langle\lambda,\rho\rangle) = \sigma^\frown\tau$ and $\varphi(\langle\lambda,\gamma\rangle) = 0^{|\tau|} = |\tau|$ then $\psi(\langle\lambda,\langle\gamma,\rho\rangle\rangle) = \langle\sigma,\tau\rangle$ and $|\langle\gamma,\rho\rangle| = 2|\gamma| + |\rho|$. Then by the additive optimality of φ, definition of $\langle\ ,\ \rangle$, and Lemma 1.2.12 we have

$$\mathrm{KP}(\sigma,\tau) \leq^{+} \mathrm{K}_{\psi}(\langle\sigma,\tau\rangle) \leq^{+} \mathrm{KP}(\sigma^\frown\tau) + 2\cdot\mathrm{KP}(|\tau|) \leq^{+} \mathrm{KP}(\sigma^\frown\tau) + 4\log_2|\tau|$$

for all σ and τ. Then let c be a specific constant such that $\mathrm{KP}(\sigma,\tau) \leq \mathrm{KP}(\sigma^\frown\tau) + 4\log_2|\tau| + c$ for all σ and τ. Also let d be a specific constant large enough for Theorem 1.2.16 to hold. Since $t > 0$ we can fix a $k \in \mathbb{N}$ large enough that $(1-t)k > 4\log_2 k + c + d$. Finally let e be a constant such that $\mathrm{KP}(\sigma^\frown\tau) \geq \mathrm{KP}(\sigma) - e$ for all σ and all $\tau \in \{0,1\}^{\leq k}$. Such a constant must exist since $\{0,1\}^{\leq k}$ is a finite set.

Building X. We will build a sequence $\langle\sigma_n\rangle$ such that $\sigma_{n+1} \supset \sigma_n$ for each n and then we will let $X = \bigcup_n \sigma_n$, the unique element of $\{0,1\}^{\mathbb{N}}$ such that $\sigma_n \subset X$ for each n. Pick σ_0 to be of length k such that $\mathrm{KP}(\sigma_0) \geq k$. For each $n \geq 0$, if $\mathrm{KP}(\sigma_n) \geq f(\sigma_n) + tk + e$, then define $\sigma_{n+1} = {\sigma_n}^\frown 0^k$. Otherwise, apply Lemma 1.2.13 to find a τ of length k such that $\mathrm{KP}(\tau|\langle\sigma_n, \mathrm{KP}(\sigma_n)\rangle) \geq k$; define $\sigma_{n+1} = {\sigma_n}^\frown\tau$.

Verifying that X is KP-f-complex. It is sufficient to show that $\mathrm{KP}(\sigma_n) \geq f(\sigma_n)$ for all n, since for any σ such that $\sigma_n \subset \sigma \subset \sigma_{n+1}$ the complexity of σ is within a constant of the complexity of σ_n, while $f(\sigma) < f(\sigma_n) + tk$. So let $n \geq 0$ be arbitrary. If $\mathrm{KP}(\sigma_n) \geq f(\sigma_n) + tk + e$ then

$$\mathrm{KP}(\sigma_{n+1}) \geq \mathrm{KP}(\sigma_n) - e \geq f(\sigma_n) + tk + e - e \geq f(\sigma_{n+1})$$

since $|\sigma_{n+1}| = |\sigma_n| + k$. On the other hand if $\mathrm{KP}(\sigma_n) < f(\sigma_n) + tk + e$ then by induction we can at least assume that $\mathrm{KP}(\sigma_n) \geq f(\sigma_n)$. In our construction we will have picked τ such that $\mathrm{KP}(\tau|\langle\sigma_n, \mathrm{KP}(\sigma_n)\rangle) \geq k$ and defined $\sigma_{n+1} = \sigma_n{}^\frown\tau$. Applying Theorem 1.2.16 to the earlier inequality we proved, we have

$$\begin{aligned}\mathrm{KP}(\sigma_{n+1}) = \mathrm{KP}(\sigma_n{}^\frown\tau) &\geq \mathrm{KP}(\sigma_n, \tau) - 4\log_2|\tau| - c \\ &\geq \mathrm{KP}(\sigma_n) + \mathrm{KP}(\tau|\langle\sigma_n, \mathrm{KP}(\sigma_n)\rangle) - 4\log_2|\tau| - c \\ &\geq f(\sigma_n) + k - 4\log_2 k - c - d \\ &> f(\sigma_n) + k - (1-t)k = f(\sigma_n) + tk > f(\sigma_{n+1}).\end{aligned}$$

This shows that X is KP-f-complex.

Verifying that X is not strongly KP-f-complex. Recall that f is strongly unbounded by assumption. Suppose that there are only finitely many n such that $\mathrm{KP}(\sigma_n) \leq f(\sigma_n) + tk + d$. Then there is an N such that $\sigma_{n+1} = \sigma_n{}^\frown 0^k$ for all $n > N$. In this case we have $X = \sigma_N{}^\frown 0^{\mathbb{N}}$, which is a recursive point in $\{0,1\}^{\mathbb{N}}$. Then since f is recursive and strongly unbounded, the function $g(n) = f(X \upharpoonright n)$ is recursive and unbounded as well. However, a recursive X cannot have its prefix-free complexity bounded from below by a recursive function. This leads to a contradiction, and so there are infinitely many n such that $\mathrm{KP}(\sigma_n) \leq f(\sigma_n)+tk+d$. Therefore $\liminf_{n\to\infty}(\mathrm{KP}(X \upharpoonright n)-f(n)) < \infty$ and so X is not strongly KP-f-complex. □

Note that we cannot eliminate the condition that f be strongly unbounded: if $f(\sigma) < c$ for some c and all σ, then every $X \in \{0,1\}^{\mathbb{N}}$ becomes trivially both KP-f-complex and Solovay KP-f-complex. However, Theorem 4.3.2 is an improvement over previously known versions of the result which required f to grow faster than a linear function as well. Here f can grow very slowly so long as it does grow eventually. The proof used in the above theorem is also a completely new technique.

Weak separations and KA

In some specific cases, weakly separating KA-f-complex and strongly KA-f-complex can be done quite simply. To do so, we harness the fact that if X is Martin-Löf random then it is KA-g-complex but not strongly KA-g-complex, where g is the function $g(\sigma) = |\sigma|$. Let us look at such an example.

Example 4.3.3. Let $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ be defined by $f(\sigma) = |\sigma|/2$ for all σ. Then if X is Martin-Löf random, $Y = X \oplus 0^{\mathbb{N}}$ is KA-f-complex but not strongly KA-f-complex.

Proof. Let δ be the r.e. continuous semimeasure used to define KA. Define a new r.e. continuous semimeasure γ by $\gamma(\sigma \oplus 0^{|\sigma|-1}) = \gamma(\sigma \oplus 0^{|\sigma|}) = \delta(\sigma)$ for all σ and $\gamma(\tau) = 0$ otherwise. From this definition and Definition 1.2.17 it follows that $\mathrm{KA}(\sigma \oplus 0^{|\sigma|-1}) =^{+} \mathrm{KA}(\sigma \oplus 0^{|\sigma|}) =^{+} \mathrm{KA}(\sigma)$ for every σ. In particular, if X is Martin-Löf random and $Y = X \oplus 0^{\mathbb{N}}$ then we have $\mathrm{KA}(Y \upharpoonright n) =^{+} n/2$ for each n. Then Y is KA-f-complex but not strongly KA-f-complex. $\square$

Staiger [47] showed that a similar argument works for any f of the form $f(\sigma) = \epsilon|\sigma|$ for recursive $0 < \epsilon < 1$. This method of splicing 0's into a Martin-Löf random will work so long as f is length-invariant. Dropping the length-invariance condition, we need to adopt a new technique. We will construct a recursive tree that acts similar to the mapping of an arbitrary X to the sequence $X \oplus 0^{\mathbb{N}}$.

We start with the integer-valued case.

Theorem 4.3.4. Let $f : \{0,1\}^{\mathbb{N}} \to \mathbb{N}$ be recursive and convex. There is an $X \in \{0,1\}^{\mathbb{N}}$ which is KA-f-complex but not strongly KA-f-complex.

Proof. Define a partial recursive tree-map $T :\subseteq \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ as follows. First let $T(\lambda) = \lambda$. Then for each σ such that $T(\sigma)$ is defined, effectively search for some $\tau \supseteq T(\sigma)$ such that $f(\tau) = f(T(\sigma))$ and $\{\tau^\frown 0, \tau^\frown 1\} \subset \mathcal{I}(f)$. If such a τ is found define $T(\sigma^\frown 0) = \tau^\frown 0$ and $T(\sigma^\frown 1) = \tau^\frown 1$. Note that if this process halts and $T(\sigma^\frown 0) = \tau^\frown 0$ and $T(\sigma^\frown 1) = \tau^\frown 1$ then τ satisfied $f(T(\sigma^\frown 0)) = f(T(\sigma^\frown 1)) = f(\tau)+1 = f(T(\sigma))+1$. Then T is clearly a partial recursive tree-map. So we simply ask whether or not T is total.

Case 1: T is total. Letting δ be the r.e. continuous semimeasure used to define KA, define a new r.e. continuous semimeasure γ by

$$\gamma(\tau) = \begin{cases} \sup\{\delta(\sigma) \mid \tau \subseteq T(\sigma)\} & \text{if } \tau \subseteq T(\sigma) \text{ for some } \sigma \\ 0 & \text{otherwise} \end{cases}$$

for each τ.

Then applying Definition 1.2.17 to γ we have $\mathrm{KA}(T(\sigma)) = \mathrm{KA}(\sigma)$ for all σ. Letting X be any Martin-Löf random, consider $Y = \bigcup_n T(X \upharpoonright n)$. If $Y \upharpoonright n \in \mathcal{I}(f)$, then careful consideration of the construction shows that $Y \upharpoonright n = T(X \upharpoonright m)$ for some m. Moreover, since $f(T(\lambda)) = f(\lambda)$ and $f(T(X \upharpoonright k+1)) = f(T(X \upharpoonright k))+1$ for all k, by induction we have $f(Y \upharpoonright n) = f(T(X \upharpoonright m)) = f(\lambda) + m$. Then we have for all $Y \upharpoonright n \in \mathcal{I}(f)$ the equality

$$\begin{aligned} \mathrm{KA}(Y \upharpoonright n) &= \mathrm{KA}(T(X \upharpoonright m)) \\ &=^{+} \mathrm{KA}(X \upharpoonright m) =^{+} m =^{+} f(Y \upharpoonright n) - f(\lambda) =^{+} f(Y \upharpoonright n). \end{aligned}$$

By Corollary 4.1.5 this implies that X is KA-f-complex. It is clear though that X cannot be strongly KA-f-complex since we have tight equality infinitely often.

Case 2: T is not total. Pick σ such that $T(\sigma)$ is defined but T is not defined on any extension of σ. We define a sequence $\langle\sigma_n\rangle$ as follows, starting with $\sigma_0 = \sigma$. Inductively assume that $f(\sigma_n) \le f(\sigma)$. If $f(\sigma_n\widehat{\ }0) \le f(\sigma)$ then let $\sigma_{n+1} = \sigma_n\widehat{\ }0$. Otherwise, since $T(\sigma\widehat{\ }i)$ is undefined we must have $f(\sigma_n\widehat{\ }1) \le f(\sigma)$. Then let $\sigma_{n+1} = \sigma_n\widehat{\ }1$.

By induction $f(\sigma_n) \le f(\sigma)$ for all n. So let $X = \bigcup_n \sigma_n$. X is recursively constructed and so $\mathrm{KA}(X \upharpoonright n) \le^+ 0$ for all n. It is trivial that $\mathrm{KA}(X \upharpoonright n) \ge^+ f(X \upharpoonright n)$ for all n and so X is KA-f-complex. But on the other hand X cannot be strongly KA-f-complex since $\mathrm{KA}(X \upharpoonright n) - f(X \upharpoonright n) \le^+ 0$ for all n as well. This completes the proof. $\square$

While the above proof does not work for convex real-valued functions, it does at least prove the same result for all strongly convex functions. The question for all convex functions is still open, as for non-convex functions.

Corollary 4.3.5. Let $f : \{0,1\}^{\mathbb{N}} \to [0,\infty)$ be recursive and strongly convex. There is an X which is KA-f-complex but not strongly KA-f-complex.

Proof. Apply Theorem 3.4.17 to f and then apply Theorem 4.3.4 to the resulting function. $\square$

Chapter 5

Strong separations and extracting randomness

So far we have established so-called **weak separations** of several partial randomness notions. Among these weak separations, two stand out:

1. In Theorem 4.3.2 we proved that under the correct conditions on f we can construct an X which is KP-f-complex (equivalently dwt-f-random) but not strongly KP-f-complex (equivalently Solovay dwt-f-random).

2. We also established in Theorem 4.3.4 that for many f an X can be made which is KA-f-complex (equivalently pwt-f-random) and not strongly KA-f-complex (equivalently Solovay pwt-f-random).

The latter result was proved as follows: we built a recursive perfect tree T that branches as fast as f grows; the natural mapping of $\{0,1\}^{\mathbb{N}}$ onto T is such that any Martin-Löf random Y maps to an X with the desired properties. In that case, though, the constructed X still easily computes Y, using the recursive inverse of the functional which generated X in the first place. This raises the question of the possibility of **strong separation**:

Question 5.0.6. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. Is there an X which is KA-f-complex but does not compute any Y which is strongly KA-f-complex?

The same question can of course be asked for KP in place of KA. Here are two cases where the answer is known:

1. X is Martin-Löf random if and only if $\mathrm{KA}(X \restriction n) \geq^+ n$ for all n. So if the function f is given by $f(\sigma) = |\sigma|$ for all σ, then any Martin-Löf

random is KA-f-complex. On the other hand no $X \in \{0,1\}^{\mathbb{N}}$ can be strongly KA-f-complex (since $\mathrm{KA}(\sigma) \leq^+ |\sigma|$ for all σ). So the notions of KA-f-complex and strongly KA-f-complex are strongly separated.

2. Let f be as above but consider KP instead. X is Martin-Löf random if and only if it is Solovay random; stated in our language, this means that no X can be dwt-f-random without additionally being Solovay dwt-f-random. Then the strong separation of dwt-f-randomness and Solovay dwt-f-randomness is not possible.

Other than the above cases, nothing is known about the possibility of strong separations between dwt/pwt-f-randomness and Solovay dwt/pwt-f-randomness. In this chapter, we look at a more tractable question of strong separation:

Question 5.0.7. For which recursive $f, g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ does there exist an X which is KA-f-complex but for which no $Y \leq_T X$ is KA-g-complex?

This question is related, since if Y is strongly KA-f-complex, then there is a (not necessarily recursive) function g such that $\mathrm{KA}(Y \upharpoonright n) \geq g(Y \upharpoonright n)$ for all n and $\lim_{n\to\infty}(g(Y \upharpoonright n) - f(Y \upharpoonright n)) = \infty$. The issue is that this lower bound g is not effectively given. By looking only at recursive such lower bounds, the problem becomes easier to approach. Our main result is the following, where the exact meaning of these terms will be made precise later:

Theorem 5.0.8. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive, convex, length-invariant, and strongly unbounded such that $f \not\approx h$ where $h(\sigma) = |\sigma|$ for all σ. There is an X which is KA-f-complex but no $Y \leq_T X$ is KA-g-complex for any recursive g such that $f \ll_r g$.

We will prove the analogous strong separation result for KP in place of KA as well. The proof of the theorem will use an adaptation of the method used by Miller [31] to prove the existence of a Turing degree of effective Hausdorff dimension 1/2.

Since the statement of the theorem is somewhat technical, examples illustrate the type of strong separations that follow from the theorem. Several examples are:

1. There is an X_1 such that $\mathrm{KA}(X_1 \upharpoonright n) \geq^+ n/2$ for all n but no $Y \leq_T X_1$ satisfies $\mathrm{KA}(Y \upharpoonright n) \geq^+ n/2 + 2\log_2 n$ for all n.

2. There is an X_2 such that $\mathrm{KA}(X_2 \upharpoonright n) \geq^+ \sqrt[3]{n}$ for all n but no $Y \leq_T X_2$ satisfies $\mathrm{KA}(Y \upharpoonright n) \geq^+ \sqrt{n}$ for all n.

3. There is an X_3 such that $\mathrm{KA}(X_3 \upharpoonright n) \geq^+ \log_2 n$ for all n but no $Y \leq_T X_3$ satisfies $\mathrm{KA}(Y \upharpoonright n) \geq^+ \log_2 n + 2\log_2 \log_2 n$ for all n.
4. There is an X_4 such that $\mathrm{KA}(X_4 \upharpoonright n) \geq^+ n - 2\log_2 n$ for all n but no $Y \leq_T X_4$ is Martin-Löf random.

Much of this chapter will be spent creating the tools and techniques necessary to prove Theorem 5.0.8.

No general theorem is possible in the non-length-invariant case

One should note in the statement of Theorem 5.0.8 that we are requiring f to be length-invariant. One might ask if this condition is truly necessary. The other goal of this chapter is to show that the condition cannot be removed.

Example 5.0.9. There is a recursive, convex, strongly unbounded, non-length-invariant f satisfying: if X is KA-f-complex, then there is a $Y \leq_T X$ which is KA-$(2f)$-complex, that is $\mathrm{KA}(Y \upharpoonright n) \geq^+ 2f(Y \upharpoonright n)$ for all n.

So we can actually extract quite a bit of randomness from this X, although as it turns out this is not really the same as extracting information.

5.1 Forcing with optimal f-covers

5.1.1 Optimal f-covers in the length-invariant case

In Section 3.4 we studied a generalization of the optimal covers Miller [31] first made use of. In brief, we showed that when a recursive f is convex, optimal f-covers always exist and can be found by looking at boundedly optimal f-covers. Since the strong separations we prove in this chapter hold only for length-invariant functions, we can simplify some of our notation.

Definition 5.1.1. Consider an arbitrary length-invariant strongly recursive function $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ (since any recursive function is equivalent to a strongly recursive one). The function f induces a recursive function $f^* : \mathbb{N} \to [0,\infty)$ by $f^*(n) = f(0^n)$ for all n. The function f^* tells us all values of f, since $f(\sigma) = f^*(|\sigma|)$ for any σ.

Consider the increasing set of f. We can define the similar set

$$\mathcal{I}^*(f) = \{n \mid (\forall m < n)(f^*(m) < f^*(n))\}.$$

Note that $\sigma \in \mathcal{I}(f)$ if and only if $|\sigma| \in \mathcal{I}^*(f)$; this set is recursive, as $\mathcal{I}(f)$ is.

Finally, recall that f is convex if $\mathrm{dwt}_f(\sigma) \le \mathrm{dwt}_f(\{\sigma^\frown 0, \sigma^\frown 1\})$ for all σ. This is equivalent to writing $2^{-f^*(n)} \le 2^{-f^*(n+1)+1}$ or $f^*(n+1) \le f^*(n) + 1$ for all n. So say that a function $g : \mathbb{N} \to [0,\infty)$ is ***convex*** if there is a convex and length-invariant $h : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ such that $g = h^*$. In other words, our length-invariant f is convex exactly when f^* is. Recall by Lemma 3.4.15 that length-invariant $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ is convex if and only if it is strongly convex; so in the case of functions $f : \mathbb{N} \to [0,\infty)$ we do not need to make a distinction between convex and strongly convex.

In this part of the chapter, all functions considered are length-invariant. So for the sake of simplicity, we ignore the distinction between f^* and f, between $\mathcal{I}(f)$ and $\mathcal{I}^*(f)$, and between the notions of convex for functions on $\{0,1\}^{<\mathbb{N}}$ and on $\mathbb{N}$. For example, for $f : \mathbb{N} \to [0,\infty)$, to say that X is dwt-f-random would mean that X is dwt-$\tilde{f}$-random for $\tilde{f} : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ defined by $\tilde{f}(\sigma) = f(|\sigma|)$ for all σ. There is also an obvious way to talk about equivalence for functions $f, g : \mathbb{N} \to [0,\infty)$.

Definition 5.1.2. Let $f, g : \mathbb{N} \to [0,\infty)$ be recursive functions. We say that ***f is equivalent to g***, denoted $f \approx g$, if $|f(n) - g(n)| \le^+ 0$ for all n. We say that ***f is weakly equivalent to g***, denoted $f \sim g$, if $f(n) \le^+ g(n)$ for all $n \in \mathcal{I}(f)$ and $g(n) \le^+ f(n)$ for all $n \in \mathcal{I}(g)$.

For any $f, g : \mathbb{N} \to [0,\infty)$ we can define $f', g' : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ by $f'(\sigma) = f(|\sigma|)$ and $g'(\sigma) = g(|\sigma|)$ for all σ. Then the above definitions are such that $f' \approx g'$ if and only if $f \approx g$ and $f' \sim g'$ if and only if $f \sim g$.

It follows from Claim 3.1.2 that if f and g are equivalent, they are interchangeable in all randomness and complexity notions. For example if $f \sim \mathrm{id}_{\mathbb{N}}$ where $\mathrm{id}_{\mathbb{N}}(n) = n$ for all n, then dwt-f-randomness and pwt-f-randomness coincide with Martin-Löf randomness.

The case when $f \sim \mathrm{id}_{\mathbb{N}}$ is of the least interest to us, since Martin-Löf randomness is well understood. Also, the identity $\mathrm{id}_{\mathbb{N}}$ grows as fast as possible for a convex function; if some convex f is not equivalent to $\mathrm{id}_{\mathbb{N}}$, then it lags behind $\mathrm{id}_{\mathbb{N}}$ eventually.

Lemma 5.1.3. Let $f : \mathbb{N} \to [0,\infty)$ be recursive and convex. If $f \not\sim \mathrm{id}_{\mathbb{N}}$ then $\lim_{n\to\infty} n - f(n) = \infty$.

Proof. This is obvious. □

Definition 5.1.4. Let $f : \mathbb{N} \to \mathbb{N}$ be recursive and convex and $g : \mathbb{N} \to [0,\infty)$ be recursive. We say that ***f is recursively strongly dominated***

by g, denoted $f \ll_r g$, if $\sum_{n\in\mathcal{I}(f)} 2^{f(n)-g(n)} < \infty$ and this sum converges to a recursive number.

For general recursive convex $f : \mathbb{N} \to [0,\infty)$ let $\Phi(n,k)$ be the rational-valued function approximating $f(n)$ to within 2^{-k}. Letting $f' : \mathbb{N} \to \mathbb{N}$ be the function $f'(n) = \min\{f'(n-1), \Phi(n,0)\}$ for all n, we say that $f \ll_r g$ if and only if $f' \ll_r g$. Note that f' is convex as well by essentially the same argument as Lemma 3.4.17.

If $f \ll_r g$ and there are recursive $f', g' : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ such that $f'(\sigma) = f(|\sigma|)$ and $g'(\sigma) = g(|\sigma|)$ for all σ, then it follows from Lemma 3.3.18 that $f' \ll g'$ as well. However, the converse does not hold as we are requiring a sum to not only converge but be recursive as well.

For any $f : \mathbb{N} \to [0,\infty)$, the functions

- $g_1(n) = (1+\epsilon)f(n)$ for fixed recursive $\epsilon > 0$,
- $g_2(n) = f(n) + \sqrt{f(n)}$, and
- $g_3(n) = f(n) + 2\log_2 f(n)$

are all examples of functions which recursively strongly dominate f.

By adding this extra condition of having a recursive sum, we also get the following strengthening of Lemma 3.3.16.

Corollary 5.1.5. Let $f, g : \mathbb{N} \to [0,\infty)$ be recursive and such that $f \ll_r g$, and let A be such that $|\sigma| \in \mathcal{I}(f)$ for all $\sigma \in A$. There is a recursive function $p : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{dwt}_g(A \cap \{0,1\}^{>p(k)}) \le 2^{-k} \cdot \mathrm{pwt}_f(A)$ for all k.

Proof. Looking at the proof of Lemma 3.3.16, it is clear that a function p with the above property exists. Since $\sum_n 2^{f(n)-g(n)}$ is recursive, we can find such a p recursively. The same p will then work for all such sets A. □

Let us now restate for length-invariant functions the useful properties of optimal f-covers that we originally proved throughout Section 3.4.

Lemma 5.1.6. Let $f : \mathbb{N} \to [0,\infty)$ be recursive and convex, let $\sigma \in \{0,1\}^{\mathbb{N}}$ and let $A \subseteq \{0,1\}^{<\mathbb{N}}$ be recursively enumerable.

1. There is an r.e. set $\mathbb{A}_f(A)$, called the approximate f-cover of A, and a set $\mathbb{O}_f(A)$, called the optimal f-cover of A, such that

   (a) $[\mathbb{O}_f(A)] = [\mathbb{A}_f(A)] \supseteq [A]$,

   (b) $\mathbb{O}_f(A) = \widehat{\mathbb{A}_f(A)}$, and

(c) $\mathrm{vwt}_f(A) = \mathrm{pwt}_f(\mathbb{O}_f(A)) = \mathrm{dwt}_f(\mathbb{A}_f(A))$.

2. If $\sigma \in \mathcal{I}(f)$ and $\mathrm{dwt}_f(\sigma) > \mathrm{vwt}_f(A)$ then $[\sigma] \nsubseteq [\mathbb{O}_f(A)]$.

3. If $[\sigma] \nsubseteq [\mathbb{O}_f(A)]$ then $\mathbb{O}_f(A \cap [\sigma]^{<\mathbb{N}}) = \mathbb{O}_f(A) \cap [\sigma]^{<\mathbb{N}}$.

Proof. These are respectively restatements of the following: Definition 3.4.23 and Theorem 3.4.12, Lemma 3.4.21, and Lemma 3.4.24. □

5.1.2 Defining f-conditions

The forcing conditions used by Miller [31] are effectively closed sets whose measures have small effective Hausdorff dimension. Our forcing conditions are very closely related, but instead we will need that the measure of an f-condition is not dwt-g-random for any g such that $f \ll_r g$.

Definition 5.1.7. Let $f : \mathbb{N} \to [0, \infty)$ be recursive and convex. An ***f*-condition** is an ordered pair $\langle \sigma, S \rangle$ such that

1. $\sigma \in \{0,1\}^{<\mathbb{N}}$,

2. $S \subseteq [\sigma]^{<\mathbb{N}}$ is r.e., and

3. $P_{\langle \sigma, S \rangle} = [\sigma] \setminus [\mathbb{O}_f(S)] \neq \varnothing$.

In other words, S cannot be so large that the optimal f-cover of S covers all of $[\sigma]$. $P_{\langle \sigma, S \rangle}$, the part of $[\sigma]$ not covered by the optimal f-cover of S, is effectively closed since $[\mathbb{O}_f(S)] = [\mathbb{A}_f(S)]$ is effectively open.

We can put a partial order on the f-conditions.

Definition 5.1.8. Let $f : \mathbb{N} \to [0, \infty)$ be recursive and convex. If $\langle \sigma, S \rangle$ and $\langle \tau, T \rangle$ are f-conditions, we say that $\boldsymbol{\langle \tau, T \rangle}$ ***extends*** $\boldsymbol{\langle \sigma, S \rangle}$, denoted $\langle \tau, T \rangle \preccurlyeq \langle \sigma, S \rangle$, when $P_{\langle \tau, T \rangle} \subseteq P_{\langle \sigma, S \rangle}$.

5.1.3 Properties of f-conditions

In this section we will develop the basic properties of f-conditions and their corresponding effectively closed sets. For example, we would like to know in what ways an f-condition can be extended.

First we show that any f-condition has trivial extensions. Namely, if $\tau \supset \sigma$ and $[\tau] \nsubseteq [\mathbb{O}_f(S)]$ as well, we can immediately form an extending condition.

Lemma 5.1.9. Let $f : \mathbb{N} \to [0, \infty)$ be recursive and convex. If $\langle \sigma, S\rangle$ is an f-condition and $\tau \supseteq \sigma$ is such that $[\tau] \nsubseteq [\mathbb{O}_f(S)]$, then $\langle \tau, S \cap [\tau]^{<\mathbb{N}}\rangle$ is an f-condition and $\langle \tau, S \cap [\tau]^{<\mathbb{N}}\rangle \preccurlyeq \langle \sigma, S\rangle$.

Proof. Applying part (2) of Lemma 5.1.6 we know that

$$\mathbb{O}_f(S \cap [\tau]^{<\mathbb{N}}) = \mathbb{O}_f(S) \cap [\tau]^{<\mathbb{N}}$$

where the enumeration of $S \cap [\tau]^{<\mathbb{N}}$ is inherited from S. From this it follows that $[\tau] \nsupseteq [\mathbb{O}_f(S \cap [\tau]^{<\mathbb{N}})]$ and so $\langle \tau, S \cap [\tau]^{<\mathbb{N}}\rangle$ is an f-condition. Also

$$\begin{aligned} P_{\langle \tau, S\cap[\tau]^{<\mathbb{N}}\rangle} &= [\tau] \setminus [\mathbb{O}_f(S \cap [\tau]^{<\mathbb{N}})] \\ &= [\tau] \setminus [\mathbb{O}_f(S)] \subseteq [\sigma] \setminus [\mathbb{O}_f(S)] = P_{\langle \sigma, S\rangle} \end{aligned}$$

and so $\langle \tau, S \cap [\tau]^{<\mathbb{N}}\rangle \preccurlyeq \langle \sigma, S\rangle$ as required. □

An immediate corollary of this is that any f-condition has infinitely many extensions: if $[\sigma] \nsubseteq [\mathbb{O}_f(S)]$ then there are arbitrarily long $\tau \supset \sigma$ such that $[\tau] \nsubseteq [\mathbb{O}_f(S)]$ as well.

We already noted that if $\langle \sigma, S\rangle$ is an f-condition, the set $P_{\langle \sigma, S\rangle}$ is necessarily non-empty. The following lemma proves a much stronger result: this set has non-zero measure as well.

Lemma 5.1.10. Let $f : \mathbb{N} \to [0, \infty)$ be a recursive convex function such that $f \not\equiv \mathrm{id}_{\mathbb{N}}$. If $\langle \sigma, S\rangle$ is an f-condition, then $\mu(P_{\langle \sigma, S\rangle}) > 0$.

Proof. If $\mathbb{O}_f(S)$ is a finite set, then $P_{\langle \sigma, S\rangle}$ is a non-empty clopen set and so has positive measure. So we may assume that $\mathbb{O}_f(S)$ is an infinite set. Since f is not equivalent to the identity function $\mathrm{id}_{\mathbb{N}}$, $(\forall c)(\exists n)(f(n) < n - c)$. Combined with the fact that f is convex, $(\exists^\infty n)(f(n+1) < f(n) + 1)$.

Let $k = |\sigma|$. Since $\mathbb{O}_f(S) \neq \{\sigma\}$, $\mathrm{vwt}_f(S) \leq \mathrm{dwt}_f(\sigma) = 2^{-f(k)}$. Additionally, $|\tau| > |\sigma| = k$ for all $\tau \in \mathbb{O}_f(S)$. Using induction on the definition of convex, it follows that $f(|\tau|) \leq f(k) + |\tau| - k$ for all $\tau \in \mathbb{O}_f(S)$. Moreover since $\mathbb{O}_f(S)$ is infinite it contains arbitrarily long strings; so for many $\tau \in \mathbb{O}_f(S)$ we will actually have $f(|\tau|) < f(k) + |\tau| - k$ as well. Then observe that

$$\begin{aligned} \mu([\mathbb{O}_f(S)]) &= \sum_{\tau \in \mathbb{O}_f(S)} 2^{-|\tau|} \\ &< \sum_{\tau \in \mathbb{O}_f(S)} 2^{-k+f(k)-f(|\tau|)} \\ &= 2^{-k+f(k)} \mathrm{dwt}_f(\mathbb{O}_f(S)) = 2^{-k+f(k)} \mathrm{vwt}_f(S) \leq 2^{-k} = \mu([\sigma]). \end{aligned}$$

From this it follows that $\mu([\sigma] \setminus [\mathbb{O}_f(S)]) > 0$ as required. □

For completeness, let us illustrate why f must not be equivalent to the function $\mathrm{id}_{\mathbb{N}}$: Consider the function $f = \mathrm{id}_{\mathbb{N}}$, $\sigma = \lambda$ and $S = \{0^k \mid k \geq 1\}$. Then while $[\sigma] \not\supseteq [\mathbb{O}_f(S)]$ and so $\langle \sigma, S\rangle$ is an f-condition, we also have $[\sigma] \setminus [\mathbb{O}_f(S)] = \{1^{\mathbb{N}}\}$ which has measure 0.

The fact that f-conditions correspond to effectively closed sets of positive measure provides us with a second way to extend f-conditions, one which allows us to extend a great number of finite conditions simultaneously.

Lemma 5.1.11. Let $f : \mathbb{N} \to [0, \infty)$ be a recursive, convex, and strongly unbounded function such that $f \not\approx \mathrm{id}_{\mathbb{N}}$. Let $\langle \sigma_1, S_1\rangle, \dots, \langle \sigma_n, S_n\rangle$ be f-conditions such that $\mu(P_{\langle \sigma_1, S_1\rangle} \cap \cdots \cap P_{\langle \sigma_n, S_n\rangle}) > 0$. Then there is an f-condition $\langle \tau, T\rangle$ such that $\langle \tau, T\rangle \preccurlyeq \langle \sigma_i, S_i\rangle$ for $1 \leq i \leq n$.

Proof. Clearly $\sigma_i \asymp \sigma_j$ for each i and j; so define $\sigma = \bigcup_{i \leq n} \sigma_i$, the shortest element of $\{0,1\}^{<\mathbb{N}}$ such that $\sigma_i \subseteq \sigma$ for each i. Then let

$$P = P_{\langle \sigma_1, S_1\rangle} \cap \cdots \cap P_{\langle \sigma_n, S_n\rangle} = [\sigma] \setminus [\mathbb{O}_f(S_1) \cup \cdots \cup \mathbb{O}_f(S_n)].$$

$\mu(P)$ is positive by assumption, so fix a $b \in \mathbb{N}$ such that $\mu(P) \geq 2^{-b}$. For each $m \in \mathcal{I}(f) \cap \{k \mid k > b\}$ define

$$D_m = \{\tau \supseteq \sigma \mid (|\tau| = m) \wedge \neg(\exists \rho \subseteq \tau)(\rho \in \mathbb{O}_f(S_1) \cup \cdots \cup \mathbb{O}_f(S_n)\}.$$

If $|\tau| = m$ but $\tau \notin D_m$ then either $\tau \not\supseteq \sigma$ or $[\tau] \subseteq [\mathbb{O}_f(S_i)]$ for some i. Either of these possibilities implies $P \cap [\tau] = \varnothing$. Since this is true for any τ of length m, we have $P \subseteq [D_m]$. Therefore $2^{-b} \leq \mu(P) \leq \#(D_m)2^{-m}$ and so $\#(D_m) \geq 2^{m-b}$.

Next, for each $\tau \in D_m$ define $T_\tau = \{\sigma \supseteq \tau \mid \sigma \in \mathbb{A}_f(S_1) \cup \cdots \cup \mathbb{A}_f(S_n)\}$. Since $f \not\approx \mathrm{id}_{\mathbb{N}}$ the sequence $\langle k - f(k) \mid k \in \mathbb{N}\rangle$ tends to positive infinity. For fixed m observe that

$$\begin{aligned} n \geq \sum_{i=1}^{n} \mathrm{vwt}_f(S_i) &= \sum_{i=1}^{n} \mathrm{dwt}_f(\mathbb{O}_f(S_i)) \\ &\geq \mathrm{dwt}_f(\mathbb{O}_f(S_1) \cup \cdots \cup \mathbb{O}_f(S_n)) \\ &\geq \sum_{\tau \in D_m} \mathrm{dwt}_f((\mathbb{O}_f(S_1) \cup \cdots \cup \mathbb{O}_f(S_n)) \cap [\tau]^{<\mathbb{N}}) \\ &\geq \sum_{\tau \in D_m} \mathrm{vwt}_f(T_\tau). \end{aligned}$$

Note that $\mathrm{vwt}_f(T_\tau) \leq 2^{-f(m)}$ for all $\tau \in D_m$. Assume that every $\tau \in D_m$ also satisfies $\mathrm{vwt}_f(T_\tau) = 2^{-f(m)}$. Then we have

$$n \geq \sum_{\tau \in D_m} \mathrm{vwt}_f(T_\tau) = \sum_{\tau \in D_m} 2^{-f(m)} \geq 2^{m-b} 2^{-f(m)} = 2^{(m-f(m))-b}.$$

But $n < 2^{(m-f(m))-b}$ for some $m \in \mathcal{I}(f)$, which is a contradiction. So for this m and some $\tau \in D_m$ we must have $\mathrm{dwt}_f(\tau) > \mathrm{vwt}_f(T_\tau)$. Then $[\tau] \notin [\mathbb{O}_f(T_\tau)]$ by Lemma 3.4.21. Since T_τ is an r.e. set, $\langle \tau, T_\tau \rangle$ is an f-condition. For each $1 \le i \le n$ we have

$$P_{\langle \tau, T_\tau \rangle} \subseteq [\tau] \smallsetminus [\mathbb{O}_f(S_1) \cup \cdots \cup \mathbb{O}_f(S_n)] \subseteq P \subseteq [\sigma_i] \smallsetminus [\mathbb{O}_f(S_i)] = P_{\langle \sigma_i, S_i \rangle}$$

and so $\langle \tau, T_\tau \rangle \preccurlyeq \langle \sigma_i, S_i \rangle$ for all $1 \le i \le n$ as required. □

The final property we wish to verify for f-conditions is that, while the measure is positive, the measure itself has restricted partial randomness. For this to make sense, we need the following definition.

Definition 5.1.12. For a number $x \in [0,1]$ define $x^\star \in \{0,1\}^{\mathbb{N}}$ as follows. For a fixed binary digit expansion of x, $x^\star(n)$ is defined to be the $n+1$ place binary digit of x. If x has two possible binary digit expansions, then we prefer the one ending in all 0's rather than ending in all 1's.

Now we can say what it means for the measure to have restricted partial randomness.

Lemma 5.1.13. *Let $f : \mathbb{N} \to \mathbb{N}$ be recursive, convex, and strongly unbounded, let $\langle \sigma, S \rangle$ be an f-condition, and $g : \mathbb{N} \to [0, \infty)$ be such that $f \ll_r g$. Then $\mu(P_{\langle \sigma, S \rangle})^\star$ is not dwt-g-random.*

Proof. For convenience, let $A = \mathbb{A}_f(S)$; note that if A is finite this problem becomes trivial, and so we may assume that A is infinite. We may also assume that A is enumerated without repetition. We will show that $\mu([A])^\star$ is not dwt-g-random. This will prove the stated result, since g is length-invariant, $\mu(P_{\langle \sigma, S \rangle}) = \mu([\sigma]) - \mu([A])$, and $\mu([\sigma])$ is a diadic rational.

Let p be the recursive function guaranteed by Corollary 5.1.5; note that $\mathrm{dwt}_g(A \cap \{0,1\}^{>p(k)}) \le 2^{-k}$ for each k, since $\mathrm{pwt}_f(A) \le 1$. We will define a dwt-g-test $\langle B_k \rangle$ by building sets B_k^t uniformly in k and t and letting $B_k = \bigcup_t B_k^t$.

Defining B_k^t. Define $a_1 = \mu([A^1])$ and $b_1 = 2\mu([A^1])$; note that $b_1 - a_1 = 2^{-|\tau_1|}$ where $A^1 = \{\tau_1\}$. For each $t > 1$ let τ_t be the unique element of $A^t \smallsetminus A^{t-1}$. If $|\tau_t| \le p(k+1)$ then define $a_t = b_t = b_{t-1}$. Otherwise, if $|\tau_t| > p(k+1)$ and $|\tau_{t-1}| \le p(k+1)$ then define $a_t = \mu([A^t])$ and $b_t = \mu([A^t]) + 2^{-|\tau_t|}$. Lastly, if $|\tau_t| > p(k+1)$ and $|\tau_{t-1}| > p(k+1)$ as well, then define $a_t = b_{t-1}$ and $b_t = a_t + 2^{-|\tau_t|} = b_{t-1} + 2^{-|\tau_t|}$.

Now define $B_{k,0} = \varnothing$. For each t such that $a_t = b_t$, define $B_k^t = B_k^{t-1}$. For t such that $a_t \ne b_t$ define $B_k^t = B_k^{t-1} \cup \{\rho_1, \rho_2\}$ where each ρ_i is picked as follows: $\rho_1 = a_t^\star \upharpoonright |\tau_t|$ and ρ_2 is the lexicographical successor of ρ_1 in $\{0,1\}^{|\tau_t|}$. If such a lexicographical successor does not exist, then let $\rho_2 = \rho_1$. Note that if $a_t \le x \le b_t$ then $x^\star \in [\{\rho_1, \rho_2\}]$.

Proving that $\langle B_k \rangle$ is a dwt-g-test. It is clear that B_k^t is defined uniformly in k and t. Therefore the sequence $\langle B_k \rangle$ is uniformly r.e. Additionally

$$\operatorname{dwt}_g(B_k) \le \sum_{\substack{\tau \in A \\ |\tau| > p(k+1)}} 2 \cdot 2^{-g(|\tau|)} = 2 \cdot \operatorname{dwt}_g(A \cap \{0,1\}^{>p(k+1)}) \le 2^{-k}.$$

This completes the verification that $\langle B_k \rangle$ is a dwt-g-test.

Proving that $\langle B_k \rangle$ covers $\mu([A])^\star$. Since $\{0,1\}^{\le p(k+1)}$ is finite, let u be the largest stage such that $|\tau_{u-1}| \le p(k+1)$, or 1 if no such stage exists. In the construction we will have $a_u = \mu([A^u])$ and $b_u = \mu([A^u]) + 2^{-|\tau_u|}$. For all $t > u$ it holds that $|\tau_t| > p(k+1)$ and $|\tau_{t-1}| > p(k+1)$, and so $a_t = b_{t-1}$ and $b_t = a_t + 2^{-|\tau_t|}$. Note that $b_u - \mu([A_u]) = 2^{-|\tau_u|}$ by definition. Then for $t > u$

$$b_t - \mu([A_t]) \ge (b_{t-1} + 2^{-|\tau_t|}) - (\mu([A_{t-1}]) + 2^{-|\tau_t|}) = b_{t-1} - \mu(A_{t-1}) \ge 2^{-|\tau_u|}$$

by induction on t. Then in the limit

$$\mu([A]) + 2^{-|\tau_u|} = \lim_{t\to\infty} \mu([A_t]) + 2^{-|\tau_u|} \le \lim_{t\to\infty} b_t$$

and so there must be some t such that $\mu([A]) < b_t$. Then for such a t it holds that $a_u = \mu([A_u]) \le \mu([A]) < b_t$ and so $\mu([A])^\star \in [B_k]$. Therefore $\mu([A])^\star$ is not dwt-g-random, as required. $\square$

5.2 Extracting additional complexity is difficult

In this section we will prove the main theorem by building a sequence of f-conditions whose corresponding effectively closed sets have non-empty intersection. To this end, below we introduce the starting f-condition.

Lemma 5.2.1. Let $f : \mathbb{N} \to [0, \infty)$ be recursive and convex and let

$$S = \{\sigma \in \{0,1\}^{<\mathbb{N}} \mid \mathrm{KA}(\sigma) \le f(|\sigma|) - f(0)\}.$$

Then $\langle \lambda, S \rangle$ is an f-condition and every $X \in P_{\langle \lambda, S\rangle}$ is KA-f-complex.

Proof. First we show that $\langle \lambda, S \rangle$ is an f-condition. S is clearly r.e. by definition. Note that

$$\operatorname{vwt}_f(S) \le \operatorname{dwt}_f(\widehat{S}) = \sum_{\sigma \in \widehat{S}} 2^{-f(|\sigma|)} \le \sum_{\sigma \in \widehat{S}} 2^{-\mathrm{KA}(\sigma) - f(0)} < 2^{-f(0)}$$

where this last inequality follows from Lemma 1.2.18. This shows that $\mathrm{vwt}_f(S) < \mathrm{dwt}_f(\lambda)$ and so $[\lambda] \nsubseteq [\mathbb{O}_f(S)]$ by Lemma 3.4.21. Therefore $\langle \lambda, S \rangle$ is an f-condition. Lastly, if $X \in P_{\langle \lambda, S \rangle}$ then $X \notin [\mathbb{O}_f(S)] \supseteq [S]$. Therefore $\mathrm{KA}(X \upharpoonright n) > f(n) - f(0)$ for all n, and so X is KA-f-complex. □

The following lemma outlines how we extend from one f-condition to the next, and is the key argument for the main theorem.

Lemma 5.2.2. Let $f : \mathbb{N} \to \mathbb{N}$ be a recursive, convex, and strongly unbounded function such that $f \not\preccurlyeq \mathrm{id}_{\mathbb{N}}$. Let $g : \mathbb{N} \to [0, \infty)$ be recursive such that $f \ll_r g$ and let $e, a \in \mathbb{N}$ be arbitrary. For any f-condition $\langle \sigma, S \rangle$ there is an f-condition $\langle \sigma', S' \rangle \preccurlyeq \langle \sigma, S \rangle$ such that if $X \in P_{\langle \sigma', S' \rangle}$ then either

1. $\Phi_e^X(m) \uparrow$ for some m, or if $\Phi_e^X \in \{0,1\}^{\mathbb{N}}$ then
2. there is a k such that $\mathrm{KP}(\Phi_e^X \upharpoonright k) \leq g(k) - a$.

Proof. Fix $b \in \mathbb{N}$ such that $2^{-b} < \mu(P_{\langle \sigma, S \rangle})$ and let φ be the prefix-free partial recursive function used to define KP. We define a new partial recursive prefix-free $R :\subseteq \{0,1\}^{<\mathbb{N}} \to \{0,1\}^{<\mathbb{N}}$ which will compress Φ_e^X if it is total. We define R only on strings $\langle \lambda, \rho \rangle$ and leave R undefined otherwise.

Defining $R(\langle \lambda, \rho \rangle)$ for each ρ. First attempt to compute $\pi = \varphi(\langle \lambda, \rho \rangle)$ and let $m = |\pi|$ if this computation halts. Now simultaneously begin enumerating the sets $T_\tau = \{\nu \supseteq \sigma \mid \tau \subseteq \Phi_e^\nu\}$ for each $\tau \in 2^{m-b}$. T_τ is the set of extensions of σ computing τ via the e^{th} recursive functional.

Let $\tilde{\pi} \in [0,1]$ be the unique rational number whose $(n+1)$th binary digit is $\pi(n)$ for all $n < |\pi|$ and is 0 otherwise. As we enumerate each T_τ, search for some $\tau \in 2^{m-b}$ such that $\mu(P_{\langle \sigma, S \cup T_\tau \rangle}) < \tilde{\pi}$. This search can be performed effectively, since each such measure is a right r.e. real number. When the first such τ is found, the machine halts, outputting $R(\langle \lambda, \rho \rangle) = \tau$. Defined in this way, R is both partial recursive and prefix-free.

Defining σ' and S'. Since b is fixed, we can pick constants c and d such that $\mathrm{KP}(\tau) \leq \mathrm{K}_R(\tau) + c$ and $\mathrm{KP}(\tau^\frown \rho) \leq \mathrm{KP}(\tau) + d$ for all τ and $\rho \in \{0,1\}^b$. By Lemma 5.1.10 the string $\mu(P_{\langle \sigma, S \rangle})^\star$ is not KP-g-complex and so there is an $\widetilde{m} \in \mathbb{N}$ such that $\mathrm{KP}(\mu(P_{\langle \sigma, S \rangle})^\star \upharpoonright \widetilde{m}) \leq g(\widetilde{m}) - a - c - d$. In fact infinitely many such values exist, so we may pick one sufficiently large that $\mu(P_{\langle \sigma, S \rangle})^\star \upharpoonright \widetilde{m} \neq 0^{\widetilde{m}}$ since $\mu(P_{\langle \sigma, S \rangle}) > 0$. Letting $m = \widetilde{m} + b$ note that

$$\mathrm{KP}(\mu(P_{\langle \sigma, S \rangle})^\star \upharpoonright m) \leq \mathrm{KP}(\mu(P_{\langle \sigma, S \rangle})^\star \upharpoonright \widetilde{m}) + d \leq g(\widetilde{m}) - a - c.$$

Now pick ρ to be a minimal description of $\mu(P_{\langle \sigma, S \rangle})^\star \upharpoonright m$; that is $\varphi(\langle \lambda, \rho \rangle) = \mu(P_{\langle \sigma, S \rangle})^\star \upharpoonright m$ and $|\rho| = \mathrm{KP}(\mu(P_{\langle \sigma, S \rangle})^\star \upharpoonright m)$. How we define σ' and S' depends on whether or not $R(\langle \lambda, \rho \rangle)$ is defined.

Case 1: $R(\langle\lambda,\rho\rangle)$ is defined. Let $R(\langle\lambda,\rho\rangle) = \tau$. Then following along with the construction note that $\pi = \varphi(\langle\lambda,\rho\rangle) = \mu(P_{\langle\sigma,S\rangle})^\star \upharpoonright m$. Then since R halted we have $\mu(P_{\langle\sigma,S\cup T_\tau\rangle}) < \tilde{\pi} \le \mu(P_{\langle\sigma,S\rangle})$. From this it follows that the set $P_{\langle\sigma,S\rangle} \setminus P_{\langle\sigma,S\cup T_\tau\rangle}$ has non-zero measure and so is non-empty.

Then there must exist some $\sigma' \in T_\tau$ such that $[\sigma'] \nsubseteq [\mathbb{O}_f(S)]$; otherwise $[\mathbb{O}_f(S)] \supseteq [S \cup T_\tau]$ and $[\mathbb{O}_f(S)] \supseteq [\mathbb{O}_f(S\cup T_\tau)]$. Define $S' = S \cap [\sigma']^{<\mathbb{N}}$. By Lemma 5.1.9 $\langle\sigma',S'\rangle$ is an f-condition and $\langle\sigma',S'\rangle \preccurlyeq \langle\sigma,S\rangle$. For any $X \in P_{\langle\sigma',S'\rangle}$ suppose that $\Phi_e^X \in \{0,1\}^{\mathbb{N}}$; of course $X \supset \sigma'$ in this case and $\sigma' \in T_\tau$. Then $\tau \subseteq \Phi_e^X$ by definition of T_τ. In particular $\tau = \Phi_e^X \upharpoonright \tilde{m}$. Then our construction has ensured that

$$\begin{aligned}
\mathrm{KP}(\Phi_e^X \upharpoonright \widetilde{m}) = \mathrm{KP}(\tau) &\le \mathrm{K}_R(\tau) + c \\
&\le |\rho| + c \\
&= \mathrm{KP}(\mu(P_{\langle\sigma,S\rangle})^\star \upharpoonright m) + c \le g(\widetilde{m}) - a
\end{aligned}$$

as required.

Case 2: $R(\langle\lambda,\rho\rangle)$ is undefined. Again, follow the construction of $R(\langle\lambda,\rho\rangle)$. The computation $\pi = \varphi(\langle\lambda,\rho\rangle) = \mu(P_{\langle\sigma,S\rangle})^\star \upharpoonright m$ halts and $\pi \ne 0^m$. Then since R is not defined we must have $\mu(P_{\langle\sigma,S\cup T_\tau\rangle}) \ge \tilde{\pi} > 0$ for every $\tau \in \{0,1\}^{m-b}$; so each such $\langle\sigma,S\cup T_\tau\rangle$ is an f-condition extending $\langle\sigma,S\rangle$. Note also that $\mu(P_{\langle\sigma,S\rangle}) \le \tilde{\pi} + 2^{-m}$, since otherwise π would be different in some bit. Then since $P_{\langle\sigma,S\rangle} \supseteq P_{\langle\sigma,S\cup T_\tau\rangle}$ for each $\tau \in \{0,1\}^{m-b}$, we have $\mu(P_{\langle\sigma,S\rangle} \setminus P_{\langle\sigma,S\cup T_\tau\rangle}) \le 2^{-m}$ for each such τ as well. Then

$$\begin{aligned}
\mu\left(\bigcap_{\tau\in\{0,1\}^{m-b}} P_{\langle\sigma,S\cup T_\tau\rangle}\right) &= \mu\left(P_{\langle\sigma,S\rangle} \setminus \bigcup_{\tau\in\{0,1\}^{m-b}} \left(P_{\langle\sigma,S\rangle} \setminus P_{\langle\sigma,S\cup T_\tau\rangle}\right)\right) \\
&\ge \mu(P_{\langle\sigma,S\rangle}) - \sum_{\tau\in\{0,1\}^{m-b}} \mu(P_{\langle\sigma,S\rangle} \setminus P_{\langle\sigma,S\cup T_\tau\rangle}) \\
&> 2^{-b} - 2^{m-b}2^{-m} = 0.
\end{aligned}$$

Apply Lemma 5.1.11 to find an f-condition $\langle\sigma',S'\rangle$ extending $\langle\sigma,S\cup T_\tau\rangle$ for all $\tau \in \{0,1\}^{m-b}$. Clearly $\langle\sigma',S'\rangle \le \langle\sigma,S\rangle$ as well.

Now let $X \in P_{\langle\sigma',S'\rangle}$. If Φ_e^X were total, we would have $X \in [T_\tau]$ for $\tau = \Phi_e^X \upharpoonright (m-b)$. But $X \in [\sigma'] \setminus [\mathbb{O}_f(S')] \subseteq [\sigma'] \setminus [T_\tau]$ and so $X \notin [T_\tau]$. Therefore Φ_e^X is not total. □

Finally we put together all of the lemmas and prove the main theorem.

Theorem 5.2.3. Let $f : \mathbb{N} \to [0,\infty)$ be recursive, convex, and strongly unbounded such that $f \not\gg \mathrm{id}_{\mathbb{N}}$. There is a KA-$f$-complex X such that if $g : \mathbb{N} \to \mathbb{N}$ is recursive, $f \ll g$, and $Y \le_T X$, then Y is not KP-g-complex.

Proof. It is sufficient to prove the result for integer-valued f, by definition of recursive strong domination.

We will build a sequence of f-conditions $\langle\langle\sigma_i, S_i\rangle \mid i \in \mathbb{N}\rangle$ such that $\langle\sigma_{i+1}, S_{i+1}\rangle \preccurlyeq \langle\sigma_i, S_i\rangle$ for each i. X will be chosen to belong to $\bigcap_n P_{\langle\sigma_n, S_n\rangle}$, which must be non-empty since it is the intersection of non-empty closed sets. Let $\langle\sigma_0, S_0\rangle$ be the f-condition $\langle\lambda, S\rangle$ described in Lemma 5.2.1.

There is a non-effective enumeration $\langle g_i \mid i \in \mathbb{N}\rangle$ of all recursive $g : \mathbb{N} \to \mathbb{N}$ with the property that $f \ll_r g$. Let $R_{e,a,i}$ denote the requirement

$$\Phi_e^X \text{ is total } \Rightarrow (\exists k)(\mathrm{KP}(\Phi_e^X \upharpoonright k) \leq g_i(k) - a).$$

If we can satisfy each requirement $R_{e,a,i}$ then X will be as desired. Since there are countably many such requirements, we simply apply Lemma 5.2.2 at each stage to find an extending condition satisfying the next requirement. □

Note that since $\mathrm{KP}(\sigma) \geq^+ \mathrm{KA}(\sigma)$ for all σ, this implies the originally stated theorem for both KA and for KP.

5.3 The non-length-invariant case

Finally, we turn to the non-length-invariant case. We would like to create a function which satisfies all of the conditions of Theorem 5.2.3 except for that of length-invariance, and yet which fails to allow for any sort of reasonable strong separation. We first need to extend the meaning of recursive strong domination to non-length-invariant functions.

Definition 5.3.1. Let $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ be recursive and convex and let $g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. We say that $f \ll_r g$ if there is a recursive function $h : \mathbb{N} \to \mathbb{N}$ such that $\sum_n 2^{-h(n)} < \infty$ is a recursive real and $g(\sigma) \geq f(\sigma) + h(f(\sigma))$ for all $\sigma \in \mathcal{I}(f)$.

More generally, let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive and convex. Letting $f' : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ be the function constructed from f in Lemma 3.4.17, we say that $f \ll_r g$ if $f' \ll_r g$.

Then it follows from the proof of Lemma 3.3.18 that for $f, g : \mathbb{N} \to [0,\infty)$ we have $f \ll_r g$ if and only if $f' \ll_r g'$ where $f', g' : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ are defined by $f'(\sigma) = f(|\sigma|)$ and $g'(\sigma) = g(|\sigma|)$ for all σ.

Theorem 5.3.2. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be defined as follows: $f(\lambda) = 0$, $f(0^k) = k/2$ for all k, and $f(0^{k\frown}1^\frown\sigma) = k/2 + (|\sigma| + 1)/(2k+2)$ for all k and all $\sigma \in \{0,1\}^{<\mathbb{N}}$. Let $g : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be given by $g(\sigma) = 2f(\sigma)$ for all σ.

Then f is recursive, convex, not equivalent to the function $h(n) = n$ for all n, and $f \ll_r g$; yet any X which is KA-f-complex Turing computes some Y which is KA-g-complex.

Proof. It is clear that f is recursive and not equivalent to the function $h(n) = n$. Next we will see that f is convex. If $\sigma = 0^k$ then

$$\begin{aligned}\mathrm{dwt}_f(\{\sigma^\frown 0, \sigma^\frown 1\}) &= 2^{-(k+1)/2} + 2^{-k/2-1/(2k+2)} \\ &= 2^{-k/2}(2^{-1/2} + 2^{-1/(2k+2)}) > 2^{-k/2} = \mathrm{dwt}_f(\sigma).\end{aligned}$$

If $\sigma = 0^{k\frown}1^\frown\rho$ then

$$\begin{aligned}\mathrm{dwt}_f(\sigma) &= 2^{-k/2-(|\rho|+1)/(2k+2)} \\ &< 2^{-k/2-(|\rho|+2)/(2k+2)+1} \\ &= 2\cdot 2^{-k/2-(|\rho|+2)/(2k+2)} = \mathrm{dwt}_f(\{\sigma^\frown 0, \sigma^\frown 1\}).\end{aligned}$$

Together this shows that f is convex.

Next, let $f' : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ be the function defined by $f'(\sigma) = \lfloor f(\sigma)\rfloor$ for all σ. Actually, f' is the function constructed by applying Lemma 3.4.17 to f. Then $g(\sigma) = 2f(\sigma) \geq 2f'(\sigma) = f'(\sigma) + \mathrm{id}_{\mathbb{N}}(f'(\sigma))$ for all σ. Then since $\sum_n 2^{-\mathrm{id}_{\mathbb{N}}(n)} = \sum_n 2^{-n} < \infty$, we see that $f \ll_r g$.

Now let X be KA-f-complex. Then $X \neq 0^{\mathbb{N}}$ since $f(0^k) = k$ for all k yet $\mathrm{KA}(0^k) \leq c$ for all k and some c. So $X = 0^{k\frown}1^\frown Z$ for some $Z \in \{0,1\}^{\mathbb{N}}$. Then by definition of f we have

$$\mathrm{KA}(0^{k\frown}1^\frown(Z \upharpoonright n)) \geq^+ (n+1)/(k+1)$$

for all n. Finally we wish to show that $Y = 0^{2k\frown}1^\frown Z$ is KA-$2f$-complex. For each n we have

$$\mathrm{KA}(0^{2k\frown}1^\frown(Z\upharpoonright n)) \geq^+ (n{+}1)/(k{+}1) \geq^+ 2(n{+}1)/(2k{+}1) = 2f(0^{2k\frown}1^\frown(Z\upharpoonright n))$$

which completes the proof. $\square$

Note that this construction is ultimately not a construction about extracting complexity. In fact, it is simply a matter of moving a string to another part of $\{0,1\}^{\mathbb{N}}$ where the complexity requirement to be KA-g-complex is much lower.

Chapter 6

Lower bounds on Kolmogorov complexity

In Chapter 4 we looked at sequences X which have measurable amounts of information: that is there is a recursive function f such that $\mathrm{KP}(X \upharpoonright n) \geq^{+} f(X \upharpoonright n)$ for all n. As long as f is unbounded along X such an X has what amounts to measurable information content. In this section we will look instead at properties pertaining to having trivial information content.

6.1 K-triviality, low for K, and NCR

The following are two related ways in which an $X \in \{0,1\}^{\mathbb{N}}$ can have minimal or trivial information content.

Definition 6.1.1. $X \in \{0,1\}^{\mathbb{N}}$ is ***K-trivial*** if $\mathrm{KP}(X \upharpoonright n) =^{+} \mathrm{KP}(n)$ for all n. $X \in \{0,1\}^{\mathbb{N}}$ is ***low for K*** if $\mathrm{KP}^X(\sigma) =^{+} \mathrm{KP}(\sigma)$ for all σ.

It is clear that if X is recursive then X is both K-trivial and low for K. The more interesting case is when X is not recursive but satisfies either of these conditions. The original construction of a non-recursive K-trivial is attributed to Solovay.

From the definitions, it can easily be shown that if X is low for K then it must also be K-trivial. To see this, just look at $\mathrm{KP}^X(X \upharpoonright n)$ for all n. In fact, the following equivalence holds.

Theorem 6.1.2 (Nies and Hirschfeldt, see Nies [32]; Hirschfeldt, Nies, and Stephan [19])**.** For $X \in \{0,1\}^{\mathbb{N}}$ the following are equivalent:

1. X is K-trivial,

2. X is low for K,

3. there exists an $A \geq_T X$ such that A is Martin-Löf random relative to X, and

4. every A is Martin-Löf random if and only if it is Martin-Löf random relative to X.

We will need just the following other basic result about K-trivials for this chapter.

Theorem 6.1.3 (Chaitin [7]). If X is K-trivial then $X \leq_T \varnothing'$.

The final concept we will study in this chapter is that of NCR, the class of X which are not random with resepect to any continuous measure.

Definition 6.1.4. A measure μ on $\{0,1\}^{\mathbb{N}}$ is ***continuous*** if it has no atoms, that is $\mu(\{X\}) = 0$ for all $X \in \{0,1\}^{\mathbb{N}}$. Since a measure μ is determined by the measure that it places on basic open sets, we look also at the function $\mu^* : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ defined by $\mu^*(\sigma) = \mu([\sigma])$ for all σ. Such a function can be rationally approximated by a function $f : \{0,1\}^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}$ such that $|f(\sigma,n) - \mu^*(\sigma)| < 2^{-n}$ for all σ and n. Since functions $f : \{0,1\}^{<\mathbb{N}} \times \mathbb{N} \to \mathbb{Q}^{\geq 0}$ can be effectively coded as subsets of $\mathbb{N}$ or equivalently as elements of $\{0,1\}^{\mathbb{N}}$, we say that X is a ***representation*** of a measure μ if it codes a function f which rationally approximates R_μ. It is not hard to see that every measure on $\{0,1\}^{\mathbb{N}}$ has infinitely many representations. A measure is called ***recursive*** if it has a recursive representation.

Let μ be a measure with representation R_μ. A ***μ-test relative to R_μ*** is a uniformly effectively open relative to R_μ sequence $\langle A_n \rangle$ such that $\mu(A_n) \leq 2^{-n}$ for all n. X is ***μ-random relative to R_μ*** if there is no μ-test relative to R_μ $\langle A_n \rangle$ such that $X \in A_n$ for all n. X is ***μ-random*** if it is μ-random relative to R_μ for some representation R_μ of μ.

NCR is the set of all X such that X is not μ-random for any continuous measure μ.

Reimann and Slaman [35] first introduced and studied NCR. While they showed that any non-recursive X is μ-random for some measure such that $\mu(\{X\}) = 0$, for many non-recursive sequences this constructed measure will necessarily have atoms; so NCR has non-recursive members.

6.2 Oracles which compute lower bounds for KP

While it is clear that Lemma 1.2.15 and Lemma 1.2.14 relativize to KP^X, we would also like to ensure that Schnorr's theorem relativizes. We follow the standard proof here.

Theorem 6.2.1 (relativized Schnorr's theorem)**.** Let μ be a measure with representation R_μ. Then $\mathrm{KP}^{R_\mu}(X \upharpoonright n) \geq^+ -\log_2 \mu([X \upharpoonright n])$ for all n if and only if X is μ-random relative to R_μ.

Proof. First suppose that X is μ-random relative to R_μ. Define for each n

$$A_n = \{\sigma \mid \mathrm{KP}^{R_\mu}(\sigma) < -\log_2 \mu([\sigma]) - n\}.$$

$\langle [A_n] \rangle$ is a uniformly effectively open relative to R_μ sequence and we have

$$\mu([A_n]) \leq \sum_{\sigma \in A_n} \mu([\sigma]) \leq \sum_{\sigma \in A_n} 2^{-\mathrm{KP}^{R_\mu}(\sigma) - n} < 2^{-n}$$

for all n. Therefore $\langle [A_n] \rangle$ is a μ-test relative to R_μ, and so there is an n such that $X \notin [A_n]$ and so $\mathrm{KP}^{R_\mu}(X \upharpoonright k) \geq -\log_2 \mu([X \upharpoonright k]) - n$ for all k.

Next assume that X is not μ-random relative to R_μ, and let $\langle A_n \rangle$ be a μ-test relative to R_μ such that $X \in A_n$ for all n. Let $\langle B_n \subseteq \{0,1\}^{<\mathbb{N}} \mid n \in \mathbb{N} \rangle$ be uniformly r.e. relative to R_μ such that $A_n = [B_n]$ for each n. Define a set $A \subseteq \{0,1\}^{<\mathbb{N}} \times \mathbb{N}$ by

$$A = \{(\sigma, -\log_2 \tilde{\mu}(\sigma) - n) \mid \sigma \in B_{2n}\}$$

where $\tilde{\mu}(\sigma)$ is defined effectively in R_μ such that $\mu([\sigma]) \geq \tilde{\mu}(\sigma) \geq \frac{1}{4}\mu([\sigma])$ and $\tilde{\mu}(\sigma)$ has the form 2^{-k} for some $k \in \mathbb{N}$. This last requirement is only to guarantee that $A \subseteq \{0,1\}^{<\mathbb{N}} \times \mathbb{N}$ as required.

Since $\langle B_n \rangle$ is uniformly r.e. relative to R_μ, the set A is r.e. relative to R_μ as well. Moreover,

$$\begin{aligned}
\sum_{(\sigma,n) \in A} 2^{-n} &= \sum_k \sum_{\sigma \in B_{2k}} 2^{-(-\log_2 \tilde{\mu}(\sigma) - k)} \\
&= \sum_k \sum_{\sigma \in B_{2k}} 2^k \tilde{\mu}(\sigma) \\
&\leq \sum_k \sum_{\sigma \in B_{2k}} 2^k \mu([\sigma]) \leq \sum_k 2^k \mu(A_{2k}) \leq \sum_k 2^{-2k+k} = \sum_k 2^{-k} < \infty.
\end{aligned}$$

Then A is a Kraft-Chaitin set relative to R_μ and so there is a constant h such that $\mathrm{KP}^{R_\mu}(\sigma) \leq n + h$ for all $(\sigma, n) \in A$. Therefore

$$\mathrm{KP}^{R_\mu}(\sigma) \leq -\log_2 \tilde{\mu}(\sigma) - k + h \leq -\log_2 \mu([\sigma]) - k + h + 2$$

for all $\sigma \in A_{2k}$. Since $X \in [A_{2k}]$ for all k, it follows that for all n we have $\mathrm{KP}^{R_\mu}(X \upharpoonright n) \not\geq^{+} -\log_2 \mu([X \upharpoonright n])$. $\square$

The following is the key new result we will use in this chapter to prove results about K-trivials and NCR.

Theorem 6.2.2. *$X \geq_T \varnothing'$ if and only if there is an $f \leq_T X$ which is monotonic, unbounded, and $\mathrm{KP}(n) \geq f(n)$ for all n.*

Proof. For this proof let ψ denote the function used to define KP. Since ψ is partial recursive, $\varnothing'$ can effectively find $\max\{\psi(\langle\lambda,\sigma\rangle) \mid |\sigma| \leq n\}$ for each n, and compute a function f as required.

So suppose that X computes such an f. Recall that $\{\varphi_e\}$ is an effective enumeration of all partial recursive functions. Let $\varphi_{e,s}(n)\downarrow$ mean that $\varphi_e(n)\downarrow$ and that the Turing machine performing this computation requires s or fewer steps for the computation to halt; otherwise write $\varphi_{e,s}(n)\uparrow$. Define a partial function $N :\subseteq \{0,1\}^{<\mathbb{N}} \to 2^{<\mathbb{N}}$ by

$$N(\langle\lambda,\sigma\rangle) = \begin{cases} s & \text{if } \sigma = 1^{n\frown}0 \text{ and } \varphi_{n,s-1}(n)\uparrow \text{ and } \varphi_{n,s}(n)\downarrow \\ \uparrow & \text{otherwise} \end{cases}$$

for each σ. N is prefix-free and partial recursive. So for some constant k and for all s

$$\begin{aligned} \mathrm{KP}(s) &\leq \min\{|\sigma| \mid N(\langle\lambda,\sigma\rangle) = s\} + k \\ &= \min\{n+1 \mid \varphi_n(n)\downarrow \text{ in exactly } s \text{ steps}\} + k. \end{aligned}$$

In particular if $\varphi_n(n)$ halts in exactly s steps then $\mathrm{KP}(s) \leq n+k+1$. Since f is monotonic and unbounded X must also compute the function $g : \mathbb{N} \to \mathbb{N}$ defined by

$$\begin{aligned} g(n) &= \min\{t \mid f(t) > n+k+1\} \\ &= \min\{t \mid (\forall s \geq t)(f(s) > n+k+1)\}. \end{aligned}$$

Then for any $s > g(n)$ we have $\mathrm{KP}(s) \geq f(s) > n+k+1$ and so $\varphi_n(n)$ does not halt in exactly s steps. So for any n we have $\varphi_n(n)\downarrow$ if and only if $\varphi_{n,g(n)}(n)\downarrow$. Then g is a modulus function for $\varnothing'$, and it is clear that $g \geq_T \varnothing'$. Therefore $X \geq_T \varnothing'$ as well. $\square$

If we replace all Turing machines with oracle machines, the above result relativizes to the following.

Theorem 6.2.3. *$A' \leq_T X \oplus A$ if and only if there is some $f \leq_T X \oplus A$ which is monotonic, unbounded, and $\mathrm{KP}^A(n) \geq f(n)$ for all n.*

6.3 Applications to NCR and K-trivials

The results in the previous section provide new proofs for several known facts. The following result was originally proved by Barmpalias, Greenberg, Montalban, and Slaman [2]. The original proof is given in terms of measures, rather than Kolmogorov complexity. The following argument is a much simpler proof of the same fact.

Corollary 6.3.1. If X is K-trivial then X is in NCR.

Proof. By Theorem 6.1.3, $X \leq_T \varnothing'$. Fix a continuous measure μ with representation R_μ. Since μ is continuous the function $f : \mathbb{N} \to [0, \infty)$

$$f(n) = \min\{-\log_2 \mu([\sigma]) \mid \sigma \in \{0,1\}^n\}$$

is monotonic, unbounded, and $f \leq_T R_\mu$. Then there also exists a function $f^* : \mathbb{N} \to \mathbb{N}$ which is recursive in R_μ, monotonic, unbounded, and has $f^*(n) \leq^+ f(n)$ for all n. If X is μ-random relative to R_μ

$$\mathrm{KP}(n) =^+ \mathrm{KP}(X \upharpoonright n) \geq^+ \mathrm{KP}^{R_\mu}(X \upharpoonright n) \geq^+ f(n) \geq^+ f^*(n)$$

for all n by Theorem 6.2.1. Then $X \leq_T \varnothing' \leq_T R_\mu$ by Theorem 6.2.2.

However, it is clear that a non-atomic point computable in R_μ cannot be μ-random relative to R_μ, so we have a contradiction. Since μ and R_μ were arbitrary, X is in NCR. □

The following is another known result with an alternate proof provided by Theorem 6.2.2. Nies [32] proved the much stronger result that every K-trivial is superlow.

Corollary 6.3.2. If A is low for K then A is generalized low, that is $A' =_T \varnothing' \oplus A$.

Proof. Obviously $A' \geq_T \varnothing' \oplus A$.

On the other hand, let $f \leq_T \varnothing'$ be a lower bound for KP, as guaranteed by Theorem 6.2.2. Since A is low for K we have

$$\mathrm{KP}^A(n) \geq^+ \mathrm{KP}(n) \geq^+ f(n)$$

for all n. Since $f \leq_T \varnothing' \oplus A$ is trivially true, we have $A' \leq_T \varnothing' \oplus A$ by Theorem 6.2.3. Therefore X is generalized low. □

Finally, the third corollary of Theorem 6.2.2 is a new result. Van Lambalgen [50] proved that $X \oplus Y$ is Martin-Löf random if and only if X is Martin-Löf random relative to Y and Y is Martin-Löf random relative to X. It is known, due to Day and Reimann [11], that this theorem also extends to a similar one for $\mu \times \mu$-randomness relative to a representation R_μ. A natural next question asked by Day and Reimann is: For which X and Y does there exist a measure μ such that $X \oplus Y$ is $\mu \times \mu$-random and $\mu(\{X\}) = \mu(\{Y\}) = 0$? We prove a new case in which X and Y cannot be made relatively random, although it is closely related to a case discussed by Day and Reimann.

Theorem 6.3.3. If X is K-trivial and $Y \leq_T \varnothing'$ then there is no measure μ such that $X \oplus Y$ is $\mu \times \mu$-random and $\mu(\{X\}) = \mu(\{Y\}) = 0$.

Proof. By van Lambalgen's theorem, this is equivalent to saying that there is no measure μ and representation R_μ such that $\mu(\{X\}) = \mu(\{Y\}) = 0$, X is μ-random relative to R_μ and Y is μ-random relative to $R_\mu \oplus X$.

Let X be K-trivial and let $Y \leq_T \varnothing'$. Let μ be a measure such that X is μ-random relative to R_μ and $\mu(\{X\}) = 0$. For all n we have

$$\mathrm{KP}(n) =^+ \mathrm{KP}(X \upharpoonright n) \geq^+ \mathrm{KP}^{R_\mu}(X \upharpoonright n) \geq f(n)$$

where f is defined by $f(n) = -\log_2 \mu([X \upharpoonright n])$ for each n. The function f is non-decreasing, unbounded, and recursive in $X \oplus R_\mu$; there is also an integer-valued f^* with the same properties, found by rounding a rational approximation of f to the nearest integer. By Theorem 6.2.3 we must have $X \oplus R_\mu \geq_T \varnothing' \geq_T Y$. Then Y cannot be μ-random relative to $X \oplus R_\mu$ without also being an atom. $\square$

Chapter 7

Conclusion

7.1 Hausdorff dimension and partial randomness

Many authors have primarily been interested in partial randomness for its application to the study of effective Hausdorff dimension. For any $0 \le s \le 1$ let $f_s : \{0,1\}^{\mathbb{N}} \to [0,\infty)$ be the function defined by $f(\sigma) = s \cdot |\sigma|$ for all σ. Several of the following equivalences are well known.

Theorem 7.1.1. For any $X \in \{0,1\}^{\mathbb{N}}$ the following numbers are equal:

1. $\liminf_{n\to\infty} \frac{\mathrm{KP}(X\upharpoonright n)}{n}$,
2. $\liminf_{n\to\infty} \frac{\mathrm{KA}(X\upharpoonright n)}{n}$,
3. $\sup\{s \mid X$ is dwt-f_s-random$\}$,
4. $\sup\{s \mid X$ is Solovay dwt-f_s-random$\}$,
5. $\sup\{s \mid X$ is pwt-f_s-random$\}$, and
6. $\sup\{s \mid X$ is Solovay pwt-f_s-random$\}$.

Proof. Let $d_1, \ldots, d_6$ be the numbers above. Note that every f_s is convex. It is clear that $d_1 = d_2$, since the two complexities differ on the scale of a logarithm. It is clear that $d_3 \ge d_4 \ge d_5 \ge d_6$ due to implications proven in Chapters 3 and 4. For any $t > d_1$, it is clearly the case that X is not KP-f_t-complex, and therefore not dwt-f_t-random either; so $t \ge d_3$ as well. Therefore $d_1 \ge d_3$. Finally, if $t > d_6$ then X is not Solovay pwt-f_t-random. Therefore for infinitely many n we have $\mathrm{KA}(X \upharpoonright n) \le^+ t \cdot n$ and so $t \ge d_2$ as well. Therefore $d_6 \ge d_2$ and so all six numbers coincide. □

Of course we can define a similar supremum for any other notion of f-randomness; all such quantities must be equal.

Definition 7.1.2. For any X we define the ***effective Hausdorff dimension*** of X to be the number satisfying Theorem 7.1.1, denoted $\dim(X)$.

While we are using this as the definition, this result is actually a theorem of Mayordomo [30], who proved that effective Hausdorff dimension coincides with the first quantity in Theorem 7.1.1. A longstanding open question was whether or not a sequence X of effective Hausdorff dimension 1 must compute a Martin-Löf random. This was settled by Greenberg and Miller [17], who constructed a sequence X such that $\dim(X) = 1$ but X computes no Martin-Löf random. In fact, this result can also be seen as a corollary of Theorem 5.2.3.

Theorem 7.1.3. There is an X such that $\mathrm{KA}(X \upharpoonright n) \geq^+ n - 2\log_2 f(n)$ for all n, and so $\dim(X) = 1$, yet X does not compute a Martin-Löf random.

Proof. Let $f : \mathbb{N} \to [0, \infty)$ be defined by $f(n) = n - 2\log_2 n$ for all n, and let $g : \mathbb{N} \to [0, \infty)$ be defined by $g(n) = n$ for all n. Then $f \ll_r g$ and so Theorem 5.2.3 implies that such an X exists. Clearly $\dim(X) = 1$ as well. □

7.2 Partial randomness and DNR

The following definition is due to Kjos-Hanssen, Merkle, and Stephan [20].

Definition 7.2.1. $X \in \{0,1\}^{\mathbb{N}}$ is ***complex*** if there is a recursive and unbounded function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{KP}(X \upharpoonright n) \geq f(n)$ for all n. X is ***autocomplex*** if there is a recursive relative to X and unbounded function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{KP}(X \upharpoonright n) \geq f(n)$ for all n.

Looking at these definitions, it is clear that X is complex if and only if there is a recursive strongly unbounded length-invariant $f : \{0,1\}^{<\mathbb{N}} \to [0, \infty)$ such that X is KP-f-complex. Higuchi, Hudelson, Simpson, and Yokoyama [18] have shown that X is autocomplex if and only if X is KP-f-complex for some recursive $f : \{0,1\}^{<\mathbb{N}} \to [0, \infty)$ which is unbounded along X.

These notions are also connected to the following definition.

Definition 7.2.2. A function $f : \mathbb{N} \to \mathbb{N}$ is ***diagonally non-recursive (DNR)*** if for each n either $\varphi_n(n) \uparrow$ or $\varphi_n(n) \neq f(n)$. A function f is ***recursively bounded DNR*** if it is DNR and there is a recursive $g : \mathbb{N} \to \mathbb{N}$ such that $f(n) \leq g(n)$ for all n.

The following connects DNR functions with complex and autocomplex sequences, and therefore with partial randomness.

Definition 7.2.3. We say that X ***wtt-computes*** Y if there is an index e and a recursive function $g : \mathbb{N} \to \mathbb{N}$ such that $Y(n) = \varphi_e^X(n) = \varphi_e^{X \upharpoonright g(n)}(n)$ for all n; in other words the oracle machine φ_e^X has recursively bounded use of the oracle.

Theorem 7.2.4 (Kjos-Hanssen, Merkle, and Stephan [20])**.** Let $X \in \{0,1\}^{\mathbb{N}}$. X is autocomplex if and only if X computes a DNR function. Moreover X is complex if and only if X wtt-computes a recursively bounded DNR function.

Our interest in recursively bounded DNR functions is that they have previously been separated in a similar fashion to our strong separation of notions of f-randomness. While the result is somewhat similar in statement, the proof methods are quite different and use bushy trees.

Theorem 7.2.5 (Ambos-Spies, Kjos-Hanssen, Lempp, and Slaman [1])**.** For any recursive function $f : \mathbb{N} \to \mathbb{N}$ there exists an X which computes a recursively bounded DNR function, but does not compute a DNR function bounded by f.

Then we can say the following by combining the proofs of Theorems 7.2.4 and 7.2.5. For a length-invariant recursive $f_1 : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ there is a recursive $g_1 : \mathbb{N} \to \mathbb{N}$ such that every KP-f_1-complex X computes a DNR function bounded by g_1. Then there exists a Y which computes a DNR function bounded by some recursive g_2, but which does not compute a DNR function bounded by g_1; therefore Y also does not compute any Z which is KP-f_1-complex. Moreover, Y is complex and so KP-f_2-complex for some length-invariant recursive function f_2. Putting together the pieces, we have that for any length-invariant recursive f_1 there exists a length-invariant recursive f_2 and some X which is KP-f_2-complex but does not compute any KP-f_1-complex Y.

While this is true, it does not say what the function f_2 would look like for a fixed function f_1. In fact, f_2 would have to grow much, much slower than f_1 for this approach to work. Instead, we can apply Theorem 5.2.3 to get a much sharper result.

Corollary 7.2.6. Let $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ be recursive, convex, and length-invariant; let g be the function $g(\sigma) = f(\sigma) - 2\log_2 f(\sigma)$ for all σ. Then there exists an X such that X is KP-g-complex but no $Y \leq_T X$ is KP-f-complex.

Moreover, Theorem 5.2.3 when translated back to DNR functions provides a new method of separating different classes of diagonally non-recursive functions, one not using bushy trees. That is, Theorem 7.2.5 can be seen as a corollary of Theorem 5.2.3 and Theorem 7.2.4

DNR and lower bounds for KP

The second connection between complex sequences and this dissertation concerns Theorem 6.2.2. That result stated that X computes a monotonic unbounded lower bound for the function $\mathrm{KP}(n) : \mathbb{N} \to \mathbb{N}$ if and only if X also computes the halting problem.

We can easily see that known results about autocomplex points imply something similar.

Theorem 7.2.7. X is autocomplex if and only if there exists an $f : \mathbb{N} \to \mathbb{N}$ such that $f \leq_T X$, f is unbounded, and $\mathrm{KP}(n) \geq f(n)$ for all n.

Proof. Let X be autocomplex. Kjos-Hanssen, Merkle, and Stephan [20] showed that X is autocomplex if and only if there is a function $g : \mathbb{N} \to \mathbb{N}$ which is recursive in X such that $\mathrm{KP}(g(n)) \geq n$ for all n. Without loss of generality g can be picked to be nondecreasing. Then define f by

$$f(n) = \begin{cases} m & \text{if } g(m) = n \\ 0 & \text{otherwise} \end{cases}$$

for each n. Since g is nondecreasing $f \leq_T X$ as well. Note that f is unbounded and $\mathrm{KP}(n) \geq f(n)$ for all n as required.

On the other hand if there is an $f \leq_T X$ as above, then define g by

$$g(n) = \text{the least } m \text{ such that } f(m) \geq n$$

for each n. Then $\mathrm{KP}(g(n)) \geq n$ for each n and so X is autocomplex. □

So *autocomplex* and *computing the halting problem* can be characterized in terms of computing lower bounds for KP; the difference lies in the type of lower bound that is allowed. Other types of lower bounds for KP are possible as well. Any X can easily compute a lower bound on KP if that lower bound is not required to tend to infinity or even to be unbounded. There is another possibility, which is that a lower bound tends to infinity eventually, but not in a monotonic fashion.

Question 7.2.8. Can we characterize the set of X such that there exists an $f \leq_T X$ satisfying $\mathrm{KP}(n) \geq f(n)$ for all n and $\lim_{n\to\infty} f(n) = \infty$?

7.3 Open problems

New meaningful notions of partial randomness

Each variant of partial randomness that is introduced in Chapters 3 and 4 can be thought of as generalizing a definition or a characterization of Martin-Löf randomness. The exceptions to this are Solovay pwt-f-randomness, Solovay vwt-f-randomness, and their equivalents. While Solovay vwt-f-randomness cannot exist, Solovay pwt-f-randomness still proves an interesting subject. A natural question to ask is what other partial randomness definitions would be worth studying. For instance, let KM mean monotone complexity, another variant of Kolmogorov complexity mentioned before. Just as with KP and KA, any X is Martin-Löf random if and only if $\mathrm{KM}(X \upharpoonright n) \geq^+ n$ for all n. Then we could define X to be ***KM-f-complex*** if $\mathrm{KM}(X \upharpoonright n) \geq^+ f(X \upharpoonright n)$ for all n, for a recursive function $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$.

Question 7.3.1. Is there a natural test-based variant of f-randomness, that is a variant defined in terms of modified vwt, pwt, or dwt-f-tests, which characterizes KM-f-complex?

In a certain sense this has been answered positively by Yokoyama, who proved that such a test-based notion must exist; the existence does not help describe the type of test in a natural way.

In fact, it is even unknown whether or not KM-f-complex differs from KA-f-complex. Gacs [15] proved that $\mathrm{KA}(\sigma)$ and $\mathrm{KM}(\sigma)$ do not agree up to an additive constant for all σ; this can be extended to also show that there is an X such that $\lim_{n\to\infty} \mathrm{KM}(X \upharpoonright n) - \mathrm{KA}(X \upharpoonright n) = \infty$. However, for any Martin-Löf random it necessarily holds that $\mathrm{KM}(X \upharpoonright n) =^+ \mathrm{KA}(X \upharpoonright n)$ for all n. Then we can ask:

Question 7.3.2. Does there exist a recursive function $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ and some X which is KM-f-complex but not KA-f-complex?

Another question is motivated as follows. Martin-Löf randomness can be characterized not only in terms of supermartingales, but also martingales. A ***martingale*** is a supermartingale such that strict equality holds in each inequality from the definition. A (super)martingale δ ***strongly succeeds*** on X if $\liminf_{n\to\infty} \delta(X \upharpoonright n) = \infty$. Then X is Martin-Löf random if and only if no r.e. (super)martingale (strongly) succeeds on X. Similarly we can define f-martingales and strong success for f-(super)martingales.

Question 7.3.3. Let $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ be recursive. What is the set of all X such that no r.e. f-(super)martingale (strongly) succeeds on X?

It is possible that restricting to f-martingales or using strong success would not change the notion of f-random, but it is unclear to what extent techniques and arguments which work for martingales and supermartingales translate to f-martingales and f-supermartingales.

Implications and non-implications for partial randomness

There are also several implications that are unknown in general among the variants of partial randomness introduced in Chapters 3 and 4. For example:

Question 7.3.4. Is there a direct and simple proof that pwt-f-randomness is always equivalent to BC pwt-f-randomness?

Of course, in the convex case this question is answered positively, as a result of the equivalence between pwt-f-randomness and vwt-f-randomness. Even without convexity, this result holds due to a corollary proved by Higuchi, Hudelson, Simpson, and Yokoyama [18]. However that proof is not direct but an indirect result about propogation of randomness.

We are also interested in whether the separation of vwt-f-randomness and pwt-f-randomness can be improved.

Question 7.3.5. Is there a length-invariant $f : \{0,1\}^{<\mathbb{N}} \to [0,\infty)$ such that vwt-f-randomness and pwt-f-randomness do not coincide?

Another interesting observation is that both Theorems 4.3.2 and 4.3.4 require stronger conditions than just convex. Additionally Theorem 4.3.2 requires f to be length-invariant while Theorem 4.3.4 does not.

Question 7.3.6. For which recursive $f : \{0,1\}^{<\mathbb{N}} \to \mathbb{N}$ does there exist an X which is KP-f-complex but not strongly KP-f-complex? For which recursive f does there exist an X which is KA-f-complex but not strongly KA-f-complex?

As noted earlier, if f is the function $f(\sigma) = |\sigma|$ for all σ, then KP-f-complex and strongly KP-f-complex are both equivalent to Martin-Löf random. Therefore for at least some functions such a separation will never be possible. However for the same function f, KA-f-complex is equivalent to Martin-Löf random while no X can be strongly KA-f-complex. Therefore it is possible that for every function f such a weak separation can be done.

Strong separations

There are several other strong separations that would be desirable to achieve. First, can we improve the statement of Theorem 5.2.3 by weakening the hypotheses? We demonstrated with Theorem 5.3.2 one way in which the theorem cannot be strengthened. Other possibilities include the following.

Question 7.3.7. In Theorem 5.2.3, can $f \ll_r g$ be replaced with $f \ll g$? Can it be replaced with the condition that $\lim_{n\to\infty}(g(n) - f(n)) = \infty$?

It is also plausible that we could strongly separate KP-f-complex from KA-f-complex for some function f.

Question 7.3.8. For what f does there exist an X which is KP-f-complex but does not compute any Y which is KA-f-complex?

Current techniques seem to fail for answering this question. The strong separation that has been established uses effectively closed sets. However, it can be shown that every effectively closed set which contains a KP-f-complex sequence also contains a KA-f-complex sequence. Equivalently:

Theorem 7.3.9. Let $P \subseteq \{0,1\}^{\mathbb{N}}$ be effectively closed, that is there is an r.e. set $A \subseteq \{0,1\}^{<\mathbb{N}}$ such that $P = \{0,1\}^{\mathbb{N}} \setminus [A]$. Then if some $X \in P$ is dwt-f-random, there is a $Y \in P$ such that Y is pwt-f-random.

Proof. Let X be dwt-f-random and suppose that $\langle A_n \rangle$ is a pwt-f-test such that $P \subseteq [A_n]$ for each n. Then by compactness and the enumerability of the A_n, there is a recursive function $g : \mathbb{N} \to \mathbb{N}$ such that $P \subseteq [A_n^{g(n)}]$ for each n. Then $B_n = \widehat{A_n^{g(n)}}$ has the property that $\mathrm{dwt}_f(B_n) \leq \mathrm{pwt}_f(A_n)$ for each n. Since $\langle B_n \rangle$ is uniformly r.e. as well, it is a dwt-f-test. But then $X \in [B_n]$ for each n and so X is not dwt-f-random, which is a contradiction. Therefore there is no pwt-f-test such that $P \subseteq [A_n]$ for each n. In particular, since universal pwt-f-tests exist, some $Y \in P$ is pwt-f-random. □

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