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# LIVŠIC THEOREM FOR COCYCLES WITH VALUE IN $GL(N, \mathbb{Q}_P)$

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# Abstract

We extend the well-known results of Livšic theorem on the regularity of measurable solutions to $GL(n, \mathbb{Q}_p)$ valued cocycles, where $\mathbb{Q}_p$ is the p-adic field, which is a non-archimedean field. To prove the main result, we give the approximation of Lyapunove exponents by the Lyapunov exponents at periodic points and prove the uniformly boundedness of the cocycle if it is uniformly bounded at periodic points under the non-archimedean condition.

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Chapter 1

# Introduction

## 1.1 Livšic Theorem

Let $f : X \to X$ be a continuous map on a compact metric space $X$ and let $G$ be a group and let $A : X \to G$ be a Hölder continuous function. Assume there exists a $f$-invariant ergodic measure on $X$. We are interesting in the sufficient condition of existing a Hölder continuous function $P : X \to G$ such that

$$A(x) = P(fx)P(x)^{-1} \text{ for all } x \in X. \tag{1.1.1}$$

We call this kind of function $A$ coboundary.

For the cocycle generated by a coboundary function $A$,

$$\mathcal{A}(x, m) = A(f^{m-1}(x)) \cdots A(f(x))A(x),$$

it must have trivial periodic data, i.e. for all $q \in X, m \in \mathbb{N}$ with $f^m q = q$

$$\mathcal{A}(q, m) = A(f^{m-1}(q)) \cdots A(f(q))A(q) = \text{Id}.$$

We are interesting in the question whether this necessary condition is also the sufficient condition.

In [1, 2], A. Livšic first obtained the positive answer to real valued cocycles over Anosov $\mathbb{Z}$-actions. He proved for a compact Riemannian manifold $M$, a topologically transitive $C^1$ Anosov diffeomorphism $f$ of $M$, $0 < \alpha < 1$, and any

$\alpha$-Hölder real-valued function $A$ satisfying $\Sigma_{i=0}^{n-1} A(f^k x) = 0$ whenever $f^n x = x$, there exists an $\alpha$-Hölder function $P$ on $M$ such that $A(x) = P(fx) - P(x)$ holds.

The question turns to more difficult when the group $G$ is non-abelian. In [3], V.Nicţică and A. Török extended Livsic Theorem to cocycles with values in the group of $C^K$-diffeomorphisms of a compact closed manifold with stably trivial tangent bundle($4 \leq K < \infty$). The strategy of the proof was similar to the one used by Livsic, but the situation is more complicated due to the absence of invariant metrics. They replaced those by the quasi-invariance properties which are introduced by themselves and enough for their purpose. In [4], they considered Livsic theorem with the transformations which need not be hyperbolic and cocycles which take values in certain non-abelian non-compact groups. i.e. continuous map $f : X \to X$ is locally eventually onto or minimal and the group $G$ has a left invariant metric with all sectional curvatures non-positive, the theorem applies when G is $\mathbb{R}^n$, $E(2)$ or is commutative.

In [5], H.Bercovici and V.Nicţică proved that a banach algebra valued cocycle which is sufficiently close to $Id$, is a coboundary. In [6], M.Nicol and M.Pollicott showed that, when $G$ is a connected non-compact semisimple Lie group and satisfies a pinching condition, then the Livsic theorem holds, the theorem applies to $SL(2, \mathbb{R})$. In [7], R. De la Llave and A. Windsor extened the Livsic theorem to cocycles taking values in Lie group and Diffeomorphism group under the localization assumption. In [8], A.Katok and V.Nicţică introduced the center bunching condition and also proved Livsic theorem for cocycles taking values in Lie group and Diffeomorphism group. In [9], B. Kalinin proved a remarkable result for any $GL(n, \mathbb{R})$ valued cocycle by using the approximation of Lyapunov exponents by the Lyapunov exponents at periodic points and the approximation of growth of the cocycle along orbits.

There are also some results in the general case of cohomology of two cocycles. In [10], W.Parry and M.Pollicott studies the cohomology equation $B(x) = P(f(x))A(x)P(x)^{-1}$ not only for coboundary but a general cocycle $A$ with values in $\mathbb{R}$, $\mathbb{C}^d$ and $U(d)$(the $d$-dimensional unitary group). They first proved the result for a subshift of finite type $f : X \to X$ and then pushed down their result to hyperbolic diffeomorphisms. In [11], K.Schmidt introduced the assumption "bounded distortion" to control the growth of the cocycle. Under this assumption , he proved that, if $A$ and $B$ have the same periodic data, then they are Hölder cohomologous.

There are also some related papers, we refer [12, 13, 14, 15].

In this disseration, we extend Kalinin's theorem to arbitrary $GL(n, \mathbb{Q}_p)$ valued cocycle. We give the a different construction of Lyapunov norm than Kalinin's method and obtain a stronger estimate of Lyapunov exponents in the non-archimedean case. Further more, the result enable us to extend the positive answer to the sufficient condition question to any $GL(\mathcal{W})$ cocycles, where $\mathcal{W}$ is a vector space over a local field.

## 1.2 Formulation of Result

In this section, we point out the main result of our research.

**Theorem 1.** *Let $f$ be a topological transitive anosov diffeomorphism of a compact metric space $X$ and let $A : X \to GL(n, \mathbb{Q}_p)$ be an $\alpha$-Hölder function generating the cocycle $\mathcal{A}(x, m)$. If the periodic data of cocycle $A$ satisfies the condition*

$$\mathcal{A}(q, m) = A(f^{m-1}q) \cdots A(f(q))A(q) = Id$$

*for any periodic point $q \in X$ such that $q = f^m q$.*

*Then there exists an $\alpha$-Hölder function $P : X \to GL(n, \mathbb{Q}_p)$ such that*

$$A(x) = P(fx)P(x)^{-1} \text{ for all } x \in X.$$

**Remark 1.2.1.** Theorem 1 holds for any $GL(\mathcal{W})$ valued cocycles, where $\mathcal{W}$ is a vector space of a local field. We know a local field is isomorphic either to $\mathbb{R}$ or $\mathbb{C}$ or a finite extension of p-adic field $\mathbb{Q}_p$ or a field of formal power series in one variable over a finite field. In [9] Boris Kalinin proved the real number case and in [16] G. Margulis showed the Multiplicative ergodic theorem for any $GL(\mathcal{W})$ valued cocycles. Then for the complex case, the proof should be identical to Kalinin's proof, for the non-archimedean case, since we have the Multiplicative ergodic theorem, the proof should be identical to this dissertation.

## 1.3 Outline of Proof

In the proof we first estimate the Lyapunov exponents by the Lyapunov exponents at periodic points, which is summarized as the following theorem.

**Theorem 2.** *Let $f : X \to X$ be an anosov diffeomorphism of a compact metric space $X$, let $\mathcal{A}$ be a Hölder $GL(n, Q_p)$ cocycle over $f$, and let $\mu$ be an ergodic invariant measure for $f$. Suppose the Lyapunov exponents are $\chi_1 \leq \chi_2 \leq \ldots \leq \chi_n$(with multiplicities) of $\mathcal{A}$ with respect to $\mu$, then for any $\epsilon > 0$, there exists a periodic point $q \in X$ for which the Lyapunov exponents $\chi_1^{(q)} \leq \chi_2^{(q)} \leq \ldots \leq \chi_n^{(q)}$ of $\mathcal{A}$ satisfy $|\chi_i - \chi_i^{(q)}| < \epsilon$ for $i = 1, \ldots, n$.*

**Remark 1.3.1.** Both of the papers [17, 18] considered the approximation of Lyapunov exponents. [17] gave the estimate of Lyapunov exponents of a hyperbolic ergodic measure for a diffeomorphism; [18] showed the Lyapunov exponents of $GL(m, \mathbb{R})$ valued Hölder cocycle can be approximated by the Lyapunov exponents at periodic points by a different approach other than [9].

To complete the proof of theorem 2, we introduce the Lyapunove exponents for $GL(n, \mathbb{Q}_p)$-valued cocycles and construct the Lyapunov norm on the cocycles, which is the key part of the proof.

And then we apply theorem 2 and give the uniformly boundedness of the cocycle if the cocycle is uniformly bounded at any periodic points.

**Theorem 3.** *Let $f$ be a topological transitive measure-preserving anosov diffeomorphism of a compact metric space $X$ and let $A : X \to GL(n, \mathbb{Q}_p)$ be an $\alpha$-Hölder function generating the cocycle $\mathcal{A}(x, m)$. If $\mathcal{A}(q, m)$ is uniformly bounded for all periodic points $q = f^m q$ and $m \in \mathbb{Z}$, then the cocycle $\mathcal{A}(x, m)$ is uniformly bounded for all $x \in X$ and $m \in \mathbb{Z}$.*

**Remark 1.3.2.** In the proof of theorem 2 and 3, we use the Anosov closing lemma and the local product structure. So these two results can be extended to any smooth hyperbolic systems which satisfy the Anosov closing lemma and the local product structure, such as hyperbolic automorphisms of tori and nilmanifolds and locally maximal hyperbolic sets. We can also apply this to symbolic dynamical systems such as subshift of finite type.

With theorem 2 and 3, we can use the classic method which is the extension along the dense orbit of a transitive point to complete the proof of theorem 1.

# Chapter 2

# The field of p-adic numbers

In this chapter, we review some basic definitions and theories of the field of p-adic numbers and the vector space over the field of p-adic numbers.

## 2.1 p-adic number

**Definition 2.1.1.** Let $p$ be any prime number. For any nonzero integer $a$, let $\text{ord}_p(a)$ be the highest power of $p$ which divides $a$, i.e., the greatest $m$ such that $a \equiv 0 (\text{mod } p^m)$. Now for any rational number $x = \frac{a}{b}$, define $\text{ord}_p(x) = \text{ord}_p(a) - \text{ord}_p(b)$. We have

$$\text{ord}_p(x) = n \text{ if } x = p^n \cdot \frac{c}{d} \text{ and } p \nmid cd \tag{2.1.1}$$

Further define a map $| \ |_p$ on $\mathbb{Q}$ as follows:

$$|x|_p = \begin{cases} \frac{1}{p^{\text{ord}_p(x)}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

It is easy to check $| \ |_p$ is a norm on $\mathbb{Q}$, more precisely it satisfies the following conditions:

1. $|x|_p = 0$ if and only if $x = 0$.
2. $|x \cdot y|_p = |x|_p \cdot |y|_p$.

3. $|x+y|_p \leq \max(|x|_p, |y|_p) \leq |x|_p + |y|_p$.

A norm satisfies the condition (3) is called non-archimedean.

**Proposition 2.1.1.** *Every triangle is isosceles for a non-archimedean norm.*

*Proof.* Let $\| \ \|$ be a non-archimedean norm on a field $F$. The non-archimedean triangle inequality says:$\|x-y\| \leq \max(\|x\|, \|y\|)$, if $\|x\| \neq \|y\|$,without loss of generality, we can suppose $\|x\| < \|y\|$. Then

$$\|x-y\| \leq \|y\|.$$

But

$$\|y\| = \|x-(x-y)\| \leq \max(\|x\|, \|x-y\|).$$

Since $\|y\|$ is not $\leq \|x\|$, we must have $\|y\| \leq \|x-y\|$, so $\|y\| = \|x-y\|$.Thus, if the two sides $x$ and $y$ are not equal, the longer one have the same length as the third side. $\square$

Let $S$ be the set of Cauchy sequences $\{a_n\}$ of rational numbers such that given $\epsilon > 0$, there exists an $N$ such that $|a_n - a_{n'}|_p < \epsilon$ if both $n, n^{'} > N$. We call two such Cauchy sequences $\{a_n\}$ and $\{b_n\}$ equivalent if $|a_n - b_n|_p \to 0$ as $n \to \infty$.

Actually, Cauchy sequences can be characterized much more simply when the absolute value is non-archimedian.

**Proposition 2.1.2.** *A sequence $\{a_n\}$ of rational number is a Cauchy sequence with respect to non-archimedian absolute value $| \ |_p$ if and only if we have*

$$\lim_{n\to\infty} |a_{n+1} - a_n|_p = 0$$

**Definition 2.1.2.** We define the field of p-adic numbers to be set of equivalence classes of Cauchy sequences, i.e.,

$$\mathbb{Q}_p = S/I$$

where $I$ is the maximal ideal of $S$.

Actually,

$$I = \{\{a_n\} : \lim_{n\to\infty} |a_n|_p = 0\}$$

It makes sense to extend the absolute value $| \, |_p$ to $\mathbb{Q}_p$.

**Definition 2.1.3.** If $\lambda \in \mathbb{Q}_p$ is an element of $\mathbb{Q}_p$, and $\{a_n\}$ is any Cauchy sequence representing $\lambda$, we define

$$|\lambda|_p = \lim_{n\to\infty} |a_n|_p$$

We have following theorem to show that $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ with respect to the non-archimedian absolute value $| \, |_p$.

**Theorem 4** ([19]Theorem 3.2.11)**.** *For each prime $p \in \mathbb{Z}$ there exists a field $\mathbb{Q}_p$ with a non-archimedian absolute value $| \, |_p$, such that:*

1. *there exists an inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, and the absolute value induced by $| \, |_p$ on $\mathbb{Q}$ via this inclusion is the p-adic absolute value;*

2. *the image of $\mathbb{Q}$ under this inclusion is dense in $\mathbb{Q}_p$ with respect to the absolute value $| \, |_p$;*

3. *$\mathbb{Q}_p$ is complete with respect to the absolute value $| \, |_p$.*

**Remark 2.1.1.** The field of real numbers $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the regular absolute value $| \, |$.

Now we know the set of values of $\mathbb{Q}$ and of $\mathbb{Q}_p$ under $| \, |_p$ is the same, more precisely speaking, the two sets

$$\{x \in \mathbb{R}_+ : x = |\lambda|_p \; forsome \; \lambda \in \mathbb{Q}\}$$

and

$$\{x \in \mathbb{R}_+ : x = |\lambda|_p \; forsome \; \lambda \in \mathbb{Q}_p\}$$

are both equal to the set $\{p^n : n \in \mathbb{Z}\} \cup \{0\}$. Or we can restate it as a proposition:

**Proposition 2.1.3.** *For each $x \in \mathbb{Q}_p$, there exists an integer $ord_p(a)$ such that*

$$|x|_p = \begin{cases} \frac{1}{p^{ord_p(x)}} & if \; x \neq 0 \\ 0 & if \; x = 0 \end{cases}$$

We will end this section with a property of $\mathbb{Q}_p$ about its compactness.

**Proposition 2.1.4.** *$\mathbb{Q}_p$ is locally compact, so are $\mathbb{R}$ and $\mathbb{C}$.*

## 2.2 Normed Vector Space over $\mathbb{Q}_p$

In order to consider $GL(n, \mathbb{Q}_p)$ valued cocycles, we need to understand the $\mathbb{Q}_p$-vector space. Clearly, any property of a vector space over a complete field will hold for $\mathbb{Q}_p$-vector space.

**Definition 2.2.1.** Let $V$ be a vector space over $\mathbb{Q}_p$. A norm on $V$ is a function

$$\| \ \| : V \to \mathbb{R}_+$$

satisfying the following conditions:

1. $\|v\| = 0$ if and only if $v = 0$,
2. for any two vectors $v, w \in V$, we have $\|v + w\| \leq \|v\| + \|w\|$,
3. for any $v \in V$ and any $\lambda \in \mathbb{Q}_p$, we have $\|\lambda v\| = |\lambda|_p \|v\|$.

A vector space $V$ which has a norm $\| \ \|$ is called a normed vector space over $\mathbb{Q}_p$.

It is less clear that it is a good idea to introduce the notion of non-archimedian norms. For example, consider the norm on $V = \mathbb{Q}_p \times \mathbb{Q}_p$ given by

$$\|(x, y)\| = \sqrt{|x|_p^2 + |y|_p^2}$$

It is easy to check that this is a norm, but that it does not satisfy

$$\|(x + x^{'}, y + y^{'})\| \leq \max\{\|(x, y)\|, \|(x^{'}, y^{'})\|\}.$$

But this norm is still non-archimedian in the sense that given two vectors it may not be possible to find an integer multiple of one which is bigger than the other. In fact, this suggests that normed vector spaces over non-archimedian complete fields are automatically non-archimedian in any reasonable sense.

When the vector space is finite-dimensional, we have the following property,

**Theorem 5** ([19]Theorem 5.2.1)**.** *Let $V$ be a finite-dimensional vector space over $\mathbb{Q}_p$. Then any two norms on $V$ are equivalent. Moreover, $V$ is complete with respect to the metric induced by any norm.*

There is one property of finite-dimensional $\mathbb{Q}_p$-vector space that is worth pointing out. It is about the compactness.

**Proposition 2.2.1.** *Let $V$ be a finite-dimensional vector space over $\mathbb{Q}_p$. Then $V$ is locally compact.*

We refer [20, 21, 19] for more details of the field of p-adic numbers.

Chapter 3

# Cocycles over Anosov action and Lyapunov Exponents

In this chapter, we study the Anosov diffeomorphisms, cocycles and the Lyapunov exponents for cocycles. Then we introduce the Lyapunov norm.

## 3.1 Anosov diffeomorphisms

Through out this dissertation, we consider cocycles over rank one Anosov actions, so it is necessary to study the Anosov diffeomorphism and its properties.

We consider a $C^1$ diffeomorphism $f : M \to M$ on a compact differentiable manifold $M$. We will denote by $TM$ the tangent bundle of $M$, and by $Df : TM \to TM$ the derivative of $f$.

**Definition 3.1.1.** The diffeomorphism $f$ is said to be an Anosov diffeomorphism if there exists a smooth Riemannian metric $\| \cdot \|$ on $M$, a number $\zeta \in (0, 1)$, and a continuous splitting $TM = E^s \oplus E^u$ of the tangent bundle into $Df$-invariant sub-bundles $E^s$ and $E^u$, such that

$$\|Dfv\| \leq \zeta\|v\|, \ v \in E^s;$$

$$\|Df^{-1}v\| \leq \zeta\|v\|, \ v \in E^u$$

Let $d_M$ be the distance on $M$ induced by the Riemannian metric in definition

3.1.1. We have the following theorem

**Theorem 6** (Stable and Unstable Manifolds Theorem [8] Theorem 1.8.3)**.** *Let $M, f, E^s, E^u, \zeta$ be as in definition 3.1.1. Then for each $x \in M$ there exists a pair of embedded $C^1$-discs $W^s_{loc}(x), W^u_{loc}(x)$, called the local stable manifold and the local unstable manifold at $x$, such that:*

1. $T_x W^s_{loc}(x) = E^s(x), T_x W^u_{loc}(x) = E^u(x)$;
2. $f(W^s_{loc}(x)) \subset W^s_{loc}(fx), f^{-1}(W^u_{loc}(x)) \subset W^u_{loc}(f^{-1}x)$;
3. *for any $\mu \in (\zeta, 1)$, there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$d_M(f^n(x), f^n(y)) \leq C\mu^n d_M(x, y), \text{ for } y \in W^s_{loc}(x)$$

$$d_M(f^{-n}(x), f^{-n}(y)) \leq C\mu^n d_M(x, y), \text{ for } y \in W^u_{loc}(x);$$

4. *there exists a constant $\beta > 0$ such that for all $x \in M$*

$$W^s_{loc}(x) = \{y \in M | f^n(y) \in B(f^n(x), \beta)\}$$

$$W^u_{loc}(x) = \{y \in M | f^{-n}(y) \in B(f^{-n}(x), \beta)\},$$

*for all $n > 0$, where $B(x, \beta)$ is the ball of radius $\beta$ centered at $x$ in $M$.*

*The local stable (unstable) manifold can be extended to global stable (unstable) manifolds $W^s(x)$ and $W^u(x)$*

$$W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(W^s_{loc}(f^n(x))),$$

$$W^u(x) = \bigcup_{n=0}^{\infty} f^{n}(W^u_{loc}(f^{-n}(x))),$$

*which are well defined, smooth injectively immersed and also given by*

$$W^s(x) = \{y \in M | d_M(f^n(x), f^n(y)) \to 0, \text{ as } n \to \infty\},$$

$$W^u(x) = \{y \in M | d_M(f^{-n}(x), f^{-n}(y)) \to 0, \text{ as } n \to \infty\},$$

*These global manifolds are the leaves of global foliations $W^s$ and $W^u$ of $M$.*

**Theorem 7** (Local Product Structure)**.** *Given an Anosov diffeomorphism $f : M \to M$, there are constants $\Delta > 0, K > 0$ such that for any $x, y \in M$ with $d_M(x,y) < \Delta$, the intersection $W^s_{loc}(x) \cap W^u_{loc}(y)$ contains exactly one element, whose distance to both $x$ and $y$ is at most $Kd_M(x,y)$.*

**Definition 3.1.2.** Let $f : M \to M$ be a diffeomorphism and $\epsilon > 0$. A sequence $\{x_0, x_1, \cdots, x_{m-1}\}$ of points in $M$ is called a periodic $\epsilon$-pseudo-orbit for $f$ if

$$d_M(f(x_i), x_{i+1}) \leq \epsilon. \ for \ i = 0, 1, \cdots, m-1,$$

where $x_n = x_0$. A periodic orbit is $\epsilon$-pseudo-orbit for $\epsilon = 0$.

**Theorem 8** (Anosov Closing Lemma)**.** *Let $f : M \to M$ be an Anosov diffeomorphism, there are constants $K > 0, \epsilon_0 > 0$, such that given a periodic $\epsilon$-pseudo-orbit $\{x_0, x_1, \cdots, x_{m-1}\}$ with $\epsilon < \epsilon_0$, there is a periodic orbit of the same length $\{x, f(x), \cdots, f^{m-1}(x)\}, f^m(x) = x$, such that*

$$d_M(x_k, f^k(x)) \leq K\epsilon, 0 \leq k \leq m.$$

By the well-known results about Anosov diffeomorphism, Stable and unstable manifolds theorem, Local product structure and Anosov closing lemma, we can prove the following proposition which is used in the proof of the main results of this dissertation.

**Proposition 3.1.1.** *Let $f : M \to M$ be an Anosov diffeomorphism on compact manifold $M$, let $\Delta, K, C, \zeta$ as above. Fix $\lambda$ such that $e^{-\lambda} \in (\zeta, 1)$. There exists a constant $c$ with the following property. If $x \in M$ and $m \in \mathbb{N}$ are such that $d_M(f^m x, x) < \Delta/K = \delta_0$, there exist $p$ and $y \in M$ with the following properties:*

1. $f^m q = q$;
2. $d_M(f^i x, f^i y) \leq \delta e^{-\lambda i}$ *for* $i = 0, 1, \cdots, m$;
3. $d_M(f^i y, f^i q) \leq \delta e^{-\lambda(m-i)}$ *for* $i = 0, 1, \cdots, m$;
4. $d_M(f^i x, f^i q) \leq \delta e^{-\lambda \min\{i, m-i\}}$ *for* $i = 0, 1, \cdots, m$.

*where* $\delta = cd_M(f^m x, x)$.

In [9], a homeomorphism satisfying the proposition above was called a homeomorphism with closing property and Kalinin gave the following definition

**Definition 3.1.3.** Two orbit segments $x, fx, \cdots, f^m x$ and $y, fy, \cdots, f^m y$ are called exponentially $\delta$ close with exponent $\lambda > 0$ if for every $i = 0, \cdots, m$ we have

$$\mathrm{d}(f^i x, f^i y) \leq \delta \cdot e^{-\lambda \min\{i, m-i\}}.$$

From this definition, we know in the Proposition 3.1.1, the orbit segments $x, fx, \cdots, f^m x$ and $q, fq, \cdots, f^m q$ are exponentially $\delta$ close with exponent $\lambda$.

In this dissertation we will continue using this concept and the closing property.

## 3.2 Cocycles

Cocycles play a crucial role in many questions about rigidity of various smooth actions, existence of invariant structures and other important properties of the action.

### 3.2.1 General Definition

**Definition 3.2.1.** Let $X$ be a topological space. Let $\alpha : X \times G \to X$ be a continuous (or discrete) action of a continuous (or discrete) group $G$ on $X$. If $H$ is a topological group then a cocycle over the action $\alpha$ with values in $H$ is a continuous (or measurable) function $\mathcal{A} : X \times G \to H$ satisfying:

$$\mathcal{A}(x, g_1 g_2) = \mathcal{A}(\alpha(x, g_2), g_1)\mathcal{A}(x, g_2).$$

for any $g_1, g_2 \in G$.

**Definition 3.2.2.** Two cocycles $\mathcal{A}_1$ and $\mathcal{A}_2$ are called continuously (or smoothy, measurably etc.) cohomologous if there exists a continuous (or smooth, measurable) map $P : X \to H$ such that

$$\mathcal{A}_2(x, g) = P(gx)\mathcal{A}_1(x, g)P(x)^{-1}$$

for all $g \in G, x \in X$. The map $P$ is called transfer map.

Given an action $\alpha : X \times G \to X$ and a cocycle $\mathcal{A} : X \times G \to H$, we can construct an extension $\alpha_{\mathcal{A}}$ of $\alpha$ to the trivial $H$-bundle $X \times H$ over $X$ defined by

$$\alpha_{\mathcal{A}}((x, h), g) = (\alpha(x, g), \mathcal{A}(x, g)h).$$

The extension $\alpha_{\mathcal{A}}$ is an action of $G$ on $X \times H$.

The induced actions $\alpha_{\mathcal{A}_1}$ and $\alpha_{\mathcal{A}_2}$ of two continuous (or smooth, measurable) cohomologous cocycles $\alpha_{\mathcal{A}_1}$ and $\alpha_{\mathcal{A}_2}$ are topologically equivalent in the sense that there exists a continuous (or smooth, measurable) conjugacy map $h$ on $X \times H$ such that

$$\alpha_{\mathcal{A}_2}(g) = h \circ \alpha_{\mathcal{A}_1} \circ h^{-1}$$

where $h(x, h) = (x, P(x)h)$.

**Definition 3.2.3.** A cocycle is cohomologous to a constant cocycle (cocycle not depending on x) if there exists a homomorphism $\pi : G \to H$ and a transfer map P such that

$$\mathcal{A}(x, g) = P(gx)\pi(g)P(x)^{-1} \tag{3.2.1}$$

In particular, a cocycle is a coboundary if it is cohomologous to a trivial cocycle $\pi(g) = id_H$ i.e.

$$\mathcal{A}(x, g) = P(gx)P(x)^{-1} \tag{3.2.2}$$

the equation (3.2.2) is called Livshits's cohomology equation.

**Definition 3.2.4.** An action $\alpha$ is $\mathcal{C}_H^{a,b}$-cocycle rigid if any $C^a$ cocycle over $\alpha$ with values in $H$ is cohomologous to a constant cocycle via a $C^b$ transfer map.

**Definition 3.2.5.** An action $\alpha$ is $\mathcal{C}_H^{a,b}$-cocycle stable if any class of cohomologous $C^a$ cocycles over $\alpha$ with values in $H$ is closed and the transfer map for two cohomologous cocycles is of class $C^b$ where $a, b > 0$ or $a, b \in \{\infty, \omega\}$.

For more detailed discussion of cocycles, we refer the book [22, 23].

### 3.2.2 Livšic Theorem

In the problem that we try to solve, we consider cocycles over rank one Anosov actions. We need to show that the necessary condition that the cocycle has trivial periodic data is also the sufficient condition of coboundary. A. Livšic first obtained the result for real valued cocycle.

**Theorem 9** (Livšic). *Let $M$ be a compact Riemannian manifold, $f$ a topologically transitive $C^1$ Anosov diffeomorphism of $M$. Then for any $\alpha$-Hölder continuous real valued cocycle $\mathcal{A}$ satisfying*

$$\Sigma_{k=0}^{m-1}\mathcal{A}(f^k(q)) = 0 \textit{ whenever } f^m(q) = q,$$

*there exists an $\alpha$-Hölder continuous function $P : M \to \mathbb{R}$ such that*

$$\mathcal{A}(x) = P(f(x)) - P(x)$$

*holds.*

B. Kalinin obtained the positive result for $GL(n, \mathbb{R})$-valued cocycles, which can be apply to any connected Lie group $G$ through adjoint representation.

**Theorem 10** (B.Kalinin). *Let $f$ be a topologically transitive $C^1$ Anosov diffeomorphism of a compact matric space $X$. Let $A : X \to GL(n, \mathbb{R})$ be an $\alpha$-Hölder continuous function such that $\forall p \in X, n \in \mathbb{N}$ with $f^m(q) = q$*

$$A(f^{m-1}(q)) \cdots A(f(q))A(q) = Id$$

*Then there exists an $\alpha$-Hölder continuous function $P : X \to GL(n, \mathbb{R})$ such that for all $x \in X$*

$$A(x) = P(f(x))P(x)^{-1}.$$

### 3.2.3 $GL(n, \mathbb{Q}_p)$-valued Cocycle

Now we study the $GL(n, \mathbb{Q}_p)$-valued cocycles.

**Definition 3.2.6.** Let $f : X \to X$ is an invertible measurable transformation of a measure space X which preserves a finite measure. A function $A : X \times \mathbb{Z} \to$

$GL(n,\mathbb{Q}_p)$ is called a cocycle with value in $GL(n,\mathbb{Q}_p)$ over $f$ or simply a cocycle if for each $x \in X$, we have $A(x,0) = Id$ and given $m,k \in \mathbb{Z}$,

$$\mathcal{A}(x,m+k) = \mathcal{A}(f^k x,m)\mathcal{A}(x,k). \tag{3.2.3}$$

Given a measurable function $A : X \to GL(n,\mathbb{Q}_p)$,we define a cocycle by

$$\mathcal{A}(x,m) = \begin{cases} A(f^{m-1}(x)) \cdots A(f(x))A(x) & \text{if } m > 0 \\ Id & \text{if } m = 0 \\ A(f^{-m}(x))^{-1} \cdots A(f^{-2}(x))^{-1}A(f^{-1}(x))^{-1} & \text{if } m < 0 \end{cases}$$

We call the map $A$ the generator of the cocycle $\mathcal{A}$ and we say that the cocycle $\mathcal{A}$ is generated by the function $A$. Note that $A(x) = \mathcal{A}(x,1)$ and $\mathcal{A}(x,-m) = \mathcal{A}(f^{-m}(x),m)^{-1}$.

Now, we introduce a metric on $GL(n,\mathbb{Q}_p)$ as follow

$$\mathrm{d}_{GL(n,\mathbb{Q}_p)}(A,B) = \|A - B\|, \tag{3.2.4}$$

where

$$\|A\| = \sup\{|a_{ij}|_p : A = (a_{ij}) \in GL(n,\mathbb{Q}_p)\}.$$

which coincides with the operator norm

$$\|A\| = \sup\{\|Au\| \cdot \|u\|^{-1} : 0 \neq u \in \mathbb{Q}_p^n\},$$

where

$$\|u\| = \sup\{|u_i|_p : u = (u_i) \in \mathbb{Q}_p^n\}.$$

It is easy to check that $\|\cdot\|$ is a non-archimedean norm i.e. $\|A+B\| \leq \max\{\|A\|,\|B\|\}$, for $A,B \in GL(n,\mathbb{Q}_p)$.

A cocycle $\mathcal{A}$ over a homeomorphism $f$ of a compact metric space $X$ is called $\alpha$-hölder if its generator $A : X \to GL(n,\mathbb{Q}_p)$ is hölder continuous with exponent $\alpha$,i.e. for all $x,y \in X$

$$\|A(x) - A(y)\| \leq c\mathrm{d}(x,y)^{\alpha}.$$

## 3.3 Lyapunov exponents for cocycles

In this section we study the Lyapunov exponents for both $GL(n, \mathbb{R})$ valued Cocycle and $GL(n, \mathbb{Q}_p)$ valued Cocycle.

### 3.3.1 Lyapunov exponents for $GL(n, \mathbb{R})$ valued Cocycle

**Theorem 11** (Multiplicative ergodic theorem [24], [25]). *Let $f$ be an invertible ergodic measure-preserving transformation of a Lebesgue probability measure space $(X, \mu)$.Suppose that the $GL(n, \mathbb{R})$ valued cocycle $\mathcal{A}$ is integrable, i.e.* $\log \|A(x)\| \in L^1(X, \mu)$ *and* $\log \|A(x)^{-1}\| \in L^1(X, \mu)$.*Then there exists $\mathcal{R} \subset X$ of full measure such that $f(\mathcal{R}) = \mathcal{R}$ and for every $x \in \mathcal{R}$, there exist numbers $\chi_1 < \cdots \chi_r$, and an A-invariant Lyapunov decomposition of $\mathbb{R}^n$ such that*

1.
$$\mathbb{R}^n = \bigoplus_{1 \leq i \leq r} E_i(x);$$
*and $A(x)E_i(x) = E_i(f(x))$.*

2. *for each $i, 1 \leq i \leq r$,*
$$\lim_{m \to \pm\infty} \frac{1}{m} \log \frac{\|\mathcal{A}(x, m)u\|}{\|u\|} = \chi_i,$$
*uniformly in $0 \neq u \in E_i(x)$.*

3. *Let dim $E_i(x) = k_i$,then*
$$\sum_{i=1}^{r} k_i \chi_i = \lim_{m \to \pm\infty} \frac{1}{m} \log det \mathcal{A}(x, m)$$

**Remark 3.3.1.** The point $x \in X$ satisfying the property 3 above is said to be a forward and backward regular point for $\mathcal{A}$, and the set $\mathcal{R}$ is said to be the set of regular points for $\mathcal{A}$. For every regular point $x \in X, 1 \leq i, j \leq r$ with $i \neq j$, and every distinct vectors $v, w \in E_i$, we have

$$\lim_{m \to \pm\infty} \frac{1}{m} \log \sin \sphericalangle(E_i(f^m(x)), E_j(f^m(x))) = 0;$$

that means, the angles between any two spaces $E_i(x)$ and $E_j(x)$ can grow at most subexponentially along the orbit of x, and

$$\lim_{m\to\pm\infty}\frac{1}{m}\log\sin\sphericalangle(\mathcal{A}(x,m)v,\mathcal{A}(x,m)w)=0.$$

**Definition 3.3.1.** The numbers $\chi_1,\cdots\chi_r$ are called the Lyapunov exponents of cocycle $\mathcal{A}$, and the number $k_i$ is the multiplicity of the Lyapunov exponent $\chi_i$ and also the dimension of the space $E_i(x)$.

### 3.3.2 Lyapunov exponents for $GL(n,\mathbb{Q}_p)$ valued Cocycle

**Theorem 12** (Multiplicative ergodic theorem [16] 5.2.1)**.** *Let $X$ be a locally compact $\sigma$-compact space with finite regular Borel measure $\mu$,let $f$ be an invertible ergodic measure-preserving transformation of $(X,\mu)$.Suppose that the $GL(n,\mathbb{Q}_p)$ valued cocycle $\mathcal{A}$ is integrable, i.e. $\log\|A(x)\|\in L^1(X,\mu)$ and $\log\|A(x)^{-1}\|\in L^1(X,\mu)$.Then*

1. *There exist numbers $r\in\mathbb{N}^+,k_i\in\mathbb{N}^+,\chi_i\in\mathbb{R}$, and measurable maps $E_i: X\to GR_{k_i}(\mathbb{Q}_p^n), 1\le r\le n, 1\le i\le r$, with the following preperties:*
   (a) *for almost all $x\in X$,*
   $$\mathbb{Q}_p^n=\bigoplus_{1\le i\le r}E_i(x);$$
   *and $A(x)E_i(x)=E_i(f(x))$.*
   (b) *for almost all $x\in X$ and for each $i.1\le i\le r$,*
   $$\lim_{m\to\pm\infty}\frac{1}{m}\log\frac{\|\mathcal{A}(x,m)u\|}{\|u\|}=\chi_i, \tag{3.3.1}$$
   *uniformly in $0\ne u\in E_i(x)$.*
   (c) $\sum_{i=1}^{r}k_i\chi_i=\frac{1}{\mu(X)}\int_X\log|\mathcal{A}(x,m)|d\mu(x).$
   (d) *$\chi_i<\chi_j$,if $1\le i<j\le r$.*
2. *For almost all $x\in X$ and for each $0\ne u\in\mathbb{Q}_p^n$, the following limits exist:$\chi^+(x,u)=\lim_{m\to+\infty}\log\|\mathcal{A}(x,m)u\|/m$ and $\chi^-(x,u)=\lim_{m\to-\infty}\log\|\mathcal{A}(x,m)u\|/m$.* $\blacksquare$

*Furthermore, for almost all $x \in X$ and for each $a \in \mathbb{R}$*

$$\{0\} \cup \{0 \neq u \in \mathbb{Q}_p^n \mid \chi^+(x,u) \leq a\} = \bigoplus_{\chi_i \leq a} E_i(x)$$

*and*

$$\{0\} \cup \{0 \neq u \in \mathbb{Q}_p^n \mid \chi^-(x,u) \geq a\} = \bigoplus_{\chi_i \geq a} E_i(x)$$

**Remark 3.3.2.** This theorem holds not only for $GL(n, \mathbb{Q}_p)$ valued cocycle, actually it holds for all $GL(\mathcal{W})$ valued cocycles, where $\mathcal{W}$ denotes a vector space over $K$ of dimension $n$ and $K$ denotes a local field. In case $K = \mathbb{R}$, the multiplicative ergodic theorem is due to Oseledec. In [26], A.Katok and R. Spatzier also discussed the Lyapunov exponents in the non-archimidean directions.

**Remark 3.3.3.** In the $\mathbb{Q}_p$ case, we do not have the definition for angle between the spaces $E_i(x)$ and $E_j(x)$ since the lack of inner product within $\mathbb{Q}_p$ vectors, but we still have the similar control of the grow rate of the distance between those spaces, say, the distance between the spaces $E_i(x)$ and $E_j(x)$ can grow at most subexponentially along the orbit of regular point $x \in X$.

**Definition 3.3.2.** The numbers $\chi_1, \cdots \chi_r$ are called the Lyapunov exponents of cocycle $\mathcal{A}$, and the number $k_i$ is the multiplicity of the Lyapunov exponent $\chi_i$ and also the dimension of the space $E_i(x)$. The set of points that satisfy the properties above is called regular and $f$-invariant and denoted with $\mathcal{R}^\mu$.

## 3.4 Lyapunov norm

In this section, we will introduce Lyapunov norm both for $GL(n, \mathbb{R})$ valued cocycles and $GL(n, \mathbb{Q}_p)$ valued cocycles.

### 3.4.1 Lyapunov norm in $\mathbb{R}$ Case

Let $\mathcal{A}$ be a $GL(n, \mathbb{R})$ valued cocycle over a measurable transformation $f : X \to X$, for $x \in X$, consider a family of inner products $< \cdot, \cdot > = < \cdot, \cdot >_x$.

**Proposition 3.4.1.** *For each $\epsilon > 0$, each regular point $x \in X$ and $i = 1, \cdots, r$, the formula*

$$< u, v >'_{x,i} = \sum_{m \in \mathbb{Z}} < \mathcal{A}(x,m)u, \mathcal{A}(x,m)v > e^{-2\chi_i(x)m - 2\epsilon|m|}$$

*determines a scalar product on $E_i(x)$.*

For a fixed $\epsilon > 0$, and a regular point $x \in X$, we introduce a new inner product on $\mathbb{R}^n$ by

$$< u, v >'_x = \sum_{i=1}^{r} < u_i, v_i >'_{x,i}$$

where $u_i$ and $v_i$ are the projections of the vectors $u$ and $v$ over $E_i(x)$ along $\bigoplus_{j \neq i} E_j(x)$.

**Definition 3.4.1.** We call $< \cdot, \cdot >'_x$ the Lyapunov inner product at $x$ and the corresponding norm $\| \cdot \|'_x$ the Lyapunov norm at $x$.

**Proposition 3.4.2.** *Let $< \cdot, \cdot >'_x$ be a Lyapunov inner product at $x$. Then*

1. *it depends Borel measurably on $x$ on the set of regular points in $X$;*

2. *for every regular point $x \in X$ and $i \neq j$, the spaces $E_i(x)$ and $E_j(x)$ are orthogonal with respect to the Lyapunov inner product.*

**Definition 3.4.2.** A coordinate change $C_\epsilon : X \to GL(n, \mathbb{R})$ is called a Lyapunov change of coordinates if for each regular point $x \in X$ and $u, v \in \mathbb{R}^n$, it satisfies

$$< u, v >_x = < C_\epsilon(x)u, C_\epsilon(x)v >'_x$$

**Theorem 13** (Oseledets-Pesin Reduction Theorem [24] Theorem 3.5.5)**.** *Let $f : X \to X$ be an invertible measure preserving transformation of the Lebesgue space $(X, \mu)$ and $\mathcal{A}$ a measurable cocycle over $f$. Given $\epsilon > 0$, if $x$ is a regular point for $\mathcal{A}$ then*

1. *any Lyapunov change of coordinates $C_\epsilon$ sends the orthogonal decomposition $\bigoplus_{i=1}^{r} \mathbb{R}^{k_i(x)}$ to the decomposition $\bigoplus_{i=1}^{r} E_i(x)$ of $\mathbb{R}^n$.*

2. *the cocycle* $A_\epsilon(x) = C_\epsilon(f(x))^{-1}A(x)C_\epsilon(x)$ *has the block form*

$$A_\epsilon(x) = \begin{pmatrix} A_\epsilon^1(x) & & \\ & \ddots & \\ & & A_\epsilon^r(x) \end{pmatrix}$$

*where each block* $A_\epsilon^i(x)$ *is a* $k_i(x) \times k_i(x)$ *matrix, and the entries are zero above and below the matrices* $A_\epsilon^i(x)$*;*

3. *each block* $A_\epsilon^i(x)$ *satisfies*

$$e^{\chi_i(x)-\epsilon} \leq \|A_\epsilon^i(x)^{-1}\|^{-1} \leq \|A_\epsilon^i(x)\| \leq e^{\chi_i(x)+\epsilon};$$

4. *if* $\log\|A(x)\| \in L^1(X,\mu)$ *and* $\log\|A(x)^{-1}\| \in L^1(X,\mu)$ *holds then the map* $C_\epsilon$ *is tempered* $\mu$*-almost everywhere, i.e.,*

$$\lim_{m\to\pm\infty} \frac{1}{m}\log\|C_\epsilon(f^m(x))\| = \lim_{m\to\pm\infty} \frac{1}{m}\log\|C_\epsilon(f^m(x))^{-1}\| = 0$$

*and the spectra of* $\mathcal{A}$ *and* $\mathcal{A}_\epsilon$ *coincide* $\mu$*-almost everywhere.*

### 3.4.2 Lyapunov norm in $\mathbb{Q}_p$ Case

In $\mathbb{Q}_p$ case, we do not have inner product structure, but we can still define Lyapunov norm in the norm sense.

For a fixed $\epsilon > 0$ and a regular point $x$ we introduce the $\epsilon$-Lyapunov norm as follows. For $v \in E_i(x)$ and $i = 1, \cdots, r$, we define

$$\|v\|_{x,i}' = \sum_{m\in\mathbb{Z}} \|\mathcal{A}(x,m)v\| \exp\left(-\chi_i(x) - \epsilon|m|\right) \tag{3.4.1}$$

Observe that the series in (3.4.1) converges. In fact by (3.3.1), for each $\epsilon > 0$, any regular point $x$, and any $0 \neq v \in E_i(x)$, there exists an integer $m_0 = m_0(x,v,\epsilon) > 0$ such that for $|m| > m_0$,

$$\|\mathcal{A}(x,m)v\| \leq \exp\left(\chi_i(x)m + \epsilon|m|/2\right)\|v\|$$

Therefore

$$\begin{aligned}\|v\|'_{x,i} &= \sum_{m\in\mathbb{Z}} \|\mathcal{A}(x,m)v\| \exp\left(-\chi_i(x) - \epsilon|m|\right) \\ &\le \sum_{|m|\le m_0} \|\mathcal{A}(x,m)v\| \exp\left(-\chi_i(x) - \epsilon|m|\right) \\ &+ \sum_{|m|>m_0} \exp\left(-\epsilon|m|/2\right)\|v\| < +\infty\end{aligned}$$

For $v \in \mathbb{Q}_p^n$, define

$$\|v\|'_x = \sup_{1\le i\le r} \|v_i\|'_{x,i},$$

where $v_i$ is the projection of the vectors $v$ over $E_i(x)$ along $\bigoplus_{j\neq i} E_j(x)$.

For fixed $\epsilon > 0$ and regular point $x$, we call the norm $\| \ \|_x$ the Lyapunov norm. We now summarize some important propositions of the Lyapunov norm.

**Proposition 3.4.3.** *The Lyapunov norm satisfies the following properties:*

1. *For $v \in E_i(x)$, then $\mathcal{A}(x,m)v \in E_i(f^m x)$ and*

$$e^{(m\chi_i - \epsilon m)}\|v\|'_{x,i} \le \|\mathcal{A}(x,m)v\|'_{f^m x,i} \le e^{(m\chi_i + \epsilon|m|)}\|v\|'_{x,i} \quad \forall m \in \mathbb{Z}. \qquad (3.4.2)$$

2. *We have*

$$e^{m\chi - \epsilon m} \le \|\mathcal{A}(x,m)\|_{f^m(x)\leftarrow x} \le e^{m\chi + \epsilon m} \quad \forall m \in \mathbb{N}, \qquad (3.4.3)$$

*where $\chi$ is the maximal Lyapunov exponent,or say $\chi_r$ and $\|\cdot\|_{f^m\leftarrow x}$ is defined as follows, for any matrix $A$ and any regular points $x, y$*

$$\|A\|_{y\leftarrow x} = \sup\{\frac{\|Av\|'_y}{\|v\|'_x} : 0 \neq v \in \mathbb{Q}_p^n\}.$$

3. *There exists a positive measurable function $K_\epsilon : \mathcal{R}^\mu \to \mathbb{R}$ such that*

   *(a)*

$$K_\epsilon(x)e^{-\epsilon|m|} \le K_\epsilon(f^m(x)) \le K_\epsilon(x)e^{\epsilon|m|} \quad \forall m \in \mathbb{Z}; \qquad (3.4.4)$$

*(b)*

$$\|v\| \le \|v\|_x^{'} \le K_\epsilon(x)\|v\| \quad \forall v \in \mathbb{Q}_p^n. \tag{3.4.5}$$

*Further more, for any matrix $A$ and regular points $x$ and $y$,we have*

$$K_\epsilon(x)^{-1}\|A\| \le \|A\|_{y\leftarrow x} \le K_\epsilon(y)\|A\| \tag{3.4.6}$$

*Proof.* Since $\mathcal{A}(x, m+1) = \mathcal{A}(f(x), m)A(x), \forall v \in E_i(x)$, we have

$$\begin{aligned}
\|A(x)v\|_{fx,i}^{'} &= \sum_{m\in\mathbb{Z}} \|\mathcal{A}(f(x), m)A(x)v\| e^{-\chi_i(x)m-\epsilon|m|} \\
&= \sum_{m\in\mathbb{Z}} \|\mathcal{A}(x, m+1)v\| e^{-\chi_i(x)m-\epsilon|m|} \\
&= \sum_{k\in\mathbb{Z}} \|\mathcal{A}(x, k)v\| e^{-\chi_i(x)k-\epsilon|k|} e^{\chi_i+\epsilon(|k|-|k-1|)} \\
&= e^{\chi_i+\epsilon(|k|-|k-1|)}\|v\|_{x,i}^{'},
\end{aligned}$$

Therefore,

$$e^{\chi_i(x)-\epsilon}\|v\|_{x,i}^{'} \le \|A(x)v\|_{fx,i}^{'} \le e^{\chi_i(x)+\epsilon}\|v\|_{x,i}^{'},$$

which implies inequality (3.4.2).

By the inequality (3.4.2), we have

$$\begin{aligned}
\|\mathcal{A}(x, m)v\|_{f^m x}^{'} &= \sup_{1\le i\le r} \|\mathcal{A}(x, m)v\|_{f^m x,i}^{'} \\
&\le \sup_{1\le i\le r} e^{(m\chi_i+\epsilon m)}\|v\|_{x,i}^{'} \\
&\le e^{m\chi+\epsilon m} \sup_{1\le i\le r} \|v\|_{x,i}^{'} \\
&= e^{m\chi+\epsilon m}\|v\|_x^{'}
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{A}(x, m)v\|_{f^m x}^{'} &= \sup_{1\le i\le r} \|\mathcal{A}(x, m)v\|_{f^m x,i}^{'} \\
&\ge \sup_{1\le i\le r} e^{(m\chi_i-\epsilon m)}\|v\|_{x,i}^{'}
\end{aligned}$$

$$\geq e^{m\chi - \epsilon m} \|v\|'_{x,r}$$

Therefore, the operator norm

$$\|\mathcal{A}(x,m)\|_{f^m(x)\leftarrow x} = \sup\{\frac{\|\mathcal{A}(x,m)\|'_{f^m(x)}}{\|v\|'_x} : 0 \neq v \in \mathbb{Q}_p^n\}$$

satisfies the inequality (3.4.3).

To prove (3), first for $v \in \mathbb{Q}_p^n, v = v_1 + \cdots + v_r$,we have $\|v\| \leq \max \|v_i\| \leq \sup \|v_i\|'_{x,i} = \|v\|'_x$, this implies the first part of the inequality (3.4.5).

To prove the other part of the inequality (3.4.5), we need to introduce a coordinate change $C_\epsilon : X \to GL(n, \mathbb{Q}_p)$ such that for any regular point $x$ and $v_i \in E_i$ we have $\|v_i\| = \|C_\epsilon(x)v_i\|'_{x,i}$ and let $D_i = C_\epsilon(x)^{-1}E_i$, then $C_\epsilon(x)$ send the decomposition $\bigoplus_{i=1}^{r} D_i(x)$ to the decomposition $\bigoplus_{i=1}^{r} E_i(x)$. Further more, we have $\|v\| = \|C_\epsilon(x)v\|'_x, \forall v \in \mathbb{Q}_p^n$.

Followed by the spirit of Oseledets-Pesin Reduction Theorem, we know the induced cocycle $A_\epsilon(x) = C_\epsilon(f(x))^{-1}A(x)C_\epsilon(x)$ will have the same Lyapunov exponents on $D_i(x)$ as $A(x)$ on $E_i(x)$ and therefore $C_\epsilon$ is tempered almost everywhere, i.e.

$$\lim_{m\to\pm\infty} \frac{1}{m} \log \|C_\epsilon(f^m(x))\| = 0$$

$$\lim_{m\to\pm\infty} \frac{1}{m} \log \|C_\epsilon(f^m(x))^{-1}\|) = 0.$$

With the following lemma we can prove (3),

**Lemma 3.4.1** (Tempering Kernel Lemma [24] Lemma 3.5.7)**.** *Let $f : X \to X$ be a measurable transformation. If $K : X \to \mathbb{R}$ is a positive measurable function tempered on some subset $Z \subset X$, then for any $\epsilon > 0$, there exists a positive measurable function $K_\epsilon : Z \to \mathbb{R}$ such that $K(x) \leq K_\epsilon(x)$ and for $x \in Z$*

$$e^{-\epsilon} \leq \frac{K_\epsilon(f(x))}{K_\epsilon(x)} \leq e^{\epsilon}.$$

Since we have $\|v\|'_x \leq \|C_\epsilon(x)^{-1}\| \cdot \|v\|$, apply the Tempering Kernel Lemma to the positive function $K(x) = \|C_\epsilon(x)^{-1}\|$, then we can find a function $K_\epsilon$ satisfying the desired inequalities.

By inequalities (3.4.5),we have

$$\|Av\| \leq \|Av\|_y^{'} \leq K_\epsilon(y)\|v\| \quad \forall v \in \mathbb{Q}_p^n,$$

$$\|v\| \leq \|v\|_x^{'} \leq K_\epsilon(x)\|v\| \quad \forall v \in \mathbb{Q}_p^n,$$

which implies the inequality (3.4.6). □

For any $N > 0$ we can define the following sets of regular points:

$$\mathcal{R}_{\epsilon,N}^{\mu} = \{x \in \mathcal{R}^{\mu}, K_\epsilon(x) \leq N\} \tag{3.4.7}$$

so $\mu(\mathcal{R}_{\epsilon,N}^{\mu}) \to 1$ as $N \to \infty$. We can assume that the set $\mathcal{R}_{\epsilon,N}^{\mu}$ is compact and that Lyapunov splitting and Lyapunov norm are continuous on $\mathcal{R}_{\epsilon,N}^{\mu}$.

Chapter 4

# Approximation of Lyapunov Exponents

## 4.1 Introduction

In this chapter, we want to give the approximation of Lyapunov exponents by the Lyapunov exponents at periodic points. We first give the estimate of the operator norm of $\mathcal{A}$ with respect to the standard norm and the Lyapunov norm along a orbit which is close to a regular orbit. Then we give the approximation of the largest Lyapunov exponent of cocycle $\mathcal{A}$ by the largest Lyapunov exponent at a periodic point, in which we use a cone argument. At the end, we apply this approximation to the natural extension of cocycle $\mathcal{A}$ to complete the proof.

## 4.2 Estimate of Operator Norm

We start to prove theorem 2 with a lemma that gives the estimate of the operator norm of $\mathcal{A}$ with respect to the standard norm and Lyapunov norm along a orbit segment close to a regular one,$x, fx, \cdots, f^m x$, where $x, f^m x \in \mathcal{R}^{\mu}_{\epsilon,N}$.

**Lemma 4.2.1.** *Let $\mathcal{A}$ be an $\alpha$-Hölder $GL(n, \mathbb{Q}_p)$ cocycle over a homeomorphism $f$ of a compact metric space $X$ and let $\mu$ be an ergodic measure for $f$ with the largest Lyapunov exponent $\chi$. Then for any positive $\lambda$ and $\epsilon$ satisfying $\epsilon - \alpha\lambda < 0$, there exists $c > 0$ such that for any $m \in \mathbb{N}$, any regular point $x$ with both $x$ and*

*$f^m x$ in $\mathcal{R}^{\mu}_{\epsilon,N}$, and any point $y \in X$ such that the orbit segments $x, fx, \cdots, f^m x$ and $y, fy, \cdots, f^m y$ are exponentially $\delta$ close with exponent $\lambda$ we have*

$$\|\mathcal{A}(y,m)\|_{f^m(x)\leftarrow x} \le e^{Nc\delta^\alpha} e^{m(\chi+\epsilon)} \le e^{2m\epsilon+Nc\delta^\alpha}\|\mathcal{A}(x,m)\|_{f^m(x)\leftarrow x}, \tag{4.2.1}$$

$$\|\mathcal{A}(y,m)\| \le N e^{Nc\delta^\alpha} e^{m(\chi+\epsilon)} \le N^2 e^{2m\epsilon+Nc\delta^\alpha}\|\mathcal{A}(x,m)\|. \tag{4.2.2}$$

*The constant $c$ depends only on the cocycle $\mathcal{A}$ and the number $\epsilon - \alpha\lambda$.*

*Proof.*

$$\begin{aligned}
&\|\mathcal{A}(y,m)\|_{f^m x\leftarrow x} = \|A(f^{m-1}y)\cdots A(fy)A(y)\|_{f^m x\leftarrow x}\\
&= \|A(f^{m-1}x)[A(f^{m-1}x)^{-1}A(f^{m-1}y)]\cdots A(x)[A(x)^{-1}A(y)]\|_{f^m x\leftarrow x}\\
&= \sup \frac{\|A(f^{m-1}x)[A(f^{m-1}x)^{-1}A(f^{m-1}y)]\cdots A(x)[A(x)^{-1}A(y)]v\|_{f^m x}}{\|v\|_x}\\
&= \sup \prod_{1\le i\le m} \frac{\|A(f^{i-1}x)[A(f^{i-1}x)^{-1}A(f^{i-1}y)]\cdots A(x)[A(x)^{-1}A(y)]v\|_{f^i x}}{\|[A(f^{i-1}x)^{-1}A(f^{i-1}y)]\cdots A(x)[A(x)^{-1}A(y)]v\|_{f^{i-1}x}}\times\\
&\frac{\|[A(f^{i-1}x)^{-1}A(f^{i-1}y)]A(f^{i-2}x)\cdots A(x)[A(x)^{-1}A(y)]v\|_{f^{i-1}x}}{\|A(f^{i-2}x)[A(f^{i-2}x)^{-1}A(f^{i-2}y)]\cdots A(x)[A(x)^{-1}A(y)]v\|_{f^{i-1}x}}\\
&\le \|A(f^{m-1}x)\|_{f^m x\leftarrow f^{m-1}x}\|A(f^{m-1}x)^{-1}A(f^{m-1}y)\|_{f^{m-1}x\leftarrow f^{m-1}x}\\
&\cdots\|A(x)\|_{fx\leftarrow x}\|A(x)^{-1}A(y)\|_{x\leftarrow x}\\
&\le e^{m(\chi+\epsilon)}\prod_{i=0}^{m-1}\|A(f^i x)^{-1}A(f^i y)\|_{f^i x\leftarrow f^i x} \qquad (by \quad (3.4.3))
\end{aligned} \tag{4.2.3}$$

To estimate the product above, we consider $D_i = A(f^i x)^{-1}A(f^i y)$-Id. The norm of $D_i$ satisfies

$$\|D_i\| \le \|A(f^i x)^{-1}\|\|A(f^i y) - A(f^i x)\|$$

Since $A(x)$ is $\alpha$-hölder continuous on the compact space $X$, so $\|A(x)^{-1}\|$ is uniformly bounded and

$$\|A(x) - A(y)\| \le c_1 \cdot \mathrm{d}(x,y)^\alpha,$$

therefore

$$\|D_i\| \leq c_1 \cdot \mathrm{d}(f^i x, f^i y)^\alpha \leq c_1 \cdot (\delta e^{-\lambda \min\{i, m-i\}})^\alpha. \tag{4.2.4}$$

Since $x, f^m x \in \mathcal{R}^\mu_{\epsilon,N}$, by (3.4.4) and (3.4.7),we have

$$K_\epsilon(f^i x) \leq N e^{\epsilon \min\{i, m-i\}}.$$

By (3.4.6), we have

$$\begin{aligned}\|D_i\|_{f^i x \leftarrow f^i x} &\leq K_\epsilon(f^i x)\|D_i\| \\ &\leq N e^{\epsilon \min\{i,m-i\}} c_1 (\delta e^{-\lambda \min\{i,m-i\}})^\alpha \\ &= N c_1 \delta^\alpha e^{(\epsilon - \alpha\lambda)\min\{i,m-i\}}.\end{aligned}$$

Thus

$$\begin{aligned}\|A(f^i x)^{-1} A(f^i y)\|_{f^i x \leftarrow f^i x} &= \|D_i + \mathrm{Id}\|_{f^i x \leftarrow f^i x} \\ &\leq \|D_i\|_{f^i x \leftarrow f^i x} + 1 \\ &\leq N c_1 \delta^\alpha e^{(\epsilon - \alpha\lambda)\min\{i,m-i\}} + 1 \qquad (4.2.5)\end{aligned}$$

By (4.2.3) and (4.2.5), we can conclude

$$\begin{aligned}\log \|\mathcal{A}(y, m)\|_{f^m x \leftarrow x} &\leq m(\chi + \epsilon) + \sum_{i=0}^{m-1} \log \|A(f^i x)^{-1} A(f^i y)\|_{f^i x \leftarrow f^i x} \\ &\leq m(\chi + \epsilon) + \sum_{i=0}^{m-1} \log(N c_1 \delta^\alpha e^{(\epsilon - \alpha\lambda)\min\{i,m-i\}} + 1)\end{aligned}$$

since $\log(1 + x) \leq x$ for $x \geq 0$ and the series $\sum_{i=0}^{m-1} e^{(\epsilon - \alpha\lambda)\min\{i,m-i\}}$ is uniformly bounded if $\epsilon - \alpha\lambda < 0$, the inequality above turns into

$$\begin{aligned}\log \|\mathcal{A}(y, m)\|_{f^m x \leftarrow x} &\leq m(\chi + \epsilon) + N c_1 \delta^\alpha \sum_{i=0}^{m-1} e^{(\epsilon - \alpha\lambda)\min\{i,m-i\}} \\ &\leq m(\chi + \epsilon) + N c \delta^\alpha.\end{aligned}$$

The constant $c$ depends only on cocycle $\mathcal{A}$ and $\epsilon - \alpha\lambda$. Using (3.4.3), we conclude that

$$\|\mathcal{A}(y,m)\|_{f^m(x)\leftarrow x} \le e^{Nc\delta^\alpha} e^{m(\chi+\epsilon)} \le e^{2m\epsilon+Nc\delta^\alpha} \|\mathcal{A}(x,m)\|_{f^m(x)\leftarrow x},$$

By (3.4.6) we can conclude that

$$\begin{aligned}
\|\mathcal{A}(y,m)\| &\le K_\epsilon(x)\|\mathcal{A}(y,m)\|_{f^m x\leftarrow x} \\
&\le Ne^{Nc\delta^\alpha} e^{m(\chi+\epsilon)} \\
&\le Ne^{2m\epsilon+Nc\delta^\alpha} \|\mathcal{A}(x,m)\|_{f^m(x)\leftarrow x} \\
&\le Ne^{2m\epsilon+Nc\delta^\alpha} K_\epsilon(f^m x)\|\mathcal{A}(x,m)\| \\
&\le N^2 e^{2m\epsilon+Nc\delta^\alpha} \|\mathcal{A}(x,m)\|.
\end{aligned}$$

since $x, f^m x \in \mathcal{R}^\mu_{\epsilon,N}$.

This completes the proof of Lemma 4.2.1. □

## 4.3 Estimate of the Largest Lyapunov Exponent

Now we give the approximation of the largest Lyapunov exponent of cocycle $\mathcal{A}$ by the following proposition by which we can prove the theorem 2.

**Proposition 4.3.1.** *Let $\mathcal{A}$ be an $\alpha$-Hölder $GL(n,\mathbb{Q}_p)$ cocycle over a homeomorphism $f$ of a compact metric space $X$, where $f$ satisfies the closing property with exponent $\lambda$. Let $\mu$ be an ergodic invariant measure for $f$ with the largest Lyapunov exponent $\chi$ and second largest Lyapunov exponent $\xi$. For any positive $N$ and $\epsilon < \epsilon_0 = \min\{\alpha\lambda, (\chi-\xi)/2\}$ and sufficiently large $m$, there exists $\delta > 0$ such that if periodic orbit $q, fq, \cdots, f^m q$ is exponentially $\delta$ close to an orbit segment of $x, fx, \cdots, f^m x$ with respect to $\lambda$, with $x, f^m x \in \mathcal{R}^\mu_{\epsilon,N}$, then $|\chi - \chi^q| \le 2\epsilon$, where $\chi^q$ is the largest Lyapunov exponent of $\mathcal{A}$ at $q$.*

*Proof.* In this proof we consider the case when $\chi$ is not the only Lyapunov exponent of $\mathcal{A}$ with respect to $\mu$. Otherwise the argument becomes simpler.

Apply Lemma 4.2.1 to $y = q$, by (4.2.2) we have

$$\begin{aligned}
\chi^q &\le \frac{1}{m}\log\|\mathcal{A}(q,m)\| \\
&\le \frac{1}{m}\log(Ne^{Nc\delta^\alpha} e^{m(\chi+\epsilon)})
\end{aligned}$$

$$
\begin{aligned}
&= \chi + \epsilon + \frac{1}{m}(\log(N) + Nc\delta^\alpha) \\
&\leq \chi + 2\epsilon
\end{aligned}
$$

for sufficiently small $\delta$ and sufficiently large $m$, this implies the upper bound of $\chi^q$.

For the approximation of $\chi$ from below, we need to introduce cones $C_i$ of $\mathbb{Q}_p^n$ such that if we split $\mathbb{Q}_p^n = E(f^i x) \oplus F(f^i x)$, where $E(f^i x)$ is the Lyapunov space at $f^i x$ with respect to $\chi$ and $F(f^i x)$ is the Lyapunov space at $f^i x$ with respect to the Lyapunov exponents less than $\chi$, then for any $v \in C_i$, the Lyapunov norm of the projection of $v$ on $E(f^i x)$ is no less than the Lyapunov norm of the projection of $v$ on $F(f^i x)$. More precisely, for any $v \in \mathbb{Q}_p^n$ and splitting $\mathbb{Q}_p^n = E(f^i x) \oplus F(f^i x)$, we have $v = v_E + v_F$ with $v_E \in E(f^i x)$ and $v_F \in F(f^i x)$. then we denote

$$C_i = \{v \in \mathbb{Q}_p^n : \|v_F\|'_{f^i x} \leq \|v_E\|'_{f^i x}\}$$

and

$$C_i^\eta = \{v \in \mathbb{Q}_p^n : \|v_F\|'_{f^i x} \leq (1-\eta)\|v_E\|'_{f^i x}\}$$

for $i = 0, \cdots, m$

We claim the following lemma

**Lemma 4.3.1.** *Let all the notations be as above, if orbit segments $y, fy, \cdots, f^m y$ and $x, fx, \cdots, f^m x$ are exponentially $\delta$ close with exponent $\lambda$, and $x, f^m x \in \mathcal{R}^\mu_{\epsilon,N}$, then for $i = 0, \cdots, m-1$, we have*

$$A(f^i y)(C_i) \subset C_{i+1}^\eta$$

*and*

$$\|(A(f^i y)v)_E\|'_{f^{i+1}x} \geq e^{\chi-\epsilon}\|v_E\|'_{f^i x}$$

*for any $v \in C_i$.*

*Proof.* Fix $i$ and let $D_i = A(f^i y)A(f^i x)^{-1} - \mathrm{Id}$, then we have

$$A(f^i y) = A(f^i y)A(f^i x)^{-1}A(f^i x) = (D_i + \mathrm{Id})A(f^i x).$$

Since $A$ is $\alpha$-Hölder continuous on the compact space $X$,

$$\begin{aligned}\|D_i\| &\leq \|A(f^i y) - A(f^i x)\|\|A(f^i x)^{-1}\| \\ &\leq c_1 \mathrm{d}(f^i y, f^i x)^{\alpha}.\end{aligned} \tag{4.3.1}$$

For any $v \in C_i, v = v_E + v_F$, then

$$\|v\|_{f^i x}^{'} \leq \|v_E\|_{f^i x}^{'} + \|v_F\|_{f^i x}^{'} \leq 2\|v_E\|_{f^i x}^{'}.$$

Let $u = A(f^i x)v = u_E + u_F$, then

$$u_E = A(f^i x)v_E \in E(f^{i+1}x)$$

and

$$u_F = A(f^i x)v_F \in F(f^{i+1}x).$$

By (3.4.2), we have

$$e^{\chi+\epsilon}\|v_E\|_{f^i x}^{'} \geq \|u_E\|_{f^{i+1}x}^{'} = \|A(f^i x)v_E\|_{f^{i+1}x}^{'} \geq e^{\chi-\epsilon}\|v_E\|_{f^i x}^{'}$$

and

$$\|u_F\|_{f^{i+1}x}^{'} = \|A(f^i x)v_F\|_{f^{i+1}x}^{'} \leq e^{\xi+\epsilon}\|v_F\|_{f^i x}^{'}$$

Consider

$$\begin{aligned}w &= A(f^i y)v \\ &= (D_i + \mathrm{Id})A(f^i x)v \\ &= (D_i + \mathrm{Id})u \\ &= D_i u + u \\ &= w_E + w_F \\ &= (D_i u)_E + u_E + (D_i u)_F + u_F\end{aligned}$$

By (3.4.6), since $x, f^m x \in \mathcal{R}_{\epsilon,N}^{\mu}$ and $y, fy, \cdots, f^m y$ and $x, fx, \cdots, f^m x$ are exponentially $\delta$ close with exponent $\lambda$, we have

$$\begin{aligned}
\|D_i u\|'_{f^{i+1}x} &\leq \|D_i\|_{f^{i+1}x \leftarrow f^{i+1}x} \|u\|'_{f^{i+1}x} \\
&\leq K_\epsilon(f^{i+1}x)\|D_i u\| \|A(f^i x)v\|'_{f^{i+1}x} \\
&\leq N e^{\epsilon \min\{i+1, m-i-1\}} c_1 \mathrm{d}(f^i y, f^i x)^\alpha e^{\chi+\epsilon} \|v\|'_{f^i x} \\
&\leq N c_1 \delta^\alpha e^{(\epsilon-\alpha\lambda)\min\{i,m-i\}} e^{\chi+2\epsilon} 2 \|v_E\|'_{f^i x} \\
&\leq N c \delta^\alpha e^{\chi+2\epsilon} \|v_E\|'_{f^i x}
\end{aligned}$$

Now we give the estimate of $w_E = (D_i u)_E + u_E$ and $w_F = (D_i u)_F + u_F$, for sufficiently small $\delta$, $\|D_i u\|'_{f^{i+1}x}$ can be very close to 0, which means both $\|(D_i u)_E\|'_{f^{i+1}x}$ and $\|(D_i u)_F\|'_{f^{i+1}x}$ are very close to 0. At the meanwhile, both $\|(D_i u)_E\|$ and $\|(D_i u)_F\|$ are small enough such that

$$\|\mathcal{A}(x,m)(D_i u)_{E/F}\| < \|\mathcal{A}(x,m) u_{E/F}\|$$

for any $m \in \mathbb{Z}$ and $x \in \mathcal{R}^\mu$. Hence

$$\begin{aligned}
\|w_E\|'_{f^{i+1}x} &= \|(D_i u)_E + u_E\|'_{f^{i+1}x,r} \\
&= \sum_{m\in\mathbb{Z}} \|\mathcal{A}(f^{i+1}x,m)[(D_i u)_E + u_E]\| \exp(-\chi - \epsilon|m|) \\
&= \sum_{m\in\mathbb{Z}} \max\{\|\mathcal{A}(f^{i+1}x,m)(D_i u)_E\|, \|\mathcal{A}(f^{i+1}x,m)u_E\|\} \exp(-\chi - \epsilon|m|) \\
&= \sum_{m\in\mathbb{Z}} \|\mathcal{A}(f^{i+1}x,m)u_E\| \exp(-\chi - \epsilon|m|) \\
&= \|u_E\|'_{f^{i+1}x,r} \\
&= \|u_E\|'_{f^{i+1}x}
\end{aligned}$$

since for non-archimedean norm, $\|x+y\| = \max\{\|x\|, \|y\|\}$ if $\|x\| \neq \|y\|$. Similarly we can get $\|w_F\|'_{f^{i+1}x} = \|u_F\|'_{f^{i+1}x}$.

Therefore, we have

$$e^{\chi-\epsilon}\|v_E\|'_{f^i x} \leq \|w_E\|'_{f^{i+1}x} \leq e^{\chi+\epsilon}\|v_E\|'_{f^i x}. \tag{4.3.2}$$

So we get the inequality

$$\|(A(f^iy)v)_E\|'_{f^{i+1}x} \geq e^{\chi-\epsilon}\|v_E\|'_{f^ix}.$$

Now we can estimate

$$\begin{aligned}
\|w_E\|'_{f^{i+1}x} - \|w_F\|'_{f^{i+1}x} &= \|u_E\|'_{f^{i+1}x} - \|u_E\|'_{f^{i+1}x} \\
&\geq e^{\chi-\epsilon}\|v_E\|'_{f^ix} - e^{\xi+\epsilon}\|v_F\|'_{f^ix} \\
&\geq e^{\chi-\epsilon}\|v_E\|'_{f^ix} - e^{\xi+\epsilon}\|v_E\|'_{f^ix} \\
&= (e^{\chi-\epsilon} - e^{\xi+\epsilon})\|v_E\|'_{f^ix} \\
&\geq \frac{e^{\chi-\epsilon} - e^{\xi+\epsilon}}{e^{\chi+\epsilon}}\|w_E\|'_{f^{i+1}x} \\
&> \eta\|w_E\|'_{f^{i+1}x}
\end{aligned}$$

Hence $A(f^iy)v = w \in C^\eta_{i+1}$ for $\eta < \frac{e^{\chi-\epsilon}-e^{\xi+\epsilon}}{e^{\chi+\epsilon}}$, so $A(f^iy)(C_i) \subset C^\eta_{i+1}$. $\square$

Apply Lemma 4.3.1 to the orbit segment $q, fq, \cdots, f^mq$, which is the periodic orbit exponentially $\delta$ close to $x, fx, \cdots, f^mx$ with exponent $\lambda$, then $A(f^iq)C_i \subset C^\eta_{i+1} \subset C_{i+1}$, which implies $\mathcal{A}(q,m)C_0 \subset C^\eta_m$. Since the Lyapunov splitting and Lyapunov norm are continuous on the set of regular points $\mathcal{R}^\mu_{\epsilon,N}$, if $\delta$ is sufficiently small, $x$ and $f^m(x)$ are close enough, then the cones $C^\eta_0$ and $C^\eta_m$ are close enough such that $C^\eta_m \subset C_0$. Hence $\mathcal{A}(q,m)C_0 \subset C_0$, we can conclude that for $v \in C_0$ and sufficiently close $x, f^mx$,

$$\begin{aligned}
\|\mathcal{A}(q,m)v\|'_{f^mx} &\geq \|(\mathcal{A}(q,m)v)_E\|'_{f^mx,r} \\
&= \|(\mathcal{A}(q,m)v)_E\|'_{f^mx} \\
&\geq e^{m(\chi-\epsilon)}\|v_E\|'_x \\
&\geq \frac{1}{2}e^{m(\chi-\epsilon)}\|v\|'_x \\
&\geq \frac{1}{3}e^{m(\chi-\epsilon)}\|v\|'_{f^mx}
\end{aligned}$$

Since $C_0$ is invariant under the transformation of cocycle $\mathcal{A}(q,m)$, $\mathcal{A}(q,km)v \in C_0$ for any $v \in C_0$ and any $k \in \mathbb{N}$. Now we can give the estimate of $\chi^q$ from below, for $v \in C_0$ and sufficiently large number $m$,

$$\begin{aligned}\chi^q \geq \chi(v) &= \lim_{k\to+\infty} \frac{1}{km} \log \|\mathcal{A}(q,km)v\|'_{f^m x} \\ &\geq \lim_{k\to+\infty} \frac{1}{km} \log((\frac{1}{3}e^{m(\chi-\epsilon)})^k \|v\|'_{f^m x}) \\ &\geq \chi - \epsilon - \frac{\log 3}{m} + \lim_{k\to+\infty} \frac{\|v\|'_{f^m x}}{km} \\ &\geq \chi - 2\epsilon.\end{aligned}$$

This completes the proof of Proposition 4.3.1. □

## 4.4 Proof of Theorem 2

We denote $\bigwedge^k \mathcal{A}$ be the natural extension of cocycle $\mathcal{A}$ to the $k$-fold exterior power of $\mathbb{Q}_p^n$, which is $\bigwedge^k \mathbb{Q}_p^n$. By the paper of M.S.Ragunatan [27], we know the largest Lyapunov exponent of $\bigwedge^k \mathcal{A}$ is $\sum_{n-k+1\leq i\leq n} \chi_i$, where $\chi_1 \leq \cdots \leq \chi_n$ are the Lyapunov exponents of $\mathcal{A}$ with multiplicities.

For $0 < \epsilon < \min\{\alpha\lambda, (\chi-\xi)/2\}$ and let $\mathcal{R} = \bigcap_{1\leq i\leq n} \mathcal{R}^{\mu,i}_{\epsilon,N}$, where $\mathcal{R}^{\mu,i}_{\epsilon,N}$ is the regular set of cocycle $\bigwedge^i \mathcal{A}$ satisfying (3.4.7). We can choose big enough $N$ such that $\mu(\mathcal{R}) > 0$ and pick a non-periodic and non-isolated point $x \in \mathcal{R}$, since $f$ is a measure-preserving homeomorphism, by Poincaré recurrence theorem, there exists $f^m x$ with $m$ growing infinity, returning to $\mathcal{R}$ arbitrarily close to $x$. Then by closing property, there exists a periodic point $q = f^m q$ so that $q, fq, \cdots, f^m q$ is exponentially $\delta$ close to $x, fx, \cdots, f^m x$ with exponent $\lambda$ for any $\delta > 0$. Then apply the proposition 4.3.1 to $p$ and get

$$|\sum_{n-k+1\leq i\leq n} \chi_i - \sum_{n-k+1\leq i\leq n} \chi_i^q| \leq 2\epsilon$$

for all $i = 1, \cdots, n$. This implies $|\chi_i - \chi_i^q| \leq 2\epsilon$ for $i = 1, \cdots, n$ and completes the proof of theorem 2.

Chapter 5

# Uniformly Boundedness

## 5.1 Introduction

In this chapter we will show that the cocycle $\mathcal{A}$ is uniformly bounded if it is uniformly bounded at any periodic points. We first give the estimate of the growth of the norm of cocycle when the Lyapunov exponents of the cocycle $\mathcal{A}$ at all periodic points are bounded, where we need to apply the subadditive ergodic theorem. Then we show that the cocycle along a dense orbit is uniformly bounded, which will complete the proof of theorem 3.

## 5.2 Boundedness of the Growth of the Norm of Cocycle

First, we have following proposition

**Proposition 5.2.1.** *Let $f$ be a transitive homeomorphism of a compact metric space $X$ satisfying the closing property and let $A : X \to GL(n, \mathbb{Q}_p^n)$ be an $\alpha$-Hölder function generating the cocycle $\mathcal{A}(x, m)$. If for any periodic point $q \in X$ such that $q = f^m q$, the Lyapunov exponents of $\mathcal{A}(q, m)$ at $q$ satisfy $\chi_{min} \le \chi_i^q \le \chi_{max}$. Then for any $\epsilon > 0$ there exists a constant $c(\epsilon)$ such that for almost all $x \in X$ and $m \in \mathbb{N}$,*

$$\|\mathcal{A}(x, m)\| \le c(\epsilon) e^{m(\chi_{max}+\epsilon)} \tag{5.2.1}$$

*and*

$$\|\mathcal{A}(x,m)^{-1}\| \leq c(\epsilon)e^{-m(\chi_{min}-\epsilon)}. \tag{5.2.2}$$

*Proof.* To prove this proposition, we need to apply the following result

**Lemma 5.2.1** ([28],Proposition 3.4)**.** *Let $f : X \to X$ be a continuous map of a compact metric space. Let $a_m : X \to \mathbb{R}$, $m \geq 0$, be a sequence of continuous functions such that*

$$a_{m+k}(x) \leq a_m(f^k x) + a_k(x)$$

*for every $x \in X, m, k \geq 0$, and such that there is a sequence of continuous functions $b_m : X \to \mathbb{R}, m \geq 0$, satisfying*

$$a_m(x) \leq a_m(f^k x) + a_k(x) + b_k(f^m x)$$

*for every $x \in X, m, k \geq 0$.*

*If $\inf_m \frac{1}{m}\int_X a_m d\mu < 0$ for every ergodic f-invariant measure, then there exists $M \geq 0$ such that $a_M(x) < 0$ for every $x \in X$.*

Then we can apply the lemma 5.2.1 in the following way.

Let $a_m(x) = \log\|\mathcal{A}(x,m)\| - m(\chi_{max}+\epsilon)$, then

$$\begin{aligned} a_{m+k}(x) &= \log\|\mathcal{A}(x,m+k)\| - (m+k)(\chi_{max}+\epsilon) \\ &= \log\|\mathcal{A}(f^k x,m)\mathcal{A}(x,k)\| - (m+k)(\chi_{max}+\epsilon) \\ &\leq \log\|\mathcal{A}(f^k x,m)\| + \log\|\mathcal{A}(x,k)\| - (m+k)(\chi_{max}+\epsilon) \\ &= a_m(f^k x) + a_k(x) \end{aligned}$$

By the subadditive ergodic theorem, we have

$$\inf_m \frac{1}{m}\int_X a_m d\mu = \lim_{m\to\infty}\frac{1}{m}a_m = \chi - \chi_{max} - \epsilon < 0$$

Let $b_k = \log\|\mathcal{A}(x,k)^{-1}\| + k(\chi_{max}+\epsilon)$, then

$$\begin{aligned} a_m &= \log\|\mathcal{A}(x,m)\| - m(\chi_{max}+\epsilon) \\ &= \log\|\mathcal{A}(f^m x,k)^{-1}\mathcal{A}(x,m+k)\| - m(\chi_{max}+\epsilon) \\ &\leq \log\|\mathcal{A}(f^m x,k)^{-1}\| + \log\|\mathcal{A}(x,m+k)\| + k(\chi_{max}+\epsilon) - (m+k)(\chi_{max}+\epsilon) \end{aligned}$$

$$\leq a_m(f^k x) + a_k(x) + b_k(f^m x)$$

Now we can apply Lemma 5.2.1 to $a_m$ and conclude that there exists a positive number $M_\epsilon$ such that $a_{M_\epsilon}(x) < 0$ for every $x \in X$, which implies

$$\|\mathcal{A}(x, M_\epsilon)\| \leq e^{M_\epsilon(\chi_{max}+\epsilon)}.$$

And pick $c(\epsilon)$ such that $\|\mathcal{A}(x, m)\| \leq c(\epsilon)$ for $m = 1, \cdots, M_\epsilon - 1$, then for any $m \in \mathbb{N}$, suppose $m = aM_\epsilon + k$, we have

$$\begin{aligned}\|\mathcal{A}(x, m)\| &\leq \|\mathcal{A}(f^k x, aM_\epsilon)\|\|\mathcal{A}(x, k)\| \\ &\leq c(\epsilon)e^{aM_\epsilon(\chi_{max}+\epsilon)} \\ &\leq c(\epsilon)e^{m(\chi_{max}+\epsilon)}.\end{aligned}$$

for almost all $x \in X$.

The other argument can be proved similarly if we apply the Lemma 5.2.1 to

$$a_m(x) = \log \|\mathcal{A}(x, m)^{-1}\| + m(\chi_{min} - \epsilon)$$

and this completes the proof. □

## 5.3 Boundedness of Cocycle along Dense Orbit

Since $f$ is transitive, there exists a dense orbit $\mathcal{O} = \{f^i z\}_{i\in\mathbb{Z}}$. By the continuity of cocycle $\mathcal{A}$ , if we can show the $\|\mathcal{A}(z, m)\|$ is uniformly bounded, then $\|\mathcal{A}(x, m)\|$ is uniformly bounded in $x \in X$ and $m \in \mathbb{Z}$. For any two points $f^{k_1}z, f^{k_2}z$ with $k_1 < k_2$ in the dense orbit $\mathcal{O}$ such that $\mathrm{d}(f^{k_1}z, f^{k_2}z) < \delta_0$. Let $x = f^{k_1}z$, and $m = k_2 - k_1$, then $\mathrm{d}(x, f^m x) < \delta_0$, by the closing property, there exists periodic point $q \in X$ with $q = f^m q$ and point $y \in X$ such that

$$\mathrm{d}(f^i y, f^i q) \leq \delta e^{-\lambda(m-i)}$$

and

$$\mathrm{d}(f^i y, f^i x) \leq \delta e^{-\lambda i}$$

for $i = 0, \cdots, m$.

Now we show that there exists constant $c > 0$ such that $\|\mathcal{A}(x,m)\| \leq c$ for $x, y, q, m$ as above if there exists constant $c^{'}$ such that $\|\mathcal{A}(q,m)\| \leq c^{'}$ for all $q \in X$ and $m \in \mathbb{Z}$. We first prove $\mathrm{d}_{GL(n,\mathbb{Q}_p)}(\mathcal{A}(x,m), \mathcal{A}(q,m))$ is uniformly bounded for all $x \in \mathcal{O}$. Below we give the estimate of $\mathrm{d}_{GL(n,\mathbb{Q}_p)}(\mathcal{A}(y,m), \mathcal{A}(q,m))$ and $\mathrm{d}_{GL(n,\mathbb{Q}_p)}(\mathcal{A}(x,m), \mathcal{A}(y,m))$ by considering the values of $\|\mathcal{A}(y,m)\mathcal{A}(q,m)^{-1} - \mathrm{Id}\|$ and $\|\mathcal{A}(x,m)^{-1}\mathcal{A}(y,m) - \mathrm{Id}\|$.

Let $D_i = A(f^i y)A(f^i q)^{-1} - \mathrm{Id}$, then

$$\begin{aligned}
\|D_i\| &= \|A(f^i y)A(f^i q)^{-1} - \mathrm{Id}\| \\
&\leq \|A(f^i y) - A(f^i q)\| \|A(f^i q)^{-1}\| \\
&\leq c_1 \mathrm{d}(f^i y - f^i q)^{\alpha} \\
&\leq c_1 \delta^{\alpha} e^{-\alpha\lambda(m-i)}.
\end{aligned}$$

Now let we give the estimate of $\|\mathcal{A}(y,m)\mathcal{A}(q,m)^{-1} - \mathrm{Id}\|$

$$\begin{aligned}
\mathcal{A}(y,m)\mathcal{A}(q,m)^{-1} &= \mathcal{A}(fy, m-1)A(y)A(q)^{-1}\mathcal{A}(fq, m-1)^{-1} \\
&= \mathcal{A}(fy, m-1)(D_0 + \mathrm{Id})\mathcal{A}(fq, m-1)^{-1} \\
&= \mathcal{A}(fy, m-1)D_0\mathcal{A}(fq, m-1)^{-1} + \mathcal{A}(fy, m-1)\mathcal{A}(fq, m-1)^{-1} \\
&\cdots \\
&= \sum_{i=0}^{m-1} \mathcal{A}(f^{i+1}y, m-i-1)D_i\mathcal{A}(f^{i+1}q, m-i-1)^{-1} + \mathrm{Id}
\end{aligned}$$

Since there exists a positive constant $c^{'}$ such that $\|\mathcal{A}(q,m)\| \leq c^{'}$ for all periodic number $q \in X$ with $q = f^m q$, then the Lyapunov exponents of cocycle $\mathcal{A}$ at $q$ are all 0, by the Proposition 5.2.1, we know

$$\|\mathcal{A}(x,m)\| \leq c(\epsilon)e^{m\epsilon} \text{ and } \|\mathcal{A}(x,m)^{-1}\| \leq c(\epsilon)e^{m\epsilon} \tag{5.3.1}$$

for all $x \in X, m \in \mathbb{N}$.

Hence by using (5.3.1) we can conclude that

$$\begin{aligned}
\|\mathcal{A}(y,m)\mathcal{A}(q,m)^{-1}-\mathrm{Id}\| &\le \|\sum_{i=0}^{m-1}\mathcal{A}(f^{i+1}y,m-i-1)D_i\mathcal{A}(f^{i+1}q,m-i-1)^{-1}\| \\
&\le \max_{0\le i\le m-1}\{\|\mathcal{A}(f^{i+1}y,m-i-1)D_i\mathcal{A}(f^{i+1}q,m-i-1)^{-1}\|\} \\
&\le \sup_{0\le i\le m-1}\|\mathcal{A}(f^{i+1}y,m-i-1)\|\|D_i\|\|\mathcal{A}(f^{i+1}q,m-i-1)^{-1}\| \\
&\le \sup_{0\le i\le m-1} c(\epsilon)^2e^{2(m-i-1)\epsilon}c_1\delta^\alpha e^{-\alpha\lambda(m-i)} \\
&= \sup_{0\le i\le m-1} c(\epsilon)^2e^{(m-i)(2\epsilon-\alpha\lambda)}c_1\delta^\alpha e^{-2\epsilon} \\
&\le c_2\delta^\alpha
\end{aligned}$$

for sufficiently small $\epsilon$ such that $2\epsilon-\alpha\lambda<0$, this inequality holds for all $m\in\mathbb{N}$.

Similarly we can show that

$$\|\mathcal{A}(y,m)^{-1}\mathcal{A}(x,m)-\mathrm{Id}\|\le c_2\delta^\alpha.$$

Now we can give the estimate of $\mathrm{d}_{GL(n,\mathbb{Q}_p)}(\mathcal{A}(y,m),\mathcal{A}(q,m))$ and $\mathrm{d}_{GL(n,\mathbb{Q}_p)}(\mathcal{A}(x,m),\mathcal{A}(y,m))$.

$$\begin{aligned}
\mathrm{d}_{GL(n,\mathbb{Q}_p)}(\mathcal{A}(y,m),\mathcal{A}(q,m)) &= \|\mathcal{A}(y,m)-\mathcal{A}(q,m)\| \\
&= \|\mathcal{A}(y,m)\mathcal{A}(q,m)^{-1}-\mathrm{Id}\|\|\mathcal{A}(q,m)\| \\
&\le c^{'}c_2\delta^\alpha \\
&\le c_3\delta^\alpha
\end{aligned}$$

which implies $\|\mathcal{A}(y,m)\|$ is uniformly bounded. Then

$$\begin{aligned}
\mathrm{d}_{GL(n,\mathbb{Q}_p)}(\mathcal{A}(x,m),\mathcal{A}(y,m)) &= \|\mathcal{A}(x,m)-\mathcal{A}(y,m)\| \\
&= \|\mathcal{A}(y,m)\|\|\mathcal{A}(y,m)^{-1}\mathcal{A}(x,m)-\mathrm{Id}\| \\
&\le c_4c_2\delta^\alpha \\
&\le c_5\delta^\alpha.
\end{aligned}$$

By this we obtain that $\|\mathcal{A}(x,m)\|$ is uniformly bounded for all $x\in\mathcal{O}$ and $m\in\mathbb{Z}^+$

in the setting. Actually,

$$\begin{aligned}\|\mathcal{A}(x,m)-\mathcal{A}(q,m)\| &\le \max\{\|\mathcal{A}(x,m)-\mathcal{A}(y,m)\|,\|\mathcal{A}(y,m)-\mathcal{A}(q,m)\|\}\\ &\le \max\{c_3,c_5\}\delta^{\alpha}\\ &= c_6\delta^{\alpha}. \qquad (5.3.2)\end{aligned}$$

Similarly we can show that $\|\mathcal{A}(x,m)\|$ is uniformly for all $x \in \mathcal{O}$ and $m \in \mathbb{Z}^-$.

## 5.4 Proof of Theorem 3

Now we need to show that for any $m^{'} \in \mathbb{Z}$ and any $z \in \mathcal{O}$, $\|\mathcal{A}(z,m^{'})\|$ is uniformly bounded. Since $\mathcal{O}$ is dense in $X$, we denote its $\delta_0$-net by $\mathcal{O}_K = \{f^k(z)\}_{k\in[-K,K]}$, and let $c_7 = \max_{k\in[-K,K]} \|\mathcal{A}(z,k)\|$, then there exists $f^k z \in \mathcal{O}_K$ such that $\mathrm{d}(f^k z, f^{m^{'}} z) < \delta_0$, let $x = f^k z$ and $m = m^{'} - k$, then $\|\mathcal{A}(x,m)\|$ and $\|\mathcal{A}(z,k)\|$ are uniformly bounded, so is

$$\|\mathcal{A}(z,m^{'})\| \le \|\mathcal{A}(x,m)\|\|\mathcal{A}(z,k)\|.$$

This completes the proof of Theorem 3.

Chapter 6

# Proof of Main Result

## 6.1 Introduction

In this section we use the classic way to prove the theorem 1. Which means, we will define the function along the dense orbit of a transitive point in the space $X$ and the extend this function to the whole space $X$.

## 6.2 Proof of the Main Result

Let $\phi_m : \mathcal{O} \to GL(n, \mathbb{Q}_p^n)$ be a function such that $\phi_m(x) = \mathcal{A}(x, m)$. Since we have the trivial periodic data assumption, by the theorem 3, $\|\phi_m(x)\|$ is uniformly bounded. Now we show that there exists a constant $C$ such that $\|\phi_m(x) - \mathrm{Id}\| \leq C\mathrm{d}(f^m x, x)^\alpha$, actually by (5.3.2)

$$\begin{aligned}
\|\phi_m(x) - \mathrm{Id}\| &= \|\mathcal{A}(x, m) - \mathrm{Id}\| \\
&= \|\mathcal{A}(x, m) - \mathcal{A}(q, m)\| \\
&\leq c_6 \delta^\alpha \leq C\mathrm{d}(f^m x, x)^\alpha.
\end{aligned}$$

Apply the same argument to $f^k x$, we get $\|\phi_m(f^k x) - \mathrm{Id}\| \leq C\mathrm{d}(f^{m+k}x, f^k x)^\alpha$.

Since the orbit of $x$ is dense in $X$, we can get the $\alpha$-Hölder continuous function $P : X \to GL(n, \mathbb{Q}_p^n)$ such that $P(f^m x) = \phi_m(x)$ for $m \in \mathbb{Z}$. The Hölder continuity is given by the following argument

$$\begin{aligned}
\|P(f^{m+k}x) - P(f^k x)\| &= \|\phi_{m+k}(x) - \phi_k(x)\| \\
&= \|\mathcal{A}(x, m+k) - \mathcal{A}(x, k)\| \\
&= \|\mathcal{A}(f^k x, m)\mathcal{A}(x, k) - \mathcal{A}(x, k)\| \\
&\leq \|\mathcal{A}(f^k x, m) - \mathrm{Id}\| \|\mathcal{A}(x, k)\| \\
&= \|\phi_m(f^k x) - \mathrm{Id}\| \|\mathcal{A}(x, k)\| \\
&\leq C' \mathrm{d}(f^{m+k}x, f^k x)^\alpha.
\end{aligned}$$

And it is easy to check that $A(x) = P(fx)P(x)^{-1}$ on a dense set, hence everywhere. This completes the proof of Theorem 1.

# Bibliography


[1] LIVŠIC, A. N. (1972) "Cohomology of dynamical systems," *Math. USSR Izvestija*, **6**, pp. 1278–1301.

[2] ——— (1971) "Homology properties of Y-systems," *Math. Zametki*, **10**, pp. 758–763.

[3] NICŢICĂ, V. and A. TÖRÖK (1995) "Cohomology of dynamical systems and rigidity of partially hyperbolic actions of higher-rank lattices," *Duke Math. J.*, **79**(3), pp. 751–810.

[4] ——— (1998) "Regularity of the transfer map for cohomologous cocycles," *Ergodic Theory Dynamical Systems*, **18**(5), pp. 1187–1209.

[5] BERCOVICI, H. and V. NICŢICĂ (1998) "A banach algebra version of the Livšic theorem," *Discrete and Continuous Dynamical Systems*, **4**(3), pp. 523–534.

[6] NICOL, M. and M. POLLICOTT (2001) "Livšic's theorem for semisimple Lie group," *Ergodic Theory Dynamical Systems*, **21**(5), pp. 1501–1509.

[7] DE LA LLAVE, R. and A. WINDSOR (2010) "Livšic theorems for non-commutative groups including groups of diffeomorphisms, and invariant geometric structures," *Ergodic Theory Dynamical Systems*, **30**, pp. 286–300.

[8] KATOK, A. and V. NICŢICĂ *Differentiable Rigidity of Higher Rank Abelian Group Actions*, to be published.

[9] KALININ, B. (2011) "Livšic theorem for matrix cocycles," *Ann. of Math.*, **173**(2), pp. 1025–1042.

[10] PARRY, W. and M. POLLICOTT (1997) "The Livšic cocycle equation for compact Lie group extensions of hyperbolic systems," *J. London Math. Soc.*, **56**(2), pp. 405–416.

[11] Schmidt, K. (1999) "Remarks on Livšic theory for non-abelian cocycles," vol. 19, pp. 703–721.

[12] Walkden, C. P. (2000) "Livšic regularity theorems for twisted cocycle equations over hyperbolic systems," *J. London Math.Soc.*, **61**(1), pp. 286–300.

[13] ——— (1997) "Livšic Theorems for Hyperbolic Flows," *Trans. Amer. Math. Soc.*, **352**(3), pp. 1299–1313.

[14] PARRY, W. (1999) "Livšic periodic point theorem for non-abelian cocycles," *Ergodic Theory Dynamical Systems*, **19**, p. 687C701.

[15] Llave, R. D. L. (1997) "Analytic regularity of solutions of Livšic's cohomology equation and some applications to analytic conjugacy of hyperbolic dynamical systems," *Ergodic Theory Dynamical Systems*, **17**, p. 649C662.

[16] Margulis, G. A. (1991) *Discrete Subgroups of Semisimple Lie Groups*, 1st ed., Springer-Verlag.

[17] Wang, Z. and W. Sun (2009) "Lyapunov exponents of hyperbolic measures and hyperbolic periodic orbits," *Trans. Amer. Math.*, **361**, pp. 1543–1579.

[18] Dai, X. (2010) "Exponential closing property and approximation of Lyapunov exponents of linear cocycles," *Forum Math.*

[19] Gouvêa, F. (1993) *p-adic Numbers, An Introduction*, 1st ed., Springer.

[20] Katok, S. (2007) *p-adic Analysis Compared with Real*, 1st ed., American Mathematical Society.

[21] Robert, A. (2000) *A Course in p-adic Analysis*, 1st ed., Springer.

[22] Katok, A. and B. Hasselblatt (1996) *Introduction to the modern theory of dynamical systems*, 1st ed., Cambridge University Press.

[23] ——— (2002) *Handbook of Dynamical Systems*, 1st ed., Elsevier North-Holland.

[24] Barreira, L. and Y. Pesin (2007) *Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov exponents*, 1st ed., Cambridge University Press.

[25] Oseledec, V. (1968) "Multiplicative ergodic theorem. Lyapunov characteristic numbers for dynamical systems," *Trans. Moscow Math. Soc.*, **19**(4), p. 197C221.

[26] KATOK, A. and R. SPATZIER (1996) "Invariant measures for higher rank hyperbolic abelian action," *Ergodic Theory Dynamical Systems*, **16**, pp. 751–778.

[27] RAGHUNATHAN, M. S. (1979) "A proof of Oseledec's multiplicative ergodic theorem," *Israel J. of Math.*, **32**(4), pp. 356 – 362.

[28] HERTZ, F. R. (2007) "Global rigidity of certain abelian actions by toral automorphisms," *Journal of Moden Dynamics*, **1**(3), pp. 425–442.

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