COEXISTENCE OF HYPERBOLIC AND NON-HYPERBOLIC

BEHAVIOR IN SMOOTH DYNAMICAL SYSTEMS

A Dissertation in Mathematics
by
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Abstract

We investigate the essential coexistence of hyperbolic and non-hyperbolic behavior in dynamical systems, which was numerically observed in classical mechanics but theoretically remains an important open problem. We present some recent developments of such coexistence phenomenon in the category of smooth conservative systems in both discrete and continuous-time cases.

In the discrete-time case, we show that there is a smooth volume preserving diffeomorphism of a 4-dimensional compact smooth manifold, which is close to the identity map and has nonzero Lyapunov exponents on an open and dense subset of positive but not full volume while having zero Lyapunov exponents on its complement. Moreover, this subset consists of countably many connected components, on each of which the diffeomorphism is isomorphic to a Bernoulli automorphism.

We demonstrate an essential coexistence phenomenon in the continuous-time case by constructing a smooth volume preserving flow on a 5-dimensional compact smooth manifold that has nonzero Lyapunov exponents almost everywhere on an open and dense subset of positive but not full volume and is ergodic on this subset while having zero Lyapunov exponents on its complement. The latter is a union of 3-dimensional invariant submanifolds and on each of these submanifolds the flow is linear with Diophantine frequency vector. Thus the flow exhibits a KAM-type phenomenon.
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Dedication

To my family and friends for their love and support.
Chapter 1

Introduction and Overview

1.1 Lyapunov exponents

The Lyapunov exponent is an important characteristic of a dynamical system that measures the rate of separation of infinitesimally close trajectories. To be precise, we consider a smooth system $f^t$ with $t \in \mathbb{Z}$ or $\mathbb{R}$ acting on a compact smooth Riemannian manifold $M$. The Lyapunov exponent of $f^t$ at a point $x \in M$ of a vector $v \in T_x M$ is defined by the formula

$$\lambda(x, v, f^t) = \limsup_{t \to \infty} \frac{1}{t} \log \|d_x f^t v\|. \quad (1.1)$$

We shall mainly investigate the conservative systems. More precisely, let us consider a smooth conservative dynamical system $(M, f^t, \mu)$, that is, the system $f^t : M \to M$ preserves a smooth measure $\mu$ — a measure which is equivalent to the Riemannian volume on $M$. Such a measure is also called a “volume”.

Presence of nonzero Lyapunov exponents on a subset of positive volume indicates some sort of instability while presence of zero Lyapunov exponents on a subset of positive volume describe a certain regular behavior of the system.
1.1.1 Hyperbolic systems with nonzero Lyapunov exponents

A system \((\mathcal{M}, f^t, \mu)\) is said to have \textit{nonzero Lyapunov exponents} if

\[ \lambda(x, v, f^t) \neq 0 \]

for \(\mu\)-almost every \(x \in \mathcal{M}\) and every vector \(v \in T_x\mathcal{M}\) (except along the flow direction in the continuous-time case). It follows from Pesin theory that a conservative system is (completely nonuniformly) hyperbolic if and only if it has nonzero Lyapunov exponents (see [BP07, Pes07]). Roughly speaking, the system exhibits instability along typical trajectories, which can be described by appropriate hyperbolicity conditions. Such instability and the compactness of \(\mathcal{M}\) cause the system to exhibit the “deterministic chaos”— the appearance of stochastic behavior in purely deterministic systems.

The statistical properties of hyperbolic systems has been studied since 1960 along with fruitful results, the existence of hyperbolic systems was yet established not very long ago.

**Theorem 1.1.1.** [DP02] \textit{Any compact smooth Riemannian manifold \(\mathcal{M}\) of dimension \(\geq 2\) admits a \(C^\infty\) volume preserving diffeomorphism \(f\), which has nonzero Lyapunov exponents almost everywhere and is Bernoulli.}

A similar result for the continuous-time case was obtained in [HPT04].

1.1.2 Non-hyperbolic behavior with zero Lyapunov exponents

On the other hand, there are many systems whose behavior is quite regular, namely, all Lyapunov exponents are zero on a subset of positive volume. We emphasize that such a regular set is often persistent under small perturbations and so cannot be neglected. For instance, the celebrated KAM theory shows that any small perturbation of a completely integrable Hamiltonian systems possesses a positive volume Cantor set of invariant tori, on which the perturbed Hamiltonian flow is quasi-periodic and has all zero Lyapunov exponents.
There is also a discrete version of KAM phenomenon in the category of volume preserving diffeomorphisms.

**Theorem 1.1.2.** [CS89, Her90, Xia92, Yoc92] Let $\mathcal{M}$ be a Riemannian manifold of dimension $\geq 2$. For sufficiently large integer $r$, $\text{Diff}^r(\mathcal{M}, m)$ - the space of $C^r$ volume preserving diffeomorphisms of $\mathcal{M}$ - contains an open set $\mathcal{U}$ such that for every $P \in \mathcal{U}$, the following hold:

1. There is a family of codimension-1 invariant tori;
2. The union of these tori has positive volume;
3. On each torus, the map $P$ is $C^1$ conjugate to a Diophantine translation and has all Lyapunov exponents zero.

It should be pointed out that the set of invariant tori is nowhere dense in the manifold $\mathcal{M}$ and cannot be destroyed under small perturbations. One would expect that those invariant tori are surrounded by “chaotic sea” - an ergodic component with nonzero Lyapunov exponents thus leading to coexistence of hyperbolic and non-hyperbolic behavior.

### 1.2 The coexistence phenomenon

As observed in Section 1.1, there are systems with nonzero Lyapunov exponents displaying the chaos as well as systems with zero Lyapunov exponents showing regularity. Both behavior are robust but not generic. Thus it is natural to ask which kind of behavior prevails: the chaos or the regularity, more exactly, hyperbolic or non-hyperbolic? More interestingly, we wonder if both behavior can coexist.

**Definition 1.2.1.** We say a smooth conservative system $(\mathcal{M}, f^t, \mu)$ exhibits an essential coexistence of regular and chaotic behavior if

1. $\mathcal{M}$ can be split into two invariant disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of positive volume - the chaotic and regular regions respectively;
2. $f^t|\mathcal{A}$ has nonzero Lyapunov exponents, while all Lyapunov exponents for $f^t|\mathcal{B}$ are zero;
Furthermore, we say that essential coexistence is of type I if

3. the set $\mathcal{A}$ is dense in $\mathcal{M}$, or equivalently, the set $\mathcal{B}$ is nowhere dense in $\mathcal{M}$ and of type II otherwise.

The coexistence phenomenon is “essential” in the sense that both chaotic and regular regions are of positive volume. Indeed, a completely hyperbolic system may also exhibit the coexistence phenomenon since it might contain plenty of points at which all Lyapunov exponents are zero. However, the fact that those points form a set of zero measure makes regular behavior insignificant.

The last requirement in Definition 1.2.1 means that the chaotic region $\mathcal{A}$ and the regular region $\mathcal{B}$ cannot be topologically separated. Essential coexistence of type I reflects the situation that occurs in KAM theory and Theorem 1.1.2: the set $\mathcal{B}$ is the union of invariant tori; it has positive volume and is nowhere dense. On the other hand, there are several examples displaying essential coexistence of type II: the set $\mathcal{B}$ is a union of elliptic islands, which is open and may or may not be dense. We shall see examples of both types below.

We point out that various examples with coexistence phenomenon have been found, numerically or theoretically, in non-smooth dynamical systems, such as piecewise linear mappings, piecewise smooth mappings and billiard dynamics (see for example, the expository survey [Str91]). However, the coexistence in non-smooth systems differs substantially from that in the smooth cases due to the presence of singularities. In this dissertation, we shall only discuss the coexistence phenomenon in smooth dynamics and focus on the coexistence of type I. One can see the recent survey [CHP12] in this direction.

1.2.1 The case of diffeomorphisms

1.2.1.1 Essential coexistence of type I

As a discrete version of KAM phenomenon is described by Theorem 1.1.2, it is yet unknown what exactly happens on the complement of the regular region. As we mention above, it is expected that outside the invariant tori all the Lyapunov exponents are nonzero and the system is ergodic. It has since been an open problem to find out to what extent this picture is true.
A first step towards understanding this picture is to construct a particular example of a volume preserving diffeomorphism exhibiting the essential coexistence of type I in the spirit of Definition 1.2.1.

**Theorem 1.2.2.** [HPT10] Given $\alpha > 0$, there exist a compact smooth Riemannian manifold $\mathcal{M}$ of dimension 5 and a $C^\infty$ diffeomorphism $P : \mathcal{M} \to \mathcal{M}$ preserving the Riemannian volume $m$ such that

1. $\|P - Id\|_{C^1} \leq \alpha$ and $P$ is homotopic to $Id$;

2. $P$ is ergodic on an open, dense and connected subset $U \subset \mathcal{M}$ and $m(U) < m(\mathcal{M})$; in particular, $P$ is topologically transitive on $\mathcal{M}$; furthermore, $P|_U$ is a Bernoulli diffeomorphism;

3. the Lyapunov exponents of $P$ are nonzero for almost every $x \in U$;

4. the complement $U^c = \mathcal{M}\backslash U$ has positive volume, $P|U^c = Id$ and the Lyapunov exponents of $P$ on $U^c$ are all zero.

We emphasize that the regular region $U^c$ is not a Cantor set of invariant tori, but a Cantor set of invariant submanifolds. More precisely, $U^c$ is the direct product $N \times C$, where $N$ is a 3-dimensional smooth compact manifold and $C$ is a positive volume Cantor subset of the 2-torus $T^2$. Thus $U^c$ has codimension two.

Fayad [Fay04] constructed a volume preserving diffeomorphism exhibiting a weaker version of coexistence: only *some but not all* Lyapunov exponents are zero on the regular region. Ensuring that *all* Lyapunov exponents are zero in Theorem 1.2.2 is a substantially more difficult problem that requires a completely different set of techniques. The matter is that if all Lyapunov exponents in $U^c$ are zero, then a typical trajectory that originates in $U$ will spend a long time in the vicinity of $U^c$ where contraction and expansion rates are very small. This should be compensated by even longer periods of time that the trajectory should spend away from $U^c$ thus gaining sufficient contraction and expansion and ensuring nonzero Lyapunov exponents.

By modifying the construction in Theorem 1.2.2, one can obtain a $C^\infty$ diffeomorphism $P$ of a compact smooth Riemannian manifold of dimension 4 with the essential coexistence phenomenon (see [Chen10]). The major difference is that the
regular region $\mathcal{U}^c$ is the direct product of a 3-dimensional smooth compact manifold and a Cantor set of positive length in a circle and thus has codimension one. Furthermore, $P$ has countably many ergodic components in the chaotic region $\mathcal{U}$. This example is our first main result in the dissertation. Precise statements and proofs will be given in Chapter 3.

1.2.1.2 Essential coexistence of type II

Examples of essential coexistence phenomenon of type II have been intensely studied in the 2-dimensional diffeomorphisms.

Przytycki [Prz83] considered a smooth one-parameter family of $C^\infty$ area preserving diffeomorphisms $H_\varepsilon: \mathbb{T}^2 \to \mathbb{T}^2$, $-\varepsilon_0 \leq \varepsilon \leq \varepsilon_0$ given by

$$H_\varepsilon(x, y) = (x + y, y + h_\varepsilon(x + y)) \pmod{1} \quad (1.2)$$

where the function $h_\varepsilon$ is particularly chosen such that $H_\varepsilon$ demonstrates a route from uniform hyperbolicity to non-uniform hyperbolicity and then to coexistence of regular and chaotic behavior. More precisely,

1. for every $\varepsilon > 0$ the map $H_\varepsilon$ is an Anosov diffeomorphism and topologically conjugate to the hyperbolic automorphism given by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$;

2. the map $H_0$ has nonzero Lyapunov exponents almost everywhere and is isomorphic to a Bernoulli automorphism;

3. for every $\varepsilon < 0$ there exists an elliptic island $O_\varepsilon$ - a domain between the separatrices connecting two saddles near $0 \in \mathbb{T}^2$ (see Figure 1.1.). Further, the map $H_\varepsilon$ behaves stochastically on $S_\varepsilon = \mathbb{T}^2 \setminus O_\varepsilon$, more exactly, the Lyapunov exponents for $H_\varepsilon|_{S_\varepsilon}$ are nonzero almost everywhere and $H_\varepsilon|_{S_\varepsilon}$ is isomorphic to a Bernoulli automorphism.

Figure 1.1. The elliptic island surrounded by the separatrices
Note that the elliptic island $O_\varepsilon$ and the chaotic sea $S_\varepsilon$ are sharply separated by the separatrices, and therefore $S_\varepsilon$ is not dense in $T^2$. This example demonstrates the essential coexistence phenomenon of type II and motivates us to obtain coexistence by a large perturbation of some good hyperbolic system via the parameter exclusion technique.

We point out that it is still unknown whether there exists a stochastic sea of positive measure for a general family $H_\varepsilon$ of the form in (1.2). By the entropy formula [Pes77, BP07], this question is equivalent to the problem of whether $H_\varepsilon$ has positive metric entropy. To this end Liverani [Liv04] has shown that there exists a constant $C > 0$ and $\varepsilon_0 > 0$ such that for each $-\varepsilon_0 < \varepsilon < 0$ one can construct a $C^\infty$ area preserving diffeomorphism $\tilde{H}_\varepsilon : T^2 \to T^2$ with positive metric entropy such that

\[
\text{dist}_{C^2}(\tilde{H}_\varepsilon, H_\varepsilon) \leq e^{-C\varepsilon^{-\frac{1}{2}}}, \quad m(\{x \in T^2 : \tilde{H}_\varepsilon(x) \neq H_\varepsilon(x)\}) \leq e^{-C\varepsilon^{-\frac{1}{2}}}.
\]

This means that the new family $\tilde{H}_\varepsilon$ approaches the family $H_\varepsilon$ with an exponential rate as $\varepsilon \to 0$.

We emphasize that the positivity of metric entropy for the Chirikov-Taylor standard map remains one of the most challenging problems in dynamics. See, for example, [CHP12] for its current state.

It is also worth mentioning that the family of diffeomorphisms $H_\varepsilon : T^2 \to T^2$ can also be viewed as symplectomorphisms. The coexistence problem for symplectomorphisms is in fact open and seemingly much more difficult in higher dimension.

### 1.2.2 The case of flows

The coexistence has been numerically observed in the category of smooth flows, most of which come from simple systems of differential equations in small dimension (see [Str91]). Here we are particularly interested in

1. Hamiltonian flows: there is an example by Donnay and Liverani [DL91] of a particle moving in a special potential field on the 2-torus, which demonstrates an essential coexistence of type II – the system has positive metric entropy but is not ergodic and the chaotic sea is not dense. We believe that
one can construct an example demonstrating essential coexistence of type I for Hamiltonian flows. Ideally, one may want to achieve this by a small perturbation of some completely integrable system, but this is apparently a much more difficult problem.

2. Geodesic flows on compact Riemannian manifolds: Donnay [Don88] constructed an example of a surface on which the geodesic flow exhibits a coexistence phenomenon of type II. It is obtained by inserting a light-bulb cap into a negative curved surface. In this example the set of geodesics, which are trapped in the cap, is invariant, has positive volume and almost every point in this set has zero Lyapunov exponents. Since it has non-empty interior, the stochastic sea (the set of geodesics that leave the cap) is not dense.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{lightbulb_cap.png}
\caption{Light-bulb cap}
\end{figure}

New methods and techniques should be developed to establish essential coexistence of type I for geodesic flows since we need to do local perturbations of the Riemannian metric.

Although the essential coexistence phenomenon of type I has not yet been demonstrated for either Hamiltonian systems or geodesic flows, it has been established in the category of volume preserving flows by extending Theorem 1.2.2 to the continuous-time case (see [CHP11]). More precisely, we consider the same compact smooth Riemannian $\mathcal{M}$ as in Theorem 1.2.2, and we construct a smooth volume preserving flow $h^t$ with essential coexistence phenomenon of type I in such a way that the regular part $\mathcal{U}^c$ is still the direct product $\mathcal{N} \times C$. We stress that each 3-dimensional invariant submanifold, i.e., $\mathcal{N} \times \{y\}$ for $y \in C$, is in turn a union of 2-dimensional invariant tori on which $h^t$ is a non-identity linear flow with
Diophantine frequency vector. This fact makes our construction of the flow non-trivial. This example is our second main result, which will be present in Chapter 4.
Chapter 2

Preliminaries

We shall give a brief introduction to smooth ergodic theory and partially hyperbolic systems. One can find more details in [BP07, Pes77, Pes04].

2.1 Individual and average Lyapunov exponents

Consider a conservative system \((\mathcal{M}, f^t, \mu)\), where \(f^t\) (\(t \in \mathbb{Z}\) or \(\mathbb{R}\)) smoothly acts on a compact Riemannian manifold \(\mathcal{M}\) and preserves a smooth measure \(\mu\). We recall that the (individual) Lyapunov exponent of \(f^t\) at a point \(x \in \mathcal{M}\) of a tangent vector \(v \in T_x \mathcal{M}\) is defined as

\[
\lambda(x, v, f^t) = \limsup_{t \to \infty} \frac{1}{t} \log \|d_x f^t v\|.
\]

It is easy to see that the function \(\lambda(x, \cdot, f^t)\) attains no more than \(d\) distinct finite values on \(T_x \mathcal{M} \setminus \{0\}\), where \(d = \dim \mathcal{M}\). We denote those values of the Lyapunov exponents by \(\lambda_i(x, f^t), i = 1, \ldots, d\), counted with multiplicities and arranged in the decreasing order.

Note that the functions \(\lambda_i(x) = \lambda_i(x, f^t)\) are invariant and measurable but may not be continuous. Moreover, if we fix a typical point \(x \in \mathcal{M}\), the individual Lyapunov exponents \(\lambda_i(x, g^t)\) and \(\lambda_i(x, f^t)\) might differ a lot even though \(g^t\) is \(C^1\)-close to \(f^t\).
Now we introduce the $k$-th average Lyapunov exponent of $f^t$ given by

$$L_k(f^t) = \int_M \sum_{i=1}^k \lambda_i(x, f^t) d\mu(x). \quad (2.1)$$

It can be shown that the $k$-th average exponent $L_k(\cdot)$ is upper-semicontinuous in the $C^1$ topology. This property helps us to maintain relations between average Lyapunov exponents when we try to perturb $f^t$.

### 2.2 Foliation theory in smooth dynamics

A smooth dynamical system is often accompanied with some invariant foliations with smooth leaves. We stress that those foliations are typically not smooth but continuous, for which the Fubini’s theorem might fail. To avoid such pathological situation, we need the concept of absolute continuity. We also introduce the concept of accessibility to describe how two foliations connect with each other and spread out the whole space.

Given a subset $S \subseteq M$, we call a partition $\mathcal{P}$ of $S$ a $(\delta, q)$-foliation with smooth leaves if there exist continuous functions $\delta = \delta(x) > 0$, $q = q(x) > 0$, and an integer $k > 0$ such that for each $x \in S$:

1. there exists a smooth immersed $k$-dimensional manifold $W(x)$ containing $x$ for which $\mathcal{P}(x) = W(x)$ where $\mathcal{P}(x)$ is the element of the partition $\mathcal{P}$ containing $x$. The manifold $W(x)$ is called the global leaf of the foliation at $x$; the connected component of the intersection $W(x) \cap B(x, \delta(x))$ that contains $x$ is called the local leaf at $x$ and is denoted by $V(x)$;

2. there exists a continuous map $\phi_x : B(x, q(x)) \to C^1(D, \mathcal{M})$ (where $D$ is the $k$-dimensional unit ball) such that $V(y)$ is the image of the map $\phi_x(y) : D \to \mathcal{M}$ for each $y \in B(x, q(x))$; the number $q(x)$ is called the size of $V(x)$.

We shall simply call a $(\delta, q)$-foliation with smooth leaves as a continuous foliation. We say that a continuous foliation is absolutely continuous if for almost every $x \in S$ and almost every $y \in B(x, q(x))$ the conditional measure of the Riemannian volume $m$ on $V(y)$ (with respect to the partition of $B(x, q(x))$) by local
leaves) is absolutely continuous with respect to the leaf volume \( m_{V(y)} \) on \( V(y) \). We remark that an absolute continuous foliation allows us to apply Fubini’s theorem with respect to the partition by local leaves.

Let \( W_1 \) and \( W_2 \) be two continuous foliations with smooth leaves that are transversal to each other. Let also \( S_1 \subset S \subset \mathcal{M} \) be two subsets. We say that the pair \( W_1 \) and \( W_2 \) has the accessibility property on \( S_1 \) if any two points \( z, z' \in S_1 \) are accessible via a \((u,s)\)-path in \( S \), that is

1. there exists a collection of points \( z_1, \ldots, z_n \in S \) such that \( z = z_1, z' = z_n \) and \( z_k \in W_i(z_{k-1}) \) for \( i = 1, 2 \) and \( k = 2, \ldots, n \);
2. the points \( z_{k-1} \) and \( z_k \) can be connected by a smooth curve \( \gamma_k \subset W_i(z_{k-1}) \) in \( S \) for \( i = 1, 2 \) and \( k = 2, \ldots, n \).

The collection of the leaf-wise paths \( \gamma_k \) is called a \((u,s)\)-path and is denoted by \([z_1, \ldots, z_n]\).

### 2.3 Notions of partially hyperbolic systems

#### 2.3.1 The discrete-time case

We consider a diffeomorphism \( f \) of a compact smooth Riemannian manifold \( \mathcal{M} \). It is called uniformly partially hyperbolic on a compact invariant subset \( \Lambda \subset \mathcal{M} \) if

1. for every \( x \in \Lambda \) the tangent space at \( x \) admits an invariant splitting

\[
T_x\mathcal{M} = E^s(x) \oplus E^c(x) \oplus E^u(x)
\]

into stable \( E^s(x) = E^s_f(x) \), central \( E^c(x) = E^c_f(x) \) and unstable \( E^u(x) = E^u_f(x) \) subspaces;
2. there are numbers \( 0 < \lambda < \tilde{\lambda} \leq 1 \leq \tilde{\mu} < \mu \) such that

\[
\|dfv\| \leq \lambda\|v\|, \quad v \in E^s(x),
\]

\[
\tilde{\lambda}\|v\| \leq \|dfv\| \leq \tilde{\mu}\|v\|, \quad v \in E^c(x),
\]

\[
\mu\|v\| \leq \|dfv\|, \quad v \in E^u(x).
\]
Remark 2.3.1. It is well-known for a uniformly partially hyperbolic system \((\mathcal{M}, f)\) that

1. Stable and unstable subspaces are integrable to continuous foliations with smooth leaves — strongly stable and unstable foliations — which are uniformly transverse to each other;

2. These strong foliations are absolutely continuous;

3. The set of uniformly partially hyperbolic diffeomorphisms is \(C^1\)-open. In other words, any sufficiently small perturbation of a uniformly partially hyperbolic diffeomorphism in the \(C^1\) topology is also uniformly partially hyperbolic.

In this dissertation we need a weaker property than uniform partial hyperbolicity. Let \(S \subset \mathcal{M}\) be an invariant open subset. We say that a diffeomorphism \(f\) is pointwise partially hyperbolic on \(S\) if for every \(x \in S\) the tangent space at \(x\) admits an invariant splitting \((2.2)\) and there are continuous functions \(0 < \lambda(x) < \tilde{\lambda}(x) \leq 1 \leq \tilde{\mu}(x) < \mu(x)\) such that \((2.4)\) holds with constants \(\lambda, \tilde{\lambda}, \tilde{\mu}, \mu\) being replaced with these functions.

The notion of pointwise partial hyperbolicity is introduced and studied by Burns and Wilkinson [BW10] in the case when \(S = \mathcal{M}\). It turns out that in this case the properties in Remark 2.3.1 are still true. However, these properties may fail if \(S\) is just an open invariant subset of \(\mathcal{M}\).

2.3.2 The continuous-time case

We now consider a smooth flow \(f^t\) on \(\mathcal{M}\) which is generated by the vector field \(\mathcal{X}_f(x) = \frac{d}{dt}f^t(x)|_{t=0}\). We say that the flow is uniformly partially hyperbolic on a compact invariant subset \(\Lambda \subset \mathcal{M}\) if for every \(x \in \Lambda\) the tangent space at \(x\) admits an invariant splitting \((2.2)\) into stable \(E^s(x) = E^s_f(x)\), central \(E^c(x) = E^c_f(x)\) and unstable \(E^u(x) = E^u_f(x)\) subspaces satisfying:

1. the vector field \(\mathcal{X}_f(x)\) is contained in the central subspace \(E^c(x)\);

2. there are numbers \(0 < \lambda < \tilde{\lambda} \leq 1 \leq \tilde{\mu} < \mu\) such that for all \(t \in [0, 1]\)

\[\|df^tv\| \leq \lambda^t\|v\|, \quad v \in E^s(x),\]
\[
\tilde{\lambda}^t \|v\| \leq \|df^tv\| \leq \tilde{\mu}^t \|v\|, \quad v \in E^c(x),
\]
\[
\mu^t \|v\| \leq \|df^tv\|, \quad v \in E^u(x).
\]

Note that if a flow \(f^t\) is uniformly partially hyperbolic, then every time-\(t\) map \((t \neq 0)\) is also uniformly partially hyperbolic with the same invariant splitting.

It is easy to see that the first three properties in Remark 2.3.1 also hold for uniformly partially hyperbolic flows. We need some arguments to establish the last property. Given \(\delta > 0\), we say that a flow \(g^t\) is \((C^1, \delta)\)-close to \(f^t\) on an invariant set \(\Lambda\) if \(X_g = X_f\) outside \(\Lambda\) and \(\|X_g - X_f\| \leq \delta\). We can show that uniformly partially hyperbolic flows form an open set in the \(C^1\) topology (see Lemma B.0.8 in the Appendix B).

While the definition of uniformly partially hyperbolic flows is parallel to that of diffeomorphisms, we don’t have a satisfying analogy in defining pointwise partially hyperbolic flows but via the time-1 map instead. More precisely, given an invariant open subset \(S \subset M\) we call a flow \(f^t\) pointwise partially hyperbolic on \(S\) if its time-1 map \(f^1\) is pointwise partially hyperbolic on \(S\). Therefore, properties in Remark 2.3.1 might fail as well for the pointwise partially hyperbolic flows.

### 2.4 Integrability and Dynamical coherence

For a uniformly partially hyperbolic diffeomorphism \(f\) or flow \(f^t\) on \(\Lambda \subset M\) one can construct stable and unstable local manifolds of uniform size at every point in \(\Lambda\). Therefore, the stable and unstable distributions are uniquely integrable to the stable and unstable foliations, whose global leaves are obtained by iterating the local manifolds. As already pointed out in the previous section, this may not be true for a pointwise partially hyperbolic system on an open set \(S\). However, all pointwise partially hyperbolic systems that we construct in this dissertation will have global strongly stable and unstable transverse foliations with smooth leaves. We denote these foliations by \(W^s = W^s_f\) and \(W^u = W^u_f\) respectively.

A uniformly partially hyperbolic diffeomorphism \(f\) is called dynamically coherent if the subbundles \(E^c, E^{cs} = E^c \oplus E^s\) and \(E^{cu} = E^c \oplus E^u\) are integrable to continuous foliations with smooth leaves \(W^c, W^{cs}\) and \(W^{cu}\), called respectively the center, center-stable and center-unstable foliations. Furthermore, the foliations \(W^c\)
and $W^s$ are subfoliations of $W^{cs}$ while $W^c$ and $W^u$ are subfoliations of $W^{cu}$.

Similarly, we can define dynamical coherence for a uniformly partially hyperbolic flow. It is easy to see that a uniformly partially hyperbolic flow $f^t$ is dynamically coherent if and only if the time-1 map $f^1$ is dynamically coherent. The following theorem (see [HPS77,PS97]) shows that dynamical coherence is robust.

**Theorem 2.4.1.** Suppose that $f$ is a partially hyperbolic diffeomorphism. If the center foliation $W^c$ is smooth, then $f$ is dynamically coherent. Moreover, any diffeomorphism that is close to $f$ in the $C^1$ topology is dynamically coherent. Similar results hold for flows.

### 2.5 A key theorem

Consider a $C^2$ conservative system $(\mathcal{M}, f^t, \mu) \ (t \in \mathbb{Z} \text{ or } \mathbb{R})$ that is pointwise partially hyperbolic on an open set $S$.

We say that $f^t$ has **positive central exponents** if there is an invariant set $\mathcal{A} \subset S$ of positive measure such that for every $x \in \mathcal{A}$ and every $v \in E^c(x)$ (or $v \in E^c(x) \setminus \text{Span}\{X^f(x)\}$ in the flow case) the Lyapunov exponent $\lambda(x, v, f^t) > 0$.

The following theorem plays an important role in the proof of both Main results.

**Theorem 2.5.1.** [HPT10,CHP11] Assume that the following conditions hold:

1. $f^t$ has strongly stable and unstable $(\delta, q)$-foliations $W^s$ and $W^u$ where $\delta = \delta(x)$ and $q = q(x)$ are continuous functions on $S$;
2. the foliations $W^s$ and $W^u$ are transversal and absolutely continuous;
3. $f^t$ has the accessibility property via the foliations $W^s$ and $W^u$, i.e., $W^s$ and $W^u$ has accessibility property on $S$;
4. $f^t$ has positive central exponents.

Then $f^t$ has positive central exponents at almost every point $x \in S$. Moreover,

1. (discrete) $f^t \ (t \in \mathbb{Z})$ is ergodic and indeed Bernoulli on $S$;
2. (continuous) $f^t \ (t \in \mathbb{R})$ is an ergodic flow on $S$. 
We remark that this theorem originates from Theorem 1 in [BDP02] for uniformly partially hyperbolic diffeomorphisms. One can adapt to the case of pointwise partially hyperbolic systems with slight modifications of the proof.
Chapter 3

Coexistence in the discrete-time case

3.1 Coexistence with countably infinitely many ergodic components

We concentrate on the discrete-time case in this chapter and present our first main result on the essential coexistence phenomenon of type I.

Recall that in Theorem 1.2.2, the chaotic region $\mathcal{U}$ is the only ergodic component of the diffeomorphism $P$. In view of Pesin’s ergodic decomposition theorem, that is, a smooth conservative system with nonzero Lyapunov exponents has at most countably many ergodic components, we construct

**Theorem 3.1.1.** [Chen10] Given $\alpha > 0$, there exist a compact smooth Riemannian manifold $\mathcal{M}$ of dimension 4 and a $C^\infty$ diffeomorphism $P : \mathcal{M} \to \mathcal{M}$ preserving the Riemannian volume $\mu$ such that

1. $\|P - \text{Id}\|_{C^1} \leq \alpha$ and $P$ is homotopic to $\text{Id}$;

2. $P$ has nonzero Lyapunov exponents almost everywhere on an open and dense subset $\mathcal{U} \subset \mathcal{M}$. Moreover, $\mathcal{U}$ consists of countably infinitely many open connected components, on which $P$ is ergodic, and indeed Bernoulli;

3. The set $\mathcal{U}^c \subset \mathcal{M}$ has positive volume and is a union of 3-dimensional invariant submanifolds. Also $P|\mathcal{U}^c = \text{Id}$ and the Lyapunov exponents of $P$ on $\mathcal{U}^c$
are all zero.

This theorem is parallel and somewhat complementary to Theorem 1.2.2 in the way that the chaotic part $P|U$ has countably infinitely many ergodic components. On the other hand, the regular part $U^c$ is the direct product of a 3-dimensional compact manifold and a Cantor set of positive Lebesgue measure in a circle, and thus has codimension 1. The codimension 1 property of the invariant submanifolds makes our example closer to the “real KAM-type” picture described in Theorem 1.1.2.

Let us briefly explain the ideas of this construction. Due to the structure of the manifold $M$ and non-ergodicity of the map $P$, we need to deal with connected components of the set $U$ one by one. By modifying the construction in [HPT10], we perturb the identity map and obtain nonzero Lyapunov exponents and ergodicity near each open connected component of $U$. In this way we can construct consecutive small perturbations $P_j$, $j = 1, 2, \ldots$, which possesses hyperbolicity and ergodicity on the first $j$ connected components of $U$ and is identity elsewhere.

The main technical issue here, which does not appear in [HPT10], is how to effect this inductive argument and guarantee that the sequence $P_j$, $j = 1, 2, \ldots$, converges to the desired map $P$ in Theorem 3.1.1. Indeed, such inductive procedure relies on the special structure of the Cantor set in a circle. More precisely, this 1-dimensional Cantor set can be produced by consecutively removing disjoint open intervals from the circle (see section 3.2). Thus we can easily label the connected components of $U$, each closure of which possesses a neighborhood disjoint from other components. The inductive step from $P_{j-1}$ to $P_j$ can hence be restricted in a neighborhood of the $j$-th component of $U$, at which $P_{j-1}$ is identity. Controlling the $C^j$-norm of the difference $P_j - P_{j-1}$ carefully, one can obtain the desired map $P$ as the limit map of the sequence $P_j$.

In the following, we first describe the construction of the 4-dimensional manifold $M$ and the open and dense subset $U$, and show that the construction of the diffeomorphism $P$ can be reduced to the construction of a perturbation $H$ at some neighborhood of each connected components of $U$. Then we apply the approach in [HPT10] to obtain the perturbation map $H$ in the reduction. For the construction of $H$, we first set up the “start-up” map, then create positive central exponents and finally produce the accessibility property.
3.2 Construction of the 4-dimensional manifold

In this section, we first describe the construction of the 4-dim manifold $\mathcal{M}$ and the related sets, that is, the open dense subset $\mathcal{U}$ and its connected components $\mathcal{U}_j, j = 1, 2, \ldots$.

Pick an Anosov automorphism $A$ of the 2-torus $X = \mathbb{T}^2$ with the constant expanding rate $\eta_A > 1$ along the unstable direction. We consider the suspension flow $T^\tau$ over $A$ with the roof function being constantly 1. This flow acts on the suspension manifold

$$\mathcal{N} = X \times \mathbb{R}/\sim,$$

where “$\sim$” is the identification $(x, t + 1) \sim (Ax, t)$. See Appendix A for the differential and metric structure of the suspension manifold $\mathcal{N}$.

Set $Y = S^1 = [0, 1]/\{0 \sim 1\}$. For each $n \in \mathbb{N}$, pick finitely many non-overlapping closed intervals $C_{i_1 \ldots i_n} \subset Y$ in such a way that $C_{i_1 \ldots i_n, i_{n+1}} \subset C_{i_1 \ldots i_n}$, then we obtain a Cantor set $C \subset Y$ by letting

$$C = \bigcap_{n=1}^{\infty} \bigcup_{(i_1 \ldots i_n)} C_{i_1 \ldots i_n}.$$

Alternatively, one can obtain this Cantor set by consecutively removing disjoint open intervals from $Y$. More precisely, denote by $I_1, I_2, \ldots$ those open intervals that are removed from $Y$, then set $\mathcal{I} = \bigcup_{j=1}^{\infty} I_j$ and $C = Y \setminus \mathcal{I}$. Moreover, let us assume that $\sum_{j=1}^{\infty} |I_j| < 1$ so that the Cantor set $C$ is of positive Lebesgue measure, where $|I_j|$ is the length of the interval $I_j$.

Finally, we set

$$\mathcal{M} = \mathcal{N} \times Y, \quad \mathcal{U} = \mathcal{N} \times \mathcal{I}, \quad \text{and} \quad \mathcal{U}_j = \mathcal{N} \times I_j, \quad j = 1, 2, \ldots.$$

Clearly $\{\mathcal{U}_j\}_{j=1}^{\infty}$ are open connected components of $\mathcal{U}$. Also the complement $\mathcal{U}^c = \mathcal{N} \times C$ is of positive Riemannian volume.
3.3 Construction of the desired map \( P \)

We shall show that the desired map \( P \) of Theorem 3.1.1 can be obtained by inductive perturbations on the components \( U_j \), in other words, we need to construct a sequence of diffeomorphisms \( P_j \) converging to \( P \). Due to the special structures of the manifold \( M \) and the open dense set \( \mathcal{U} \), we can actually reduce our construction of \( P_j \) to the construction of the perturbation that changes \( P_{j-1} \) to \( P_j \), which is much simpler.

### 3.3.1 The sequence of diffeomorphisms \( P_j \)

Starting from \( P_0 = \text{Id} \) on \( M \), we intend to obtain inductively the map \( P_j \) as a small homotopic perturbation of \( P_{j-1} \) for \( j \geq 1 \). Furthermore, \( P_j \) differs from \( P_{j-1} \) only on the \( j \)-th connected component \( U_j \) of \( \mathcal{U} \). More precisely, we shall show

**Proposition 3.3.1.** Given \( \alpha > 0 \), there exists a sequence of \( C^\infty \) volume preserving diffeomorphisms \( P_j : M \rightarrow M \), \( j = 0, 1, 2, \ldots \) such that

1. \( P_0 = \text{Id} \); \( P_j = P_{j-1} \) outside \( \mathcal{U}_j \), in particular, \( P_j = \text{Id} \) outside \( \bigcup_{k=1}^{j} \mathcal{U}_k \); Also, each \( P_j \) is homotopic to \( P_{j-1} \);

2. \( P_j \) is ergodic and has nonzero Lyapunov exponents almost everywhere on each \( \mathcal{U}_k \), \( k = 1, \ldots, j \);

3. \( \| P_j - P_{j-1} \|_{C^1} \leq \alpha/2^j \).

Once such a sequence of diffeomorphisms \( \{P_j\}_{j=0}^{\infty} \) is constructed, we can take the pointwise limit

\[ P(z) = \lim_{j \rightarrow \infty} P_j(z). \]

We shall show that \( P \) is the desired map of Theorem 3.1.1.

### 3.3.2 Proof of Theorem 3.1.1

**Proof of Theorem 3.1.1.** By Proposition 3.3.1(3), we have

\[
\| P_l - P_j \|_{C^r} \leq \sum_{k=j}^{l-1} \| P_{k+1} - P_k \|_{C^{k+1}} \leq \sum_{k=j}^{l-1} \alpha_{2^{k+1}} \leq \frac{\alpha}{2^j},
\]
for any \( r \geq 1 \) and \( l > j \geq r - 1 \). It follows that \( \| P - P_j \|_{C^r} \leq \alpha/2^j \), in particular, \( \| P - Id \|_{C^r} \leq \alpha \). Hence \( P_j \) converges to \( P \) in the \( C^r \) topology, and therefore \( P \) is a \( C^\infty \) diffeomorphism since \( r \) is arbitrary. Clearly \( P \) is volume preserving.

In addition, by Proposition 3.3.1(1), \( P = P_j \) on \( \bigcup_{k=1}^{j} \mathcal{U}_k \) and \( P = Id \) outside \( \bigcup_{j=1}^{\infty} \mathcal{U}_j = \mathcal{U} \). Since every \( P_j \) is homotopic to \( Id \), \( P \) is also homotopic to \( Id \). These imply statement (1) and (3) of Theorem 3.1.1.

By Proposition 3.3.1(2), \( P_j \) is ergodic and has nonzero Lyapunov exponents almost everywhere on each \( \mathcal{U}_k \), \( k = 1, \ldots, j \), and so is \( P \). Since \( j \) is arbitrary, statement (2) of the theorem follows.

\[ \square \]

### 3.3.3 The reduction

The proof of Proposition 3.3.1 boils down to a construction of a suitable homotopic perturbation \( H_j : \mathcal{M} \to \mathcal{M} \) that changes \( P_{j-1} \) to \( P_j \), that is, \( P_j = H_j \circ P_{j-1} \) and \( H_j \) is homotopic to \( Id \), for each \( j = 1, 2, \ldots \).

Let us fix \( j \) and pick an open interval \( \tilde{I}_j \supset \tilde{T}_j \) in such a way that \( \tilde{I}_j \cap \tilde{T}_k = \emptyset \) for all \( 1 \leq k < j \). Set \( \tilde{\mathcal{U}}_j = \mathcal{N} \times \tilde{I}_j \), then \( \tilde{\mathcal{U}}_j \supset \tilde{\mathcal{U}}_j \) and \( \tilde{\mathcal{U}}_j \cap \tilde{\mathcal{U}}_k = \emptyset \) for all \( 1 \leq k < j \). Note that

1. on the set \( \tilde{\mathcal{U}}_j \), \( P_{j-1} = Id \) and hence \( H_j = P_j \).
2. outside the set \( \mathcal{U}_j \), \( P_{j} = P_{j-1} \) and hence \( H_j = Id \).

Therefore, we only need to restrict the construction of \( H_j \) on \( \tilde{\mathcal{U}}_j \) such that \( H_j \) is homotopic to \( Id \), ergodic and has nonzero Lyapunov exponents almost everywhere on \( \mathcal{U}_j \). Also \( H_j = Id \) on \( \tilde{\mathcal{U}}_j \setminus \mathcal{U}_j \), and \( \| H_j \mathcal{U}_j - Id \mathcal{U}_j \|_{C^r} \leq \alpha/2^j \). More generally, we can show that

**Proposition 3.3.2.** Set \( \mathcal{Z} = \mathcal{N} \times I \) and \( \tilde{\mathcal{Z}} = \mathcal{N} \times \tilde{I} \), where \( I, \tilde{I} \) are two arbitrary open intervals satisfying \( I \subset \tilde{I} \). Given \( \delta > 0 \) and \( r \in \mathbb{N} \), there exists a \( C^\infty \) volume preserving diffeomorphism \( H : \tilde{\mathcal{Z}} \to \tilde{\mathcal{Z}} \) such that

1. \( H \) is homotopic to \( Id \), and \( H = Id \) on \( \tilde{\mathcal{Z}} \setminus \mathcal{Z} \);
2. \( H \mathcal{Z} \) is ergodic and has nonzero Lyapunov exponents almost everywhere;
3. \( \| H - Id \|_{C^r} \leq \delta \).
One can see that Proposition 3.3.1 immediately follows from Proposition 3.3.2. After proper scalings, we may assume $I = (-1, 1)$ and $\tilde{I} = (-2, 2)$. We are going to prove this proposition in the remaining sections.

### 3.4 Construction of the perturbation map $H$

We describe the construction of the perturbation map $H$ which splits into several steps, following [HPT10]. From now on, let us fix $\delta$ and $r$ in the assumption of Proposition 3.3.2.

#### 3.4.1 Construction of $H$ via consecutive perturbations

**3.4.1.1 Step 1: The “start-up” map $T$**

Pick an open subinterval $\tilde{I} \subset I = (-1, 1)$ (for example, $\tilde{I} = (-5/8, 5/8)$) and a $C^\infty$ function $\kappa : \tilde{I} = (-2, 2) \to \mathbb{R}$ satisfying:

1. $\kappa(y) > 0$ if $y \in I$ and $\kappa(y) = 0$ if $y \in \tilde{I}\setminus I$;
2. $\kappa(y) = \kappa_0$ for $y \in \tilde{I}$, where $\kappa_0$ is a constant;
3. $\|\kappa\|_{C^r} \leq 1$.

We define a map $T : \mathcal{Z} \to \mathcal{Z}$ by

$$(x, t, y) \mapsto (\tau^{\kappa(y)}(x, t), y),$$

where $(x, t) \in \mathcal{N}$, $y \in \tilde{I}$, and $\tau^{r}$ is the suspension flow on $\mathcal{N}$.

Recall that for each $\tau \neq 0$ the map $T^{\tau}$ is uniformly partially hyperbolic with one-dimensional stable $E^{s}_{T^{\tau}}$, one-dimensional unstable $E^{u}_{T^{\tau}}$ and one-dimensional center $E^{c}_{T^{\tau}}$ subbundles, and these subbundles are integrable to smooth stable $W^{s}_{T^{\tau}}$, unstable $W^{u}_{T^{\tau}}$ and center $W^{c}_{T^{\tau}}$ foliations of $\mathcal{N}$. Moreover, we can choose a suitable Riemannian metric on $\mathcal{N}$ such that at every $(x, t) \in \mathcal{N}$, $T^{\tau}$ expands at the rate $\eta^{r}_{A}$ along the unstable direction and contracts at the rate $\eta^{r}_{A}$ along the stable direction.

By the above properties of $T^{\tau}$ and the construction of $T : \mathcal{Z} \to \mathcal{Z}$, we immediately have
Proposition 3.4.1. The map $T$ is a $C^\infty$ volume preserving diffeomorphism of $\tilde{Z}$ satisfying:

1. given $\delta_1 > 0$, one can choose the function $\kappa$ such that $\|T - \text{Id}\|_{C^r} \leq \delta_1$; moreover, $T$ is homotopic to $\text{Id}$;

2. $T$ preserves the fibers $\mathcal{N} \times \{y\}$;

3. $T$ is pointwise partially hyperbolic on $Z$ with one-dimensional stable $E^s_T$, one-dimensional unstable $E^u_T$ and two-dimensional center $E^c_T$ subspaces; the subspaces $E^s_T$ and $E^u_T$ are integrable to strongly stable and unstable foliations $W^s_T$ and $W^u_T$ with smooth leaves; these foliations are uniformly transversal and their local leaves have uniform size; in addition, these foliations are absolute continuous;

4. $T$ is uniformly partially hyperbolic on any invariant subset $\mathcal{N} \times J$ where $J \subset I$ is a closed subinterval; moreover, $T$ is dynamically coherent with the center foliation $W^c_T = W^c_T \times \tilde{I}$;

5. $T|(\tilde{Z} \setminus \mathcal{Z}) = \text{Id}$ and $dT_z = \text{Id}$ for all $z \in \tilde{Z} \setminus \mathcal{Z}$; in particular, the Lyapunov exponents of $T|(\tilde{Z} \setminus \mathcal{Z})$ are all zero;

6. for every $z = ((x,t), y) \in \mathcal{Z}$, the Lyapunov exponents of $T$ are as follows:

$$
\lambda_1(z, T) = \lambda^u(z, T) = \kappa(y) \log \eta_A > 0 = \lambda_2(z, T) = \lambda_3(z, T)
$$

$$
> \lambda_4(z, T) = \lambda^s(z, T) = -\kappa(y) \log \eta_A.
$$

where $\lambda^u(z, T)$ and $\lambda^s(z, T)$ corresponds to the direction $E^u_T$ and $E^s_T$ respectively and $\lambda_2(z, T)$ and $\lambda_3(z, T)$ corresponds to the direction of the flow $T^r$ and the $I$-direction respectively. Consequently,

$$
L_1(T) = L_2(T) = L_3(T) > 0 = L_4(T),
$$

where $L_k(\cdot)$ is the $k$-th average Lyapunov exponent given by (2.1).

We say that a diffeomorphism $F : \tilde{Z} \to \tilde{Z}$ is a gentle perturbation of $T$ if

1. $F$ is $C^1$ close to $T$;
2. $F|Z = Z$ and $F$ is pointwise partially hyperbolic in $Z$;

3. the one-dimensional strongly stable and unstable subbundles of $F$ are integrable to one-dimensional strongly stable and unstable foliations with smooth leaves on $Z$; the two-dimensional central subbundle of $F$ is integrable to a central foliation;

4. $F|(\tilde{Z}\setminus Z) = Id$.

Remark. Let $F : \tilde{Z} \to \tilde{Z}$ be a diffeomorphism that is $C^1$ close to $T$. Given any closed interval $J \subset I$, set $\Lambda = N \times J$. Assume that $F = T$ on $\tilde{Z} \setminus \Lambda$, in particular, $\Lambda$ is invariant under $F$, then $F$ is a gentle perturbation of $T$ and in fact, $F|\Lambda$ is uniformly partially hyperbolic.

### 3.4.1.2 Step 2: The second perturbation $Q$

In this step we try to create a gentle perturbation $Q$ of $T$ with nonzero Lyapunov exponents, more precisely, one negative and three positive average Lyapunov exponents. However, $Q$ is not necessarily ergodic.

We take $I_0 = (-0.5, 0.5) \subset \tilde{I} \subset I$, and $Z_0 = N \times I_0$. The following statements describe properties of the map $Q$, which will be proved in section 3.4.3.

**Proposition 3.4.2.** Given $\delta_2 > 0$, there exists a $C^{\infty}$ volume preserving diffeomorphism $Q$ of $\tilde{Z}$ satisfying:

1. $\|Q - T\|_{C^r} \leq \delta_2$ and $Q$ is homotopic to $T$;

2. $Q = T$ on the set $\tilde{Z} \setminus Z_0$; in particular, $Q$ preserves the fibers $N \times \{y\}$ if $y \in \tilde{I} \setminus I_0$, and $Q$ is a gentle perturbation of $T$.

3. $Q$ satisfies statements (3)-(5) of Proposition 3.4.1;

4. for any $z \in \tilde{Z}$ we have

$$E_Q^{\text{uty}}(z) = E_T^{\text{uty}}(z), \quad \det(dQ|E_Q^{\text{uty}}(z)) = \det(dT|E_T^{\text{uty}}(z));$$

5. the average Lyapunov exponents of $Q$ satisfy

$$L_1(Q) < L_2(Q) < L_3(Q) = L_3(T), \quad \text{and} \quad L_4(Q) = 0.$$
3.4.1.3 Step 3: The final perturbation $H$

We go on to perturb the map $Q$ to a map $H$ that is pointwise partially hyperbolic on $\mathcal{Z}$, and possesses two transversal stable and unstable foliations $W^s_H$ and $W^u_H$ of $\mathcal{Z}$. Furthermore, we will ensure that $H|\mathcal{Z}$ has the accessibility property via these two transversal foliations. We shall also show that $H$ can be constructed in such a way that $\int_\mathcal{Z} \lambda_i(z,H) dm(z) > 0$ for $i = 1, 2, 3$, and hence $H|\mathcal{Z}$ has positive central exponents. Then we can get the ergodicity of $H$ by Theorem 2.5.1.

To effect our construction of $H$, we choose two sequences of subintervals of $I$ as follows: for $n = 0, 1, 2, \ldots$, set

\begin{align*}
I_n &= (-1 + \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+1}}), \quad \bar{I}_n = (-1 + \frac{3}{2^{n+3}}, 1 - \frac{3}{2^{n+3}}), \\
\breve{I}_n &= (-1 + \frac{7}{2^{n+4}}, 1 - \frac{7}{2^{n+4}}), \quad \tilde{I}_n = (-1 + \frac{15}{2^{n+5}}, 1 - \frac{15}{2^{n+5}}).
\end{align*}

Clearly we have $I_n \subset \breve{I}_n \subset \bar{I}_n \subset \tilde{I}_n \subset \tilde{I}_{n+1}$ and $\bigcup_{n \geq 0} I_n = I$. We set

\begin{align*}
\mathcal{Z}_n &= \mathcal{N} \times I_n, \quad \breve{\mathcal{Z}}_n = \mathcal{N} \times \bar{I}_n, \\
\mathcal{Z}_n &= \mathcal{N} \times I_n, \quad \tilde{\mathcal{Z}}_n = \mathcal{N} \times \tilde{I}_n.
\end{align*}

Apparently $\mathcal{Z}_n \subset \tilde{\mathcal{Z}}_n \subset \tilde{\mathcal{Z}}_n \subset \mathcal{Z}_n$, and each of these sequences of sets exhausts $\mathcal{Z}$. We will construct a sequence of diffeomorphisms $H_n, n = 0, 1, 2, \ldots$, with the following properties:

**Proposition 3.4.3.** Given $\delta_3 > 0$, there exist two sequences of positive numbers $\delta_n$ with $\delta_n \leq \delta_3/2^n$ and $\delta_n \leq d(I_n, \partial I)^2 = 1/2^{2n+2}$, and $\theta_n$ and a sequence of $C^\infty$ volume preserving diffeomorphisms $H_n : \mathcal{Z} \to \mathcal{Z}, n = 0, 1, 2, \ldots$, such that

1. $\|H_0 - Q\|_{C^r} \leq \delta_0$, and $\|H_n - H_{n-1}\|_{C^{r+n}} \leq \delta_n$ for $n \geq 1$; moreover, $H_n$ is homotopic to $Q$;

2. $H_n(\tilde{\mathcal{Z}}_n) = \tilde{\mathcal{Z}}_n, H_n = T$ on $\mathcal{Z} \setminus \tilde{\mathcal{Z}}_n$, and $H_n = H_{n-1}$ on $\tilde{\mathcal{Z}}_{n-2}$; in particular, $H_n$ is a gentle perturbation of $T$;

3. $H_n$ satisfies statements (3)-(5) of Proposition 3.4.1;
4. for every $z \in \tilde{Z}$,

$$E_{H_n}^{ut}(z) = E_Q^{ut}(z), \quad \det(dH_n|E_{H_n}^{ut}(z)) = \det(dQ|E_Q^{ut}(z));$$

5. for all $z \in \tilde{U}_j$, $j = 0, \ldots, n$ and $i = u, s, c$,

$$\angle(E_{H_{n+1}}^i(z), E_{H_n}^i(z)) \leq \theta_j/2^n;$$

6. if the number $\delta_2 > 0$ (in Proposition 3.4.2) is sufficiently small, then each map $H_n$ is stably accessible in the following sense: let $H^2$ be a $C^2$ volume preserving diffeomorphism of $\tilde{Z}$ that is a gentle perturbation of $T$; assume for all $z \in \tilde{Z}_n$ and $i = u, s, c$,

$$\angle(E_{H_n}^i(z), E_{H_n}^i(z)) \leq \theta_n;$$

then any two points $z_1, z_2 \in \tilde{Z}_n$ are accessible via a $(u, s)_{H^2}$-path in $Z$; in particular, $H_n$ has the accessibility property on $\tilde{Z}_n$.

3.4.2 Proof of Proposition 3.3.2

We will prove the following proposition in section 3.4.4. By statement (1) and (2) of Proposition 3.4.3, we can take the uniform limit $H = \lim_{n \to \infty} H_n$, which will be the desired map in Proposition 3.3.2. The proof is essentially given in section 3.5 of [HPT10], so we just outline it here.

Proof of Proposition 3.3.2. First by Proposition 3.4.3 (1), we can show that $H_n$ converges to $H$ in the $C^k$ topology for any $k \in \mathbb{N}$, and hence $H$ is a $C^\infty$ diffeomorphism. Clearly $H$ is volume preserving. Also we can have $\|H - Id\|_{C^r} \leq \delta$ if we choose sufficiently small numbers $\delta_1, \delta_2, \delta_3$. Moreover, $H = H_n$ on $\tilde{Z}_{n-1}$, then $H$ is homotopic to $Q$, to $T$ and hence to $Id$. Also, since $H_n$ is pointwise partially hyperbolic, $H$ is also pointwise partially hyperbolic with one-dimensional strongly stable $E^s_H$ and unstable $E^u_H$ subbundles. One can show that the Lyapunov exponents $\lambda^s(z) < 0$ in the direction $E^s_H(z)$ and $\lambda^u(z) > 0$ in the direction $E^u_H(z)$ for almost every point $z \in \tilde{Z}$. 
By Proposition 3.4.3(3),(5) for any \( z \in \mathcal{Z} \), the strongly stable local manifolds \( V_{H_n}^s(z) \) have uniform size and converges in \( C^1 \) topology to a local manifold, which gives the strongly stable local manifold of \( H \) at \( z \). Similarly one can get the strongly unstable local manifold of \( H \) in this way. Hence the strongly stable \( E_{H}^s \) and unstable \( E_{H}^u \) subbundles are integrable to strongly stable \( W_{H}^s \) and unstable \( W_{H}^u \) foliations with smooth leaves, which are transversal and absolute continuous (see [BP07]).

To get the accessibility property of \( H \) via \( W_{H}^s \) and \( W_{H}^u \), one use Proposition 3.4.3 (6) to show that \( \angle(E_{H}^i(z), E_{H_n}^j(z)) \leq \theta_n \) for \( z \in \hat{Z}_n, \ i = u, s, c \), and so \( H \) has accessibility property on each \( Z_n \), hence on \( \mathcal{Z} \).

To show that \( H \) has positive central Lyapunov exponents, by Proposition 3.4.3 (4), semicontinuity of average Lyapunov exponent \( L_k(\cdot) \) and the fact \( L_3(Q) - L_2(Q) > 0 \), one can show that

\[
L_3(H) - L_2(H) = \int \lambda_3(z,H) dm(z) > 0.
\]

If follows that there is a subset \( \mathcal{A} \subset \mathcal{Z} \) of positive volume such that \( \lambda_1(z) \geq \lambda_2(z) \geq \lambda_3(z) > 0 \) for all \( z \in \mathcal{A} \). And since \( H \) preserves volume, one must have \( \lambda_4(z) < 0 \) for all \( z \in \mathcal{A} \).

Now by Theorem 2.5.1, we obtain that \( H \) has positive central exponents almost everywhere in \( \mathcal{Z} \), and \( H|\mathcal{Z} \) is ergodic and indeed, is a Bernoulli diffeomorphism.

Finally by Proposition 3.4.3 (3) and the fact that \( \delta_n \leq d(I_n, \partial I)^2 \), we have \( H = Id \) on \( \hat{Z}\setminus \mathcal{Z} \) and \( dH_z = Id \) for all \( z \in \hat{Z}\setminus \mathcal{Z} \). This completes the proof of Proposition 3.3.2. \( \square \)

### 3.4.3 Construction of The Map \( Q \): Proof of Proposition 3.4.2

In this part, using an approach similar to the ones in [HPT10, HT06], we obtain the map \( Q \) as two consecutive gentle perturbations of \( T \) as follows:

\[
T \xrightarrow{h_S} S = h_S \circ T \xrightarrow{h_Q} Q = h_Q \circ S,
\]
where \( h_S, h_Q \) are two \( C^\infty \) volume preserving diffeomorphisms of \( \hat{\mathcal{Z}} \), which are close to \( Id \). Moreover, \( h_S \) and \( h_Q \) are concentrated on disjoint small open subsets \( \Omega_S \) and \( \Omega_Q \) of \( \mathcal{Z}_0 \) respectively. If follows that \( Q = T \) outside \( \Omega_S \cup \Omega_Q \). By this construction, we shall show that \( S = h_S \circ T \) has two positive average Lyapunov exponents in the \( E^u_T \) subbundle, i.e. \( L_1(S) < L_2(S) \), and \( Q = h_Q \circ S \) has three positive average Lyapunov exponents, i.e. \( L_1(Q) < L_2(Q) < L_3(Q) \).

First note that given \( z \in \hat{\mathcal{Z}} \), we can choose a local coordinate system \( (s,u,t,y) \) associated to \( T \), i.e.

\[
F_s(z) := \frac{\partial}{\partial s} = E^s_T(z), \quad F^u(z) := \frac{\partial}{\partial s} = E^u_T(z), \quad F^t(z) := \frac{\partial}{\partial t} \tag{3.3}
\]

are the stable, unstable and central flow direction of \( T \) respectively, and

\[
F^y(z) := \frac{\partial}{\partial y} \tag{3.4}
\]

is tangent to \( \hat{I} \).

Choose periodic points \( p, p', p'' \) of the Anosov automorphism \( A \) of \( X \), which are close to each other and whose orbits are pairwise disjoint. Let \( V^s_A(p), V^u_A(p), V^s_A(p') \) and \( V^u_A(p') \), \( i = t, y \) be the stable and unstable local manifolds at these periodic points. We may assume that each intersection \( V^s_A(p) \cap V^u_A(p') \) and \( V^u_A(p') \cap V^s_A(p) \) consists of exactly one point, denoted by \( [p,p'] \) and \( [p',p] \) respectively. Let \( \gamma^i \) denote the closed quadrilateral path with the collection of points \( p, [p,p'], p', [p',p] \) and \( p \), and let

\[
\gamma(p) = V^u_A(p) \cup V^s_A(p), \quad \gamma(p') = V^u_A(p') \cup V^s_A(p').
\]

Choose sufficiently small number \( \nu > 0 \), and set for \( i = t, y \),

\[
\Omega^i(\nu) = ( \bigcup_{t \in [0, \tau(p')]} B_N(T^t(p',0), \nu) ) \times I
\]

\[
\hat{\Omega}^i(\nu) = ( \bigcup_{(x,t) \in (\gamma(p) \times [0, \tau(p)]) \cup (\gamma(p') \times [0, \tau(p')])} B_N((x,t), \nu) ) \times I
\]

\[
\Omega(\nu) = ( \bigcup_{i=t,y} \Omega^i(\nu) ) \cup ( \bigcup_{i=t,y} \hat{\Omega}^i(\nu) ).
\]
where $\tau(p)$ and $\tau(p')$ are the periods of $p$ and $p'$, and $B_N((x, t), r)$ is the ball in $N$ of radius $r$ centered at $(x, t) \in N$. Finally, we set

$$
\Omega_0 = \Omega_0(\nu) = \Omega(\nu) \cap \mathcal{Z}_0 \quad (3.5)
$$

According to sublemma 3.4.5, given $\epsilon = \delta_2/2 > 0$, where $\delta_2$ is given in Proposition 3.4.2, we can choose $\theta_0 > 0$ and an integer $k_0 > 0$ such that

$$
\theta = \pi/2k_0 < \theta_0 \quad (3.6)
$$

Now choose $\nu$ small enough to ensure that

$$
20k_0m(\Omega_0(\nu)) < 1 \quad (3.7)
$$

### 3.4.3.1 Construction of the map $S$

We shall obtain the map $S$ from $T$ via a small perturbation $h_S$, which is a small rotation in the $E^{ut}_T$ subbundle on a small subset of $\mathcal{Z}_0$.

Recall that $T((x, t), y) = (T^{\kappa_0}((x, t)), y)$ at every $z = ((x, t), y) \in \mathcal{Z}_0$, and the expansion rate in the $E^{u}_T$-direction at $z$ is the constant $\eta = \eta^{\kappa_0}_A$. Given $N_0 \geq 20k_0$, pick a point $(x_0, t_0) \in N$ and $\varepsilon_1 > 0$ such that

$$
B_N((x_0, t_0), 2\varepsilon_1) \cap \text{Proj}_N \Omega_0 = \emptyset;
$$

$$
T^{-k\kappa_0}B_N((x_0, t_0), 2\varepsilon_1) \cap B_N((x_0, t_0), 2\varepsilon_1) = \emptyset, \quad k = 1, \ldots, N_0,
$$

where $\text{Proj}_N$ is the projection of $\mathcal{Z}$ onto $N$. Now set

$$
\Omega_S = B_N((x_0, t_0), \varepsilon_1) \times I_0. \quad (3.8)
$$

One can take $\varepsilon_1$ small enough such that

$$
\Omega_S \cap \Omega_0 = \emptyset, \quad T^{-k}\Omega_S \cap \Omega_S = \emptyset, \quad k = 1, \ldots, N_0. \quad (3.9)
$$

Choose a $C^\infty$ function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

1. $\psi(r) = \psi_0 > 0$ if $0 \leq r \leq 0.9$;
2. \( \psi(r) > 0 \) if \( 0 \leq r < 1 \) and \( \psi(r) = 0 \) if \( r \geq 1 \);

3. \( \|\psi\|_{C^r} \leq 1 \).

Centered at \(((x_0, t_0), 0) \in \Omega_S\), we switch from the local Cartesian coordinate system \((u, t, y, s)\) to the local cylindrical coordinate system \((r, \theta, y, s)\), where \( u = r \cos \theta, \ t = r \sin \theta \). Given \( \tau > 0 \), define a small rotation \( h_{S, \tau} \) in the \( E^u_T \) subbundle on \( \Omega_S \) by

\[
h_{S, \tau}(r, \theta, y, s) = (r, \theta + \tau \sigma, y, s),
\]

where

\[
\sigma = \sigma(r, s, y) = 0.25 \varepsilon_1^2 \psi(r) \psi(\frac{|s|}{\varepsilon_1}) \psi(2|y|).
\]

And we can extend \( h_{S, \tau} \) to \( \hat{Z} \) by simply letting \( h_{S, \tau} = Id \) outside \( \Omega_S \). Clearly \( h_{\tau} \) is a \( C^\infty \) volume preserving diffeomorphism such that

1. \( \|h_{S, \tau} - Id\|_{C^r} \to 0 \) as \( \tau \to 0 \);

2. \( dh_{S, \tau} \) preserves \( E^u_T \) subbundle;

3. \( \det(dh_{S, \tau}|E^u_T(z)) = 1 \) for any \( z \in \hat{Z} \).

We then define \( S_{\tau} = T \circ h_{S, \tau} \). Also we set \( I'_0 = 0.9 I_0 = (-0.45, 0.45) \).

**Lemma 3.4.4.** Given \( 0 < \delta_S < \delta_2/2 \), there exists \( \tau > 0 \) such that the map \( S = S_{\tau} \) is a \( C^\infty \) volume preserving diffeomorphism of \( \hat{Z} \) satisfying:

1. \( \|S - T\|_{C^r} \leq \delta_S \) and \( S \) is homotopic to \( T \);

2. \( S = T \) outside \( \Omega_S \), in particular, \( S = T \) on the sets \( \hat{Z}\backslash Z_0 \) and \( \Omega_0 \), which indicates that \( S \) is a gentle perturbation of \( T \);

3. \( S \) satisfies statements (3)-(5) of Proposition 3.4.1;

4. for any \( z \in \hat{Z} \),

\[
E^u_S(z) = E^u_T(z), \quad \det(dS|E^u_S(z)) = \det(dT|E^u_T(z));
\]

5. for any \( y_1, y_2 \in I'_0 \),

\[
\text{Proj}_N(S((x,t), y_1)) = \text{Proj}_N(S((x,t), y_2));
\]
6. \( L_1(S) < L_1(T) \) and hence,

\[
L_1(S) < L_2(S) = L_3(S) = L_3(T) > 0 = L_4(S) = L_4(T);
\]

7. there exists a number \( \lambda_S > 0 \) and a set \( \Pi_S = \text{Proj}_{\mathcal{N}} \Pi_S \times I_0 \) such that

\[
m(\Pi_S) \geq 20k_0m(\Pi_S \cap \Omega_S) > 0,
\]

where \( m \) is the Riemannian volume on \( \hat{\mathcal{Z}} \), and for any \( z \in \Pi_S \) the map \( S \) has two positive Lyapunov exponents \( \lambda_1(z, S), \lambda_2(z, S) \geq \lambda_S \) along the \( E_{u}^{u} = E_{T}^{u} \) subbundle.

Proof. The proof is an adaptation of arguments in \cite{DHP01, HPT10} to our case. We shall just outline the proof here. Statements (1)-(5) follows easily from the construction of the map \( h_{S, \tau} \) when \( \tau \) is sufficiently small. Moreover, \( S \) is dynamical coherent by Theorem 2.4.1. It remains to show Statements (6) and (7).

For statement (6), since \( S = S_\tau = T \) on \( \hat{\mathcal{Z}} \setminus \mathcal{Z}_0 \), we only need to show that

\[
L_1(S_\tau | \mathcal{Z}_0) < L_1(T | \mathcal{Z}_0) \tag{3.11}
\]

for sufficiently small \( \tau \). It is easy to see that

\[
L_1(S_\tau | \mathcal{Z}_0) = \int_{\mathcal{Z}_0} \lambda_1(z, S_\tau) dm(z) = \int_{\mathcal{Z}_0} \log |dS_\tau(z)| E_{u}^{u}_{S_\tau}(z) | dm(z).
\]

By the fact that \( h_{S, \tau} \) and \( S_\tau \) preserve the \( E_{u}^{u} \)-subbundle, we denote by \( e_\tau(z) \) the unique number such that the vector \( v_\tau(z) = (1, e_\tau(z), 0, 0)^t \in E_{u}^{u}_{S_\tau}(z) \) for all \( z \in \mathcal{Z} \) in the \( (u, t, y, s) \) local coordinate system. Also for \( z \in \Omega_S \subset \mathcal{Z}_0 \), we have

\[
dT | E_{u}^{u}(z) = \begin{pmatrix} \eta & 0 \\ 0 & 1 \end{pmatrix}, \quad dh_\tau | E_{u}^{u}(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]

where

\[
A = A(\tau, z) = 1 - \tau r \sigma_r \sin \theta \cos \theta - \frac{\tau^2 \sigma^2}{2} - \tau^2 \sigma_r \sigma \cos^2 \theta + O(\tau^3),
\]

\[
B = B(\tau, z) = -\tau \sigma - \tau r \sigma_r, \sin^2 \theta - \tau^2 \sigma_r \sigma, \sin \theta \cos \theta + O(\tau^3),
\]
\[ C = C(\tau, z) = \tau \sigma + \tau r \sigma_r \cos^2 \theta - \tau^2 r \sigma_r \sin \theta \cos \theta + O(\tau^3), \]
\[ D = D(\tau, z) = 1 + \tau r \sigma_r \sin \theta \cos \theta - \frac{\tau^2 \sigma_r}{2} - \tau^2 r \sigma_r \sin^2 \theta + O(\tau^3), \]

and hence
\[ dS_\tau|E_T^{ut}(z) = \begin{pmatrix} \eta A & \eta B \\ C & D \end{pmatrix}. \]

Repeating the arguments of Lemma B.7 in [DHP01], one can show that
\[ L_\tau = L_1(S_\tau|Z_0) = \int_{Z_0} \log \eta dm(z) - \int_{Z_0} \log[D(\tau, z) - \eta B(\tau, z)e_\tau(S_\tau(z))]dm(z). \]

Note that \( D(0, z) = 1 \) and \( B(0, z) = 0 \), we get
\[ L_0 = \int_{Z_0} \log \eta dm(z) = L_1(T|Z_0). \]

Applying the same arguments in Lemma B.8 in [DHP01], we can show
\[ \frac{dL_\tau}{d\tau}|_{\tau=0} = 0, \quad \frac{d^2L_\tau}{d\tau^2}|_{\tau=0} < 0, \]

which immediately implies that (4.13) for sufficiently small \( \tau > 0 \).

In fact, one get that
\[ \frac{dL_\tau}{d\tau}|_{\tau=0} = -\int_{Z_0} D_\tau|_{\tau=0} dm(z) = 0 \]

and
\[ \frac{d^2L_\tau}{d\tau^2}|_{\tau=0} = \int_{Z_0} [(D_\tau)^2 - D_{\tau\tau} + 2\eta B_\tau \frac{\partial}{\partial \tau}(e_\tau(S_\tau(z)))]|_{\tau=0} dm(z). \]

Similar to Lemma B.9 in [DHP01], this integral can be written as
\[ \int_{Z_0} [(D_\tau(0, z))^2 - D_{\tau\tau}(0, z) + 2\eta B_\tau(0, z)C_\tau(0, z)] dm(z) \]
\[ + \int_{Z_0} \sum_{i=1}^{\infty} \frac{1}{\eta^2} 2B_\tau(0, z)C_\tau(0, T^{-i}(z)) dm(z). \]
The first term is bounded above by
\[-(1 - \varepsilon_1) \int_{z_0} \sigma^2 dm(z) - \frac{1}{8} \int_{z_0} r^2 \sigma^2 dm(z).\]

For the second term, note that
\[\int_{z_0} 2B_r(0, z) C_r(0, T^{-i}(z)) dm(z) \leq 4 \int_{z_0} (\sigma^2 + r^2 \sigma^2) dm(z),\]
and \(B_r(0, z) C_r(0, T^{-i}(z)) = 0\) for \(z \in Z_0 \setminus \Omega_S\) and any \(i\). Also \(B_r(0, z) C_r(0, T^{-i}(z)) = 0\) at every \(z \in \Omega_S\) for \(i = 1, \ldots, N_0 - 1\) since \(T^{-i}\Omega_S \cap \Omega_S = \emptyset\). This allows we take \(N_0 > 0\) big enough such that the second term is bounded by
\[\frac{1}{10} \int_{\Omega_S} (\sigma^2 + r^2 \sigma^2) dm(z).\]

Hence
\[\frac{d^2 L_{\tau}}{d\tau^2} |_{\tau=0} \leq -(\frac{9}{10} - \varepsilon_1) \int_{z_0} \sigma^2 dm(z) - \frac{1}{40} \int_{\Omega_S} r^2 \sigma^2 dm(z) < 0.\]

It follows that \(L_1(S_\tau) < L_1(T)\) for sufficiently small \(\tau > 0\). Moreover, since \(\det(dS_\tau|E_T^i) = \det(dT|E_T^i)\) for \(i = ut, uty\) and \(S_\tau\) is volume preserving, we have \(L_i(S_\tau) = L_i(T)\) for \(i = 2, 3\), and \(L_4(S_\tau) = 0\), and hence
\[L_1(S) < L_2(S) = L_3(S) = L_3(T) > 0 = L_4(S) = L_4(T).\]

To prove statement (7) we notice that for any \(S_\tau\) preserves the fiber \(\mathcal{N} \times \{y\}\) for any \(y \in I\), and by statement (5), we know that for any \(y_1, y_2 \in I'_0\),
\[\text{Proj}_{\mathcal{N}}(S((x,t), y_1)) = \text{Proj}_{\mathcal{N}}(S((x,t), y_2)),\]
in other words, the action of \(S_\tau\) on the fiber \(\mathcal{N} \times \{y\}\) are the same for each \(y \in I'_0\). Then by the same arguments as above, one could have \(L_1(S_\tau|\mathcal{N} \times \{y\}) < L_2(S_\tau|\mathcal{N} \times \{y\})\) for sufficiently small \(\tau\), which does not depend on \(y \in I'_0\). Moreover, for sufficiently small \(\lambda_S > 0\), the sets \(\Pi_S(y) = \{z \in \mathcal{N} \times \{y\} : \lambda_2(z, S_\tau) \geq \lambda_S\}\) are of the same form as \(\Pi_S(y) = A \times \{y\}\) for some \(A \subset \mathcal{N}\) of positive Lebesgue measure.
Then we set $\Pi_S = \mathcal{A} \times I'_0$. Clearly $\Pi_S$ is an $S_\tau$-invariant subset. Moreover, by the fact that $\Pi_S$ is also $T$-invariant and (3.9), we have

$$m(\Pi_S) \geq \sum_{k=1}^{N_0} m(\Pi_S \cap T^{-k}\Omega_S) = N_0 m(Pi_S \cap \Omega_S) \geq 20k_0 m(\Pi_S \cap \Omega_S).$$

This completes the proof of Statement (7). \qed

### 3.4.3.2 Construction of the map $Q$

Following [HPT10], we go on to perturb $S$ to the map $Q$ via a diffeomorphism $h_Q$. This time we construct $h_Q$ as a composition of rotations in $F^y$-subbundle on several pairwise disjoint cylinders in $\Pi_S$ (in Lemma 3.4.4), which gives a total rotation $\pi/2$. In this way we gain positive central exponents in both central directions $F^t$ and $F^y$ for the map $Q$. The technique we shall use here is the Rokhlin Halmos tower construction for $S$ on the measurable set $\Pi_S$, which allows us to do rotations on finitely many small cylinders.

Let $\lambda = \lambda_S$ and $\Pi = \Pi_S$ be as in Lemma 3.4.4. Given $K \in \mathbb{N}$, set

$$\Lambda' = \Lambda'(K) = \{ z \in \Pi : \frac{1}{k} \log \| dS^k(z, v) \| - \lambda \leq 0.1 \lambda, \forall v \in E^u_S(z), \| v \| = 1 \text{ and } |k| \geq 0.5 K \}$$

and also

$$\Lambda = \Lambda(K) = \bigcap_{i=0}^{k_0-1} S^{-i}\Lambda'(K),$$

where $k_0$ is given by (3.6). Since $m(\Lambda'(K)) \to m(\Pi)$ as $K \to \infty$, we also have $m(\Lambda(K)) \to m(\Pi)$ as $K \to \infty$. Remember that $\delta_1$ and $\delta_2$ are given in Proposition 3.4.1 and 3.4.2 respectively, and one can choose $K$ so large that

$$K\lambda \geq \max\{5k_0\lambda, 10 \log 2, -10k_0 \log (1 - \delta_1 - \delta_2)\},$$

$$\lambda m(\Pi) + 40 \log (1 - \delta_1 - \delta_2) m(\Pi \setminus \Lambda) > 0,$$

$$20m(\Pi \setminus \Lambda) \leq m(\Pi).$$
Set
\[ \Lambda^* = \Lambda \setminus \bigcup_{i=0}^{k_0-1} S^{-i}(\Omega_0 \cup \Omega_S) \] (3.17)
where \( \Omega_0 \) and \( \Omega_S \) are given by (3.5) and (3.9) respectively. By Lemma 3.4.4 (7), and also choosing \( \nu \) in (3.5) small enough, we have
\[ m(\Omega_S \cap \Pi) \leq m(\Pi)/20k_0, \quad m(\Omega_0 \cap \Pi) \leq m(\Pi)/20k_0 \] (3.18)
Combining above relations, we find that \( m(\Lambda^*) \geq 0.8m(\Pi) \).

Now let us use Rokhlin-Halmos towers (see [KSF82]) to approximate the measurable set \( \Pi \). More precisely, we can choose a measurable subset \( \Gamma' \subset \Pi \) such that \( S_i \Gamma' \) are pairwise disjoint for \( -K \leq i \leq 6K + k_0 - 1 \) and
\[ m \left( \bigcup_{i=-K}^{6K+k_0-1} S^i \Gamma' \right) \geq 0.9m(\Pi) \] (3.19)
Take \( \Gamma_0 \) as the set of first entries to \( \Lambda^* \) of the trajectories \( \{S^i(z)\}_{i=0}^{5K-1} \) with \( z \in \Gamma' \), i.e.
\[ \Gamma_0 = \{S^j(z) : z \in \Gamma', 0 \leq j \leq 5K - 1, S^i(z) \in \Lambda^*, S^i(z) \in \Lambda^* \text{ for } i < j\} \]
By Lemma 3.4.4 (5), one can find that \( \Gamma_0 = \text{Proj}_{\Lambda'} \Gamma_0 \times I'_0 \). Furthermore, we can approximate \( \Gamma_0 \) by finitely many disjoint cylinders \( \Sigma_{0j} \) of the form
\[ \Sigma = B_{F^u}(u_j, r'_j) \times B_{F^s}(s_j, r''_j) \times B_{F^v}((t_j, y_j), r_j) \]
where \( z_j = (u_j, s_j, t_j, y_j) \in \tilde{Z}, r'_j \geq r_j, r''_j \geq r_j \eta^0, \) and \( j = 1, \ldots, J \). Let
\[ \Gamma_i = S^i \Gamma_0, \quad \Gamma = \bigcup_{i=-K}^{K+k_0-1} \Gamma_i \] (3.20)
where \( \Gamma_i \) are pairwise disjoint for \( -K \leq i \leq K + k_0 - 1 \). Then we can also approximate \( \Gamma_i \) by \( \Sigma_{ij} = S^i \Sigma_{0j}, \) \( j = 1, \ldots, J \), which are also cylinders. Moreover, we may choose the sets \( \Sigma_{0j} \) in such a way that
\[ \Sigma_{ij} \cap \Sigma_{kl} = \emptyset, \quad (i, j) \neq (k, l), -K \leq i, k \leq K + k_0 - 1, 1 \leq j, l \leq J, \]
and $\Sigma_{ij} \cap (\Omega_0 \cup \Omega_S) = \emptyset$ for $0 \leq i \leq k_0 - 1$, $1 \leq j \leq J$. Set $\Delta_i = \bigcup_{j=1}^{J} \Sigma_{ij}$ for $i = 0, \ldots, k_0 - 1$, we can also assume that

$$m(\Gamma_i \triangle \Delta_i) \leq 0.05 \max\{m(\Gamma_i), m(\Delta_i)\}$$  \hspace{1cm} (3.21)

We need the following sublemma (see sublemma 4.5 in [HPT10]) to construct the perturbation $h_Q$.

**Sublemma 3.4.5.** For any $\epsilon > 0$, there is $\theta_0 > 0$ such that for any $0 \leq \theta \leq \theta_0$, any cylinder $\Sigma \subset \mathbb{R}^4$ of the form

$$\Sigma = B_1(z_1, s_1) \times B_2(z_2, s_2) \times B_{34}((z_3, z_4), s_3)$$

with $s_1, s_2 \geq s_3$, there exists a subcylinder $\Sigma' \subset \Sigma$ of the form

$$\Sigma' = B_1(z_1, s'_1) \times B_2(z_2, s'_2) \times B_{34}((z_3, z_4), s'_3)$$

and a $C^\infty$ map $\rho : \mathbb{R}^4 \to \mathbb{R}^4$ satisfying:

1. $\rho$ is a rotation by the angle $\theta$ in $z_3z_4$-plane on $\Sigma'$, i.e.

$$\rho(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3 \cos \theta - z_4 \sin \theta, z_3 \sin \theta + z_4 \cos \theta);$$

2. $\rho = Id$ outside $\Sigma$;

3. $m(\Sigma')/m(\Sigma) \geq 0.75$;

4. $s'_i/s_i \geq 0.9$ for $i = 1, 2, 3$;

5. $\|\rho - Id\|_{C^r} \leq \epsilon$, where $r$ is in Proposition 3.3.2.

Applying this sublemma on each cylinder $\Sigma_{ij}$, $i = 0, \ldots, k_0 - 1$, $j = 1, \ldots, J$, we obtain a map $\rho_{ij}$ and a subcylinder $\Sigma'_{ij} \subset \Sigma_{ij}$ such that $\|\rho_{ij} - Id\| \leq \delta_2/2$ and $m(\Sigma'_{ij})/m(\Sigma_{ij}) \geq 0.75$. Moreover, by (3.6, one can make $\rho_{ij}\Sigma_{ij}$ the rotation by the angle $\theta = \pi/2k_0$ along the $F^{uy}$-subspace and $\rho_{ij} = Id$ outside $\Sigma_{ij}$. With good
choices of $\Sigma'_{ij}$ one may assume that $S(\Sigma'_{ij}) = \Sigma'_{i+1,j}$ for $i = 0, \ldots, k_0 - 1$. Let

$$\Omega_Q = \bigcup_{i=0}^{k_0-1} \Delta_i, \quad \Delta'_i = \bigcup_{j=1}^{J} \Sigma'_{ij}$$

(3.22)

then $m(\Delta'_i)/m(\Delta_i) \geq 0.75$. Define $h_Q = \rho_{ij}$ on each $\Sigma_{ij}$, and $h_Q = Id$ otherwise. Clearly $h_Q$ is a $C^\infty$ volume preserving diffeomorphism and $dh_Q$ preserves $E_T^{utyt}$ subbundle with $\det(dh_Q|E_T^{utyt}(z)) = 1$ for any $z \in \tilde{Z}$. Finally define $Q = h_Q \circ S$, and we ready to show that $Q$ is the desired map in Proposition 3.4.2.

Proof of Proposition 3.4.2. We give a sketch of the proof here, see section 4.2 in [H-PT10] for more details. By the construction of $Q$, statements (1)-(4) hold immediately. To prove statement (5), set $\Delta^*_0 = \Delta'_0 \cap \Lambda$, and

$$U_1 = Q^{-K}\Delta^*_0, \quad U_2 = \Delta_0 \setminus \Delta^*_0,$$

$$U_3 = Q^{k_0}((\Delta_0 \cap \Lambda) \setminus \Delta^*_0), \quad U_4 = Q^{k_0}(\Delta_0 \setminus \Lambda)$$

and consider the first return map $\overline{Q} = Q^\beta$ on the set $U = U_1 \cup U_2 \cup U_3 \cup U_4$, where $\beta = \beta(z)$ is the first return time of $z \in U$ to $U$ under $Q$. Note that $E_\overline{Q}^{utyt}(z) = E_S^{utyt}(z) = E_T^{utyt}(z)$ for any $z \in U$.

We intend to show that

$$\int_U \left( \log \| \wedge^3 (d\overline{Q}|E_T^{utya}(z)) \| - \log \| \wedge^2 (d\overline{Q}|E_T^{utya}(z)) \| \right) dm(z) > 0$$

(3.23)

where $\wedge^k (d\overline{Q}|E_T^{utya}(z))$ is the $k$-th exterior power of $d\overline{Q}|E_T^{utya}(z)$. Indeed if this is the case, we take $\Pi' = \bigcup_{i=-\infty}^{\infty} Q^i(U)$, then for $k = 2, 3$,

$$\int_U \log \| \wedge^k (d\overline{Q}|E_T^{utya}(z)) \| dm(z) = \int_{\Pi'} \log \| \wedge^k (dQ|E_T^{utya}(z)) \| dm(z)$$

$$= \int_{\Pi'} \sum_{i=1}^{k} \lambda_i(z,Q) dm(z) = L_k(Q|\Pi'),$$

and hence $L_3(Q|\Pi') > L_2(Q|\Pi')$. Since $Q = S$ outside $\Pi'$, we obtain that $L_3(Q) > L_2(Q)$, which indicates statement (5) of Proposition 3.4.2.

To show (3.23), one can split the left-hand side into four integrals on $U_i$, $i =
1, 2, 3, 4, and estimate lower bounds for each. See [HPT10] for more details.

3.4.4 Construction of the maps $H_n$: Proof of Proposition 3.4.3

Recall that the map $Q$ in Proposition 3.4.2 is pointwise partially hyperbolic with one-dimensional stable, one-dimensional unstable and two-dimensional central sub-bundles. $Q$ is a gentle perturbation of $T$ and has positive average central Lyapunov exponents on $Z$, however, $Q|Z$ does not have the accessibility property, which is needed for the ergodicity of $Q|Z$ in view of Theorem 2.5.1.

To this end we construct $H_n$ as a small gentle perturbation of $Q$ for each $n \geq 0$, such that $H_n$ has the accessibility property on an invariant open set $\tilde{Z}_n$, and is stably accessible on an open set $Z_n$ (see (3.2)). Then the limit diffeomorphism $H$ of the sequence $H_n$ will be accessible on $Z$ since $\bigcup_{n \geq 0} Z_n = Z$.

3.4.4.1 Construction of the maps $H_n$

First we can decompose the sets $Z_n$, $\tilde{Z}_n$, $\tilde{Z}_n$ and $\tilde{Z}_n$ as follows: let $J_0 = [-0.5, 0.5]$, $\tilde{J}_0 = (-5/8, 5/8)$, $\bar{J}_0 = (-9/16, 9/16)$ and $\tilde{\bar{J}}_0 = (-17/32, 17/32)$. For $l \geq 1$, set

$$J_l = \left[1 - \frac{1}{2^l}, 1 - \frac{1}{2^{l+1}} \right], \quad \tilde{J}_l = \left(1 - \frac{3}{2^{l+1}}, 1 - \frac{3}{2^{l+3}} \right),$$

$$\bar{J}_l = \left(1 - \frac{9}{2^{l+3}}, 1 - \frac{7}{2^{l+4}} \right), \quad \tilde{\bar{J}}_l = \left(1 - \frac{20}{2^{l+4}}, 1 - \frac{15}{2^{l+5}} \right)$$

and $J_l = -J_{-l}$, $\tilde{J}_l = -\tilde{J}_{-l}$, $\bar{J}_l = -\bar{J}_{-l}$, $\tilde{\bar{J}}_l = -\tilde{\bar{J}}_{-l}$ for $l \leq -1$. Clearly, we have for all $l \in \mathbb{Z}$,

$$J_l \subset \tilde{J}_l \subset \bar{J}_l \subset \tilde{\bar{J}}_l.$$

Also note that $\tilde{J}_l \cap \tilde{J}_{l+2} = \emptyset$. Moreover,

$$I_n = \bigcup_{|l| \leq n} J_l, \quad \tilde{I}_n = \bigcup_{|l| \leq n} \tilde{J}_l, \quad \bar{I}_n = \bigcup_{|l| \leq n} \bar{J}_l, \quad \tilde{\bar{I}}_n = \bigcup_{|l| \leq n} \tilde{\bar{J}}_l, \quad \text{(3.24)}$$
where $I_n, \tilde{I}_n, \bar{I}_n, \bar{I}_n$ are given by (3.1). Hence we can write

$$Z_n = \bigcup_{|t| \leq n} \mathcal{N} \times J_t, \quad \tilde{Z}_n = \bigcup_{|t| \leq n} \mathcal{N} \times \tilde{J}_t,$$

$$\bar{Z}_n = \bigcup_{|t| \leq n} \mathcal{N} \times J_t, \quad \bar{Z}_n = \bigcup_{|t| \leq n} \mathcal{N} \times \bar{J}_t.$$

Also we take $K = (-1/8, 1 + 1/8)$, $\bar{K} = (-1/4, 1 + 1/4)$ and $\bar{K} = (-1/16, 1 + 1/16)$ for the time variable $t$.

Pick two triples of pairwise disjoint periodic points $\{p_j, p^i_j, p^y_j\}$, $j = 0, 1$, of the Anosov automorphism $A$ of $X$. We can assume that the orbits of $p_1$ and $p_2$ under $A$ are disjoint, and choose $\epsilon_0 > 0$ such that $B_X(A^i p_1, \epsilon_0) \cap B_X(A^i p_2, \epsilon_0) = \emptyset$ for $i = -1, 0, 1$. We may also assume that $p^i_j, p^y_j \in B_X(p_j, \epsilon_0/3)$. Without loss of generality, one can take $\{p_0, p^i_0, p^y_0\} = \{p, p^i, p^y\}$ as in the construction of $\Omega_0$ at the beginning of section 3.4.3. For $\epsilon_0$ sufficiently small, we have that $V^u(p_j) \cap V^s(p^i_j), \ V^s(p_j) \cap V^u(p^y_j)$ consist of exactly one point, denoted by $[p_j, p^i_j], [p^i_j, p_j]$ respectively, where $i = t, y, j = 0, 1$.

For any integer $l \neq 0$, let $\eta_-(l) = \min \{\eta^+_{j}(y) : y \in J_t\}$, set

$$\tilde{\nu}^i_u(l) = d(p^i_j, [p^i_j, p^y_j]), \quad \tilde{\nu}^i_s(l) = d(p^i_j, [p_j, p^i_j]);$$

$$\nu^i_u(l) = \tilde{\nu}^i_u(l)/\eta_-(l), \quad \nu^i_s(l) = \tilde{\nu}^i_s(l)/\eta_-(l),$$

where $j \equiv l$ mod 2, and $i = t, y$. Define rectangles in $X$:

$$\tilde{\Pi}^i_l = B_{F^u}(p^i_j, \tilde{\nu}^i_u(l)) \times B_{F^s}(p^i_j, \tilde{\nu}^i_s(l)), \quad \Pi^i_l = B_{F^u}(p^i_j, \nu^i_u(l)) \times B_{F^s}(p^i_j, \nu^i_s(l)).$$

In the case $l = 0$, choose the smallest numbers $\bar{l}_u^i$ and $\bar{l}_s^i$ such that

$$A^{-\bar{l}_u^i}[p^i, p] \in B_X(p^i, \nu/2), \quad A^{\bar{l}_s^i}[p, p^i] \in B_X(p^i, \nu/2),$$

where $\nu$ is given by (3.7), and

$$\tilde{\nu}^i_u(0) = d(p^i, A^{-\bar{l}_u^i}[p^i, p]), \quad \tilde{\nu}^i_s(0) = d(p^i, A^{\bar{l}_s^i}[p, p^i]);$$

$$\nu^i_u(0) = \tilde{\nu}^i_u(0)/\eta^+_{A}, \quad \nu^i_s(0) = \tilde{\nu}^i_s(0)/\eta^-_{A}.$$
and define rectangles centered at $p^i$ in $X$:

$$
\tilde{\Pi}_0^i = B_{F^u}(p^i, \tilde{\nu}_u^i(0)) \times B_{F^s}(p^i, \tilde{\nu}_s^i(0)), \quad \Pi_0^i = B_{F^u}(p^i, \nu_u^i(0)) \times B_{F^s}(p^i, \nu_s^i(0)).
$$

Finally, we let

$$
\tilde{\epsilon}_t(l) = \max\{\kappa(y)/2 : y \in J_l\}, \quad \epsilon_t(l) = 5\tilde{\epsilon}_t(l)/6.
$$

Fix an integer $l$ and write

$$\nu_i^t = \nu_{\lambda}^t(l), \quad \tilde{\nu}_i^t = \tilde{\nu}_{\lambda}^t(l), \quad i = t, y, \quad \lambda = u, s,$$

and $\epsilon_t = \epsilon_t(l), \quad \tilde{\epsilon}_t = \tilde{\epsilon}_t(l)$. Choose $C^\infty$ functions on $\mathbb{R}$ as follows:

1. $\phi^i, \psi^i$ satisfying:
   
   (a) $\phi^i =$const. on $(-\nu_u^i, \nu_u^i)$, $\psi^i =$const. on $(-\nu_s^i, \nu_s^i)$.

   (b) $\phi^i = 0$ outside $(-\tilde{\nu}_u^i, \tilde{\nu}_u^i)$, $\psi^i = 0$ outside $(-\tilde{\nu}_s^i, \tilde{\nu}_s^i)$.

   (c) $\int_0^{\pm \tilde{\epsilon}_t^i} \phi^i(r)dr = 0$, and $\psi^i > 0$ on $(-\tilde{\nu}_s^i, \tilde{\nu}_s^i)$.

   (d) $\|\phi^i\|_{C^{r+\bar{l}}}, \|\psi^i\|_{C^{r+\bar{l}}} \leq 1$.

2. $\xi_t, \xi_y$ satisfying:

   (a) $\xi_t =$const. on $K$, $\xi_y =$const. on $J_t$.

   (b) $\xi_t = 0$ outside $\tilde{K}$, $\xi_y = 0$ outside $\tilde{J}_t$.

   (c) $\xi_t > 0$ on $\tilde{K}$, $\xi_y > 0$ on $\tilde{J}_t$.

   (d) $\|\xi_t\|_{C^{r+\bar{l}}}, \|\xi_y\|_{C^{r+\bar{l}}} \leq 1$.

3. $\zeta_t, \zeta_y$ satisfying:

   (a) $\zeta_t =$const. on $(-\epsilon_t, \epsilon_t)$, $\zeta_y =$const. on $J_t$.

   (b) $\zeta_t = 0$ outside $(-\tilde{\epsilon}_t, \tilde{\epsilon}_t)$, $\zeta_y = 0$ outside $\tilde{J}_t$.

   (c) $\zeta_t > 0$ on $(-\tilde{\epsilon}_t, \tilde{\epsilon}_t)$, $\zeta_y > 0$ on $\tilde{J}_t$.

   (d) $\|\zeta_t\|_{C^{r+\bar{l}}}, \|\zeta_y\|_{C^{r+\bar{l}}} \leq 1$. 
Now Consider the box $\hat{\Omega}^y_I = \tilde{\Pi}^y_I \times (\frac{1}{2} - \varepsilon_t, \frac{1}{2} + \varepsilon_t) \times \tilde{J}_t$ centered at $z^y_I = (p^y_I, \frac{1}{2}, y_I)$, where $j \equiv l \mod 2$ and $y_I$ is the middle point of $J_t$. Introduce the local coordinate system $(u, t, y, s)$ originated at $z^y_I$, then

$$\hat{\Omega}^y_I = \{(u, t, y, s) : |u| \leq \hat{v}^y_u, |s| \leq \hat{v}^y_s, |t| < \varepsilon_t, y \in \tilde{J}_t\}. \quad (3.25)$$

For each $\beta > 0$, define a vector field $X^y = X^y_{l,\beta}$ supported on $\hat{\Omega}^y_I$ by

$$X^y = \beta \psi^y(s) \xi(t) \left( -\xi^t(y) \int_0^u \phi^y(r) dr, 0, \xi^y(y) \phi^y(u), 0 \right), \quad (3.26)$$

and clearly $X^y$ is constant on the subset $\Omega^y_I = \Pi^y_I \times (\frac{1}{2} - \varepsilon_t, \frac{1}{2} + \varepsilon_t) \times J_t$. We define $h^y = h^y_{l,\beta}$ on $\hat{\Omega}^y_I$ to be the time-1 map of the flow generated by $X^y$, and set $h^y = Id$ outside $\hat{\Omega}^y_I$. Since $X^y$ is divergence free, $dh^y$ preserves $E^u_T$-subbundle and $\det(dh^y|E^u_T(z)) = 1$.

Similarly, take $\hat{\Omega}^t_I = \tilde{\Pi}^t_I \times K \times \tilde{J}_t$ centered at $z^t_I = (p^t_I, \frac{1}{2}, y_I)$, and the local coordinate system $(u, t, y, s)$ is centered at $z^t_I$, then

$$\hat{\Omega}^t_I = \{(u, t, y, s) : |u| \leq \hat{v}^t_u, |s| \leq \hat{v}^t_s, |t| < 3/4, y \in \tilde{J}_t\}. \quad (3.27)$$

Define $X^t = X^t_{l,\beta}$ supported on $\Omega^t_I$ by

$$X^t = \beta \psi^t(s) \xi^t(y) \left( -\xi^t(y) \int_0^u \phi^t(r) dr, \xi^t(y) \phi^t(u), 0, 0 \right), \quad (3.28)$$

and clearly $X^t$ is constant on the subset $\Omega^t_I = \Pi^t_I \times K \times J_t$. We define $h^t = h^t_{l,\beta}$ on $\hat{\Omega}^t_I$ to be the time-1 map of the flow generated by $X^t$, and set $h^t = Id$ outside $\hat{\Omega}^t_I$. Since $X^t$ is divergence free, $dh^t$ preserves $E^u_T$-subbundle and $\det(dh^t|E^u_T(z)) = 1$.

Take the map $Q$ in Proposition 3.4.2. Given a sequence of positive numbers $\{\beta_n\}_{n \geq 0}$, start with

$$H_0 = h^t_{0,\beta_0} \circ h^y_{0,\beta_0} \circ Q,$$

and inductively let

$$H_n = h^t_{-n,\beta_n} \circ h^y_{-n,\beta_n} \circ h^t_{n,\beta_n} \circ h^y_{n,\beta_n} \circ H_{n-1}, \quad (3.29)$$

for $n \geq 1$. We are going to show that $H_n$ are the desired maps in Proposition 3.4.3.
with suitable choices of the sequence \( \{\beta_n\}_{n \geq 0} \).

### 3.4.4.2 Proof of Proposition 3.4.3

We outline the proof of Proposition 3.4.3, following [DHP01, HPT10].

First note that statements (2) and (4) and the fact that \( H_n \) is homotopic to \( Q \) follow directly from the above construction. Moreover, the original map \( T \) is uniformly partially hyperbolic on each \( \hat{\mathcal{Z}}_n \) and is dynamical coherent in view of Theorem 2.4.1. We can choose \( \{\beta_n\}_{n \geq 0} \) carefully in (3.29), and a positive sequence \( \{\delta'_{n}\}_{n \geq 0} \) with \( \delta'_{n} \leq \delta'_{n-1} / 2 \) such that

\[
\|H_0 - Q\|_{C^r} \leq \delta'_0, \quad \|H_n - H_{n-1}\|_{C^{r+n}} \leq \delta'_n
\]

then the statement (3) holds. In particular, \( H_n \) is a gentle perturbation of \( T \) and dynamical coherent. It remains to show \( H_n \) satisfies statement (5) and (6) with good choices of \( \delta_n \) and \( \theta_n \).

Denote \( W_{H_n}^c(z) \) the center manifold of \( H_n \) at the point \( z \in \hat{\mathcal{Z}} \). For any \( l \in \mathbb{Z} \), set \( z_0(l) = (q_j, \frac{1}{2}, y_l) \in \mathcal{N} \times \hat{J}_l \), where \( j \equiv l \mod 2 \), and \( y_l \) is the middle point of \( \bar{J}_l \). Let \( n = |l| \), and we denote by \( W_{H_n}^c(z_0(l), K, J_l) \) the connected component of \( W_{H_n}^c(z_0(l)) \cap (X \times K \times J_l) \) that contains \( z_0(l) \). We shall also use similar notations \( W_{H_n}^c(z_0(l), \hat{K}, \hat{J}_l) \), etc.

Now let us introduce two important maps \( \Theta \) and \( \Psi \).

Denote \( \gamma_j^i \) the quadrilateral \((u, s)_{A}\)-path of \( X \) with the collection of points \( p_j, [p_j, p_j^i], [p_j^i, p_j], [p_j^i, p_j] \) and \( p_j \), for \( i = t, y \) and \( j = 0, 1 \). Given \( l \in \mathbb{Z} \), set \( n = |l| \) and \( j \equiv l \mod 2 \), one can lift the quadrilateral \( \gamma_j^i \) to a quadrilateral \((u, s)_{H_n}\)-path \( \hat{\gamma}_j^i \) of \( \mathcal{Z} \) with the initial point \( z_1 \) by letting

\[
\begin{align*}
z_2 &= V_{H_n}^u(z_1) \cap V_{H_n}^{sc}(p_j^i, 1/2, y_l), \\
z_3 &= V_{H_n}^s(z_2) \cap V_{H_n}^{uc}(p_j^i, 1/2, y_l), \\
z_4 &= V_{H_n}^u(z_3) \cap V_{H_n}^{sc}(z_1), \\
z_5 &= V_{H_n}^s(z_4) \cap V_{H_n}^{uc}(z_1). \tag{3.31}
\end{align*}
\]

and \( \hat{\gamma}_j^i = \{z_1, \ldots, z_5\} \). This path defines a map \( \Theta^i = \Theta^i_{l,H_n} \) given by \( \Theta^i(z_1) = z_5 \). It is easy to see that \( z_5 \in V_{H_n}^c(z_1) \), and hence \( \Theta^i \) maps \( W_{H_n}^c(z_0(l), \hat{K}, \hat{J}_l) \) into...
itself. Reparameterizing the curve $V_{H_n}(z_1)$ from $z_1$ to $z_2$ by $\sigma : [0, 1] \rightarrow V_{H_n}(z_1)$ so that $\sigma(0) = z_1$ and $\sigma(1) = z_2$, we obtain a parameterized family of quadrilaterals $\tilde{\gamma}_j^i = \{z_1(\tau), \ldots, z_5(\tau)\}$, $\tau \in [0, 1]$, where $z_1(\tau) = z_1$, $z_2(\tau) = \sigma(\tau)$, and $z_i(\tau)$, $i = 3, 4, 5$ are obtained in the way similar to (4.43). Then we obtain $\Theta^i_\tau = \Theta^i_{\tau,l,H_n}$ given by $\Theta^i_\tau(z_1) = z_5(\tau)$. Clearly $\Theta^i_\tau = Id$, $\Theta^i_1 = \Theta^i$, and $\Theta^i_\tau$ depends continuously on $\tau$ and maps $W^r_{H_n}(z_0(l), \tilde{K}, \tilde{Z}_t)$ into $W^r_{H_n}(z_0(l))$.

On the other hand, given $z = ((x, t), y)$, there is a $(u, s)_T$-path connecting $z$ to $z' = ((p_j, t), y)$ whose length does not exceed $2d(x, p_j)$. This generates a map $\Psi_T = \Psi_{T,j}$ from $Z$ to $\{p_j\} \times \tilde{K} \times I$ given by $\Psi_T(z) = z'$.

Moreover, if $H^2$ is a gentle perturbation of $H_n$, and hence a gentle perturbation of $T$, by uniform partially hyperbolicity and dynamical coherence of $H_n$ and $T$, one can define $\Theta^i_{l,H^2}$, $\Theta^i_{\tau,l,H^2}$ and $\Psi_{H^2,j}$ in a similar way, $i = t, y, \tau \in [0, 1]$ and $j \equiv l \mod 2$. As long as $H^2 = T$ outside some $Z_k$ and $\angle(E^u_{H^2}(z), E^u_T(z))$ is sufficiently small for all $z \in Z_k$, $\omega = u, s, c$, the maps $\Theta^i_{l,H^2}$, $\Theta^i_{\tau,l,H^2}$ and $\Psi_{H^2,j}$ depends uniformly continuously on $H^2$, see [HPT10] for more details.

Given a set $\Gamma \subset \mathcal{Z}$ and a gentle perturbation $H^2$ of $T$, set

$$\mathcal{A}_{H^2}(\Gamma) = \{z \in \mathcal{Z} : \text{there exists } y \in \Gamma \text{ such that } y \text{ is accessible to } z \text{ via a } (u, s)_{H^2} \text{ path}\}. \quad (3.32)$$

Let $n = |\Gamma|$, and set $\epsilon_n = \min\{1/2^{n+4}, \epsilon_t(l)\}$. Also, let $i = t, y$, $\omega = u, s, c$, $\tau \in [0, 1]$.

We now show how to choose $\delta_n$ and $\theta_n$. For $n = l = 0$, choose $\theta^*_0 > 0$ such that for any gentle perturbation $H^2$ of $T$ with $\angle(E^u_{H^2}(z), E^u_T(z)) \leq 2\theta^*_0$ for $z \in \tilde{Z}_0 = \mathcal{N} \times \tilde{I}_0$, the maps $\Psi_{H^2} = \Psi_{H^2,0}$ and $\Theta_{H^2,0}$, are well defined. We assume that $\delta_2$ in Proposition 3.4.2 is so small that $\angle(E^u_{Q}(z), E^u_T(z)) \leq \theta^*_0$ and $d(\Theta^i_{t,0,Q}(z), z) \leq \epsilon_0/8$ for $z \in Z_0 = \mathcal{N} \times I_0$. Now choose $\theta^*_0$ such that $0 < \theta^*_0 \leq \theta^*_0/2$, and if $H^2$ is a gentle perturbation of $T$ with $\angle(E^u_{H^2}(z), E^u_T(z)) \leq 2\theta^*_0$ for $z \in \tilde{Z}_0$, then

$$d(\Psi_{H^2}(z), \Psi_Q(z)) \leq 1/27, \quad z \in \tilde{Z}_0 = \mathcal{N} \times \tilde{I}_0. \quad (3.33)$$

Also choose $\delta'_0$ in (3.30) so small that if $\|H_0 - Q\|_{C^r} \leq \delta'_0$, then for all $z \in \tilde{Z}_0$, we have $\angle(E^u_{H_0}(z), E^u_Q(z)) \leq \theta'_0$.

Set $\delta_0 = \min\{\delta'_0, \delta'_0\}$ and $\theta_0 = \min\{\theta'_0, \theta'_0\}$ where $\delta'_0$ and $\theta'_0$ are given by Lemma
3.4.6. For any gentle perturbation $H^z$ of $H_0$ with $\angle(E^\omega_{H^z}(z), E^\omega_{H_0}(z)) \leq \theta'_0$ for all $z \in Z_0$, we have

$$\angle(E^\omega_{H^z}(z), E^\omega_{Q}(z)) \leq 2\theta'_0 \leq \theta^*_0,$$  \hspace{1cm} (3.34)

$$\angle(E^\omega_{H^z}(z), E^\omega_{T}(z)) \leq 2\theta^*_0.$$  \hspace{1cm} (3.35)

By Lemma 3.4.6, (3.34) implies that

$$d(\Theta_{\tau_0,H_0}(z), z) \leq \epsilon_0/4$$

for all $z \in W_{H_0}(z_0(0), \bar{K}, \bar{J}_0)$. Moreover,

$$A_{H^z}(z_0(0)) \supset W^c_{H^z}(z_0(0), \bar{K}, \bar{J}_0),$$

Since the distance between $\partial \bar{J}_0$ and $\partial \bar{J}_0$ is $1/2^5$, (3.35) implies that

$$\Psi_{P_t}(\mathcal{N} \times \bar{J}_0) \subset W^c_{H^z}(z_0(0), \bar{K}, \bar{J}_0),$$

and by the fact that $z$ and $\Psi_{H^z}(z)$ are $(u,s)_{H^z}$-accessible and hence

$$A_{H^z}(z_0(0)) \supset \mathcal{N} \times \bar{J}_0 = \bar{Z}_0.$$

Proceed inductively in a similar way for $n \geq 1$, we can find $\delta'_n$ and $\theta'_n$ such that (3.30) and statements (5) and (6) of Proposition 3.4.3 hold. More precisely, if $\|H_n - H_{n-1}\| \leq \delta'_n$ then $\angle(E^\omega_{H_n}(z), E^\omega_{H_{n-1}}(z)) \leq \theta'_n$ for $z \in \bar{Z}_n$. Take $\delta_n = \min\{\delta'_n, \delta''_n\}$ and $\theta_n = \min\{\theta'_n, \theta''_n\}$ where $\delta''_n$ and $\theta''_n$ are given by Lemma 3.4.6, and also one can make $\theta'_n < \theta_{n-1}/2$. Moreover, if $H^z$ is a gentle perturbation of $H_n$ such that $\angle(E^\omega_{H^z}(z), E^\omega_{H_n}(z)) \leq \theta_n$ for $z \in \bar{Z}_n$, then $\angle(E^\omega_{H^z}(z), E^\omega_{H_{n-1}}(z)) \leq 2\theta'_n \leq \theta_{n-1}$, and hence by statement (6), $H^z$ has accessibility property on $\bar{Z}_{n-1}$. By the same argument above for $n = 0$, one can get

$$A_{H^z}(z_0(n)) \supset \mathcal{N} \times \bar{J}_l, \quad l = \pm n,$$  \hspace{1cm} (3.36)
in other words, $H^2$ has the accessibility property on $\mathcal{N} \times \tilde{J}_l$, $l = \pm n$. Note that

$$\tilde{Z}_n = \tilde{Z}_{n-1} \cup (\mathcal{N} \times \tilde{J}_n) \cup (\mathcal{N} \times \tilde{J}_{-n}),$$

and $\tilde{Z}_{n-1} \cup (\mathcal{N} \times \tilde{J}_l) \neq \emptyset$ for $l = \pm n$. Since $\tilde{Z}_n$ is connected, we obtain the accessibility of $H^2$ on $\tilde{Z}_n$. In particular, take $H^2 = H_n$, and we obtain that $H_n$ has accessibility property on $\tilde{Z}_n$.

To complete the proof of Proposition 3.4.3, it remains to show the following lemma. (Since the construction on $\mathcal{N} \times \tilde{J}_{-l} \text{ and } \mathcal{N} \times \tilde{J}_l$ are symmetric for $l \geq 1$, we just need to consider the case when $n = l \geq 0$ for this lemma.)

**Lemma 3.4.6.** Let $H_{-1} = Q$, $z_0(-1) = z_0(0)$, $\epsilon_{-1} = \epsilon_0/2$, and also $i = t, y$, $\omega = u, s, c$, $\tau \in [0, 1]$. Suppose for some $n \geq 0$, $d(\Theta^i_{\tau, n, H_{n-1}}(z), z) \leq \epsilon_{n-1}/4$ for all $z \in W^c_{H_{n-1}}(z_0(n), \tilde{K}, \tilde{J}_n)$. Then there exist $\delta''_n$, $\theta''_n > 0$ such that if $H_n$ satisfies $\|H_n - H_{n-1}\|_{C^r+n} \leq \delta''_n$, then we have

$$d(\Theta^i_{\tau, n+1, H_n}(z), z) \leq \epsilon_n/4, \text{ for all } z \in W^c_{H_n}(z_0(n+1), \tilde{K}, \tilde{J}_{n+1}). \quad (3.37)$$

Moreover, for any gentle perturbation $H^2$ of $H_n$ with

$$\angle(E^\omega_{H^2}(z), E^\omega_{H_n}(z)) \leq \theta''_n, \text{ for all } z \in \mathcal{N} \times \tilde{J}_n,$$

we have

$$A_{H^2}(z_0(n)) \supset W^c_{H^2}(z_0(n), \tilde{K}, \tilde{J}_n) \quad (3.38)$$

In particular, (4.46) holds with $H^2 = H_{n+1}$.

This is essentially the same as Lemma 5.2 in [HPT10]. The proof are parallel, hence omitted here.
Chapter 4

Coexistence in the continuous-time case

4.1 Essential coexistence for volume preserving flows

The goal of this chapter is to extend Theorem 1.2.2 to dynamical systems with continuous time thus demonstrating essential coexistence of regular and chaotic dynamics (of type I). In fact, we construct an example of a smooth volume preserving flow that exhibits what can be viewed as a counterpart of the “KAM-type” phenomenon.

Theorem 4.1.1. [CHP11] There exist a compact smooth Riemannian manifold $\mathcal{M}$ of dimension 5 and a $C^\infty$ flow $h^t : \mathcal{M} \to \mathcal{M}$ such that

1. $h^t$ preserves the Riemannian volume $m$ on $\mathcal{M}$;

2. $h^t$ ($t \neq 0$) has nonzero Lyapunov exponents (except for the exponent in the flow direction) almost everywhere on an open, dense and connected subset $\mathcal{U} \subset \mathcal{M}$; moreover, $h^t|\mathcal{U}$ is an ergodic flow;

3. the complement $\mathcal{U}^c$ has positive volume and is a union of 3-dimensional invariant submanifolds. $h^t$ is a non-identity linear flow with Diophantine frequency vector on each invariant submanifold, and $h^t$ has zero Lyapunov exponents on $\mathcal{U}^c$. 
We emphasize that it is Statement (3) that makes our construction of the flow $h^t$ nontrivial and of “KAM-type” on $U^c$. Indeed, let us consider the suspension flow $H^t$ over $P$ with a constant roof function, where $P$ is the diffeomorphism in Theorem 1.2.2. It is easy to verify that $H^t$ satisfies Statements (1)–(2), however, $H^t$ is just a periodic flow on $U^c$.

We split the proof of this theorem into several steps. We construct the manifold $\mathcal{M}$ and the open set $\mathcal{U}$ in Section 4.2 and introduce the “start-up” flow $f^t$ that satisfies Statements (3) and (4) of the theorem in Section 4.3. In Section 4.4.2 we construct a volume preserving flow $g^t$, which is a small perturbation of $f^t$ and does not affect the action of $f^t$ on the set $U^c$. The flow $g^t$ has nonzero Lyapunov exponents on a subset of positive volume in $\mathcal{U}$. Finally, in Section 4.5 we construct the desired flow $h^t$ as a small perturbation of the flow $g^t$.

In our construction of flows $g^t$ and $h^t$ we use the perturbation techniques developed in [HPT10] for the case of diffeomorphisms. However, there is a principle difference between the discrete-time and continuous-time cases. In particular, to effect our construction we make perturbations of the vector fields that generate the required flows and we need to make sure that these perturbations produce the same or similar effects on the time-1 maps of the flows as in [HPT10]. This is made possible due to the crucial fact that the “start-up” flow $f^t$ has a global cross-section and we need to make sure that our perturbations are done in such a way that both flows $g^t$ and $h^t$ preserve this cross-section. This is achieved by using specific formulae for perturbations of the vector fields.

### 4.2 Construction of the 5-dimensional manifold

Let $A$ be an Anosov automorphism of the 2-torus $X = T^2$ with expanding rate $\eta > 1$ along the unstable direction. Consider the suspension flow $S^t$ on the suspension manifold $\mathcal{N} = X \times \mathbb{R} / \sim$, with the identification $(x, \tau + 1) \sim (Ax, \tau)$. See Appendix A for more details of the geometric structure of $\mathcal{N}$.

Set $Y = T^2$ and $\mathcal{M} = \mathcal{N} \times Y$. To effect our construction we choose:

(A1) a Cantor set $C \subset Y$ of positive measure whose complement $U = Y \setminus C$ is a non-empty open and connected set;
(A2) an open square $U_0$ such that $\overline{U_0} \subset U$;

(A3) an open neighborhood $U_1$ of $U_0$ such that $\overline{U_1} \subset U$, whose choice will be specified in Subsection 4.5.1.

Finally we set $\mathcal{U} = \mathcal{N} \times U$ and $\mathcal{U}^c = \mathcal{N} \times C$.

4.3 Construction of the “start-up” flow $f_t$

Note that the action of the suspension flow on $\mathcal{N}$ is given by the formula $S^t(x, \tau) = (x, \tau + t\kappa)$. Given $\kappa > 0$, one can make a constant change of speed on the suspension flow, that is, $S^t_\kappa(x, \tau) = (x, \tau + t\kappa)$. On the other hand, given $\alpha \in \mathbb{T}^2$, let $T^t_\alpha : \mathcal{N} \to \mathcal{N}$ be a linear flow defined by $(x, \tau) \mapsto (x + t\alpha, \tau)$. It preserves each level set $X \times \{\tau\}$.

We now choose a $C^\infty$ function $\kappa : Y \to \mathbb{R}$ such that

\begin{enumerate}[(k1)]
  \item $\kappa(y) > 0$ for $y \in U$ and $\kappa(y) = 0$ for $y \in C$;
  \item $\kappa(y) = 1$ for all $y \in U_1$;
  \item $\|\kappa\|_{C^1} \leq 1$,
\end{enumerate}

and a $C^\infty$ map $\alpha : Y \to \mathbb{R}^2$ such that

\begin{enumerate}[(a1)]
  \item $\alpha(y) = 0$ for $y \in U_1$;
  \item $\alpha(y) = \alpha_0$ for all $y \in C$ where $\alpha_0$ is a Diophantine vector;
  \item $\sup_{y \in Y} \|\alpha(y)\| \leq \bar{\alpha}$, where $\bar{\alpha}$ is a positive number determined in Subsection 4.5.2;
\end{enumerate}

We define the “start-up” flow $f^t$ on $\mathcal{M}$ by the formula

$$f^t((x, \tau), y) = ((x + t\alpha(y), \tau + t\kappa(y)), y)$$ (4.1)

where $(x, \tau) \in \mathcal{N}$ and $y \in Y$. The following proposition describes the properties of the flow $f^t$ and its proof follows immediately from the definitions.

**Proposition 4.3.1.** The following statements hold:
1. \( f^t \) is a \( C^\infty \) volume preserving flow;

2. \( f^t \) preserves each fiber \( \mathcal{N} \times \{y\} \), on which \( f^t \) is the composition of the scaled suspension flow \( S^t_{\kappa(y)} \) and the linear flow \( T^t_{\alpha(y)} \). In particular, \( f^t \) is exactly the suspension flow \( S^t \) on \( \mathcal{N} \times \{y\} \) for \( y \in \overline{U_1} \);

3. \( f^t \) is pointwise partially hyperbolic on \( U \) with one-dimensional stable \( E^s_f \), one-dimensional unstable \( E^u_f \) and 3-dimensional center \( E^c_f \) subbundles. \( E^s_f \) and \( E^u_f \) are integrable to strongly stable and unstable foliations \( W^s_f \) and \( W^u_f \) with smooth leaves, which are absolutely continuous, uniformly transversal and have local leaves of uniform size;

4. \( f^t \) is uniformly partially hyperbolic on \( \mathcal{N} \times A \) where \( A \subset U \) is a compact subset, and hence \( f^t \) is dynamically coherent with the central foliation \( W^c_f = W^c_{S^t} \times Y \).

5. \( f^t \) preserves every two-dimensional torus \( X \times \{\tau\} \times \{y\} \) (\( \tau \in [0, 1] \), \( y \in C \) are fixed) and acts on it as a linear flow with a Diophantine frequency vector; moreover, \( f^t|U^c \) has all zero Lyapunov exponents;

6. for every \( z = ((x, \tau), y) \in \mathcal{M} \) the Lyapunov exponents of \( f^t \) are as follows:

\[
\lambda_1(z, f^t) = \kappa(y) \log \eta \geq 0 = \lambda_2(z, f^t) = \lambda_3(z, f^t) = \lambda_4(z, f^t) \\
\geq \lambda_5(z, f^t) = -\kappa(y) \log \eta.
\]

Moreover, if \( z \in U \), then \( \lambda_1(z, f^t) = \lambda^u(z, f^t) > 0 \) corresponds to the \( E^u_f(z) \) subspace, \( \lambda_5(z, f^t) = \lambda^s(z, f^t) < 0 \) corresponds to the \( E^s_f(z) \) subspace, and \( \lambda_2(z, f^t), \lambda_3(z, f^t) \) and \( \lambda_4(z, f^t) \) correspond to the flow direction and two directions in \( Y \) respectively.

We say that a \( C^\infty \) flow \( \psi^t \) is a gentle perturbation of \( f^t \) if the following conditions hold:

1. \( \psi^t \) is \( C^1 \)-close to \( f^t \);

2. \( \psi^t(U) = U \) and \( \psi^t \) is pointwise partially hyperbolic on \( U \);
3. the one-dimensional strongly stable and unstable subbundles for $\psi^t$ are integrable to strongly stable and unstable foliations with smooth leaves on $U$; the 3-dimensional central subbundle of $\psi^t$ is integrable to a central foliation;

4. $\psi^t|U^\epsilon = f^t|U^\epsilon$.

Remark. Let $\psi^t$ be a flow on $M$ which is $C^1$-close to $f^t$. Assume that there is an open $\psi^t$-invariant set $V$ such that $\overline{V} \subset U$ and $\psi^t|\overline{V} = f^t|\overline{V}$, then $\psi^t$ is a gentle perturbation of $f^t$ and in fact, $\psi^t$ is uniformly partially hyperbolic on $\overline{V}$.

### 4.4 Removing zero exponents

In this section we will construct a gentle perturbation $g^t$ of the original flow $f^t$, with positive central Lyapunov exponents on a set of positive volume but is not necessarily ergodic. Then we shall perturb $g^t$ to the desired flow $h^t$ of Theorem 4.1.1 in Section 4.5.

Given $z \in M$, there is a local Cartesian coordinate system $(u,s,\tau,a,b)$ (see Appendix A) such that

$$F^u(z) := \frac{\partial}{\partial u} = E^u_f(z), \quad F^s(z) := \frac{\partial}{\partial s} = E^s_f(z), \quad F^\tau(z) := \frac{\partial}{\partial \tau} = E^\tau_f(z)$$

are the unstable, stable and flow directions of $f^t$ respectively, and

$$F^a(z) := \frac{\partial}{\partial a} = E^a_f(z), \quad F^b(z) := \frac{\partial}{\partial b} = E^b_f(z)$$

are the other two central directions tangent to $Y$.

The following statement describes properties of the flow $g^t$.

**Proposition 4.4.1.** Given $\delta_g > 0$, there is a $C^\infty$ volume preserving flow $g^t$ on $M$ such that

1. $g^t$ is $(C^1, \delta_g)$-close to $f^t$, i.e., $\|X_f - X_g\| \leq \delta_g$, where $X_f$ and $X_g$ are the vector fields of the flows $f^t$ and $g^t$ respectively;

2. $g^t = f^t$ outside $N \times U_0$, and hence $g^t$ is a gentle perturbation of $f^t$ and satisfies Statements (3)-(5) of Proposition 4.3.1;
3. \( g^t \) preserves the subbundle \( E^\omega_f \), \( \omega = uab, uab\tau \). Moreover,

\[
\det(dg^t|E^\omega_f(z)) = \det(df^t|E^\omega_f(z)), \text{ for all } z \in \mathcal{M}
\]  

(4.2)

4. the average Lyapunov exponents of \( g^t \) satisfy

\[
L_5(g^t) = 0 < L_1(g^t) < L_2(g^t) < L_3(g^t) = L_4(g^t).
\]  

(4.3)

To prove this proposition, we extend the approach in [HPT10] to the case of flows and obtain the flow \( g^t \) as a result of two consecutive perturbations. First, we perturb the start-up flow \( f^t \) to a flow \( \tilde{g}^t \) by adding a rotational vector field \( \tilde{X}_R \) to the vector field \( X_f \). This produces two positive average Lyapunov exponents for the flow \( \tilde{g}^t \) in the \( E^{u\alpha} \) subbundle, i.e., \( L_1(\tilde{g}^t) < L_2(\tilde{g}^t) \). Next, we perturb \( \tilde{g}^t \) to the desired flow \( g^t \) by adding another rotational vector field \( X_R \) to the vector field \( X_{\tilde{g}} \) for the flow \( \tilde{g}^t \). As a result the flow \( g^t \) has three positive average Lyapunov exponents in the \( E^{uab} \) subbundle, i.e., \( L_1(g^t) < L_2(g^t) < L_3(g^t) \).

The vector fields \( \tilde{X}_R \) and \( X_R \) are chosen to be supported on disjoint open subsets \( \tilde{\Omega}_R \) and \( \Omega_R \) of \( \mathcal{N} \times U_0 \) respectively such that \( \tilde{X}_R = 0 \) outside \( \tilde{\Omega}_R \), \( X_R = 0 \) outside \( \Omega_R \) and \( ||\tilde{X}_R||_{C^1}, ||X_R||_{C^1} < \delta_g/2 \). Since \( \mathcal{N} \times U_0 \) is invariant under \( f^t \), we have that \( g^t = \tilde{g}^t = f^t \) outside \( \mathcal{N} \times U_0 \).

Our construction utilizes the following crucial feature of the flow \( f^t \): the set

\[
\Pi_0 = X \times \{0\} \times U_0
\]  

(4.4)

is a global cross-section of \( f^t|\mathcal{N} \times U_0 \), and the time-1 map restricted to \( \Pi_0 \) is exactly the Poincaré return map of \( f^t \) to \( \Pi_0 \). Furthermore, we make the construction of vector fields \( \tilde{X}_R \) and \( X_R \) in such a way that \( \Pi_0 \) is also a global cross-section for both flows \( \tilde{g}^t \) and \( g^t \) with the time-1 maps to be the Poincaré return map to \( \Pi_0 \). This fact allows us to apply arguments similar to those in [HPT10] to our flow case by focusing on the time-1 maps.
4.4.1 Construction of the flow $\tilde{g}^t$

In this section we construct the flow $\tilde{g}^t$ by perturbing the vector field $X_f$ inside the set $\mathcal{N} \times U_0$.

To effect our construction we choose distinct periodic points $q, p^a, p^b$ and $p^\tau$ of the Anosov automorphism $A$ of $X$, which are close to each other. Let $V_A^u(q), V_A^u(p^i)$ and $V_A^s(p^i), i = a, b, \tau$ be the stable and unstable local manifolds at these periodic points. We may assume that each intersection $V_A^u(q) \cap V_A^s(p^i)$ and $V_A^u(p^i) \cap V_A^s(q)$ consists of a single point, which we denote by $[q, p^i]$ and $[p^i, q]$ respectively. Let $\gamma_i$ denote the closed quadrilateral path with the collection of points $q, [q, p^i], p^i, [p^i, q]$ and $q$, and let

$$\gamma(q) = V_A^u(q) \cup V_A^s(q), \quad \gamma(p^i) = V_A^u(p^i) \cup V_A^s(p^i).$$

Choose $\nu > 0$ and set for $i = a, b, \tau$,

$$\Omega^i_0(\nu) = \left( \bigcup_{t \in [0, \iota(p^i)]} B_N(f^t(p^i, 0), \nu) \right) \times U_0,$$

$$\widehat{\Omega}^i_0(\nu) = \left( \bigcup_{(x, \tau) \in (\gamma(q) \times [0, \iota(q)]) \cup (\gamma(p^i) \times [0, \iota(p^i)])} B_N((x, \tau), \nu) \right) \times U_0,$$

$$\Omega_0 = \Omega_0(\nu) = \left( \bigcup_{i=a,b,\tau} \Omega^i_0(\nu) \right) \cup \left( \bigcup_{i=a,b,\tau} \widehat{\Omega}^i(\nu) \right),$$

where $\iota(q)$ and $\iota(p^i)$ are the periods of $q$ and $p^i$ respectively, and $B_N((x, \tau), r)$ is the ball in $\mathcal{N}$ of radius $r$ centered at the point $(x, \tau) \in \mathcal{N}$. We choose sufficiently small number $\nu$ to ensure that the measure of the set $\text{Proj}_{\Pi_0} \Omega_0$ is much smaller than the measure of $\Pi_0$, where $\Pi_0$ is given by (4.4).

To construct the rotational vector field $\tilde{X_R}$ we choose a $C^\infty$ function $\psi : \mathbb{R} \to [0, 1]$ such that

1. $\psi = 1$ on $(-0.9, 0.9)$;
2. $\psi > 0$ on $(-1, 1)$ and $\psi = 0$ outside $(-1, 1)$;
3. $\|\psi\|_{C^1} \leq 10$. 


Observe that \( N \times U_0 \) is invariant under the flow \( f^t \) and that
\[
f^t((x, \tau), y) = ((x, \tau + t), y)
\]
for \( ((x, \tau), y) \in N \times U_0 \). In other words, \( f^t|N \times U_0 \) is the product of the suspension flow on \( N \) and the identity map on \( U_0 \). It follows that \( \Pi_0 \) is a global cross-section for the flow \( f^t|N \times U_0 \), and the time-1 map restricted to \( \Pi_0 \) is \( f^1 = A \times Id \) and is exactly the Poincaré return map of \( f^t \) to \( \Pi_0 \). We call a set \( \Pi \times [\tau_1, \tau_2] \subset \Pi_0 \times \mathbb{R} / \sim = N \times U_0 \) a tube if
\[
f^t(\Pi \times \{\tau_1\}) \cap (\Pi \times \{\tau_1\}) = \emptyset \quad \text{for all } t \in [0, \tau_2 - \tau_1].
\]
It is easy to check that \( \Pi \times [\tau_1, \tau_2] \) is a tube if and only if the sets \( \Pi, f^1(\Pi), \ldots, f^l(\Pi) \) are pairwise disjoint, where \( l = \lfloor \tau_2 - \tau_1 \rfloor \). Choose a non-periodic point \( z_0 = (x_0, 0, y_0) \in \Pi_0 \setminus \text{Proj}_{\Pi_0} \Omega_0 \), where \( x_0 \) is a non-periodic point of the Anosov automorphism \( A \), \( y_0 \) is the center of square \( U_0 \) and \( \Omega_0 \) is the set given by 4.5. The local Cartesian coordinate in \( \Pi_0 \) originated at \( z_0 \) is given by \((u, s, a, b)\). We shall also consider the \( ua \)-cylindrical coordinate \((r, \theta, s, b)\), where \( u = r \cos \theta \), \( a = r \sin \theta \).

Given \( \varepsilon > 0 \), one can choose a \( ua \)-cylinder \( B \subset \Pi_0 \) centered at \( z_0 \) of size \( \varepsilon \), i.e.,
\[
B = B^{ua}(z_0, \varepsilon) \times B^a(z_0, \varepsilon) \times B^b(z_0, \varepsilon)
\]
(4.6)
\[
= \{(u, s, a, b) : u^2 + a^2 \leq \varepsilon^2, |s| \leq \varepsilon, |b| \leq \varepsilon\}
\]
\[
= \{(r, \theta, s, b) : r \leq \varepsilon, |s| \leq \varepsilon, |b| \leq \varepsilon\}.
\]

Given a sufficiently large \( N_0 \geq 20k_0 \) (the number \( k_0 \) is given by Lemma 4.4.5), we can choose \( \varepsilon \) so small that \( f^i(B) \cap B = \emptyset \) for \( i = 1, \ldots, N_0 \). Consider the tube
\[
\tilde{\Omega}_R = B \times [0, 1/2].
\]
(4.7)
Since \( z_0 \notin \text{Proj}_{\Pi_0} \Omega_0 \), we can further reduce \( \varepsilon \) to ensure that \( B \cap \text{Proj}_{\Pi_0}(\Omega_0) = \emptyset \). Hence \( \tilde{\Omega}_R \cap \Omega_0 = \emptyset \).
Given $\beta > 0$, define a $C^\infty$ rotational vector field $\tilde{X}_R = \tilde{X}_{R,\beta}$ on $\mathcal{M}$ as follows:

\[
\tilde{X}_{R,\beta}(z) = \begin{cases} 
\beta \tilde{\psi}(z) \frac{\partial}{\partial \theta}, & z \in \tilde{\Omega}_R, \\
0, & z \in \mathcal{M} \setminus \tilde{\Omega}_R,
\end{cases}
\]  

(4.8)

where

\[
\tilde{\psi}(z) = \tilde{\psi}(r, \theta, s, b, \tau) = \psi \left( \frac{r^2}{\varepsilon^2} \right) \psi \left( \frac{s}{\varepsilon} \right) \psi \left( \frac{b}{\varepsilon} \right) \psi \left( \frac{\tau - (1/4)}{1/4} \right).
\]

It is easy to see that $\| \tilde{\psi} \frac{\partial}{\partial \theta} \| \leq c$ where $c > 0$ is a constant, which is independent of $\varepsilon$. Hence, $\| \tilde{X}_{R,\beta} \| \to 0$ as $\beta \to 0$. Furthermore, $\tilde{X}_{R,\beta}$ is divergence free. Let $\tilde{g}^t = \tilde{g}^t_\beta$ be the flow generated by the vector field

\[
X_{\tilde{g}} = X_{\tilde{g},\beta} = X_f + \tilde{X}_{R,\beta}.
\]

**Proposition 4.4.2.** There exists $\beta > 0$ such that $\tilde{g}^t = \tilde{g}^t_\beta$ is a $C^\infty$ volume preserving flow with the following properties:

1. $\tilde{g}^t$ is $(C^1, \delta_g/2)$-close to $f^t$, i.e., $\| X_f - X_{\tilde{g}} \|_{C^1} \leq \delta_g/2$, where $X_f$ and $X_{\tilde{g}}$ are the vector fields corresponding to flows $f^t$ and $\tilde{g}^t$ respectively;

2. $\tilde{g}^t = f^t$ outside $N \times U_0$, and hence $\tilde{g}^t$ is a gentle perturbation of $f^t$ and satisfies Statements (3)-(5) of Proposition 4.3.1;

3. $\tilde{g}^t$ preserves the subbundle $E^\omega_f$, $\omega = ua, uab, uab\tau$. Moreover,

\[
\det(df^t|E^\omega_f(z)) = \det(d\tilde{g}^t|E^\omega_f(z)) \quad \text{for all } z \in \mathcal{M};
\]

(4.9)

4. the average Lyapunov exponents of $\tilde{g}^t$ satisfy

\[
0 < L_1(\tilde{g}^t) < L_2(\tilde{g}^t) = L_3(\tilde{g}^t) = L_4(\tilde{g}^t) > 0 = L_5(\tilde{g}^t).
\]

(4.10)

5. $\Pi_0$ is a global cross-section of the flow $\tilde{g}^t|N \times U_0$, and the time-1 map $\tilde{g}^1$ is the Poincaré return map of $g^t$ to $\Pi_0$. Furthermore, there exist $\lambda > 0$ and a
\( \tilde{g}^1 \)-invariant set \( \Pi \subset \Pi_0 \) such that

\[
m(\Pi) \geq 20k_0m(\Pi \cap B) > 0
\]

and for any \( z \in \Pi \) the flow \( \tilde{g}^t \) has two positive Lyapunov exponents \( \lambda_1(z, \tilde{g}^t) > \lambda_2(z, \tilde{g}^t) \geq \lambda \) along the \( E^u_f \) subbundle.

**Proof.** Statements (1) and (2) are easy corollaries of the construction of the flow \( \tilde{g}^t \). To prove Statement (3) we shall first show that \( d\tilde{g}^t \) preserves the subbundles \( E^u_f \). It suffices to check that for any smooth vector field \( \mathcal{X} \in E^u_f \) and any \( z \in \tilde{\Omega}_R \), the Lie bracket \([\mathcal{X}_{\tilde{g}}(z), \mathcal{X}(z)] \in E^u_f(z)\). Indeed, we have

\[
\mathcal{X}_{\tilde{g}}(z) = \frac{\partial}{\partial \tau} + \beta \tilde{\psi}(z) \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \tau} + \beta \tilde{\psi} \left( -a \frac{\partial}{\partial u} + u \frac{\partial}{\partial a} \right),
\]

and the direct calculation yields

\[
\left[ \mathcal{X}_{\tilde{g}}, \frac{\partial}{\partial \omega} \right] = \beta \left( \frac{\partial}{\partial \omega} \frac{\partial}{\partial u} - \frac{\partial}{\partial \omega} \frac{\partial}{\partial a} \right) \in E^u_f, \; \omega = u, a.
\]

Similarly, one can show that \( d\tilde{g}^t \) preserves the subbundles \( E^{ua}_f \) and \( E^{uab}_f \). Next, consider the variational differential equations

\[
\frac{d}{dt} f^t = D\mathcal{X}_f df^t, \quad \frac{d}{dt} \tilde{g}^t = D\mathcal{X}_{\tilde{g}} d\tilde{g}^t.
\]

The determinants along \( E^\omega_f \) with \( \omega = ua, uab, uab, \tau \) satisfy

\[
\frac{d}{dt} \det(df^t|E^\omega_f) = \text{div}(\mathcal{X}_f|E^\omega_f) \det(df^t|E^\omega_f), \quad \frac{d}{dt} \det(d\tilde{g}^t|E^\omega_f) = \text{div}(\mathcal{X}_{\tilde{g}}|E^\omega_f) \det(d\tilde{g}^t|E^\omega_f). \tag{4.11}
\]

Direct calculations show that \( \tilde{X}_R = \mathcal{X}_{\tilde{g}} - \mathcal{X}_f \) is divergence free along \( E^\omega_f \) and thus \( \text{div}(\mathcal{X}_f|E^\omega_f) = \text{div}(\mathcal{X}_{\tilde{g}}|E^\omega_f) \). Therefore, using (4.11) and the fact that \( \det(df^0|E^\omega_f) = \det(d\tilde{g}^0|E^\omega_f) = 1 \) we find that

\[
\det(d\tilde{g}^t|E^\omega_f(z)) = \det(df^t|E^\omega_f(z))
\]
and Statement (3) follows.

It remains to prove Statements (4) and (5). We need the following lemma showing that $\Pi_0$ is a global cross-section for the flow $\tilde{g}^t|N \times U_0$.

**Lemma 4.4.3.** Given $z \in B$, the $r$-, $s$-, $b$- and $\tau$-coordinates of $\tilde{g}^t(z)$ and $f^t(z)$ are the same for $t \in [0, 1/2]$. Consequently, $\tilde{g}^{\frac{1}{2}}(B) = f^{\frac{1}{2}}(B)$ and $\Pi_0$ is a global cross-section for the flow $\tilde{g}^t|N \times U_0$.

**Proof of the lemma.** Let us compare the orbit segments of $\tilde{g}^t(z)$ and $f^t(z)$ for $t \in [0, 1/2]$. Note that for any smooth function $\varphi$ and any vector field $X$ we have that

$$\frac{d}{dt}\varphi(F^t(z)) = L_X\varphi|_{F^t(z)},$$

where $L_X(\cdot)$ is the Lie derivative and $F^t$ is the flow that is generated by $X$. This implies that for $\omega = r, s, b,$

$$\frac{d}{dt}(\tilde{g}^t(z)) = L_{X bars}(\tilde{g}^t(z)) = \frac{d}{dt}(f^t(z));$$

$$\frac{d}{dt}(\omega(\tilde{g}^t(z))) = L_{X \omega}(\omega(\tilde{g}^t(z))) = \frac{d}{dt}(\omega(f^t(z))).$$

Under the same initial condition at $t = 0$, we get that the $r$, $s$, $b$- and $\tau$-coordinates of $\tilde{g}^t(z)$ and $f^t(z)$ are the same. Since $B$ has the cylindrical structure, we obtain that $\tilde{g}^t(z) \in B \times \tau(f^t(z)) \subset B \times \{t\}$. In particular, $\tilde{g}^{\frac{1}{2}}(B) = f^{\frac{1}{2}}(B) = B \times \{\frac{1}{2}\}$.

Since $X_{\tilde{g}} = X_f$ outside $\tilde{\Omega}_R = B \times [0, 1/2]$, we have that $\tilde{g}^1(\Pi_0) = f^1(\Pi_0) = \Pi_0$. In other words, $\Pi_0$ is also a global cross-section for $\tilde{g}^t|N \times U_0$. This completes the proof of the lemma.

It follows from the lemma that $\Pi_0$ is a global cross-section for the flow $\tilde{g}^t|N \times U_0$ and the time-1 map $\tilde{g}^1$ restricted to $\Pi_0$ is the Poincaré return map of $\tilde{g}^t$ on $\Pi_0$. Therefore, (4.10) is equivalent to

$$L_4(\tilde{G}) = 0 < L_1(\tilde{G}) < L_2(\tilde{G}) = L_3(\tilde{G}),$$

(4.12)

where $\tilde{G} = G_{\beta} = \tilde{g}^1|\Pi_0$. In fact, by (4.9), we have that $L_k(\tilde{G}) = L_k(f^1|\Pi_0)$ for
\( k = 2, 3, 4 \), and thus we only need to show that
\[
L_1(\tilde{G}) < L_1(f^1|\Pi_0). \tag{4.13}
\]

To this end we need the following lemma.

**Lemma 4.4.4.** For all \( z \in \Pi_0 \) the derivative of \( \tilde{G} = g^1|\Pi_0 \) along \( E_{ua}^\alpha \) has the form
\[
\frac{d\tilde{G}_\beta(z)|_{E_{ua}^\alpha}(z)}{d\beta} = \begin{pmatrix}
\eta A(\beta, z) & \eta B(\beta, z) \\
C(\beta, z) & D(\beta, z)
\end{pmatrix},
\tag{4.14}
\]

where
\[
A = A(\beta, z) = 1 - \beta r\sigma, \sin \theta \cos \theta - \frac{\beta^2 \sigma^2}{2} - \beta^2 r\sigma \sigma_r \cos^2 \theta + O(\beta^3),
\]
\[
B = B(\beta, z) = -\beta \sigma - \beta r\sigma, \sin^2 \theta - \beta^2 r\sigma \sigma_r \sin \theta \cos \theta + O(\beta^3),
\]
\[
C = C(\beta, z) = \beta \sigma + \beta r\sigma, \cos^2 \theta - \beta^2 r\sigma \sigma_r \sin \theta \cos \theta + O(\beta^3),
\]
\[
D = D(\beta, z) = 1 + \beta r\sigma, \sin \theta \cos \theta - \frac{\beta^2 \sigma^2}{2} - \beta^2 r\sigma \sigma_r \sin^2 \theta + O(\beta^3).
\]

**Proof of the lemma.** The desired relation (4.14) is apparent for \( z \in \Pi_0 \setminus B \) since \( \tilde{G} = f^1 \) and \( \sigma = 0 \) on \( \Pi_0 \setminus B \). Given \( z = (r, \theta, s, b, 0) \in B \) in the \( ua \)-cylindrical coordinate, by Lemma 4.4.3, we have that \( \tilde{g}^1(z) = (r, \theta + \theta(t), s, b, t) \) where \( \theta(t) = \beta \int_0^t \tilde{\psi}(\tilde{g}^1z)dt \) for \( 0 \leq t \leq 1/2 \). In particular, the coordinate of \( \tilde{g}^1 \) is \( (r, \theta + \beta \sigma, s, b, 1/2) \), where
\[
\sigma = \sigma(r, s, b) = \frac{1}{4} \psi \left( \frac{r^2}{\varepsilon^2} \right) \psi \left( \frac{b}{\varepsilon} \right) \psi \left( \frac{s}{\varepsilon} \right) \int_{-1}^1 \psi(t)dt.
\]

Back in the Cartesian coordinate \( (u, a, s, b, \tau) \), we obtain that
\[
\tilde{g}^1(z) = (u_1, a_1, s, b, 1/2)
\]
\[
:= (u \cos(\beta \sigma) - a \sin(\beta \sigma), u \sin(\beta \sigma) + a \cos(\beta \sigma), s, b, 1/2),
\]

and hence
\[
\tilde{G}(z) = \tilde{g}^1(z) = f^1 \tilde{g}^1(z) = (u_1, a_1, s, b, 1) = (\eta u_1, a_1, \eta^{-1} s, b, 0).
\]
The last equality follows from (A.1) and the fact that \( \tilde{g}^1(z_0) = f^1(z_0) \), where \( z_0 \) is the center of \( B \). Since \( \tilde{G} \) preserves the \( E^u_\beta \) subbundle, we have that

\[
d\tilde{G}_\beta(z)|E^u_\beta(z) = \begin{pmatrix} \eta A(\beta, z) & \eta B(\beta, z) \\ C(\beta, z) & D(\beta, z) \end{pmatrix},
\]

where

\[
A(\beta, z) = \frac{\partial u_1}{\partial u}, \quad B(\beta, z) = \frac{\partial u_1}{\partial a}, \quad C(\beta, z) = \frac{\partial a_1}{\partial u}, \quad D(\beta, z) = \frac{\partial a_1}{\partial a}.
\]

Then

\[
A = \frac{\partial u_1}{\partial u} = \cos(\beta \sigma) + [-u \sin(\beta \sigma) - a \cos(\beta \sigma)]\beta \sigma_u
\]
\[
= \cos(\beta \sigma) - \beta r \sin(\theta + \beta \sigma) \sigma_r \cos \theta
\]
\[
= 1 - \beta r \sigma_r \sin \theta \cos \theta - \beta^2 \frac{\sigma^2}{2} - \beta^2 r \sigma_r \cos^2 \theta + O(\beta^3).
\]

Similarly, we can obtain the formulae for \( B, C \) and \( D \).

This lemma allows us to follow the line of argument in the proof of Lemma 4.1 in [HPT10] to establish (4.13). For reader’s convenience we outline this argument here.

Denote by \( e_\beta(z) \) the unique number such that the vector \( v_\beta(z) = (1, e_\beta(z))^t \in E^u_{\tilde{G}_\beta}(z) \) for all \( z \in \Pi_0 \). One can show that

\[
L_\beta = L_1(\tilde{G}_\beta) = \int_{\Pi_0} \log \eta \ dm(z)
\]
\[
- \int_{\Pi_0} \log[D(\beta, z) - \eta B(\beta, z)e_\beta(\tilde{G}_\beta(z))] dm(z).
\]

Note that \( L_0 = L_1(f^1|\Pi_0) \), and we shall show that

\[
\frac{dL_\beta}{d\beta}|_{\beta=0} = 0, \quad \frac{d^2L_\beta}{d\beta^2}|_{\beta=0} < 0, \tag{4.15}
\]

which immediately implies that (4.13) holds for all sufficiently small \( \beta > 0 \).
To show (4.15) observe that
\[
\frac{dL_\beta}{d\beta} \bigg|_{\beta=0} = - \int_{\Pi_0} D_\beta \bigg|_{\beta=0} \ dm(z) = 0
\]
thus proving the first relation in (4.15). To prove the second relation note that
\[
\frac{d^2L_\beta}{d\beta^2} \bigg|_{\beta=0} = \int_{\Pi_0} \left[ (D_\beta)^2 - D_{\beta\beta} + 2\eta B_\beta \frac{\partial}{\partial \beta}(e_\beta(\bar{G}_\beta(z))) \right] \bigg|_{\beta=0} \ dm(z).
\]
This integral can be written as
\[
\int_{\Pi_0} \left[ (D_\beta(0, z))^2 - D_{\beta\beta}(0, z) + 2\eta B_\beta(0, z)C_\beta(0, z) \right] \ dm(z)
+ \sum_{i=1}^{\infty} \int_{\Pi_0} \frac{1}{\eta} 2B_\beta(0, z)C_\beta(0, f^{-i}(z)) \ dm(z).
\]
The first term is bounded from above by
\[-(1 - \varepsilon) \int_{\Pi_0} \sigma^2 dm(z) - \frac{1}{8} \int_{\Pi_0} r^2 \sigma^2 r dm(z).\]
To estimate the second term note that
\[
\int_{\Pi_0} 2B_\beta(0, z)C_\beta(0, f^{-i}(z)) dm(z) \leq 4 \int_{\Pi_0} (\sigma^2 + r^2 \sigma^2) dm(z)
\]
and that
\[B_\beta(0, z)C_\beta(0, f^{-i}(z)) = 0\]
for all \(z \in \Pi_0 \setminus B\) and all \(i\). Moreover,
\[B_\beta(0, z)C_\beta(0, f^{-i}(z)) = 0\]
for every \(z \in B\) and \(i = 1, \ldots, N_0 - 1\) since \(f^i(B) \cap B = \emptyset\). This allows us to take \(N_0 > 0\) large enough to ensure that the second term is bounded by
\[
\frac{1}{10} \int_{B} (\sigma^2 + r^2 \sigma^2) \ dm(z).
\]
Hence,
\[ \frac{d^2 L_\beta}{d \beta^2} |_{\beta=0} \leq -\left( \frac{9}{10} - \varepsilon \right) \int_{\Pi_0} \sigma^2 dm(z) - \frac{1}{40} \int_{\Pi_0} r^2 \sigma_\gamma^2 dm(z) < 0. \]

This completes the proof of the inequality (4.13) thus guaranteeing that for any sufficiently small \( \lambda > 0 \) the level set
\[ \Pi = \{ z \in \Pi_0 : \lambda_1(z, \tilde{G}) \geq \lambda_2(z, \tilde{G}) > \lambda \} \]
has positive measure. It is also invariant under \( \tilde{G} \). Since \( f^i(B) \cap B = \emptyset \) for \( i = 1, \ldots, N_0 \), we obtain that the sets \( \tilde{g}^i(\Pi \cap B) = \Pi \cap \tilde{g}^i(B) = \Pi \cap f^i(B) \) corresponding to different \( i \) are pairwise disjoint subsets of \( \Pi \). This implies that
\[ m(\Pi) \geq N_0 m(\Pi \cap B) \geq 20k_0 m(\Pi \cap B) > 0 \]
thus completing the proof of Proposition 4.4.2.

\[ \square \]

4.4.2 Construction of the flow \( g^t \)

We perturb the flow \( \tilde{g}^t \) to a flow \( g^t \) by adding a vector field \( \mathcal{X}_R \) to the vector field \( \mathcal{Z}_{\tilde{g}} \). We obtain \( \mathcal{X}_R \) as a sum of rotational vector fields in the \( ab \)-direction along several pairwise disjoint tubes so that the total rotation is \( \pi/2 \). This ensures positive Lyapunov exponents along the \( E_u^{ab} \) subbundle for the flow \( g^t \).

Note that there is \( M_0 > 0 \) such that for any flow \( F^t \) that is sufficiently \( C^1 \)-close to the flow \( f^t \)
\[ \|F^1 - f^1\|_{C^1} \leq M_0 \|\mathcal{X}_F - \mathcal{X}_f\|_{C^1}. \] (4.16)

According to Lemma B.0.5, \( M_0 \) depends only on the Riemannian metric and the start-up flow \( f^t \).

Let \( \lambda \) and \( \Pi \) be as in Statement (5) of Proposition 4.4.2. Given \( K > 0 \), let
\[ \Lambda' = \Lambda'(K) = \{ z \in \Pi : \left| \frac{1}{k} \log \|d\tilde{g}^k(z, v)\| - \lambda \right| \leq 0.1 \lambda, \] (4.17)
for all \( v \in E_u^{a}(z), \|v\| = 1 \) and all \( |k| \geq 0.5 K \}. \]
Let
\[ \Lambda = \Lambda(K) = \bigcap_{i=0}^{k_0-1} \tilde{g}^{-i}\Lambda'(K), \] (4.18)
where \( k_0 \) is given by Lemma 4.4.5. Since \( m(\Lambda'(K)) \rightarrow m(\Pi) \) as \( K \rightarrow \infty \) and hence, \( m(\Lambda(K)) \rightarrow m(\Pi) \) as \( K \rightarrow \infty \), one can choose \( K \) so large that
\[ K\lambda \geq \max\{5k_0\lambda, 10\log 2, -10k_0\log(1 - M_0\delta_g)\}. \] (4.19)
\[ \lambda m(\Pi) + 40\log(1 - M_0\delta_g)m(\Pi \setminus \Lambda) > 0, \] (4.20)
\[ 20m(\Pi \setminus \Lambda) \leq m(\Pi). \] (4.21)

Set
\[ \Lambda^* = \Lambda \setminus \bigcup_{i=0}^{k_0-1} \tilde{g}^{-i}(\Proj_{\Pi_0}(\Omega_0 \cup \tilde{\Omega}_R)) \] (4.22)
where \( \Omega_0 \) and \( \tilde{\Omega}_R \) are given by (4.5) and (4.7) respectively. If the number \( \nu \) is chosen small enough, Statement (5) of Proposition 4.4.2, allows us to assume that
\[ m(\Proj_{\Pi_0}\Omega_0 \cap \Pi) \leq m(\Pi)/20k_0, \] (4.23)
\[ m(\Proj_{\Pi_0}\tilde{\Omega}_R \cap \Pi) = m(B \cap \Pi) \leq m(\Pi)/20k_0. \]

Combining (4.21), (4.22) and (4.23), we find that
\[ m(\Lambda^*) \geq 0.8m(\Pi). \] (4.24)

By Statement (5) of Proposition 4.4.2, the set \( \Pi \) is invariant under the time-1 map of the flow \( \tilde{g}^1 \).

We will approximate the set \( \Pi \) by constructing an appropriate Rokhlin-Halmos tower (see [KSF82]) for the map \( \tilde{g}^1 \). More precisely, we choose a measurable subset \( \Gamma' \subset \Pi \) such that the sets \( \tilde{g}^i(\Gamma') \) are pairwise disjoint for \( -K \leq i \leq 6K + k_0 - 1 \) and
\[ m\left( \bigcup_{i=-K}^{6K+k_0-1} \tilde{g}^i\Gamma' \right) \geq 0.9m(\Pi). \] (4.25)

Consider the set \( \Gamma_0 \) of first entries of orbits \( \{\tilde{g}^i(z)\}_{i=0}^{5K-1} \) (with \( z \in \Gamma' \)) to the set
Λ*. More precisely, set

\[ \Gamma_0 = \{ \tilde{g}^j(z) : z \in \Gamma', \ 0 \leq j \leq 5K - 1, \ \tilde{g}^j(z) \in \Lambda^*, \ \tilde{g}^j(z) \not\in \Lambda^* \text{ for } i < j \}, \]

and let

\[ \Gamma_i = \tilde{g}^i(\Gamma_0), \quad \Gamma = \bigcup_{i=-K}^{K+k_0-1} \Gamma_i. \tag{4.26} \]

Clearly, the sets \( \{ \Gamma_i \} \) are pairwise disjoint for \(-K \leq i \leq K+k_0-1\). We then approximate \( \Gamma_0 \) by finitely many disjoint \( ab \)-cylinders \( B_{0j} \) of the form:

\[
B_{0j} = B^u(z_j, r_j') \times B^s(z_j, r_j'') \times B^{ab}(z_j, r_j) \\
= \{(u_j, s_j, a_j, b_j) : |u_j| \leq r_j', |s_j| \leq r_j'', a_j^2 + b_j^2 \leq r_j^2\}, \\
= \{(u_j, s_j, \rho_j, \varphi_j) : |u_j| \leq r_j', |s_j| \leq r_j'', \rho_j \leq r_j\},
\]

where \( r_j', r_j'', r_j > 0 \) for \( j = 1, \ldots, J \) and \( z_j = (u_j, s_j, a_j, b_j) = (u_j, s_j, \rho_j, \varphi_j) \in \Pi_0 \) is the center of \( B_{0j} \). For \( i = -K, \ldots, K + k_0 - 1 \) set

\[ B_{ij} = \tilde{g}^i(B_{0j}), \quad \Delta_i = \bigcup_{j=1}^J B_{ij}. \tag{4.27} \]

We can choose the sets \( B_{0j} \) in such a way that

1. \( B_{ij} \cap B_{kl} = \emptyset \) for \((i, j) \neq (k, l)\) with \(-K \leq i, k \leq K+k_0-1\) and \(1 \leq j, l \leq J\);
2. for each \( i = 0, 1, \ldots, k_0\)

\[ m(\Gamma_i \Delta \Delta_i) \leq 0.05 \max\{m(\Gamma_i), m(\Delta_i)\}; \tag{4.28} \]

3. \( B_{ij} \cap \text{Proj}_{\Pi_0}(\Omega_0 \cup \tilde{\Omega}_R) = \emptyset \) for \( 0 \leq i \leq k_0 - 1, 1 \leq j \leq J\).

The last property implies that the set \( B_{ij} = \tilde{g}^i(B_{0j}) = f^i(B_{0j}) \) lies in a neighborhood around \( f^i(z_j) \) and hence is still an \( ab \)-cylinder if the numbers \( r_j, r_j', r_j'' \) are chosen small enough.

We need the following lemma.
Lemma 4.4.5. Given $\delta > 0$, there is $\theta_0 = \theta_0(\delta) > 0$ such that for any $\theta \in [0, \theta_0]$ and any tube $T = C \times [0, 1/2]$, where $C \subset \Pi_0$ is an ab-cylinder of the form

$$C = B^u(z, r') \times B^a(z, r'' \times B^{ab}(z, r),$$

there exist a subtube $T' = C' \times [1/40, 19/40] \subset T$, where $C' \subset C$ is a cylinder of the form

$$C' = B^u(z, r'_0) \times B^a(z, r''_0 \times B^{ab}(z, r_0),$$

and a $C^\infty$ vector field $\mathcal{X} = \mathcal{X}_{T, \theta}$ on $\mathcal{M}$ such that

1. $\mathcal{X}$ is a rotation vector field with speed $\theta$ in ab-plane, i.e.,

$$\mathcal{X}(z) = \mathcal{X}(u, s, a, b, \tau) = \theta(-b \frac{\partial}{\partial a} + a \frac{\partial}{\partial b}), \quad z \in T';$$

2. $\mathcal{X} = 0$ outside $T$;

3. $m(C')/m(C) \geq 0.75$;

4. $r_0/r, r'_0/r', r''_0/r'' \geq 0.9$;

5. $\|\mathcal{X}\|_{C^1} < \delta$.

Moreover, let $k_0 > 0$ be such that

$$\bar{\theta} := \frac{2\pi}{k_0 \int_{-1}^{1} \psi(t)dt} < \theta_0(\delta_g/2), \quad (4.29)$$

where $\psi(t)$ is the function in Section 4.4.1 and $\delta_g$ is given by Proposition 4.4.1. Then $\|\mathcal{X}_{T, \bar{\theta}}\| \leq \delta_g/2$.

Proof of the lemma. Given $0 < \alpha < 1$, we define a subcylinder

$$C_\alpha = B^u(z, \alpha r') \times B^a(z, \alpha r'') \times B^{ab}(z, \alpha r) \subset C.$$

By (A.2), the measure of $C_\alpha$ and $C$ is induced by the flat metric $du^2 + ds^2 + da^2 + db^2$, and hence the ratio $m(C_\alpha)/m(C)$ depends only on $\alpha$ but not on the cylinder $C$. It follows that $m(C_\alpha)/m(C) \to 1$ as $\alpha \to 1$. 


Fix $\alpha > 0.9$ such that $m(C_\alpha)/m(C) > 0.75$, and set $C' = C_\alpha$. Let us choose a $C^\infty$ function $\xi : \mathbb{R} \to [0, 1]$ satisfying:

1. $\xi = 1$ on $(-\alpha, \alpha)$;
2. $\xi > 0$ on $(-1, 1)$ and $\xi = 0$ outside $(-1, 1)$;
3. $\|\xi\|_{C^1} \leq \frac{2}{1-\alpha}$.

We introduce the \textit{ab}-cylindrical coordinate $(u, s, \rho, \varphi)$, and define a $C^\infty$ rotational vector field $X = X_{T, \theta}$ by the formula

$$X_{T, \theta}(z) = \begin{cases} \theta \tilde{\xi}(z) \frac{\partial}{\partial \varphi}, & z \in T, \\ 0, & z \in \mathcal{M}\setminus T, \end{cases}$$

(4.30)

where

$$\tilde{\xi}(z) = \tilde{\xi}(u, s, \rho, \varphi, \tau) = \xi \left( \frac{u}{r'} \right) \xi \left( \frac{s}{r''} \right) \xi \left( \frac{\rho}{r} \right) \psi \left( \frac{\tau - 1/4}{1/4} \right)$$

and $\psi$ is the smooth function in Section 4.4.1. Note that $\|\tilde{\xi} \frac{\partial}{\partial z}\| \leq c$ where $c > 0$ depends only on $\alpha$ but not on the choice of the cylinder $C$. Thus for any $\delta > 0$, there is $\theta_0 = \theta_0(\delta) > 0$ such that $\|X\|_{C^1} < \delta$ for any $\theta \in [0, \theta_0]$. \hfill \square

Consider the tubes $T_{ij} = B_{ij} \times [0, 1/2]$. Applying Lemma 4.4.5 with $T = T_{ij}$, we obtain a vector field $X_{R, ij} = X_{T_{ij}, \bar{\theta}}$ such that $\|X_{R, ij}\| \leq \delta_\delta/2$, where $\bar{\theta}$ is given by (4.29). Moreover, there is a sub-cylinder $B_{ij}' \subset B_{ij}$ such that $m(B_{ij}')/m(B_{ij}) \geq 0.75$. Furthermore, by Lemma 4.4.5, we may assume that $\bar{\eta}(B_{ij}') = B_{ij}'$ for $i = 1, \ldots, k_0$. Finally, let

$$\Delta_i' = \bigcup_{j=1}^J B_{ij}', \quad \Omega_R = \bigcup_{i=0}^{k_0-1} \bigcup_{j=1}^J T_{ij},$$

(4.31)

and define the vector field $X_R$ by

$$X_R = \sum_{i=0}^{k_0-1} \sum_{j=1}^J X_{R, ij}.$$

(4.32)
We obtain a new flow \( g' \) generated by the vector field \( X_g = X_{\tilde{g}} + X_R \). Clearly, \( g' \) is a \( C^\infty \) volume preserving flow since \( X_R \) is divergence free. We shall show that the flow \( g' \) has all the desired properties stated in Proposition 4.4.1.

**Proof of Proposition 4.4.1.** Statements (1) and (2) follow immediately from the construction of the flow \( g' \) and Statement (3) can be proved in the same way as Statement (3) of Proposition 4.4.2.

We shall prove Statement (4). We need the following statement whose proof is very similar to the proof of Lemma 4.4.3.

**Lemma 4.4.6.** Given \( z \in B_{ij} \), the \( \rho_j \), \( u_j \), \( s_j \) and \( \tau \) coordinates of \( g'(z) \) and \( f'(z) \) are the same for \( t \in [0,1/2] \). Consequently, \( g_{1/2}^1(B_{ij}) = f_{1/2}^1(B_{ij}) \) and hence \( \Pi_0 \) is a global cross-section for the flow \( g^1|N \times U_0 \).

By the lemma, the time-1 map \( g^1 \) restricted to \( \Pi_0 \) is the Poincaré return map of \( g^t \) on \( \Pi_0 \). Therefore, (4.3) is equivalent to

\[
0 < L_1(G) < L_2(G) < L_3(G) > 0 = L_4(G),
\]

where \( G = g^1|\Pi_0 \). In fact, by (4.2) and (4.9) we have for \( k = 3, 4 \) that

\[
L_k(G) = L_k(f^1|\Pi_0) = L_k(\tilde{g}^1|\Pi_0) = L_k(\tilde{G}).
\]

Hence, we only need to show that

\[
L_2(G) < L_3(G). \tag{4.34}
\]

We follow the argument in Section 4.2 in [HPT10] and give a sketch of the proof of (4.34).

Set \( \Delta^*_0 = \Delta'_0 \cap \Lambda \), where \( \Delta'_0 \) and \( \Lambda \) are given by (4.31) and (4.18) respectively, and

\[
U_1 = G^{-K}\Delta^*_0, \quad U_2 = \Delta_0 \setminus \Delta^*_0, \quad U_3 = G^k_0((\Delta_0 \cap \Lambda) \setminus \Delta^*_0), \quad U_4 = G^k_0(\Delta_0 \setminus \Lambda).
\]
Consider the first return map $G = G^R$ on the set

$$U = U_1 \cup U_2 \cup U_3 \cup U_4 \subset \Pi_0,$$

where $R = R(z)$ is the first return time of $z \in U$ to $U$ under $G$. Note that the flow $g^t$ preserves the $E_f^{uab}$-subbundle, and so does $G$.

We intend to show that

$$\int_U (\log \| \wedge^3 (dG|E_f^{uab}(z)) \| - \log \| \wedge^2 (dG|E_f^{uab}(z)) \|) \, dm(z) > 0, \quad (4.35)$$

where

$$\wedge^k (dG|E_f^{uab}(z)) : \wedge^k (E_f^{uab}(z)) \to \wedge^k (E_f^{uab}(z))$$

is the $k$-th exterior power of $dG|E_f^{uab}(z)$. Indeed, assuming that (4.35) holds, consider the $G$-invariant set

$$\Pi' = \bigcup_{i=\infty}^{\infty} G^i(U) \subset \Pi_0.$$

For $k = 2, 3$ we have that

$$\int_U \log \| \wedge^k (dG|E_f^{uab}(z)) \| \, dm(z) = \int_{\Pi'} \log \| \wedge^k (dG|E_f^{uab}(z)) \| \, dm(z)$$

$$= \int_{\Pi'} \sum_{i=1}^{k} \lambda_i(z, G) \, dm(z) = L_k(G|\Pi')$$

and hence, (4.35) implies that $L_2(G|\Pi') < L_3(G|\Pi')$. Since $G = \tilde{G}$ outside $\Pi'$, we obtain that $L_2(G) < L_3(G)$.

To show (4.35) we split the integral over $U$ into four integrals $I_1$, $I_2$, $I_3$ and $I_4$ over some domains $U_1$, $U_2$, $U_3$ and $U_4$ respectively, and we obtain lower bounds for each of them. Namely, we shall show that

$$I_1 \geq 0.85K\lambda \cdot 0.7m(\Delta_0), \quad I_2 \geq k_0 \log(1 - M\delta_g) \cdot 0.25m(\Delta_0), \quad I_3 \geq 0, \quad I_4 \geq 2\log(1 - M\delta_g)m(\Pi\setminus \Lambda). \quad (4.36)$$

The lower bounds for $I_2$, $I_3$ and $I_4$ can be obtained using arguments in the proof.
of Lemma 4.2 in [HPT10]. However, the proof of the lower bound for $I_1$ in our continuous case requires substantial changes and we shall present it here. We need the following Lemma.

**Lemma 4.4.7.** Let $z \in U_1 = G^{-K}(\Delta_0^*).$ Then for any $v \in E_{f}^{uab}(z)$, we have

$$\|d_z \mathcal{G}(v)\| \geq \frac{\sqrt{2}}{2} \|v\| e^{0.9K\lambda} \tag{4.37}$$

**Proof of the lemma.** Note that for any $z \in G^{-K}(\Delta_0^*), the first return time $R(z)$ is at least $2K + k_0$. Set

$$z_1 = G^K(z), \quad z_2 = G^{k_0}(z_1) = G^{K+k_0}(z), \quad z_3 = \mathcal{G}(z) = G^{R(z)}(z).$$

Since the orbit segments $\{g^t(z)\}_{0 \leq t \leq K}$ from $z$ to $z_1$ and $\{g^t(z_2)\}_{0 \leq t \leq R(z) - K - k_0}$ from $z_2$ to $z_3$ are outside the set $\Omega_R$, we have that

$$z_1 = G^K(z) = \tilde{G}^K(z), \quad z_3 = G^{R(z) - K - k_0}(z_2) = \tilde{G}^{R(z) - K - k_0}(z_2).$$

On the other hand, since $z_1 \in \Delta_0^* = \Delta_0^* \cap \Lambda$, we can assume that $z \in B_{0j}$ for some $j$, and by our construction, we have $G^i(z_1) \in B_{ij}$ for $i = 1, \ldots, k_0 - 1$, and every cylinder $B_{ij}$ is inside a local coordinate neighborhood of its center. Therefore, we write $z_1 = (u, s, a, b, 0) \in B_{0j}$, and apply the similar arguments as in the proof of Lemma 4.4.4, we have that

$$G(z_1) = f^{\frac{1}{2}} g^{\frac{1}{2}}(z_1) = f^{\frac{1}{2}}(u, s, a \cos \phi - b \sin \phi, a \sin \phi + b \cos \phi, 1/2)$$

$$= (u, s, a \cos \phi - b \sin \phi, a \sin \phi + b \cos \phi, 1)$$

$$= (\eta u, \eta^{-1} s, a \cos \phi - b \sin \phi, a \sin \phi + b \cos \phi, 0),$$

where

$$\phi = \frac{1}{4} \hat{\theta} \int_{-1}^{1} \psi(t) dt = \frac{\pi}{2k_0}.$$ 

Repeating such calculations for $G^1(z_1), G^2(z_1), \ldots, G^{k_0-1}(z_1)$, we obtain that

$$z_2 = G^{k_0}(z_1) = (\eta^{k_0} u, \eta^{-k_0} s, a \cos(k_0 \phi) - b \sin(k_0 \phi), a \sin(k_0 \phi) + b \cos(k_0 \phi), 0)$$

$$= (\eta^{k_0} u, \eta^{-k_0} s, -b, a, 0).$$
This formula means that $d_zG^{k_0}$ is non-contracting along $E_f^{uab}$ subbundle and rotates the vector in $E_f^{ab}$ by the angle $\pi/2$.

To obtain (4.37), we write $v = v^ua + v^b \in E_f^{uua}(z) \oplus E_f^b(z)$ and consider the following two cases:

1. if $\|v^b\| \leq \frac{\sqrt{2}}{2}\|v\|$, since $d_zG^K = d_z\tilde{G}^K$ and $z \in G^{-K}\Delta'_0 \subset \tilde{G}^{-K}\Lambda'$, by (4.17) and (4.18), we find that

$$\|d_zG^K v\| = \|d_z\tilde{G}^K v\| \geq \|d_z\tilde{G}^K v^{ua}\| \geq \|v^{ua}\| e^{0.9K\lambda} \geq \frac{\sqrt{2}}{2}\|v\| e^{0.9K\lambda},$$

and hence

$$\|d_zG v\| = \|d_z\tilde{G}^R(z)^{-K-k_0}d_zG^{k_0}d_zG^K v\| \geq \|d_z\tilde{G}^K v\| \geq \frac{\sqrt{2}}{2}\|v\| e^{0.9K\lambda}.$$ 

2. if $\|v^b\| \geq \frac{\sqrt{2}}{2}\|v\|$, since $d_zG^{k_0}$ rotates the vector in $E_f^{ab}$ by the angle $\pi/2$, we have

$$d_zG^{K+k_0}v^b = d_zG^{k_0}(d_z\tilde{G}^K v^b) \in E_f^{uua}(z_2)$$

Since $z_2 \in \Lambda'$, by (4.17) we obtain

$$\|d_zG v\| \geq \|d_zG v^b\| = \|d_z\tilde{G}^R(z)^{-K-k_0}d_zG^{K+k_0}v^b\|$$

$$\geq \|d_zG^{K+k_0}v^b\| e^{0.9K\lambda} \geq \frac{\sqrt{2}}{2}\|v\| e^{0.9K\lambda}.$$

By the lemma and (4.19), we find that

$$\log \|d_zG(v)\| \geq 0.9K\lambda - 0.5\log 2 + \log \|v\| \geq 0.85K\lambda + \log \|v\|,$$

for any $z \in U_1$ and $v \in E_f^{uab}(z)$. Hence,

$$\log \|\wedge^3 (d\overline{G}|E_f^{uab}(z))\| - \log \|\wedge^2 (d\overline{G}|E_f^{uab}(z))\| \geq 0.85K\lambda.$$
On the other hand, it is proved in [HPT10] that \( m(U_1) \geq 0.7m(\Delta_0) \). Therefore,

\[
I_1 = \int_{U_1} \left( \log \| \wedge^3 (dG|E_f^{uab}(z)) \| - \log \| \wedge^2 (dG|E_f^{uab}(z)) \| \right) \, dm(z) \\
\geq 0.85K\lambda \cdot 0.7m(\Delta_0).
\]

It follows from (4.36) that

\[
\int_{U} \left( \log \| \wedge^3 (dG|E_f^{uab}(z)) \| - \log \| \wedge^2 (dG|E_f^{uab}(z)) \| \right) \\
\geq 0.595\lambda K m(\Delta_0) + 0.25k_0 \log(1 - M_0\delta_g)m(\Delta_0) \\
+ 2\log(1 - M_0\delta_g)m(\Pi \setminus \Lambda) \\
\geq 0.57\lambda K m(\Delta_0) + 2\log(1 - M_0\delta_g)m(\Pi \setminus \Lambda) \\
\geq 0.0627\lambda m(\Pi) - 0.05\lambda m(\Pi) = 0.0127\lambda m(\Pi) > 0.
\]

The last two inequalities follow from (4.19), (4.20) and Sublemma 4.4 in [HPT10] that states that \( m(\Delta_0) \geq 0.11K^{-1}m(\Pi) \). This completes the proof of Statement (4) of Proposition 4.4.1.

\[\square\]

### 4.5 Accessibility and Ergodicity

Notice that the flow \( g^t \) has positive central exponents on a set of positive volume but is not necessarily ergodic. We shall perturb \( g^t \) to a flow \( h^t \) that is pointwise partially hyperbolic on the open set \( U \) and still has positive central exponents. Furthermore, we will ensure that the flow \( h^t \) possesses two transversal strongly stable and unstable foliations \( W^s_h \) and \( W^u_h \) of \( U \) and satisfies the accessibility property on \( U \) via these two foliations. In view of Theorem 2.5.1, \( h^t \) is indeed the desired flow in our Main Theorem.

We shall follow the arguments in [HPT10] and make some necessary modifications for the flow case. We choose two sequences of open subsets \( U_n, \tilde{U}_n \subset U \), \( n = 1, 2, \ldots \) such that

(A4) \( U_0 \subset \tilde{U}_1 \);

(A5) \( \tilde{U}_n \subset \overline{U_n} \subset U_n \subset \overline{U_n} \subset U \) and \( \bigcup_{n \geq 1} U_n = U \);
(A6) $\tilde{U}_n$ and $U_n$ are connected sets for any $n \geq 1$.

Set

$$U_n = N \times U_n, \quad \tilde{U}_n = N \times \tilde{U}_n.$$ (4.38)

We will construct a sequence of flows $\{h^t_n\}_{n \geq 0}$, whose limit is the desired flow $h^t$.

The following statement is proven in Subsection 5.2.

**Proposition 4.5.1.** Given $\delta_h > 0$, one can find a sequence of positive numbers $\{\delta_n\}$ with $\delta_n \leq \min\{\delta_h/2^n, d(C, U_n)^2\}$ as well as a sequence of $C^\infty$ divergence free vector fields $X_n$ on $\mathcal{M}$, generating a sequence of volume preserving flows $h^t_n$, such that for $n \geq 0$

1. $X_0 = X_g$, and hence $h^0_t = g^t$;
2. $\|X_{n+1} - X_n\|_{C^{n+1}} \leq \delta_n$;
3. $X_n = X_f$ on $\mathcal{M} \setminus U_n$, and $X_{n-1} = X_{n-1}$ on $U_{n-2}$; in particular, each flow $h^t_n$ is a gentle perturbation of $f^t$ and hence satisfies Statements (3)-(5) of Proposition 4.3.1;
4. for every $z \in \mathcal{M}$, we have

$$E^{uabr}_{h^t_n}(z) = E^{uabr}_g(z), \quad \det(dh^t_n|E^{uabr}_{h^t_n}(z)) = \det(dg^t|E^{uabr}_g(z));$$
5. for all $z \in U_j$, $j = 1, \ldots, n$ and $\omega = u, s, c$,

$$\angle(E^{\omega}_{h^t_n}(z), E^{\omega}_{h^t_{n-1}}(z)) \leq \delta_j/2^{n-j};$$
6. if the number $\delta_g$ in Proposition 4.4.1 is sufficiently small, then each flow $h^t_n$ is stably accessible in the following sense: let a flow $h^t_n$ be a gentle perturbation of the flow $f^t$, and assume that for all $z \in U_n$ and $\omega = u, s, c$,

$$\angle(E^{\omega}_{h^t_n}(z), E^{\omega}_{h^t_n}(z)) \leq \delta_n,$$

then any two points $z_1, z_2 \in \tilde{U}_n$ are accessible via a $(u, s)h^t_n$-path in $\mathcal{U}$; in particular, $h^t_n$ has accessibility property in $\tilde{U}_n$. 
Statements (1)-(3) imply that the limit vector field $\mathcal{X}_h = \lim_{n \to \infty} \mathcal{X}_n$ exists. Moreover,

$$\|\mathcal{X}_n - \mathcal{X}_k\|_{C^{k+1}} \leq \sum_{j=k}^{n-1} \|\mathcal{X}_{j+1} - \mathcal{X}_j\|_{C^{j+1}} \leq \sum_{j=k}^{n-1} \delta_j \leq \delta_h / 2^{k-1}$$

for any $n \geq k \geq 0$. It follows that $\mathcal{X}_n$ converges to $\mathcal{X}_h$ uniformly in the $C^{k+1}$ topology. Since $k$ is arbitrary, $\mathcal{X}_h$ is a $C^\infty$ vector field. We shall show that the flow $h^t$ generated by $\mathcal{X}_h$ has all the desired properties.

### 4.5.1 Construction of the sets $U_n$ and $\tilde{U}_n$

We view the 2-torus $Y$ as the square $[0, 8] \times [0, 8]$ whose opposite sides are identified. For each $n \geq 1$, consider the partition of $Y$ into squares

$$\tilde{Z}_{ij}^{(n)} = \left[ \frac{i}{2^n}, \frac{i+1}{2^n} \right] \times \left[ \frac{j}{2^n}, \frac{j+1}{2^n} \right], \quad i, j = 0, 1, \ldots, 2^{n+3} - 1.$$

Without loss of generality we shall assume that the square $U_0$, constructed in Section 3, is contained in some $\tilde{Z}_{i_0j_0}^{(1)}$ so that

$$d(U_0, \tilde{Z}_{i_0j_0}^{(1)}) \geq 1/2^4 \quad \text{and} \quad d(C, \tilde{Z}_{i_0j_0}^{(1)}) > 2,$$

where $C$ is the Cantor set constructed in Section 3. Consider the open squares

$$Z_{ij}^{(n)} = \left( \frac{i}{2^n} - \frac{1}{2^{n+2}}, \frac{i+1}{2^n} + \frac{1}{2^{n+2}} \right) \times \left( \frac{j}{2^n} - \frac{1}{2^{n+2}}, \frac{j+1}{2^n} + \frac{1}{2^{n+2}} \right),$$

$$\tilde{Z}_{ij}^{(n)} = \left( \frac{i}{2^n} - \frac{1}{2^{n+5}}, \frac{i+1}{2^n} + \frac{1}{2^{n+5}} \right) \times \left( \frac{j}{2^n} - \frac{1}{2^{n+5}}, \frac{j+1}{2^n} + \frac{1}{2^{n+5}} \right).$$

Clearly, these squares have the same center as $\tilde{Z}_{ij}^{(n)}$ and $\tilde{Z}_{ij}^{(n)} \subset \tilde{Z}_{ij}^{(n)} \subset Z_{ij}^{(n)}$. For $n \geq 1$ consider the set

$$Y_n = \{ y \in Y : d(y, C) \geq 1/2^{n-2} \}.$$
Since \( U_0 \subset Y_1 \), we let \( Y'_n \) be the connected component of \( Y_n \) that contains \( U_0 \). Finally, consider the sets

\[
\tilde{U}_1 = \hat{Z}^{(1)}_{i_{0}j_{0}}, \quad U_1 = Z^{(1)}_{i_{0}j_{0}} \text{ and } \tilde{U}_1 = \tilde{Z}^{(1)}_{i_{0}j_{0}},
\]

and for \( n > 1 \),

\[
\tilde{U}_n = \bigcup_{Z_{ij}^{(n)} \cap Y'_n \neq \emptyset} \hat{Z}_{ij}^{(n)}, \quad U_n = \bigcup_{Z_{ij}^{(n)} \cap Y'_n \neq \emptyset} Z_{ij}^{(n)}, \quad \tilde{U}_n = \bigcup_{Z_{ij}^{(n)} \cap Y'_n \neq \emptyset} \tilde{Z}_{ij}^{(n)}.
\]

It is clear that the sets \( U_n \) and \( \tilde{U}_n \) satisfy Conditions (A4)-(A6).

Let \( \hat{Z}_n = \{ \hat{Z}_{ij}^{(n)} : \hat{Z}_{ij}^{(n)} \subset \tilde{U}_n \} \) and \( \tilde{Z}_n = \{ \tilde{Z}_{ij}^{(n)} : \tilde{Z}_{ij}^{(n)} \subset \hat{Z}_n \} \). Relabeling elements of \( \tilde{Z}_n \), we shall denote them by \( Z_{1ij}^{(n)}, \ldots, Z_{k_nij}^{(n)} \), and we shall use the notations \( \hat{Z}_l^{(n)} \) and \( \tilde{Z}_l^{(n)} \) for the corresponding squares contained in \( Z_l^{(n)} \). Thus we have

\[
U_n = U_{n-1} \cup \bigcup_{l=1}^{k_n} Z_l^{(n)}.
\]

Clearly the collection of sets \( \{ \hat{Z}_l^{(n)} : n = 1, 2, \ldots, l = 1, \ldots, k_n \} \) forms a countable partition of \( U \) up to a set of measure 0 while the collection of sets \( \{ Z_l^{(n)} : n = 1, 2, \ldots, l = 1, \ldots, k_n \} \) forms a cover of \( U \) of multiplicity at most 4. The following lemma is proved in [HPT10].

**Lemma 4.5.2.** There is a labeling of the squares \( \{ Z_l^{(n)} \} \) by integers from 1 to 8 such that for any \( y \in U \), the labels of the squares \( Z_l^{(n)} \) containing \( y \) are all different. In particular, \( Z_1^{(1)} \) can be labeled by 1.

### 4.5.2 Construction of vector fields \( \chi_n \)

The constructions are similar to Section 5.2 in [HPT10], with a slightly modification on the collection of periodic points. We need the following preparations before we construct the vector fields \( \chi_n \).

Let \( q_j, j = 1, \ldots, 8 \) be eight periodic points of the Anosov automorphism \( A \).
whose orbits are pairwise disjoint. There is $\epsilon_0 > 0$ such that

$$B_X(A^i q_j, \epsilon_0) \cap B_X(A^j q_j', \epsilon_0) = \emptyset$$

whenever $j \neq j'$ and $i = -1, 0, 1$. For each $q_j$ we choose three periodic points $p_j \in B_X(A^i q_j, \epsilon_0/3)$ for $A, i = a, b, \tau$, whose orbits are pairwise disjoint. Denote by $[q_j, p_j] = V^u_A(q_j) \cap V^s_A(p_j)$, $i = a, b, \tau$, where $V^u_A$ and $V^s_A$ are the stable and unstable local manifolds respectively. For $i = a, b, \tau$ and $j = 1, \ldots, 8$, consider the closed quadrilateral $(u, s)_A$-path $\gamma_j^i$ with the collection of points $q_j$, $[q_j, p_j]$, $p_j$, $[p_j, q_j]$, and $q_j$. Without loss of generality, we shall assume that $q_1 = q$, $p_1^i = p^i$ and $\gamma_1^i = \gamma^i$ for $i = a, b, \tau$ where $q$, $p^i$ and $\gamma^i$ are chosen as in the beginning of Section 4.4.1.

For $j = 1, \ldots, 8$ and $i = a, b, \tau$, we have

$$A^{\iota(q_j)}(q_j) = q_j, \quad A^{\iota(p_j^i)}(p_j^i) = p_j^i,$$

where $\iota(q_j)$ and $\iota(p_j^i)$ are periods of $q_j$ and $p_j^i$ respectively. There exists $\bar{\alpha}(j, i) > 0$ such that for any $\alpha \in Y = T^2$ with $\|\alpha\| \leq \bar{\alpha}(j, i)$, the Anosov affine map $A + \alpha$ has a $\iota(q_j)$-periodic point $q_j(\alpha)$ close to $q_j$ and a $\iota(p_j^i)$-periodic point $p_j^i(\alpha)$ close to $p_j^i$. Moreover, we can choose the number $\bar{\alpha}$ (in Condition (\alpha3) at the beginning of Section 3) to be less than $\min\{\bar{\alpha}(j, i) : j = 1, \ldots, 8, i = a, b, \tau\}$ such that any two points from the set of periodic points

$$\{q_j(\alpha), p_j^i(\alpha) : j = 1, \ldots, 8, i = a, b, \tau, \|\alpha\| \leq \bar{\alpha}\}$$

are disjoint.

Given $n \geq 1$ and $l = 1, \ldots, k_n$, let $j$ be the label of $Z^{(n)}_l$ in Lemma 4.5.2, and $y_0(n, l) = (a_0(n, l), b_0(n, l))$ the center of $Z^{(n)}_l$. We take the points associated to $Z^{(n)}_l$ as follows:

$$q(n, l) = q_j(\alpha(y_0(n, l))), \quad p^i(n, l) = p_j^i(\alpha(y_0(n, l))), \quad i = a, b, \tau. \quad (4.39)$$

Recall that $\eta$ is the expanding rate of $A$ along its unstable direction, and the function $\kappa : Y \to \mathbb{R}$ is given at the beginning of Section 3. For $n \geq 1$ let us choose
a square $Z^{(n)}_i \in Z_n$. For the case $n > 1$, write $q = q(n, l)$ and $p^i = p^i(n, l)$ for simplicity, and let $\eta_-(n, l) = \min\{\eta^{(n)}(y) : y \in Z^{(n)}_i\}$, then set

$$
\alpha^i_u = \alpha^i_u(n, l) = d(p^i, [p^i, q]), \quad \alpha^i_s = \alpha^i_s(n, l) = d(p^i, [q, p^i]);
$$

(4.40)

$$
\tilde{\alpha}^i_u = \tilde{\alpha}^i_u(n, l) = \alpha^i_u(n, l)/\eta_-(n, l), \quad \tilde{\alpha}^i_s = \tilde{\alpha}^i_s(n, l) = \alpha^i_s(n, l)/\eta_-(n, l).
$$

and the rectangles in $X$

$$
\Pi^i(n, l) = B_{F^a}(p^i, \alpha^i_u) \times B_{F^a}(p^i, \alpha^i_s), \quad \tilde{\Pi}^i(n, l) = B_{\bar{F}^a}(p^i, \tilde{\alpha}^i_u) \times B_{\bar{F}^a}(p^i, \tilde{\alpha}^i_s).
$$

We shall assume that the rectangles $\Pi^i(n, l)$, $n \geq 1$, $l = 1, \ldots, k_n$ and $i = a, b, r$ are pairwise disjoint if the number $\tilde{\alpha}$ is chosen sufficiently small. Finally, we let

$$
\epsilon_r = \epsilon_r(n, l) = \min\{\kappa(y)/2 : y \in Z^{(n)}_i\}, \tilde{\epsilon}_r = \tilde{\epsilon}_r(n, l) = 5\epsilon_r(n, l)/6. \quad (4.41)
$$

In the case $n = 1$, we have $Z^{(1)}_1 = U_1$ and $q(1, 1) = q_1$, $p^i(1, 1) = p^i_1$ since the function $\alpha = 0$ on $U_1$. Choose $l_u^i$ and $l_s^i$ such that

$$
A^{-l_u^i}([p^i_1, q_1]) \subseteq B_X(p^i_1, \nu/2), \quad A^{l_s^i}([q_1, p^i_1]) \subseteq B_X(p^i_1, \nu/2),
$$

where $\nu$ is given in (4.5). Then we set

$$
\alpha^i_u = \alpha^i_u(1, 1) = d(p^i_1, A^{-l_u^i}[p^i_1, q_1]), \quad \alpha^i_s = \alpha^i_s(1, 1) = d(p^i_1, A^{l_s^i}[q_1, p^i_1])
$$

with other quantities and sets to be defined in a similar way.

In addition to the squares $\hat{Z}^{(n)}_{ij}$, $\tilde{Z}^{(n)}_{ij}$ and $\hat{Z}^{(n)}_{ij}$ constructed in the previous subsection, we need to consider the following squares:

$$
\hat{Z}^{(n)}_{ij} = \left(\frac{i}{2^n} - \frac{1}{2^{n+3}}, \frac{i + 1}{2^n} + \frac{1}{2^{n+3}}\right) \times \left(\frac{j}{2^n} - \frac{1}{2^{n+3}}, \frac{j + 1}{2^n} + \frac{1}{2^{n+3}}\right);
$$

$$
\text{and} \quad \tilde{Z}^{(n)}_{ij} = \left(\frac{i}{2^n} - \frac{1}{2^{n+4}}, \frac{i + 1}{2^n} + \frac{1}{2^{n+4}}\right) \times \left(\frac{j}{2^n} - \frac{1}{2^{n+4}}, \frac{j + 1}{2^n} + \frac{1}{2^{n+4}}\right).
$$
as well as the following intervals:

\[ I_n = J_n = \left( -\frac{3}{2^{n+2}}, \frac{3}{2^{n+2}} \right), \quad \tilde{I}_n = \tilde{J}_n = \left( -\frac{5}{2^{n+3}}, \frac{5}{2^{n+3}} \right), \]

and

\[ K = (-1/4, 1 + 1/4), \quad \tilde{K} = (-1/8, 1 + 1/8), \quad \bar{K} = (-1/16, 1 + 1/16). \]

Note that we have that

\[ \hat{Z}^{(n)}_{ij} \subset \tilde{Z}^{(n)}_{ij} \subset \bar{Z}^{(n)}_{ij} \subset \bar{\tilde{Z}}^{(n)}_{ij} \subset \bar{\hat{Z}}^{(n)}_{ij} \]

and similar relations for \( I_n \) and \( J_n \).

Fix \( n \geq 1 \) and \( l = 1, \ldots, k_n \), and write \( \alpha^i_\omega = \alpha^i_\omega(n,l) \), \( \tilde{\alpha}^i_\omega = \tilde{\alpha}^i_\omega(n,l) \) for \( i = a, b, \tau, \omega = u, s \), and \( \epsilon_\tau = \epsilon_\tau(n,l), \check{\epsilon}_\tau = \check{\epsilon}_\tau(n,l) \). We choose functions as follows:

1. \( \phi^i \) and \( \psi^i \) are \( C^\infty \) functions on \( \mathbb{R} \) such that
   
   (a) \( \phi^i \) = const. on \( (-\check{\alpha}^i_u, \tilde{\alpha}^i_u) \) and \( \psi^i \) = const. on \( (-\check{\alpha}^i_s, \tilde{\alpha}^i_s) \);
   
   (b) \( \phi^i(r) = 0 \) for \( |r| \geq \alpha^i_u \), \( \psi^i(r) = 0 \) for \( |r| \geq \alpha^i_s \);
   
   (c) \( \int_{0}^{\pm \alpha^i_u} \phi^i(r) dr = 0 \), and \( \psi^i(r) > 0 \) for any \( |r| < \alpha^i_s \);
   
   (d) \( \| \phi^i \|_{C^\alpha}, \| \psi^i \|_{C^\alpha} \leq 1 \).

2. \( \xi_\tau \) and \( \xi_Y \) are \( C^\infty \) functions supported on \( K \) and \( I_n \) respectively such that
   
   (a) \( \xi_\tau \) = const. on \( \tilde{K} \), and \( \xi_Y \) = const. on \( \tilde{I}_n \);
   
   (b) \( \xi_\tau(r) > 0 \) for \( r \in K \), and \( \xi_Y(r) > 0 \) for \( r \in I_n \);
   
   (c) \( \xi_\tau(r) = 0 \) for \( r \notin K \), and \( \xi_Y(r) = 0 \) for \( r \notin I_n \);
   
   (d) \( \| \xi_\tau \|_{C^\alpha}, \| \xi_Y \|_{C^\alpha} \leq 1 \).

3. \( \zeta_\tau \) and \( \zeta_Y \) are \( C^\infty \) functions supported on \( (-\epsilon_\tau, \epsilon_\tau) \) and \( I_n \) respectively such that
   
   (a) \( \zeta_\tau \) = const. on \( (-\check{\epsilon}_\tau, \check{\epsilon}_\tau) \), and \( \zeta_Y \) = const. on \( \tilde{I}_n \);
(b) \( \zeta_r(r) > 0 \) for \( r \in (-\epsilon, \epsilon) \), and \( \zeta_Y(r) > 0 \) for \( r \in I_n \);

(c) \( \zeta_r(r) = 0 \) for \( r \not\in (-\epsilon, \epsilon) \), and \( \zeta_Y(r) = 0 \) for \( r \not\in I_n \);

(d) \( \|\zeta_r\|_{C^a}, \|\zeta_Y\|_{C^a} \leq 1 \).

Now we are ready to define the sequence of vector fields \( \mathcal{X}_n \). Given \( n \geq 1, l = 1, \ldots, k_n \) and \( i = a, b, \tau \), take the Cartesian coordinate system \( z = (u, s, \tau, a, b) = (x, t, a, b) \) with the origin at \( (p^i(n, l), 1/2, y_0(n, l)) \). In this coordinate system the interval \( K \) is in the symmetric form \((-3/4, 3/4)\). Take the boxes for \( i = a, b \)

\[
\Omega^i = \Omega_{n,l}^i = \{(x, \tau, y) : x \in \Pi^i(n, l), \ |\tau| \leq \epsilon(n, l), \ y \in Z_l^{(n)}\},
\]

and

\[
\Omega^\tau = \Omega_{n,l}^\tau = \{(x, \tau, y) : x \in \Pi^\tau(n, l), \ \tau \in K, \ y \in Z_l^{(n)}\}.
\]

By the construction of rectangles \( \Pi^i(n, l) \), we have that \( \Omega^i(n, l) \cap \Omega^i(n', l') = \emptyset \) if \((i, n, l) \neq (i', n', l')\). Similarly, we can choose \( \hat{\Omega}_i \), \( i = a, b, \tau \) by taking \( \hat{\Pi}^i, \hat{\epsilon}, \hat{K} \) and \( \hat{Z}_l^{(n)} \). Next we define three divergence free vector fields

\[
\mathcal{X}^a = \mathcal{X}_{n,l}^a = \zeta_Y(b)\zeta_r(b)\psi^a(s) \left( -\xi_Y(a) \int_0^u \phi^a(r) dr, 0, 0, \xi_Y(a)\phi^a(u), 0 \right),
\]

\[
\mathcal{X}^b = \mathcal{X}_{n,l}^b = \zeta_Y(a)\zeta_r(b)\psi^b(s) \left( -\xi_Y(b) \int_0^u \phi^b(r) dr, 0, 0, \xi_Y(b)\phi^b(u) \right),
\]

\[
\mathcal{X}^\tau = \mathcal{X}_{n,l}^\tau = \zeta_Y(a)\zeta_r(b)\psi^\tau(s) \left( -\xi_r(\tau) \int_0^u \phi^\tau(r) dr, 0, \xi_r(\tau)\phi^\tau(u), 0, 0 \right).
\]

Clearly each vector field \( \mathcal{X}_{n,l}^i \) vanishes outside the corresponding box \( \Omega_{n,l}^i \), and are constant on the smaller box \( \hat{\Omega}_{n,l}^i \). Finally, we set

\[
\mathcal{X}_n = \sum_{l=1}^{k_n} (\mathcal{X}_{n,l}^a + \mathcal{X}_{n,l}^b + \mathcal{X}_{n,l}^\tau), \quad \mathcal{X}_n = \mathcal{X}_g + \sum_{k=1}^n \beta_k \mathcal{X}_k, \quad (4.42)
\]

where the sequence of small positive numbers \( \{\beta_n\} \) is determined inductively to ensure Statements (2) and (5) of Proposition 4.5.1. Let \( h_n^i \) be the flow on \( M \) generated by the vector fields \( \mathcal{X}_n \).
4.5.3 Proof of Proposition 4.5.1

Statements (1)-(4) follow directly from our construction. It remains to show how to choose the sequence of positive numbers $\delta_n$ such that $h_n^t$ satisfies Statements (5) and (6) of the proposition. Note that these two statements only concern about those invariant subbundles $E^\omega$ and foliations $W^\omega$, $\omega = u, s, c, cs, cu$, which are the same for the flow and its time-1 map. Therefore, the choice of $\delta_n$ and related arguments are similar to the diffeomorphism case in [HPT10]. We shall outline the proof here.

For any gentle perturbation $h_z^t$ of $f^t$, we denote by $W^c_{h_z^t}(z)$ the center manifold of $h_z^t$ at the point $z \in \mathcal{M}$. Given a square $Z_i^{(n)}$ with the center $y_0(n,l)$, let $q(n,l), p^i(n,l), i = a, b, \tau$ be the associated periodic points given by (4.39), and $z_0 = z_0(n,l) = (q(n,l), 1/2, y_0(n,l))$. We denote by $W^c_{h_z^t}(z_0, K, Z_i^{(n)})$ the connected component of $W^c_{h_z^t}(z_0) \cap (X \times K \times Z_i^{(n)})$ that contains $z_0$. We will also use similar notations $W^c_{h_z^t}(z_0, \tilde{K}, \tilde{Z}_i^{(n)})$, etc.

Next we will introduce two important families of maps $\Theta$ and $\Psi$ for a gentle perturbation $h_z^t$ of $f^t$.

Fix $n \geq 1$ and $l = 1, \ldots, k_n$, we take the collection of points $q = q(n,l), p^i = p^i(n,l), i = a, b, \tau$. Consider a quadrilateral $(u, s)_{h_z^t}$-path $\tilde{\gamma}^i = \{z_1, \ldots, z_5\}$ with initial point $z_1$ defined by

\[
\begin{align*}
    z_2 &= V^n_{h_z^t}(z_1) \cap V^{cs}_{h_z^t}(p^i, 1/2, a_0, b_0), \\
    z_3 &= V^s_{h_z^t}(z_2) \cap V^{cu}_{h_z^t}(p^i, 1/2, a_0, b_0), \\
    z_4 &= V^n_{h_z^t}(z_3) \cap V^{cs}_{h_z^t}(z_1), \\
    z_5 &= V^s_{h_z^t}(z_4) \cap V^{cu}_{h_z^t}(z_1). 
\end{align*}
\]

This path defines a map $\Theta^i = \Theta^i_{n,l,h_z^t}$ given by $\Theta^i(z_1) = z_5$. It is easy to see that $z_5 \in V^n_{h_z^t}(z_1)$, and $\Theta^i$ maps $W^c_{h_z^t}(z_0, K, Z_i^{(n)})$ into itself. Reparameterizing the curve on $V^n_{h_z^t}(z_1)$ from $z_1$ to $z_2$ by $\sigma : [0, 1] \to V^n_{h_z^t}(z_1)$ so that $\sigma(0) = z_1$ and $\sigma(1) = z_2$, we obtain a parameterized family of quadrilaterals $\tilde{\gamma}^i(\vartheta) = \{z_1(\vartheta), \ldots, z_5(\vartheta)\}$, $\vartheta \in [0, 1]$, where $z_1(\vartheta) = z_1, z_2(\vartheta) = \sigma(\vartheta)$, and $z_k(\vartheta), k = 3, 4, 5$ are obtained in the way similar to (4.43). Then we define $\Theta^i_\vartheta = \Theta^i_{\vartheta,n,l,h_z^t}$ given by $\Theta^i_\vartheta(z_1) = z_5(\vartheta)$.
Clearly \( \Theta_0^i = Id, \Theta_1^i = \Theta_i, \Theta_2^i \) maps \( W_{h_z}(z_0, K, Z_l^{(n)}) \) into \( W_{h_z}(z_0) \) and depends continuously on \( \theta \in [0, 1] \).

On the other hand, given \( z = ((x, \tau), y) \in \mathcal{U} \), there is a \( (u, s) \)-path \( \gamma_f(z) \) connecting \( z \) to \( z' = ((q, \tau), y) \) whose length does not exceed \( 2d(x, q) \). This generates a map \( \Psi_f = \Psi_{f,n,l} \) from \( \mathcal{U} \) to \( \{q\} \times K \times G \) given by \( \Psi_f(z) = z' \). Furthermore, given a gentle perturbation \( h_z^i \) of \( f^i \) and a point \( z \in Z_l^{(n)} \), we can find a \( (u, s) \)-path \( \gamma_{h_z}(z) \), which is close to \( \gamma_f(z) \) and connect \( z \) to a point \( z' = z'(h_z^i) \in W_{h_z}(z_0, K, Z_l^{(n)}) \) and we can then define \( \Psi_{h_z} = \Psi_{h_z,n,l} \) by \( \Psi_{h_z}(z) = z'(h_z^i) \).

Note that the maps \( \Psi_{h_z,n,l}, \Theta_{n,l,h_z}^i \) and \( \Theta_{\theta,n,l,h_z}^i \); \( i = a, b, \tau \) depend continuously on \( h_z \) as long as \( h_z^i \) is a gentle perturbation of \( f^i \) with \( h_z^i = f^i \) outside some fixed \( \mathcal{U}_n \) and with \( \angle(E_{h_z}(z), E_f(z)) \) sufficiently small for all \( z \in \mathcal{U}_n \) and \( \omega = u, s, c \). Moreover, the continuity is uniform with respect to \( z \).

Given a set \( \Gamma \subset \mathcal{M} \) and a gentle perturbation \( h_z^i \) of \( f^i \), set

\[
\mathcal{A}_{h_z}(\Gamma) = \{ z \in \mathcal{M} : \text{there exists } y \in \Gamma \text{ such that } y \text{ is accessible to } z \text{ via a } (u, s)_{h_z^i} \text{ - path}. \} \quad (4.44)
\]

For \( n \geq 1 \) denote by \( \epsilon_n = \min\{1/2^{n+5}, \bar{\epsilon}_r(n, l), l = 1, \ldots, k_n \} \), where \( \bar{\epsilon}_r(n, l) \) is defined by (4.41).

We now briefly describe how to choose the sequence \( \{\delta_n\} \). See [HPT10] for more details. Recall that \( U_1 = Z_1^{(1)}, \tilde{U}_1 = \tilde{Z}_1^{(1)} \), and \( U_1 = N \times U_1, \tilde{U}_1 = N \times \tilde{U}_1 \). Choose \( \theta_0 > 0 \) such that the families of maps \( \Psi_{h_z} \) and \( \Theta_{h_z}^i \) are well-defined for any gentle perturbation \( h_z^i \) of \( f^i \) with \( \angle(E_{h_z}(z), E_f(z)) \leq 2\theta_0 \) for \( \omega = u, s, c \). We assume that the number \( \delta_g \) in Proposition 4.4.1 is so small that the flow \( h_0^i = g \) satisfies \( \angle(E_{h_0}(z), E_f(z)) \leq \theta_0 \) and \( d(\Theta_{\theta,1,1,h_0}(z), z) \leq \epsilon_1/4 \) for \( z \in N \times U_0, \theta \in [0, 1] \) and \( i = a, b, \tau \).

Now choose \( \theta_1^i \) with \( 0 < \theta_1^i \leq \theta_0/2 \) such that \( d(\Psi_{h_z}(z), \Psi_{h_0}(z)) \leq 1/2^8 \) if \( \angle(E_{h_z}(z), E_{h_0}(z)) \leq 2\theta_0^i \) for all \( z \in N \times Z_1^{(1)} \). Also choose \( \delta_1^i > 0 \) such that if \( \|X_1 - X_0\| \leq \delta_1^i \), then \( \angle(E_{h_1}(z), E_{h_0}(z)) \leq \theta_1^i \). Finally set \( \theta_1 = \min\{\theta_1^i, \theta_1'' \} \) and \( \delta_1 = \min\{\delta_1^i, \delta_1'', \delta_1 \} \), where \( \delta_1'' \) and \( \theta_1'' \) are given by Lemma 4.5.3 below. We can show

1. \( d(\Psi_{h_1}(z), \Psi_{h_0}(z)) \leq 1/2^8 \) for all \( z \in N \times Z_1^{(1)} \);
2. \( d(\Theta^i, z, \tilde{U}_n, \tilde{U}_{n-1}) \leq \epsilon_n/4 \) for all \( z \in W_{h^i_n}(z_0(l), K, Z^{(l)}_n) \), \( i = a, b, \tau, \vartheta \in [0, 1] \) and \( l = 1, \ldots, k_n \);

3. \( A_{h^i_n}(z_0(l)) \supset W_{h^i_n}(z_0(l), K, Z^{(l)}_1) \) for any gentle perturbation \( h^i_n \) of \( f^l \), close to \( h^i_1 \), with \( \angle(E^\omega_{h^i_n}(z), E^\omega_{h^i_1}(z)) \leq \theta'_l, \omega = u, s, c \) and \( z \in N \times Z^{(l)}_1 \).

Moreover, the above statements imply that \( A_{h^i_n}(z_0(1)) \supset N \times Z^{(1)}_1 \), in particular, \( h^i_1 \) has accessibility property on \( N \times Z^{(1)}_1 \).

Proceeding inductively, we can choose \( \delta_n \) such that Statements (5) and (6) of Proposition 4.5.1 hold. Furthermore, we have for \( i = a, b, \tau, \vartheta \in [0, 1] \) and \( l = 1, \ldots, k_{n+1} \),

1. \( d(\Psi_{h^i_n}(z), \Psi_{h^i_{n-1}}(z)) \leq 1/2^{n+7} \) for all \( z \in N \times Z^{(n)}_{l} \);

2. \( d(\Theta^i, \vartheta, n, l, h^i_{n-1}(z), z) \leq \epsilon_n/4 \) for all \( z \in W_{h^i_{n-1}}(z_0(n), l, K, Z^{(n+1)}_l) \);

3. \( A_{h^i_n}(z_0(n, l)) \supset W_{h^i_n}(z_0(n), l, K, Z^{(n)}_l) \) for any gentle perturbation \( h^i_n \) of \( f^l \), close to \( h^i_n \), with \( \angle(E^\omega_{h^i_n}(z), E^\omega_{h^i_n}(z)) \leq \delta_n, \omega = u, s, c \) and \( z \in N \times Z^{(n)}_l \).

Therefore, \( A_{h^i_n}(z_0(n, l)) \supset N \times Z^{(n)}_l \) for all \( l = 1, \ldots, k_{n+1} \). In other words, \( h^i_n \) has the accessibility property on \( N \times Z^{(n)}_l \).

Note that

\[
\tilde{U}_n = \tilde{U}_{n-1} \cup \left( \bigcup_{l=1}^{k_n} N \times Z^{(n)}_l \right),
\]

and the intersection of any two sets among \( \tilde{U}_{n-1} \) and \( N \times Z^{(n)}_l, l = 1, \ldots, k_n \) contains a nonempty open set whenever they intersect. Since \( \tilde{U}_n \) is connected, we obtain accessibility of \( h^i_n \) on \( \tilde{U}_n \). In particular, \( h^i_n \) has the accessibility property on \( \tilde{U}_n \) when we apply that \( h^i_n = h^i_n \).

### 4.5.4 A technical lemma

The proof of Proposition 4.5.1 heavily relies on the following technical statements.

**Lemma 4.5.3.** Suppose for some \( n > 0 \), \( d(\Theta^i, \vartheta, n, l, h^i_{n-1}(z), z) \leq \epsilon_n/4 \) for all \( i = a, b, \tau, \vartheta \in [0, 1] \), \( z \in W_{h^i_{n-1}}(z_0(n), l, K, Z^{(n)}_l) \), \( l = 1, \ldots, k_n \). Then there are \( \delta'_n, \theta'_n > 0 \) such that
1. If $\|X_n - X_{n-1}\|_{C^n} \leq \delta''_n$, then we have

$$d(\Theta^i_{\varphi,n+1,l,h_n}(z), z) \leq \epsilon_{n+1}/4, \text{ as } z \in W^c_{\varphi}(z_0(n + 1), K, Z_l^{(n+1)}),$$

(4.45)

for all $i = a, b, \tau$, $\varphi \in [0, 1]$, and $l = 1, \ldots, k_{n+1}$;

2. For any gentle perturbation $h^i\circ$ of $f^i$, close to $h^i_n$ and with

$$\angle(E^\omega_{h_5}(z), E^\omega_{h_n}(z)) \leq \theta''_n, \text{ for all } z \in N \times Z_l^{(n)}, \omega = u, s, c$$

we have

$$A_{h_5}(z_0(n, l)) \supset W^c_{h_5}(z_0(n, l), K, \tilde{Z}_l^{(n)}) \text{ for all } l = 1, \ldots, k_n.$$

(4.46)

In particular, (4.46) holds with $h^i\circ = h^i_n$.

This lemma is indeed a variation of Lemma 5.2 in [HPT10] to the flow case. One can prove it in a similar fashion provided we have the following sublemma, which is parallel to Sublemma 5.3 in [HPT10].

Sublemma 4.5.4. For each $n > 0$, there exists $\delta''_n > 0$ such that if $\|X_n - X_{n-1}\|_{C^n} = \beta_n\|\hat{X}_n\|_{C^n} \leq \delta''_n$, then for all $l = 1, \ldots, k_n$, we have

1. $\Theta^a((q, 1/2, a, 0)) = (q, 1/2, a', 0)$ with $a' < a$ for any $a \in I_n$;

2. $\Theta^b((q, 1/2, a, b)) = (q, 1/2, a, b')$ with $b' < b$ for any $a \in I_n$, $b \in J_n$;

3. $\Theta^\tau((q, \tau, a, b)) = (q, \tau', a, b)$ with $\tau' < \tau$ for any $a \in I_n$, $b \in J_n$ and $\tau \in K$,

where $q = q(n, l)$ is given by (4.39), and $\Theta^i = \Theta^i_{n,l,h_n}$ for $i = a, b, \tau$. The square $Z_l^{(n)}$ is parameterized as $(a, b) \in I_n \times J_n$. In particular, the center $y_0(n, l)$ of $Z_l^{(n)}$ is of coordinate $(0, 0)$.

Proof. The proof is similar to that of [DHP01] Lemma B.4 and [HPT10] Sublemma 5.3, while we will use the language of vector fields here.

We prove the first statement. Consider the coordinate system $(u, s, \tau, a, b)$ in $\Omega^a_{n,l}$ with the origin at $(p^a(n, l), 1/2, y_0(n, l))$. Write $q = q(n, l)$, $p^a = p^a(n, l)$ and
\( y_0 = y_0(n, l) \), and we may assume that the local coordinates of the points \( q, [q, p^a], [p^a, q] \) and \( p^a \) are \((u_0, s_0), (0, s_0), (u_0, 0)\) and \((0, 0)\) respectively, where \( u_0 = \alpha_n^a(n, l) \) and \( s_0 = \alpha_n^a(n, l) \) given by (4.40).

Consider the case \( n > 1 \) first. Note that the vector field \( \mathcal{X}_n \) inside \( \Omega^a(n, l) \) is exactly \( \mathcal{X}_f + \beta_n \mathcal{X}_{n,i}^a \). For any \( a_1 = a \in I_n, b \in J_n, \) and \( \tau \in (1/2 - \epsilon, 1/2 + \epsilon) \), choose the point \( z_1 = (q, \tau, a_1, b) = (u_0, s_0, \tau, a_1, b) \). Note that under the original flow \( f^t \), we have a closed quadrilateral \((u, s)_{f^t}\)-path \( \gamma = \{z_1, z_2, z_3, z_4, z_5\} \), where

\[
\begin{align*}
\bar{z}_2 &= ([q, p^a], \tau, a_2, b) = (0, s_0, \tau, a_2, b), \\
\bar{z}_3 &= (p^a, \tau, a_3, b) = (0, 0, \tau, a_3, b), \\
\bar{z}_4 &= ([p^a, q], \tau, a_4, b) = (u_0, 0, \tau, a_4, b), \\
\bar{z}_5 &= ([q, p^a], \tau, a_5, b) = (u_0, s_0, \tau, a_5, b) = \bar{z}_1,
\end{align*}
\]

and \( a_k = a_1 = a \) for \( k = 1, 2, 3, 4, 5 \).

Let us compare the vector field \( \mathcal{X}_n = \mathcal{X}_f + \beta_n \mathcal{X}_{n,i}^a \) on each leg \( L_k = [z_k, z_{k+1}] \) for \( k = 1, 2, 3, 4 \). In fact, \( \mathcal{X}_{n,i}^a \equiv 0 \) on legs \( L_1 \) and \( L_4 \). Since the \( u \)-component of every point on the leg \( L_2 \) is 0, the \( u \)-component of the vector field \( \mathcal{X}_{n,i}^a \) is 0, and the \( a \)-component does not depend on the \( u \)-coordinate. On the leg \( L_3 = [z_3, z_4] \), the \( u \)-component of \( \mathcal{X}_{n,i}^a \) is negative, zero at two endpoints \( z_3 \) and \( z_4 \), while the \( a \)-component is positive, with the value smoothly changing from a constant to zero.

Now choose the point \( z_1 = z_1 \), and we have the quadrilateral \((u, s)_{f^t}\)-path \( \gamma = \{z_1, z_2, z_3, z_4, z_5\} \). By the above comparison, the \( \tau \)- and \( b \)-coordinate are the same for each \( z_k, k = 1, 2, 3, 4, 5 \). We can choose sufficiently small \( \delta_n^a > 0 \) such that if \( ||\mathcal{X}_n - \mathcal{X}_{n-1}||_{C^a} \leq \delta_n^a \), then the variation of \( u \)-coordinate of points on the leg \([z_3, z_4]\) is so small that the image of this leg under the flow \( h^t \) is contained in \( \Omega^a \). Now let \( a_k \) be the \( a \)-coordinate of \( z_k, k = 1, 2, 3, 4, 5 \), since the \( a \)-component of \( \mathcal{X}_n \) are the same for all points on \( L_1, L_2 \) and \( L_4 \), while it changes from a constant to zero along the unstable leg \( L_3 \), then we have \( a_1 = a_2 = a_3 > a_4 = a_5 \). This shows Statement (1) for the case \( n > 1 \).

In the case \( n = 1 \) similar arguments can be used with the following modification, and we will obtain \( a_1 = a_2 \geq a_3 > a_4 = a_5 \). This completes the proof of Statement (1). Statements (2) and (3) can be proved in the similar way.
4.6 Proof of Theorem 4.1.1

Since each $X_n$ is divergence free, so is $X_n^h$, and hence $h^t$ is volume preserving. The first statement of Theorem 4.1.1 follows.

Note that $h^t = f^t$ on $U^c$ and is of the form

$$h^t((x, \tau), y) = ((x + t\alpha_0, \tau), y)$$

for each $z = ((x, \tau), y) \in U^c = \mathcal{N} \times C$, where $\alpha_0$ is a Diophantine vector. Hence, $h^t$ preserves each 3-dimensional submanifold $\mathcal{N} \times \{y\}$, $y \in C$, and $h^t|\mathcal{N} \times \{y\}$ is a non-identity linear flow with the frequency vector $\alpha_0$. Thus Statements (3) of Theorem 4.1.1 follows.

It remains to prove the second statement. By Proposition 4.4.1, each diffeomorphism $h^t_n$ is pointwise partially hyperbolic on $U$ and uniformly partially hyperbolic on $\overline{U}_n$. By Theorem B.0.1 in the Appendix B, if the sequence $\delta_n$ decreases sufficiently fast, the limit flow $h^t$ is pointwise partially hyperbolic on $U$.

We now claim that the one-dimensional strongly stable $E_s^h$ and unstable $E_u^h$ subbundles are integrable to invariant strongly stable $W^s_h$ and unstable $W^u_h$ foliations with smooth leaves, which are transversal and absolutely continuous. Recall that the “start-up” flow $f^t$ has strongly stable and unstable local manifold $V^s_f(z)$ and $V^u_f(z)$ respectively at each $z \in U$. Moreover, these local manifolds are of uniform size, say larger than a certain number $4r > 0$. By Proposition 4.5.1(3), $h_n^t|U_n^c = f^t|U_n^c$, and thus $V^\omega_{h_n}(z) = V^\omega_f(z)$ for all $z \in U \setminus U_h$, $\omega = s, u$. On the other hand, each $h_n^t$ is a perturbation of $h_{n-1}^t$ on the compact set $U_n$, on which both $h_n^t$ and $h_{n-1}^t$ are uniformly partially hyperbolic if $\delta_n$ is sufficiently small. Furthermore, let $r_n$ be the size of $V^\omega_{h_n}(z)$ for $z \in U_n$, one can have $r_n/r_{n-1} \geq 2^{-1/2^n}$, and thus by induction we would have the size of local manifolds of $h_n^t|U_n$ is bigger than $r$.

Therefore, given $z \in U$, we obtain that the size of $V^\omega_{h_n}(z)$ has a lower bound $r > 0$, which is independent of $z$ and $n$. Write each $V^s_{h_n}(z)$ in the coordinate chart given by $f^t$, i.e.,

$$V^s_{h_n}(z) = \exp_z \{(v, \psi^s_{h_n}(v)) : v \in B^s_f(0, r)\},$$

where

$$\psi^s_{h_n} : B^s_f(0, r) \to E^s_f(z)$$
is a $C^1$ map which satisfies $\psi_{s,h_n}^s(0) = 0$ and $d\psi_{s,h_n}^s(0) = 0$. The $C^1$-norm of each $\psi_{s,h_n}^s$ is small provided $\delta_g, \delta_n$ are sufficiently small. By Proposition 4.5.1(5), for any $n > k$, $\omega = s,u,c$, and any $z \in U_k \subset U$,

$$\angle(E_{h_n}^\omega(z), E_{h_k}^\omega(z)) \leq \delta_k(1 - 1/2^{n-k}) < \delta_k. \quad (4.47)$$

It follows that the sequence $\psi_{s,h_n}^s$ converges to a map $\psi_{h}^s$ in the $C^1$ topology, and hence we obtain a local manifold through $z$ for $h^t$ by

$$V_{h}^s(z) = \exp_z\{(v, \psi_{h}^s(v)) : v \in B_f^s(0, r)\}.$$  

Apparently $T_zV_{h}^s(z) = E_h^s(z)$, and hence $V_{h}^s(z)$ is a strongly stable manifold of size at least $r$. Similarly we can obtain the strongly unstable local manifold for $h^t$ in this fashion. Since the time-1 map $h^1$ is nonuniformly partially hyperbolic on $U$, by Theorem 8.6.1 in [BP07], we obtain that the strongly stable and unstable foliations are absolutely continuous.

Next we will show that the flow $h^t$ has the accessibility property on $U$ via its invariant foliations $W_{h}^s$ and $W_{h}^u$. Indeed, taking the limit as $n \to \infty$ in (4.47) we obtain that $\angle(E_{h}^\omega(z), E_{h_k}^\omega(z)) \leq \delta_k$ on $U_k$. Hence, by Proposition 4.5.1(6), the flow $h^t$ has the accessibility property on each $\tilde{U}_k$. Since $k$ is arbitrary, we obtain that the flow $h^t$ has the accessibility property on $U$.

To show that the flow $h^t$ has positive central Lyapunov exponent, we first recall that the average Lyapunov exponents of the flow $g^t$ are arranged as in (4.3). Set $c = L_3(g^t) - L_2(g^t) > 0$. By the upper semicontinuity of $L_i(\cdot)$, we choose the number $\delta_h > 0$ in Proposition 4.5.1 so small that $L_2(h^t) < L_2(g^t) + c/2$. On the other hand, it follows from Proposition 4.5.1(4) that

$$L_4(h^t_n) = \int_M \det(dh^t_n|E^{uabt}_{h_n}(z))dm = \int_M \det(dg^t|E^{uabt}_g(z))dm = L_4(g^t).$$

Taking the limit as $n \to \infty$ we obtain $L_4(h^t) = L_4(g^t) = L_3(g^t)$. Therefore,

$$L_4(h^t) - L_2(h^t) = \int_M (\lambda_3(z, h^t) + \lambda_4(z, h^t))dm(z) \geq c/2 > 0,$$

then there is a subset $A \subset U$ such that $\lambda_3(z, h^t) + \lambda_4(z, h^t) > 0$ for all $z \in
\( \mathcal{A} \), and thus \( \lambda_2(z, h^t) \geq \lambda_3(z, h^t) \geq \frac{1}{2} [\lambda_3(z, h^t) + \lambda_4(z, h^t)] > 0 \) for all \( z \in \mathcal{A} \).

Since the center subspace \( E^c_h(z) \) is 3-dimensional and the flow direction \( \text{Span}\{\mathcal{X}_h\} \) corresponds to zero exponent, we conclude that \( \lambda_2(z, h^t) \) and \( \lambda_3(z, h^t) \) corresponds to vectors in \( E^c_h(z) \setminus \text{Span}\{\mathcal{X}_h\} \). Thus the flow \( h^t \) has positive central Lyapunov exponent.

By Theorem 2.5.1, we obtain that \( h^t \) has positive central exponents at almost every point in \( \mathcal{U} \), and \( h^t|\mathcal{U} \) is an ergodic flow.
We specify the differential and metric structure of the suspension manifold $\mathcal{N}$ and the 5-dimensional manifold $\mathcal{M}$ in Chapter 3 and 4. Associated to the Anosov automorphism $A$ of $X = \mathbb{T}^2$, one can find smooth local charts $(U_x, \phi_x)$ around each $x \in X$ such that

$$\phi_x : U_x \to (-u_0(x), u_0(x)) \times (-s_0(x), s_0(x))$$

satisfies $\phi_x(x) = (0,0)$ and

$$\phi_{Ax} \circ A \circ \phi_x^{-1}(u,s) = (\eta u, \eta^{-1}s),$$

where $u_0(x), s_0(x) > 0$ are sizes of charts depending on $x$, and $\eta > 1$ is the expanding rate along the unstable direction. In fact, $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial s}$ are the unstable and stable directions of $A$ respectively.

Recall that the suspension manifold $\mathcal{N}$ is the quotient space $X \times \mathbb{R}/ \sim$ with the equivalent relation

$$(x, \tau + 1) \sim (Ax, \tau),$$

and let $\pi : X \times \mathbb{R} \to \mathcal{N}$ be the natural projection. Following [PMe82] there is a natural differential structure on $\mathcal{N}$ with atlas $(U^1_{(x,\tau)}, \phi^1_{(x,\tau)})$ for $\tau \in (-1/4, 3/4)$
and \((U^2_{(x,\tau)}, \phi^2_{(x,\tau)})\) for \(\tau \in (1/4, 5/4)\), where

\[
U^1_{(x,\tau)} = \pi(U_x \times (-1/4, 3/4)), \quad \phi^1_{(x,\tau)}(\pi(\phi^{-1}_x(u, s), \tau)) = (u, s, \tau);
\]

\[
U^2_{(x,\tau)} = \pi(U_x \times (1/4, 5/4)), \quad \phi^2_{(x,\tau)}(\pi(\phi^{-1}_x(u, s), \tau)) = (u, s, \tau).
\]

It is easy to verify that

\[
\phi^i_{(x',\tau')} \circ \phi^i_{(x,\tau)}(\pi(\phi^{-1}_x(u, s), \tau)) = (\phi^i_{x'} \circ \phi^{-1}_x(u, s), \tau), \quad i = 1, 2;
\]

\[
\phi^1_{(x',\tau')} \circ \phi^2_{(x,\tau)}(\pi(\phi^{-1}_x(u, s), \tau)) = (\phi^1_{Ax'} \circ A \circ \phi^{-1}_x(u, s), \tau - 1).
\]

In particular,

\[
\phi^1_{(x,\tau')} \circ \phi^2_{(x,\tau)}(\pi(\phi^{-1}_x(u, s), \tau)) = (\eta u, \eta^{-1} s, \tau - 1). \tag{A.1}
\]

There are three subbundles \(E^u\), \(E^s\) and \(E^\tau\) on \(\mathcal{N}\) generated by independent vector fields \(d\pi(\frac{\partial}{\partial u})\), \(d\pi(\frac{\partial}{\partial s})\) and \(d\pi(\frac{\partial}{\partial \tau})\) respectively. By [?], we can choose the Riemannian metric on \(\mathcal{N}\) which has the local representation \(\eta^2 \tau du^2 + \eta^{-2} \tau ds^2 + d\tau^2\).

Under this metric, the suspension flow \(S^t : \mathcal{N} \to \mathcal{N}\) satisfies

\[
\|dS^t v\| = \eta^t \|v\|, \quad v \in E^u;
\]

\[
\|dS^t v\| = \eta^{-t} \|v\|, \quad v \in E^s;
\]

\[
\|dS^t v\| = \|v\|, \quad v \in E^\tau.
\]

For the 2-torus \(Y = \mathbb{T}^2\), we choose the local coordinate \((a, b)\) centered at each \(y \in Y\). Given \(z = (x, \tau, y) \in \mathcal{M} = \mathcal{N} \times Y\), we can hence choose a local coordinate system \((u, s, \tau, a, b)\), endowed with the product Riemannian metric \(\eta^2 \tau du^2 + \eta^{-2} \tau ds^2 + d\tau^2 + da^2 + db^2\). In particular, the metric on the cross-section \(X \times \{0\} \times Y\) is given by the flat metric

\[
du^2 + ds^2 + da^2 + db^2 \tag{A.2}
\]
Appendix B

Small perturbations of pointwise partially hyperbolic systems

Let $\mathcal{M}$ be a compact smooth Riemannian manifold and $\mathcal{S} \subset \mathcal{M}$ an open subset. Let also $H^t$ be the flow on $\mathcal{M}$ that is pointwise partially hyperbolic on $\mathcal{S}$. Further, let $\mathcal{U}_n \subset \mathcal{S}$, $n \geq 1$ be a sequence of open subsets such that:

1. $\mathcal{U}_n \subset \mathcal{U}_{n+1} \subset \mathcal{U}_{(n+1)+1}$ and $\bigcup \mathcal{U}_n = \mathcal{S}$;
2. each $\mathcal{U}_n$ is $H^t$-invariant;
3. $H^t|_{\mathcal{U}_n}$ is uniformly partially hyperbolic.

The goal of this Appendix is to prove the following statement.

**Theorem B.0.1.** There exists a sequence of positive numbers $\varepsilon_n$ such that if smooth vector fields $\mathcal{X}_n$ on $\mathcal{M}$ satisfy

$$\mathcal{X}_0 = \mathcal{X}_H, \quad \mathcal{X}_n = \mathcal{X}_{n-1} \text{ on } \mathcal{M} \setminus \mathcal{U}_n$$

and

$$\|\mathcal{X}_n - \mathcal{X}_{n-1}\|_{C^1} \leq \varepsilon_n,$$

then for every $n \geq 1$ the corresponding flow $h^t_n$ is uniformly partially hyperbolic on the invariant set $\mathcal{U}_n$ and hence pointwise partially hyperbolic on $\mathcal{S}$. Moreover, the limit vector field $\mathcal{X} = \lim_{n \to \infty} \mathcal{X}_n$ is of class $C^1$ and generates a pointwise partially hyperbolic flow $h^t$ on $\mathcal{S}$. 
We need the following technical statements.

**Lemma B.0.2.** Given a sequence of positive numbers \( \{a_n\}_{n \geq 1} \) satisfying \( \sum_{n=1}^{\infty} a_n \leq \frac{1}{4} \), we have

\[
\prod_{n=1}^{\infty} (1 + a_n) \leq 1 + 2 \sum_{n=1}^{\infty} a_n, \quad \prod_{n=1}^{\infty} (1 - a_n) \geq 1 - 2 \sum_{n=1}^{\infty} a_n.
\]

**Lemma B.0.3** (Gronwall’s inequality). Let \( \eta(t) \) be a nonnegative \( C^1 \) function on \([0, T]\) satisfying

\[
\eta'(t) \leq \phi(t) \eta(t) + \psi(t),
\]

where \( \phi(t) \) and \( \psi(t) \) are nonnegative integrable functions, then for all \( 0 \leq t \leq T \),

\[
\eta(t) \leq e^{\int_0^t \phi(s) ds} [\eta(0) + \int_0^t \psi(s) ds].
\]

**Lemma B.0.4.** Set \( K := 2 \|\mathcal{X}_H\|_{C^1} \). If \( \varepsilon_n < \frac{K}{2n+1} \), then \( \|\mathcal{X}_n\|_{C^1} \leq K \) for all \( n \geq 0 \). Moreover, given a flow \( F^t \) with \( \|\mathcal{X}_F\|_{C^1} \leq K \), we have for any \( x \in M \) and \( t \in \mathbb{R}^+ \) that,

\[
e^{-tK} \leq m(d_x F^t) \leq \|d_x F^t\| \leq e^{tK}. \quad \text{(B.1)}
\]

In particular,

\[
e^{-K} \leq \min_{0 \leq t \leq 1} m(d_x F^t) \leq \max_{0 \leq t \leq 1} \|d_x F^t\| \leq e^K. \quad \text{(B.2)}
\]

**Proof.** Since \( \|\mathcal{X}_H\|_{C^1} = K/2 \), we find that

\[
\|\mathcal{X}_n - \mathcal{X}_H\|_{C^1} \leq \sum_{k=1}^{n} \|\mathcal{X}_k - \mathcal{X}_{k-1}\|_{C^1} \leq \sum_{k=1}^{n} \varepsilon_k < K/2.
\]

This implies that \( \|\mathcal{X}_n\|_{C^1} \leq K \).

Let \( F^t \) be a flow and \( \mathcal{X}_F \) the corresponding vector field. Consider the variational differential equation

\[
\frac{d}{dt} d_x F^t = D\mathcal{X}_F(F^t(x))d_x F^t,
\]

for any \( x \in M \) and \( t \in \mathbb{R}^+ \). Then

\[
\frac{d}{dt} \|d_x F^t\| \leq \|\frac{d}{dt} d_x F^t\| \leq \|D\mathcal{X}_F\| d_x F^t \leq K\|d_x F^t\|.
\]
Since $\|d_x F^0\| = 1$, by Lemma B.0.3, we obtain that $\|d_x F^t\| \leq e^{tK}$. Noting that $m(d_x F^t) = \|d_x F^{-t}\|^{-1}$ and the flow $F^{-t}$ corresponds to the vector field $-X_F$, we get that $m(d_x F^t) \geq e^{-tK}$.

Let $F^t$ and $G^t$ be flows on $\mathcal{M}$ and $X_F$ and $X_G$ the corresponding vector fields. Assume that $\|X_F\|_{C^1}, \|X_G\|_{C^1} \leq K$, where $K$ is given in Lemma B.0.4.

**Lemma B.0.5.** Set

$$M := e^{3K}\|X_F\|_{C^2} + e^{2K}, \quad \varepsilon_{F,G} := \|X_G - X_F\|_{C^1}.$$  

Then for $t \in [0, 1]$,

$$\rho_{C^1}(G^t, F^t) \leq 2tM\varepsilon_{F,G}, \quad (B.3)$$

where

$$\rho_{C^1}(G^t, F^t) = \max_{x \in \mathcal{M}}(\text{dist}(G^t(x), F^t(x)) + \|d_x F^t - d_x G^t\|)$$

is the distance between the flows in the $C^1$ topology.

**Proof.** Consider the family of flows $F^t(\tau)$ generated by the family of vector fields $(1 - \tau)X_F + \tau X_G$ with $\tau \in [0, 1]$. Given $x \in \mathcal{M}$ and $t \in [0, 1]$, the curve $c_t = c_t^\tau : \tau \mapsto F^t(\tau)(x)$ is of length

$$L(c_t) = \int_0^1 \left\| \frac{\partial c_t}{\partial \tau} \right\| \, d\tau = \int_0^1 \left\| \frac{\partial}{\partial \tau} F^t(\tau)(x) \right\| \, d\tau,$$

and hence,

$$\frac{d}{dt} L(c_t) \leq \int_0^1 \left\| \frac{\partial}{\partial \tau} \frac{d}{dt} F^t(\tau)(x) \right\| \, d\tau$$

$$\leq \|X_G - X_F\| + [(1 - \tau)\|D X_F\| + \tau \|D X_G\|] \int_0^1 \left\| \frac{\partial c_t}{\partial \tau} \right\| \, d\tau$$

$$\leq \varepsilon_{F,G} + K L(c_t).$$

Recall that $L(c_0) = 0$, $c_t(0) = F^t x$ and $c_t(1) = G^t x$. By Lemma B.0.3, we obtain

$$\text{dist}(F^t x, G^t x) \leq L(c_t) \leq t e^{tK} \varepsilon_{F,G} \leq tM \varepsilon_{F,G}. \quad (B.4)$$
On the other hand,

\[
\frac{d}{dt} \|d_x F^t - d_x G^t\| \leq \|D X_F(F^t x) d_x F^t - D X_G(G^t x) d_x G^t\| \\
\leq \|D X_F(F^t x) d_x F^t - D X_F(F^t x) d_x G^t\| \\
+ \|D X_F(F^t x) d_x G^t - D X_F(G^t x) d_x G^t\| \\
+ \|D X_G(G^t x) d_x G^t - D X_G(G^t x) d_x G^t\| \\
\leq \|D X_F\| \|d_x F^t - d_x G^t\| + \|D^2 X\| \text{dist}(F^t x, G^t x) \|d_x G^t\| \\
+ \|D X_F - D X_G\| \|d_x G^t\| \\
\leq K \|d_x F^t - d_x G^t\| + M e^{-K \varepsilon_{F,G}},
\]

where in the last inequality we use Lemma B.0.4 and the inequalities (B.4). By Lemma B.0.3, we obtain

\[
\|d_x F^t - d_x G^t\| \leq t M e^{(t-1)K} \varepsilon_{F,G} \leq t M \varepsilon_{F,G}. \tag{B.5}
\]

We obtain (B.3) by combining (B.4) and (B.5).

\[\square\]

Given flows \(F^t\) and \(G^t\) and invariant distributions \(E_F\) and \(E_G\) on \(S\) respectively, let

\[
\Delta_{F^t, G^t, E_F, E_G}(x) = \max \left\{ \left| \frac{m(d_x F^t|E_F(x))}{m(d_x G^t|E_G(x))} - 1 \right|, \left| \frac{m(d_x G^t|E_G(x))}{m(d_x F^t|E_F(x))} - 1 \right| \right\}, \tag{B.6}
\]

\[
\delta_{F^t, G^t} = \|G^t - F^t\|_{C^1}, \quad \theta_{E_F, E_G}(x) = \angle(E_F(x), E_G(x)).
\]

**Lemma B.0.6.** Assume that \(\|X_F\|_{C^1} \leq K\), then

\[
\Delta_{F^t, G^t, E_F, E_G}(x) \leq e^K [\delta_{F^t, G^t} + C e^K \theta_{E_F, E_G}(x)]
\]

for any \(x \in S\) and \(t \in [0, 1]\), where \(C > 0\) is a constant which depends only on the Riemannian metric of \(M\).

**Proof.** Similarly to the proof of Lemma B.0.4, we can show that if \(\|X_F\|_{C^1} \leq K\) then for any \(x \in S\) and \(t \in [0, 1]\),

\[
e^{-K} \leq m(d_x F^t) \leq \|d_x F^t\| \leq e^K.
\]
We have that
\[
\left| \frac{\|d_x G^t|E_G(x)\|}{\|d_x F^t|E_F(x)\|} - 1 \right| \leq \frac{1}{m(d_x F^t)} \left[ \|d_x G^t - d_x F^t\| + C\|d_x F^t\|\angle(E_G(x), E_F(x)) \right]
\]
for some constant \(C > 0\) depending only on the Riemannian metric of \(M\). Dividing both sides of the inequality by \(\|d_x F^t|E_F(x)\|\) and noting that \(\|d_x F^t|E_F(x)\| \geq m(d_x F^t)\), we obtain that

\[
\left| \frac{\|d_x G^t|E_G(x)\|}{\|d_x F^t|E_F(x)\|} - 1 \right| \leq e^K[\delta_{F,G} + C e^K \theta_{E,F,G}(x)].
\]

Similarly, one can show that \(\left| \frac{\|d_x G^t|E_G(x)\|}{m(d_x F^t|E_F(x))} - 1 \right|\) admits the same upper bound.

\[\blacksquare\]

**Lemma B.0.7.** A flow \(F^t\) is uniformly partially hyperbolic on a compact invariant subset \(\Lambda \subset S\) if and only if the time-1 map \(F^1|\Lambda\) is uniformly partially hyperbolic.

**Proof.** See [HPS77]. \[\blacksquare\]

**Lemma B.0.8.** Suppose that \(F^t\) is uniformly partially hyperbolic on a compact invariant subset \(\Lambda \subset S\). Pick numbers \(0 < \lambda < \bar{\lambda} \leq 1 \leq \bar{\mu} < \mu\) such that
\[
\lambda \geq \lambda(F^1, \Lambda) = \sup_{x \in \Lambda} \|d_x^s F^1\|, \quad \bar{\lambda} \leq \bar{\lambda}(F^1, \Lambda) = \inf_{x \in \Lambda} m(d_x^s F^1),
\]
\[
\bar{\mu} \geq \bar{\mu}(F^1, \Lambda) = \sup_{x \in \Lambda} \|d_x^c F^1\|, \quad \mu \leq \mu(F^1, \Lambda) = \inf_{x \in \Lambda} m(d_x^c F^1),
\]
where \(d_x^\omega F^t = d_x F^t|E^\omega_F(x), \omega = s, c, u\). Given \(\Delta > 0\), there is \(\varepsilon = \varepsilon(\Delta, \lambda, \bar{\lambda}, \bar{\mu}, \mu)\) such that if \(\|X_G - X_F\|_{C^1} < \varepsilon\) and \(X_G = X_F\) on \(S\setminus \Lambda\), then \(G^t|\Lambda\) is also a uniformly partially hyperbolic flow and
\[
\Delta_{F^t,G^t}^\omega(x) := \Delta_{F^t,G^t,E^\omega_F, E^\omega_G}(x) \leq \Delta t, \quad \omega = s, c, u, \ x \in \Lambda, \ t \in [0.5, 1]. \quad (B.7)
\]
In particular,

\[ 1 - \Delta \leq \frac{\lambda(G^1, \Lambda)}{\lambda(F^1, \Lambda)}, \quad \frac{\bar{\lambda}(G^1, \Lambda)}{\bar{\lambda}(F^1, \Lambda)}, \quad \frac{\bar{\mu}(G^1, \Lambda)}{\bar{\mu}(F^1, \Lambda)}, \quad \frac{\mu(G^1, \Lambda)}{\mu(F^1, \Lambda)} \leq 1 + \Delta. \]  

(B.8)

Proof. Consider the time-1 map \( F^1 \). By [Pes04], there is \( \varepsilon < \Delta e^{-K}/4M \) depending on \( \Delta, \lambda, \bar{\lambda}, \tilde{\mu}, \mu \) such that if \( \|X_G - X_F\|_{C^1} < \varepsilon \) and \( X_G = X_F \) on \( S \setminus \Lambda \), then \( G^1|\Lambda \) is uniformly partially hyperbolic on \( \Lambda \) with

\[ \sup_{x \in \Lambda} \angle(E^\omega_G(x), E^\omega_F(x)) < \frac{\Delta}{4Ce^{2K}}. \]  

(B.9)

By Lemma B.0.7, the flow \( G^t \) is uniformly partially hyperbolic on \( \Lambda \) with the same invariant distributions as its time-1 map \( G^1 \). Moreover, it follows from Lemma B.0.5 and B.0.6 that

\[ \Delta_{F^t, G^t, E^\omega_{F^t}, E^\omega_{G^t}}(x) \leq \frac{\Delta t}{2} + \frac{\Delta}{4} \leq \Delta t, \quad \omega = s, c, u, \quad x \in \Lambda, \quad t \in [0.5, 1]. \]

In particular,

\[ \|d^c_x G^1\| \leq \|d^c_x F^1\|(1 + \Delta) \leq \lambda(1 + \Delta), \]

and hence \( \frac{\lambda(G^1, \Lambda)}{\lambda(F^1, \Lambda)} \leq 1 + \Delta \). Other inequalities in (B.8) can be shown in a similar fashion.

We shall now specify how to choose the sequence of numbers \( \varepsilon_n \) in the theorem. First choose four sequences of numbers \( 0 < \lambda_n < \bar{\lambda}_n \leq 1 \leq \tilde{\mu}_n < \mu_n \) such that

1. \( \lambda_n \geq \lambda(H^1, \mathcal{U}_n), \quad \bar{\lambda}_n \leq \bar{\lambda}(H^1, \mathcal{U}_n), \quad \tilde{\mu}_n \geq \tilde{\mu}(H^1, \mathcal{U}_n), \quad \mu_n \leq \mu(H^1, \mathcal{U}_n); \)

2. \( \lambda_n, \tilde{\mu}_n \) are strictly increasing while \( \bar{\lambda}_n, \mu_n \) are strictly decreasing.

For all \( x \in \mathcal{S} \), let

\[ \gamma(x) = \min \left\{ \frac{\min\{1, m(d^c_x H^1)\}}{\|d^c_x H^1\|}, \quad \frac{m(d^c_x H^1)}{\max\{1, \|d^c_x H^1\|\}} \right\}, \]

and choose a strictly decreasing sequence of numbers \( \gamma_n \) such that

\[ 0 < \gamma_n \leq \inf_{x \in \mathcal{U}_n} \frac{\gamma(x) - 1}{8}. \]  

(B.10)
Now choose a sequence of positive numbers $\Delta_n$ such that
\[
\max\left\{\frac{\tilde{\lambda}_{n+1}}{\lambda_n}, \frac{\mu_{n+1}}{\mu_n}\right\} \leq 1 - \Delta_n < 1 + \Delta_n \leq \min\left\{\frac{\lambda_{n+1}}{\lambda_n}, \frac{\tilde{\mu}_{n+1}}{\mu_n}\right\}; \tag{B.11}
\]
\[
\Delta_n < \frac{1}{2^{n+2}}, \sum_{k=n}^{\infty} \Delta_k < \gamma_n. \tag{B.12}
\]
Finally, choose
\[
\varepsilon_n < \frac{1}{2} \min\{\frac{K}{2^{n+1}}, \varepsilon(\Delta_n, \lambda_n, \tilde{\lambda}_n, \tilde{\mu}_n, \mu_n)\},
\]
where $\varepsilon(\Delta, \lambda, \tilde{\lambda}, \tilde{\mu}, \mu)$ is given by Lemma B.0.8.

Proof of Theorem B.0.1. First we shall show that for every $n > 0$ the map $h_n^t$ is uniformly partially hyperbolic on $\mathcal{U}_n$. It is clearly true for $h_0$ and we shall use induction assuming that $h_k^t|\mathcal{U}_k$ for $k = 1, \ldots, n - 1$ are uniformly partially hyperbolic. By Lemma B.0.6, we obtain that
\[
1 - \Delta_k \leq \frac{\lambda(h_k^1, \mathcal{U}_k)}{\lambda(h_{k-1}^1, \mathcal{U}_k)}, \frac{\tilde{\lambda}(h_k^1, \mathcal{U}_k)}{\tilde{\lambda}(h_{k-1}^1, \mathcal{U}_k)}, \frac{\tilde{\mu}(h_k^1, \mathcal{U}_k)}{\tilde{\mu}(h_{k-1}^1, \mathcal{U}_k)}, \frac{\mu(h_k^1, \mathcal{U}_k)}{\mu(h_{k-1}^1, \mathcal{U}_k)} \leq 1 + \Delta_k.
\]
Note that
\[
\lambda(h_k^1, \mathcal{U}_{k+1}) \leq \max\{\lambda(h_{k+1}^1, \mathcal{U}_{k+1}), \lambda(h_k^1, \mathcal{U}_k)\} \\
\leq \max\{\lambda_{k+1}, \lambda(h_k^1, \mathcal{U}_k)\} \\
\leq \max\{\lambda_{k+1}, \lambda(h_{k-1}^1, \mathcal{U}_k)(1 + \Delta_k)\}.
\]
The fact that $\lambda(h_0^1, \mathcal{U}_1) \leq \lambda_1$ and the choice of $\Delta_n$ in (B.11) guarantee that
\[
\lambda_n' := \lambda(h_{n-1}^1, \mathcal{U}_n) \leq \lambda_n.
\]
Similarly, we have
\[
\tilde{\lambda}_n' := \tilde{\lambda}(h_{n-1}^1, \mathcal{U}_n) \geq \tilde{\lambda}_n, \quad \tilde{\mu}_n' := \tilde{\mu}(h_{n-1}^1, \mathcal{U}_n) \leq \tilde{\mu}_n, \\
\mu_n' := \mu(h_{n-1}^1, \mathcal{U}_n) \geq \mu_n.
It follows that
\[ \varepsilon_n \leq \varepsilon(\Delta_n, \lambda_n, \tilde{\lambda}_n, \tilde{\mu}_n, \mu_n) \leq \varepsilon(\Delta_n, \lambda'_n, \tilde{\lambda}'_n, \tilde{\mu}'_n, \mu'_n). \]
Since \( \|\mathcal{X}_n - \mathcal{X}_{n-1}\|_{C^1} \leq \varepsilon_n \), by Lemma B.0.8, we obtain that \( h^t_n|\mathcal{U}_n \) is uniformly partially hyperbolic.

Next we shall show that \( \mathcal{X} = \lim_{n \to \infty} \mathcal{X}_n \) exists and is smooth. In fact, \( \{\mathcal{X}_n\} \) is a Cauchy sequence in the \( C^1 \) topology since for any \( n, m \in \mathbb{N} \),
\[
\|\mathcal{X}_{n+m} - \mathcal{X}_n\|_{C^1} \leq \sum_{l=1}^{m} \|\mathcal{X}_{n+l} - \mathcal{X}_{n+l-1}\|_{C^1} \leq \sum_{l=1}^{m} \varepsilon_{n+l} \leq \frac{K}{2^{n+1}}.
\]
Hence \( \mathcal{X} = \lim_{n \to \infty} \mathcal{X}_n \) exists and is \( C^1 \).

It remains to show that the flow \( h^t \) generated by \( \mathcal{X} \) is pointwise partially hyperbolic on \( \mathcal{S} \). First we construct invariant distributions for \( h^t \). Given \( x \in \mathcal{S} \), we have
\[
\angle(E_h^\omega(x), E_h^\omega(x)) \leq \frac{\Delta_n}{4Ce^{2K}} < \frac{1}{2^{n+4}Ce^{2K}}, \quad \omega = s, c, u.
\]
Hence the sequence of subspaces \( E_{h_n}^\omega(x) \) is Cauchy and it converges to
\[
E_h^\omega(x) = \lim_{n \to \infty} E_{h_n}^\omega(x),
\]
which is clearly \( dh^t \)-invariant for all \( t \in \mathbb{R}^+ \).

Now we would like to estimate \( \Delta_{h^1_n,H^1_n}^\omega(x) \). Fix \( x \in \mathcal{U}_n \setminus \overline{\mathcal{U}_{n-1}} \), we have
\[
\begin{align*}
\Delta_{h^1_k,H^1_{k-1}}^\omega(x) &= \begin{cases} 
0, & k < n, \\
\Delta_k, & k \geq n.
\end{cases}
\end{align*}
\]
Note that
\[
\frac{\|d_Z h^1_k\|}{\|d_Z H^1\|} = \prod_{k=1}^{l} \frac{\|d_Z h^1_k\|}{\|d_Z h^1_{k-1}\|}, \quad \frac{m(d_Z h^1_k)}{m(d_Z H^1)} = \prod_{k=1}^{l} \frac{m(d_Z h^1_k)}{m(d_Z h^1_{k-1})},
\]

and $\sum \Delta_k < 1/4$, we obtain by Lemma B.0.2,

$$\Delta_{h^1, H^1}^\omega(x) \leq \prod_{k=1}^l (1 + \Delta_{h_k^1, h_{k-1}^1}^\omega(x)) - 1 \leq \prod_{k=n}^\infty (1 + \Delta_k) - 1 \leq 2 \sum_{k=n}^\infty \Delta_k.$$ 

Letting $l \to \infty$, we have

$$\Delta_{h^1, H^1}^\omega(x) \leq 2 \sum_{k=n}^\infty \Delta_k, \ \omega = s, c, u, \ x \in U_n \setminus \overline{U_{n-1}}.$$

Therefore,

$$\frac{\|d_x^s h^1\|}{\min\{1, m(d_x^c h^1)\}} \leq \frac{1 + 2 \sum_{k=n}^\infty \Delta_k}{1 - 2 \sum_{k=n}^\infty \Delta_k \min\{1, m(d_x^c H^1)\}} \frac{\|d_x^s H^1\|}{\min\{1, m(d_x^c H^1)\}} < 1$$

Similarly, one can show that $m(d_x^c h^1) > \max\{1, \|d_x^c h^1\|\}$. It follows that $h^1$ is pointwise partially hyperbolic on $S$, and so is the flow $h^t$ by definition.
Bibliography


Vita
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Jianyu Chen was born in Lianjiang, Fuzhou, Fujian Province, China in 1985. He is the second child in his family. Most of his childhood was spent with his aunt and uncle at Mawei Harbor - a small town located southeast of Fuzhou.

In 1997, he returned to his hometown Lianjiang to pursue his study at the local high schools. In 2003, he was admitted to Yuanpei program in Peking University, Beijing, in which he studied Mathematics and received his Bachelor degree in July 2007.

From 2007 to 2012, Jianyu was a Ph.D student in the Department of Mathematics at Penn State University. Under the influence of the Center for Dynamical Systems and Geometry, he became interested in the theory of dynamical systems shortly after he arrived at Penn State. Since 2008, he had been doing research under the supervision of Prof. Yakov Pesin in the areas of dynamical systems and fractal geometry.