INTEGRATED MODELS AND METHODOLOGIES FOR PARAMETER AND TOLERANCE DESIGNS

A Dissertation in
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by
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ABSTRACT

Most products are mixtures or assemblies of multiple components. These components require determining the means (nominal values) and acceptable tolerances of their relevant quality characteristics. The process of setting the means and tolerances of the components characteristics is called parameter and tolerance designs. These designs are found to be most effective when conducted simultaneously (integrated designs). In the classical models of integrated parameter and tolerance designs, the objective is to minimize the total costs while meeting the customer requirements and expectations using available technological capabilities. Total costs can be divided into two categories. The first category is the manufacturing costs, which include production and internal failure costs, i.e. costs incurred before the product is shipped to the customer. The second category is the quality losses or external failure costs, which include warranty costs, loss of market share, etc., i.e. costs incurred after the product is shipped to the customer. The major limitation of most existing integrated parameter and tolerance designs models is the use of the quadratic loss function. It is used to capture all external failure costs incurred by the manufacturer, customer, and the society as a whole as a function of the deviation of the product’s quality characteristic from its ideal value using a single proportionality constant. Despite the merits of its idea, the quadratic loss function suffers from its simplistic form and the difficulty of estimating its proportionality constant.

In this research, integrated parameter and tolerance designs models are proposed that use warranty costs instead of the quadratic loss function for estimating external failure costs. The advantage of using warranty costs is that they are related to product
reliability, which can be modeled fairly accurately using empirical product failure data. However, since warranty costs are only part of the external failure costs, the first set of proposed integrated models run the risk of underestimating the external failure costs and jeopardizing the accuracy of the models results. This issue is addressed by proposing a second set of integrated parameter and tolerance designs models, which incorporate in them microeconomic and marketing concepts, such as pricing and demand functions. In these integrated models, instead of listing and estimating all external failure costs other than the warranty costs, these external failure costs are modeled as part of the customer’s decision of the amount he or she is willing to pay for the product. That is, the customer sets the purchase price based on a tradeoff between the amount of gained satisfaction and the perceived risk of using the product. As a result, the objective of these integrated models is to maximize the profit, which is the difference between the selling price and the total cost of the product, as opposed to minimizing total cost. After surveying pricing and demand models from the microeconomic and marketing literature, it is found that most pricing models use product manufacturing and quality costs in a very crude manner. Therefore, not only do the proposed integrated parameter and tolerance designs models provide a better representation of the external failure costs, but they also provide a better representation of manufacturing and quality costs of the product in the microeconomic and marketing decisions models. That is, the proposed integrated models bridge the gap between marketing and manufacturing operations and decisions by eliminating unnecessary assumptions and poor links from both areas.

Several numerical examples are presented to validate the proposed models and show their flexibility and applicability to a wide range of problems. For the first set of
integrated models, in which microeconomic and marketing decisions are not considered, the examples include the cases of linear and nonlinear relationships between the product’s main quality characteristic and the quality characteristics of its components as well as the case of finite manufacturing processes. For the second set of integrated models, in which microeconomic and marketing decisions are considered, the numerical examples include the cases of static market analysis approach for a general demand function, a monopoly, and a duopoly competition as well as the case of a dynamic market analysis approach. In the different numerical examples, a variety of optimization and analysis techniques, such as nonlinear programming, integer programming, response surface methodology, simulation, and sensitivity analysis, are employed to show the methodology of handling the different design problems.
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Chapter 1

Introduction

1.1 Problem Statement

In today’s economy, fierce competition requires manufacturers to strive for perfection and operate at the highest levels of efficiency when it comes to fulfilling and exceeding customer requirements and expectations. This cannot be achieved without taking full advantage of the enormous amounts of data collected about the marketplace and customer behavior as well as the financial resources and technological capabilities of the firm in making critical decisions about the products being manufactured. This requires proper data analysis, model building and finding solutions while taking into consideration every issue that could impact the success of the product.

The focus of this research is on parameter and tolerance designs, which are part of the product design phase of new product development. The different phases of new product development are presented in Figure 1.1.
Typically, new product lifecycle starts with surveying customer preferences and market information (Park et al., 2008). Then, product design and process design are conducted. After that, actual production, sales, and after-sales services follow. It is important to emphasize that not only does product design highly depend on the information collected about the marketplace and customer preferences, but it also greatly influences the later phases of process design, production, sales, and after-sales services. Thus, the key idea is that, even though the actual implementation of the different phases of new product development occurs sequentially, the decision making process and the mathematical models used in product design ought to encompass all the information related to customer preferences, market structure, selling price, manufacturing capabilities and costs, product reliability, and warranty costs as well as any other information that could contribute to the success and profitability of the product. This idea is the main premise of this research.
As shown in Figure 1.1, product design consists of three main steps that are conventionally followed in sequence as advocated by Dr. Genichi Taguchi (Park et al., 2008). These three steps of product design are system design, parameter design, and tolerance design. System design is the process of developing a prototype of the product detailing its features, components, assembly, functionality, etc. Parameter design is the process of determining the means (nominal values) for all the quality characteristics of the components of the final product, which are typically optimized to achieve the functional requirements of the product with the least variability and maximum robustness against uncontrollable noise factors. Tolerance design is the final process of assigning tolerances to the quality characteristics of the product and its components with the objective of minimizing manufacturing costs in order to further reduce the variability, if the variability could not be satisfactorily minimized by parameter design (Chandra, 2001). It is worth mentioning that process design also consists of the same three steps of system design, parameter design, and tolerance design. In system design, the manufacturing process with all the steps required for production is developed based on available resources and technologies. In parameter design, the operating conditions and machine settings for the product manufacturing process are determined. Lastly, in tolerance design, tolerances are assigned to the manufacturing process conditions and settings (Park et al., 2008).

In the product design phase, the sequential application of system design, parameter design, and tolerance design has been criticized as being suboptimal. For instance, there are several studies that argue that adjusting the tolerances of the components characteristics in tolerance design after their mean values have been set in
parameter design could affect the optimality of these mean values (e.g., Li et al., 1999; Jeang, 1999; Cho et al., 2000; etc.). These studies focus on the integration of parameter and tolerance designs with the objective of minimizing costs. That is, it is suggested that parameter and tolerance designs ought to be conducted simultaneously in order to achieve an optimal design. Furthermore, there are other studies that argue that an integrated approach, which utilizes all the information related to the manufacturing and marketing operations during product development, would result in a more profitable outcome for the firm (e.g., Bagajewicz, 2005; Karmarkar et al., 1997; etc.). Often, a product is manufactured based on what is best or optimal from the point of view of the manufacturer and then the marketing team is asked to try and sell the product (Bagajewicz, 2005). Other times, a product is designed with all the functional requirements, parameters and tolerances based on market information and what the customer wants, while manufacturing process selection and costs involved are ignored. These practices can either result in unnecessary costs and wasted resources with little return on investment or they can result in lost opportunities and the failure to meet expectations. Thus, it seems more logical to include marketing decisions and the voice of the customer in all the stages of product development along with full consideration of the manufacturing capabilities and costs in order to achieve maximum efficiency and profitability.

The only downside of such approaches, which call for more holistic methodologies for the implementation of product design, is the added complexity of the models and the difficulty to solve them. Then again, with the rapid improvements in the technological and computer capacities, optimization algorithms, data mining, etc., solving
massive numerical problems and complex models can be performed more easily today than ever before.

The problem with previous studies that attempt to provide integrated models for parameter and tolerance designs (or product development in general), as presented in more detail in the literature review of Chapter 2, is that there tends to be an emphasis by some studies on manufacturing issues while having a very simplistic treatment of marketing issues. On the other hand, there are other studies that have the exact opposite problem where there tends to be an emphasis on marketing issues while crudely representing manufacturing issues. Therefore, the main contribution of this research is to propose integrated parameter and tolerance designs models that emphasize the importance of both manufacturing and marketing issues. However, before presenting the detailed objectives of this research in section 1.3, a brief description of the basic concepts related to the proposed models in this research is presented. These basic concepts include manufacturing costs, quality losses, warranty models, reliability models, customer utility and demand functions, and market structure types.

1.2 Basic Concepts Related to This Research

1.2.1 Manufacturing Costs

In manufacturing and quality engineering literatures, manufacturing costs are typically modeled as a function of the natural process tolerances of the quality characteristics of the product components. The tighter the tolerances, the higher the
manufacturing costs and vice versa. This is because it takes more effort and perhaps slower production rate as well as more skilled workers, more expensive production and monitoring equipments with high precision, etc., in order to maintain tighter tolerances (Jeang, 2001). The relationship between manufacturing costs and the natural process tolerances of the components characteristics could be presented in the form of a tolerance-cost table, which lists the level of tolerance and the cost associated with it (Li et al., 1999), or it could be presented in functional form using, for instance, the reciprocal or the exponential relationships given by (Wu et al., 1998):

\[ MC = d_0 + d_1 t^{-d_2} \]  \hspace{1cm} (1.1) \\

and

\[ MC = d_0 + d_t \exp(-td_2) \]  \hspace{1cm} (1.2) \\

respectively, where \( t \) is the natural process tolerance of the component characteristic and \( d_0, d_1 \) and \( d_2 \) are model parameters. These formulations are typical in the literature and many studies define manufacturing costs only as a function of the natural process tolerances of the components characteristics. However, there are other studies (e.g. Jeang, 2003) that include a separate internal failure costs term, i.e. scrap and rework costs, in addition to the tolerance costs in the manufacturing costs relationship since they are affected by the means as well as the natural process tolerances of the components characteristics. If the mean of the main quality characteristic of the product is set at the midpoint within the allowable design tolerance or very close to it, then the internal failure costs could be modeled as a function of the natural process tolerances of the components characteristics alone and lumped into the tolerance-cost relationships mentioned above.
(e.g. equations (1.1) and (1.2)). However, if the mean of the main product characteristic is allowed to vary considerably from the midpoint of the allowable design tolerance, then the internal failure costs must be related to both the means and the natural process tolerances of the components characteristics.

One of the most common ways to relate the mean and the natural process tolerance (or variance) of a product characteristic to the allowable design tolerances is to use process capability indices. Typical process capability indices include $C_p$, $C_{pk}$ and $C_{pm}$, which are defined for a normally distributed quality characteristic of a product as follows (Chandra, 2001):

$$C_p = \frac{USL - LSL}{6\sigma}, \quad (1.3)$$

$$C_{pk} = \frac{\min\left[(\mu - LSL), (USL - \mu)\right]}{3\sigma}, \quad (1.4)$$

and

$$C_{pm} = \frac{USL - LSL}{6\sqrt{\sigma^2 + (\mu - x_0)^2}}, \quad (1.5)$$

where $USL$ and $LSL$ are the upper and lower specification limits, respectively, and their difference is the allowable design tolerance. Also, $\sigma$, $\mu$ and $x_0$ are the standard deviation, mean, and the ideal value of the product characteristic, respectively. If the quality characteristic of the product has a non-normal distribution, it is suggested to replace $6\sigma$ in the denominator of equation (1.3) with $(X_u - X_l)$ and $3\sigma$ in the denominator of equation (1.4) with the minimum of $(X_u - X_{0.5})$ and $(X_{0.5} - X_l)$, where
$X_p$ is the $p \times 100$ percentile of the product characteristic and values for $u$ and $l$ include 1 and 0, 0.99865 and 0.00135, or 0.995 and 0.005 (Chandra, 2001; Clements, 1989; and McCormack et al., 2000). It is worth mentioning that six-sigma companies require their process capability indices to be around two.

In microeconomic and marketing literatures, on the other hand, manufacturing costs, in general, are not related to the means and the natural process tolerances of the product or components characteristics as in the manufacturing and quality engineering literatures. Instead, manufacturing costs are either modeled as a function of some attribute, such as quality level (Teng et al., 1996), product reliability (Huang, et al., 2007), etc., or defined as the sum of fixed and variable costs (Miller, 2006). Here, fixed costs are the costs that are independent of the production rate, which include costs related to investment costs in facilities, heavy machinery, etc. Conversely, variable costs are the costs that depend on the production rate, which include the costs of raw materials, personnel, electricity, etc.

1.2.2 Quality Losses

In quality engineering literature, the most widely used formulation for estimating the quality losses (external failure costs) is the quadratic loss function, which was first developed by Dr. Taguchi. It is arrived at by employing the Taylor series expansion and ignoring higher order terms. The final form of the quadratic loss function for a nominal-the-best (N-type) quality characteristic $X$ of a product is given by (Chandra, 2001):

$$L(X) = k(X - x_0)^2, \quad LSL \leq X \leq USL$$ (1.6)
where \( k \) is a constant that represents the monetary loss incurred by the society after the product is shipped due to a unit deviation of the product characteristic from its ideal value \( x_0 \). Also, \( LSL \) and \( USL \) are the lower and upper specification limits of the product characteristic, respectively. The quadratic loss function states that quality losses increase (quadratically) as the difference between the value of the product characteristic and its ideal value increases, while there are no losses incurred if the value of the product characteristic is equal to its ideal value. The main advantage for the use of the quadratic loss function is this ability to capture the effect of the product characteristic deviation from its ideal value in monetary value. The result is that any deviation from the ideal value incurs losses, even if the product meets all the specifications and tolerances. This is a big improvement over the “goal-post mentality” that assumes all products that meet the specifications are identical in terms of quality (Chandra, 2001).

The expected value of the quadratic loss function (equation (1.6)) is given by:

\[
E[L(X)] = k \left[ \sigma^2 + (\mu - x_0)^2 \right]
\]

(1.7)

where \( \mu \) and \( \sigma^2 \) are the mean and variance of the product characteristic \( X \). One of the most significant and practical results of equation (1.7) is that the expected quality losses are directly proportional to the variance and the deviation of the mean from the ideal value of the product characteristic. Thus, the quadratic loss function suggests that setting the mean equal to the ideal value and minimizing variability of the product characteristic should be the goal of any manufacturing process in order to achieve high quality products. That is, for a product with one main quality characteristic \( X \) that is a function
of the quality characteristics of the product components $X_1, X_2, \ldots, X_k$, the optimization model is given by:

$$\text{Minimize } \sigma^2(\mu_1, \mu_2, \ldots, \mu_k; \sigma_1^2, \sigma_{12}, \ldots, \sigma_{(k-1)k}, \sigma_k^2) \quad (1.8a)$$

$$\text{Subject to: } \mu(\mu_1, \mu_2, \ldots, \mu_k; \sigma_1^2, \sigma_{12}, \ldots, \sigma_{(k-1)k}, \sigma_k^2) = x_0 \quad (1.8b)$$

where $\mu$, $\sigma^2$, and $x_0$ are the mean, variance, and ideal value of the main product characteristic, respectively. Also, $\mu_1, \mu_2, \ldots, \mu_k$ and $\sigma_1^2, \sigma_{12}, \ldots, \sigma_{(k-1)k}, \sigma_k^2$ are the means and variance/covariance terms of the components characteristics. Here, $\mu_1, \mu_2, \ldots, \mu_k$ are the decision variables, whereas $\sigma_1^2, \sigma_{12}, \ldots, \sigma_{(k-1)k}, \sigma_k^2$ are given. This optimization model is referred to as Robust Parameter Design RPD (Chandra, 2001).

The case where the ideal value $x_0$ is in the midpoint of the design tolerance, which again is the difference between USL and LSL, is called a balanced design tolerance. Consequently, an unbalanced design tolerance is the case where the ideal value $x_0$ is not in the midpoint of the design tolerance. In addition, the quadratic loss function in its general form is established for an N-type quality characteristic as shown in equation (1.6) with symmetric losses at the LSL and USL, i.e. the scrap and rework costs are equal for $X \leq LSL$ and $X \geq USL$. However, this has been extended to the case of asymmetrical losses at the LSL and USL for an N-type quality characteristic as follows (Chandra, 2001):

$$L(X) = \begin{cases} k_1(X - x_0)^2, & LSL \leq X \leq x_0 \\ k_2(X - x_0)^2, & x_0 \leq X \leq USL \end{cases} \quad (1.9)$$
where \( k_1 \) and \( k_2 \) are the monetary values of the loss to society for a unit deviation either below or above the ideal value \( x_0 \), respectively. The cases of a smaller-the-better (S-type) and larger-the-better (L-type) quality characteristics are also considered. For these cases, the quadratic loss functions are given, respectively, by (Chandra, 2001):

\[
L(X) = kX^2, \quad X \leq USL \tag{1.10}
\]

and

\[
L(X) = \frac{k}{X^2}, \quad X \geq LSL \tag{1.11}
\]

The main limitation of the quadratic loss function is that it tries to lump all costs associated with the deviation of the product from its ideal value into a single vaguely defined parameter. The proportionality constant \( k \) of the quadratic loss function is supposed to capture the loss to society as a whole due to the imperfection of the product in monetary value. This loss could mean the liability and warranty costs incurred by the manufacturer, the inconvenience and waste of time that the customer is faced with when having to fix or replace the product, loss of future sales and the decline in the market share of the firm, any environmental or unforeseen problems, lawsuits, etc. (Hassan, 2009). As it can be seen, even if it was possible to list all possible losses to society, similar to those mentioned and others, putting a dollar value on each of these losses is practically impossible in most cases. Moreover, these losses tend to be related to many factors that could be interdependent. Thus, lumping all these losses into a single parameter \( k \) is too simplistic and restrictive.

To Dr. Taguchi’s credit, however, it is mentioned in Taguchi et al. (1990) that the quadratic loss function (QLF) “is a simple approximation, to be sure, not a law of
nature”. The authors continue to say that “actual field data cannot be expected to vindicate QLF precisely, and if your corporation has a more exacting way of tracking the costs of product failure, use it” (Taguchi et al., 1990).

Despite the limitations of the quadratic loss function, it is still widely used in the literature perhaps for no reason other than not having a practical replacement.

It is worth mentioning that the quadratic loss function focuses on conformance quality, whereas many microeconomic and marketing studies focus on performance quality (Karmarkar et al., 1997). Thus, in the microeconomic and marketing literatures, quality levels are commonly defined based on ordinal scales. For instance, the quality levels are defined as grades or classes, i.e., grade A, grade B; class 1, class 2; etc. (Teng et al., 1996), or other more descriptive classifications, such as worse than average, average, better than average, etc. (Menezes et al., 1992). Then, quality losses are estimated in relation to the defined quality levels.

1.2.3 Warranty Models

Generally, a warranty can be defined as “a contract or an agreement under which the manufacturer of a product or service must agree to repair, replace, or provide service when the product fails or the service does not meet the customer’s requirements before a specified time (length of warranty)” (Elsayed, 1996). Thus, depending on the type of the product and the type of service desired by the customer, manufacturers could offer a wide range of warranty policies. To the customer, the warranty policy can serve as a signal for the quality of the product in addition to the obvious purpose of protection against faulty
products. To the manufacturer, the warranty policy can be used as an advertising tool as well as protection against the customer’s misuse of the product by providing clear guidelines and detailed rights and responsibilities for both the customer and the manufacturer (Monga et al., 1998).

Warranty policies could be for a finite period of time or lifetime warranties; they could require the manufacturer to repair, replace, or provide a pro-rated or a lump-sum rebate; or they could be a combination of the simpler warranty policies with specific rules and requirements. Moreover, besides specifying the type of warranty policy to offer to the customer, the manufacturer has to decide the length of the warranty period and the cost of the warranty as well (Elsayed, 1996). Despite the fact that there are various types of warranty models, only one simple warranty model, which is called the minimal repair warranty model, is considered in this research to show how warranty models are implemented in the proposed models. Other warranty models, which can be found in Elsayed (1996) and Blichke et al. (1996), can be implemented in a similar manner. More recent advances on the topic of warranty and its relation to reliability can be found in Murthy (2006).

The minimal repair warranty model considered in this research was first developed by Barlow et al. (1960). In this model, “it is assumed that the failure rate of the product remains unchanged after a repair” (Nguyen et al. (1984)). This is typically the case for a minor repair, which brings the system back to its original condition just before the failure, or the replacement of a component that is a small part of a larger system such that the reliability of the system remains practically unchanged due to the degradation of its other components.
For the minimal repair warranty model, the expected number of failures during the warranty period \([0, w]\) is given by (Nguyen et al. (1984)):

\[
M(w) = \int_{0}^{w} h(\tau) d\tau = -\ln R(w)
\]  

(1.12)

where \(h(\tau)\) is the hazard rate function and \(R(w)\) is the reliability at the end of the warranty period \(w\). Thus, the total warranty costs \(WC\) can be found by:

\[
WC = C_r M(w) = -C_r \ln R(w)
\]

(1.13)

where \(C_r\) is the expected repair cost per failure.

### 1.2.4 Reliability Models

A formal definition of reliability is given by Leemis (2009) and it states that “the reliability of an item is the probability that it will adequately perform its specified purpose for a specified period of time under specified environmental conditions”. Thus, it follows from the definition that the main random variable of traditional reliability models is time to failure \(T\). However, other random variables or parameters can also be included in the reliability models resulting in reliability models with covariates. The studies by Deleveaux (1997), Blue (2001), Zhang (2006) and Hassan (2009) try to relate the mean and variance of the main quality characteristic of the product to warranty costs through the use of reliability models with covariates. The reliability models that are considered in this research are the ones developed by Blue (2001) and Hassan (2009), which are based
on the Weibull lifetime distribution. The conditional reliability for a product with a single main N-type quality characteristic is given by (Blue, 2001):

\[
R_x(\tau | x) = \exp \left[ -\left( a + b(x - x_0)^2 \right) \tau^x \right]
\]  
(1.14)

where \( a \), \( b \), and \( c \) are model parameters, \( x \) and \( x_0 \) are the observed and ideal values of the product characteristic, respectively, and \( \tau \) is the time to failure. It is to be noted that the model parameters, i.e. \( a \), \( b \), and \( c \), are all nonnegative. The parameter \( a \), when \( a > 0 \), captures the rate at which the reliability of an item with an observed product characteristic that is equal to the ideal value, i.e. \( x = x_0 \), decreases over time. However, when \( a = 0 \) and \( x = x_0 \), the reliability is equal to one at all times, which means that the item never fails. This situation is very unlikely especially for consumer products, where used items tend to be less reliable than new items. Thus, the parameter \( a \) provides the flexibility in the model to capture the wear of an item over time, even if its quality characteristic equals the ideal value. In contrast, the quadratic loss function assumes the external failure costs of an item with a product characteristic that is equal to its ideal value is zero. The parameter \( b \) captures the intensity at which the reliability is penalized for deviating from the ideal value. Lastly, the parameter \( c \) is a scaling parameter for the time \( \tau \) used in the model.

Analogous to the asymmetrical loss function case of an N-type quality characteristic (equation (1.9)), if experience or life testing shows that the reliability is affected differently when the value of the product characteristic is greater than its ideal value \( x_0 \) compared to when it is smaller than \( x_0 \), then an asymmetric conditional
reliability function can be used. That is, for an N-type product characteristic, the asymmetric conditional reliability function is given by:

\[
R_N(\tau | x) = \begin{cases} 
\exp\left( -\left( a_1 + b_1 (x-x_0) \right) \tau^c \right), & x \leq x_0 \\
\exp\left( -\left( a_2 + b_2 (x-x_0) \right) \tau^c \right), & x > x_0 
\end{cases}
\]  
(1.15)

where \(a_1, b_1, c_1, a_2, b_2, \) and \(c_2\) are model parameters, \(x\) and \(x_0\) are the observed and ideal values of the product characteristic, respectively, and \(\tau\) is the time to failure. Moreover, similar to equation (1.14), the conditional reliabilities for S-type and L-type product characteristics, respectively, are given by (Hassan, 2009):

\[
R_s(\tau | x) = \exp\left( -(a + bx^2) \tau^c \right)
\]  
(1.16)

and

\[
R_L(\tau | x) = \exp\left( -\left( a + \frac{b}{x^2} \right) \tau^c \right)
\]  
(1.17)

The unconditional reliabilities for normally distributed N-type and S-type product characteristics, respectively, are given by (Blue, 2001; Hassan, 2009):

\[
R_N(\tau) = \frac{1}{\sqrt{1 + 2b\sigma^2 \tau^c}} \exp\left[ -\left( a + \frac{b(\mu-x_0)^2}{1 + 2b\sigma^2 \tau^c} \right) \tau^c \right]
\]  
(1.18)

and

\[
R_s(\tau) = \frac{1}{\sqrt{1 + 2b\sigma^2 \tau^c}} \exp\left[ -\left( a + \frac{b\mu^2}{1 + 2b\sigma^2 \tau^c} \right) \tau^c \right]
\]  
(1.19)
where $\mu$ and $\sigma^2$ are the mean and variance of the product characteristic $X$. As for the normally distributed L-type product characteristic $X$, the unconditional reliability can be approximated by:

$$R_e(\tau) \approx \frac{1}{\sqrt{1 + 2b \left( \frac{\sigma^2}{\mu^2} \right) \tau^c}} \exp \left[ - \left( a + \frac{b \left( \frac{1}{\mu} \right)^2}{1 + 2b \left( \frac{\sigma^2}{\mu^4} \right) \tau^c} \right) \tau^c \right]$$

(1.20)

The approximation in equation (1.20) is fairly accurate ($\leq 5\%$ error) for the case when the ratio $\mu/\sigma$ is large ($\mu/\sigma \geq 10$) (Hassan, 2009). This result is evident from a Taylor series expansion of $1/X$ around $\mu$, which is given by:

$$\frac{1}{X} = \frac{1}{\mu} - \frac{1}{\mu^2} (X - \mu) + \frac{1}{\mu^3} (X - \mu)^2 + \text{Remainder}$$

(1.21)

where the second- and higher-order terms in equation (1.21) can often be neglected when the ratio $\mu/\sigma$ is large. For example, the second-order term in equation (1.21) is simply a chi-squared random variable with one degree of freedom divided by $\mu(\mu/\sigma)^2$. This means that $1/X$ can be approximated with a first-order Taylor series expansion, which results in the distribution of $1/X$ to be approximately normal with mean $1/\mu$ and variance $\sigma^2/\mu^4$.

1.2.5 Customer Utility and Demand Functions

The level of customer satisfaction with respect to product attributes and price can be captured by the customer’s utility function, where customer utility simply means
customer satisfaction. Even though customer utility can be a highly subjective concept and differs from one customer to another, customer utility functions can still be useful in capturing customer satisfaction trends and reactions to variations in product price and attributes (Miller, 2006). One of the major phenomena that the concept of customer utility is based upon is the law of diminishing marginal utility, where marginal utility refers to the satisfaction a customer achieves from consuming an additional unit of the product or commodity. The law of diminishing marginal utility states that “as an individual consumes more of a particular commodity, the total level of utility or satisfaction derived from that consumption usually increases. Eventually, however, the rate at which it increases diminishes as more is consumed” (Miller, 2006).

The key idea that is relevant to this research is that customer utility functions can be used as a measure of customer satisfaction relative to variations in product price and quality. An intuitive relationship between customer satisfaction, on one hand, and product price and quality, on the other, is that as the product price decreases and its quality increases, customer satisfaction increases. In addition, the idea presented by the law of diminishing marginal utility suggests that as the quality increases, the rate at which customer satisfaction increases drops and, at some point, the customer would be willing to pay less and less for further increases in quality. For instance, it is logical to say that increasing the reliability of a refrigerator such that its useful life is improved from 7 to 17 years is more satisfying to the customer than improving the useful life from 25 to 35 years or, perhaps less realistically, from 40 to 50 years, even though the improvement in useful life is ten years for all cases.
Alternatively, demand functions can be used to serve a similar purpose as the customer utility functions. Since demand is directly affected by customer preferences and utility, demand functions can be related to product price and quality to represent customer and market behaviors. In this research, both customer utility and demand functions are used to model the effects of price and quality on customer or market behaviors.

It is worth mentioning that, in elementary microeconomics, the relationship between price and quantity demanded is commonly represented by a demand curve. When there is a change in price only, the quantity demanded changes based on the demand curve, i.e. moving along the demand curve. However, when there is a change in factors other than price, such as customer preferences, product quality, market size, etc., the whole demand curve shifts (Miller, 2006). Demand curves are nothing but graphical representations of demand functions.

1.2.6 Market Structures

In economic theory, there are several types of market structures. A monopoly, a monopolistic competition, an oligopoly, and a perfect competition are examples of the types of market structures. Pure monopoly and perfect competition market structures are idealistic cases and fairly rare in real life. That is why “the vast majority of firms in modern economies are either oligopolistic firms or monopolistic competitors” (Friedman, 1983). However, for completeness, the four types of market structure and their characteristics are discussed briefly in this section. It is important to appreciate the complex relationships among product price, demand, and quality as well as other factors
and dynamics in a given market structure in addition to the discrepancies across the different types of market structures. Market structures and their characteristics can be found in almost any microeconomic book (e.g. Miller, 2006; Boyes et al., 2011; etc.).

1.2.6.1 Perfect Competition

The perfect competition market structure has the following characteristics:

1. The numbers of producers and buyers are large.
2. The products are identical (homogenous), i.e. the quality is set.
3. Entering and exiting the market is easy.
4. Producers are price takers, i.e. the price is set.
5. The demand curve is horizontal, i.e. perfect demand elasticity.
6. Advertisement is not used, in general, since there is no product differentiation.
7. Any quantity produced can be sold at the market price.

In perfect competition, the price and quality level of the product are set by the market since individual producers constitute a very small portion of the market such that their individual effects on market price and quality level are negligible. Thus, the task for the producer is to determine the quantity of product to produce while minimizing cost, subject to achieving a product that has a quality level that is commonly accepted or expected in the market. This decision about the quantity of the product is made independently from other producers’ decisions since the number of producers in the
market is large. Here, it is assumed that the customer has perfect knowledge and clear expectation of the quality of the product that is to be purchased at the market set price. Again, it is unlikely to find a market in real life that possesses all the characteristics of a perfect competition market. However, well-established products, such as wheat and milk, could be examples of a market that is close to a perfect competition.

1.2.6.2 Monopoly

The monopoly market structure has the following characteristics:

1. There is a single producer.
2. The product has no close substitute.
3. Entering the market is extremely difficult.
4. The demand curve is downward sloping but not perfectly inelastic.

A monopoly can be established and maintained when there is a huge barrier of entry in the market by competitors. Barriers of entry include government regulations, patents, economies of scale, high initial fixed costs, ownership of a rare resource, etc. (Miller, 2006). Unlike a perfect competitor, whose main decision is the optimal output level, a monopolist needs to simultaneously determine both the optimal selling price and the optimal output level in order to maximize profit.
1.2.6.3 Monopolistic Competition

The monopolistic competition market structure has the following characteristics:

1. The numbers of producers and buyers are large.
2. The products offered in the market are differentiated (heterogeneous) but they are close substitutes of each other.
3. Entering and exiting the market is relatively easy.
4. The demand curve is downward sloping but it is not as steep as that in a monopoly.
5. Advertisement is used since there is product differentiation.

Since the demand curve is downward sloping, a monopolistically competitive firm has some control over pricing unlike a perfectly competitive firm, which is a price taker. However, this control over pricing does not reach the level of control enjoyed by a monopolist. Moreover, similar to the perfect competition market structure, since there are many producers in the market and entry into the market is relatively easy, decisions about pricing and output level are made independently from other producers’ decisions and reactions.

1.2.6.4 Oligopoly

The oligopoly market structure has the following characteristics:

1. The number of producers is small but the number of buyers is large.
2. Producers’ price, quantity and quality decisions in the market are interdependent.

3. The products offered in the market could be identical or differentiated.

4. Entering and exiting the market is fairly difficult.

5. The demand curve is downward sloping and it can exhibit a discontinuity (kinked demand curve).

6. Advertisement is used in the case of differentiated products.

In this market structure, since the whole market is composed of a few competitive producers, decisions about product price, quality, quantity, etc., are made by each producer while considering not only what is best for their firms, but also how the other competitors are likely to react. Thus, game theory is used extensively in the modeling and analysis of oligopolistic market structures. Moreover, reaction functions are employed by each producer to model the strategies of the competitors in response to their actions.

One of the key characteristics of oligopolistic markets is that the dynamics, rules, and interactions among the producers in these markets could be very different from one case to another (Kuenne, 1998). As a result, the models used in the analysis of oligopolistic markets tend to be highly dependent on the assumptions and scenarios considered in these markets. For instance, Banker et al. (1998) developed a model that is based on a duopoly competition, i.e. an oligopoly with two competitors, with each competitor offering a single product. The scenario laid out for the analysis starts with the two competitors simultaneously specifying the quality levels of their products. The two competitors then observe each other’s chosen quality levels. After that, based on the
quality levels observed, the two competitors set their prices. Finally, based on the chosen prices and quality levels, the quantities demanded from each competitor are determined using the demand function. According to this specific scenario, a model is developed and analyzed. The model and solution changes considerably if a completely different scenario is considered. Identical versus differentiated products, single period versus multi-period analysis, cooperative versus non-cooperative competition, price versus output quantity as decision variable, etc., are all examples of market considerations that could entirely change the analysis and solution of the oligopoly competition problem. In spite of that, there is still a common classification of models that are used in the analysis of oligopolistic market structures (Puu et al., 2002). For instance, the model developed by Banker et al. (1998) is considered a Bertrand oligopoly since the decision variable is the selling price, which was first proposed by Bertrand (1883). Similar classifications include Cournot, Hotelling, Chamberlin, and Stackelberg oligopolies. Cournot (1838) was the first to analyze and discuss oligopoly market structures. His model assumed the competitors produced identical products and the decision variable was the production quantity. Bertrand (1883) criticized the model by Cournot (1838) and changed the decision variable to the selling price instead of the production quantity. Hotelling (1929) considered a duopoly, which was similar to the model by Bertrand (1883) with identical products and selling price as the decision variable. However, Hotelling (1929) also included transportation costs. Thus, in the model, each producer enjoys a local monopoly whereas competition arises at the boundaries where the two producers’ sum of price and transportation cost are equal. A producer can capture more of the market share by lowering the product price. Chamberlin (1932) considered product differentiation, i.e.
heterogeneous but close substitutes products, as opposed to identical products. It is assumed in this model that customers have their preferences among the products in the market and only shift to another product if their preferred product becomes too expensive compared to the substitute product. Stackelberg (1934) considered the dynamic case where producers know the reaction functions of each other, i.e. observe and then react to each other over a multi-period time horizon.

Again, besides the classification that is based on the basic characteristics of the few classical oligopolistic models, i.e. Cournot oligopoly, Bertrand oligopoly, etc., there is no single model or framework that can always be used in the analysis of oligopolistic market structures. Instead, models are developed for each specific case based on the assumptions and dynamics of the oligopoly problem at hand.

1.3 Research Objectives

The overall objective of the research in this dissertation is developing new comprehensive models for the integration of parameter and tolerance designs that are easily implementable in practice and capture real phenomena related to both manufacturing and marketing operations of the product. Moreover, the focus is on replacing vague concepts, such as the loss to society in the quadratic loss function, and replacing them with variables based on empirically measurable data, such as those related to customer satisfaction and behavior, warranty policies, product reliability, process selection and robustness, etc. Thus, in this dissertation, two sets of integrated parameter and tolerances designs models are proposed. First, integrated designs models are
developed with the objective of minimizing manufacturing and warranty costs. For these models, it is assumed that all functional requirements and specifications are predetermined and fixed. Second, integrated designs models are developed with the objective of maximizing the profit. The profit is the difference between the selling price and the total cost of manufacturing and warranty costs. Pricing theory and customers behavior (captured by customer’s utility or demand functions) along with the ideas developed in the first set of models, i.e. integrated parameter and tolerance designs without microeconomic and marketing considerations, are used in this second set of models. Furthermore, the procedure of implementing and solving the integrated designs models under different conditions and assumptions is illustrated through several examples. The first set of models considers the cases of linear and nonlinear relationships between the main quality characteristic of the product and the quality characteristics of its components as well as the case of finite manufacturing processes. The second set of models, on the other hand, considers static market analysis approach for the cases of a general demand function, a monopoly, and a duopoly competition as well as the case of a dynamic market analysis approach. Sensitivity analysis is conducted on some of the critical parameters in the different models both to verify the correct behavior of the models and to examine the robustness of the models against variations in their parameters.
1.4 Dissertation Outline

The literature review for each of the two sets of proposed models, i.e. the integrated parameter and tolerance designs without and with microeconomic and marketing considerations, is presented in Chapter 2. Then, model development and numerical examples are presented for the two sets of proposed models in Chapters 3 and 4, respectively. Finally, Chapter 5 presents the research contributions and suggestions for future research.
Chapter 2

Literature Review

In this chapter, a detailed survey of a number of studies, which are relevant to this research, is presented. First, the studies that focus on parameter and tolerance designs models, which do not involve microeconomic and marketing considerations, such as market structure types, product pricing and demand models, etc., are discussed in section 2.1. Then, the studies that focus on microeconomic and marketing considerations and their connection to product design and development are discussed in section 2.2.

2.1 Integrated Parameter and Tolerance Designs without Microeconomic and Marketing Considerations

The economic parameter and tolerance designs and their integration based on manufacturing and quality costs considerations have been studied by many researchers (e.g., Jeang et al., 2002a; Jeang, 2003; Parkinson, 2000; Kim et al., 2000a and 2000b; etc.). In the following, a detailed literature review of some of the studies in this area is presented.

Zhang et al. (2010) consider the problem of incorporating manufacturing costs into Robust Parameter Design (RPD). A product having \( N \) main quality characteristics (functional requirements) is considered, where each of the quality characteristics is a function of \( k_j \) components quality characteristics (\( j = 1, 2, ..., N \)). The mathematical model of the RPD is essentially a minimization problem of the sum of the variances of
the main product characteristics with the means of these main characteristics set equal to their respective ideal values. This RPD model is similar to that considered in equation (1.8). However, the major difference is that a constraint is added to restrict manufacturing costs, which is a function of the natural process tolerances of the components characteristics. These tolerances, in turn, are incorporated into the variance equation using six-sigma design, i.e.  \( \sigma_i^2 = \frac{t_i^2}{36} \), where \( \sigma_i^2 \) and \( t_i \) are the variance and the natural process tolerance of component \( i \), respectively, for \( i = 1, 2, \ldots, k_j \) and \( j = 1, 2, \ldots, N \). Thus, the RPD model determines the means and the natural process tolerances of the components characteristics that minimize the variance, achieve zero-bias, are within tolerance limits, and do not exceed a given maximum manufacturing costs for each of the main product characteristics. The main issue with this model is that it requires determining the maximum allowable manufacturing costs. This approach could be suboptimal since it is possible that very little improvement might be added with too much manufacturing costs as long as these costs are less than the maximum allowable manufacturing costs.

Wu et al. (1998) propose a systematic procedure for allocating tolerances to the quality characteristics of the product components based on the minimization of manufacturing costs and quality losses. The model considers both symmetric and asymmetric quality losses. Also, an assumption is made that the main quality characteristic of the product is set at its ideal value. The tolerance of the main product characteristic \( t_r \) is related to the tolerances of the components characteristics \( t_i \), for \( i = 1, \ldots, n \), using two methods, namely, the worst-case method:
\[
 t_r = \sum_i \left[ \frac{\partial G}{\partial X_i} t_i \right]
\]

(2.1)

and the statistical (probabilistic) method:

\[
 t_r = \left( \sum_i \left[ \left( \frac{\partial G}{\partial X_i} \right)^2 t_i^2 \right] \right)^{0.5}
\]

(2.2)

where \( G \) is the function that relates the main product characteristic to the components characteristics \( X_i \), for \( i = 1, \ldots, n \). The manufacturing costs are directly related to the tolerances using data that are fitted to two models in functional form, i.e. the reciprocal function:

\[
 C(t) = a_1 + a_2 e^{-a_3 t}
\]

(2.3)

and the exponential function:

\[
 C(t) = a_4 + a_5 e^{-a_6 t}
\]

(2.4)

where \( a_1, a_2, a_3, a_4, a_5, \) and \( a_6 \) are model parameters. The quality losses, represented by the quadratic loss function, are related to the tolerances using the relationship between the standard deviation and the tolerance. For instance, for a normally distributed product characteristic \( X \), \( t = 3\sigma \), which covers 99.7% of the distribution of \( X \). Finally, other constraints are added that restrict the maximum allowable tolerance of the main product characteristic based on functional requirements and impose minimum limits on the tolerances of the components characteristics based on the manufacturing process capabilities. The main limitation of the proposed model is the use of the quadratic loss function to estimate quality losses. Again, the vagueness of the proportionality constant in the quadratic loss function compromises the usefulness of the whole model since it has
a profound impact on the final optimal solution yet it is, at best, just a very rough estimate of the quality costs. Also, the model should be modified if the zero-bias assumption were to be relaxed.

Jeang (1999) considers the problem of finding the optimal tolerances of the components quality characteristics of a product by minimizing the total cost $TC$, which is calculated as the sum of manufacturing costs and quality losses. The manufacturing costs term is assumed to be a function of only the tolerances of the components characteristics. The quality losses term, which is again modeled by the quadratic loss function, is also transformed into a function of only the tolerances of the components characteristics. This is done by assuming that the mean of the main quality characteristic of the product can be set equal to its ideal value so that the bias term in the expected quadratic loss function vanishes. The model uses $\sigma_y^2 = \frac{t_y^2}{9C_{pm}}$ for the variance of the main product characteristic, where $C_{pm}$ is the process capability index and $t_y = \sum_{i=1}^{n} t_i$ is the worst-case scenario tolerance for the main product characteristic. Thus, given a set of tolerances of the components characteristics, the total cost is calculated. Then, using Response Surface Methodology (RSM), a relationship is found between the response, which is the total cost $TC$, and the tolerances of the components characteristics, i.e. $TC = f(t_1, t_2, ..., t_n)$, such that this function is directly optimized to find the set of tolerances for the components characteristics that result in the minimum $TC$. Some of the limitations of this study include the model’s restriction to zero-bias and balanced tolerances, which are not necessarily optimal based on cost considerations. Also, the use
of the quadratic loss function and having to estimate its proportionality constant is yet another limitation.

Zhang et al. (2008) use the dual response surface methodology to relate the mean and standard deviation of the main quality characteristic of a product to the tolerances of the quality characteristics of the product components. Then, the optimal tolerances are obtained by minimizing the total cost, which again is the sum of manufacturing costs and quality losses. Manufacturing costs are related to the tolerances using discrete data without fitting these data to a continuous function. Quality losses are modeled using the quadratic loss function, which is the main drawback of the model. The study also considers two cases with respect to the mean setting of the main product characteristic. First, the zero-bias case is considered, which is handled in the model by adding a constraint that requires the mean of the product characteristic to be equal to its ideal value. The other case is when the mean is allowed to be set at a value that is different from the ideal value. In this situation, a constraint is added that limits the mean deviation from the ideal value with a specified maximum allowable bias value.

Li et al. (1999) also propose the integration of parameter and tolerance designs. Using a detailed example from the field of chemical engineering, a comparison is made between the sequential approach of parameter and tolerance designs as advocated by Dr. Taguchi and three different integration methodologies of parameter and tolerance designs. The three integration methodologies are essentially three different approaches for the evaluation of the quality losses term by relating the mean and variance of the main quality characteristic of the product to the means and variances of the quality characteristics of its components. The first method is the use of Taylor series expansion,
which is only viable when there is a known functional relationship between the main product characteristic and the components characteristics. When this functional relationship is unknown, the second methodology is used, which is based on Monte Carlo simulation. In this approach, a number of random realizations of the components characteristics are created and are directly used to evaluate the quality losses. If the system is too large and prohibitively costly to use the Monte Carlo simulation methodology, the third methodology is used, which is the orthogonal-array (OA) Monte Carlo simulation. In this method, design of experiments is used to reduce the number of simulation runs required to a manageable size and obtain an approximate solution with reasonable accuracy. The main limitation of these methodologies is again the use of the quadratic loss function to estimate quality losses. Moreover, even though the authors claim that no functional relationship is required between the main product characteristic and the components characteristics when using the two simulation methodologies, the function that relates the main product characteristic and the components characteristics still appears in the formulation of these two methods.

Jeang (2001) combines quality losses and manufacturing costs in the simultaneous parameter and tolerance designs. This paper makes the case that the strategy of aiming for zero-bias and minimum variability might not be optimal when asymmetric quality losses and unbalanced tolerances are considered in the optimization model that minimizes the total costs. Here, the total costs consist of three main components. The first component is the manufacturing costs, which are assumed to be independent of the mean of the product characteristic and only affected by its natural process tolerance. The second component is the quality losses (external failure costs), which are estimated using
the quadratic loss function. The third component is the internal failure costs, which consist of scrap and rework costs and depend on both the mean and the natural process tolerance of the product characteristic. One unique feature of the model in this study is that it considers the allowable design tolerance as a decision variable unlike all other studies surveyed, which assume the allowable design tolerance to be given. However, this study unlike conventional parameter design, considers only a single quality characteristic of the product without it being a function of different components characteristics. The goal of the model is to determine the optimal mean, the natural process tolerance, and design tolerances of the single product characteristic such that the total costs are minimized. Thus, the models developed are relatively limited, despite considering the three types of quality characteristics, namely, the nominal-the-best (N-type), the larger-the-better (L-type), and the smaller-the-better (S-type) quality characteristics. Another limitation of the model is that the quality losses are still very simplistic and involve the use of the quadratic loss function even though the paper develops very detailed manufacturing costs models based on asymmetric scrap and rework costs as well as material and inspection costs.

Cho et al. (2000) provide a comprehensive model for integrated parameter and tolerance designs. First, response surface methodology is used to find a relationship between the mean and variance of the main quality characteristic of the product in terms of the controllable factors. Based on that, closed-form solutions of optimal tolerances for the main product characteristic are found by minimizing the total costs of manufacturing and product quality. One of the key features of the model is that zero-bias is not required in the solution but rather the objective is to simultaneously optimize bias and variability
such that the minimum total costs are achieved. Another feature of the model is the use of asymmetric quality losses where the cost of rework is different from the cost of scrap. This allows for considering unbalanced tolerances. One of the limitations of the model is the fact that the model does not consider individual tolerances of the controllable factors as decision variables but rather assumes that they all have predetermined values. Thus, it is possible that better solutions could be achieved if the tolerances of the controllable factors were allowed to vary while constraining the tolerances of the main product characteristic with maximum allowable values. The use of the quadratic loss function is yet another limitation of the model. However, the authors defend the use of the quadratic loss function by saying that its use is justified in situations where “there is little information known about the functional relationship between quality and cost, or where there is no direct evidence to refute a quadratic representation” (Cho et al., 2000).

Feng et al. (1997) propose a stochastic integer programming model for the selection of the natural process tolerances and manufacturing processes that minimize manufacturing costs. One of the main issues raised in this study is the use of cost functions by other researchers for relating manufacturing costs to tolerances, which treat tolerances as continuous variables and require the estimation of model parameters. The problem with these functions is that the uncertainty in the model parameters and the regression errors cannot be avoided. Thus, it is suggested instead to treat tolerances as discrete variables based on available manufacturing processes. This, however, requires knowing and listing all available processes and their variances and costs, which could be cumbersome. One of the limitations of the proposed model is ignoring the quality costs in the optimization model’s objective function. This is done in order to avoid having to
estimate the proportionality constant of the quadratic loss function used by the majority of the studies in this area. As a result, the optimal trade-off between manufacturing costs and quality losses cannot be captured by the model. Instead, the model finds the tolerances that achieve the minimum manufacturing costs based on several constraints including one that caps the overall variance of the main product characteristic.

As it can be seen from the literature review of this section, the main idea of the integrated parameter and tolerance designs is to determine the optimal means and tolerances of the quality characteristics of product components that minimize the total costs by specifying the best trade-off between manufacturing costs and quality losses. Again, the reason for the integrated approach is that if the effects of the means and tolerances on the total costs are not independent, which is most likely to be the case in real life applications, then, sequential parameter and tolerance designs would almost always result in a suboptimal design (Li et al., 1999). This is because any modification in the tolerances in the second step (tolerance design) could result in losing the optimality of the means achieved in the first step (parameter design).

For achieving the minimum total costs, the dilemma is that tight tolerances on the quality characteristics of product components tend to increase manufacturing costs while decrease quality losses, whereas loose tolerances have the opposite effect on manufacturing costs and quality losses. Moreover, the choice of the means of the components characteristics has a profound effect on quality losses since they influence both the variance and the amount of deviation of the mean of the main product characteristic from its ideal value, which in turn could indirectly affect the manufacturing costs as well. These influences that the means and the tolerances of the components
characteristics have on the total costs are what the integrated models try to capture and optimize.

The main limitation of previous works in this area is the use of the quadratic loss function to model quality losses. Estimating the proportionality constant of the quadratic loss function is practically impossible due to the vagueness of its definition. Another major limitation of many previous studies is defining manufacturing costs as a function of the natural process tolerances alone, while ignoring internal failure costs incurred due to the mean settings, which could be significant. To overcome this problem, these studies impose a constraint in their models to limit the stack-up tolerance of the main product characteristic, which essentially eliminates internal failure costs if the process capability index is high enough. Even with the use of this approach, almost all studies surveyed fail to incorporate the process capability index properly in their models.

2.2 Integrated Parameter and Tolerance Designs with Microeconomic and Marketing Considerations

The integration of microeconomic and marketing considerations with product design and manufacturing has been studied by many researchers (e.g., Banker et al., 1998; Monga et al., 1998; Whitnack et al., 2008; Wu et al., 2009; etc.). In the following, a detailed literature review of some of the studies in this area is presented.

Bagajewicz (2005) suggests that product design should not be conducted without taking financial considerations into account. That is, advertising, pricing and consumer behavior along with financial risks and market types should be considered in the initial stages of product design and optimization. Ultimately, profit needs to be maximized
while taking into consideration all the stages of the product’s lifecycle as well as all the processes and operations associated with it. The logic behind this idea is that product design sets the product’s attributes, such as composition, structure, functionality, etc. However, each combination of product attributes has certain costs associated with it, which could be different from any other combination. This requires the selling price of the product to be greater than all the costs if any profit were to be achieved. Selling price in turn greatly influences the demand for the product, which is ultimately related to the customer’s satisfaction and preferences. Then again, customer preferences and expectations are the basis for product design. This logic is depicted in Figure 2.1, which shows the decisions and influences that affect product design.

![Figure 2.1: Decisions and influences affecting product design.](image)
Following this logic, it can be seen that product design is directly related to the market. However, too often, the product is developed to achieve highest possible quality and then it is left to the marketing team to find a market and optimal price for the product. Thus, the key idea here is that the best product in terms of quality is not necessarily the most profitable and in order to maximize profit, an integrated framework must be considered in which marketing and pricing decisions are involved in the initial stages of product design. The author also presents an empirical study in which results show that there tends to be an optimal combination of price and quality levels in a given market. This is as opposed to the two extreme cases of cheap with undesirably low quality product and high quality product but prohibitively expensive, where both result in losses to the manufacturer. The mathematical model for this study and its detailed derivations are better presented in Bagajewicz (2007).

Bagajewicz (2007) presents a product design model for the case where there exists a product that has a well-established market. The task is to design a new more profitable product that substitutes the existing product in the market. The new product composition and selling price are the decision variables and maximizing the net present value is the objective of the optimization model. The net present value is the difference between revenues and total costs. The revenues term is defined as the discounted value of the product of demand and the selling price. The total costs term is defined as the discounted value of the sum of manufacturing costs and fixed costs. In order to find the demand for the new product, a pricing model that relates demand to price and product composition is developed. This is achieved by solving a maximization problem where the demands for the existing and new product are to be specified such that the customer’s
utility is maximized, subject to a constraint that is related to the customer’s budget limitations. This optimization model is given by (Bagajewicz, 2007):

\[
\begin{align*}
\text{Maximize} & \quad u(d_1, d_2) = \left[ (\alpha d_1)^\rho + (\beta d_2)^\rho \right]^{1/\rho} \\
\text{Subject to:} & \quad p_1 d_1 + p_2 d_2 \leq Y
\end{align*}
\] (2.5a)

(2.5b)

where \(d_1\) and \(d_2\) are the demands for the new and existing products, respectively, \(\alpha\) is a measure of consumer knowledge about the relative quality of the new and existing products, \(\beta\) is a measure of consumer preference of existing product over new product, \(\rho\) is a model parameter, \(p_1\) and \(p_2\) are the prices for the new and existing products, respectively, \(Y\) is the consumer budget limitation. The solution to this maximization problem is found to be (Bagajewicz, 2007):

\[
p_1 d_1 - \left( \frac{\alpha}{\beta} \right)^\rho p_2 \left[ \frac{Y - p_1 d_1}{p_2} \right]^{1-\rho} d_1^\rho = 0
\] (2.6)

Moreover, the way the consumer preferences are captured using \(\beta\) is by first identifying the product properties that are important and desired by the consumer. These product properties are related to product composition using a process such as Quality Function Deployment (QFD). Then, \(\beta\) is defined as (Bagajewicz, 2007):

\[
\beta = \frac{H_2}{H_1} = \frac{\sum_j w_{2j} y_{2j}}{\sum_j w_{1j} y_{1j}}
\] (2.7)

where \(H_1\) and \(H_2\) are the preference functions for the new and existing products, respectively, \(y_{ij}\) is the contribution of property \(j\) to the preference function of product \(i\)
\( y_j \in [0,1] \), \( w_j \) is the weight given to property \( j \) of product \( i \) that determines its relative importance with respect to other properties \( \left( w_j \in [0,1], \sum_j w_j = 1 \right) \). Then, using this model, the new product composition and its selling price are determined such that the net present value is maximized. It is to be noted that the resultant product from this model is the most profitable product and it could be different from the best product in terms of quality. The best product in terms of quality is achieved by minimizing \( \beta \). Limitations of the proposed model include not considering tolerance design and neglecting the type of competition and market structure.

Karmarkar et al. (1997) also try to bridge the gap between manufacturing and marketing by first recognizing that the notion of quality is used differently in manufacturing and marketing literatures. Conformance to specifications is the prevalent definition of quality in manufacturing applications whereas the performance or class of the product is what defines quality in the field of marketing. Then, the basic framework for an integrated product design is suggested. First, the customer’s utility function is defined in terms of both price and quality level. Next, manufacturing costs are defined in terms of the quality level. Then, either profit, which is the difference between selling price and the manufacturing costs, is maximized, for the case of a monopolistic market, subject to the customer’s utility being greater than some minimum value, or customer’s utility is maximized, for the case of perfect competition, subject to achieving some nonnegative profit. The case of an oligopolistic competition is also addressed by considering production quantity in the model. For a product with a single normally
distributed L-type quality characteristic, the quality level is defined in terms of the mean (performance quality proportional to $\mu$) and variance (conformance quality proportional to $\frac{1}{\sigma^2}$) of the product characteristic. Using this representation of the quality level, simple functions for the expected customer utility and manufacturing costs are proposed, which are given by (Karmarkar et al., 1997):

$$U(p, \mu, \sigma) = a\mu - b\sigma^2 - p$$

(2.8)

and

$$C(\mu, \sigma) = c\mu^2 + \frac{d}{\sigma^2}$$

(2.9)

respectively, where $p$ is the selling price, $\mu$ is the mean of the product characteristic, $\sigma^2$ is the variance of the product characteristic, $a$, $b$, $c$ and $d$ are model parameters. With this basic framework, the authors then list and discuss several issues that can be considered that add to the complexity and flexibility of the model. One of these issues is related to quality attributes that are observable by the consumer and how to relate them to the design specifications, which are mostly unobservable by the consumer but essential to the manufacturer. That is, the manufacturer needs to know how to translate or map consumer needs and desires into product specifications, such as mean settings and tolerances, for manufacturing purposes. It is suggested that Quality Function Deployment (QFD) can be used to address this issue. Another issue is customer perception of the product and its quality as oppose to true quality. This matter involves advertising, competition, market uncertainty, etc. Other issues that are discussed by the authors are different market types and competitions as well as heterogeneous customers and the
tradeoffs that need to be made in these different situations. One limitation of the models proposed in this study is the simplistic form of the manufacturing costs function, even though the manufacturing costs are related to the mean and variance of the product characteristic. Another limitation is that, since the focus is mainly on microeconomic and marketing issues, such as different market structures, customer homogeneity, customer preference uncertainty, etc., the models neither involve warranty costs nor parameter and tolerance designs explicitly. As a result, for determining the optimal mean and variance of the product characteristic, the tradeoff is only between manufacturing costs and selling price, both of which increase (decrease) as the quality level of the product increases (decreases).

Teng et al. (1996) study optimal price and quality decisions for a new product with a dynamic market analysis approach. An optimization model is considered where the product selling price and quality level are the decision variables and the objective is to maximize the total present value of profit $\pi$ over the planning time horizon $L$ as follows (Teng et al., 1996):

$$\text{Maximize} \quad \pi = \int_{0}^{L} e^{-\delta \tau} \left[ p(\tau) - C(q(\tau), Q(\tau)) \right] \hat{Q}(p(\tau), q(\tau), Q(\tau)) d\tau \quad (2.10)$$

where $\delta$ is the discount rate, $\tau$ is time, $p(\tau)$ is the selling price at time $\tau$, $C(\cdot)$ is the manufacturing cost per item, $q(\tau)$ is the quality level at time $\tau$, $Q(\tau)$ is the cumulative sales at time $\tau$, and $\hat{Q}(\cdot)$ is the sales rate, which is analogous to the customer demand function and it is formulated such that the sales rate increases as the selling price
decreases and the quality level increases. Examples of the sales rate function \( \dot{Q}(\cdot) \) used in the study include (Teng et al., 1996):

\[
\dot{Q}(p(\tau), q(\tau)) = a_0 - a_1 p + a_2 q + a_3 \frac{q}{p},
\]

(2.11)

\[
\dot{Q}(p(\tau), q(\tau)) = b - \frac{p}{q},
\]

(2.12)

and

\[
\dot{Q}(p(\tau), q(\tau), Q(\tau)) = p[(e_1 + e_2 q)(M - Q) + (e_3 + e_4 q)(M - Q) Q],
\]

(2.13)

where \( p, q, \) and \( Q \) are the product’s selling price, quality level, and cumulative sales, respectively; \( a_0, a_1, a_2, a_3, b, e_1, e_2, e_3, \) and \( e_4 \) are model parameters; “\( M \) is the total number of potential purchasers over the life cycle of the product” (Teng et al., 1996).

When solving the optimization problem using the proposed formulation, not only does the solution provide the optimal price and quality level, but it also provides the optimal strategy of how price and quality level should vary during the predetermined planning horizon \( L \). Some of the interesting results presented are that sometimes price and quality level should be set low (high) initially and then both be raised (lowered) simultaneously and some other times price should be lowered and quality level should be raised continuously. These different situations depend on conditions that are related to the type of product, type of competition, time horizon, product demand, etc. Even though the model includes learning effects due to experience and captures the influences of demand diffusion and saturation, the quality of the product is not well defined and it is only represented by a random variable called quality level. It is assumed that the customer can
easily identify the quality level of the product. However, it is not shown how the quality is measured and what the underlying decisions are for achieving a certain level of quality.

Voros (2006) borrows from Teng et al. (1996) the same ideas of price and quality affecting product demand and learning effects due to experience but makes a number of modifications to the model by Teng et al. (1996). First, the demand for the product is a function of time in addition to price and quality. The idea is that the demand for the product drops over time even if the price and quality stay the same. This phenomenon is called quality inflation and it is due to the increasing number of competitors and imitators over time, even if the product started in a monopolistic environment. Second, the learning effects are modeled as a function of both experience and investment in training. Third, quality attributes are divided into strategic and non-strategic attributes. Strategic attributes require development paths, such as increasing the efficiency of a car’s fuel consumption, while non-strategic attributes can be bought and do not require development paths, such as adding diamonds to a watch. Even with this classification of quality attributes, similar to the model by Teng et al., (1996), quality is still modeled as a random variable without any elaboration as to how to achieve a certain level of quality for manufacturing purposes.

There are several studies (e.g., DeCroix, 1999; Lin et al., 2005; Jaykumar et al., 2010; etc.) that consider the case where the product warranty can be used as the main signal for the quality of the product. That is, longer warranty period and better warranty policy are seen by the customer as indications for a higher quality and more reliable product. There are different justifications for replacing quality attributes with warranty. One justification is that “certain attributes of quality may be difficult to quantify” (Lin et
Another justification is given by Huang et al. (2007) who argue that “of course other attributes, such as style, size, performance, etc., will also influence consumer demand, but these factors are suppressed in our analysis to allow us to focus on the variables of interest” (Huang et al., 2007). In the following, some of the ideas and methodologies related to these studies, which use warranty as the main signal for quality, are presented in more detail.

Huang et al. (2007) make the case that marketing and engineering decisions ought to be made simultaneously. They argue that selling price is related to reliability and warranty. The higher the reliability, the higher the selling price and perhaps the better warranty terms for the customer. However, higher reliability means higher manufacturing costs but less warranty related costs. Higher manufacturing costs could be accounted for by higher selling price, which in turn negatively affects sales volume. Thus, in order to maximize the profit, the manufacturer has to consider both manufacturing issues, which influence product’s reliability, and marketing issues, which are related to price and warranty, in the development of a new product. The model and the solution approach in this study are similar to those in the study by Teng et al. (1996). However, the main difference between the two studies is replacing the quality level of the product with the product reliability. Moreover, it is assumed that the customer uses warranty information to assess product reliability. Thus, the selling price $p(\tau)$, reliability parameter $\theta$, and the length of the warranty period $W(\tau)$ are the decision variables of the proposed optimization model and maximizing the expected discounted profit $\pi$ over the planning time horizon $L$ is the objective function, which is given by (Huang et al., 2007):
Maximize \[ \pi = \int_0^T e^{-\delta \tau} \left[ p(\tau) - C(\theta, Q(\tau)) - \omega(\theta, W(\tau)) \right] q(p(\tau), \theta, W(\tau), Q(\tau)) d\tau \] (2.14)

where \( \delta \) is the discount rate, \( \tau \) is time, \( C(\cdot) \) is the manufacturing cost per item, \( Q(\tau) \) is the cumulative sales at time \( \tau \), \( \omega(\cdot) \) is the total expected warranty cost during the warranty period \( W(\tau) \), and \( q(\cdot) \) is the sales rate. The demand for the product, which is captured by the sales rate \( q(\cdot) \), is modeled using a modified version of the displaced log-linear demand function proposed by Glickman et al. (1976) as follows (Huang et al., 2007):

\[ q(p(\tau), W(\tau), Q(\tau)) = k_1(W(\tau) + k_2)^\alpha p(\tau)^{-\beta} \left[ 1 - \frac{Q(\tau)}{Q_M} \right]^{\psi} + \frac{Q(\tau)}{Q_M} \] (2.15)

where \( p, W, Q(\tau), Q_M \) are the product’s selling price, length of warranty period, cumulative sales up to time \( \tau \), and maximum sales potential, respectively. Also, \( k_1, k_2, \alpha, \beta, \) and \( \psi \) are model parameters. Again, the sales rate increases as the selling price decreases and the length of the warranty period increases. Furthermore, as the cumulative sales increases, the sales rate decreases due to market diffusion and saturation. It is to be noted that sales rate is not a function of the reliability parameter \( \theta \) explicitly. This is because product reliability is not observed by the customer. Instead, only the length of the warranty period is used by the customer to assess the reliability of the product. While the concept of product reliability used in this study is a better-defined concept than that of product quality level used by Teng et al. (1996) and others, it is still only represented by a single parameter \( \theta \). It is unclear how the product needs to be designed in order to achieve
a certain level of reliability. Thus, from a practical point of view, the way product reliability is modeled in this study is in effect as vague as product quality level used in the other studies.

Ladany et al. (2007) study the problem of finding the optimal length of the warranty period while maximizing the total net profit for a given planning period. The total net profit $\pi$ is given by (Ladany et al., 2007):

$$\pi = Q(R - c) - K$$

(2.16)

where $Q$ is the total quantity of the product sold during the planning period, $R$ is the expected revenue per item, $c$ and $K$ are the variable and fixed costs of manufacturing and servicing the product, respectively. Here, $Q$ is related to the selling price $P$ and the length of the warranty period $w$ using the Cobb-Douglas-type demand function given by (Ladany et al., 2007):

$$Q = \alpha P^{-\beta} w^{\gamma}$$

(2.17)

where $\alpha$, $\beta$, and $\gamma$ are model parameters. Furthermore, it is assumed that $P$ is related to $w$ according to the linear relationship given by (Ladany et al., 2007):

$$P = a + bw$$

(2.18)

where $a$ and $b$ are model parameters. In the proposed model, it is assumed that a longer warranty period is an indication of higher quality, which results in higher demand for the product for a given selling price. In addition, the length of the warranty period is related to the lifetime distribution of the product. It is assumed that there are both lower and upper specification limits for this lifetime distribution and that the manufacturer wants to align the length of the warranty period with the lower specification limit of the product’s
lifetime. The upper specification limit is predetermined while the lower specification limit is to be found. If the product fails within the warranty period, i.e. the lifetime of the product is less than the lower specification limit, the manufacturer incurs warranty costs. On the other hand, if the product survives beyond the upper specification limit, it is still considered an undesirable situation from the point of view of the manufacturer since this causes a reduction in product demand. Thus, the expected revenue per item \( R \) is given by (Ladany et al., 2007):

\[
R = r_L P \int_0^{LSL} f(x) \, dx + P \int_{LSL}^{USL} f(x) \, dx + r_U P \int_{USL}^\infty f(x) \, dx
\]  
(2.19)

where \( f(x) \), \( LSL \), and \( USL \) are the probability density function, lower specification limit, and upper specification limit for the lifetime of the product, respectively. Moreover, \( r_L P \), \( P \), and \( r_U P \) are the expected revenues per item if the product fails before \( LSL \), between \( LSL \) and \( USL \), and after \( USL \), respectively, with \( r_L \leq 1 \) (\( r_L \) is allowed to be negative) and \( 0 \leq r_U \leq 1 \). The general solution of this problem is derived using Response Modeling Methodology (RMM) to model the quantile function for a variety of lifetime distributions. One of the main limitations that can be seen in the proposed model is the fact that the only quality characteristic is the lifetime distribution of the product, which is only a function of time. The effect of manufacturing processes and product components and their specifications on the lifetime of the product is not modeled explicitly. As an extension to the model by Ladany et al. (2007), the effect of inspection error is studied by Chen et al. (2010).
Sinha et al. (2011) study price and warranty competition in a duopoly market structure. It is assumed that the two firms produce differentiated but substitutable products. Each firm tries to maximize its total expected profit during a given planning horizon while considering not only its own decisions about the product selling price and the length of its warranty period, but also the competitor’s price and warranty decisions as well. The total expected profit $\pi_i$ for firm $i$ during the planning horizon is given by (Sinha et al., 2011):

$$\pi_i = D_i \left[ p_i - \nu - c_i \left( \lambda_i w_i \right)^\kappa \right]$$

(2.20)

for $i = 1, 2$, where $D_i$ is the total demand during the planning horizon for product $i$, $p_i$ is the selling price, $\nu$ is the manufacturing cost per item, $c_i \left( \lambda_i w_i \right)^\kappa$ is the warranty cost per item since a Weibull lifetime distribution with scale parameters $\lambda_i$ and shape parameter $\kappa$ is assumed for each of the two products. $D_i$ is based on the demand function proposed by Banker et al. (1998), which is given by (Banker et al., 1998):

$$D_i = k_i \alpha - \beta p_i + \gamma p_j + \theta x_i - \eta x_j$$

(2.21)

for $i, j = 1, 2$ and $i \neq j$, where $p$ is the selling price of the product, $x$ is the quality level, $k_i, \alpha, \beta, \gamma, \theta,$ and $\eta$ are nonnegative model parameters with $k_1 + k_2 = 1, \beta > \gamma,$ and $\theta > \eta$. Sinha et al. (2011) modify equation (2.21) by replacing the quality levels $x_i$ with the length of the warranty periods $w_i$ and ignoring the parameters $k_i$, for $i = 1, 2$. With this model, Nash-Bertrand equilibrium is derived for the cases of price competition alone, warranty competition alone, and simultaneous price and warranty competitions.
Moreover, for each of these three competition cases, the multi-period dynamics of the iterative interactions between the two competitors from the start of the competition until reaching Nash-Bertrand equilibrium are studied. Furthermore, the case of coordination between the two firms, as oppose to competition, is also considered. It is to be noted that, while this study is quite detailed in terms of the important aspects of the duopoly market structure, product design and its connection to product price and warranty are not considered in any of the proposed models.

It can be seen from the literature review of this section that critical decisions related to the selling price, quality level, warranty policy, production rate, etc., of the product are made in the initial stages of product development based on customer and market behaviors as well as manufacturing costs. Customer and market behaviors are assessed in a fairly detailed manner in most of the studies. However, manufacturing costs are modeled with very crude representations and simple equations, which compromise the accuracy and optimality of the results obtained. Thus, it can be seen that the proposed models of the studies reviewed in this section make many assumptions about the production process, whereas the studies reviewed in section 2.1, i.e. integrated parameter and tolerance designs without microeconomic and marketing considerations, make many assumptions about the customer and market behaviors.

In this research, new integrated parameter and tolerance designs models are proposed to address the main limitations of the previous studies. In Chapter 3, integrated parameter and tolerance designs models are considered without discussing microeconomic and marketing issues, such as pricing, market structure, customer utility
and demand functions, etc. Microeconomic and marketing issues are considered in the integrated parameter and tolerance designs models of Chapter 4.
Chapter 3

Integrated Parameter and Tolerance Designs without Microeconomic and Marketing Considerations

In this chapter, new integrated parameter and tolerance designs models are developed that resolve the main limitations of previous models discussed in the literature review of section 2.1. Most importantly, the quadratic loss function that is used by all studies to estimate quality losses is replaced with the warranty costs model. This is because the warranty costs model is better defined conceptually when compared to the quadratic loss function and is related to measurable quality attributes, such as product reliability. With the use of the warranty and reliability models presented in sections 1.2.3 and 1.2.4, respectively, the deviation of the product quality characteristic from its ideal value is related to warranty costs in an analogous manner as the way it is related to external failure costs in the quadratic loss function.

3.1 Model Development

In this model, $X$ is the main quality characteristic of a product and it is a function of the quality characteristics of $k$ components $X_1, X_2, \ldots, X_k$. The functional form $X = e\left( X_1, X_2, \ldots, X_k \right)$, which defines the relationship between the main product characteristic and the components characteristics, is assumed to be known. Starting with the functional form $X = e\left( X_1, X_2, \ldots, X_k \right)$, Chandra (2001) shows that the mean and
variance of $X$ can be approximated using a Taylor series expansion and ignoring higher order terms as follows:

$$
\mu = e \left( \mu_1, \mu_2, \ldots, \mu_k \right) + \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{k} h_{ij} \left( \mu_1, \mu_2, \ldots, \mu_k \right) \sigma_{ij}
$$

(3.1)

$$
\sigma^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} g_i \left( \mu_1, \mu_2, \ldots, \mu_k \right) g_j \left( \mu_1, \mu_2, \ldots, \mu_k \right) \sigma_{ij}
$$

(3.2)

where $\mu_1, \mu_2, \ldots, \mu_k$ are the means of the $k$ components characteristics,

$$
\sigma_{ij} = COV \left( X_i, X_j \right) = \rho_{ij} \sigma_i \sigma_j, \quad \text{if } i \neq j
$$

$$
= \sigma_i^2, \quad \text{if } i = j
$$

(3.3)

$$
g_i \left( \mu_1, \mu_2, \ldots, \mu_k \right) = \frac{\partial e \left( X_1, X_2, \ldots, X_k \right)}{\partial X_i} \bigg|_{\mu_1, \mu_2, \ldots, \mu_k},
$$

(3.4)

$$
h_{ij} \left( \mu_1, \mu_2, \ldots, \mu_k \right) = \frac{\partial^2 e \left( X_1, X_2, \ldots, X_k \right)}{\partial X_i \partial X_j} \bigg|_{\mu_1, \mu_2, \ldots, \mu_k}
$$

(3.5)

$COV \left( X_i, X_j \right)$ and $\rho_{ij}$ are the covariance and correlation between $X_i$ and $X_j$, respectively, for $i, j = 1, 2, \ldots, k$.

In this research, the quality characteristics of the $k$ components $X_1, X_2, \ldots, X_k$ are assumed to have a joint multivariate normal distribution. As a result of this assumption, the marginal distribution for each of the $k$ components characteristics is normal with means $\mu_1, \mu_2, \ldots, \mu_k$ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_k^2$. That is $X_i \sim N \left( \mu_i, \sigma_i^2 \right)$ for $i = 1, 2, \ldots, k$. Moreover, any linear combination of the form $X = \sum_{i=1}^{k} v_i X_i$ has a
univariate normal distribution with exact mean and variance, which are given by (Hogg, 2006):

\[ \mu = \sum_{i=1}^{k} v_i \mu_i \]  

(3.6)

and

\[ \sigma^2 = \sum_{i=1}^{k} \sum_{j=1}^{k} v_i v_j \sigma_{ij} \]  

(3.7)

respectively, where \( \mu_i \) is the mean of the quality characteristic of component \( i \) for \( i = 1, 2, \ldots, k \) and \( \sigma_{ij} \) is the covariance term as defined in equation (3.3). It is worth mentioning that it is quite common in practice for the \( k \) components characteristics \( X_1, X_2, \ldots, X_k \) to be independent. That is why the majority of the studies in the literature assume that \( X_1, X_2, \ldots, X_k \) are independent in their models. Therefore, if the independence of \( X_1, X_2, \ldots, X_k \) is assumed, the mean and variance of the main product characteristic \( X \) are reduced from equations (3.1) and (3.2) to:

\[ \mu = e \left( \mu_1, \mu_2, \ldots, \mu_k \right) + \frac{1}{2} \sum_{i=1}^{k} h_i \left( \mu_1, \mu_2, \ldots, \mu_k \right) \sigma_i^2 \]  

(3.8)

and

\[ \sigma^2 = \sum_{i=1}^{k} \left[ g_i \left( \mu_1, \mu_2, \ldots, \mu_k \right) \right]^2 \sigma_i^2 \]  

(3.9)

where \( g_i(.) \) and \( h_i(.) \) are defined in equations (3.4) and (3.5), respectively.

Given the relationship between the main product characteristic and the components characteristics, the idea of an integrated parameter and tolerance designs model is to specify the means and the natural process tolerances (or equivalently the
variances instead of the natural process tolerances) of the components characteristics such that the total cost $TC$, which is the sum of manufacturing costs $MC$ and warranty costs $WC$, is minimized while satisfying functional requirements and customer preferences. Thus, the objective function of this optimization model that integrates parameter and tolerance designs is given by:

$$\text{Minimize} \quad TC = MC + WC \quad (3.10)$$

The manufacturing costs $MC$ consist of the natural process tolerance costs and internal failure costs as follows:

$$MC = C_r(t_1, t_2, ..., t_k) + C_f(\mu, \tau, \text{LSL,USL}) \quad (3.11)$$

where $C_r(t_1, t_2, ..., t_k)$ is the combined natural process tolerance costs for all the $k$ components characteristics, $t_1, t_2, ..., t_k$ are the natural process tolerances of the $k$ components characteristics, $C_f(\mu, \tau, \text{LSL,USL})$ is the internal failure costs due to scrap and rework, $\mu$, $\tau$, $\text{LSL}$ and $\text{USL}$ are the mean, the natural process tolerance, the lower and upper specification limits of the main product characteristic $X$, respectively.

The warranty costs $WC$ during the warranty period $w$ are a function of the product reliability $R(w)$ and they are estimated using warranty models (section 1.2.3). The product reliability, in turn, is a function of the warranty period $w$ and the value of the main product characteristic. Thus,

$$R(w) = E_x[R(w|x)] \quad (3.12)$$

where $R(w|x)$ is the conditional reliability and it is estimated using reliability models similar to the ones presented in section 1.2.4.
It is to be noted that, since the marginal distribution for each of the $k$ components characteristics is normal and the range of all possible values of these characteristics is infinite, it is conventional to define the natural process tolerances of the components characteristics as:

$$ t_i = 6\sigma_i $$

(3.13)

such that, $t_i$ contains 99.73% of all possible values of the quality characteristic of component $i$, for $i = 1, 2, \ldots, k$. As a result of equation (3.13), the natural process tolerance costs term in equation (3.11), i.e. $C_T(t_1, t_2, \ldots, t_k)$, is in effect a relationship between the costs and the variances of the $k$ components characteristics. That is why the natural process tolerances and the variances of the components characteristics can be used interchangeably as decision variables in the optimization model. Moreover, the natural process tolerance $t$ of the main product characteristic $X$ can be defined using either the worst-case relationship or the probabilistic relationship, where the use of the worst-case relationship results in tighter constraints on the tolerances of the components characteristics and smaller failure rates (larger $C_{\rho_k}$) than the use of the probabilistic relationship. The worst-case relationship for $t$ is defined as (Wu et al., 1998):

$$ t = \sum_{i=1}^{k} |g_i(\mu_1, \mu_2, \ldots, \mu_k)|t_i $$

(3.14)

where $g_i(.)$ is given by equation (3.4), $\mu_i$ and $t_i$ are the mean and the natural process tolerance of the quality characteristic of component $i$, respectively, for $i = 1, 2, \ldots, k$. The probabilistic relationship for $t$, on the other hand, is derived starting with the relationship
for the variance of the main quality characteristic $X$ of the product, i.e. equation (3.2), which can be rewritten as:

$$
\sigma^2 = \sum_{i=1}^{k} \left[ g_i (\mu_i, \mu_2, \ldots, \mu_k) \right]^2 \sigma_i^2 \\
+ 2 \sum_{i<j} g_i (\mu_i, \mu_2, \ldots, \mu_k) g_j (\mu_i, \mu_2, \ldots, \mu_k) \rho_{ij} \sigma_i \sigma_j 
$$

(3.15)

for $i, j = 1, 2, \ldots, k$. Then, replacing the standard deviations $\sigma_1, \sigma_2, \ldots, \sigma_k$ of the components characteristics with the natural process tolerances $t_1, t_2, \ldots, t_k$ using equation (3.13), equation (3.15) becomes:

$$
\sigma^2 = \sum_{i=1}^{k} \left[ g_i (\mu_i, \mu_2, \ldots, \mu_k) \right]^2 \left( \frac{t_i}{6} \right)^2 \\
+ 2 \sum_{i<j} g_i (\mu_i, \mu_2, \ldots, \mu_k) g_j (\mu_i, \mu_2, \ldots, \mu_k) \rho_{ij} \left( \frac{t_i}{6} \right) \left( \frac{t_j}{6} \right) 
$$

Finally, $\sigma$ is replaced with $t/G$ to give:

$$
t = \frac{G}{6} \left( \sum_{i=1}^{k} \left[ g_i (\mu_i, \mu_2, \ldots, \mu_k) \right]^2 \left( \frac{t_i}{6} \right)^2 \\
+ 2 \sum_{i<j} g_i (\mu_i, \mu_2, \ldots, \mu_k) g_j (\mu_i, \mu_2, \ldots, \mu_k) \rho_{ij} \left( \frac{t_i}{6} \right) \left( \frac{t_j}{6} \right) \right)^{0.5} 
$$

(3.16)

where $G$ is the coefficient that makes 100% or at least 99.73% of all possible values of the main product characteristic $X$ fall within the range $G\sigma$, i.e. $t = G\sigma$. Examples for $G$ values include 6 for a normal distribution and $\sqrt{12}$ for a uniform distribution (Chandra, 2001). If the distribution of $X$ is unknown, empirical or simulated data can be used, based on $X = e (X_1, X_2, \ldots, X_k)$ and known $X_1, X_2, \ldots, X_k$ distributions, to estimate the natural process tolerance $t$, which is the range of 100% or close to 100% of all
possible values of $X$. Then, $G$ is calculated as the ratio of $t$ and $\sigma$, where $\sigma$ is calculated from equation (3.2) or (3.15).

Besides the objective function (equation (3.10)), several constraints are added to the optimization model in order to insure proper solutions. The means and standard deviations of the components characteristics are constrained by minimum and maximum values as follows:

$$\mu_{i_{\min}} \leq \mu_i \leq \mu_{i_{\max}}$$

$$\sigma_{i_{\min}} \leq \sigma_i \leq \sigma_{i_{\max}}$$

for $i = 1, 2, \ldots, k$. These constraints imply that even though the choices for the means and standard deviations of the components characteristics are generally flexible, they are still expected to be within certain ranges (Jeang et al., 2002b). The constraints in (3.17) are imposed based on customer preferences and feasible process ranges while the constraints in (3.18) are imposed based on available manufacturing processes and their capabilities. Again, since the standard deviations and natural process tolerances of the components characteristics are related as in equation (3.13), the constraints in (3.18) can be equivalently presented as:

$$t_{i_{\min}} \leq t_i \leq t_{i_{\max}}$$

for, $i = 1, 2, \ldots, k$. Furthermore, another constraint is added to insure that the resultant natural process tolerance $t$ of the main product characteristic $X$ is not too wide from the customer’s point of view. This constraint, which is typically referred to as the stack-up constraint for $t$, is defined as:
\[ t \leq \frac{2\min\{(\mu - LSL), (USL - \mu)\}}{C_{pk\min}} \]  

(3.20)

where \( \mu, C_{pk\min}, LSL, \) and \( USL \) are the mean, minimum allowable process capability index, the lower and upper specification limits, respectively, for the main product characteristic. Equation (3.20) is derived from the fact that, in order to guarantee the achievement of an actual \( C_{pk} \) that is at least equal to \( C_{pk\min} \), \( \mu \) should be constrained as follows:

\[ LSL + C_{pk\min} \frac{t}{2} \leq \mu \leq USL - C_{pk\min} \frac{t}{2} \]  

(3.21)

Solving for \( t \), the constraint in (3.21) becomes:

\[ t \leq \frac{2(\mu - LSL)}{C_{pk\min}} \]  

(3.22a)

and

\[ t \leq \frac{2(USL - \mu)}{C_{pk\min}} \]  

(3.22b)

which is equivalent to equation (3.20). Figure 3.1 illustrates the effect of imposing the constraint of equation (3.20), where the distribution of the main product characteristic \( X \) is normal and \( C_{pk\min} = 1 \). Figures 3.1a and 3.1b show, respectively, the minimum and maximum values of the mean of the main product characteristic that can be chosen while the actual \( C_{pk} \) is still at least equal to \( C_{pk\min} \). Also, Figure 3.1c shows a general case where the mean of the main product characteristic falls within the minimum and maximum values such that the actual \( C_{pk} \) is greater than \( C_{pk\min} \).
Figure 3.1: Illustration of the stack-up constraint for $t$. a) The extreme case where $\mu$ is as small as possible while maintaining $C_{pk,\text{actual}} \geq C_{pk,\text{min}}$, b) the other extreme case where $\mu$ is as large as possible while maintaining $C_{pk,\text{actual}} \geq C_{pk,\text{min}}$, c) a general case where $C_{pk,\text{actual}} > C_{pk,\text{min}}$. (Note: in the figures, the distribution of the main product characteristic $X$ is normal and $C_{pk,\text{min}} = 1$).
If the ideal value $x_0$ of the main product characteristic $X$ is set at the midpoint between the lower and upper specification limits, i.e. the allowable design tolerance $(T = USL - LSL)$ is balanced, then, the stack-up constraint for $t$ of equation (3.20) reduces to:

$$t \leq \frac{T - 2|\mu - x_0|}{C_{pk\min}}$$  \hspace{1cm} (3.23)

where $\mu$ and $C_{pk\min}$ are the mean and the minimum allowable process capability index, respectively, for $X$. Again, equation (3.23) is derived in a similar manner as equation (3.20) using the following constraint on $\mu$:

$$x_0 - \frac{T}{2} + C_{pk\min} \frac{t}{2} \leq \mu \leq x_0 + \frac{T}{2} - C_{pk\min} \frac{t}{2}$$  \hspace{1cm} (3.24)

Solving for $t$, the constraint in (3.24) becomes:

$$t \leq \begin{cases} 
\frac{T + 2(\mu - x_0)}{C_{pk\min}}, & \text{if } \mu < x_0 \\
\frac{T - 2(\mu - x_0)}{C_{pk\min}}, & \text{if } \mu > x_0 
\end{cases}$$  \hspace{1cm} (3.25)

which is equivalent to equation (3.23).

### 3.2 Numerical Examples

In this section, several numerical examples are used to illustrate the methodology of setting up and solving a variety of problems with different assumptions and challenges based on the general optimization model for integrating parameter and tolerance designs developed in section 3.1. Moreover, sensitivity analysis is conducted to validate the
model and its solutions as well as to provide some insight into the effects of model parameters and constraints on the solution. In addition, the solutions are compared to the solutions that are achieved from conducting parameter and tolerance designs in sequence (two-stage design) as oppose to the integrated design developed in section 3.1.

### 3.2.1 A Linear Relationship between the Main Quality Characteristic of the Product and the Quality Characteristics of its Components

In this example, the main quality characteristic $X$ of the product is linearly related to the quality characteristics $X_1, X_2, \ldots, X_k$ of its $k$ components, i.e. $X = \sum_{i=1}^{k} v_i X_i$, for $i=1,2,\ldots,k$. As a result, $X$ is normally distributed with the mean and variance given by equations (3.6) and (3.7), respectively. Again, $X_1, X_2, \ldots, X_k$ have a joint multivariate normal distribution. In addition, the manufacturing and warranty costs are modeled using:

$$MC = d_0 + \frac{d_1}{t_1} + \frac{d_2}{t_2} + \ldots + \frac{d_k}{t_k} + C_{f1} P(X \leq LSL) + C_{f2} P(X \geq USL)$$

(3.26)

and

$$WC = -C_r \ln[R(w)]$$

(3.27)

where

$$R(w) = \frac{1}{\sqrt{1+2b\sigma^2w'}} \exp \left[ -\left( \frac{a + b(\mu - x_0)^2}{1+2b\sigma^2w'} \right) w' \right]$$

(3.28)

Here, $d_l$, for $l=0,1,\ldots,k$, are the tolerance-costs parameters, $t_i$, for $i=1,2,\ldots,k$, are the natural process tolerances of the $k$ components characteristics, $C_{f1}$ and $C_{f2}$ are the
internal failure costs per item when $X$ is less than $LSL$ and $X$ is greater than $USL$, respectively, $LSL$, $USL$, $\mu$, $\sigma^2$, and $x_0$ are the lower specification limit, upper specification limit, mean, variance, and ideal value for the main product characteristic, respectively, $C_r$ is the warranty repair cost per item, $a$, $b$, and $c$ are the reliability model’s parameters, and $w$ is the length of the warranty period. Therefore, the optimization model for integrating parameter and tolerance designs for this example has an objective function (equation 3.10) that is calculated by summing equations (3.26) and (3.27) subject to the constraints (3.17), (3.18) or (3.19), and (3.20).

### 3.2.1.1 Baseline Example

A simple numerical example for this model is presented for $X = X_1 + X_2$. Thus, $X$ is normally distributed with mean $\mu_1 + \mu_2$ and variance $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$, i.e. $X \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$. The optimization model seeks to determine the optimal means and natural process tolerances (or variances instead of the natural process tolerances according to equation (3.13)) of the two components characteristics that minimize the total cost. This is a nonlinear optimization problem that can be solved using programs, such as LINGO or Excel’s Solver. The data required for the model and the values used in the baseline example are shown in Table 3.1.
The correlation coefficient between the two components characteristics \( \rho = 0 \)

The tolerance-costs parameters
\[
\begin{align*}
    d_0 &= 10 \\
    d_1 &= 3 \\
    d_2 &= 5
\end{align*}
\]

The internal failure cost per item \( C_{f_1} = C_{f_2} = $20 \)

The reliability model’s parameters
\[
\begin{align*}
    a &= 0.001 \\
    b &= 0.025 \\
    c &= 0.8
\end{align*}
\]

The ideal value of the main product characteristic \( x_0 = 20 \)

The length of the warranty period \( w = 12 \)

The warranty repair cost per item \( C_r = 100 \)

The minimum and maximum values for the means and standard deviations of the two components characteristics
\[
\begin{align*}
    8 \leq \mu_1 &\leq 10 \\
    10 \leq \mu_2 &\leq 12 \\
    0.05 \leq \sigma_1 &\leq 0.6 \\
    0.05 \leq \sigma_2 &\leq 0.6
\end{align*}
\]

The lower and upper specification limits for the main product characteristic
\[
\begin{align*}
    LSL &= 18.5 \\
    USL &= 21.5
\end{align*}
\]

The minimum allowable process capability index \( C_{pk_{\text{min}}} = 1.33 \)

<table>
<thead>
<tr>
<th>Table 3.1: The data required for the model of this section and their numerical values used in the baseline example.</th>
</tr>
</thead>
<tbody>
<tr>
<td>The correlation coefficient between the two components characteristics</td>
</tr>
<tr>
<td>The tolerance-costs parameters</td>
</tr>
<tr>
<td>The internal failure cost per item</td>
</tr>
<tr>
<td>The reliability model’s parameters</td>
</tr>
<tr>
<td>The ideal value of the main product characteristic</td>
</tr>
<tr>
<td>The length of the warranty period</td>
</tr>
<tr>
<td>The warranty repair cost per item</td>
</tr>
<tr>
<td>The minimum and maximum values for the means and standard deviations of the two components characteristics</td>
</tr>
<tr>
<td>The lower and upper specification limits for the main product characteristic</td>
</tr>
<tr>
<td>The minimum allowable process capability index</td>
</tr>
</tbody>
</table>

Using Excel’s Solver, the optimal means, standard deviations, natural process tolerances, and total cost for the baseline example are found to be: \( \mu_1^* = 9 \) \( \mu_2^* = 11 \) \( \mu^* = 20 \) \( \sigma_1^* = 0.242 \) \( \sigma_2^* = 0.287 \) \( \sigma^* = 0.376 \) \( t_1^* = 1.454 \) \( t_2^* = 1.724 \) \( t^* = 2.256 \) (using the probabilistic relationship), and \( TC^* = $18.21 \) per item. It is to be noted that this optimal solution could be one of many local optimal solutions and not necessarily the global optimal solution (Hillier et al., 2005). This is because the optimization problem is
nonlinear and the solution might depend on the initial values used to start the optimization algorithm. This issue can be thoroughly investigated by studying the convexity of the nonlinear problem’s objective function and constraints as well as considering different initial values or even alternative optimization algorithms. However, this matter is not pursued in this dissertation.

3.2.1.2 Sensitivity Analysis

Sensitivity analysis can be conducted by varying the assumptions and parameters of the baseline example and studying the model behavior and solution in response to these changes in model assumptions and parameters. Again, the purpose of conducting sensitivity analysis is to validate the model and its solution as well as to gain some insight into the effects of model assumptions, parameters, and constraints on the solution.

The effect of the correlation between \( X_1 \) and \( X_2 \) can be studied by considering values for the correlation coefficient \( \rho \) other than zero, where \(-1 \leq \rho \leq 1\). Table 3.2 shows the optimal means, standard deviations, the natural process tolerance of the main product characteristic, process capability index, and total cost for several values of the correlation coefficient \( \rho \) (the optimal natural process tolerances \( t_1 \) and \( t_2 \) can be found from the optimal standard deviations \( \sigma_1 \) and \( \sigma_2 \) using equation (3.13)).
Table 3.2: Effect of the value of the correlation coefficient $\rho$ for $X_1$ and $X_2$ on the optimal solution of the baseline example.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\mu_1^*$</th>
<th>$\mu_2^*$</th>
<th>$\mu^*$</th>
<th>$\sigma_1^*$</th>
<th>$\sigma_2^*$</th>
<th>$\sigma^*$</th>
<th>$t^*$</th>
<th>$C_{pk}^*$</th>
<th>$TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.564</td>
<td>0.578</td>
<td>0.256</td>
<td>1.534</td>
<td>1.96</td>
<td>14.24</td>
</tr>
<tr>
<td>-0.5</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.317</td>
<td>0.351</td>
<td>0.335</td>
<td>2.013</td>
<td>1.49</td>
<td>16.69</td>
</tr>
<tr>
<td>0</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.242</td>
<td>0.287</td>
<td>0.376</td>
<td>2.256</td>
<td>1.33</td>
<td>18.21</td>
</tr>
<tr>
<td>0.5</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.193</td>
<td>0.24</td>
<td>0.376</td>
<td>2.256</td>
<td>1.33</td>
<td>19.31</td>
</tr>
<tr>
<td>0.9</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.169</td>
<td>0.217</td>
<td>0.376</td>
<td>2.256</td>
<td>1.33</td>
<td>20.05</td>
</tr>
</tbody>
</table>

It can be seen from Table 3.2 that the optimal total cost $TC^*$ always increases as $\rho$ increases. Moreover, the optimal means $\mu_1^*$, $\mu_2^*$, and $\mu^*$ for the baseline example are independent of the correlation coefficient $\rho$. However, as $\rho$ increases, the optimal standard deviation $\sigma^*$ and natural process tolerance $t^*$ for the main product characteristic increase until the constraint for $t$ given by equation (3.20) becomes binding at a value of $C_{pk} = 1.33$, where the values for $\sigma^*$ and $t^*$ stay the same even as $\rho$ increases. It is to be noted that, in order to maintain $\sigma^*$ and $t^*$ at constant values as $\rho$ increases and after $C_{pk_{\text{min}}} = 1.33$ is imposed, $\sigma_1^*$ and $\sigma_2^*$ need to continue to decrease because of the effect $\rho$ has on the equation for $\sigma$, i.e. $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$. If the baseline example is modified with $LSL = 18$ and $USL = 21$ such that the ideal value $x_0$ of the main product characteristic is not at the midpoint between the $LSL$ and $USL$ (unbalanced design tolerance), then the values for $\mu_1^*$, $\mu_2^*$, and $\mu^*$ decrease with the increase in the value of $\rho$, as shown in Table 3.3.
This decrease in the value of $\mu^*$ and its shift toward the midpoint between the $LSL$ and $USL$ as $\rho$ increases is accompanied by a continuous increase in $\sigma^*$ while the constraint for $t$ (equation (3.20)) is always binding at the value of $C_{pk} = 1.33$. Again, $\sigma_1^*$ and $\sigma_2^*$ decrease with the increase in $\rho$ because of the effect $\rho$ has on the equation for $\sigma$. It is worth noting that the value of $C_{pk}$ can also be maintained at 1.33 by keeping $\mu = x_0 = 20$ and $\sigma = 0.251$ for all values of $\rho$. However, this requires $\sigma_1$ and $\sigma_2$ to decrease more drastically as $\rho$ increases than the decrease shown in Table 3.3, which would be more expensive.

Table 3.4 presents the effect of unbalanced design tolerance on the optimal solution of the baseline problem. Again, a design tolerance is unbalanced when the ideal value $x_0$ of the main product characteristic is not at the midpoint between the lower and upper specification limits, i.e. $x_0 - LSL \neq USL - x_0$.
Table 3.4: Effect of unbalanced design tolerance \((x_0 - LSL \neq USL - x_0, \text{ with } x_0 = 20)\) on the optimal solution of the baseline example.

<table>
<thead>
<tr>
<th>LSL</th>
<th>USL</th>
<th>(\mu_1^*)</th>
<th>(\mu_2^*)</th>
<th>(\mu^*)</th>
<th>(\sigma_1^*)</th>
<th>(\sigma_2^*)</th>
<th>(\sigma^*)</th>
<th>(t^*)</th>
<th>(C_{pk}^*)</th>
<th>(TC^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.5</td>
<td>20.5</td>
<td>8.857</td>
<td>10.857</td>
<td>19.71</td>
<td>0.127</td>
<td>0.151</td>
<td>0.197</td>
<td>1.182</td>
<td>1.33</td>
<td>22.38</td>
</tr>
<tr>
<td>18</td>
<td>21</td>
<td>8.948</td>
<td>10.948</td>
<td>19.9</td>
<td>0.178</td>
<td>0.211</td>
<td>0.276</td>
<td>1.659</td>
<td>1.33</td>
<td>19.04</td>
</tr>
<tr>
<td>18.5</td>
<td>21.5</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.242</td>
<td>0.287</td>
<td>0.376</td>
<td>2.256</td>
<td>1.33</td>
<td>18.21</td>
</tr>
<tr>
<td>19</td>
<td>22</td>
<td>9.052</td>
<td>11.052</td>
<td>20.1</td>
<td>0.178</td>
<td>0.211</td>
<td>0.276</td>
<td>1.659</td>
<td>1.33</td>
<td>19.04</td>
</tr>
<tr>
<td>19.5</td>
<td>22.5</td>
<td>9.143</td>
<td>11.143</td>
<td>20.29</td>
<td>0.127</td>
<td>0.151</td>
<td>0.197</td>
<td>1.183</td>
<td>1.33</td>
<td>22.38</td>
</tr>
</tbody>
</table>

As it can be seen in Table 3.4, when \(x_0 = 20\) shifts from the midpoint between \(LSL\) and \(USL\), \(\mu^*\) tends to be different from \(x_0\) and \(\sigma^*\) decreases in order to satisfy the constraint on \(t\) (equation (3.20)). This, in turn, results in an increase in \(TC^*\).

The constraint on \(t\) given by equation (3.20) is imposed, as mentioned previously, to insure that the optimal solution, which minimizes the total costs, is not obtained at an excessively wide value of \(t\) (low \(C_{pk}\)) that would be unsatisfactory to the customer. Table 3.5 shows the effect of varying the value of \(C_{pk_{\min}}\) in equation (3.20) on the optimal solution of the baseline example.
Table 3.5: Effect of varying $C_{pk_{min}}$ on the optimal solution of the baseline example.

<table>
<thead>
<tr>
<th>$C_{pk_{min}}$</th>
<th>$\mu_1^*$</th>
<th>$\mu_2^*$</th>
<th>$\mu^*$</th>
<th>$\sigma_1^*$</th>
<th>$\sigma_2^*$</th>
<th>$\sigma^*$</th>
<th>$t^*$</th>
<th>$C_{pk}^*$</th>
<th>$TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.67</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.243</td>
<td>0.288</td>
<td>0.377</td>
<td>2.261</td>
<td>1.327</td>
<td>18.21</td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.243</td>
<td>0.288</td>
<td>0.377</td>
<td>2.261</td>
<td>1.327</td>
<td>18.21</td>
</tr>
<tr>
<td>1.33</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.242</td>
<td>0.287</td>
<td>0.376</td>
<td>2.256</td>
<td>1.33</td>
<td>18.21</td>
</tr>
<tr>
<td>1.67</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.193</td>
<td>0.229</td>
<td>0.299</td>
<td>1.796</td>
<td>1.67</td>
<td>18.57</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.161</td>
<td>0.191</td>
<td>0.25</td>
<td>1.5</td>
<td>2</td>
<td>19.32</td>
</tr>
</tbody>
</table>

It can be seen from Table 3.5 that $C_{pk_{min}}$ values less than 1.327 result in the constraint in equation (3.20) to be nonbinding. This means that the optimal solution without imposing the constraint in equation (3.20) or when the constraint is nonbinding is $TC^* = $18.21, which is achieved at $C_{pk}^* = 1.327$. For $C_{pk_{min}}$ values greater than 1.327, on the other hand, the constraint in equation (3.20) is binding. This results in $C_{pk}^* = C_{pk_{min}}$ and $TC^*$ increasing with the increase in $C_{pk_{min}}$. It is worth emphasizing that the results in Table 3.5 do not suggest that $TC^*$ increases monotonically with the increase in the value of $C_{pk}$ used. In order to observe the effect of varying $C_{pk}$ on the optimal solution, the constraint in equation (3.20) should be changed to equality. Figure 3.2 illustrates the difference between the results obtained in Table 3.5 and the results that would be obtained if the constraint in equation (3.20) is changed to equality.
Figure 3.2: Illustration of the difference between the effect of varying $C_{pk}$ on the optimal solution versus varying $C_{pk,min}$ in the model.

Figure 3.2 shows that $C^*_{pk} = 1.327$ is the optimal solution for the baseline problem without imposing the constraint of equation (3.20). Furthermore, any deviation of $C_{pk}$ from this value in either direction results in an increased $TC^*$. Also, $C^*_{pk}$ is constant at 1.327 when the constraint of equation (3.20) is nonbinding. It is to be noted that the value of $C^*_{pk} = 1.327$ could be affected by other parameters in the model. However, similar trends should be expected as in Table 3.5. As shown in Table 3.6, where the baseline example is modified with $LSL = 18$ and $USL = 21$, the value of $C^*_{pk}$ when the constraint of equation (3.20) is nonbinding is 0.925 instead of 1.327.
Besides the shifts in the optimal means due to the unbalanced design tolerance, Tables 3.5 and 3.6 exhibit similar trends.

One of the most important model parameters, which is of high interest, even for the models developed in Chapter 4, is the length of the warranty period \( w \). Table 3.7 presents the effect of the length of the warranty period on the optimal solution of the baseline example.

<table>
<thead>
<tr>
<th>( w )</th>
<th>( \mu_1^* )</th>
<th>( \mu_2^* )</th>
<th>( \mu^* )</th>
<th>( \sigma_1^* )</th>
<th>( \sigma_2^* )</th>
<th>( \sigma^* )</th>
<th>( t^* )</th>
<th>( C_{pk}^* )</th>
<th>( TC^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.242</td>
<td>0.287</td>
<td>0.376</td>
<td>2.256</td>
<td>1.33</td>
<td>16.84</td>
</tr>
<tr>
<td>12</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.242</td>
<td>0.287</td>
<td>0.376</td>
<td>2.256</td>
<td>1.33</td>
<td>18.21</td>
</tr>
<tr>
<td>18</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.219</td>
<td>0.259</td>
<td>0.339</td>
<td>2.036</td>
<td>1.473</td>
<td>19.33</td>
</tr>
<tr>
<td>24</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.203</td>
<td>0.241</td>
<td>0.315</td>
<td>1.889</td>
<td>1.588</td>
<td>20.25</td>
</tr>
<tr>
<td>30</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.191</td>
<td>0.227</td>
<td>0.297</td>
<td>1.782</td>
<td>1.683</td>
<td>21.04</td>
</tr>
</tbody>
</table>

Table 3.7: Effect of the length of the warranty period \( w \) on the optimal solution of the baseline example.
Table 3.7 shows that, not surprisingly, the optimal total cost $TC^*$ increases as the length of the warranty $w$ increases. Moreover, as $w$ increases, $C_{pk}^*$ increases by decreasing $\sigma^*$ and $t^*$.

The results shown above are only a sample of the full sensitivity analysis that can be conducted for every model assumption and parameter. There are obviously other parameters and many interactions among these parameters and model constraints that could be studied in a similar manner as the results shown above. However, since the idea is to only show the methodology and some of the interesting results of conducting sensitivity analysis, the full sensitivity analysis results are not presented.

3.2.1.3 Comparison between Sequential and Integrated Parameter and Tolerance Designs

The sequential parameter and tolerance designs approach is a two-stage design approach. In the first stage, parameter design is conducted using the Robust Parameter Design (RPD) model of equation (1.8). Thus, for the simple example considered so far, i.e. $X = X_1 + X_2$, RPD determines the optimal $\mu_1$ and $\mu_2$ values for a given set of values for $\sigma_1$ and $\sigma_2$, such that, $\mu = \mu_0$ and $\sigma$ is minimized. Typically, for economical reasons, $\sigma_1$ and $\sigma_2$ are chosen to be large, i.e. low grade processes (Jeang, 2003). Then, in the second stage, tolerance design is conducted by determining the optimal values for $\sigma_1$ and $\sigma_2$ (or equivalently $t_1$ and $t_2$ according to equation (3.13), hence the name tolerance design) that minimize the total cost. Thus, this tolerance design uses the same objective
function and constraints as the integrated parameter and tolerance design developed in this chapter except that \( \mu_1 \) and \( \mu_2 \) are predetermined from the RPD model.

Since \( X \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2) \), the mean \( \mu \) and variance \( \sigma^2 \) of \( X \) are independent. Thus, the only situation where the integrated approach is superior to the sequential approach is when the design tolerance for \( X \) is unbalanced such that \( x_0 \) is not at the midpoint between \( LSL \) and \( USL \). Table 3.8 presents the effect of unbalanced design tolerance on the optimal solution of the sequential parameter and tolerance design, where \( x_0 = 20 \).

<table>
<thead>
<tr>
<th>LSL</th>
<th>USL</th>
<th>( \mu_1^* )</th>
<th>( \mu_2^* )</th>
<th>( \mu^* )</th>
<th>( \sigma_1^* )</th>
<th>( \sigma_2^* )</th>
<th>( \sigma^* )</th>
<th>( t^* )</th>
<th>( C_{pk}^* )</th>
<th>( TC^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>17.5</td>
<td>20.5</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.081</td>
<td>0.096</td>
<td>0.125</td>
<td>0.752</td>
<td>1.33</td>
<td>25.90</td>
</tr>
<tr>
<td>18</td>
<td>21</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.162</td>
<td>0.192</td>
<td>0.251</td>
<td>1.504</td>
<td>1.33</td>
<td>19.31</td>
</tr>
<tr>
<td>18.5</td>
<td>21.5</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.242</td>
<td>0.287</td>
<td>0.376</td>
<td>2.256</td>
<td>1.33</td>
<td>18.21</td>
</tr>
<tr>
<td>19</td>
<td>22</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.162</td>
<td>0.192</td>
<td>0.251</td>
<td>1.504</td>
<td>1.33</td>
<td>19.31</td>
</tr>
<tr>
<td>19.5</td>
<td>22.5</td>
<td>9</td>
<td>11</td>
<td>20</td>
<td>0.081</td>
<td>0.096</td>
<td>0.125</td>
<td>0.752</td>
<td>1.33</td>
<td>25.90</td>
</tr>
</tbody>
</table>

Table 3.8: Effect of unbalanced design tolerance (\( x_0 - LSL \neq USL - x_0 \), with \( x_0 = 20 \)) on the optimal solution of the sequential parameter and tolerance designs.

Comparing Tables 3.4 and 3.8, it can be seen that, when the design tolerance is unbalanced, the integrated approach results in better optimal solutions than the sequential approach. Moreover, \( \mu^* \) is restricted to being always equal to \( x_0 \) in the sequential approach. This is in contrast to the integrated approach, where \( \mu^* \) can shift from \( x_0 \) if minimizing the total cost requires that.
3.2.2 A Nonlinear Relationship between the Main Quality Characteristic of the Product and the Quality Characteristics of its Components

There are many forms of nonlinear relationships between the main quality characteristic $X$ of the product and the quality characteristics of its $k$ components that can be considered for analysis. However, since the purpose of this example is to present the solution and analysis methodology for the nonlinear relationship case, a simple nonlinear relationship is chosen for analysis. Thus, in this example, the relationship between the main product characteristic $X$ and the characteristics of only two product components $X_1$ and $X_2$ is given by $X = X_1 X_2$. Due to this nonlinear relationship, the distribution of $X$ is treated as being unknown. Nonetheless, using equations (3.1) and (3.2), the mean and variance of $X$ can be approximated as $\mu \approx \mu_1 \mu_2 + \rho \sigma_1 \sigma_2$ and $\sigma^2 \approx \mu_1^2 \sigma_1^2 + \mu_2^2 \sigma_2^2 + 2 \rho \mu_1 \mu_2 \sigma_1 \sigma_2$, respectively. Again, $X_1$ and $X_2$ have a joint multivariate normal distribution with a correlation coefficient $\rho$. In addition, the manufacturing and warranty costs are modeled using:

$$MC = d_0 + \frac{d_1}{t_1} + \frac{d_2}{t_2} + C_{f1} P(X \leq LSL) + C_{f2} P(X \geq USL)$$  \hspace{1cm} (3.29)$$

and

$$WC = -C_r \ln \left( E_x \left[ R(w|x) \right] \right)$$  \hspace{1cm} (3.30)$$

where

$$R(w|x) = \exp \left[ - \left( a + b(x - x_0)^2 \right) w^p \right]$$  \hspace{1cm} (3.31)$$
Here, $d_l$, for $l = 0,1,2$, are the tolerance-costs parameters, $t_i$, for $i = 1,2$, are the natural process tolerances of the two components characteristics, $C_{f1}$ and $C_{f2}$ are the internal failure costs per item when $X$ is less than $LSL$ and $X$ is greater than $USL$, respectively, $LSL$, $USL$, $x$ and $x_0$ are the lower specification limit, upper specification limit, observed value, and ideal value for the main product characteristic, respectively, $C_r$ is the warranty repair cost per item, $a, b$, and $c$ are the reliability model’s parameters, and $w$ is the length of the warranty period. It is to be noted that the unconditional reliability could not be represented in closed-form as in equation (3.28) due to the nonlinear relationship between the main product characteristic and the two components characteristics. That is why the conditional reliability is used in the model instead (equation (3.31)). Therefore, the optimization model for integrating parameter and tolerance designs has an objective function (equation 3.10) that is calculated by summing equations (3.29) and (3.30) subject to the constraints (3.17), (3.18) or (3.19), and (3.20).

3.2.2.1 Baseline Example

The data required for the model and the values used in the baseline example are shown in Table 3.9.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>The correlation coefficient between the two components characteristics</td>
<td>$\rho = 0$</td>
</tr>
<tr>
<td>The tolerance-costs parameters</td>
<td>$d_0 = 10$</td>
</tr>
<tr>
<td></td>
<td>$d_1 = 6$</td>
</tr>
<tr>
<td></td>
<td>$d_2 = 10$</td>
</tr>
<tr>
<td>The internal failure cost per item</td>
<td>$C_{f1} = C_{f2} = $20</td>
</tr>
<tr>
<td>The reliability model’s parameters</td>
<td>$a = 0.0001$</td>
</tr>
<tr>
<td></td>
<td>$b = 0.001$</td>
</tr>
<tr>
<td></td>
<td>$c = 0.8$</td>
</tr>
<tr>
<td>The ideal value of the main product characteristic</td>
<td>$x_0 = 100$</td>
</tr>
<tr>
<td>The length of the warranty period</td>
<td>$w = 12$</td>
</tr>
<tr>
<td>The warranty repair cost per item</td>
<td>$C_r = 100$</td>
</tr>
<tr>
<td>The minimum and maximum values for the means and standard deviations of</td>
<td>$8 \leq \mu_1 \leq 10$</td>
</tr>
<tr>
<td>the two components characteristics</td>
<td>$10 \leq \mu_2 \leq 12$</td>
</tr>
<tr>
<td></td>
<td>$0.05 \leq \sigma_1 \leq 0.3$</td>
</tr>
<tr>
<td></td>
<td>$0.05 \leq \sigma_2 \leq 0.3$</td>
</tr>
<tr>
<td>The lower and upper specification limits for the main product characteristic</td>
<td>$\begin{cases} LSL = 90 \ USL = 110 \end{cases}$</td>
</tr>
<tr>
<td></td>
<td>$T = 20$</td>
</tr>
<tr>
<td>The minimum allowable process capability index</td>
<td>$C_{pk,\text{min}} = 1.33$</td>
</tr>
</tbody>
</table>

Table 3.9: The data required for the model and their numerical values used in the baseline example.

Again, since the unconditional reliability cannot be represented in closed-form due to the nonlinear relationship $X = X_1X_2$, which results in an unknown distribution for $X$, the total cost, which is the objective function of the optimization model (equation 3.10), cannot be represented in closed-form either. As a result, conventional nonlinear optimization algorithms cannot be used directly. Instead, the well-known Central Composite Design (CCD) used in Response Surface Methodology (RSM) is employed to
find an approximate second-order objective function that relates the total cost to the
decision variables. This objective function is then minimized to achieve an approximate
optimal solution (Jeang, 2003). In this example, since the decision variables are the
means and standard deviations of the two components characteristics, i.e., $\mu_1, \mu_2, \sigma_1,$
and $\sigma_2,$ a four factor CCD is used. The coded CCD for four factors as generated by
Minitab is given in Appendix A. Again, based on equation (3.13), the natural process
tolerances $t_1$ and $t_2$ could have been used instead of $\sigma_1$ and $\sigma_2$ as decision variables.
Table 3.10 shows a portion of the calculations for one replication used in the CCD
approach.

| $\mu_1$ | $\mu_2$ | $\sigma_1$ | $\sigma_2$ | $x_1$ | $x_2$ | $x$ | $R(w|x)$ | $TC$ |
|---------|---------|------------|------------|-------|-------|-----|-----------|------|
| 8.5     | 10.5    | 0.113      | 0.113      | 8.492 | 10.770| 91.46| 0.587     | 87.03|
| 9.5     | 10.5    | 0.113      | 0.113      | 9.396 | 10.657| 100.14| 0.999     | 33.79|
| 8.5     | 11.5    | 0.113      | 0.113      | 8.623 | 11.430| 98.56| 0.984     | 35.29|
| 9.5     | 11.5    | 0.113      | 0.113      | 9.425 | 11.632| 109.63| 0.508     | 101.52|
| 8.5     | 10.5    | 0.238      | 0.113      | 8.417 | 10.598| 89.20| 0.426     | 114.25|
| 9.5     | 10.5    | 0.238      | 0.113      | 9.293 | 10.554| 98.08| 0.973     | 31.78|
| 8.5     | 11.5    | 0.238      | 0.113      | 8.286 | 11.642| 96.47| 0.912     | 38.19|
| 9.5     | 11.5    | 0.238      | 0.113      | 9.724 | 11.552| 112.33| 0.329     | 140.07|
| 8.5     | 10.5    | 0.113      | 0.238      | 8.587 | 10.518| 90.33| 0.505     | 94.31|

Table 3.10: A portion of CCD calculations for the example of this section
Table 3.10 shows the uncoded values for $\mu_1$, $\mu_2$, $\sigma_1$, and $\sigma_2$, which are obtained from their allowable ranges provided in Table 3.9. Then, for each of the specific values of $\mu_1$, $\mu_2$, $\sigma_1$, and $\sigma_2$, random sample values, $x_1$ and $x_2$, of the components characteristics are generated, where $X_i \sim N(\mu_i, \sigma_i^2)$ for $i = 1, 2$, and the resultant main product characteristic $x$ is calculated using the relationship $x = x_1 x_2$. After that, $R(w|x)$ is found using equation (3.31) and the total cost is calculated by summing equations (3.29) and (3.30). After replicating the CCD calculations several times, a second-order model is fitted to the total cost $TC$ as a function of $\mu_1$, $\mu_2$, $\sigma_1$, and $\sigma_2$. The second-order model obtained for this example is:

$$
TC = 28259 - 3149\mu_1 - 2527\mu_2 - 176\sigma_1 - 321.3\sigma_2 \\
+ 87.79\mu_1^2 + 56.54\mu_2^2 + 416\sigma_1^2 + 619.1\sigma_2^2 \\
+ 141.4\mu_1\mu_2 - 3.956\mu_1\sigma_1 + 3.802\mu_2\sigma_2 \\
+ 3.755\mu_1\sigma_1 + 0.725\mu_2\sigma_2 + 53.13\sigma_1\sigma_2
$$

(3.32)

with $R^2 \approx 68\%$. Now, the optimization model for integrating parameter and tolerance designs consists of an objective function that is given by equation (3.32) and is subject to the constraints (3.17), (3.18) or (3.19), and (3.20). Using Excel’s Solver, the optimal means, standard deviations, natural process tolerances, and total cost are found to be:

$\mu_1^* = 9.567$, $\mu_2^* = 10.45$, $\mu^* = 100$, $\sigma_1^* = 0.162$, $\sigma_2^* = 0.193$, $\sigma^* = 2.506$, $t_1^* = 0.972$, $t_2^* = 1.159$, $t^* = 15.037$ (using the probabilistic relationship), $TC^* = $29.25 per item.

When the correlation coefficient $\rho$ between the two components characteristics is nonzero, the solution procedure is slightly modified to account for this correlation between $X_1$ and $X_2$. This is done by generating correlated random samples $x_1$ and $x_2$ in...
the CCD calculations instead of the independent random samples used above \((x_1\) and \(x_2\) columns in Table 3.10). Correlated random samples are obtained by first generating two independent and identically distributed random samples from the standard normal distribution, i.e. \(Z_1, Z_2 \sim N(0,1)\), (Kroese et al., 2011). Then, \(X_1\) and \(X_2\) become correlated random variables using the relationship:

\[
\begin{bmatrix}
X_1 \\
X_2
\end{bmatrix} =
\begin{bmatrix}
\mu_1 \\
\mu_2
\end{bmatrix} +
\begin{bmatrix}
\sigma_1 & 0 \\
\rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2
\end{bmatrix}
\begin{bmatrix}
Z_1 \\
Z_2
\end{bmatrix}
\]  

(3.33)

where \(\mu_1\), \(\mu_2\), \(\sigma_1\), \(\sigma_2\), and \(\rho\) are the means, standard deviations, and the correlation coefficient of \(X_1\) and \(X_2\), respectively. Also, the matrix \(C = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{bmatrix}\) is obtained from the Cholesky decomposition as shown in Appendix B. After generating correlated random samples, the same calculations and solution procedure are followed as in the case for \(\rho = 0\) shown above. It is worth noting that the same steps can be followed to generate more than two correlated random variables using the appropriate \(C\) matrix, which is obtained from the covariance matrix \(\Sigma\) using the relationship \(\Sigma = CC^T\).

3.2.2.2 Sensitivity Analysis

Conducting sensitivity analysis for this example is not as straightforward as it is shown for the example of section 3.2.1.2. This is because using the second-order fitted relationship of equation (3.32) to conduct sensitivity analysis might not be appropriate since this fitted relationship is only valid for the example parameters given in Table 3.9.
Therefore, any changes in the parameters of the original example could result in a different fitted relationship between the total cost $TC$ and the decision variables $\mu_1$, $\mu_2$, $\sigma_1$, and $\sigma_2$. Nonetheless, sensitivity analysis can be conducted for a given parameter by including this parameter as a decision variable in the fitted relationship for the total cost $TC$. Another way to conduct sensitivity analysis, which is shown by Jeang (2003), is to repeat the whole solution procedure for different values of the parameter of interest.

If the main product characteristic $X$ is assumed to be normally distributed, sensitivity analysis can be conducted in a similar manner as in section 3.2.1.2. This is because equation (3.28) can be used for the unconditional reliability. As a result, the model objective function, which is the total cost, is represented in closed-form by summing the warranty and manufacturing costs of equations (3.28) and (3.29), respectively. The normality assumption for $X$ is quite appropriate for the parameter values considered in this example as shown in Appendix C. It is to be noted that, even though $X$ is assumed to be normally distributed, its mean and variance are not independent since $\mu \equiv \mu_1 \mu_2 + \rho \sigma_1 \sigma_2$ and $\sigma^2 \equiv \mu_2^2 \sigma_1^2 + \mu_1^2 \sigma_2^2 + 2 \rho \mu_1 \mu_2 \sigma_1 \sigma_2$.

Table 3.11 presents the effect of the correlation coefficient $\rho$ on the optimal solution of the baseline example.
Table 3.11: Effect of the value of the correlation coefficient $\rho$ for $X_1$ and $X_2$ on the optimal solution of the baseline example.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\mu^*_1$</th>
<th>$\mu^*_2$</th>
<th>$\mu^*$</th>
<th>$\sigma^*_1$</th>
<th>$\sigma^*_2$</th>
<th>$\sigma^*$</th>
<th>$t^*$</th>
<th>$C^*_{pk}$</th>
<th>$TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td>10</td>
<td>10.01</td>
<td>99.991</td>
<td>0.3</td>
<td>0.3</td>
<td>1.342</td>
<td>8.0527</td>
<td>2.481</td>
<td>20.26</td>
</tr>
<tr>
<td>-0.5</td>
<td>8.354</td>
<td>11.97</td>
<td>100</td>
<td>0.209</td>
<td>0.3</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>24.79</td>
</tr>
<tr>
<td>0</td>
<td>8.333</td>
<td>12</td>
<td>100</td>
<td>0.144</td>
<td>0.218</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>29.06</td>
</tr>
<tr>
<td>0.5</td>
<td>8.332</td>
<td>12</td>
<td>100</td>
<td>0.117</td>
<td>0.179</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>32.33</td>
</tr>
<tr>
<td>0.9</td>
<td>8.332</td>
<td>12</td>
<td>100</td>
<td>0.103</td>
<td>0.16</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>34.57</td>
</tr>
</tbody>
</table>

As it can be seen from Table 3.11, the optimal total cost $TC^*$ increases as $\rho$ increases. Moreover, for $\rho = -0.9$, even though $\sigma_1$ and $\sigma_2$ reach their maximum values of 0.3 (given in Table 3.9), a low value for $\sigma^*$ is achieved with a $C^*_{pk} = 2.481$. Then, as $\rho$ increases, $\sigma^*_1$ and $\sigma^*_2$ decrease to maintain $\sigma^*$ at 2.506 ($C^*_{pk} = C^*_{pk_{min}} = 1.33$).

Table 3.12 shows the effect of unbalanced design tolerance, i.e. $x_0 - LSL \neq USL - x_0$, with $x_0 = 100$, on the optimal solution of the baseline example.

<table>
<thead>
<tr>
<th>LSL</th>
<th>USL</th>
<th>$\mu^*_1$</th>
<th>$\mu^*_2$</th>
<th>$\mu^*$</th>
<th>$\sigma^*_1$</th>
<th>$\sigma^*_2$</th>
<th>$\sigma^*$</th>
<th>$t^*$</th>
<th>$C^*_{pk}$</th>
<th>$TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>85</td>
<td>105</td>
<td>8.178</td>
<td>12</td>
<td>98.138</td>
<td>0.099</td>
<td>0.152</td>
<td>1.72</td>
<td>10.32</td>
<td>1.33</td>
<td>35.68</td>
</tr>
<tr>
<td>87.5</td>
<td>107.5</td>
<td>8.25</td>
<td>12</td>
<td>98.995</td>
<td>0.123</td>
<td>0.187</td>
<td>2.132</td>
<td>12.789</td>
<td>1.33</td>
<td>31.05</td>
</tr>
<tr>
<td>90</td>
<td>110</td>
<td>8.333</td>
<td>12</td>
<td>100</td>
<td>0.144</td>
<td>0.218</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>29.06</td>
</tr>
<tr>
<td>92.5</td>
<td>112.5</td>
<td>8.41</td>
<td>12</td>
<td>100.92</td>
<td>0.121</td>
<td>0.182</td>
<td>2.11</td>
<td>12.661</td>
<td>1.33</td>
<td>31.22</td>
</tr>
<tr>
<td>95</td>
<td>115</td>
<td>8.483</td>
<td>12</td>
<td>101.79</td>
<td>0.098</td>
<td>0.146</td>
<td>1.702</td>
<td>10.214</td>
<td>1.33</td>
<td>36.09</td>
</tr>
</tbody>
</table>

Table 3.12: Effect of unbalanced design tolerance ($x_0 - LSL \neq USL - x_0$, with $x_0 = 100$) on the optimal solution of the baseline example.
It can be seen from Table 3.12 that, the more \( x_0 \) shifts from the midpoint between the \( LSL \) and \( USL \), the higher are the optimal total cost \( TC^* \) and the difference between \( \mu^* \) and \( x_0 \). Moreover, in order to maintain \( C_{pk}^* \) at 1.33, \( \sigma^* \) decreases as \( x_0 \) shifts from the midpoint between the \( LSL \) and \( USL \).

The effect of the value of \( C_{pk,\text{min}} \) on the optimal solution of the baseline example is shown in Table 3.13.

<table>
<thead>
<tr>
<th>( C_{pk,\text{min}} )</th>
<th>( \mu_1^* )</th>
<th>( \mu_2^* )</th>
<th>( \mu^* )</th>
<th>( \sigma_1^* )</th>
<th>( \sigma_2^* )</th>
<th>( \sigma^* )</th>
<th>( t^* )</th>
<th>( C_{pk}^* )</th>
<th>( TC^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.67</td>
<td>8.329</td>
<td>12</td>
<td>99.952</td>
<td>0.174</td>
<td>0.264</td>
<td>3.032</td>
<td>18.191</td>
<td>1.094</td>
<td>28.45</td>
</tr>
<tr>
<td>1</td>
<td>8.329</td>
<td>12</td>
<td>99.952</td>
<td>0.174</td>
<td>0.264</td>
<td>3.032</td>
<td>18.191</td>
<td>1.094</td>
<td>28.45</td>
</tr>
<tr>
<td>1.33</td>
<td>8.333</td>
<td>12</td>
<td>100</td>
<td>0.144</td>
<td>0.218</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>29.06</td>
</tr>
<tr>
<td>1.67</td>
<td>8.333</td>
<td>12</td>
<td>100</td>
<td>0.115</td>
<td>0.173</td>
<td>1.996</td>
<td>11.976</td>
<td>1.67</td>
<td>31.23</td>
</tr>
<tr>
<td>2</td>
<td>8.333</td>
<td>12</td>
<td>100</td>
<td>0.096</td>
<td>0.145</td>
<td>1.667</td>
<td>10</td>
<td>2</td>
<td>34.01</td>
</tr>
</tbody>
</table>

Table 3.13: Effect of varying \( C_{pk,\text{min}} \) on the optimal solution of the baseline example.

Table 3.13 shows that \( C_{pk}^* \) is achieved at a value of 1.094 when \( C_{pk,\text{min}} \) is low. As \( C_{pk,\text{min}} \) increases, the constraint on \( t \) (equation (3.20)) becomes binding, where \( C_{pk}^* = C_{pk,\text{min}} \). Thus, \( TC^* \) is initially constant at 28.45 when \( C_{pk}^* = 1.094 \). Then, \( TC^* \) increases with the increase in \( C_{pk,\text{min}}^* \).

Lastly, the effect of the length of the warranty period \( w \) on the optimal solution of the baseline example is presented in Table 3.14.
Table 3.14: Effect of the length of the warranty period $w$ on the optimal solution of the baseline example.

<table>
<thead>
<tr>
<th>$w$</th>
<th>$\mu_1^*$</th>
<th>$\mu_2^*$</th>
<th>$\mu^*$</th>
<th>$\sigma_1^*$</th>
<th>$\sigma_2^*$</th>
<th>$\sigma^*$</th>
<th>$t^*$</th>
<th>$C_{pk}^*$</th>
<th>$TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>8.333</td>
<td>12</td>
<td>100</td>
<td>0.144</td>
<td>0.218</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>29.06</td>
</tr>
<tr>
<td>24</td>
<td>8.333</td>
<td>12</td>
<td>100</td>
<td>0.144</td>
<td>0.218</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>32.13</td>
</tr>
<tr>
<td>36</td>
<td>8.331</td>
<td>12</td>
<td>99.972</td>
<td>0.133</td>
<td>0.201</td>
<td>2.312</td>
<td>13.869</td>
<td>1.438</td>
<td>34.61</td>
</tr>
<tr>
<td>48</td>
<td>8.331</td>
<td>12</td>
<td>99.976</td>
<td>0.124</td>
<td>0.187</td>
<td>2.151</td>
<td>12.907</td>
<td>1.546</td>
<td>36.54</td>
</tr>
<tr>
<td>60</td>
<td>8.332</td>
<td>12</td>
<td>99.978</td>
<td>0.117</td>
<td>0.177</td>
<td>2.035</td>
<td>12.21</td>
<td>1.634</td>
<td>38.15</td>
</tr>
</tbody>
</table>

Table 3.14 shows that the optimal total cost $TC^*$ increases as $w$ increases due to the higher warranty costs. Moreover, as $w$ increases, the value of $C_{pk\min} = 1.33$ becomes insufficient at a value of $w$ between 24 and 36. This causes $C_{pk}^*$ values to start increasing with a continuous decrease in the values of $\sigma^*$ and $t^*$.

3.2.2.3 Comparison between Sequential and Integrated Parameter and Tolerance Designs

For the example considered in this section, i.e. $X = X_1X_2$, as in section 3.2.1.3, Robust Parameter Design (RPD) is used in the first stage of the sequential parameter and tolerance designs to determine the optimal $\mu_1$ and $\mu_2$ values for a given set of values for $\sigma_1$ and $\sigma_2$, such that, $\mu = x_0$ and $\sigma$ is minimized. Again, for economical reasons, $\sigma_1$ and $\sigma_2$ are chosen to be large, i.e. low grade processes (Jeang, 2003). Then, in the second stage, tolerance design is conducted by determining the optimal values for $\sigma_1$ and $\sigma_2$ (or equivalently $t_1$ and $t_2$ according to equation (3.13)) that minimize the total cost. Thus,
this tolerance design uses the same objective function and constraints as the integrated parameter and tolerance design developed in this chapter except that $\mu_1$ and $\mu_2$ are predetermined from the RPD model.

Once again, the normality assumption for $X = X_1 X_2$ is assumed to be valid, as verified in Appendix C. However, the mean $\mu = \mu_1 \mu_2 + \rho \sigma_1 \sigma_2$ and variance $\sigma^2 = \mu_1^2 \sigma_1^2 + \mu_2^2 \sigma_2^2 + 2 \rho \mu_1 \mu_2 \sigma_1 \sigma_2$ of $X$ are not independent. Thus, unlike the example presented in section 3.2.1.3, the integrated approach is superior to the sequential approach in cases other than the case of an unbalanced design tolerance, i.e. $x_0 - LSL \neq USL - x_0$.

Table 3.15 presents the effect of the correlation coefficient $\rho$ on the optimal solution of the sequential parameter and tolerance designs.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>$\mu_1^*$</th>
<th>$\mu_2^*$</th>
<th>$\mu^*$</th>
<th>$\sigma_1^*$</th>
<th>$\sigma_2^*$</th>
<th>$\sigma^*$</th>
<th>$t^*$</th>
<th>$C_{pk}^*$</th>
<th>$TC^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.9</td>
<td>10</td>
<td>10.01</td>
<td>100</td>
<td>0.3</td>
<td>0.3</td>
<td>1.342</td>
<td>8.0531</td>
<td>2.484</td>
<td>20.26</td>
</tr>
<tr>
<td>-0.5</td>
<td>9.996</td>
<td>10.01</td>
<td>100.01</td>
<td>0.236</td>
<td>0.262</td>
<td>2.503</td>
<td>15.017</td>
<td>1.33</td>
<td>25.04</td>
</tr>
<tr>
<td>0</td>
<td>10</td>
<td>10</td>
<td>100</td>
<td>0.162</td>
<td>0.192</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>29.35</td>
</tr>
<tr>
<td>0.5</td>
<td>9.995</td>
<td>10</td>
<td>99.965</td>
<td>0.128</td>
<td>0.16</td>
<td>2.498</td>
<td>14.985</td>
<td>1.33</td>
<td>32.67</td>
</tr>
<tr>
<td>0.9</td>
<td>9.992</td>
<td>10</td>
<td>99.933</td>
<td>0.112</td>
<td>0.144</td>
<td>2.49</td>
<td>14.938</td>
<td>1.33</td>
<td>34.96</td>
</tr>
</tbody>
</table>

Table 3.15: Effect of the correlation coefficient $\rho$ on the optimal solution of the sequential parameter and tolerance designs.

A comparison between Tables 3.11 and 3.15 shows that, besides the case for $\rho = -0.9$, where the maximum values of $\sigma_1^*$ and $\sigma_2^*$ of 0.3 are reached, the integrated approach always results in better optimal solutions than the sequential approach. In
addition, Table 3.16 presents the effect of unbalanced design tolerance on the optimal solution of the sequential parameter and tolerance designs, where \( x_0 = 100 \).

<table>
<thead>
<tr>
<th>LSL</th>
<th>USL</th>
<th>( \mu_1^* )</th>
<th>( \mu_2^* )</th>
<th>( \mu^* )</th>
<th>( \sigma_1^* )</th>
<th>( \sigma_2^* )</th>
<th>( \sigma^* )</th>
<th>( t^* )</th>
<th>( C_{pk}^* )</th>
<th>TC*</th>
</tr>
</thead>
<tbody>
<tr>
<td>85</td>
<td>105</td>
<td>10</td>
<td>10</td>
<td>100</td>
<td>0.081</td>
<td>0.096</td>
<td>1.253</td>
<td>7.5188</td>
<td>1.33</td>
<td>40.98</td>
</tr>
<tr>
<td>87.5</td>
<td>107.5</td>
<td>10</td>
<td>10</td>
<td>100</td>
<td>0.121</td>
<td>0.144</td>
<td>1.88</td>
<td>11.278</td>
<td>1.33</td>
<td>32.44</td>
</tr>
<tr>
<td>90</td>
<td>110</td>
<td>10</td>
<td>10</td>
<td>100</td>
<td>0.162</td>
<td>0.192</td>
<td>2.506</td>
<td>15.038</td>
<td>1.33</td>
<td>29.35</td>
</tr>
<tr>
<td>92.5</td>
<td>112.5</td>
<td>10</td>
<td>10</td>
<td>100</td>
<td>0.121</td>
<td>0.144</td>
<td>1.88</td>
<td>11.278</td>
<td>1.33</td>
<td>32.44</td>
</tr>
<tr>
<td>95</td>
<td>115</td>
<td>10</td>
<td>10</td>
<td>100</td>
<td>0.081</td>
<td>0.096</td>
<td>1.253</td>
<td>7.5188</td>
<td>1.33</td>
<td>40.98</td>
</tr>
</tbody>
</table>

Table 3.16: Effect of unbalanced design tolerance \(( x_0 - LSL \neq USL - x_0, \text{ with } x_0 = 100 \) on the optimal solution of the sequential parameter and tolerance designs.

Comparing Tables 3.12 and 3.16, it can be seen that the integrated approach always results in better optimal solutions than the sequential approach. Moreover, the better results of the integrated approach are more profoundly exhibited as the shift of \( x_0 \) from the midpoint between \( LSL \) and \( USL \) becomes larger. Also, since \( \mu_1^* \) and \( \mu_2^* \) are obtained from the first stage of the sequential approach, they are independent of the \( LSL \) and \( USL \).

### 3.2.3 Dealing with Discrete Tolerance-Costs Data Using Integer Programming

This example addresses the situation where the combined natural process tolerance cost, i.e. tolerance-costs \( C_T(t_1, t_2, ..., t_k) \) that appear in equation (3.11) for manufacturing costs, is provided in the form of a table as opposed to a continuous
function. This is a more realistic situation in real life problems since typically there are only a finite number of available manufacturing processes. Consequently, using the continuous tolerance-costs function in the optimization model could result in manufacturing processes that do not exist. Table 3.17 presents an example of a tolerance-costs data set for the quality characteristics of two components that constitute a given product.

<table>
<thead>
<tr>
<th>Component ( i )</th>
<th>Process ( l )</th>
<th>( \sigma_{il} = \frac{t_{il}}{6} )</th>
<th>Unit Cost ( D_{il} ) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.1</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.2</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.3</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.1</td>
<td>55</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.2</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.25</td>
<td>25</td>
</tr>
</tbody>
</table>

Table 3.17: Example of a tolerance-costs data list for the quality characteristics of two components of a product.

Table 3.17 shows \( k = 2 \) components each of which having \( m_i = 3 \) available processes for their manufacturing. Also, the standard deviation of the quality characteristic of component \( i \) that manufacturing process \( l \) is capable of achieving and the corresponding cost per unit produced \( D_{il} \) are shown in Table 3.17 as well.

The optimization model for integrating parameter and tolerance designs for this example still consists of an objective function (equation 3.10) that is calculated by
summing equations (3.11) and (3.12). However, the tolerance-costs term \( C_T(t_1, t_2, \ldots, t_k) \) from equation (3.11) is modeled using:

\[
C_T(t_1, t_2, \ldots, t_k) = \sum_{i=1}^{k} \sum_{l=1}^{m_i} D_{il} Y_{il}
\]

where \( D_{il}, \) for \( i = 1,2,\ldots,k \) and \( l = 1,2,\ldots,m_i, \) is the cost per unit of component \( i \) produced by process \( l, \) and \( Y_{il} = 1 \) if process \( l \) is selected to manufacture component \( i \) and \( Y_{il} = 0 \) otherwise. In addition, this optimization model is subject to the constraints (3.17) and (3.20). However, equations (3.14) and (3.16) for the natural process tolerance \( t \) of the main quality characteristic \( X \) of the product need to be modified as follows:

\[
t_{\text{worst-case}} = \sum_{i=1}^{k} \left( g_i(\mu_1, \ldots, \mu_k) \sum_{l=1}^{m_i} t_{il} Y_{il} \right)
\]

\[
t_{\text{prob.}} = \frac{G}{6} \left( \sum_{i=1}^{k} \left[ g_i(\mu_1, \mu_2, \ldots, \mu_k) \right]^2 \left( \sum_{l=1}^{m_i} t_{il} Y_{il} \right)^2 \right)^{0.5} + 2 \sum_{i < j} \left( g_i(\mu_1, \mu_2, \ldots, \mu_k) g_j(\mu_1, \mu_2, \ldots, \mu_k) \mu_q \left( \sum_{l=1}^{m_i} t_{il} Y_{il} \right) \left( \sum_{l=1}^{m_j} t_{jl} Y_{jl} \right) \right)
\]

where \( t_{\text{prob.}} = G \sigma \). Moreover, another constraint is added to the model to insure that only one manufacturing process is selected for a given component. Thus, for \( i = 1,2,\ldots,k, \)

\[
\sum_{l=1}^{m_i} Y_{il} = 1
\]

The optimization algorithm for this model depends on the relationship between the main product characteristic and the components characteristics, i.e. \( X = e(X_1, X_2, \ldots, X_k). \) If this relationship is linear, the model can be solved using
programs, such as LINGO or Excel’s Solver as in section 3.2.1. On the other hand, if the relationship \( X = e(X_1, X_2, ..., X_k) \) is nonlinear and the distribution for \( X \) is unknown, the model can be solved using the same solution methodology as in section 3.2.2. However, the standard deviations \( \sigma_1, \sigma_2, ..., \sigma_k \) of the \( k \) components characteristics are treated as continuous variables when fitting the second-order relationship (using the CCD approach) between the total cost \( TC \) and the means \( \mu_1, \mu_2, ..., \mu_k \) and standard deviations \( \sigma_1, \sigma_2, ..., \sigma_k \) of the \( k \) components characteristics. Then, \( \sigma_1, \sigma_2, ..., \sigma_k \) are replaced with

\[
\sum_{i=1}^{m} \sigma_{il} Y_{il} \quad \text{or} \quad \sum_{i=1}^{m} \frac{t_{il} Y_{il}}{6}, \quad \text{for} \quad l = 1, 2, ..., k \quad \text{and} \quad l = 1, 2, ..., m_i, \quad \text{and the model is optimized.}
\]

### 3.3 Discussion

The integrated parameter and tolerance designs models proposed in this chapter are flexible enough to handle a wide range of real-life problems. This is evident from the different variety of models, numerical examples, and solution methodologies presented in this chapter. However, it is important to realize that the proposed models are by no means exhaustive or rigidly applicable to every real-life problem. The models used in the numerical examples represent only a sample of the different reliability, warranty, and manufacturing costs models that can be used. Furthermore, there are other issues that can be considered within the same framework of the proposed models. These issues include setup costs, monitoring and control costs, maintenance costs, measurement error considerations, updating model parameters (adaptive models), etc. Again, the focus of this chapter has been mainly to show the merit of using integrated models for parameter
and tolerance designs and the benefit of using well-defined and empirically measurable concepts in the models. Consequently, it is left to the designers and engineers to select the appropriate model components that best fit their application.

It is to be noted that one area of concern about the proposed models of this chapter is the fact that finding a solution for the models could be time consuming or even impossible when the size of the problem is large. This concern is especially relevant when there is a nonlinear relationship between the main quality characteristic of the product and the quality characteristics of the components. Obviously, this issue can be addressed, at least partially, with advances in computer-related technologies, better optimization algorithms, etc. However, even if the size of the problem is a concern, it is worth emphasizing that the usefulness of the proposed models is not limited to solving the full model for a new product. Instead, the proposed models can be used to study the economic viability (return-on-investment decisions) of certain adjustments in the design, manufacturing processes, warranty policies, etc., of existing products as well. That is, the proposed models that integrate all major parameters, decision variables, and constraints can be used to study the effects of changing a small number of model parameters, constraints, or variables of interest.

Another issue that is important to mention is the viability of replacing the quadratic loss function with warranty costs. As stated previously, one of the main contributions of the models developed in this chapter for integrating parameter and tolerance designs is replacing the quadratic loss function with warranty costs. Again, the quadratic loss function suffers from its simplistic formulation making it difficult to estimate external failure costs accurately. Warranty costs, on the other hand, are based on
solid empirical and financial grounds where they are related to the quality characteristics of the product through the use of warranty and reliability models. However, the quadratic loss function by definition encompasses all external failure costs, part of which is the warranty costs. Thus, there exists the risk of underestimating the external failure costs when the quadratic loss function is replaced with warranty costs. This underestimation is compensated for by including the constraint on $t$ (equation (3.20)) and the specification of $C_{pk\min}$ in the proposed models. The minimum allowable process capability index $C_{pk\min}$ introduces the voice of the customer in the proposed models through specifying the minimum level of conformance quality required by the customer. Then again, determining the appropriate value for $C_{pk\min}$ that is satisfactory to the customer and both technologically and economically viable for the manufacturer could be a challenge. Addressing the issue of underestimating external failure costs when the quadratic loss function is replaced with warranty costs is the topic of Chapter 4.
Chapter 4
Integrated Parameter and Tolerance Designs with Microeconomic and Marketing Considerations

One of the key features of the integrated parameter and tolerance designs models of Chapter 3 is the replacement of the quadratic loss function with warranty costs. Due to the simple form of the quadratic loss function and the vague definition of its loss to society parameter, estimating external failure costs using the quadratic loss function can be quite a challenge in real-life problems. Warranty models are both conceptually and mathematically sound and can be developed using reliability models and empirical data. If the warranty costs are the only major external failure costs to the manufacturer and any other losses, such as loss of customer’s trust, loss of market-share, lawsuits, etc., are so minimal or inconsequential that they can be neglected, then replacing the quadratic loss function with the warranty costs is justified. However, if external failure costs other than warranty costs cannot be neglected, then replacing the quadratic loss function with the warranty costs alone results in underestimated external failure costs, which would affect the appropriateness of the proposed models and their solutions. Therefore, this chapter introduces integrated parameter and tolerance designs models that address external failure costs other than the ones covered under warranty by modeling their effects on customer utility and product demand instead of treating them as actual costs that require listing and estimation in monetary values. This is achieved by introducing microeconomic and marketing concepts, such as pricing models, market structure types, consumer and
competitor behaviors, etc., into the integrated models and relating the customer utility function or the demand function to the purchase price and product quality.

It is typical to assume that the price a customer is willing to pay for a given product increases as the quality of the product increases. The key idea is that the increase in purchase price for a higher quality product relative to another lower quality one reflects, among other decision making criteria, the customer’s assessment of external failure costs other than the warranty costs covered by the manufacturer. That is why, a four-year-old used car with a mileage of 60,000 miles, for instance, can be as attractive as a new car of the same model and options if its purchase price is significantly lower than that of the new car, even for a customer who can afford the new car. This is because the difference in price compensates for the obvious difference in quality (especially the perceived reliability) making the two cars comparable in terms of customer utility. Again, this approach is quite different from the approach of the quadratic loss function that tries to list all possible losses to the manufacturer, customer, and the society as a whole and estimate their monetary costs as a function of conformance quality.

4.1 Model Development

In the development of new products, information about the customer and the market has to be known beforehand since this information greatly influences the analysis approach and solution methodology. This is because different rules, dynamics, and challenges are exhibited depending on the type of the market structure, i.e. monopoly, oligopoly, perfect competition, etc., and depending on considering a static or a dynamic
market analysis approach. For instance, for the case of a monopoly in a static market, Karmarkar et al. (1997) suggest maximizing the profit while satisfying a certain level of customer utility. On the other hand, for the case of a perfect competition also in a static market, it is suggested to maximize customer’s utility while maintaining a nonnegative profit. These two models are given by (Karmarkar et al., 1997):

\[
\text{Maximize} \quad \pi = P - TC \quad (4.1a)
\]

Subject to: \[ U \geq u_0 \quad (4.1b) \]

and

\[
\text{Maximize} \quad U \quad (4.2a)
\]

Subject to: \[ \pi = P - TC \geq 0 \quad (4.2b) \]

where \( P \) is the selling price of an item, \( TC \) is the total cost of producing an item, \( U \) is the expected customer utility, \( u_0 \) is the minimum acceptable customer utility. However, in the case of a dynamic market, models (4.1) and (4.2) are not sufficient. This is because the effects of market dynamics, i.e. product expansion and saturation in the market, learning effects, imitators and innovators influences, price variations, etc., over time cannot be captured by these models. Instead, the model by Teng et al. (1996), for instance, can be used. This model is given by equation (2.10) and it is shown here again for convenience (Teng et al., 1996):

\[
\text{Maximize} \quad \pi = \int_0^L e^{-\delta \tau} \left[ p(\tau) - C(q(\tau), Q(\tau)) \right] Q\left( p(\tau), q(\tau), Q(\tau) \right) d\tau \quad (4.3)
\]

where \( \pi \) is the total present value of profit over the planning time horizon \( L \), \( \delta \) is the discount rate, \( p(\tau) \) is the selling price at time \( \tau \), \( C(\cdot) \) is the manufacturing cost per
item, \( q(\tau) \) is the quality level at time \( \tau \), \( Q(\tau) \) is the cumulative sales at time \( \tau \), and \( \dot{Q}(\cdot) \) is the sales rate. Therefore, in this chapter, several examples of model development for integrating parameter and tolerance designs are presented such that the characteristics of the variety of market structures and the difference between static and dynamic market analysis approaches are highlighted.

While different product and market situations require different analysis approaches, as substantiated above, the integrated parameter and tolerance designs models that are developed in this chapter have several common characteristics and are based on similar real-life observations. Thus, the idea is to establish the following concepts and relationships in the proposed models:

1. The goal is to maximize the profit, which is the difference between the selling price and the total cost.
2. The selling price is affected by the customer behavior (captured by customer utility or demand functions), quality level, and total cost.
3. Customer utility and product demand decrease with selling price and increase with quality level.
4. Quality level is defined by product attributes and product specifications.
5. The product attributes are used by the customer as signals for the quality level of the product. These attributes include, among many others, the length of the warranty period and the process capability index.
6. The product specifications are the means, variances, and the natural process tolerances of the quality characteristics of product components.
7. Product attributes and specifications along with the selling price are the decision variables of the integrated models.

8. Total cost is the sum of manufacturing and warranty costs. These costs depend on the quality level of the product, which in turn depends on product attributes and specifications.

For simplicity, in all of the examples of this chapter, the relationship between the main quality characteristic $X$ of the product that is being designed and the quality characteristics of the product’s only two components $X_1$ and $X_2$ is given by $X = X_1 + X_2$. Thus, similar to the baseline example of section 3.2.1.1, $X \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2)$, where $X_1$ and $X_2$ have a multivariate normal distribution with a correlation coefficient $\rho$. In addition, the manufacturing and warranty costs are modeled using:

$$MC = d_0 + \frac{d_1}{t_1} + \frac{d_2}{t_2} + C_{f1}P(X \leq LSL) + C_{f2}P(X \geq USL) \quad (4.4)$$

and

$$WC = -C_r \ln\left[R(w)\right] \quad (4.5)$$

where

$$R(w) = \frac{1}{\sqrt{1+2b\sigma^2w^\rho}} \exp\left[-\left(a + \frac{b(\mu_x^2)}{1+2b\sigma^2w^\rho}\right)w^\rho\right] \quad (4.6)$$

Here, $d_l$, for $l = 0, 1, 2$, are the tolerance-costs parameters, $t_i$, for $i = 1, 2$, are the natural process tolerances of the two components characteristics, $C_{f1}$ and $C_{f2}$ are the internal
failure costs per item when $X$ is less than $LSL$ and $X$ is greater than $USL$, respectively, $LSL$, $USL$, $\mu$, $\sigma^2$, and $x_0$ are the lower specification limit, upper specification limit, mean, variance, and ideal value for the main product characteristic, respectively, $C_r$ is the warranty repair cost per item, $a$, $b$, and $c$ are the reliability model’s parameters, and $w$ is the length of the warranty period. Moreover, the values of the model parameters and constraints, which are common among all of the examples presented in this chapter, are given in Table 4.1.

<table>
<thead>
<tr>
<th>The correlation coefficient between the two components characteristics</th>
<th>$\rho = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The tolerance-costs parameters</td>
<td>$d_0 = 10$</td>
</tr>
<tr>
<td></td>
<td>$d_1 = 3$</td>
</tr>
<tr>
<td></td>
<td>$d_2 = 5$</td>
</tr>
<tr>
<td>The internal failure cost per item</td>
<td>$C_{f1} = C_{f2} = $20</td>
</tr>
<tr>
<td>The reliability model’s parameters</td>
<td>$a = 0.001$</td>
</tr>
<tr>
<td></td>
<td>$b = 0.025$</td>
</tr>
<tr>
<td></td>
<td>$c = 0.8$</td>
</tr>
<tr>
<td>The ideal value of the main product characteristic</td>
<td>$x_0 = 20$</td>
</tr>
<tr>
<td>The warranty repair cost per item</td>
<td>$C_r = 100$</td>
</tr>
<tr>
<td>The minimum and maximum values for the means and standard deviations of the two components characteristics</td>
<td>$8 \leq \mu_1 \leq 10$</td>
</tr>
<tr>
<td></td>
<td>$10 \leq \mu_2 \leq 12$</td>
</tr>
<tr>
<td></td>
<td>$0.05 \leq \sigma_1 \leq 0.6$</td>
</tr>
<tr>
<td></td>
<td>$0.05 \leq \sigma_2 \leq 0.6$</td>
</tr>
<tr>
<td>The lower and upper specification limits for the main product characteristic</td>
<td>$LSL = 18.5$</td>
</tr>
<tr>
<td></td>
<td>$USL = 21.5$</td>
</tr>
<tr>
<td></td>
<td>$T = 3$</td>
</tr>
</tbody>
</table>

Table 4.1: The values of the model parameters and constraints that are common among all the examples of the integrated models presented in this chapter.
It is worth mentioning that the constraint on the natural process tolerance $t$ of the main product characteristic $X$, which is given by equation (3.20) and used in the models of Chapter 3, is unnecessary for the models of this chapter since the voice of the customer is captured by the customer utility or demand functions.

4.1.1 Example 1: Using a Demand Function to Capture Total Sales during a Given Planning Horizon

This example follows a similar modeling approach as the study by Ladany et al. (2007), which is one of the simplest cases where pricing models and demand functions are used for warranty decisions. The objective function of the integrated parameter and tolerance designs model is given by:

$$
\text{Maximize } \pi = (P - MC - WC)Q \tag{4.7}
$$

where $\pi$ is the total profit during a given planning horizon, $P$ is the selling price of an item, $MC$ and $WC$ are the manufacturing and warranty costs per item, which are given by equations (4.4) and (4.5), respectively, and $Q$ is the total quantity demanded (total sales) during the planning horizon, which is given by (Ladany et al., 2007):

$$
Q = \alpha P^{-\beta} w^\gamma \tag{4.8}
$$

where $\alpha$ is a proportionality constant, $P$ is again the selling price, $w$ is the length of the warranty period, which is used as the only signal for the product quality level observed by the customer, $\beta$ ($> 1$) and $\gamma$ ($0 > \gamma > 1$) are the selling price elasticity and the length of the warranty period elasticity, respectively (Huang et al., 2007). It can be seen from equation (4.8) that $Q$ increases as $P$ decreases and $w$ increases. Moreover, as $\beta$ and $\gamma$
increase, the effects of the changes in $P$ and $w$ on $Q$ intensify. Figure 4.1 illustrates the
relationship given by equation (4.8), where $\beta$ and $\gamma$ are held constant.

![Diagram showing the change in quantity demanded $Q$ as a function of the selling price $P$ and the length of the warranty period $w$. The diagram illustrates three series labeled $P = 35$, $P = 40$, and $P = 45$.]

Figure 4.1: Illustration of the change in quantity demanded $Q$ as a function of the selling price $P$ and the length of the warranty period $w$ ($\beta = 2.5$ and $\gamma = 0.3$).

The decision variables of this optimization model are the selling price $P$ and the
length of the warranty period $w$ along with the means $\mu_i$ and variances $\sigma_i$ (or the
natural process tolerances $t_i$ since $t_i = 6\sigma_i$), for $i = 1, 2$, for the two components
c characteristics. These decision variables when compared to the only decision variable of
the model by Ladany et al. (2007), which is the length of the warranty period $w$, clearly
show the significant difference between the two models. The model by Ladany et al.
(2007) seeks the optimal value of $w$, which is related to $P$, $Q$, and the lifetime
distribution of the product with given deterministic equations, such that profit is
maximized. Parameter and tolerance designs, which are the focus of the models in this dissertation, are not considered in their model.

The values of the model parameters and constraints used in this example, besides the ones that are already shown in Table 4.1, are given in Table 4.2.

<table>
<thead>
<tr>
<th>The total demand function parameters</th>
<th>$\alpha = 100,000,000$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\beta = 2.5$</td>
</tr>
<tr>
<td></td>
<td>$\gamma = 0.3$</td>
</tr>
<tr>
<td>The minimum and maximum price values</td>
<td>$0 \leq P \leq 60$</td>
</tr>
<tr>
<td>The minimum and maximum values for the length of the warranty period</td>
<td>$3 \leq w \leq 72$</td>
</tr>
<tr>
<td>The minimum and maximum total product quantity demanded during the planning horizon</td>
<td>$0 \leq Q \leq 200,000$</td>
</tr>
</tbody>
</table>

Table 4.2: The values of the model parameters and constraints that are specific to the example of this section.

Using Excel’s Solver, the optimal selling price, length of the warranty period, means, standard deviations, natural process tolerances, process capability index, quantity demanded, and total profit during the planning horizon for this example are found to be: $P^* = 38.25$, $w^* = 47.56$, $\mu_1^* = 9$, $\mu_2^* = 11$, $\mu^* = 20$, $\sigma_1^* = 0.170$, $\sigma_2^* = 0.201$, $\sigma^* = 0.263$, $t_1^* = 1.019$, $t_2^* = 1.208$, $t^* = 1.581$ (using the probabilistic relationship), $C_{pk}^* = 1.898$, $Q^* = 35196$, and $\pi^* = \$538,551.89$.

In this example, two of the key parameters that appear in the integrated model are the selling price elasticity $\beta$ and the length of the warranty period elasticity $\gamma$. These two parameters represent the sensitivity of the demand for the product (the customer’s
decision to purchase the product) to variations in the values of the selling price and the length of the warranty period, respectively. Table 4.3 presents the effect of varying the value of $\beta$ on the optimal solution while holding other model parameters constant as shown in Tables 4.1 and 4.2.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\sigma_1^*$</th>
<th>$\sigma_2^*$</th>
<th>$\sigma^*$</th>
<th>$t^*$</th>
<th>$C_{pk}^*$</th>
<th>$p^*$</th>
<th>$w^*$</th>
<th>$Q^*$</th>
<th>$\pi^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.153</td>
<td>0.181</td>
<td>0.237</td>
<td>1.419</td>
<td>2.114</td>
<td>50.11</td>
<td>72</td>
<td>143647</td>
<td>3599295.44</td>
</tr>
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<td>2.1</td>
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<td>0.237</td>
<td>1.419</td>
<td>2.114</td>
<td>47.84</td>
<td>72</td>
<td>107085</td>
<td>2439252.15</td>
</tr>
<tr>
<td>2.2</td>
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<td>0.237</td>
<td>1.419</td>
<td>2.114</td>
<td>45.94</td>
<td>72</td>
<td>79513</td>
<td>1660277.31</td>
</tr>
<tr>
<td>2.3</td>
<td>0.153</td>
<td>0.181</td>
<td>0.237</td>
<td>1.419</td>
<td>2.114</td>
<td>44.33</td>
<td>72</td>
<td>58854</td>
<td>1134365.85</td>
</tr>
<tr>
<td>2.4</td>
<td>0.159</td>
<td>0.188</td>
<td>0.246</td>
<td>1.478</td>
<td>2.029</td>
<td>41.50</td>
<td>61.49</td>
<td>45023</td>
<td>778441.74</td>
</tr>
<tr>
<td>2.5</td>
<td>0.170</td>
<td>0.201</td>
<td>0.263</td>
<td>1.581</td>
<td>1.898</td>
<td>38.25</td>
<td>47.56</td>
<td>35196</td>
<td>538551.89</td>
</tr>
<tr>
<td>2.6</td>
<td>0.181</td>
<td>0.215</td>
<td>0.281</td>
<td>1.684</td>
<td>1.781</td>
<td>35.58</td>
<td>37.28</td>
<td>27438</td>
<td>375460.09</td>
</tr>
<tr>
<td>2.7</td>
<td>0.192</td>
<td>0.228</td>
<td>0.298</td>
<td>1.789</td>
<td>1.677</td>
<td>33.34</td>
<td>29.56</td>
<td>21347</td>
<td>263562.21</td>
</tr>
<tr>
<td>2.8</td>
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<td>0.242</td>
<td>0.316</td>
<td>1.896</td>
<td>1.583</td>
<td>31.43</td>
<td>23.68</td>
<td>16584</td>
<td>186161.76</td>
</tr>
<tr>
<td>2.9</td>
<td>0.215</td>
<td>0.255</td>
<td>0.334</td>
<td>2.004</td>
<td>1.497</td>
<td>29.80</td>
<td>19.14</td>
<td>12870</td>
<td>132231.07</td>
</tr>
<tr>
<td>3</td>
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<td>0.269</td>
<td>0.352</td>
<td>2.112</td>
<td>1.420</td>
<td>28.38</td>
<td>15.62</td>
<td>9979</td>
<td>94404.35</td>
</tr>
</tbody>
</table>

Table 4.3: Effect of the selling price elasticity $\beta$ on the optimal solution.

It can be seen from Table 4.3 that as $\beta$ increases, $P^*$ decreases. Moreover, as $\beta$ keeps increasing, at a certain point, $w^*$ starts to decrease and the optimal variances and tolerances start to increase as well thereby lowering $C_{pk}^*$. The effects $\beta$ has on the optimal solution can be explained fairly intuitively. As $\beta$ increases, higher prices are penalized more severely by lower product demand. As the selling price is forced to
decrease, offering long warranty periods becomes more difficult and financially unjustifiable. As a result of the shorter warranty periods in addition to the lower selling prices, it becomes unnecessary to maintain high levels of $C_{pk}$ values. Therefore, $C_{pk}$ is lowered by increasing the variances.

The effect of varying the value of $\gamma$ on the optimal solution while holding other model parameters constant is shown in Table 4.4.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\sigma_1^*$</th>
<th>$\sigma_2^*$</th>
<th>$\sigma^*$</th>
<th>$t^*$</th>
<th>$C_{pk}^*$</th>
<th>$P^*$</th>
<th>$w^*$</th>
<th>$Q^*$</th>
<th>$\pi^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>0.315</td>
<td>0.373</td>
<td>0.488</td>
<td>2.930</td>
<td>1.024</td>
<td>26.46</td>
<td>3.842</td>
<td>33975</td>
<td>359611.84</td>
</tr>
<tr>
<td>0.175</td>
<td>0.286</td>
<td>0.340</td>
<td>0.444</td>
<td>2.666</td>
<td>1.125</td>
<td>27.82</td>
<td>6.09</td>
<td>33618</td>
<td>374048.17</td>
</tr>
<tr>
<td>0.2</td>
<td>0.255</td>
<td>0.303</td>
<td>0.396</td>
<td>2.378</td>
<td>1.262</td>
<td>29.53</td>
<td>9.838</td>
<td>33323</td>
<td>393677.17</td>
</tr>
<tr>
<td>0.225</td>
<td>0.227</td>
<td>0.269</td>
<td>0.352</td>
<td>2.112</td>
<td>1.420</td>
<td>31.53</td>
<td>15.62</td>
<td>33241</td>
<td>419284.39</td>
</tr>
<tr>
<td>0.25</td>
<td>0.204</td>
<td>0.242</td>
<td>0.316</td>
<td>1.896</td>
<td>1.583</td>
<td>33.68</td>
<td>23.68</td>
<td>33518</td>
<td>451497.74</td>
</tr>
<tr>
<td>0.275</td>
<td>0.185</td>
<td>0.219</td>
<td>0.287</td>
<td>1.722</td>
<td>1.742</td>
<td>35.92</td>
<td>34.22</td>
<td>34175</td>
<td>490966.70</td>
</tr>
<tr>
<td>0.3</td>
<td>0.170</td>
<td>0.201</td>
<td>0.263</td>
<td>1.581</td>
<td>1.898</td>
<td>38.25</td>
<td>47.56</td>
<td>35196</td>
<td>538551.89</td>
</tr>
<tr>
<td>0.325</td>
<td>0.157</td>
<td>0.186</td>
<td>0.244</td>
<td>1.463</td>
<td>2.051</td>
<td>40.70</td>
<td>64.06</td>
<td>36572</td>
<td>595388.84</td>
</tr>
<tr>
<td>0.35</td>
<td>0.153</td>
<td>0.181</td>
<td>0.237</td>
<td>1.419</td>
<td>2.114</td>
<td>41.76</td>
<td>72</td>
<td>39640</td>
<td>662170.53</td>
</tr>
<tr>
<td>0.375</td>
<td>0.153</td>
<td>0.181</td>
<td>0.237</td>
<td>1.419</td>
<td>2.114</td>
<td>41.76</td>
<td>72</td>
<td>44114</td>
<td>736890.85</td>
</tr>
<tr>
<td>0.4</td>
<td>0.153</td>
<td>0.181</td>
<td>0.237</td>
<td>1.419</td>
<td>2.114</td>
<td>41.76</td>
<td>72</td>
<td>49091</td>
<td>820042.71</td>
</tr>
</tbody>
</table>

Table 4.4: Effect of the length of the warranty period elasticity $\gamma$ on the optimal solution.

Table 4.4 shows that as $\gamma$ increases, $P^*$, $w^*$, and $C_{pk}^*$ keep increasing and the optimal variances and tolerances keep decreasing until the maximum value of $w$ is reached (at $\gamma = 0.3355$), where they stay constant even with further increases in $\gamma$. On
the other hand, it is interesting to see that $Q'$ decreases as $\gamma$ increases until it reaches a minimum at around $\gamma = 0.225$, where it then starts increasing again. These results can be explained by the fact that higher $\gamma$ values imply better customer responsiveness to longer warranty periods. As a result, higher prices can be charged to cover the expenses of the longer warranty periods and achieve higher profitability.

It is to be noted that the optimal means for the main product characteristic and the two components characteristics are not shown in Tables 4.3 and 4.4. This is because the optimal means are always given by $\mu_1^* = 9$, $\mu_2^* = 11$, and $\mu^* = 20$ since they are independent of the values of $\beta$ and $\gamma$. It is worth mentioning, however, that the values of these optimal means are expected to vary with changes in $\beta$ and $\gamma$ if the design tolerance was unbalanced, i.e. $x_0 - LSL \neq USL - x_0$, or if $\rho \neq 0$ for $X_1$ and $X_2$.

Furthermore, it can be seen from the results presented above that the constraints on the selling price and the total quantity demanded, which are shown in Table 4.2, are never binding and the same results could be achieved without including these constraints in the optimization model. Nonetheless, it is still recommended to include these constraints in the model since not only do they insure achieving reasonable solutions, but also it might be more realistic to expect constrained selling price and quantity demanded in real-life situations.
4.1.2 Example 2: Using the Length of the Warranty Period and the Process Capability Index as Signals for the Quality Level of the Product

In this example, a modified version of the model proposed by Karmarkar et al. (1997) for a monopolist in a static market (equation (4.1)) is used. This example is chosen to show how utility functions, as opposed to demand functions, can be used to capture customer preferences with respect to product price and quality level. The objective function and the minimum customer utility constraint are given by (Karmarkar et al., 1997):

Maximize \( \pi = P - MC - WC \) \hspace{1cm} (4.9a)

Subject to: \( U \geq u_0 \) \hspace{1cm} (4.9b)

where \( \pi \) is the profit per item, \( P \) is the selling price of an item, \( MC \) and \( WC \) are the manufacturing and warranty costs per item, which are given by equations (4.4) and (4.5), respectively, \( U \) is the expected customer utility, and \( u_0 \) is the minimum acceptable customer utility that results in a purchase of the product. The expected customer utility \( U \) is given by:

\[
U(P, w, C_{pk}) = P_{\text{max}} - P - \frac{E}{(A_1 w)^{A_2} + (B_1 C_{pk})^{B_2}} \tag{4.10}
\]

where \( P \) is again the selling price of an item, \( w \) is the length of the warranty period, \( C_{pk} \) is the process capability index, \( P_{\text{max}} \) is the maximum price the customer is willing to pay for the product, and \( E \), \( A_1 \), \( A_2 \), \( B_1 \), and \( B_2 \) are the model parameters, which capture the effects of \( w \) and \( C_{pk} \) on \( U \) . It is worth mentioning that equation (4.10) is quite different from the equation used by Karmarkar et al. (1997), where the expected customer utility is
related to the mean $\mu$ and variance $\sigma^2$ of the product quality characteristic in addition to the selling price $P$ (e.g. equation (2.8)). While it is reasonable, in theory, to expect the customer utility to be related to $\mu$ and $\sigma^2$, it could be fairly difficult in practice to capture this relationship. Instead, it is easier to relate the customer utility to $w$ and $C_{pk}$ as in equation (4.10). For example, it is much harder for customers to provide the price that they are willing to pay for a product, such as an energy-efficient lamp, in terms of the values of its inner dimensions and tolerances than it is to do so in terms of $w$ and the maximum expected proportion of defects of the product (conveyed by the value of $C_{pk}$).

Setting $u_0 = 0$ as in the model by Karmarkar et al. (1997), the minimum customer utility constraint (equation (4.9b)) becomes:

$$P \leq P_{\text{max}} - \frac{E}{(A_1 w)^{A_1} + (B_1 C_{pk})^{B_1}}$$  \hspace{1cm} (4.11)

which can be replaced with an equality constraint since, in the optimal solution, the right-hand-side of the constraint would never be greater than the selling price (the left-hand-side). This is because maximizing the profit forces the selling price to be set at the highest acceptable value (Karmarkar et al., 1997).

Again, equation (4.11) suggests that the customer uses $w$ and $C_{pk}$ as signals for the quality of the product. The greater the values of $w$ and $C_{pk}$, the higher the value of $P$ that the customer is willing to pay for the product. However, as $w$ and $C_{pk}$ keep increasing, $P$ gets asymptotically closer to $P_{\text{max}}$. Figure 4.2 shows the relationship
represented by equation (4.11) when the constraint is binding (equality constraint), where

\[ P_{\text{max}} = 32, \; E = 29, \; A_1 = 0.1, \; A_2 = 0.7, \; B_1 = 1, \text{ and } B_2 = 0.9. \]
The decision variables of this optimization model are the selling price $P$, the length of the warranty period $w$, and the process capability index $C_{pk}$ along with the means $\mu_i$ and variances $\sigma_i$ (or the natural process tolerances $t_i$), for $i=1,2$, for the two components characteristics. Table 4.5 shows the values of the model parameters and constraints used in this example besides the ones that are already shown in Table 4.1.

| The expected customer utility model parameters | $P_{\text{max}} = 32$  
$E = 29$  
$A_1 = 0.1$  
$A_2 = 0.7$  
$B_1 = 1$  
$B_2 = 0.9$ |
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>The minimum and maximum price values</td>
<td>$0 \leq P \leq 60$</td>
</tr>
<tr>
<td>The minimum and maximum values for the length of the warranty period</td>
<td>$3 \leq w \leq 72$</td>
</tr>
<tr>
<td>The minimum acceptable customer utility</td>
<td>$u_0 = 0$</td>
</tr>
</tbody>
</table>

Table 4.5: The values of the model parameters and constraints that are specific to the example of this section.

Using Excel’s Solver, the optimal selling price, length of the warranty period, means, standard deviations, natural process tolerances, process capability index, and the profit per item for this example are found to be: $P^* = 24.40$, $w^* = 26.05$, $\mu_1^* = 9$, $\mu_2^* = 11$, $\mu^* = 20$, $\sigma_1^* = 0.162$, $\sigma_2^* = 0.192$, $\sigma^* = 0.251$, $t_1^* = 0.971$, $t_2^* = 1.150$, $t^* = 1.505$ (using the probabilistic relationship), $C_{pk}^* = 1.993$, and $\pi^* = $3.51.
4.1.3 Example 3: A Duopoly Competition

One of the most interesting market structure types in terms of product competition and market dynamics is the case of an oligopoly competition. In order to show how the proposed integrated parameter and tolerance designs models can be developed for this type of market structure, a representative example of a duopoly competition is chosen. A duopoly competition is an oligopoly market structure with only two firms. In this market structure, the two firms consider the reaction of the customers as well as the reaction of their competitor when making pricing and quality decisions for their products. Similar to one of the models considered by Sinha et al. (2011), the two firms are involved in a simultaneous price and warranty competition. However, it is assumed in this example that one firm is designing a new product (product 1) to be introduced in the market while the other firm already has an existing product (product 2) in the market. This is different from the model by Sinha et al. (2011) where the two products exist in the market, i.e. product design is not considered, and both follow known Weibull lifetime distributions. In addition, it is assumed in this example that the customer assesses the quality level of the two products only by the length of the warranty periods offered with these products. Also, even though the two products are differentiated, they are considered to be close substitutes of each other.

In this example, an integrated parameter and tolerance designs model is developed for the new product in the duopoly competition described above. The objective function of the integrated model is given by:

$$\text{Maximize} \quad \pi_1 = (P_1 - MC - WC)Q_1$$  \hspace{1cm} (4.12)
where $\pi_i$ is the total profit of the firm with the new product during a given planning horizon, $P_i$ is the selling price of an item, $MC$ and $WC$ are the manufacturing and warranty costs per item, which are given by equations (4.4) and (4.5), respectively, and $Q_i$ is the total demand for the new product during the planning horizon. The total demand $Q_i$ for product $i$, where again $i=1$ refers to the new product and $i=2$ refers to the existing product, during the planning horizon is given by (Banker et al., 1998; Sinha et al., 2011):

$$Q_i = \alpha - \beta P_i + \gamma P_j + \theta w_i - \eta w_j$$

(4.13)

for $i, j = 1, 2$ and $i \neq j$, where $P$ is the selling price of the product, $w$ is the length of the warranty period, $\alpha, \beta, \gamma, \theta$, and $\eta$ are nonnegative model parameters with $\beta > \gamma$ and $\theta > \eta$. The decision variables for maximizing $\pi_1$ (equation (4.12)) are the selling prices $P_1$ and $P_2$, the length of the warranty periods $w_1$ and $w_2$, the means $\mu_1$ and $\mu_2$, and variances $\sigma_1$ and $\sigma_2$ (or the natural process tolerances $t_1$ and $t_2$) of the quality characteristics of the new product components.

At first glance, it seems that the model is complete. However, if this model is optimized the way it is, the optimal solution most definitely would be achieved at the highest and lowest possible values of $P_2$ and $w_2$, respectively. This is unrealistic since the competitor firm with the existing product is simultaneously maximizing its profit $\pi_2$. Thus, it is necessary to include the optimality conditions for the competitor as well.

The total profit $\pi_2$ for the firm with the existing product during the planning horizon is given by (Sinha et al., 2011):
\[ \pi_2 = \left( P_2 - n + \varepsilon (\lambda w_2)^\kappa \right) Q_2 \] (4.14)

where \( P_2 \) is the selling price of an item of the existing product, \( n \) is the manufacturing costs per item, \( \varepsilon \) is the minimum-repair warranty costs per item, \( (\lambda w_2)^\kappa \) is the expected number of failures during the warranty period \( w_2 \) since the existing product is assumed to have a Weibull lifetime distribution with scale parameter \( \lambda \) and shape parameter \( \kappa \), and \( Q_2 \) is given by equation (4.13). Moreover, in order to achieve Nash equilibrium for the simultaneous price and warranty competition between the two firms, it is important to add to the optimization model the following best response functions for \( P_2 \) and \( w_2 \) of the firm with the existing product, which are derived from the first-order conditions \( \frac{\partial \pi_2}{\partial P_2} = 0 \) and \( \frac{\partial \pi_2}{\partial w_2} = 0 \) (Sinha et al., 2011):

\[ P_2 = \frac{\alpha + \nu \beta + \gamma P_1 + \theta w_2 + \varepsilon (\lambda w_2)^\kappa \beta - \eta w_1}{2 \beta} \] (4.15)

\[ w_2 = \left[ \frac{(P_2 - \nu) \theta}{\varepsilon \lambda^\kappa (\theta w_2 + Q_2 \kappa)} \right]^{\frac{1}{\kappa - 1}} \] (4.16)

where, again, \( Q_2 \) is given by equation (4.13). These best response functions are guaranteed to provide a unique maximum value for the profit \( \pi_2 \) if the determinant of the Hessian of \( \pi_2 \) is negative, i.e. (Sinha et al., 2011):
\[
\begin{vmatrix}
\frac{\partial^2 \pi_2}{\partial P_2^2} & \frac{\partial^2 \pi_2}{\partial P_2 \partial w_2} \\
\frac{\partial^2 \pi_2}{\partial w_2 \partial P_2} & \frac{\partial^2 \pi_2}{\partial w_2^2}
\end{vmatrix} = \begin{bmatrix}
-2\beta & \theta + \beta \epsilon \kappa \lambda w_2^{\kappa-1} \\
\theta + \beta \epsilon \kappa \lambda w_2^{\kappa-1} & -\epsilon \kappa \lambda w_2^{\kappa-2}[2\theta w_2 + Q_2(\kappa - 1)]
\end{bmatrix} < 0
\]

or

\[
\left(\theta + \beta \epsilon \kappa \lambda w_2^{\kappa-1}\right)^2 > 2\beta \epsilon \kappa \lambda w_2^{\kappa-2}[2\theta w_2 + Q_2(\kappa - 1)]
\] (4.17)

Using Excel’s Solver along with the values of the model parameters and constraints used in this example, which are presented in Tables 4.1 and 4.6, as well as the equality constraints given by equations (4.15) and (4.16), the optimal selling prices, length of the warranty periods, means, standard deviations, natural process tolerances, process capability index of the new product, total demands, and total profits during the planning horizon for this example are found to be: \( P_1^* = 52.27 \), \( P_2^* = 49.72 \), \( w_1^* = 72 \), \( w_2^* = 56.89 \), \( \mu_1^* = 9 \), \( \mu_2^* = 11 \), \( \mu^* = 20 \), \( \sigma_1^* = 0.152 \), \( \sigma_2^* = 0.181 \), \( \sigma^* = 0.237 \), \( t_1^* = 0.915 \), \( t_2^* = 1.085 \), \( \epsilon^* = 1.419 \) (using the probabilistic relationship), \( C_{\mu k}^* = 2.11 \), \( Q_1^* = 626 \), \( Q_2^* = 489 \), \( \pi_1^* = $17,034.91 \) and \( \pi_2^* = $9,551.99 \).
The total demand function parameters
\[
\begin{align*}
\alpha &= 1000 \\
\beta &= 25 \\
\gamma &= 10 \\
\theta &= 10 \\
\eta &= 5
\end{align*}
\]

The manufacturing and warranty costs per item of the existing product (product 2)
\[
\begin{align*}
\nu &= 15 \\
\varepsilon &= 100
\end{align*}
\]

The scale and shape parameters of the Weibull lifetime distribution of the existing product
\[
\begin{align*}
\lambda &= 0.005 \\
\kappa &= 1.5
\end{align*}
\]

The minimum and maximum price values for the two products
\[
0 \leq P_1 \leq 60
\]
\[
0 \leq P_2 \leq 60
\]

The minimum and maximum values for the length of the warranty periods of the two products
\[
3 \leq w_1 \leq 72
\]
\[
3 \leq w_2 \leq 72
\]

The minimum and maximum total demands for the two products during the planning horizon
\[
0 \leq Q_1 \leq 2000
\]
\[
0 \leq Q_2 \leq 2000
\]

Table 4.6: The values of the model parameters and constraints that are specific to the example of this section.

### 4.1.4 Example 4: A Dynamic Market Analysis Approach

In this example, unlike the other examples considered so far where the market analysis approach is static, the selling price and the length of the warranty period are allowed to vary during the planning horizon. Thus, this example is a case of dynamic market analysis approach. The model considered here is based on the work by Huang et al. (2007). It is assumed that the selling price and the length of the warranty period can be modified every six months during a sixty-month ($L = 60$) planning horizon. The objective function for the integrated parameter and tolerance designs model is to maximize the total discounted profit $\pi$ over the planning horizon $L$, which is given by (Huang et al., 2007):
Maximize \[ \pi = \int_0^L e^{-\delta \tau} \left[ P(\tau) - MC - WC \right] q(\tau) d\tau \] (4.18)

where \( \delta \) is the discount rate, \( \tau \) is time, \( P(\tau) \) is the selling price of an item, \( MC \) and \( WC \) are the manufacturing and warranty costs per item, which are given by equations (4.4) and (4.5), respectively, and \( q(\tau) \) is the demand rate. It is worth emphasizing that, here, the \( MC \) and \( WC \) are related to the means and variance of the product components characteristics, unlike the model proposed by Huang et al. (2007), where \( MC \) and \( WC \) are related to a reliability parameter \( \theta \). As mentioned in the literature review of section 2.2, the main limitation of the model by Huang et al. (2007) is that it is unclear how the product ought to be designed in order to achieve a certain level of reliability using only the parameter \( \theta \).

The demand rate is modeled using a modified version of the displaced log-linear demand function proposed by Glickman et al. (1976), as follows (Huang et al., 2007):

\[ q(\tau) = \frac{\partial Q(\tau)}{\partial \tau} = \alpha P(\tau)^{-\beta} \left( w(\tau) + \xi \right)^\gamma \left[ 1 - \frac{Q(\tau)}{Q_M} \right] \left[ \psi + \frac{Q(\tau)}{Q_M} \right] \] (4.19)

where \( P(\tau), w(\tau), Q(\tau), \) and \( Q_M \) are the product’s selling price, length of warranty period, cumulative demand up to time \( \tau \), and maximum demand potential, respectively. Also, \( \alpha, \beta, \xi, \gamma, \) and \( \psi \) are model parameters. The decision variables of the optimization model, besides the means \( \mu_i \) and variances \( \sigma_i \) (or the natural process tolerances \( t_i \)), for \( i = 1, 2 \), of the two components characteristics, are the selling prices \( P_j \) and the lengths of the warranty periods \( w_j \) with \( j = 1, 2, ..., 10 \) for each of the ten half years (every six months for sixty months) as well as the cumulative demand at the end of
the planning horizon $Q_L$. The reason for using $Q_L = Q(60)$ as a decision variable is because $q(\tau)$ depends on the value of $Q(\tau)$, as shown in equation (4.19). Thus, after finding $q(60)$ from equation (4.19), backward induction is used to find all values of $Q(\tau)$ using (Huang et al., 2007):

$$Q(\tau - 6) = Q(\tau) - 6q(\tau)$$  \hspace{1cm} (4.20)

for $\tau = 0, 6, 12, \ldots, 54$ with $q(\tau)$ calculated from equation (4.19), such that, $Q(0) = Q_0$. The values of the model parameters and constraints used in this example, besides the ones that are already shown in Table 4.1, are given in Table 4.7.

<table>
<thead>
<tr>
<th>The discounted rate</th>
<th>$\delta = 0.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>The demand rate function parameters</td>
<td>$\alpha = 10,000,000$</td>
</tr>
<tr>
<td></td>
<td>$\beta = 2.5$</td>
</tr>
<tr>
<td></td>
<td>$\xi = 0.1$</td>
</tr>
<tr>
<td></td>
<td>$\gamma = 0.3$</td>
</tr>
<tr>
<td></td>
<td>$\psi = 0.03$</td>
</tr>
<tr>
<td>The initial demand (added to the model as an equality constraint)</td>
<td>$Q_0 = 2000$</td>
</tr>
<tr>
<td>The total demand potential</td>
<td>$Q_M = 30,000$</td>
</tr>
<tr>
<td>The minimum and maximum price values for the ten half years ($j = 1, 2, \ldots, 10$)</td>
<td>$0 \leq P_j \leq 60$</td>
</tr>
<tr>
<td>The minimum and maximum values for the length of the warranty periods for the ten half years ($j = 1, 2, \ldots, 10$)</td>
<td>$3 \leq w_j \leq 72$</td>
</tr>
</tbody>
</table>

Table 4.7: The values of the model parameters and constraints that are specific to the example of this section.
Using Excel’s Solver, the optimal means, standard deviations, natural process tolerances, and process capability index for this example are found to be: \( \mu_1^* = 9 \), \( \mu_2^* = 11 \), \( \sigma_1^* = 0.175 \), \( \sigma_2^* = 0.208 \), \( \sigma^* = 0.272 \), \( t_1^* = 1.051 \), \( t_2^* = 1.246 \), \( t^* = 1.63 \) (using the probabilistic relationship), and \( C_{pk}^* = 1.84 \). Moreover, the optimal selling prices, lengths of the warranty periods, cumulative demand, and discounted profit for each of the ten half years are given in Table 4.8.

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( P^* )</th>
<th>( w^* )</th>
<th>( q^* )</th>
<th>( Q^* )</th>
<th>( \Delta \pi^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>2000</td>
<td>–</td>
</tr>
<tr>
<td>6</td>
<td>31.09</td>
<td>34.54</td>
<td>1285.32</td>
<td>9712</td>
<td>54166.47</td>
</tr>
<tr>
<td>12</td>
<td>43.27</td>
<td>52.95</td>
<td>707.78</td>
<td>13959</td>
<td>34146.07</td>
</tr>
<tr>
<td>18</td>
<td>49.29</td>
<td>63.02</td>
<td>524.48</td>
<td>17106</td>
<td>17438.24</td>
</tr>
<tr>
<td>24</td>
<td>53.41</td>
<td>70.23</td>
<td>407.92</td>
<td>19553</td>
<td>8479.40</td>
</tr>
<tr>
<td>30</td>
<td>55.45</td>
<td>72</td>
<td>332.30</td>
<td>21547</td>
<td>4042.84</td>
</tr>
<tr>
<td>36</td>
<td>57.09</td>
<td>72</td>
<td>268.07</td>
<td>23155</td>
<td>1887.28</td>
</tr>
<tr>
<td>42</td>
<td>58.15</td>
<td>72</td>
<td>218.36</td>
<td>24465</td>
<td>871.61</td>
</tr>
<tr>
<td>48</td>
<td>57.50</td>
<td>64.64</td>
<td>182.02</td>
<td>25558</td>
<td>399.08</td>
</tr>
<tr>
<td>54</td>
<td>52.38</td>
<td>40.17</td>
<td>161.56</td>
<td>26527</td>
<td>178.29</td>
</tr>
<tr>
<td>60</td>
<td>44.29</td>
<td>26.77</td>
<td>162.19</td>
<td>27500</td>
<td>76.65</td>
</tr>
</tbody>
</table>

Table 4.8: The optimal solution for each of the ten half year periods.

### 4.2 Discussion

Similar to the proposed models of Chapter 3, it is important to note that the purpose of the integrated parameter and tolerance designs models, which are presented in
the examples of section 4.1, is not to serve as comprehensive models for new product
development to be used blindly by designers and practitioners. Instead, the main goal of
these integrated models is to present the viability and flexibility of the general framework
of introducing microeconomic and marketing decisions at the early stages of new product
development. It is left to the designer and practitioner to use the right model elements that
best fit their real-life applications.

In this chapter, besides the new expected utility function given by equation (4.10),
there is no attempt or claim to develop new microeconomic and marketing models. Thus,
it can be noticed from the examples of section 4.1 that the aspects of the integrated
parameter and tolerance models related to microeconomic and marketing modeling, i.e.
demand functions, market structures, equilibrium conditions for the duopoly competition,
etc., follow closely existing models from the microeconomic and marketing literature. A
conscious decision is made not to alter these existing models except when it is necessary
to fit the general framework of the proposed models and to better convey the purpose of
using them. This is to show that there is already a well-established field of study related
to pricing decisions and modeling customer preferences, which leaves no room for poor
representation of customer and market behaviors as it is with the use of the quadratic loss
function. Again, while the idea of the quadratic loss function is a novel one, where it is
used to capture the cost of low product quality, modeling the cost of low product quality
as the cost of warranty and the relatively low product selling price due to low quality is a
more logical and practical approach than trying to list and estimate all possible external
failure costs.
The optimal solutions obtained using the integrated parameter and tolerance designs models of section 4.1 are determined such that the maximum profitability is achieved while taking virtually all costs, manufacturing capabilities, and market dynamics as well as the voice of the customer into consideration. As a result, there is no need to set $C_{pk}$ to an arbitrarily high value, such as $C_{pk} = 2$ used by six-sigma companies, or to impose a lower-bound on $C_{pk}$ as it is done by introducing the constraint for the natural process tolerance $t$ (equation (3.20)) in the models of Chapter 3.

It is worth mentioning that, even though sensitivity analysis is conducted for two model parameters from the example of section 4.1.1, full sensitivity analysis is not pursued for any of the examples of section 4.1. This is because the focus of the examples is not on drawing specific results and conclusions as much as it is on presenting the variety of scenarios, challenges, and solution approaches for the integrated parameter and tolerance designs models with microeconomic and marketing considerations. Despite that, it is important to note that it is highly recommended in practice to conduct sensitivity analysis, at least for the main model parameters, since this analysis both validates the appropriateness of the proposed models and provides additional insights into the models and their solutions. Obviously, for the examples of section 4.1, sensitivity analysis can easily be conducted for any model parameter of interest in a similar manner as it is conducted for the two parameters of the example of section 4.1.1.
Chapter 5

Research Contributions and Directions for Future Research

It is concluded from this research that, in order to develop a product that is competitive and successful in the marketplace, the decisions related to both marketing and manufacturing operations must be considered simultaneously in an efficient and optimal manner. This is because the main decisions from both these areas are interdependent. Setting the selling price and positioning the product in the right market environment require the knowledge of manufacturing capabilities and costs. On the other hand, designing the product and the manufacturing processes require the knowledge of customer preferences and market constraints. Despite that, existing design models from each of the two fields of marketing and manufacturing lack the required rigor in assessing the other field’s input to their models. More specifically, it is found that virtually all existing integrated parameter and tolerance designs models use the quadratic loss function to capture the voice of the market and customer. The quadratic loss function is well-known to suffer from its vague definition, simplistic form, and difficult to estimate proportionality constant. On the other hand, manufacturing and quality costs are grossly underrepresented in most existing pricing models, where simple functions and parameters, such as categorical grades and classes for the product quality levels or fixed and variable costs for the product total costs, are used.

In this research, new integrated parameter and tolerance designs models are proposed that address the main limitations of existing models and bridge the gap between
manufacturing and marketing operations and decisions. The contributions of this research are highlighted in section 5.1 and the directions for future research are presented in section 5.2.

5.1 Research Contributions

In this section, the main contributions of this research are presented as follows:

1. One of the most important contributions of this research is proposing integrated parameter and tolerance designs models that eliminate the quadratic loss function and replace it with warranty costs. Warranty costs can be estimated more easily and accurately using warranty and reliability models, which are well-defined and are based on measurable empirical data. This greatly simplifies the task of communicating with top management and making critical decisions about product design and marketing. Moreover, the warranty and reliability models as well as the manufacturing costs used in the proposed integrated models are related to product attributes and specifications, i.e. means and tolerances of all the quality characteristics of the product and its components, length of the warranty period, process capability index, etc., thereby establishing a logical and concise framework for integrating parameter and tolerance designs.

2. The first set of integrated models proposed in this research assumes that the external failure costs other than warranty costs are negligible. For the cases where these external failure costs cannot be neglected, a second set of
integrated parameter and tolerance designs models are proposed. For these models, marketing tools, such as pricing models, market structure analysis, product demand and customer utility functions, etc., are used. These proposed integrated models are based on the idea that external failure costs other than warranty costs can be estimated as the increase (decrease) in the price the customer is willing to pay for the product relative to the amount of increase (decrease) in product quality. In other words, the opportunity costs of using low quality products are considered as opposed to actual costs and losses. Consequently, the benefits of the second set of the proposed integrated parameter and tolerance designs models is twofold. First, the issue of the external failure costs other than warranty costs is addressed. Second, existing pricing and competition models from the microeconomic and marketing fields are improved by introducing in them better models for the manufacturing and warranty costs of the product, which are related to product attributes and specifications.

3. The flexibility of the general framework of the proposed integrated parameter and tolerance designs models and its applicability to a wide range of problems are clearly demonstrated using a variety of numerical examples and solution methodologies. The numerical examples include the cases of linear and nonlinear relationships between the main quality characteristic of the product and the quality characteristics of its components as well as the case of finite manufacturing processes. Moreover, numerical examples for the cases of static market analysis approach for a general demand function, a
monopoly, and a duopoly competition as well as the case of a dynamic market analysis approach are also considered. In these different numerical examples, a variety of optimization and analysis techniques, such as nonlinear programming, integer programming, response surface methodology, simulation, and sensitivity analysis, are employed to show the methodology of handling the different design problems.

4. There are several specific contributions that can be mentioned, which are consequences of the more technical part of this research. These contributions include:

   a. Improving the stack-up constraint for the allowable design tolerance $t$ of the main product characteristic by relating $t$ to the means and the natural process tolerances of the product components characteristics using the process capability index $C_{pk}$ (equation (3.20)). Most existing integrated parameter and tolerance designs models either fail to properly include the effects of the means of the components characteristics in the stack-up constraint for $t$ or simply ignore them altogether.

   b. Proposing a new customer utility function that is related to the product selling price $P$, the length of the warranty period $w$, and the process capability index $C_{pk}$. Not only does this new utility function include more than one metric of product quality, i.e. $w$ and $C_{pk}$, but also the quality metrics $w$ and $C_{pk}$ better resonate with the customer when
assessing the product quality as opposed to relating product quality
directly to product specifications, such as means and variances of the
quality characteristics of the product and its components. Details about
the formulation of this new utility function are found in section 4.1.2.

5.2 Directions for Future Research

In this section, several directions for future research are suggested as follows:

1. The methodology of implementing and validating the integrated parameter
   and tolerance designs models proposed in this dissertation have been shown
   using simple and fictitious numerical examples. An area of future research
   would be to see how the proposed models perform using real-life problems.

2. The proposed models can be extended and modified to include other issues
   that have not been considered, such as fixed costs, inventory costs, quality
   inflation, advertisement, cooperation of rival firms, transportation, etc.

3. The product demand and customer utility are represented in the proposed
   models using deterministic functions, i.e. the expected demand and expected
   utility are used. These functions can probably better represent the customer
   and market behaviors by replacing them with stochastic functions that follow
   certain probability distributions, as suggested by Lin et al. (2005).

4. In real life problems, the costs of monitoring and maintaining the means of
   the quality characteristics at their intended nominal values can be significant.
   These costs are not considered in the proposed models. Thus, the proposed
models can be modified to include these costs and to allow for some level of deviation from the optimal values. The implications of these modifications should be taken into consideration, especially their effects on the reliability model.

5. Another area of future research is to analyze the model complexity and compare different optimization algorithms, heuristics, etc., in order to identify ways to simplify the problem and its solution without sacrificing real-life representation.

6. In the proposed models, the length of the warranty period is the main metric used as the signal for product quality to the customer. A direction of future research can be to study other metrics that affect the customer’s purchasing decisions and patterns. Product effectiveness, weight, brightness, toxicity, scent, durability, texture, quietness, etc., are examples of product metrics. These metrics are likely to vary greatly from one product (or one market) to another.

7. In order to take full advantage of the proposed models, the Graphic User Interface (GUI) capabilities of software programs, such as MATLAB, can be used to program the process of importing and reading data, fitting models, optimizing, and conducting sensitivity analysis in a user-friendly manner. This way, the procedure of implementing the proposed models by designers and practitioners and communicating their results to top management is greatly simplified.
Appendix A

Minitab Output of the Central Composite Design (CCD)

One replication of the Central Composite Design (CCD) generated by Minitab for the example of section 3.2.2.1 is shown below:

Central Composite Design

Factors: 4 Replicates: 1
Base runs: 31 Total runs: 31
Base blocks: 1 Total blocks: 1

Two-level factorial: Full factorial

Cube points: 16
Center points in cube: 7
Axial points: 8
Center points in axial: 0

Alpha: 2

Design Table

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Appendix B

Cholesky Decomposition of a Two-by-Two Covariance Matrix

Given a covariance matrix $\Sigma$ that is positive definite, the Cholesky decomposition is defined as the matrix $C$ such that $\Sigma = CC^T$ (Kroese et al., 2011). For a two-by-two covariance matrix, let:

$$\Sigma = \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = CC^T \quad (B.1)$$

Then,

$$s_1 = a^2$$
$$s_2 = ab$$
$$s_3 = ab$$
$$s_4 = b^2 + c^2 \quad (B.2)$$

Solving for $a$, $b$, and $c$,

$$a = \sqrt{s_1}$$
$$b = \frac{s_2}{\sqrt{s_1}}$$
$$c = \sqrt{s_4 - \frac{s_2^2}{s_1}} \quad (B.3)$$

Thus, for a covariance matrix given by:

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix} \quad (B.4)$$
the Cholesky decomposition is given by:

\[ C = \begin{bmatrix} \sigma_1 & 0 \\ \rho \sigma_2 & \sqrt{1 - \rho^2 \sigma_2} \end{bmatrix} \]  

(B.5)
Appendix C

Normality Check for $X = X_1 X_2$

In section 3.2.2.2, it is assumed that $X = X_1 X_2$ is normally distributed so that sensitivity analysis can be conducted in a straightforward manner as in section 3.2.1.2. In order to check the appropriateness of the normality assumption, a comparison is made between random samples that are generated for $X = X_1 X_2$, where $X_1$ and $X_2$ have a joint multivariate normal distribution with a correlation coefficient $\rho$, and a normally distributed $X$ with mean and variance $\mu = \mu_1 \mu_2 + \rho \sigma_1 \sigma_2$ and $\sigma^2 = \mu_1^2 \sigma_1^2 + \mu_2^2 \sigma_2^2 + 2 \rho \mu_1 \mu_2 \sigma_1 \sigma_2$, respectively. Figure C.1 shows the Probability Density Functions (PDFs) for $X = X_1 X_2$ and the normally distributed $X$, where the PDFs are obtained using histograms of the randomly generated samples for $\mu_1 = 9$, $\mu_2 = 11$, $\sigma_1 = 0.3$, $\sigma_2 = 0.3$, and $\rho = 0.8$ (different values for $\mu_1$, $\mu_2$, $\sigma_1$, $\sigma_2$, and $\rho$ were used to generate the random samples and similar results were obtained). It can be seen from Figure C.1 that the random sample of $X = X_1 X_2$ has the appropriate bell-shaped PDF, which is almost identical in shape to the PDF of the random sample of the normally distributed $X$. Moreover, the values of the means and standard deviations of the two random samples are also very similar.
Figure C.1: Probability Density Functions (PDFs) of $X = X_1 X_2$ and the normally distributed $X$.

The normality of $X = X_1 X_2$ is also checked using the Anderson-Darling (AD) test and the normal probability plot. The hypotheses for the AD test are $H_0$: the distribution of $X$ is normal versus $H_1$: the distribution of $X$ is not normal. The p-value of this test ought to be less than the predetermined significance level $\alpha$ (typically, $\alpha = 0.05$ or $0.1$) for $H_0$ to be rejected. Figures C.2 and C.3 show the normal probability plots and the p-values for the two random samples generated for $X = X_1 X_2$ and the normally distributed $X$. Since the p-value in Figure C.2 is 0.614, $H_0$ cannot be rejected and $X = X_1 X_2$ can be assumed normal.
Figure C.2: Normal probability plot for $X = X_1X_2$.

Figure C.3: Normal probability plot for the normally distributed $X$.
Bibliography


Vita

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Jawad S. Hassan is currently in the process of finishing his dual-title Ph.D. degree in Industrial Engineering and Operations Research with a minor in Statistics from The Pennsylvania State University. After graduation, he will join the department of Industrial and Management Systems Engineering at Kuwait University as an assistant professor. Jawad received his B.S. and M.S. degrees in Mechanical Engineering from The Pennsylvania State University in 2003 and 2005, respectively. His M.S. thesis is titled: “Numerical Study of the Effect of High Free-Stream Turbulence on Flat-Plate Film Cooling Effectiveness using FLUENT”, and it was under the supervision of Professor Savas Yavuzkurt. With a teaching assistantship from the Mechanical and Nuclear Engineering Department at The Pennsylvania State University and after spending a year in its Mechanical Engineering Ph.D. program, Jawad was offered a full scholarship from Kuwait University to pursue his M.S. and Ph.D. degrees in Industrial Engineering. He received his dual-title M.S. degree in Industrial Engineering and Operations Research with a Quality Engineering option from the Pennsylvania State University in 2009. His second M.S. thesis is titled: “External Failure Cost Estimation using Reliability models: an Alternative to Taguchi’s Loss Function”, and it was under the supervision of Professor M. Jeya Chandra.