ASSORTMENT PLANNING OF VERTICALLY DIFFERENTIATED PRODUCTS UNDER CONSUMER CHOICE

A Dissertation in
Business Administration
by
Mrinmay Deb

© 2013 Mrinmay Deb

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

May 2013
The dissertation of Mrinmay Deb was reviewed and approved* by the following:

Dr. Susan H. Xu  
Professor of Management Science and Supply Chain Management  
Dissertation Adviser, Chair of Committee

Dr. Terry P. Harrison  
Professor of Supply Chain and Information Systems

Dr. Saurabh Bansal  
Assistant Professor of Supply Chain Management

Dr. Min Ding  
Professor of Marketing

Dr. John Eugene Tyworth  
Head of the Department of Supply Chain & Information Systems

*Signatures are on file in the Graduate School.
Abstract

Product variety decision has been analyzed in the Marketing, Economics and Operations Management (OM) literature. While both Marketing and Economics literature have made significant contributions in analyzing product line decisions of a manufacturer, these studies do not consider discrete consumer choice explicitly. The OM literature has analyzed assortment decisions of a retailer taking into account consumer choice, but the primary focus is on horizontally differentiated products. This dissertation investigates the assortment and pricing decisions of a retailer selling vertically (quality) differentiated products under discrete consumer choice. Thus, this research contributes to the OM literature by extending the assortment planning problem from horizontally differentiated products to vertically differentiated products, and to the Marketing literature by explicitly incorporating consumer choice.

The first essay investigates the assortment planning problem of a single category of vertically differentiated products under consumer choice. Consumers have heterogeneous preferences for quality and the retailer is aware of the distribution of the preference. The consumer choice process is as follows: A typical consumer independently chooses the product (including the outside option) that maximizes her utility and does not substitute with another available product if the first choice product is out of stock. This simple choice process leads to stochastic demand of each item, and hence inventory cost plays a role in retailer’s assortment and pricing decisions. The optimal assortment retailer offers under demand uncertainty is found to be a subset of the optimal assortment offered in case demand of each product is known with certainty. Thus inventory cost limits the number of products retailer offers. This finding significantly reduces the computational effort required to find the optimal assortment. We also provide an algorithm to identify the set of products within which the search
for the optimal assortment can be restricted.

The second essay investigates the optimal pricing and assortment decisions of two categories, where each category has several quality differentiated products. Consumers have heterogeneous preference for quality in both categories. Depending on the way consumers choose products from the two categories, they are categorized into two types, cherry pickers and basket shoppers. A cherry picker evaluates products in each category independently and purchases a product only if it maximizes her utility compared to other products in that category and the outside option. In contrast, a basket shopper evaluates both categories together, and selects a configuration comprising of one product from each category, only if the total utility of the configuration is higher than the utilities from other configurations and the outside option. We show that in presence of the two consumer types, retailer offers the same variety in each category that she would offer when there are only cherry pickers. The variety level offered in each category is the category-profit maximizing set of products. Moreover, retailer offers certain high quality products in the two categories category at the same prices which she offers to cherry pickers. The pricing policy for the remaining lower quality products, however, depends on the relative proportion of the two types and the correlation between the two sensitivities.

The last essay investigates assortment and pricing decisions of a co-production system, most noticeably present in semiconductor manufacturing, where the quality of the manufactured units vary. A firm operating in such a system tests each unit after production, classifies the units into quality buckets or bins, and then sells all units in a bin at the same price. We investigate how many bins firm should offer, the quality levels of those bins and their prices. These decisions are made keeping into consideration inventory and demand of the bins. Units in different bins cannot be manufactured independently due to variability in output quality. Further, the demands of the bins, which are generated from consumers’ utility maximization, are also interdependent. We show that firm chooses the optimal quality levels and the
batch size such that the demands of all offered bins are met in full. The optimal quality levels are unevenly spaced across the support of the distribution of quality. An easier strategy to offer an assortment with equidistant bins can be significantly sub-optimal. We also find that a limited number of carefully selected bins provide almost all potential revenue from the market. Finally, we discuss practical considerations for pricing, technology, and optimal product offerings, and explain the proliferation of bins witnessed in the last decade in processor manufacturing.
4 Assortment Planning for Hi-Tech Products: Theoretical and Practical Considerations

4.1 Introduction ........................................... 78
4.2 Literature Review ..................................... 82
4.3 Model Preliminaries .................................. 84
  4.3.1 Manufacturing Process ............................ 84
  4.3.2 Pricing ........................................... 86
  4.3.3 Consumer Choice Model and Demand .......... 88
  4.3.4 Firm’s Optimization Problem .................. 89
4.4 Optimal Quality and Quantity for Fixed Number of Bins .......... 90
  4.4.1 Optimal Quantity for a given Quality .......... 90
  4.4.2 Optimal Quality Levels ........................ 92
  4.4.3 Optimal Number of Bins ....................... 95
  4.4.4 Counterfactual for the Prevalence of Equidistant Bins ..... 98
4.5 Numerical Analysis ................................... 99
  4.5.1 Numerical Design ................................ 99
  4.5.2 An Illustrative Example ......................... 101
    4.5.2.1 Base Case .................................. 101
    4.5.2.2 Comparative Static Analysis for $\sigma_X$ and $\sigma_G$ .. 102
  4.5.3 Synthesis of Other Comparative Analysis ........ 104
    4.5.3.1 Optimal Solution vis-à-vis Equidistant Bins ..... 104
    4.5.3.2 Effect of Change in Number of Bins on Optimal Quality .. 105
    4.5.3.3 Benefit of adding Bins ...................... 106
4.6 Discussion and Conclusions .................................. 107

Bibliography ................................................. 109

Appendix A. Proofs for Chapter 2 ................................... 119
Appendix B. Proofs for Chapter 3 .................................. 136
Appendix C. Proofs for Chapter 4 .................................. 145
# List of Figures

1.1 Three bins created from the distribution of clock speed ........................................... 6  
2.1 Plot of $\Pi(p)$ ................................................................................................................. 23  
2.2 CPU time (in seconds) vs. $n$ (number of products in $\Omega$) ........................................ 24  
2.3 Optimal assortments for convex and concave cost cases ............................................... 27  
2.4 Lowest increasing convex curve on quality-cost ......................................................... 28  
2.5 Optimal assortment of Example 1 obtained by the Increasing Convex Envelope Algorithm .......................................................... 31  
2.6 Optimal assortment in Example 2 .................................................................................. 33  
2.7 CPU time (in seconds) vs. $n$ ....................................................................................... 34  
2.8 % Savings in computation time vs. $n$ .......................................................................... 40  
2.9 Impact of Error in Variance of Demand ........................................................................ 41  
2.10 Impact of Error in Estimation of Quality ..................................................................... 43  
3.1 Lowest increasing convex curve on quality-cost ............................................................. 57  
3.2 Configurations selected by cherry pickers ....................................................................... 60  
3.3 Optimal configurations for basket shoppers .................................................................. 62  
3.4 Selection of Configurations ............................................................................................ 64  
3.5 Setting price for Basket Shoppers .................................................................................. 66  
3.6 Plots of Example 4 .......................................................................................................... 68  
3.7 Selection of Configurations in Example 4 ...................................................................... 69  
3.8 An example of two categories with parameters: $\mathbf{q} = [161, 709], \mathbf{c} = [9.14, 177.25]$, $\mathbf{q'} = [1109, 1649, 1763], \mathbf{c'} = [174.4017, 385.5929, 440.75]$ ....................................................... 73  
3.9 Two Categories of Example 5 ......................................................................................... 75  
4.1 Three bins created from the distribution of clock speed ............................................... 80  
4.2 Co-production system with $n$ output categories ............................................................ 85
List of Tables

2.1 Numerical Experiment Design ........................................... 39
2.2 Effect of error in variances under arbitrary and convex costs ....... 42
2.3 Effect of error in quality under convex and arbitrary costs ........ 44
3.1 Cutoffs of Example 5 ...................................................... 74
4.1 Bin Specifications and prices for AMD processors ...................... 79
4.2 Experimental Design with Normal distribution of Yield and Preference 100
4.3 Optimal Solution of Base Case .......................................... 101
4.4 Impact of change of $\sigma_X$ ............................................. 102
4.5 Impact of change of $\sigma_G$ ............................................. 103
4.6 Optimality Gap of Equidistant Bins .................................. 104
B.1 Output of Algorithm 1 ..................................................... 144
C.1 Experimental Design with Uniform distribution of Yield and Preference 157
Acknowledgments

I am heartily thankful and deeply indebted to all members of my committee for their guidance and support throughout the course of my dissertation work. I am extremely grateful to Dr. Susan H. Xu for her thoughtful supervision and caring guidance. I have learnt from her the insights of doing rigorous analytical work. Her support, patience and encouragement has given me the strength to move my dissertation forward, realize my own potential and improve myself at every step. The guidance I received from her made my Ph.D. fruitful and enlightening. I thank Dr. Terry P. Harrison for his encouragement and valuable inputs during my dissertation work. I am grateful to Dr. Saurabh Bansal for his guidance on a part of my dissertation. His enthusiasm, passion, tenacity and strong commitment to research was a great lesson for me.

I heartily thank all professors from whom I took excellent courses which helped me greatly in my research. Especially, I would like to thank Dr. Tom M. Cavalier and Dr. Terry L. Friesz (Industrial & Manufacturing Engineering), Dr. Arkady Templeman (Mathematics), Dr. John Liechty (Marketing), Dr. Murali Haran (Statistics), Dr. Vijay Krishna (Economics), and last but not least our Dr. Susan H. Xu, Dr. Russell Barton and Dr. Terry P. Harrison.

I would like to show my deep gratitude to Alice Young, Jane Jones, Beth Bower, and Teresa Lehman because they were like a second family here and their help was unparallel. I thank Kate, Valery, Paolo, Andrew, Natalia, Yaru, Yang, Hiroko and other fellow Ph.D. students in SC&IS, for making my Ph.D. experience rich with knowledge. Most importantly, none of this would have been possible without the patience and encouragement of my parents back home, and the unconditional love and support of my dear friends in State College, Kiranmoy, Kaustav, Sanghamitra and Souvik to name a few.
Chapter 1

Introduction

A retailer’s assortment is defined by the set of products carried in each store at each point in time. The goal of assortment planning is to determine an assortment that maximizes sales or gross margin, subject to various operational and strategic considerations. Operational considerations include limited budget for purchase of products and limited shelf space for displaying products, while strategic considerations may involve the need to carry certain trendy products in order to combat competition and increase the attractiveness of the store. The assortment carried by a retailer has a great impact on sales and gross margin, and hence assortment planning has received high priority in both academics and practice. The academic approach to the assortment planning problem rests on the formulation of an optimization problem, which finds the optimal set of products to offer from a large pool of available alternatives. Decisions for each product are interdependent because products are linked in considerations such as shelf space availability, common vendors (brands), joint replenishment policies etc. The assortment decision also depends to a great extent on the knowhow about the various aspects of consumer choice. Specifically, how do consumers choose their most preferred product from the available alternatives. This choice often depends on the nature of differentiation among the alternatives (horizontal vs. vertical [Anderson et al. 1992]). Horizontally differentiated products are different in features that can not be ordered. In that case, each of the products is ranked first for some of the consumers (e.g., shirts of different color). On the other hand, vertically differentiated products can be ordered according to their feature/quality from the highest to the lowest (e.g., external hard drives with different
Another important aspect of consumer choice is how consumers substitute if certain products of their choice are not carried by the retailer, or are not available during store visit. Consumers who repeatedly shop for a daily consumable might switch to another available product if the product of choice is not available at the time of store visit. This is called stock-out based substitution behavior. On the other hand, consumers sometimes visit stores with some particular product in mind, and if that product is out of stock or not carried at the time of store visit, consumers may choose a product from the ones available on the shelf. This is called assortment based substitution behavior. It is important for retailers to take into account these two aspects, viz. consumer choice and consumer substitution behavior, while deciding on the assortment.

A vast body of the literature in operations management focuses on assortment planning decisions of horizontally differentiated products under consumer choice, incorporating both assortment based and/or stock-out based substitution behavior (Kök et al. 2008) provides a detailed review of the literature on assortment planning). However, few papers focus on the assortment planning of vertically (quality) differentiated products (Bharghava and Choudhary 2001, Honhon and Pan 2012). In an attempt to contribute to this part of the literature, we investigate the assortment planning problem of vertically differentiated products through an analytical model which explicitly incorporates consumer choice. There are three essays in this dissertation:

1. Assortment Planning of Vertically Differentiated Products under Consumer Choice

2. Assortment Planning of Multi-category Products with Multiple Consumer Types

3. Assortment Planning for Hi-Tech Products: Theoretical and Practical Considerations
The first essay deals with the assortment planning problem of a retailer selling vertically differentiated products of a single category. Consumers are heterogeneous in their willingness to pay for quality and consumer behavior is specified by the vertical choice model ([Berry and Pakes 2007]). A typical consumer independently chooses the product (including the outside option) that maximizes her utility, and does not substitute with another available product if her preferred product is out of stock. This simple choice process leads to stochastic demand of each item, and hence inventory cost plays an important role in the assortment and pricing decisions. Due to inventory risk, this situation is referred as the risky case. We show that the optimal assortment retailer would offer in the risky case is a subset of the optimal assortment she would offer in the riskless case. The riskless case refers to a situation where individual item demand is deterministic, and hence there is no inventory risk. The intuition behind the subset result is as follows: while, adding variants to the assortment increases the likelihood that consumers will purchase something from the assortment; more choice alternatives results in thinning of the total demand, increases variability of demand for each variant, and in turn increases inventory costs. Thus inventory cost puts an upper limit to the number of products that can be offered. Using this result the search for the optimal assortment can be restricted only to the products that constitute the riskless assortment. We develop a simple procedure, called the Increasing Convex Envelope Algorithm, to find the riskless assortment. Since retailers often have to choose from a very large number of products what stock keeping units (SKU) to offer, a reduction in the number of products to search from is helpful. Through a numerical study we demonstrate that the savings in computational time to find the optimal assortment is significantly high when retailer has to choose from a large number of products. We find that the optimal assortment is highly sensitive to the accuracy in the estimate of the quality value, but robust w.r.t. errors in estimation of the variance of demand. We also find that the optimal prices are adjusted up from the optimal riskless prices so as to reduce the effect of inventory cost.
Most of the literature in operations management deal with the assortment planning problem of a single category. However, many retailers carry products of several categories, and consumers purchase products from multiple categories in a single purchase occasion. Purchase of products with multiple components also involve multi-category choice (e.g., box and frame in a modular furniture, memory drive and processor in a personal computer, flight and hotel in a travel itinerary). Multi-category assortment planning need to take into account cross-category interactions, since one category’s optimal decisions depends on the decisions of the other categories. In the second essay we investigate the optimal pricing and assortment decisions of a retailer selling products in two categories, where each category has several quality differentiated variants (e.g., processors with varying clock speeds, flights with one or more stops). Consumers have heterogenous preference for quality in both categories. Further, depending on the way consumers choose products from the two categories, they are categorized into two types, cherry pickers and basket shoppers. A cherry picker evaluates products in each category independently and purchases a product only if it maximizes her utility compared to other products in that category and the outside option. In contrast, a basket shopper evaluates both categories together, and selects a configuration comprising of one product from each category, only if the total utility of the configuration is higher than the utilities from other configurations and the outside option. Both cherry pickers and basket shoppers have correlated sensitivities to product qualities in the two categories. The population is a mix of these two types, and retailer needs to set optimal prices and assortments in the two categories in such a way that consumers will select products and configurations according to their own type. First, we consider the special case where there is only one type of consumer, and determine the pricing and assortment decisions in each category for each consumer type. We find that the retailer will offer the same variety in each category to the two types. Further, retailer will charge the same price for all products except for the lowest quality product offered in each category, which is priced lower for the
basket shoppers than for the cherry pickers. Based upon these results we find that in presence of the two consumer types, retailer offers the same variety in each category as she would offer to each type when individually present. Moreover, retailer offers certain high quality products in the two categories category at the same prices which she offers to cherry pickers. The pricing policy for the remaining lower quality products, however, depends on the relative proportion of the two types and the correlation between the two sensitivities. If cherry pickers are relatively low in proportion compared to basket shoppers, retailer has more flexibility in setting these prices. We find that in this case, retailer sets a higher discount for the low quality products so as to attract basket shoppers. On the other hand, if cherry pickers are relatively high in proportion, retailer does not have the liberty to discount the prices by as much in case when cherry pickers are low in proportion. We also find that if consumers have independent sensitivities for quality in the two categories, retailer may markup prices in one category, while sell products in the other category at a discount.

The last essay investigates assortment and pricing decisions in co-production systems where the quality of the manufactured units vary, most noticeably present in the computer processor manufacturing industry. The manufacturing process of processors is complex since each processor contains billions of transistors that together determine the clock speed of the processor. Invariably, every unit manufactured under identical conditions has a different clock speed. The firms test each unit for its clock speed and categorize it in a set of discrete versions or “bins,” each having a distinct quality level. All units in a bin are sold at the same price. The quality levels of the units within a bin vary but bear the quality tag at which the bin is sold in market. For example, the three versions of the Athlon processors shown in Figure 1.1 are produced on the same manufacturing line simultaneously. Every unit produced with a clock speed between 2.8-2.9 GHz is classified as the 2.8 GHz processor, etc.
These specifications determine both the supply side and demand side. The market shares of the processors are determined by the quality specifications and the selling prices in the market. The specifications also determine the yield of the three versions: the yield of 3.0 GHz processors is the fraction of processors with a clock speed more than 3.0 GHz, the yield of 2.9 GHz processors is the fraction of processors with a clock speed between 2.9 and 3.0 GHz, etc. These yields, in turn, strongly impact the profitability of the product line. A mismatch between supply and demand occurs if the specifications are not carefully selected. Finding the appropriate specifications of bins is non-trivial because firms seldom have full control over fractions in which various bins are produced and over the demands of the bins.

Motivated by these observations and challenges, we investigate the assortment planning problem of a co-production system. Specifically, how many bins a firm should offer, the quality levels of those bins and their prices. We investigate the problem analytically and complement the analysis with practical considerations that exist in this industry. We show that in the optimal assortment and manufacturing policy, a firm will choose the optimal batch size and offer bins such that the firm can fulfill the demands of all bins offered. We find that, unlike the standard assortment planning problem, the problem of finding the optimal quality levels is not well behaved.
for arbitrary distributions of quality and consumer preferences and, therefore, requires extensive numerical search. The optimal quality levels are unevenly spaced across the support of the distribution of quality. Therefore, an easier strategy to offer an assortment with equidistant bins can be significantly suboptimal. Improvements in manufacturing technology do not always benefit the firm though the reasons are different for improvement in average quality level produced (technology push) and in the spread of the quality levels (better process control). Finally, we discuss practical considerations for pricing, technology, and optimal product offerings, and explain the proliferation of bins witnessed in the last decade in the processor industry.
Chapter 2
Assortment Planning of Vertically Differentiated Products under Consumer Choice

2.1. Introduction

Assortment planning is both a strategic and tactical exercise in retail stores, and is often key to profitability. [Kök et al. 2008] describes the assortment planning process: It begins with forecasting the sales of each product category for a future planning period ranging from a several month season to a fiscal year. Then, scarce store shelf space and inventory purchase dollars are allocated to each product category based on the sales projections, along with other possible strategic considerations. Finally, given these resource allocations, the number of SKUs to be carried in each product category is determined. At this point, retailer chooses certain SKUs from a large number of available variants of a product category (such as personal computers), or subcategories (such as laptops and desktops). The SKUs within a product category/subcategory are substitutable, and compete for limited budget and shelf space. Therefore, selecting which SKUs to offer is an important tactical decision for retailers. An example of the challenge involved in choosing SKUs of flat TV at Best Buy is given in [Kök et al. 2008] as follows: "As an example, in flat panel TV’s, Best Buy might carry 82 different SKUs. By contrast, the number of potential SKUs is much larger, comprising of 8 diagonal widths (e.g., 19”, 25”, 32”, 35”, 40”, etc.), 5 screen types (plasma, LCD, projection, etc.), 7 resolutions (analog, 480i, 720p, 1080i, etc.) and 9 major vendors
(Sony, Panasonic, Pioneer, etc.) for a total of \(8 \times 5 \times 7 \times 9 = 2520\) potential SKUs. It is left to the buyer through a largely manual process to determine which 82 out of these 2520 SKUs will be carried by Best Buy”.

An important factor that influences assortment decision is the knowhow about consumer choice. For instance, what is a proper model to describe how consumers choose products from the offered assortment? How do consumers substitute if certain variants of their choice are not carried by the retailer, or are not available during store visit? Consumer choice model depends on the nature of differentiation amongst variants (horizontal or vertical differentiation, [Anderson et al. 1992]). Horizontally differentiated products are different in features that can not be ordered. In that case, each of the products is ranked first for some of the consumers (e.g., shirts of different color). On the other hand, vertically differentiated products can be ordered according to their feature/quality from the highest to the lowest (e.g., external hard drives with different capacity). A higher quality product is more desirable than a lower quality product for any consumer, however consumers have different willingness to pay for quality. Depending on the availability of products, consumer substitution are primarily two types. Stock-out based substitution is switching to an available variant when the favorite product is carried in the store, but is stocked-out at the time of shopping (occurs during repeat purchases like food items). Assortment-based substitution, is the switch to an available variant when the favorite product is not carried in the store (occurs during one time purchases like apparel).

In this chapter we address assortment planning problem of a retailer who offers an assortment of of quality differentiated SKUs belonging to a single product category (e.g., computers with varying memory capacity and processors speeds, external hard drives drive with different memory sizes, flat screen televisions with different screen sizes). Each variant has a quality specification, and all else equal, customers prefer a product with a higher quality level to a product with a lower quality level. However, consumers differ in their willingness to pay for a particular quality. The retailer is
unaware of a particular consumer’s willingness to pay, but knows the distribution of consumer valuations as a population. We assume that consumers are utility maximizers and choose their most preferred product from based on quality and price only (e.g., consumers prefer a SanDisk 320GB external hard drive over a Western Digital 160GB hard drive, if the former is available at a lower or equal price). Thus, consumer choice is specified by the well known vertical choice model (also used in [Akçay et al.(2010)], [Bharghava and Choudhary 2001], [Honhon and Pan 2012]). We note that choice based solely on price and quality exists when consumers also have brand considerations. Often consumers would select the brand first and then choose a variant in that brand based on the price and quality/feature (e.g., a consumer chooses Nike as the brand for footwear, and then chooses a particular shoe from the different quality grades of shoes). The vertical choice model is also applicable to such situations of hierarchical choice.

Further, we consider consumer substitution behavior to be assortment based: consumer chooses her favorite from what she sees on the shelf and buys it if it is better than her no purchase option. In this case, there may be other products she may have preferred, but she didn’t see them either because the retailer didn’t carry them or they are stocked out. If a product runs out of stock or is not carried, customers who prefer that product as their first choice, do not switch to another available product, but simply buy nothing. This is a restrictive assumption because some demand of the out-of-stock product may spill over to the ones that are still in stock. However, we believe that it is useful to characterize the assortment planning problem and its solution for the simpler case addressed in this paper before introducing stockout-based substitution as well as assortment-based substitution together. Finally, we consider that consumers choose independently, in the sense that there is no trend of purchasing only certain products. With these assumptions on the choice process, the individual item demand is stochastic even when a fixed number of consumers visit the store (To see this assume $S$ customers visit the store. Then, the number of them who choose
product \( i \in \{1, \ldots, n\} \) as their first choice is a binomial random variable with mean \( S\alpha_i(p) \), where \( \alpha_i(p) \) is the first choice probability given by the vertical choice model. We refer to this as the \textit{risky} case, since inventory risk comes into consideration in deciding the optimal assortment and prices.

The retailer sources products from multiple manufacturers, and the quality levels and unit purchasing costs are exogenously determined. In a single-period model, we determine the retailer’s optimal pricing and assortment decisions, assuming consumers choose according to the vertical choice model and no stock-out based substitution. Thus, we analyze the price-dependent newsvendor problem coupled with an assortment decision.

Our key results are the following. First, in the \textit{risky} case where retailer faces stochastic demand, we show that the joint pricing and assortment problem in presence of inventory considerations can be reduced to a problem of finding only the optimal prices. Assuming unmet demands become lost sales, we show that our problem is well behaved in the sense that there is a unique vector of prices that is optimal. The optimal prices in turn determine the optimal assortment, since the assortment is set of products with non zero purchase probability. Next, we show that the optimal assortment in the \textit{risky} case is a subset of the optimal assortment in the \textit{riskless} case. The \textit{riskless} case refers to a situation where individual item demand is deterministic, and hence there is no inventory related risk for the retailer. The intuition behind the subset result is as follows: while, adding variants to the assortment increases the likelihood that consumers will purchase something from the assortment; more choice alternatives results in thinning of the total demand, increases variability of demand for each variant, and in turn increases inventory costs. This inventory cost puts an upper limit to the number of products that can be offered. Such result, involving variety-cost trade off is also found in the literature focussing on horizontally differentiated assortment (see [van Ryzin and Mahajan 1999], [Li 2007], [Maddah and Bish 2007] and [Maddah et al. 2011]). We also find that the optimal risky price is adjusted up
from the riskless price in such a way as to reduce the effect of uncertainty cost.

The main implication of the subset result is that the search space for the optimal assortment is reduced. Using this result, retailer needs to find the optimal prices of only those products which constitute the riskless assortment, which consists of a certain set of products. These optimal prices in turn determine the optimal assortment. Thus the computational effort of determining the optimal assortment in the risky case can be significantly reduced (reported in §2.5). In order to find the riskless assortment, we develop the *Increasing Convex Envelope Algorithm*, which is a polynomial time algorithm. We also provide a simple graphical representation of the riskless assortment: it is the set of products whose costs form the *lowest increasing convex curve* on the quality-cost graph.

The papers that are most closely related to this work and consider pricing and/or assortment decisions of multiple substitutable products under a vertical choice model are [Akçay et al.(2010)], [Bharghava and Choudhary 2001] and [Honhon and Pan 2012].

[Akçay et al.(2010)] finds the joint dynamic pricing of a fixed assortment of multiple perishable products when products can be differentiated vertically as well as horizontally. In their vertical model they show that the optimal prices exhibit quality monotonicity: the optimal price of a high quality product is always higher than that of a lower quality product. They develop a polynomial-time and exact algorithm to find the optimal prices based on an inventory aggregation property. We do not consider dynamic pricing, but focus on product selection under demand uncertainty in a single period setting.

[Bharghava and Choudhary 2001] consider the problem of selecting and pricing vertically differentiated products when demand is deterministic, quality levels are exogenous and there are no fixed costs. Their work addresses two special cases. They find the condition on cost and quality under which the optimal assortment contains only the highest quality product, and the condition when it contains all the products.
We consider a similar setting (the riskless case in our paper), but we determine the optimal assortment for any arbitrary marginal cost and quality, with the help of an efficient algorithm. Thus our result is more general than theirs. Further, unlike their setup, we analyze the optimal assortment under demand uncertainty.

[Honhon and Pan 2012] assumes a fixed cost of offering a product and find the pricing and assortment decisions under a general distribution of consumer types and deterministic demand. They consider two scenarios, one in which the retailer can set prices and the other in which prices are exogenously given. They develop algorithms to find the optimal assortment in each scenario. Although we do not consider fixed cost and generic consumer preferences, we explicitly consider demand uncertainty that arises from a consumer choice, and compare its impact on the optimal prices and optimal assortment vis-a-vis the deterministic case. We find that under stochastic demand, the optimal prices increase and we also characterize the set of products, within which the search for the optimal assortment can be restricted.

The rest of this chapter is organized as follows. §4.2 reviews the related literature. In §2.3.1 and §2.3.2 we present the vertical choice model and the consumer demand models, respectively. In §2.4.1 we present the retailer’s optimization problem in the risky case, and the riskless case in §2.4.2. We then state our main result on the optimal assortment in §2.4.3, and discuss the implications and importance of the result in §2.4.4. Finally, §2.5 reports a numerical study to demonstrate the savings in computational effort due to the subset result, and tests the robustness of this result w.r.t. changes in model parameters such as quality and demand variance. We conclude the paper in §2.6. All proofs are given in Appendix A.

2.2. Literature Review

In this section we review the literature on inventory, pricing, and assortment decisions of substitutable products under discrete consumer choice. We first review the
literature on pricing and product line selection of vertically differentiated products. Next, we give a brief account of the literature on inventory, pricing, and assortment decisions of horizontally differentiated products.

In the economics literature, the research on pricing and product line selection takes the point of view of a manufacturer facing deterministic demand, and studies how manufacturer sets the prices and quality levels of the products, assuming customers have heterogenous preferences for quality. Two early papers in this area are [Moorthy 1984] and [Musa and Rosen 1978]. [Moorthy 1984] considers a monopolist who sets prices and assigns quality levels to a finite number of consumer segments, assuming consumer preference is nonlinear in quality. The author finds that the optimal price increases with quality so as to make each consumer segment indifferent between the segment assignment and the assignment of the immediate lower neighbor. The author also shows that if preferences are nonlinear, a monopolist could aggregate segments in order to mitigate the cannibalization effect. [Musa and Rosen 1978] consider a continuous distribution of customer heterogeneity and a linear utility function. Assuming convex production cost, they show that a monopolist charges a higher price than the marginal cost, lowers the quality level, and sells a wide range of quality as compared to that under competition. A number of papers, which we do not review here, consider the problem of offering vertically differentiated products in a competitive setting (see, for example, [Jing 2006], [Moorthy 1984], [Rhee 1996], [Shaked and Sutton 1982] and [Shugan 1989]). Except [Carlton and Dana 2008], this stream of the literature mostly ignores demand uncertainty and hence inventory decisions. [Carlton and Dana 2008] characterize the optimal product line for a monopolist in the presence of ex-ante sunk costs and uncertain demand. The authors show that when demand is certain, a single product is produced; but when demand is uncertain, selling a range of product qualities is more profitable for the firm and also more efficient for a social planner. They also show that the highest quality product generates the greatest consumer surplus, yields the highest absolute
margin and the lowest percentage margin; whereas, the lowest quality product gives the lowest absolute margin but the highest percentage margin. In contrast to these set of papers, our work takes the point of view of a retailer who caters to heterogenous consumers and faces demand uncertainty. The retailer selects an assortment from a finite number of exogenously available quality differentiated products, as opposed to a manufacturer who sets the quality levels of the product line. Moreover, retailer’s assortment decision is made taking into consideration inventory cost in addition to heterogeneity in consumer preferences. Although, [Bharghava and Choudhary 2001] and [Honhon and Pan 2012] analyze the retailer’s assortment and pricing decision, they do not consider demand uncertainty as we do, and hence inventory cost plays no role in assortment decision.

A vast body of the literature in operations management focuses on inventory, pricing, and/or variety decisions of horizontally differentiated products ([Kök et al. 2008] provides a detailed review of the literature in this area). The papers in this area can be broadly classified into three categories and we only mention the representative papers. The first category focuses on inventory and assortment planning decisions assuming prices are exogenously given (see for example [Cachon et al. 2005], [Li 2007], [Mahajan and van Ryzin 2001], [Smith and Agrawal 2000], [van Ryzin and Mahajan 1999]). The major findings in this category concerns the structure of the optimal assortment. With a multinomial logit (MNL) choice model and normally distributed demand, [van Ryzin and Mahajan 1999] finds that the optimal assortment consists of the most popular products. [Cachon et al. 2005] include consumer search and show that the retailer might stock unprofitable products. The second category focuses on inventory and pricing decisions assuming fixed assortment ([Aydin and Porteus 2008], [Cattani et al. 2003], [Petruzzi and Dada 1999] ). The last category focuses on assortment and pricing decisions under MNL choice model assuming infinite inventory availability. Two papers are reviewed here in order to compare with the manufacturer’s pricing and product line literature. [Aydin and Ryan 2002] show that
the optimal prices can be characterized by equal profit margins with the expected profit being unimodal in the common margin. [Hopp and Xu 2005] show that the optimal assortment for a risk-averse retailer is composed of the variants with the highest price markups markups.

The only paper that consider joint pricing, inventory and assortment decision using a MNL choice model in a newsvendor setting is [Maddah and Bish 2007]. They find that optimal prices are characterized by approximately equal margins, quite similar to the equal profit margin result in their infinite inventory case. In contrast, our results show that in the vertical model, the optimal margin of a product increases with its quality level. The optimal assortment in [Maddah and Bish 2007] consists of a set of products with the highest popularity or reservation prices. However, our results show that the optimal assortment consists of a mix of high and low quality products and we find it with the help of an efficient algorithm.

2.3. The Model

2.3.1 Consumer Choice Model

Consider that a consumer has to choose from a set of vertically differentiated products \( \Omega = \{1, 2, ..., n\} \) offered by the retailer. Let product \( i \) have a quality level \( q_i \) (common to all consumers) and product quality be universally ordered such as \( q_n > q_{n-1} > ... > q_1 \). The products are differentiated in quality, i.e., if any two products \( i \) and \( j \) have the same price, then a consumer would prefer product \( i \) over product \( j \) if \( q_i > q_j \). A consumer can also choose the no-purchase option, which is denoted as the product \( i = 0 \) in the assortment. We assume a typical consumer’s utility from the purchase of product \( i \) at price \( p_i \) can be expressed by the linear random utility model: \( U_i = \theta q_i - p_i \), where \( \theta \) is an independent random draw from a uniform distribution between 0 and 1, denoted by \( U[0,1] \). Here, \( \theta \) denotes consumer affinity towards quality; a higher
θ means a higher willingness to pay for a given quality. We normalize the value of the no-purchase option to zero, i.e., $U_0 = \theta q_0 - p_0 = 0$, by setting $q_0 \equiv p_0 \equiv 0$. A consumer of type $\theta$ chooses product $i$ if and only if $U_i \geq U_k, k \neq i, i = 0, 1, 2, \ldots, n$, or equivalently, $\theta q_i - p_i \geq \theta q_k - p_k \Rightarrow \theta \geq \frac{p_i - p_k}{q_i - q_k}, k \neq i, i = 0, 1, 2, \ldots, n$.. Following [Akçay et al.(2010)], the choice probabilities under the vertical choice model is given by

$$
\alpha_i(p) = \begin{cases} 
1 - \frac{p_i - p_{i-1}}{q_i - q_{i-1}}, & i = n, \\
\frac{p_{i+1} - p_i}{q_{i+1} - q_i} - \frac{p_i - p_{i-1}}{q_i - q_{i-1}}, & i = 1, 2, ..., n-1,
\end{cases}
$$

where, $p = (p_1, p_2, \ldots p_n)$ is the vector of prices. The probability that a customer will choose the outside option is given by

$$
\alpha_0(p) = 1 - \sum_{j=1}^{n} \alpha_j(p) = \frac{p_1}{q_1}.
$$

As argued in [Akçay et al.(2010)], in order to ensure that $\alpha_i(p) \geq 0$, it is sufficient to restrict the prices to the set of quality aligned prices $\tilde{P}$, given by

$$
\tilde{P} = \left\{ p : 1 \geq \frac{p_n - p_{n-1}}{q_n - q_{n-1}} \geq \frac{p_{n-1} - p_{n-2}}{q_{n-1} - q_{n-2}} \geq \ldots \geq \frac{p_2 - p_1}{q_2 - q_1} \geq \frac{p_1}{q_1} \geq 0 \right\}
$$

In (2.2), the prices $\{p_i\}$ form an increasing convex mapping of the quality values $\{q_i\}$ over the interval $[0,1]$. From (2.1) it can be seen that a demand substitution of product $i$ due to a change in $p_i$ can happen only to the adjacent products, which are gross substitutes ($\frac{\partial^2 \alpha_i(p)}{\partial p_j} > 0, j = i - 1, i + 1$) and this substitution occurs at a fixed rate. Also, the choice probability of product $i$ is decreasing at a constant rate in its own price ($\frac{\partial \alpha_i(p)}{\partial p_i} < 0$).
2.3.2 Consumer Demand Model

The economics and marketing literature mostly assumes that demand faced by a multi-product manufacturer is deterministic, since the manufacturer caters the market at the aggregate demand level. In contrast, at a retail store demand for each product is generated by a consumer choice process and hence is stochastic. To substantiate this we make two assumptions concerning the consumer choice process (these assumptions are also made in [Smith and Agrawal 2000] and [van Ryzin and Mahajan 1999]). First, we assume that consumers visit the store with the knowledge of the assortment but they are uninformed about the actual inventory level of each product. Upon store visit, consumers inspect the prices and quality levels of the offered products and choose their most preferred product with the probability given by the vertical choice model in (2.1). Finally, we assume that if the first choice product is out of stock, unmet demand is lost, and consumers prefer to go elsewhere rather than selecting another alternative from the assortment. Hence the products are not substitutes in the sense that consumers will dynamically substitute one variant for another if their first choice is out of stock. Rather, it is only a consumer’s initial choice that is influenced by the set of alternatives offered. These assumptions may be justified in certain settings. For example, customers of a catalog retailer do not know the inventory status of items prior to ordering. In a store setting, if customers choose based on inspection of ”floor models”, the status of the on-hand inventory is also not readily observable. Often by inspecting the assortment consumers learn about products and identify the one that they like best. If that product is out of stock, the newly-informed customer prefers to go elsewhere to obtain the preferred variant rather than settling for another alternative (occurs in one time purchases like apparel). While these assumptions are somewhat restrictive, they provide a reasonable starting point and simplify analysis. Assuming that a customer’s choice also depends on on-hand inventory introduces a complex demand process and requires sample path analysis ( [Mahajan and van Ryzin
With these rather simplistic assumptions on the consumer choice process, individual item demand can be quite complex as noted in the following. Let $D$ be the (discrete) random number of consumers who visit the store during the selling season, and $D$ has mean $\mu$ and variance $\sigma^2$. Let $D_i(p)$ be the total number of consumers whose first choice is product $i$, and let $\alpha_i(p)$ as given in (2.1), denote the probability that an arriving customer initially prefers product $i$. The demand of product $i$ is $D_i(p) = \sum_{i=1}^{D} Z_i$, where $Z_i$ is an i.i.d Bernoulli random variable with mean $E[Z_i] = \alpha_i(p)$. Thus the p.m.f. of $D_i(p)$ is given by

$$Pr(D_i(p) = d_i|D = d) = \binom{d}{d_i} \alpha_i(p)^{d_i}(1 - \alpha_i(p))^{d-d_i} Pr(D = d), \quad (2.3)$$

where, $Pr(D = d)$ is the p.m.f. of the store traffic. For arbitrary distribution on $D$, the exact distribution of $D_i(p)$ in (2.3) is difficult to derive. [Smith and Agrawal 2000] noted that the exact distribution of $D_i(p)$ in (2.3) has a fairly simple form only for certain choices of the distribution of $D$: if $D$ follows a Poisson distribution with rate $\lambda$, then $D_i(p)$ also follows Poisson distribution with rate $\lambda \alpha_i(p)$; and if $D$ follows a negative binomial distribution, so does $D_i(p)$. However, discrete store traffic complicates the analysis, since the exact demand distribution for $D_i(p)$ in (2.3) is complex for arbitrary distributions of $D$, and consequently the inventory-price optimization becomes a complex non-linear integer-programming problem. We therefore, work with a continuous approximation of the demand [Aydin and Porteus 2008] and [Li 2007]). We assume the demand of $i$ is given by the multiplicative form

$$D_i(p) = \mu \alpha_i(p) \varepsilon_i, \quad i = 1, 2, \ldots, n. \quad (2.4)$$

In (2.4), $\varepsilon_i$'s are correlated random variables; each $\varepsilon_i$ has unit mean and finite variance $\sigma^2_i$, and a distribution function that has increasing failure rate (IFR). The IFR as-
sumption implies that demand is stochastically decreasing in price in the hazard rate order. Most common distributions, such as uniform, normal, logistic, chi-squared, exponential and Laplace are IFR distributions. (2.4) says that the realized demand for a product is a multiplicative random perturbation of the expected demand $\mu \alpha_i(p)$. The use of multiplicative randomness implies that the coefficient of variation of the demand for each product is constant. We assume without loss of generality (using normalization) that $\mu = 1$. Under the model in (2.4), if item $i$ is not chosen by any consumer ($\alpha_i(p) = 0$), then $D_i(p) = 0$. Also, both mean demand $\alpha_i(p)$ and variance $\alpha_i^2(p)\sigma_i^2$, are decreasing in its own price $p_i$. Thus the retailer can use pricing as a lever to control demand uncertainty. In order to ensure that the probability of negative demand is negligible, we assume $\sigma_i \ll 1$, $\forall i$.

While the demand model in (2.3) admittedly captures the demand generated by consumer choice process, and could be useful for computing the prices and inventory levels exactly, using the continuous approximation in (2.4) helps in analytical tractability and finding structural insights into the properties of the optimal prices and assortments.

2.4. The Retailer’s Problem

2.4.1 The Risky Case

The retailer is a multi-item newsvendor, who sources from (multiple) suppliers and sells the products in a single selling period. Demand of each product is stochastic as given in (2.4), hence we refer this to be the risky case. Unmet demands become lost sales, and leftover inventory has no value (suitable for fashion and seasonal items). Let $\Omega = \{1, 2, ..., n\}$ be the set of products available to the retailer. Throughout the remainder of the paper, let $p = (p_1, \ldots, p_n)$, denote the vector of prices and $y = (y_1, \ldots, y_n)$, the vector of stock levels. Each product has a quality level $q_i$ and
unit purchasing cost $c_i$, which are exogenously given. The quality levels are ordered as $q_1 < q_2 < \ldots < q_n$. We do not assume that the unit costs necessarily increase in quality (this is plausible when there are multiple sources). Let $F_i(p, x)$ and $f_i(p, x)$ be the cumulative distribution function (c.d.f.) and the probability density function (p.d.f.) of $D_i(p)$ respectively, given the price vector $p$.

The retailer jointly determines the optimal set of products to offer, $S^* \subseteq \Omega$, the optimal prices $p_i^*, i \in S^*$, and the optimal inventory levels $y_i^*, i \in S^*$. This can be reduced to only joint pricing and inventory problem in the following sense. Note that, given a price vector $p$, the set of products with $\alpha_i(p) > 0$, $i \in \Omega$, constitute the offered assortment. This implies, the optimal assortment is automatically determined once retailer to finds the optimal prices and inventory levels. Therefore, retailer needs to solve the following:

\[
\max_{y \geq 0, p} \Pi(p, y) = \max_{y \geq 0} \left( \sum_{i=1}^{n} \left[ p_i E[\min(y_i, D_i(p))] - c_i y_i \right] \right),
\]

\[
\text{s.t. } 1 \geq \frac{p_n - p_{n-1}}{q_n - q_{n-1}} \geq \ldots \geq \frac{p_2 - p_1}{q_2 - q_1} \geq \frac{p_1}{q_1} \geq 0.
\]

(2.5)-(2.6) is a constrained multi-item joint inventory-pricing problem. The profit function in (2.5) is the usual multi-item newsvendor profit, and the constraint in (2.6) ensures non negative choice probabilities $\alpha_i(p) \geq 0$, $i \in \Omega$. [Petruzzi and Dada 1999] and [Whitin 1959] showed that for the single item newsvendor, the joint pricing and inventory problem is solved by first obtaining the optimal inventory level, $y^*(p)$, for a given price $p$ (the critical fractile solution), and then substituting $y^*(p)$ back into the profit function $\Pi(p, y)$ to obtain the induced profit function $\Pi(p, y^*(p))$. The optimal price $p^*$ is found by maximizing the induced profit function and consequently the optimal inventory level $y^*(p^*)$. Since we assume no stock-out based substitution, the inventory level of a product depends on its demand only. Therefore, for a given assortment, we can extend the solution method in [Whitin 1959] to find optimal price
and inventory levels of the multiple items. That is, for a given \( p \) satisfying (2.6), the optimization problem in (2.5)-(2.6) can be separated into \( n \) single product newsvendor problems, each concave in \( y_i \), and the optimal inventory level of product \( i \), \( y_i^*(p) \), is given by the usual critical fractile solution:

\[
F_i(p, y_i^*(p)) = 1 - \frac{c_i}{p_i}, \quad i \in \Omega. \quad (2.7)
\]

Next, substituting (2.7) in (2.5), the profit function becomes the following

\[
\Pi(p) = \sum_{i=1}^{n} p_i \int_{0}^{y_i^*(p)} xf_i(x, p) dx. \quad (2.8)
\]

Retailer maximizes the induced profit function in (2.8), subject to the constraint (2.6), in order to determine the optimal price vector \( p^* \) (see for example, [Porteus 2002]). In order to further simplify (2.8), we use the inverse demand function \( z_j(p, t) \) defined as

\[
F_i(p, z_i(p, t)) = t, \quad i = 1, 2, ..., n. \quad (2.9)
\]

With this definition, \( z_i(p, t) \) is the demand corresponding to the \( t \) fractile of \( F_i \), given prices \( p \), and thus \( y_i^*(p) = z_i(p, (1 - c_i/p_i)) \) (see [Aydin and Porteus 2008], [Lariviere and Porteus 2001]). Using this representation, the induced profit function in (2.8) can be written in the following form

\[
\Pi(p) = \sum_{i=1}^{n} \Pi_j(p) = \sum_{i=1}^{n} p_i \int_{0}^{1 - \frac{c_i}{p_i}} z_i(p, t) dt. \quad (2.10)
\]

In the rest of the paper we will use (2.10) to show structural results and analyze the optimal solution.

To summarize, the solution to the original joint inventory, price and assortment optimization problem is obtained by the following steps. Retailer first solves the
following in order to determine the optimal price vector \( \mathbf{p}^* \)

\[
\max_{\mathbf{p}} \Pi(\mathbf{p}) = \max \left( \sum_{i=1}^{n} p_i \int_{0}^{1} \frac{\epsilon_i}{p_i} z_i(p, t) dt \right),
\]

s.t. \( \frac{p_n - p_{n-1}}{q_n - q_{n-1}} \geq \cdots \geq \frac{p_2 - p_1}{q_2 - q_1} \geq \frac{p_1}{q_1} \geq 0. \)

(2.12)

Next, the optimal assortment is obtained as the set of products which have strictly positive choice probability: \( S^* = \{ i \in \Omega | \alpha_i(\mathbf{p}^*) > 0 \} \). Finally, the optimal inventory levels are obtained using (2.7). Therefore, price as the only decision variable in solving the retailer’s problem. Following proposition show structural properties of the optimization problem in (2.11)-(2.12).

**Proposition 2.4.1.** \( \Pi(\mathbf{p}) \) given in (2.11) is strictly pseudoconcave in \( \mathbf{p} \).

Proof in Appendix A. Figures 2.1(a) and 2.1(b) illustrates the plot of \( \Pi(\mathbf{p}) \) for two examples. By Proposition 2.4.1, the optimization problem in (2.11)-(2.12) turns out to be is a well defined constrained generalized convex program ( [Bazaraa et al. 2006], [Cambini and Martein 2009]), that can be solved by standard nonlinear
solvers. Moreover, since \( \Pi(p) \) is strictly pseudoconcave, the optimal price vector \( p^* \) is unique ( [Bazaraa et al. 2006], [Cambini and Martein 2009]). Consequently, the optimal assortment obtained as \( S^* = \{ i \in \{1, 2, ..., n\} | \alpha_i(p^*) > 0, \} \) and the optimal inventory vector \( y^* \), obtained using (2.7) are also unique. Proposition 2.4.1 generalizes the single-product version of this result in [Petruzzi and Dada 1999] using a similar approach. Proposition 2.4.1 implies that \( \Pi(p) \) is also jointly quasiconcave in the prices ( [Bazaraa et al. 2006]).

However, in order to obtain the optimal assortment retailer has to determine optimal prices of all \( n \) products, which can pose high computational burden if \( n \) is very large. We substantiate this using a numerical example represented in Figure 2.2.

![Figure 2.2: CPU time (in seconds) vs. \( n \) (number of products in \( \Omega \))](image)

We considered ten cases where the number of products varied from \( n = 20 \) to \( n = 110 \) in steps of 10, and assumed demand of each product is Normally distributed. For each case, the quality vector \( q = (q_1, ..., q_n) \) is set by randomly choosing \( q_i \in [10, 1000], \ i = 1, ..., n; \) the costs as \( c_i = \text{rand}(0,1) \times q_i, \ i = 1, ..., n. \) Further we assume the error terms of (2.4) are normally distributed with unit mean and standard deviation \( \sigma_i = \text{rand}(0,1) \times 0.3, \ i = 1, ..., n. \) Fig 2.2 shows how rapidly the CPU time

\( \)
to find the optimal prices increases with $n$. Implemented in Matlab 9 on a Dell Workstation with a 3.0GHz processor speed, we found that for $n = 100$ the time required is about 5 minutes, and for $n = 110$, it is as high as 16 minutes.

Moreover, solving (2.11)-(2.12) also requires estimates of the variance of the random error terms ($\sigma_i^2$) as well as the estimates of the quality levels ($q_i$’s) for all $n$ products. Therefore, a reduction of the problem size might be highly beneficial. We show in Proposition 2.4.6 that indeed the retailer can reduce the problem size, i.e., in order to obtain $S^*$, retailer needs to find the optimal prices only of a subset of the $n$ products, which is called the riskless optimal assortment. In the following subsection we therefore, introduce the riskless case, and then provide a algorithm to find the riskless optimal assortment.

2.4.2 The Riskless Case

The riskless case refers to situation when retailer faces deterministic demand and hence inventory is no longer a decision. This would be the case when retailer has perfect information about each consumer’s sensitivity to quality ($\theta$), and therefore knows exactly the probability with which each consumer will choose a product. In this case, demand of each product is simply the proportion of consumers who prefer that product the most, i.e., $D_i(p) = \mu \alpha_i(p)$, $i = 1, 2 \ldots, n$. ([Li 2007], [Netessine and Rudi 2003]). As mentioned in §2.3.2, we assume without loss of generality that $\mu = 1$.

The profit in the riskless case, $\Pi_0(p)$, is simply the sum of the revenues earned from each product:

$$\Pi_0(p) = \sum_{i=1}^{n} (p_i - c_i)\alpha_i(p).$$

(2.13)
The retailer determines the optimal prices, \( p_0^* = (p_{01}^*, \ldots, p_{0n}^*) \) by solving the following:

\[
\max_{\mathbf{p}} \Pi_0(\mathbf{p}) = \max_{\mathbf{p}} \sum_{i=1}^{n} (p_i - c_i) \alpha_i(\mathbf{p}), \tag{2.14}
\]

\[
s.t., 1 \geq \frac{p_n - p_{n-1}}{q_n - q_{n-1}} \geq \ldots \geq \frac{p_2 - p_1}{q_2 - q_1} \geq \frac{p_1}{q_1} \geq 0. \tag{2.15}
\]

The optimal riskless assortment can then be obtained as: \( S_0^* = \{ i \in \Omega \mid \alpha_i(p_0^*) > 0 \} \). We show that the profit function in (2.14) is concave in \( \mathbf{p} \) (see proof of Lemma 2.4.2 in Appendix), and the solution to the unconstrained optimization problem in (2.14) is given in the following.

**Lemma 2.4.2.** The optimal solution to the unconstrained problem in (2.14) is

\[
p_{0i}^* = \frac{q_i + c_i}{2}, \quad i \in \Omega. \tag{2.16}
\]

The optimal solution to the constrained problem (2.14)-(2.15), must satisfy \( \alpha_i(p_0^*) \geq 0, \quad i \in \Omega \). Substituting \( p_{0i}^* = \frac{q_i + c_i}{2}, \quad i = 1, \ldots, n \), in the choice probabilities in (2.1) we get

\[
\alpha_i(p_0^*) = \begin{cases} 
\frac{1}{2} \left( 1 - \frac{c_i - c_{i-1}}{q_i - q_{i-1}} \right) = \frac{\alpha_i(c)}{2}, & \text{if } i = 0, \\
\frac{1}{2} \left( \frac{c_{i+1} - c_i}{q_{i+1} - q_i} - \frac{c_i - c_{i-1}}{q_i - q_{i-1}} \right) = \frac{\alpha_i(c)}{2}, & \text{if } i = 2, \ldots, n - 1.
\end{cases} \tag{2.17}
\]

From (2.17) it is clear that, \( \alpha_i(p_0^*) \geq 0 \) if and only if \( \alpha_i(c) \geq 0 \). Thus, any feasible assortment \( S = \{ j_1, \ldots, j_m \} \subset \Omega \) must consist of products such that their marginal costs \( \{ c_{j_i} \}, j_i \in S \), form an increasing convex mapping of the quality values \( \{ q_{j_i} \}, j \in S \). In other words, the marginal costs belong to the set \( \tilde{C} \) defined as:

\[
\tilde{C} = \left\{ \{ c_{j_i} \} : 1 \geq \frac{c_{j_m} - c_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}} \geq \ldots \geq \frac{c_{j_2} - c_{j_1}}{q_{j_2} - q_{j_1}} \geq \frac{c_{j_1}}{q_{j_1}} \geq 0 \right\}. \tag{2.18}
\]

The condition in (2.18) can be depicted on a quality-cost graph: the marginal cost
function of a feasible assortment $S$, defined as $c(q_i) = c_i$, $i \in S$, is increasing convex in $q$, and is bounded above by the line $C(q) = q$. Figure 2.3(a) illustrates the quality-cost plot of the products in $\Omega \cup \{0\}$ (product 0 is the outside option), when the costs of the products in $\Omega$ increase convexly in quality. Similarly, Figure 2.3(b) shows the case when the costs of the products in $\Omega$ increase concavely in quality. In case of convex marginal cost, any subset of $\Omega$ satisfies the condition in (2.18), and in case of concave marginal cost, only the assortment $S = \{0, n\}$ satisfies (2.18). Similarly, if marginal cost is a linear function of quality, $S = \{0, n\}$ is the only assortment that satisfies (2.18). For any arbitrary cost function, there could be multiple feasible assortments $S \subseteq \Omega$, whose costs satisfy the condition in (2.18).

From the above discussion it is clear that when the marginal cost function is linear or concave, the optimal assortment is $S^* = \{0, n\}$, since it is the only feasible assortment. However, in case when marginal cost is convex or any arbitrary function of quality, there could be multiple assortments that satisfy the condition in (2.18). In such cases, the optimal assortment may be found by enumeration, however, that becomes computationally burdensome as $n$ increases. We derive the structure of the optimal assortment for any arbitrary cost function in Proposition 2.4.3. However, before stating it, we define the lowest increasing convex curve on the quality-cost plot.
of the products in $\Omega$ as follows:

**Definition 1.** : The lowest increasing convex curve on the quality-cost plot of $\Omega = \{1, 2, \ldots, n\}$ is formed by the products $S = \{j_1, \ldots, j_m\}$ with $q_{j_1} < q_{j_2} < \ldots < q_{j_m}$ and $c_{j_1} < c_{j_2} < \ldots < c_{j_m}$, such that

$$1 > \frac{c_{j_m} - c_{j_{m-1}}}{q_{j_m} - q_{j_{m-1}}} > \ldots > \frac{c_{j_2} - c_{j_1}}{q_{j_2} - q_{j_1}} > \frac{c_{j_1}}{q_{j_1}} > 0,$$

(2.19)

$$\frac{c_k - c_{j_i}}{q_k - q_{j_i}} > \frac{c_{j_{i+1}} - c_k}{q_{j_{i+1}} - q_k}, \text{ for } i = 1, \ldots, m - 1, \text{ and } k = j_i + 1, \ldots, j_{i+1} - 1.$$

(2.20)

$$\frac{c_k}{q_k} > \frac{c_{j_1}}{q_{j_1}}, k = 1, \ldots, j_1, \text{ and } \frac{c_k - c_{j_m}}{q_k - q_{j_m}} > 1, k = j_m + 1, \ldots, n.$$

(2.21)

The first condition of the definition, (2.19), simply implies that the costs $\{c_{j_i}\}, j_i \in S$, form an increasing convex mapping of the quality values $\{q_{j_i}\}$. The second condition, (2.20)-(2.21), implies that any product in $\Omega$, which is not in $S$, has a higher $c/q$ ratio than some a product on $S$ with a lower quality level.

![Figure 2.4: Lowest increasing convex curve on quality-cost](image)

Figure 2.4 illustrates the lowest increasing convex curve on the quality-cost. Here the curve consists of the products $\{4, 6, 10, 17, 23, 24\}$. Note that (2.21) is satisfied
since the products $\{1, 2, 3\}$ has higher cost-quality ratio than the first product on the curve which is $\{4\}$. Also (2.20) is satisfied since products $\{7, 8, 9\}$ has higher cost-quality ratios than product $\{6\}$, which lies on the curve. It is easy to see that (2.19) implies the products on the lowest increasing convex curve have increasing cost-quality ratios: $\frac{c_{j_m}}{q_{j_m}} > \ldots > \frac{c_{j_2}}{q_{j_2}} > \frac{c_{j_1}}{q_{j_1}}$. With this definition on hand we now state the optimal assortment in the riskless case.

**Proposition 2.4.3.** : The products on the lowest increasing convex cost curve of the quality-cost plot constitutes the optimal riskless assortment.

Proposition 2.4.3 states that among the feasible assortments $S \subseteq \Omega$, which form an increasing convex curve on the quality-cost plot of $\Omega$, the lowest one gives the highest profit. To see the intuition, note that the optimal margin of product $i$ is $p_i^\ast - c_i = (q_i - c_i)/2$, which can be represented as the vertical distance between the point $(q_i, c_i)$ and the line $C(q) = q$ on the quality-cost plot. Thus, lower the increasing convex curve, the higher are the profit margins of the products on that curve. In the cases of linear and concave marginal costs, the lowest increasing convex curve consists of the products $\{0, n\}$, and in case of convex cost, it consists of $\Omega \cup \{0\}$. Thus Proposition 2.4.3 provides a simple graphical insight into the structure of the optimal riskless assortment. For the more general case, when cost is any arbitrary function of quality, we developed the *Increasing Convex Envelope Algorithm*, which finds the lowest increasing convex cost curve on the quality-cost plot of $\Omega$ for any given cost function.

The algorithm, formally given below, checks iteratively which set of products satisfy the inequalities in (2.18). In the first iteration the algorithm assumes $S_0^\ast = \{0, 1, 2, \ldots, n\}$ and then finds the products $j \in S_0^\ast$ which do not satisfy the inequalities in (2.18). These products are removed from $S_0^\ast$ and $S_0^\ast$ is updated. In the next iteration, it checks if the remaining products satisfy the inequalities in (2.18), removes those products which do not satisfy (2.18) and updates $S_0^\ast$. This procedure is repeated until
all the remaining products satisfy (2.18). Thus when the algorithm stops, the cost-
quality curve of the assortment obtained forms the lowest increasing convex curve on
the quality-cost plot of $\Omega$. The algorithm is formally stated as follows.

**Algorithm 1**: Increasing Convex Envelope Algorithm

\begin{verbatim}
Inputs: $c, q, S_0^* = \{1, \ldots, n\}$, $m = |S_0^*|$, $B = \phi$

Step 1: 
if $m > 1$ then
  for $j = 1$ to $m$ do
    if $\frac{c_{j+1} - c_j}{q_{j+1} - q_j} \geq 1 \text{ OR } \frac{c_{j+1} - c_j}{q_{j+1} - q_j} \leq \frac{c_j - c_{j-1}}{q_j - q_{j-1}}$ then
      $B = B \cup \{j\}$
    end if
  end for
  if $B = \phi$ then
    return $S_0^*$
    Exit
  else
    $S_0^* := S_0^* \setminus B, m := |S_0^*|, c := c \setminus c(B), q := q \setminus q(B), B := \phi$
    GOTO Step 1
  end if
end if
\end{verbatim}

**Lemma 2.4.4.**: The Increasing Convex Envelope Algorithm gives the lowest increasing convex curve of the quality-cost plot.

**Corollary 2.4.5.** The complexity of the Increasing Convex Envelope Algorithm is $O(n^2)$.

The following example illustrates the working of the Increasing Convex Envelope Algorithm.

**Example 1.** Let $\Omega = \{1, 2, \ldots, 8\}$, $c = [6.817, 7.8608, 50, 18, 45, 40, 100, 200]$ and $q = [23, 38, 65, 80, 100, 150, 213, 388]$. The Increasing Convex Envelope Algorithm gives $S_0^* = \{2, 4, 8\}$. 
Fig 2.5 shows the cost-quality plot of the products in $\Omega$. The algorithm starts with $S^*_0 = \{0, 1, 2, ..., 8\}$. The $\frac{c_j+1-c_j}{q_j+1-q_j}$ ratios of the products in $S^*_0$ are given by

$$[0, 0.2963, 0.0695, 1.560, -2.133, 1.35, -0.1, 2.539, -0.571]$$

In the first iteration the algorithm eliminates products $\{1, 3, 5, 7\}$ from $S^*_0$ and updates $S^*_0$ to $\{0, 2, 4, 6, 8\}$. The $\frac{c_j+1-c_j}{q_j+1-q_j}$ ratios of the products in $S^*_0 = \{0, 2, 4, 6, 8\}$ are $[0.206, 0.241, 0.314, 0.252]$. Clearly, $\{6\}$ is eliminated because it violates the condition in (2.18). Hence the new $S^*_0$ is $\{0, 2, 4, 8\}$. In the third iteration no product is eliminated and hence the optimal assortment is found: $S^*_0 = \{0, 2, 4, 8\}$. Note in Figure 2.5 that there are no products that lie below the (blue) line joining the points $\{0, 2, 4, 8\}$.

The optimal prices can be obtained from (2.16) as $p^* = [22.9304, 49.0, 244.0]$ and the choice probabilities from (2.1) as $\alpha(p^*) = [0.0173, 0.0124, 0.3669]$. The optimal profit from $S^*_0$ is 53.4762.

Figure 2.5: Optimal assortment of Example 1 obtained by the Increasing Convex Envelope Algorithm

[Bharghava and Choudhary 2001] previously studied a setting similar to the riskless case. They provide conditions under which the optimal assortment contains
all n products, i.e., \( S_0^* = \{0, 1, 2, \ldots, n\} \) and conditions under which it contains only the product with the highest quality, i.e., \( S_0^* = \{0, n\} \). In contrast, our algorithm provides an efficient way to identify the optimal solution for any \((c, q)\). Our algorithm is similar in terms of computational time and complexity to the Zero Fixed Cost Algorithm in [Honhon and Pan 2012]. However, with the help of our algorithm we provide a graphical insight into the structure of the optimal riskless assortment, which is more easy to conceptualize.

### 2.4.3 Risky Case: Optimal Assortment

Using the Increasing Convex Envelope Algorithm, the retailer can find the most profitable set of products, if demand were deterministic (riskless case). However, in reality seldom this is the case, and as discussed in §2.3.2, the retailer actually faces stochastic demand (risky case). We now state our main result of the paper, which is on the optimal assortment of the risky case \((S^*)\).

**Proposition 2.4.6.** The optimal assortment in the risky case is a subset of the optimal riskless assortment: \( S^* \subseteq S_0^* \).

We show an example before discussing the implications and intuition of Proposition 2.4.6 in §2.4.4.

**Example 2.** Consider again the problem given in Example 1. Additionally, assume the random errors \( \varepsilon_i, i = 1, \ldots, 8 \), are normally distributed, with unit mean and variances \( \sigma_i = 0.131 i = 1, \ldots, 8 \). The optimal assortment obtained by solving (2.11)-(2.12) is \( S^* = \{0, 8\} \).

The optimal risky prices obtained by solving (2.11)-(2.12) are

\[
p^* = [21.5962, 35.6808, 61.0329, 75.1174, 93.8967, 140.8451, 208, 364.3192]
\]
Substituting $p^*$ in (2.1) we get, $\alpha(p^*) = [0, 0, 0, 0, 0, 0, 0, 0.61]$, hence $S^* = \{0, 8\}$. Recall that optimal riskless assortment in Example 1 was found to be $S_0^* = \{0, 2, 4, 8\}$. Figure 2.6 shows the cost-quality plot of $\Omega$ and highlights $S^*$.

![Figure 2.6: Optimal assortment in Example 2](image)

Proposition 2.4.6 has important implication in terms of reduction of the search space for the optimal assortment, and reduction of the number of parameters that needs to be known. To see this recall from the discussion in §2.4.1, that the optimal assortment is obtained from the optimal price vector $p^*$ as $S^* = \{i \in \Omega | \alpha_i(p^*) > 0\}$. By Proposition 2.4.6, we have $S^* \subseteq S_0^*$, meaning that $S^*$ can be obtained from the optimal risky prices of the products in $S_0^*$. Therefore, retailer needs to solve the constrained optimization (2.11)-(2.12) only for the products in $S_0^*$, which is a smaller size problem. Moreover, the effort required to find the optimal assortment without Proposition 2.4.6 involves the estimates of the variances of the random error terms $\varepsilon_i$’s, for a very large numbers of slow movers, which is typically the case in many product lines. Also, estimating the quality levels $q_i$’s, of all products is a tricky task, since the quality is often measured over several dimensions such as design, feature, functionalities, etc. With the result of Proposition 2.4.6, retailer needs the estimates
of quality, demand variance and costs of fewer number of products.

To give an example of the usefulness of Proposition 2.4.6 in terms of reduction in computation time to find the optimal assortment, we re-consider the numerical example shown in Figure 2.2. Using the same set of cost and quality parameters as before, we first find the riskless assortment with the *Increasing Convex Envelope Algorithm*, and then determine the optimal prices of the products in the riskless assortment by solving (2.11)-(2.12). Figure 2.7 shows the total CPU time as well as the break up of CPU times to find the lowest increasing cost curve and the optimal prices of the products on that curve (which in turn determines the optimal assortment of the risky case). Compared to the CPU time of 16 minutes required to solve (2.11)-(2.12) with a problem size $n = 110$ (see Figure 2.2), we now require only a fraction of a second to find the optimal assortment (see Figure 2.7). A more comprehensive study to demonstrate the savings in computation time is given in §2.5.

![Figure 2.7: CPU time (in seconds) vs. $n$](image-url)
2.4.4 Discussion

A central issue in retail merchandising is to determine what variants to stock within any given category of merchandise, and how much to stock? Product variety has been studied in the economics and marketing literature (see [Lancaster 1990] for a survey), but such assortment questions have not been addressed directly. In particular, the economics literature focuses primarily on variety at the market level (e.g., [Shugan 1989]), or variety in product line design (e.g., [Musa and Rosen 1978]). The marketing literature emphasizes on demand estimation aspect as well as pricing and variety decisions (e.g., [Moorthy 1984]), but ignores inventory considerations. In contrast, retail assortment planning requires detailed decisions on variety and inventory levels within each merchandize category and at each store level. Understanding assortment benefits and costs at the category-store level requires more detailed modeling of both the choice process and the inventory cost. While, adding variants to an assortment increases the likelihood that consumers will purchase something from the assortment; more choice alternatives results in thinning of the total demand, increases variability of demand for each variant, and in turn increases inventory costs. Several papers in the operations management literature has analyzed this aspect of variety-cost trade-off, by developing models that account for operational costs as well as consumer choice. With a focus on a horizontally differentiated merchandise (e.g., shirts of different color, ice cream with different flavors), each variant having the same exogenously given price and same unit cost, [van Ryzin and Mahajan 1999] show that the optimal assortment has a simple popular set structure, with items having the highest mean reservation prices. [Li 2007] extended this result to the case of unequal prices and show that the optimal assortment consists of first \( k \) products with highest profit rate, which is a combined measure of the profit margin, overage cost, and demand variability. [Maddah and Bish 2007] and [Maddah et al. 2011] consider a merchandize with items having distinct endogenous prices, and also find that the optimal assortment has items with
the largest popularity. Thus, even when the retailer has control over both the price and the variety level, inventory costs still restrict the variety level.

In our work, which focuses on a vertically differentiated merchandize, we find support for this result. With Proposition 2.4.6, one could consider only the products in the lowest increasing convex curve, which is the optimal assortment in absence of demand uncertainty, and confidently obtain the optimal assortment. Thus, there exists an upper limit on the variety level, that is, inventory cost limits the breadth level of the assortment. Further, even when the number of available variants is quite large, one can find the optimal assortment within a very short time, for any given cost and quality parameters, using the Increasing Convex Envelope Algorithm and the subset result in Proposition 2.4.6.

However, our result is in contrast to the economics literature which investigates product line decisions of a firm in presence of ex ante sunk costs and demand uncertainty. [Sheshinski and Dreze 1976] show that firms facing demand uncertainty will use a range of production technologies to meet demand and allocate their production across these production technologies to minimize costs given the realization of demand. [Carlton and Dana 2008] also show that in presence of sunk costs and uncertain demand, the firm produces multiple quality products as compared to a single product when demand is certain. Therefore, in these papers sunk cost is the driving force for product line proliferation. In our model although the retailer does not incur fixed costs associated with offering a product, unlike the sunk cost of the manufacturer, there are inventory costs associated with uncertain demand. Therefore, the retailer’s ex ante decision is to stock fewer number of products.

Our results of the riskless case can be compared with the economics and marketing literatures that have have focused exclusively on heterogeneous consumer preferences and asymmetric information as the reasons for product variety. An important part of this literature looks at the product line and pricing decisions of a monopolist engaging in second degree price discrimination when consumers valuations are correlated with
their preference for product quality ([Eaton and Lipsey 1989], [Lancaster 1990], [Musa and Rosen 1978], [Johnson and Myatt 2003], and [Shepard 1991]). Similar to our result, [Carlton and Dana 2008] also show that if demand were certain, then the highest quality product, which has the highest absolute margin will be produced by the firm. However, with our Increasing Convex Envelope Algorithm we can find the optimal assortment for any arbitrary marginal cost function and any quality vector.

Next we discuss how the risky prices compares with the riskless prices, which is an important question in the literature on the single item joint pricing and inventory problem.

Lemma 2.4.7. Optimal prices in the risky case are higher than optimal riskless prices: $p_i^* > p_{0i}^* = \frac{c_i + q_i}{2}$.

Lemma 2.4.7 supports previous results on the joint inventory and pricing problem for a single product ([Mills 1959], [Karlin and Carr 1962], [Petruzzi and Dada 1999]). [Petruzzi and Dada 1999] showed that in the single-product case with multiplicative uncertainty, the firms optimal price under stochastic demand is greater than the optimal price when demand is deterministic. For an additive demand function, [Mills 1959] finds that $p_i^* < p_{0i}^*$, on the other hand, for the multiplicative demand case, [Karlin and Carr 1962] prove that $p_i^* > p_{0i}^*$. [Young 1978] generalizes these two results by proving that (i) $p_i^* < p_{0i}^*$ if the demand variance and demand coefficient of variation are both non decreasing in $p_i$ and (ii) $p_i^* > p_{0i}^*$ if the demand variance and coefficient of variation are both non increasing in $p_i$. In our demand model, the variance of an item $i$ (given as $\alpha_i^2(p)\sigma_i^2$) is decreasing while the coefficient of variation $\sigma_i$ is constant, therefore we have $p_i^* > p_{0i}^*$. The intuition behind higher prices can also be explained as follows. If price of $i$ is set at its riskless price, then the revenue from $i$ is maximized, however inventory costs reduce the net profit. To reduce the effect of inventory cost, the variance $(\alpha_i^2(p)\sigma_i^2)$ needs to be reduced. This can be achieved by increasing $p_i$, since a higher $p_i$ reduces the choice probability $\alpha_i(p)$.
2.5. Numerical Study

The purpose of the numerical study is threefold. First, we conduct a numerical study to compare the computational time of the two ways to find the optimal assortment. The straightforward way is to solve for the optimal prices of all available products, which in turn gives the assortment as the set of products with non zero purchase probability. The other way, which we propose as an implication of Proposition 2.4.6, is to first find the riskless assortment using the Increasing Convex Envelope Algorithm, and then determine the optimal prices of the products in the riskless assortment. The results show that our proposed way becomes quickly beneficial in terms of reduced computation time as the number of products gets large. Next, we investigate the sensitivity of the optimal assortment w.r.t. perturbations in the quality values. Estimating the quality level of an item is a tricky task since quality is often measured along several dimensions such as design, feature, style, functionality etc. Because our model allows for only one dimension of quality, which can be considered as the combined measure of multiple quality aspects, we analyze how possible error in the quality estimate affects the optimal assortment. Again, estimating variances of the random error terms ($\varepsilon_i$’s) for all products may be challenging because many product lines have a large number of slow movers. Therefore, we also analyze how errors in variance estimates affects the optimal assortment.

In the numerical analysis we assume that the multiplicative random errors $\varepsilon_i$’s of the demand model given in (2.4) are normally distributed with unit mean and standard deviation $\sigma_i$. The normal demand assumption is consistent with literature ([van Ryzin and Mahajan 1999], [Maddah et al. 2011]). However, all the directional results hold for distributions that are IFR (e.g., uniform). Under normal demand, the induced profit function in (2.11) assumes the expression: $\Pi(p) = \sum_{j=1}^{n} p_j(1 - \phi(z_j)\sigma_j) - c_j\alpha_j(p)$, where, $z_j = \Phi^{-1}(1 - \frac{c_j}{p_j})$ is the critical fractile, and $\phi(.)$ and $\Phi(.)$ are the probability density function and the cumulative distribution function of the
standard normal distribution, respectively (see [Silver Pyke and Peterson 1998], pp. 404-408). The optimal risky prices $p^* = (p_1^*, \ldots, p_n^*)$ are found by solving the following:

$$\max_p \Pi(p) = \sum_{j=1}^n \mu \left[ p_j (1 - \phi(z_j) \sigma_j) - c_j \right] \alpha_j(p),$$  \hspace{1cm} (2.22)

s.t., $1 \geq \frac{p_{n-1}}{q_{n-1}} \geq \frac{p_{n-2}}{q_{n-2}} \geq \ldots \geq \frac{p_2}{q_2 - q_1} \geq \frac{p_1}{q_1} \geq 0.$ \hspace{1cm} (2.23)

The optimal assortment is found as $S^* = \{ i \in \{1, 2, \ldots, n\} | \alpha_i(p^*) > 0 \}$. We generate total 500 instances, in each instance the number of products $n$ is randomly chosen between 1 and 20. Next, the quality, cost and standard deviation of the error terms are generated as summarized in Table 2.1.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Symbol</th>
<th>Parameter values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quality</td>
<td>Exact quality</td>
<td>$q = (q_1, \ldots, q_n)$</td>
<td>$q_i = \text{rand}(0, 1) \times (b - a), i = 1, \ldots, n$, where $a$ is a random number in $[0, 10]$, and $b$ a random number in $[10, 1000]$</td>
</tr>
<tr>
<td></td>
<td>Erroneous quality</td>
<td>$\tilde{q} = (\tilde{q}_1, \ldots, \tilde{q}_n)$</td>
<td>$\tilde{q}<em>i = q_i + \text{rand}(0, 1) \times (q</em>{i+1} - q_i), i = 1, \ldots, n - 1.$</td>
</tr>
<tr>
<td>Cost</td>
<td>Convex cost</td>
<td>$c = (c_1, \ldots, c_n)$</td>
<td>$c_i = \frac{q_i^2}{Nq_n}, i = 1, \ldots, n$, where $N$ is a random integer in $[1, 10]$</td>
</tr>
<tr>
<td></td>
<td>Arbitrary cost</td>
<td>$c' = (c'_1, \ldots, c'_n)$</td>
<td>$c'_i = \text{rand}(0, 1) \times q_i, i = 1, \ldots, n.$</td>
</tr>
<tr>
<td>Std. Deviation</td>
<td>Exact Std. Deviation</td>
<td>$\sigma = (\sigma_1, \ldots, \sigma_n)$</td>
<td>$\sigma_i = \text{rand}(0, 1) \times 0.3, i = 1, \ldots, n.$</td>
</tr>
<tr>
<td></td>
<td>Erroneous Std. Deviation</td>
<td>$\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$</td>
<td>$\tilde{\sigma}_i = \text{rand}(0, 1) \times \sigma_i, i = 1, \ldots, n.$</td>
</tr>
</tbody>
</table>

Table 2.1: Numerical Experiment Design

For each $n$, we have two sets of quality, cost and standard deviation of the error terms. Thus for each $n$, we have 8 scenarios, and hence a total of $500 \times 8 = 4000$ cases. We first set the exact quality levels $q = (q_1, \ldots, q_n)$, and then generate the erroneous quality levels $\tilde{q}_1, \ldots, \tilde{q}_n$, where the error in any $q_i$ is incorporated by perturbing its
value: \( \tilde{q}_i = q_i + rand(0,1) \times (q_{i+1} - q_i) \). We assume two types of costs, one where cost is convex in quality, and in the other cost is an arbitrary function of quality. Convex cost is set by \( c_i = \frac{q_i^2}{N q_m} \), \( i = 1, \ldots, n \), and the arbitrary cost by \( c_i = rand(0,1) \times q_i \), \( i = 1, \ldots, n \).

Finally, the standard deviation of the error terms, \( (\sigma_1, \ldots, \sigma_n) \) are set by choosing \( \sigma_i = rand(0,1) \times 0.3 \), \( i = 1, \ldots, n \), and these values are then perturbed to obtain the erroneous standard deviations \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_n \).

### 2.5.1 Results

#### 2.5.1.1 Savings in Computation Time

In each of the 4000 scenarios, we first find the total time \( (t_1) \) required to find the optimal prices (by solving (2.22)-(2.23) ) and then to find the optimal assortment \( (S^*) \) using (2.1). Next, for each scenario we find the total time required \( (t_2) \) to first obtain the the riskless assortment \( (S^*_0) \) using the Increasing Convex Envelope Algorithm, and then to solve (2.22)-(2.23) with the products in \( S^*_0 \). After obtaining \( t_1 \) and \( t_2 \), we compute the % difference as \( \frac{t_1-t_2}{t_1} \). Figure 2.8 shows the average % difference across different values of \( n \), the total number of available products.

![Figure 2.8: % Savings in computation time vs. \( n \)](image-url)
Clearly, our proposed way of finding the optimal assortment becomes quite beneficial as the number of products increases. Often retailers need to select the assortment from a large pool of available products, and this result helps to reduce the computational effort by restricting the search to a particular set of products.

2.5.1.2 Impact of Error in Estimating Demand Variance

Estimating variances of the random error terms ($\varepsilon_i$’s) for all products may be challenging because many product lines have a large number of slow movers, so we analyze how errors in variance estimates affects the optimal assortment. Figure 2.9 shows that the percentage of cases where the optimal assortment changed when the variances of the error terms were altered is high and increases as $n$ changes. However, it shows that the percentage error in profit, due to the error in variance estimation are reasonably contained (5-6%).

![Figure 2.9: Impact of Error in Variance of Demand](image)

In Table-2.2 we show some instances of the optimal assortment in the risky case ($S^*$), as well as the assortment when variances of the errors are altered ($S^*_{er}$), assuming convex and arbitrary costs, respectively.
Table 2.2: Effect of error in variances under arbitrary and convex costs

We observed that for small values of $n$, $n < 10$, we were able to get $S^* \equiv S^*_{er}$ for both arbitrary and convex costs. Although the assortments were same the price vector of $S^*_{er}$ is different from that of $S^*$, because the variances are different in both assortments. Thus, in these cases, there were difference in the optimal profits, although the gap is reasonably contained. The managerial implication is that for small assortment size, errors in estimates of variances have less effect on the stability of the assortment.

2.5.1.3 Impact of Error in Quality Values

Estimating quality levels of products is a tricky task. For example, the quality of an iPad may not solely be a function of the memory size and whether or not it has 3G capabilities, it might depend on other factors like internal circuit design, functionalities, etc. Because our model allows for only one dimension for quality, we combine all of these features into a single one measure. This might be an erroneous quality estimate, and so we want to see how errors in quality estimate affects the
optimal assortment. Figure 2.10 shows that the percentage of cases where the optimal assortment changed when the quality levels were altered, and the percentage error in profit, are both high and increases as $n$ changes.

![Figure 2.10: Impact of Error in Estimation of Quality](image)

In Table-2.3 we show some instances of the optimal assortments of the risky case ($S^*$), as well as the assortment when the quality estimates are altered ($S_{er}^*$), assuming convex and arbitrary costs, respectively. We observed that as a result of altering the quality levels, the assortment changed ($S^* \neq S_{er}^*$) for both arbitrary and convex cost and this was found for all values of $n$. Also, in these cases, the % error in the optimal profit was significantly high. Thus, the assortment decision is very sensitive to the accuracy in the estimate of the quality value. With the availability of survey systems to capture data on consumer perception towards product features and quality, and with the availability of analytics techniques to assess product quality from the data, we believe, our results shows a pointer to the area that retailers should focus on for maximizing category profit.
Table 2.3: Effect of error in quality under convex and arbitrary costs

2.6. Conclusion

In this chapter we investigated the pricing, inventory and assortment decisions under both demand heterogeneity and demand uncertainty. In both cases, risky and riskless, we showed that optimal prices increase with quality so as to form an increasing convex mapping of the quality levels. In the riskless case, the optimal assortment consists of those products whose costs form the lowest increasing convex curve on the quality-cost graph and for any marginal cost function, our Increasing Convex Envelope Algorithm identifies the optimal assortment in polynomial time. In the risky case, we showed that the optimal pricing problem is a generalized convex program and can be solved with standard solvers. We also showed that the optimal assortment in the risky case is a subset of the optimal assortment in the riskless case and that the optimal risky prices are higher. To the best of our knowledge, this is the first attempt to characterize the optimal assortment under demand uncertainty for vertically differentiated products. One line of analysis for future research is derive optimal prices
and optimal assortment considering stock-out based substitution. In such model, one might need to approximate the objective function to find a tractable solution (as in [Smith and Agrawal 2000], and [Mahajan and van Ryzin 2001]). Another simplifying assumption we made was to ignore fixed costs and competition among products. In a more general setting, there would be multiple products within a category that compete for shelf space and budgetary allocations. Such a general model could be used to analyze the breadth and depth of the optimal assortment.
Chapter 3

Assortment Planning of Multi-category Products with Multiple Consumer Types

3.1. Introduction

In many settings, consumer choice involves purchase of products from multiple categories on a single purchase occasion. For example, consumers may choose several items in a hardware store, or a predefined set of concert tickets from a performing arts theatre. In each category consumers are offered several variants, often differentiated in quality (e.g., portable generators of different wattage). Purchase of products with multiple components also involve multi-category choice. Examples of such products are modular furniture (box frame, head board and corner poles), personal computers (hard drive, processor and accessories) and travel itineraries (flight and hotel). In these products each component belongs to a particular category, and in each component, there are several quality differentiated variants from which consumers can choose (e.g., processors with varying clock speeds, flights with one or more stops).

Some consumers evaluate products in each category independently and purchase a product only if it maximizes their utility compared to other products in that category and the outside option. For example, price-sensitive travelers may buy only the flight tickets from an online travel agency, and book the hotel through some other channel. Again, some consumers configuring a computer, may buy the essential components, such as processor, hard disk etc., from an e-tailer, and buy the accessories, such as mouse, speakers from some other store if that store offers a better deal. Such a behavior, commonly called cherry picking, may be understood as the behavior
of “selecting the best or most desirable” ([Fox and Hoch(2005)]). Cherry pickers are selective about which products or services they purchase and at what locations and prices. Therefore they distribute their shopping across stores to take advantage of discounts and different selections. [Levy and Weitz(2004)] define cherry pickers as customers who visit a store and buy only merchandise sold at deep discounts. In a retail context, cherry-picking can occur within a single store visit, as when consumers literally stand at an end-cap display of Mt. Rainier cherries and pick through every one, choosing only the largest and plumpest ([Fox and Hoch(2005)]). [Fox and Hoch(2005)] compare cherry pickers with single store shoppers and find that consumers with lower income and larger shopping baskets are more likely to engage in cherry picking.

On the other hand there are some consumers who buy products from multiple categories at one location on a single purchase occasion. For example, travelers who did not find time to do an extensive search for the best deals, might buy both flight and hotel from the same online travel agency. In a retail context, consumers who perceive a high disutility if required to visit multiple stores, buy all items they need from a single store. These consumers are called basket shoppers ([Bell and Latin(1998)]), since they are concerned with maximizing their basket value. A basket shopper will choose to buy from a store when her total basket utility at that store is higher than other alternatives, even if the store is not her first choice in all categories.

In reality the consumer population is a mix of basket shoppers and cherry pickers. Presence of these two types influences the pricing and assortment decisions for multi-category retailers. Retailers may wish to increase prices in one category and reduce in the other in such a way that the total basket utility is still high for basket shoppers. However, increasing prices in that one category will discourage cherry pickers. Similarly, offering deep discount in one category might be attract cherry pickers, but doing so might result in loss of revenue from basket shoppers. Retailers often adopt an efficient assortment strategy, which primarily seeks to find the profit maximizing
level of variety by eliminating low-selling products. However, if a retailer reduces
variety in all categories based on single-category analyses, the store might become
less attractive particularly for basket shoppers who wish to purchase from multiple
categories. Retailers are often aware of the proportions of these two types at the
aggregate level, but not the exact type of a particular consumer. Moreover, it might
be costly or even infeasible for retailers to identify the type of every customer and
customize the assortment and product prices accordingly. Rather retailers have to
offer "one price and variety for all". Therefore, in order to decide the assortment to
be offered and the product prices, it is important for retailers selling multi category
products to understand the purchase behaviors of these two types of consumers.

In this chapter we analyze the assortment and pricing problem of a retailer selling
products in two different categories to two types of consumers, cherry pickers and
basket shoppers. On each category there are several quality differentiated products
from which consumers can choose. A cherry picker evaluates products in each cate-
gory independently and purchases a product only if it maximizes her utility compared
to other products in that category and the outside option. In contrast, a basket shop-
per evaluates both categories together, and selects a configuration (a configuration is
comprised of one product from each category), only if the total utility of the config-
uration is higher than the utilities from other configurations and the outside option.
Both cherry pickers and basket shoppers are assumed to have correlated sensitivities
to the product qualities in the two categories. [Duvvuri et al. (2007)] report studies
from marketing that indicate consumers sensitivities are indeed strongly correlated
across product categories, both in unrelated product categories (e.g., ketchup, peanut
butter, stick margarine, toilet tissue) and in complementary products (e.g., spaghetti
and sauce).

In order to analyze the assortment and pricing decisions in presence of basket
shoppers and cherry pickers, we first consider two special cases where there is only
type of consumer, and determine the optimal prices and variety level offered in each
category. We find that when there are only cherry pickers, retailer can determine the optimal prices and assortments of the two categories separately, as in a single-category situation. The optimal prices in each category convexly increase in their quality, and the variety level offered in each category is the profit maximizing set of products. The optimal assortment of a category has a simple graphical representation; the products in the optimal assortment form the lowest increasing convex curve on the quality-cost graph of that category. On the other hand, when there are only basket shoppers, retailer offers only a certain number of configurations. The products which make these configurations are the same products offered to cherry pickers in each category. Thus we establish that regardless of the consumer type, retailer continues to offer the same variety in each category. Further, retailer will charge the same price for all products except for the lowest quality product offered in each category, which is priced lower for the basket shoppers than for the cherry pickers.

Based on these results we find that in presence of the two consumer types, retailer offers the same variety in each category as she would offer to each type when individually present. Further, retailer offers certain high quality products in the two categories at the same prices which she offers to cherry pickers. The pricing policy for the remaining lower quality products, however, depends on the relative proportion of the two types and the correlation between the two sensitivities. If cherry pickers are relatively low in proportion compared to basket shoppers, retailer has more flexibility in setting these prices. We find, numerically, that in this case, retailer sets a higher discount for the low quality products so as to attract basket shoppers. On the other hand, if cherry pickers are relatively high in proportion, retailer does not have the liberty to change the prices by as much in case when cherry pickers are low in proportion. We also find that if consumers have independent sensitivities for quality in the two categories, retailer may markup prices in one category, while sell products in the other category in discount.

The rest of this chapter is organized as follows. In §3.2 we review literature on
multi-category assortment planning and in §3.3 we describe the model setup. Subsequently, in §3.4.1 we consider the assortment and pricing decisions of the retailer assuming there are only cherry pickers, and in §3.4.2 we analyze the same assuming only basket shoppers. In §3.5 we present the retailer’s problem where there are both type of consumers, and conclude with a discussion on the nuances in setting prices when there are both consumer types. All proofs are given in Appendix B.

3.2. Literature Review

The assortment planning problem focuses on which subset of a given set of products to offer. [Pentico (2008)] reviews the body of work on the assortment problem in this sense. Our work is related to the subset of this research that deals with assortment planning in retail operations. Such problems have been studied in the economics, marketing and operations literature. However, most of the literature in operations deal with the assortment planning problem of a single category. Our two-category assortment planning problem is akin to multiple-category assortment planning studied in the marketing literature. See [Russell et al. (1997)] for a review. Assortment planning for multiple-categories has received scant attention in operations management. We review the few papers on multi-category assortment planning here. [Agrawal and Smith (2003)] deal with a problem of multi-category assortment planning in the absence of pricing. In their model, products are complementary in the sense that there exist customers who wish to purchase a set of products, and if one product is out-of-stock, then the demand for other products in the customers set may also be lost. [Cachon and Kök(2007)] also study assortment planning for complementary products. Specifically, they consider multiple competing firms, each of which is selling multiple categories whose demands are complementary. [Cachon and Kök(2007)] focus on the equilibrium assortments that arise in their competition model. We focus on a single retailer instead of modeling the competition among several firms. [Rodríguez and Ay-
din (2011)] focus on the assortment and pricing decisions of a configurable product with two complementary components, and analyze how the optimal prices and the optimal assortment of one category depends on the optimal prices and assortment of the other category. They use the multinomial logit model to represent consumer choice and assume demand is stochastic. Their results about assortment selection underscore the importance of the variants surplus, which is a measure of the variants profitability that depends not only on the customers utility from the variant, but also on the variants unit purchase cost, unit underage and overage costs, service level and demand variability. They show that, when choosing from two variants of a given component, the firm should pick the one with the higher surplus. However, they show that, when choosing from two variants that belong to different components, it is not enough to compare the variants surpluses. In such a case, one must take into account the complementarity between the demands of two components. They find that one could choose from two variants that belong to different components by comparing the attractiveness of the new configurations that are enabled by the addition of each variant. We focus on vertically differentiated variants in each category, however do not consider demand uncertainty. Additionally, we assume there are more than one consumer types present at the same time, and analyze its impact on assortment and pricing decisions.

3.3. Model Basics

We consider two categories \( I, J \), and there are several quality differentiated variants available in each category. Let the available variants in category-1 be \( I = \{1, 2, \ldots, n\} \), and the quality levels be ordered as \( q_1 < \ldots < q_n \). Similarly, the available variants in category-2 are \( J = \{1', 2', \ldots, m\} \), with quality levels \( q_1' < \ldots < q_m' \). Let \( p = \{p_1, \ldots, p_n\} \) be the price vector of the products in category-\( I \) and \( p' = \{p_1', \ldots, p_m'\} \) be the price vector of the products in category-\( J \).
3.3.1 Behavior of Basket Shoppers and Cherry Pickers

We assume consumers are utility maximizers and consumer utility is specified by a linear random utility model. Let $U_i = \theta_1 q_i - p_i$ be the utility of a consumer who buys only variant $i \in \mathcal{I}$, and let $U'_j = \theta_2 q'_j - p'_j$ be the utility from buying only $j \in \mathcal{J}$. In these utility models, the random variables $\theta_i, i = 1, 2$, capture consumer sensitivity towards quality in the two categories. We assume for $i = 1, 2$, $\theta_i \in [\theta_{iL}, \theta_{iH}]$, $\theta_{iL} \geq 0, \theta_{iH} = 1$, with probability density function $f_i(\theta_i)$, distribution function $F_i(\theta_i)$, and inverse hazard function $\eta_i(\theta_i)$. We also assume both $\theta_1$ and $\theta_2$ has increasing hazard rates. This assumption is satisfied by most common distributions such as uniform, normal etc. The no purchase option in each category is represented by product 0, with $q_0 \equiv q'_0 = 0$ and $p'_0 \equiv p_0 = 0$.

A cherry picker chooses variants from both the categories only if both categories are attractive compared to their respective outside option. In other words, a cherry picker will select configuration $[i, j], i \in \mathcal{I}, j \in \mathcal{J}$, if and only if the utility from product $i$, $U_i > \max\{0, U_k, k \neq i\}$; and also the utility from product $j$, $U'_j > \max\{0, U'_l, l \neq j\}$. Thus the probability of choosing a configuration $[i, j], i \in \mathcal{I}, j \in \mathcal{J}$, is given by

$$\alpha_{[i,j]}(p, p') = Pr\left(\theta_1 q_i - p_i \geq \max_{k \in \mathcal{I}}\{0, \theta_1 q_k - p_k\}, \theta_2 q'_j - p'_j \geq \max_{l \in \mathcal{J}}\{0, \theta_2 q'_l - p'_l\}\right) . \quad (3.1)$$

Note that a cherry picker can choose only $i \in \mathcal{I}$ or, only $j \in \mathcal{J}$. A cherry picker will choose only $i \in \mathcal{I} (j \in \mathcal{J} )$, if she finds a best pick $i \in \mathcal{I} (j \in \mathcal{J} )$, but does not find any $j \in \mathcal{J}(i \in \mathcal{I} )$ attractive. Thus the probability of choosing $[i, 0], i \in \mathcal{I}$ is

$$\alpha_{[i,0]}(p, p') = Pr\left(\theta_1 q_i - p_i \geq \max_{k \in \mathcal{I}}\{0, \theta_1 q_k - p_k\}, 0 > \max_{j \in \mathcal{J}}\{\theta_2 q'_j - p'_j\}\right) , \quad (3.2)$$

and the probability of choosing $[0, j], j \in \mathcal{J}$ is

$$\alpha_{[0,j]}(p, p') = Pr\left(0 > \max_{i \in \mathcal{I}}\{\theta_1 q_k - p_k\}, \theta_2 q'_j - p'_j \geq \max_{l \in \mathcal{J}}\{0, \theta_2 q'_l - p'_l\}\right) . \quad (3.3)$$
On the other hand a basket shopper chooses configuration \([i, j]\) if the total utility from \([i, j]\) is nonnegative and greater than the utility of other configurations. In other words, if the prices \(p, p'\) are offered, then a basket shopper selects \([i, j]\), \(i \in I, j \in J\), with probability

\[
\alpha_{[i,j]}^b(p, p') = Pr \left( \theta_1 q_i + \theta_2 q'_j - (p_i + p'_j) \geq \max_{k \in I, l \in J} \{0, \theta_1 q_k + \theta_2 q'_l - (p_k + p'_l)\} \right). \tag{3.4}
\]

It may be noted that for any given prices, \(p, p'\), \(\alpha_{[i,j]}^b(p, p') \geq \alpha_{[i,j]}^c(p, p')\), i.e., basket shoppers are more likely to choose a configuration \([i, j]\), \(i \in I, j \in J\), than cherry pickers, since basket shoppers are concerned with a net positive utility from the configurations. The exact expressions of the choice probabilities in (3.1) and (3.4) depends on the distribution functions of the two random variables \(\theta_i, i = 1, 2\), and the correlation structure between them. The two random variables could be perfectly correlated \((\theta_1 = \theta_2 = \theta)\) on one extreme, or independently distributed on the other extreme. If \(\theta_1\) and \(\theta_2\) are independent then (3.1) is simply the product of the probabilities of choosing \(i \in I\) and \(j \in J\).

### 3.3.2 Retailers Problem

In reality the customer population is a mix of basket shoppers and cherry pickers and retailer is not be able to distinguish between the consumer types. In such a situation retailer has to set the optimal prices and determine the optimal assortments in each category such that customers will select configurations according to their type. Note that it is possible to set the selling price of a product so high that no customer buys it, that is, such that the product has a zero purchase probability. In this case, removing that product does not affect the demand for other products and therefore, the optimal assortment never includes a product with zero purchase probability. Therefore we regard prices as the only decision variable in this problem and define the corresponding assortment as the set of the products with positive purchasing probability given the
prices (this is also noted in [Deb and Xu(2012)] and [Honhon and Pan(2012)])]. The retailer determines optimal prices $p^* = (p_1^*, \ldots, p_n^*)$ and $p'^* = (p'_1^*, \ldots, p'_m^*)$ by solving the following.

$$\max_{p, p'} \Pi(p, p') = \gamma \Pi^c(p, p') + (1 - \gamma) \Pi^b(p, p'), \quad (3.5)$$

where, $0 \leq \gamma \leq 1$ is the proportion of cherry pickers, and $1 - \gamma$ is that of basket shoppers. The first term of (3.5) is the profit from cherry pickers and is given by

$$\Pi^c(p, p') = \sum_{i \in I} \sum_{j \in J} \alpha^c_{i,j}(p, p')(p_i + p'_j - c_i - c'_j) + \sum_{i \in I} \alpha^c_{i,0}(p, p')(p_i - c_i) + \sum_{j \in J} \alpha^c_{0,j}(p, p')(p'_j - c'_j). \quad (3.6)$$

The second term of (3.5) is the profit from basket shoppers:

$$\Pi^b(p, p') = \sum_{i \in I} \sum_{j \in J} \alpha^b_{i,j}(p, p')(p_i + p'_j - c_i - c'_j). \quad (3.7)$$

We show that $\Pi^c(p, p')$ and $\Pi^b(p, p')$ are each bi-concave and so $\Pi(p, p')$ in (3.5) is a bi-concave function (see Lemma B.0.7 in Appendix B). Biconvex programs are global optimization problems and are typically solved by specially designed algorithms that rely on iteratively solving subproblems, where each subproblem finds a subset of the decision variables. Depending on the parameters in the problem and the underlying structure of the subproblems, the global optima may or may not be found easily (see [Gorski et al.(2007)] for a comprehensive survey of biconvex functions and solution methods of biconvex programming). In Algorithm 3 (Appendix B) we outline the Alternate Convex Search Algorithm of solving (3.5) and show the proof of its convergence. We found that the run time of this algorithm becomes significantly high as the problem size increases. Implemented in Matlab® 9 on a Dell Workstation with a 3.0GHz processor, a problem which had to find prices of five products in each
category assuming uniformly distributed and perfectly correlated sensitivities in the
two categories, took about 26 minutes. Therefore, solving (3.5) for any arbitrary
problem size and arbitrary set of cost and quality vectors, and arbitrary distribution
the two sensitivities may become too computation intensive. In §3.4 we consider
two special cases where there are only basket shoppers and only cherry pickers. We
compare the pricing and assortment decisions of these two special cases, and find
properties of the problem that helps bring a reduction in the problem size when there
both types are present. Specifically, we show that retailer needs to solve for prices of
only a certain low quality products in the two categories, which can be found with the
help of the Alternate Convex Search Algorithm, while prices of the remaining products
are fixed.

3.4. Assortment and Pricing with One Consumer
Type

3.4.1 Assortment and Pricing with only Cherry Pickers

In this section we analyze the assortment and pricing decisions assuming there are
only cherry pickers (i.e., when $\gamma = 1$). The profit in this case as given in (3.6) can be
further simplified as follows:

$$
\Pi^c(p, p') = \sum_{i \in I} (p_i - c_i) \left( \sum_{j \in J} \alpha_{[i,j]}^c(p, p') + \alpha_{[i,0]}^c(p,p') \right) 
+ \sum_{j \in J} (p_j' - c_j') \left( \sum_{i \in I} \alpha_{[i,j]}^c(p, p') + \alpha_{[0,j]}^c(p,p') \right). 
$$

(3.8)

In (3.8), the term $\sum_{j \in J} \alpha_{[i,j]}^c(p, p') + \alpha_{[i,0]}^c(p,p')$ denotes the fraction of cherry pickers
who select $i \in I$ in combination with other products from category $J$, including the
no-purchase option in category $\mathcal{J}$. Using (3.1) this term can be written as follows

$$
\sum_{j \in \mathcal{J}} \alpha_{i,j}^c(p, p') + \alpha_{i,0}^c(p, p') = Pr\left(\theta_1 q_i - p_i \geq \max_{k \in \mathcal{I}} \{0, \theta_1 q_k - p_k\}, \theta_2 \leq \theta_2 \leq 1\right),
$$

$$
= Pr\left(\theta_1 q_i - p_i \geq \max_{k \in \mathcal{I}} \{0, \theta_1 q_k - p_k\}\right) \equiv \alpha_i^c(p), \quad (3.9)
$$

where, $\alpha_i^c(p)$ denotes the probability of choosing $i \in \mathcal{I}$ when there is only category-$\mathcal{I}$, and is given by ( [Akčay et al.(2010), Honhon and Pan(2012)]),

$$
\alpha_i^c(p) = \begin{cases} 
F_1\left(\frac{p_{i+1} - p_i}{q_{i+1} - q_i}\right) - F_1\left(\frac{p_i - p_{i-1}}{q_i - q_{i-1}}\right), & i = 1, \ldots, n - 1, \\
1 - F_1\left(\frac{p_i - p_{i-1}}{q_i - q_{i-1}}\right), & i = n.
\end{cases} \quad (3.10)
$$

Similarly, it can be shown that the fraction of cherry pickers who select $j \in \mathcal{J}$ in combination with other products from category $\mathcal{I}$ is given by $\sum_{i \in \mathcal{I}} \alpha_{i,j}^c(p, p') + \alpha_{0,j}^c(p, p') \equiv \alpha_j^c(p')$. Therefore, (3.8) can be written as

$$
\Pi^c(p, p') = \Pi(p) + \Pi(p'), \quad (3.11)
$$

where $\Pi(p) = \sum_{i \in \mathcal{I}} (p_i - c_i)\alpha_i^c(p)$, and $\Pi(p') = \sum_{j \in \mathcal{J}} (p_j' - c_j')\alpha_j^c(p')$. Hence the problem separates to two independent problems: $\max_{p,p'} \Pi^c(p, p') = \max_p \Pi(p) + \max_{p'} \Pi(p')$, where each one finds the optimal prices of a single category. Due to the separability of the problem, the optimal prices and assortments in each category are same as those of a single category problem as addressed in [Deb and Xu(2012)] and [Honhon and Pan(2012)]. In Proposition 3.4.1 we formally state the optimal prices and assortments in the two categories. However, before that we recall the concept of cost-curve of the products in a category discussed in Chapter 2. The cost-curve of the products in any category, is formed by joining the $n$ points $(c_1, q_1), \ldots, (c_n, q_n)$ on the quality-cost graph. Further for any arbitrary cost curve, the definition of the lowest increasing convex cost curve on the quality-cost plot is given in Definition 1.
Figure 3.1: Lowest increasing convex curve on quality-cost

Figure 3.1 (reproduced from Figure 2.4 in Chapter 2) illustrates the lowest increasing convex curve on the quality-cost for an example. Given any arbitrary cost curve, the lowest increasing convex curve can be found using the Increasing Convex Envelope Algorithm discussed in Chapter 2. With a brief review of these concepts, we now formally state the optimal prices and assortments in Proposition 3.4.1 below.

Proposition 3.4.1. (a) The optimal prices offered to cherry pickers are

1. Category $I$: $p_i^* = x_i q_1, p_i^* = p_{i-1}^* + x_i (q_i - q_{i-1}), i = 2, \ldots, n$, where $x_i, i = 1, \ldots, n$, are obtained by solving $x_i = \eta_1(x_i) + \frac{c_i - c_{i-1}}{q_i - q_{i-1}}, i = 1, \ldots, n$.

2. Category $J$: $p_j^* = x_j q'_1, p_j^* = p_{j-1}^* + x_j (q'_j - q'_{j-1}), j = 2, \ldots, m$, where $x_j, j = 1, \ldots, m$, are obtained by solving $x_j = \eta_2(x_j') + \frac{c'_j - c'_{j-1}}{q'_j - q'_{j-1}}, j = 1, \ldots, m$.

(b) The optimal assortments offered to cherry pickers are

1. Category $I$: $S_I^* = \{i_1, \ldots, i_k\}$, where $\{i_1, \ldots, i_k\}$ constitute the lowest increasing convex cost curve of category $I$.

2. Category $J$: $S_J^* = \{j_1, \ldots, j_l\}$, where $\{j_1, \ldots, j_l\}$ constitute the lowest increasing convex cost curve of category $J$. 
A cherry picker evaluates each category separately, and selects a configuration if each product in the configuration is attractive on its own. Due to this retailer can set optimal prices and assortments in each category separately. As given in part (a) of Proposition 3.4.1, the optimal prices of the variants in each category increase convexly in the quality levels. Also, the price markup of any variant over the price of the variant with next lower quality level, increases with the quality difference between the two variants. It may be noted from part (a), that the optimal price of a product can be determined independent of the optimal prices of the other products in the category.

Part (b) of Proposition 3.4.1 states that the optimal assortment in a category is the set of products which form the lowest increasing convex cost curve on the cost-quality graph of a category. These set of products maximize the category profit ( Deb and Xu (2012)).

[Honhon and Pan(2012)] showed that the optimal assortment of a single category in absence of fixed costs, is independent of the distribution of consumer sensitivity to quality. Our result extends this to the case where there are more than one category with different distributions of consumer sensitivity in each category. Further we establish that the two optimal assortments are independent of the correlation between the two sensitivities. This is admittedly a strong result that gives a simple guideline when retailers have to choose assortments in more than one category.

An important point we discuss is how the market shares of the configurations, and the market share of products in the two categories depend on the correlation between the two sensitivities. On one extreme the two sensitivities are independently distributed, and on the other they are perfectly correlated. In case when the two sensitivities $\theta_1$ and $\theta_2$ are independently distributed, the choice probability of any configuration $[i,j]$ given in (3.1) assumes the product form, i.e., $\alpha_{i,j}[p,p'] = \alpha_i^c(p)\alpha_j^c(p')$. Thus, as long as prices are set such that cherry pickers find positive utility from each product in category-$\mathcal{I}$, and each product in category-$\mathcal{J}$, all possible configurations will be selected. Figure 3.2(a) illustrates how cherry pickers choose configurations.
when \( \theta_1 \) and \( \theta_2 \) are independently distributed. This Figure depicts an example where the two optimal assortments are \( S_I^* = \{1, 2, 3\} \), \( S_J^* = \{1', 2', 3', 4'\} \), and the optimal cutoffs are \( x_{iu} \), \( u = 1, 2, 3 \), \( x'_{jv} \), \( v = 1, 2, 3, 4 \). A cherry picker with sensitivity \( \theta_1 \) in category-\( I \) selects \( i_u \in \{1, 2, 3\} \) iff \( x_{iu} + 1 \geq \theta_1 \geq x_{iu} \). Similarly, in category-\( J \), she selects \( j_v \in \{1', 2', 3', 4'\} \) iff \( x'_{jv} + 1 \geq \theta_2 \geq x'_{jv} \). Since no two cutoffs are the same in any category, each product in the two categories have non negative demand. Hence, all the \( 3 \times 4 \) configurations are selected.

Now we consider the case when \( \theta_1 \) and \( \theta_2 \) are perfectly correlated (\( \theta_1 = \theta_2 = \theta \)). In this case, a cherry picker will select a configuration \([i_u, j_v]\) iff \( x_{iu} + 1 \geq \theta \geq x_{iu} \) and \( x'_{jv} + 1 \geq \theta \geq x'_{jv} \), or equivalently, iff \( \min\{x_{iu+1}, x'_{jv+1}\} \geq \theta \geq \max\{x_{iu}, x'_{jv}\} \). Suppose \( x_{iu}, u = 1, 2, 3 \), and \( x'_{jv}, v = 1, 2, 3, 4 \), are related as follows: \( x_1 < x_2 < x_3 \). Cherry pickers with sensitivity \( \theta \), where \( x_1 < \theta < x_2 \), selects \([i_1, 0]\). Further, cherry pickers with sensitivity \( \theta \), where \( x_2 < \theta < x_3 \), selects \([i_2, j_1]\). It can be noted that cherry pickers with sensitivity \( \theta < x'_1 \) do not select any product from category \( J \), since they do not derive positive utility from any product from category \( J \). Only cherry pickers those with sensitivity \( \theta > x'_1 \) select products from both categories. Clearly, in this case \([i_1, j_1]\) is not selected by any cherry picker (but \([i_1, j_1]\) is selected when the sensitivities are independent).

Therefore, the market share of the configurations depend on the correlation between the two sensitivities. However, from (3.9) it can be observed that the market share of a product in a category only depends on the distribution of consumer sensitivity in that category, and is not affected by the correlation between the two sensitivities. We formally state this observation in the remark below. The correlation between \( \theta_1 \) and \( \theta_2 \) affects the market shares of the configurations, but not the market shares of the products in each category.

The market share of a particular product in a category is the fraction of consumers who select that product along with products from the other category. Depending on
the correlation between $\theta_1$ and $\theta_2$, certain configurations get selected, and the total market share of that product gets distributed among those configurations.

![Diagram illustrating configurations selected by cherry pickers](image)

(a) $\theta_1$ and $\theta_2$ are independently distributed

(b) $\theta_1$ and $\theta_2$ are perfectly correlated

Figure 3.2: Configurations selected by cherry pickers

### 3.4.2 Assortment and Pricing with only Basket Shoppers

In this section we analyze the assortment and pricing decisions assuming there are only basket shoppers (i.e., when $\gamma = 0$). A basket shopper is one who evaluates the total utility from a configuration, and selects the one that gives maximum utility among all possible configurations. The quality level of a configuration is simply the sum of the quality of the two components that goes into it. Let $\Omega$ denote the set of all $n \times m$ configurations, which are ranked according to increasing quality
levels. A basket shopper’s choice of a configuration \([i, j]\) is the same as choosing a product with quality \(q_i + q'_j\) from a single category that has \(m \times n\) quality differentiated products, that has highest quality \(q_n + q'_m\) and lowest quality \(q_1 + q'_1\). Consequently, the set of configurations chosen by basket shoppers is that which forms the lowest increasing convex cost curve on the quality-cost graph of \(\Omega\) ([Honhon and Pan(2012)], [Deb and Xu(2012)]). This curve can be obtained using the Increasing Convex Cost Algorithm of [Deb and Xu(2012)]. It turns out, as the following Lemma states, that the lowest increasing convex cost curve on the quality-cost graph of \(\Omega\) is constructed with products which constitute the lowest increasing convex cost curves of the two categories.

**Lemma 3.4.2.** The optimal configurations offered to basket shoppers are \(S^* = \{[i_1, j_1], \ldots, [i_k, j_l]\}\), where, \(\{i_1, \ldots, i_k\}\) constitutes the lowest increasing convex cost curve of category \(\mathcal{I}\), and \(\{j_1, \ldots, j_l\}\) constitutes that of category \(\mathcal{J}\).

Lemma 3.4.2 states that the products offered to basket shoppers in a category are those which also constitute the optimal assortment offered to cherry pickers in that category. Thus, retailer offers the category level best variety in each category to both basket shoppers and cherry pickers. Our result supports the common practice of category management (CM) followed by multi category retailers. CM is an example of a decentralized regime for controlling assortment in multiple categories, where each category manager is entrusted with maximizing profit for his or her assigned category. CM is used in practice, because it is easily manageable, although it might ignore, the impact of cross-category interactions. We establish that regardless of the consumer type, retailer continues to offer the same variety in each category. Further, the variety level of each category is the profit maximizing set of products. We illustrate the result of Lemma 3.4.2 with the following example, and show how the optimal configurations for basket shoppers can be determined, given the cost and quality levels of the products in the two categories.
Example 3. Consider two categories, $\mathcal{I} = \{1, 2, 3, 4, 5\}$, and $\mathcal{J} = \{1', 2'\}$. Let $q = (560, 688, 723, 844, 925)$ and $c = (169.52, 255.86, 282.55, 385.05, 462.5)$ be the cost and quality vectors of category $\mathcal{I}$, and $q' = (1084, 1644)$ and $c' = (238.25, 548)$ be those of category $\mathcal{J}$. Both categories have convex cost curves, such that all products form the respective lowest increasing convex cost curve of the the cost-quality graph of that category. There are 10 possible configurations: $\Omega = \{[5, 2'], [4, 2'], [3, 2'], [2, 2'], [1, 2'], [5, 1'], [4, 1'], [3, 1'], [2, 1'], [1, 1']\}$, in the order of decreasing quality levels. The lowest increasing convex cost curve of the cost-quality graph of $\Omega$ can be obtained using the Increasing Convex Cost Algorithm as $S^* = \{[1, 1'], [1, 2'], [2, 2'], [3, 2'], [4, 2'], [5, 2']\}$. Clearly, $S^*$ is formed with $\{1, \ldots, 5\}$ and $\{1', 2'\}$, which are the lowest increasing convex cost curves of each category. Figure 3.3 illustrates this example.

Figure 3.3: Optimal configurations for basket shoppers.
An immediate implication of Lemma 3.4.2 is the following

**Corollary 3.4.3.** The minimum number of configurations offered to basket shoppers is \( \max\{l, k\} \), where \( k = |S_I^*_p| \) and \( l = |S_J^*_p| \).

Corollary 3.4.3 quantifies the minimum number of configurations retailer should offer so that all products of the optimal assortments of the two categories are represented in the assortment for basket shoppers. As an example confirming Corollary 3.4.3, a search for a two day round trip, from SCE to DTT on the Expedia website, showed up 44 flights, 216 hotels, and 238 flight+hotels.

We now discuss the optimal prices offered to basket shoppers. In presence of only *basket shoppers*, retailer’s problem is akin to a single category problem that has \( m \times n \) quality differentiated products. Therefore, retailer can determine the optimal prices of the \( m \times n \) configurations using Proposition 3.4.1. Let \( \overline{p} = (\overline{p}_{1,1}, \ldots, \overline{p}_{k,l}) \) be the vector of the optimal prices of the configurations. Further, the optimal price of any configuration can be arbitrarily split into two prices of the two variants that go into the configuration. Consequently, an interesting question to ask is how should the price of a configuration be split into the prices of its components. Let the optimal prices of the configurations be split into the prices of the components as \( \bar{p}_{iu}, u = 1, \ldots, k \) and \( \bar{p}'_{jv}, v = 1, \ldots, l \). Recall that the optimal prices offered to cherry pickers are \( p_{iu}^*, u = 1, \ldots, k \), in category-\( I \), and \( p'_{jv}^*, v = 1, \ldots, l \), in category-\( J \). Lemma 3.4.4 gives the relation of prices paid by basket shoppers and cherry pickers for the pairs they select.

**Lemma 3.4.4.** The price paid by basket shopper and cherry picker are related by:

\[
\bar{p}_{i1} + \bar{p}'_{j1} \leq p_{i1}^* + p_{j1}^*, \quad \text{and} \quad \bar{p}_{iu} + \bar{p}'_{jv} = p_{iu}^* + p'_{jv}^*, \quad (i_u, j_v) \in S^*, u > 1, v > 1.
\]

Lemma 3.4.4 states that basket shoppers and cherry pickers pay the same price for all configurations except the one with the lowest quality, i.e., the configuration \([i_1, j_1]\). Thus retailer can offer the same prices to basket shoppers as she would offer
to cherry pickers for all the high quality configurations \( ([i_u, j_u] \in S^*, u > 1, v > 1) \). The intuition behind this result is that cherry pickers with higher levels of sensitivity in the two categories select the same high quality configurations which basket shoppers also select. These configurations are \( ([i_u, j_u] \in S^*, u > 1, v > 1) \), and therefore the same price can be charged to the two types for these configurations.

However, at lower levels of sensitivity, cherry pickers may not select configurations that basket shoppers select. Therefore, prices for lower quality configurations must be reduced in order to induce basket shoppers. In order to set the price for the configuration \([i_1, j_1]\), it is sufficient for retailer to lower the price of the higher of the two prices \(p^*_1\) and \(p'^*_1\), such that the total price \(p^*_1 + p'^*_1\) reduces to match basket shopper’s expectation. Suppose (w.l.o.g.), \(p'^*_1 = \max\{p^*_1, p'^*_1\}\). In §3.4.1 we noted that the prices of any product can be determined independent of the other products in the category. Therefore, changing \(p'^*_1\) does not affect the optimal prices \(p'^*_2, \ldots, p'^*_l\) in category-\(J\). By setting the price to some \(p < p'^*_1\), retailer can induce basket shoppers to select some configurations which would not be selected otherwise. However, with the gain in market share of configurations, there is also loss in profit margin. This trade off between market share and profit margin determines how much to reduce the price \(p'^*_1\).

Figure 3.4: Selection of Configurations

Suppose \(p'^*_1\) is lowered to \(p\), or equivalently the cutoff for \(j_1\), which is \(x'_{j_1} (= p'^*_1 / q'^*_1)\) is lowered to \(x (= p / q'^*_1)\), such that \(x \in (x_i, x'_{j_1})\) (see Fig 3.4). Let the market share of the configuration \([i, j_1]\) with the price \(p\) be denoted by \(\alpha_{[i, j_1]}(p^*, p, p'^*)\). Then, the
revenue earned from configuration \([i, j_1]\) is 
\[ \Pi(p) = \alpha_{[i, j_1]}^b(p^*, p, p^*) (p^*_i + p - c_i - c'_{j_1}) \]
Suppose instead, \(x'_{j_1}\) is lowered to \(x\), where \(x \in (x_{i-1}, x_i]\). Then the configuration \([i - 1, j_1]\) is also selected along with \([i, j_1]\). In this case the total revenue earned is 
\[ \Pi(p) = \alpha_{[i, j_1]}^b(p^*, p, p^*) (p^*_i + p - c_i - c'_{j_1}) + \alpha_{[i-1, j_1]}^b(p^*, p, p^*) (p^*_{i-1} + p - c_{i-1} - c'_{j_1}) \]
In this fashion, the revenue function can be written as

\[
\Pi(p) = \left\{ \begin{array}{ll}
\alpha_{[i, j_1]}^b(p^*, p, p^*) (p^*_i + p - c_i - c'_{j_1}) & , p \in (q'_{j_1} x_i, q'_{j_1} x'_{j_1}], \\
\sum_{k=i-1}^i \alpha_{[k, j_1]}^b(p^*, p, p^*) (p^*_k + p - c_k - c'_{j_1}) & , p \in (q'_{j_1} x_{i-1}, q'_{j_1} x_i], \\
..... & ,..., \\
..... & ,..., \\
\sum_{k=i_1}^i \alpha_{[k, j_1]}^b(p^*, p, p^*) (p^*_k + p - c_k - c'_{j_1}) & , p \in (q'_{j_1} x_{i_1}, q'_{j_1} x'_{j_1}], 
\end{array} \right. 
\tag{3.12}
\]
We show that \(\Pi(p)\) in (3.12) is pseudoconcave in each of the intervals of \(p\) and that it is continuous across the intervals (see Lemma B.0.6 in Appendix B). Hence the optimal price \(p^*\) can obtained by solving \(\max_p \Pi(p)\), where \(p\) can be in any of the intervals, i.e., \(p \in [q'_{j_1} x_{i_1}, q'_{j_1} x'_{j_1}]\).

The question is in which of these intervals does \(p^*\) lie. Note that the profit margin obtained from any configuration \([k, j_1], k \in \{i_1, \ldots, i\}\), given by \((p^*_k + p - c_k - c'_{j_1})\) reduces as \(p\) reduces, while its market share increases. Since the revenue is the product of profit margin and market share, it is not profitable to lower \(p\) to a point such that \(p^*_k + p - c_k - c'_{j_1} < 0\). Substituting \(p = x q'_{j_1}\), we can rewrite the condition \(p^*_k + p - c_k - c'_{j_1} > 0\), as \(x > \frac{c'_{j_1} - (p^*_k - c_k)}{q'_{j_1}}\), where \(k \in \{i_1, \ldots, i\}\). [Honhon and Pan(2012)] showed that the profit margins are increasing in the quality levels, i.e., \(p^*_1 - c_{i_1} < \ldots < p^*_i - c_i\). This implies \(\frac{c'_{j_1} - (p^*_k - c_k)}{q'_{j_1}} < \ldots < \frac{c'_{j_1} - (p^*_1 - c_{i_1})}{q'_{j_1}}\). Therefore, \(x'_{j_1}\) could be lowered until \(x\), where \(x = \min_{k=(i_1, \ldots, i)} \{ \frac{c'_{j_1} - (p^*_k - c_k)}{q'_{j_1}} \}\). Fig 3.5 illustrates the interval \([x, x_i]\) in which the search for \(p^*\) can be restricted.
Further, the exact interval in which $p^*$ lies can be identified as follows. Note that since $\Pi(p)$ in (3.12) is pseudoconcave and continuous across the intervals, we have $d\Pi(p)/dp > 0$ for $p < p^*$, and $d\Pi(p)/dp < 0$ for $p > p^*$. This implies, if $d\Pi(p)/dp > 0$ at $p = x_kq'_j$, i.e., if $\Pi(p)$ is increasing at the right boundary of the interval $[x_{k-1}, x_k]$, then $p^*$ does not lie in the interval $[x_{k-1}q'_j, x_kq'_j]$, and the search can be moved to the next interval $[x_k, x_{k+1}]$. On the other hand, if $d\Pi(p)/dp < 0$ at $p = x_kq'_j$, i.e., if $\Pi(p)$ is decreasing at the right boundary of the interval $[x_{k-1}, x_k]$, then $\Pi(p)$ will further decrease across the next intervals $[x_k, x_{k+1}], \ldots, [x_i, x_{j1}]$. Using this property, formalize the pricing policy as follows
Algorithm 2: Prices for Basket Shoppers

Step 1: Find $S^*_I = \{i_1, \ldots, i_k\}$ using Increasing Convex Envelope Algorithm and $(x_i, p^*_i), i = i_1, \ldots, i_k$ using Proposition 3.4.1. Also find, $S^*_J = \{j_1, \ldots, j_l\}$ and $(x'_j, p'^*_j), j = j_1, \ldots, j_l$. Suppose (w.l.o.g.), $p'^*_j1 = \max \{p'^*_i, p'^*_j\}$.

Step 2: Find $L = \{i_1, \ldots, i\} \subset S^*_I$ such that $x_{i_1} < \ldots < x_{i_k} < x_{i_1}'$.

Step 3: Finding the interval $[x, x_i]$ for $k = i_2, \ldots, i$ do

if $x_k < \frac{c'_i - (p'^*_i - c_k)}{q'_{i_1}}$ then

$k = k + 1$;

else

$x := \frac{c'_i - (p'^*_i - c_k)}{q'_{i_1}}; K := k$.

end if

end for

Step 4: Finding $p^*$

for $y = x_K, \ldots, x_i$ do

if $\frac{d\Pi(p)}{dp} \bigg|_{p=yq'_{i_1}} < 0$ then

$p^* = \arg \max_{q'_{xK} < p < q'_{xK+1}} \sum_{k=K}^{i} a_{b[k,j_1]}(p^*, p, p'^*)(p^*_k + p - c_k - c'_1)$.

end if

end for

Following example illustrates Algorithm 1.

Example 4. Suppose category $\mathcal{I}$ has 7 products and category $\mathcal{J}$ has 4 products as depicted in Figure 3.6(a). Let the quality and cost vectors of category $\mathcal{I}$ be

$q = [271, 380, 713, 720, 864, 985, 1000]$ and $c = [18.36, 36.1, 127.09, 129.6, 186.62, 242.55, 250]$, respectively. The quality and cost vectors of category $\mathcal{J}$ are $q' = [1095, 1252, 1614, 1625]$, and $c' = [405.82, 530.53, 881.69, 893.75]$, respectively. Each category has a convex cost curve, such that all products in the category form the lowest increasing cost curve (see Figure 3.6(a)). Further, the sensitivity to quality in the two categories are both normally distributed with mean 5 and standard deviation 1.
Now we show the steps of Algorithm 1 for this problem.

**Step 1:** We have $S^*_I = \{1, \ldots, 7\}$ and $S^*_J = \{1', \ldots, 4'\}$. The optimal cutoffs are obtained using Proposition 3.4.1(a) as $\mathbf{x}^* = [3.92, 3.93, 3.95, 3.97, 3.97, 3.98, 3.991]$ and $\mathbf{x}'^* = [3.97, 4.05, 4.08, 4.11]$. Further, the optimal prices of category $I$ and of category $J$ are $\mathbf{p}^* = [1062.75, 1491.91, 2809.21, 2837.01, 3409.73, 3892.40, 3952.32]$; and $\mathbf{p}^{'*} = [4350.14, 4986.21, 6465.63, 6510.87]$, respectively.
**Step 2:** Note that \( x_1^* < x_2^* < x_3^* < x_4^* < x_1'^* < x_2'^* < x_3'^* < x_4'^* \). Figure 3.7 illustrates the cutoffs. Clearly, \( L = \{1, 2, 3, 4\} \).

![diagram](image1.png)

**Figure 3.7:** Selection of Configurations in Example 4

**Step 3:** Finding the interval \([x, x_4]\): For \( k = 1, 2, 3, 4 \), we find the values of \( \frac{c'_k - (p_k - c_k)}{q_k} \), and compare with the values of \( x_1^*, x_2^*, x_3^*, x_4^* \). We have, \( x_1^* = 3.9216 > \frac{c'_1 - (p_1 - c_1)}{q_1} = -0.58, x_2^* = 3.9372 > \frac{c'_2 - (p_2 - c_2)}{q_2} = -0.9589, x_3^* = 3.9559 > \frac{c'_3 - (p_3 - c_3)}{q_3} = -2.0782 \) and \( x_4^* = 3.9706 > \frac{c'_4 - (p_4 - c_4)}{q_4} = -2.1102 \). So, \( x = x_1^* \) and \( K := 1 \).

**Step 4:** Finding \( p^* \): Starting with the interval \([x_1^*, x_2^*]\), we can write (3.12) in the explicit form as: \( \Pi(p) = [F(x_1^*) - F(x_2^*)](p_1^* + p - c_4 - c'_1) + [F(x_1^*) - F(x_3^*)](p_3^* + p - c_3 - c'_1) + [F(x_2^*) - F(x_3^*)](p_2^* + p - c_2 - c'_1) + [F(x_2^*) - F(x_4^*)](p_4^* + p - c_1 - c'_1) \). Note that \( d\Pi(p)/dp \) at \( p = q_1 x_1^* \) is \( -[F(x_1^*) - F(p/q_1)] - f(p/q_1')(p_2^* + p - c_2 - c'_1)/q_1 < 0 \). Therefore, \( p^* \) lies in the interval \([x_1^* q_1, x_2^* q_1]\), and \( p^* = x_1^* q_1' = 3.9216 \times 1095 = 4294.1 \). Figure 3.6(b) shows the plot of \( \Pi(p) \) in the range \( p \in [q_1 x_1^*, q_1' x_2^*] \).

To summarize, in this section we showed that retailer offers the same variety in each category to basket shoppers as she offers to cherry pickers. These products are the two optimal assortments \( S_1^* = \{i_1, \ldots, i_k\} \) and \( S_2^* = \{j_1, \ldots, j_l\} \). Further, we find that basket shoppers can be charged the same price charged to cherry pickers for all configurations except the one with the lowest quality, i.e., the configuration \([i_1, j_1]\), which should be priced lower for basket shoppers. Thus retailer’s problem for basket shoppers is significantly reduced. Retailer only needs to find the price of the lowest
quality product of one product category. Further, the usefulness of Algorithm 1 is in terms of reduction of the search length of this price. With these results on hand, we now analyze the assortment and pricing decisions when both consumer types are present.

### 3.5. Assortment and Pricing with Basket Shoppers and Cherry Pickers

In this section we analyze the pricing and assortment decisions in presence of both types consumers. In §3.4.1 and §3.4.2, we discussed the pricing and assortment policy in presence of only one type of consumers. We found that retailer offers identical variety in the category level to basket shoppers and cherry pickers (Lemma 3.4.2). That is, in each category retailer offers the set of products that form the lowest increasing convex cost curve on the cost-quality graph of that category. In presence of the two consumer types, if a product belonging one of the two optimal assortments is not offered, then retailer loosed revenue from cherry pickers who buys that product alone. Therefore, it can be argued that in presence of both types of consumers retailer has no incentive to offer products that do not belong to the optimal assortments in each category. In other words, retailer offers $S^*_I = \{i_1, \ldots, i_k\}$ in category-$I$, and $S^*_J = \{j_1, \ldots, j_l\}$ in category-$J$. We also found in Lemma 3.4.4 that the prices offered to basket shoppers and cherry pickers are identical for all higher quality configurations, i.e., $\bar{p}_{iu} + \bar{p}'_{jv} = p^*_i + p'^*_j, (i_u, j_v) \in S^*, u > 1, v > 1$, and lower for the lowest quality configuration $[i_1, j_1]$. As indicated in Algorithm 1, the lower quality configurations which are not selected are: $[i_2, j_1], \ldots, [i, j_1]$. In this case retailer has the flexibility of changing prices of multiple products from both categories, i.e, prices of $\{i_1, \ldots, i\} \subset S^*_I$, as well as the price $p'^*_j$. By changing these prices retailer can induce basket shoppers to select some of the configurations which would not be selected otherwise.
But at the same time there could be loss in revenue from cherry pickers who would buy those product individually. Therefore, depending on the relative proportion of the two types and the trade off between revenues from the two types, retailer can determine the prices. We formalize the pricing and assortment decision as follows.

**Proposition 3.5.1.** (a) The optimal assortments offered to cherry pickers and basket shoppers are

1. Category $\mathcal{I}$: $S_\mathcal{I}^* = \{i_1, \ldots, i_k\}$, where $\{i_1, \ldots, i_k\}$ constitute the lowest increasing convex cost curve of category $\mathcal{I}$.

2. Category $\mathcal{J}$: $S_\mathcal{J}^* = \{j_1, \ldots, j_l\}$, where $\{j_1, \ldots, j_l\}$ constitute the lowest increasing convex cost curve of category $\mathcal{J}$.

(a). The optimal prices in the two categories are

1. For products $i, \ldots, i_k \in S_\mathcal{I}^*$ the optimal prices are $p^*_{iu}$, $u = 2, \ldots, k$, and for products $j_2, \ldots, j_l \in S_\mathcal{J}^*$ the optimal prices are $p^*_{jv}$, $v = 2, \ldots, l$, as given in Proposition 3.4.1(a).

2. The prices of $i_1, i_2, \ldots, i \in S_\mathcal{I}^*$ and the price of $j_1 \in S_\mathcal{J}^*$ are obtained by solving

$$
\Pi(p, p_{i_1}, \ldots, p_i) = \begin{cases} 
(1 - \gamma)\alpha^b_{[i,j_1]}(p, p_{i_1}, \ldots, p_{i_k})(p_i + p - c_i - c'_{j_1}) + \gamma \alpha^c_{j_1}(p)(p - c'_{j_1}) \\
+ \gamma \sum_{k=1}^i \alpha^c_k(p)(p_k - c_k), & p \in (q'_{j_1} x_i, q'_{j_1} x_{i_1}], \\
(1 - \gamma)\sum_{k=1}^i \alpha^b_{[k,j_1]}(p, p_{i_1}, \ldots, p_{i_k})(p_k + p - c_k - c'_{j_1}) + \gamma \alpha^c_{j_1}(p')(p - c'_{j_1}) \\
+ \gamma \sum_{k=1}^i \alpha^c_k(p)(p_k - c_k), & p \in (q'_{j_1} x_{i-1}, q'_{j_1} x_{i}], \\
\vdots \\
(1 - \gamma)\sum_{k=1}^i \alpha^b_{[k,j_1]}(p, p_{i_1}, \ldots, p_{i_k})(p_k + p - c_k - c'_{j_1}) + \gamma \alpha^c_{j_1}(p')(p - c'_{j_1}) \\
+ \gamma \sum_{k=1}^i \alpha^c_k(p)(p_k - c_k), & p \in (q'_{j_1} x_{i_1}, q'_{j_1} x_{i_2}], 
\end{cases}
$$

Since, (3.5) is bi-concave, it follows that $\Pi(p, p_{i_1}, \ldots, p_i)$ is also bi-concave, and the Alternate Convex Search Algorithm can be used to solve for the optimal prices.
be noted that due to the revenue terms from basket shoppers, the problem in (3.13) is no longer separable into individual category problems. Therefore, depending on the correlation between the two sensitivities, the optimal solution varies. To illustrate this, Figure 3.8(a) and Figure 3.8(b) shows the plot of the function in (3.13) for an example, where there are two prices to be determined, one from each category. Figure 3.8(a) shows the plot of the profit function assuming perfectly correlated sensitivities, while in Figure 3.8(b) show the plot when there is no correlation between the two sensitivities. The optimal solutions for this example is found to be (i) $x_{i1}^* = x_{j1}^* = 0.423$ when correlation is one, and (ii) $x_{i1}^* = 0, x_{j1}^* = 0.58$ when correlation is zero.
(a) Plot of $\Pi(x_{i1}, x'_{j1})$ assuming correlation=1, $\theta \sim U(0,1)$, proportion of cherry pickers, $\gamma = 10\%$

(b) Plot of $\Pi(x_{i1}, x'_{j1})$ assuming zero correlation, $\theta \sim U(0,1)$, proportion of cherry pickers, $\gamma = 10\%$

Figure 3.8: An example of two categories with parameters: $q = [161, 709]$, $c = [9.14, 177.25]$, $q' = [1109, 1649, 1763]$, $c' = [174.4017, 385.5929, 440.75]$. 

Taking clue from this simple case it can be said that, unlike the pricing policy in presence of one consumer type, the pricing decisions of the lower quality products identified in Proposition 3.5.1, in presence of basket shoppers and cherry pickers, is
rather complicated. With the help of the following example we illustrate the various factors which influence the prices of the lower quality products in the two categories.

**Example 5.** Suppose category-$I$ has 4 products and category-$J$ has 2 products. Let the quality vector of category $I$ be $q = [56, 548, 795, 939]$, and the cost vector be $c = [0.66, 63.95, 134.61, 187.8]$. Let the quality vector of category $J$ be $q' = [1676, 1856]$, and the cost vector be $c' = [302.69, 371.2]$. Each category has convex cost curve, such that all products in the category form the lowest increasing cost curve, i.e., $S^*_I = [1, 2, 3, 4]$ and $S^*_J = [1', 2']$ (see Figure 3.9(a)). The configurations selected by basket shoppers are $S^* = [2, 1'], [3, 1'], [4, 1'], [4, 2']$, which is the lowest increasing cost curve of the configurations (see Figure 3.9(b)). Further, we assume the sensitivity to quality in the two categories are both uniformly distributed between 0 and 1. We also assume two levels of the proportion of cherry pickers: $\gamma = \{10\%, 90\\%\}$.

The solution of this problem is as follows. First we find the optimal cutoffs in the two categories using Proposition 3.4.1: $x_1 = 0.506 < x_2 = 0.5643 < x_1' = 0.5903 < x_3 = 0.643 < x_4 = 0.6847 < x_2' = 0.6903$. Clearly, in this case retailer can change the cutoff $x_2$ in order to induce basket shoppers to select $[2, 2']$ which is not selected. Further, retailer can also change the cutoff $x_1'$ in order to increase the market share of $[2, 1']$. Therefore, we resolve the problem in order to find the cutoffs $x_1, x_2, x_1'$, assuming $x_3 = 0.643, x_4 = 0.6847, x_2' = 0.6903$ as fixed. Table 3.1 reports the solutions for different values of $\gamma$ and different correlations.

<table>
<thead>
<tr>
<th></th>
<th>Correlation=1</th>
<th>Correlation=0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>0.420</td>
<td>0.506</td>
</tr>
<tr>
<td>$x_2$</td>
<td>0.422</td>
<td>0.562</td>
</tr>
<tr>
<td>$x_1'$</td>
<td>0.422</td>
<td>0.565</td>
</tr>
</tbody>
</table>

Table 3.1: Cutoffs of Example 5

From Table 3.1, we find under the perfect correlation case, that when the propor-
Figure 3.9: Two Categories of Example 5

tion of basket shoppers is as high as 90%, retailer reduces \(x_2\) from 0.5643 to 0.422, and \(x'_1\) from 0.5903 to 0.422 and thereby increases the market share of \([2, 1']\) from 5.03% to 21%. On the other hand, when the proportion of cherry pickers is as high as 90%, \(x_2\) is reduced by only 0.0023 (from 0.5643 to 0.562), and \(x'_1\) is reduced by only 0.0243 (from 0.5903 to 0.565). In this case, the market share of \([2, 1']\) increases to only 7.8%. Clearly, when the proportion of cherry pickers is high, a high discount in prices \(p_2\) and \(p_1'\) is not profitable, since retailer looses revenue on the cherry pickers who buy these items individually. This trend is also observe when correlation between
the sensitivities is zero. Therefore the pricing policy largely depends on the relative proportion of the two consumer types. When cherry pickers are a majority, retailer may not discount the prices as much as she would when the proportion of cherry pickers is low.

We also find from Table 3.1, that retailer takes different pricing decision when correlation between the sensitivities is zero than under the perfect correlation case. For example, when the proportion of cherry pickers is only 10%, retailer chooses to increase the cutoff $x'_1$ from 0.5903 to 0.5945, while reduces $x_2$ by a large amount (0.5643 to 0.061). Thus the market share of $[2, 1']$ is reduced from 5.03% to 4.73%, but market share for cherry pickers buying product 2 is increased from about 3% to 53.3%. Since consumes preferences in the two categories are independently distributed, retailer has more flexibility: an increase of price in one category is possible while price in the other category is reduced. If however, consumes preferences in the two categories are perfectly correlated, the direction of change of prices is the same.

### 3.6. Conclusion

In this chapter we analyzed the pricing and assortment decisions of two vertically differentiated categories, in presence of two consumer types, basket shoppers and cherry pickers. A cherry picker evaluates products in each category independently and purchases a product only if it maximizes her utility compared to other products in that category and the outside option. In contrast, a basket shopper evaluates both categories together, and selects a configuration comprising of one product from each category, only if the total utility of the configuration is higher than the utilities from other configurations and the outside option. Both cherry pickers and basket shoppers have correlated sensitivities to product qualities in the two categories. We show that regardless of the consumer type, retailer offers the category-profit-maximizing variety level in each category. We also show that retailer offers certain high quality...
products in the two categories category at the same prices which she offers to cherry pickers. The pricing policy for the remaining lower quality products, however is complicated, and depends on factors like the relative proportion of the two types and the correlation between the two sensitivities. If cherry pickers are relatively low in proportion compared to basket shoppers, retailer has more flexibility in setting these prices. We find, numerically, that in this case, retailer sets a higher discount for the low quality products so as to attract basket shoppers. On the other hand, if cherry pickers are relatively high in proportion, retailer does not have the liberty to change the prices by as much in case when cherry pickers are low in proportion. We also find that if consumers have independent sensitivities for quality in the two categories, retailer may markup prices in one category, while sell products in the other category in discount.
Chapter 4
Assortment Planning for Hi-Tech Products: Theoretical and Practical Considerations

4.1. Introduction

The computer processor industry has evolved rapidly in scale and scope in the last two decades. In 1998, the total size of the computer industry in the U.S.A. was $62 billion [Goeree 2008]. Intel, the leading computer processor company, sold the Pentium series of computer processors in a small number of versions at different clock speeds. The electronic circuits on these processors were separated by more than 100 nanometers of distance [Bohr 2009]. In contrast, the sales of tablet computer, just one of many computers present in households today, is estimated to exceed $35 billion in 2012 [Statistica 2012]. The processors are now manufactured using an 18 nanometer technology which requires an extremely high degree of precision [Li et al. 2011]. Intel launches a new processor line with a new design every alternate year, often in a large number of versions with distinct quality levels.

These technical advances in design and manufacturing have been accompanied with interesting business problems for Intel, AMD, and IBM. The manufacturing process has become extremely complex with each processor containing billions of transistors that together determine the clock speed of the processor. The firms find it increasingly difficult to lay each transistor correctly and obtain every unit with the exact same clock speed. Effectively, every unit manufactured under identical
conditions has a different clock speed. The firms test each unit for its clock speed and categorize it in a set of discrete products, versions, or “bins,” each having a distinct quality level. All units in a bin are sold at the same price. The quality levels of units within a bin vary but always exceed the quality tag at which the bin is sold in the market. For example, the three versions of the Athlon processors shown in Figure 4.1 are produced on the same manufacturing line simultaneously. Every unit produced with a clock speed between 2.8-2.9 GHz is classified as the 2.8 GHz processor, etc. This simultaneous manufacturing of various versions of a product is called co-production.

The computer processor industry is highly competitive (AMD resorted to selling most of its manufacturing capacity in 2009 to stay afloat), manufacturing capacity is expensive (a fabrication facility costs ∼ $4-5 billion) and only an appropriate assortment enables firms to use the capacity judiciously while meeting the demands. Developing the appropriate assortment of bins is non-trivial because firms seldom have control over fractions in which various bins are produced and over the demands of the bins. This may lead to a frequent mismatch between supply and demand if the bins are not carefully selected.

Motivated by these observations and challenges, in this chapter we determine (i) the optimal number of bins that a firm should offer, (ii) the optimal quality levels of these bins, and (iii) the drivers of this decision. A first glance may suggest that this problem is a standard assortment planning problem where offering too many or too few products has different drawbacks. If a firm offers only a single version, it meets the requirements of only a small number of customers and loses the remaining

<table>
<thead>
<tr>
<th>Processor Specification</th>
<th>Clock Speed tag</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Athlon X2 5800</td>
<td>3.0 GHz</td>
<td>$112</td>
</tr>
<tr>
<td>Athlon X2 5600</td>
<td>2.9 GHz</td>
<td>$102</td>
</tr>
<tr>
<td>Athlon X2 5400</td>
<td>2.8 GHz</td>
<td>$87</td>
</tr>
</tbody>
</table>

Table 4.1: Bin Specifications and prices for AMD processors
demand. By offering a large number of versions, the firm can cater to the needs of more customers that have varying willingness to pay and can extract maximum rent from the customers. However, offering more versions is typically costly due to the additional testing equipment/setup required.

The simultaneous production of all bins in a co-production system introduces many novel features to the classical assortment planning problem. Firms cannot control the ratios in which various bins can be manufactured and stored. These ratios have cascading effects that must be taken into account while determining the quality level of each bin. As an example, consider the distribution of clock speeds of AMD processors produced in a single run of a wafer manufacturing process, as shown in Figure 4.1. After manufacturing, AMD tests all processors to determine their actual clock speeds and classifies them into three bins with specifications 2.8 GHz, 2.9 GHz, and 3.0 GHz.

![Figure 4.1: Three bins created from the distribution of clock speed](image)

These specifications determine both the supply side and demand side. The demands of the processors are determined by the quality specifications and the selling prices in the market (a higher clock speed translates to a higher price, see Table 4.1). The specifications also determine the yield of the three versions: the yield of 3.0 GHz processors is the fraction of processors with a clock speed more than 3.0 GHz, the yield of 2.9 GHz processors is the fraction of processors with a clock speed between
2.9 and 3.0 GHz, etc. These yields, in turn, strongly impact the profitability of the product line. As an illustration, suppose the fraction of the 3 GHz processor is 0.05. Thus, an average of $1/0.05=20$ units must be manufactured before one unit of this version is obtained. The price of the 3 GHz processor must be set sufficiently high in order to recover its own manufacturing cost, as well as the cost of unsold units of the other two versions if the two versions have relatively low demands. However, a high price would reduce the demand of this version. Instead, a version at 2.95 GHz would have a higher yield and a lower net manufacturing cost. But this version will command a lower price than the 3 GHz version. To summarize, the decision of what quality levels to offer requires a careful analysis and an understanding of the complex dynamics of manufacturing technology and market characteristics.

We note that even though the discussion so far has been in the context of computer processors, co-production systems exist widely in many high-tech industries such as the most semiconductor based industries including hard drive, mother board and graphics card industry, biochemical, and chemical industries wherein products of varying quality are produced simultaneously in a single production run. Our results are generic and are applicable to these industries for managing co-production through a judicious product assortment planning.

We investigate the problem analytically and complement the analysis with practical considerations that exist in this industry. We incorporate a utility maximizing consumer choice model and determine the optimal batch size and optimal quality levels of the bins. We show that in the optimal assortment and manufacturing policy, a firm will choose the optimal batch size and the optimal quality levels such that demands of all the offered bins are fulfilled. We also find that the optimal quality levels of the bins are unevenly spaced across the support of the distribution of quality. In contrast, an easier and more appealing strategy to offer an assortment with equidistant bins can be significantly suboptimal. A small number of carefully selected non-equidistant bins can be almost equally effective. Finally, we discuss
practical considerations for pricing, technology, and optimal product offerings, and explain the proliferation of bins witnessed in the last decade in the processor industry.

The remainder of this chapter is organized as follows. In §4.2 we review the relevant literature; §4.3 presents the model; §4.4 presents the solution and properties of the solution; in §4.5 we do a numerical study to complement our analytical results; and §4.6 concludes with a discussion and implications of the results for practice and future extensions. All proofs are given in Appendix C.

4.2. Literature Review

Our research focus is relevant to three major streams of literature: 1) literature on co-production systems, 2) literature on product line design, and 3) literature on assortment planning.

Co-production literature typically assumes that the quality specifications of the versions are pre-determined and then focuses on the downstream manufacturing and lot-sizing decisions. [Gerchak et al. 1996], [Gerchak et al. 1994], and [Hsu and Bassok 1999] discussed the problem of determining the optimal production lot size under uncertain demands for all versions for a single period horizon. [Bitran and Leong 1992] and [Bitran and Gilbert 1994] developed heuristics to solve the multi-period manufacturing problem. Recently, [Huang and Song 2010] showed that a modified base-stock policy is optimal for the multi-period problem. This literature assumes that the firm does not have the flexibility to change the demand of any version by changing its quality specification or price. The only exception that we found was [Tomlin and Wang 2008] who evaluated the flexibility of changing prices of bins to modify their demands after these demands have been observed. They considered fixed quality levels. Our work is different from this body of literature in the following two ways.

First, we model the decision of determining the number of versions and their qual-
ity levels at a tactical level. This decision is dependent on the firm’s manufacturing technology that is characterized by the distribution of the quality levels of the units manufactured. Firms usually have an option of selecting/improving technology while planning a new product line. Our model enables us to make recommendations regarding investments for adoption/changes in technology to maximize profit. Second, our model incorporates the heterogeneity in customers’ valuation for each version offered at a specific price using a consumer choice model, and determines the optimal assortment of versions for a specific customer population whose aggregated characteristics are known to the firm.

The literature on product line design has focused on determining the optimal number and quality levels of different versions in the presence of fixed and/or variable development costs. Models in this literature such as the ones in [Moorthy and Png 1992], and [Krishnan and Zhu 2006] among others have typically assumed that the market segments are exogenous and homogeneous, i.e., each customer in a particular market segment evaluates a particular version exactly the same. This literature further assumes that the firm can manufacture all versions independently and in exact quantities as needed, and that segment sizes are independent of the prices. However, the simultaneous production of all bins in a co-production process restricts the supply of various bins in a production lot to specific ratios. This constraint must be taken into account to determine the specifications of various bins.

A vast body of literature in operations management focuses on assortment planning (see [Köksal et al. 2008] for a detailed review). Most literature in this area focuses on assortment planning of horizontally differentiated products assuming prices are exogenously given (see, for example, [Cachon et al. 2005], [Li 2007], [Mahajan and van Ryzin 2001], [Smith and Agrawal 2000], and [van Ryzin and Mahajan 1999]). The major finding in these papers is that the optimal assortment consists of the most popular products or products with the highest reservation prices.

Only a handful of papers focus on assortment planning for vertically differentiated
products ( [Akcay et al. 2010], [Bharghava and Choudhary 2001], and [Honhon and Pan 2012]). [Akcay et al. 2010] finds the joint dynamic pricing of a fixed assortment of multiple perishable products when products can be differentiated vertically as well as horizontally. We do not consider dynamic pricing, but focus on product selection under quality distribution in a single period setting. [Bharghava and Choudhary 2001] consider the problem of selecting and pricing vertically differentiated products when demand is deterministic, quality levels are exogenous and no fixed costs are incurred. Unlike their setup, we analyze the optimal assortment when the supplies of various products cannot be obtained in independent quantities. [Honhon and Pan 2012] assume a fixed cost of offering a product and find the pricing and assortment decisions under a general distribution of consumer types and deterministic demand. Similarly, we also consider a fixed cost of adding bins and explicitly consider the interdependence of quantities supplied for each product that arises in co-production systems and analyze its impact on the optimal prices and optimal assortment.

4.3. Model Preliminaries

We consider a profit-maximizing firm operating a co-production system where a single production run results in units with different quality levels. These units are classified into different versions based on the units’ tested performance levels. In the following discussion we use the terms ‘bin’ and ‘version’ interchangeably. Our model setup consists of three components (i) the manufacturing process, (ii) the pricing policy, and (iii) the consumer choice model, which we discuss next.

4.3.1 Manufacturing Process

The random quality or the performance level of a unit is denoted by the random variable $X$ which has an increasing generalized failure rate (IGFR), mean $\mu_X$ and
standard deviation $\sigma_X$. Most common distributions such as the normal, uniform, and exponential distributions are IGFR. The cumulative density function (cdf) of this distribution is defined over the range $[0, x_H < \infty]$ and is denoted by $F_X(x)$. The upper level $x_H$ is the limitation of the technology of the product (e.g., maximum clock speed possible with the current manufacturing process). We assume that $F_X(x)$ is continuous and twice differentiable to ensure that the first order conditions for expressions with the cdf can be evaluated.

![Diagram](image)

**Figure 4.2: Co-production system with n output categories**

After a production run, the units are tested and classified into $n$ discrete quality levels, $x = (x_1, x_2, \ldots, x_n)$, where $x_L \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq x_H$. The two limits $x_L$ and $x_H$ are exogenously given. The lower level $x_L$ represents the minimum manufactured quality level. If, upon testing, a unit is found to have a quality level higher than $x_k$ but lower than $x_{k+1}$, i.e. $x \in [x_k, x_{k+1})$, then the unit is labeled with the quality level $x_k$. Figure 4.2 shows this binning process. We also assume that the firm will not sell a bin at quality level $x_L$, which is the worst quality level produced.

The fraction of units in the bin of quality $x_k$ is given by $\beta_k(x) = F(x_{k+1}) - F(x_k)$. Consequently, when the production batch size is $Q$, the supply quantities are:

$$Q_k(Q, x) = \begin{cases} Q\beta_k(x) = Q [F(x_{k+1}) - F(x_k)], & k = 1, \ldots, n - 1, \\ Q\beta_n(x) = Q[1 - F(x_n)]. \end{cases} \quad (4.1)$$

We note that it is possible to model supply fractions is to consider random yields,
i.e., the supply of bin $k$ is a binomial random variable with mean $Q \beta_k(x)$. However, motivated by two observations/practice, we allow yields to be deterministic in our model. First, for new technology products, firms typically simulate the distribution of the clock speeds and then make assortment decisions based on the yields obtained from setting different quality cut-off points. [Datta et al. 2006 ] have described this process at Intel. Second, at AMD, the supply chain department obtained estimates of the yield from the manufacturing units and made inventory decisions assuming these deterministic yields. In the words of a manager, “AMD manufactures millions of processors and the assumption of fixed yields is acceptable due to the law of large numbers”. Finally, similar to the prior literature such as [Hsu and Bassok 1999], and [Gerchak et al. 1996 ] on random quality production processes, we assume that the manufacturing cost for all units is identical at $c$.

### 4.3.2 Pricing

We assume a semi-flexible pricing policy motivated by prior empirical and theoretical findings. Empirical data collected from Intel and AMD by [Datta et al. 2006 ], reproduced in Figure 3(a), confirms that semi-conductor firms follow a pricing policy that is convexly increasing in the quality levels (clock speeds). A popular technology news medium noted in 2006 with Figure 3(b) that “The Intel curve has leveled off even more from last week to the point where it is almost a straight line” [Raby 2006]. The deviations from the convex pricing curves in this Figure suggests that even when firms do not strictly adhere to pricing curves, they are likely use these curves to obtain a first-cut assortment and then modify it for market reasons.
Based on these evidences, we assume that the firm follows an increasing convex price curve $p(x) = x^a$ such that $\frac{\partial^2 p(x)}{\partial x^2} > 0$. Consequently, the quality levels $x_k$ determine the prices $p_k = p(x_k)$, $k = 1, \ldots, n$. Note that firm can endogenize the pricing decision by selecting a specific value of $a$. [Akçay et al. 2010] and [Honhon and Pan 2012] also show that when pricing quality differentiated products in a single category, a retailer sets prices that convexly increase in quality levels. Our assumption on convex pricing curve is consistent with these two papers.
4.3.3 Consumer Choice Model and Demand

Demand of each bin is driven by a consumer choice process. We assume consumers choose independently, and their choice is based on prices and quality only. The random utility from buying bin \( k \) is \( U_k = \theta x_k - p(x_k) \), where \( \theta \) is the random variable that denotes a customer’s marginal value for per unit of quality. We assume \( \theta \) is an IGFR random variable that has a cumulative distribution function \( G(.) \) and the support of \( \theta \) is \([θ_1, θ_2]\). Utility maximization gives the choice probabilities, denoted by \( \alpha_k(x) \)s, which are also known as the choice probabilities of the vertical choice model [Akçay et al. 2010, Honhon and Pan 2012]:

\[
\alpha_k(x) = \begin{cases} 
G \left( \frac{p(x_1)}{x_1} \right), & k = 0 \\
G \left( \frac{p(x_{k+1}) - p(x_k)}{x_{k+1} - x_k} \right) - G \left( \frac{p(x_k) - p(x_{k-1})}{x_k - x_{k-1}} \right), & k = 1, \ldots, n - 1 \\
1 - G \left( \frac{p(x_k) - p(x_{k-1})}{x_k - x_{k-1}} \right), & k = n.
\end{cases}
\] 

The index \( k = 0 \) denotes the “No purchasing” option. The fraction \( \alpha_0(x) \) denotes the fraction of customers who do not buy any bin. Note that \( \frac{p(x_i) - p(x_{i-1})}{x_i - x_{i-1}} \) denotes the slope of the price curve between two points \((x_i, p(x_i))\) and \((x_{i-1}, p(x_{i-1}))\). Since firm follows an increasing convex price curve as discussed in §4.3.2, it follows that \( \frac{p(x_n) - p(x_{n-1})}{x_n - x_{n-1}} \geq \ldots \geq \frac{p(x_2) - p(x_1)}{x_2 - x_1} \geq \frac{p(x_1)}{x_1} \). This further implies the choice probabilities in (4.2) are non-negative.

We assume the total market size is fixed with size \( S \). The demand of bin \( k \) is thus the proportion of customers who choose \( k \) as their first choice bin. In other words, the demand of \( k \) is \( D_k(x) = S\alpha_k(x) \). This proportional split demand model can be justified since the firm caters to the aggregate market. Several papers in the operations literature use the proportional split demand models, e.g., [Li 2007], and [Honhon and Pan 2012]. In essence, we assume assortment based substitution behavior, i.e., consumers do not switch to another available product if their most preferred product is out-of-stock. It is likely that some but not all customers might have second or third
choices that they may buy if their preferred product is out of stock. But we do not consider stock-out based substitution for ease of analysis, as the problem of finding the optimal quality levels becomes challenging even with assortment based substitution assumption. The inclusion of stock-out based substitution will constitute an interesting extension to this kind of research.

4.3.4 Firm’s Optimization Problem

With the aforementioned manufacturing, pricing, and demand models, the profit-maximizing firm determines the optimal number of bins \( n^* \), the optimal quality specifications \( \mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*) \), and the optimal production quantity \( Q^* \). Manufacturing costs are assumed to be linear in the lot size \( Q \) with a marginal cost of \( c \). In addition, a fixed setup cost \( \bar{c} \) occurs for testing and sorting a new bin; specifically, this cost corresponds to additional tests that must be setup on the testing equipment to sort the units into \( n \) bins instead of \( n-1 \) bins. Thus the firm’s optimization problem is given as:

\[
\max_{n \leq N, \mathbf{x}, Q} \Pi(\mathbf{x}, Q, n) = \sum_{k=1}^{n} p(x_k) \min\{Q\beta_k(\mathbf{x}), S\alpha_k(\mathbf{x})\} - cQ - n\bar{c}, \quad (4.3)
\]

subject to:

\[
x_L \leq x_1 < \ldots < x_i < x_{i+1} < \ldots x_n \leq x_H. \quad (4.4)
\]

The first term in (4.3) is the total sale revenue from all bins. The sale quantity of each bin \( k \) is equal to the minimum of its supply \( Q\beta_k(\mathbf{x}) \) and its demand \( S\alpha_k(\mathbf{x}) \). The second and third terms are the total production cost and the cost of introducing \( n \) bins, respectively. Constraint (4.4) ensure that the quality levels \( x_1, \ldots, x_n \) are in increasing order between \( x_L \) and \( x_H \).

The number of bins could be \( n \in 1, 2, \ldots, N \). We take advantage of this fact to first solve the joint optimization problem (4.3)-(4.4) for the optimal quantity and the
optimal quality levels for a given \( n \), and then determine the optimal \( n \) that gives the highest profit. In the following section, we solve for the optimal quantity and optimal quality levels for a given number of bins \( n \).

### 4.4. Optimal Quality and Quantity for Fixed Number of Bins

Given an \( n \), the firm needs to determine the optimal quality vector \( \mathbf{x}^* = (x^*_1, \ldots, x^*_n) \), and the optimal quantity \( Q^* \) by solving (4.3)-(4.4). Since the problem is deterministic, the sequence of decisions does not affect the optimal solutions. So, we first optimize the production quantity \( Q \), given a quality vector \( \mathbf{x} \) in §4.1, and obtain \( Q^*(\mathbf{x}) \). Once we have expressed the optimal quantity in term of the quality levels, we obtain \( \mathbf{x}^* \) in §4.2

#### 4.4.1 Optimal Quantity for a given Quality

Let \( \Omega = \{1, \ldots, n\} \), where \( n \) is the fixed number of bins. Given the quality vector \( \mathbf{x} = (x_1, \ldots, x_n) \), such that, \( x_H \geq x_n > \ldots > x_1 \geq x_L \), the optimal quantity \( Q^*(\mathbf{x}) \) is found by solving the following problem:

\[
\max_{Q(\mathbf{x}) \geq 0} \Pi(Q(\mathbf{x})) = \sum_{k=1}^{n} p(x_k) \min\{Q(\mathbf{x})\beta_k(\mathbf{x}), S\alpha_k(\mathbf{x})\} - cQ(\mathbf{x}) - \bar{c}n. \tag{4.5}
\]

The profit function \( \Pi(Q(\mathbf{x})) \) in (4.5) is a piece-wise linear function of \( Q(\mathbf{x}) \) with the breakpoints \( \{Q_1, \ldots, Q_n\} \) (Figure 4). Given \( \mathbf{x} \), the bins can be ordered by an increasing value of \( \frac{\alpha_{ik}(\mathbf{x})}{\beta_{ik}(\mathbf{x})} \) for which the breakpoint is given as \( Q_k = S\frac{\alpha_{ik}(\mathbf{x})}{\beta_{ik}(\mathbf{x})} \). Each breakpoint characterizes the threshold of manufacturing quantity where the supply of one additional bin fully meets its demand. The slope of \( \Pi(Q(\mathbf{x})) \) for any \( Q \) between any two breakpoints \( Q_k \) and \( Q_{k+1} \) is the marginal profit from an additional unit. A positive
(negative) slope indicates that the firm will make a profit (loss) from the next unit and should manufacture it. Any breakpoint where the slope of $\Pi(Q(x))$ changes from positive to negative is the optimal quantity since beyond this point the firm will incur a loss on every additional unit.

We explain the intuition behind how to find the optimal quantity. First, we define the margin from product $i$, for a given quality vector $x$, as $p(x_i)\beta_i(x) - c$, $i \in \{1, \ldots, n\}$. The margin is the expected revenue that an unit is sold at price $p(x_i)$, minus its cost of manufacturing. Let the set $I_x$ be the largest set for which $\sum_{i \in I_x} p(x_i)\beta_i(x) < c$. This means the total revenue earned from the products in $I_x$ is less than the unit manufacturing cost $c$, which, in turn, implies that (i) the revenue of each product in $I_x$ is less than the unit production cost, and (ii) the total revenue earned from the products of any subset of $I_x$ is also less than the unit production cost. The products in $I_x$ are unprofitable in the sense that each product of $I_x$ has a negative margin. Because of the a net negative profit from these products, the firm is better off by choosing the optimal batch size $Q^*(x)$ in such a way that these products are produced in the least possible quantity.

The complementary set $J_x = \Omega \setminus I_x$ comprises of bins that are profitable. Clearly, the firm should produce just enough to meet the demands of the bins in the set $J_x$.
and then stop further production. For example, in Figure 4, if \( J_x = \{ i_1, i_2 \} \) for a given \( x \), then after the demands of bins \( \{ i_1 \} \) and \( \{ i_2 \} \) are met in full, the firm stops production. We formalize this intuition in Lemma 4.4.1.

**Lemma 4.4.1.** Let the set \( I_x \) be the largest set of bins such that the \( \sum_{i \in I_x} p(x_i) \beta_i(x) < c \), and let \( J_x = \Omega \setminus I_x \). The optimal production quantity is given by \( Q^*(x) = S \cdot \max_{j \in J_x} \left\{ \frac{\alpha_j(x)}{\beta_j(x)} \right\} \).

At the optimal quantity \( Q^*(x) \) given in Lemma 4.4.1, the supply quantity of every bin \( i \in I_x \) is less than the corresponding demand. On the other hand, at the optimal quantity the demands of the bins in the set \( J_x = \Omega \setminus I_x \) are fully met and some of them may even have some excess inventory.

### 4.4.2 Optimal Quality Levels

Now that we have the solution for optimal quantity, we solve for the optimal quality levels. Substituting \( Q^*(x) = S \cdot \max_{j \in J_x} \left\{ \frac{\alpha_j(x)}{\beta_j(x)} \right\} \) into (6), we can rewrite the optimization problem in (6) as the following:

\[
\max_{x_H \geq x_n > \ldots > x_1 \geq x_L} \Pi(x) = \sum_{i \in I_x} p(x_i) \beta_i(x) Q^*(x) + \sum_{j \in J_x} Sp(x_j) \alpha_j(x) - cQ^*(x) - \bar{c}n, \quad (4.6)
\]

The solution to (4.6) gives the optimal quality vector \( x^* = (x_1^*, \ldots, x_n^*) \), which, in turn, determines the set of bins with fully met demand \( (J_x^*) \), the set of bins with unmet demand \( (I_x^*) \). Further we obtain the optimal quantity \( Q^*(x^*) \) using Lemma 4.4.1. Note that if the optimal quality vector \( x^* \) is set such that \( I_x^* = \Omega \), then \( J_x^* = \emptyset \), and by Lemma 4.4.1, \( Q^*(x^*) = 0 \). That is, if it is optimal for the firm to not completely meet the demand of any bin, then the firm should not produce anything, and the profit will be zero. Thus, the firm can do better by setting \( x^* \) in a way so as to fully
meet demand of at least one bin. This observation leads to the interesting question: is it optimal for a firm to fully meet the demands of all bins \( (I_x^* = \emptyset) \) or fully meet the demands of some bins \( (I_x^* \subset \Omega) \), and what would decide this? The next Proposition formalizes the answer.

**Proposition 4.4.2.** The optimal quality vector \( x^* \) is set such that the demands of all offered bins are met in full \( (I_x^* = \emptyset) \).

The intuition of the result is as follows. We discussed earlier (in §4.1) that when firm makes the manufacturing quantity decision for given set of bins with quality levels \( x = (x_1, \ldots, x_n) \), then firm will choose the optimal quantity so as not to fully meet the demand of the bins which are unprofitable (those in \( I_x \)). These bins incur losses. Therefore, when the firm has the opportunity to determine the quality levels, it will set the quality levels so that these bins are profitable \( (I_x^* = \emptyset) \). From a managerial perspective, the firm’s problem is to maximize revenue while minimizing the supply-demand mismatch. The optimal quality levels determine the optimal quantity, which, in turn, determines the supply quantity of each bin as well as the demand of each bin, through prices. Therefore, the optimal quality levels are determined such that the firm can fully match the demands of all bins it offers. Using Proposition 4.4.2, the optimal quantity in Lemma 4.4.1 can be rewritten as

\[
Q^*(x) = S \times \max_{j=1,\ldots,n} \left\{ \frac{\alpha_j(x)}{\beta_j(x)} \right\},
\]

and the optimization problem in (4.6) as

\[
\max_{x_L \geq x_1 \geq \ldots \geq x_n \geq x_H} \Pi(x) = S \left( \sum_{j=1}^{n} p(x_j) \alpha_j(x) - c \max_{j=1,\ldots,n} \left\{ \frac{\alpha_j(x)}{\beta_j(x)} \right\} \right) - cn.
\]

It is clear from (4.7) that the optimal quality levels are independent of the market size. This independence implies that the relative location and the shape of the distri-
tribution curves for the quality levels and consumer preferences completely determine the assortment for specific pricing curves. The market size does not affect the optimal solution. The market size only increases the number of units produced, with the quality specifications remaining intact. Put together, firm will set the quality levels such that all demands will always be met in full, and that the technical specifications of the bins offered will not change when the market size changes. Firms that typically run focus groups/studies to estimate the distribution of willingness to pay ($\theta$ in our model) and to estimate the market size will use the findings of the studies for different purposes – estimates for the distribution of customers’ preferences will determine the bin specifications; the market size will determine the manufacturing quantity.

For any arbitrary choice of yield distribution $F$ and distribution of consumer preference $G$, the profit function in (4.7) is not concave or even quasiconcave in each $x_i$, and therefore will not be jointly concave or quasiconcave. To see this, we note that $\Pi(x)$ given in (4.7) is not concave unless the first term $\sum_{j=1}^{n} p(x_j) \alpha_j(x)$, and the second term $c \max_{j=1,...,n} \left\{ \frac{\alpha_j(x)}{\beta_j(x)} \right\}$, are both concave. To provide a confirmation of non-concavity of $\Pi(x)$ given in (4.7), we show that the second term is not concave. We have assumed that the yield $X$ and consumer preferences $\theta$ are IGFR distributions, and most common IGFR distributions have concave or log-concave cdfs. Therefore, $G(.)$ is non-decreasing and pseudoconcave in $\theta \in [\theta, \bar{\theta}]$ and $F(.)$ non-decreasing and pseudoconcave in $x \in [x_L, x_H]$. The choice probabilities $\alpha_j(x), j = 1, \ldots, n$, and the supply fractions $\beta_j(x), j = 1, \ldots, n$ are then difference concave (d.c.) functions by definition. The ratio of d.c. functions is also d.c.(see [Hartman 1959]). Therefore, the second term in (4.7) is a difference convex function, which is not concave or even quasi-concave.

See Figures 4.5(a) and 4.5(b) for two examples of the profit function where the firm creates and sells three bins. From these figures and several others we investigated, although the objective function should apparently be unimodal in the domain \{$(x_1, x_2) : x_2 \geq x_1 \geq 0$\} (but not necessarily jointly), we were unable to verify this re-
result analytically. The practical implication of non-concavity or non-quasiconcavity is that a comprehensive numerical search is required to obtain the true optimal quality levels. Regardless of this complexity, we establish that the optimal quality vector is set in such a way that the demands of all offered bins are met in full.

4.4.3 Optimal Number of Bins

We now look at the factors that govern the optimal number of bins and use these results to explain the recent proliferation of bins in the computer processor industry. Co-production systems, at least in a theoretical sense, produce infinite levels of qual-
ity. The classification of these levels into specific bins determines the yields of the bins and the prices of the bins split the market into various segments that seek the bins offered. It appears natural that creating grades of demand and supply both will lead to mismatches between the demand and supply, and that these mismatches will increase further as the number of bins increases. It appears that the firm should offer a fixed number of bins to restrict these mismatches. The next Proposition shows that this is not the case.

**Proposition 4.4.3.** *In the absence of the fixed cost \( \bar{c} \), the firm should offer an infinite number of bins.*

The underlying intuition of Proposition 4.4.3 is that if there is no extra cost for determining the exact quality specification of the unit, then the firm can determine the exact quality level of each unit, sell it to the customer who derives highest value from it and extract the maximum premium from the customer.

Consequently at a fixed classification cost \( \bar{c} \), the firm’s total cost of offering the product line also increases with the number of bins. The net effect of the increasing trend in both benefits and costs is that the total profit first increases (benefits dominate the cost) and then decreases (costs dominate the benefit) with the number of bins. The higher the fixed cost \( \bar{c} \) is, the fewer will be the number of bins that the firm will offer. Proposition 4.4.3 considered one extreme case. In the other extreme when the testing process for classification is destructive in nature, the testing cost is at least as high as the selling price, and the firm should just offer one bin without conducting any tests beyond the ones that ensure that the unit is functional.

Therefore, as the testing cost \( \bar{c} \) decreases, the firm will find it optimal to offer more (equal or more in a strict sense) bins than before. This conclusion provides an economic explanation for why the number of bins offered in each product line in the computer processor industry has increased drastically in the last decade. In the last decade, the cost of testing equipment has decreased drastically, enabling firms to test
every processor more carefully and offer a larger number of bins with small quality differences.

We also note that recent research in marketing, e.g. [Iyengar and Lepper 2000], has established empirically that in the presence of too many choices customers get confused and may not make a purchase at all. If such an effect is present for a firm’s product line, then the maximum number of bins should be determined after comparing the potential benefit from offering more bins that fit customers’ needs better with the potential cost of customer confusion.

Before we discuss some practical issues in determining the optimal quality of bins in §4.4.4, we analyze the case when firm can endogenize the pricing decision. We assume that firm follows a price curve \( p(x) = x^a, a > 1 \). [Honhon and Pan 2012] and other papers in assortment planning which consider endogenous pricing, explicitly consider prices of all products as decision variables. As opposed to that we assume firm has partial control on setting the prices in the sense that firm chooses what price curve to follow, which in turn determines the optimal prices of the bins. [Honhon and Pan 2012] finds that the optimal prices convexly increase in the quality levels and are unique. We show in Proposition 4.4.4 that there exists an optimal value of the exponent \( a \), and hence firm can uniquely determine the optimal prices which convexly increase in the quality of the bins. Substituting \( p(x_i) = x_i^a, i = 1, \ldots, n \), in (4.7) we can rewrite the profit function for any given quality vector \( x \) as:

\[
\Pi(a) = S \left( \sum_{j=1}^{n} x_j^a \alpha_j(x, a) - c \max_{j=1, \ldots, n} \left\{ \frac{\alpha_j(x, a)}{\beta_j(x)} \right\} \right) - cn, \tag{4.8}
\]

where, \( \alpha_j(x, a), j = 1, \ldots, n \), are the same choice probabilities given in (4.2), with \( p(x_i) = x_i^a, i = 1, \ldots, n \).

**Proposition 4.4.4.** \( \Pi(a) \) in (4.8) is pseudoconcave in \( a \).

Proposition 4.4.4 implies that firm can set the optimal prices as \( p_i^* = x_i^{a_i^*} \), where
\( a^* \) is the solution to \( \max_{a>1} \Pi(a) \). Our result in Proposition 4.4.2, which says that the optimal quality vector is set so as to fully meet the demands of all offered bins, also holds when firm follows the optimal pricing policy as in Proposition 4.4.4.

4.4.4 Counterfactual for the Prevalence of Equidistant Bins

The discussion in §4.4.2 suggests that determining the “true” optimal quality levels might be computationally demanding task. Even when the quality levels are determinable, they might be extremely uneven. As an alternative, the firm could just offer equidistant bins. We now show analytically (and numerically in §4.5) that offering equidistant bins, i.e., when the difference between any two adjacent quality levels of the optimal quality vector are equal, is not optimal in general. Let \( y_i = x_i - x_{i-1} \) denote the gap between any two consecutive quality levels, and let, \( y_1 = x_1 - x_L, y_2 = x_2 - x_1, \ldots, y_n = x_n - x_{n-1} \), and let \( y = (y_1, \ldots, y_n) \). We can write the optimization problem of (4.7) as a function of \( y \) as follows:

\[
\max_{y_i \geq 0, i=1,\ldots,n} \Pi(y) = S \left( \sum_{j=1}^{n} p(y, x_L) \alpha_j(y) - c \cdot \max_{j=1,\ldots,n} \left\{ \frac{\alpha_j(y)}{\beta_j(y)} \right\} \right) - \bar{c}n, \tag{4.9}
\]

where, \( \beta_j(y), j = 1, \ldots, n \), and \( \alpha_j(y), j = 1, \ldots, n \) are obtained by substituting \( x_i = x_L + \sum_{k=1}^{i} y_k, i = 1, \ldots, n \), into (4.1) and (4.2) respectively. The question is: Does the optimal solution to (4.9), say \( y^* \), have the property \( y^*_1 = \ldots = y^*_n \), and if so, under what conditions? We answer this question using some concepts of majorization theory ([Marshall and Olkin 1979]). In Appendix C we review those concepts in detail; however, for ease of exposition, we present only three key results here to support our claim that equidistant bins are not optimal, in general.

First, if a vector \( y^1 \) majorizes another vector \( y^2 \) (denoted \( y^1 \succ y^2 \)), the components of \( y^1 \) are more spread out than the components of \( y^2 \). Second, any vector that has equal components is the least spread out vector, and all other vectors majorize it.
Finally, if $f$ is Schur-concave on $\mathbb{R}^n$, then the optimal solution to $\max_u f(u)$ is obtained when all components of $u$ are equal. With these three points, we argue that offering equidistant bins may not be always optimal.

Since the profit function in (4.7) is not jointly concave or quasiconcave in $x$ for arbitrary choices of $F$ and $G$, it follows that (4.9) is not jointly concave or quasiconcave in $y$ and hence is not Schur-concave. Therefore, the optimal solution for the quality levels is not guaranteed to be a vector with equal components. Only for specific choices on $F$ and $G$, which makes (4.9) Schur-concave, the optimal solution will have equal components; that is, equidistant bins will be optimal. However, in general it is non-trivial to come up with the specifications of $F$ and $G$ that make (4.9) Schur-concave. We leave determining these specifications for future research, as might help a firm identify conditions when equidistant bins are optimal or nearly optimal.

### 4.5. Numerical Analysis

So far we discussed analytical results on the optimal quality and quantity decisions. The purpose of our numerical analysis is to complement our analytical findings with insights such as how does the quality of the bins change when parameters of the quality distribution and the distribution of consumer preference change. We also use the numerical study to compare the profit from the optimal assortment with the profit from equidistant bins.

#### 4.5.1 Numerical Design

We consider a scenario where both manufacturing yield $X$ and consumer preference $G$ are normally distributed, and we call this as the NYP scenario. Our assumption of normal distribution on the yield distribution is consistent with the empirical finding in [Datta et al. 2006]. Also, consistent with [Honhon and Pan 2012], we assume
consumer preference to be normally distributed. We allow both mean ($\mu_X$) and
standard deviation ($\sigma_X$) of the yield curve to have five levels: $\mu_X = \{5, 6, 7, 8, 9\}$
and $\sigma_X = \{0.5, 0.6, 0.7, 0.8, 0.9\}$. The mean ($\mu_G$) and standard deviation ($\sigma_G$) of the
distribution of consumer preference also have five levels: $\mu_G = \{6, 7, 8, 9, 10\}$ and
$\sigma_G = \{0.9, 1.05, 1.2, 1.35, 1.5\}$. We assume the price curve to be $p(x) = x^a$, and consider
four levels of the index $a = 1.01, 1.2, 1.5, 2$. A value of $a = 1.01$ represents almost linear
pricing, whereas $a = 2$ represents strictly convex pricing. The above factorial design
yielded 2,500 total instances. Table 4.2 reports the factors and their values for normal
distribution of yield and preferences.

<table>
<thead>
<tr>
<th>Factor description</th>
<th>Symbol</th>
<th>Factor values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Yield</td>
<td>$\mu_X$</td>
<td>${5, 6, 7, 8, 9}$</td>
</tr>
<tr>
<td>Spread Yield</td>
<td>$\sigma_X$</td>
<td>${0.5, 0.6, 0.7, 0.8, 0.9}$</td>
</tr>
<tr>
<td>Mean Preference</td>
<td>$\mu_G$</td>
<td>${6, 7, 8, 9, 10}$</td>
</tr>
<tr>
<td>Spread Preference</td>
<td>$\sigma_G$</td>
<td>${0.9, 1.05, 1.2, 1.35, 1.5}$</td>
</tr>
<tr>
<td>Index of price curve</td>
<td>$a$</td>
<td>${1.01, 1.2, 1.5, 2}$</td>
</tr>
</tbody>
</table>

Table 4.2: Experimental Design with Normal distribution of Yield and Preference

The remaining parameters, unit production cost $c$, cost of classification $\bar{c}$, market
size $S$, and lower limit of quality $x_L$, are set as follows. Unit production cost is $c = 1$
For the classification cost, we fix $\bar{c} = 0$. We also fix the market size to $S = 100$. The
lower quality limit is set as $x_L = A + 0.001$, when yield follows uniform distribution,
and $x_L = \mu_X - 3\sigma_X + 0.001$, for normal yield\(^1\). In each instance we search for the critical
points of the profit function, starting with a randomly perturbed equidistant quality
vector. The optimal solution is found as the critical point with the maximum profit
value (the code is written in Matlab\textsuperscript{®} 9, and we implement it on a Dell Workstation
with a 3.0GHz processor).

\(^1\)We also consider a scenario where both yield and preferences are uniformly distributed (the UYP
scenario) and found that all the results of NYP scenario also holds there. The numerical design for
this scenario is given in the Appendix.
4.5.2 An Illustrative Example

First we discuss a base case in §4.5.2.1, and highlight the salient features of the optimal solution. Subsequently in §4.5.2.2 we do a comparative static analysis over the base cases when parameters of the yield distribution and distribution of consumer preferences change.

4.5.2.1 Base Case

For the base case, we consider both manufacturing yield $X$ and consumer preference $\theta$ are normally distributed, with parameters $\mu_X = 6, \sigma_X = 0.9$ and $\mu_\theta = 8, \sigma_\theta = 0.9$. Unit manufacturing cost is $c = 1$ and cost of adding bins is $\tau = 0$. Prices follow the curve $p(x) = x^2$ and the total market size is $S = 100$. The optimal quality levels for various bins are

<table>
<thead>
<tr>
<th># of bins</th>
<th>Optimal quality vector: $x^*$</th>
<th>Profit: $\Pi(x^*)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>(6.33548, 8.699998, 8.699999)</td>
<td>3610.683609</td>
</tr>
<tr>
<td>4</td>
<td>(6.33548, 8.69998, 8.699999, 8.7)</td>
<td>3610.68380</td>
</tr>
<tr>
<td>5</td>
<td>(6.33549, 8.69980, 8.69983, 8.69986, 8.69991)</td>
<td>3610.68940</td>
</tr>
<tr>
<td>6</td>
<td>(6.33548, 8.699997, 8.6999977, 8.6999980, 8.6999984, 8.6999989)</td>
<td>3610.69958</td>
</tr>
</tbody>
</table>

Table 4.3: Optimal Solution of Base Case

We make three observations from Table 4.3. First, as the number of bins increase, the consecutive quality levels get very close to each other and become practically identical (for $n = 4$, we get $x^*_3 = 8.69999$, $x^*_4 = 8.7$). It is possible that consumers may not be able to distinguish extremely small differences in the product line, so for practical purpose, firm may club these two values and just offer the product at one quality level (say 8.7). Second, with the introduction of a new product, the quality levels of the existing bins do not change by a large amount (when $n$ changed from
n = 3 to n = 4, the value of $x^*_2$ changed from 8.699998 to 8.69998). As new bins are added, the quality levels of those new bins lie on the higher end, i.e., higher than the existing highest quality bin. Finally, we observe that none of the optimal solutions is the equidistant vector. For example, when $n = 6$, the equidistant quality vector is $\mathbf{x} = (4.072, 4.843, 5.614, 6.386, 7.157, 7.928)$.

4.5.2.2 Comparative Static Analysis for $\sigma_X$ and $\sigma_G$

Now we analyze the effect of change in variability of the yield distribution and change in variability of consumer preference distribution on the optimal assortment length, which is defined as the gap between the highest and lowest quality bins. The assortment length is a measure of the dispersion in the quality level of the bins. As bins are more unevenly spaced, the assortment length increases. We assume the process variability takes values $\sigma_X = 0.6, 0.7, 0.8, 0.9$, and the variability of consumer heterogeneity takes values $\sigma_G = 0.9, 1.05, 1.2$. For each case we solve the problem keeping all other parameters same as in the base case. We report the assortment length in Table 4.4 and Table 4.5 as these two parameters vary.

<table>
<thead>
<tr>
<th>$\sigma_X$</th>
<th>n = 3</th>
<th>n = 4</th>
<th>n = 5</th>
<th>n = 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>1.534</td>
<td>1.534</td>
<td>1.534</td>
<td>1.534</td>
</tr>
<tr>
<td>0.7</td>
<td>1.797</td>
<td>1.797</td>
<td>1.797</td>
<td>1.797</td>
</tr>
<tr>
<td>0.8</td>
<td>2.075</td>
<td>2.075</td>
<td>2.075</td>
<td>2.075</td>
</tr>
<tr>
<td>0.9</td>
<td>2.364</td>
<td>2.364</td>
<td>2.364</td>
<td>2.364</td>
</tr>
</tbody>
</table>

Table 4.4: Impact of change of $\sigma_X$

In Table 4.4 we observe that for each $n$, the assortment length increases as the process variability amplifies. The managerial intuition is that as there is a wider disparity in the offered output quality, the firm wants to extract additional revenues
from heterogeneous customers by segmenting the offered output into different bins that are widely apart.

| Assortment length \((=x_n^* - x_i^*)\) as \(\sigma_G\) varies |
|---|---|---|---|---|
| \(\sigma_G\) | \(n = 3\) | \(n = 4\) | \(n = 5\) | \(n = 6\) |
| 0.9 | 2.3645 | 2.36451 | 2.364518 | 2.3645189 |
| 1.05 | 2.5175 | 2.51758 | 2.51737 | 2.519140 |
| 1.2 | 2.6598 | 2.65984 | 2.65977 | 2.6597606 |

Table 4.5: Impact of change of \(\sigma_G\)

A similar trend, i.e., increase in assortment length is observed in Table 4.5 as the variability in consumer heterogeneity increases. In other words as consumer preference become more and more diverse, the firm has more freedom to offer an assortment with wider variation in quality. Thus firm can extract additional revenues by segmenting the heterogeneous customer base. We analyzed the impact of change of \(\sigma_X\) and \(\sigma_G\) on the assortment length in several other numerical instances, like we did on this base case, and found similar trends. In Appendix C we report those results in details.

To summarize, we find that the optimal quality levels of the bins are unevenly spaced, and the dispersion between the quality of the bins increase as the process becomes more variable, or the consumer preferences become more diverse.

In the following section we outline some more observations and trends that were observed throughout the numerical instances, and we provide managerial intuition behind them.
4.5.3 Synthesis of Other Comparative Analysis

4.5.3.1 Optimal Solution vis-á-vis Equidistant Bins

In our numerical results we investigated the nature of the optimal quality vector, in particular whether it has equally spaced or unevenly spaced bins. In each numerical instance we evaluate the optimality gap between the profit from equidistant bins \( \Pi^{equi} \) and the optimal profit \( \Pi^* \) as \( OG = \frac{\Pi^* - \Pi^{equi}}{\Pi^{equi}} \). In Table 4.6 we report the average of these percentage gaps under the UYP and NYP scenarios. In both scenarios, the optimality gap increases with the number of bins. Therefore, the benefit from determining the optimal assortment justifies the effort required. Further, these results suggest that the common wisdom that if the firm must offer multiple bins, then offering equidistant bins is as good as any other assortment does not hold true, and it indeed reduces the firm’s profit.

<table>
<thead>
<tr>
<th>Number of bins</th>
<th>Average Optimality Gap under UYP scenario</th>
<th>Average Optimality Gap under NYP scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>12.78%</td>
<td>40%</td>
</tr>
<tr>
<td>4</td>
<td>18.77%</td>
<td>44.7%</td>
</tr>
<tr>
<td>5</td>
<td>27.26%</td>
<td>45%</td>
</tr>
<tr>
<td>6</td>
<td>38.96%</td>
<td>53%</td>
</tr>
</tbody>
</table>

Table 4.6: Optimality Gap of Equidistant Bins

Yet in many practical cases we observe equidistant or nearly equidistant product offerings. For example, AMD Phenom II processor comes in six different speeds, 2.7-3.3GHz, with increments of 0.1 GHz. And, Intel Pentium Core2 Duo (E550 series) comes at five different speeds, 2.5GHz, 2.7GHz, 2.8GHz, 3.0GHz and 3.2GHz. In these cases, the optimal assortment is perhaps determined by incorporating a minimum-gap constraint between two consequent bins. This type of requirements may arise due to marketing considerations and consumer side considerations. Consumers may not be able to distinguish two bins if the distinction between the bins is not sufficiently high,
and in the absence of sufficient differentiation, advertising and selling these bins may be difficult.

4.5.3.2 Effect of Change in Number of Bins on Optimal Quality

In this part, we analyze how the spacings between bins are affected as number of bins change. In the base case described in §4.5.2.1, we observed that as the number of bins increase, the new bins get staggered to the right end, i.e., the quality of the new bins are added after the existing highest quality bin. Further the quality levels of the existing bins are not affected much as new bins are added. We found a similar trend in all other numerical instances. In each instance, we measure the percentage change of the quality of an existing bin as a new bin is added to its base case. That is, in each instance, we measure the percentage change in $x^*_1, \ldots, x^*_5$ as $n$ changes from $n = 3$ to $n = 4, 5, 6$. Figure 4.6 shows the percentage change in the five optimal quality levels, averaged across all instances.

![Figure 4.6: % Change in the quality level vs. bins under NYP scenario](image)

We observe that the percentage change is minuscule for all the five quality levels (0.005% - 0.025%). A managerial interpretation of this observation is that as firm gets opportunity to introduce new bins, a reason for which could be reduction in the cost of binning, then firm does not change the quality levels of the existing offerings, rather
the new bins are added at the high end. This is consistent with several observations in practice, where as product lines are revised, firms often retain the existing product offerings, and new products are introduced so as to create demand in newer segments of the market.

4.5.3.3 Benefit of adding Bins

Finally, we investigate how does the profit increase as firm gets opportunity to offer more bins. At an aggregated level (i.e., we find the average profit across all numerical instances for each $n$), we find that the rate of increase in profit reduces as firm introduces more and more bins (see Figure 4.7). From a strict sense there is increase in profit, but concavely. This phenomenon together with the marginal benefit of introducing more bins restrict the number of bins that firm should eventually offer.

![Profit vs. number of bins](image)

**Figure 4.7:** Average profit vs. number of bins under NYP scenario

To summarize, in our numerical study we found that the optimality gap of equidistant bins increase as more bins are added. We also found that the quality levels of the existing bins do not change much as more bins are added. Further, we found that
4.6. Discussion and Conclusions

This paper discusses analytical and practical aspects of assortment planning in co-production systems, which are widely present in the manufacturing of high-tech products. The units manufactured on these systems have different quality levels due to the random variations in complex manufacturing operations. The units are classified and sold in different bins based on their actual performance levels. The key analytical finding is that optimal assortment of bins and quantity for a firm is such that the demands of all bins are met in full. From a practical standpoint, however, the optimal quality levels of the bins might be very close to each other. In the absence of strong differentiation, advertising and selling these bins may be difficult. Further, the underlying optimization problem is not well behaved and might require sophisticated numerical search techniques to find the optimal assortment. We investigated the optimality of equidistant bins that appear intuitively appealing but found numerically that the firm could leave a lot of money on the table with such bins. A reasonable balance between maximizing profit from a purely economic perspective and offering adequately differentiated bins might be reached by first finding the optimal assortment analytically and then modifying this assortment for marketing considerations.

A key conclusion of this research is that the decrease in the testing cost of semiconductor devices enables firms to offer a larger number of bins. Since this cost continues to fall further as more sophisticated high-precision equipment becomes available, the product variety will continue to increase in the future. Other market characteristics will further drive this proliferation or restrict it. Different appliances and gadgets operate at different ampere levels, can tolerate different levels of heat emission. These operating characteristics lead to demands for bins at a large number of
specific quality levels. Computer processor firms will continue to expand their offerings as the operating characteristics of devices become more diverse. At the same time, empirical evidence such as [Iyengar and Lepper 2000] suggests that offering too many choices for a specific need may confuse the customers and may result in a loss of sales. This phenomenon will restrict the number of bins that a firm could eventually offer. Empirical work that documents and formalizes the interplay of these factors in the computer processor industry will constitute interesting research. In our numerical analysis we found that as firm gets opportunity to introduce new bins, then firm does not change the quality levels of the existing offerings too much, rather the new bins are added at the high end.

We find that offering equidistant bins is optimal in extremely restricted situations and, in general, optimally set bins will not be equidistant. However, we observe in practice that firms sometimes offer equidistant bins. While this fact may be due to the fact that the problem of determining optimal number/quality levels of bins is extremely challenging as we showed earlier, other reasons related to customer expectations and industry norms may also be at play. We are not aware of any work that documents the response of customers who face choices between products that are not equi-spaced or regularly spaced. Firms may prefer to offer equi-spaced bins because customers may prefer such choice sets. Similarly, firms typically invest in manufacturing technology which will result in an overhaul of the bin offerings. However, to ease the version management process, firms may prefer to specify versions in specific increments only. Understanding these factors using a survey or an experiment will be useful.

In the future, revisiting the model formulation to determine the optimal assortment while considering the possibility of downward substitution will be useful. Let

\[ z_1 = \min(Q\beta_1(x), S\alpha_1(x)), \quad z_2 = \min(Q(\beta_1(x) + \beta_2(x)), S\alpha_2(x)) \ldots \text{ and so on, then the} \]
firm’s problem can be rewritten as follows:

\[
\begin{align*}
\max_{x,Q,n} \Pi(x,Q,n) &= \sum_{i=1}^{n} p(x_i) \min \{ Q \sum_{k=1}^{i} \beta_k(x) - \sum_{k=1}^{i-1} z_k, S \alpha_i(x) \} - cQ - n\bar{\epsilon}, \\
\text{s.t., } x_L &\leq x_1 < \ldots < x_i < x_{i+1} < \ldots < x_n \leq x_H, \\
\alpha_i(x) &\geq 0, \quad i = 1, \ldots, n.
\end{align*}
\]

(4.10)  
(4.11)  
(4.12)

Intuitively, it appears that incorporating the flexibility of downward substitution will encourage the firm to offer products that are more differentiated than before when the substitution was not possible. Downward substitution enables the firm to reduce the supply-demand mismatch, and the firm will take advantage of this flexibility to offer products at higher quality levels to extract the high premium of the high-end customers. However, these conjectures are difficult to prove analytically. A numerical analysis is likely to play an important role in such analysis.
Bibliography


agement. 7. 330-346.


93(3) 748-774.


[Honhon and Pan(2012)] Honhon, D., X. A. Pan. 2012. Assortment Planning for Verti-


Appendix A

Proofs for Chapter 2

1. Proposition 2.4.1 is proved with the help of Lemmas A.0.1 and A.0.2.

2. Proposition 2.4.3 is proved with the help of Lemma A.0.3.

3. Proposition 2.4.6 is proved with the help of Lemma A.0.4.

1. Proofs of Lemmas A.0.1, A.0.2, A.0.3 and A.0.4

In this section of the Appendix we state and prove Lemmas A.0.1, A.0.2, A.0.3 and A.0.4 which are used to prove Propositions 2.4.1, 2.4.3, and 2.4.6.

Let $\nabla_i g(p)$ denote the first partial of $g$ w.r.t. $p_i$, $\nabla_{ii}^2 g(p)$ the second partial w.r.t. $p_i$, and $\nabla_{ji}^2 g(p)$ the cross partial w.r.t. $p_i$ and $p_j$, respectively.

**Lemma A.0.1.** For any $i \in \{1, 2, \ldots, n\}$ we have the following:

(a) $\nabla_i \alpha_i(p) < 0$, $\nabla_{ii}^2 \alpha_i(p) = 0 = \nabla_{ji}^2 \alpha_i(p)$

(b) $\nabla_i \alpha_k(p) > 0$, $k = i - 1, i + 1$; $\nabla_i \alpha_k(p) = 0$, $k \neq i - 1, i, i + 1$

(c) \[
\frac{\partial \alpha_k(p)}{\partial q_i} = \begin{cases} 
-\frac{p_i - p_{i-1}}{(q_i - q_{i-1})^2}, & k = i - 1, \\
\frac{p_{i+1} - p_i}{(q_{i+1} - q_i)^2} + \frac{p_i - p_{i-1}}{(q_i - q_{i-1})^2}, & k = i, \\
-\frac{p_i + 1 - p_i}{(q_{i+1} - q_i)^2}, & k = i + 1.
\end{cases}
\]

(d) $\nabla_i z_k(p, t) = \frac{z_k(p, t)}{\alpha_k(p)} \frac{\partial \alpha_k(p)}{\partial q_i}$.

(e) $\nabla_{ii}^2 z_k(p, t) = 0$, $\nabla_{ji}^2 z_k(p, t) = 0$.

(f) $\frac{\partial z_k(p, t)}{\partial q_i} = \frac{z_k(p, t)}{\alpha_k(p)} \frac{\partial \alpha_k(p)}{\partial q_i}$, $k = i - 1, i, i + 1$; $\frac{\partial z_k(p, t)}{\partial q_i} = 0$, $k \neq i - 1, i, i + 1$.
(g) \( u = -\sum_{k=i-1}^{i+2} \left[ p_k \left( \frac{\nabla \alpha_k(p)}{\alpha_k(p)} \times \nabla \alpha_i(p) + \frac{\nabla \alpha_{i+1}(p)}{\alpha_{i+1}(p)} \times \nabla \alpha_k(p) \right) \times \frac{p_k - c_k}{p_k} z_k(p, t) \right] \) < 0

Proof. (a) From (2.1) we have \( \alpha_i(p) = \frac{p_i + 1 - p_i}{q_i + 1 - q_i} - \frac{p_i - p_i - 1}{q_i - q_i} \). Therefore, \( \nabla_j \alpha_i(p) = -\frac{q_i + 1 - q_i - 1}{(q_i + 1 - q_i)(q_i - q_i - 1)} < 0 \), \( \nabla^2 \alpha_i(p) = 0 \), \( \nabla^2 \alpha_i(p) = 0 \).

(b) From (2.1) we have \( \alpha_{i+1}(p) = \frac{p_i + 2 - p_i + 1}{q_i + 2 - q_i + 1} - \frac{p_i + 1 - p_i}{q_i + 1 - q_i} \) and \( \alpha_{i-1}(p) = \frac{p_i - p_i - 1}{q_i - q_i - 1} - \frac{p_i - p_i - 2}{q_i - q_i - 2} \). Therefore, \( \nabla_i \alpha_{i-1}(p) = \frac{1}{q_i - q_i} > 0 \) and \( \nabla_i \alpha_{i+1}(p) = \frac{1}{q_i + 1 - q_i} > 0 \).

(c) Differentiating \( \alpha_i(p) = \frac{p_i + 1 - p_i}{q_i + 1 - q_i} - \frac{p_i - p_i - 1}{q_i - q_i - 1} \), \( \alpha_{i+1}(p) = \frac{p_i + 2 - p_i + 1}{q_i + 2 - q_i + 1} - \frac{p_i + 1 - p_i}{q_i + 1 - q_i} \) and \( \alpha_{i-1}(p) = \frac{p_i - p_i - 1}{q_i - q_i - 1} - \frac{p_i - p_i - 2}{q_i - q_i - 2} \), each w.r.t. \( q_i \) we get the result.

(d) Recall that \( z_k(p, t) \) is implicitly defined in (2.9) as \( F_k(p, z_k(p, t)) = t \), i.e., \( z_k(p, t) \) is the demand corresponding to the \( t \) fractile of \( F_k \). Since the demand model in (2.4) is a multiplicative one, we can write \( z_k(p, t) = \alpha_k(p) z_k \). So, \( F_k(p, z_k(p, t)) = t \Rightarrow Pr(z_k(p, t) = \alpha_k(p) z_k) = t \Rightarrow Pr(\frac{z_k(p, t)}{\alpha_k(p)}) = t \Rightarrow G_k(\frac{z_k(p, t)}{\alpha_k(p)}) = t \), where \( G_k(.) \) is the c.d.f. of \( z_k \). Differentiating \( G_k(\frac{z_k(p, t)}{\alpha_k(p)}) = t \) w.r.t. \( p_i \) we get

\[
g_k \left( \frac{z_k(p, t)}{\alpha_k(p)} \right) \left[ -z_k(p, t) \frac{\nabla \alpha_k(p)}{\alpha_k(p)} + \nabla_i z_k(p, t) \frac{1}{\alpha_k(p)} \right] = 0. \quad (A.1)
\]

Simplifying (A.1) we get \( \nabla_i z_k(p, t) = \frac{g_k(p, t) \nabla \alpha_k(p)}{\alpha_k(p)} \).

(e) Differentiating \( \nabla_i z_k(p, t) = \frac{g_k(p, t) \nabla \alpha_k(p)}{\alpha_k(p)} \) w.r.t. \( p_j \) we get

\[
\nabla^2_{j} z_k(p, t) = \nabla_j z_k(p, t) \frac{\nabla \alpha_k(p)}{\alpha_k(p)} + z_k(p, t) \left( \frac{g_k(p, t) \nabla^2 \alpha_k(p) - \nabla_j \alpha_k(p) \nabla_j \alpha_k(p)}{\alpha_k(p)} \right)
\]
\[
= z_k(p, t) \frac{\nabla_j \alpha_k(p) \nabla \alpha_k(p)}{\alpha_k(p)} + z_k(p, t) \left( \frac{g_k(p, t) \nabla^2 \alpha_k(p) - \nabla_j \alpha_k(p) \nabla_j \alpha_k(p)}{\alpha_k(p)} \right)
\]
\[
= z_k(p, t) \frac{\nabla^2 \alpha_k(p)}{\alpha_k(p)}. \quad (A.2)
\]

From part (a) we know \( \nabla^2 \alpha_k(p) = 0 \), hence in (A.2) we have \( \nabla^2 \alpha_k(p) = 0 \). Similarly it can be showed that \( \nabla^2 \alpha_k(p) = 0 \).
(f) Differentiating \( G_k \left( \frac{z_k(p,t)}{\alpha_k(p)} \right) = t \) w.r.t. \( q_i \) we get

\[
g_k \left( \frac{z_k(p,t)}{\alpha_k(p)} \right) \left[ \frac{z_k(p,t)}{\alpha_k(p)} \cdot \frac{\partial \alpha_k(p)}{\partial q_i} + \frac{\partial z_k(p,t)}{\partial q_i} \cdot \frac{1}{\alpha_k(p)} \right] = 0.
\]

Therefore, \( \frac{\partial z_k(p,t)}{\partial q_i} = \frac{z_k(p,t)}{\alpha_k(p)} \cdot \frac{\partial \alpha_k(p)}{\partial q_i} \).

(g) First we expand the summation in \( u \)

\[
u = - \sum_{k=i-1}^{i+2} \left[ p_k \left( \frac{\nabla_i \alpha_{i-1}(p)}{\alpha_{i-1}(p)} \times \frac{\nabla_{i+1} \alpha_i(p)}{\alpha_i(p)} + \frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)} \times \frac{\nabla_{i+1} \alpha_{i+1}(p)}{\alpha_{i+1}(p)} \right) \times \int_0^{p_k - c_k} z_k(p,t) dt \right]
\]

\[
= -p_{i-1} \left( \frac{\nabla_i \alpha_{i-1}(p)}{\alpha_{i-1}(p)} \times \frac{\nabla_{i+1} \alpha_i(p)}{\alpha_i(p)} + \frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)} \times \frac{\nabla_{i+1} \alpha_{i+1}(p)}{\alpha_{i+1}(p)} \right) \times \int_0^{1 - \frac{c_i}{R_{i-1}}} z_{i-1}(p,t) dt
\]

\[
- p_{i+1} \left( \frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)} \times \frac{\nabla_{i+1} \alpha_i(p)}{\alpha_i(p)} + \frac{\nabla_i \alpha_{i+2}(p)}{\alpha_{i+2}(p)} \times \frac{\nabla_{i+1} \alpha_{i+1}(p)}{\alpha_{i+1}(p)} \right) \times \int_0^{1 - \frac{c_i}{R_{i+1}}} z_{i+1}(p,t) dt
\]

\[
- p_{i+2} \left( \frac{\nabla_i \alpha_{i+2}(p)}{\alpha_{i+2}(p)} \times \frac{\nabla_{i+1} \alpha_i(p)}{\alpha_i(p)} + \frac{\nabla_i \alpha_{i+2}(p)}{\alpha_{i+2}(p)} \times \frac{\nabla_{i+1} \alpha_{i+2}(p)}{\alpha_{i+2}(p)} \right) \times \int_0^{1 - \frac{c_i}{R_{i+2}}} z_{i+2}(p,t) dt
\]

\(\text{(A.3)}\)

We will show that each of the four terms (rows) in (A.3) are negative. First, consider the first row of (A.3). From Lemma A.0.1(b) we know that \( \nabla_i \alpha_{i-1}(p) > 0, \nabla_{i+1} \alpha_i(p) > 0, \nabla_i \alpha_{i+1}(p) > 0 \) and \( \nabla_{i+1} \alpha_{i-1}(p) = 0 \). Thus, the first term in (A.3) is

\[
- p_{i-1} \left( \frac{\nabla_i \alpha_{i-1}(p)}{\alpha_{i-1}(p)} \times \frac{\nabla_{i+1} \alpha_i(p)}{\alpha_i(p)} \right) \int_0^{1 - \frac{c_i}{R_{i-1}}} z_{i-1}(p,t) dt < 0.
\]

Now consider the last term of (A.3). From Lemma A.0.1(b) we know that \( \nabla_{i+1} \alpha_{i+2}(p) > 0 \) and \( \nabla_i \alpha_{i+2}(p) = 0 \). Thus, the last term in (A.3) is

\[
- p_{i+2} \left( \frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)} \times \frac{\nabla_{i+1} \alpha_{i+2}(p)}{\alpha_{i+2}(p)} \right) \int_0^{1 - \frac{c_i}{R_{i+2}}} z_{i+2}(p,t) dt < 0.
\]
Now consider the second and third terms of (A.3). The second term has a positive term within parenthesis $\frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)}$, and a negative term $\frac{\nabla_i \alpha_i(p)}{\alpha_i(p)}$. In the third term of (A.3) the positive term in parenthesis is $\frac{\nabla_{i+1} \alpha_{i+1}(p)}{\alpha_{i+1}(p)}$, and the negative term is $\frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)}$. We show that the positive terms in parenthesis is greater than the negative terms in parenthesis, hence the second and third terms are both negative. In particular we show, $\frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)} > -\frac{\nabla_i \alpha_i(p)}{\alpha_i(p)}$ and $\frac{\nabla_{i+1} \alpha_{i+1}(p)}{\alpha_{i+1}(p)} > -\frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)}$. Rearranging $\frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)} > -\frac{\nabla_i \alpha_i(p)}{\alpha_i(p)}$ and $\frac{\nabla_{i+1} \alpha_{i+1}(p)}{\alpha_{i+1}(p)} > -\frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)}$, we get $\frac{\nabla_i \alpha_{i+1}(p)}{\alpha_i(p)} > \frac{\nabla_{i+1} \alpha_{i+1}(p)}{\alpha_{i+1}(p)}$. This inequality holds only if $\frac{\nabla_i \alpha_{i+1}(p)}{\alpha_i(p)} > \frac{\nabla_{i+1} \alpha_{i+1}(p)}{\alpha_{i+1}(p)}$. Substituting $\nabla_i \alpha_i(p) = -\frac{q_i+1-q_i-1}{(q_i+1-q_i)(q_i+q_i-1)}$, $\nabla_{i+1} \alpha_i+1(p) = -\frac{q_i+2-q_i}{(q_i+2-q_i)(q_i+1-q_i)}$, $\nabla_{i+1} \alpha_{i+1}(p) = \nabla_i \alpha_{i+1}(p) = \frac{1}{q_{i+1}-q_i}$ into $\frac{\nabla_i \alpha_{i+1}(p)}{\alpha_i(p)} > \frac{\nabla_{i+1} \alpha_{i+1}(p)}{\alpha_{i+1}(p)}$ we get, $\frac{(q_{i+1}-q_i)}{(q_{i+2}-q_i)(q_{i+1}+q_i-1)} < 1$. Simplifying this we get

$$-(q_{i+1} - q_i)(q_{i+2} - q_i-1) < 0.$$ 

Lemma A.0.2. If $p$ satisfies $\nabla_i \Pi(p) = 0$ and $F_1$, $i = 1, 2, ..., n$ are IFR then $\nabla_1 y^*_i(p) < 0$.

Proof. This Lemma is borrowed from Proposition 1 in [Aydin and Porteus 2008]. We refer to the paper for proof of this Lemma.

Lemma A.0.3. Let $\Pi_0^*(c, q)$ be the optimal profit value in riskless case, given as

$$\Pi_0^*(c, q) = \sum_{i=1}^{n} (p_0^* - c_i) \alpha_i(p_0^*). \tag{A.4}$$

Then, $\Pi_0^*(c, q)$ in (A.4) is strictly $c$-decreasing, i.e., $\frac{\partial \Pi_0^*(c, q)}{\partial c_i} < 0$, $i = 1, \ldots, n$.

Proof. Substituting $p_0^* = \frac{q_i+c_i}{2}$, $i = 1, \ldots, n$, and $\alpha_i(p_0^*) = \frac{\alpha_i(c)}{2}$ (see (2.17)), into (A.4) we get

$$\Pi_0^*(c, q) = \sum_{i=1}^{n} \frac{1}{4} (q_i - c_i) \alpha_i(c) \tag{A.5}$$
Taking the partial derivative of $\Pi_i^*(c, q)$ in (A.5) w.r.t. $c_i$, $i = 1, \ldots, n$, we get,

$$\frac{\partial \Pi_i^*(c, q)}{\partial c_i} = \frac{1}{4} \left[ -\alpha_i(c) + (q_i - c_i) \frac{\partial \alpha_i(c)}{\partial c_i} + (q_{i+1} - c_{i+1}) \frac{\partial \alpha_{i+1}(c)}{\partial c_i} + (q_i - c_i) \frac{\partial \alpha_{i-1}(c)}{\partial c_i} \right], i = 1, \ldots, n. (A.6)$$

Substituting $\frac{\partial \alpha_i(c)}{\partial c_i} = \frac{-(q_{i+1} - q_i)}{(q_i - q_{i-1})} \frac{\partial \alpha_{i+1}(c)}{\partial c_i} = \frac{1}{q_i - q_{i-1}}$ and $\frac{\partial \alpha_{i-1}(c)}{\partial c_i} = \frac{1}{q_i - q_{i-1}}$ into (A.6) and upon simplification we get

$$\frac{\partial \Pi_i^*(c, q)}{\partial c_i} = -\frac{1}{2} \left[ \frac{c_i - c_{i-1}}{q_i - q_{i-1}} - 1 + 1 \right] = -\frac{\alpha_i(c) + 1}{2} < 0, i = 1, \ldots, n. (A.7)$$

\[ \square \]

**Lemma A.0.4.** : Let $\Pi^*(c, q)$ be the optimal profit value in the risky case, given as

$$\Pi^*(c, q) = \max_{p \in \mathbb{P}} \Pi(p) = \max_{p \in \mathbb{P}} \sum_{i=1}^{n} p_i \int_{0}^{q_i} \frac{p_i - c_i}{p_i} z_i(p, t) dt. (A.8)$$

Let $r_i = \frac{c_i}{q_i}$ be the cost-to-quality ratio of product $i$ and $r = (r_1, \ldots, r_n)$ denote the vector of cost-to-quality ratios. Then, $\Pi^*(c, q)$ in (A.8) is strictly $r$-decreasing, i.e., $\frac{\partial \Pi^*(c, q)}{\partial r_i} < 0, i = 1, \ldots, n.$

**Proof.** Using chain rule we have, $\frac{\partial \Pi^*(c, q)}{\partial r_i} = \frac{\partial \Pi^*(c, q)}{\partial q_i} \cdot \frac{\partial q_i}{\partial r_i}$ and $\frac{\partial \Pi^*(c, q)}{\partial c_i} = \frac{\partial \Pi^*(c, q)}{\partial q_i} \cdot \frac{\partial q_i}{\partial c_i}$. By Envelope theorem we can write $\frac{\partial \Pi^*(c, q)}{\partial c_i} \bigg|_{p=p^*} = \frac{\partial \Pi(p)}{\partial c_i} \bigg|_{p=p^*}$, and we can find the partial derivative of $\Pi(p)$ w.r.t. $c_i$, using Leibnitz rule:

$$\frac{\partial \Pi(p)}{\partial c_i} = \frac{\partial}{\partial c_i} \left( \sum_{i=1}^{n} p_i \int_{0}^{q_i} \frac{p_i - c_i}{p_i} z_i(p, t) dt \right) = -z_i(p, 1 - c_i/p_i)$$

Therefore, $\frac{\partial \Pi^*(c, q)}{\partial r_i} \bigg|_{p=p^*} = [\frac{\partial \Pi(p)}{\partial c_i} \bigg|_{p=p^*} = -y_i^+(p^*)] < 0$ (Recall that $z_i(p, t)$ is the demand corresponding to the $t$ fractile of $F_i$, given prices $p$, so when $t = 1 - \frac{c_i^*}{p_i^*}$, the critical fractile, we get $z_i(p, 1 - c_i^*/p_i^*) = y_i^+(p)$. Also, $y_i^+(p^*) = 0$ only
if \( p^*_i = c_i \). It is now easy to see that 
\[
\frac{\partial \Pi^*(c,a)}{\partial r_i} = \frac{\partial \Pi^*(c,a)}{\partial c_i} = q_i \frac{\partial \Pi^*(c,a)}{\partial c_i} = -q_i y^*_i(p^*) < 0.
\]

### 2. Proofs of Propositions 2.4.1, 2.4.3 and 2.4.6

**Proof of Proposition 2.4.1**

**Proof.** Using the definition of a strictly pseudoconcave function ( [Cambini and Martein 2009], pp. 244) we will show that

(a) Whenever, \( \nabla \Pi(p) = 0 \), we have \( \nabla^2 \Pi(p) \) is negative definite.

(b) If \( p \) satisfies \( \nabla \Pi(p) = 0 \), then \( \Pi(p) \) has a strict local maximum at \( p \).

We will use results from Lemma A.0.1 and A.0.2 in completing the proof of Proposition 2.4.1.

**Part (a):** We show that when \( \nabla \Pi(p) = 0 \), \( \nabla^2 \Pi(p) \) is a tri-diagonal matrix with negative diagonal terms, \( \nabla^2_{ii} \Pi(p) < 0 \), \( i = 1, \ldots, n \), and negative sub diagonals, for any \( i \in \{1, \ldots, n\} \), \( \nabla^2_{ji} \Pi(p) < 0 \), \( j = i - 1, i + 1 \) and \( \nabla^2_{ji} \Pi(p) = 0 \), for \( j \neq i - 1, i + 1 \).

Differentiating the profit function in (2.11) w.r.t. \( p_i \) we get

\[
\nabla_i \Pi(p) = \int_0^{p_i-c_i} \frac{p_i-c_i}{p_i} z_i(p,t) dt + \frac{c_i}{p_i} y^*_i(p) + \sum_{k=1}^n \left[ p_k \int_0^{p_k-c_k} \nabla_i z_k(p,t) dt \right],
\]

\[
= \int_0^{p_i-c_i} \frac{p_i-c_i}{p_i} z_i(p,t) dt + \frac{c_i}{p_i} y^*_i(p) + \sum_{k=1}^n \left[ p_k \nabla_i \alpha_k(p) \int_0^{p_k-c_k} \nabla_i z_k(p,t) dt \right]. \tag{A.9}
\]

Further differentiating (A.9) w.r.t. \( p_i \) yields

\[
\nabla^2_{ii} \Pi(p) = 2 \int_0^{p_i-c_i} \nabla_i z_i(p,t) dt + \frac{c_i}{p_i} \nabla_i y^*_i(p) + \frac{c_i}{p_i} \nabla_i z_i(p,t) + \sum_{k=1}^n \left[ p_k \int_0^{p_k-c_k} \nabla^2_{ii} z_k(p,t) dt \right]. \tag{A.10}
\]
Substituting $\nabla_i z_i(p,t) = \frac{ze_i(p,t)\nabla_i a_i(p)}{a_i(p)}$ and $\nabla_i^2 z_k(p,t) = 0$ from Lemma A.0.1 into (A.10), we get.

$$\nabla_i^2 \Pi(p) = 2\frac{\nabla_i a_i(p)}{a_i(p)} \int_0^{p_i - c_i} \nabla_i z_i(p,t)dt + \frac{c_i}{p_i} \nabla_j y_i^*(p) + \frac{c_i}{p_i} \frac{\nabla_i a_i(p)}{a_i(p)} z_i(p,t). \quad (A.11)$$

By Lemma A.0.1(b), the first and last term of (A.11) are negative, and by Lemma A.0.2, we have $\nabla_i y_i^*(p) < 0$ whenever $\nabla_i \Pi(p) = 0$. Therefore, when $\nabla_i \Pi(p) = 0$, $i = 1,2,\ldots,n$, we have $\nabla_i^2 \Pi(p) < 0$, $i = 1,2,\ldots,n$. Now we differentiate (A.9) w.r.t. $p_j$, $j \neq i$, and get

$$\nabla_j^2 \Pi(p) = \int_0^{p_i - c_i} \nabla_j z_i(p,t)dt + \frac{c_i}{p_i} \nabla_j y_i^*(p) + \int_0^{p_j - c_j} \nabla_i z_j(p,t)dt + \frac{c_j}{p_j} \nabla_i y_j^*(p) + \sum_{k=1}^n \left[ p_k \int_0^{p_k - c_k} \nabla_j^2 z_k(p,t)dt \right]. \quad (A.12)$$

Recalling that $z_i(p,t)$ is the demand corresponding to the $t$ fractile of $F_i$, we note that when $t = 1 - \frac{c_i}{p_i}$, we have demand of $i$ is equal to its optimal inventory, i.e., $y_i^*(p) = z_i(p,\frac{p_i - c_i}{p_i})$. Differentiating $y_i^*(p) = z_i(p,\frac{p_i - c_i}{p_i})$ w.r.t. $p_j$, $j \neq i$, we get $\nabla_j y_i^*(p) = \nabla_j z_i(p,\frac{p_i - c_i}{p_i}) = y_i^*(p)\frac{\nabla_j a_i(p)}{a_i(p)}$. Using this relation and substituting from Lemma A.0.1, the relations $\nabla_i z_j(p,t) = \frac{z_j[p,r]\nabla_i a_j(p)}{a_j(p)}$, $\nabla_j z_i(p,t) = \frac{z_i[p,r]\nabla_j a_i(p)}{a_i(p)}$, and $\nabla_j^2 z_k(p,t) = 0$, we rewrite (A.12) as:

$$\nabla_j^2 \Pi(p) = \frac{\nabla_j a_i(p)}{a_i(p)} \int_0^{p_i - c_i} z_i(p,t)dt + \frac{c_i}{p_i} \nabla_j y_i^*(p) \frac{\nabla_j a_i(p)}{a_i(p)}$$

$$+ \frac{\nabla_j a_j(p)}{a_j(p)} \int_0^{p_j - c_j} z_j(p,t)dt + \frac{c_j}{p_j} \nabla_j y_j^*(p) \frac{\nabla_j a_j(p)}{a_j(p)}$$

$$= \frac{\nabla_j a_i(p)}{a_i(p)} \left[ \int_0^{p_i - c_i} z_i(p,t)dt + \frac{c_i}{p_i} \nabla_j y_i^*(p) \right] + \frac{\nabla_j a_j(p)}{a_j(p)} \left[ \int_0^{p_j - c_j} z_j(p,t)dt + \frac{c_j}{p_j} \nabla_j y_j^*(p) \right]. \quad (A.13)$$
Now we do some algebraic manipulation on (A.13). Adding and subtracting the terms
\[
\frac{\nabla_j \alpha_i(p)}{\alpha_i(p)} \sum_{k=1}^{n} \left[ p_k \frac{\nabla_j \alpha_k(p)}{\alpha_k(p)} \int_0^t z_k(p, t) \, dt \right]
\]
and \[
\frac{\nabla_i \alpha_j(p)}{\alpha_j(p)} \sum_{k=1}^{n} \left[ p_k \frac{\nabla_i \alpha_k(p)}{\alpha_k(p)} \int_0^t z_k(p, t) \, dt \right]
\]
to (A.13) we can write
\[
\nabla^2_{ji} \Pi(p) = \frac{\nabla_j \alpha_i(p)}{\alpha_i(p)} \left[ \int_0^t \frac{p_k - c_i}{p_i} z_i(p, t) \, dt + \frac{c_i}{p_i} y_i^* (p) \right] + \sum_{k=1}^{n} \left[ \frac{\nabla_i \alpha_k(p)}{\alpha_k(p)} \int_0^t \frac{p_k - c_k}{p_k} z_k(p, t) \, dt \right]
\]
\[
+ \frac{\nabla_i \alpha_j(p)}{\alpha_j(p)} \left[ \int_0^t \frac{p_j - c_i}{p_j} z_j(p, t) \, dt + \frac{c_i}{p_j} y_j^* (p) \right] + \sum_{k=1}^{n} \left[ \frac{\nabla_j \alpha_k(p)}{\alpha_k(p)} \int_0^t \frac{p_k - c_k}{p_k} z_k(p, t) \, dt \right]
\]
\[
- \sum_{k=1}^{n} \left[ p_k \left( \frac{\nabla_i \alpha_k(p)}{\alpha_k(p)} \times \frac{\nabla_j \alpha_i(p)}{\alpha_i(p)} + \frac{\nabla_i \alpha_j(p)}{\alpha_j(p)} \times \frac{\nabla_j \alpha_k(p)}{\alpha_k(p)} \right) \frac{p_k - c_k}{p_k} z_k(p, t) \, dt \right]
\]
\[
= \frac{\nabla_i \alpha_j(p)}{\alpha_j(p)} \nabla_j \Pi(p) + \frac{\nabla_j \alpha_i(p)}{\alpha_i(p)} \nabla_i \Pi(p)
\]
\[
- \sum_{k=1}^{n} \left[ p_k \left( \frac{\nabla_i \alpha_k(p)}{\alpha_k(p)} \times \frac{\nabla_j \alpha_i(p)}{\alpha_i(p)} + \frac{\nabla_i \alpha_j(p)}{\alpha_j(p)} \times \frac{\nabla_j \alpha_k(p)}{\alpha_k(p)} \right) \frac{p_k - c_k}{p_k} z_k(p, t) \, dt \right],
\]
(A.14)

where, the last step in (A.14) follows from substituting the expressions of \( \nabla_i \Pi(p) \) and \( \nabla_j \Pi(p) \) from (A.9). Note that when \( \nabla_i \Pi(p) = 0, \ i = 1, \ldots, n \), then (A.14) becomes
\[
\nabla^2_{ji} \Pi(p) = - \sum_{k=1}^{n} \left[ p_k \left( \frac{\nabla_i \alpha_k(p)}{\alpha_k(p)} \times \frac{\nabla_j \alpha_i(p)}{\alpha_i(p)} + \frac{\nabla_i \alpha_j(p)}{\alpha_j(p)} \times \frac{\nabla_j \alpha_k(p)}{\alpha_k(p)} \right) \frac{p_k - c_k}{p_k} z_k(p, t) \, dt \right].
\]
(A.15)

From Lemma A.0.1(a) we know that \( \nabla_j \alpha_i(p) = \nabla_i \alpha_j(p) = 0, \ j \neq i - 1, i + 1 \), so in (A.15) we have \( \nabla^2_{ji} \Pi(p) = 0 \), for \( j \neq i - 1, i + 1 \).

We now show that the two sub diagonal terms are negative, i.e., \( \nabla^2_{ji} \Pi(p) < 0, \ j = i - 1, i + 1 \). For brevity we only show \( \nabla^2_{i+1,i} \Pi(p) < 0 \). From Lemma A.0.1(a) we know that for any \( i, \nabla_i \alpha_k(p) \neq 0 \) when \( k = i - 1, i, i + 1 \); and similarly, for any \( j, \nabla_j \alpha_k(p) \neq 0 \) when \( k = j - 1, j, j + 1 \). So, when \( j = i + 1 \), then index \( k \) in (A.15) takes values
We can write $\nabla^2_{i+1,i} \Pi(p)$ in (A.15) as

$$
\nabla^2_{i+1,i} \Pi(p) = - \sum_{k=i-1}^{i+2} \left( \frac{\nabla_i \alpha_k(p)}{\alpha_k(p)} \times \frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)} + \frac{\nabla_i \alpha_{i+1}(p)}{\alpha_{i+1}(p)} \times \frac{\nabla_i \alpha_k(p)}{\alpha_k(p)} \right) \int_0^{p_k-c_k} z_k(p, t) dt.
$$

(A.16)

From Lemma A.0.1(g) we see that $\nabla^2_{i+1,i} \Pi(p) = u^0$.

**Part (b)**: A function $g(t)$ is said to have a strict local maxima at $t = t_0$ if $g'(t_0) = 0$ and $g''(t_0) < 0$ ([Bazaraa et al. 2006]). The necessary condition for $p = (p_1, \ldots, p_n)$ to be a maximizer of $\Pi(p)$ is $\nabla_i \Pi(p) = 0$, $i = 1, 2, \ldots, n$. We show that there exists a unique $p$ such that $\nabla_i \Pi(p) = 0$, $i = 1, 2, \ldots, n$, and $p$ is a strict local maxima of $\Pi(p)$. The proof is by induction. When $n = 1$, the desired result follows from part (a), because we have $\frac{d^2 \Pi(p)}{dp^2} < 0$ at $p$ satisfying $\frac{d \Pi(p)}{dp} = 0$. Now suppose the statement holds for $n - 1$ products, i.e., suppose for a given $p_1$, there exists, $p^*(p_1) = (p^*_2(p_1), p^*_3(p_1), \ldots, p^*_n(p_1))$ such that

$$
\nabla_i \Pi(p_1, p^*_2(p_1), p^*_3(p_1), \ldots, p^*_n(p_1)) = 0, \text{ for } i = 2, 3, \ldots, n,
$$

(A.17)

and $p^*_2(p_1), p^*_3(p_1), \ldots, p^*_n(p_1)$ maximizes $\Pi(p^*(p_1))$. Now let all the $n$ prices be decision variables. Note that the maximization of $\Pi(p)$ over $(p_1, \ldots, p_n)$ is equivalent to maximizing $\Pi(p_1, p^*(p_1))$ over $p_1$, i.e.,

$$
\max_{p_1, p_2, \ldots, p_n} \Pi(p_1, p_2, \ldots, p_n) \Leftrightarrow \max_{p_1} \Pi(p_1, p^*(p_1)).
$$

(A.18)

Let the optimal value function be $\Pi^*(p_1) = \max_{p_1} \Pi(p_1, p^*(p_1))$. By Envelope theorem we can write

$$
\frac{d \Pi^*(p_1)}{dp_1} = \nabla_1 \Pi(p_1, p^*(p_1)).
$$

(A.19)
Further differentiating (A.19) w.r.t. $p_1$ we get

$$\frac{d^2 \Pi^*(p_1)}{dp_1^2} = \nabla_1^2 \Pi(p_1, p^*(p_1)) + \sum_{j=2}^{n} \frac{dp_j^*(p_1)}{dp_1} \nabla_{j1} \Pi(p_1, p^*(p_1)).$$  \hspace{1cm} (A.20)

We show that when $\frac{d \Pi^*(p_1)}{dp_1} = 0$, we have $\frac{d^2 \Pi^*(p_1)}{dp_1^2} < 0$. When $\frac{d \Pi^*(p_1)}{dp_1} = 0$, by (A.19) we have $\nabla_1 \Pi(p_1, p^*(p_1)) = 0$. This together with (A.17) implies

$$\nabla_i \Pi(p) = 0, \text{ for } i = 1, 2, \ldots, n.$$  \hspace{1cm} (A.21)

By part (a) of this proof we know that when (A.21) holds, we have

$$\nabla_1^2 \Pi(p_1, p^*(p_1)) < 0, \text{ } \nabla_2^2 \Pi(p_1, p^*(p_1)) < 0, \text{ } \nabla_{j1}^2 \Pi(p_1, p^*(p_1)) = 0, \text{ } j = 3, 4, \ldots, n.$$  \hspace{1cm} (A.22)

Using (A.22) we can rewrite (A.20) as

$$\frac{d^2 \Pi^*(p_1)}{dp_1^2} = \nabla_1^2 \Pi(p_1, p^*(p_1)) + \frac{dp_2^*(p_1)}{dp_1} \nabla_{21} \Pi(p_1, p^*(p_1)).$$  \hspace{1cm} (A.23)

Now we will show that $\frac{dp_2^*(p_1)}{dp_1} > 0$, and hence from (A.23) we will have $\frac{d^2 \Pi^*(p_1)}{dp_1^2} < 0$. Using (A.9) we write the expression for $\nabla_1 \Pi(p) = 0$ as follows

$$\nabla_1 \Pi(p) = 0 \Rightarrow 2 \int_0^{p_1-c_1} \nabla_1 z_1(p, t) dt + \frac{c_1}{p_1} \nabla_1 y_1^*(p) + p_1 \int_0^{p_1-c_1} \nabla_1 z_1(p, t) dt = -p_2 \int_0^{p_2-c_1} \nabla_1 z_2(p, t) dt.$$  \hspace{1cm} (A.24)

Differentiating (A.24) w.r.t. $p_1$ we get

$$2 \int_0^{p_1-c_1} \nabla_1 z_1(p, t) dt + \frac{c_1}{p_1} \nabla_1 y_1^*(p) + \frac{c_1}{p_1} \nabla_1 z_1(p, t) = -\frac{dp_2(p_1)}{dp_1} \int_0^{p_2-c_1} \nabla_1 z_2(p, t) dt.$$  \hspace{1cm} (A.25)

By Lemma A.0.1(d) we have $\nabla_1 z_1(p, t) < 0$ and $\nabla_1 z_2(p, t) > 0$. From Lemma A.0.2, we know $\nabla_1 y_1^*(p) < 0$ when $\nabla_1 \Pi(p) = 0$. Using these results, we note that the left
Proof of Proposition 2.4.3

Proof. The idea of proof is straightforward: we show that the profit from the products which lie on the lowest increasing cost curve (LICC) is higher than any other feasible assortment (a feasible assortment is one, in which the cost of the products increase convexly in the quality levels as given in (2.18)).

Let $S = \{j_1, j_2, \ldots, j_m\} \subseteq \Omega$ and let $c_S = (c_{j_1}, \ldots, c_{j_m})$ be the cost vector, and $q_S = (q_{j_1}, \ldots, q_{j_m})$ be the quality vector of $S$. Let $\{(c_{j_1}, q_{j_1}), (c_{j_2}, q_{j_2}), \ldots, (c_{j_m}, q_{j_m})\}$ constitute the lowest increasing convex curve (LICC) on the quality-cost plot of the products in $\Omega$. Let $S' = \{k_1, \ldots, k_l\}$ be another feasible assortment, with cost vector $c_{S'} = (c_{k_1}, \ldots, c_{k_l})$ and quality vector $q_{S'} = (q_{k_1}, \ldots, q_{k_l})$. See Figure A.1(a) for an illustration. We will show that for any feasible assortment $S'$, such that $S' \neq S$, we have $\Pi^*(c_{S'}, q_{S'}) < \Pi^*(c_S, q_S)$, where $\Pi^*(c_{S'}, q_{S'})$ and $\Pi^*(c_S, q_S)$ denotes the optimal profits from the assortments $S'$ and $S$ respectively.

Since $S'$ is feasible, we have $q_{k_1} < q_{k_2} < \ldots < q_{k_1}, c_{k_1} < c_{k_2} < \ldots < c_{k_l}$ and $\frac{c_{k_1}}{q_{k_1}} < \ldots < \frac{c_{k_l}}{q_{k_l}}$. Moreover, $S' \neq S$ and the products of $S$ constitute the LICC, so, $\exists k_i \in S'$ (at least one), such that $c_{k_i} > c_{j_i}$ and $\frac{c_{k_i}}{q_{k_i}} > \frac{c_{j_i}}{q_{j_i}}$, for some $j_i \in S$. Note that, $|S'| = l$ and $|S| = m$. Without loss of generality we assume $l \neq m$ and $q_{k_i} \neq q_{j_i}, \forall k_i \in S', \forall j_i \in S$.

We add $l$ fictitious products, $\{u_1, u_2, \ldots, u_l\}$ to $S$, and obtain $\bar{S} = S \cup \{u_1, u_2, \ldots, u_l\}$. Now, $|\bar{S}| = l + m$ (see Figure A.1(b) ). The quality levels of those $l$ fictitious product are $q_{u_1} = q_{k_1}, \ldots, q_{u_l} = q_{k_l}$. Let $q_{\bar{S}} = (q_1, q_2, \ldots, q_{l+m})$ be the quality vector of $\bar{S}$, such that for each $i \in \bar{S}$, $q_i \in (q_{j_1}, \ldots, q_{j_m}) \cup (q_{u_1}, \ldots, q_{u_l})$ and $q_i - 1 < q_i < q_{i+1}$, i.e., $q_{\bar{S}}$ is ordered. Suppose, for some $i \in \bar{S}$, $q_i = q_{u_i}$ , then $\exists j_i, j_{i+1} \in S$, such that $q_{j_1} < \ldots < q_{j_i} < q_{u_i} < q_{j_{i+1}} < \ldots < q_{j_m}$. Let $c_{u_i}$ be the cost of a fictitious product.
$u_i \in \{u_1, u_2, \ldots, u_l \}$, and let $c_{ui}$ be given by the relation: $\frac{c_{ji+1} - c_{qi}}{q_{ji+1} - q_{qi}} - \frac{c_{ui} - c_{ji-1}}{q_{ui} - q_{ji-1}} = 0$. This implies that (i) the point $(q_{ui}, c_{ui})$, corresponding to the fictitious product $u_i$, lies on the line joining $(q_{ji}, c_{ji})$ and $(q_{ji+1}, c_{ji+1})$ on the cost-quality graph, and (ii) $\alpha_{ui}(c_{\bar{S}}) = 0$. Therefore, by adding the $l$ fictitious products to $S$, the optimal profit of $S$ does not change, i.e., we have $\Pi^*(c_S, q_S) = \Pi^*(c_{\bar{S}}, q_{\bar{S}})$. Also, the products of $\bar{S}$ still constitute the LICC.

Similarly, we add $m$ fictitious products, $\{v_1, v_2, \ldots, v_m \}$ to $S'$, and obtain $\bar{S} = S' \cup \{v_1, v_2, \ldots, v_m \}$. Now we have, $|\bar{S}| = l + m$. The quality levels of those $m$ fictitious product are $q_{v_1} = q_{j_1}, \ldots, q_{v_m} = q_{j_m}$. Let $q_{\bar{S}} = (q_1, q_2, \ldots, q_{i+m})$ be the quality vector of $\bar{S}$, such that for each $i \in \bar{S}$, $q_i \in (q_{k_1}, \ldots, q_{k_l}) \cup (q_{v_1}, \ldots, q_{v_m})$ and $q_{i-1} < q_i < q_{i+1}$. Thus we have $q_{\bar{S}} = q_{\bar{S}}$ (see Fig A.1(b)). Suppose, for some $i \in \bar{S}$, $q_i = q_{v_i}$ and $\exists k_i, k_{i+1} \in S'$, such that $q_{k_i} < \ldots < q_{k_i} < q_{v_i} < q_{k_{i+1}} < \ldots < q_{k_l}$. Let $c_{v_i}$ be the cost of a fictitious product $v_i$, and let $c_{v_i}$ be given by the relation: $\frac{c_{k_i+1} - c_{v_i}}{q_{k_i+1} - q_{v_i}} - \frac{c_{v_i} - c_{k_i-1}}{q_{v_i} - q_{k_i-1}} = 0$, implying that $\alpha_{v_i}(c_{\bar{S}}) = 0$. Therefore, by adding the $m$ fictitious products to $S'$, the optimal profit of $S'$ does not change, i.e., we have $\Pi^*(c_{S'}, q_{S'}) = \Pi^*(c_{\bar{S}'}, q_{\bar{S}'})$. By assumption, products in $S'$ does not constitute LICC, so products in $\bar{S}'$ also does not constitute LICC. Thus, $\exists i \in \bar{S}'$ (at least one), such that $c_i > c_j$, $\frac{c_i}{q_i} > \frac{c_j}{q_j}$, for some $j \in \bar{S}$. This implies that $c_{\bar{S}} \leq c_{\bar{S}'}$. By Lemma A.0.3 we know that the optimal
profit function of any feasible assortment in the riskless case is strictly \(c\)-decreasing. Therefore, \(\Pi^*(c_S, q_S) < \Pi^*(c_{S'}, q_{S'})\), or equivalently, \(\Pi^*(c_S, q_{S'}) < \Pi^*(c_S, q_S)\). 

**Proof of Proposition 2.4.6**

**Proof.** In order to show that the optimal assortment in the risky case \(S^*\), is a subset of the optimal assortment in the riskless case \(S^*_0\), we show that \(k \in S^*_0 \Rightarrow k \in S^*\).

The proof is by contradiction. We assume that \(k \notin S^*_0\), but \(k \in S^*\) and \(S^*\) is the optimal assortment in the risky case. We then show that there exists an assortment \(S = \{S^* \setminus k\} \cup \{j_x\}\), \(j_x \in S^*_0\), which has a higher profit than \(S^*\), leading to a contradiction to the assumption.

![Figure A.2: Optimal risky assortment and LICC of quality-cost graph](image)

Let \(S^*_0 = \{j_1, \ldots, j_m\}\). In \(S^*_0\) we have, \(q_{j_1} < \ldots < q_{j_m}\) and \(\frac{c_{j_1}}{q_{j_1}} < \ldots < \frac{c_{j_m}}{q_{j_m}}\). Let \(k \notin S^*_0\) and \(S^* = \{j_1, \ldots, j_i, k\}\), \(i < m\). In \(S^*\) we have \(q_{j_1} < \ldots < q_{j_i} < q_k\) and \(\frac{c_{j_1}}{q_{j_1}} < \ldots < \frac{c_{j_i}}{q_{j_i}} < \frac{c_k}{q_k}\). Let \(r_{S^*} = (\frac{c_{j_1}}{q_{j_1}}, \ldots, \frac{c_{j_i}}{q_{j_i}}, \frac{c_k}{q_k})\) be the vector of cost-to-quality ratios of the products in \(S^*\). We consider two cases of the possible position of \(\{k\}\) on the cost-quality graph as shown in Figure A.2(a) and Figure A.2(b).

**Case I:** The position of \(\{k\}\) on the cost-quality graph is such that \(q_k > q_{j_m}\), \(c_k > c_{j_m}\), \(\frac{c_k}{q_k} > \frac{c_{j_m}}{q_{j_m}}\). Figure A.2(a) illustrates this case. We choose \(j_x \in \{j_{i+1}, \ldots, j_m\}\) and...
construct a new assortment $S = \{j_1, \ldots, j_i, j_x\}$. Let $r_S = \left(\frac{c_{j_1}}{q_{j_1}}, \ldots, \frac{c_{j_i}}{q_{j_i}}, \frac{c_{j_x}}{q_{j_x}}\right)$ be the vector of cost-to-quality ratios of the products in $S$. Since $\frac{c_k}{q_k} > \frac{c_{jm}}{q_{jm}} > \frac{c_{jx}}{q_{jx}}$, we have, $r_{S^*} \geq r_S$.

**Case II:** The position of $\{k\}$ on the cost-quality graph is such that $q_{jy} < q_k < q_{jm}$, $c_{jy} < c_k < c_{jm}$ and $\frac{c_{jy}}{q_{jy}} < \frac{c_k}{q_k}$, where, $j_y \in \{j_{i+1}, \ldots, j_m\}$. Figure A.2(b) illustrates this case. We choose $j_x \in \{j_{i+1}, \ldots, j_y\}$ and construct a new assortment $S = \{j_1, \ldots, j_i, j_x\}$. Let $r_S = \left(\frac{c_{j_1}}{q_{j_1}}, \ldots, \frac{c_{j_i}}{q_{j_i}}, \frac{c_{jx}}{q_{jx}}\right)$ be the vector of cost-to-quality ratios of the products in $S$. Since $\frac{c_k}{q_k} > \frac{c_{jy}}{q_{jy}} > \frac{c_{jx}}{q_{jx}}$, we have, $r_{S^*} \geq r_S$.

In either cases above, by replacing $k \in S^*$ with $j_x \in S^*_0$, which has a lower cost-to-quality ratio than $k$, we get the new assortment $S = \{S^* \setminus k\} \cup \{j_x\}$ such that $r_{S^*} > r_S$. Let $\Pi^*(c_{S^*}, q_{S^*})$ and $\Pi^*(c_S, q_S)$, denote the optimal profits in the risky case from the assortments $S^*$ and $S$, respectively. By lemma A.0.4 we know that the optimal profit from an assortment decreases if the cost-to-quality ratio of any product belonging to the assortment increases. Therefore, $\Pi^*(c_{S^*}, q_{S^*}) < \Pi^*(c_S, q_S)$, and we obtain a contradiction that $S^*$ is the optimal assortment in the risky case. 

4. Proofs of Lemmas 2.4.2,2.4.4,2.4.7 and Corollary 2.4.5

**Proof of Lemma 2.4.2**

*Proof.* We have from (2.14), $\Pi_0(p) = \sum_{i=1}^{n} (p_i - c_i)\alpha_i(p)$. The Hessian is of $\Pi_0(p)$ is given by

$$
\frac{\partial^2 \Pi_0(p)}{\partial p_j \partial q_i} = \begin{cases} 
2 \frac{\partial \alpha_i(p)}{\partial p_i} = \frac{-2(q_{i+1} - q_i)}{(q_{i+1} - q_i)(q_i - q_{i-1})} < 0, & j = i, \\
\frac{\partial \alpha_i(p)}{\partial p_j} = \frac{1}{q_j - q_i} > 0, & j = i + 1, \\
\frac{\partial \alpha_i(p)}{\partial p_j} = \frac{1}{q_i - q_j} > 0, & j = i - 1, \\
0, & j \neq i - 1, i, i + 1.
\end{cases}
$$

(A.26)
We refer to [Akçay et al.(2010)], for the proof that the Hessian given in (A.26) is a strictly diagonally dominant matrix with negative diagonal elements and nonnegative sub diagonals and therefore, $\Pi_{0}(p)$ is strictly concave in $p$. The global optima is obtained from $\nabla \Pi_{0}(p) = 0$, which is the following system of equations.

\begin{align*}
\sum_{j=1}^{2} (p_j - c_j) \frac{\partial \alpha_j(p)}{\partial p_1} + \alpha_1(p) &= 0, \\
\sum_{j=i-1}^{i+1} (p_j - c_j) \frac{\partial \alpha_j(p)}{\partial p_i} + \alpha_i(p) &= 0, \quad j = 2, \ldots, n - 1, \\
\sum_{j=n-1}^{n} (p_j - c_j) \frac{\partial \alpha_j(p)}{\partial p_n} + \alpha_n(p) &= 0. \quad (A.27)
\end{align*}

Note that

\begin{align*}
\sum_{j=1}^{2} (p_j - c_j) \frac{\partial \alpha_j(p)}{\partial p_1} &= \left( \frac{p_2 - p_1}{q_2 - q_1} - \frac{p_1}{q_1} \right) - \left( \frac{c_2 - c_1}{q_2 - q_1} - \frac{c_1}{q_1} \right) = \alpha_1(p) - \alpha_1(c), \\
\sum_{j=i-1}^{i+1} (p_j - c_j) \frac{\partial \alpha_j(p)}{\partial p_i} &= \left( \frac{p_{i+1} - p_i}{q_{i+1} - q_i} - \frac{p_i - p_{i-1}}{q_i - q_{i-1}} \right) - \left( \frac{c_{i+1} - c_i}{q_{i+1} - q_i} - \frac{c_i - c_{i-1}}{q_i - q_{i-1}} \right) = \alpha_i(p) - \alpha_i(c), \\
&\quad i = 2, \ldots, n - 1, \\
\sum_{j=n-1}^{n} (p_j - c_j) \frac{\partial \alpha_j(p)}{\partial p_n} &= \left( 1 - \frac{p_n - p_{n-1}}{q_n - q_{n-1}} \right) - \left( 1 - \frac{c_n - c_{n-1}}{q_n - q_{n-1}} \right) = \alpha_n(p) - \alpha_n(c). \quad (A.28)
\end{align*}

where, $\alpha_i(c), \ i = 1, \ldots, n$ is obtained by substituting $p = c$ into the choice probabilities given in (2.1). Substituting (A.28) into (A.27) we get,

\begin{align*}
2\alpha_i(p) - \alpha_i(c) &= 0, \quad i = 1, \ldots, n. \quad (A.29)
\end{align*}

From (2.1) we know $\sum_{i=1}^{n} \alpha_i(p) = 1 - \frac{p_1}{q_1}$. Summing both sides of (A.29) we get $\sum_{i=1}^{n} \alpha_i(p) = \sum_{i=1}^{n} \frac{\alpha_i(c)}{2}$, which upon simplification gives $p_1 = \frac{q_1 + c_1}{2}$. Substituting $p_1 = \frac{q_1 + c_1}{2}$ in the relation $\alpha_2(p) = \frac{\alpha_2(c)}{2}$, we get $p_2 = \frac{q_2 + c_2}{2}$. In this fashion we get $p_{ni} = \frac{q_i + c_i}{2}, \ i = 1, 2, \ldots, n$. \qed
Proof of Lemma 2.4.4

Proof. When the algorithm stops, from the stopping criteria, it is easy to see that the set of selected products $S = \{j_1, j_2, \ldots, j_m\}$, with $q_{jm} > q_{jm-1} > \ldots > q_{j_1}$ satisfy

$$1 > \frac{c_{jm} - c_{jm-1}}{q_{jm} - q_{jm-1}} > \ldots > \frac{c_{j_{m-1}} - c_{jm}}{q_{j_{m-1}} - q_{jm}} > \frac{c_{jm}}{q_{jm}} > 0.$$ 

We show by induction that all the eliminated products lie above the increasing convex curve which joins the points $\{(c_{j_1}, q_{j_1}), (c_{j_2}, q_{j_2}), \ldots, (c_{jm}, q_{jm})\}$ on the equality-cost graph, therefore the algorithm generates the lowest increasing convex curve. Let the total number of rounds run by algorithm at the stopping time be $l + 1$. Consider two arbitrary adjacent products $j_i, j_{i+1} \in S$, with $q_{ji+1} > q_{ji}$. Let the products eliminated during round $l$ be $\{l_1, l_2, \ldots, l_l\}$, where $q_{l_i} < q_{l_1} < q_{l_2} < \ldots < q_{l_l} < q_{ji+1}$. Since $\{l_1, l_2, \ldots, l_l\}$ are eliminated, it must be that,

$$\frac{c_{l_1} - c_{l_i}}{q_{l_1} - q_{l_i}} \geq \frac{c_{l_2} - c_{l_1}}{q_{l_2} - q_{l_1}} \geq \ldots \geq \frac{c_{l_{i+1}} - c_{l_i}}{q_{l_{i+1}} - q_{l_i}}.$$ 

Using the relation $a \geq b > 0 \Rightarrow \frac{a}{b} \geq \frac{a - c}{b - c} \geq \frac{b - d}{a - d}$, $a, b, c, d > 0$, and applying it to the above inequalities we obtain the following

$$\frac{c_k - c_{j_i}}{q_k - q_{j_i}} \geq \frac{c_{l_{i+1}} - c_k}{q_{l_{i+1}} - q_k}, \quad k = l_1, l_2, ..., l_l.$$ 

This implies that products $l_1, l_2, ..., l_l$ either lie on the line segment joining the points $(c_{j_i}, q_{j_i})$ and $(c_{j_{i+1}}, q_{j_{i+1}})$ (i.e., when equality holds) on the quality-cost graph or lie above it. Now we show that the products eliminated during round $l - 1$ also lie above the line segment joining the points $(c_{j_i}, q_{j_i})$ and $(c_{j_{i+1}}, q_{j_{i+1}})$. Since products $l_1, l_2, ..., l_l$ are eliminated in round $l$, they were not eliminated in round $l - 1$. Consider an arbitrary product $k \in \{l_1, l_2, ..., l_l\}$ which is eliminated during round $l - 1$. Let $k_i$ and $k_{i+1}$ denote the two products in $\{l_1, l_2, ..., l_l\}$ that are adjacent to product $k$, i.e,
\[ q_k < q_k < q_{k+1}. \] Since \( k \) is eliminated, product \( k \) lies on or above the line segment with end points \((c_{k+1}, q_{k+1})\) and \((c_k, q_k)\), i.e.,

\[
\frac{c_{k+1} - c_k}{q_{k+1} - q_k} \leq \frac{c_k - c_{k_i}}{q_k - q_{k_i}}
\]

Since \( k_i \) and \( k_{i+1} \) are also eliminated, both the points \((c_{k_i+1}, q_{k_i+1})\) and \((c_{k_i}, q_{k_i})\) lie on or above the line segment with end points \((c_{j_i}, q_{j_i})\) and \((c_{j_i+1}, q_{j_i+1})\), which further implies that the line segment with end points \((c_{k_i+1}, q_{k_i+1})\) and \((c_{k_i}, q_{k_i})\) lies on or above the line segment with end points \((c_{j_i}, q_{j_i})\) and \((c_{j_i+1}, q_{j_i+1})\). Therefore, the product \( k \) also lies on or above the line segment with end points \((c_{j_i}, q_{j_i})\) and \((c_{j_i+1}, q_{j_i+1})\). Hence, we prove by induction that the algorithm gives the the lowest increasing convex curve and by Proposition 2.4.3, we obtain the optimal assortment.

\[ \square \]

**Proof of corollary 2.4.5**

**Proof.** In the worst case, the algorithm checks \( n \) inequalities and finds one product to be removed from \( S^* \) in the first iteration. In the second iteration it checks \( n - 1 \) inequalities and finds one product to be removed from \( S^* \) and so on. Therefore the total number of operations needed to find \( S^* \) is \( n + (n-1) + (n-2) + \ldots + n - (n-1) = n^2 \). Therefore the complexity of the algorithm is \( O(n^2) \).

\[ \square \]

**Proof of Lemma 2.4.7**

**Proof.** By Proposition 2.4.1 we know that \( \Pi(p) \) is strictly pseudoconcave, and there is a unique vector \((p_1, \ldots, p_n)\) that satisfy \( \nabla_i \Pi(p) = 0 \). We show that \( \nabla_i \Pi(p) = 0 \big|_{p_i = \frac{c_i + q_i}{2}} > 0 \), which implies \( p_i^* > \frac{c_i + q_i}{2} \). Substituting \( p_i = \frac{c_i + q_i}{2} \) in (A.9) we get,

\[
\nabla_i \Pi(p) \bigg|_{p_i = \frac{c_i + q_i}{2}} = \int_0^1 \frac{2c_i}{c_i + q_i} z_i(p, t) dt + \frac{2c_i y_i^*(p)}{c_i + q_i} + \sum_{k = \{i-1, i+1\}} p_k \frac{\nabla_k \alpha_k(p)}{\alpha_k(p)} \int_0^{p_k - c_k} \frac{p_k - c_k}{p_k} z_k(p, t) dt > 0.
\]

\[ \square \]
Appendix B

Proofs for Chapter 3

Lemma B.0.5. \( \Pi(p) = \sum_{i \in I} (p_i - c_i) \alpha_i^c(p) \) given in (3.11) is strictly pseudoconcave.

Proof. Let \( x_i = \frac{p_i - p_i - 1}{q_i - q_i - 1} \), \( i = 1, \ldots, n \), denote the cutoffs on the support of \( \theta_i \); and let \( x = (x_1, \ldots, x_n) \). Note that there is a one-to-one mapping between \( p \) and \( x \). Using this transformation, we can express the prices as:

\[
p_i = \sum_{k=1}^{i} x_k (q_k - q_{k-1}), \quad i = 1, \ldots, n,
\]  

(B.1)

and the choice probabilities in (4.2) as

\[
\alpha_i^c(x) = F_1(x_{i+1}) - F_1(x_i), \quad i = 1, \ldots, n - 1, \quad \alpha_n^c(x) = 1 - F_1(x_n).
\]  

(B.2)

Using Eq. (B.1) and (B.2) we can rewrite the problem \( \max_p \Pi(p) = \sum_{i \in I} (p_i - c_i) \alpha_i^c(p) \), as

\[
\max_x \Pi(x) = \sum_{i=1}^{n} [1 - F_1(x_i)] (x_i (q_i - q_{i-1}) - c_i + c_{i-1}),
\]  

(B.3)

s.t., \( x_1 \leq x_2 \leq \ldots \leq x_n \).

(B.4)

which finds the optimal cutoffs of category-\( I \). Without loss of generality, we assume that the prices are set such that \( x_1 \leq x_2 \leq \ldots \leq x_n \), because for any set of prices that does not satisfy this condition, there exists a set of prices that does, with the same purchase probability for each product and the same total expected profit. Differenti-
ating (B.3) w.r.t. \( x_i, \ i = 1, ..., n \), we have

\[
\frac{\partial \Pi(x)}{\partial x_i} = -f_1(x_i)[x_i(q_i - q_{i-1}) - (c_i - c_{i-1})] + [1 - F_1(x_i)](q_i - q_{i-1}). \tag{B.5}
\]

From (B.5), we get the first order conditions (FOC) as follows

\[
0 = f_1(x_i)(q_i - q_{i-1}) \left[ -x_i + \frac{c_i - c_{i-1}}{q_i - q_{i-1}} + \eta_1(x_i) \right], \ i = 1, ..., n. \tag{B.6}
\]

Differentiating (B.5) further w.r.t. \( x_k \), we have

\[
\left. \frac{\partial^2 \Pi(x)}{\partial x_k \partial x_i} \right|_{FOC} = \begin{cases} 
0, & k \neq \{i, i-1\}, \\
-f_1(x_i)(q_i - q_{i-1})[-1 + \eta'_1(x_i)] > 0, & k = i - 1, \\
f_1(x_i)(q_i - q_{i-1})[-1 + \eta'_1(x_i)] < 0, & k = i.
\end{cases} \tag{B.7}
\]

Since the Hessian matrix is negative definite at the FOC, \( \Pi(x) \) in B.3 is strictly pseudoconcave. Similarly, it can be shown that \( \Pi(p') = \sum_{j \in J} (p'_j - c'_j)a'_j(p') \) is strictly pseudoconcave.

\qquad \Box

**Proof of Proposition 3.4.1**

**Proof.** Proof. We show the proof for category-\( \mathcal{I} \) only, since optimal prices and assortments for the other category can be obtained in a similar fashion.

**Part (a)** By Lemma B.0.5, we know that the profit from category-\( \mathcal{I} \), given in B.3 is strictly pseudoconcave. Hence the solutions to the F.O.C of \( \Pi(x) \) (given in (B.6)) uniquely determine the optimal cutoffs, i.e.,

\[
x_i = \eta_1(x_i) + \frac{c_i - c_{i-1}}{q_i - q_{i-1}}, i = 1, ..., n.
\]
Substituting the optimal cutoffs in (B.1), the optimal prices can be obtained as follows

\[ p^*_1 = x_1 q_1, p^*_i = p^*_{i-1} + x_i (q_i - q_{i-1}), i = 2, \ldots, n. \]

**Part (b):** By definition \( S^*_I = \{i_1, \ldots, i_k : c_{iu} (p^*) > 0, u = 1, \ldots, k \} \). Therefore, the optimal cutoffs must satisfy \( x_{i_1} < \ldots < x_{i_k} \), which implies \( 1 > \frac{c_{ik} - c_{i_k-1}}{q_{ik} - q_{i_k-1}} > \ldots > \frac{c_{i_2} - c_{i_1}}{q_{i_2} - q_{i_1}} \). [Deb and Xu(2012)] show that \( \{i_1, \ldots, i_k\} \) form the lowest increasing cost curve of the quality-cost plot of the products of category-\( I \).

**Proof of Lemma 3.4.2**

**Proof.** The pairs selected by basket shoppers form the lowest increasing cost curve of the cost-quality graph of all the pairs. In other words, by (2.19) of Definition 1, the cost and quality levels of the pairs \( S^* = \{[i_1, j_1], \ldots, [i_k, j_k]\} \) follow the relation:

\[
0 \leq \frac{c_{i_1} + c'_{j_1}}{q_{i_1} + q'_{j_1}} < \ldots < \frac{(c_{i_k} + c'_{j_k}) - (c_{i_1} + c'_{j_1})}{(q_{i_k} + q'_{j_k}) - (q_{i_1} + q'_{j_1})} \leq 1. \tag{B.8}
\]

Also, the cost and quality levels must follow the first condition of (2.21) given in Definition 1, i.e.,

\[
\frac{c_u + c'_v}{q_u + q'_v} > \frac{c_{i_1} + c'_{j_1}}{q_{i_1} + q'_{j_1}}, \forall u \in I, v \in J, \text{ such that } q_u + q'_v < q_{i_1} + q'_j. \tag{B.9}
\]

We prove by contradiction, i.e., we suppose \( i_1 \) does not lie on the lowest increasing cost curve of category \( I \). The fact that \( i_1 \) does not lie on the lowest increasing cost curve, implies that it is dominated by some \( k \in I \), i.e., \( q_k > q_{i_1} \), \( c_k < c_{i_1} \) and so \( \frac{c_k}{q_k} < \frac{c_{i_1}}{q_{i_1}} \) (see Lemma 4 of [Honhon and Pan(2012)]). Suppose (w.l.o.g) \( \frac{c_k}{q_k} < \frac{c_{i_1}}{q_{i_1}} < \frac{c'_{j_1}}{q'_{j_1}} \). Using the relation \( \frac{c_k}{q_k} < \frac{c_{i_1}}{q_{i_1}} < \frac{c'_{j_1}}{q'_{j_1}} \), we obtain \( \frac{c_k + c'_{j_1}}{q_k + q'_{j_1}} < \frac{c_{i_1} + c'_{j_1}}{q_{i_1} + q'_{j_1}} \), which violates (B.9), and implies that
Proof. The profit from basket shoppers given in (3.5) can be written as,

\[
\Pi^b(p,p') = \alpha^b_{[i_1,j_1]}(p,p')(p_{i_1} + p'_{j_1} - c_{i_1} - c'_{j_1}) + \sum_{u>1} \sum_{v>1} \alpha^b_{[i_u,j_v]}(p,p')(p_{i_u} + p'_{j_v} - c_{i_u} - c'_{j_v}),
\]

(B.10)

since only the configurations \((i_u,j_v)\) \(\in S^*\) are chosen with non negative probability. We first show that the configurations \((i_u,j_v)\) \(\in S^*\), \(u > 1, v > 1\), are also chosen by cherry pickers. Recall from Proposition 3.4.1 that the two optimal assortments offered to cherry pickers are \(S^*_I = \{i_1, \ldots, i_k\}\) and \(S^*_J = \{j_1, \ldots, j_l\}\). Given the optimal cutoffs \(x_{i_u}, u = 1, \ldots, k, x'_{j_v}, v = 1, \ldots, l\), as in Proposition 3.4.1(a), all cherry pickers with sensitivity \(\theta_1 \geq x_{i_1}\) select products from category-\(I\). Similarly if their sensitivity is \(\theta_2 \geq x'_{j_1}\), they select products from category-\(J\). Therefore, cherry pickers with sensitivities \(\theta_1 \geq \max\{x_{i_1}, x'_{j_1}\}\) and \(\theta_2 \geq \max\{x_{i_1}, x'_{j_1}\}\), select products from both categories, i.e., they select the configurations \(S = \{[i_u,j_v], u > 1, v > 1\}\). Since cherry pickers derive positive utility from each component of a configuration in \(S\), basket shoppers also derive positive utility from these the configurations. Therefore, the optimal prices offered to cherry pickers for the products in \(S = \{[i_u,j_v], u > 1, v > 1\}\), are also optimal for basket shoppers, i.e., \(p_{i_u} + p'_{j_v} = p^*_{i_u} + p^*_j, (i_u,j_v) \in S^*, u > 1, v > 1\).

Now consider the configuration \([i_1,j_1]\). Let \(\pi^b = \alpha^b_{[i_1,j_1]}(p,p')(p_{i_1} + p'_{j_1} - c_{i_1} - c'_{j_1})\) be the revenue earned from basket shoppers who select \([i_1,j_1]\), and \(\pi^c = \alpha^c_{[i_1,j_1]}(p,p')(p_{i_1} + p'_{j_1} - c_{i_1} - c'_{j_1})\) the revenue from cherry pickers. We know \(\alpha^b_{[i_1,j_1]}(p,p') \geq \alpha^c_{[i_1,j_1]}(p,p')\), for any given \(p = (p_{i_1}, \ldots, p_{i_k})\) and \(p' = (p'_{j_1}, \ldots, p'_{j_l})\), since basket shoppers are more likely to choose a configuration than cherry pickers. This implies, \(\pi^b \geq \pi^c\). Moreover, \(\pi^b\) and \(\pi^c\) are both pseudoconcave in the combined price \(p_{i_1} + p'_{j_1}\). So, \(\pi^b \geq \pi^c\) implies

Proof of Lemma 3.4.4

Proof. The profit from basket shoppers given in (3.5) can be written as,
that the price $\bar{p}_{i1} + \bar{p}'_{j1}$ which maximizes $\pi^b$, does not exceed the price $p^*_i + p'^*_j$ that maximizes $\pi^c$, i.e., $\bar{p}_{i1} + \bar{p}'_{j1} \leq p^*_i + p'^*_j$.

**Lemma B.0.6.** $\Pi(p)$ given in (3.12) is pseudoconcave in each of the intervals of $p$ and is continuous across the intervals.

We write (3.12) in the explicit form as follows (see Figure B.1)

\[
\Pi(p) = \begin{cases} 
\Pi_1(p) = [F(x_{i+1}) - F(p/q'_{j1})](p^*_i + p - c_i - c'_j) & p \in (q'_{j1} x_i q'_{j1} x'_i], \\
\Pi_2(p) = [F(x_{i+1}) - F(x_i)](p^*_i + p - c_i - c'_j) \\
& + [F(x_i) - F(p/q'_{j1})](p^*_{i-1} + p - c_{i-1} - c'_j) & p \in (q'_{j1} x_{i-1} x'_j x_i], \\
\Pi_k(p) = [F(x_{i+1}) - F(x_i)](p^*_i + p - c_i - c'_j) + ... \\
& + [F(x_{i-2}) - F(p/q'_{j1})](p^*_{i+1} + p - c_{i+1} - c'_j) & p \in (q'_{j1} x_{i+1} x'_j x_{i-2}] 
\end{cases} 
\] (B.11)

We show that $\Pi(p)$ is continuous at the junction of any two intervals. Consider the two intervals $[q'_{j1} x_{i-1}, q'_{j1} x_i]$ in which $\Pi_2(p)$ is defined as in (3.12), and the interval $[q'_{j1} x_i, q'_{j1} x'_j]$ in which $\Pi_1(p)$ is defined as in (3.12). Note that limiting value of $\Pi_2(p)$ as $p$ approaches $x_i q'_{j1}$ from left is $\lim_{p \to x_i q'_{j1}} \Pi_2(p) = [F(x_{i+1}) - F(x_i)](p^*_i + x_i q'_{j1} - c_i - c'_j)$, which is the same as the limiting value of $\Pi_1(p)$ as $p$ approaches $x_i q'_{j1}$ from the right, i.e., $\lim_{p \to x_i q'_{j1}} \Pi_1(p) = [F(x_{i+1}) - F(x_i)](p^*_i + x_i q'_{j1} - c_i - c'_j)$. In this fashion, the continuity can be shown considering any pair of adjacent intervals.

![Figure B.1: Selection of Configurations](image-url)
Lemma B.0.7. Eq. (3.5), $\Pi(p, p') = \Pi^c(p, p') + \Pi^b(p, p')$, is a bi-concave function of $p$ and $p'$.

Proof. A function $f(x, y) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ is biconcave if $f(., y)$ is concave (or quasiconcave) in $x$ for each $y \in \mathbb{R}^m$, and $f(x,.)$ is concave (or quasiconcave) in $y$ for each $x \in \mathbb{R}^n$. Sum of bi-concave functions is also biconcave ([Gorski et al.(2007)]). We will show that each of $\Pi^c(p, p')$ and $\Pi^b(p, p')$ are bi-concave. First note that, $\Pi^c(p, p')$ given in (3.5) can be written as $\Pi^c(p, p') = \Pi(p) + \Pi(p')$ (see (3.11)), which is strictly pseudoconcave in $p$ given $p'$, and strictly pseudoconcave in $p'$ given $p$ (by Lemma B.0.5). Hence $\Pi^c(p, p')$ is bi-concave. Now we show (3.7), which is $\Pi^b(p, p') = \sum_{i \in I} \sum_{j \in J} \alpha_{i,j}(p, p')(p_i + p'_j - c_i - c'_j)$ is also bi-concave. Let $\bar{\mathbf{p}}(p, p') = (\bar{p}_{nm}, \ldots, \bar{p}_{11})$ denote the vector of the prices of the $n \times m$ configurations, where any element of $\mathbf{p}(p, p')$ is given by $\bar{p}_{ij} = p_i + p'_j$. A basket shopper’s choice probability for a configuration $[i, j]$ is the same as choosing a product with quality $q_i + q'_j$ from a single category that has $m \times n$ quality differentiated products with prices $\bar{\mathbf{p}}(p, p')$ ([Deb and Xu(2012]) and [Honhon and Pan(2012)]). Using this notation, we can write $\Pi^b(p, p') \equiv \Pi(\bar{\mathbf{p}}(p, p'))$, where $\Pi(\bar{\mathbf{p}}(p, p'))$ is the profit function of the single category that has $m \times n$ quality differentiated products and price vector $\bar{\mathbf{p}}(p, p')$. Again by Lemma B.0.5, we have $\Pi(\bar{\mathbf{p}}(p, p'))$ is strictly pseudoconcave in $\bar{\mathbf{p}}$. Since $\bar{\mathbf{p}}$ is linear in $p$ and $p'$, it follows that, $\Pi^b(\bar{\mathbf{p}}(p, p'))$ is pseudoconcave in $\mathbf{p}$, given $p'$, and vice-versa (the composition of a pseudoconcave function with a linear function is pseudoconcave, [Boyd and Vandenberghe (2004)]). Since sum of bi-concave functions is also biconcave ([Gorski et al.(2007)]), we have the result.

Alternate Convex Search Algorithm
In Lemma B.0.7 we showed that $\Pi(p, p')$ in (3.5) is strictly pseudoconcave in $p$ given $p'$, and strictly pseudoconcave in $p'$ given $p$. Note that for a given $p'$, $\mathbf{p}(p') = (p_1(p'), \ldots, p_n(p'))$ can be found by solving the problem: $\max_p \Pi(p, .)$, which is a well defined generalized convex program in $p$. Similarly, for any given $p$, $\mathbf{p}'(p)$ =
(p_1', \ldots, p_m') can be found by solving the problem: \( \max_{p'} \Pi(p, p') \), which is again a generalized convex program. Taking advantage of this generalized convex substructure of the problem, we outline an algorithmic procedure to find the optimal solution.

**Algorithm 3: Alternate Convex Search Algorithm**

**Step 1:** Choose an arbitrary \( p'_k = (p'_{1k}, \ldots, p'_{mk}) \), and an \( p_k = (p_{1k}, \ldots, p_{nk}) \). Set \( k = 0 \).

**Step 2:** Solve for fixed \( p'_k \), the following problem

\[
\max_{p} \Pi(p, p'_k) \tag{B.12}
\]

If there exists an optimal solution \( p^* = (p^*_1, \ldots, p^*_n) \) to the problem in (B.12), set \( p_{k+1} = p^*_k \), otherwise STOP.

**Step 3:** Solve for fixed \( p_{k+1} \), the following problem

\[
\max_{p'} \Pi(p_{k+1}, p') \tag{B.13}
\]

If there exists an optimal solution \( p''^* = (p''^*_1, \ldots, p''^*_m) \) to the problem in (B.13), set \( p'_{k+1} = p''^*_k \), otherwise STOP.

if \( |\Pi(p_{k+1}, p'_k) - \Pi(p_k, p'_k)| < \epsilon \) then

STOP

else

Augment \( i \) by 1 and go back to Step 2.

end if

**Proof of Convergence of Alternate Convex Search Algorithm**

*Proof:* We review some technical results on biconvex optimization from [Gorski et al. (2007)] to do the proof. An optimization problem of the form \( \min \{f(x, y) : (x, y) \in B\} \) is said to be a biconvex optimization problem or biconvex for short, if the feasible
set $B$ is biconvex on $X \times Y$, and the objective function $f$ is biconvex on $B$. The algorithmic procedures to solve biconvex programs rely on the basic idea that the variable set can be divided into disjoint blocks. In every step, only the variables of an active block are optimized, while those of the other blocks are fixed. For example, for fixed $y$, the problem $\min \{ f(x, y), x \in B_y \}$, and for fixed $x$, the problem $\min \{ f(x, y), y \in B_x \}$, are two subproblems. If the subproblems are convex, then efficient convex minimization methods can be used to solve those subproblems. The paper describes the Alternate Convex Search (ACS) method in Algorithm 4.1, which can be used to solve biconvex programs when the subproblems have convex structures. The ACS algorithm generates a sequence of candidate solutions $\{ z_i = (x_i, y_i) \}_{i \in \mathbb{N}}$ and a sequence of function values $\{ f(z_i) \}_{i \in \mathbb{N}}$. In such algorithms, it is possible that when the function values $\{ f(z_i) \}_{i \in \mathbb{N}}$ converge, the candidate solutions $\{ z_i \}_{i \in \mathbb{N}}$ may not converge. Theorem 4.5 shows that if $f : B \to \mathbb{R}$ is bounded below, then the sequence of function values $\{ f(z_i) \}_{i \in \mathbb{N}}$ generated by ACS converges monotonically, and Theorem 4.9 gives the conditions under which $\{ z_i \}_{i \in \mathbb{N}}$ converges. The main condition in Theorem 4.9 that guarantees the convergence of $\{ z_i \}_{i \in \mathbb{N}}$ is that the subproblems must be uniquely solvable. If $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ are closed, and $f : X \times Y \to \mathbb{R}$ is continuous, and $f$ is strictly (pseudo)convex in $y$ for fixed $x$ and vice versa, then the condition of Theorem 4.9 is automatically guaranteed.

The sets $\tilde{P}$ and $\tilde{P}'$ are both compact sets. Moreover, $\Pi(p, p')$ is continuous and bounded below because $\Pi(., .)$ maps the compact space $p \times p'$ to $\mathbb{R}$. Also we have $\Pi(., p')$ is strictly-pseudoconcave in $p'$, and vice versa. Therefore, all the conditions for convergence of $\{ \Pi(p_k, p_k') \}_{k \in \mathbb{N}}$ and $\{ p_k, p_k' \}_{k \in \mathbb{N}}$ are satisfied, and Algorithm 1 converges to the optimal solution. \hfill \Box

**Example 6.** Suppose category-1 has $n = 3$ variants and category-2 has $m = 2$ variants. The quality and cost vectors are $q_1 = (262, 348, 466), c_1 = (73.6524, 129.9339, 233)$, and $q_2 = (160, 814), c_2 = (15.7248, 407)$, respectively. We assume convex cost in each
category, and consumer sensitivity to be uniformly distributed: \( \theta \sim U[1, 5] \). The optimal solution to (3.5) is obtained by Algorithm 1 in two steps and shown in table below.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( p_1^* )</th>
<th>( p_2^* )</th>
<th>( p_3^* )</th>
<th>( p_1'^* )</th>
<th>( p_2'^* )</th>
<th>( \Pi(p, p') )</th>
<th>FOC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>92.02</td>
<td>190.66</td>
<td>360.42</td>
<td>56.19</td>
<td>799.70</td>
<td>0</td>
<td>0.1514</td>
</tr>
<tr>
<td>2</td>
<td>237.00</td>
<td>333.54</td>
<td>466.00</td>
<td>148.01</td>
<td>814.00</td>
<td>636.0025</td>
<td>0.0139</td>
</tr>
<tr>
<td>3</td>
<td>259.87</td>
<td>347.46</td>
<td>466.00</td>
<td>154.86</td>
<td>807.13</td>
<td>633.0809</td>
<td>0.0002</td>
</tr>
<tr>
<td>4</td>
<td>262.00</td>
<td>347.98</td>
<td>465.99</td>
<td>159.94</td>
<td>813.89</td>
<td>639.8887</td>
<td>0.0001</td>
</tr>
<tr>
<td>5</td>
<td>262.00</td>
<td>348.00</td>
<td>466.00</td>
<td>159.99</td>
<td>813.99</td>
<td>639.9998</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

Table B.1: Output of Algorithm 1

The optimal prices \( p^* = (262, 348, 466) \) and \( p'^* = (160, 814) \) give the optimal assortments: \( S_1^* = \{ i \in \{1, 2, 3\} : \alpha_{i|j}(p^*, p'^*) > 0 \} = \{3\} \) and \( S_2^* = \{ j \in \{1', 2'\} : \alpha_{i|j}(p^*, p'^*) > 0 \} = \{2'\} \).
Appendix C

Proofs for Chapter 4

1. Proof of Lemma 4.4.1

We first show that \( \Pi(Q(x)) = \sum_{k=1}^{n} p(x_k) \min\{Q(x)\beta_k(x), S\alpha_k(x)\} - cQ(x) - \bar{c}n \) is piece-wise linear in \( Q(x) \) with breakpoints \( \{Q_1, \ldots, Q_n\} \). The concavity follows from linearity. Then we show that the optimal solution occurs at a breakpoint beyond which \( \Pi(Q(x)) \) decreases monotonically. Note that when \( Q(x) \) is very small such that \( \min\{Q(x)\beta_k(x), S\alpha_k(x)\} = Q(x)\beta_k(x), \forall k \in \Omega \), we can write

\[
\Pi(Q(x)) = Q(x) \left( \sum_{k=1}^{n} p(x_k)\beta_k(x) - c \right) - \bar{c}n.
\]

For such small values of \( Q(x) \), the slope of \( \Pi(Q(x)) \), given by \( \sum_{k=1}^{n} p(x_k)\beta_k(x) - c \) denotes the marginal profit from an additional unit (see Fig C.1). Clearly, if \( \sum_{k=1}^{n} p(x_k)\beta_k(x) - c > 0 \), then the firm starts to produce. As \( Q(x) \) increases, at \( Q(x) = Q_1 \), the supply of \( i_1 \in \Omega \) matches its demand, i.e., \( Q_1\beta_{i_1}(x) = S\alpha_{i_1}(x) \), while the supply of the products in \( \Omega\{i_1\} \) are still strictly less than their respective demands, \( Q_1\beta_k(x) < S\alpha_k(x), k \in \Omega\{i_1\} \). As \( Q(x) \) increases further, say at \( Q_2 > Q_1 \), the supply of \( i_2 \in \Omega\{i_1\} \), matches its demand, i.e., \( Q_2\beta_{i_2}(x) = S\alpha_{i_2}(x) \), while the supply of the products in \( \Omega\{i_1, i_2\} \) are still strictly less than their respective demands. Since \( Q_2 > Q_1 \), we have \( Q_2\beta_{i_2}(x) > Q_1\beta_{i_1}(x) = S\alpha_{i_1}(x) \). In other words, the supply of \( i_1 \) at \( Q_2 \) exceeds its demand while the supply of \( i_2 \) matches its demand. Thus, \( Q_2 = S \cdot \frac{\alpha_{i_2}(x)}{\beta_{i_2}(x)} = S \max_{k \in \{i_2, i_1\}} \left\{ \frac{\alpha_k(x)}{\beta_k(x)} \right\} \). In this fashion, for a given \( x \), the breakpoints can be defined as \( Q_k = S \max_{k \in \{i_1, \ldots, i_k\}} \left\{ \frac{\alpha_k(x)}{\beta_k(x)} \right\}, k = 2, \ldots, n \). At any breakpoint \( Q_k \), the demands of \( \{i_1, \ldots, i_k\} \) are fully met while the products in \( \Omega\{i_1, \ldots, i_k\} \) have unmet demand. Now, we can write \( \Pi(Q(x)) \) in the following piece-wise linear form.
\[
\Pi(Q(x)) = \begin{cases} 
Q(x) \left[ \sum_{k \in \Omega} p(x_k) \beta_k(x) - c \right] - \bar{c}n, & 0 \leq Q(x) < Q_1, \\
Sp(x_{i_1}) \alpha_{i_1}(x) + Q(x) \left[ \sum_{k \in \Omega \setminus \{i_2\}} p(x_k) \beta_k(x) - c \right] - \bar{c}n, & Q_1 \leq Q(x) < Q_2, \\
\sum_{k=1}^2 Sp(x_{i_k}) \alpha_{i_k}(x) + Q(x) \left[ \sum_{k \in \Omega \setminus \{i_1, i_2\}} p(x_k) \beta_k(x) - c \right] - \bar{c}n, & Q_2 \leq Q(x) < Q_3, \\
\ldots & \\
\ldots & \\
\sum_{k=1}^{n-1} Sp(x_{i_k}) \alpha_{i_k}(x) + Q(x) \left[ \sum_{k \in \Omega \setminus \{i_1, \ldots, i_{n-1}\}} p(x_k) \beta_k(x) - c \right] - \bar{c}n, & Q_{n-1} \leq Q(x) < Q_n, \\
\sum_{k \in \Omega} Sp(x_k) \alpha_k(x) - cQ(x) - \bar{c}n, & Q(x) \geq Q_n. 
\end{cases}
\]

Figure C.1 shows the breakpoints \( \{Q_1, \ldots, Q_n\} \) and the slopes of \( \Pi(Q(x)) \) in each segment between breakpoints. The slope of \( \Pi(Q(x)) \) between \([Q_i, Q_{i+1}]\), is given by \( \sum_{k \in \Omega \setminus \{i_1, \ldots, i_i\}} p_k \beta_k(x) - c \), and denotes the marginal profit from an additional unit produced in excess of \( Q_i \). Suppose the slope of \( \Pi(Q(x)) \) between \([Q_i, Q_{i+1}]\) is negative. Note that \( \sum_{k \in \Omega \setminus \{i_1, \ldots, i_{i-1}\}} p_k \beta_k(x) - c < 0 \) implies, \( \sum_{k \in \Omega \setminus \{i_1, \ldots, i_{i+1}\}} p_k \beta_k(x) - c < 0 \). Thus, if the slope of \( \Pi(Q(x)) \) between \([Q_i, Q_{i+1}]\) is negative, the slope between \([Q_{i+1}, Q_{i+2}]\) is
also negative. Similarly, the slopes between \([Q_{i+2}, Q_{i+3}], \ldots, [Q_{n-1}, Q_n]\) are all negative. This means \(\Pi(Q(x))\) monotonically decreases after the breakpoint \(Q_i\). The optimal quantity is then \(Q^*(x) = Q_i = S \times \max_{k \in \{i_1, i_2, \ldots, i_{k}\}} \{\frac{\alpha_k(x)}{\beta_k(x)}\}\). Clearly, in order to find the breakpoint which is optimal for a given \(x\), we need to identify the set of products \(\{i_1, \ldots, i_{k}\}\). Given \(x\), we define \(I_x = \Omega \setminus \{i_1, \ldots, i_{k}\}\), as the largest set such that the total revenue from the products in \(I_x\), given as \(\sum_{i \in I_x} p \pi_i(x)\), is less than the marginal cost of production \(c\). Therefore, we can write \(Q^*(x) = S \times \max_{j \in \Omega \setminus I_x} \{\frac{\alpha_j(x)}{\beta_j(x)}\}\).

2. Proof of Proposition 4.4.2

We first show two results that will be useful.

1. \(\frac{\partial \alpha_i(x)}{\partial x_i} \leq 0, \frac{\partial \alpha_i(x)}{\partial x_i+1} \geq 0\)

2. \(\frac{\partial \alpha_i-1(x)}{\partial x_i} \geq 0, \frac{\partial \alpha_i(x)}{\partial x_i-1} \leq 0\).

From (4.2) we have

\[
\alpha_i(x) = G \left( \frac{p(x_i+1) - p(x_i)}{x_i+1 - x_i} \right) - G \left( \frac{p(x_i) - p(x_i-1)}{x_i - x_i-1} \right),
\]

\[
\alpha_{i-1}(x) = G \left( \frac{p(x_i) - p(x_i-1)}{x_i - x_i-1} \right) - G \left( \frac{p(x_i-1) - p(x_i-2)}{x_i-1 - x_i-2} \right),
\]

\[
\alpha_{i+1}(x) = G \left( \frac{p(x_{i+2}) - p(x_{i+1})}{x_{i+2} - x_{i+1}} \right) - G \left( \frac{p(x_{i+1}) - p(x_i)}{x_{i+1} - x_i} \right).
\]

Let \(\theta_i(x) = \frac{p(x_{i+1}) - p(x_i)}{x_{i+1} - x_i}, k = i - 1, i, i + 1\). Differentiating \(\alpha_i(x) = G(\theta_{i+1}(x)) - G(\theta_i(x))\) w.r.t. \(x_k, k = i - 1, i, i + 1\) we get \(\frac{\partial \alpha_i(x)}{\partial x_i} = g(\theta_{i+1}(x)) \frac{\partial \theta_{i+1}(x)}{\partial x_i} - g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_i}, \frac{\partial \alpha_i(x)}{\partial x_{i+1}} = g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_{i+1}}\) and \(\frac{\partial \alpha_i(x)}{\partial x_i-1} = -g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_{i-1}}\). Similarly, \(\frac{\partial \alpha_{i+1}(x)}{\partial x_i} = -g(\theta_{i+1}(x)) \frac{\partial \theta_{i+1}(x)}{\partial x_i}, \frac{\partial \alpha_{i+1}(x)}{\partial x_{i+1}} = g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_{i+1}}\), and \(\frac{\partial \alpha_{i-1}(x)}{\partial x_i} = g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_{i-1}}\). Noting that \(\frac{\partial \theta_i(x)}{\partial x_{i+1}} = \frac{p(x_{i+1}) - p(x_i) + p'(x_i)(x_{i+1} - x_i)}{(x_{i+1} - x_i)^2} \geq 0\) and \(\frac{\partial \theta_i(x)}{\partial x_i-1} = \frac{p(x_i) - p(x_{i-1}) + p'(x_i)(x_i - x_{i-1})}{(x_i - x_{i-1})^2} \geq 0\) (since \(p(x)\) is increasing convex), we have the desired result.

We now use the results in points 1 and 2 above to show that the firm can do better by fully meeting demand of at least one product rather than having unmet demand for
all products, which leads to zero profit. Suppose firm sets $x = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)$ such that $J_x = \{k\}$, $I_x = \{1, 2, \ldots, k-1, k+1, \ldots, n\}$. So, when quality vector is set at $x$ only the demand of $k$ is fully met (see Figure C.2). The optimal quantity at $x$ is $Q^*(x) = S^{\alpha_k(x)}_{\beta_k(x)}$, and the optimal profit is

$$
\Pi(x) = S^{\alpha_k(x)}_{\beta_k(x)} \left( \sum_{i \in \Omega \setminus \{k\}} p(x_i) \beta_i(x) - c \right) + Sp(x_k)\alpha_k(x) - \bar{c}n \tag{C.2}
$$

$$
= S^{\alpha_k(x)}_{\beta_k(x)} \left( \sum_{i \in \Omega} p(x_i) \beta_i(x) - c \right) - \bar{c}n. \tag{C.3}
$$

From the last expression of (C.2) it can be easily checked that $\Pi(x) > 0$. Since $k+1 \in I_x$, it has unmet demand. This means,

$$
\frac{\alpha_{k+1}(x)}{\beta_{k+1}(x)} > \frac{\alpha_k(x)}{\beta_k(x)}. \tag{C.4}
$$

Figure C.2: Optimal quantity with $x$
Now we show (i) there exists a quality vector $x'$ such that, $J_{x'} = \{k, k+1\}$, i.e., both $k$ and $k+1$ have their demands fully met, and (ii) by setting the quality vector at $x'$ firm can earn higher profit than by setting the quality vector at $x$. At $x'$, there could be either of the two cases: (i) Case I: $\frac{\alpha_{k+1}(x')}{\beta_{k+1}(x')} < \frac{\alpha_k(x')}{\beta_k(x')}$ (see Figure C.3(a)) or, (ii) Case II: $\frac{\alpha_{k+1}(x')}{\beta_{k+1}(x')} > \frac{\alpha_k(x')}{\beta_k(x')}$ (see Figure C.3(b)). In each of these cases, the following holds,

\[
\text{case I: } \sum_{i \in \Omega \setminus \{k, k+1\}} p(x'_i) \beta_i(x') = \sum_{i \in \Omega \setminus \{k, k+1\}} p(x'_i) \beta_i(x') + p(x'_{k+1}) \beta_{k+1}(x') > c, \tag{C.5}
\]
\[
\text{case II: } \sum_{i \in \Omega \setminus \{k\}} p(x'_i) \beta_i(x') = \sum_{i \in \Omega \setminus \{k, k+1\}} p(x'_i) \beta_i(x') + p(x'_{k+1}) \beta_{k+1}(x') > c. \tag{C.6}
\]
On the other hand, when the quality vector is \( \mathbf{x} \) (see Figure C.2(a)), we have
\[
\sum_{i \in \Omega \setminus \{k\}} p(x_i)\beta_i(\mathbf{x}) = \sum_{i \in \Omega \setminus \{k, k+1\}} p(x_i)\beta_i(\mathbf{x}) + p(x_{k+1})\beta_{k+1}(\mathbf{x}) < c. \tag{C.7}
\]

There could be possibly several ways to set \( \mathbf{x}' \) such that (C.5) or (C.6) is achieved. One way (C.5) can be achieved is by setting \( \mathbf{x}' = (x_1, \ldots, x'_k, x_{k+1}, \ldots, x_n) \), where \( x'_k \neq x_k \), such that \( p(x'_k)\beta_k(\mathbf{x}') > p(x_{k+1})\beta_{k+1}(\mathbf{x}) \). On the other hand, one way (C.6) can be achieved is by setting \( \mathbf{x}' = (x_1, \ldots, x_k, x'_{k+1}, \ldots, x_n) \), where \( x'_{k+1} \neq x_{k+1} \), such that \( p(x'_{k+1})\beta_{k+1}(\mathbf{x}') > p(x_{k+1})\beta_{k+1}(\mathbf{x}) \).

We only show the analysis for the first case, represented in Figure C.3(a). In this case, as mentioned above, we set \( \mathbf{x}' = (x_1, \ldots, x'_k, x_{k+1}, \ldots, x_n) \), where \( x'_k < x_k < x_{k+1} \) and \( p(x'_k)\beta_k(\mathbf{x}') > p(x_{k+1})\beta_{k+1}(\mathbf{x}) \). Since the yield distribution \( F \) is assumed to be IGFR, by arbitrarily reducing \( x_k \) to \( x'_k \), the value of \( p(x_k)\beta_k(\mathbf{x}) \) can be increased until we have \( p(x'_k)\beta_k(\mathbf{x}') > p(x_{k+1})\beta_{k+1}(\mathbf{x}) \). To see this argument, note that the derivative of \( r_k(\mathbf{x}) = p(x_k)\beta_k(\mathbf{x}) \) w.r.t. \( x_k \) is \( \frac{dr_k(\mathbf{x})}{dx_k} = p'(x_k)[F(x_{k+1}) - F(x_k)] - f(x_k)p(x_k) \). Clearly, if \( \frac{p'(x_k)}{p(x_k)} > (\frac{<c}{<c}) \frac{f(x_k)}{F(x_{k+1}) - F(x_k)} \), then \( \frac{dr_k(\mathbf{x})}{dx_k} > (\frac{<c}{<c})0 \). Since \( p(x_k) = x_k^a, a > 1 \), we have \( \frac{p'(x_k)}{p(x_k)} = \frac{a}{x_k} \). So, when \( a < \frac{x_k f(x_k)}{F(x_{k+1}) - F(x_k)} \), the value of \( p(x_k)\beta_k(\mathbf{x}) \) increases as \( x_k \) decreases (Comment: The function \( \frac{x_k f(x_k)}{F(x_{k+1}) - F(x_k)} \) is the generalized hazard rate of the yield curve when \( x \) lies in \([x_k, x_{k+1}]\) (by Eq. (3) in Navar and Oruiz (1996)). We checked that most common IGFR distributions follow the condition \( a < \frac{x_k f(x_k)}{F(x_{k+1}) - F(x_k)} \), for all \( x_i > 0 \), even with \( a = 4 \).

With the above argument, when we have \( x'_k < x_{k+1} \) and \( p(x'_k)\beta_k(\mathbf{x}') > p(x_{k+1})\beta_{k+1}(\mathbf{x}) \), it implies \( \beta_k(\mathbf{x}') > \beta_{k+1}(\mathbf{x}) \). By Technical Result 1, for any \( i \), \( \frac{\alpha_{k+1}(\mathbf{x}'_i)}{\beta_{k+1}(\mathbf{x}')^i} \leq 0 \). Thus, when \( x'_k < x_k \), we have \( \alpha_{k+1}(\mathbf{x}') \geq \alpha_{k+1}(\mathbf{x}) \). Also, \( \beta_{k+1}(\mathbf{x}') \equiv \beta_{k+1}(\mathbf{x}) = F(x_{k+2}) - F(x_{k+1}) \). With these relations, we can write \( \frac{\alpha_{k+1}(\mathbf{x}')}{\beta_{k+1}(\mathbf{x}')^i} \geq \frac{\alpha_{k+1}(\mathbf{x})}{\beta_{k+1}(\mathbf{x})^i} \). But by (C.4), \( \frac{\alpha_{k+1}(\mathbf{x})}{\beta_{k+1}(\mathbf{x})^i} > \frac{\alpha_k(\mathbf{x})}{\beta_k(\mathbf{x})^i} \). So, it follows that \( \frac{\alpha_{k+1}(\mathbf{x}')}{\beta_{k+1}(\mathbf{x}')^i} > \frac{\alpha_k(\mathbf{x}')}{\beta_k(\mathbf{x}')^i} \). Thus we have \( \max_{i \in \{k, k+1\}} \left\{ \frac{\alpha_i(\mathbf{x}')}{\beta_i(\mathbf{x}')^i} \right\} > \frac{\alpha_k(\mathbf{x})}{\beta_k(\mathbf{x})^i} \), in
other words, \( Q^*(x') > Q^*(x) \). The optimal profit with \( x' \) is

\[
\Pi(x') = S \frac{\alpha_k(x')}{\beta_k(x')} \left( \sum_{i \in \Omega \setminus \{k+1\}} p(x'_i) \beta_i(x') - c \right) + S p(x_{k+1}) \alpha_{k+1}(x') + S p(x'_k) \alpha_k(x') - \bar{c} n,
\]

\[
= S \frac{\alpha_k(x')}{\beta_k(x')} \left( \sum_{i \in \Omega \setminus \{k+1\}} p(x'_i) \beta_i(x') - c \right) + S p(x_{k+1}) \alpha_{k+1}(x') - \bar{c} n. \tag{C.8}
\]

Now consider the difference in profits: \((C.8)-(C.2)\) gives us

\[
\Pi(x') - \Pi(x) = S \frac{\alpha_k(x')}{\beta_k(x')} \left( \sum_{i \in \Omega \setminus \{k+1\}} p(x'_i) \beta_i(x') - c \right) - S \frac{\alpha_k(x)}{\beta_k(x)} \left( \sum_{i \in \Omega \setminus \{k\}} p(x_i) \beta_i(x) - c \right) + S p(x_{k+1}) \alpha_{k+1}(x') - S p(x_k) \alpha_k(x). \tag{C.9}
\]

The first term in \((C.9)\) is positive, since \( \sum_{i \in \Omega \setminus \{k+1\}} p(x'_i) \beta_i(x') > c \) (see Figure C.3(a)). The second term is negative since \( \sum_{i \in \Omega \setminus \{k\}} p(x_i) \beta_i(x) < c \) (see Figure C.2(a)). Since \( \frac{\alpha_{k+1}(x')}{\beta_{k+1}(x')} > \frac{\alpha_k(x)}{\beta_k(x)} \) and \( \beta_k(x') > \beta_{k+1}(x') \), it follows that \( \alpha_{k+1}(x') > \alpha_k(x) \), and so the sum of the third and fourth terms in \((C.9)\) is nonnegative. Therefore, we have \( \Pi(x') > \Pi(x) \).

In this fashion we can change the quality vector such that more products fall in \( J_x \) and the profit increases. \( \square \)

3. First Order Conditions of \( \Pi(x) \)

The first order conditions of \( \Pi(x) \) shown in this part are solved as a system of equations to determine the critical points of the profit function. Let \( \max_{j=1, \ldots, n} \left\{ \frac{\alpha_j(x)}{\beta_j(x)} \right\} = \frac{\alpha_k(x)}{\beta_k(x)} \).

Differentiating \( \Pi(x) = S \left( \sum_{j=1}^n p(x_j) \alpha_j(x) - c \max_{j=1, \ldots, n} \left\{ \frac{\alpha_j(x)}{\beta_j(x)} \right\} \right) - \bar{c} n \) w.r.t \( x_i \) we get \( k \neq \{i - 1, i, i + 1\} \):

\[
\frac{\partial \Pi(x)}{\partial x_i} = -g(\theta_{i+1}(x)) \frac{\partial \theta_{i+1}(x)}{\partial x_i} [p(x_{i+1}) - p(x_i)] - g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_i} [p(x_i) - p(x_{i-1})] + p'(x_i) \alpha_i(x). \tag{C.10}
\]
\[ k = i : \]

\[
\frac{\partial \Pi(x)}{\partial x_i} = -g(\theta_{i+1}(x)) \frac{\partial \theta_{i+1}(x)}{\partial x_i} [p(x_{i+1}) - p(x_i)] - g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_i} [p(x_i) - p(x_{i-1})] + p'(x_i) \alpha_i(x) \\
- \frac{\beta_i(x) g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_i} - \alpha_i(x) f(x_i)}{(\beta_i(x))^2} \] (C.11)

\[ k = i - 1 : \]

\[
\frac{\partial \Pi(x)}{\partial x_i} = -g(\theta_{i+1}(x)) \frac{\partial \theta_{i+1}(x)}{\partial x_i} [p(x_{i+1}) - p(x_i)] - g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_i} [p(x_i) - p(x_{i-1})] + p'(x_i) \alpha_i(x) \\
- \frac{\beta_{i-1}(x) g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_i} - \alpha_i(x) f(x_i)}{(\beta_{i-1}(x))^2} \] (C.12)

\[ k = i + 1 : \]

\[
\frac{\partial \Pi(x)}{\partial x_i} = -g(\theta_{i+1}(x)) \frac{\partial \theta_{i+1}(x)}{\partial x_i} [p(x_{i+1}) - p(x_i)] - g(\theta_i(x)) \frac{\partial \theta_i(x)}{\partial x_i} [p(x_i) - p(x_{i-1})] + p'(x_i) \alpha_i(x) \\
+ \frac{\beta_{i+1}(x) g(\theta_i(x)) \frac{\partial \theta_{i+1}(x)}{\partial x_i}}{(\beta_{i+1}(x))^2} \] (C.13)

4. **Brief Review of Majorization Theory:**

We review some concepts of majorization from [Marshall and Olkin 1979]. Let \( u^{(1)} \) indicate the largest element in \( u \in \mathbb{R}^n \), \( u^{(2)} \) the second-largest element, and so on. That is, \( u^{(1)} \geq \ldots \geq u^{(n)} \). Let \( u^{(1)} \) indicate the largest element in \( u \in \mathbb{R}^n \), \( u^{(2)} \) the second-largest element, and so on. That is, \( u^{(1)} \geq \ldots \geq u^{(n)} \).

1. A vector \( u \) is said to majorize the vector \( v \) (denoted \( u \succ v \)) if \( \sum_{i=1}^k u^{(i)} \geq \sum_{i=1}^k v^{(i)} \), \( 1 < k < n - 1 \), and \( \sum_{i=1}^n u^{(i)} = \sum_{i=1}^n v^{(i)} \) ([Marshall and Olkin 1979], pp. 7). This means, the components of \( u \) are more spread out than the components of \( v \).
2. For any vector \( u \in \mathbb{R}^n \) let \( s = \sum_{i=1}^{n} u_i \), and let \( 1_s \in \mathbb{R}^n \) denote the vector with equal elements given by \( 1_i = \frac{s}{n} \). Then \( u \succ 1_s \) (\[Marshall and Olkin 1979\], pp. 7). This simply means that the sum-\(s\) uniform vector \( 1_s \), which has equal components is the least spread out vector, and so all other vectors majorize it.

3. A function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) is called Schur-convex if and only if (i) it is symmetric (i.e., \( f(u) = f(\Pi u), \forall u \in \mathbb{R}^n \) and for all permutations \( \Pi \)), and (ii) \( \frac{\partial f}{\partial u_i} < \frac{\partial f}{\partial u_j} \) for all \( u_i < u_j \) (\[Marshall and Olkin 1979\], 3.A.4). An example of schur-convex function is \( \max(u) = u^{(1)} \). Note that the max function is symmetric. If \( f \) is Schur-convex on \( \mathbb{R}^n \), then \( -f \) is Schur-concave on \( \mathbb{R}^n \) and vice versa.

4. If a function is symmetric and convex, or symmetric and quasiconvex, then it is Schur-convex (\[Marshall and Olkin 1979\], 3.C.2, 3.C.3).

5. If \( f \) is Schur-convex (concave), then \( u \succ v \Rightarrow f(u) \geq (\leq)f(v) \) (\[Marshall and Olkin 1979\], 3.A.1). Schur-convex functions translate the ordering of vectors to a scalar ordering. If \( f \) is Schur-concave on \( \mathbb{R}^n \), then the optimal solution to \( \max_u f(u) \) is obtained when all components of \( u \) are equal (Lemma 2, \[Ben-Haim and Dvorkind 2004\]) This is true because any vector \( u \in \mathbb{R}^n \) majorizes the uniform vector \( 1_s \), and since \( f \) is Schur-concave, \( u \succ 1_s \) implies \( f(u) \leq f(1_s) \). Consequently, if \( f \) is not Schur-concave, then \( 1_s \) is not necessarily optimal.

With the help brief background material above we will explain whether offering equidistant bins is optimal. We can rewrite the optimization in (4.7) as a function of \( y = (y_1, \ldots, y_n) \), as given in (C.15). Since the profit function in (4.7) is not jointly concave or quasiconcave in \( x \) for arbitrary choices of \( F \) and \( G \), it follows that (C.15) is also not jointly concave or quasiconcave in \( y \). The function in (C.15) is symmetric (i.e., for any permutation \( \Pi y \), the function \( \Pi(y) \) remains same), but since it is not quasiconcave, it is not Schur-concave either. Therefore, it is not guaranteed that the optimal solution will be a vector with equal components.
5. Proof of Proposition 4.4.4

Proof. We will use the following result to prove Proposition 4.4.4.

TECHNICAL LEMMA 1: Let \( f(\varphi_1, \ldots, \varphi_n) \) be defined on \( \mathbb{R}^n \), and \( f \) has the property that for each \( i = 1, \ldots, n \), \( \frac{\partial^2 f}{\partial \varphi_i^2} \leq 0 \), whenever \( \frac{\partial f}{\partial \varphi_i} = 0 \). Suppose \( \varphi_i, i = 1, \ldots, n \), are increasing and strictly convex in \( a \in \mathbb{R} \). Then the composite function \( F(a) = f(\varphi(a)) \) is pseudoconcave in \( a \).

Proof of TECHNICAL LEMMA 1: Differentiating \( F(a) = f(\varphi(a)) \) w.r.t. \( a \) we get

\[
\frac{dF}{da} = \sum_{i=1}^{n} \frac{\partial f}{\partial \varphi_i} \frac{d\varphi_i}{da}.
\]

Since \( \varphi_i, i = 1, \ldots, n \), are increasing and strictly convex \( (\frac{d\varphi_i}{da} > 0, \frac{d^2\varphi_i}{da^2} > 0) \), \( \frac{dF}{da} = 0 \) implies, \( \frac{\partial f}{\partial \varphi_i} = 0, i = 1, \ldots, n \). Differentiating \( \frac{dF}{da} = \sum_{i=1}^{n} \frac{\partial f}{\partial \varphi_i} \frac{d\varphi_i}{da} \) w.r.t. \( a \) we get

\[
\frac{d^2F}{da^2} = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial \varphi_i^2} \left( \frac{d\varphi_i}{da} \right)^2 + \sum_{i=1}^{n} \frac{\partial f}{\partial \varphi_i} \frac{d^2\varphi_i}{da^2}.
\]

Clearly, when \( \frac{dF}{da} = 0 \), we have

\[
\left. \frac{d^2F}{da^2} \right|_{\frac{dF}{da}=0} = \left. \sum_{i=1}^{n} \frac{\partial^2 f}{\partial \varphi_i^2} \left( \frac{d\varphi_i}{da} \right)^2 + \sum_{i=1}^{n} \frac{\partial f}{\partial \varphi_i} \frac{d^2\varphi_i}{da^2} \right|_{\frac{\partial f}{\partial \varphi_i}=0, i=1,\ldots,n}
= \sum_{i=1}^{n} \frac{\partial^2 f}{\partial \varphi_i^2} \left. \frac{d\varphi_i}{da} \right|_{\frac{\partial f}{\partial \varphi_i}=0, i=1,\ldots,n} \times \left. \left( \frac{d\varphi_i}{da} \right)^2 \right|_{\frac{\partial f}{\partial \varphi_i}=0, i=1,\ldots,n} \leq 0.
\]

Using the above result we will show that (4.8) is pseudoconcave in \( a \).

Let \( \max_{j=1,\ldots,n} \left\{ \frac{\alpha_j(x,a)}{\beta_j(x)} \right\} = \frac{\alpha_k(x,a)}{\beta_k(x)} \). Then (4.8) can be further simplified as

\[
\Pi(a) = S \left( \sum_{j=1}^{n} x_j^a \alpha_j(x,a) - c \frac{\alpha_k(x,a)}{\beta_k(x)} \right) - \bar{c} n,
\] (C.15)
where,

$$\alpha_j(x, a) = \begin{cases} G\left(\frac{x_{j+1} - x_{j}}{x_{j+1} - x_{j-1}}\right) - G\left(\frac{x_{j} - x_{j-1}}{x_{j} - x_{j-1}}\right) & j = 1, \ldots, n - 1, \\ 1 - G\left(\frac{x_{n} - x_{n-1}}{x_{n} - x_{n-1}}\right) & j = n. \end{cases} \quad (C.16)$$

Define \( \varphi_j = \frac{x_{j} - x_{j-1}}{x_{j} - x_{j-1}} \), \( j = 1, \ldots, n \), and let \( \varphi = (\varphi_1, \ldots, \varphi_n) \). Using this change of variable, we can write \( x_{j} = \sum_{i=1}^{j} \varphi_i(x_i - x_{i-1}), j = 1, \ldots, n, \) and (C.15) as follows

$$\Pi(\varphi) = \sum_{j=1}^{n} S\left(\sum_{i=1}^{j} \varphi_i(x_i - x_{i-1})\right) \times (G(\varphi_{j+1}) - G(\varphi_j)) - c G(\varphi_{k+1}) - G(\varphi_k) \beta_k(x) - cn. \quad (C.17)$$

Differentiating (C.17) w.r.t. \( \varphi_j, j = 1, \ldots, n \), we get

$$\frac{d^2 \Pi}{d\varphi_j^2} \bigg|_{d\Pi=0} = g(\varphi_j)(x_j - x_{j-1}) \eta'(\varphi_j) \leq 0, j = 1, \ldots, n. \quad (C.18)$$

Note that, \( \frac{d^2 \varphi_j(a)}{da^2} = \frac{x_{j}^2 \log x_{j}^2 - x_{j-1}^2 \log x_{j-1}^2}{x_{j} - x_{j-1}} > 0 \), so, \( \varphi_j, j = 1, \ldots, n \), are increasing convex in \( a \). Therefore, by the Technical Lemma 1, we have (4.8) is pseudoconcave in \( a \).

6. Proof of Proposition 4:

We show (by contradiction) that firm is better off by introducing an additional bin so that there is some positive demand for the new bin, rather than no demand for it.

Let there be \( n \) bins at quality levels \( x = (x_1, \ldots, x_n) \), and let a new bin be introduced at quality level \( x_k \), where \( x_{n-1} < x_k < x_n \). The quality vector with \( n + 1 \) bins is \( x' = (x_1, \ldots, x_{n-1}, x_k, x_n) \). Suppose \( x_k \) is set such that \( \alpha_k(x') = 0 \), i.e., there is no demand for bin \( k \) ( \( x_k \) is the solution to the equation \( \alpha_k(x') = G\left(\frac{p(x_n) - p(x_{k-1})}{x_n - x_k}\right) - G\left(\frac{p(x_k) - p(x_{n-1})}{x_k - x_{n-1}}\right) = 0 \). Note that by introducing \( k \), only the demand of \( n - 1 \) is reduced \( (\alpha_{n-1}(x') = \)
\[ G(\frac{p(x_k)-p(x_{n-1})}{x_k-x_{n-1}}) - G(\frac{p(x_{n-1})-p(x_{n-2})}{x_{n-1}-x_{n-2}}) < \alpha_{n-1}(x) = G(\frac{p(x_n)-p(x_{n-1})}{x_n-x_{n-1}}) - G(\frac{p(x_{n-1})-p(x_{n-2})}{x_{n-1}-x_{n-2}}). \]

Since \( \alpha_k(x') = 0 \), we have \( \max_{j=1, \ldots, n-1, k, n} \{ \frac{\alpha_j(x')}{\beta_j(x')} \} = \max_{j=1, \ldots, n} \{ \frac{\alpha_j(x)}{\beta_j(x)} \} \), i.e., by introducing \( k \) with no demand there is no additional quantity to be produced. When \( \bar{c} = 0 \), we can write the difference of the optimal profits from \( x' \) and \( x \) as follows.

\[
\Pi(x') - \Pi(x) = Sp(x_{n-1})[\alpha_{n-1}(x') - \alpha_{n-1}(x)] + p(x_k)\alpha_k(x') < 0. \tag{C.19}
\]

Therefore, firm is better off \( n + 1 \) bins, such that there is demand for all \( n + 1 \) bins. With this logic, one can argue that there could be theoretically infinite bins. □

7. Numerical Design for UYP scenario:

We also consider a scenario where both yield and preferences are uniformly distributed (the UYP scenario). In this scenario, let the consumer preference distribution be \( G \sim U(C; D) \) and the yield distribution be \( X \sim U(A; B) \). We varied \( A \) and \( B \), such that both mean and standard deviation of yield has five levels: the mean of yield distribution \( (A + B)/2 \) varied from 2.0 to 6.0 (step size 1.0), and the standard deviation \( \frac{B-A}{\sqrt{12}} \) varied from \( \frac{0.25}{\sqrt{12}} \) to \( \frac{1.5}{\sqrt{12}} \) (step size \( \frac{0.25}{\sqrt{12}} \)). Similarly, for the preference distribution we varied \( C \) and \( D \), such that the mean \( (C + D)/2 \) varied from 5.0 to 9.0 (step size 1.0), and the standard deviation \( \frac{D-C}{\sqrt{12}} \) varied from \( \frac{2.5}{\sqrt{12}} \) to \( \frac{4.5}{\sqrt{12}} \) (step size \( \frac{0.5}{\sqrt{12}} \)). We assumed the price curve \( p(x) = x^a \), and considered four levels of the index \( a = 1.01, 1.2, 1.5, 2 \). A value of \( a = 1.01 \) represents almost linear pricing, whereas \( a = 2 \) represents strictly convex pricing. The above factorial design yielded 2,500 total instances. Table C.1 reports the factors and their values for uniform distribution of yield and preferences. □

8. Effect of Change in Process Technology on Optimal Quality

In this section we explore the impact of process technology on the firm’s product line and profit. Specifically we analyze the effect of change in mean on the optimal profit and the impact of change in the variability of the yield distribution on the optimal
Table C.1: Experimental Design with Uniform distribution of Yield and Preference

<table>
<thead>
<tr>
<th>Factor description</th>
<th>Symbol</th>
<th>Factor values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Yield</td>
<td>$(A + B)/2$</td>
<td>${2, 3, 4, 5, 6}$</td>
</tr>
<tr>
<td>Spread Yield</td>
<td>$(B - A)/2$</td>
<td>${0.5, 0.75, 1, 1.25, 1.5}$</td>
</tr>
<tr>
<td>Mean Preference</td>
<td>$(C + D)/2$</td>
<td>${5, 6, 7, 8, 9}$</td>
</tr>
<tr>
<td>Spread Preference</td>
<td>$(D - C)/2$</td>
<td>${2.5, 3, 3.5, 4, 4.5}$</td>
</tr>
<tr>
<td>Index of price curve</td>
<td>$a$</td>
<td>${1.01, 1.2, 1.5, 2}$</td>
</tr>
</tbody>
</table>

assortment length, which is defined as the distance between the highest and lowest quality bins. We arbitrarily chose 50 base cases from the NYP scenario. Each of these base cases has multiple associated cases, where only the process variability $\sigma_X$ changed while other parameters were constant. We measured the percentage change in optimal assortment length over the base case as $\sigma_X$ changed, and this was measured on each case for $n = 3, 4, 5, 6$. 
Figure C.4: % Change in Length of Assortment with change in Variability of Process (NYP scenario)
Figures C.4(a)-C.4(c) show the scatter plot of percentage change in the assortment length and percentage change in the variability of the yield across different values of $n$. In these figures each point represents a numerical case. We observe that the product line length increases in the process variability, and this trend is observed for any number of offerings. The managerial intuition is that as there is a wider disparity in the offered output quality, the firm wants to extract additional revenues from heterogeneous customers by segmenting the offered output into different bins that are widely apart. Further, when the product line increases in depth, i.e., as the number of offerings increase, the variation in the length of assortment also widens with output variability (see Figures C.4(a) and C.4(c)). In this case firm has two levers, one is the more number of offerings, and the other a highly variable quality output, and together these give firm more flexibility to capture the heterogeneity in consumer preference. We also observed across all instances, that the optimal profit increases in the mean quality of the yield distribution, ceteris paribus.

9. Effect of Change in Consumer Heterogeneity on Optimal Quality

To analyze the impact of change in consumer heterogeneity on the optimal quality levels, we arbitrarily chose 50 base cases from the NYP scenario. Each of these base cases has multiple associated cases, where only the variability of the distribution of consumer preference ($\sigma_G$) changed, while other parameters were constant. We measured the percentage change in optimal assortment length over the base case as $\sigma_G$ changed, and this was measured on each case for $n = 3, 4, 5, 6$. Figures C.5(a)-C.5(d) show the scatter plots, where each point represents a numerical case. We observe that variation in product line length increases as the variability in consumer preferences increase, a trend similar to the effect of change in process variability. The managerial intuition is that as the heterogeneity in consumer base increases, the firm will extract additional revenues from these heterogeneous customers by segmenting the offered output into different bins that are widely apart.
Figure C.5: % Change in Length of Assortment with change in Variability of Consumer Heterogeneity
VITA
Mrinmay Deb

Contact Information
483A Business Building                               Email:  mrinmay@psu.edu
Department of Supply Chain & Information Systems      Work:    (814) 865-0607
The Smeal College of Business                        Cell:     (814) 321-7826
The Pennsylvania State University
University Park, PA 16802

Education
• Ph.D. in Business Administration                   2013 The Pennsylvania State University
• B.E (Honors) in Mechanical Engineering             2003 National Institute of Technology, Durgapur, India

Research Interests
Retail Assortment Planning, Consumer Choice Modeling, Pricing of Substitutable Products

Working Papers