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MODELS OF STRATEGIC AND PAIRWISE TRADE

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ABSTRACT

This dissertation consists of three chapters**.**

In Chapter 1, a new type of market game is formulated: the strategy space generalizes the usual Cournot quantities with limit prices. Under mild market-thickness conditions, symmetric Nash equilibria coincide with price-taking equilibria. In the case of two goods, a price-taking equilibrium is a Walrasian equilibrium. In the case of multiple goods, a price-taking equilibrium is competitive, but subject to a cash-in-advance constraint.

Chapter 2 considers periodic, complete-participation trade in the Lagos-Rocheteau (Econometrica, 2009) Mode. Lagos-Rocheteau is part of the literature that applies a search model to asset trade in the over-the-counter market. The only friction in their model is a cost of agents getting into contact with other agents. Therefore, as an alternative to their investor-dealer random meetings, a centralized competitive market which occurs periodically is studied. This arrangement preserves the main tension in their paper: a tradeoff between a portfolio that maximizes current utility and one that is good on average. For calibrated versions of the model, it is shown that this market must occur only infrequently in order for investors to be as well off as they in the Lagos-Rocheteau setup.

In Chapter 3, employment agents (called as em-agents) are introduced as a third type of agent in world of a labor market with search frictions. Each type of agent matches pairwise with the other two types through two independent matching processes. Each process depends on the ratio of the agent's own type to the other type in the match (market tightness). Job matches can be formed directly between a worker and a firm, or indirectly through an em-agent. It is shown that there is a unique steady-state equilibrium in which em-agents are active. The presence of em-agents enhances the welfare of workers. More generally, the welfare of workers decreases as the outside option of em-agents or firms increases. When job matches are heterogeneous in productivity, jobs filled through em-agents have higher average productivity than ones done directly.

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Dedication

To my parents, my wife, Huan and my son, Alan.

CHAPTER 1

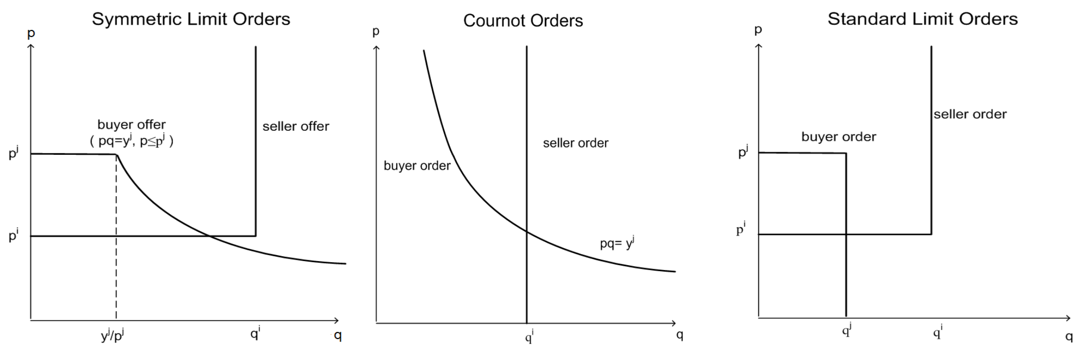
A Market Game with Symmetric Limit Orders

1.1. Introduction

A substantial literature focuses on providing a strategic foundation for competitive equilibrium using variants of a Cournot quantity game, sometimes called a market game or a trading-post game. We contribute to this literature by studying a market game with a new kind of limit order. The first diagram of figure 1 illustrates the strategies in a setting with one good and a numeraire. An agent who wants to sell the good submits a supply curve specifying the maximum amount of the good to be sold if the price is not less than that named. An agent who wants to buy the good submits a demand curve specifying the maximum expenditure on the good if the price is not higher than that named. These are called symmetric limit orders because within a two-object market, the outcome would be the same if the roles of the good and the numeraire were reversed.

The strategy space of our market game nests that of the Cournot type quantity game, whose typical strategy is shown in the second diagram of figure 1.1, where a seller offers some amount of the good and a buyer offers some amount of the numeraire. The limit orders studied in the previous literature are displayed in the third diagram of figure 1.1—see Dubey (1982), Simon (1984) and Mertens (2003). They are not symmetric in the above sense, do not nest the Cournot quantity offers, and do not determine a unique price for arbitrary strategies.

Figure 1.1: Strategy Space



In this chapter, it is shown that mild market-thickness conditions imply that the set of symmetric Nash equilibria coincide with the set of price-taking equilibria. In an economy with two goods, any price-taking equilibrium is, by definition, a Walrasian equilibrium. For the case with multiple goods, we consider price-taking equilibria subject to a cash-in-advance constraint. There are three market thickness conditions. First, there are two small-endowment, nonstrategic agents (from now on called special agents) in each market; they submit small Cournot offers as in the second diagram of figure 1.1 shows.[1] Second, we assume that any price-taking equilibrium of the economy without special agents has trade in each market. Third, there are at least two (regular) agents of each type, where a type is determined by preferences and endowments.

There are two widely recognized issues that need to be dealt with in order to have any or all equilibria of a market game be a Walrasian equilibrium. One is market thickness and the other is the liquidity problem that prevents the proceeds of sales in a two-object market from being used to make offers on another two-object market. Dubey and Shubik

[1]This is the way we embed the Dubey and Shubik (1978) refinement, a refinement that prevents no-trade from being an equilibrium in the game.

(1978) deal with the first, but require a large market in the sense of a limit under replication. For a variant of the Cournot type market game that permits short sales, Peck and Shell (1990) show that there is always a Nash equilibrium allocation that is arbitrarily close to a competitive equilibrium allocation. However, they use an overall budget constraint and assume that the game is shut down if anyone exceeds his budget constraint. Dubey, Sahi and Shubik (1993) study retrading in a market game with a continuum of agents and find that if enough repeated rounds of trade are permitted, then Nash outcomes are Walrasian. With a finite number of agents, Ghosal and Morelli (2004) show that competitive equilibrium allocations can be approximated arbitrarily closely under repeated retrading when trade is myopic. In addition, they show that the same sequence of trades that approximates a Pareto optimal allocation under myopic retrading is a subgame perfect equilibrium under far-sighted retrading under some information structure. However, they also point out that there is a huge multiplicity of equilibria with far-sighted retrading and that not all subgame perfect equilibrium allocations are close to the Pareto frontier. This chapter does not deal with the liquidity issue. Instead, it provides new results concerning market thickness.

The model environment is set out in section 2. The two-good model is analyzed in section 3. A many-good version is analyzed in section 4. Concluding remarks appear in section 5. All the proofs appear in the appendix.

1.2. Environment

Consider an exchange economy with I agents who trade in 1, 2,..., m, $m+1$ goods, the last one being the numeraire good. Each agent $i \in I = \{1,...,I\}$ has an initial endowment $\omega^i \in \mathbb{R}_+^{m+1} \backslash \{0\}$ and a twice differentiable, strictly concave, strictly increasing utility function: $u^i : \mathbb{R}_+^{m+1} \longrightarrow \mathbb{R}_+$ which satisfies $u_j^i(c^i) = \infty$ if $c_j^i = 0$, and $u^i(0) = 0$. Also, $\sum_{i=1}^I \omega^i \gg 0$; i.e., every named commodity is present in the aggregate.

For each $j \in M = \{1,...,m\}$, there are two special agents: one has a small endowment of good j and only gets utility from the numeraire; the other is endowed with a small amount of the numeraire and only gets utility from good j. These agents don't act strategically. They are assumed to make Cournot offers of their endowments.

1.3. The two-good model

In this section, we provide sufficient conditions for coincidence between the set of symmetric Nash equilibria and the set of Walrasian equilibria.

1.3.1. The Game

Information is complete and agents move simultaneously. Let

$$P_L = \frac{\epsilon_2}{\epsilon_1 + \sum_{i=1}^I \omega_1^i},\ P_H = \frac{\epsilon_2 + \sum_{i=1}^I \omega_2^i}{\epsilon_1},$$

and let $P = [P_L, P_H]$.

The strategy set of agent i is

$$S^i = \left\{ \begin{array}{c} \{(q^i, p^i, y^i)\} \mid 0 \le q^i \le \omega_1^i,\ 0 \le y^i \le \omega_2^i, \\ q^i y^i = 0,\ p^i \in P. \end{array} \right\}, \tag{1.1}$$

where q^i is a quantity of good 1, y^i is a quantity of the numeraire, and p^i is a price of good 1 in terms of the numeraire. Let $S = S^1 \times \cdots \times S^I$ and let $S^{-i} = S^1 \times \cdots \times S^{i-1} \times S^{i+1} \times$

$\cdots \times S^I$. Also, let s, s^i, and s^{-i} stand for elements of S, S^i, and S^{-i}, respectively. One special agent offers $\{(\epsilon_1, P_L, 0)\}$ and the other offers $\{(0, P_H, \epsilon_2)\}$, for some $\epsilon = (\epsilon_1, \epsilon_2) > 0$.

Payoffs are expressed in terms of a market-clearing price, denoted $p(s)$, which is determined as follows. Given $s \in S$, demand and supply correspondences for good 1 are well defined for each $p \in P$. Demand is

(1.2) $$Q^d(p) = [(\epsilon_2 + \sum_{\{i \mid p^i > p\}} y^i)/p,\ (\epsilon_2 + \sum_{\{i \mid p^i \geq p\}} y^i)/p],$$

while supply is

(1.3) $$Q^s(p) = [\epsilon_1 + \sum_{\{i \mid p^i < p\}} q^i,\ \epsilon_1 + \sum_{\{i \mid p^i \leq p\}} q^i].$$

Then, excess demand, denoted $E(p)$, is

$$E(p) = \left\{\ q \mid \exists\ q_d \in Q^d(p), \text{ and } q_s \in Q^s(p), \text{ such that } q = q_d - q_s\ \right\}.$$

Lemma 1.1: $\forall s \in S$, $\exists$ a unique $p(s)$ such that $0 \in E(p(s))$. Moreover, $p(s) \in P$.

Now we can express payoffs. Let $E_d = \max\left\{\sup Q^d(p(s)) - \inf Q^s(p(s)),\ 0\right\}$, residual demand for sellers with $p^i = p(s)$, and let $E_s = \max\left\{\sup Q^s(p(s)) - \inf Q^d(p(s)),\ 0\right\}$, residual supply for buyers with $p^i = p(s)$. The amount of good 1 agent i buys is

$$(1.4) \qquad \phi^i(s) = \begin{cases} \frac{y^i}{p(s)} & \text{if } p^i > \ p(s) \\ \min\{\frac{E_s}{\sum_{\{k \ | \ p^k = p(s)\}} y^k} y^i, \ \frac{y^i}{p(s)}\} & \text{if } p^i = \ p(s) \text{ and } y^i > 0 \\ 0 & \text{otherwise} \end{cases} ,$$

and the amount agent i sells is

$$(1.5) \qquad \varphi^i(s) = \begin{cases} q^i & \text{if } p^i < p(s) \\ \min\{\frac{E_d}{\sum_{\{k \ | \ p^k = p(s)\}} q^k} q^i, \ q^i\} & \text{if } p^i = \ p(s) \text{ and } q^i > 0 \\ 0 & \text{otherwise} \end{cases} .$$

Notice that a buyer who specifies a price higher than $p(s)$ sells the amount of the numeraire offered. If there is excess demand at $p(s)$, an order that named $p(s)$ shares residual supply at $p(s)$ with other orders specifying the same price according to the shares of the expenditures offered. Other orders are not executed. A sell order that specifies a price lower than $p(s)$ gets fully executed. If there is excess supply at $p(s)$, an order naming $p(s)$ shares residual demand with other orders specifying the same price, according to the share of quantities supplied. Other orders are not executed.

Then, the allocation of agent i is $c^i(s) = (c^i_1(s), c^i_2(s))$, where

$$(1.6) \qquad c^i_1(s) = \omega^i_1 + \phi^i(s) - \varphi^i(s) \text{ and } c^i_2(s) = \omega^i_2 + (\varphi^i(s) - \phi^i(s))p(s).$$

Hence, the payoff for agent i is $u^i(c^i(s)) = u^i(c^i(s^{-i},s^i))$. In terms of these payoffs, we have

Definition 1.1: A pure strategy Nash equilibrium is $s \in S$, such that

$$s^i \in \arg\max_{x \in S^i} u^i(c^i(s^{-i}, x)) \tag{1.7}$$

for each $i \in I$.

1.3.2. Coincidence

Let $\mathbb{W}(\epsilon)$ and $\mathbb{N}(\epsilon)$ denote the sets of Walrasian and outcomes of symmetric Nash equilibria, respectively. To get coincidence between these sets requires more than the Dubey-Shubik refinement. To see why, suppose there is only one regular agent. In the game, this agent chooses a price and an offer to maximize his utility given the residual demand of special agents, which is $\frac{\epsilon_2}{p} - \epsilon_1$. Hence, the unique Nash equilibrium is the same as the monopoly solution for a monopolist who faces that demand and it is not a Walrasian equilibrium. To get coincidence, we impose two additional conditions.

A1: At least one element of $\mathbb{W}(0)$ has trade.

A2: There are at least two regular agents of each type, where a type is determined by preferences and endowments.

Proposition 1.1: Assume A1 and A2. $\exists\ \epsilon_0 \gg 0$ such that if $\epsilon \in (0,\epsilon_0)$, then $\mathbb{W}(\epsilon) = \mathbb{N}(\epsilon)$.

Without special agents, no trade is a Nash equilibrium and, therefore, $\mathbb{N}(0) \subseteq \mathbb{W}(0)$ fails. Assumption A1 is used to construct a small enough ϵ_0 so that $\mathbb{W}(\epsilon) \subseteq \mathbb{N}(\epsilon)$ can be established. The main idea is to propose as a candidate equilibrium in the game that regular agents name the Walrasian price and quantities. If there is trade in $\mathbb{W}(0)$ and ϵ is sufficiently small, then no regular agent is induced to sacrifice trade with other regular agents in order to act like a monopolist relative to the special agents. As for A2, it plays the same role in establishing $\mathbb{W}(\epsilon) \supseteq \mathbb{N}(\epsilon)$ as two agents play in the demonstration that a competitive outcome is achieved if two agents engage in Bertrand competition.

1.4. Many goods

In this section, we provide sufficient conditions for coincidence between the set of symmetric Nash equilibria and the set of price-taking equilibria under a cash-in-advance (CIA) constraint. We begin by defining the latter.

Definition 1.2: $\langle p, \{c^i\}_{i=1}^I \rangle$ is a CIA price-taking equilibrium of economy $\langle \{\omega^i\}_{i=1}^I, \{u^i\}_{i=1}^I, \epsilon \rangle$ if $\{c^i\}_{i=1}^I$ is feasible and if c^i maximizes $u^i(c^i)$ subject to

$$c_{m+1}^i = \omega_{m+1}^i + \sum_{j=1}^m (\omega_j^i - c_j^i) p_j,$$

and

$$\sum_{\{j \mid c_j^i > \omega_j^i, \, j \in M\}} (c_j^i - \omega_j^i) p_j \leq \omega_{m+1}^i,$$

where $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_m)$, and in each market, one special agent supplies ϵ_j amount of the good, while the other special agent offers ϵ_j amount of numeraire to exchange for the good.

We assume that $c^i(\omega^i, p)$ is continuously differentiable and there exists a CIA price-taking equilibrium.

1.4.1. The Game

As above, the equilibrium notion is Nash equilibrium. Let

$$P_{L,j} = \frac{\epsilon_j}{\epsilon_j + \sum_{i=1}^{I} \omega_j^i},\ P_{H,j} = \frac{\epsilon_j + \sum_{i=1}^{I} \omega_{m+1}^i}{\epsilon_j},$$

and let $P_j = [P_{L,j},\, P_{H,j}]$, $j \in M$.

The strategy set of agent i is

$$S^i = \left\{ \begin{array}{c} \{(q_j^i, p_j^i, y_j^i)\}_{j=1}^m \mid p_j^i,\, y_j^i \in \mathbb{R}_+, \\ q_j^i y_j^i = 0,\, 0 \le q_j^i \le \omega_j^i, \\ p_j^i \in P_j,\, j \in M,\ \sum_{j=1}^m y_j^i \le \omega_{m+1}^i. \end{array} \right\} \tag{1.8}$$

where q_j^i is a quantity of the good j, y_j^i is a quantity of numeraire spent on good j, and p_j^i is a price of good j in terms of the numeraire and $\sum_{j=1}^m y_j^i \le \omega_{m+1}^i$ requires that total spending on goods cannot exceed the endowment of the numeraire. Let $S = S^1 \times \cdots \times S^I$, and let $S^{-i} = S^1 \times \cdots \times S^{i-1} \times S^{i+1} \times \cdots \times S^I$. Also, let s, s^i, and s^{-i} stand for elements of S, S^i, and S^{-i}, respectively. We assume that in each market j, one special agent offers $\{(\epsilon_j,\, P_{L,j},\, 0)\}$, and the other offers $\{(0,\, P_{H,j},\, \epsilon_j)\}$ for some $\epsilon_j > 0$, $j \in M$,

Payoffs are expressed in terms of market-clearing prices, denoted $p_j(s)$, $j \in M$, which are determined as follows. Given $s \in S$, demand and supply correspondences for good j are well-defined for each $p_j \in P_j$. Demand for good j is

$$Q^d_j(p_j) = [(\epsilon_j + \sum\nolimits_{\{i \ | \ p^i_j > p_j\}} y^i_j)/p_j,\ (\epsilon_j + \sum\nolimits_{\{i \ | \ p^i_j \geq p_j\}} y^i_j)/p_j], \tag{1.9}$$

while supply for good j is

$$Q^s_j(p_j) = [\epsilon_j + \sum\nolimits_{\{i \ | \ p^i_j < p_j\}} q^i_j,\ \epsilon_j + \sum\nolimits_{\{i \ | \ p^i_j \leq p_j\}} q^i_j]. \tag{1.10}$$

Then, excess demand, denoted $E_j(p_j)$, is

$$E_j(p_j) = \left\{q | \ \exists \ q_d \in Q^d_j(p_j), \text{ and } q_s \in Q^s_j(p_j), \text{ such that } q = q_d - q_s. \ \right\}$$

Lemma 1.2: $\forall s \in S$, $\exists$ a unique $p_j(s)$ such that $0 \in E_j(p_j(s))$. Moreover, $p_j(s) \in P_j$.

The proof of Lemma 1.2 is the same as that of Lemma 1.1.

Now we can express payoffs. Let

$$E_{d,j} = \max\left\{\sup Q^d_j(p_j(s)) - \inf Q^s_j(p_j(s)),\ 0\right\}$$

be residual demand for sellers of good j with $p^i_j = p_j(s)$, and let

$$E_{s,j} = \max\left\{\sup Q^s_j(p_j(s)) - \inf Q^d_j(p_j(s)), 0\right\}$$

be residual supply for buyers of good j with $p^i_j = p_j(s)$. The amount of good j agent i buys is

$$(1.11) \qquad \phi_j^i(s) = \begin{cases} \frac{y_j^i}{p_j(s)} & \text{if } p_j^i > p_j(s) \\ \min\{\frac{E_{s,j}}{\sum_{\{k \mid p_j^k = p_j(s)\}} y_j^k} y_j^i, \ \frac{y_j^i}{p_j(s)}\} & \text{if } p_j^i = p_j(s) \text{ and } y_j^i \neq 0 \\ 0 & \text{otherwise} \end{cases},$$

and the amount of good j agent i sells is

$$(1.12) \qquad \varphi_j^i(s) = \begin{cases} q_j^i & \text{if } p_j^i < p_j(s) \\ \min(\frac{E_{d,j}}{\sum_{\{k \mid p_j^k = p_j(s)\}} q_j^k} q_j^i, \ q_j^i) & \text{if } p_j^i = p_j(s) \text{ and } q_j^i \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

As in the two goods case, a buyer who specifies a price higher than $p_j(s)$ sells the amount of the numeraire offered. If there is excess demand at $p_j(s)$, an order that named $p_j(s)$ shares residual supply at $p_j(s)$ with other orders specifying the same price according to the shares of the expenditures offered. Other orders are not executed. A sell order that specifies a price lower than $p_j(s)$ gets fully executed. If there is excess supply at $p_j(s)$, an order naming $p_j(s)$ shares residual demand with other orders specifying the same price, according to the share of quantities supplied. Other orders are not executed.

Then, the allocation of agent i is $c^i(s) = \big(c_1^i(s), \cdots, c_{m+1}^i(s)\big)$, where

$$(1.13) \qquad c_j^i(s) = \omega_j^i + \phi_j^i(s) - \varphi_j^i(s),\ j \in M,$$

and

$$c^i_{m+1}(s) = \omega^i_{m+1} + \sum_{j=1}^{m}(\varphi^i_j(s) - \phi^i_j(s))p_j(s). \tag{1.14}$$

Hence, the payoff for agent i is $u^i(c^i(s)) = u^i(c^i(s^{-i},s^i))$. In terms of these payoffs, we have

Definition 1.3: A pure strategy Nash equilibrium is $s \in S$, such that

$$s^i \in \arg\max_{x \in S^i} u^i(c^i(s^{-i}, x)) \tag{1.15}$$

for each $i \in I$.

1.4.2. Coincidence

Let $\mathbb{W}_C(\epsilon)$ and $\mathbb{N}_C(\epsilon)$ denote the sets of price-taking equilibria and outcomes of symmetric Nash equilibria, respectively. To get coincidence between these sets, we impose similar conditions as in the two goods case. The proof is also similar, except that we need to handle the possibility of simultaneous deviation in several markets. The following assumption replaces A1.

A3: Any element of $\mathbb{W}_C(0)$ has trade in each market.

Proposition 1.2: Assume that A2 and A3 hold. $\exists\ \epsilon_0 \gg 0$ such that if $\epsilon \in (0,\epsilon_0)$, then $\mathbb{W}_C(\epsilon) = \mathbb{N}_C(\epsilon)$.

1.5. Conclusion

Our one-order-per-good restriction and the restriction that an agent is not allowed to sell and buy the good in the same market are without loss of generality. They obviously do not matter for $\mathbb{W}(\epsilon) \supseteq \mathbb{N}(\epsilon)$. As for $\mathbb{W}(\epsilon) \subseteq \mathbb{N}(\epsilon)$, only net trades and the net effect of deviations is relevant.

However, it must be admitted that our market game shares some undesirable properties with other limit order models, even though it nests the Cournot quantity game. First, given the strategies of others, one agent's payoff is not continuous in the agent's strategy. For example, if an active seller raises his price, then he may sell nothing. Second, at equilibrium, the best response of an agent is not unique. For example, in the two goods case, given that all other agents submit their Walrasian quantities and the competitive price, the best response for an agent who wants to sell can be the Walrasian quantity and any price that does not exceed the Walrasian price.

1.6. Appendix

1.6.1. Proof of Lemma 1.1

The demand and supply correspondences for each agent are obviously upper hemicontinuous and bounded. Following Aumann (1976), it follows that $Q^d(p)$ and $Q^s(p)$ are upper hemicontinuous. Since $Q^d(p)$ and $Q^s(p)$ are upper hemicontinuous and convex valued, these properties hold for $E(p)$.

For any p_1, p_2, monotonicity is defined by comparing any two elements of $Q^s(p_1)$ and $Q^s(p_2)$, or any two elements of $Q^d(p_1)$ and $Q^d(p_2)$. Then $Q^s(p)$ is non-decreasing in p, and $Q^d(p)$ is strictly decreasing in p. $E(p)$ is strictly decreasing in p.

Define $\Theta = \{p : \exists\ a_p \in E(p) \text{ and } a_p \leq 0\}$. Then $\inf \Theta$ is a candidate for $p(s)$.

First, Θ is not empty. Let $P_H = \frac{\epsilon_2 + \sum_{i=1}^{I} \omega_2^i}{\epsilon_1}$. Because $0 < max(Q^d(P_H)) \leq \frac{\epsilon_2 + \sum_{i=1}^{I} \omega_2^i}{P_H}$, and $\min Q^s(P_H) \geq \epsilon_1$, $\forall\ a \in E(P_H)$, $a \leq 0$. Hence, $P_H \in \Theta$, which implies that $\inf \Theta \leq P_H$. If $0 \in E(P_H)$, then because $\forall\ p < P_H$, $\forall a \in E(p)$, $a > 0$, we have $\inf \Theta = P_H$.

Let $P_L = \frac{\epsilon_2}{\epsilon_1 + \sum_{i=1}^{I} \omega_1^i}$. For $\forall p < P_L$, $\min Q^d(p) > \frac{\epsilon_2}{P_L}$, and $\max(Q^s(P_L)) \leq \epsilon_1 + \sum_{i=1}^{I} \omega_1^i$. Therefore, $\forall\ a \in E(P_L)$, $a > 0$. By monotonicity, $\inf \Theta \geq P_L$. If $0 \in E(P_L)$, because $\forall\ p > P_L$, $\forall a \in E(p)$, $a < 0$, we have $\inf \Theta = P_L$.

Now assume $P_L < \inf \Theta < P_H$. We know that given $p \in [P_L,\ P_H]$, $E(p)$ is not empty.

Suppose $\forall\ a \in E(\inf \Theta)$, $a < 0$. By the definition of $\inf \Theta$, $\forall P_L \leq p < \inf \Theta$, there is no $a_p \in E(p)$ such that $a_p \leq 0$. Now consider a price sequence $\{p^n\}$ such that $P_L \leq p^n < \inf \Theta$ and $p^n \to \inf \Theta$ as $n \to \infty$. Then there is $a \in E(p^n)$ with $a > 0$. Take any $a_{p^n} \in E(p^n)$ such that $a_{p^n} > 0$. Because $E(p^n)$ and $\{a_{p^n}\}$ are bounded, there is a convergent subsequence of $\{a_{p^n}\}$, denoted $\{a_{p^{n,m}}\}$, and an associated correspondence $\{p^{n,m}\}$. Obviously, $a_{p^{n,m}} \longrightarrow \hat{a} \geq 0$, by upper hemicontinuity of $E(p)$, $a \in E(\inf \Theta)$. Contradiction.

Suppose now $\forall a \in E(\inf \Theta)$, $a > 0$. Consider a price sequence $\{p^n\}$ such that $P_h \geq p^n > \inf \Theta$ and $p^n \to \inf \Theta$ as $n \to \infty$. By decreasing of $E(p)$, there is $a \in E(p^n)$ with $a \leq 0$. Take any $a_{p^n} \in E(p^n)$ such that $a_{p^n} \leq 0$. Because $E(p^n)$ and $\{a_{p^n}\}$ are bounded, there is a convergent subsequence of $\{a_{p^n}\}$, denoted $\{a_{p^{n,m}}\}$, and an associated correspondence $\{p^{n,m}\}$. Obviously, $a_{p^{n,m}} \longrightarrow \hat{a} \leq 0$, by upper hemicontinuity of $E(p)$, $a \in E(\inf \Theta)$. Contradiction.

Together, $0 = E(\inf \Theta)$ or there is $a_1 \geq 0$ and $a_2 \leq 0$ such that a_1, $a_2 \in E(\inf \Theta)$. By the convexity of $E(p)$, $0 \in E(\inf \Theta)$.

By definition, $\forall p < \inf \Theta, 0 \notin E(p)$. Since $\inf E(\inf \Theta) \leq 0$, by monotonicity, $\forall p > \inf \Theta$, and $\forall a \in E(p)$, $a < \inf E(\inf \Theta) \leq 0$. This proves the uniqueness of $p(s)$.

1.6.2. Proof of Proposition 1.1

There are three steps in the proof. In the first step, assumption A1 is used to construct endowments for the special agents. The second step establishes $\mathbb{W}(\epsilon) \subseteq \mathbb{N}(\epsilon)$, while the third step establishes $\mathbb{N}(\epsilon) \subseteq \mathbb{W}(\epsilon)$.

1. Construction of ϵ_0.

Given the twice differentiable, strictly concave, strictly increasing utility function which satisfies $u^i(0) = 0$, it is easy to check that assumptions $(i) - (iv)$ in Katzner (1968) are satisfied here, hence, by Katzner (1968) or Debreu (1970), the demand function for regular agents are continuously differentiable. The sum of special agents's demand function, $\frac{\epsilon_2}{p} - \epsilon_1$ is obviously continuously differentiable. Hence, the following remark in Debreu (1970) hold here. Define $\mathbb{H}$ to be the set of strictly positive vectors in R^2, such that the sum of the components is unity and let $v^0 = \left\{\omega^1, \cdots, \omega^I, 0\right\}$, where 0 stands for $\epsilon = 0$. Then there is an open neighborhood V of v^0 and Z continuously differentiable functions $g_1, \cdots, g_Z$ from V to $\mathbb{H}$ such that for every v in V, the set of equilibrium price,

$W(v)$, consists of Z distinct elements $g_1(v), \cdots, g_Z(v)$. Because any Walrasian allocation $\{c^i(v)\}_{i=1}^I$ is a continuously differentiable function of price, we come to the same conclusion for them. In particular, because demand is homogeneous of degree zero, $\mathbb{W}(\epsilon)$ consists of Z distinct elements, which are continuous functions of ϵ as ϵ converges to zero.

Hence, there are a finite number of Walras equilibria in the economy without special agents. By Proposition 17.F.5 in Mas-Colell, Whinston and Green (1995), we know that A1 implies that there is trade in any Walras equilibrium $\langle p, \{c^i\}_{i=1}^I\rangle$ of the economy without special agents.

Take the $z-th$ such equilibrium, $\langle p, \{c^i\}_{i=1}^I\rangle$. There is an agent i with $c_1^i < \omega_1^i$. Because $u^i(c^i) > u^i(\omega_1^i, \omega_2^i)$, by continuity of u^i, there is $\epsilon_{z2} > 0$ such that, $u^i(c^i) > u^i(\omega_1^i, \omega_2^i + \epsilon_{z2})$. Similarly, there is an agent k with $c_1^k > \omega_1^k$ and $\epsilon_{z1} > 0$ such that, $u^k(c^k) > u^k(\omega_1^k + \epsilon_{z1}, \omega_2^k)$. Let $\delta_z = \min\{u^k(c^k) - u^k(\omega_1^k + \epsilon_{z1}, \omega_2^k), u^i(c^i) - u^i(\omega_1^i, \omega_2^i + \epsilon_{z2})\}$. Let $\tilde{\epsilon}_{0,1} = \min_z \{\epsilon_{z1}\}_{z=1}^Z$, $\tilde{\epsilon}_{0,2} = \min_z \{\epsilon_{z2}\}_{z=1}^Z$, and let $\delta = \frac{1}{2}\min\{\delta_z\}_{z=1}^Z$, where Z is the number of Walras equilibrium.

By the continuity of Walras equilibrium, there is sufficient small $\epsilon_{0,1} \in \left(0, \tilde{\epsilon}_{0,1}\right)$, $\epsilon_{0,2} \in \left(0, \tilde{\epsilon}_{0,2}\right)$, such that, $\forall\ \epsilon_1 \in (0, \epsilon_{0,1})$, $\epsilon_2 \in (0, \epsilon_{0,2})$, in any Walrasian equilibrium $\langle p(\epsilon), \{c^i(\epsilon)\}_{i=1}^I\rangle$ of the economy $\langle\{\omega^i\}_{i=1}^I, \{u^i\}_{i=1}^I, \epsilon\rangle$, we have, $|u^i(c^i(\epsilon)) - u^i(c^i)| < \delta$, where c^i is the limit of $c_1^i(\epsilon)$ as ϵ converges to zero, and ϵ_0 is uniform for all Walras equilibria.

2. $W(\epsilon) \subseteq \mathbf{N}(\epsilon)$.

For $0 \ll \epsilon \ll \epsilon_0$, let $\langle p(\epsilon), \{c^i(\epsilon)\}_{i=1}^I \rangle$ be a Walras equilibrium. Then, let

$$s^i = \begin{cases} (\omega_1^i - c_1^i(\epsilon), p(\epsilon), 0) & \text{if } \omega_1^i - c_1^i(\epsilon) \geq 0 \\ (0,\ p(\epsilon), -[\omega_1^i - c_1^i(\epsilon)]\, p(\epsilon)) & \text{if } \omega_1^i - c_1^i(\epsilon) < 0 \end{cases}.$$

We claim that s is a Nash equilibrium. It suffices to show that given s^{-i}, there is no profitable deviation for agent i.

(a) Suppose $\omega_1^i - c_1^i(\epsilon) \geq 0$.

Let B_p^i be the competitive budget set of agent i when the price is p. Let $\widetilde{p}$ denote the market clearing price implied by s^{-i} and deviating action of i and let $\widetilde{c}^i$ be payoff of i. There are three cases according to the relationship between $p(\epsilon)$ and $\widetilde{p}$.

First, $\widetilde{p} = p(\epsilon)$. Then $\widetilde{c}^i \in B_{\widetilde{p}}^i = B_{p(\epsilon)}^i$, where the set inclusion follows from the way payoffs are defined for the game. However, from the definition of Walras equilibrium, $u^i\left(c_1^i(\epsilon)\right) \geq u^i\left(\widetilde{c}^i\right)$, which implies that the deviation is not profitable.

Second, $\widetilde{p} < p(\epsilon)$. Consider the possibility that the deviation is such that agent i is a buyer at $\widetilde{p}$. By the definition of s^{-i}, only one special agent is willing to sell the good at a price lower than $p(\epsilon)$. And, by assumption A1, there is another regular agent, say agent k, whose demand is greater than the special agent's supply at $\widetilde{p} < p(\epsilon)$. This implies excess demand at $\widetilde{p}$, which is a contradiction. If the deviation is such that agent i remains a seller, then we have $B_{\widetilde{p}}^i \cap \left\{\omega_1^i \geq \widetilde{c}_1{}^i\right\} \subset B_{p(\epsilon)}^i \cap \left\{\omega_1^i \geq \widetilde{c}_1{}^i\right\} \subset B_{p(\epsilon)}^i$, which contradicts profitability of the deviation.

Third, $\widetilde{p} > p(\epsilon)$. Any deviation that leads agent i to become a buyer is not profitable, because $B_{\widetilde{p}}^i \cap \left\{\omega_1^i < \widetilde{c}_1{}^i\right\} \subset B_{p(\epsilon)}^i \cap \left\{\omega_1^i < \widetilde{c}_1{}^i\right\} \subset B_{p(\epsilon)}^i$. Therefore, suppose the deviation is such that agent i is still a seller. Then, the only buyer is a special agent. It follows that either there is excess supply at the higher price or agent i is the agent described in A1. If the former, then we have a contradiction to $\widetilde{p} > p(\epsilon)$; if the latter, then condition A1 implies that agent i is worse off.

Thus, we have ruled out a profitable deviation if $\omega_1^i - c_1^i(\epsilon) \geq 0$. The case $\omega_1^i - c_1^i(\epsilon) < 0$ is analogous.

3. $\mathbb{W}(\epsilon) \supseteq \mathbb{N}(\epsilon)$

Suppose there is a symmetric pure strategy Nash equilibrium which is not in $\mathbb{W}(\epsilon)$. There are two conditions for $\mathbb{W}(\epsilon)$: a price p and an allocation is in $\mathbb{W}(\epsilon)$ if the allocation is feasible and if each person maximizes utility taking the price p as given. Because a Nash equilibrium is feasible, if it not in $\mathbb{W}(\epsilon)$, then someone's marginal rate of substitution is not equal to the price. Call this person i. We produce a utility-improving deviation, denoted x.

Let $\widetilde{p}$ denote the market clearing price in a game implied by s^{-i} and x, and let

$$Y^{-i}(\widetilde{p}, x) = \sum\nolimits_{\{k \mid k \neq i,\, p^k > \widetilde{p}\}} y^k + \sum\nolimits_{\{k \mid k \neq i,\, p^k = \widetilde{p},\, y^k > 0\}} \left(\omega_2^k - c_2^k(x)\right),$$

and

$$Q^{-i}(\widetilde{p}, x) = \sum\nolimits_{\{i \mid k \neq i,\, p^k < \widetilde{p}\}} q^k + \sum\nolimits_{\{i \mid k \neq i,\, p^i = \widetilde{p},\, q^i > 0\}} \left(\omega_1^k - c_1^k(x)\right).$$

(a) Person i does not trade in the Nash equilibrium.

First, suppose $u_1^i(\omega^i) > p u_2^i(\omega^i)$ and consider $x = (0, P_H, \eta)$ for some $\eta > 0$. If $\widetilde{p} \leq p$, then agent i is better-off. Therefore, suppose $\widetilde{p} > p$. Then $Y^{-i}(p, s^i) \geq Y^{-i}(\widetilde{p}, x)$ and

$Q^{-i}(\widetilde{p},x) \leq Q^{-i}(p,s^{-i})$. Therefore,

$$\begin{aligned}\widetilde{p} &= \frac{Y^{-i}\left(\widetilde{p},x\right)+\epsilon_2+\eta}{Q^{-i}(\widetilde{p},x)+\epsilon_1} \leq \frac{Y^{-i}\left(p,s^i\right)+\epsilon_2+\eta}{Q^{-i}(p,s^i)+\epsilon_1} \\ &\leq \frac{Y^{-i}\left(p,s^i\right)+\epsilon_2}{Q^{-i}(p,s^i)+\epsilon_1}+\frac{\eta}{\epsilon_1} = p+\frac{\eta}{\epsilon_1},\end{aligned}$$

and

$$\begin{aligned}u^i(c^i(x)) &= u^i(\omega_1^i+\frac{\eta}{\widetilde{p}},\omega_2^i-\eta) \\ &\geq u^i(\omega_1^i+\frac{\eta}{p+\frac{\eta}{\epsilon_1}},\omega_2^i-\eta) \triangleq U(\eta).\end{aligned}$$

It follows that

$$U^{'}(0) = \frac{u_1^i(\omega^i))}{p} - u_2^i(\omega^i) > 0.$$

Therefore, there exists $\eta > 0$ such that $x = (0, P_H, \eta)$ is a profitable deviation. The case $u_1^i(\omega^i) < pu_2^i(\omega^i)$ is analogous.

(b) $\omega_1^i > c_1^i$.

Since selling less is always a feasible deviation without lowering the price, the equilibrium cannot be such that $u_1^i\left(c^i\right) > pu_2^i\left(c^i\right)$. Hence, we must have $u_1^i\left(c^i\right) < pu_2^i\left(c^i\right)$. Let p^i be the price in agent i's equilibrium action. First, $p^i > p$ contradicts $\omega_1^i > c_1^i$. Second, consider $p^i < p$. If there is excess demand at p, then i can sell more without affecting the price, a profitable deviation. The same is true or there is one other active seller who named p. It remains to consider no excess demand at p and no other active seller who named p. Then a buyer could profitably deviate by naming a price lower than p. We conclude that $p^i = p$.

Now consider a deviation by agent i of the form $x = (\omega_1^i - c_1^i + \eta,\, p - \eta^2, 0)$ for some $\eta < \omega_1^i - c_1^i$. If the implied market clearing price satisfies $\widetilde{p} \leq p - \eta^2$, then $Y^{-i}(\widetilde{p}, x) \geq Y^{-i}(p, s^i)$. Moreover, because i's symmetric seller no longer sells, we have $Q^{-i}(p, s^i) > Q^{-i}(\widetilde{p}, x) + \eta$. Therefore,

$$\widetilde{p} \geq \frac{Y^{-i}(\widetilde{p}, x) + \epsilon_2}{Q^{-i}(\widetilde{p}, x) + \epsilon_1 + \omega_1^i - c_1^i + \eta} > \frac{Y^{-i}(p, s^i) + \epsilon_2}{Q^{-i}(p, s^i) + \epsilon_1 + \omega_1^i - c_1^i} = p,$$

where the first inequality holds at equality if $\widetilde{p} < p - \eta^2$ and may hold strictly if $\widetilde{p} = p - \eta^2$. But, this contradicts $\widetilde{p} \leq p - \eta^2$. Hence, $\widetilde{p} > p - \eta^2$ and agent i is able to sell η more by deviating to x. Then,

$$\begin{aligned} u^i(c^i(x)) &= u^i(c_1^i - \eta, \omega_2^i + \left(\omega_1^i - c_1^i + \eta\right)\widetilde{p}) \\ &> u^i(c_1^i - \eta, \omega_2^i + \left(\omega_1^i - c_1^i + \eta\right)\left(p - \eta^2\right)) \triangleq U(\eta). \end{aligned}$$

It follows that

$$U'(0) = -u_1^i\left(c^i\right) + p u_2^i\left(c^i\right) > 0.$$

Therefore, x is a profitable deviation.

The case in which trader i is a buyer is analogous.

1.6.3. Proof of Proposition 1.2

Similar to the proof of Proposition 1.1, There are three steps in the proof. In the first step, assumption A3 is used to construct endowments for the special agents. The second step establishes $\mathbb{W}_C(\epsilon) \subseteq \mathbb{N}_C(\epsilon)$, while the third step establishes $\mathbb{N}_C(\epsilon) \subseteq \mathbb{W}_C(\epsilon)$.

1. Construction of ϵ_0.

By assumption, demand functions for regular agents are continuously differentiable. The sum of special agents's demand function, $\sum_{j=1}^{m}\left(\frac{\epsilon_j}{p} - \epsilon_j\right)$ is obviously continuously differentiable. Because every commodity is desired by each regular agent, assumption (A) in Debreu (1970) is also satisfied here. Hence, the following remark in Debreu (1970) hold here. Define $\mathbb{H}$ to be the set of strictly positive vectors in R^{m+1}, such that the sum of the components is unity and let $v^0 = \left\{\omega^1, \cdots, \omega^I, 0\right\}$, where 0 stands for $\epsilon = 0$. Then there is an open neighborhood V of v^0 and Z continuously differentiable functions $g_1, \cdots, g_Z$ from V to $\mathbb{H}$ such that for every v in V, the set of equilibrium price, $W(v)$, consists of Z distinct elements $g_1(v), \cdots, g_Z(v)$. Because any price-taking equilibrium allocation $\{c^i(v)\}_{i=1}^I$ is a continuously differentiable function of price, we come to the same conclusion for them. In particular, because demand is homogeneous of degree zero, $\mathbb{W}(\epsilon)$ consists of Z distinct elements, which are continuous functions of ϵ as ϵ converges to zero. Hence, there are a finite number of price-taking equilibria in the economy without special agents. Each one has trade by A3.

Take the $z-th$ such equilibrium, $\langle p, \{c^i\}_{i=1}^I\rangle$. For each agent i, let $T^i = \{\ j \mid c_j^i \neq \omega_j^i, j \neq m+1\}$. Take any non-empty subset $\sigma \subset T$, and take a vector $\epsilon(\sigma) \gg 0$. Let the endowment of agent i become

$$\omega_j^i\left(\epsilon\left(\sigma\right)\right)=\begin{cases}\omega_{m+1}^i+\sum_{j\in\sigma,\ c_j^i<\omega_j^i}\epsilon_j\left(\sigma\right) & \text{if } j=m+1\\ \omega_j^i+\epsilon_j\left(\sigma\right) & \text{if } j\in\sigma \text{ and } c_j^i>\omega_j^i\\ \omega_j & \text{otherwise}\end{cases}.$$

Given p, denote $\tilde{c}^i\left(\epsilon\left(\sigma\right)\right)$ to be the allocation that maximizes the utility of agent i, subjects to the CIA constraint and the following constraints: $\tilde{c}_j^i\left(\epsilon\left(\sigma\right)\right)=\omega_j^i\left(\epsilon\left(\sigma\right)\right)$ for all $j\in\sigma$. Since $u^i(c^i)>u^i(\omega^i)$, there is $\epsilon^i\left(\sigma\right)\gg 0$ such that, $u^i(c^i)>u^i(\tilde{c}^i\left(\epsilon^i\left(\sigma\right)\right))$. Let $\delta^i(\sigma)=u^i(c^i)-u^i(\tilde{c}^i\left(\epsilon^i\left(\sigma\right)\right))$. Let $\delta_z=\min_{i,\sigma\subset T^i}\left\{\delta^i(\sigma)\right\}$, $\epsilon_{z,j}=\min_{i,\sigma\subset T^i}\left\{\epsilon_j^i\left(\sigma\right)\right\}$. Let $\tilde{\epsilon}_{0,j}=\min_z\left\{\epsilon_{z,j}\right\}_{z=1}^Z$, $j\in M$, and let $\delta=\frac{1}{2}\min\left\{\delta_z\right\}_{z=1}^Z$, where Z is the number of price-taking equilibrium.

By the continuity of price-taking equilibrium, there is sufficient small $\epsilon_{0,j}\in\left(0,\ \tilde{\epsilon}_{0,j}\right)$, $j\in M$, such that, $\forall\ \epsilon_j\in(0,\ \epsilon_{0,j})$, in any price-taking equilibrium $\langle p(\epsilon),\{c^i(\epsilon)\}_{i=1}^I\rangle$ of the economy $\langle\{\omega^i\}_{i=1}^I,\ \{u^i\}_{i=1}^I,\ \epsilon\rangle$, we have, $|u^i(c^i\left(\epsilon\right))-u^i(c^i)|<\delta$, where c^i is the limit of $c_1^i\left(\epsilon\right)$ as ϵ converges to zero, and ϵ_0 is uniform for all price-taking equilibria.

2. $\mathbb{W}_C\left(\epsilon\right)\subseteq\mathbb{N}_C\left(\epsilon\right)$

For $0\ll\epsilon\ll\epsilon_0$, let $\langle p(\epsilon),\ \{c^i(\epsilon)\}_{i=1}^I\rangle$ be a price-taking equilibrium. Then, define

$$(q_j^i,p_j,y_j^i)=\begin{cases}(\omega_j^i-c_j^i(\epsilon),p_j(\epsilon),0) & \text{if } \omega_j^i-c_j^i(\epsilon)\geq 0\\ (0,\ p_j(\epsilon),-\left[\omega_j^i-c_j^i(\epsilon)\right]p_j(\epsilon)) & \text{if } \omega_j^i-c_j^i(\epsilon)<0\end{cases},$$

and let $s^i=\{(q_j^i,\ p_j,\ y_j^i)\}_{j=1}^m$. We claim that s is a Nash equilibrium. It suffices to show that given s^{-i}, there is no profitable deviation for agent i. Let $\widetilde{p}$ denote the market clearing price implied by s^{-i} and deviating action of i and let $\widetilde{c}^i$ be payoff of i. There are two cases according to the relationship between $p(\epsilon)$ and $\widetilde{p}$.

First, $\widetilde{p} = p(\epsilon)$. Then following the way payoffs are defined for the game, $\widetilde{p}\widetilde{c}^i = \widetilde{p}\omega^i = p(\epsilon)c_1^i(\epsilon)$. However, from the definition of price-taking equilibrium, $u^i\left(c_1^i(\epsilon)\right) \geq u^i\left(\widetilde{c}^i\right)$, which implies that the deviation is not profitable.

Second, $\widetilde{p} \neq p(\epsilon)$. There are several facts that don't depend on whether the deviation is related to multiple market or not.

(a) Suppose $c_j^i(\epsilon) < \omega_j^i$.

A outcome such as $\left\{\widetilde{c}_j^{\,i} \neq c_j^i(\epsilon),\ \widetilde{p}_j = p_j(\epsilon)\right\}$ or $\left\{\widetilde{c}_j^i < \omega_j^i,\ \widetilde{p}_j > p_j(\epsilon)\right\}$ will be ruled out below. Then a deviation leads to $\left\{\widetilde{c}_j^i \geq \omega_j^i,\ \widetilde{p}_j > p_j(\epsilon)\right\}$ is also ruled out, because it ends up with less with less numeraire than $\left\{\widetilde{c}_j^i \geq \omega_j^i,\ \widetilde{p}_j = p_j(\epsilon)\right\}$. By similar reason, $\left\{\widetilde{c}_j^i \leq \omega_j^i,\ \widetilde{p}_j < p_j(\epsilon)\right\}$ is ruled out. There is no deviation leads to $\left\{\widetilde{c}_j^i > \omega_j^i,\ \widetilde{p}_j < p_j(\epsilon)\right\}$. Only one special agent is willing to sell the good j if $\widetilde{p}_j < p_j(\epsilon)$. And by the construction of ϵ_0, there is excess demand at $\widetilde{p}_j < p_j(\epsilon)$.

(b) Suppose $c_j^i(\epsilon) > \omega_j^i$.

A deviation leads to$\left\{\widetilde{c}_j^{\,i} \neq c_j^i(\epsilon),\ \widetilde{p}_j = p_j(\epsilon)\right\}$ or $\left\{\widetilde{c}_j^i > \omega_j^i,\ \widetilde{p}_j < p_j(\epsilon)\right\}$ will be ruled out below. Then a deviation leads to $\left\{\widetilde{c}_j^i \leq \omega_j^i,\ \widetilde{p}_j < p_j(\epsilon)\right\}$ is also ruled out, because it ends up with less with less numeraire than $\left\{\widetilde{c}_j^i \leq \omega_j^i,\ \widetilde{p}_j = p_j(\epsilon)\right\}$. By similar reason, $\left\{\widetilde{c}_j^i \geq \omega_j^i,\ \widetilde{p}_j > p_j(\epsilon)\right\}$ is ruled out. There is no deviation leads to $\left\{\widetilde{c}_j^i < \omega_j^i,\ \widetilde{p}_j > p_j(\epsilon)\right\}$. The only buyer of good j is a special agent if $\widetilde{p}_j > p_j(\epsilon)$. And by the construction of ϵ_0, there is excess supply at $\widetilde{p}_j > p_j(\epsilon)$.

(c) Suppose $c_j^i(\epsilon) = \omega_j^i$.

A deviation leads to$\left\{\widetilde{c}_j^{\,i} \neq c_j^i(\epsilon),\ \widetilde{p}_j = p_j(\epsilon)\right\}$ will be ruled out below. Then a deviation leads to $\left\{\widetilde{c}_j^i \geq \omega_j^i,\ \widetilde{p}_j > p_j(\epsilon)\right\}$ is also ruled out, because it ends up with less with less numeraire than $\left\{\widetilde{c}_j^i \geq \omega_j^i,\ \widetilde{p}_j = p_j(\epsilon)\right\}$. By similar reason, $\left\{\widetilde{c}_j^i \leq \omega_j^i,\ \widetilde{p}_j < p_j(\epsilon)\right\}$ is ruled out. Similar to (b), There is no deviation leads to $\left\{\widetilde{c}_j^i < \omega_j^i,\ \widetilde{p}_j > p_j(\epsilon)\right\}$. And there is no deviation leads to $\left\{\widetilde{c}_j^i > \omega_j^i,\ \widetilde{p}_j < p_j(\epsilon)\right\}$ by the same reason as (a).

It is left to rule out one of the three outcomes:

$$\left\{\ \widetilde{p}_j = p_j(\epsilon),\ \widetilde{c}_j^{\,i} \neq c_j^i(\epsilon)\ \right\},$$

$$\left\{\ c_j^i(\epsilon) < \omega_j^i,\ \widetilde{c}_j^i < \omega_j^i,\ \widetilde{p}_j > p_j(\epsilon)\ \right\},$$

and

$$\left\{\ c_j^i(\epsilon) > \omega_j^i,\ \widetilde{c}_j^i > \omega_j^i,\ \widetilde{p}_j < p_j(\epsilon)\right\},$$

which is done as below.

Let $T_1 = \{\ j \mid \widetilde{p}_j > p_j(\epsilon)\}$, $T_2 = \{\ j \mid \widetilde{p}_j < p_j(\epsilon)\}$, and let $T = T_1 \cup T_2$. Consider the case that an agent i satisfies $c_j^i(\epsilon) < \omega_j^i$, $j \in T_1$, and $c_j^i(\epsilon) > \omega_j^i$, $j \in T_2$. By the construction of ϵ_0, there is no profitable deviation such that, agent i sacrifices trade with other regular agents in market $j \in T$, while he is still a seller in market $j \in T_1$, and he is still a buyer in market $j \in T_2$.

Hence, s is a Nash equilibrium.

3. $\mathbb{W}_C(\epsilon) \supseteq \mathbb{N}_C(\epsilon)$

Suppose there is a symmetric pure strategy Nash equilibrium which is not in $\mathbb{W}_C(\epsilon)$. There are two conditions for $\mathbb{W}_C(\epsilon)$: a price p and an allocation is in $\mathbb{W}_C(\epsilon)$ if the allocation is feasible and if each person maximizes utility under CIA constraint, taking the price p as given. Because a Nash equilibrium is feasible, if it not in $\mathbb{W}_C(\epsilon)$, then someone doesn't maximizes his utility under CIA constraint, taking the price as givens. Call this person i. We produce a utility-improving deviation, denote x.

Let $\widetilde{p}$ denote the market clearing price in a game implied by s^{-i} and x, and let

$$Y_j^{-i}(\widetilde{p}_j, x) = \sum\nolimits_{\left\{k \ | \ k \neq i, \ p_j^k > \widetilde{p}_j\right\}} y^k + \sum\nolimits_{\left\{k \ | \ k \neq i, \ p_j^k = \widetilde{p}_j, \ y^k > 0\right\}} \widetilde{p}_j \left(c_j^k(x) - \omega_j^k\right),$$

and

$$Q_j^{-i}(\widetilde{p}_j, x) = \sum\nolimits_{\left\{i \ | \ k \neq i, \ p_j^k < \widetilde{p}_j\right\}} q^k + \sum\nolimits_{\left\{i \ | \ k \neq i, \ p_j^k = \widetilde{p}_j, \ q^i > 0\right\}} \left(\omega_j^k - c_j^k(x)\right).$$

Let $\Gamma_b^i = \{j \mid c_j^i > \omega_j^i, \ j \in M\}$, $\Gamma_s^i = \{j \mid c_j^i < \omega_j^i, \ j \in M\}$, and let $\Gamma_e^i = \{j \mid c_j^i = \omega_j^i, j \in M\}$. Let $u_j^i = \frac{\partial u^i(c^i)}{\partial c_j^i}$.

(a) Since selling less of good h is always a feasible deviation without lowering the price of the good, there is no $h \in \Gamma_s^i$, such that $u_{m+1}^i < \frac{u_h^i}{p_h}$. Similarly, since buying less of good h is always a feasible deviation without raising the price of the good, there is no $h \in \Gamma_b^i$, such that $u_{m+1}^i > \frac{u_h^i}{p_h}$.

(b) Suppose there exists $l \in \Gamma_b^i$ and $h \in \Gamma_e^i$ such that $\frac{u_h^i}{p_h} > \frac{u_l^i}{p_l}$. Replace the $h-th$ element of s^i with $(0, P_{H,h}, \eta)$ and replace the $l-th$ element of s^i with $(0, p_l^i, (c_l^i(s) - \omega_l^i)\, p_l - \eta)$ for some $\eta > 0$. Denote the new strategy of agent i to be x. Obviously, $\widetilde{p_l} \leq p_l$ and $\widetilde{p_j} = p_j$ for $j \neq h,\ l$. If $\widetilde{p_h} \leq p_h$, then agent i is better-off. Therefore, suppose $\widetilde{p_h} > p_h$. then $Y_h^{-i}(p_h, s^i) \geq Y_h^{-i}(\widetilde{p_h}, x)$ and $Q_h^{-i}(p_h, s^i) \leq Q_h^{-i}(\widetilde{p_h}, x)$. Therefore,

$$\begin{aligned}
\widetilde{p}_h &= \frac{Y_h^{-i}(\widetilde{p}_h, x) + \epsilon_h + \eta}{Q_h^{-i}(\widetilde{p}_h, x) + \epsilon_h} \leq \frac{Y_h^{-i}(p_h, s^i) + \epsilon_h + \eta}{Q_h^{-i}(p_h, s^i) + \epsilon_h} \\
&\leq \frac{Y_h^{-i}(p_h, s^i) + \epsilon_h}{Q_h^{-i}(p_h, s^i) + \epsilon_h} + \frac{\eta}{\epsilon_h} = p_h + \frac{\eta}{\epsilon_h},
\end{aligned}$$

and

$$\begin{aligned}
u^i(c^i(x)) &= u^i(\cdots, \omega_h^i + \frac{\eta}{\widetilde{p}}, \cdots, \omega_l^i + \frac{(c_l^i(s) - \omega_l^i)\, p_l - \eta}{\widetilde{p}_j}, \cdots) \\
&\geq u^i(\cdots, \omega_h^i + \frac{\eta}{p + \frac{\eta}{\epsilon_1}}, \cdots, \omega_l^i + \frac{(c_l^i(s) - \omega_l^i)\, p_l - \eta}{p_l}, \cdots) \triangleq U(\eta).
\end{aligned}$$

It follows that

$$U^{'}(0) = \frac{u_h^i}{p_h} - \frac{u_l^i}{p_l} > 0.$$

Therefore, there exists $\eta > 0$ such x is a profitable deviation.

Hence, $\forall l \in \Gamma_b^i$ and $\forall h \in \Gamma_e^i$, $\frac{u_l^i}{p_l} \geq \frac{u_h^i}{p_h}$.

(c) Suppose there exists $h \in \Gamma_e^i$ such that $u_{m+1}^i > \frac{u_h^i}{p_h}$. For some $\eta > 0$, replace the $h-th$ element of s^i with $(\eta, P_{L,h}, 0)$. Denote the new strategy of agent i to be x. Obviously, $\widetilde{p}_j = p_j$ for $j \neq h$. If $\widetilde{p}_h \geq p_h$, then agent i is better-off. Therefore, suppose $\widetilde{p}_h < p_h$. Then $Y_h^{-i}(p_h, s^i) \leq Y_h^{-i}(\widetilde{p}_h, x)$ and $Q_h^{-i}(p_h, s^i) \geq Q_h^{-i}(\widetilde{p}_h, x)$. Therefore,

$$\begin{aligned}
\widetilde{p}_h &= \frac{Y_h^{-i}(\widetilde{p}_h, x) + \epsilon_h + \eta}{Q_h^{-i}(\widetilde{p}_h, x) + \epsilon_h} \geq \frac{Y_h^{-i}(p_h, s^i) + \epsilon_h + \eta}{Q_h^{-i}(p_h, s^i) + \epsilon_h} \\
&\geq \frac{Y_h^{-i}(p_h, s^i) + \epsilon_h}{Q_h^{-i}(p_h, s^i) + \epsilon_h} + \frac{\eta}{\epsilon_h} = p_h + \frac{\eta}{\epsilon_h},
\end{aligned}$$

and

$$
\begin{aligned}
u^i(c^i(x)) &= u^i(\cdots, \omega_h^i - \eta, \cdots, c_{m+1}^i(s) + \eta \widetilde{p}_h) \\
&\geq u^i(\cdots, \omega_h^i - \eta, \cdots, c_{m+1}^i(s) + \eta \left(p_h + \frac{\eta}{\epsilon_1}\right)) \triangleq U(\eta).
\end{aligned}
$$

It follows that

$$
U'(0) = -u_h^i + p_h u_{m+1}^i > 0.
$$

Therefore, there exists $\eta > 0$ such that x is a profitable deviation.

Hence, $\forall \in \Gamma_e^i$, $u_{m+1}^i \leq \frac{u_h^i}{p_h}$.

(d) Suppose $\sum_{\{j \mid c_j^i > \omega_j^i \; j \in M\}} (c_j^i - \omega_j^i) p_j < \omega_{m+1}^i$ and there exists $h \in \Gamma_e^i$ such that $\frac{u_h^i}{p_h} > u_{m+1}^i$. Keeping i's CIA constraint satisfied, replace the $h-th$ element of s^i with $(0, P_{H,h}, \eta)$ for some $\eta > 0$. Denote the new strategy of agent i to be x. Obviously, $\widetilde{p}_j = p_j$ for $j \neq h$. If $\widetilde{p}_h \leq p_h$, then agent i is better-off. Therefore, suppose $\widetilde{p}_h > p_h$. Then $Y_h^{-i}(p_h, s^i) \geq Y_h^{-i}(\widetilde{p}_h, x)$ and $Q_h^{-i}(p_h, s^i) \leq Q_h^{-i}(\widetilde{p}_h, x)$. Therefore,

$$
\begin{aligned}
\widetilde{p}_h &= \frac{Y_h^{-i}(\widetilde{p}_h, x) + \epsilon_h + \eta}{Q_h^{-i}(\widetilde{p}_h, x) + \epsilon_h} \leq \frac{Y_h^{-i}(p_h, s^i) + \epsilon_h + \eta}{Q_h^{-i}(p_h, s^i) + \epsilon_h} \\
&\leq \frac{Y_h^{-i}(p_h, s^i) + \epsilon_h}{Q_h^{-i}(p_h, s^i) + \epsilon_h} + \frac{\eta}{\epsilon_h} = p_h + \frac{\eta}{\epsilon_h},
\end{aligned}
$$

and

$$\begin{aligned} u^i(c^i(x)) &= u^i(\cdots, \omega_h^i + \frac{\eta}{\widetilde{p}}, \cdots, c_{m+1}^i(s) - \eta) \\ &\geq u^i(\cdots, \omega_h^i + \frac{\eta}{p + \frac{\eta}{\epsilon_1}}, \cdots, c_{m+1}^i(s) - \eta) \triangleq U(\eta). \end{aligned}$$

It follows that

$$U'(0) = \frac{u_h^i}{p_h} - u_{m+1}^i > 0.$$

Therefore, there exists $\eta > 0$ such that x is a profitable deviation.

Hence, if $\sum_{\{j \ | \ c_j^i > \omega_j^i \ j \in M\}} (c_j^i - \omega_j^i) p_j < \omega_{m+1}^i$, then $\forall h \in \Gamma_e^i$, $\frac{u_h^i}{p_h} \leq u_{m+1}^i$.

(e) Suppose there exists $h \in \Gamma_s^i$ such that $u_{m+1}^i > \frac{u_h^i}{p_h}$.

Let p_h^i be the price of of good h in agent i's equilibrium action. First, $p_h^i > p_h$ contradicts $h \in \Gamma_s^i$. Second, consider $p_h^i < p_h$. If there is excess demand of good h at p_h, then i can sell more good h without affecting the price, a profitable deviation. The same is true or there is one other active seller who named p_h. It remains to consider no excess demand at p_h and no other active seller who named p_h. Then a buyer of good h could profitably deviate by naming a price lower than p_h. We conclude that $p_h^i = p_h$.

Replace the $h-th$ element of s^i with $(\omega_h^i - c_h^i(s) + \eta, p_h - \eta^2, 0)$ for some $\eta < \omega_h^i - c_h^i$. Denote the new strategy of agent i to be x. Let x be a deviation by agent i. Obviously, $\widetilde{p}_j = p_j$ for $j \neq h$. If $\widetilde{p}_h \leq p_h - \eta^2$, then $Y_h^{-i}(\widetilde{p}_h, x) \geq Y_h^{-i}(p_h, s^i)$. Moreover, because i's symmetric seller no longer sells, we have $Q_h^{-i}(p_h, s^i) > Q_h^{-i}(\widetilde{p}_h, x) + \eta$. Therefore,

$$\begin{aligned} \widetilde{p}_h &\geq \frac{Y_h^{-i}(\widetilde{p}_h, x) + \epsilon_h}{Q_h^{-i}(\widetilde{p}_h, x) + \epsilon_h + \omega_h^i - c_h^i(s) + \eta} \\ &> \frac{Y_h^{-i}(p_h, s^i) + \epsilon_h}{Q_h^{-i}(p_h, s^i) + \epsilon_h + \omega_h^i - c_h^i(s)} = p_h, \end{aligned}$$

where the first inequality holds at equality if $\widetilde{p_h} < p_h - \eta^2$ and may hold strictly if $\widetilde{p_h} = p_h - \eta^2$. Hence, $\widetilde{p_h} > p_h - \eta^2$ and agent i is able to sell η more of good h by deviating to x. Then

$$\begin{aligned}
& u^i(c^i(x)) \\
= \; & u^i\begin{pmatrix} \cdots, c^i_h(s) - \eta, \cdots, \\ c^i_{m+1}(s) - (\omega^i_h - c^i_h(s))\, p_h + (\omega^i_h - c^i_h(s) + \eta)\, \widetilde{p_h} \end{pmatrix} \\
> \; & u^i\begin{pmatrix} \cdots, c^i_h(s) - \eta, \cdots, \\ c^i_{m+1}(s) - (\omega^i_h - c^i_h(s))\, p_h + (\omega^i_h - c^i_h(s) + \eta)\, (p_h - \eta^2) \end{pmatrix} \triangleq U(\eta).
\end{aligned}$$

It follows that

$$U^{'}(0) = -u^i_h + p_h u^i_{m+1} > 0.$$

Therefore, there exists $\eta > 0$ such that x is a profitable deviation.

Hence, $\forall \in \Gamma^i_s$, $u^i_{m+1} \leq \frac{u^i_h}{p_h}$.

(f) Suppose there exists $h, l \in \Gamma^i_b$ such that $\frac{u^i_h}{p_h} > \frac{u^i_l}{p_j}$.

Let p^i_h be the price of of good h in agent i's equilibrium action. First, $p^i_h < p_h$ contradicts $h \in \Gamma^i_b$. Second, consider $p^i_h > p_h$. If there is excess supply of good h at p_h, then i can buy more good h without affecting the price, a profitable deviation. The same is true or there is one other active buyer who named p_h. It remains to consider no excess supply at p_h and no other active buyer who named p_h. Then a seller of good h could profitably deviate by naming a price higher than p_h. We conclude that $p^i_h = p_h$.

Replace the $h-th$ element of s^i with $(0, p_h+\eta^2, p_h\left[c_h^i(s)-\omega_h^i\right]+\eta)$ and replace the $l-th$ element of s^i with $(0, p_l^i, p_l\left[c_l^i(s)-\omega_l^i\right]-\eta)$ for some $\eta < p_h\left[c_h^i(s)-\omega_h^i\right]$. Denote the new strategy of agent i to be x. Let agent i deviate to submit x. Obviously, $\widetilde{p_l} \le p_l$ and $\widetilde{p_j} = p_j$ for $j \neq h,\, l$. If $\widetilde{p_h} \ge p_h+\eta^2$, then $Q_h^{-i}(p_h, s^i) \le Q_h^{-i}(\widetilde{p_h}, x)$. Moreover, because i's symmetric buyer on longer buys, we have $Y_h^{-i}\left(\widetilde{p_h}, x\right)+\eta < Y_h^{-i}\left(p_h, s^i\right)$. Therefore,

$$\begin{aligned}\widetilde{p_h} &\le \frac{Y_h^{-i}\left(\widetilde{p_h}, x\right)+\epsilon_h+p_h\left[c_h^i(s)-\omega_h^i\right]+\eta}{Q_h^{-i}(\widetilde{p_h}, x)+\epsilon_h}\\ &< \frac{Y_h^{-i}\left(p_h, s^i\right)+\epsilon_h+p_h\left[c_h^i(s)-\omega_h^i\right]}{Q_h^{-i}(p_h, s^i)+\epsilon_h} = p_h,\end{aligned}$$

where the first inequality holds at equality if $\widetilde{p_h} > p_h+\eta^2$ and may hold strictly if $\widetilde{p_h} = p_h+\eta^2$. Hence, $\widetilde{p_h} < p_h+\eta^2$ and agent i is able to spend η more on good h by deviating to x. Then

$$\begin{aligned}&u^i(c^i(x))\\ =\;& u^i(\cdots, \omega_h^i+\frac{p_h\left[c_h^i(s)-\omega_h^i\right]+\eta}{\widetilde{p_h}}, \cdots, \omega_l^i+\frac{p_l\left[c_l^i(s)-\omega_l^i\right]-\eta}{\widetilde{p_l}}, \cdots)\\ >\;& u^i(\cdots, \omega_h^i+\frac{p_h\left[c_h^i(s)-\omega_h^i\right]+\eta}{p_h+\eta^2}, \cdots, \omega_l^i+\frac{p_l\left[c_l^i(s)-\omega_l^i\right]-\eta}{p_l}, \cdots) \triangleq U(\eta).\end{aligned}$$

It follows that

$$U^{'}(0) = \frac{u_h^i}{p_h} - \frac{u_l^i}{p_l} > 0.$$

Therefore, there exists $\eta > 0$ such that x is a profitable deviation.

Hence, $\forall\; h,\, l \in \Gamma_b^i$, $\frac{u_h^i}{p_h} = \frac{u_l^i}{p_j}$.

(g) Suppose $\sum_{\{j \mid c_j^i > \omega_j^i \; j \in M\}} (c_j^i - \omega_j^i) p_j < \omega_{m+1}^i$ and there exists $h \in \Gamma_b^i$ such that, $\frac{u_h^i}{p_h} > u_{m+1}^i$.

Let p_h^i be the price of of good h in agent i's equilibrium action. First, $p_h^i < p_h$ contradicts $h \in \Gamma_b^i$. Second, consider $p_h^i > p_h$. If there is excess supply of good h at p_h, then i can buy more good h without affecting the price, a profitable deviation. The same is true or there is one other active buyer who named p_h. It remains to consider no excess supply at p_h and no other active buyer who named p_h. Then a seller of good h could profitably deviate by naming a price higher than p_h. We conclude that $p_h^i = p_h$.

Keeping agent i's CIA constraint satisfied, replace the $h - th$ element of s^i with $(0, p_h + \eta^2, p_h [c_h^i(s) - \omega_h^i] + \eta)$ for some $\eta < p_h [c_h^i(s) - \omega_h^i]$. Denote the new strategy of agent i to be x. Let agent i deviate to submit x. Obviously, $\widetilde{p_j} = p_j$ for $j \neq h$. If $\widetilde{p_h} \geq p_h + \eta^2$, then $Q_h^{-i}(p_h, s^i) \leq Q_h^{-i}(\widetilde{p_h}, x)$. Moreover, because i's symmetric buyer on longer buys, we have $Y_h^{-i}(\widetilde{p_h}, x) + \eta < Y_h^{-i}(p_h, s^i)$. Therefore,

$$\begin{aligned} \widetilde{p_h} &\leq \frac{Y_h^{-i}(\widetilde{p_h}, x) + \epsilon_h + p_h [c_h^i(s) - \omega_h^i] + \eta}{Q_h^{-i}(\widetilde{p_h}, x) + \epsilon_h} \\ &< \frac{Y_h^{-i}(p_h, s^i) + \epsilon_h + p_h [c_h^i(s) - \omega_h^i]}{Q_h^{-i}(p_h, s^i) + \epsilon_h} = p_h, \end{aligned}$$

where the first inequality holds at equality if $\widetilde{p_h} > p_h + \eta^2$ and may hold strictly if $\widetilde{p_h} = p_h + \eta^2$. Hence, $\widetilde{p_h} < p_h + \eta^2$ and agent i is able to spend η more on good h by deviating to x. Then

$$\begin{aligned} u^i(c^i(x)) &= u^i(\cdots, \omega_h^i + \frac{p_h\,[c_h^i(s) - \omega_h^i] + \eta}{\widetilde{p_h}}, \cdots, c_{m+1}^i(s) - \eta) \\ &> u^i(\cdots, \omega_h^i + \frac{p_h\,[c_h^i(s) - \omega_h^i] + \eta}{p_h + \eta^2}, \cdots, c_{m+1}^i(s) - \eta) \triangleq U(\eta). \end{aligned}$$

It follows that

$$U^{'}(0) = \frac{u_h^i}{p_h} - u_{m+1}^i > 0.$$

Therefore, there exists $\eta > 0$ such that x is a profitable deviation.

Hence, If $\sum_{\{j \ \mid \ c_j^i > \omega_j^i \ j \in M\}} (c_j^i - \omega_j^i) p_j < \omega_{m+1}^i$, then $\forall h \in \Gamma_b^i$, $\frac{u_h^i}{p_h} \leq u_{m+1}^i$.

(a)-(g) imply that

(1) If $\sum_{\{j \ \mid \ c_j^i > \omega_j^i \ j \in M\}} (c_j^i - \omega_j^i) p_j < \omega_{m+1}^i$, then, $\frac{u_j^i}{p_j} = u_{m+1}^i$, $j \in M$ or

(2) If $\sum_{\{j \ \mid \ c_j^i > \omega_j^i \ j \in M\}} (c_j^i - \omega_j^i) p_j = \omega_{m+1}^i$, then, $\frac{u_j^i}{p_j} = \frac{u_h^i}{p_h} \geq u_{m+1}^i$, $\forall j,\ h \in \Gamma_b^i$; $\frac{u_h^i}{p_h} = u_{m+1}^i$, $\forall h \in \Gamma_s^i$; and, $u_{m+1}^i \leq \frac{u_l^i}{p_l} \leq \frac{u_h^i}{p_h}$, $\forall h \in \Gamma_b^i$, $\forall\ l \in \Gamma_e^i$.

The above conditions are sufficient for utility maximization. Therefore, each person maximizes utility under CIA constraint, taking the price p as given.

CHAPTER 2

Periodic, Complete-participation Trade in the Lagos-Rocheteau Model

2.1. Introduction

Lagos-Rocheteau (2009) study a continuous-time economy with non-atomic measures of two kinds of infinitely-lived agents, investors and dealers. There are two divisible goods, an asset and a numeraire. The aggregate quantity of the asset is exogenous. Preferences are quasi-linear. In particular, investors have additively separable preferences over the two goods: they get strictly concave (flow) utility from the asset, which is subject to heterogeneous preference shocks which arrive according to a Poisson process; they get linear utility from the numeraire. Dealers have the same preferences except that they care only about the numeraire. Investors meet dealers pairwise and at random according to another Poisson process, while each dealer is also in continuous contact with all other dealers. The allocation problem in this economy concerns how the asset is distributed among the investors.

Lagos and Rocheteau is part of a literature that is meant to be a theory of transaction costs (bid-ask spreads) in some markets for securities (see Duffie et al, 2005; Vayanos and Weill 2005)—perhaps, what are called over-the-counter markets. However, the model contains none of the frictions that seem to give rise to actual over-the-counter markets. The main frictions would seem to be (i) the desire to trade large quantities and the attendant concern about the effect on the price; (ii) the fact that trades are agreed to before they are executed which leads participants to care about the credibility of their counter-parties; and (iii) asymmetric information about the securities being traded. Instead, in a meeting between an investor and a dealer in Lagos and Rocheteau, there is a known competitive price of the asset which the current trade does not affect; trade is *quid pro quo* with immediate execution made possible by the assumed quasi-linear preferences; there is symmetric information in two senses—the asset is a uniform object and the dealer knows the current preference realization and portfolio of the investor. In addition, each such meeting is a one-shot meeting so that there is no possibility of a continuing relationship between the investor and the dealer. The only friction is that an investor is in contact with no one except dealers: he meets one dealer at random. In other words, one meeting (with a dealer), which occurs at random, is free; all others are infinitely costly. This seems an extreme view of markets in securities and leads us to consider an alternative specification of costly contacts.

We adopt all the assumptions of Lagos and Rocheteau except the assumption about who is in contact with whom. We assume that there is a centralized competitive market among investors which occurs periodically. There are no dealers. For us, the infrequency of the meeting is a stand-in for the cost of operating such a market.

Periodic, complete-participation trade also maintains the main tension in the Lagos-Rocheteau model: the mismatch between the current preference type and the asset holding, which is caused by infrequent trade. Also, a periodic market is more reasonable than one that meets continuously for an economy with a large finite number of agents.

For calibrated versions of the model, we ask how frequently this market must occur in order that investors are as well off as they in the Lagos-Rocheteau setup. For example, if there is one dealer per 100 investors in their model, which implies that an investor meets a dealer once every 10 days on average, then our investors are as well off if their centralized market meets once every 26.5 days. The much lower trade frequency does not make investors worse off because they avoid the dealer pay-off in the Lagos-Rocheteau model which arises from the Nash bargain between the investor and the dealer.

The plan of this chapter is as follows. The environment is set out in section 2.2. Competitive equilibrium is analyzed in section 2.3. In section 2.4, we make the welfare comparisons. All the proofs appear in the appendix.

2.2. Environment

Time υ is continuous, starts at $\upsilon = 0$, and goes on forever. There is a unit measure of infinitely lived agents. There is one asset and one (numeraire) good and preferences are quasi-linear. The asset is durable, perfectly divisible, and in fixed supply $A \in R_+$. Instantaneous utility from asset is $u_l(a)$, where $a \in R_+$ represents the asset holding of an agent, $l \in L = \{1,..., L\}$ denotes a preference type. The function $u_l(a)$ is twice continuously differentiable, strictly increasing, and strictly concave, with $u_l^{'}(0) = \infty$ and $u_l^{'}(\infty) = 0$. Utility from the good is $c \in R$, where c is the net consumption of the good ($c < 0$ if an agent produces more of the good than he consumes). Production and consumption of good occur instantaneously. However, when an agent produces or consumes good, he does not have time to consume asset. There is preference shock to the consumption of asset for each agent, which has a Poisson arrival rate δ, and is independent across agents. Conditional on the arrival of the preference shock, the probability that an agent draws preference type l is $\pi_l > 0$, with $\sum_{l=1}^{L} \pi_l = 1$. Given a time path of asset holding $\{a(\upsilon)\}_{\upsilon=0}^{\infty}$ and consumptions of the good at time $t \in \{T_{\alpha_1}, T_{\alpha_2}, ...,\}$, expected discounted life time utility for an agent with preference type l at $\upsilon = 0$ is [1]

$$E_l \int_0^{\infty} e^{-r\upsilon} u_{k(\upsilon)}(a(\upsilon))d\upsilon + \sum_{i=1}^{\infty} e^{-T_{\alpha_i} r} c(T_{\alpha_i}), \tag{2.1}$$

where r is the discounting rate, and $k(\upsilon) \in L$ denotes the agent's preference type at time υ. The expectation operator, E_l, is over the random variable $k(\upsilon)$, which is indexed by l to indicate that it is conditional on $k(0) = l$. By assumption, $Pr[k(\upsilon) = j | k(t) = l] = [1 - e^{-\delta(\upsilon - t)}]\pi_j + e^{-\delta(\upsilon - t)} I_{\{j=l\}}$, $\forall \upsilon \geq t$.

Lagos-Rocheteau (2009) assumes that asset adjustment takes place according to a Poison process, hence T_{α_i} is a random variable. Here we assume that it occurs at regular intervals for a fixed period of time Δ, that is, $T_{\alpha_{i+1}} - T_{\alpha_i} = \Delta$, or $\{T_{\alpha_1}, T_{\alpha_2}, ...,\} = \{0, \Delta, 2\Delta, ...\} \equiv \Lambda$.

Let $Z = L \times R_+$, and let $z = (l, a) \in Z$ be a combination of a preference type l and an asset holding a. State of the economy is a distribution over Z. The joint distribution of preference type and asset holding is described by a probability space (Z, Z_β, μ), where Z_β is the Borel σ-algebra over Z, and μ is a probability measure on Z_β that represents the distribution of agents across asset holdings and preference types. Marginal distribution

[1]Consumption of the good is purely atomic, with atoms of size e^{-tr} at $t \in \Lambda$. Consumption of asset may not occur at $t \in \Lambda$, which has zero measure in time.

of preference type, $\mu_t(l,\,\cdot)$, $l \in L$, satisfies $\sum_{l=1}^{L}\mu_0(l,\,\cdot) = 1$. Assume that there is some atomless marginal distribution for asset holding at $t = 0$, satisfying $\int z_2 d\mu_0(\cdot,\, z_2) = A$.

2.3. Competitive equilibrium

Suppose there is a competitive asset market at each $t \in \Lambda$. Denote $p(t)$ as asset price, let $a_l(t)$ be a post-trade asset holding for an agent with a type $z = (l,\, z_2)$ at the begin to time t, where l is his preference type, and z_2 is his asset holding.

Expected discounted utility flow from asset holding $a_l(t)$ over the time interval $[t,\, t + \Delta]$ for an agent with preference type l at time t is

$$U_l(a_l(t)) = E_l \int_t^{t+\Delta} e^{-r(\upsilon - t)} u_{k(\upsilon)}(a_l(t)) d\upsilon. \tag{2.2}$$

Given a time path of price $\{p(t)\}_{t=0}^{\infty}$, utility maximization problem for an agent with a type $z = (l,\, z_2)$ is to choose a time path of asset holding $\{a_{k(t)}(t)\}_{t\in\Lambda}$ to maximize

$$\sup_{\{a_{k(t)}(t)\}_{t\in\Lambda}} \left\{ \begin{array}{c} U_l(a_l(0)) - p(0)\left[a_l(0) - z_2\right] \\ + \sum_{t\in\Lambda}^{\infty} e^{-tr} \sum_{j}^{L} \left\{[1 - e^{-\delta t)}]\pi_j + e^{-\delta t} I_{\{j=l\}}\right\} \\ * \left\{U_j(a_j(t)) - p(t)\left[a_j(t) - a_{k(t-\Delta)}(t-\Delta)\right]\right\} \end{array} \right\}. \tag{2.3}$$

At time t, market clearing condition requires

$$\sum_{l=1}^{L} \mu_t(l, \cdot) a_l(t) = A, \tag{2.4}$$

where

$$\mu_t(l,\cdot) = \left\{[1-e^{-\delta t}]\pi_l + e^{-\delta t}\mu_0(l,\cdot)\right\}, \forall t \in \Lambda. \quad (2.5)$$

Definition 2.1: A competitive equilibrium is a time-path $\{\mu_t(l,\cdot)\}_{t\in\Lambda}$, $\{a_l(t)\}_{t\in\Lambda}$, the asset holding of an agent who leaves the market with preference type l at time t, and a price sequence $\{p(t)\}_{t\in\Lambda}$, that solves problem (2.3) for each agent and satisfies (2.4) for all t, given initial distributions $\langle\{\mu_0(l,\cdot)\},\, \mu_0\rangle$.

The following lemma characterizes the sufficient conditions for a time path of asset holding to be optimal.

Lemma 2.1: Denote $q(t) = \left[p(t) - e^{-\Delta r}p(t+\Delta)\right]$, if $\{a_l(t)\}_{t\in\Lambda}$ satisfies

$$U_l'(a_l(t)) = q(t), \text{ for all } t \in \Lambda, \quad (2.6)$$

and

$$\lim_{\upsilon\to\infty} e^{-\Delta r\upsilon}p(\upsilon)a_{k(\upsilon)}(\upsilon) = 0, \quad (2.7)$$

then it is optimal.

Knowing the time path of asset holding for each preference type, the measure of agent with a type $(j,\, a_l(t))$ can be derived for any point of time,

$$\mu_\upsilon(j,\, a_l(t)) = \mu_t(l,\cdot)\left\{[1-e^{-\delta(\upsilon-t)}]\pi_j + e^{-\delta(\upsilon-t)}I_{\{j=l\}}\right\}, \forall \upsilon \in (t, t+\Delta]. \quad (2.8)$$

Proposition 2.1: There exists a unique competitive equilibrium. For any $\langle\{\mu_0(l,\cdot)\}, \mu_0\rangle$, equilibrium allocations $\{\mu_t(l,\cdot)\}_{t\in\Lambda}$, $\{a_l^*(t)\}_{t\in\Lambda}$, converge to a unique steady-state allocation $\langle\{a_l^*\}, \{\pi_l\}\ \rangle$ that satisfy:

$$U_l^{'}(a_l^*) = q, \tag{2.9}$$

$$\sum_{l=1}^{L} \pi_l a_l^* = A, \tag{2.10}$$

$$\mu_\upsilon(j,\, a_l^*) = \pi_l \left\{[1 - e^{-\delta(\upsilon - t)}]\pi_j + e^{-\delta(\upsilon-t)} I_{\{j=l\}}\right\},\ \forall \upsilon \in (t, t+\Delta]. \tag{2.11}$$

The steady-state average welfare is a function of Δ

$$W(\Delta) = \sum\nolimits_{\upsilon=0}^{\infty} e^{-\upsilon\Delta r} \sum_{l=1}^{L} \pi_l U_l(a_l^*) \tag{2.12}$$

Lemma 2.2: The steady-state average welfare is non-increasing in Δ, or

$$\frac{\partial W(\Delta)}{\partial \Delta} \leq 0. \tag{2.13}$$

Since agents are not able to adjust asset holdings continuously, under preference shock, there is misallocation between asset holding and preference type. Misallocation is worsen when market opens less frequently.

2.4. Compare welfare with Lagos-Rocheteau model

In this section, we compare the welfare in our model with that in the Lagos-Rocheteau model. A free entry version of their model is more relevant here because which takes into account of the resource cost associated with dealership system. We first duplicate equilibrium conditions of the Lagos-Rocheteau model as below.

2.4.1. Steady-state equilibrium in the Lagos-Rocheteau model

Let $\alpha(\nu)$ be the Poisson arrival rate with which an investor meets a dealer according to a Poisson process. Assume that $\alpha(\nu)$ is a continuously differentiable function of the measure of dealer ν entering the market. Let $\gamma > 0$ represent the ongoing costs of running the dealership.

Suppose in one of the steady-states[2], ν^* measure of dealers enters the market, then the measure of investors with asset holding $\tilde{a}_i$ and preference type j is

$$n_{ij} = \frac{\delta\pi_i\pi_j + \alpha(\nu^*)\pi_i 1_{\{i=j\}}}{\alpha(\nu^*) + \delta}. \tag{2.14}$$

Average instantaneous utility of the whole economy is [3]

[2]Multiple steady-state equlibria are possible, see Lagos-Roceteau(2009).
[3]The utility comes from the good are canceled out.

$$\sum_i \sum_j \frac{\delta\pi_i\pi_j + \alpha(\nu^*)\pi_i 1_{\{i=j\}}}{\alpha(\nu^*)+\delta} u_j(\tilde{a_i}), \tag{2.15}$$

where $\{\tilde{a_i}\}$ satisfies first order conditions

$$\bar{u}_{i^\sim}'(\tilde{a_i}) = q = rp, \tag{2.16}$$

market clearing condition

$$\sum_i \pi_i \tilde{a_i} = A, \tag{2.17}$$

and free-entry condition

$$\alpha(\nu^*)/\nu^*\Phi = \gamma, \tag{2.18}$$

where

$$\Phi = \sum_{i,j} n_{ij}\phi_{ij}, \tag{2.19}$$

and

$$\phi_{ij} = \eta \frac{\bar{u}_{i^\sim}(\tilde{a_i}) - \bar{u}_{i^\sim}(\tilde{a_j}) - q\left[\tilde{a_i} - \tilde{a_j}\right]}{r + \alpha(\nu^*)(1-\eta)}, \tag{2.20}$$

while η is the bargaining power of a dealer.

The function $\bar{u}_{\tilde{i}}(\cdot)$ is defined as

$$\bar{u}_{\tilde{i}}(a) = \frac{(r + \alpha(\nu^*)(1-\eta))u_i(a) + \delta\bar{u}(a)}{r + \alpha(\nu^*)(1-\eta) + \delta}. \tag{2.21}$$

Average (steady-state) welfare is

$$\tilde{W} = \left(\sum\nolimits_i \sum\nolimits_j \frac{\delta\pi_i\pi_j + \alpha(\nu^*)\pi_i 1_{\{i=j\}}}{\alpha(\nu^*) + \delta} u_j(\tilde{a}_i) - \nu^*\gamma\right)\frac{1}{r}. \tag{2.22}$$

In case of multiple steady-state equilibria, the steady-state equilibrium with highest welfare is selected.

2.4.2. Numerical results

Our welfare comparison is based on the fact that, there is Δ^*, such that,

$$W(\Delta^*) = \tilde{W} \tag{2.23}$$

Note that $W(0) > \tilde{W}$ is obvious. Since no trade is always an option for an agent in the Lagos-Rocheteau model, $W(\infty) \leq \tilde{W}$. Existence of Δ^* is then obvious because $W(\Delta)$ is continuous, non-increasing in Δ,

To do the calibration, we first parametrize the models. We then ask how frequently our market must occur in order that investors are as well off as they in the Lagos-Rocheteau setup. We also show how our numerical results vary when we adjust some of the parameters.

There are two unknown functions in the model. Lagos-Rocheteau (2007) assumes that $u_l(a) = \varepsilon_l \frac{a^{1-\sigma}}{1-\sigma}$ and $\alpha(\nu) = v^{\theta}$. Let $\varepsilon = \{1{,}2\}$ and $\pi = \{.5, .5\}$.

Following Lagos-Rocheteau (2009), we normalize the stock of assets by setting $A = 1$. As Lagos-Rocheteau does, we let a unit of time correspond to a day and take the rate of time preference to be 10 percent per year, i.e., $r = 0.1/360$. They set $\delta = 1/7$ so that investors receive one preference shock every week on average. And they assume that dealers and investors have equal bargaining power, i.e., $\eta = .5$.

As for the parameters of utility function, following Kydland and Prescott (1982), or Mehra and Prescott (1985), we use a number between 1 and 2 for σ, that is, $\sigma = 1.5$.

To determine the parameter of search function, we follow Shimer (2005) and let $\theta = \eta$. Because, from Hosios (1990), we know that the externality of dealers' entering is internalized if and only if the elasticity of the matching technology $\alpha(\nu)$ coincides with dealers' bargaining power.

The only thing left is a value for the dealers' ongoing cost. We adjust the ongoing cost for dealers such that, there is one dealer per 100 investors in steady-state equilibrium of Lagos-Rocheteau (2007). This means that, a dealer meets about 10 investors each day. Then $\gamma = .443$, $\tilde{W} = -10787$ and $\Delta^* = 26.445$ satisfies equation (2.33). $W(\infty) = -10800$, which means that in Lagos-Rocheteau (2007), the total gain from asset trading is $\tilde{W} - W(\infty) + \frac{\nu^* \gamma}{r} = 28.948$. The cost of trade for investors is 15.948, which accounts for 55% in the gain. If in steady-state equilibrium of Lagos-Rocheteau (2007), there is one dealer per 1000 investors. Then $\gamma = .9168$, $\tilde{W} = -10798$ and $\Delta^* = 70.75$. In this case, the total gain from trade is 5.3. The cost of trade for investors is 3.3, which accounts for 62% in the gain.

As table 2.1 below shows, given ε_1, changing ε_2 does not change Δ^* too much, even though ongoing cost changes a lot.

Table 2.1–(γ, Δ^*) as a function of ε_2

ε_2	γ	Δ^*
1/2	.222	26.485
2	.443	26.445
3	1.338	26.485
5	3.581	26.440
10	9.950	26.425

Table 2.2 below shows that, fix $\varepsilon = \{1, 2\}$, changing σ does not change Δ^* too much neither.

Table 2.2– (γ, Δ^*) as a function of σ

σ	γ	Δ^*
1.1	.6045	26.47
1.25	.531	26.435
1.5	.443	26.445
1.75	.379	26.410
1.9	.3505	26.480

Now, fix other parameters, changing bargaining power η does change Δ^* a lot. Denote Δ_{LR} as how many days on average an investor in Lagos-Rocheteau needs to wait before meeting with a dealer. We find that the difference between our model and the Lagos-Rocheteau model exists in similar way, that is, $\frac{\Delta^*}{\Delta_{LR}} \in (2, 3)$.

Table 2.3– (γ, Δ^*) as a function of η

η	γ	Δ^*	Δ_{LR}	$\frac{\alpha(\nu^*)}{\nu^*}$
.25	.462	7.5	3.2	31.6
.5	.443	26.5	10	10
.55	.353	35	12.6	7.94
.6	.263	47	15.8	6.31

As η rises, we expect γ to increase. But θ rises as well, which substantially lowers the number of business a dealer has per unit of time, as the value of $\alpha(\nu^*)/\nu^*$ shows. The later effect dominates, so γ decreases.

2.5. Appendix

2.5.1. Proof of Lemma 2.1

Given price path $\{p(t)\}_{t\in\Lambda}$, let $\{a^*_{k(t)}(t)\}_{t=0}^{\infty}$ satisfy (2.6) and (2.7), and let $\{a_{k(t)}(t)\}_{t=0}^{\infty}$ be any feasible asset sequence. It is sufficient to show that the difference, called as D, between the objective function in the problem (2.3) evaluated at $\{a^*_{k(t)}(t)\}_{t=0}^{\infty}$ and at $\{a_{k(t)}(t)\}_{t=0}^{\infty}$ is nonnegative.

$$\begin{aligned} D \;=\; & U_l(a_l^*(0)) - p(0)\left[a_l^*(0) - z_2\right] - \left\{U_l(a_l(0)) - p(0)\left[a_l(0) - z_2\right]\right\} \\ & + \lim_{T\to\infty} \sum\nolimits_{t\geq 1, t\in\Lambda}^{T} e^{-tr} \sum\nolimits_{j}^{L} \left[[1-e^{-\delta t)}]\pi_j + e^{-\delta t} I_{\{j=l\}}\right] \\ & * \left\{ \begin{array}{l} \left[U_j(a_j^*(t)) - p(t)\left[a_j^*(t) - a^*_{k(t-\Delta)}(t-\Delta)\right]\right] \\ - \left[U_j(a_j(t)) - p(t)\left[a_j(t) - a_{k(t-\Delta)}(t-\Delta)\right]\right] \end{array} \right\} \end{aligned}$$

Because utility function $u_i(a)$ is strictly concave, so does $U_l(a)$ and they are continuously differentiable. Hence,

$$\begin{aligned} D \;\geq\; & \left[U_l'(a_l^*(0)) - p(0)\right]\left[a_l^*(0) - a_l(0)\right] + \lim_{T\to\infty} \sum\nolimits_{t=\Delta}^{T} e^{-tr} \sum\nolimits_{j}^{L} \left[[1-e^{-\delta t)}]\pi_j + e^{-\delta t} I_{\{j=l\}}\right] \\ & * \left[\left[U_j'(a_j^*(t)) - p(t)\right]\left[a_j^*(t) - a_j(t)\right] + p(t)\left[a^*_{k(t-\Delta)}(t-\Delta) - a_{k(t-\Delta)}(t-\Delta)\right]\right] \end{aligned}$$

Rearranging terms gives,

$$\begin{aligned} D &\geq [U_l'(a_l^*(0)) - p(0)]\,[a_l^*(0) - a_l(0)] + \sum\nolimits_j^L \left[[1 - e^{-\delta\Delta)}]\pi_j + e^{-\delta\Delta} I_{\{j=l\}}\right] \\ &\quad * \left\{ e^{-\Delta r}\left[U_j'(a_j^*(\Delta)) - p(\Delta)\right]\left[a_j^*(\Delta) - a_j(\Delta)\right] + p(\Delta)e^{-\Delta r}\left[a_l^*(0) - a_l(0)\right]\right\} \\ &\quad + \lim_{T\to\infty} \sum\nolimits_{t\geq 2\Delta}^T e^{-tr} \sum\nolimits_j^L \left[[1 - e^{-\delta t)}]\pi_j + e^{-\delta t} I_{\{j=l\}}\right] \\ &\quad * \left[\left[U_j'(a_j^*(t)) - p(t)\right]\left[a_j^*(t) - a_j(t)\right] + p(t)\left[a_{k(t-\Delta)}^*(t-\Delta) - a_{k(t-\Delta)}(t-\Delta)\right]\right] \end{aligned}$$

With $U_l'(a_l^*(0)) - \left[p(0) - p(\Delta)e^{-\Delta r}\right] = 0$,

$$\begin{aligned} D &\geq e^{-\Delta r} \sum\nolimits_j^L \left[[1 - e^{-\Delta\delta)}]\pi_j + e^{-\delta\Delta} I_{\{j=l\}}\right] \{\left[U_j'(a_j^*(\Delta)) - p(\Delta)\right]\left[a_j^*(\Delta) - a_j(\Delta)\right] \\ &\quad + \lim_{T\to\infty} \sum\nolimits_{t\geq 2\Delta}^T e^{-tr} \sum\nolimits_{i=1}^L \left[[1 - e^{-\delta t)}]\pi_i + e^{-\delta t} I_{\{i=j\}}\right] \\ &\quad * \left[\left[U_i'(a_i^*(t)) - p(t)\right]\left[a_i^*(t) - a_i(t)\right] + p(t)\left[a_{k(t-\Delta)}^*(t-\Delta) - a_{k(t-\Delta)}(t-\Delta)\right]\right]\} \end{aligned}$$

Run the same procedure for each $j \in L$, then

$$D \geq \lim_{T\to\infty} e^{-\Delta T r} \sum\nolimits_j^L \left[[1 - e^{-\Delta T\delta)}]\pi_j + e^{-\delta T\Delta} I_{\{j=l\}}\right] \left[U_{k(T)}'(a_{k(T)}^*(T)) - p(T)\right]\left[a_{k(T)}^*(T) - a_{k(T)}(T)\right]$$

With $U_{k(T)}'(a_{k(T)}^*(T)) - p(T) = p(T+\Delta)e^{-\Delta r} \geq 0$, the above equation leads to

$$\begin{aligned} D &\geq -\lim_{T\to\infty} e^{-\Delta T r} \sum\nolimits_j^L \left[[1 - e^{-\Delta T\delta)}]\pi_j + e^{-\delta T\Delta} I_{\{j=l\}}\right] p(T)\left[a_{k(T)}^*(T) - a_{k(T)}(T)\right] \\ &\geq -\lim_{T\to\infty} \sum\nolimits_j^L \pi_j e^{-\Delta T r} p(T) a_j^*(T) \end{aligned}$$

where the last line uses the fact that $a_{k(t)}(t) \geq 0$, for all t. It then follows (2.7) that $D \geq 0$, establishing the desired result. Q.E.D.

2.5.2. Proof of Proposition 2.1

For all t, the distribution of $\{\mu_t(l,\cdot)\}_{l=1}^{L}$ is unique and given by (2.5).

Because the utility function $u_i(a)$ is strictly concave, so does $U_i(a)$. Hence given $\{p(t)\}$, there is unique solution to equation (2.6). Denote the solution as $a_l(t)$, which is continuous and strictly decreasing in $q(t) = \left[p(t) - e^{-\Delta r}p(t+\Delta)\right]$, so as $\sum_{l=1}^{L}\mu_t(l,\cdot)a_l(t)$. Apply market clearing condition (2.4), then there is unique $q(t)$ such that $\sum_{l=1}^{L}\mu_t(l,\cdot)a_l(q(t)) = A$. Given this $q(t)$, there is unique $\{a_l(t)\}_{l=1}^{L}$ that solves (2.6). Finally, given $\{a_l(t)\}_{l=1}^{L}$, the distribution of $\mu_\upsilon(a_l(t),\, j)$ is given by (2.8).

From (2.5), $lim_{t\longrightarrow\infty}\mu_t(l,\cdot) = \pi_l$ for each l. By similar argument as above, there is a unique, time-invariant q that clears the asset market, such that $\sum_{l=1}^{L}\pi_l a_l(q) = A$. Given this q, (2.4) implies a unique set of time-invariant optimal asset holdings $\{a_l\}_{l=1}^{L}$. Finally, given $\{a_l\}_{l=1}^{L}$, the distribution of $\mu_\upsilon(j,\, a_l)$ is given by (2.11).

2.5.3. Proof of Lemma 2.2

Changing variables, equation (2.2) can be written as $U_l(a) = E_l\displaystyle\int_0^\Delta e^{-r\upsilon}u_{k(\upsilon)}(a)d\upsilon$. Combined with $Pr[k(\upsilon) = j|k(t) = l] = [1-e^{-\delta(\upsilon-t)}]\pi_j + e^{-\delta(\upsilon-t)}I_{\{j=l\}}, \forall\upsilon \geq t$, we have, $U_l(a) = \displaystyle\int_0^\Delta \sum_j \left\{\left[1-e^{-\delta\upsilon}\right]e^{-r\upsilon}\pi_j u_j(a)\right\}d\upsilon + \int_0^\Delta e^{-r\upsilon}e^{-\delta\upsilon}u_l(a)d\upsilon = \sum_j \pi_j u_j(a)\int_0^\Delta \left[e^{-r\upsilon} - e^{-(\delta+r)\upsilon}\right]d\upsilon + u_l(a)\int_0^\Delta e^{-(\delta+r)\upsilon}d\upsilon$.

Denote $\bar{u}(a) = \sum_j \pi_j u_j(a)$, then

$$
\begin{aligned}
U_l(a) &= \bar{u}(a)\left[-\frac{1}{r}e^{-r\upsilon} + \frac{1}{\delta+r}e^{-(\delta+r)\upsilon}\right]_0^{\Delta} + u_l(a)\left[-\frac{1}{\delta+r}e^{-(\delta+r)\upsilon}\right]_0^{\Delta} \\
&= \bar{u}(a)\left[-\frac{1}{r}e^{-r\Delta} + \frac{1}{\delta+r}e^{-(\delta+r)\Delta} - \frac{1}{\delta+r} + \frac{1}{r}\right] + u_l(a)\left[-\frac{1}{\delta+r}e^{-(\delta+r)\Delta} + \frac{1}{\delta+r}\right] \\
&= \frac{\left[-\frac{\delta+r}{r}e^{-r\Delta} + e^{-(\delta+r)\Delta} + \frac{\delta}{r}\right]\bar{u}_l(a) + u_l(a)\left[1 - e^{-(\delta+r)\Delta}\right]}{\delta+r}
\end{aligned}
$$

Since $\int_0^{\Delta}\sum_j\left\{\left[1-e^{-\delta\upsilon}\right]e^{-r\upsilon}\pi_j u_j(a)\right\}d\upsilon > 0$ for all δ, r, $\Delta \in R_+$, $-\frac{\delta+r}{r}e^{-r\Delta} + e^{-(\delta+r)\Delta} + \frac{\delta}{r} > 0$ or $-(\delta+r)e^{-\delta\Delta} + \delta e^{-(\delta+r)\Delta} + r > 0$ for all δ, r, $\Delta \in R_+$. Changing the position of r and δ, then $-(\delta+r)e^{-\delta\Delta} + \delta e^{-(\delta+r)\Delta} + r > 0$,for all δ, r, $\Delta \in R_+$.

With equation (2.12),

$$
\begin{aligned}
\frac{\partial W(\Delta)}{\partial\Delta} &= \frac{\sum_l \pi_l\left\{\bar{u}^{'}(a_l^*)\left[-\frac{\delta+r}{r}e^{-r\Delta} + e^{-(\delta+r)\Delta} + \frac{\delta}{r}\right] + u_l^{'}(a_l^*)\left[1 - e^{-(\delta+r)\Delta}\right]\right\}\frac{\partial a_l^*}{\partial\Delta}}{\left(1-e^{-r\Delta}\right)(\delta+r)} \\
&+\frac{\sum_l \pi_l\left\{\bar{u}(a_l^*)\left[(\delta+r)e^{-r\Delta} - (\delta+r)e^{-(\delta+r)\Delta}\right] + u_l(a_l^*)\left[(\delta+r)e^{-(\delta+r)\Delta}\right]\right\}}{(\delta+r)(1-e^{-r\Delta})} \\
&-re^{-r\Delta}\frac{\sum_l \pi_l\left\{\bar{u}(a_l^*)\left[-\frac{\delta+r}{r}e^{-r\Delta} + e^{-(\delta+r)\Delta} + \frac{\delta}{r}\right] + u_l(a_l^*)\left[1 - e^{-(\delta+r)\Delta}\right]\right\}}{(1-e^{-r\Delta})^2(\delta+r)}
\end{aligned}
$$

By equation (2.9),

$$
\begin{aligned}
\frac{\partial W(\Delta)}{\partial\Delta} &= \sum_{l=1}^{L}\pi_l q(\Delta)\frac{\partial a_l^*}{\partial\Delta} + e^{-r\Delta}\frac{\sum_l \pi_l\left\{\bar{u}(a_l^*)\left[1-e^{-\delta\Delta}\right] + u_l(a_l^*)\left[e^{-\delta\Delta}\right]\right\}}{(1-e^{-r\Delta})^2}(1-e^{-r\Delta}) \\
&+\frac{e^{-r\Delta}\sum_l \pi_l\left\{\bar{u}(a_l^*)\left[(\delta+r)e^{-r\Delta} - re^{-(\delta+r)\Delta} - \delta\right] + u_l(a_l^*)\left[re^{-(\delta+r)\Delta} - r\right]\right\}}{(1-e^{-r\Delta})^2(\delta+r)}
\end{aligned}
$$

Since $\sum_{l=1}^{L}\pi_l a_l^* = A$, $\sum_{l=1}^{L}\pi_l\frac{\partial a_l^*}{\partial\Delta} = 0$, and $\sum_{l=1}^{L}\pi_l q(\Delta)\frac{\partial a_l^*}{\partial\Delta} = 0$,

$$\frac{\partial W(\Delta)}{\partial \Delta} = \frac{e^{-r\Delta}\sum_l \pi_l \left\{\bar{u}(a_l^*)\left[1-e^{-r\Delta}-e^{-\delta\Delta}+e^{-(\delta+r)\Delta}\right]+u_l(a_l^*)\left[e^{-\delta\Delta}-e^{-(\delta+r)\Delta}\right]\right\}}{(1-e^{-r\Delta})^2(\delta+r)}$$
$$+\frac{e^{-r\Delta}\sum_l \pi_l \left\{\bar{u}(a_l^*)\left[(\delta+r)e^{-r\Delta}-re^{-(\delta+r)\Delta}-\delta\right]+u_l(a_l^*)\left[re^{-(\delta+r)\Delta}-r\right]\right\}}{(1-e^{-r\Delta})^2(\delta+r)}$$

Or

$$\frac{\partial W(\Delta)}{\partial \Delta} = \frac{e^{-r\Delta}\sum_l \pi_l\bar{u}(a_l^*)\left[1-e^{-r\Delta}+e^{-r\Delta}-\frac{\delta}{(\delta+r)}-\left(e^{-\delta\Delta}-e^{-(\delta+r)\Delta}+\frac{r}{(\delta+r)}e^{-(\delta+r)\Delta}\right)\right]}{(1-e^{-r\Delta})^2}$$
$$+\frac{e^{-r\Delta}\sum_l \pi_l u_l(a_l^*)\left[e^{-\delta\Delta}-e^{-(\delta+r)\Delta}+\frac{re^{-(\delta+r)\Delta}}{(\delta+r)}-\frac{r}{(\delta+r)}\right]}{(1-e^{-r\Delta})^2}$$

Or

$$\frac{\partial W(\Delta)}{\partial \Delta} = \frac{e^{-r\Delta}\sum_l \pi_l\bar{u}(a_l^*)\left[(\delta+r)-\delta-(\delta+r)e^{-\delta\Delta}+(\delta+r)e^{-(\delta+r)\Delta}-re^{-(\delta+r)\Delta}\right]}{(1-e^{-r\Delta})^2(\delta+r)}$$
$$+\frac{e^{-r\Delta}\sum_l \pi_l u_l(a_l^*)\left[(\delta+r)e^{-\delta\Delta}-(\delta+r)e^{-(\delta+r)\Delta}+re^{-(\delta+r)\Delta}-r\right]}{(1-e^{-r\Delta})^2(\delta+r)}$$

So

$$\frac{\partial W(\Delta)}{\partial \Delta} = \frac{e^{-r\Delta}\sum_l \pi_l \left\{\begin{array}{c}\bar{u}(a_l^*)\left[r-(\delta+r)e^{-\delta\Delta}+\delta e^{-(\delta+r)\Delta}\right]\\ +u_l(a_l^*)\left[(\delta+r)e^{-\delta\Delta}-\delta e^{-(\delta+r)\Delta}-r\right]\end{array}\right\}}{(1-e^{-r\Delta})^2(\delta+r)}$$
$$= -\frac{\left[-(\delta+r)e^{-\delta\Delta}+\delta e^{-(\delta+r)\Delta}+r\right]e^{-r\Delta}\sum_l \pi_l\left[u_l(a_l^*)-\bar{u}(a_l^*)\right]}{(1-e^{-r\Delta})^2(\delta+r)}$$

$\forall l,\, j \in L$, $\pi_l \pi_j u_j(a_l^*) + \pi_j \pi_l u_l(a_j^*) = \pi_j \pi_l \left[u_j(a_l^*) + u_l(a_j^*)\right]$ and $\pi_l \pi_j u_l(a_l^*) + \pi_j \pi_l u_j(a_j^*) = \pi_j \pi_l \left[u_l(a_l^*) + u_j(a_j^*)\right]$. Without lost of generalization, assume that $a_j^* \geq a_l^*$, then

$$\begin{aligned}
& \pi_j \pi_l \left[u_l(a_l^*) + u_j(a_j^*)\right] - \pi_j \pi_l \left[u_j(a_l^*) + u_l(a_j^*)\right] \\
\geq \; & \pi_j \pi_l \left[u_j^{'}(a_j^*) \left(a_j^* - a_l^*\right) + u_l^{'}(a_l^*) \left(a_l^* - a_j^*\right)\right] \\
\geq \; & \pi_j \pi_l \left[\left[u_j^{'}(a_j^*) - u_l^{'}(a_l^*)\right] \left(a_j^* - a_l^*\right)\right].
\end{aligned}$$

Equation (2.9) is the same as $\frac{\bar{u}'(a_l^*)\left[-\frac{\delta+r}{r}e^{-r\Delta} + e^{-(\delta+r)\Delta} + \frac{\delta}{r}\right] + u_l'(a_l^*)\left[1 - e^{-(\delta+r)\Delta}\right]}{\delta+r} = q$ for all l, $j \in L$, then

$$\begin{aligned}
& \frac{\bar{u}'(a_l^*)\left[-\frac{\delta+r}{r}e^{-r\Delta} + e^{-(\delta+r)\Delta} + \frac{\delta}{r}\right] + u_l'(a_l^*)\left[1 - e^{-(\delta+r)\Delta}\right]}{\delta+r} \\
= \; & \frac{\bar{u}'(a_j^*)\left[-\frac{\delta+r}{r}e^{-r\Delta} + e^{-(\delta+r)\Delta} + \frac{\delta}{r}\right] + u_j'(a_j^*)\left[1 - e^{-(\delta+r)\Delta}\right]}{\delta+r}
\end{aligned}$$

$a_j^* \geq a_l^*$ then implies $\bar{u}'(a_j^*)\left[-\frac{\delta+r}{r}e^{-r\Delta} + e^{-(\delta+r)\Delta} + \frac{\delta}{r}\right] \leq \bar{u}'(a_l^*)\left[-\frac{\delta+r}{r}e^{-r\Delta} + e^{-(\delta+r)\Delta} + \frac{\delta}{r}\right]$, so, $u_j'(a_j^*) \geq u_l'(a_l^*)$.

Hence, $\pi_j \pi_l \left[u_l(a_l^*) + u_j(a_j^*)\right] - \pi_j \pi_l \left[u_j(a_l^*) + u_l(a_j^*)\right] \geq \pi_j \pi_l \left[\left[u_j^{'}(a_j^*) - u_l^{'}(a_l^*)\right] \left(a_j^* - a_l^*\right)\right] \geq 0$, $\forall l,\, j \in L$.

Since $\sum_l \pi_l u_l(a_l^*) = \sum_j \sum_l \pi_l \pi_j u_l(a_l^*)$ and $\sum_l \pi_l \bar{u}(a_l^*) = \sum_j \sum_l \pi_l \pi_j u_j(a_l^*)$,

$$\sum_l \pi_l \left[u_l(a_l^*) - \bar{u}(a_l^*)\right] \geq \sum_j \sum_l \pi_j \pi_l \left[\left[u_j^{'}(a_j^*) - u_l^{'}(a_l^*)\right] \left(a_j^* - a_l^*\right)\right] \geq 0$$

Therefore, $\frac{\partial W(\Delta)}{\partial \Delta} \leq 0$. Q.E.D.

CHAPTER 3

Employment Agents in the Labor Market

3.1. Introduction

This chapter present a continuous time search model to study the role of em-agents in the labor market. There are three types of agents: workers, firms and em-agents. A flow of consumption good is produced once a worker and a firm form a match. Such a match can be reached directly between a worker and a firm or indirectly through an em-agent, who forms a pair with a worker first and then search for a firm by himself. Each type of agent matches pairwise with the other two types through two independent matching processes. [1]Each process depends on the ratio of the agent's own type to the other type in the match (market tightness). After pairwise matching, the term of contract is determined by a bargaining process. There is free entry for firms and em-agents. Agents are able to commit fully.

It is shown that there is unique steady-state equilibrium in which em-agents are active. In equilibrium, without assuming an em-agent has a better matching technology than a worker, an em-agent finds a vacancy faster than a worker. In addiction, workers are better-off when there present em-agents. More generally, the welfare of workers decreases as the outside option of em-agents or firms increases. When job matches are heterogeneous

[1]For our purpose, we disregard the match between a firm and a headhunter if the headhunter has not matched with a worker before.

in productivity, jobs filled through em-agents have higher average productivity than ones done directly.

Two common assumptions in random search literature are relevant to our results. They are free entry and constant-return aggregate matching functions (or match probability is a function of market tightness). Mortensen and Pissarides (1999) points out that constant returns to scale in the matching technology is typically assumed in models of search in the labor market and Cobb-Douglas matching function with constant returns is the most common specification in the applied literature. Hall (2005) also takes it as a standard specification. See also Pissarides (1985), Mortensen and Pissarides (1994), Den Haan, Ramey, and Watson (2000), Shimer (2007), etc. Usually, the previous literature studies an economy with two types of agent. One exception is Wasmer and Weil (2004), where they assume match probability between entrepreneurs and workers, or between financiers and entrepreneurs is a monotonic function of the associated market tightness. For their purpose, a matching between workers and financiers is not relevant. Some other literature considers subtypes in the matching. For example, Moen (1997) assumes a finite number of submarkets indexed by different wages and uses the labor market tightness in each submarket to determine the matching probability. Other literature on directed search has similar setup, for example, Shi (2009) models a continuum of submarkets, also see Acemoglu and Shimer (1999), Menzio (2007) etc. In this chapter, these two assumptions are applied to an economy which has three types of agent and has three kinds of matching.

By assuming constant-returns aggregate matching functions and free entry, our model can be viewed as an extension of Rubinstein and Wolinsky (1987), where they argue that the role of middlemen is to reduce the time-preference losses that occur when agents must search for trading partners. Following Rubinstein and Wolinsky (1987), there is a lot of literature the provide alternative explanation of the role of middlemen in trading process with search frictions. Biglaiser (1993) analyzes how middlemen help to solve the moral hazard problem in exchange environment. Li (1998) presents a model where agents endogenously choose whether to become middlemen by investing in a technology that allows them to identify quality, which is private information of producers, hence, middlemen improve efficiency by increasing people's incentive to produce high-quality output as well as bringing high-quality goods from producers to customers. Johri and Leach (2002) assume that goods are heterogeneous and that workers have idiosyncratic tastes, so that the way in which goods are allocated to workers matters. Middlemen hold inventories, so that they are better able to match people with goods. Shevchenko (2004) considers an economy with a variety of goods and heterogeneous tastes, there is a role for middlemen who hold inventories of many goods to help solve the double coincidence problem. Masters (2007) shows how allowing for goods to be divisible at the point of consumption and incorporating productive heterogeneity lead to the emergence of middlemen in a equilibrium search environment.

This chapter differs from the previous literature on middlemen in several respects. First, given constant-returns aggregate matching functions and free entry, market tightness determines match probability in this chapter, while there is no role for it in the previous literature. Second, in this chapter, the presence of middlemen enhances social welfare, while welfare effect of middlemen is uncertain, or depends on parameters in the preview literature. Third, Rubinstein and Wolinsky (1987) restricts to consider symmetric equilibrium and there is no uniqueness of equilibrium. What is more, Rubinstein and

Wolinsky (1987) assumes that middlemen can meet buyers more effectively than producers can meet buyers directly, here we have modeled how middlemen are more efficient in matching.

The rest of this chapter is organized as follows. The model is set out in section 3.2, where we also characterize the equilibrium. Section 3.3 consists of extensions. Section 3.4 are concluding remarks while all the proofs appear in section 3.5.

3.2. The model

3.2.1. Environment

Time is continuous. There are three types of risk-neutral agent: workers, em-agents and firms. All agents have a common discounting rate ρ. Job matches can be formed directly between a worker and a firm, or indirectly through an em-agent. A job match products y unit of good per unit of time.

The measure of worker is fixed to be one. There is free entry for firms and em-agents. A Firm suffers ongoing cost η_f to create and hold a vacancy. It costs an em-agent η_h per unit of time in searching for a worker. It costs an em-agent-worker pair nothing to search for a vacancy. It is assumed that an em-agent-worker pair can not search for a worker, and an em-agent can not search for a vacancy. At any point of time, any match with a worker is faced with an exogenous separation shock with Poisson arrival rate δ.

Before we introduce matching processes, let's describe the state of the economy. Denote the measure of worker who has not matched with an em-agent or a firm as μ. Let v be the measure of vacancy, π_n be the measure of em-agent who engages in searching for a worker, and π be the measure of em-agent who has matched with a worker and is searching for a vacancy, or the measure of em-agent-worker pair. A measure over vector (μ, v, π_n, π) defines the state of the economy.

There are three matching processes in this economy. Assume that the flow of new matches between workers and em-agents is determined by a matching function $M_{wh}(\mu, \pi_n)$. Similarly, denote the matching function between em-agent-worker pairs and firms, and between workers and firms, as $M_{hf}(\pi, v)$ and $M_{wf}(\mu, v)$, respectively. All these three matching functions share some properties as below. For example, $M_{wh}(\mu, \pi_n)$ is non-negative, strictly increasing in both arguments, homogeneous of degree one, and satisfying $M_{wh}(0, \pi_n) = M_{wh}(\mu, 0) = 0$. Denote $\theta_{wh} \equiv \frac{\pi_n}{\mu}$, the Poisson arrival rate at which a worker finds an em-agent, $\alpha_{wh}(\theta_{wh}) \equiv \frac{M_{wh}(\mu, \pi_n)}{\mu} \equiv M_{wh}(1, \theta_{wh})$, is positive and strictly increasing in "market tightness", θ_{wh}, from the point of view of a worker. It is also assumed that $\alpha_{wh}(\theta_{wh})$ satisfies $\alpha_{wh}(0) = 0$ and $\alpha_{wh}(\infty) = \infty$. Similarly, the rate at which an em-agent finds a worker, $\alpha_{hw}(\theta_{hw}) \equiv M_{wh}(\theta_{hw}, 1)$, is positive, strictly increasing in $\theta_{hw} \equiv \frac{\mu}{\pi_n}$ and satisfies $\alpha_{hw}(0) = 0$ and $\alpha_{hw}(\infty) = \infty$.

After two different types of agent match, they engage in a Nash bargaining to determine their payoffs. Assume that, in any meeting, a firm has bargaining power $\beta \in (0, 1)$, while the other party (an em-agent-worker pair or a single worker) has bargaining power $(1 - \beta)$. That is, neither an em-agent-worker pair nor a single worker has bargain advantage over each other. For simplicity, in a meeting between an em-agent and a worker, it is assumed that the em-agent makes take-it-or-leave-offer to the worker.

3.2.2. Equilibrium

In this section, we focus on steady-states of the economy, where (μ, v, π_n, π) is a constant.

For a worker, V_U denotes his value when he hasn't formed a match or a pair, V_E denotes his value after he has directly formed a job match with a firm, and V_H denotes his value after he has formed a pair with an em-agent. In a meeting between a worker and an em-agent, the em-agent offers a value to the worker such that the worker is indifferent between accepting the offer and keeping search, $V_H = V_U$.

For a firm, denote J_E to be the value of a job filled directly, J_H to be the value of a job filled with an em-agent-worker pair, and J_U to be the value of a vacancy. Free entry implies that $J_U = 0$.

Now consider the values for an em-agent. After an em-agent-worker pair meets with a firm, the promised value from the em-agent to the worker is already sunk, which is not taken into account in the bargaining between an em-agent-worker pair and a firm. Considering his gross value only, denote W_E to be his value after he has reached a contract with a firm, denote W_H to be his value when he is searching for a vacancy, and denote W_U to be his value when he is searching for a worker. Free entry implies that $W_U = 0$.

The value function V_U satisfies the continuous-time Bellman equation

$$\rho V_U = \alpha_{wf} \left(V_E - V_U \right). \tag{3.1}$$

While the value function V_E satisfies $\rho V_E = \omega + \delta \left(V_U - V_E \right)$, where ω is the wage for workers. Then J_E satisfies $\rho J_E = y - \omega - \delta J_E$. These two equations lead to

$$y + \delta V_U = \left(\rho + \delta \right) \left(V_E + J_E \right). \tag{3.2}$$

Total surplus from a job match is $V_E + J_E - V_U - J_U$. The bargaining between a worker and a firm implies that $V_E - V_U = (1-\beta)(V_E + J_E - V_U - J_U)$, or

$$J_E = \frac{\beta}{1-\beta}(V_E - V_U). \tag{3.3}$$

The value function W_H satisfies

$$(\rho+\delta) W_H = \alpha_{hf}(W_E - W_H). \tag{3.4}$$

To divide between an em-agent and a firm, total surplus from a job match between a vacancy and an em-agent-worker pair is $\int_0^\infty e^{-(\rho+\delta)\tau} y d\tau = \frac{y}{\rho+\delta}$. The bargaining between an em-agent and a firm solves $\max_{W_E} [W_E - W_H]^{1-\beta} \left[\frac{y}{\rho+\delta} - W_E\right]^\beta$, which has a solution $W_E - W_H = (1-\beta)\left[\frac{y}{\rho+\delta} - W_H\right]$, or

$$W_E = (1-\beta)\frac{y}{\rho+\delta} + \beta W_H. \tag{3.5}$$

Hence,

$$J_H = \beta\left[\frac{y}{\rho+\delta} - W_H\right]. \tag{3.6}$$

Free entry of em-agents means

$$\eta_h = \alpha_{hw} \left[W_H - V_U \right], \tag{3.7}$$

while the free entry condition for a firm is

$$\eta_f = \alpha_{fh} J_H + \alpha_{fw} J_E, \tag{3.8}$$

where the first term is a firm's expected value from the matching with an em-agent-worker pair, and the second term comes from the matching with a worker directly.

The transition of μ satisfies

$$\mu \left(\alpha_{wh} + \alpha_{wf} \right) = (1 - \mu) \delta, \tag{3.9}$$

while the law of motion for π_n and π is

$$\pi_n \alpha_{hw} = \pi \left(\alpha_{hf} + \delta \right). \tag{3.10}$$

A steady-state equilibrium of the economy is a $(\mu,\, v,\, \pi_n,\, \pi)$ and a $(V_U,\, V_H,\, J_H,\, J_E,\, W_H,\, W_E)$ that satisfy equations (3.1)-(3.10).

It is easy to check that $V_E = (y + \delta V_U) \frac{1-\beta}{\rho+\delta} + \beta V_U$ is an increasing function of V_U, which is generally true except when there is match specific heterogeneity.

3.2.3. Steady-state without em-agents

We first analyze the steady-state equilibrium of an economy that prohibits the business of em-agents, which is a well-known world. Denote v^0, μ^0, α^0_{wf}, V^0_U, and V^0_E as the variables in the steady-state equilibrium without em-agents.

Proposition 3.1 There exists unique steady-state equilibrium.

In steady-state, $V^0_U = \frac{\alpha^0_{wf}(1-\beta)y}{\rho\alpha^0_{wf}(1-\beta)+\rho(\rho+\delta)}$ and V^0_E are increasing in α^0_{wf}, hence the welfare of workers (and the economy, because firms gain their outside options) is increasing in α^0_{wf}.

Proposition 3.2 α^0_{wf}, V^0_E, and V^0_U are decreasing in η_f.

3.2.4. Steady-state with em-agents

Denote α^*_{wf}, V^*_E, etc. as the variables in the steady-state equilibrium with em-agents.

Proposition 3.3 There exists a unique steady-state equilibrium. In this steady state, em-agents are active ($\pi > 0$ and $\pi_n > 0$).

Why is there a role for em-agents? Because if there are none and em-agents are not prevented from entering, then entrants match at an infinite rate, and earn enough to cover entry cost.

Proposition 3.4 The steady state is such that $\alpha^*_{hf} \geq \alpha^*_{wf} > \alpha^0_{wf}$, $V^*_U > V^0_U$ and $V^*_E > V^0_E$. Also, α^*_{wf}, V^*_E and V^*_U are decreasing in η_f and η_h.

In the steady-state equilibrium with em-agents, an em-agent is more efficient in finding a vacancy than a worker. The existence of em-agents enhances the welfare of workers.

3.3. Extensions

In this section, we consider several extensions to see how our main results vary with assumptions.

3.3.1. Nash bargaining between em-agents and workers

Consider a general Nash bargaining in a meeting between an em-agent and a worker, where the em-agent has bargaining power $\beta \in (0,\ 1)$. The bargaining between an em-agent and a worker solves $\max\limits_{V_H \geq 0} \left\{ [V_H - V_U]^{1-\beta} [W_H - V_H]^{\beta} \right\}$. The solution is $V_H - V_U = (1-\beta)[W_H - V_U]$, or the promised value for the worker is $V_H = (1-\beta)W_H + \beta V_U$. Then $V_U = \frac{\alpha_{wh}(1-\beta)W_H + \alpha_{wf} y \frac{1-\beta}{\rho+\delta}}{\rho + \alpha_{wh}(1-\beta) + \alpha_{wf}\rho\frac{1-\beta}{\rho+\delta}}$. $W_H = \frac{\alpha_{hf}(1-\beta)\frac{y}{\rho+\delta}}{\delta+\rho+\alpha_{hf}(1-\beta)}$ is the same as before. The free entry condition for an em-agent changes to $\eta_h = \beta\alpha_{hw}[W_H - V_U]$.

Proposition 3.5 In the steady-state equilibrium with em-agents, $\alpha^*_{hf} \geq \alpha^*_{wf} > \alpha^0_{wf}$, $V^*_U > V^0_U$ and $V^*_E > V^0_E$.

3.3.2. Different bargaining power

Now consider the case when an em-agent and a worker have different bargaining powers in the bargaining with a firm.

Assume that, in a meeting between a firm and a worker, the firm has bargaining power $\beta \in (0,\ 1)$, while in a meeting between a firm and an em-agent-worker pair, the firm has bargaining power $\gamma \in (0,\ 1)$. Then in equilibrium $W^*_H = \frac{\alpha_{hf}(1-\gamma)\frac{y}{\rho+\delta}}{\delta+\rho+\alpha_{hf}(1-\gamma)}$, $\eta_f = \frac{\alpha^*_{fh}\gamma y}{\rho+\delta+(1-\gamma)\alpha^*_{hf}} + \frac{\alpha^*_{fw}\beta y}{\alpha^*_{wf}(1-\beta)+\rho+\delta}$, other equations are the same. The following results is obvious.

Proposition 3.6 In the steady-state equilibrium with em-agents, $\alpha^*_{hf}(1-\gamma) \geq \alpha^*_{wf}(1-\beta) > \alpha^0_{wf}(1-\beta)$, $V^*_U > V^0_U$ and $V^*_E > V^0_E$.

Hence, if a worker has bargaining advantage over an em-agent-worker pair in the bargaining with a firm, then in equilibrium, an em-agent needs to search for a firm sufficiently faster than a worker to compensate for it.

3.3.3. Workers have outside options

Suppose now an unmatched worker produces $b < y$ amount of good per unit of time. Then in equilibrium $V_U^* = \frac{b+\alpha_{wf}^*(1-\beta)y}{\rho\alpha_{wf}^*(1-\beta)+\rho(\rho+\delta)}$, and $\alpha_{hf}^* > \alpha_{wf}^*$. Other results don't change.

3.3.4. When an em-agent's payment to a worker is conditional on a job match

Suppose now when an em-agent meets with a worker, he doesn't actually pay the worker the promised value. This payment is conditional on a job match formed between the em-agent-worker pair and a firm in the future. That is, $\alpha_{hf}V_H = V_U$. To divide between an em-agent and a firm, total surplus from a job match between a vacancy and an em-agent-worker pair is now $\frac{y}{\rho+\delta} - V_H$. The Nash bargaining means that

$$W_E = (1-\beta)\frac{y}{\rho+\delta} + \beta W_H + \beta V_H. \tag{3.11}$$

Hence,

$$J_H = \beta\left[\frac{y}{\rho+\delta} - V_H - W_H\right]. \tag{3.12}$$

The value function W_H satisfies

$$(\rho + \delta)\, W_H = \alpha_{hf}\, (W_E - V_H - W_H)\,. \tag{3.13}$$

Hence, $W_H = \frac{\alpha_{hf}(1-\beta)\left(\frac{y}{\rho+\delta}-V_H\right)}{\rho+\delta+\alpha_{hf}(1-\beta)}$. Free entry condition for an em-agent requires $\eta_h = \alpha_{hw} W_H$. Then $V_U = \frac{\alpha_{wf}(1-\beta)y}{(\rho+\delta)\left(\rho+\alpha_{wh}-\frac{\alpha_{wh}}{\alpha_{hf}}\right)+\alpha_{wf}\rho(1-\beta)}$.

The following results are obviously.

Proposition 3.7 In the steady-state equilibrium with em-agents, $\alpha^*_{wf} > \alpha^0_{wf}$, $V^*_U > V^0_U$ and $V^*_E > V^0_E$.

This alternative assumption leads an em-agent to a better bargaining position than a worker in the bargaining with a firm, where an em-agent is assumed to have higher outside option than a worker. This advantage of em-agents makes $\alpha^*_{hf} \geq \alpha^*_{wf}$ not to hold again. However, welfare result still holds.

3.3.5. Match specific heterogeneity

Now consider the case when there is match specific heterogeneity in productivity. Specifically, when a vacancy is filled with a worker or an em-agent-worker pair, its productivity is randomly drown from a uniform distribution over [0, 1]. Then the value function V_U satisfies a new continuous-time Bellman equation

$$\rho V_U = \alpha_{wf} \int_{\phi}^{1} \left(\max\{V_E\,(\upsilon)\,,\ V_U\} - V_U\right) d\upsilon, \tag{3.14}$$

where $V_E(\upsilon)$ denotes the value of a worker if his match productivity is υ, and ϕ is the cutoff productivity, with which a worker is willing to form a job match with a firm. Then $V_E(\phi) = V_U$, or

$$V_U = \frac{\phi}{\rho + \delta}. \tag{3.15}$$

Similarly, the value function W_H satisfies

$$(\rho + \delta) W_H = \alpha_{wf} \int_{\varphi}^{1} (\max\{W_E(\upsilon), \; W_H\} - W_H) \, d\upsilon, \tag{3.16}$$

where $W_E(\upsilon)$ denotes the value of an em-agent-worker pair if his job match has productivity υ, and φ is the cutoff productivity with which an em-agent is willing to reach a contract with a firm. Then

$$W_H = \frac{\varphi}{\rho + \delta}. \tag{3.17}$$

The free entry condition for a firm is now

$$\eta_f = \alpha_{fh} \int_{\varphi}^{1} J_H(\upsilon) d\upsilon + \alpha_{fw} \int_{\phi}^{1} J_E(\upsilon) d\upsilon. \tag{3.18}$$

Proposition 3.8 With match specific heterogeneity, the unique steady-state equilibrium is defined by

$$\eta_h = \alpha^*_{hw} \left[W^*_H - V^*_U \right], \tag{3.19}$$

$$\eta_f = \frac{(\delta+\rho)\,\beta\alpha^*_{fh}\frac{(1-\varphi^*)^2}{2(\delta+\rho)}}{\delta+\rho+\alpha^*_{hf}(1-\varphi^*)(1-\beta)} + \frac{(\rho+\delta)\,\alpha^*_{fw}\beta\frac{(1-\phi^*)^2}{2(\rho+\delta)}}{\rho+\delta+\alpha^*_{wf}(1-\phi^*)(1-\beta)}, \tag{3.20}$$

$$\pi^*\left(\alpha^*_{hf}(1-\varphi^*)+\delta\right) = \pi^*_n\alpha^*_{hw}, \tag{3.21}$$

$$\mu = \frac{\delta}{\alpha^*_{wh}+\alpha^*_{wf}(1-\phi^*)+\delta}, \tag{3.22}$$

where $W^*_H = \frac{\alpha^*_{hf}(1-\beta)\frac{(1-\varphi^*)^2}{2(\delta+\rho)}}{\delta+\rho+\alpha^*_{hf}(1-\varphi^*)(1-\beta)} = \frac{\varphi^*}{\delta+\rho}$ and $V^*_U = \frac{(1-\beta)\alpha^*_{wf}\frac{(1-\phi)^2}{2\rho}}{\rho+\delta+\alpha^*_{wf}(1-\phi^*)(1-\beta)} = \frac{\phi^*}{\delta+\rho}$.

The proof of Proposition 3.8 is the same as that of proposition 3.3, noting that φ^* and $\alpha^*_{hf}(1-\varphi^*)$ (ϕ^* and $\alpha^*_{wf}(1-\phi^*)$) are strictly increasing functions of α^*_{hf} (α^*_{wf}).[2]

Proposition 3.9 With match specific heterogeneity, the steady-state equilibrium is such that $\alpha^*_{hf} \geq \alpha^*_{wf} > \alpha^0_{wf}$, $\varphi^* \geq \phi^*$, $V^*_U > V^0_U$. φ^* and ϕ^* are decreasing in η_f. ϕ^* is also decreasing in η_h, but φ^* is increasing in η_h.

The proof is similar to that of Proposition 3.4. $\varphi^* \geq \phi^*$ implies jobs filled through em-agents have higher average productivity than ones done directly, or $\frac{1+\varphi^*}{2} \geq \frac{1+\phi^*}{2}$. Overall productivity is also higher when em-agents present.

[2]Since $\frac{\phi^*}{(\delta+\rho)(1-\phi^*)}$ is increasing in ϕ^*.

3.4. Conclusion

We have shown that most of our results don't vary when we alter some assumptions of the model. On the other hand, relaxing some of the assumptions raise interesting questions for further research. For example, we might allow em-agents to searching for more than a worker, or allow an em-agent and a firm to form a pair and search for a worker, or both. We may also assume that a worker is unable to make commitment to em-agents.

Even though constant-return-to-scale aggregate matching function and free entry are widely assumed in the random search literature, this chapter shows how we can get some interesting results if we assume these two in an economy with three types of agent. Our match technology might therefore be used in other setups.

3.5. Appendix

3.5.1. Proof of Proposition 3.1

The steady-state is determined by $\eta_f = \frac{\alpha^0_{fw}\beta y}{\alpha^0_{wf}(1-\beta)+\rho+\delta}$ and $\mu^0 = \frac{\delta}{\delta+\alpha^0_{wf}}$.

In the steady-state without em-agents, we have $\rho V_U = \alpha_{wf}(V_E - V_U)$, or $V_E = \frac{\rho+\alpha_{wf}}{\alpha_{wf}}V_U$. Since $J_E = \frac{\beta}{1-\beta}(V_E - V_U)$, $J_E = \frac{\beta}{1-\beta}\left(\frac{\rho+\alpha_{wf}}{\alpha_{wf}}V_U - V_U\right) = \frac{\beta}{1-\beta}\frac{\rho}{\alpha_{wf}}V_U$. $y + \delta V_U = (\rho+\delta)(V_E + J_E)$ implies $V_E = (y+\delta V_U)\frac{1-\beta}{\rho+\delta} + \beta V_U$ and $(\rho+\delta)\left(\frac{\rho+\alpha_{wf}}{\alpha_{wf}}V_U + \frac{\beta}{1-\beta}\frac{\rho}{\alpha_{wf}}V_U\right) = y + \delta V_U$ or $\frac{(\rho+\delta)}{\alpha_{wf}}\left(\rho + \alpha_{wf} + \frac{\beta\rho}{1-\beta}\right)V_U - \delta V_U = y$, or $\frac{(\rho+\delta)}{\alpha_{wf}(1-\beta)}(\rho + \alpha_{wf}(1-\beta))V_U - \delta V_U = y$, or $\left(\frac{\rho(\rho+\delta)}{\alpha_{wf}(1-\beta)} + \rho\right)V_U = y$, so $V_U = \frac{\alpha_{wf}(1-\beta)y}{\rho\alpha_{wf}(1-\beta)+\rho(\rho+\delta)}$. Then $J_E = \frac{\beta}{1-\beta}\frac{\rho}{\alpha_{wf}}$, $V_U = \frac{\beta}{1-\beta}\frac{\rho}{\alpha_{wf}}\frac{\alpha_{wf}(1-\beta)y}{\rho\alpha_{wf}(1-\beta)+\rho(\rho+\delta)} = \frac{\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}$. $\eta_f = \alpha_{fw}J_E$ implies that $\eta_f = \frac{\alpha_{fw}\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}$.

Obviously, $\frac{\alpha_{fw}\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}$ is a strictly deceasing function of α_{wf} or θ_{wf}, if $\theta_{wf}=0$, $\frac{\alpha_{fw}\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}=\infty$, if $\theta_{wf}=\infty$, $\frac{\alpha_{fw}\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}=0$. Intermediate Value Theorem implies that there is unique θ^0_{wf} or α^0_{wf} such that $\eta_f=\frac{\alpha_{fw}\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}$. Given this unique θ^0_{wf} or α^0_{wf}, there is unique μ^0 satisfying $\mu=\frac{\delta}{\delta+\alpha^0_{wf}}$, which leads to the uniqueness of υ^0 and V^0_U.

3.5.2. Proof of Proposition 3.2

Since $\frac{\alpha_{fw}\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}$ is a strictly decreasing function of θ_{wf}, as η_f increases, θ^0_{wf} or α^0_{wf} decrease, so does V^0_U. Hence, μ^0 and α^0_{fw} increase. Since $(1-\mu^0)\,\delta=\mu^0\alpha^0_{wf}$ and μ^0 increases, $\mu^0\alpha^0_{wf}$ decreases. $\upsilon^0\alpha^0_{fw}=\mu^0\alpha^0_{wf}$ implies υ^0 is decreasing in η_f.

3.5.3. Proof of Proposition 3.3

We first show that steady-state (if exists) is a (μ^*, υ^*, π^*_n, π^*) satisfies the following four equations. We then prove that the four equations have a solution by using the intermediate value theorem to eliminate variables one at a time. This also depends on establishing the relevant monotonicity at each stage.

$$\eta_h=\alpha^*_{hw}\left[W_H-V^*_U\right], \tag{B3.1}$$

$$\eta_f=\frac{\alpha^*_{fh}\beta y}{\rho+\delta+(1-\beta)\,\alpha^*_{hf}}+\frac{\alpha^*_{fw}\beta y}{\alpha^*_{wf}\,(1-\beta)+\rho+\delta}, \tag{B3.2}$$

(B3.3)
$$\pi^* \left(\alpha^*_{hf} + \delta\right) = \pi^*_n \alpha^*_{hw},$$

(B3.4)
$$\mu^* = \frac{\delta}{\alpha^*_{wh} + \alpha^*_{wf} + \delta},$$

where $W^*_H = \frac{(1-\beta)\alpha^*_{hf}}{\rho+\delta+(1-\beta)\alpha^*_{hf}} \frac{y}{\rho+\delta}$ and $V^*_U = \frac{\alpha^*_{wf}(1-\beta)y}{\rho\alpha^*_{wf}(1-\beta)+\rho(\rho+\delta)}$.

Similar to the proof of Proposition 3.1, it is easy to show that $V_U = \frac{\alpha_{wf}(1-\beta)y}{\rho\alpha_{wf}(1-\beta)+\rho(\rho+\delta)}$, $V_E = \frac{\left(\rho+\alpha_{wf}\right)(1-\beta)y}{\rho\alpha_{wf}(1-\beta)+\rho(\rho+\delta)}$, $J_E = \frac{\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}$. Equation (3.4) and (3.5) implies that $(\rho+\delta)\, W_H = \alpha_{hf}\left((1-\beta)\frac{y}{\rho+\delta} + \beta W_H - W_H\right)$, or $W_H = \frac{\alpha_{hf}(1-\beta)}{\delta+\rho+\alpha_{hf}(1-\beta)}\frac{y}{\rho+\delta}$. Equation (B3.1) comes from equations (3.7). Equations (3.3), (3.6) and (3.8) imply that $\eta_f = \alpha_{fh}J_H + \alpha_{fw}J_E = \frac{\alpha_{fh}\beta y}{\rho+\delta+(1-\beta)\alpha_{hf}} + \frac{\alpha_{fw}\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}$, that is equation (B3.2). Other two equation are obvious.

Now let's establish the relevant monotonicities, which also implies the uniqueness of equilibrium.

Since $\frac{\alpha_{fw}\beta y}{\alpha_{wf}(1-\beta)+\rho+\delta}$ ($\frac{\alpha_{fh}\beta y}{\rho+\delta+(1-\beta)\alpha_{hf}}$) is strictly decreasing function of θ_{wf} (θ_{hf}), equation (B3.2) defines θ_{hf} as a strictly decreasing function of θ_{wf}. Equation (B3.1) then defines θ_{hw} as a strictly increasing function of θ_{wf}. So equation (B3.4) defines μ as a function of θ_{wf}, π_n, π, and v can then be determined. Since $\pi_n\alpha_{hw} = \mu\alpha_{wh} = \frac{\alpha_{wh}\delta}{\alpha_{wh}+\alpha_{wf}+\delta}$ is a strictly decreasing function of θ_{wf}, π_n is a strictly decreasing function of θ_{wf}. Since $v\alpha_{fw} = \mu\alpha_{wf} = \frac{\alpha_{wf}\delta}{\alpha_{wh}+\alpha_{wf}+\delta}$ is a strictly increasing function of θ_{wf}, v is a strictly increasing function of θ_{wf}. Since $\pi\alpha_{hf} = v\alpha_{fh}$ is a strictly increasing function of θ_{wf}, π is a strictly increasing function of θ_{wf}. $\pi\left(\alpha_{hf}+\delta\right) = v\alpha_{fh} + \pi\delta$ is also a strictly increasing function of

θ_{wf}, so $\pi_n\alpha_{hw}-\pi\left(\alpha_{hf}+\delta\right)$ is a strictly decreasing function of θ_{wf}. Hence, the steady-state (if exists) is unique.

Now let's use the intermediate value theorem to prove the existence of solution.

Let $\theta_{wf}=\theta_{wf}^0$, then $\theta_{hf}=\infty$, $W_H=\frac{y}{\rho+\delta}>V_U^0$. Similar to the proof of Proposition 3.1, there is θ_{hf}^0, or (α_{hf}^0, α_{fh}^0 and W_U^0) such that $\eta_f=\frac{\alpha_{fh}\beta y}{\rho+\delta+(1-\beta)\alpha_{hf}}$, so $\theta_{wf}=\infty$, $V_U=\frac{y}{\rho}>W_U^0$. W_H-V_U is a strictly decreasing function of θ_{wf}. By intermediate value theorem, there is unique $\theta_{wf}^1\in\left(\theta_{wf}^0,\ \infty\right)$, and $\theta_{hf}^1\in\left(\theta_{hf}^0,\ \infty\right)$, such that $\eta_f=\frac{\alpha_{fh}^1\beta y}{\rho+\delta+(1-\beta)\alpha_{hf}^1}+\frac{\alpha_{fw}^1\beta y}{\alpha_{wf}^1(1-\beta)+\rho+\delta}$ and $W_U^1=V_U^1$.

If $\theta_{wf}>\ \theta_{wf}^1$, then $\theta_{hf}<\theta_{hf}^1$ and $W_H<V_U$, $\eta_h=\alpha_{hw}\left[W_H-V_U\right]$ can not hold. It suffices to consider $\theta_{hf}>\theta_{hf}^1$, and $\theta_{wf}\in\left(\theta_{wf}^0,\ \theta_{wf}^1\right)$. $\forall\ \theta_{wf}\in\left(\theta_{wf}^0,\ \theta_{wf}^1\right)$, there is unique $\theta_{hf}\in\left(\theta_{hf}^1,\ \infty\right)$ such that equation (B3.2) satisfies and $W_H>V_U$, then there is unique θ_{wh} such that $\eta_h=\alpha_{hw}\left[W_H-V_U\right]$.

If θ_{wf} is sufficient close to θ_{wf}^0, then $\theta_{hf}\to\infty$, $h\to 0$. Define $\alpha_{hw}\equiv\eta_h/\left[W_H-V_U\right]<\infty$. $\mu=\frac{\delta}{\alpha_{wh}+\alpha_{wf}+\delta}$, π_n, v are determined. Obviously, $\pi_n\alpha_{hw}>\pi\left(\alpha_{hf}+\delta\right)$, where $\pi\alpha_{hf}\to 0$. On the other hand, if θ_{wf} is sufficient close to θ_{wf}^1, then $\alpha_{hw}=\eta_h/\left[W_H-V_U\right]\to\infty$, so $\pi_n\to 0$ and $\pi_n\alpha_{hw}\to 0$, therefore $\pi_n\alpha_{hw}<\pi\left(\alpha_{hf}+\delta\right)$.

By intermediate value theorem, there is $\theta_{wf}^*\in\left(\theta_{wf}^0,\ \theta_{wf}^1\right)$, $\theta_{hf}^*>\theta_{hf}^1$, finite θ_{hw}^*, such that equation (B3.1)-(B3.4) hold with α_{wh}^*, α_{hw}^*, α_{hf}^*, α_{fh}^*, α_{wf}^*, and α_{fw}^*. $\theta_{wf}^*>\theta_{wf}^0$ implies $\alpha_{wf}^*>\alpha_{wf}^0$.

By equation (B3.4), μ^* is known. Since θ_{hw}^* known, h_n^* is known. By equation (B3.3), π^* is known. Then v^* can be determined by θ_{wf}^*.

3.5.4. Proof of Proposition 3.4

Equation (B3.1) implies $\alpha^*_{hf} \geq \alpha^*_{wf}$. In the proof of proposition 3.3, it is shown that $\alpha^*_{wf} > \alpha^0_{wf}$, which implies $V^*_U > V^0_U$ and $V^*_I > V^0_I$.

Now, consider the case when η_f increases first.

Claim: θ_{hf} (or α_{hf}) decreases. Suppose not, then θ_{wf} decreases, by equation (3.13) α_{hw} decreases, α_{wh} increases, $\mu\alpha_{wh} = \frac{\alpha_{wh}\delta}{\alpha_{wh}+\alpha_{wf}+\delta} = \pi_n\alpha_{hw}$ increases, π_n increases. $\alpha_{wf}\mu = \frac{\alpha_{wf}\delta}{\alpha_{wh}+\alpha_{wf}+\delta} = \alpha_{fw}v$ decreases, v decreases. α_{fh} doesn't increase, $\pi\alpha_{hf} = v\alpha_{fh}$ decreases, π decreases, $\pi_n\alpha_{hw} > \pi\left(\alpha_{hf}+\delta\right)$, contradiction.

Claim: θ_{wf} (or α_{wf}) decreases. Suppose not, by equation (B3.1), α_{hw} increases. $\alpha_{wh}\mu = \frac{\alpha_{wh}\delta}{\alpha_{wh}+\alpha_{wf}+\delta} = \alpha_{hw}\pi_n$ decreases. $\alpha_{wf}\mu = \frac{\alpha_{wf}\delta}{\alpha_{wh}+\alpha_{wf}+\delta} = \alpha_{fw}v$ increases, so v increases. $\pi\alpha_{hf} = v\alpha_{fh}$ increases, π increases, and $\pi\left[\delta+\alpha_{hf}\right]$ increases, so $\pi\left[\delta+\alpha_{hf}\right] > \alpha_{hw}\pi_n$, contradiction.

Second, consider the case when η_h increases.

Claim: θ_{wf} (or α_{wf}, V_U) decreases, while θ_{hf} (or α_{hf}, W_H) increases.

First, suppose θ_{wf} (or α_{wf}, V_U) doesn't change, neither θ_{hf} (or α_{hf},W_H). Then α_{hw} increases. μ increases because α_{wh} decreases. Fixed α_{wf} leads v to increase. With θ_{hf} unchanged, π increases , so does $\pi\left[\delta+\alpha_{hf}\right]$. Decreasing α_{wh} implies that $\pi_n\alpha_{hw} = \mu\alpha_{wh} = \frac{\alpha_{wh}\delta}{\alpha_{wh}+\alpha_{wf}+\delta}$ decreases. Together, $\pi\left[\delta+\alpha_{hf}\right] > \pi_n\alpha_{hw}$, contradiction.

Second, suppose θ_{wf} (or α_{wf}, V_U) increases. Equation (B3.2) implies θ_{hf} (or α_{hf},W_H) decreases. By equation (B3.1), α_{hw} increases or α_{wh} decreases. Hence, $\alpha_{wh}\mu = \frac{\alpha_{wh}\delta}{\alpha_{wh}+\alpha_{wf}+\delta} = \alpha_{hw}\pi_n$ decreases, so does π_n. Also, $\alpha_{wf}\mu = \frac{\alpha_{wf}\delta}{\alpha_{wh}+\alpha_{wf}+\delta} = \alpha_{fw}v$ increases, so v increases. Then $\pi\alpha_{hf} = v\alpha_{fh}$, π and $\pi\left[\delta+\alpha_{hf}\right] = \pi\delta+\alpha_{fh}v$ increases. Therefore, $\pi\left[\delta+\alpha_{hf}\right] > \pi_n\alpha_{hw}$, another contradiction.

3.5.5. Proof of Proposition 3.5

(B3.1) implies $W_H \geq V_U$. First, in equilibrium, $\alpha^*_{hf} \geq \alpha^*_{wf}$. Suppose not, then $V_U > \frac{\alpha_{wh}(1-\beta)W_H+\alpha_{hf}y\frac{1-\beta}{\rho+\delta}}{\rho+\alpha_{wh}(1-\beta)+\alpha_{hf}\rho\frac{1-\beta}{\rho+\delta}}$, or $V_U > \frac{\alpha_{wh}(1-\beta)\frac{(1-\beta)\alpha_{hf}}{\rho+\delta+(1-\beta)\alpha_{hf}}\frac{y}{\rho+\delta}+\alpha_{hf}y\frac{1-\beta}{\rho+\delta}}{\rho+\alpha_{wh}(1-\beta)+\alpha_{hf}\rho\frac{1-\beta}{\rho+\delta}} = \frac{\frac{\alpha_{wh}(1-\beta)+\rho+\delta+(1-\beta)\alpha_{hf}}{\rho+\delta+(1-\beta)\alpha_{hf}}}{\rho+\alpha_{wh}(1-\beta)+\alpha_{hf}\rho\frac{1-\beta}{\rho+\delta}}\frac{(1-\beta)\alpha_{hf}y}{\rho+\delta}$. So $V_U = \frac{\alpha_{wh}(1-\beta)+\rho+\delta+(1-\beta)\alpha_{hf}}{\rho+\alpha_{wh}(1-\beta)+\alpha_{hf}\rho\frac{1-\beta}{\rho+\delta}}W_H > W_H$, contradiction.

Second, $\alpha^*_{wf} > \alpha^0_{wf}$. Suppose not, then $\alpha^0_{fw} \leq \alpha^*_{fw}$ and $\frac{\alpha^*_{fw}\beta y}{\alpha^*_{wf}(1-\beta)+\rho+\delta} \geq \frac{\alpha^0_{fw}\beta y}{\alpha^0_{wf}(1-\beta)+\rho+\delta} = \eta_f$. However, we have $\eta_f > \frac{\alpha^*_{fw}\beta y}{\alpha^*_{wf}(1-\beta)+\rho+\delta}$, contradiction. $V^*_U = \frac{\alpha^*_{wh}(1-\beta)W^*_H+\alpha^*_{wf}y\frac{1-\beta}{\rho+\delta}}{\rho+\alpha^*_{wh}(1-\beta)+\alpha^*_{wf}\rho\frac{1-\beta}{\rho+\delta}} > \frac{\alpha^*_{wh}(1-\beta)V^*_U+\alpha^*_{wf}y\frac{1-\beta}{\rho+\delta}}{\rho+\alpha^*_{wh}(1-\beta)+\alpha^*_{wf}\rho\frac{1-\beta}{\rho+\delta}}$, so $V^*_U > \frac{\alpha^*_{wf}y\frac{1-\beta}{\rho+\delta}}{\rho+\alpha^*_{wf}\rho\frac{1-\beta}{\rho+\delta}} = \frac{\alpha^*_{wf}(1-\beta)}{\rho(\rho+\delta)+\rho\alpha^*_{wf}(1-\beta)}y > V^0_U$. Then $V^*_E > V^0_E$. This completes the proof.

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