NUMERICAL STUDIES OF SEISMICALLY INDUCED SLOPE DEFORMATION USING SMOOTHED PARTICLE HYDRODYNAMICS METHOD

A Dissertation in
Civil Engineering
by
Wei Chen

© 2012 Wei Chen

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

December 2012
The dissertation of Wei Chen was reviewed and approved* by the following:

Tong Qiu  
Assistant Professor of Civil Engineering  
Dissertation Advisor  
Chair of Committee

Daniel G. Linzell  
Shaw Professor of Civil Engineering

Derek Elsworth  
Professor of Energy and Geo-Environmental Engineering

Prasenjit Basu  
Assistant Professor of Civil Engineering

Shelley M. Stoffels  
Associate Professor of Civil Engineering

Peggy A. Johnson  
Professor of Civil Engineering  
Head of the Department of Civil and Environmental Engineering

*Signatures are on file in the Graduate School
ABSTRACT

There has been growing interest in improving current procedures for estimating seismically-induced deformations of natural and man-made slopes due to recently frequent earthquake events and the resulted damaged to infrastructure systems. The aim of this study is to develop a numerical model to effectively and reliably assess seismically-induced slope deformations that typically involve large deformations and complex soil constitutive behaviors. A numerical model based on the meshfree Smoothed Particle Hydrodynamics (SPH) method has been developed by implementing various advanced constitutive models into the SPH formulations. The developed model is validated by two readily available and well-documented experiments: axisymmetric collapses of granular columns and model slope tests on a shaking table. For the former, the non-dilatant Drucker-Prager (D-P) constitutive relationship with perfect plasticity is used. The developed model precisely reproduces the experimentally-observed three regimes of flow patterns based on the initial aspect ratio of the granular column. In addition to the flow patterns, the simulated final deposit height and run-out distance along with the non-deformed region after the collapse of granular columns are in excellent agreement with experimental data in the literature. For the latter, a constitutive model that combines the strain-softening viscoplasticity and Modified Kondner and Zelasko (MKZ) rule is implemented and utilized to account for the effects of wave propagation in the sliding mass, cyclic nonlinear behavior of soil, and progressive reduction in shear strength during sliding, which are not explicitly considered in various Newmark-type analyses widely used in the current research and practice in geotechnical earthquake engineering. The initiation of slope failure and subsequent progressive development of
the sliding surface are successfully captured by the developed SPH model. A localized
shear band along the failure surface and a bulge near the toe of the model slope are
observed in the simulations, showing a good agreement with the experimental
observations. The simulated failure mode, displacement time histories, and acceleration
response spectra at several monitor locations along the model slope also agree well with
the experimental recordings.

Based on the validated SPH model, a parametric study is followed to investigate the
effects of spatial parameters including both particle spacing and smooth length on the
accuracy of SPH simulations. The parametric study also investigates the effects of
material strength and shear modulus along with boundary conditions on the seismically-
induced slope deformations, providing insights into the mechanisms of earthquake-
induced slope deformations. It is thus suggested that the proposed SPH model is an
effective tool for assessing the seismic performance of soil slopes. It may be also used to
advance the computational capability of modeling geotechnical engineering phenomena
involving large deformations.
# TABLE OF CONTENTS

LIST OF FIGURES .............................................................................................................................. vii

ACKNOWLEDGEMENTS ........................................................................................................................ xii

Chapter 1 Introduction ......................................................................................................................... 1

1.1 Motivation ...................................................................................................................................... 1
1.2 Pseudostatic Analysis ...................................................................................................................... 3
1.3 Permanent Displacement Analysis ............................................................................................... 5
  1.3.1 Newmark rigid-block analysis ................................................................................................. 5
  1.3.2 Derivatives of Newmark analysis ............................................................................................. 8
1.4 Stress-Deformation Analysis and Advanced Computational Methods ....................................... 9
  1.4.1 Arbitrary Lagrangian Eulerian method (ALE) ........................................................................ 10
  1.4.2 Discrete Element Method (DEM) .......................................................................................... 12
  1.4.3 Mesh-free Methods ................................................................................................................. 14
1.5 Dissertation Scope and Layout ..................................................................................................... 18
References ........................................................................................................................................... 20

Chapter 2 Numerical Simulations for Large Deformation of Granular Materials
  Using Smoothed Particle Hydrodynamics Method ............................................................................. 26

2.1 Introduction .................................................................................................................................... 27
2.2 Numerical Implementation ............................................................................................................. 29
  2.2.1 Governing and constitutive equations for soil ........................................................................ 29
  2.2.2 SPH formulations .................................................................................................................... 32
  2.2.3 Time integration ....................................................................................................................... 37
2.3 Model Validation ............................................................................................................................ 39
2.4 Numerical Simulation for 3-D Granular Flows ............................................................................. 42
  2.4.1 Granular flow patterns .......................................................................................................... 44
  2.4.2 Final runout distance .............................................................................................................. 49
  2.4.3 Final deposit height ................................................................................................................. 50
  2.4.4 Non-deformed region ............................................................................................................. 51
2.5 Conclusions .................................................................................................................................... 54
References ........................................................................................................................................... 56

Chapter 3 Simulation of Earthquake-induced Slope Deformation Using SPH
  Method ................................................................................................................................................. 59

3.1 Introduction .................................................................................................................................... 60
3.2 SPH Formulations .......................................................................................................................... 65
3.3 Constitutive Model ........................................................................................................................ 71
  3.3.1 Finite strain consideration ..................................................................................................... 72
  3.3.2 Isotropic strain softening viscoplasticity .............................................................................. 74
  3.3.3 Nonlinear cyclic behavior of soil ........................................................................................... 82
3.4 Explicit Time Integration ............................................................... 89
3.5 Model Calibration ........................................................................... 91
  3.5.1 Calibration of viscous parameters in the Peirce viscoplastic model ... 92
  3.5.2 Calibration of strain softening parameters .......................... 98
  3.5.3 Calibration of curve-fitting parameters in MKZ model .......... 102
  3.5.4 Calibration of rate-dependent stiffness of clay .................... 104
3.6 Model Validation ........................................................................... 108
  3.6.1 Slope failure mode ............................................................... 109
  3.6.2 Acceleration response spectra .......................................... 113
  3.6.3 Slope deformation ............................................................... 115
3.7 Conclusions .................................................................................. 119
References .......................................................................................... 122

Chapter 4 Parametric Study ........................................................................ 131
4.1 Introduction .................................................................................... 131
4.2 Viscoplastic Parameters ............................................................... 132
4.3 Spatial Parameters in SPH Method .................................................... 136
  4.3.1 Particle-spacing effects ........................................................ 136
    4.3.1.1 Rate-independent model ........................................... 137
    4.3.1.2 Rate-dependent model ............................................. 144
  4.3.2 Smoothing-length effects ....................................................... 147
    4.3.2.1 Rate-independent model ........................................... 148
    4.3.2.2 Rate-dependent model ............................................. 149
4.4 Effects of Soil Properties on Slope Deformations ................................. 150
  4.4.1 Residual strength and plastic modulus ..................................... 151
  4.4.2 Peak strength ............................................................................... 160
  4.4.3 Shear modulus ............................................................................... 161
4.5 Non-reflecting Boundary .................................................................. 164
  4.5.1 Mathematical formulation ....................................................... 167
  4.5.2 Validation of quiet boundary against FEM ......................... 171
  4.5.3 Modeling slope test with quiet boundary ..................................... 175
4.6 Conclusions .................................................................................. 178
References .......................................................................................... 180

Chapter 5 Conclusions and Recommendations ............................................... 182
5.1 Conclusions .................................................................................. 182
5.2 Recommendations for Future Work ..................................................... 186
  5.2.1 Solid-fluid couplings in saturated soils ................................... 186
  5.2.2 Correction factors for Newmark analysis ............................. 187
  5.2.3 Coupled SPH-FEM method for soil-structure interactions .......... 188
References .......................................................................................... 189

Appendix A SPH Simulation of Collapse of Two-dimensional Granular Column .... 190
Appendix B SPH Simulation of an Elastic Cantilever Beam ............................ 195
LIST OF FIGURES

Figure 1-1. Pseudostatic slope stability analysis.........................................................4
Figure 1-2. Analogy between potential sliding mass and block on an inclined plane.................................................................6
Figure 1-3. Illustration of Newmark analysis: (a) acceleration time history; (b) sliding velocity time history; (c) sliding displacement time history..............7
Figure 1-4. High speed impact of a metal bar: (a) A quarter model of a metal bar impacting a rigid wall; (b) Updated Lagrangian solution; (c) ALE solution.................................................................13
Figure 1-5. Comparison of approximations using (a) grid-based method; (b) meshfree method..................................................................................................................15
Figure 1-6. High speed impact of a metal bar with contours showing the Von Mises stress in the bar: (a) ALE solution; (b) SPH solution....................................................................17
Figure 2-1. Particle approximation based on kernel function $W$ in influence domain $\Omega$ with radius $kh$ .................................................................................................33
Figure 2-2. Illustration of test setup and geometry: (a) initial condition; (b) final profile after collapse (after Lube et al. 2005).................................40
Figure 2-3. Comparison of SPH simulation for unidirectional collapse in 3-D and 2-D conditions: (a) initial configuration; (b) isometric view of final profile in 3-D simulation; (c) side view of final profile in 3-D simulation; (d) final profile in 2-D simulation..................................................................................................................41
Figure 2-4. Comparison of SPH simulation and experiment with $a = 0.55$: (a) initial configuration of sand column; (b) simulated final profile after collapse; (c) side view of simulated final profile; (d) experimental final profile (Lube et al. 2004)...........................................................................................................45
Figure 2-5. Comparison of SPH simulation and experiment with $a = 0.9$: (a) initial configuration of sand column; (b) simulated final profile after collapse; (c) side view of simulated final profile; (d) experimental final profile (Lube et al. 2004)...........................................................................................................47
Figure 2-6. Comparison between SPH simulation and experiment with $a = 2.75$: (a) initial configuration of sand column; (b) simulated flow during collapse; (c) experimental flow during collapse (Lube et al. 2004); (d) simulated final profile after collapse; (e) side view of simulated final profile; (f)
experimental final profile (Lube et al. 2004)……………………….48

Figure 2-7. Comparison of normalized final runout distance between SPH simulations and experiments from Lube et al. (2004)………………………………49

Figure 2-8. Comparison of normalized final deposit height between SPH simulations and experiments from Lube et al. (2004)…………………………………50

Figure 2-9. Accumulative equivalent plastic strain after collapse of sand column with $a = 0.9$: (a) side view; (b) isometric view………………………………….52

Figure 2-10. Accumulative equivalent plastic strain after collapse of soil column with $a = 2.75$: (a) side view; (b) isometric view…………………………53

Figure 2-11. Comparison of the radius at the base of non-deformed region between SPH simulations and experiments from Lube et al. (2004) for

$\frac{a}{a} < 1.7$………………………………………………………………..54

Figure 3-1. Particle approximation based on kernel function $W$ in influence domain $\Omega$ with radius $\alpha$……………………………………………………………67

Figure 3-2. Drucker-Prager yield model with yield surface evolution in $\left(-I_1, \sqrt{J_2}\right)$ plane…………………………………………………………….77

Figure 3-3. Hysteresis loops and backbone curves………………………………….84

Figure 3-4. Flow chart of computation for material response…………………….88

Figure 3-5. Initial configuration of model slope (after Wartman 1999)………………….92

Figure 3-6. Simulation of vane shear test: (a) initial configuration of vane shear test (only half of the soil simulated is presented for better visual effect); (b) cross section of $A - A'$ showing yield region in the clay after a blade rotation of 110 degrees………………………………………………………94

Figure 3-7. Schematic diagram of ghost boundary condition in SPH………………….95

Figure 3-8. Strain rate effects on peak shear strength of model soft clay…………….97

Figure 3-9. Geometric configuration of strip footing………………………………….99

Figure 3-10. Comparison of average pressure beneath the footing in ABAQUS and SPH…………………………………………………………………………100

Figure 3-11. Comparison of undrained shear strength vs. peripheral displacement between SPH simulation and laboratory vane shear test (experimental data digitized from Wartman et al. 2005)………………………………102
Figure 3-12. Modulus reduction and damping curves (experimental curves digitized from Wartman et al. 2001)………………………………………………….104

Figure 3-13. Acceleration time history of frequency sweep test………………….106

Figure 3-14. Comparison of acceleration response spectra of simulated and recorded motions at accelerometer No. 7……………………………………..…107

Figure 3-15. Time histories of input motion: (a) acceleration; (b) displacement……109

Figure 3-16. Comparison of simulated failure mode and deformed shape with model slope test: (a) $t = 13.4$ s; (b) $t = 20.2$ s; (c) $t = 32.0$ s; (d) final deformed profile and sliding surface from model slope test (from Wartman et al. 2005 with permission)…………………………………………….111

Figure 3-17. Simulated failure mode and deformed shape by rate-independent model with corrected peak strength: (a) $t = 13.4$ s; (b) $t = 20.2$ s; (c) $t = 32.0$ s …………………………………………………………………………………..112

Figure 3-18. Comparison of simulated and recorded acceleration response spectral at: (a) accelerometer No. 6; (b) accelerometer No. 7…………………...…114

Figure 3-19. Comparison of simulated and recorded horizontal displacement time histories at various locations along model slope…………………………….116

Figure 3-20. Displacement time histories at potentiometer No. 11 converge as SPH particle size decreases……………………………………………………….118

Figure 3-21. Comparison of displacement time histories simulated using viscoplastic model and rate-independent model…………………………………….119

Figure 4-1. Effects of viscous parameters on peak vane shear strength………….133

Figure 4-2. Effects of viscous parameters on displacement time history at location No. 11……………………………………………………………………..134

Figure 4-3. Effects of viscous parameters on displacement time history at location No. 12……………………………………………………………………..134

Figure 4-4. Comparison of failure modes and deformed shapes in SPH simulations and model slope tests: (a) $\mu_1 = 100$ and $\mu_2 = 0.07$; (b) $\mu_1 = 5$ and $\mu_2 = 0.2$; (c) final deformed shape observed in model slope tests (Wartman 2005)…………………………………………………………..135

Figure 4-5. Simulated pressures beneath the footing for different mesh sizes using FEM……………………………………………………………………138
Figure 4-6. Simulated pressures beneath the footing for different particle spacing using rate-independent strain-softening SPH model

Figure 4-7. Simulated pressures beneath the footing for different particle spacing using rate-independent perfect-plastic model in SPH

Figure 4-8. Localized shear bands simulated using rate-independent model with different particle spacing: (a) $\Delta x = 0.8$ cm; (b) $\Delta x = 0.5$ cm; (c) $\Delta x = 0.2$ cm

Figure 4-9. Displacement time histories simulated by the rate-independent model with different particle spacing

Figure 4-10. Simulated pressures beneath the footing for different particle spacing using strain-softening viscoplastic model in SPH

Figure 4-11. Localized shear bands simulated by viscoplastic model with different particle spacing: (a) $\Delta x = 0.8$ cm; (b) $\Delta x = 0.5$ cm; (c) $\Delta x = 0.2$ cm; (d) $\Delta x = 0.1$ cm

Figure 4-12. Displacement time histories simulated by the viscoplastic model with different particle spacing

Figure 4-13. Displacement time histories simulated using the rate-independent model with different smoothing lengths

Figure 4-14. Displacement time histories simulated using the viscoplastic model with different smoothing lengths

Figure 4-15. Illustrative diagram of residual strength and plastic modulus

Figure 4-16. A simplified strain-softening relation used in parametric study

Figure 4-17. Undrained shear strength vs. peripheral displacement in vane shear test simulated with different (a) Residual strength; (b) Plastic modulus

Figure 4-18. Post-failure residual strength in parametric study

Figure 4-19. Effects of residual strength on slope displacement

Figure 4-20. Shear failure region simulated with different residual strengths: (a) 2.68 kPa; (b) 2.10 kPa; (c) 1.77 kPa; (d) 1.40 kPa

Figure 4-21. Plastic modulus used in parametric study

Figure 4-22. Effects of plastic modulus on slope displacement
Figure 4-23. Effects of peak strength on slope displacement………………………..161
Figure 4-24. Simulated permanent displacements at various period ratios………..164
Figure 4-25. Illustration of different boundary conditions: (a) free boundary (free transmission of wave energy); (b) rigid boundary (reflection of wave energy); (c) quiet boundary (absorption of wave energy)……………166
Figure 4-26. A viscous boundary absorbing waves (a) a shear wave traveling through $A-A'$ plane in a medium; (b) a viscous boundary equivalent to the medium on the right side of $A-A'$ plane……………………………..168
Figure 4-27. An infinite half space subjected to a distributed impulse load……….172
Figure 4-28. Triangular impulse of the distributed load……………………………..172
Figure 4-29. Comparison of displacement time histories simulated by SPH and FEM at the monitor locations (a): MN1; (b) MN2………………………………………………………174
Figure 4-30. Boundary condition for model slope test……………………………...175
Figure 4-31. Comparison of simulated displacement time histories with non-reflecting boundary and rigid boundary in SPH model (a): location No. 11; (b) location No. 12…………………………………………………………176
Figure 4-32. Comparison of simulated deformed slopes with (a): rigid boundary; (b): non-reflecting boundary in SPH……………………………………………178
Figure A-1. Development of failure surface during the collapse of granular column…………………………………………………………………191
Figure A-2. Comparison between experiment and SPH simulation of collapse of aluminum bar column in 2-D plane strain condition: (a) simulated slope profile after collapse and contour of accumulated plastic strain; (b) slope profiles after collapse in experiment and simulation……………..193
Figure B-1. A plane cantilever beam subjected to an impulse…………………..197
Figure B-2. Time history of horizontal displacement on the top of the beam simulated by SPH with: (a) artificial stress method; (b) conservative smoothing method………………………………………………………198
ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my advisor, Dr. Tong Qiu, for his most valuable instruction, guidance, patience, and most importantly, his friendship during my graduate studies at Penn State and Clarkson. He encourages me to not only grow as a numerical modeler but also as a researcher and an independent thinker. His standard of excellence inspires me to strive for a high level of professionalism. I would also like to offer my genuine appreciation to my committee members, Dr. Daniel Linzell, Dr. Derek Elsworth, Dr. Prasenjit Basu and Dr. Shelley Stoffels for reviewing the manuscript and providing valuable perspective on my work. Special thanks are due to Dr. Elsworth for inviting me to deliver a lecture in his class on the numerical methods in geomechanics.

I would like to thank Dr. Joseph Wartman of the University of Washington for providing the data of his laboratory tests performed at the University of California, Berkeley. I would also like to show my gratefulness to Dr. Kerop Janoyan of Clarkson University for his support and kindness.

This dissertation would never have been completed without the support and devotion of my family and friends. I would like to thank my wife, Li Liu, for her unwavering love, quiet patience and persistent encouragement. You are the reason I managed to get through the most difficult time in my life and stay encouraged to pursue our dream. I heartily thank my parents for their faith in me and their continuous support. Many thanks are owing to my friend Jared Wright for reviewing this manuscript. I am thankful to Lynsey Reese for her help and the amazing ‘SuperWei’ T-shirt that she gave me. I would
like to thank my friends Dr. Michael Gangone and Henry Wason IV at Clarkson University. I also extend my thanks to Yin Go, Yanbo Huang and Yibing Xiang.

Finally, I offer my regards to all of those who supported me in any respect during the completion of the dissertation.
CHAPTER 1

INTRODUCTION

1.1 Motivation

Landslides are one of the most damaging hazards generated by earthquakes, particularly in hilly and mountainous terrains. Large earthquakes are capable of triggering thousands of landslides within land areas approaching 100,000 km$^2$ around the quake (Keefer 1984). For example, the 1994 earthquake ($M = 6.7$) in Northridge, California, triggered more than 11,000 landslides over an area of about 10,000 km$^2$ (Harp and Gibson 1995). The recent 2008 Wenchuan earthquake ($M = 8.0$) reportedly triggered more than 50,000 geohazards in the forms of landslides, rockfalls, and debris flows (Huang and Xu 2008). Earthquake-induced landslides pose a significant threat to human lives and infrastructure systems (e.g., roads, pipelines, utilities). Therefore, predicting the location and shaking conditions needed to trigger landslides is a key element in regional seismic hazard assessment (Jibson et al. 1998).

Methods for assessing the performance of a slope during earthquakes fall into three categories (Jibson 2011): (1) pseudostatic analysis, (2) permanent displacement analysis, and (3) stress-deformation analysis. Pseudostatic analysis, used mainly for preliminary or screening analysis due to its crude characterization of the physical processes (Jibson 2011), can only indicate the possibility of slope failure. As a significant improvement to pseudostatic analysis, permanent displacement analysis can estimate the consequences
after the incidence of slope failure. It provides a more quantitative measure to evaluate the performance of slopes during earthquakes. Both pseudostatic analysis and permanent displacement analysis are based on highly simplified geometric and material models.

However, these two methods are unable to reliably evaluate earthquake-induced slope deformations under complex geological conditions. To fill these needs, stress-deformation analysis, usually performed using computational methods such as the Finite Element Method (FEM) and Finite Difference Method (FDM), has been extensively applied to critical projects and complex earth structures. This approach can account for complex soil behaviors and geometric conditions. The continuum-scale grid based methods (e.g. FEM and FDM) have difficulties in modeling large deformations due to severe grid distortion and entanglement. As a result, stress-deformation analysis is currently limited to estimating relatively small seismically-induced slope deformations (Jibson 2011). The drawback of grid-based methods in dealing with large deformations therefore considerably impedes their application in the analysis of earthquake-induced slope deformations. Therefore, the permanent displacement analysis prevails in the current geotechnical engineering practice.

In order to overcome these drawbacks in stress-deformation analysis, various methods have been proposed. Arbitrary Lagrangian Eulerian (ALE) method is a remedy to FEM in dealing with highly deforming materials on a continuum scale. In ALE method, the mesh moves in an arbitrary manner which is independent of the motion of the material being analyzed. Since the mesh is not connected to the material, state variables on ALE grids must be properly mapped from the original grids. This arbitrary meshing scheme can effectively avoid severe distortion of purely Lagrangian grid.
Nevertheless, this method encounters tremendous challenges when applied to modeling ultimate deformations under dynamic loading (Liu and Liu 2004) and when complex material models are used. Another method to handle large deformations is the Discrete Element Method (DEM) (Cundall and Strack 1979) which is a particle method formulated on a micro-structural scale. As opposed to FEM that treats material as a continuum, DEM treats material as discrete particles and is widely accepted as an alternative to FEM in solving engineering problems in granular and discontinue materials. DEM is a computationally intensive method and, at present, is restricted to small-scale simulations.

This study is motivated by the lack of an efficient, effective and reliable method capable of modeling earthquake-induced slope deformations. A more detailed review of current approaches in evaluating seismic slope stability is presented during this chapter.

1.2 Pseudostatic Analysis

As the earliest attempt at analyzing seismic effects on slopes, pseudostatic analysis has been commonly used in engineering practice. Pseudostatic analysis calculates a factor of safety (FS) using a limit equilibrium method in which the seismic shaking is represented by a constant inertial force applied on a sliding mass. As shown in Figure 1-1, the horizontal inertial force is expressed by the product of pseudostatic coefficient $k$ and weight of the sliding mass $W$. The pseudostatic coefficient is defined as

$$k = \frac{a_h}{g} \quad (1-1)$$
where $a_h$ is the horizontal acceleration of ground motion and $g$ is the acceleration of gravity.

Figure 1-1. Pseudostatic slope stability analysis

A common procedure in pseudostatic analysis is to iteratively conduct limit equilibrium analysis with different values of $k$ until the FS approaches to unity. The resulting pseudostatic coefficient is called the yield coefficient $k_y$. The slope is considered unsafe if the horizontal acceleration of ground motion exceeds the yield acceleration (i.e., $a_y \geq k_y g$).

A major weakness of pseudostatic analysis is that it assumes the seismic force is constant and acts in one direction. This method tends to be over conservative in many situations. However, it has been shown that pseudostatic analysis is not conservative for soils that are susceptible to pore pressure build up or losing more than 15% of their peak shear strength during seismic shaking (Kramer 1996). Another limitation of pseudostatic analysis is that it is unable to predict the consequences of slope instability. The analysis
can be used to evaluate the stability of slopes, but it is unable to assess the movement of sliding mass after limit equilibrium is exceeded.

### 1.3 Permanent Displacement Analysis

For engineering practice, it is of particular interest to predict the deformation of natural and man-made slopes under seismic shaking. The predicted seismically-induced permanent deformation is a useful design index as it indicates the potential damage to the slope or an earth structure founded on the slope. In order to overcome the drawbacks of pseudostatic analysis, Newmark (1965) proposed a displacement-based procedure to evaluate the serviceability of slopes under earthquake shaking. Newmark analysis and its various derivatives are widely used in the current geotechnical engineering practice to evaluate earthquake-induced slope deformations.

#### 1.3.1 Newmark rigid-block analysis

The Newmark rigid-block method analogizes an earth mass sliding along a shear failure surface to a rigid block sliding over an inclined plane. This analogy is illustrated in Figure 1-2. The permanent slope deformation induced by earthquakes is estimated by the permanent displacement of the rigid block sliding along the inclined plane under a base acceleration. The block has a yield or critical acceleration \( a_y \). This yield acceleration can be determined through the aforementioned pseudostatic analysis.
Figure 1-2. Analogy between potential sliding mass and block on an inclined plane

Figure 1-3 presents the essential idea of Newmark rigid-block analysis. Figure 1-3(a) shows an acceleration time history. The sliding is initiated after the ground acceleration exceeds the critical acceleration as shown in Figure 1-3(a). The acceleration values of the record that exceed the yield acceleration are integrated to produce the relative velocity time history of the sliding mass as shown in Figure 1-3(b). The relative velocity time history is subsequently integrated as a function of time to obtain the cumulative displacement shown in Figure 1-3(c). Displacement continues and the permanent displacement accumulates until the inertial forces fall below the yield resistance, and the velocities of the sliding mass and the slip surface coincide. The Newmark integration routine has been implemented into readily available computer software (e.g., Jibson and Jibson 2003).

Newmark analysis is based on the following simplifying assumptions: (1) the soil behaves in a rigid (i.e., neglecting wave propagation in the soil), perfectly plastic manner; (2) displacement occurs along a single, well defined slip surface; and (3) the soil does not undergo degradation of strength and stiffness during shaking (Wartman et al. 2003). Therefore, Newmark-type analyses are more applicable to shallow landslides in more brittle materials rather than to deeper landslides in softer materials (Jibson 2007). As
shallow landslides are the predominant failure mechanism in earthquake-induced landslides, Newmark-type analysis has been widely used to estimate earthquake-induced slope deformations.

Figure 1-3. Illustration of Newmark analysis: (a) acceleration time history; (b) sliding velocity time history; (c) sliding displacement time history
1.3.2 Derivatives of Newmark analysis

Various modifications to the sliding block procedure have been proposed. The method proposed by Makdisi and Seed (1978) is a widely used analysis that accounts for internal deformations and wave propagation in sliding masses. This method consists of two steps. First, a one-dimensional dynamic analysis of the earth slope is performed by representing the slope as a multiple degree-of-freedom system. Based on the calculated horizontal acceleration histories at different degrees of freedom, an average acceleration history of the sliding mass is developed. This average acceleration is commonly referred to as the horizontal equivalent acceleration (HEA). Second, the resulting HEA is used as the acceleration time history in Newmark’s rigid-block analysis for estimating permanent displacements of a sliding mass above the potential failure surface. This method, in which the dynamic response analysis and displacement analysis are conducted independently, is referred to as decoupled analysis.

On the other hand, the dynamic response and sliding displacements of a slope are modeled in a simultaneous manner in the coupled analysis method which was proposed by Lin and Whitman (1983) and Rathje and Bray (1999, 2000). It models the sliding mass above a potential sliding surface as a multiple degree-of-freedom system. The sliding effects on the dynamic system are accounted for by introducing the sliding force at the sliding interface into the dynamic motions. As a result, the permanent displacements of the sliding mass are obtained directly by integrating the 1-D dynamic equations.

The Newmark method and its various derivatives discussed above have been validated against laboratory tests performed on shaking tables or centrifuge (e.g. Yegian
and Lahaf 1992; Wartman 1999, Wartman et al. 2003, 2005), and against case histories of earthquake-induced landslides (e.g., Pradel et al. 2005). After comparing Newmark predicted slope deformation with a well documented case history of earthquake-induced landslide movement during the Northridge earthquake \((M = 6.7)\), Pradel et al. (2005) concluded that Newmark-type sliding block analysis can result in reasonable estimates of seismic displacements for landslides using site-specific geotechnical analyses. However, uncertainties in groundwater level, ground motion characteristics, and shear strength of on-site soil should be considered when performing this type of analysis to estimate seismic slope performance (Pradel et al. 2005). Therefore, deformation predicted by Newmark-type analysis is merely a useful index of how a slope is likely to perform during seismic shaking (Jibson et al. 1998).

1.4 Stress-Deformation Analysis and Advanced Computational Methods

Using highly simplified geometry and material models, the analytical procedures described above are not physically precise and unable to reliably evaluate slope performance under complex geological conditions (e.g., topography, soil profile, and seismic shaking). To overcome these limitations, computational techniques such as the finite element method (FEM) (e.g., Finn et al. 1986; Crosta et al. 2005) and finite difference method (FDM) (e.g., Bathurst and Simac 1994) have recently been used to evaluate earthquake-induced slope deformations. Adaptive for complex geological conditions and capable of simulating complex soil behaviors, numerical simulation can provide insight into the mechanism of earthquake-induced slope deformations,
particularly the initiation and subsequent progressive movement of seismic slope failure. To simulate these phenomena with high fidelity, it is imperative to establish a numerical model with the capability of handling post-failure large deformation which is a key feature in earthquake-induced slope failures. The majority of current stress-deformation analysis of seismic slope deformation is limited to small deformation cases due to the limitation of classical FEM and FDM in solving large deformations.

Computational methods that are widely used to simulate large-deformation problems can be grouped into three categories: (1) grid-based continuum scale method improved by adaptive techniques, such as adaptive meshing and Arbitrary Lagrangian Eulerian (ALE); (2) particle-based micro-structure scale method such as DEM; and (3) mesh-free continuum scale method, such as the smoothed particle hydrodynamics (SPH) method and element free Galerkin Method (EFG).

1.4.1 Arbitrary Lagrangian Eulerian method (ALE)

Grid-based Lagrangian numerical methods formulated at a continuum scale, such as FEM and FDM, generally have difficulty in modeling large deformations because of a severely distorted and entangled mesh that may result in numerical error and failure of convergence. An adaptive remeshing procedure (e.g., Khoei and Lewis 1999) has been used to remediate this issue in the Lagrangian approach. This remeshing technique completely rezones the model, which is computationally expensive and technically formidable for three dimensional problems. Another alternative is the description of material movement in an Eulerian viewpoint in which the mesh is stationary and the material flows through the mesh. Eulerian approach is largely used in computational fluid
dynamics and is well suited for high-deformation flows. This method, however, becomes problematic in modeling free surface flow and tracing material response. The Level Set Method (LSM) (e.g., Osher and Fedkiw 2002) has recently been used to capture free-surface and interfaces of fluid-like materials on a fixed Eulerian grid. A surface function dependent on material velocity in LSM enables this method to produce more smooth and realistic solutions on free surface flows than conventional Eulerian methods.

Arbitrary Lagrangian-Eulerian (ALE) method (e.g., Belytschko et al. 2000; Donea et al. 1982; Huges et al. 1981) is a very effective alternative and is currently becoming a standard numerical approach for solving large-deformation problems. ALE allows the mesh to move independently from the motion of material. Although the mesh may move in an arbitrary fashion, it typically deforms with the material while incrementally smoothing the distorted mesh. The state variables are then mapped from the distorted mesh to the smoothed mesh. As a result, the ALE algorithm is usually performed in conjunction with the remeshing procedure. Figure 1-4 shows how ALE can improve the simulation of an impact of a metal bar by allowing mesh smoothing. Only a quarter of the metal bar impacting a rigid wall is shown. As shown in Figure 1-4 (b), the mesh at the impacting region is highly distorted in the updated Lagrangian method. It will yield inaccurate results and may lead to convergence difficulties. As compared to the updated Lagrangian method, the mesh at the impacting region in the ALE analysis remains well-shaped and will significantly improve solution convergence and quality. The remeshing in ALE is distinct from earlier adaptive remeshing techniques. As opposed to complete remeshing techniques, ALE does not alter the topology of the mesh, maintaining element type and connectivity. As a consequence, the application of ALE is usually limited to
geometries where the material motion is relatively predictable. To preserve high-quality mesh upon extreme deformation is still a challenging task for ALE. Another hurdle of ALE method is in modeling the path dependent behavior of plastic flow. As plastic behaviors are path or history dependent, the relative motion between mesh and material must be properly accounted for in the material constitutive equations, which imposes a significant burden on the application of ALE to advanced material modeling.

**Figure 1-4.** High speed impact of a metal bar: (a) A quarter model of a metal bar impacting a rigid wall; (b) Updated Lagrangian solution; (c) ALE solution
1.4.2 Discrete Element Method (DEM)

Another method widely used in geotechnical engineering to solve large-deformation problems is the discrete element method (DEM) (Cundall and Strack 1979) that approximates geomaterials at a micro-mechanical level. In this approach, the material (e.g., sand and rock) is modeled as a collection of discrete particles, each of which represents an individual grain of the material. The motion of each discrete particle is calculated separately using Newton’s Second law, and the interaction forces between each pair of the particles are obtained through simple mechanical contact models utilizing springs, dashpots and frictional sliders. DEM analysis, which can capture discontinuous and heterogeneous material behaviors from a micro-scale perspective, has been used to simulate a wide variety of problems involving granular materials. Successful application of DEM in landslides has been documented in literatures (e.g., Cleary & Campbell 1993; Campbell et al. 1995). However, as each particle in DEM represents a real solid grain, this method requires a tremendous number of discrete particles for a large-scale problem, and is hence usually restricted to small-scale and short-duration simulations.

1.4.3 Mesh-free Methods

Continuum-scale numerical methods that do not require a mesh (i.e., meshfree) are considered as more desirable and efficient for the simulation of large-deformation and large-scale problems such as seismically-induced slope failures in geotechnical engineering. The major difference between the classical methods (e.g. FEM and FDM) and meshfree methods is the absence of grids in the latter. One advantage of meshfree methods is the elimination of mesh reliance by constructing approximations entirely in
terms of nodes that have no topological connection among them. Meshfree methods can be viewed as an extension of classical grid-based methods to scattered node configurations without fixed connectivity. A common feature of all meshfree methods is the use of an influence domain as shown in Figure 1-5. Unlike in a grid-based method where variable approximations are dependent on the mesh, computational nodes in a meshfree method interact with each other on the basis of influence domain. Each node carries an influence domain throughout the entire calculation, and the nodes that fall in the influence domain are updated at each timestep. Therefore, large deformations can be easily handled in a meshfree method due to its adaptive nature.

One of the earliest meshfree methods is the Smoothed Particle Hydrodynamics (SPH) method developed by Lucy (1977) and Gingold and Monaghan (1977) for astrophysical applications. SPH utilizes the principle of inverse distance weighting to approximate field quantities in an influence domain. The weighting function in SPH serves the similar purpose as the shape function in FEM. The SPH method has been widely used to simulate free surface flows and multiphase flows (e.g., Monaghan 1994; Monaghan et al. 2003; Monaghan and Kocharyan 1995) and flow through porous media (e.g., Zhu et al. 1999). More recently, SPH method has been used to simulate the elastic response of solids (e.g., Libersky et al. 1993; Gray et al. 2001) and elasto-plastic behavior of geomaterials (e.g., Bui et al. 2008; Chen and Qiu 2012a, 2012b).
It was found by Swegle et al. (1995) that SPH usually suffers tension instability when particles are under tensile stress situations. Substantial improvements have been made to remediate this numerical instability, enabling this method to be applied more broadly. Libersky et al. (1997) successfully modeled high-velocity impact of nonlinear solids by using a conservative smoothing technique. Gray et al. (2001) proposed an artificial stress method to reduce numerical instability in the simulation of elastic large-deformation problems. More recently, total Lagrangian (Bonet and Kulasegaram 2001) and updated Lagrangian corrections (Vidal et al. 2007) in the SPH method have been proven stable and robust for solid mechanics. Figure 1-6 shows an application of ALE
and SPH to the simulation of a metal bar impacting a rigid wall. The comparison of solutions provided by SPH and ALE shows that SPH is capable of producing satisfactory results in modeling large deformation problems in solids. SPH method has also been applied to simulate geotechnical engineering problems involving large deformations in a phenomenological manner (e.g., Bui et al. 2008). However, very simple plastic constitutive models were used in these studies and the accuracy of SPH method in simulating complex soil behaviors remains unknown.

Other meshfree methods that have been widely used in solid mechanics include the element free Galerkin (EFG) (Belytschko et al. 1994) and Reproducing Kernel Particle Method (RKPM) (Liu et al. 1995). These methods are formulated based on weak forms of differential equations, and are not truly meshfree as a background mesh is required for numerical integrations. They are consequently more computationally expensive than SPH method.
Figure 1-6. High speed impact of a metal bar with contours showing the Von Mises stress in the bar: (a) ALE solution; (b) SPH solution
1.5 Dissertation Scope and Layout

The objective of this research is to develop and validate a new numerical model for simulating seismically-induced slope deformations. The numerical model presented in this study is intended to advance our capability in simulating dynamic slope failures and to provide insights into slope deformations caused by earthquakes. As a relatively mature and efficient meshfree method, the SPH method is utilized in this research to treat large deformations in geomaterials. The motivation of conducting this study and a brief review of current literature were presented previously in Chapter 1.

Chapter 2 presents the development of a 3-D SPH model and its application to the simulation of granular materials under large deformations. The developed model is validated against well-documented experiments of axisymmetric collapse of granular columns. This chapter is based on a paper published in the *International Journal of Geomechanics*, ASCE, 2012.

Chapter 3 presents the development, calibration, and validation of a SPH model for the simulation of seismically-induced slope deformations under undrained condition. The capability of SPH method in modeling complex material behaviors is examined in this chapter. An advanced constitutive model that combines the strain softening viscoplasticity and cyclic nonlinearity is implemented into the 3-D SPH code presented in Chapter 2. The developed SPH model accounts for the cyclic nonlinear behavior of soil, progressive reduction in shear strength, and strain-rate dependency of soil properties (e.g. stiffness and strength) during dynamic loading. The developed SPH model is then used to simulate a readily available and well-documented model slope test on a shaking table.
This chapter is based on a manuscript submitted to the *International Journal for Numerical and Analytical Methods in Geomechanics*.

Chapter 4 is an extension of the previous chapter, investigating the effects of several key parameters in the SPH model on seismically-induced slope deformations. These parameters encompass both SPH parameters and material properties, such as the particle spacing, radius of the SPH influence domain, peak and residual strengths, as well as fundamental frequency. The parametric study discusses (1) how the spatial parameters in the SPH method (e.g., particle spacing and influence domain) impact the accuracy of SPH simulations, and (2) how the soil properties (e.g., peak and residual strengths) influence earthquake-induced slope deformations. A non-reflecting boundary condition in the SPH method is also presented in this chapter.

Chapter 5 draws final conclusions and provides recommendations for future research on modeling dynamic behaviors of geomaterials under large deformations.
References


CHAPTER 2

NUMERICAL SIMULATIONS FOR LARGE DEFORMATION OF GRANULAR MATERIALS USING SMOOTHED PARTICLE HYDRODYNAMICS METHOD


KEY WORDS: cohesionless soils; granular media; landslides; numerical models; particles; plasticity;

Abstract

The application of Smoothed Particle Hydrodynamics (SPH) method to the simulation of granular materials under large deformation is presented. The Drucker-Prager constitutive model with non-associated flow rule is implemented into the SPH formulations to model the granular flow in a continuum framework. The developed model is validated by experiments of the collapse of 2-D granular columns as reported in literature. Simulations of the collapse of 3-D axisymmetric sand columns with various aspect ratios are also conducted. Numerical results of the granular flow pattern, final
runout distance, final deposit height, and non-deformed region are in good agreement with the experimental observations as reported in literature. It is suggested that despite being a continuum-scale model, the developed SPH model can be used to effectively simulate large deformation and dense flow of granular materials, and geomaterials in general if proper constitutive models are implemented. The developed model thus may find applications in various problems involving dense granular flow and large deformations such as landslides and debris flow.

2.1 Introduction

The study of how granular materials or bulk solids deform and flow under large deformation is of importance to many fields such as geotechnical engineering, pharmaceutical industry, mining industry, and various industrial processes involving fluidized beds. Computational simulation can provide insight into the mechanisms of granular flow and assistance to industrial design. Two numerical methods have been widely used to solve engineering problems involving granular materials, namely the discrete element method (DEM) at the particle scale and the finite element method (FEM) at the continuum scale. Although DEM can capture the micromechanics of granular materials and handle large deformations with ease due to its Lagrangian nature, it is limited to small-scale problems due to its computational cost since each particle in DEM represents a real solid particle/grain. On the other hand, FEM is efficient for large-scale problems and has been frequently used in geotechnical engineering. One problem associated with FEM, however, is the difficulty in dealing with large deformation which may lead to severe distortion of meshes, inaccurate results, and failure of convergence.
Advanced techniques such as adaptive remeshing (e.g., Khoei and Lewis 1999) have been used to remediate this problem. However, these remeshing techniques become problematic when complex constitutive models are employed (Bui et al. 2008). Overall, continuum-scale numerical methods that don’t require a mesh (i.e., meshfree) are considered as more desirable for the simulation of problems involving both large scale and large deformations. In recent years, several meshfree methods tracking materials by a set of particles instead of grids have been developed. A detailed discussion of various meshfree methods is presented by Liu and Liu (2004). Among these meshfree continuum-scale methods, the Smoothed Particle Hydrodynamics (SPH) method is a relatively mature one. Originally developed for astrophysical applications by Lucy (1977) and Gingold and Monaghan (1977), SPH method has been widely used to simulate free surface flows and multiphase flows (e.g., Monaghan 1994, Monaghan et al. 2003, Monaghan and Kocharyan 1995). More recently, SPH method has been used to simulate elastic response of solids (e.g., Libersky et al. 1993; Gray et al. 2001) and elasto-plastic behavior of geomaterials in two dimensional conditions (e.g., Bui et al. 2008).

Following the work of Bui et al. (2008) in applying SPH method to the simulation of granular materials in 2-D conditions, this paper presents a 3-D SPH model for granular materials and demonstrates its accuracy and stability for simulating granular materials under large deformation. The Drucker-Prager constitutive model with non-associated flow rule is implemented into the SPH formulations to model the granular flow in a continuum framework. The developed model is validated by experiments of the collapse of 2-D granular columns as reported in literature. Simulations of the collapse of 3-D axisymmetric sand columns with various aspect ratios are also conducted. Numerical
results of the granular flow pattern, final runout distance, final deposit height, and non-deformed region are compared with the experimental observations as reported in literature. In the following sections, the numerical implementation of the SPH model is first presented, followed by model validation and numerical simulation of granular flows in 3-D conditions.

2.2 Numerical Implementation

2.2.1 Governing and constitutive equations for soil

The formulations for the governing and constitutive equations of soil used in this study are based on the work of Bui et al. (2008). This section presents the essential idea of these formulations. The governing equations for solids consist of the linear momentum and continuity equations expressed as (Bui et al. 2008)

\[
\frac{Dv^\alpha}{Dt} = \frac{1}{\rho} \frac{\partial \sigma^{\alpha\beta}}{\partial x^\beta} + b^\alpha \quad (2-1)
\]

\[
\frac{D\rho}{Dt} = -\rho \frac{\partial v^\alpha}{\partial x^\alpha} \quad (2-2)
\]

Where \(\alpha\) and \(\beta\) denote the Cartesian components \(x, y, z\) with the Einstein convention applied to repeated indices. For convenience, Greek symbols \(\alpha\) and \(\beta\) are used to denote coordinate directions, whereas \(i\) and \(j\) are used to denote particle indices. In the above equations, \(\rho\) is the soil density; \(v\) is the velocity; \(\sigma\) stands for the total stress tensor; \(b\) is the acceleration caused by external force (i.e., gravity force in this study); and \(D/Dt\) is the material derivative. The stress tensor consists of an isotropic hydrostatic
pressure \( P \) and a deviatoric stress \( S^{\alpha\beta} \). In the traditional SPH formulations for fluid flow, the fluid hydrostatic pressure \( P \) is generally calculated based on the fluid density through an equation of state, assuming that the fluid is weakly compressible (e.g., Monaghan 1994, Morris et al. 1997). The deviatoric stress \( S^{\alpha\beta} \) is typically considered as purely viscous and depends on the fluid models (Bui et al. 2008). Previous applications of SPH method in solid dynamics (e.g., Gray et al. 2001) also employed various forms of equation of state and the Hooke’s law to calculate the mean principal stress and the deviatoric stress, respectively. As proposed by Bui et al (2008), such equation of state is not necessary in simulating soils because the mean principal stress can be calculated directly from soil constitutive relationships. In this paper, the stress tensor is calculated by the following elasto-plastic stress-strain relation proposed by Bui et al. (2008)

\[
\dot{\sigma}^{\alpha\beta} = 2G \dot{\varepsilon}^{\alpha\beta} + \left( K - \frac{2}{3} G \right) \delta^{\alpha\beta} \dot{\varepsilon}^{\alpha\beta} - \dot{\lambda} \left[ \frac{K - \frac{2}{3} G}{\sigma_{mn}} \frac{\partial g}{\partial \sigma^{mn}} \delta^{\alpha\beta} \sigma_{\alpha\beta} + 2G \frac{\partial g}{\partial \sigma^{\alpha\beta}} \right]
\]

(2-3)

where \( K \) and \( G \) are the bulk modulus and shear modulus, respectively; \( \dot{\lambda} \) is the rate of change in plastic multiplier which determines the magnitude of plastic strain; \( m \) and \( n \) are dummy indices; \( \delta \) is Kronecker’s delta; and \( \dot{\varepsilon} \) is the total strain rate tensor defined as

\[
\dot{\varepsilon}^{\alpha\beta} = \frac{1}{2} \left( \frac{\partial v^{\alpha}}{\partial x^{\beta}} + \frac{\partial v^{\beta}}{\partial x^{\alpha}} \right)
\]

(2-4)

The total strain rate tensor \( \dot{\varepsilon}^{\alpha\beta} \) can also be expressed as the sum of an elastic strain rate tensor \( \dot{\varepsilon}^{\alpha\beta}_e \) and a plastic strain rate tensor \( \dot{\varepsilon}^{\alpha\beta}_p \)

\[
\dot{\varepsilon}^{\alpha\beta} = \dot{\varepsilon}^{\alpha\beta}_e + \dot{\varepsilon}^{\alpha\beta}_p
\]

(2-5)
\begin{equation}
\dot{\varepsilon}_{\alpha\beta}^{e} = \frac{\dot{S}_{\alpha\beta}}{2G} + \frac{1-2\nu}{3E} \dot{\sigma}^{\gamma\gamma} \delta_{\alpha\beta} \tag{2-6}
\end{equation}

\begin{equation}
\dot{\varepsilon}_{p}^{\alpha\beta} = \dot{\lambda} \frac{\partial g}{\partial \sigma^{\alpha\beta}} \tag{2-7}
\end{equation}

where $\dot{S}_{\alpha\beta}$ is the deviatoric stress rate tensor; $\nu$ is the Poisson’s ratio; $E$ is the Young’s modulus; $G$ is the shear modulus; and $\dot{\sigma}^{\gamma\gamma}$ is the sum of the three normal stress components $\dot{\sigma}^{\gamma\gamma} = \dot{\sigma}^{xx} + \dot{\sigma}^{yy} + \dot{\sigma}^{zz}$. Similarly in Equation (2-3), $\dot{\varepsilon}_{p}^{\gamma\gamma} = \dot{\varepsilon}^{xx} + \dot{\varepsilon}^{yy} + \dot{\varepsilon}^{zz}$. The Drucker-Prager constitutive model with non-associated flow rule is used herein to model the granular materials. The plastic flow potential function $g$ can be expressed as

\begin{equation}
g = \sqrt{J_2} + \alpha_2 I_1 \tag{2-8}
\end{equation}

where $I_1$ and $J_2$ are the first and second stress invariants, respectively; and $\alpha_2$ is related to the dilation angle which is assumed to be zero for simplicity. This assumption precludes plastic volumetric strain after yielding. The ramification of this simplification on the model behavior is discussed later in this paper. The yield criteria can be expressed as

\begin{equation}
f(I_1, J_2) = \sqrt{J_2} + \alpha_1 I_1 - k_c \tag{2-9}
\end{equation}

where $\alpha_1$ and $k_c$ are Drucker-Prager’s constants and are related to the cohesion $c$ and friction angle $\phi$ of the Mohr-Coulomb’s failure criteria. In this paper, the cohesion is considered to be zero for granular materials; therefore, $c = 0$ and $k_c = 0$. The parameter $\alpha_1$ can be obtained through
\[ \alpha_i = \frac{2 \sin \phi}{\sqrt{3(3 - \sin \phi)}} \]  

(2-10a)

for 2-D plane strain conditions and

\[ \alpha_i = \frac{\tan \phi}{\sqrt{9 + 12 \tan^2 \phi}} \]  

(2-10b)

for 3-D conditions.

Substituting Equations (2-5) through (2-8) into (2-3) and considering contributions from rigid body rotation (Libersky et al., 1993), the final form of stress rate vs. strain rate for large deformation of granular materials can be obtained as

\[ \dot{\sigma}^{\alpha \beta} = 2G \dot{\varepsilon}^{\alpha \beta} + \left( K - \frac{2}{3}G \right) \dot{\varepsilon}^{\alpha \gamma} \delta^{\beta \gamma} - \dot{\lambda} \frac{G}{\sqrt{J_2}} S^{\alpha \beta} + \sigma^{\alpha \gamma} \dot{\omega}^{\beta \gamma} + \sigma^{\beta \gamma} \dot{\omega}^{\alpha \gamma} \]  

(2-11)

where \( \dot{\omega} \) is the rotation rate tensor defined as

\[ \dot{\omega}^{\alpha \beta} = \frac{1}{2} \left( \frac{\partial \dot{v}^\alpha}{\partial x^\beta} - \frac{\partial \dot{v}^\beta}{\partial x^\alpha} \right) \]  

(2-12)

The rate of change of plastic multiplier \( \dot{\lambda} \) can be calculated as

\[ \dot{\lambda} = \frac{3 \alpha_i K \dot{\varepsilon}^{\gamma \gamma}}{G} + \frac{S^{\alpha \beta} \dot{\varepsilon}^{\alpha \beta}}{\sqrt{J_2}} \]  

(2-13)

### 2.2.2 SPH formulations

In SPH simulations, the computational domain is discretized into a finite number of particles, each representing a certain volume and mass of the material (fluid or solid) and carrying simulation parameters such as velocity, acceleration, density, and pressure/stress.
Therefore, SPH method is a continuum-scale numerical method. The material properties
\( f(x) \) at any point \( x \) in the simulation domain are then calculated according to an
interpolation process over its neighboring particles that are within an influence domain \( \Omega \)
through

\[
f(x) = \int_{\Omega} f(x') W(x - x', h) dx'
\]

(2-14)

where \( W \) is the kernel or smoothing function, which is essentially a weighting function.

\[ f(x) = \frac{\int_{\Omega} f(x') W(x - x', h) dx'}{\int_{\Omega} W(x - x', h) dx'} = 1 \]

(2-15)

\[ \text{Figure 2-1. Particle approximation based on kernel function } W \text{ in influence domain } \Omega \text{ with radius } kh \]

Figure 2-1 presents the essential idea of this interpolation process. The kernel function \( W \)
satisfies the unity condition

\[ \int_{\Omega} W(x - x', h) dx' = 1 \]
the delta condition

$$\lim_{h \to 0} W(x - x', h) = \delta(x - x')$$  \hspace{1cm} (2-16)$$

and the compact condition

$$W(x - x', h) = 0 \text{ for } |x - x'| \geq \kappa h$$  \hspace{1cm} (2-17)$$

where $h$ is the smoothing length of the smoothing function $W$; $\kappa$ is a constant; and $\kappa h$ defines the effective (non-zero) length of the smoothing function (i.e., the radius of the influence domain). The continuous integral representation of the field variable $f(x)$ in Equation (2-14) can be further approximated by the summation over the neighboring particles as

$$f(x) = \sum_{i=1}^{N} f(x_i) W(x - x_i, h) V_i = \sum_{i=1}^{N} f(x_i) W(x - x_i, h) \frac{m_i}{\rho_i}$$  \hspace{1cm} (2-18)$$

where $V_i$, $m_i$ and $\rho_i$ are the volume, mass and density of particle $i$, respectively; and $N$ is the number of particles within the influence domain. The spatial derivative of field variable $f(x)$ can be approximated through the differential operations on the kernel function

$$\frac{\partial f(x)}{\partial x} = \sum_{i=1}^{N} \frac{m_i}{\rho_i} f(x_i) \frac{\partial W(x - x_i, h)}{\partial x_i}$$  \hspace{1cm} (2-19)$$

It is indicated by Equations (2-14) through (2-19) that the efficiency and accuracy of SPH simulations depend on the kernel function. Among various kernel functions presented in literature, the cubic spline kernel function proposed by Monaghan and Lattanzio (1985) has been proven to be accurate and efficient. This kernel function is employed in the
current study and can be expressed as

\[
W(r, h) = \alpha_D \times \begin{cases} 
1 - 1.5q^2 + 0.75q^3 & 0 \leq q \leq 1 \\
0.25 \times (2 - q)^3 & 1 \leq q \leq 2 \\
0 & q \geq 2 
\end{cases} \tag{2-20}
\]

where \( \alpha_D = 10/(7\pi h^2) \) and \( 1/(\pi h^2) \) for 2-D and 3-D conditions, respectively. The normalized distance \( q = |r_{ij}|/h \), where \( r_{ij} \) is the distance vector between particles \( i \) and \( j \). The kernel function drops to zero for \( |r_{ij}| \geq 2h \), implying that the influence domain in Equation (2-20) has a radius of \( 2h \) (see Figure 2-1).

As an important parameter in SPH method, the smoothing length \( h \) has significant impacts on the overall numerical behavior (e.g., accuracy and efficiency). SPH particles interact with each other only if they are within the influence domain; otherwise, they are independent from each other (see Figure 2-1). Therefore, larger smoothing length (i.e., larger influence domain) generally results in a smoother or more continuous behavior as the SPH particles are more interdependent with each other; whereas smaller smoothing length (i.e., smaller influence domain) generally yields more discrete behaviors as the SPH particles are more independent from each other. Choosing an appropriate smoothing length for this study thus warrants careful consideration of two criteria. First, SPH method requires a sufficient number of particles within the influence domain to yield accurate and reliable solutions due to the interpolation process and continuum nature of the method. Second, the discrete behavior of granular materials under large deformation (e.g., granular flow) needs to be preserved. Therefore, a balance between the two criteria needs to be achieved. According to Liu and Liu (2004), a smoothing length of \( h = 1.2 \times \Delta x \), where \( \Delta x \)
is the initial particle spacing, can support a sufficient and necessary number of particles that are within the influence domain to provide satisfactory results. This value is used throughout this paper.

Various partial differential equations have been implemented into the SPH framework to simulate complex behaviors of fluids and solids. For example, SPH method has been widely used to solve the Navier-Stokes equations in simulating free surface and multiphase flows (e.g., Monaghan 1994, Monaghan et al. 2003, Monaghan and Kocharyan 1995) and flow through porous media (Zhu et al. 1999). More recently, SPH method has been used to simulate the elastic response of solids (e.g., Libersky et al. 1993; Gray et al. 2001) and elasto-plastic behavior of geomaterials in 2-D conditions (e.g., Bui et al. 2008). By using a kernel function, SPH method interpolates the stress tensor, velocity vector and density of SPH particles to generate smooth continuous interpolation fields. The resultant interpolation fields can be substituted into the governing equations such as the linear momentum (e.g., Equation (2-1)), continuity (e.g., Equation (2-2)), and stress rate equations (e.g., Equation (2-11)) to produce a set of partial differential equations (PDE) that can be solved using explicit time integration. A detailed procedure on the numerical implementation of elasto-plastic constitutive equations into the SPH framework was presented by Bui et al. (2008), and hence is not presented herein.

Boundary conditions in SPH method are an important and yet challenging topic due to the Lagrangian nature of the method. Two types of boundary methods are widely used: the ghost nodes method (Libersky et al. 1993) and the repulsive force method (Monaghan 1994). In the ghost nodes method, layers of artificial particles are fixed on boundaries, carrying the same stress and density as the real SPH particles within the influence domain.
The velocities of these artificial particles are commonly determined by the no-penetration and non-slip conditions. Field variables, except the position variables, carried by the artificial particles are involved in the SPH computations. Within the geotechnical engineering context, this method has been used in 2D simulation of fluid flow through porous media (e.g., Morris et al. 1997 and Zhu et al. 1999) and fluid-solid interaction (e.g., Potapov et al. 2001). For 3-D simulations, this approach requires a large number of boundary particles, resulting in a lower computational efficiency. The repulsive force method proposed by Monaghan (1994) involves placing a line of ghost particles along the boundary to produce a highly repulsive distance-dependent force (e.g., Lennard-Jones force) to the SPH particles near the boundary, thus preventing these particles from unphysically penetrating the boundary. The main advantage of this method is its apparent computational efficiency; therefore, this method is used in the current study to model the boundary conditions between granular materials and solid boundaries.

2.2.3 Time integration

As shown in the previous sections, the governing and constitutive equations are PDEs with respect to time and space. These PDEs can be solved numerically using many readily available techniques, such as the predictor-corrector, leap-frog (LF) and Runge-Kutta (RK) methods. As a second-order accurate numerical scheme, LF method is efficient and is adopted herein to solve the PDEs. In this method, the particle velocities and positions are offset by half a time step as shown in the following equations

\[
v_{n+1/2} = v_{n-1/2} + \Delta t \cdot \left( \frac{Dv}{Dt} \right)_n
\]  

(2-21a)
\[ \rho_{n+1/2} = \rho_{n-1/2} + \Delta t \cdot \left( \frac{D\rho}{Dt} \right)_n \]  
\[ \sigma_{n+1/2} = \sigma_{n-1/2} + \Delta t \cdot \left( \frac{D\sigma}{Dt} \right)_n \]  
\[ r_{n+1} = r_n + \Delta t \cdot v_{n+1/2} \]

At the start of each time step, the density, velocity and stress of each particle are predicted at half a time step, whereas the position is calculated at the integer time step. Stress tensor is substituted into the Drucker-Prager yield criteria (i.e., Equation (2-9)) to evaluate the yielding status for each particle at the end of each time step. The time step is controlled by a combination of the Courant condition and a viscous condition (Monaghan 1992). The speed of sound, \(c\) is given by (Liu 2003)

\[ c = \sqrt{\frac{4G}{3\rho} + \frac{K}{\rho}} \]

The explicit integration of stress tensor may lead to a stress state that lies outside of the yield surface when yielding occurs. This phenomenon violates the principals of plasticity since stresses after yielding must stay within or on the yield surface. This numerical error can be reduced using any nonlinear iteration method such as the implicit Runge-Kutta method. However, these implicit methods require the assembly of coefficient matrix and are generally not efficient for models consisting of large number of particles. Therefore, an explicit approach is adopted in this study. The explicit correction method proposed by Chen (1990) and recently applied in SPH method by Bui et al (2008) is employed to scale the stresses exceeding the yield strength of granular materials back to the yield surface.
2.3 Model Validation

Prior to applying the developed SPH model to simulate large deformation of granular materials in 3-D conditions, two benchmark studies on the collapse of 2-D granular columns are conducted to validate the developed model. The first is related to the laboratory test conducted by Bui et al. (2008). In this test aluminum bars with length of 50 mm and density of 2650 Kg/m$^3$ were placed into a rectangular box with a dimension of 200 mm $\times$ 100 mm $\times$ 50 mm. The bars were aligned along the width of the box. The plane strain condition was largely enforced as the bars were constrained from rotation geometrically (i.e., only translational motion) due to the same size in bar length and width of the box (i.e., 50 mm). Excellent agreement was observed between the experiments and 2-D SPH simulations conducted by Bui et al. (2008) with regard to the final slope profile and the undisturbed region within the stockpile after the collapse. The results produced in this benchmark simulation are presented in Appendix A. The SPH model developed in this paper is also validated against the experiments conducted by Bui et al. (2008) and similarly excellent agreement is also observed; therefore, details of this validation are not presented herein.
Detailed experimental investigation on the granular flow of the collapse of 3-D granular columns was conducted by Lube et al. (2005). Instead of using aluminum bars to enforce the plain strain condition, Lube et al. (2005) used real granular materials (e.g., sand, sugar, and rice). Figure 2-2 presents the test setup. A column of granular materials is confined in a rectangular box. The column has an initial height of $h_i$ and an initial basal length of $d_i$. Upon releasing the gate instantaneously via a pulley system, the granular column collapses resulting in a final profile with a height of $h_f$ and a runout distance of $d_{\infty}$. Lube et al. (2005) observed an undisturbed region remaining at its initial height after the collapse if the initial aspect ratio $a = h_i/d_i$ is less than 1.8. The final runout distance $d_{\infty}$ for the case of $a < 1.8$ can be estimated through the following non-dimensional equation

$$\frac{d_{\infty} - d_i}{d_i} = 1.6a$$  \hspace{1cm} (2-23)
Figure 2-3. Comparison of SPH simulation for unidirectional collapse in 3-D and 2-D conditions: (a) initial configuration; (b) isometric view of final profile in 3-D simulation; (c) side view of final profile in 3-D simulation; (d) final profile in 2-D simulation

Figure 2-3 shows the SPH simulation of a test conducted by Lube et al. (2005). As presented in Figure 2-3(a), the initial height, basal length, and width of the sand column are 4.0 cm, 8.0 cm, and 20 cm, respectively. Based on a parametric study, the initial SPH particle spacing is chosen to be 2 mm for the optimal computational accuracy and efficiency. Lube et al. (2005) reported an angle of repose of 30° for the sand and this angle is used as the friction angle in the SPH model. Parametric study indicates that the final profile and runout distance are not sensitive to the values of bulk modulus and Poisson’s ratio of the sand. Therefore, typical values for loose sands are used: namely $K = 5$ MPa and $\nu = 0.3$. SPH simulations were conducted in 3-D condition as shown in Figures 2-3(b) and 2-3(c) and 2-D condition as shown in Figure 2-3(d). As indicated in Figure 2-3, the
final runout distances for 3-D and 2-D conditions are approximately 13.9 cm and 17.5 cm, respectively. The runout distance from the experiment can be estimated as 14.4 cm based on Equation (2-23). Thus, the errors for 3-D and 2-D simulations are 3.5% and 21.5%, respectively. An undisturbed region is observed remaining at its initial height after the collapse as shown in Figures 2-3(b), 2-3(c) and 2-3(d), which is consistent with the experimental observations. The good agreement between 3-D simulation and experiment with regard to the final runout distance indicates that the developed SPH model is capable of capturing granular flow during the collapse of sand columns. The larger error in 2-D simulation indicates that despite that the column is confined in a rectangular box and referred as 2-D granular column by Lube et al. (2005), the stress and deformation conditions within the sand flow during the collapse are of three-dimensional nature. The application of 2-D model is, however, acceptable when the plain strain condition is enforced in experiments such as the one conducted by Bui et al. (2008) as discussed previously. The benchmark studies discussed above indicate that the developed SPH model, despite being a continuum-scale method, is capable of simulating dense granular flow during the collapse of granular column with satisfactory accuracy.

2.4 Numerical Simulation for 3-D Granular Flows

Numerical simulations are performed to simulate the experiments conducted by Lube et al. (2004) on the collapse of initially vertical 3-D axisymmetric columns of various granular materials including dry grains of salt, sand, sugar, and rice. It was observed in the experiments that the flow behavior of granular columns depends on the aspect ratio \( a = h_i / r_i \), where \( h_i \) and \( r_i \) are the initial height and radius of the granular column,
respectively. Interestingly, it was observed that the final extent (i.e., runout distance) of the
deposit and the time for emplacement are essentially quantitatively independent of any
friction coefficient. The frictional effects between individual grains on the granular flow
only play a role in the last instant of the flow, as it comes to an abrupt halt. After
conducting a series of collapse experiments with different grains, Lube et al (2004)
concluded that the final runout distance $r_e$ and final height of the collapsed column $h_e$
can be expressed solely in terms of the aspect ratio and initial radius, suggesting that the
runout distance is independent of material properties including friction angle, bulk
modulus, and Poisson’s ratio.

The developed SPH model is utilized to simulate these experiments. As previously
discussed, the simulated runout distance and final profile are practically independent of the
bulk modulus and Poisson’s ratio used for the granular materials. This is consistent with
the experimental observation of Lube et al. (2004). Therefore, typical values of bulk
modulus and Poisson’s ratio of loose sands, namely 5 MPa and 0.3 are used, respectively.
However, the friction angle plays an important role in the numerical predictions of the final
runout distance and deposit profile. Lube et al. (2004) reported an angle of repose of $30^\circ$
for the sand used in their experiments and this angle is used as the friction angle in the
current SPH simulations. This choice of friction angle is generally consistent with its
physical meaning in the continuum framework.

Granular flows during the collapse of soil columns with various aspect ratios ranging
from 0.225 to 20 are simulated using the developed SPH model and the results are
presented herein. The initial radius $r_i$ for all the simulations is 0.1 m. SPH particles with a
spacing of 5 mm are initially placed on a regular cubic lattice that is larger than the column.
After the particles reach equilibrium under their self weight, those that are outside of the column region are deleted and the simulation of column collapse begins. This approach may lead to an imperfect representation of the curved surface along the column, particularly at low resolutions. However, this approach guarantees equal spacing and uniform mass distribution among SPH particles during the initialization. More than 25,000 SPH particles are used to represent the sand column in the simulations conducted. The final runout distance and deposit profile, as well as the non-deformed region of deposits, are investigated and compared with the experimental observations and the results are presented below.

2.4.1 Granular flow patterns

Lube et al. (2004) observed that depending on the aspect ratio $a$, different flow patterns exist during the collapses. For granular columns with low aspect ratios, only the materials in outer region of the initial column are in motion and a static axisymmetric region remains undisturbed; whereas for columns with high aspect ratios, the entire upper free surface starts to flow immediately after the constraint vertical walls are removed. The threshold value of aspect ratio separating the two flow patterns is found to be 1.7 based on a series of experiments.
Figure 2-4. Comparison of SPH simulation and experiment with \( a = 0.55 \): (a) initial configuration of sand column; (b) simulated final profile after collapse; (c) side view of simulated final profile; (d) experimental final profile (Lube et al. 2004)

The flow with aspect ratio \( a < 1.7 \) can be further subdivided into two sub-regimes. For \( 0 < a < 0.74 \), it was observed that a circular undisturbed area at the top of the column remains at the initial height. Figure 2-4 presents the comparison of SPH simulation and experimental results for a sand column with \( a = 0.55 \). The simulated final profile after the collapse is shown in Figures 2-4(b) and 2-4(c). A circular static inner region remaining at the initial column height is clearly demonstrated. Good qualitative agreement is observed between the simulated final profile and the experimental results in Figure 2-4.
Lube et al. (2004) observed that for $0.74 < a < 1.7$, the outer region of the column starts to flow after removal of vertical boundaries, and the inner region remains unmoved during the early stage of the collapse. In contrast to the case of $a < 0.74$, the inner circular region is eroded gradually by outer moving particles, leaving a sharp cone at the center. Figure 2-5 presents the comparison of SPH simulation and experimental results for a sand column with $a = 0.9$. The cone-shaped final profile is reproduced by the SPH simulation. The experiments of Lube et al. (2004) indicate that the tip of the final cone remains at the initial height. In the SPH simulations, however, the conical region is eroded in the last stage of the granular flow, leading to a 15% decrease in the final height as shown in Figure 2-5(c). Several factors may have contributed to this discrepancy. First, a zero dilation angle of the sand is assumed in the constitutive model for simplicity, which leads to a weaker soil in the SPH model. Second, SPH formulation is based on an interpolation process as discussed earlier, which requires an adequate number of SPH particles to produce smooth and reliable results. Therefore, the limited number of SPH particles that exist in the cone tip region may not be adequate to produce accurate and reliable results. Last, the cone tip in the experiment is very sharp as indicated in Figure 2-5(d). The behavior of very few sand particles in the cone tip region may not be adequately modeled using the continuum-scale approach presented in this paper.
Figure 2-6 presents the comparison of SPH simulation and experimental results for a sand column with $a = 2.75$. For sand columns with $a > 1.7$, Lube et al. (2004) observed that the entire upper surface of the column starts to flow and the height of column decreases immediately. At the base of the column, a flow front is developed and accelerates to a high velocity, leaving a discontinuity band at the column foot as shown in Figures 2-6(b) and 2-6(c). After the flow front at the base of column stops, the upper surface is still in motion until all the particles reach a new equilibrium, forming a steep and smooth cone at the final stage as shown in Figures 2-6(d), 2-6(e) and 2-6(f). These complex flow patterns are accurately reproduced by the SPH simulations as shown in Figure 2-6.
Figure 2-6. Comparison between SPH simulation and experiment with $a = 2.75$: (a) initial configuration of sand column; (b) simulated flow during collapse; (c) experimental flow during collapse (Lube et al. 2004); (d) simulated final profile after collapse; (e) side view of simulated final profile; (f) experimental final profile (Lube et al. 2004)
2.4.2 Final runout distance

The experiments of Lube et al. (2004 and 2005) and SPH simulations conducted in this study indicate that the final runout distance of the collapsed granular column primarily depends on the aspect ratio $a$. Based on a regression analysis on the experiment results, Lube et al. (2004) provided the following best-fit equations to link $r_\infty$, $r_i$ and $a$

\[
\frac{r_\infty - r_i}{r_i} = 1.24a \quad \text{for} \quad a < 1.7 \quad (2-24a)
\]

\[
\frac{r_\infty - r_i}{r_i} = 1.6a^{0.5} \quad \text{for} \quad a > 1.7 \quad (2-24b)
\]

For the large number of experiments conducted with various granular materials including sand, salt, rice, couscous, and sugar, Lube et al. (2004) reported that the regression
coefficients for Equation (2-24a) and (2-24b) are 0.975 and 0.988, respectively. Figure 2-7 presents a comparison of the normalized final runout distance \((r_\infty - r_i)/r_i\) between SPH simulations and Equation (2-24). An excellent agreement is observed.

![Figure 2-7](image)

**Figure 2-7.** Comparison of the normalized final runout distance between SPH simulations and Equation (2-24).

### 2.4.3 Final deposit height

The final height of granular deposit after collapse is dependent on the aspect ratio and initial height. Lube et al. (2004) observed that the final cone height remains at its initial height for \(a < 1.7\); therefore, the normalized final deposit height, \(h_\infty / r_i\), can be expressed as

\[
\frac{h_\infty}{r_i} = a
\]

\[
\frac{h_\infty}{r_i} = 0.88 a^{1/6}
\]

![Figure 2-8](image)

**Figure 2-8.** Comparison of normalized final deposit height between SPH simulations and experiments from Lube et al. (2004)
\[ \frac{h_e}{r_i} = a \quad \text{for} \quad 0 \leq a < 1.7 \quad \text{(2-25a)} \]

While for \( 1.7 < a < 10 \), the normalized final deposit height is represented by the following power-law relationship

\[ \frac{h_e}{r_i} = 0.88 a^{1/6} \quad \text{for} \quad 1.7 < a < 10 \quad \text{(2-25b)} \]

Figure 2-8 presents a comparison of the normalized final deposit height between SPH simulations and Equation (2-25). An excellent agreement is observed between SPH simulation and experiments for the ranges of \( a < 0.74 \) and \( a > 1.7 \). For the range of \( 0.74 < a < 1.7 \), however, the numerically predicted final deposit height is lower than the values predicted by Equation (2-25a). The factors that may have contributed to this discrepancy are discussed earlier in this paper.

### 2.4.4 Non-deformed region

As discussed previously, a static non-deformed region is observed after the column collapse for aspect ratios less than 1.7. In the experiments by Lube et al. (2004), this non-deformed region was identified by dying the initial granular column and subsequent visual observation of the dyed granular materials after the collapse. In the SPH simulations, the non-deformed region is determined by SPH particles with zero accumulative equivalent plastic strain after the collapse. The contour of plastic strain for a sand column with \( a = 0.9 \) is shown in Figure 2-9. The SPH particles highlighted in deep red represent largest plastic deformation, whereas the particles in deep blue indicate zero plastic strain. By comparing Figure 2-9 with experimental observations, it is found that plastic
deformation (i.e., disturbance) actually occurred in the regions that were considered undisturbed in the experiments. For comparison, the contour of plastic strain for sand column with \( a = 2.75 \) is shown in Figure 2-10. As indicated in the contour plots, the overall plastic strain is larger and penetrates deeper in Figure 2-10 as compared to Figure 2-9, with the undisturbed region shrinking towards the center of the sand deposit.

![Figure 2-9](image1.png) (a)

![Figure 2-9](image2.png) (b)

**Figure 2-9.** Accumulative equivalent plastic strain after collapse of sand column with \( a = 0.9 \): (a) side view; (b) isometric view

Lube et al. (2004) provided the following equation for the radius at the base of non-deformed region \( r_a \)

\[
 r_a = 9.6 - 0.585 h_i
\]  \hspace{1cm} (2-26)

for \( a < 1.7 \). Figure 2-11 presents a comparison of \( r_a \) between SPH simulations and Equation (2-26). As indicated in Figure 2-11, the simulated values of \( r_a \) as determined by
the equivalent plastic strain are in good agreement with the experimental results, but are generally smaller than those estimated by Equation (2-26). A possible explanation is that the experimental identification of non-deformed region through visual observation of dyed sand particles is not as accurate as numerical simulation through the computed accumulative plastic strain. Very small deformation or particle rearrangement may not be visually identifiable in the experiments, resulting in a larger non-deformed region observed than simulated.

\[ a = 2.75 \] (a) side view; (b) isometric view

**Figure 2-10.** Accumulative equivalent plastic strain after collapse of soil column with
54

Figure 2-11. Comparison of the radius at the base of non-deformed region between SPH simulations and experiments from Lube et al. (2004) for \( a < 1.7 \)

### 2.5 Conclusions

The application of Smoothed Particle Hydrodynamics (SPH) method to the simulation of granular materials under large deformation is presented. The Drucker-Prager constitutive model with non-associated flow rule is implemented into the SPH formulations to model the granular flow in a continuum framework. The developed model is validated by the experiments of the collapse of 2-D granular columns as reported in literature. Excellent agreement is observed between the model simulations and experimental observations with regard to the final runout distance after the collapse. Simulations of the collapse of 3-D axisymmetric sand columns with various aspect ratios are also conducted. Numerical results of the granular flow pattern, final runout distance,
final deposit height, and non-deformed region are in good agreement with the experimental observations as reported in literature. The simulated non-deformed region within a sand column during the collapse is smaller than the experimental observation. The likely cause is that the numerical simulations are able to capture small deformations through a direct evaluation of the plastic strain inside the sand column; whereas small deformation or particle rearrangement may not be visually identifiable in the experiments.

This study indicates that despite being a continuum-scale model, the developed SPH model can be used to effectively simulate large deformation and dense flow of granular materials, and geomaterials in general if proper constitutive models are implemented. This is due to the fact that SPH method is a particle-based meshfree numerical method. In SPH method, the field variables of a SPH particle are calculated through an interpolation process over its neighboring particles within its influence domain that is determined by the smoothing length. Larger smoothing length generally results in a smoother or more continuous behavior as the SPH particles are more interdependent with each other; whereas smaller smoothing length generally yields more discrete behaviors as the SPH particles are more independent from each other. By choosing an appropriate smoothing length, the developed SPH model is capable of preserving the discrete characteristics of dense granular flow for granular materials under large deformation. Therefore, the developed model may find applications in various problems involving dense granular flow such as landslides and debris flow.
References


CHAPTER 3

SIMULATION OF EARTHQUAKE-INDUCED SLOPE
DEFORMATION USING SPH METHOD


KEY WORDS: cyclic nonlinear behavior; elasto-plasticity; viscoplasticity; large deformation; meshless method; progressive failure; seismic loading

Abstract

This paper presents the development, calibration, and validation of a smoothed particle hydrodynamics (SPH) model for the simulation of seismically induced slope deformation under undrained condition. A constitutive model that combines the isotropic strain softening viscoplasticity and the modified Kondner and Zelasko rule is implemented into SPH formulations. The developed SPH model accounts for the effects of wave propagation in the sliding mass, cyclic nonlinear behavior of soil, and progressive reduction in shear strength during sliding, which are not explicitly considered in various Newmark-type analyses widely used in the current research and practice in geotechnical earthquake engineering. Soil parameters needed for the developed model
can be calibrated using typical laboratory shear strength tests, and experimental or empirical shear modulus reduction curve and damping curve. The strain-rate effects on soil strength are considered. The developed SPH model is validated against a readily available and well documented model slope test on a shaking table. The model simulated slope failure mode, acceleration response spectra, and slope deformations are in excellent agreement with the experimental data. It is thus suggested that the developed SPH model may be utilized to reliably simulate earthquake-induced slope deformations. This paper also indicates that if implemented with appropriate constitutive models, SPH method can be used to model large-deformation problems with high fidelity.

3.1 Introduction

Landslides are one of the most damaging hazards induced by earthquakes, posing significant threat to human lives and infrastructure systems (e.g., roads, pipelines, utility lifelines). Therefore, predicting the location and shaking conditions needed to trigger landslides is a key element in regional seismic hazard assessment (Jibson et al. 1998). For engineering practice, it is of particular interest to predict the deformation of natural and man-made slopes under seismic shaking. The Newmark method (Newmark 1965) has been widely used to evaluate the seismic performance of slopes. Newmark analyses are based on the following simplifying assumptions: (1) the soil behaves in a rigid manner (i.e., neglecting wave propagation in the sliding mass); (2) displacement occurs along a single, well defined slip surface; and (3) the soil does not undergo degradation of strength and stiffness during seismic shaking (Wartman et al. 2003). Various improvements to the Newmark method have been proposed. The method has been extended to nonrigid slide
mass by evaluating the horizontal equivalent acceleration (HEA) of flexible slide mass, accounting for the effect of wave propagation inside the mass (e.g., Seed and Martin 1966; Makdisi and Seed 1978; Kramer and Smith 1997; Bray and Rathje 1998; Rathje and Bray 1999, 2000). Lin and Whitman (1983) and Yegian et al. (1991) have developed simplified probabilistic design charts for estimating seismically induced deformations by combining the sliding block analysis and probabilistic analysis.

Earthquake-induced slope deformation has also been investigated using advanced elasto-plastic constitutive models and numerical techniques such as the finite element method (FEM) (e.g., Finn et al. 1986) and finite difference method (FDM) (e.g., Bathurst and Simac 1994). Numerical models are capable of providing insight into the initiation and subsequent progressive movement of seismic slope deformations. To simulate this phenomenon with high fidelity, however, two critical issues need to be addressed. The first is related to the ability of numerical models to handle post-failure large deformation, which is a key feature in earthquake-induced slope deformations. As a grid-based numerical method, FEM generally has difficulty in modeling large deformations of geomaterials. Advanced techniques such as adaptive remeshing (e.g., Khoei and Lewis 1999) and Arbitrary Lagrangian-Eulerian (ALE) method (e.g. Hughes et al. 1981) have been used to remediate this problem. However, these remeshing techniques become problematic and difficult in treating highly distorted mesh (Belytschko et al. 2000; Liu and Liu 2004). As opposed to grid-based methods, meshfree methods that track materials by a set of particles instead of grids have been developed and utilized to simulate engineering problems involving large deformations. A detailed discussion of various meshfree methods is presented by Liu and Liu (2004). Among these methods the
Smoothed Particle Hydrodynamics (SPH) method is a relatively mature one. Originally developed for astrophysical applications by Lucy (1977) and Gingold and Monaghan (1977), SPH method has been widely used to simulate free surface flows and multiphase flows (e.g., Monaghan 1994; Monaghan et al. 2003; Monaghan and Kocharyan 1995) and flow through porous media (Zhu et al. 1999). More recently, SPH method has been used to simulate the elastic response of solids (e.g., Libersky et al. 1993; Gray et al. 2001) and elasto-plastic behavior of geomaterials (e.g., Bui et al. 2008; Chen and Qiu 2012a, 2012b). Bui and Fukgawa (2011) used SPH method to simulate saturated soils, in which the solid phase is approximated using the effective stress concept while the fluid phase is assumed to be hydrostatic.

The second critical issue is related to the ability of numerical models to adequately account for the complex constitutive behaviors of geomaterials under seismic loading. These behaviors include the strain softening behavior under large deformations, the cyclic nonlinear behavior manifested as the shear modulus reduction and hysteresis damping, and the strain rate effects on soil strength and stiffness. The strain softening behavior of soils under static loading can be modeled using the isotropic softening Drucker-Prager (D-P) model (e.g., Prevost and Hoeg 1975; Chen and Mizuno 1990). The conventional D-P model utilizes a single yield surface to describe soil behavior at and beyond yielding; within the yield surface, however, soil behaves as elastic regardless of the magnitude of shear strain. Therefore, this model is not suitable for seismic conditions. To overcome this limitation, elasto-plastic models with multiple yield surfaces allowing nonlinear hysteresis behavior between them and bounding surface models (e.g., Dafalias and Popov 1976; Prevost 1979; Bardet 1986; Manzari and Dafalias 1997) have been
developed. These models, however, require more input parameters than what is typical in current geotechnical engineering practice.

Masing’s rule (Masing 1926) has been widely used to model the nonlinear behavior of soil under cyclic loading (Kramer 1996). Although Masing’s rule is mathematically robust and is appropriate for metals, it has limitations in realistically modeling the cyclic behavior of soils. For example, cyclic degradation of the initial tangent modulus is not considered (Matasovic and Vucetic 1993) and damping is generally overestimated by Masing’s rule (Ishihara 1982). Additional rules have been proposed by Pyke (1979) to address the cyclic behavior of soils under irregular cyclic loading and by Vucetic (1990) to address the cyclic behavior of clays. Approaches that based on the concept of an initial backbone curve and subsequent degraded backbone curves in conjunction with Masing’s rule or Pyke’s rule have been utilized to model clays (e.g., Idriss et al. 1978; Vucetic 1990) and liquefiable sands (e.g., Finn et al. 1977; Lee and Finn 1978; Pyke and Beikae 1993; Matasovic and Vucetic 1993).

Recently, elasto-plastic constitutive models have been utilized in conjunction with the extended Masing’s rules (Pyke 1979) to model soil cyclic behavior. Jung (2009) expanded a 1-D cyclic hyperbolic model based on the stress-strain relationship proposed by Hardin and Drnevich (1972) and Pyke’s rule (1979) into 3-D domain by introducing an octahedral shear stress and shear strain. The octahedral shear stress at failure was then defined based on the D-P yield criteria. The developed model was implemented into ABAQUS to simulate the seismic earth pressure on retaining structures, accounting for soil nonlinear behavior during cyclic loading (Jung 2009).
In addition to the aforementioned nonlinearities, soil properties (e.g., stiffness and strength) exhibit strong dependence on the rate of loading as observed by many researchers (e.g., Isenhower and Stokoe 1981; Biscontin and Pestana 2001). This rate dependence is generally modeled using viscoplasticity (or rate-dependent plasticity) models. Among these models, the formulation based on Perzyna’s theory (Perzyna 1966) has been widely used. The Perzyna-type viscoplasticity treats viscous behavior using a time-rate flow rule; therefore, it can be incorporated into conventional rate-independent plasticity models to simulate rate-dependent behaviors (e.g., Tong and Tuan, 2007).

The objective of this paper is to develop and validate a SPH model capable of simulating earthquake-induced slope deformations. A constitutive model that combines the isotropic strain softening viscoplasticity and modified Kondner and Zelasko rule (Matasovic and Vucetic 1993) is implemented into a 3-D SPH model developed by Chen and Qiu (2012a). The developed model accounts for the effects of rate-dependent shear strength, cyclic nonlinear behavior, and strain softening behavior during sliding, as well as the dynamic response of the slope and sliding mass. The developed model is validated against a readily available and well-documented model slope test on a shaking table. The simulated slope failure mode, acceleration response spectra, and slope deformations at various locations along the model slope are in excellent agreement with the experimental data. In the following sections, the model development is first presented, followed by model calibration and model validation.
3.2 SPH Formulations

In SPH method, the computational domain is discretized into a finite number of points, which are referred to as SPH particles. Each particle represents a certain volume and mass of the material simulated, carrying all field variables, such as velocities, stresses/pressures, and densities. A kernel function is used to interpolate the stress tensor, velocity vector, and density of SPH particles into smooth and continuous interpolation fields. The resultant interpolation fields can be substituted into governing equations such as the linear momentum, continuity, and stress rate equations to produce a set of ordinary differential equations (ODE) with respect only to time. These ODEs can then be solved using explicit time integration (Liu and Liu 2004). Due to its Lagrangian and adaptive nature, SPH method is advantageous for applications involving large deformations and complex geometry. Various governing equations have been implemented into the SPH framework to simulate complex behaviors of fluids and solids. A detailed description of SPH method and its formulations for geomaterials can be found in literature (e.g., Liu and Liu 2004; Monaghan 2005; Bui et al. 2008) and hence is not presented herein. Instead, the following section provides an overview of the SPH formulations for finite strain in solids.

In SPH method, field variables at any point in the simulation domain are calculated according to an interpolation process over its neighboring particles that are within an influence domain $\Omega$ as shown in Figure 3-1. A function $A(x)$ and its derivatives $\nabla A(x)$ at particle $i$ can be calculated as
\[ A(x_i) = \sum_{j=1}^{N} A(x_j)W(x_i - x_j, h)V_j = \sum_{j=1}^{N} A(x_j)W(x_i - x_j, h)\frac{m_j}{\rho_j} \quad (3-1) \]

\[ \nabla A(x_i) = \sum_{j=1}^{N} \frac{m_j}{\rho_j} A(x_j) \frac{\partial W(x_i - x_j, h)}{\partial x_j} \quad (3-2) \]

where \( V, m \) and \( \rho \) represent volume, mass, and density, respectively; \( N \) is the number of particles within \( \Omega \); \( W \) is an interpolation function which is called kernel or smoothing function. Subscript \( j \) represents a neighboring particle of particle \( i \). As shown in Figure 3-1, \( h \) is the smoothing length of the smoothing function \( W \), \( \kappa \) is a constant, and \( \kappa h \) defines the effective (non-zero) length of the smoothing function (i.e., the radius of \( \Omega \)). The kernel function \( W \) has to satisfy the unity, delta, and compact conditions (e.g., Liu and Liu 2004). The cubic spline kernel function proposed by Monaghan and Lattanzio (1985) has been proven to be accurate and efficient and is employed in our current study. The function is given by

\[ W(r, h) = \alpha_D \times \begin{cases} 1 - 1.5q^2 + 0.75q^3 & 0 \leq q \leq 1 \\ 0.25 \times (2 - q)^3 & 1 \leq q \leq 2 \\ 0 & q \geq 2 \end{cases} \quad (3-3) \]

where \( \alpha_D = \frac{10}{(7\pi h^2)} \) and \( \frac{1}{(\pi h^3)} \) for 2-D and 3-D conditions, respectively; and normalized distance \( q = \frac{|r_{ij}|}{h} \), where \( r_{ij} \) is the distance vector between particles \( i \) and \( j \). The kernel function drops to zero when \( \frac{|r_{ij}|}{h} \geq 2h \), implying that the influence domain in Equation (3-3) has a radius of \( 2h \) (see Figure 3-1). The smoothing length \( h \) has significant impacts on the overall accuracy and efficiency of the simulations. SPH particles communicate with each other only if they are within each other’s influence.
domain; otherwise, they are independent from each other (see Figure 3-1). Therefore, if $h$ is too small, the number of particles in the influence domain to interact with a given particle may be insufficient, resulting in low accuracy. On the other hand, excessively large smoothing length may smooth out particle properties in the computational domain, leading to poor solutions. Liu and Liu (2004) concluded that a value of $h$ equal to $1.2 \times \Delta x$, where $\Delta x$ is the particle initial spacing, can produce results with acceptable accuracy and efficiency for problems in fluid dynamics. The SPH simulations of dense granular flow conducted in the previous chapter found that $h = 1.2 \times \Delta x$ is adequate to produce accurate solutions for geomaterials; therefore, this value is adopted in the current work.

**Figure 3-1.** Particle approximation based on kernel function $W$ in influence domain $\Omega$ with radius $\kappa h$. 
The mass and momentum conservation equations in continuum mechanics can be expressed as

\[
\frac{D\rho}{Dt} = -\rho \frac{\partial v^\alpha}{\partial x^\alpha} \quad (3-4)
\]

\[
\frac{Dv^\alpha}{Dt} = \frac{1}{\rho} \frac{\partial \sigma^{\alpha\beta}}{\partial x^\beta} + g^\alpha \quad (3-5)
\]

where \( v \) is the velocity; \( g \) is the acceleration caused by external force (i.e., gravity force in this study); and \( \frac{D}{Dt} \) is the material derivative. For clarity throughout this paper, Greek symbols \( \alpha \) and \( \beta \) are used as superscripts to denote spatial coordinates (i.e., \( x \), \( y \), and \( z \)) and represent components of a tensor; subscripts \( i \) and \( j \) are reserved for particle indices. For example, \( \sigma^{\alpha\beta} \) denotes a component of the total stress tensor \( \sigma \); \( v_i \) denotes the velocity of particle \( i \). The Cauchy stress tensor \( \sigma \) consists of an isotropic mean stress \( p \) and a deviatoric stress tensor \( s \). In this paper, the stress is positive in tension and negative in compression. Applying the kernel interpolation in Equations (3-1) and (3-2) to Equation (3-4) (i.e., mass conservation) leads to

\[
\frac{D\rho_i}{Dt} = \sum_{j=1}^{N} m_j (v_i^\alpha - v_j^\alpha) \frac{\partial W_{ij}}{\partial x_i^\alpha} \quad (3-6)
\]

Equation (3-6) implies that the change in soil density is caused by the change of void volume in soil, which is consistent with the assumption of incompressible solid grains. Similarly, the momentum equation can be approximated in SPH as
\[
\frac{Dv^\alpha_i}{Dt} = \sum_{j=1}^{N} m_j \left( \frac{\sigma^{\alpha\beta}_{ij} + \sigma^{\alpha\beta}_{ji}}{\rho_j^2} - \Pi_{ij} \delta_{ij} \right) \frac{\partial W_{ij}}{\partial x^\alpha_i} + g^\alpha_i \quad (3-7)
\]

where \( \delta \) is Kronecker’s delta; and \( \Pi_{ij} \) is an artificial viscosity used to mitigate potential numerical instability caused by shocks in dynamic computation (Monaghan 1992). For Equation (3-7), mathematical manipulation is performed to enforce the inter-particle momentum to be equal and opposite according to Liu and Liu (2004). The artificial viscosity term is expressed as (Monaghan 1992)

\[
\Pi_{ij} = \begin{cases} 
-\alpha_{\|} c_{S_{ij}} \phi_{ij} / \rho_{ij} & v_{ij} \cdot r_{ij} < 0 \\
0 & v_{ij} \cdot r_{ij} \geq 0 
\end{cases} 
\quad (3-8)
\]

where \( \phi_{ij} = \frac{h_{ij} \cdot v_{ij} \cdot r_{ij}}{r_{ij}^2 + (0.1 \cdot h_{ij})^2} \), \( c_{S_{ij}} = \frac{c_{S_i} + c_{S_j}}{2} \), \( \rho_{ij} = \frac{\rho_i + \rho_j}{2} \), \( h_{ij} = \frac{h_i + h_j}{2} \), \( r_{ij} = r_i - r_j \), \( v_{ij} = v_i - v_j \) and \( c_S \) is the speed of sound (i.e., compression wave) which is given by Liu and Liu (2004)

\[
c_S = \sqrt{\frac{4G + K}{3\rho}} \quad (3-9)
\]

where \( G \) and \( K \) are the shear modulus and bulk modulus of soil, respectively. The parameter \( \alpha_{\|} \) is a constant and its value varies in different applications. Large values of \( \alpha_{\|} \) may lead to excessive energy dissipation; whereas small values of \( \alpha_{\|} \) may not be adequate to suppress numerical instability. Liu and Liu (2004) suggested that an optimum value of \( \alpha_{\|} \) for fluid dynamics is 0.1. Libersky et al. (1993) and Gray et al. (2001)
selected \( \alpha_{l} = 2.5 \) for their elastic dynamic problems in solids. The damping ratio induced by artificial viscosity with \( \alpha_{l} = 2.5 \) in a harmonic vibration test performed by Gray et al. (2001) was found to be less than 1.5\%. For problems involving cohesive soils, \( \alpha_{l} = 1.0 \) is found to be effective in eliminating numerical oscillations (Bui et al. 2008) and this value is used in the current study. The numerical damping induced by this artificial viscosity is considered to have a minimal effect on the overall dynamic response of the model slope to be discussed later, where the inherent material damping is significantly higher.

Standard SPH formulations can exhibit tensile instability in simulating elastic solids under tension. This instability was first studied by Swegle et al. (1995). It is not noticeable in cohesionless granular materials (Bui et al. 2008; Chen and Qiu 2012a), but becomes severe in cohesive soils as investigated in detail by Bui et al. (2008). One commonly used method to mitigate tensile instability is the conservative smoothing technique proposed by Randles and Libsersky (1996), which essentially involves averaging field variables over neighboring particles in the influence domain. This method was shown to be successful in mitigating tensile instability in a bar subjected to a tension shock wave (Libersky et al. 1997). However, the conservative smoothing technique results in excessive energy dissipation (i.e., numerical damping) based on a benchmark simulation of harmonic vibration of an elastic cantilever. For the details of this numerical experiment on the cantilever beam, please refer to Appendix B. Therefore, this technique is not suitable for the simulation of seismic performance of slopes where realistic energy dissipation (i.e., material and geometric damping) is needed. Monaghan (2000) introduced an artificial repulsive force correction term in SPH momentum equations to
mitigate tensile instability. This correction was applied to simulate the tensile and harmonic response of elastic solids, satisfactory results with slight artificial energy dissipation was obtained (Gray et al. 2001). Recent applications of this method in cohesive soils can be found in the work of Bui et al. (2008). The artificial repulsive force correction is therefore adopted in our current study and the corrected standard momentum equation is

\[
\frac{Dv_i^\alpha}{Dt} = \sum_{j=1}^{N} \left( \frac{\sigma_i^{\alpha\beta}}{\rho_i^2} + \frac{\sigma_j^{\alpha\beta}}{\rho_j^2} - \Pi_j \delta^{\alpha\beta} + \Gamma_{ij} \cdot (R_i^{\alpha\beta} + R_j^{\alpha\beta}) \right) \frac{\partial W_{ij}}{\partial x_i^\beta} + g_i^\alpha
\]  

(3-10)

where the coefficient \( \Gamma_{ij} \) is defined as

\[
\Gamma_{ij} = \frac{W_{ij}}{W(\Delta x, h)}
\]  

(3-11)

The exponent, \( \vartheta = W(0, h)/W(\Delta x, h) \), is a parameter to control the amplitude of repulsive force. The value of \( \vartheta \) should be carefully chosen in order to effectively eliminate instabilities (Gray et al. 2001). Based on our parametric study, \( \vartheta = 2.5 \) provides satisfactory results and is utilized in our current application. Parameters \( R_i^{\alpha\beta} \) and \( R_j^{\alpha\beta} \) are components of the artificial repulsive stress tensors at particles \( i \) and \( j \), respectively.

The artificial stress correction term in Equation (3-10) is invoked when any of the principal stresses at a SPH particle exhibits tension (i.e. positive principal stress) based on a procedure discussed in details by Gray et al. (2001).

### 3.3 Constitutive Model
A constitutive model, which combines the isotropic strain softening viscoplasticity and the modified Kondner and Zelasko rule (Matasovic and Vucetic 1993), is implemented to model the complex soil behaviors involved in earthquake-induced slope deformations under undrained condition. The model accounts for the strain-softening behavior under large deformation, rate-dependent shear strength, and cyclic nonlinear behavior. The implementation of the model in SPH is discussed in detail in the following sections.

3.3.1 Finite strain consideration

A commonly used approach to treat finite strain problems is to calculate a field variable for the next timestep based on its rate of change during the current timestep. It is well known that a rigid body rotation induces changes in Cauchy stress in a fixed coordinate system (Belytschko et al. 2000), even though from a constitutive point of view the stress state of the material is unchanged. Consequently, a stress rate that is invariant with respect to rigid body rotation, also referred to as the objective rate in constitutive equations, must be employed to describe the material response. Among various frame-invariant stress rates, the Jaumann rate of Cauchy stress and the Jaumann rate of Kirchhoff stress are widely used. As indicated by Crisfield (1997), Cauchy stress and Kirchhoff stress result in equivalent formulations if the volume change is small and negligible. In the current SPH model, the elastic volumetric deformation is considerably smaller than the plastic deformation and hence negligible; the plastic volumetric deformation is ignored by using a zero dilation angle as an undrained condition is
assumed. Therefore, the Jaumann rate of Cauchy stress is adopted in the current study for its simplicity and is given by

$$\dot{\sigma}_{ij}^{ab} = \dot{\sigma}_{ij}^{ab} - \dot{\omega}^a \cdot \sigma^k \sigma_{ij}^{k \beta} - \sigma^a \cdot \dot{\omega}^{\beta k}$$

(3-12)

where the embellishment ‘·’ designates the derivative with respect to time; the Jaumann rate is designated by the subscript ‘\(J\)’; superscript \(k\) is a dummy index, and \(\dot{\omega}\) is the spin rate tensor

$$\dot{\omega}_{ij}^{ab} = \frac{1}{2} \left( \frac{\partial v_i^\alpha}{\partial x_j^\beta} - \frac{\partial v_j^\beta}{\partial x_i^\alpha} \right)$$

(3-13)

Application of SPH interpolation to the spin rate tensor leads to (Liu and Liu 2004)

$$\dot{\omega}_{ij}^{ab} = \frac{1}{2} \left( \sum_{j=1}^{N} \frac{m_j}{\rho_j} (v_i^a - v_j^a) \frac{\partial W_{ij}}{\partial x_i^\alpha} - \sum_{j=1}^{N} \frac{m_j}{\rho_j} (v_j^\beta - v_i^\beta) \frac{\partial W_{ij}}{\partial x_i^\alpha} \right)$$

(3-14)

Rearrangement of Equation (3-12) leads to the stress rate in a finite strain framework (Belytschko et al. 2000)

$$\frac{d\sigma_{ij}^{ab}}{dt} = \dot{\sigma}_{ij}^{ab} + \dot{\omega}^a \cdot \sigma^k \sigma_{ij}^{k \beta} + \sigma^a \cdot \dot{\omega}^{\beta k}$$

(3-15)

where \(\frac{d}{dt}\) is the total derivative. As shown in Equation (3-15), the stress rate consists of two parts in finite strain analysis. The first is resulted from the material response due to deformation, which is represented by the frame-invariant Jaumann rate. The second is associated with rigid body rotation, represented by the last two terms on the right hand side (RHS) of Equation (3-15).

The formulations presented above are based on the framework of objective stress rate, using stress and strain measures in the current configuration. The integration of rate
equations is contingent upon the infinitesimal strain assumption between two adjacent configurations. This assumption is acceptable as long as the time increments are small enough, which will be discussed later. Although these formulations have limitations, they are widely used due to their simplicity (e.g., Randles and Libersky 1996; Gray et al. 2001; Bui et al. 2008) and hence are adopted in the current study.

### 3.3.2 Isotropic strain softening viscoplasticity

In our current work, the total deformation rate is decomposed into an elastic part and a plastic part by using an additive decomposition

\[
\dot{\varepsilon}^{ab} = \dot{\varepsilon}_e^{ab} + \dot{\varepsilon}_p^{ab}
\]  

(3-16)

where \(\dot{\varepsilon}^{ab}\), \(\dot{\varepsilon}_e^{ab}\), and \(\dot{\varepsilon}_p^{ab}\) denote components of the total rate of deformation tensor \(\dot{\varepsilon}\), elastic rate of deformation tensor \(\dot{\varepsilon}_e\), and plastic rate of deformation tensor \(\dot{\varepsilon}_p\), respectively; and \(\dot{\varepsilon}^{ab}\) can be expressed as

\[
\dot{\varepsilon}^{ab} = \frac{1}{2} \left( \frac{\partial v^a}{\partial x^b} + \frac{\partial v^b}{\partial x^a} \right)
\]  

(3-17)

For simplicity and to be consistent with typical terminologies in the infinitesimal constitutive model that is utilized in this paper, \(\dot{\varepsilon}_e\) and \(\dot{\varepsilon}_p\) are simply referred to as the elastic strain rate and plastic strain rate, respectively. In SPH formulation, the total deformation rate is expressed as (Liu and Liu 2004)

\[
\dot{\varepsilon}^{ab} = \frac{1}{2} \left[ \sum_{j=1}^{N} \frac{m_j}{\rho_j} (v_j^a - v_i^a) \frac{\partial W_{ij}}{\partial x_i^b} + \sum_{j=1}^{N} \frac{m_j}{\rho_j} (v_j^b - v_i^b) \frac{\partial W_{ij}}{\partial x_i^a} \right]
\]  

(3-18)
By applying the plastic flow rule, the plastic strain rate is defined as

\[ \dot{\varepsilon}_p^{ab} = \dot{\lambda} \frac{\partial \psi}{\partial \sigma^{ab}} \] (3-19)

where \( \psi \) is the plastic flow potential and \( \dot{\lambda} \) is the plastic multiplier which is determined by a viscoplastic function in the proposed viscoplasticity model. The generic rate-type form of stress-strain relationship for an elasto-plastic material is given by

\[ \sigma_{ij}^{ab} = 2G\dot{e}_{ij}^{ab} + K\dot{\varepsilon}_v \delta_{ij}^{ab} - \dot{\lambda} \left[ 2G \frac{\partial \psi}{\partial \sigma^{ab}} + \left( K - \frac{2}{3} G \right) \frac{\partial \psi}{\partial \sigma^{ab}} \delta_{kl}^{ab} \delta_{ij}^{kl} \right] \] (3-20)

where \( \dot{e}_{ij}^{ab} \) is a component of the deviatoric deformation rate tensor \( \dot{\varepsilon} \), \( \dot{\varepsilon}_v \) is the volumetric strain rate (i.e., \( \dot{\varepsilon}_v = \dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy} + \dot{\varepsilon}_{zz} \)), and \( \dot{\varepsilon}_{ij}^{ab} = \dot{\varepsilon}_v \delta_{ij}^{ab} - \frac{1}{3} \dot{\varepsilon}_v \delta_{ij}^{ab} \). The superscripts \( k \) and \( l \) are dummy indices.

Although strain softening flow rules violate Drucker’s postulate from classical plasticity point of view, the concept of yield surface evolution can be utilized to model the strain softening behavior of soil (Prevost and Hoeg 1975). As shown in Figure 3-2, the yield surface can evolve with plastic strain to match strain softening behaviors observed from laboratory tests. The D-P yield function with an evolving yield surface is defined as (Neto et al. 2008)

\[ f(\sigma, c) = \sqrt{J_2(s)} + \eta \cdot I_1 - \zeta \cdot c(\bar{\varepsilon}_p) \] (3-21)

where \( J_2 \) is the second stress invariant; \( s \) is the deviatoric stress tensor; \( I_1 = \sigma^{ij} \) is the first stress invariant; and \( c(\bar{\varepsilon}_p) \) is the cohesion which is treated as a function of the accumulated equivalent plastic strain \( \bar{\varepsilon}_p \). Variable \( J_2 \) can be calculated as
\[ J_2 = \frac{1}{2} s^{\alpha\beta} \cdot s^{\beta\alpha} \]  

(3-22)

Individual components of \( s \) can be calculated as

\[ s^{\alpha\beta} = \sigma^{\alpha\beta} - \frac{1}{3} I_i \varepsilon^{\alpha\beta} \]  

(3-23)

For simplicity, the associated isotropic strain softening is adopted in this study and the rate of \( \varepsilon_p \) is defined as (Neto et al. 2008)

\[ \dot{\varepsilon}_p = \frac{d\varepsilon_p}{dt} = -\dot{\lambda} \frac{\partial f}{\partial c} = \dot{\lambda} \cdot \varepsilon \]  

(3-24)

The constants \( \eta \) and \( \zeta \) in Equation (3-21) are determined as (Chen and Mizuno 1990)

\[ \eta = \frac{\tan \phi}{\sqrt{9 + 12 \tan^2 \phi}} \]  

(3-25a)

\[ \zeta = \frac{3}{\sqrt{9 + 12 \tan^2 \phi}} \]  

(3-25b)

for 2-D plane strain conditions and

\[ \eta = \frac{2 \sin \phi}{\sqrt{3(3 - \sin \phi)}} \]  

(3-26a)

\[ \zeta = \frac{6 \cos \phi}{\sqrt{3(3 - \sin \phi)}} \]  

(3-26b)

for 3-D conditions in which the D-P failure surface coincides with the outer cone of the Mohr-Coulomb hexagonal surface, where \( \phi \) is the Mohr-Coulomb friction angle.
The general non-associated flow rule is adopted herein. Accordingly, the flow potential in Equation (3-19) is

\[ \psi = \sqrt{J_2(s)} + \bar{\eta} \cdot I_1 \]  

(3-27)

where \( \bar{\eta} \) is the D-P dilation angle. Substitution of Equation (3-27) into Equation (3-20), after rearrangements, leads to the rate-type form of stress-strain relationship for the D-P material

\[ \dot{\sigma}^{\alpha\beta} = 2G\dot{\varepsilon}^{\alpha\beta} + K\dot{\varepsilon}, \delta^{\alpha\beta} - \dot{\lambda} \left( \frac{G}{\sqrt{J_2(s)}} s^{\alpha\beta} + 3\bar{\eta}K\delta^{\alpha\beta} \right) \]  

(3-28)

In conventional rate-independent models, \( \dot{\lambda} \) can be determined by the consistency condition; whereas in rate-dependent models, \( \dot{\lambda} \) is an explicit function of the equivalent
stress \( q(\sigma) \) and the yield stress \( \sigma_y \). The Peirce viscoplastic model (Peirce et al. 1984; Peric and Owen 1992), which is a close variant of the Perzyna model (Perzyna 1966) and shows better convergence than other rate-dependent formulations, is adopted in this paper. In the Peirce model, \( \dot{\lambda} \) is defined as

\[
\dot{\lambda} = \begin{cases} 
\frac{1}{\mu_1} \left( \left( \frac{q(\sigma)}{\sigma_y} \right)^{1/\mu_2} - 1 \right) f(\sigma, c) \geq 0 \\
0 & f(\sigma, c) < 0 
\end{cases}
\]  

(3-29)

Rearrangement of the above equation leads to the following viscoplastic consistency condition (Neto et al, 2008)

\[
F = \frac{1}{\mu_1 \lambda + 1} \left( \frac{1}{\mu_1 \lambda + 1} \right)^{\mu_2} - \sigma_y = 0
\]  

(3-30)

where \( \mu_1 \) and \( \mu_2 \) are two viscous parameters that need to be calibrated for a given application; the equivalent stress \( q(\sigma) \) in D-P criterion is defined as

\[
q(\sigma) = \sqrt{J_2(s)} + \eta \cdot I_1
\]  

(3-31)

and the yield stress \( \sigma_y \) is:

\[
\sigma_y = \zeta \cdot c(\bar{\varepsilon}_p)
\]  

(3-32)

A two-step implicit scheme, referred to as the return mapping technique, is widely used in FEM simulations and is adopted in the current study to update stress and strain. In this procedure, the implicit time integration of stress consists of two steps: elastic predictor
and plastic corrector. The elastic trial solution at next time step, $\sigma_{n+1,\text{trial}}$, is first obtained by using the elastic constitutive relation. The calculated trial stress is then examined against the yield function in Equation (3-21). If the trial stress lies within or on the yield surface, it is accepted as a satisfactory solution. Otherwise, the plastic corrector step involving the Newton-Raphson iteration procedure is performed to return the trial stress to the yield surface so that the Peirce viscoplastic consistency condition (i.e., Equation (3-30)) is satisfied. The method does not necessitate a fixed yield surface, making it suitable for modeling strain softening behavior.

The implicit time integration requires the rate equations to be expressed in incremental forms. The increment of Jaumann stress rate is expressed as

$$\dot{\sigma}_{j} \cdot \Delta t = \sigma_{n+1}^{\alpha \beta} - \sigma_{n}^{\alpha \beta}$$  \hspace{1cm} (3-33)

where $\sigma_{n+1}^{\alpha \beta}$ is a component of the Cauchy stress tensor $\sigma$ attributable to material response as shown in Equation (3-15). Throughout this paper, subscripts $n$ and $n+1$ represent the current time step and next time step, respectively. Substitution of Equation (3-33) into Equation (3-28) yields the following incremental form

$$\sigma_{n+1}^{\alpha \beta} = \sigma_{n}^{\alpha \beta} + 2G \Delta e_{i}^{\alpha \beta} + K \Delta \epsilon_{i}^{\alpha \beta} - \Delta \lambda \left[ \frac{G}{\sqrt{J_2(s)}} \delta_{i}^{\alpha \beta} + 3\mu K \delta_{i}^{\alpha \beta} \right]$$  \hspace{1cm} (3-34)

According to Neto et al. (2008), Equation (3-34) can be further decomposed into
\[ s_{n+1}^{\alpha \beta} = s_{n}^{\alpha \beta} + 2G\Delta e^{\alpha \beta} - \Delta \lambda \cdot \frac{G}{\sqrt{J_2(s_{n+1})}} \cdot s_{n+1}^{\alpha \beta} \]

\[ p_{n+1} = p_n + K\Delta \varepsilon_v - \Delta \lambda \cdot 3K \cdot \bar{\eta} \]

(3-35)

where \( p = \frac{1}{3} I_1 \) is the mean stress. Equation (3-35) can be simplified using the following equation (Neto et al. 2008)

\[ \frac{s_{n+1}^{\alpha \beta}}{J_2(s_{n+1})} = \frac{s_{n+1,\text{trial}}^{\alpha \beta}}{J_2(s_{n+1,\text{trial}})} \]

(3-36)

Consequently, Equation (3-35) is reduced to

\[ s_{n+1}^{\alpha \beta} = s_{n+1,\text{trial}}^{\alpha \beta} + \Delta \lambda \cdot \frac{G}{\sqrt{J_2(s_{n+1,\text{trial}})}} \cdot s_{n+1,\text{trial}}^{\alpha \beta} \]

\[ p_{n+1} = p_{n+1,\text{trial}} - \Delta \lambda \cdot 3K \cdot \bar{\eta} \]

(3-37)

where \( s_{n+1,\text{trial}} \) is the trial deviatoric stress tensor at the next time step with its component given by

\[ s_{n+1,\text{trial}}^{\alpha \beta} = s_{n}^{\alpha \beta} + 2G\Delta e^{\alpha \beta} \]

(3-38a)

and \( p_{n+1,\text{trial}} \) is the trial mean stress at the next time step given by

\[ p_{n+1,\text{trial}} = p_n + K\Delta \varepsilon_v \]

(3-38b)

Substitution of Equation (3-37) into (3-30) results in
\[ q(\sigma_{n+1}) = \sqrt{J_2(s_{n+1,\text{trial}})} - G\Delta\lambda + 3\eta \cdot \left( p_{n+1,\text{trial}} - \Delta\lambda \cdot 3K \cdot \eta \right) \]  
(3-39)

and

\[ \sigma^{n+1}_p = \zeta \cdot c(\overline{\epsilon}^{p}_{n+1}) \]  
(3-40)

By substituting Equation (3-39) and (3-40) into the incremental form of Equation (3-30), the final return mapping equation for rate-dependent D-P model is obtained

\[ F_{n+1}(\Delta\lambda) = \left[ \sqrt{J_2(s_{n+1,\text{trial}})} - G\Delta\lambda + 3\eta \cdot \left( p_{n+1,\text{trial}} - \Delta\lambda \cdot 3K \cdot \eta \right) \right] \cdot \left( \frac{\Delta\lambda}{\mu_1 \Delta\lambda + \Delta t} \right)^{\mu_2} - \zeta \cdot c(\overline{\epsilon}^{p}_{n+1}) = 0 \]  
(3-41)

where the accumulated equivalent plastic strain at the next time step, \( \overline{\epsilon}^{p}_{n+1} \), is calculated by integrating Equation (3-24)

\[ \overline{\epsilon}^{p}_{n+1} = \overline{\epsilon}^{p}_n + \zeta \cdot \Delta\lambda \]  
(3-42)

As the viscoplastic parameters \( \mu_1 \to 0 \) and/or \( \mu_2 \to 0 \), Equation (3-41) reproduces the conventional rate-independent D-P model. As such, the proposed viscoplastic model accounts for rate-dependent strength by expanding rate-independent yield surfaces according to strain rates. Equation (3-41) is a function of one independent variable \( \Delta\lambda \) that can be solved by the standard Newton-Raphson scheme. Linearization of Equation (3-41) in terms of \( \Delta\lambda \) results in

\[ F_{n+1}(\Delta\lambda) = F_{n+1}(\Delta\lambda_0) + \frac{dF_{n+1}}{d\Delta\lambda} \Delta(\Delta\lambda) \]  
(3-43)

where
\[
\frac{dF_{n+1}}{d\Delta \lambda} = \left(-G - 9\eta \cdot \overline{\eta} \cdot K\right) \left(\frac{\Delta t}{\mu_1 \Delta \lambda_0 + \Delta t}\right)^{\mu_2} - q(\sigma_{n+1}) \frac{\mu_1 \cdot \delta \cdot \Delta t^{\mu_2}}{\mu_2 \Delta \lambda_0 + \Delta t} - \zeta \frac{dc}{d\Delta \lambda} - \sigma \cdot \delta \cdot \Delta t^{\mu_2} + H_p \tag{3-44}
\]

In the above equations, \(q(\sigma_{n+1})\) is defined in Equation (3-39); \(\Delta \lambda_0\) and \(\Delta \lambda_0\) are the updated trial and initial trial for \(\Delta \lambda\), respectively; \(\Delta(\Delta \lambda)\) is the increment of \(\Delta \lambda\). Term \(\zeta \frac{dc}{d\Delta \lambda}\) defines the evolution of yield surface and is usually obtained from laboratory tests, such as unconfined uniaxial compression or tension and pure shear tests. Expanding \(\zeta \frac{dc}{d\Delta \lambda}\) by the chain rule, and using Equation (3-24), yields

\[
\frac{dc}{d\Delta \lambda} = \frac{d(\zeta \cdot c)}{d\overline{\varepsilon}_p} \cdot \frac{d\overline{\varepsilon}_p}{d\Delta \lambda} = \frac{d(\zeta \cdot c)}{d\overline{\varepsilon}_p} = H_p \tag{3-45}
\]

where \(H_p\) is the plastic modulus that is determined by the slope of the stress vs. plastic strain curve from a laboratory test. Substituting Equation (3-44) into Equation (3-43) and with some rearrangements, the Newton-Raphson solution for \(\Delta \lambda\) is given by

\[
\Delta \lambda_{n+1} = \Delta \lambda_0 + \frac{F_{n+1}(\Delta \lambda_0)}{\left(-G - 9\eta \cdot \overline{\eta} \cdot K\right) \left(\frac{\Delta t}{\mu_1 \Delta \lambda_0 + \Delta t}\right)^{\mu_2} - q(\sigma_{n+1}) \frac{\mu_1 \cdot \delta \cdot \Delta t^{\mu_2}}{\mu_2 \Delta \lambda_0 + \Delta t} - \sigma \cdot \delta \cdot \Delta t^{\mu_2} + H_p} \tag{3-46}
\]

The solution of \(\Delta \lambda\) can be obtained iteratively. The stress associated with the material response at the next time step is subsequently updated by Equation (3-33).

3.3.3 Nonlinear cyclic behavior of soil

The strain softening viscoplastic model discussed in the previous section is adequate for simulating dense granular materials under monotonic loading. For simulating these
materials under cyclic loading, however, cyclic nonlinear behavior inside the yield surface needs to be accounted for. Masing’s rule (Masing 1926) has been widely used to model the cyclic nonlinear behavior of soil (Kramer 1996). Pyke (1979) improved the original Masing’s rule by introducing the following equation

\[
\tau - \tau_{rev} = G_0 \frac{(\gamma - \gamma_{rev})}{1 + \frac{\gamma - \gamma_{rev}}{n \cdot \gamma_r}}
\]  
(3-47)

where \( \tau \) is the shear stress; \( \gamma \) is the shear strain; \( \tau_{rev} \) and \( \gamma_{rev} \) are the shear stress and shear strain at the previous load reversal, respectively; \( \gamma_r \) is the reference strain determined by the maximum shear stress and initial shear modulus \( G_0 \); parameter \( n = 1 \) for initial loading and \( n = 2 \) for subsequent loading. An important assumption in Pyke’s rule is that the initial shear modulus \( G_0 \) is constant. Hence the backbone curve in Pyke’s rule does not undergo degradation during cyclic loading. This assumption is valid for small-strain levels (Matasovic and Vucetic 1993). However, if \( \gamma \) exceeds a certain threshold value, the stress-strain cycles will follow degraded backbone curves as shown in Figure 3-3. The associated reduction in stiffness (and/or strength) may result from soil structure rearrangement and increase of pore water pressure, commonly encountered in geomaterials under cyclic loading. Therefore, the application of Pyke’s rule combined with traditional incremental theory of plasticity in finite strain analysis may lead to inaccurate prediction of permanent displacement.
Based on the work of Kondner and Zelasko (1963), Matasovic and Vucetic (1993) proposed the following modified Kondner and Zelasko (MKZ) rule to improve the accuracy of the initial backbone curve

\[ \tau = G_0 \frac{\gamma}{1 + a \cdot \left( \frac{\gamma}{n \cdot \gamma_r} \right)^b} \] (3-48)

Differentiation of Equation (3-48) with respect to \( \gamma \) and replacing \( \gamma \) with \( \gamma - \gamma_{rev} \) leads to
$$G = G_0 \frac{1 + a(1-b) \left( \frac{\gamma - \gamma_{rev}}{n\gamma_r} \right)^b}{\left( 1 + a \left( \frac{\gamma - \gamma_{rev}}{n\gamma_r} \right)^b \right)^2}$$

(3-49)

where $G$ is the tangent shear modulus; parameters $a$ and $b$ are two curve-fitting constants; $n$ has the same definition as that in Pyke’s rule. In the MKZ rule, the degradation of backbone curve can be accounted for by considering $G_0$ as a function of the mean effective stress. This approach is suitable for liquefiable sand because the degradation of $G_0$ is mainly caused by the decrease of effective stress associated with pore pressure increase during cyclic loading. The strength and stiffness degradation in soft clay may result from a change in soil structure/fabric, which may be independent from the mean effective stress. Idriss et al. (1978) proposed that a degraded backbone curve can be constructed by multiplying $G_0$ with a degradation index $\chi$ which decreases with the cyclic shear stress amplitude during cyclic loading (Vucetic 1990). In our study, the backbone curve degradation is considered to be associated with the post-peak strength softening. As shown in Figure 3-3, the degradation of backbone curve is initiated after the peak strength is mobilized and the subsequent hysteresis loops are formulated based on the degraded backbone curves. Therefore, the cyclic shear stress amplitude which was linked to the degradation index $\chi$ by Vucetic (1990) can be interpreted as the shear resistance reduction at various post-peak strain levels, which can be implemented in the strain softening D-P model. Accordingly, the degradation of backbone curve used in this study is formulated as
\[ G_0' = G_0 \cdot \chi^d = G_0 \left[ \frac{c(\bar{\varepsilon}_p)}{c(0)} \right]^d \]  

(3-50)

where \( G_0' \) and \( G_0 \) are tangent shear modulus of the degraded and initial backbone curves as shown in Figure 3-3, respectively; \( c(0) \) is the initial cohesion and \( c(\bar{\varepsilon}_p) \) is post-peak cohesion defined by D-P model as shown in Equation (3-21); and \( d \) is a curve-fitting constant. Combining Equations (3-49) and (3-50) yields the following equation that accounts for the modulus reduction linked to the strain softening behavior

\[
G = G_0 \left[ \frac{c(\bar{\varepsilon}_p)}{c(0)} \right]^d \left[ 1 + a \left( \frac{\gamma - \gamma_{rev}}{n\gamma_r} \right)^b \right] \left[ 1 + a \left( \frac{\gamma - \gamma_{rev}}{n\gamma_r} \right)^b \right]^{-2} 
\]

(3-51)

As such, the nonlinear cyclic behavior within the yield surface is modeled by following Pyke’s rule and Equation (3-51).

The MKZ rule and Pyke’s rule discussed above are constructed under 1-D condition. The 1-D MKZ rule can be extended to 3-D conditions by replacing the shear strain and shear stress with the corresponding octahedral shear stress and strain (Jung 2009)

\[
\tau_{oct} = \frac{1}{3} s^{\alpha \beta} \cdot s_{\beta \alpha} 
\]

(3-52a)

\[
\varepsilon_{oct} = 2 \sqrt{\frac{1}{3} e^{\alpha \beta} \cdot e_{\beta \alpha}} 
\]

(3-52b)

where \( \tau_{oct} \) and \( \varepsilon_{oct} \) are the octahedral shear stress and octahedral shear strain, respectively. Replacing the shear strain \( \gamma \) in Equation (51) with \( \varepsilon_{oct} \) yields the 3-D MKZ
\[ G = G_0 \left[ \frac{c(\bar{\epsilon}_p)}{c(0)} \right]^d \left( 1 + a(1 - b) \left( \frac{\epsilon_{oct} - \epsilon_{rev,oct}}{n \epsilon_{r,oct}} \right)^b \right) \left( 1 + a \left( \frac{\epsilon_{oct} - \epsilon_{rev,oct}}{n \epsilon_{r,oct}} \right)^b \right)^2 \]

(3-53)

where \( \epsilon_{rev,oct} \) is the octahedral shear strain at load reversal; \( \epsilon_{r,oct} \) is the reference octahedral shear strain which can be calculated as

\[ \epsilon_{r,oct} = \frac{\tau_{f,oct}}{G_0} = \frac{\tau_{f,oct}}{G_0 \left[ \frac{c(\bar{\epsilon}_p)}{c(0)} \right]^d} \]

(3-54)

where \( \tau_{f,oct} \) is the octahedral shear stress when the soil reaches its shear strength and can be determined by the D-P yield function

\[ \tau_{f,oct} = \sqrt{\frac{2}{3} \left( -\eta \cdot I_1 + \zeta \cdot c(\bar{\epsilon}_p) \right)} \]

(3-55)

The constitutive model used in this study is formulated mainly by coupling Equations (3-21), (3-28) and (3-53). Figure 3-4 presents the flow chart of the computation for material response.
Calculate backbone curve degradation using Equation (3-50)

Compute $\dot{\varepsilon}$ using Equations (3-10) and (3-17)

Compute $\varepsilon_{n+1}$

Check load reversal based on $\varepsilon_{n+1}$, $\varepsilon_n$, and $\varepsilon_{n-1}$

Compute $\varepsilon_{oct}$ and $\varepsilon_{rev.oct}$

Calculate $G$ using 3-D MKZ model in Equation (3-51)

Calculate trial stress $\sigma_{\text{trial}}$ using Equation (3-38)

Check consistency: Does the trial stress lie outside the yield surface?

Perform the return mapping:
1) Determine $\Delta \lambda$ and $c_{n+1}$ by iteratively solving Equation (3-46);
2) Update stress $\bar{\sigma}_{n+1}$ using Equation (3-34).

Figure 3-4. Flow chart of computation for material response
3.4 Explicit Time Integration

Equations (3-6), (3-10), (3-15) are integrated over time domain using a second order Runge-Kutta scheme which is also known as the predictor-corrector method. For simplicity, the following simplified expressions are used

\[ D = \frac{d\rho}{dt} \quad (3-56a) \]

\[ F = \frac{dv}{dt} \quad (3-56b) \]

\[ v = \frac{d\mathbf{r}}{dt} \quad (3-56c) \]

\[ E = \frac{d\sigma}{dt} \quad (3-56d) \]

where \( \mathbf{r} \) is the position vector. The density, velocity and position, along with the stress, are predicted at half time steps as

\[ \rho_{n+1/2} = \rho_n + \frac{\Delta t}{2} \cdot D_n \quad (3-57a) \]

\[ v_{n+1/2} = v_n + \frac{\Delta t}{2} \cdot F_n \quad (3-57b) \]

\[ r_{n+1/2} = r_n + \frac{\Delta t}{2} \cdot v_n \quad (3-57c) \]

\[ \sigma_{n+1/2} = \sigma_n + \frac{\Delta t}{2} \cdot E_n \quad (3-57d) \]
In the above equations and throughout this paper, subscript $n+1/2$ represents half time step. Substituting Equation (3-15) into Equation (3-57d), the stress update can be further expressed as

$$
\sigma_{n+1/2} = \sigma_n + \frac{\Delta t}{2} \cdot \sigma'_{n} + \frac{\Delta t}{2} \cdot \left( \omega_n \cdot \sigma_n + \sigma_n \cdot \omega_n^T \right)
$$

Equation (3-58) is expressed in the tensor form for simplicity. Stress tensor $\tilde{\sigma}_{n+1/2}$ is calculated using the return mapping scheme shown in Equation (3-34) with the replacement of the whole time step by the half time step. The predicted values at the half time step are subsequently used to advance the values at the current time step to the next time step

$$
\rho_{n+1} = \rho_n + \Delta t \cdot D_{n+1/2}
$$

(3-59a)

$$
v_{n+1} = v_n + \Delta t \cdot F_{n+1/2}
$$

(3-59b)

$$
r_{n+1} = r_n + \Delta t \cdot V_{n+1/2}
$$

(3-59c)

$$
\sigma_{n+1} = \sigma_n + \Delta t \cdot F_{n+1/2}
$$

(3-59d)

$$
= \sigma_n + \Delta t \cdot \sigma'_{n+1/2} + \Delta t \cdot \left( \omega_{n+1/2} \cdot \sigma_{n+1/2} + \sigma_{n+1/2} \cdot \omega_{n+1/2}^T \right)
$$

where $\tilde{\sigma}_{n+1}$ is obtained from Equation (3-34). This method to advance stress in a finite strain framework was used by Crisfield (1997) in his explicit dynamic finite element code. It is noticed that the two stress components are updated using different schemes. The stress component due to material deformation is updated through the return mapping.
scheme; whereas the stress component due to rigid body rotation is updated through the explicit time integration. The size of time step is controlled by a combination of the Courant condition and the viscous condition, which has been thoroughly discussed in abundant literatures (e.g., Monaghan 1992; Morris et al. 1997); therefore, details for determining the time step are not presented herein.

### 3.5 Model Calibration

The aforementioned constitutive model is implemented into a 3-D SPH model developed by Chen and Qiu (2012a) to simulate several well-documented model slope tests on a shaking table by Wartman (1999). This paper presents the results of our simulation on one test that yielded the largest slope deformation. Figure 3-5 presents the initial configuration of the model slope, which was comprised of soft clay (light blue) underlain by a stiff clay layer (dark gray). The two clays consisted of a mixture of 75% Kaolinite and 25% bentonite and were prepared at different water contents. Properties of the model clays including their density, shear wave velocity, and peak and residual strengths are well documented in Wartman (1999) and Wartman et al. (2001 and 2005). The soft clay has an average density of 1390 kg/m$^3$ and shear wave velocity of 7.5 m/s. The stiff clay has the same density as the soft clay, but a higher shear wave velocity of 17.7 m/s. Besides the soil properties reported above, the developed SPH model requires six additional parameters that need to be calibrated, which include the two viscous parameters in Peirce viscoplastic model, one plastic modulus in the strain softening D-P model (see Equation (3-45)), and three curve-fitting constants in the MKZ model. The stiffness of the clays can be obtained from the density and measured shear wave
velocities. However, the obtained stiffness needs to be adjusted to account for the strain-rate effects during shaking. The calibration of these parameters is discussed in the following sections.

![Initial configuration of model slope (after Wartman 1999).](image)

**Figure 3-5.** Initial configuration of model slope (after Wartman 1999).

### 3.5.1 Calibration of viscous parameters in the Peirce viscoplastic model

The Peirce viscoplastic model includes two parameters that determine the dependency of material strength on strain rate. Wartman (1999) suggested that the residual strengths of the model clays are generally not affected by strain rate; whereas the strain-rate effect on peak strength is more pronounced. Accordingly, the viscoplastic model is calibrated against the peak strengths of the model clays at different shear rates. Wartman (1999) investigated the undrained shear strength of the model clays using a laboratory-scale mechanized vane shear device with a vane blade that is 5 cm by 2.5 cm (height by diameter). During the tests, the vane blade was pushed into the model clays to the depths ranging from 16.5 to 19 cm and rotated at various angular velocities. Figure 3-6(a) presents a schematic diagram of the SPH model for the vane shear test. The SPH model consists of about 140,000 SPH particles with equal spacing of 0.2 cm and is created in an
11 cm by 10 cm (height by diameter) cylindrical container. The container in the SPH model, which represents the simulation domain, is much smaller than the container used in the physical test for better computational efficiency. The self weight of the soil above the simulation domain is considered by applying a vertical pressure as shown in Figure 3-6(a). The physical test was performed under the undrained condition where the friction angle of the soil can be assumed zero. Therefore, the confining pressure and model size are expected to have negligible effects on the numerical simulations. A cross section of the vane shear test after the vane blade has rotated for 110 degrees is shown in Figure 3-6(b). The yield region in the model clay as highlighted in Figure 3-6(b) concentrates in a narrow band along the circumference of the vane shear blade. The clay outside of the yield region is barely disturbed. Figure 3-6(b) confirms that the simulation domain is adequate to capture the failure region of the model clay.

The side and bottom boundaries, along with the vane blade, are modeled using the ghost particle technique (Zhu et al. 1999; Bui et al. 2008) which is illustrated in Figure 3-7. The ghost SPH particles carry the same mass, density, and stresses as the clay SPH particles in their influence domain. The velocity of a ghost particle is determined by the non-slip boundary condition as

\[
v_B = (1 - \hat{\beta}) \cdot v_A + \hat{\beta} \cdot v_{\text{bound}}
\]

(3-60)

where \(v_A\) and \(v_B\) are the velocities of the clay particle A and ghost particle B, respectively; \(v_{\text{bound}}\) is the velocity of the physical boundary (e.g., vane blade).
Figure 3-6. Simulation of vane shear test: (a) initial configuration of vane shear test (only half of the soil simulated is presented for better visual effect); (b) cross section of $A - A'$ showing yield region in the clay after a blade rotation of 110 degrees.
The parameter $\hat{\beta}$ is used to exclude high values of artificial velocity (unphysical) and is given by (Zhu et al. 1999)

$$\hat{\beta} = \min\left(2.0, 1.0 + \frac{d_B}{d_A}\right)$$

(3-61)

where $d_A$ and $d_B$ are the perpendicular distances between the boundary and SPH particles A and B, respectively, as shown in Figure 3-7. Each vane blade is represented by four rows of ghost particles, which results in a blade thickness significantly larger than the real physical blade. This artificial thickness is created to keep clay SPH particles separated by the blade to be outside of each other’s influence domain for the correct physics and convenience of modeling. The effects of the particle spacing and blade thickness on the accuracy of simulation are discussed later. The undrained shear strength, $S_u$, mobilized along the vane blade in SPH simulations is calculated as
\[ S_u = \frac{T}{\pi \left( \frac{h_e \cdot d_a^2}{2} + \frac{d_a^3}{6} \right)} \]  

(3-62)

where \( T \) is the torque on the blade; \( h_e \) and \( d_a \) are the height and diameter of the blade, respectively. The torque \( T \) is calculated as the sum of resisting moments on the ghost particles representing the blade

\[ T = \sum_{j=1}^{N_b} \vec{R}_j \times \vec{f}_j \]  

(3-63)

where \( N_b \) is the total number of ghost particles representing the blade; \( \vec{R}_j \) is the distance vector between ghost particle \( j \) and the center vertical axis of rotation; and \( \vec{f}_j \) is the force vector acting on ghost particle \( j \), which is determined using a similar technique proposed by Monaghan et al. (2003) and this technique is briefly discussed herein. The force component \( \alpha \) on ghost particle \( j \) due to soil particle \( i \) is given by

\[ f_{j,i}^\alpha = m_j \cdot a_{ji}^\alpha \]  

(3-64)

where \( m_j \) denotes the mass of ghost particle \( j \); \( a_{ji}^\alpha \) is the acceleration induced by soil particle \( i \) and is calculated as

\[ a_{ji}^\alpha = m_i \left( \frac{\sigma_{ij}^{\alpha\beta}}{\rho_i^2} + \frac{\sigma_{ij}^{\alpha\beta}}{\rho_j^2} \right) \frac{\partial W_{ji}}{\partial \chi_j^\beta} \]  

(3-65)

The summation over surrounding soil particles in the influence domain leads to the total force on the ghost particle \( j \).
\[ f_j^a = \sum_{i=1}^{N_j} f_{ji}^a = \sum_{i=1}^{N_j} m_j \cdot m_i \left( \frac{\sigma_i^{\alpha\beta}}{\rho_i^2} + \frac{\sigma_j^{\alpha\beta}}{\rho_j^2} \right) \frac{\partial W_{ji}}{\partial x_j^\beta} \]  

(3-66)

where \( N_j \) is the total number of soil particles in the influence domain of boundary particle \( j \).

Figure 3-8 shows the effect of shear rate on peak undrained shear strength of the stiff model clay. The shear modulus used for the clay in these simulations is 0.43 MPa, corresponding to a shear wave velocity of 17.7 m/s which is consistent with the value reported by Wartman (1999). The simulations in Figure 3-8 are based on a combination of \( \mu_1 = 100 \) and \( \mu_2 = 0.07 \). Figure 3-8 indicates that this combination produces satisfactory results in matching the relationship of peak strength versus strain rate as experimentally observed by Wartman (1999), and hence is used in our study.

![Figure 3-8. Strain rate effects on peak shear strength of model soft clay](image-url)
3.5.2 Calibration of strain softening parameters

The implemented strain-softening model in SPH is first validated by comparing with FEM solution for a strip footing problem. The static response of a soil stratum to a vertically loaded footing is widely used as a benchmark for validating material models in FEM (e.g. Chen and Mizuno 1990; Zienkiewicz and Taylor 2000; and ABAQUS 2010) and SPH (e.g. Bui et al. 2008). An example problem used by Chen and Mizuno (1990) and ABAQUS Example Manual (2010) is used for this validation. Figure 3-9 shows a schematic illustration of the problem geometry. A 3.04 m wide strip footing with a rigid and perfectly rough base is supported by a shallow stratum that is 3.66 m in depth. The stratum extends 8.84 m horizontally from the footing center. Due to the geometric symmetry, the simulation domain consists of only half of the problem geometry as shown in Figure 3-9. The soil has a shear modulus of 80 MPa, a Poisson’s ratio of 0.3, and a peak cohesion of 69 kPa. Details of the problem configuration and material properties can be found in Chen and Mizuno (1990) and ABAQUS Example Manual (2010) and are not presented herein. The same non-dilatant D-P model with strain-softening feature is used in both ABAQUS and the developed SPH model. For the SPH simulation, the rate dependence of shear strength is disabled by setting the viscous parameter \( \mu_i = 0 \). The residual cohesion of the soil is assumed to be 15 kPa. For simplicity, the post-peak plastic modulus \( H_p \) in Equation (3-46) is assigned a constant value of \(-1.0 \times 10^4\) kPa for both simulations. The same space resolution of \( dx = 0.03 \) m is used in both simulations. The rigid footing is modeled as a constant velocity boundary moving downward at a velocity of 0.1 cm/s that is chosen to minimize any inertial effect. The ABAQUS/Explicit solver was activated to handle the strong nonlinearity as a result of the large deformation and
material softening. Figure 3-10 shows the simulated footing pressure versus footing vertical displacement from ABAQUS and the developed SPH model. The pressure beneath the footing drops sharply after a peak value is reached and approaches to a residual value as the displacement increases. As shown in Figure 3-10, the SPH solution agrees very well with the ABAQUS solution. As such, the capability of the developed SPH model in capturing strain-softening material behavior is validated.

Figure 3-9. Geometric configuration of strip footing

The strain-softening D-P model requires an experimental stress-strain (plastic) curve to calibrate the plastic modulus $H_p$ in Equation (3-46), which in turn requires the determination of $\frac{dc}{d(\bar{\varepsilon}_p)}$ in Equation (3-45). The following equation of $c(\bar{\varepsilon}_p)$ for clay proposed by Lo (1972) is adopted in our study due to its simplicity.
\[ c(\bar{\varepsilon}_p) = c_f \cdot \left(1 - \frac{\bar{\varepsilon}_p}{k_1 + k_2 \cdot \bar{\varepsilon}_p}\right) \]  

where \( c(\bar{\varepsilon}_p) \), as defined in Section 3.3.2, is the post-peak undrained shear strength; \( \bar{\varepsilon}_p \) is the accumulated equivalent plastic strain; \( c_f \) is the peak undrained shear strength; and \( k_1 \) and \( k_2 \) are two softening parameters.

**Figure 3-10.** Comparison of average pressure beneath the footing in ABAQUS and SPH

Lo (1972) concluded that the strain softening behavior of many clays can be generally represented by Equation (3-67). The two softening parameters are calibrated by simulating a vane shear test conducted by Wartman (1999) using the developed SPH model as discussed in Section 3.5.1. The test was conducted at a rotational velocity of 0.4
rad/s and yielded a peak undrained shear strength of 5.02 kPa. The measured undrained shear strength versus peripheral displacement (i.e., equivalent linear displacement of the outside edge of the rotating blade) was reported by Wartman (1999) and Wartman et al. (2001 and 2005).

Figure 3-11 presents the comparison of undrained shear strength versus peripheral displacement between the laboratory test data from Wartman (2005) and SPH simulations using a combination of $k_1 = 0.12$ and $k_2 = 0.39$. The sensitivity of SPH simulation on particle spacing is also examined and illustrated in Figure 3-11. It is observed that for the three spacing values used: 1 cm, 0.5 cm, and 0.2 cm, the SPH simulations are not sensitive to the particle spacing. Figure 3-11 demonstrates that the combination of $k_1 = 0.12$ and $k_2 = 0.39$ produces satisfactory results in matching the strain softening behavior observed in the vane shear test. Therefore, this combination and Equation (3-67) are used in our study.
Figure 3-11. Comparison of undrained shear strength vs. peripheral displacement between SPH simulation and laboratory vane shear test (experimental data digitized from Wartman et al. 2005).

3.5.3 Calibration of curve-fitting parameters in MKZ model

The MKZ model discussed in Section 3.3.3 includes three curve-fitting constants $a$, $b$, and $d$. Calibration of these parameters is performed by matching the predicted and experimental shear modulus reduction curve and damping curve of the model clay. The shear modulus reduction and damping curves under 1-D condition can be determined based on the hysteresis loops at different shear strain levels as shown in Figure 3-3. Instead of SPH modeling, each hysteresis loop under 1-D condition can be analytically derived according to the following procedure. Given a shear strain level $\gamma$ that defines
the maximum shear strain in a hysteresis loop, the shear stress is calculated in an incremental form as

\[ \tau_{n+1} = \tau_n + G \cdot \Delta \gamma \]  \hspace{1cm} (3-68)

The shear modulus \( G \) for every increment in Equation (3-68) can be calculated using the 1-D MKZ model in Equation (3-51). The degradation of the backbone curve is accounted for by considering Equation (3-50) and the strength softening characteristics discussed in Section 3.5.2. The reference shear strain \( \gamma_r \) is calculated as

\[ \gamma_r = \frac{\tau_f}{G_0} \]  \hspace{1cm} (3-69)

where \( \tau_f \) is the yield shear strength under 1-D condition with \( \eta = 0 \) (i.e. friction angle is zero)

\[ \tau_f = \zeta \cdot c(\bar{\varepsilon}_p) \]  \hspace{1cm} (3-70)

The secant shear modulus \( G_{sec} \) and damping ratio (e.g., Figure 3-3) are subsequently estimated as

\[ G_{sec} = \frac{\tau_{max}}{\gamma_{max}} \]  \hspace{1cm} (3-71a)

\[ \xi = \frac{A_{loop}}{2\pi \cdot \tau_{max} \cdot \gamma_{max}} \]  \hspace{1cm} (3-71b)

where \( \tau_{max} \) and \( \gamma_{max} \) are the maximum shear stress and shear strain of a hysteresis loop, respectively; \( A_{loop} \) is the area of the hysteresis loop. The initial shear modulus of soft clay is 0.07 MPa, based on the measured shear wave velocity of 7.5 m/s. It is found that the
combination of $a = 1.1$, $b = 0.65$, and $d = 1.5$ produces the best results in matching the predicted and reported modulus reduction (i.e., $G/G_{\text{max}}$ vs. $\gamma$) and damping (i.e., $\xi$ vs. $\gamma$) curves as shown in Figure 3-12; therefore, these values are used in our study.

**Figure 3-12.** Modulus reduction and damping curves (experimental curves digitized from Wartman et al. 2001).

### 3.5.4 Calibration of rate-dependent stiffness of clay

Hammer blow tests were conducted by Wartman (1999) to measure the shear wave velocities in the model clays using the accelerometers embedded in the model slope as shown in Figure 3-5 and waves generated from an instrumented hammer. The stiffness of
the model clays can be calculated based on the measured shear wave velocity and density. However, the shear strain rates in the shaking table tests were more than 100 times larger than that of the hammer blow tests (Wartman 1999). According to Bray et al. (1999) and Wartman et al. (2001), the small strain stiffness of the model clays exhibits strain-rate dependence, with larger strain rates yielding higher stiffness. The viscoplastic strain-softening model presented in Sections 3.3.2 and 3.5.1 is capable of modeling the strain rate effects on shear strength, but not on the small strain stiffness. As a result, the measured shear wave velocities need to be corrected to account for the strain rate disparity between the test motion and hammer blow test. Wartman et al. (2001) increased the shear wave velocity of the soft clay by 20% to 50% in their 1-D ground response analysis to account for the strain-rate effect. Prior to applying full-amplitude shaking motion, a frequency sweep test was performed by Wartman (1999) on the model slope with a very low peak acceleration to assess the dynamic characteristics of the model slope. The clay stiffness can therefore be calibrated using the acceleration response spectra obtained from the frequency sweep test.

The model slope was constructed in a Plexiglas box and the slope profile was nearly uniform across the width direction. Therefore, the test is simulated in 2-D condition (i.e., plane strain) for simplicity. The model slope comprised of soft clay that measured 31.3 cm in height with a face slope 1.6:1 (horizontal:vertical) and a 3.8 cm thick stiff clay layer as shown in Figure 3-5. The back of the slope consisted of stiff clay inclined at 2.1:1 (horizontal:vertical). The initial profile of the 2-D SPH model slope is represented by about 90,000 SPH particles with a particle spacing $dx = 0.2 \text{ cm}$. A friction angle of zero degree and a Poisson’s ratio of 0.45 are used in our study to approximate the
undrained condition. Based on the vane shear test with the blade rotating at 0.3 rad/s, the peak and residual strengths for the softy clay are 2.68 kPa and 1.77 kPa, respectively. The peak and residual strengths of the stiff clay are 5.90 kPa and 4.03 kPa, respectively (Wartman 1999).

![Graph](image)

**Figure 3-13.** Acceleration time history of frequency sweep test

Figure 3-13 shows the acceleration time history used in the frequency sweep test, which is applied to the ghost SPH particles on the bottom and side boundaries as the input motion. The simulated and recorded motions at the location of accelerometer No. 7 as shown in Figure 3-5 are compared, in term of their acceleration response spectra corresponding to 5% damping. The shear wave velocities of the soft and stiff clays are varied until a good match is observed between the simulated and recorded acceleration response spectra as shown in Figure 3-14. The SPH simulation in Figure 3-14 is based on 65% and 80% increase of the shear wave velocities in the soft clay and stiff clay, respectively. The corresponding shear moduli of the soft and stiff clays are 0.21 MPa and 1.4 MPa, respectively. Figure 3-14 demonstrates that the developed SPH model with the
corrected small-strain stiffness can capture the two peaks in the recorded motion. These two peaks are due to the distinctively different dynamic characteristics of the soft-clay slope and stiff-clay base (e.g., stiffness and geometry).

**Figure 3-14.** Comparison of acceleration response spectra of simulated and recorded motions at accelerometer No. 7
3.6 Model Validation

A full-amplitude shaking table test on the model slope shown in Figure 3-5 was conducted by Wartman (1999). The strong ground motion record of the 1995 Hyogo-Ken Nanbu earthquake at Port Island, Kobe, Japan was selected as the input motion and rescaled to fall within the performance range of the shaking table. The acceleration time history of the shaking table (i.e., input motion) contained significant noise, leading to the integrated velocity and displacement deviating from the zero baseline. Typical baseline-correction techniques are found inadequate to improve the signal quality in this case. As suggested by Wartman (1999), in addition to baseline correction, the input motion signal is also processed by using the high-pass 0.5 Hz Butterworth filter and low-pass 60 Hz filter. In order to avoid phase shifts during filtering, the low-pass filter is applied both in forward and backward directions. The processed acceleration and displacement time histories of the input motion are shown in Figure 3-15. The model slope test is simulated using the developed SPH model in 2D and the calibrated parameters presented in the previous sections. The following sections compare SPH simulation results with the model slope test in terms of slope failure mode, acceleration response spectra, and slope deformation.
Figure 3-15. Time histories of input motion: (a) acceleration; (b) displacement

3.6.1 Slope failure mode

The development of failure region in the model slope is presented in Figure 3-16, where the soft clay and stiff clay are presented in light blue and dark gray, respectively. The contour of accumulated plastic strain, with red indicating higher values, is also shown in Figure 3-16. A localized shear band, as shown in Figure 3-16(a), is initiated along a deep surface in the soft clay at around 13.4 s after the start of shaking. The shear band grows progressively wider and more intensive as shown in Figures 3-16(b) and 3-16(c), suggesting a strongly localized shear failure and deformation. In addition to this surface sliding failure mode, a bulge is formed near the toe of the soft-clay slope as shown in Figure 3-16(c). The final shape of the deformed slope in the model slope test is
shown in Figure 3-16(d). It can be seen that the simulated failure mode, deformed shape, and failure surface agree well with the physical model test. The simulation results in Figure 3-16 are based on the implemented viscoplastic model that accounts for the rate-dependent shear strength in a robust fashion without any predetermination of the shear strain rate during shaking prior to simulations.

Alternatively, the rate-dependent shear strength may also be partially accounted for by correcting the peak shear strength for rate effects prior to simulations in an otherwise rate-independent elasto-plastic model. The latter approach was utilized by Wartman et al. (2001) in their Newmark slope deformation analysis on the model slope test presented herein. To investigate the effect of these two approaches in treating the rate-dependent shear strength on the overall simulated responses the model slope, the shaking table test was also simulated using a rate-independent D-P model by setting the viscous parameter $\mu_i = 0$ while increasing the peak shear strength by 20%, a value used by Wartman et al. (2001). The development of failure region in the model slope based on the rate-independent D-P model is shown in Figure 3-17. The rate-independent D-P model can also capture the failure mode, deformed shape, and failure surface observed from the physical model test. Comparing Figures 3-16 and 3-17, it can be observed that the rate-independent D-P model results in a narrower and more localized shear band. This is because in the rate-independent D-P model, the peak shear strength is increased by 20% to account for the strain rate effect. This increased peak shear strength applies to the entire soft-clay slope irrespective of different shear strain rates experienced by different regions of the soft-clay slope during the actual shaking.
Figure 3-16. Comparison of simulated failure mode and deformed shape with model slope test: (a) $t = 13.4$ s; (b) $t = 20.2$ s; (c) $t = 32.0$ s; (d) final deformed profile and sliding surface from model slope test (from Wartman et al. 2005 with permission).
Figure 3-17. Simulated failure mode and deformed shape by rate-independent model with corrected peak strength: (a) \( t = 13.4 \) s; (b) \( t = 20.2 \) s; (c) \( t = 32.0 \) s.

In the viscoplastic model, however, the shear strength of the soft clay is adjusted according to its strain rate. Therefore, the regions in the soft-clay slope that experience lower shear strain rates would have lower peak shear strengths and hence accumulate
more plastic strains in the viscoplastic model than in the rate-independent model. Consequently, the viscoplastic model results in a wider shear band than the rate-independent model. It is interesting to note that the deformed shape at the back of the soft-clay slope in the rate-independent model matches with the physical model test better.

3.6.2 Acceleration response spectra

The simulated responses at two locations along the model slope are compared with experimental records to evaluate the dynamic response of our model. As shown in Figure 3-5, accelerometer No. 6 is located within the shear band predicted by the viscoplastic model and above the shear band predicted by the rate-independent model as discussed in Section 3.6.1. Accelerometer No. 7, which has been used in the stiffness calibration (Section 3.5.3), is located on the top of the soft-clay slope. Figure 3-18 presents a comparison between the simulated and recorded acceleration response spectra (5% damping) at the locations of accelerometers No. 6 and No. 7. As shown in Figure 3-18(a), the simulate acceleration response spectrum at accelerometer No.6 closely matches with the experimental records, in terms of both the spectral amplitude and predominant frequency. High spectral amplitudes at location No. 6 occur between 5 Hz (0.2 second period) and 10 Hz (0.1 second period). Figure 3-18(b) shows similar comparison at accelerometer No. 7. The simulated spectrum provides a good match with the experiment records at low frequencies, but underestimates the response at high frequencies. The peak spectral amplitude occurs at about 5 Hz (0.2 second period) in Figure 3-18(b); whereas the peak spectral amplitude occurs at about 8.3 Hz (0.12 second period) during the low-amplitude frequency sweep test as shown in Figure 3-14. The decrease of the
predominant frequency and the resulting phase shift during the full-amplitude shaking are mainly attributed to the higher extent of shear modulus reduction simulated by SPH than that recorded in the physical testing.

Figure 3-18. Comparison of simulated and recorded acceleration response spectral at:
(a) accelerometer No. 6; (b) accelerometer No. 7.
3.6.3 Slope deformation

The displacement time histories recorded by displacement potentiometers installed at various locations in the model slope were reported by Wartman (1999). As shown in Figure 3-5, potentiometer No.11 is located on the front face of the slope and potentiometers No.16 and No.12 are installed on the top surface of the slope. The comparison between simulated and recorded horizontal displacement time histories at these three locations is shown in Figure 3-19. For clarity, both simulated and recorded displacements are processed using the low pass Butterworth filter at 1.0 Hz to remove high frequency noise. Displacements away from the slope are considered as negative. Because of the large displacements in this test, all of the three potentiometers went out of range after around 25 to 30 seconds of shaking. Therefore, the displacements recorded after gauge limits of the potentiometers were exceeded are not shown in Figure 3-19. The disparity among the three recorded displacements is generally small, indicating the slope moves in a relatively uniform fashion, which is consistent with the observed slope failure mode. As shown in Figure 3-19, the permanent displacements start to increase after 10 seconds of shaking. The displacement at location No.11 near the toe of the slope is higher than that on the top of the slope (i.e., locations No. 12 and No. 16) due to the bulge formation near the toe. It is observed that the simulated displacement time history at location No.11 has an excellent agreement with the recorded data. The simulated displacement time history at location No.12 also has a satisfactory agreement with the experimental record, despite a noticeable discrepancy between 15 and 20 seconds after the shaking. However, the recorded displacements at location No.16 are considerably smaller than the simulated values. After careful observation of the test layout as reported
by Wartman (1999), it is found that this potentiometer was installed close to a side wall of the container; therefore, its displacement is likely influenced by the side wall friction. The side wall friction is not considered in the 2-D SPH simulation, resulting in higher simulated displacements at this location.

Figure 3-19. Comparison of simulated and recorded horizontal displacement time histories at various locations along model slope

Numerical simulations of strain-softening materials usually exhibit spurious mesh-size effect, precluding convergence of results with reasonable particle spacing refinement (Loret and Prevost, 1990). To avoid this mesh-size effect, one common approach is to use length-scale parameters in strain softening models to associate the softening behavior
with mesh size or particle spacing. Another way to regularize this ill-posed problem is to introduce strain-rate dependence in the constitutive model (Loret and Prevost 1990). In the current viscoplastic model, strain-rate dependence is incorporated; therefore, additional length-scale parameters are not used. The sensitivity of simulated displacements to SPH particle spacing is shown in Figure 3-20. Model simulations with three spatial resolutions: $dx = 1\, \text{cm}$, $dx = 0.5\, \text{cm}$, and $dx = 0.2\, \text{cm}$ are examined. As shown in Figure 3-20, the simulated displacement time histories converge as the particle spacing decreases, suggesting that a default resolution of $dx = 0.2\, \text{cm}$ used in the viscoplastic model for simulating the model slope test is sufficient. It further indicates that the implemented viscoplastic strain-softening model is effective in remediating the dependence of solutions on particle spacing.

Figure 3-21 shows the comparison of the simulated displacement time histories at the location of potentiometer No.11 using the viscoplastic model and the rate-independent model with corrected peak strength. Figure 3-21 indicates that the displacement time history at this location is relatively insensitive to the constitutive model used at early stages of shaking. However, the rate-independent model with corrected peak strength yields larger permanent displacement towards the end of shaking. This observation is also valid for the simulated displacement time histories at the locations of potentiometers No. 12 and 16, which are not presented herein.
Figure 3-20. Displacement time histories at potentiometer No.11 simulated by the viscoplastic model for different particle spacing
Figure 3-21. Comparison of displacement time histories simulated using viscoplastic model and rate-independent model

3.7 Conclusions

Smoothed particle hydrodynamics method is a particle-based meshless method, suitable for the simulation of various phenomena involving large deformations due to its Lagrangian and adaptive nature. This paper presents the application of this method to the simulation of seismically induced slope failure under undrained condition. A constitutive model that combines the strain softening viscoplasticity and modified Kondner and Zelasko (MKZ) rule is implemented into SPH formulations. The developed SPH model accounts for the effects of wave propagation in the sliding mass, cyclic nonlinear behavior of soil, rate-dependent shear strength, and the progressive reduction in shear
strength during sliding, which are not explicitly considered in the Newmark-type analyses. Soil parameters needed for the developed model can be calibrated using typical laboratory shear strength tests, and experimental or empirical shear modulus reduction curve and damping curve.

The developed model is validated against a readily available and well documented model slope test on a shaking table. For the calibration of soil parameters, the plastic moduli and viscous parameters needed in the strain softening viscoplastic model are calibrated using the laboratory vane shear tests conducted at different rotational speeds, and the parameters needed for the MKZ rule are calibrated based on the experimental modulus reduction curve and damping curve. The initial shear moduli of the model clays are corrected by matching the simulated acceleration response spectra with the experimental response spectra obtained from a pre-test low-amplitude frequency sweep. The rate effect on shear strength is accounted for by means of the Peirce viscoplastic formulation. The calibrated model parameters are then used to simulate the model slope test under a full-amplitude shaking.

The simulated slope failure mode, acceleration response spectra, and slope deformations are compared with experimental data. The initiation of slope failure and subsequent progressive development of the sliding surface are successfully captured by the developed SPH model. A localized shear band along the failure surface and a bulge near the toe of the model slope are observed in the simulations, showing a good agreement with the experimental observations. The simulated displacement time histories and acceleration response spectra at several monitor locations along the model slope also agree well with the experimental recordings. The rate effect on the shear strength of the
model clays is considered in two ways. One is to use the viscoplastic model implemented in this paper and the other is to use a rate-independent elasto-plastic model with the peak shear strength increased by 20%. Simulations based on these two methods indicate that the shear band width is sensitive to the treatment of rate-dependent shear strength with the viscoplastic model yielding a wider shear band, whereas the displacement time histories at several locations along the model slope are relatively insensitive to this treatment.

SPH method has been considered as a powerful numerical technique to model large deformation and failure of geomaterials; however, its accuracy has not been fully investigated. This study indicates that SPH formulations, if implemented with appropriate constitutive models, can be used to simulate large-deformation problems in geomaterials with high fidelity. Given the limited laboratory test data available for the model clays (e.g., no triaxial test data) in geotechnical literature, the constitutive model utilized in this study has limitations. For example, the model does not directly account for excess pore water pressure generated during shear, although its effect on soil stiffness is implicitly considered through the modulus reduction curve. In the developed model, the rate-dependent soil stiffness is not directly accounted for; however, this effect is implicitly accounted for by calibrating the small-strain shear moduli against the responses from a pre-test low-amplitude frequency sweep. The good agreement observed between the simulated responses of the model slope and experimental recordings suggests that the developed model, although with limitations, has captured the essential dynamic behaviors of the model slope.
References


CHAPTER 4

PARAMETRIC STUDY

4.1 Introduction

A SPH model incorporating strain-softening viscoplasticity and cyclic nonlinearity was developed and validated against experiments in the previous chapter. The developed SPH model encompasses parameters relative to both SPH method itself and the implemented material models. In the previous chapter, these parameters are calibrated using the results of material laboratory tests prior to simulating the model slope experiment on a shaking table. The effect of these parameters on seismically-induced slope deformations remains unknown. This chapter presents a parametric study that mainly discusses (1) how the spatial parameters in SPH method (e.g., particle spacing and influence domain) and boundary condition (e.g., rigid boundary and non-reflecting boundary) impact the accuracy and consistency of SPH simulations, and (2) how the soil properties (e.g., peak and residual strengths) influence earthquake-induced slope deformations. An understanding of the effects of spatial parameters in SPH method on simulation outcome is important before extending this method to more broad engineering applications. The examination of soil properties may provide insight into the mechanism of earthquake-induced slope failures.

The parametric study presented in this chapter is mainly based on the model slope test that was performed by Wartman (1999) and has been simulated in the previous chapter.
The initial profile of the slope is shown in Figure 3-5 and the input motion is shown in Figure 3-15.

4.2 Viscoplastic Parameters

The Peirce viscoplastic model shown in Equation 3-29 includes two parameters $\mu_1$ and $\mu_2$. In Chapter 3, these two parameters are calibrated using the peak undrained shear strength obtained in laboratory vane shear tests conducted under different shear strain rates. Due to inadequate laboratory testing data at various shear rates, only three data points are used to calibrate these viscous parameters. In Chapter 3, it is found that the combination of $\mu_1 = 100$ and $\mu_2 = 0.07$ produces a good match with experiments. As shown in Figure 4-1, the combination of $\mu_1 = 5$ and $\mu_2 = 0.2$ also yields a good match with the three data points. However, the two combinations of viscous parameters result in different trends of peak shear strength versus shear strain rate as shown in Figure 4-1. Compared with the combination of $\mu_1 = 100$ and $\mu_2 = 0.07$, the combination of $\mu_1 = 5$ and $\mu_2 = 0.2$ produces higher peak shear strengths at high strain rates and lower peak shear strengths at low strain rates.

The simulated displacement time histories at the location of potentiometer No.11 based on the two combinations of viscous parameters are compared in Figure 4-2. The locations of potentiometers can be found in Figure 3-5. The two simulated horizontal displacement time histories are almost identical. Similar trend is also observed for the displacement time history at location No. 12 as shown in Figure 4-3. Figures 4-2 and 4-3 indicate that the horizontal displacements of the model slope are not sensitive to the
viscous parameters in the Peirce viscoplastic model. This observation suggests that the shear strain rates during the numerical simulations are likely to be within the range of the three data points (i.e., approximately 0.1 to 1 rad/s), where both combinations have a good match with the three data points.

Figure 4-1. Effects of viscous parameters on peak vane shear strength
Figure 4-2. Effects of viscous parameters on displacement time history at location No. 11

Figure 4-3. Effects of viscous parameters on displacement time history at location No. 12
Figure 4-4. Comparison of failure modes and deformed shapes in SPH simulations and model slope tests: (a) $\mu_1 = 100$ and $\mu_2 = 0.07$; (b) $\mu_1 = 5$ and $\mu_2 = 0.2$; (c) final deformed shape observed in model slope tests (Wartman et al. 2005)

The simulated failure modes and deformed shape based on the two combinations of viscous parameters are compared with experimental observation in Figure 4-4. Both
simulations yield good agreements with experimental observations which involve a bulge failure at the toe and the sliding failure along a localized shear band in the soft clay. However, the combination of $\mu_1 = 100$ and $\mu_2 = 0.07$ yields more satisfactory match with the experimental observations in the deformed shape at the back of the slope. In addition, this combination yields a relatively narrower shear band as shown in Figure 4-5(a); whereas the combination of $\mu_1 = 5$ and $\mu_2 = 0.2$ results in a wider failure region. The wider shear band resulted from the combination of $\mu_1 = 5$ and $\mu_2 = 0.2$ may be due to the lower peak shear strengths at low strain rates as predicted by this combination (see Figure 4-1). The combination of $\mu_1 = 100$ and $\mu_2 = 0.07$ is used for the remainder of this study.

### 4.3 Spatial Parameters in SPH Method

SPH method has two important spatial parameters which are the particle spacing and the radius of influence domain. In Chapter 3, the material rate-dependence is accounted for by using the rate-dependent viscoplastic model and rate-independent model with strain-rate-adjusted peak shear strengths. In the rate-independent model, the peak shear strength is increased by 20% throughout the simulations to account for the rate effects. The rate-dependent model is implemented in a viscoplastic fashion in which the material is treated as a viscous solid in plastic regime. Strain-softening materials, when approximated in a continuum-scale model (e.g. FEM and FDM), generally exhibit strong dependence on the spatial resolution. However, these spatial effects in SPH have not been previously studied in the literature. This section investigates the effects of the two spatial
parameters in SPH method on the accuracy of numerical solutions for both rate-
dependent and rate-independent models.

4.3.1 Particle-spacing effects

Numerical simulations of strain-softening materials usually exhibit spurious mesh-
size effect, precluding convergence of results with reasonable mesh refinement (Loret
and Prevost 1990). The dependence of strain-softening simulations on mesh size was first
observed in finite element simulations (Prevost and Hughes 1981) and extensively
discussed by Zienkiewicz and Taylor (2000) among other scholars. The sensitivity of
slope deformations on particle spacing in the rate-independent and rate-dependent models
is presented in the following sections. The smoothing length is selected to be 1.2 times
the initial particle spacing \(( h = 1.2 \times \Delta x)\) throughout this section.

4.3.1.1 Rate-independent model

Prior to the examination of particle-spacing effects on seismically-induced slope
deformations, a strip-footing simulation is presented as an introductory case to elaborate
the mesh-size effect. This example shown in Figure 3-9 is used as a benchmark problem
in validating the strain-softening SPH model in Chapter 3. The problem profile and
boundary conditions are detailed in Section 3.5.2. The soil has a shear modulus of 80
MPa and a Poisson’s ratio of 0.3. The peak and residual cohesion values of the soil are 69
kPa and 27 kPa, respectively. The plastic modulus \( H_p \) is assumed to be \(-2.0 \times 10^3\) kPa.
The rigid footing is modeled as a constant velocity boundary moving downward at the
velocity of 2.0 cm/s. Three mesh sizes: \( \Delta x = 12.2 \text{ cm}, 6.1 \text{ cm}, \) and \( 3.05 \text{ cm} \) are used to
investigate the mesh-size dependent solutions of strain-softening material. The FE analysis of this strip-footing is performed in ABAQUS (2010) in which the non-dilatant D-P model with strain-softening feature is used. The load-displacement responses of the strip footing at different mesh sizes are shown in Figure 4-5, where the reaction pressure beneath the footing is plotted against the downward displacements of the footing. Despite the same plastic modulus and shear strength being used, the FE solution is mesh-size dependent with coarse mesh resulting in larger peak pressure beneath the footing.

Figure 4-5. Simulated pressures beneath the footing for different mesh sizes using FEM

This dependence of numerical solutions on spatial discretization as a result of strain softening is also found in SPH method as shown in Figure 4-6. The rate-independent
material model in SPH is recovered by setting the viscous parameter \( \mu_i = 0 \) (refer to Equation 3-41 in Chapter 3). It shows that the most significant impacts that the particle spacing has on the load-displacement curve occur in the transition phase from the peak to residual resistance. It is important to point out that the particle-spacing dependence demonstrated in Figures 4-5 and 4-6 only arises when strain-softening material is modeled. As shown in Figure 4-7, the SPH simulated load-displacement curve for the material with perfect plasticity is independent of particle spacing for the spacing values selected.

Figure 4-6. Simulated pressures beneath the footing for different particle spacing using rate-independent strain-softening SPH model
Figure 4-7. Simulated pressures beneath the footing for different particle spacing using rate-independent perfect-plastic model in SPH

It has been shown that the SPH method, along with classical continuum-scale methods such as FEM, exhibits strong particle-spacing or mesh-size dependence when modeling strain-softening material behaviors. The following section examines how this particle-spacing dependence affects the seismically-induced slope deformations simulated by the developed SPH model. The rate-independent strain-softening model is recovered by setting the viscous parameter $\mu_i = 0$ in Equation 3-41.

A narrow band of intense shear strain is observed in the model slope tests and SPH simulations. As an essential feature of the resulted plastic deformation, the growth of this localized shear band can be indicative of the initiation and subsequent progressive
development of slope failure during seismic shaking. It is revealed that this shear band has a finite width and is dependent upon the material substructure (Loret and Provest 1990). The shear band width is hence expected to be independent of spatial discretization in a numerical simulation. Figure 4-8 shows the localized shear bands at the end of the seismic shaking simulated using three different values of particle spacing. The width of localized shear band shrinks drastically as the particle spacing decreases, indicating the strong spatial-scale dependence. In addition, the finer particle spacing also results in a more pronounced downwards sliding of the slope above the shear band. This spurious particle-spacing effect is attributed to the continuum-scale strain-softening model and not to the SPH method used in this study. With the rate-independent strain-softening model formulated on a continuum scale, a great amount of energy tends to accumulate on the smallest elements or particles (Loret and Prevost 1990). This deficiency of the current strain-softening formulation leads to the localized shear deformations that are dependent on particle spacing.

The displacement time histories of the model slope also exhibit a strong dependence on SPH particle spacing. Figure 4-9 shows a comparison of the simulated displacement time histories at the location of potentiometer No.11 using different values of particle spacing. The displacement time history has difficulty reaching a convergent solution as the particle spacing is refined, with smaller particle spacing yielding larger displacements. Similar trends are observed at the other two monitor locations and hence not shown herein.
Figure 4-8. Localized shear bands simulated using rate-independent model with different particle spacing: (a) $\Delta x = 0.8 \text{ cm}$; (b) $\Delta x = 0.5 \text{ cm}$; (c) $\Delta x = 0.2 \text{ cm}$
This undesirable particle-spacing effect in the rate-independent strain-softening model can be mitigated by a regulatory technique associating softening parameters (e.g., plastic modulus $H_p$) with the particle spacing or mesh size (Zienkiewicz and Taylor 2000). This technique has been widely used in static problems involving strain-softening materials. However, it requires the calibration of regulatory parameters for a given mesh size, leaving this method impractical and less adaptive. The viscoplastic model that accounts for strain-rate effects has been found effective in mitigating the spurious dependence of solutions on particle spacing as briefly discussed in Chapter 3. This procedure will be further discussed in the following section.
4.3.1.2 Rate-dependent model

The particle-spacing effects on the viscoplastic model are firstly illustrated in the aforementioned benchmark example of strip footing. As shown in Figure 4-10, the simulated pressure-displacement curves beneath the footing are relatively insensitive to the values of particle spacing used. The undesirable particle-spacing dependence shown in Figure 4-6 is effectively eliminated by using the rate-dependent viscoplastic model.

![Figure 4-10](image)

**Figure 4-10.** Simulated pressures beneath the footing for different particle spacing using strain-softening viscoplastic model in SPH

The model slope test is simulated using the validated viscoplastic model with the viscous parameters $\mu_1 = 100$ and $\mu_2 = 0.07$. Four different values of particle spacing are
used to investigate the length-scale effects in the seismically-induced slope deformations. The comparison of simulated shear bands and final deformed shapes is shown in Figure 4-11. The widths of the localized shear band and final deformed shapes are consistent among the four simulations. The simulated displacement time histories at the location of potentiometer No.11 are presented in Figure 4-12. As opposed to the rate-independent simulations shown in Figure 4-9 in which the solutions diverge with finer spacing, the displacement time histories simulated by the viscoplastic model exhibit a tendency of convergence as particle spacing is refined. The extent of particle-spacing dependence is considerably reduced by the viscoplastic model. All of the four simulations have good agreements with recorded displacement time histories at the selected location. The maximum discrepancy in displacement among the four simulations is less than 13%. Similar trend is observed at the other two monitor locations as well. As such, the viscoplastic model can be viewed as an effective and efficient procedure to mitigate the spurious particle-spacing dependence that is observed in the rate-independent strain-softening model.
Figure 4-11. Localized shear bands simulated by viscoplastic model with different particle spacing: (a) $\Delta x = 0.8$ cm; (b) $\Delta x = 0.5$ cm; (c) $\Delta x = 0.2$ cm; (d) $\Delta x = 0.1$ cm
Figure 4-12. Displacement time histories simulated by viscoplastic model with different particle spacing

4.3.2 Smoothing-length effects

The smoothing length $h$ as shown in Figure 3-1 defines the range of the influence domain in which two particles interact with each other. It is important for both the simulation efficiency and solution accuracy. Excessively small values of $h$ will result in insufficient number of SPH particles falling in the support domain for weighting interpolation. On the other hand, unreasonably large values of $h$ may smooth out simulation results over the computational domain, leading to poor solution. Larger values of $h$ will also significantly increase the computational burden. In some situations such as high-velocity impacts and explosion, the particle density may vary drastically in space.
The smoothing length accordingly should vary in order to maintain adequate accuracy. Adaptive smoothing length (Nelson and Papaloizou 1994) that varies in space and with time has been used in applications involving large density variation and inhomogeneities. However, this technique is not necessary in this study as the particle density does not undergo noticeable change during our simulations. Therefore, a fixed smoothing length is used throughout this study. The value of $h$ is usually selected to be 1.2 ~ 1.5 times of the initial particle spacing for the consideration of both efficiency and accuracy (Liu and Liu 2004).

The effects of smoothing length on the responses of strain-softening materials have not been studied in literature. This section will examine the sensitivity of the rate-independent and rate-dependent models to SPH smoothing length. The simulations are performed with four $h$ values: $1.2 \times \Delta x, 1.3 \times \Delta x, 1.4 \times \Delta x$, and $1.5 \times \Delta x$. The same initial particle spacing of 0.5 cm (i.e., $\Delta x = 0.5 \text{ cm}$) is used in all simulations in this section.

4.3.2.1 Rate-independent model

The simulated displacement time histories at the location of potentiometer No.11 with different smoothing lengths are presented in Figure 4-13. The smoothing length $h = 1.2 \times \Delta x$ yields the largest displacements whereas $h = 1.5 \times \Delta x$ results in the smallest displacements. The horizontal displacements increase as the smoothing length decreases, showing an apparent divergence of solutions. This trend of divergence is similar to the particle-spacing effects observed in the rate-independent strain-softening model in which the displacement increases with the decrease of particle spacing. This phenomenon may
be resulted from the energy dissipation in strain-softening media which tends to accumulate on smallest spatial quantities such as element size, particle size, and influence range.

![Figure 4-13](image)

**Figure 4-13.** Displacement time histories simulated using rate-independent model with different smoothing lengths

4.3.2.2 *Rate-dependent model*

Figure 4-14 compares the simulated displacement time histories using the strain-softening viscoplastic model with different smoothing lengths. As can be seen, the discrepancy among the four simulated displacement curves is unnoticeable, indicating the independence of the numerical solution on smoothing length.
The rate-dependent strain-softening model has been shown to be insensitive to both particle spacing and smoothing length in the SPH simulations. It therefore suggests that the viscoplastic model naturally introduces a spatial scale regularization to effectively mitigate the dependence of numerical results on spatial parameters (e.g. particle spacing and smoothing length) observed in the rate-independent strain-softening model.

![Graph](image)

**Figure 4-14.** Displacement time histories simulated using viscoplastic model with different smoothing lengths

### 4.4 Effects of Soil Properties on Slope Deformations

The effects of soil properties on seismic performance of earth slopes were preliminarily studied using Newmark analysis by Wartman et al. (2001). This section
utilizes the validated SPH model to examine the soil properties and attempts to provide some insights into seismically-induced slope deformations. These properties include soil peak strength, residual strength, and the plastic modulus in the strain-softening model. The soil shear modulus that is a contributing factor to the fundamental frequency of the model slope is investigated as well.

### 4.4.1 Residual strength and plastic modulus

The residual strength is reached at large deformations after yielding; whereas the plastic modulus reflects how fast the soil is softened toward the residual strength as the plastic strain increases. These two parameters featured in the strain-softening model used in this study are illustrated in Figure 4-15. For simplicity, the strain-softening relationship between cohesion \( c(\bar{\varepsilon}_p) \) and equivalent plastic strain \( \varepsilon_p \) shown in Equation 3-67 is replaced with a simple three-point linear relation demonstrated in Figure 4-16. In order to interpret how these two parameters impact soil behaviors, the vane shear test presented in Chapter 3 is simulated with different residual strengths and plastic moduli. The effects of residual strength on the simulated undrained strength-displacement curve are shown in Figure 4-17(a) in which the residual strength varies with a fixed plastic modulus \( H_p \). Figure 4-17(b) shows the simulated strength-displacement relations with variable \( H_p \) and constant residual strength. It clearly demonstrates that the plastic modulus mainly influences how the soil is softened to the residual strength.
Figure 4-15. Illustrative diagram of residual strength and plastic modulus

Figure 4-16. A simplified strain-softening relation used in parametric study
Figure 4-17. Undrained shear strength vs. peripheral displacement in vane shear test simulated with different: (a) Residual strength; (b) Plastic modulus
The effects of residual strength on seismic slope deformations are investigated using the strain-softening viscoplastic model in which the post-failure plastic modulus $H_p$ is set to a constant value of -6.5 kPa. Four different residual strength values as shown in Figure 4-18 are examined. The strain-softening model reduces to the perfect plastic model for the residual strength equal to 2.68 kPa. The displacement time histories at the location of potentiometer No.11 simulated with different residual strengths are compared in Figure 4-19. The displacement increases as the residual strength decreases, suggesting significant impacts of post-failure strength on seismic performance of the slope. It is also observed that the difference in the displacements occurring during the early stage of shaking (less than 12s) is not noticeable. It indicates that the residual strength starts to play an important role in slope deformation only after the sliding along the failure surface is mobilized.

It is worth to emphasize that the permanent displacement simulated with the perfect plastic model (i.e. residual strength equal to peak strength of 2.68 kPa) is about 6.3 cm. This value is considerably larger than the displacement of 3.01 cm estimated by the Newmark rigid block analysis reported by Wartman et al. (2001) in which the peak strength of 2.68 kPa was used. It therefore indicates that the seismically-induced slope displacements are attributed to both the sliding movement and material plastic shear deformations. The contribution of plastic shear strain to the slope performance is significant, which is not accounted for in the conventional Newmark analysis.

The simulated shear bands and final deformed shapes for the four residual strengths are compared in Figure 4-20. The contour in red indicates a higher intensity of plastic shear deformation whereas the contour in blue represents a lower intensity. The model with higher residual strength produces a wider shear band with less intensive plastic shear
strain as shown in Figure 4-20(a). As demonstrated in Figure 4-20(d), the localized shear band becomes much narrower and has a higher concentration of plastic shear strain as the residual strength is reduced. Figure 4-20 reveals that the shear failure localization has a strong dependence on the residual strength. A small residual strength may lead to the shear failure accumulating in a very small region. It further demonstrates that seismically-induced slope deformations are primarily associated with the localized shear band and its intensity of plastic deformation.

![Graph showing cohesion vs. equivalent plastic strain](image)

**Figure 4-18.** Post-failure residual strength in parametric study
Figure 4-19. Effects of residual strength on slope displacement
Figure 20. Shear failure region simulated with different residual strengths: (a) 2.68 kPa; (b) 2.10 kPa; (c) 1.77 kPa; (d) 1.40 kPa
In the subsequent investigation, the plastic modulus is varied while the residual strength is held constant as shown in Figure 4-21. The plastic modulus $H_p$, defined as the slope of the strain-softening curve, determines how fast the material shear resistance is softened towards its residual strength. Its effects on slope displacements at the location of potentiometer No.11 are presented in Figure 4-22. The higher value (absolute) of plastic modulus tends to yield larger displacement. However, the discrepancies among displacement time histories for different plastic moduli are not significant, with the largest difference smaller than 8%. It is due to the excessively plastic shear strain occurring in the failure region during the seismic shaking. The majority of the yielded soil has reached its residual strength at the end of the slope sliding motion, rendering the effects of the transition stage governed by the plastic modulus on the permanent deformations insignificant. It is therefore suggested that the seismically-induced slope deformation is mainly controlled by the residual strength after the sliding failure is initiated. The transition phase from peak strength to residual strength is insignificant in seismically-induced slope deformations.
Figure 4-21. Plastic modulus used in parametric study

Figure 4-22. Effects of plastic modulus on slope displacement


4.4.2 Peak strength

In this section, the effects of peak strength on slope performance are investigated by varying the peak undrained strength while keeping the residual strength and plastic modulus constant. Four peak strength values: 4.70 kPa, 3.70 kPa, 2.68 kPa, and 1.77 kPa are used in this study. The value of 2.68 kPa is the peak strength measured in the experiment (Wartman 1999) and the value of 1.77 kPa is the residual strength of the soft clay measured in the experiment. Figure 4-23 compares the simulated displacement time histories for different peak strengths at the location of potentiometer No. 11, which is in the vicinity of the toe of the slope. It shows that the displacement decreases as the peak strength increases. The seismically-induced slope deformation is the result of the initiation and subsequent progressive movement of the sliding mass along the failure surface. The peak strength plays an important role in determining the initiation of the failure surface; whereas the post-failure strength (strain softening) mainly contributes to the progressive movement of the sliding mass along the initiated failure surface. As a result, both peak strength and residual strength have significant impacts on the development of slope failure and the resulted slope deformations.
4.4.3 Shear modulus

Shear modulus of the soil comprising the model slope can affect the slope performance in two aspects. First, shear modulus can influence the material cyclic nonlinearity and post-failure behavior, thus directly affecting the slope deformation. Second, shear modulus has a direct relation with the fundamental period of the slope, which can considerably affect the dynamic response of the model slope and the sliding mass. These effects on the seismically-induced slope deformations are examined in this section.

Figure 4-23. Effects of peak strength on slope displacement
The impacts of shear modulus can be interpreted in terms of the period ratio which is defined as the ratio of the fundamental period $T_s$ of the model slope to the mean period of the input ground motion $T_m$. The fundamental period of the slope can be estimated as

$$T_s = \frac{4H_{\text{slide}}}{V_s}$$

where $H_{\text{slide}}$ is the average height of the sliding mass and $V_s$ is the shear wave velocity which is calculated as

$$V_s = \sqrt{\frac{G}{\rho}}$$

in which $G$ is the shear modulus and $\rho$ is the soil density. The initiated failure surface shown in Figure 3-16(a) indicates a height of $H_{\text{slide}} = 0.26$ m for the sliding mass. The mean period of the ground motion $T_m$ is estimated as (Rathje and Bray 2000)

$$T_m = \frac{\sum C_i^2(1/f_i)}{\sum_i C_i^2}$$

where $C_i$ is the square roots of the sum of the squared real and imaginary parts of the fast Fourier transform coefficients for the ground motion acceleration; and $f_i$ represents the discrete fast Fourier transform frequencies from 0.25 to 20 Hz (Rathje and Bray 2000). The ground acceleration used in this parametric study and shown in Figure 3-15 has the mean period $T_m = 0.18$ s.

The parametric simulations in this section are performed by varying the shear wave velocity from $V_s = 1.5$ m/s corresponding to a fundamental period of $T_s = 0.7$ s to
\( V_s = 50 \text{ m/s} \) corresponding to a fundamental period of \( T_s = 0.02 \text{ s} \). The simulated permanent displacements at the location of potentiometer No.11 for various shear moduli (i.e., shear wave velocities) are presented in Figure 4-24. The horizontal axis represents the period ratio \( T_s/T_m \), with smaller value indicating higher shear modulus (i.e., more rigid). The permanent displacement estimated by the Newmark analysis using the peak undrained strength (Wartman et al. 2001) is also shown in Figure 4-24. The simulated permanent displacement converges to the result calculated by the Newmark analysis as the period ratio decreases. It is due to the slope behaving like a rigid body when the shear modulus approaches to the infinity. The predicted displacement increases as the shear modulus decreases (i.e. period ratio increases) until a peak value is reached when the fundamental period is in the vicinity of the mean period of the input motion. After this point, the permanent displacement drops quickly. The maximum displacement does not occur at the period ratio of 1.0 (i.e., resonance). It is largely due to the plastic response in soils and the underlying stiff clay (see Figure 3-5) that may have interfered with the dynamic responses of the slope.

The permanent displacement of the model slope consists of two components: shear deformation associated with the seismic shaking and sliding deformation along the initiated failure surface. Small shear modulus can lead to more intense plastic shear strain, resulting in large shear deformations. However, the sliding deformation depends on the overall dynamic responses of the slope and the sliding mass. As the shear modulus of the model slope decreases to very small values (i.e., larger values of \( T_s/T_m \)), the dynamic amplification of the model slope decreases (deamplification may even occur), which can be used to explain the decrease of the permanent slope deformation as \( T_s/T_m \).
increases for very large values of $T_s/T_m$. The trend shown in Figure 4-24 is consistent with the findings of Rathje and Bray (1998).

Figure 4-24. Simulated permanent displacements at various period ratios

4.5 Non-reflecting Boundary

The rigid boundary condition is applied to the side walls of the container in the model slope tests discussed in Chapter 3 and the previous sections of this chapter. This boundary treatment is consistent with the physical model slope tests. Under this boundary condition, seismic waves arriving at the boundary will be reflected back into the model slope. As a result, wave energy is contained/locked within the computational domain. In geotechnical engineering practice, most problems are unbounded or the region of interest
is small compared with the surrounding medium. In static analyses, the finite
discretization of space can realistically approximate an unbounded medium by placing a
fixed displacement boundary far from the region of interest. In dynamic problems, the
wave energy should transmit through the truncated edges (i.e., non-reflecting) of the
unbounded domain as shown in Figure 4.25(a). This energy dissipation mechanism is
referred to as geometric attenuation (damping). The treatment of truncated edges as a
fixed rigid boundary in a dynamic analysis does not allow the wave transmission and
attenuation as shown in Figure 4.25(b). Therefore, numerical simulations with rigid
boundary conditions may overestimate the dynamic response and permanent
displacements of the slopes simulated.

The use of extended computational domain can reduce the wave reflection on the
rigid boundary as the material damping can dissipate some of reflected wave energy. This
method, however, poses an overwhelming computational burden. Alternatively, the far-
field condition can be approximated by a non-reflecting or quiet boundary which can
minimize the reflected energy of an incident wave arriving at the boundary as shown in
Figure 4.25(c). Various numerical treatments of non-reflecting condition have been
developed and implemented in FEM, such as viscous damping boundary, boundary
element, infinite element, and slip-element method. However, non-reflecting boundary
has not been investigated and implemented in SPH method. In this section, a non-
reflecting boundary condition is developed and implemented into the SPH model to
examine the effect of far-field condition on seismically-induced slope deformation.
Figure 4-25. Illustration of different boundary conditions: (a) free boundary (free transmission of wave energy); (b) rigid boundary (reflection of wave energy); (c) quiet boundary (absorption of wave energy)
4.5.1 Mathematical formulation

The most widely used nonreflecting boundary was the one developed by Lysmer and Kuhlemeyer (1969), in which a viscous force is applied on the boundary to absorb incoming waves. This viscous boundary is based on the assumption of elastic body waves traveling orthogonally to the truncated edges. A brief description of this viscous boundary is presented herein, followed by its implementation in SPH method.

Considering a plane wave traveling along $x$-axis, it consists of two body waves which are the longitudinal wave (or $p$-wave) and shear wave (or $s$-wave). They can be written in the forms of D’Alembert solution. For the longitudinal wave,

$$u_x = f(x \pm c_p t), \quad u_y = u_z = 0$$  \hspace{1cm} (4-4)

where $c_p$ is the p-wave velocity and is defined as

$$c_p = \sqrt{\frac{\lambda + 2G}{\rho}}$$  \hspace{1cm} (4-5a)

$$\lambda = K - \frac{2}{3}G$$  \hspace{1cm} (4-5b)

where $G$ is shear modulus; $\lambda$ is Lame constant; and $K$ is bulk modulus.

Similarly, for the shear wave

$$u_y = f(x \pm c_s t), \quad u_x = u_z = 0$$  \hspace{1cm} (4-6a)

or

$$u_z = f(x \pm c_s t), \quad u_x = u_y = 0$$  \hspace{1cm} (4-6b)

where $c_s$ is the shear wave velocity
\[ c_s = \frac{\sqrt{G}}{\sqrt{\rho}} \]  

(4-7)

Figure 4-26. A viscous boundary absorbing waves (a) a shear wave traveling through \( A - A' \) plane in a medium; (b) a viscous boundary equivalent to the medium on the right side of \( A - A' \) plane.

In the solutions of longitudinal and shear waves, \( f(x - ct) \) represents the wave moving in the direction of positive \( x \); whereas \( f(x + ct) \) represents the wave traveling in the direction of negative \( x \). Assuming a shear wave \( u_y \) propagating from the left to the right side in an infinite medium shown in Figure 4-26 (a), the shear wave has the form of \( u_y = f(x - c_s t) \). The stress at \( A - A' \) plane is
\[ \sigma_{xy} = \sigma_{yx} = G \frac{\partial u_y}{\partial x} = G \frac{\partial f(x - c_s t)}{\partial x} \]  

(4-8)

Since the particle velocity is calculated from

\[ v_y = \dot{u}_y = \frac{\partial f(x - c_s t)}{\partial t} = \frac{\partial f}{\partial x} (-c_s) \]  

(4-9)

the shear stress can be further written as

\[ \sigma_{xy} = \sigma_{yx} = -\frac{1}{c_s} G \cdot v_y = -\rho c_s v_y \]  

(4-10)

The half-infinite medium on the right side of plane \( A - A' \) can be replaced with a viscous boundary idealized as a dashpot demonstrated in Figure 4-26(b). In order to make the dashpot equivalent to the half-infinite medium on the right side, the stress \( \hat{\sigma}_{xy} \) applied by the dashpot on the truncated boundary \( A - A' \) must be identical to the shear stress shown in Equation (4-10).

\[ \hat{\sigma}_{xy} = -\rho c_s v_y \]  

(4-11a)

Similarly, the shear stress on the boundary induced by shear wave \( u_z \) is

\[ \hat{\sigma}_{xz} = -\rho c_z v_z \]  

(4-11b)

and the normal stress induced by the longitudinal wave \( u_x \) is

\[ \hat{\sigma}_{xx} = -\rho c_p v_x \]  

(4-11c)

Accordingly, the resulting force \( F \) on the unit area of viscous boundary \( A - A' \) is

\[ F_i = \hat{\sigma}_{ij} \cdot \vec{n}_j \]  

(4-12)
where $\hat{\sigma}_j$ is the stress tensor on the boundary, and $\vec{n}_j$ is the unit normal vector on the boundary. Assuming $\vec{n}_j = (1,0,0)$ on boundary $A - A'$ and expanding Equation (4-12) yield

$$F_x = \hat{\sigma}_{xx} \cdot \vec{n}_x = -\rho c_p v_x$$  \hspace{1cm} (4-13a)  

$$F_y = \hat{\sigma}_{yx} \cdot \vec{n}_y = -\rho c_s v_y$$  \hspace{1cm} (4-13b)  

$$F_z = \hat{\sigma}_{zx} \cdot \vec{n}_z = -\rho c_s v_z$$  \hspace{1cm} (4-13c)  

The above derivations assume the wave propagating in the positive direction of $x$. If the viscous boundary is located on the left side in the medium, the three stress components become

$$\hat{\sigma}_{xx} = \rho c_p v_x$$  \hspace{1cm} (4-14a)  

$$\hat{\sigma}_{yy} = \rho c_s v_y$$  \hspace{1cm} (4-14b)  

$$\hat{\sigma}_{zz} = \rho c_s v_z$$  \hspace{1cm} (4-14c)  

The normal vector on the left boundary is $\vec{n}_j = (-1,0,0)$, and therefore the unit forces on the left viscous boundary have the same form as Equation (4-13). The viscous forces on the boundaries in other directions can be obtained in the similar manner. A more general form of viscous forces on the quiet boundary can be written as normal and shear tractions

$$t_n = -\rho c_p v_n$$  \hspace{1cm} (4-15a)  

$$t_s = -\rho c_s v_s$$  \hspace{1cm} (4-15b)  

where $t_n$ and $t_s$ are the force tractions on normal and shear directions of the boundary, respectively; $v_n$ and $v_s$ are the normal and shear components of the velocity at the
boundary. In SPH method, the ghost particle boundary presented in Chapter 3 is replaced with the aforementioned viscous force boundary in Equation (4-15) to absorb the waves arriving at the truncated edges of the computational domain.

4.5.2 Validation of quiet boundary against FEM

The implemented quiet boundary in SPH is validated against a benchmark problem in ABAQUS (2010). This benchmark problem is an infinite half space subjected to a vertical pulse load, in which the plain strain condition is assumed. Figure 4-27 shows the profile of this benchmark case. A vertical symmetric boundary is used on the left side to reduce computational cost. The symmetric boundary in SPH is approximated by the method described in Bui et al (2008). In the SPH simulation, the viscous forces are applied on the right and bottom boundaries to absorb the incident wave energy and approximate the infinite medium. The FEM model utilizes the infinite elements (ABAQUS 2010) to represent the right and bottom regions. The elastic medium has a bulk modulus of 7.3 GPa and a shear modulus of 2.8 GPa. The material density is 2842 kg/m³. The gravity is neglected in this benchmark problem for simplicity. The applied triangular impulse load is shown in Figure 4-28.
Distributed Impulse Load

Elastic Infinite Medium

$K = 7.3 \text{ GPa}$

$G = 2.8 \text{ GPa}$

Monitor Node: MN1

Monitor Node: MN2

Symmetric Boundary

Figure 4-27. An infinite half space subjected to a distributed impulse load

Figure 4-28. Triangular impulse of the distributed load
The vertical displacement time histories at the two monitor locations simulated by the SPH model are compared with the FEM solutions in Figure 4-29. The monitor node MN1 is positioned on the symmetric boundary and 0.5 m below the surface of the elastic medium as shown in Figure 4-27. The other monitor node MN2 is placed 0.2 m away from the bottom and the right boundary. Figure 4-29 demonstrates that the vertical displacement responses at both monitor locations obtained using the SPH model have an excellent agreement with that simulated by ABAQUS, validating the effectiveness of the implemented quiet boundary in SPH method. It is worth noting that the displacements simulated by FEM are slightly larger than those simulated by SPH as the artificial viscosity in SPH dissipates extra energy. The quiet boundary that is essentially a viscous force boundary works quiet well for the waves approaching to the boundary from an orthogonal direction. For waves with the angle of incidence lower than 30°, the energy absorption is not perfect (Cohen and Jennings 1983). As the quiet boundaries in SPH are formulated based on the elastic wave assumption, they should be placed at a reasonable distance from the region of interest in the presence of nonlinear material response.
Figure 4-29. Comparison of displacement time histories simulated by SPH and FEM at the monitor locations (a): MN1; (b) MN2
4.5.3 Modeling slope test with quiet boundary

The validated quiet boundary formulation in SPH is applied to simulate the model slope test presented in the previous sections. As shown in Figure 4-30, the quiet boundaries are enforced on the two side boundaries of the computational model to replace the ghost particle boundaries (i.e. rigid boundaries or displacement boundaries) used previously. The bottom of the slope model is still approximated by the ghost particle method. The use of ghost particles on the bottom is motivated by two reasons: 1) from the perspective of computational implementation, it is more convenient and accurate to treat the input seismic motion as a velocity or displacement boundary; 2) the ground motion at a certain depth below the ground surface can be easily obtained from earthquake records, which can be enforced by applying the corresponding velocity or displacement history at the bottom of numerical models.

![Figure 4-30. Boundary condition for model slope test](image)

No.11

No.12

Soft clay

Stiff clay

Non-reflecting boundary

Non-reflecting boundary
Figure 4-31. Comparison of simulated displacement time histories with non-reflecting boundary and rigid boundary in SPH model (a): location No. 11; (b) location No. 12
Figure 4-31 compares the displacement time histories at the locations of the potentiometer No.11 and No.12 simulated with non-reflecting boundary and rigid boundary. The model using rigid boundary yields slightly larger displacements toward the end of shaking than the model with non-reflecting boundary. The deformed shape and shear band developed after shaking are shown in Figure 4-32. The simulated deformed shapes agree well with each other. The plastic failure band simulated with non-reflecting boundary is slightly less intensive than that simulated with rigid boundary. It shows that the quiet boundary prevents the wave reflection that may have impacts on the overall dynamic response of the model slope. Nevertheless, the discrepancy of simulated slope deformations between these two boundary conditions is negligible, indicating a negligible effect of the rigid boundary on the overall seismic performance of the model slope. As the model slope is underlain by a stiff clay base, this base can damp out incident and reflected waves and effectively reduce the influence of reflected waves at the rigid boundaries on the dynamic response of model slope. The use of this underlying base to minimize boundary effects was also discussed by Wartman et al. (2001 and 2005).
Figure 4-32. Comparison of simulated deformed slopes with (a): rigid boundary; (b): non-reflecting boundary in SPH

4.6 Conclusions

The parametric study in this chapter has been largely devoted to investigating the effects of spatial parameters and the boundary conditions in SPH method along with material properties on seismically-induced slope deformations. It has been demonstrated that the rate-independent strain-softening SPH model exhibits spurious dependence on spatial parameters including both the particle spacing and influence length. These spatial effects preclude the convergence of SPH solutions as the particle spacing decreases. This undesirable feature in strain-softening model is found to be effectively mitigated by the strain-softening viscoplastic model. This model can also account for the rate-dependent
shear strength automatically. It is thus suggested that the implemented strain-softening viscoplastic SPH model can be applied to robustly and reliably predict seismically-induced slope deformations.

The advantage of SPH model over the conventional Newmark analysis is evident as the contribution of the plastic shear deformation to the permanent slope displacement is reflected in this study. It has been demonstrated that both peak strength and residual strength of soils have significant impacts on the development of shear failure and the slope deformations. The failure region is more localized as the residual strength decreases, therefore resulting in larger slope deformations. These findings suggest that earthquake-induced slope deformation is a strongly progressive process.

The permanent displacement of the model slope consists of two components: shear deformation associated with the seismic shaking and sliding deformation along the initiated failure surface. The SPH predicted slope deformations converge to the Newmark solution as the shear modulus approaches to very large values. Small shear modulus can lead to more intense plastic shear strain, resulting in large shear deformations. However, the sliding deformation depends on the overall dynamic responses of the slope and sliding mass. As the shear modulus of the model slope decreases to very small values, the dynamic amplification of the model slope decreases (deamplification may even occur), which may lead to smaller seismically-induced slope deformations.

A quiet boundary is implemented in this study to minimize the wave reflections on the side boundaries of the computational domain. It has been observed that the displacement response of the soft clay in the model slope test is barely influenced by the boundary condition. It is because the underlying stiff clay that behaves as a buffer layer, effectively eliminating the impacts of reflected waves on the overall slope performance.
References


CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

In recent years, considerable interest has been shown in improving current procedures for estimating seismically-induced deformations of natural and man-made slopes. These phenomena generally involve large deformations that are computationally challenging and intensive for conventional grid-based numerical methods commonly used in geotechnical engineering. The Smoothed Particle Hydrodynamics (SPH) method, being a particle-based meshless method and formulated on a continuum framework, is suitable for the simulation of various large-scale and large-deformation phenomena due to its Lagrangian and adaptive nature. A SPH model that is capable of modeling large deformations in geomaterials subjected to seismic loading is developed and validated against experiments readily available in the literature. Based on the developed and validated SPH model, a parametric study is conducted to investigate the effects of various parameters on the performance of the SPH model and provide some insights into seismically-induced slope deformations.

A 3-D SPH model is developed in Chapter 2 to simulate dense dry granular flow. The non-dilatant Drucker-Prager (D-P) constitutive relationship with perfect plasticity is implemented in the SPH formulations to model the behaviors of dense granular materials. The collapses of granular columns with various aspect ratios are simulated with the
developed SPH model. The numerical simulations precisely reproduce the experimentally-observed three regimes of flow patterns based on the initial aspect ratio of the granular column. In addition to the flow patterns, the simulated final deposit height and run-out distance along with the non-deformed region after the collapse of granular columns are in excellent agreement with experimental data in the literature. It also reveals that the simulated behavior of dense granular flow is merely dependent on the friction angle in the D-P constitutive model. Material properties such as shear modulus, density, and Poisson’s ratio have negligible impact on the granular flow patterns. This observation is consistent with the findings of Iverson and Vallance (2001). This chapter demonstrates that the developed SPH model can simulate large deformations associated with dense granular flow, despite it being a continuum-scale method.

In Chapter 3, this SPH model is extended to accommodate more complex soil behaviors which include cyclic nonlinearity, strain softening, and strain-rate dependence under seismic loading. A constitutive model that combines the strain-softening viscoplasticity and modified Kondner and Zelasko (MKZ) rule is implemented into the SPH formulations. The strain-softening feature is applied to model the shear strength reduction with shear deformations; whereas the viscoplasticity is employed to consider strain-rate effects on soil shear strength. The cyclic nonlinearity of soils when the stress states are within the yield surface is treated with the MKZ rule. The strain-softening and rate-dependent features in SPH method are validated by a benchmark FEM solution and laboratory vane shear tests. The validated SPH model is subsequently used to simulate a readily available and well-documented model slope test on a shaking table.
The simulated slope failure mode, acceleration response spectra, and slope deformations are compared with experimental data. The initiation of slope failure and subsequent progressive development of the sliding surface are successfully captured by the developed SPH model. A localized shear band along the failure surface and a bulge near the toe of the model slope are observed in the simulations, showing a good agreement with the experimental observations. The simulated displacement time histories and acceleration response spectra at several monitor locations along the model slope also agree well with the experimental recordings. The rate effect on the shear strength of the model clays is considered in two ways: a rate-dependent viscoplastic model implemented in this study and a rate-independent elasto-plastic model with the peak shear strength increased by 20%. Simulations based on these two methods indicate that the shear band width is sensitive to the treatment of rate-dependent shear strength, with the viscoplastic model yielding a wider shear band. The developed SPH model has been proved capable of estimating earthquake-induced slope deformations in a reliable and robust fashion. It further concludes that if proper constitutive models are used, SPH method can be used to effectively model complex nonlinear soil behaviors under large deformations with high accuracy.

The parametric study in Chapter 4 demonstrates that the behavior of strain-softening materials simulated by the rate-independent SPH model is spuriously dependent on spatial parameters including the particle spacing and the size of SPH influence domain. This fictitious spatial-scale sensitivity is found to be effectively mitigated by the strain-softening rate-dependent viscoplastic model that treats strain-rate induced strength increase in an autonomous manner. The parametric study also demonstrates the
advantages of the developed SPH model over conventional Newmark-type analyses in estimating seismically-induced slope deformations which consist of two components: plastic shear deformation associated with the seismic shaking and sliding deformation along the initiated failure surface. The SPH model successfully captures the significant contribution of the plastic shear deformations to the total permanent slope deformations, which cannot be considered in Newmark-type analyses. It is observed that the peak strength determines the initiation of material yielding, and therefore affects the initiation and propagation of the failure surface. The residual strength, on the other hand, mainly contributes to the progressive deformations once the peak strength is mobilized. Therefore both the peak and residual strength have significant impacts on the development of shear failure and the slope deformations. However, the transition phase between the peak strength and residual strength is found to have negligible influence on the seismic slope performance for the simulations conducted.

The effects of soil shear modulus on the seismic performance of the model slope are also investigated in this chapter. The SPH predicted slope deformations converge to the Newmark solution as the shear modulus approaches to very large values. The predicted displacement increases as the shear modulus decreases until a peak value is reached when the fundamental period is in the vicinity of the mean period of the input motion. Beyond this point, the permanent displacement drops quickly. A non-reflecting boundary, which is essentially a viscous force boundary, is first developed and implemented into the SPH method and validated against a FEM benchmark problem. It demonstrates that the developed non-reflecting boundary for SPH method can efficiently minimize the wave
reflection on the side boundaries of the computational domain and reduce its effects on the overall dynamic response of the system.

5.2 Recommendations for Future Work

5.2.1 Solid-fluid couplings in saturated soils

This dissertation focuses on the SPH modeling of single-phase soils including dry granular materials as discussed in Chapter 2 and the total stress analysis of clays in Chapter 3. Saturated soils are porous media with the voids filled with fluids. The volumetric deformation of soil skeleton tends to induce pore water pressure that may have considerable effects on the overall mechanical behavior of the whole system. The seismically-induced pore water pressure in the simulated slope and its effects on soil behavior in this study are indirectly accounted for by using the stiffness and strength degradation in a total stress framework. In order to directly account for the effect of seismically-induced pore pressure, a two-phase SPH model that is capable of simulating solid-fluid couplings in porous media is needed. The solid-fluid interaction in saturated soils has been extensively investigated by FEM (e.g., Zienkiewicz and Taylor 2000) where the Biot’s effective stress principle and transient seepage assumption are utilized to govern the interaction between solid and fluid phases. The treatment of solid-fluid interactions in a continuum scale using FEM is only capable of handling small deformations and slow fluid flow. Another approach that is based on a coupled computational fluid dynamics (CFD) treatment of the fluid phase and discrete-scale
The treatment of the solid phase (e.g., discrete element method) has been recently proposed (e.g., El Shamy 2004; Potapov et al 2001).

In light of the proven capability of SPH in modeling large deformations of geomaterials in this study, a two-phase SPH model that is solely based on the continuum framework can be developed. The locally averaged Navier-stokes equations (Jackson 2000) can be applied to both the solid and fluid phases in which the two phases are coupled through fluid pressure gradient and viscous drag force. The solid phase can be modeled by SPH while the fluid phase may be treated either by Lagrangian method (e.g. SPH) or Eulerian method (e.g. CFD). This coupled solid-fluid model formulated in a continuum framework may efficiently deal with large-scale and large-deformation problems. It may be found applicable to many geomechanical phenomena such as submarine landslides, liquefaction, and sand production during oil recovery. A preliminary study of coupled fluid-solid interaction modeled by SPH can be found in Chen and Qiu (2011).

### 5.2.2 Correction factors for Newmark analysis

The parametric study in Chapter 4 shows that the earthquake-induced slope displacement consists of two components: material deformation associated with the plastic shear strain and sliding deformation along the initiated failure surface. The conventional Newmark analysis is only capable of predicting the sliding deformation. As a result, Newmark-type procedure may underestimate the entire slope displacement as shown in section 4.4.1. According to Wartman et al. (2001), Newmark-type procedure may also over predict the slope displacement under certain situations such as when the
frequency ratio \( \left( \frac{F_{\text{motion}}}{F_{\text{slope}}} \right) \) is high. Therefore, based on the developed SPH model in this dissertation, more comprehensive parametric studies may be performed to assess the relative contribution of material plastic deformation to the overall slope displacement. This potential study may be used to provide a chart of correction factors for Newmark solutions under various conditions that includes different frequency ratios, material types, and slope profiles.

5.2.3 Coupled SPH-FEM method for soil-structure interactions

In addition, SPH can be applied in conjunction with FEM to model soil-structure interaction in the presence of large deformations. FEM is a proven and widely used method in modeling structural behaviors. However, FEM is less effective in modeling large deformations in geomaterials as compared to SPH. The coupled FEM-SPH can utilize the desirable features of these two methods. As SPH particles are essentially interpreted as computational nodes with the absence of a mesh, the SPH method can be coupled with FEM through the contact algorithm in which the penalty approach is applied on the SPH particles penetrating the FEM mesh. The combined SPH/FEM method can be applied to various geotechnical problems such as soil mixing, pile driving, and underground explosion.

Finally, SPH method is relatively expensive as the particle information in the influence domain is updated at every calculation cycle. A parallel computing algorithm for SPH method is needed for computationally intensive problems.
References


APPENDIX A

SPH SIMULATION OF COLLAPSE OF TWO-DIMENSIONAL GRANULAR COLUMN

 Prior to the application of SPH in 3-D stress conditions, a benchmark study involving non-cohesive soil column flow in 2-D plain strain condition is investigated. The initial rectangular soil column consisting of 5000 SPH particles is constrained in a box as shown in Figure A-1 (a). The aspect ratio of this column is 0.5. The granular flow is triggered by deactivating the boundary nodes on the right wall after initial equilibrium is established. The laboratory test for 2D soil column collapse was conducted by Bui et al. (2008) with aluminum bars. In this test, aluminum bars with length of 50mm and density of 2650 kg/m$^3$ were arranged into a rectangular box with the size of 200mm×100mm×50mm. Aluminum bars were used in the test in order to meet the plane strain condition. In order to validate the developed SPH model against the published results in the literature, the material properties used in this benchmark problem are chosen to be the same as that used by Bui et al. (2008). A direct shear test was conducted to determine the strength parameters of the aluminum bar assemblage. It was found that the friction angle is about 19.8$^\circ$, and the average bulk modulus is approximately 0.7 MPa. The Poisson’s ratio is assumed to be 0.3. It is worth to point out that the material properties such as shear modulus and Poisson’s ratio have been demonstrated in Chapter 3 to render negligible effects on dry dense granular flow. The simulated granular flow is highly dependent on
the friction angle. For more details regarding this experiment, please refer to Bui et al. (2008). The average particle spacing in the SPH model is 2.5 mm.

**Figure A-1.** Development of failure surface during the collapse of granular column

After releasing the granular column, only particles on the edge start to move. As indicated in Figure A-1 (b), an initial failure surface is developed immediately after granular flow starts. The failure surface propagates toward the upper free surface as material flows down (Figure A-1 (c), (d) and (e)). The plastic strain is accumulated gradually with the propagation of the failure surface. After all motion has ceased, a flat undisturbed area at the upper surface remains at its initial height. SPH simulation is compared with experimental results in Figure A-2. The simulated slope profile after collapse and the contour of accumulated plastic strain are shown in Figure A-2 (a). The
particles in dark blue indicate non-deformed region with zero accumulated plastic strain, which qualitatively agrees well with the experiment observation presented in Bui et al. (2008). The simulation result is also consistent with the conclusions drew from the experiments on collapses of 2D granular columns conducted by Lube et al. (2005). It is found that if the ratio of the initial height to basal length is less than 1.8, an undisturbed region remaining at its initial height is preserved after collapse of the column. This phenomenon is exactly reproduced in our simulation with the undisturbed region clearly shown in Figure A-2 (a). The comparison of final deposit profiles presented in Figure A-2 (b) indicates an excellent agreement on the final deformed shape between the SPH simulation and experiment.

In summary, this benchmark simulation shows that the current model combing SPH formulation and Drucker-Prager plasticity with zero dilatancy angle is capable of predicting collapses of dense granular columns with satisfactory accuracy.
Figure A-2. Comparison between experiment and SPH simulation of collapse of aluminum bar column in 2-D plane strain condition: (a) simulated slope profile after collapse and contour of accumulated plastic strain; (b) slope profiles after collapse in experiment and simulation.
References


APPENDIX B

SPH SIMULATION OF AN ELASTIC CANTILEVER BEAM

The standard SPH formulation of solid mechanics may exhibit numerical instability when the material is under tension. This instability is called tensile instability and can result in SPH particles forming nonphysical clumps. After first studied by Swegle et al. (1995), there have been several attempts to remove the instability by correcting the kernel functions (e.g. Johnson and Biessel 1997; Schussler and Schmitt 1981). However, these corrected kernel interpolations were not successful in all cases (Morris 1996). Recently, two most commonly used and effective methods to remediate the tensile instability are conservative smoothing approach (Randles and Libersky 1996) and artificial stress method (Monaghan 2000).

The conservative smoothing approach adds a stabilizing dissipation to the momentum equation and mass conservation equation. The corrected velocity and density with the conservative smoothing approach are expressed as

\[
\hat{v}_i = v_i + \alpha_{cs} \left[ \sum_{j \neq i} \frac{m_j}{\rho_j} \frac{v_j W_{ij}}{r_{ij}} - v_i \right] \tag{B-1}
\]
\[ \hat{\rho}_i = \rho_i + \alpha_{cs} \left[ \frac{\sum_{j \neq i} m_j W_{ij}}{\sum_{j \neq i} \rho_j} - \rho_i \right] \]  

(B-2)

where the subscript \( i \) and \( j \) represent the indices of SPH particles; the embellishment \( \wedge \) represents the corrected values with the conservative smoothing; \( \alpha_{cs} \) is a positive coefficient that is usually smaller than 0.5. In addition to removing the tensile instability, this approach can also replace the artificial viscosity (see Equation 3-8) to control numerical oscillation in the presence of shock (Randles and Libsersky 1996).

The artificial stress method (Monaghan 2000) introduces a repulsive force term in the momentum equation to prevent particle clumping caused by the tensile stress. The formulation of this method has been presented in Equation 3-10, and therefore is not shown herein.

Despite the effectiveness in removing the tensile instability, these two methods tend to induce artificial energy dissipation. The simulation of seismic performance of earth slopes in this study needs the realistic treatment of material and geometrical damping under dynamic loads. Therefore, it is imperative to perform a quantitative study examining the energy dissipated by these two methods. The free vibration of an elastic plane cantilever beam is simulated by these two methods to investigate the energy dissipation. As shown in Figure B-1, the cantilever has a length of \( L = 1.0 \text{ m} \) and width of \( H = 0.2 \text{ m} \). The elastic modulus is \( E = 1.92 \times 10^8 \text{ Pa} \), and the density is \( \rho = 8000 \text{ kg/m}^3 \). The Poisson’s ratio is assumed to be \( \nu = 0.3 \). There is no material damping in this case. A transverse impulse is applied on the top of the beam to trigger the motion.
The analytical solution of the natural frequency for a plane beam with one end free and one end fixed is (Gray et al 2001)

\[ w = \beta_n^2 \sqrt{\frac{EH^2}{12\rho(1-v^2)}} \]  \hspace{1cm} (B-3)

where \( \beta_n \) is determined by the characteristic equation

\[ \cos(\beta_n L) \cosh(\beta_n L) = -1 \]  \hspace{1cm} (B-4)

The first mode is \( \beta_n L = 1.875 \). The fundamental period of this plane cantilever is \( T_s = 0.19 \)s based on Equation (B-3).

In the artificial stress method, both the repulsive force term and the artificial viscosity are required to stabilize the SPH simulation (see Equation 3-10). The exponential coefficient for the repulsive force \( \vartheta = 4.0 \) and the coefficient in the artificial viscosity term \( \alpha_{II} = 2.5 \) (see Equation 3-8) are found adequate to mitigate the tensile instability in this case. The conservative smoothing approach has no necessity to use the artificial
viscosity. The dissipative coefficient $\alpha_{cs} = 0.05$ is the sufficient value that can successfully remove the tensile instability.

**Figure B-2.** Time history of horizontal displacement on the top of the beam simulated by SPH with: (a) artificial stress method; (b) conservative smoothing method

The comparison of displacement time history on the top of the beam simulated by the artificial stress method and conservative smoothing approach is shown in Figure B-2. The simulated period is about 0.2s that agrees well with the analytical solution. The artificial
stress method produces the negligible energy dissipation with a damping of 0.5%. However, the smoothing conservative approach dissipates considerable amount of energy, resulting in a damping of 4.5%.

In summary, the conservative smoothing approach excessively dissipates energy in the case of free vibration, and therefore it is not suitable for evaluating earthquake-induced slope deformation. This benchmark problem justifies the selection of the artificial stress method to remediate the tensile instability in Chapter 3.
References


VITA

Wei Chen

EDUCATION

Ph.D. in Civil Engineering, The Pennsylvania State University, University Park, PA

B.S. in Engineering Mechanics, Hohai University, Nanjing, China

PROFESSIONAL EXPERIENCE

2012 – Present                      FEA Structural Engineer
                                      2H Offshore Inc., Houston, TX

2010 – 2012                         Research / Teaching Assistant
                                      The Pennsylvania State University, University Park, PA

2008 – 2010                         Teaching Assistant
                                      Clarkson University, Potsdam, NY

SELECTED PUBLICATIONS


