The Pennsylvania State University

The Graduate School

Department of Economics

ESSAYS ON CONTESTS AND BARGAINING

A Dissertation in

Economics

by

Jun Xiao

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2012

The dissertation of Jun Xiao was reviewed and approved* by the following:

Vijay Krishna

Professor of Economics

Director of Graduate Studies, Dissertation Adviser, Chair of Committee

Kalyan Chatterjee

Distinguished Professor of Economics and Management Science

Edward Green

Professor of Economics

Adam Slawski

Assistant Professor of Economics

*Signatures are on file in the Graduate School

ABSTRACT

Chapter 1. Asymmetric All-Pay Contests with Heterogeneous Prizes.

This chapter studies complete-information, all-pay contests with asymmetric players competing for multiple heterogeneous prizes. In these contests, each player chooses a performance level or "score". The first prize is awarded to the player with the highest score, the second, less valuable prize to the player with the second--highest score, etc. Players are asymmetric in that they incur different constant costs per-unit of score. The prize sequence is assumed to be either quadratic or geometric. I show that each such contest has a unique Nash equilibrium and exhibit an algorithm to construct the equilibrium. I then apply the main result to study: (a) the issue of tracking students in schools, (b) the incentive effects of "superstars", and (c) the optimality of winner-take-all contests.

Chapter 2. Grouping Players in All-Pay Contests.

This chapter considers a situation with one designer and students with two types of abilities. The designer has a fixed budget of prize money, and he can assign the students into two classrooms and divide the prize money into prizes for each classroom. The assigned students in each classroom choose their scores to compete for the prizes in an all-pay contest as in Chapter 1. If the unit of prize is small, and if the school has to distinguish different ranks and wants to maximize the total score, it is optimal to group students with similar abilities together.

Chapter 3. Bargaining Order in a Multi-Person Bargaining Game.

This chapter studies a complete-information bargaining game with one buyer and multiple sellers of different "sizes" or bargaining strengths. The bargaining order is determined by the buyer. If the buyer can commit to a bargaining order, there is a unique subgame perfect equilibrium outcome where the buyer bargains in order of increasing size -- from the smallest to the largest. If the buyer cannot commit to a bargaining order and the sellers are sufficiently different, there is also a unique subgame perfect equilibrium outcome again with the order of increasing size.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

ACKNOWLEDGEMENTS

I am most grateful to my adviser Vijay Krishna. This dissertation has also benefited from suggestions made by Kalyan Chatterjee, Edward Green, Tymofiy Mylovanov, Neil Wallace. I highly appreciate their comments and advice.

Chapter 1

Asymmetric All-Pay Contests with Heterogeneous Prizes

1.1 Introduction

The winner of the 2011 US Open tennis tournament was awarded a prize of \$1.65M. The runner-up won \$800K whereas those in joint third position—that is, the losing semi-finalists—won \$400K each. This prize sequence was convex—the difference in the prizes for winner and the runner-up was greater than the difference in the prizes for the runner-up and the semi-finalists. In fact, at this tournament the prize for a particular rank was roughly twice the prize for the next rank.[1] In research and development competitions, the winner may win a major contract while other participants receive smaller contracts. Similar examples include the competition among students for grades; the competition among employees for different promotion opportunities, etc. The key characteristics common to these contests are: heterogeneous prizes awarded solely on the basis of relative performance; convex prize sequences; participants with possibly different abilities; and sunk costs of participants' investments.

This chapter studies complete-information all-pay contests in which participants

[1] Similarly, at the 2011 US Open golf tournament, the winner received \$1.44M, the runner-up \$865K, the four players tied for third-place received \$364K each—the average of the prizes for positions 3 to 6.

with differing abilities compete for heterogeneous prizes. The participants have different but constant marginal costs of performance and the prize sequence is either *quadratic* (the second-order difference in prizes is a positive constant) or *geometric* (the ratio of successive prizes is a constant). Each player chooses a costly performance level—or "score", and the player with the highest performance receives the highest prize, the player with the second-highest performance receives the second highest prize and so on (the prizes may be allocated randomly in the case of a tie). A player's payoff is his winnings, if any, minus his cost of performance. Costs are incurred regardless of whether he wins a prize or not.

My main result is that such contests have a *unique* equilibrium, and I provide an algorithm to compute the equilibrium. The uniqueness result relies essentially on the fact that no two participants have exactly the same costs—when two or more participants have the same cost, there may be multiple equilibria (see Example 1.1 below). Moreover, as the example illustrates, different equilibria may lead to different allocations and different total expected score/effort/performance. In many applications, the total expected score is the objective of the designer or planner and so when there are multiple equilibria, it is difficult to compare different designs. Our result demonstrates, however, that the uniqueness of equilibrium is a generic property and so, in cases in which multiplicity occurs, the result can be used to select an equilibrium as a limit of the sequence of unique equilibria of arbitrarily close contests.

The fact that the unique equilibrium can be explicitly constructed allows us to address some interesting questions concerning competitions where relative performance is the key. Here are three examples. First, consider the issue of tracking students in schools. The tracking system typically identifies the students' abilities and groups students with similar abilities together.[2] Consider a situation in which a school wants to allocate a group of students with different abilities into different classrooms in order to maximize the students' total effort. Should the school track the students, i.e., group students with similar abilities together, or, should the school not track the students, i.e., group students with different abilities together? We demonstrate, in an example below, that the answer depends on the returns to education for the lower-ranked students in

[2]Students are generally placed into academic, general, or vocational tracks (see Shaw, 2000).

each classroom. In particular, if the returns are not too small, tracking is *better* than not tracking, but if the returns are small, tracking is *worse* than not tracking.

Second, consider a situation in which the designer of the contest has some fixed total amount as prize money. Is a winner-take-all prize structure—in which the whole amount is won by the highest-ranking participant—optimal (in the sense of maximizing total performance) or should the total amount be split into two or more prizes? In an interesting paper, Moldovanu and Sela (2001) have shown that when the participants are ex ante symmetric and the costs are linear, then a winner-take-all prize structure is indeed optimal. We show below that this result does not hold in our model when participants are asymmetric. It should be noted that their model is one of incomplete information whereas the model in this chapter is one of complete information.

Third, consider a situation in which the set of contestants consists of one "superstar" of very high ability (very low cost) and a group of players of moderate ability. Brown (2011) has exhibited what is known as the "Tiger Woods" effect—the presence of a superstar in the contest causes the other players to decrease their effort levels. We show below that Brown's (2011) theoretical result relies on the assumption that the other players are symmetric.[3] Suppose we have a situation in which there is a group of asymmetric players, say, with one player who is a "star" but not a superstar. What happens if a "superstar" with very high ability replaces the weakest player? It turns out that in this case, the entry of the superstar may actually increase the effort of existing players.

Literature There is a substantial literature on all-pay contests and, closely related, all-pay auctions. Since a very nice survey of the whole field can be found in the book by Konrad (2009), in what follows, we discuss only the work that is directly related to this chapter.

Complete-information all-pay auctions can be shown to be isomorphic to all-pay contests. Complete-information all-pay auctions with a single prize were analyzed by Baye, Kovenock, and de Vries (1996). The case of multiple prizes with symmetric players was considered by Barut and Kovenock (1998). Both of these papers pro-

[3]Brown's (2011) analysis is in the context of a Tullock game.

vide conditions under which there is a unique equilibrium and also demonstrate the possibility of multiple—actually a continuum of—equilibria.

The various studies of all-pay contests with multiple prizes differ along two dimensions: the structure of the sequence of prizes, $v^1 \geq v^2 \geq ... \geq v^m$, and the players' cost functions $c_i(s)$. Clark and Riis (1998) study contests in which the prizes are homogeneous while players are asymmetric but with linear (constant marginal) costs. They show that under these conditions there is a unique equilibrium. Siegel (2010) shows uniqueness also assuming a constant prize sequence but allowing for very general, possibly nonlinear, cost functions. Bulow and Levin (2006) consider situations in which prizes are different, assuming that the prize sequence is *arithmetic*, that is, the difference in successive prizes is a constant. Costs are assumed to be linear but may differ across players. Again, uniqueness obtains. González-Díaz and Siegel (2010) extend the work of Bulow and Levin (2006) by allowing for some special kinds of nonlinear costs. None of these papers, however, consider the case of convex prize sequences, the distinguishing feature of this chapter. The table below provides an "at-a-glance" comparison of the various models along the two dimensions.

	PRIZE SEQUENCE	COSTS
Clark and Riis (1998)	Homogeneous $v^k = v^{k+1}$	Different linear
Siegel (2010)	Homogeneous $v^k = v^{k+1}$	Arbitrary
Bulow and Levin (2006)	Arithmetic $v^k - v^{k+1} = \beta$	Different linear
González-Díaz and Siegel (2010)	Arithmetic $v^k - v^{k+1} = \beta$	Nonlinear $\gamma_i c(s)$
This chapter	Quadratic $(v^k - v^{k+1}) - (v^{k+1} - v^{k+2}) = \beta$	Different linear
This chapter	Geometric $v^k = \alpha v^{k+1}$	Different linear

Table 1.1. All-Pay Contest Models

Our main result relies on an algorithm to construct a Nash equilibrium. We show that the algorithm results in only one equilibrium and there are no other equilibria. The key element of this algorithm is that the upper support (the least upper bound of the support) of a weaker player's strategy is a best response to the strategies in a contest in which only players stronger than him participate. This feature allows us to start with a set of strong players and determine the upper support of the next strongest player, and therefore determine his equilibrium payoff. Then, we can derive his strategy and move on to determine the upper support of another, still weaker player.

The equilibrium in this chapter cannot be constructed by the existing methods. This is because the equilibrium has two differences from the equilibria in the literature. First, the highest scores chosen by different players could be different. In contrast, the highest scores are the same in the contests, called simple, studied by Siegel (2010). This difference makes it hard to obtain equilibrium payoffs in the way that Siegel (2009) does.[4] Since the algorithm by Siegel (2010) starts with equilibrium payoffs, it cannot be used in my setting. Second, there could be gaps in the support of a player's equilibrium strategy. There is no gap in the contests studied by Bulow and Levin (2006), so their algorithm cannot be used here either.

The rest of this chapter is organized as follows. Section 1.1.1 gives a simple example illustrating how to construct a Nash equilibrium and why the equilibrium is unique. Section 1.2 introduces the general model. Section 1.3 discusses equilibrium properties and Section 1.4 exhibits an algorithm and shows that it constructs the unique equilibrium. Section 1.5 applies our results to study whether winner-take-all contests are optimal and the effect of "superstars". Finally, Section 1.6 concludes.

1.1.1 An Example

Let us start with a simple example. Consider a situation with three players competing for two prizes worth \$4 and \$1, respectively. Each player chooses a "score" (or performance level) $s \geq 0$. The players incur constant marginal costs of performance: $c_1 = 4$ for player 1, $c_2 = 6$ for player 2, and $c_3 = 7$ for player 3. The player with the highest

[4] Because the prizes are heterogeneous in this paper, players' *reach* (Siegel, 2009) is not well defined, so his result on equilibrium payoffs does not apply here.

score receives the first prize of \$4, the one with second-highest receives the second prize of \$1 and the one with lowest score receives \$0. If two or more players choose the same score, then the prizes are allocated among them, perhaps randomly, in a way that the expected prize accruing to each is positive. A player's payoff is the value of his prize less the cost of his performance.

In what follows, we construct a mixed strategy Nash equilibrium of this contest (it is easy to see that there cannot be a pure strategy equilibrium). It is assumed that no player's strategy assigns positive probability to any score $s > 0$ (this is a general property of equilibria and will be established later).

- Let $\bar{s} = 4/7$ be the upper support of all the players' strategies. Then their equilibrium payoffs must be $u_1 = 4 - 4 \times 4/7 = 12/7$, $u_2 = 4 - 6 \times 4/7 = 4/7$ and $u_3 = 4 - 7 \times 4/7 = 0$. This is because by choosing $\bar{s}$ a player wins the first prize for sure.

- For each $s \leq \bar{s}$, consider the following three-equation system in three variables G_1, G_2 and G_3 :

$$\begin{aligned} 4G_2G_3 + (G_2(1 - G_3) + (1 - G_2)G_3) - c_1 s &= u_1 \\ 4G_1G_3 + (G_1(1 - G_3) + (1 - G_1)G_3) - c_2 s &= u_2 \\ 4G_1G_2 + (G_1(1 - G_2) + (1 - G_1)G_2) - c_3 s &= u_3 \end{aligned}$$

  The first equation says that the mixed strategies G_2 and G_3 (cumulative distribution functions) for players 2 and 3 keep player 1 indifferent among any score $s \leq \bar{s}$, that is, his payoff from choosing any s is the same as his equilibrium payoff u_1. The other two equations are analogous.

- For $s \leq \bar{s}$, let $\hat{G}_1(s), \hat{G}_2(s), \hat{G}_3(s)$ be the solution to the system of equations above. Figure 1.1 depicts the three functions. Two facts are worth noticing. First, $\hat{G}_1(s)$ reaches zero at $\underline{s}_1 = 0.05$. Second, $\hat{G}_3$ is not monotone so cannot be a legitimate mixed strategy.[5]

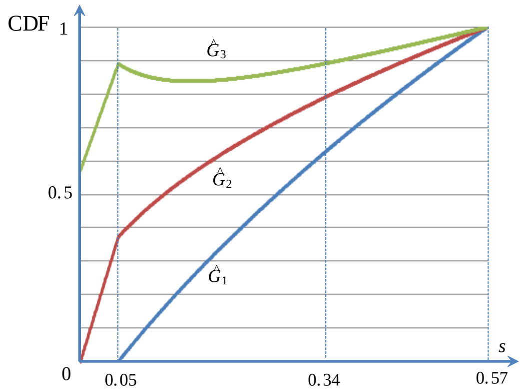

Figure 1.1. Nonmonotone Solution

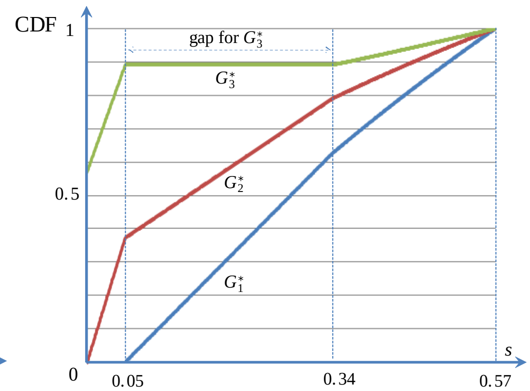

Figure 1.2. Gap

- Define G_3^* to be the smallest monotone function G that satisfies $G \geq \hat{G}_3$. As depicted in Figure 1.2, G_3^* is constant over the interval $[0.05, 0.34]$. This will be the equilibrium strategy for player 3 and if he uses G_3^*, then this means that player 3 never chooses a score in this interval, that is, there is a *gap* in the support of his mixed strategy. Thus only players 1 and 2 choose scores $s \in [0.05, 0.34]$. For $s \in [0.05, 0.34]$, let G_1^* and G_2^* be the solution to the system

$$\begin{aligned} 4G_2G_3^* + \left(G_2\left(1 - G_3^*\right) + \left(1 - G_2\right)G_3^*\right) - c_1 s &= u_1 \\ 4G_1G_3^* + \left(G_1\left(1 - G_3^*\right) + \left(1 - G_1\right)G_3^*\right) - c_2 s &= u_2 \end{aligned}$$

  Notice that this is the same as the system above except that we have fixed player 3's strategy to be G_3^*.

- For scores $s < \underline{s}_1 = 0.05$, only players 2 and 3 are active. For $s \in [0, 0.05]$, let G_2^* and G_3^* be the solution to the system

$$\begin{aligned} G_3 - c_2 s &= u_2 \\ G_2 - c_3 s &= u_3 \end{aligned}$$

[5]Throughout, by monotone, we mean non-decreasing.

- To complete the construction of the equilibrium strategies, for $s \in [0.34, 0.57]$, let $G_i^* = \hat{G}_i$.

Figure 1.2 illustrates the equilibrium strategies (G_1^*, G_2^*, G_3^*). Why does player 3 not choose a score in the gap? For any score s in the gap, let us compare $G_i(s)$ and $G_i^*(s)$ for all the players. Since $G_3(s)$ is lower than $G_3^*(s)$, this means that the strategy G_3^* for player 3 is "less aggressive" than G_3. Therefore, if player 3 switched from G_3 to G_3^* while players 1 and 2 continued to play G_1 and G_2, this would cause their payoffs to increase. As a result, to maintain their payoffs u_1 and u_2, both players 1 and 2 would have to become more aggressive. That is, for any s in the gap, $G_1(s) < G_1^*(s)$ and $G_2(s) < G_2^*(s)$. As a result, player 3's payoff from playing any s in the gap would be less than u_3 after the change from (G_1, G_2) to (G_1^*, G_2^*), hence he would not deviate to any score in the gap.[6] Section 1.3 and 1.4 show that no other deviation is profitable.

Moreover, for any Nash equilibrium (G_1^*, G_2^*, G_3^*), if we start the algorithm with $\bar{s}_1^*$, the upper support of G_1^*, the algorithm constructs a unique Nash equilibrium according to Section 1.4. Therefore, there is no other equilibrium with the same maximum score. Are there other equilibria with different maximum scores? Suppose there is and the upper support is $\bar{s}_1^* + \varepsilon$. If we start the algorithm with $\bar{s}_1^* + \varepsilon$, the strategies we construct is G_i^* shifted by ε. If $\varepsilon > 0$, then the lower support of player 2's strategy is ε. Consequently, player 3 would not choose a score between 0 and ε and so player 2 would prefer to deviate to a score above 0, which is a contradiction. If $\varepsilon < 0$, then the strategies of both players 2 and 3 would have a mass point at 0 and this too is a contradiction.

1.2 Model

Consider a complete-information, all-pay contest with n players in $\mathcal{N} = \{1, 2, ..., n\}$. There are $m \leq n$ monetary prizes in amounts $v^1 > v^2 > ... > v^m > 0$ to be awarded. The ordered set of prizes $(v^k)_{k=1}^m$ is called a *prize sequence.*

[6] If costs are linear and the prize sequence is arithmetic, then there cannot be any gaps (Bulow and Levin, 2006).

Players choose their *scores* $s_i \geq 0$ simultaneously and independently. The player with the highest score wins the highest prize, v^1; the player with the second-highest score wins the second prize, v^2; and so on. In case of a tie, prizes are awarded in a way, perhaps randomly, that all tying players have a positive expected prize.[7]

The cost of score s for player i is $c_i s$, where $0 < c_1 < ... < c_n$. Thus, players are *strictly* ranked according to "ability" and such a cost structure will be referred to as one with *distinct linear costs*. Player 1 is the strongest player in the sense that his marginal cost is lowest, player 2 is the second-strongest, etc. In general, if $i < j$, I will say that player i is *stronger* than player j and equivalently, that player j is *weaker* than player i.

Player i's payoff is $v^k - c_i s_i$ if he chooses score s_i and wins the kth prize. All players are risk-neutral.

Definition 1 (QPS) $\left(v^k\right)_{k=1}^m$ *is a* quadratic prize sequence *(QPS) if*

$$\left(v^k - v^{k+1}\right) - \left(v^{k+1} - v^{k+2}\right) = \beta$$

for $k = 1, ..., m-3$ *where* $\beta > 0$ *is a constant.*

If we normalize the lowest prize $v^m = 1$, then $v^k = (m-k+1)\left[(m-k)\beta + 2\right]/2$.

Definition 2 (GPS) $\left(v^k\right)_{k=1}^m$ *is a* geometric prize sequence *(GPS) if* $m = n$ *(so that* $v^n > 0$*) and* $v^k = \alpha v^{k+1}$ *for* $k < n$*, where* $\alpha > 1$ *is a constant.*

In a GPS, the number of prizes must be the same as the number of players. If we normalize $v^n = 1$, then $v^k = \alpha^{n-k}$.

A profile of strategies constitutes a Nash equilibrium if each player's (mixed) strategy assigns a probability of one to the set of his best responses against the strategies of other players.[8] The main result of this chapter is:

[7] In many tournaments (for example, in golf), ties are resolved by a sharing of the prizes. As an example, if players i and i' tie with the second-highest score, then each receives $(v^2 + v^3)/2$. Our formulation allows this kind of sharing.

[8] The same definition is used by Siegel (2010).

Theorem 1.1 *Every all-pay contest with a quadratic or a geometric prize sequence and distinct linear costs has a unique Nash equilibrium.*

Section 1.3 and 1.4 focus on the proof of this theorem, and the equilibrium strategies are constructed in Section 1.4.

The following example shows that the conclusion of the theorem may fail if costs are not distinct.

Example 1.1 *Suppose that there are four players ($n = 4$) competing for two prizes ($m = 2$) worth $v^1 = 3$ and $v^2 = 1$. The players' costs are:* $c_1 = 1/10$, $c_2 = 1$, $c_3 = c_4 = 6/5$.

There are at least two Nash equilibria. First, there is a "type-asymmetric" equilibrium—player 3 chooses positive scores while player 4, with the same costs as those of player 3, always chooses zero. The equilibrium strategies are:

$$
\begin{aligned}
G_1^* &= s/2 - 3/8, \quad s \in [3/4, 11/4] \\
G_2^* &= \begin{cases} 6s/5, & s \in [0, 3/4) \\ s/20 + 69/80, & s \in [3/4, 11/4] \end{cases} \\
G_3^* &= s + 1/4, \quad s \in [0, 3/4] \\
G_4^* &= 1
\end{aligned}
$$

Second, there is a "type-symmetric" equilibrium, in which the equilibrium strategies are:

$$
\begin{aligned}
G_1^{**} &= s/2 - 3/8, \quad s \in [3/4, 11/4] \\
G_2^{**} &= \begin{cases} (s + 1/4)^{-1/2}\, 6s/5, & s \in [0, 3/4) \\ s/20 + 69/80, & s \in [3/4, 11/4] \end{cases} \\
G_3^{**} &= G_4^{**} = (s + 1/4)^{1/2}, \quad s \in [0, 3/4]
\end{aligned}
$$

While the two equilibria are payoff equivalent, the allocations of prizes in the two are different—the probabilities with which players 2, 3 and 4 win the different prizes are not the same. More important, the total expected score (or performance levels) are also different. This is significant because in many applications, this may be the appropriate objective function of the planner.

If we consider a sequence of contests in which only player 4's costs are perturbed so that $c_4^t < c_3$ and $c_4^t \uparrow c_4$, then for each t, Theorem 1.1 implies that there is a unique equilibrium. The corresponding sequence of equilibria converges to the type-asymmetric equilibrium identified above.

1.3 Equilibrium Properties

In this section, I first study several properties of Nash equilibria of asymmetric contests with price sequences that satisfy either of the two conditions stated above—QPS or GPS. Equilibria typically involve mixed strategies.

We begin with the observation that a contest with QPS or GPS has at least one equilibrium. Siegel (2009) established the existence of an equilibrium when the prizes are homogeneous but his proof is readily adapted to include the kinds of prize sequences considered here. In the interests of space, I omit the minor details.[9]

The following properties of an all-pay contest are either well-known or easily derivable from known results in the literature[10]. In *any* equilibrium:

- No player chooses a score $s > 0$ with positive probability.
- Player $i > m + 1$ chooses score 0 with probability one.
- At least two players choose each s between 0 and the highest score chosen by any player.

Since the first bullet above implies that there is no pure strategy equilibrium, a Nash equilibrium (henceforth, equilibrium) consists of a set of cumulative distribu-

[9] If we replace "the probability of winning" by "probability of winning one prize", the proofs for Tie Lemma and Zero Lemma of Siegel (2009) are still true here. If we replace i's reach by the v^1/c_i, the proof of Corollary 1 of Siegel (2009) is also true here.

[10] See Bulow and Levin (2006) and Siegel (2009, 2010).

tion functions $(G_i^*)_{i=1}^n$, where G_i^* represents i's mixed strategy. Let $(g_i^*)_{i=1}^n$ denote the corresponding densities, provided that they exist.

Let $\mathcal{P}(s)$ denote the set of players who, in equilibrium, choose scores both above and below s, that is,

$$\mathcal{P}(s) = \{i \mid G_i^*(s) \in (0,1)\}$$

Let $\mathcal{A}(s)$ denote the set of players that have positive densities around s, that is,

$$\mathcal{A}(s) = \{i \mid \text{there exist } s_l \to s \text{ such that for all } l,\ g_i^*(s_l) > 0\}$$

We refer to $\mathcal{A}(s)$ as the set of *active* players at s, and to $\mathcal{P}(s)$ as the set of *participating* players at s. Note that $\mathcal{A}(s) \subseteq \mathcal{P}(s)$ but if there is a gap containing s in the support of G_i^*, then i is in $\mathcal{P}(s)$, but not in $\mathcal{A}(s)$. The properties in the following lemmas are specific to the contests with QPS or GPS.

Lemma 1.1 (Stochastic Dominance) *For any player $i < n$, $G_i^*(s) \leq G_{i+1}^*(s)$; if $i, i+1 \in \mathcal{P}(s)$, then $G_i^*(s) < G_{i+1}^*(s)$.*

The following lemma establishes that at any point s that is in the interiors of the equilibrium supports of two players, the densities associated with the equilibrium strategies can also be ordered. Of course, this implies that the supports of their mixed strategies must differ.

Lemma 1.2 (Ordered Densities) *If players $i, i+1 \in \mathcal{A}(s)$, and s is an interior point of the supports of both G_i^* and G_{i+1}^*, $g_i^*(s) > g_{i+1}^*(s)$.*

1.4 Algorithm

We first introduce an algorithm that constructs a unique set of strategies. Second, I show that this set of strategies is actually the unique equilibrium.

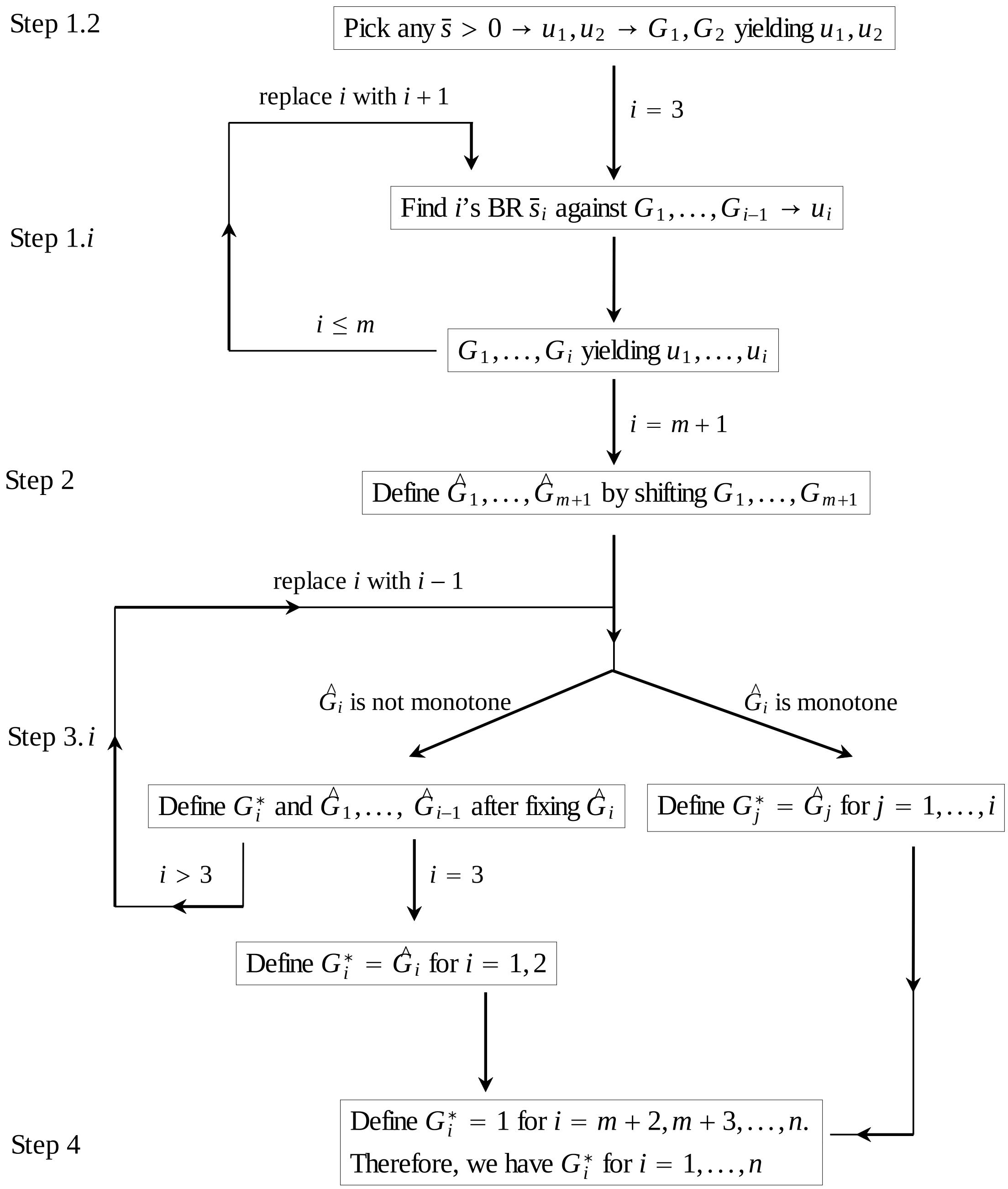

Figure 1.3. Algorithm

The algorithm is schematically represented in Figure 1.3, and explained after that. The equilibrium construction in the example of Section 1.1.1 is a special case of this algorithm. For the general case, there is a complication if the upper supports of equilibrium strategies differ, and Step 1 of the algorithm deals with this complication.

Step 1.1 To start the algorithm, pick any $\bar{s}_2 > 0$.

Step 1.2 Suppose only players 1 and 2 compete for v^1 and v^2. Suppose that both choose scores only from $[0, \bar{s}_2]$. In that case, their payoffs must be $u_i = v^1 - c_i\bar{s}_2$. This is because by choosing $\bar{s}_2$, a player wins the first prize for sure. For $s \leq \bar{s}_2$, there exist unique $G_1(s) \leq 1$ and $G_2(s) \leq 1$ that solve the system[11]:

$$\begin{aligned} G_2 v^1 + (1 - G_2) v^2 - c_1 s &= u_1 \\ G_1 v^1 + (1 - G_1) v^2 - c_2 s &= u_2 \end{aligned}$$

Extend the solution for $s < \bar{s}_2$ until $s = \underline{s}_1 > 0$ such that $G_1(\underline{s}_1) = 0$. Since $c_2 > c_1$, $G_2(\underline{s}_1) > 0$. The functions G_1, G_2 are now well-defined for $s \in [\underline{s}_1, \bar{s}_2]$. We call G_1, G_2 the *pseudo* strategies yielding u_1 and u_2.

Repeat the following step for $i = 3, ..., m+1$.

Step 1.i Suppose players $1, 2, ..., i-1$ use the strategies $G_1, G_2, ..., G_{i-1}$ determined in Step 1.$(i-1)$. Let $\bar{s}_i$ be the infimum of all scores $s \in [\underline{s}_{i-2}, \bar{s}_{i-1}]$ that maximize player i's payoff against $G_1, G_2..., G_{i-1}$. If player i chooses scores from only $[0, \bar{s}_i]$, then his payoff must be u_i.

For $s \geq \bar{s}_i$, $G_1, G_2, ..., G_{i-1}$ remain the same because i does not choose above $\bar{s}_i$. However, for $s \leq \bar{s}_i$, the strategies of players $1, 2, ..., i-1$ are different from those determined in Step 1.$(i-1)$. For $s \leq \bar{s}_i$, there exist unique $G_1(s), G_2(s), ..., G_i(s) \leq 1$ that solve the system of i equations: for $j \leq i$

$$W(\mathbf{G}_{-j}, \mathbf{v}) - c_j s = u_j \tag{1.1}$$

[11]The verification of this and other claims can be found in Appendix A.3.

where $\mathbf{v} = (v^1, v^2, ..., v^i)$ and $W(\mathbf{G}_{-j}, \mathbf{v})$ represents the expected winnings of player j when the others use strategies $\mathbf{G}_{-j} = (G_1, ..., G_{j-1}, G_{j+1}, ..., G_i)$. Extend the solution for $s < \bar{s}_i$ until $s = \underline{s}_1 > 0$ such that $G_1(\underline{s}_1) = 0$. Since $c_{j+1} > c_j$, $G_{j+1}(\underline{s}_1) > G_j(\underline{s}_1) > 0$. Since $G_1(\underline{s}_1) = 0$, player 1 does not choose below $\underline{s}_1$.

For $s \leq \underline{s}_1$, let $G_1(s) = 0$. For $s < \underline{s}_1$, substitute $G_1(s) = 0$ into (1.1) for $j > 1$. Again, find the unique solution $G_2(s), ..., G_i(s) \leq 1$ for the resulting system. Extend this solution until $s = \underline{s}_2 > 0$ such that $G_2(\underline{s}_2) = 0$.

For $s \leq \underline{s}_2$, let $G_1(s) = G_2(s) = 0$. Similarly, for $s < \underline{s}_2$, substitute $G_1(s)$ and $G_2(s)$ into (1.1) for $j > 2$. Again, find the unique solution $G_3(s), ..., G_i(s) \leq 1$ for the resulting system. Extend this solution until $s = \underline{s}_3 > 0$ such that $G_3(\underline{s}_3) = 0$ and continue in this manner.

Consequently, we have constructed functions $G_1, ..., G_i$ for $s \in [\underline{s}_{i-1}, \bar{s}]$, where $G_{i-1}(\underline{s}_{i-1}) = 0$. We call $G_1, ..., G_i$ as the *pseudo strategies* yielding $u_1, ..., u_i$.

Step 2 Define $\hat{G}_i$ for $i = 1, ..., m+1$ as $\hat{G}_i(s) = G_i(s - \underline{s}_m)$. Note first that, $\hat{G}_i$ is continuous and lies in $[0, 1]$ for $i = 1, 2, ..., m+1$, and $\hat{G}_m(0) = 0$. Second, the payoffs associated with the pseudo strategies are $u_i^* = u_i + c_i \underline{s}_m$ for $i = 1, ..., m+1$ and $u_{m+1}^* = 0$. Third, $\hat{G}_i$ may be decreasing at some scores and so it may not be a legitimate mixed strategy.

We say that a continuous function $G(s)$ has a *dent* over (s', s'') if i) $G(s') = G(s'')$; ii) $G(s) \leq G(s')$ for $s \in (s', s'')$. Figure 1.4 illustrates a dent (s', s'') for function G.

Next, we fix the non-monotonicity of the pseudo strategies $\hat{G}_1, ..., \hat{G}_{m+1}$ by replacing these with monotone functions in a way that yields the *same* payoffs. Repeat the following steps for $i = m+1, m, ..., 3$.

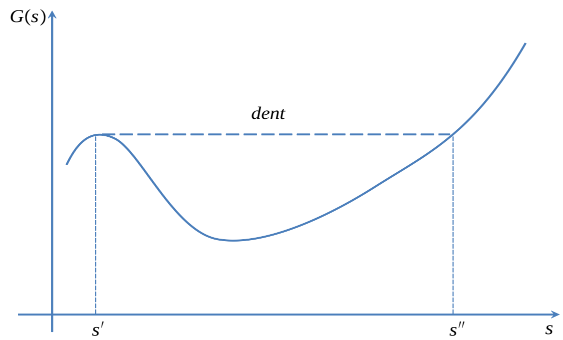

Figure 1.4. Dent

Step 3.i If $\hat{G}_i$ is monotone, let $G_j^* = \hat{G}_j$ for all $j \leq i$ and move to Step 4. Otherwise, let G_i^* be the smallest monotone function that lies on or above $\hat{G}_i$.

Find all the dents of $\hat{G}_i$, and it can be verified that $\hat{G}_i$ has a finite number of dents. Pick any dent of $\hat{G}_i$, denote it as (s', s''). For any $s \in (s', s'')$, let $G_i^*(s) = \hat{G}_i(s')$ and substitute it into the system: for $j \in \mathcal{P}(s) \setminus \{i\}$,

$$W(\mathbf{G}_{-j}, \mathbf{v}) - c_j s = u_j^* \tag{1.2}$$

where W represents j's expected winnings in this contest, and $\mathbf{G} = (G_{i'})_{i' \in \mathcal{P}(s)}$, $\mathbf{v} = (v^k)_{k \in \mathcal{P}(s)}$ and $\mathcal{P}(s)$ is the participating players at s such that $\hat{G}_i(s) \in (0,1)$. There exists a unique $G_j(s) \in [0,1]$ for $j \in \mathcal{P}(s) \setminus \{i\}$ that solves the system above. Therefore, G_j is defined over all the dents of $\hat{G}_i$. For $j = 1, ..., i-1$, *re-define* $\hat{G}_j(s) = G_j(s)$ over all the the dents of $\hat{G}_i$, and let $\hat{G}_j(s)$ remain the same if s is not contained in any dent of $\hat{G}_i$. We call $\hat{G}_1, ..., \hat{G}_{i-1}$ defined in this step as the pseudo strategies *after* fixing $\hat{G}_i$'s non-monotonicity. Note that $\hat{G}_1, ..., \hat{G}_{i-1}$ after fixing $\hat{G}_i$ are *different* from those after fixing $\hat{G}_{i+1}$.

Step 3.3 It can be verified that $\hat{G}_1, \hat{G}_2$ after fixing $\hat{G}_3$ are both monotone. Define $G_i^* = \hat{G}_i$ for $i \leq 2$ and move to Step 4.

Step 4 So far, we have defined G_i^* for $i = 1, ..., m$. Let $G_i^*(s) = 1$ for $i = m+1, ..., n$ and for all s. Now all G_i^* for $i = 1, ..., n$ over $[0, \bar{s} - \underline{s}_m]$ have been defined. The algorithm ends.

Next, let us introduce some properties of the algorithm.

Lemma 1.3 (Finiteness) *The algorithm ends in a finite number of steps.*

If the algorithm starts with a different value $\bar{s}' \neq \bar{s}$, it can be verified that the corresponding pseudo strategies constructed in Step 1 are the same functions with a horizontal shift. Therefore, after the shift in Step 2, $\hat{G}_1, ..., \hat{G}_{m+1}$ are the same as in the case starting with $\bar{s}$, which leads to the following lemma.

Lemma 1.4 (Determinateness) *The algorithm uniquely determines $(G_i^*)_{i\in\mathcal{N}}$, and $(G_i^*)_{i\in\mathcal{N}}$ is independent of the initial value $\bar{s}$.*

The lemma above implies that $u_1^*, ..., u_{m+1}^*$ and the upper support of G_1^* are uniquely determined. If the algorithm starts with the upper support of G_1^*, there would be no shift in Step 2 and the pseudo strategies constructed in Step 1.i would yield the payoffs $u_1^*, ..., u_i^*$. Let $\underline{s}_j$ and $\bar{s}_j^*$ be the lower and upper supports of j's pseudo strategy. The following lemma implies that the algorithm finds the upper supports of the equilibrium strategies.

Lemma 1.5 (Upper Support) *The upper support of $i+1$'s equilibrium strategy is the infimum of $i+1$'s best responses in $\left[\underline{s}_{i-1}, \bar{s}_i^*\right]$ against the pseudo strategies yielding $u_1^*, ..., u_i^*$.*

Let us explain the idea used to prove this lemma. Consider a contest with only three players. For $s < \bar{s}_3^*$, let us compare equilibrium strategies G_1^*, G_2^* with the pseudo strategies $\hat{G}_1$ and $\hat{G}_2$ yielding u_1^* and u_2^*. Since 3 is absent at s, if the pseudo strategies are the same as G_1^*, G_2^*, 1 and 2 would have higher payoffs than u_1^* and u_2^*. Therefore, $\hat{G}_1$ and $\hat{G}_2$ have more competition than G_1^*, G_2^* do. As a result, player 3's payoff at s against G_1^*, G_2^* is higher than his payoff against $\hat{G}_1$ and $\hat{G}_2$. Therefore, 3's payoff at

$s < \bar{s}_3^*$ is lower than u_3^* when he is facing $\hat{G}_1$ and $\hat{G}_2$. Notice that 3's payoff at $\bar{s}_3^*$ is u_3^* when he is facing $\hat{G}_1$ and $\hat{G}_2$, so $s < \bar{s}_3^*$ is never a best response against $\hat{G}_1$ and $\hat{G}_2$. Hence $\bar{s}_3^*$ is the infimum of 3's best responses in $[\underline{s}_1, \bar{s}_2^*]$ against $\hat{G}_1$ and $\hat{G}_2$.

The following lemma shows that the algorithm finds the gaps in the supports of the equilibrium strategies.

Lemma 1.6 (Gap vs. Dent) *The following two statements are equivalent:*

i) There is a gap (s_i', s_i'') in the support of i's equilibrium strategy.

ii) $\hat{G}_i$ has a dent over (s_i', s_i''), where $\hat{G}_i$ is player i's pseudo strategy after fixing $\hat{G}_{i+1}$'s non-monotonicity.

Let us briefly explain the idea used to prove this lemma. Consider a simple case when the lemma above is violated. In this case, equilibrium strategy G_i^* has a gap (s_i', s_i'') and $\hat{G}_i$ is higher than G_i^* at a score s in this gap, moreover, no other players have a gap containing (s_i', s_i''). Similar to the idea for Lemma 1.5, pseudo strategies $(\hat{G}_l)_{l\in\mathcal{A}(s)}$ give player i a higher payoff than equilibrium strategies $(G_l^*)_{\mathcal{A}(s)\setminus\{i\}}$ do, which is a contradiction because they should also give i the same payoff. Figure 1.5 illustrates that a dent of $\hat{G}_i$ coincides with a gap of G_i^*.

Lemma 1.7 (Nested Gaps) *Suppose i, j both choose above and below s and $i < j$ in an equilibrium. If the support of i's equilibrium strategy has a gap (s_i', s_i'') containing s, the support of j's equilibrium strategy also has a gap (s_j', s_j''), and $s_j' < s_i'$ and $s_j'' > s_i''$.*

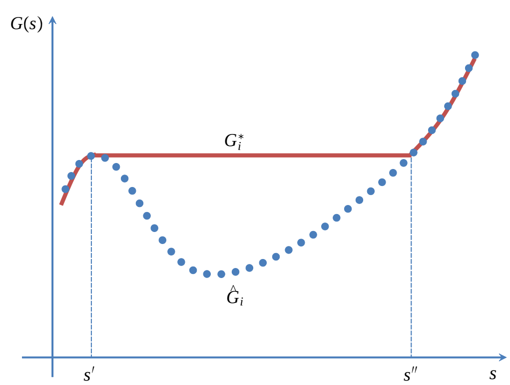

Figure 1.5. Gap vs Dent

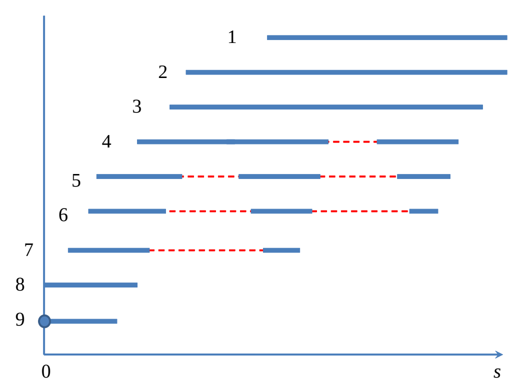

Figure 1.6. Nested Gaps

Figure 1.6 illustrates the supports of equilibrium strategies required by the lemma above.

Using Lemmas 1.1 to 1.7, we can show that the algorithm constructs the unique Nash equilibrium for every all-pay contest with a quadratic or a geometric prize sequence and distinct linear costs. Therefore, Theorem 1.1 is established.

Corollary 1.1 *If $c_i - c_j$ converges to 0 for players $i, j < m+2$, $u_i^* - u_j^*$ also converges to zero and $G_i^* - G_j^*$ pointwise converges to zero.*

As illustrated in Example 1.1, there could be multiple equilibria if costs are not distinct. This corollary allows us to select an equilibrium as a limit of the sequence of unique equilibria of nearby contests with distinct costs. Moreover, the selected equilibrium has $m+1$ players who choose scores above 0, and, among these players, the players with the same cost use the same strategy.

1.5 Applications

1.5.1 Tracking in Schools

Student tracking systems in schools have been frequently questioned (see Lockwood and Cleveland, 1998). These systems typically identify the students' abilities and group students with similar abilities together. Assuming that a school's objective is to maximize the students' total effort/performance, should the school track the students, i.e., group students with similar abilities together, or, should the school not track the students, i.e., group students with different abilities together?

The following example demonstrates that the answer depends on the returns to education for the lower-ranked students in each classroom. In particular, if the returns are not too small, tracking is *better* than not tracking, but if the returns are small enough, tracking is *worse* than not tracking. In all of the examples that follow, I use Corollary 1.1 to select a unique equilibrium in cases where there are possibly many equilibria (the multiplicity arises because of ties in the costs of different players).

Example 1.2 *Consider a school with two classrooms with four seats in each classroom. Suppose there are four H-type students of high ability and four L-type students of low ability. The H-type students have a marginal cost of $c_H = 1$ and the L-type students have a marginal cost of $c_L = 2$. In each classroom, the four students choose effort levels (scores) to compete in an all-pay contest with two prizes: v^1 and $v^2 = 4 - v^1$, where $v^1 \in (8/3, 4)$.*[12]

We compare two scenarios. In scenario one, students are tracked so that four H-type students are assigned to one classroom and four L-type students are assigned to the other classroom. In the classroom with H-type students, each student gets an equilibrium payoff of 0, which implies that the total expected cost equals the total value of prizes, $v^1 + v^2 = 4$. Since all the students have the same marginal cost in this classroom, the total expected effort is the total expected cost divided by the marginal cost: $4/c_H$. Similarly, the total expected effort of the classroom with L-type students is $4/c_L$. Therefore, the total expected effort of all the students is $\Pi_{Track} = 6$.

Now consider scenario two in which each classroom is mixed and contains two H-type students and two L-type students. It can be verified that only the H-type students choose positive effort levels in the equilibrium, and the equilibrium strategies are

$$G_H^* = s/\left(2v^1 - 4\right), \qquad G_L^* = 1.$$

The resulting total expected effort of all the students is $\Pi_{Mixed} = 4v^1 - 8$.

Thus, $\Pi_{Track} > \Pi_{Mixed}$ if $v^2 > 0.5$ and $\Pi_{Track} < \Pi_{Mixed}$ if $v^2 < 0.5$.

Why does the value of the lower prize matter? Compared to not tracking, tracking has an advantage of facilitating greater competition in each classroom by assigning students of similar abilities together. However, tracking also has a disadvantage. It does not use the highest prizes to motivate the best students. As a result, if the value of the lower prize is not too small, the advantage dominates, and tracking is better than not tracking. Now suppose the value of the lower prize is very small. If the students are tracked, only half of the prize money is used to motivate H-type students.

[12] The prizes represent the discounted future returns to education. Moreover, $v^1 \in (8/3, 4)$ ensures that the prize sequence is QPS.

However, if the students are not tracked, most of the prize money is used to motivate H-type students. Hence, the disadvantage of tracking may dominate its advantage, and tracking could be worse than not tracking.

1.5.2 Winner-Take-All?

Consider a situation in which the designer of a contest has some fixed amount of prize money, and he wants to choose the optimal prize structure to maximize the total expected score (performance). Is it optimal for the designer to adopt a winner-take-all prize structure, in which the whole amount is won by the highest-ranking player, or, should the total amount be split into two or more prizes?

Moldovanu and Sela (2001) consider a contest with incomplete information, and they find that winner-take-all prize structure is optimal if the players are symmetric and the costs are linear. However, this result does not hold in our model if the players are asymmetric. The following example demonstrates that the total expected score can actually be higher if the total amount is split into two prizes.[13]

Example 1.3 *Consider a contest with three players with costs* $c_1 = 2, c_2 = c_3 = 3$. *The total amount of prize money is* 4.

First, consider the contest with *one* prize of value 4. Player 3 always chooses 0, and players 2 and 3 compete for the prize. The equilibrium payoffs are 4/3 for player 1, and 0 for the others. The equilibrium strategies are

$$
\begin{aligned}
G_1^* &= 3s/4 \text{ for } s \in [0, 4/3] \\
G_2^* &= s/2 + 1/3 \text{ for } s \in [0, 4/3] \\
G_3^* &= 1
\end{aligned}
$$

The total expected score is 1.11.

Second, consider a contest with *two* prizes: $v^1 = 3$ and $v^2 = 1$. The equilibrium payoffs (in the equilibrium selected as a limit of equilibria of contests for which $c_3 <$

[13]Cohen and Sela (2008) demonstrate in a three-player all-pay auction that a small asymmetry of players' valuations may lead to a similar result.

$c_2 = 3$) are 1 for player 1 and 0 for others. Players 2 and 3 use the same strategy in this equilibrium, and the equilibrium strategies are

$$
\begin{aligned}
G_1^{**} &= \frac{1}{2\sqrt{s+1}}\left(3\sqrt{2}s - 2\sqrt{s+1} + \sqrt{2}\right) \text{ for } s \in \left[\sqrt{13}/9 - 2/9, 1\right] \\
G_2^{**} &= G_3^{**} = \begin{cases} \sqrt{2}\sqrt{s+1} - 1 & \text{for } s \in \left[\sqrt{13}/9 - 2/9, 1\right] \\ 3s & \text{for } s \in [0, \sqrt{13}/9 - 2/9) \end{cases}
\end{aligned}
$$

Total expected score is 1.19.

Therefore, the total expected score with two prizes 1.19 is *larger* than that with one prize, 1.11.

1.5.3 Effects of Superstars

Consider a situation in which the set of contestants consists of one "superstar" of very high ability (very low costs) and a group of players of moderate ability. Brown (2011) exhibits, in a Tullock game, what is known as the "Tiger Woods" effect—the presence of a superstar in the contest causes average players to decrease their effort levels. The effect of a superstar can also be studied in our model. In the two examples below, I show that Brown's theoretical result relies on the assumption that the other players are symmetric. In particular, Example 1.4 studies the situation in which the other players are symmetric, and exhibits the same phenomenon as in Brown (2011). However, Example 1.5 illustrates that, if the other players are asymmetric, the entry of a superstar may actually increase the expected scores (effort levels) of other players.

Why does the asymmetry of the other players matter? The presence of a superstar has two effects: first, it reduces the expected winnings of other players and therefore discourages competition; second, it increases the competition for the top prizes and motivates the other players with strong abilities. If the other players are symmetric, the second effect is small, so the presence of a superstar discourages competition. However, if some of the other players have similar abilities with the superstar, the second effect may dominate the first, so the presence of a superstar may lead to more competition.[14]

Example 1.4 *Consider a contest with three players and two prizes:* $v^1 = 3, v^2 = 1$.

First, suppose that the contest does not have a superstar. Let the set of players $\mathcal{N} = \{2, 3, 4\}$ with costs $c_2 = c_3 = c_4 = 1$. Since players are symmetric, and the marginal cost is 1, the total expected score of all the players equals their expected winnings minus their total payoff. The total expected winnings are just 4 and the total expected payoff in equilibrium is just 0. Thus the expected score of each player is 1.33.

Now suppose that we introduce a superstar with cost $c_1 = 0.1$ who displaces player 4 in the contest. Now $\mathcal{N} = \{1, 2, 3\}$. The equilibrium payoffs are $u_1^* = 2.7$, $u_2^* = u_3^* = 0$, and the equilibrium strategies are

$$\begin{aligned} G_1^* &= \frac{2}{\sqrt{0.4s + 14.8}} \left(s - 0.5\sqrt{0.4s + 14.8} + 1\right) \text{ for } s \in (0.95, 3) \\ G_2^* &= G_3^* = \begin{cases} 0.5\sqrt{0.4s + 14.8} - 1 & \text{for } s \in [0.95, 3] \\ s & \text{for } s \in [0, 3) \end{cases} \end{aligned}$$

Therefore, the expected score of 2 or 3 is 0.55.

Therefore, player 1's presence *reduces* the expected score of 2 or 3 from 1.33 to 0.55.

Example 1.5 *Consider a contest with three players and two prizes* $v^3 = 3, v^2 = 1$.

As above, first suppose that the contest does not have a superstar and the set of players $\mathcal{N} = \{2, 3, 4\}$ with costs $c_2 = c_3 = 1$ and $c_4 = 2$. The equilibrium payoffs in this case are $u_2^{**} = u_3^{**} = 1$, $u_4^{**} = 0$, and the equilibrium strategies are

$$\begin{aligned} G_3^{**} &= G_2^{**} = s/2 \text{ for } s \in [0, 2] \\ G_4^{**} &= 1 \end{aligned}$$

Therefore, the expected score of 2 or 3 is 1.

[14] Cohen and Sela (2008) demonstrate, in a three-player contest, that a small asymmetry in players' valuations may lead to a similar result.

Now suppose that a superstar with cost $c_1 = 0.8$ displaces the weakest player, player 4. The new equilibrium payoffs are $u_1^* = 0.6$, $u_2^* = u_3^* = 0$, and the equilibrium strategies are

$$\begin{aligned} G_1^* &= \frac{2(s - 0.5\sqrt{3.2s + 6.4} + 1)}{\sqrt{3.2s + 6.4}} \text{ for } s \in [0.38, 3] \\ G_2^* &= G_3^* = \begin{cases} 0.5\sqrt{3.2s + 6.4} - 1 & \text{for } s \in [0.38, 3] \\ s & \text{for } s \in [0, 0.38) \end{cases} \end{aligned}$$

Therefore, the expected score of 2 or 3 is 1.07.

In this case, player 1's presence *increases* the expected score of 2 or 3 from 1 to 1.07.

Moreover, if we fix the prizes and c_2, c_3, c_4 as above and decrease c_1 from 1 to 0, the increase in 2 or 3's expected score caused by player 1's presence decreases, and eventually this increase becomes negative and player 1's presence decreases 2 or 3's expected score.

1.6 Conclusion

In this chapter, I studied a *complete* information model of all-pay contests with asymmetries among players and (two classes) of convex prize sequences. While it would be desirable to study a similar environment under *incomplete* information, the problems associated with multiple prizes and asymmetric players under incomplete information are well known from auction theory. For instance, even with symmetric players very little is known about discriminatory (pay-as-you-bid) auctions for the sale of multiple units. Similar difficulties arise when considering all-pay auctions with multiple prizes.[15] The complete information setting allows us to study environments that, as yet, cannot be studied under an incomplete information setting.

I hope to explore some extensions of the model. First, it would be interesting to

[15] Studies of similar cases have shown that there is a unique equilibrium in asymmetric all-pay auctions with two players (Amann and Leininger, 1996; Lizzeri and Persico, 2000), but little is known about the case with more than two players.

relax the assumption of linear cost functions. The algorithm introduced here determines the support of the equilibrium strategies by "shifting" the pseudo strategies in Step 2 and this is the step that requires the linearity of costs. If costs were nonlinear such a shift would not preserve the differences in the upper supports of equilibrium strategies. A different algorithm would be needed and may rely on a priori knowledge of the upper supports.

Second, one would like to investigate, more generally, what an optimal prize sequence looks like. Must it be convex? Does the uniqueness result hold for general convex (possibly non-quadratic and non-geometric) prize sequences? These and other questions will be explored in subsequent work.

Chapter 2

Grouping Players in All-Pay Contests

2.1 Introduction

Tracking systems are very common in schools[1]. What the systems do is to identify students' abilities and group them according to their abilities. The best students are usually grouped together and take more advanced courses. Some countries have tracking systems, for example, US and Germany. However, tracking is not allowed in Chinese schools since 1990s. Then, we may ask, should we group students with similar abilities together, or should we mix students with different abilities? This chapter suggests that if the school wants to maximize the total score of students, it is optimal to group students with similar abilities together.

In particular, consider a group of students with different types. The students of one type have a higher cost of scores. The designer has a fixed budget of prize money and wants to maximize the total score of students. The designer can assign the students into one or two classrooms and divide the prize money into prizes for the two classrooms. In each classroom, the assigned students compete for the prizes in an all-pay contest as in Chapter 1. What is the optimal way to group the students into the contests? Should

[1] Salmans (1998) provides a description of the current tracking systems.

the designer separate the students, or should he mix the students with different costs, or should he put all the student into one classroom? If the unit of prize is small and the school has to distinguish different ranks, separating is better than mixing with any composition and better than putting all students into one classroom. Similar questions are also relevant in other competition scenarios. For example, consider a company wants to maximize the total profit by assigning salespeople of different abilities to different products. Should the company assign the salespeople with similar abilities to the same product, or should it mix salespeople with different abilities?

There is a large literature on contest designs[2], but few considers grouping players in multiple contests. Moldovanu and Sela (2006) consider contest designs for ex ante symmetric players in a model with incomplete information. They find that a single static contest is optimal if the designer maximizes expected total effort; and it is optimal to split the players in two divisions and let the winners compete in the final if the designer maximizes the expected highest effort and there are sufficiently many competitors. Since we want to consider the tracking system, the contest design question has to be consider with asymmetric players. There is a long discussion on the tracking controversy, but little theoretic work has been done. For example, Vanderhart (2006) finds evidence of a positive correlation between the diversity of achievement and ability-grouping. Jackson (2009) studies ability-grouping in Trinidad and Tobago, and finds that being assigned to a school with higher-achieving peers has large positive effects on examination performance. Lyle (2008) finds that mixing students in West Point improves the education outcomes. The literature on segregation in schools is also related. Those papers focus on the income gap between groups with different demographics such as gender and color[3]. Grouping in that literature means assigning students according to their demographics, while it means assigning students according to their abilities in this chapter

The rest of the chapter is organized as follows. Section 2.2 explains the model and presents the main result, and Section 2.3 provides analysis of equilibria and the proof of the main result. An alternative situation is considered in Section 2.4 and it is shown

[2] Konrad (2009) provide a nice review.

[3] See, for instance, Cooley (2009a) and Graham, Imbens and Ridder (2010).

that the main result may not hold, finally Section 2.5 concludes the chapter.

2.2 Model

The model builds on Chapter 1. Consider a situation with complete information. There are $2n$ players with type H and $2n$ players with type L. The players have constant marginal costs of score, c_H for players of type H and c_L for players of type L. We assume that $c_H > c_L > 0$. The planner has a fixed budget of prize money, V. In order to maximize the students' total score, the school divides the students into the two classrooms, and prize money into non-negative prizes in the contests. Let $n_i, m_i \geq 0$ be the numbers of type H and type L students in classroom i, and $v_i^1 \geq v_i^2 \geq ... \geq v_i^{n_1+m_1} \geq 0$ be the prize sequence in classroom i. Therefore, an action by the planner is a combination of a student allocation (n_1, m_1, n_2, m_2) and two prize sequences $\{v_1^1, v_1^2, ..., v_1^{n_1+m_1}\}$ and $\{v_2^1, v_2^2, ..., v_2^{n_2+m_2}\}$ such that $n_1+n_2 = m_1+m_2 = 2n$ and $\sum_{i,k} v_i^k = V$.

Given the prize sequence and student allocation in a classroom, the students in the classroom compete in an all-pay contest as in Chapter 1. In particular, the players in a classroom choose their scores $s_i \geq 0$ simultaneously. The player with the highest score wins the highest prize; the player with the second-highest score wins the second prize; and so on. In case of a tie, prizes are awarded in a way, perhaps randomly, that all tying players have a positive expected prize. Player j's payoff is $v^k - c_j s_j$ if he chooses score s_j and wins the kth prize. All players are risk-neutral. We only consider Nash equilibrium in this chapter and it will be referred as to equilibrium.

Let us introduce two technical assumptions. First, there is a smallest unit of prizes, $\varepsilon > 0$. As a result, V/ε should be an integer. Second, the school has to *distinguish* different ranks in each classroom. That is, if a classroom has more than one student, there should be no identical prizes in the classroom. Because of these two assumptions, the designer cannot assign all the money to one classroom, and we say that the prize money for classroom i is maximized if the prize money for the other classroom is minimized at the prize sequence $\{0, \varepsilon, ..., \varepsilon (n_{i'} + m_{i'})\}$, where $n_{i'} + m_{i'}$ is the number of students in the other classroom. Here is the main result of this chapter.

Theorem 2.1 *If i) the unit of prize is small, ii) the school has a fixed budget, and iii) the school has to distinguish different ranks, the optimal contests are such that a) the students are separated, b) the last prizes are* 0 *in both classroom, and c) the prize money for the low-cost classroom is maximized.*

2.3 Equilibrium

This section contains the analysis of equilibrium and the proof of Theorem 2.1.

Lemma 2.1 *In any equilibrium of an all-pay contest, the players with the same type receive the same payoff.*

Proof. Suppose otherwise, and $c_i = c_j$ but $u_i > u_j$. If j chooses slightly above the upper support of G_i, j's payoff can be above or very close to u_i therefore higher than u_j. This is a contradiction. ■

Let us also name three categories of student assignments. First, *separating* means that all the student of type H are assigned to one classroom and the students of type H are assigned to another classroom. Second, *pooling* means that all the students are assigned to one classroom. Finally, *mixing* means that there are students of both types in each classroom.

Lemma 2.2 *The maximum total score with separating assignments is*

$$\Pi_{seperating} = \frac{V - W_H}{c_L} + \frac{W_H}{c_H}.$$

The maximum is obtained if the prize money to the low-cost classroom is maximized, and the last prizes are 0*, which means the prize sequence for the H-type classroom is* 0, ε, 2ε, ..., $(2n-2)\,\varepsilon$, $(2n-1)\,\varepsilon$ *and the last prize in the L-type classroom is* 0*.*

Proof. Each classroom has students of the same type. Their payoff should be the same: u_L, u_H.

The total score is
$$\frac{V-W_H-2nu_L}{c_L}+\frac{W_H-2nu_H}{c_H}$$
where W_H is the the total expected winning for the students of H type. If the last prizes are 0, $u_L=0$, $u_H=0$. Moreover, W_H is minimized at prizes: $0,\varepsilon,2\varepsilon,...,(2n-2)\varepsilon,(2n-1)\varepsilon$, and the minimum is $\varepsilon\sum_{i=1}^{2n}(i-1)=\varepsilon n(2n-1)$. Therefore,

$$\Pi_{seperating}=\frac{V-\varepsilon n(2n-1)}{c_L}+\frac{\varepsilon n(2n-1)}{c_H}$$

■

Lemma 2.3 *The total score with the pooling assignment is smaller than* $\Pi_{seperating}$.

Proof.
$$\Pi_{pooling}=\frac{V-W_H-2nu_L}{c_L}+\frac{W_H}{c_H}$$
where $W_H\geq\varepsilon\sum_{i=0}^{2n-1}i=n\varepsilon(2n-1)$ and $u_L\geq v^{2n}(c_H-c_L)/c_H\geq\varepsilon 2n(c_H-c_L)/c_H$.

The last $2n{+}1$ prizes in the optimal prize sequence are $0,\varepsilon,...,2n\varepsilon$. $u_L\geq 2n\varepsilon(c_H-c_L)/c_H$

$$\Pi_{pooling}\leq\frac{V-n\varepsilon(2n-1)-\varepsilon 2n2n(c_H-c_L)/c_H}{c_L}+\frac{n\varepsilon(2n-1)}{c_H}$$

$$\begin{aligned}
&\Pi_{seperating}-\Pi_{pooling}\\
\geq\;&\frac{V-\varepsilon n(2n-1)}{c_L}+\frac{\varepsilon n(2n-1)}{c_H}\\
&-\left(\frac{V-n\varepsilon(2n-1)-\varepsilon 2n2n(c_H-c_L)/c_H}{c_L}+\frac{n\varepsilon(2n-1)}{c_H}\right)\\
=\;&4n^2\frac{\varepsilon}{c_Lc_H}(c_H-c_L)>0
\end{aligned}$$

■

Lemma 2.4 *The total score with any mixed assignment is smaller than* $\Pi_{seperating}$.

The proof is similar to those for Lemma 2.2 and 2.3, and it is contained in Appendix because it is relatively long. It is easy to see that Lemma 2.2 to 2.4 imply Theorem 2.1.

2.4 Exogenous Prizes

This section consider the situation in which the prizes are exogenous. We are going to demonstrate, in an example, that separating may not be optimal.

Example 2.1 *Suppose that there are four students with lower cost* $c_L = 1$ *and the other four with* $c_H = 2$. *There are two classrooms and there are two prizes* v^1 *and* $v^2 = 4 - v^1$ *for each class. We also assume that the prize sequence is convex, so* $v^1 > 8/3$.

Since the last prizes are zero in both classes, if the students are separated, the total score is

$$\Pi_{seperating} = 4/c_H + 4/c_L = 6$$

If the students are mixed evenly, we can verify that the H-cost students choose 0 and the strategy of L-cost student is $G_L(s) = s/(2v^1 - 4)$. Therefore, the total score is

$$\Pi_{mixing} = 4v^1 - 8$$

Hence, $\Pi_{seperating} < \Pi_{mixing}$ if $v_1 > 3.5$; $\Pi_{seperating} = \Pi_{mixing}$ if $v_1 = 3.5$; and $\Pi_{seperating} > \Pi_{mixing}$ if $8/3 < v^1 < 3.5$.

2.5 Conclusion

This chapter considers contest designs when the designer can assign players to different contests and also decide the prizes in each contests. If the designer wants to maximize the total score, it is optimal to group players with similar abilities together. However, it may be optimal to mix players with different abilities if the prizes are exogenous. Besides total score, the designer may have other criteria. For example, the designer may want to minimize the probability of a weaker player winning a higher prize, or to minimize the payoff gap between different types. These criteria may produce different results, and we leave them to future work.

Chapter 3

Bargaining Order in a Multi-Person Bargaining Game

3.1 Introduction

Consider a scenario in which a real estate developer must acquire land from multiple sellers. The sellers' lots are of different sizes with a larger lot giving a higher flow of payoffs to its owner. Such situations are quite common. For example, in Chongqing, China, the construction of a retail mall required 280 separate negotiations. The project was suspended for three years because one out of the 280 residents refused to sell his property to the developer.[1] Columbia University's expansion plan in West Manhattanville is another prominent example. The 6.3-billion, 17-acre project acquired land from 67 property owners. The whole negotiation lasted for a long period from 2002 to 2010, and the negotiation on the last three properties alone took more than three years.[2] What should the buyer (developer) do when she needs to purchase land from multiple sellers who own different sized lots? In particular, which seller should she bargain with first, the one with a large lot or a small lot? This chapter examines the corresponding non-cooperative bargaining game. The results suggest that the buyer

[1] The negotiation started in 2004, and eventually the owner sold the house in 2007. See in French (2007) or "Nail house in Chongqing demolished", *China Daily,* April 3, 2007.

[2] See more details in Williams (2008), Chung (2009) and Sieff (2010).

should bargain with the seller of the smallest lot first, especially when the sizes of the lots are quite different. This chapter does not try to explain the delay in the examples above. Delay happened even when there is only one seller left in the first example, perhaps due to incomplete information.[3]

While the model studied here is couched in the language of a single developer negotiating with multiple sellers, it is applicable to a variety of other bargaining scenarios. For example, consider an airline that must bargain with two separate unions, pilots and flight attendants, in order to end a strike. Both unions are necessary for the airline to operate but their outside options differ.[4] Which union should the firm negotiate with first? A similar question can be asked about the negotiation between a manufacturer and a group of upstream suppliers producing parts at different costs.[5] The key characteristics common to these scenarios are: the one-to-many aspect of the negotiation; the fact that an agreement with *all* sellers is necessary to reap any economic gains; and, finally, the "size" differences among the sellers.

In this chapter, bargaining strength is measured by the size of the outside/inside option[6] available to a seller when bargaining with the buyer. A seller with a large lot is stronger than a seller with smaller lot in the sense that, in equilibrium, the price received by the large seller is higher than that received by the small seller. There are other notions of bargaining strength, of course. For instance, one may measure bargaining strength by how patient a seller is and different sellers may have different discount rates. Alternatively, it may have to do with the likelihood of making first offers (as in Li (2010)).

It is useful to begin with a simple example. Consider a scenario with one developer and two farmers. All parties share a discount factor of $\delta = 2/3$. Farmer 1 owns a large lot of land that produces 1/8 units of harvest each period; and farmer 2 owns a small

[3] See, for instance, Admati and Perry (1987) for how incomplete information can lead to delay.

[4] An outside option is the payoff that a player receives if he leaves the negotiation. This paper focuses on inside options, but our qualitative results would not be affected if sellers had outside options instead.

[5] Bargaining between a manufacturer and its upstream suppliers is discussed, for example, in Blanchard and Kermer (1997) and Bedrey (2009).

[6] An inside option is the payoff received by a seller while negotiations are ongoing (see Muthoo (1999)).

lot of land that produces 1/18 units of harvest each period. A lot does not produce any harvest once it is sold to the developer. The developer must purchase both lots to build a mall that produces 1/3 units of profit each period. It is easy to see that the present value of all harvests is $v_1 = 3/8$ for farmer 1, $v_2 = 1/6$ for farmer 2, and the present value of all profits of the mall is 1.

Negotiations are sequential and in any period, the developer negotiates with only one farmer. The developer first offers a price, which the farmer may accept or reject. If the offer is accepted, the developer proceeds to negotiate with the other farmer in the next period (in a standard two-player alternating offer bargaining game). If the offer is rejected, the farmer makes a counter-offer in the next period, which the developer may accept or reject. If the developer accepts this offer, she proceeds to negotiate with the other farmer. If the developer rejects the offer, she picks a farmer, which could be the same as in the previous period, and negotiates with him in the same fashion, and so on. We would like to know which farmer the developer should bargain with first.

Because the developer can pick any remaining farmer to negotiate with, there is no restriction on the choice of bargaining orders. However, there is a unique subgame perfect equilibrium outcome, where the developer bargains with farmer 2 until he agrees and then with farmer 1, and farmer 2 sells his land in period 1 at a price of $p_2 = 1/5$, and farmer 1 sells his land in period 2 at a price of $p_1 = 5/8$.[7] The prices are explained below. The payment to the first farmer is a sunk cost to the buyer, so after farmer 2 sells his land, the surplus is $1 - v_1$, which is the difference between the value of the mall and the value of farmer 1's land. Farmer 1 gets the same $\delta/(1+\delta)$ of the surplus as in Rubinstein bargaining game,[8] which implies that his selling price is $v_1 + \frac{\delta}{1+\delta}(1 - v_1) = 5/8$. Excluded the price for farmer 1, the remaining value of mall is $\delta(1 - p_1)$ evaluated in the first period. As a result, the surplus for farmer 2 and the buyer is $\delta(1 - p_1) - v_2$, which is also the difference between the remaining value of the mall and the value of farmer 2's land. Similarly, farmer 2 and the buyer split this surplus as in Rubinstein bargaining game, therefore the price for farmer 2 is

[7] The numbers in the example are calculated according to the equilibrium derived later in the paper and are given only to illustrate the main findings. Technical details and calculations are postponed to Example 2.

[8] See Rubinstein (1982).

$v_2 + \frac{\delta}{1+\delta}\left[\delta\left(1-p_1\right) - v_2\right] = 1/5.$

Section 3.3 shows that any other bargaining order is not optimal. To demonstrate the main idea, consider a deviation in which the developer chooses a "wrong" bargaining order: bargaining with farmer 1 in the first period. The unique equilibrium outcome of the resulting subgame is the following. Both the offer and counter-offer are rejected in the first two periods; the developer chooses to bargain with farmer 2 in the third period and the farmer 2 sells his land in the third period at a price of $p_2 = 1/5$; farmer 1 then sells his land in the fourth period at a price of $p_1 = 5/8$.

Notice that while the selling prices in the two orders are the *same*, there is a two-period delay if the bargaining starts with farmer 1. The prices must be the same because the bargaining game is also a proper subgame of the game where the developer chooses farmer 1 in the first period. Since each has a unique subgame perfect equilibrium outcome, the prices must agree. However, why does one order lead to a delay? To see why, consider what would happen if, when the bargaining starts with farmer 1, the developer offers farmer 1 the present value of the price he would get in period 4 plus the appropriate compensation for the foregone harvest in three periods; that is, $p_1' = 1/8 + \delta/8 + \delta^2/8 + \delta^2 p_1 = 0.45$. Farmer 1 would agree to this, and in the resulting negotiation with farmer 2, the price would be $p_2' = 1/2$. However, the present value of the two payments $p_1' + \delta p_2' = 0.78$ is greater than δ, the value of the mall to the developer, which is realized in period 2. While this shows that one particular deviation will not break the delayed equilibrium, the formal proof in Section 3.3 shows that no such deviation can prevent delay.

Even though the bargaining game is of complete information, the "wrong" bargaining order may cause delay in agreement. Several papers have described different reasons for delay in complete information bargaining. The reason discussed above is similar to Cai (2000). On the other hand, the reasons in Haller and Holden (1990) and Fernandez and Glazer (1991) are different from those identified here. Their models are standard two-person alternating bargaining games between a labor union and a firm, but the labor union can choose between production and strike when an offer is rejected. In an equilibrium with delay, the firm would rather wait several periods to avoid the "bad" equilibrium in which the union strikes once disagreement occurs. In our paper, production (building the mall) is not allowed while the bargaining is going on. Harvests are different from the production and the farmers receive them for sure during the bargaining.

Our model builds on Cai (2000) by introducing endogenous bargaining order and asymmetric sellers. His model is the extreme case of our game when all the farmers receive no harvest. The bargaining order is fixed and rotates among the sellers in his paper. He finds multiple stationary subgame perfect equilibrium outcomes, and delay can happen in some of them. In contrast, the sellers are asymmetric and the bargaining order is endogenous in our game, this results in a *unique* subgame perfect equilibrium outcome.

Few papers discuss bargaining orders as part of equilibrium. However, there are several papers with similar setups that allow endogenous bargaining orders but in a restricted way. Perry and Reny (1993) allow each player to decide when to make an offer, which implicitly allows for different bargaining orders. Stole and Zwiebel (1996), Noe and Wang (2004) and Bedrey (2009) study bargaining orders in finite bargaining games. Chatterjee and Kim (2005) focus on the orders in which the buyer cannot switch to another seller before an agreement. The literature on agenda formation[9] also discusses orders, but the orders have a different meaning: sequences of different issues or tasks. Contrary to our paper, this literature suggests the most important issue should be discussed first.[10]

Li (2010) also allows endogenous bargaining order, but his paper is very different from ours. In his paper, a seller's bargaining strength is measured his likelihood to make the first offer in each bargaining. He finds many equilibria and any selling order can be sustained. In our paper, a seller's bargaining strength is measured by the size of his outside/inside option, and the bargaining game has a unique equilibrium outcome.

The seller holdout problem[11] also has complementary sellers as in our setup. In this literature, it is argued that incomplete information often leads to holdout. However,

[9] See, for example, Fershtman (1990), Winter (1997), Reinhard and Matthias (2001) and Flamini (2007).

[10] See, for example, Winter (1997) and Flamini (2007).

[11] See, for example, Mailath and Postelwaite (1990), Menezes and Pitchford (2004) and Chowdhury and Sengupta (2008).

holdout is not important in our setup because our bargaining game is of complete information.[12]

The rest of the paper is organized as follows. Section 3.2 introduces the bargaining game where the buyer can commit to a bargaining order, and analyzes the equilibrium in the case of two sellers. Section 3.3 introduces the bargaining game in which the buyer cannot commit to a bargaining order, and examines the two-seller case of such a game. Section 3.4 extends the previous two sections to N-seller case. Section 3.5 discusses alternative assumptions and applications. Finally, Section 3.6 concludes the paper.

3.2 Bargaining with Commitment

Our model is a non-cooperative, infinite-horizon and complete-information bargaining game with endogenous bargaining order and asymmetric sellers. The game has $N+1$ players including one buyer, B, and a set of sellers, $\{1, 2, \cdots, N\}$. Each seller (he) has one lot of land, and the buyer (she) must purchase every lot in order to build a mall. In other words, the lots are perfect complementary for the buyer.

All the players share the same discount factor $\delta \in (0, 1)$.[13] Seller i's land is of value v_i, which is the present value of a constant flow of harvests. Therefore, seller i's harvest for one period is of value $v_i (1 - \delta)$, and is received at the end of each period while the land is still in his possession. In the literature of bargaining, the harvest is referred as to seller i's inside option. It is assumed that $v_1 > v_2 > \cdots > v_N > 0$. If every unit area of land is equally productive, the assumption implies that seller i's land is of larger size than seller $(i+1)$'s. The mall produces a constant profit each period, and the present value of the profits is normalized to 1.

[12] There is no holdout when the equilibrium is unique in Section 3.2 and 3.3.1. In Section 3.3.2, when the sellers are similar and the buyer cannot commit to a bargaining order, holdout may arise.

[13] If the sellers are different only in their discount factors, it can be verified that the buyer would be indifferent among different orders because they all give her the same payoff.

3.2.1 Timing

The buyer first announces a bargaining order represented by an infinite sequence of sellers, $i_1, i_2, ...$ where $i_t \in \{1, 2, \cdots, N\}$ for all t. Starting with the first seller in the sequence, the buyer bargains with one seller over the price in each round of bargaining. Each round has one or two periods. In the first period, the buyer suggests a price to the seller, the seller then decides to either accept it or reject it. If the seller accepts, the round ends with only one period. Otherwise, the seller suggests another price in the second period, which the buyer must either accept or reject. If an agreement is reached, the buyer pays the seller the agreed price right away and the seller leaves the game permanently before the harvests of the period are realized. If the seller's suggestion is rejected, the bargaining moves to the next round in which the buyer bargains with the second seller in the sequence in the same fashion. At the end of each period, every remaining seller receives a harvest from his land.

Note that the bargaining order cannot be revised afterwards, which is relaxed in the next section. Hereafter, the above game is referred as to the "N-seller game with commitment", or the "N-seller game" where no ambiguity results. The bargaining order in the N-seller game specifies only the order of sellers before the first agreement. After the first agreement, the game has only $N-1$ sellers and the buyer chooses an order for the resulting $(N-1)$-seller game, and so on.

Let $\Gamma(j, (i_1, i_2, ...))$ denote the two-seller game with the *fixed* bargaining order $i_1, i_2, ...$, where player $j \in \{i_1, B\}$ makes the first offer. Bargaining order $i_1, i_2, ...$ means that the buyer bargains with seller i_t in period $2t+1$ if no agreement has been achieved. For example, bargaining order $1, 1, ...$ means the buyer bargains with seller 1 until he agrees. Let $\Gamma(i)$ represent the one-seller game between seller i and the buyer who makes the first offer. Figure 3.1 demonstrates the game tree for the two-seller game.

Two features of the model are very important. First, the buyer bargains with only one seller at a time. This bargaining is very popular in reality, especially when it is costly to communicate with all the sellers at the same time. In the example by Coase (1960), a railway company has to bargain with the farmers along a railway track. It is difficult to make simultaneous offers to multiple farmers when they are located far away from each other. Moreover, this assumption is commonly used in the recent bargaining studies such as Cai (2000, 2003) and Noe and Wang (2004). Second, the payments are made immediately after the corresponding agreements. In other words, the contracts between the buyer and sellers are cash-offer contracts. These contracts are widely used by real estate developers;[14] moreover, they are also studied in recent bargaining literature[15]. However, there are also situations that do not satisfy these assumptions, and the consequences of relaxing the assumptions are discussed in Section 3.5.

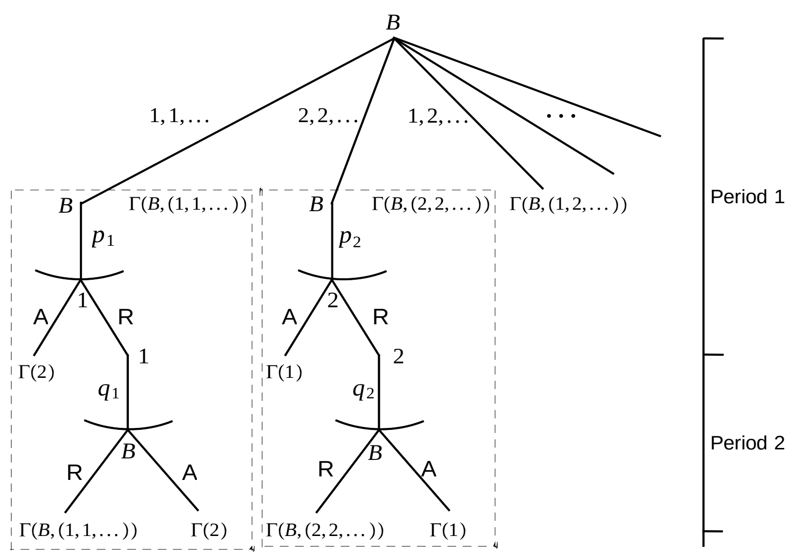

Figure 3.1. Bargaining with Commitent

3.2.2 Payoffs

An outcome is denoted as $(p_1, p_2, ..., p_N, t_1, t_2, ..., t_N)$, where seller i sells his land at price p_i in period t_i. If seller i never sells his land, t_i is infinity and p_i is zero. The

[14] See "Nail house in Chongqing demolished", *China Daily,* April 3, 2007.

[15] See Krishna and Serrano (1996) and Cai (2000, 2003).

present value of t periods of seller i's harvests is denoted as

$$H_{i,t} \equiv v_i (1 - \delta) \sum_{s=1}^{t} \delta^{s-1}.$$

Given an outcome, seller i's payoff is

$$\pi_i = H_{i,t_i-1} + \delta^{t_i-1} p_i \tag{3.1}$$

where the first term is the present value of the harvests before the land is sold, and the second term is present value of the payment from the buyer. Since there is no harvest when the buyer owns the land, the buyer's payoff is

$$\pi_B = \delta^{\max(t_1,\cdots,t_N)-1} - \sum_{i=1}^{N} \delta^{t_i-1} p_i \tag{3.2}$$

where the first term is the present value of the mall and the second term is the present value of the payments to the sellers.

The assumption that the buyer cannot reap the harvests from the land represents the fact that the buyer cannot fully utilize the land as the sellers do before the mall is built. Take the land purchasing case in Chongqing for example. The sellers received utilities by living in their houses, but the buyer could not get all of those utilities even if she owns the houses.

3.2.3 Equilibrium

Throughout the paper, "equilibrium" refers to subgame perfect equilibrium. We only consider the two-seller case for the rest of this section and in Section 3.3, and all the analysis is extended to the N-seller case in Section 3.4.

Since the payment to the first seller is a sunk cost to the buyer, the subgame after the first purchase is the one-seller game between the buyer and the remaining seller. Therefore, let us first examine the one-seller game with land value v_i.

In the one-seller game, the buyer has to bargain with the only seller every period, so the game is simply a two-person alternating bargaining game with inside options only available to the seller. The unique equilibrium of such a game is characterized in Lemma 3.1, which is a straightforward adaptation of Proposition 6.1 in Muthoo (1999).

Lemma 3.1. *Let* (p_i^1, q_i^1) *be the solution to*

$$p_i = H_{i,1} + \delta q_i \tag{3.3}$$

$$1 - q_i = \delta(1 - p_i). \tag{3.4}$$

In the one-seller game between the buyer and seller i*, there is a unique equilibrium in which*

i) the seller offers price q_i^1 *and accepts price no less than* p_i^1*,*

ii) the buyer offers price p_i^1 *and accepts price no more than* q_i^1*.*

Similar to Proposition 6.1 in Muthoo (1999), the seller and the buyer should be indifferent between accepting and rejecting the other player's offer in the unique equilibrium. (3.3) means that the buyer offers such that the seller is indifferent between accepting and rejecting, and (3.4) means that the seller offers such that the buyer is indifferent between accepting and rejecting.

The superscript denotes the number of sellers in the game where the variables are considered, but it is omitted where no ambiguity results. (3.3) and (3.4) implies that the equilibrium price is

$$p_i^1 = v_i + \frac{\delta}{1+\delta}(1 - v_i), \tag{3.5}$$

which is always higher than seller's value of the land v_i. Moreover, the equilibrium payoffs are $v_i + \frac{\delta}{1+\delta}(1 - v_i)$ for the seller and $\frac{1}{1+\delta}(1 - v_i)$ for the buyer, so they split the surplus $1 - v_i$ as in the Rubinstein bargaining game.

Lemma 3.1 characterizes the unique equilibrium after the first purchase in the two-seller game, so we focus only on the strategies *before* the first purchase for the rest of this section.

Lemma 3.2. *Let (p_2^2, q_2^2) be the solution of the following equations*

$$p_2 = H_{2,1} + \delta q_2 \tag{3.6}$$

$$\delta\left(1 - p_1^1\right) - q_2 = \delta\left(\delta\left(1 - p_1^1\right) - p_2\right) \tag{3.7}$$

If

$$\delta v_1 + (1 + \delta) v_2 \leq \delta, \tag{3.8}$$

the following strategies constitute an equilibrium in the game $\Gamma(B, (2, 2, ...))$:

i) seller 2 suggests price q_2^2 and accepts price no less than p_2^2,

ii) the buyer suggests price p_2^2 to seller 2 and accepts price no more than q_2^2 from seller 2.

Proof: According to (3.6), seller 2 is indifferent between accepting and rejecting. In particular, if seller 2 accepts p_2 in the current period, his payoff is the left hand side of (3.6). If seller 2 rejects p_2, he receives a harvest at the end of current period and accepts q_2 in the next period, then his payoff is the right hand side of (3.6).

Similarly, the buyer is indifferent between accepting and rejecting according to (3.7). In particular, if the buyer accepts q_2, she pays q_2 to seller 2 in the current period, pays p_1^1 to seller 1 and receives the value of the mall in the next period, so her payoff is the left hand side of (3.7). If the buyer rejects q_2, she pays p_2 to seller 2 in the next period, pays p_1^1 to seller 1 and receives the value of the mall two periods later, then her payoff is the right hand side of (3.7).

As a result, neither seller 2 nor the buyer would deviate in the subgame $\Gamma(B, (2, 2, ...))$, so the lemma is proved. ■

(3.6) and (3.7) imply that the equilibrium price for seller 2 is

$$p_2^2 = v_2 + \frac{\delta}{1+\delta}\left[\delta(1 - p_1^1) - v_2\right]. \tag{3.9}$$

Substituting (3.5) and (3.9) into (3.1) and (3.2) gives the equilibrium payoffs for the buyer and seller 2

$$\pi_B^* = \frac{1}{1+\delta}\left[\delta\frac{1}{1+\delta}(1-v_1)-v_2\right] \tag{3.10}$$

$$\pi_2^* = v_2+\frac{\delta}{1+\delta}\left[\delta\frac{1}{1+\delta}(1-v_1)-v_2\right] \tag{3.11}$$

Since Lemma 3.1 shows that seller 1 sells at price p_1^1 in period 2, his equilibrium payoff is

$$\pi_1^* = v_1+\delta\frac{\delta}{1+\delta}(1-v_1) \tag{3.12}$$

After the first purchase, the surplus for seller 1 is $\frac{\delta}{1+\delta}(1-v_1)$ evaluated in period 2 as in Lemma 3.1. Therefore, with the surplus for seller 1 excluded, the mall worth $\frac{1}{1+\delta}(1-v_1)$ in period 2 or $\delta\frac{1}{1+\delta}(1-v_1)$ in period 1. As a result, the agreement with seller 2 produces a surplus of $\delta\frac{1}{1+\delta}(1-v_1)-v_2$. It is easy to see from (3.10) and (3.11) that the buyer and seller 2 also split the surplus $\frac{\delta}{1+\delta}(1-v_1)-v_2$ as in the Rubinstein bargaining game, where (3.8) guarantees that this surplus is not negative. Therefore, (3.8) also implies that seller 2 and the buyer are not worse off by participating the bargaining. Intuitively, (3.8) requires that the land values cannot be too large; otherwise the mall is not profitable for the buyer and the early seller.

Proposition 3.1. *In the two-seller game with commitment, if the mall is profitable*[16] *as in (3.8), there is a unique equilibrium outcome and a unique equilibrium bargaining order where the buyer bargains with the smaller seller until an agreement is reached.*

Proof: A sketch of the proof is presented below, and the full proof is in Appendix C.7.1. The proof consists of five claims. First, in an equilibrium with agreement in the first period, any equilibrium payoff for the buyer can be reached. Second, consider the equilibria of the subgames with either bargaining order $1, 1, ...$ or $2, 2, ...$ In these equilibria, the supremum of the buyer's equilibrium payoffs, $\bar{\pi}_B$, can be approached. Third, if the buyer commits to order $2, 2, ...$, the corresponding subgame has a unique

[16]Because the sellers have to agree sequentially, the mall may not be profitabe (for the buyer and first seller) even if it is efficient to build it ($v_1 + v_2 < 1$).

equilibrium with an agreement in the first period; when the buyer commits to order $1, 1, ...$, the resulting subgame has either a unique equilibrium with agreement in the first period or equilibria with no agreement at all. Fourth, if the buyer chooses order $1, 1, ...$, her payoff is less than $\bar{\pi}_B$. Fifth, only if the buyer chooses order $2, 2, ...$, the buyer's equilibrium payoff reaches $\bar{\pi}_B$.

When δ approaches 1, seller i's harvest in each period $v_i (1 - \delta)$ and the mall's profit in each period $1 - \delta$ converge to 0. It is easy to see that Proposition 3.1 holds even for δ close to 1.

It is surprising that the equilibrium outcome[17] is unique because multiple equilibrium outcomes are prevalent in similar bargaining games.[18] For example, if the buyer follows a bargaining order that alternates between the sellers, there are multiple equilibrium outcomes by the same analysis in Theorem 1 of Cai (2000). In our model, uniqueness results because the bargaining order is endogenous. Since the buyer does not choose the alternating order in the equilibria, the multiple equilibrium outcomes do not arise.

It is important to understand the intuition behind this choice. Consider the extreme case in which δ is close to 1. Then, according to (3.5) and (3.10), the buyer and the seller split the surplus evenly in each bargaining. Suppose the selling order is small-large, where seller 2 sells in the first period and seller 1 sells in the second. Since δ is close to 1, the total surplus of the mall is approximately $1 - v_1 - v_2$. The surplus for the second period is $1 - v_1$, half of which goes to the last seller according to (3.5). What is left is the surplus for the first period, $(1 - v_1)/2 - v_2$, half of which is the buyer's payoff according to (3.10). Suppose the selling order is large-small, where seller 1 sells in the first period and seller 2 sells in the second. The surplus for the first period is $(1 - v_2)/2 - v_1$, which is lower than in the small-large order. Since the buyer receives a

[17] There are multiple equilibria in our bargaining game, but all of them have the same outcome. As in the proof of Claim 3.3, there is no agreement in $\Gamma(B, (1, 1, ...))$ if $\delta v_2 + (1 + \delta) v_1 > \delta$. Therefore, given that any offer is rejected, it is seller 1's equilibrium strategy to offer any price no less than v_1. As a result, the subgame $\Gamma(B, (1, 1, ...))$ has many equilibria, so does the whole game.

[18] However, several papers also demonstrate unique subgame perfect equilibrium in multi-person bargaining games. See Jun (1987), Chae and Yang (1988) and Krishna and Serrano (1996).

half of the surplus for the first period, her payoff is lower than in the small-large order. This intuition also works for any δ, which is the content of Claim 3.4.

In our model explicated above, the sellers' heterogeneity allows the selling orders to affect the surplus after the first purchase. In contrast, it might not be the case in which the sellers' heterogeneity does not allow this, and there could be multiple equilibrium outcomes. For example, Li (2010) considers the heterogeneity in sellers' probabilities to make the first offer, so no matter who sells first the surplus after the first purchase is the same, and he finds multiple equilibrium outcomes with different bargaining orders.

Example 3.1. Consider a two-seller game with $\delta = 2/3$, $v_1 = 3/8$ and $v_2 = 1/6$. Proposition 3.1 implies that the buyer bargains with seller 2 until an agreement is reached. In the resulting subgame, the equilibrium strategies before the first purchase are

	accepts	offers
Seller 2	$\geq 1/5$	13/60
Buyer	$\leq 13/60$ from seller 2	1/5 to seller 2

where the prices are calculated according to Lemma 3.2. After the first purchase, the equilibrium strategies are given in Lemma 3.1. As a result, seller 2 sells at price 1/5 in period 1 and seller 1 sells at price 5/8 in period 2. According to (3.1) to (3.2), the payoffs are 13/24 for seller 1, 1/5 for seller 2 and 1/20 for the buyer.

3.3 Bargaining without Commitment

The assumption that the buyer can commit to a bargaining order is relaxed in this section. In particular, the buyer does not fix a bargaining order in the beginning of the game. Instead, the buyer picks only one remaining seller at the beginning of each round, and bargains in the same pattern as in the case with commitment. If no agreement is reached in a round, the bargaining moves to the next round where the buyer picks a remaining seller and bargains with him in the same fashion. Note that the buyer can choose any remaining seller to bargain with, so there is no restriction on the bargaining order. From now on, the above game is referred as to the "N-seller game without commitment", or the "N-seller game" where no ambiguity results.

We use $\hat{\Gamma}(j, j')$ to denote the subgame in which no seller has sold his land and player j suggests a price to player j' in the first period. $\hat{\Gamma}(i)$ represents the one-seller game between seller i and the buyer who makes the first offer. Figure 3.2 demonstrates the game tree for the two-seller game.

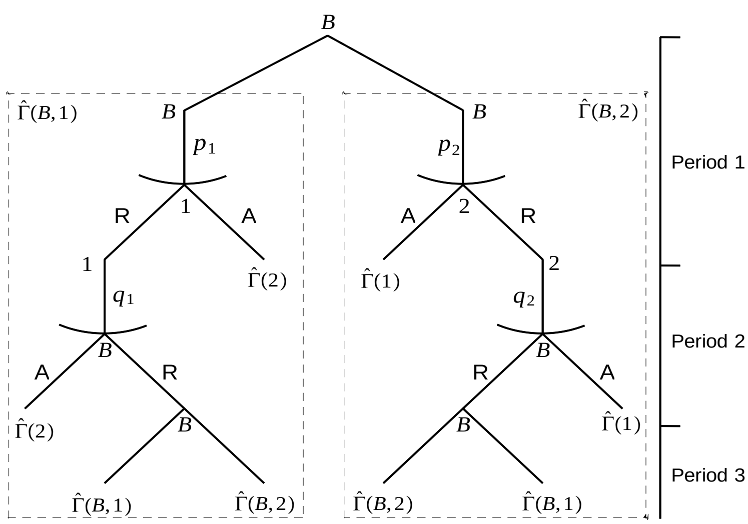

Figure 3.2. Bargaining without Commitment

Given an equilibrium, a bargaining order in the N-seller game without commitment is an infinite sequence of sellers, where the buyer bargains with the tth seller if no agreement has been reached after the buyer bargains with the first $t-1$ sellers in the sequence. After the first agreement, the buyer follows a bargaining order in the resulting $(N-1)$-seller game, and so on.

In the rest of this section, the two-seller game is discussed in two cases. In the first case where the sellers have sufficiently different sizes, we have a unique equilibrium outcome. This case is discussed in the following three steps. We first discuss the subgame in which the buyer bargains with seller 2 first; then study the subgame where the buyer bargains with seller 1 first; finally compares the two scenarios and summarizes the case. In the second case where the sellers have similar sizes, we have multiple equilibrium outcomes and examples are provided.

3.3.1 Case 1: Sufficiently Different Sizes

Lemma 3.3 characterizes all the equilibria, and Lemma 3.4 shows the corresponding outcome is unique. Since the payment to the first seller is a sunk cost to the buyer, the subgame after the first purchase is a one-seller game between the buyer and the remaining seller. Since we already know the unique equilibrium in the one-seller game, the following lemma focuses on the strategies *before* the first purchase.

Lemma 3.3. *Let (p_2^2, q_2^2) be the solution of (3.6) and (3.7), and (p_1^2, q_{B1}^2) be the solution of the following equations*

$$\begin{aligned} p_1 &= H_{1,3} + \delta^3 p_1^1 && (3.13) \\ \delta\left(1 - p_2^1\right) - q_{B1} &= \delta\left(\delta\left(1 - p_1^1\right) - p_2^2\right) && (3.14) \end{aligned}$$

If (3.8) and

$$v_1 - v_2 > \frac{\delta}{1+\delta} - \frac{1+2\delta}{1+\delta} v_2, \tag{3.15}$$

for any (q_1^2, p_{B1}^2) such that $q_1^2 > q_{B1}^2$ and $p_{B1}^2 < p_1^2$, the strategies below constitute an equilibrium in the game $\hat{\Gamma}(B, 2)$,

i) seller 2 suggests price q_2^2 and accepts price no less than p_2^2,

ii) seller 1 suggests price q_1^2 and accepts price no less than p_1^2,

iii) the buyer bargains with seller 2 until an agreement is reached; suggests price p_2^2 to seller 2 and p_{B1}^2 to seller 1; accepts price no more than q_2^2 from seller 2 and no more than q_{B1}^2 from seller 1.

Proof: Notice that the outcome is the same as in Lemma 3.2, so the equilibrium payoffs are π_B^*, π_2^* and π_1^* given in (3.10) to (3.12). The left hand side of (3.14) is the buyer's payoff if she accepts price q_{B1}^2 from seller 1, and the right hand side is $\delta\pi_B^*$, so the buyer is indifferent between accepting q_{B1}^2 and rejecting it. As a result, the buyer accepts price no higher than q_{B1}^2 from seller 1. However, (3.15) implies $q_{B1}^2 < v_1$, so seller 1 cannot afford any price that the buyer would accept, therefore seller 1 suggests q_1^2 which is below the buyer's threshold of acceptance, q_{B1}^2, and does not deviate.

The left hand side of (3.13) is seller 1's payoff if he accepts p_1^2. If seller 1 rejects p_1^2, he gets π_1^* with two periods of delay, which is the right hand side of (3.13). Hence, the equation means seller 1 is indifferent between accepting p_1^2 and rejecting it. As a result, seller 1 accepts price no less than p_1^2. However, (3.15) implies that accepting any price above p_1^2 is not profitable for the buyer, therefore the buyer offers p_{B1}^2, which is below seller 1's threshold of acceptance, and does not deviate.

As in (3.14), the buyer is indifferent between accepting q_2^2 from seller 2 and rejecting it. So the buyer would not change her threshold of acceptance, q_2^2, and seller 2 would not change his offer q_2^2.

As in (3.13), seller 2 is indifferent between accepting p_2^2 from seller 2 and rejecting it. So the buyer would not deviate from her threshold of acceptance, p_2^2, and seller 2 would not deviate from his offer p_2^2. ■

Proposition 3.2. *In the two-seller game without commitment, if the mall is profitable as in (3.8) and the sellers' sizes are significantly different as in (3.15), there is a unique equilibrium outcome and a unique bargaining order where the buyer bargains with the smaller seller until an agreement is reached.*

The proof of this lemma is in Appendix C.7.2. It is an extension of the uniqueness proof for the Rubinstein bargaining game.[19] In our model, seller 2 always sells first because the buyer gets a negative payoff otherwise. Given the price accepted by seller 2, everything is known according to the unique equilibrium of Rubinstein bargaining game. As a result, the price for seller 2 is the only parameter to be determined for the set of possible outcomes and therefore for the set of possible payoffs as well. Once the set of possible payoffs is characterized by the single parameter, the rest of the proof is parallel to the proof for Rubinstein bargaining game.

Condition (3.15) requires that, fix the size of the smaller lot, the larger lot should

[19]See Shaked and Sutton (1984).

be sufficiently different. Similar to the discussion of (3.8) after Lemma 3.2, the buyer and seller 1 split a surplus of $\delta\frac{1}{1+\delta}(1-v_2)-v_1$ if seller 1 sells his land first. However, (3.15) implies that the surplus is negative, so seller 1 does not sell first otherwise the buyer would have a negative payoff. Therefore, both the offer and counter-offer would be rejected when the buyer bargains with seller 1 first, and $\hat{\Gamma}(B,1)$ would have a delay of two periods in the equilibria. (3.15) guarantees the unique equilibrium outcome. If this condition is violated, there are multiple equilibrium outcomes with different bargaining orders as will be shown in Case 2.

Let us explain the implications of Lemma 3.3 and Proposition 3.2. Lemma 3.3 characterizes all the equilibria, which share the same outcome $(p_1^1, p_2^2, 2, 1)$ and the same bargaining order $2, 2, \ldots$. Since the equilibrium outcome is unique according to Proposition 3.2, all the strategies are uniquely determined by backward induction except q_1^2 and p_{B1}^2. As a result, Lemma 3.3 also describes all the equilibria of $\hat{\Gamma}(B,2)$.

The game $\hat{\Gamma}(B,1)$ is a proper subgame of $\hat{\Gamma}(B,2)$, so the equilibria and the equilibrium outcome are inherited from Lemma 3.3. In particular, if there is no agreement in period 1 or 2, then the buyer chooses seller 2 to bargain with and seller 2 sells in period 3 and seller 1 sells in period 4. As a result, there is a delay of two periods and the payoffs are

$$\begin{aligned}
\pi_1^{*\prime} &= H_{1,2}+\delta^2\pi_1^* \\
\pi_2^{*\prime} &= H_{2,2}+\delta^2\pi_2^* \\
\pi_B^{*\prime} &= \delta^2\pi_B^*
\end{aligned}$$

It is easy to see that Proposition 3.2 holds when δ is close to 1. Lemma 3.3 implies that delay can happen if the "wrong" order was chosen, which is also true in the game with commitment as discussed after Proposition 3.1. However, it can be avoid if the buyer chooses the bargaining order in which the buyer bargains with the smallest remaining seller until an agreement is reached.

Example 3.2. Consider Example 3.1 in the context of no commitment. Proposition 3.2 implies that the buyer always bargains with seller 2 before the first purchase. The strategies of an equilibrium are given below, and only the strategies before the first purchase are described.

	Accepts	Offers
Seller 1	≥ 0.45	3/4 (rejected)
Seller 2	$\geq 1/5$	13/60
Buyer	$\leq 3/10$ from seller 1	3/8 to seller 1 (rejected)
Buyer	$\leq 13/60$ from seller 2	1/5 to seller 2

The selling prices and payoffs are the same as in Example 1. Suppose the buyer bargains with seller 1 first, then seller 1's offer 3/4 and the buyer's offer 3/8 would be rejected in the first two periods, and the buyer would bargain with seller 2 in the next round.

3.3.2 Case 2: Similar Sizes

When the sellers have similar sizes, (3.15) could be violated. The outcome in Lemma 3.3 is still an equilibrium outcome when (3.15) is violated, but there are other equilibria with different outcomes and bargaining orders. Two other equilibria are described in the following two lemmas.

First, if we exchange seller 1 and 2 in Lemma 3.3, it gives us another set of equilibria.

Lemma 3.4. *Let* $(p_1^{2\prime}, q_1^{2\prime}, p_2^{2\prime}, q_{B2}^{2\prime})$ be the solution to

$$\begin{aligned} p_1 &= H_{1,1} + \delta q_1 \\ \delta\left(1 - p_2^1\right) - q_1 &= \delta\left(\delta\left(1 - p_2^1\right) - p_1\right) \\ p_2 &= H_{2,3} + \delta^3 p_2^1 \\ \delta\left(1 - p_1^1\right) - q_{B2} &= \delta\left(\delta\left(1 - p_2^1\right) - p_1\right). \end{aligned}$$

If (3.8), for any $(q_2^{2\prime}, p_{B2}^{2\prime})$ *such that* $q_2^2 > q_{B2}^{2\prime}$ *and* $p_{B2}^2 < p_2^{2\prime}$*, the strategies below constitute an equilibrium in the two-seller game without commitment:*

i) seller 1 accepts price no less than $p_1^{2\prime}$ *and suggests price* $q_1^{2\prime}$*,* *ii) seller 2 accepts price no less than $p_2^{2\prime}$ and suggests price $q_2^{2\prime}$,*

iii) the buyer bargains with seller 1 before the first purchase; accepts price no more than $q_1^{2\prime}$ from seller 1 and no more than $q_{B2}^{2\prime}$ from seller 2; and suggests $p_1^{2\prime}$ to seller 1 and $p_{B2}^{2\prime}$ to seller 2.

The proof is the same as the proof for Lemma 3.3. In the equilibrium, seller 1 sells in period 1 and seller 2 sells in period 2, and the bargaining order is $1, 1, ...$

Second, there is another set of equilibria in which no agreement is reached in the first two periods.

Lemma 3.5. *In the two-seller game without commitment, if (3.8) is satisfied, the following strategies constitute an equilibrium: the buyer chooses to bargain with seller 1 in the first period; everyone follows the strategies in Lemma 3.3 in* $\hat{\Gamma}(B, 1)$*; everyone follows the strategies in Lemma 3.4 in* $\hat{\Gamma}(B, 2)$.

Proof: Lemma 3.3 implies that the strategies in $\hat{\Gamma}(B, 2)$ in an equilibrium, and Lemma 3.4 implies that the strategies in $\hat{\Gamma}(B, 1)$ is also an equilibrium. By backward induction, the buyer chooses seller 1 to bargain with first because the buyer's payoff in $\hat{\Gamma}(B, 1)$ is higher than in $\hat{\Gamma}(B, 2)$. ■

As a result, there is no agreement in the first two periods and seller 2 and seller 1 sell in period 3 and 4 respectively, and the corresponding bargaining order is $1, 2, 2, ...$

Using the equilibrium payoffs in Lemma 3.3, 3.4 and 3.5 as punishments, we can construct many other equilibria; moreover, there is a continuum of equilibria without delay.[20] However, it is difficult to find the full characterization of equilibria even for the two-seller game. The difficulty is briefly explained below. In order to find the full characterization, we need to find the minimum and maximum for each player's payoffs

[20] A similar analysis is used in the three-person alternating bargaining game. See, for instance, Herrero (1984) and Osborne and Rubinstein (1990). Since the equilibria with delay also demonstrate different bargaining orders as in Lemma 3.4 and Lemma 3.5, they are not presented for the consideration of space.

as in the three-person alternating bargaining game.[21] Both the selling prices and the length of delay could affect the bounds. For example, seller 2's minimum payoff could be reached by a low selling price with shorter delay or by a higher selling price but with longer delay. Moreover, the two factors interact with each other, which makes the problem even harder. Since the maximum length of delay is very likely to be increasing in δ, the difficulty remains even when δ approaches 1.

3.4 N-Seller Game

In this section, all the results in Section 3.2 and 3.3 are extended to the N-seller case by induction on the number of sellers. After the first purchase in the N-seller game, the resulting subgame is a $(N-1)$-seller game, so we only need to study the strategies before the first purchase. Recall that the analysis on the two-seller game also focuses on the strategies before the first purchase, so the analysis in this section is parallel to Section 3.2 and 3.3. However, in order to illustrate the conditions that are needed for our results in the N-seller case, two propositions are presented below while their proofs and the discussion of equilibrium prices and payoffs are included in the Appendices C.7.3 and C.7.4.

Proposition 3.3. *In the N-seller game with commitment, if the mall is profitable as in*

$$\sum_{i=1}^{N}\left(\frac{\delta}{1+\delta}\right)^{N-i} v_i < \left(\frac{\delta}{1+\delta}\right)^{N-1}, \tag{3.16}$$

there is a unique equilibrium outcome and a unique equilibrium bargaining order where the buyer bargains with the smallest remaining seller until an agreement is reached.

Consider the N-seller game without commitment, Proposition 3.4 extends the analysis in Section 3.3.1 to the N-seller case.

[21] See, for instance, Herrero (1984) and Osborne and Rubinstein (1990).

Proposition 3.4. *In the N-seller game without commitment, if the mall is profitable as in (3.16) and the sellers' sizes are significantly different as in*

$$\left(1-\frac{\delta}{\delta+1}\right)(v_{n-1}-v_n)+\sum_{i=1}^{n-1}\left((v_i-v_{i+1})\sum_{j=n-i}^{n-1}\left(\frac{\delta}{1+\delta}\right)^j\right)$$
$$> \left(\frac{\delta}{\delta+1}\right)^{n-1}-v_n\sum_{i=1}^{n}\left(\frac{\delta}{1+\delta}\right)^{i-1} \tag{3.17}$$

for $n=2,...,N$, there is a unique equilibrium outcome and a unique bargaining order where the buyer bargains with the smallest remaining seller until an agreement is reached.

Condition (3.17) ensures that seller i sells first when the remaining sellers are $1,2,...,i$. If it is violated, there could multiple equilibria with different outcomes and different bargaining orders as in Case 2 of the two-seller game. Other equilibria can be constructed similarly as in Lemma 3.4 and 3.5. Besides the difficulties explained in Section 3.3, it is even harder to find a full characterization of the equilibria in the N-seller game. The larger number of sellers allows a much larger number of possible bargaining orders, and there is a large number of possible selling orders given each bargaining order. Moreover, when (3.17) is violated for some but not all n smaller than N, the full characterization is even more challenging.

3.5 Discussion and Applications

3.5.1 Finite vs. Infinite Horizons

If the bargaining game without commitment has a finite horizon, it has a unique equilibrium outcome even when the sellers' sizes are similar. In particular, suppose the game without commitment has a finite horizon of $2T+N-1$ periods where $T>N$.[22] Backward induction finds the unique equilibrium outcome of this game, where the buyer

[22] For the consideration of space, the formal discussion is put in Appendix D.

always bargains with the smallest remaining seller. Moreover, when T goes to infinity, the equilibrium outcome of this game converges to an equilibrium outcome in the game with no commitment and infinite horizon, and the limit outcome is the one associated with the bargaining order of increasing size. In this sense, we say that although the game without commitment could have multiple equilibrium outcomes when the sellers are very similar, only one outcome is "consistent" with the equilibrium outcome in game of finite horizon, and this outcome is associated with the bargaining order of increasing size.

3.5.2 Inside vs. Outside Options

It is natural to use inside options-ongoing profits from farming to represent the sizes of sellers. However, outside options can also be used to represent the different sizes of sellers. When a seller receives an offer, he can also sell his land in an outside market besides accepting and rejecting the offer. In this setting, our qualitative results would not be affected. However, if a seller sells his land in the market, the on-going bargaining ends, but since the buyer still needs the land, he would start a new bargaining with the new owners, which makes the situation more complicated. As a result, this chapter focuses only on the bargaining game with inside options.

3.5.3 Cash-Offer vs. Contingent Contracts

This chapter considers cash-offer contracts, which are prevalent in real estate business. However, there are other bargaining situations where contingent contracts could be used. Under a contingent contract, the payments are not made until all the sellers have agreed. In contrast to our paper, if the buyer uses contingent contracts in our model, the surplus remains the same after the agreements. Therefore, multiple equilibria with different bargaining orders could arise,[23] and our results do not hold.

[23] See also Li (2010)

3.5.4 Simultaneous vs. Sequential Offers

In many situations, it is hard or impossible for the buyer to make simultaneous offers to the sellers. However, if the buyer can make simultaneous offers to all the sellers, the sellers agree in the first period because this game is of complete information. Therefore, our results do not apply.

3.5.5 Applications and Extensions

Besides land purchasing and the two other examples in the introduction, the model is also applicable to voting scenarios. For example, when a country wants to join a trade organization, it has to get permission from all the respective existing members. The members have different attitudes toward the entry, and the member that likes the entry least corresponds to the seller with largest size in the model. As a result, the applicant should start with the member who favors the entry most.

One of the most interesting extensions is to make the sizes the sellers' private information and examine the equilibrium bargaining order. Moreover, the model has the potential for more applications under some other modifications. For example, it can be extended to the case where the buyer needs not to purchase from all the sellers.[24] This modification accommodates the situation when the developer can change[25] the shape of the mall. Besides, it also fits the voting situations where winning requires not only a minimum number of votes but also all the votes from voters with veto rights. Moreover, it would be interesting to explore the situation where the bargaining order or the offers could be confidential,[26] as this chapter only presents a game of complete information.

[24] Chowdhury and Sengupta (2008) also consider this feature.

[25] For example, Yardley (2008) reports a boutique supermarket was altered from its original design and built around a tiny house whose owner refused to sell.

[26] See also Noe and Wang (2004) and Chowdhury and Sengupta (2008).

3.6 Conclusion

This chapter studies a non-cooperative, complete-information and infinite-horizon bargaining game with one buyer and multiple sellers heterogeneous in their sizes. The bargaining order in this game is endogenously determined, and three different cases are considered. First, when the buyer can commit to a bargaining order, there is a unique equilibrium outcome where the buyer bargains with the smallest remaining seller until an agreement is reached. Second, when the buyer cannot commit to a bargaining order and the sellers are significantly different, there is also a unique equilibrium outcome associated with the same bargaining order. Third, when the sellers are similar to each other, the game without commitment still has an equilibrium with the same bargaining order, but there could be other equilibrium outcomes with different bargaining orders. However, only the equilibrium outcome, associated with the order of increasing size, is consistent with the equilibrium outcome in the finite-horizon bargaining game with no commitment.

Appendix A

Proofs for Chapter 1

A.1 Preliminaries

This appendix proves two important results, Claims A.12 and A.15, which are used to prove the unique solution of (1.1) and (1.2).

Claim A.1 *$W(\mathbf{G}_{-i}, \mathbf{v})$ is symmetric in the variables of $\mathbf{G}_{-i}$; it is linear in $\mathbf{v}$; and it is strictly increasing in each variable of $\mathbf{G}_{-i}$, G_j, if $G_j \in (0,1)$.*

Proof. It is easy to see that $W(\mathbf{G}_{-i}, \mathbf{v})$ is symmetric in the variables in $\mathbf{G}_{-i}$, and that it is linear in $\mathbf{v}$.

To see that W is increasing in G_j, notice

$$W(\mathbf{G}_{-i}, \mathbf{v}) = G_j W(\mathbf{G}_{-i,j}, \mathbf{v}_{-k'}) + (1 - G_j) W(\mathbf{G}_{-i,j}, \mathbf{v}_{-k''})$$

where k' is the lowest prize in $\mathbf{v}$ and k'' is the highest prize in $\mathbf{v}$

$$\begin{aligned} dW/dG_j &= W(\mathbf{G}_{-i,j}, \mathbf{v}_{-k'}) - W(\mathbf{G}_{-i,j}, \mathbf{v}_{-k''}) \\ &= W(\mathbf{G}_{-i,j}, \mathbf{v}_{-k'} - \mathbf{v}_{-k''}) \\ &= \sum_{l=k''}^{k'-1} (v^l - v^{l+1}) P_i^l(s) \end{aligned}$$

where $P_i^l(s)$ is i's probability of winning the lth prize if i chooses s and the players in $\mathcal{N}\backslash\{i,j\}$ choose strategies in $\mathbf{G}_{-i,j}$. Since $P_i^l(s)$ is non-negative and $v^l - v^{l+1} > 0$, $dW/dG_j > 0$ where the strict inequality comes from the fact that $P_i^l(s)$ cannot be 0 for all l. ■

$\mathbf{D}_j \equiv \mathbf{1}_j\mathbf{1}_j' - \mathbf{I}_j$, where $\mathbf{1}_j$ is a j-dimensional vector of ones. The diagonal entries of $\mathbf{D}_j$ are zeros, and all the other entries are 1. $\mathbf{B}_j$ is $\mathbf{D}_j$ with the entry at position $(1,1)$ replaced with 1.

Claim A.2 $\det \mathbf{D}_j = (j-1)(-1)^{j-1}$.

Proof. We use induction in this proof. When $j = 3$, it is easy to see that

$$\det \mathbf{D}_j = (j-1)(-1)^{j-1} \tag{A.1}$$

$$\det \mathbf{B}_j = (-1)^{j-1} \tag{A.2}$$

Suppose the two equations above are true for $j-1$. Expand $\det \mathbf{D}_j$ according the first column, we get a sum of $j-1$ terms of alternating signs. For the j_1th term, put its j_1th column to left and move columns 1 to $j_1 - 1$ one position to the right. Then, each term is $-\det \mathbf{B}_{j-1}$, and we have

$$\det \mathbf{D}_j = -(j-1)\det \mathbf{B}_{j-1} \tag{A.3}$$

Expand $\det \mathbf{B}_j$ according to the first column, we get a sum of j terms of alternating signs. For the (j_1+1)th term and $1 \leq j_1 \leq j-1$, put its j_1th column to left and move columns 1 to $j_1 - 1$ one position to the right. Then, each of the last $j-1$ term is $-\det \mathbf{B}_{j-1}$, so

$$\begin{aligned} \det \mathbf{B}_j &= \det \mathbf{D}_{j-1} - (j-1)\det \mathbf{B}_{j-1} \\ &= \det \mathbf{D}_{j-1} - \det \mathbf{D}_j \end{aligned} \tag{A.4}$$

where the second equality comes from (A.3). Therefore, (A.3) and (A.4) imply (A.1) and (A.2) are also true for j. ■

If a non-zero entry of $\mathbf{D}_j$ is replace with 0, then we say that the resulting matrix has an *off-diagonal zero* at position (j_1, j_2).

Denote $\mathcal{M}_j$ as the set of all $j \times j$ matrices such that i) it has at most j off-diagonal zeros, ii) each column has at most one off-diagonal zero.

Claim A.3 *If i) $\mathbf{A}_j \in \mathcal{M}_j$, ii) it has j off-diagonal zeros and iii) each row has an off-diagonal zero,* $\det \mathbf{A}_j$ *is* 0 *or has sign* $(-1)^{j-1}$.

Proof. Suppose the off-diagonal zero in row 1 is at column j_2, where $j_2 \neq 1$. Add all other rows to row 1 and divide it by $j-2$, then we get a row of ones. It easy to see that column j_2 does not have an off-diagonal zero. Suppose the off-diagonal zero in row j_2 is at column j_3.

Deduct column j_2 from column j_3, then column j_3 becomes zeros except -1 in row j_3.

Expand the determinant according to column j_3, we get $-\det \mathbf{A}^1_{j-1}$ where $\mathbf{A}^1_{j-1}$ is a $(j-1) \times (j-1)$ matrix with ones in the first row[1]. Moreover, two column of $\mathbf{A}^1_{j-1}$ has no off-diagonal zeros and any other column has one off-diagonal zero. Suppose the two columns without off-diagonal zeros are column j'_1 and j'_2.

Multiply row 1 by $j-3$ and deduct all others rows from it, then the first row has two off-diagonal zeros at column j'_1 and j'_2. Hence the resulting matrix is in $\mathcal{M}_{j-1}$, so its determinant is either 0 or of the sign $(-1)^{j-2}$. As a result, $-\det \mathbf{A}^1_{j-1}$ is 0 or has sign $(-1)^{j-1}$. ■

Claim A.4 *If i) $\mathbf{A}_j \in \mathcal{M}_j$, ii) it has j off-diagonal zeros, iii) at least one row has no off-diagonal zero,* $\det \mathbf{A}_j$ *is* 0 *or has the sign* $(-1)^{j-1}$.

Proof. Denote the row without an off-diagonal zero as row j_1. Suppose row j_2 is a row with an off-diagonal zero. Add all the other rows to row j_2, then divide it by $j-1$, then row j_2 only has ones.

[1] The rest of the Appendix uses elementary operation of matrix, and we use supscripts to index the matrices in a sequence of such operations.

Deduct row j_1 from row j_2, we get a row of zeros except 1 at column j_1.

Expand the determinant according to row j_2, we have $(-1)^{j_1+j_2} \det \mathbf{A}^1_{j-1}$ where $\mathbf{A}^1_{j-1}$ is the (j_2, j_1) minor matrix of $\mathbf{A}_j$.

It is easy to see that $j_1 \neq j_2$. Suppose $j_1 > j_2$. Move column j_2 of $\mathbf{A}_{j-1}$ to the left and shift the column 1 to $j_2 - 1$ to the right by one position. We have $(-1)^{j_1+j_2} (-1)^{j_2-1} \det \mathbf{A}^2_{j-1}$. Move row $j_1 - 1$ to the top and shift all the rows above row $j_1 - 1$ down by one position, we have $(-1)^{j_1+j_2} (-1)^{j_2-1} (-1)^{j_1-2} \det \mathbf{A}^3_{j-1} = -\det \mathbf{A}^3_{j-1}$. The first row of $\mathbf{A}^3_{j-1}$ only has ones, and each column has at most one off-diagonal zero. If $j_1 < j_2$, we get the same result similarly.

Multiply row 1 of $\mathbf{A}^3_{j-1}$ by $j - 3$ and deduct all the other rows from it, the resulting matrix has one off-diagonal zero in each column. Therefore, this matrix is in $\mathcal{M}_{j-1}$, and has a determinant that is either 0 or of the sign $(-1)^{j-2}$. Since $\det \mathbf{A}_j$ has the opposite sign of the determinant of the resulting matrix, $\det \mathbf{A}_j$ is 0 or has the sign $(-1)^{j-1}$. ∎

Off-Diagonal Condition: Column j_1 has an off-diagonal zero if there is a column with an off-diagonal zero in row j_1.

Claim A.5 *If i) $\mathbf{A}_j \in \mathcal{M}_j$, ii) it has less than j off-diagonal zeros, iii) it does not satisfy the off-diagonal condition,* $\det \mathbf{A}_j$ *is* 0 *or has the sign* $(-1)^{j-1}$.

Proof. Since $\mathbf{A}_j$ does not satisfy the off-diagonal condition, there is a column, j_2, such that i) column j_2 has an off-diagonal zero in row j_1, ii) column j_1 has no off-diagonal zero.

Deduct column j_2 from column j_1, and and then column j_1 becomes zeros except a 1 in row j_2.

The following analysis is similar to Claim A.3. Expand the determinant according to column j_1, and then we have $(-1)^{j_1+j_2} \det \mathbf{A}^1_{j-1}$, where $\mathbf{A}^1_{j-1}$ is the (j_2, j_1) minor matrix of $\mathbf{A}_j$. Recall that $j_2 \neq j_1$, so first consider $j_2 < j_1$. Move row $j_1 - 1$ to the top and column $j_2 - 1$ to the right. The determinant becomes $-\det \mathbf{A}^2_{j-1}$. Note that the entry at $(1, 1)$ in $\mathbf{A}^2_{j-1}$ is the entry at (j_1, j_2) in $\mathbf{A}_j$, which is 0 by assumption.

Therefore, $\mathbf{A}_{j-1}^2 \in \mathcal{A}_{j-1}$, so $\det \mathbf{A}_j$ is either 0 or of the sign $(-1)^{j-1}$. If $j_1 > j_2$, we can get the same result similarly. ■

Claim A.6 *If i)* $\mathbf{A}_j \in \mathcal{M}_j$, *ii) it has less than* j *off-diagonal zeros, iii) it satisfies the off-diagonal condition,* $\det \mathbf{A}_j$ *is* 0 *or has the sign* $(-1)^{j-1}$.

Proof. First, we claim that column j_1 has an off-diagonal zero if row j_1 has an off-diagonal zero. To see why, suppose otherwise. Then, row j_1 has an off-diagonal zero at column j_2 and column j_1 does not. Column j has an off-diagonal zero in row j_1, and column j_1 has no off-diagonal zero, which contradicts the off-diagonal condition.

As a result, if column j_1 has no off-diagonal zero, row j_1 has no off-diagonal zero. Denote $\mathcal{J}$ as $\{1, 2, ..., j\}$ and $\mathcal{H}$ as the columns with an off-diagonal zero, then $\mathcal{J} \backslash \mathcal{H}$ is a set of rows without off-diagonal zeros.[2]

Pick any row with an off-diagonal zero and add all the other rows with off-diagonal zeros to it. The resulting row is either $\hat{\jmath}$ or $\hat{\jmath} - 1$, where $\hat{\jmath}$ is the number of rows with off-diagonal zeros. Moreover, in this row, $\hat{\jmath} - 1$ is at the columns in $\mathcal{H}$ and $\hat{\jmath}$ is at the columns in $\mathcal{J} \backslash \mathcal{H}$.

Pick a row in $\mathcal{J} \backslash \mathcal{H}$ and add the rest to this row, the resulting row has entries equal $\bar{\jmath}$ or $\bar{\jmath} - 1$, where $\bar{\jmath} = \# (\mathcal{J} \backslash \mathcal{H})$. Moreover, $\bar{\jmath}$ is in the columns in $\mathcal{H}$, and $\bar{\jmath} - 1$ is in the columns in $\mathcal{J} \backslash \mathcal{H}$.

Add the row with $\hat{\jmath}$ and $\hat{\jmath} - 1$ to the one with $\bar{\jmath}$ and $\bar{\jmath} - 1$, and divide it by $\bar{\jmath} + \hat{\jmath} - 1$. The resulting row has only ones.

This row of ones replaces a row with one off-diagonal zero. Deduct a row without an off-diagonal zero from this row of ones, then we get a row of zeros except one entry as 1. Similar to Claim 3 and 4, if we expand the determinant according to this row and move some rows and columns, $\det \mathbf{A}_j$ becomes $-\det \mathbf{A}_{j-1}$, where $\mathbf{A}_{j-1} \in \mathcal{M}_{j-1}$. Hence, $\det \mathbf{A}_j$ *is* 0 *or has the sign* $(-1)^{j-1}$. ■

Claim A.7 *If* $\mathbf{A}_j \in \mathcal{M}_j$, $\det \mathbf{A}_j = 0$ *or* $\det \mathbf{A}$ *has the sign* $(-1)^{j-1}$.

Proof. By induction.

[2] There might be more than one such set.

It is easy to verify that the statement is true for $j' = 2$. Suppose the statement is true for $j' = j - 1$, Claim A.3 to A.6 show that it is also true for $j' = j$. Therefore the claim is true if integer j is bigger than 1. ■

Claim A.8 *$\det H_j$ has sign $(-1)^{j-1}$, where H_j is a $j \times j$ matrix with zero diagonal entries and $h_{j_1,j_2} = \sum_{l=1}^{j} h_l - h_{j_1} - h_{j_2}$, where $h_l > 0$ for any l.*

Proof. Column 1 of H_j is a sum of $j-1$ vectors, $\sum_{l=2}^{j} h_l \mathbf{1}_{-1,-l}$, where $\mathbf{1}_{-j_1,-j_2}$ is a column vector with ones except two zeros in row j_1 and j_2. Therefore, $\det H_j = \sum_{l=2}^{j} \det H_l^1$, where H_l^1 is a $j \times j$ matrix with column 1 as $h_l \mathbf{1}_{-1,-l}$ and the other columns the same as in H_j. Note that column 1 in H_l^1 only contains 0 or h_l.

For any H_l^1, its second column is $\sum_{l=1,3,...,j} h_l \mathbf{1}_{-2,-l}$, so $\det H_l^1$ also equals a sum of $j-1$ determinants of $j \times j$ matrices. Moreover, the first two columns of these matrices only have one h_l.

Repeat this step for the other columns until $\det H_j$ become a sum of determinants of $j \times j$ matrices that have zero or the same h_l in each column. Moreover, these determinants have other properties. First, if column j_2 of these matrices has only h_l, then it is $h_l \mathbf{1}_{-l,-j_2}$. Second, column j_2 cannot have h_{j_2}.

Denote $\mathcal{J} = \{1, ..., j\}$ and

$$\mathfrak{K}_j = \left\{ (\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_j) \mid \mathcal{J}_l \cap \mathcal{J}_{l'} = \varnothing,\ \cup_{l=1}^{j} \mathcal{J}_l = \mathcal{J},\ \text{and } \#\mathcal{J}_l < j \right\}$$

$(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_j)$ is a j-set partition of $\mathcal{J}$ except that $\mathcal{J}_l$ can be empty. For any $(\mathcal{J}_1, ..., \mathcal{J}_j) \in \mathfrak{K}_j$, replace the entry of (j_1, j_2) in $\mathbf{D}_j$ with 0 if $j_2 \in \mathcal{J}_{j_1}$, and denote the resulting matrix as $\mathbf{A}_{\mathcal{J}_1,...,\mathcal{J}_j}$. $\det H_j$ is a polynomial of order j, and each term has the same order. That is

$$\det \mathbf{H}_j = \sum_{(\gamma_1,...,\gamma_j)} \left(\eta_{\gamma_1,...,\gamma_j} \prod_{l=1}^{j} h_l^{\gamma_l} \right)$$

where the sum is over the set $\left\{ (\gamma_1, ..., \gamma_j) \mid \gamma_l < j, \ \sum_{l=1}^{j} \gamma_l = j \text{ and } \gamma_l \in \mathbb{Z}_+ \right\}$. Denote $\mathfrak{K}_{\gamma_1,...,\gamma_j} = \{(\mathcal{J}_1, \mathcal{J}_2, ..., \mathcal{J}_j) \in \mathfrak{K}_j \mid \#\mathcal{J}_l = \gamma_l\}$. Then, $\eta_{\gamma_1,...,\gamma_j} = \sum_{(\mathcal{J}_1,...,\mathcal{J}_j)} \det \mathbf{A}_{\mathcal{J}_1,...,\mathcal{J}_j}$, where the sum is over the set $\mathfrak{K}_{\gamma_1,...,\gamma_j}$.

For each $(\mathcal{J}_1, ..., \mathcal{J}_j)$ in $\mathfrak{K}_{\gamma_1,...,\gamma_j}$, $\mathbf{A}_{\mathcal{J}_1,...,\mathcal{J}_j}$ is in $\mathcal{A}_j$, so $\eta_{\gamma_1,...,\gamma_j}$ is either 0 or has sign $(-1)^{j-1}$ by Claim A.7. As a result, $\det H_j$ either is 0 or has sign $(-1)^{j-1}$. Now we are going to show $\det H_j \neq 0$ by proving that one of the coefficients is not zero.

Consider the coefficient of $h_1^{j-2} h_2 h_3$ in $\det H_j$, it is a sum of determinants and one of them is associated with $(\mathcal{J}_1, ..., \mathcal{J}_j)$ such that $\mathcal{J}_2 = \{3\}, \mathcal{J}_3 = \{1\}$ and $\mathcal{J}_1 = \mathcal{J} \setminus (\mathcal{J}_2 \cup \mathcal{J}_3)$. Such determinant has j off-diagonal zeros: one at column 3 to row 2, another at column 1 row 3 and all the others are in the first row. The resulting matrix has zeros in the first row except a 1 in column 3. Expand this determinant according to the first row, we get a $(j-1) \times (j-1)$ determinant. Switch the first two row and we get $-\det \mathbf{D}_{j-1}$ which is not zero. ■

Claim A.9 $\det \hat{H}_j$ *has sign* $(-1)^{j-1}$, *where* $\hat{H}_j$ *is a* $j \times j$ *matrix with zeros diagonal entries and* $\hat{h}_{j_1,j_2} = \sum_{l=1}^{j'} h_l - h_{j_1} - h_{j_2}$, *where* $h_l > 0$ *for any* l *and* $j' \geq j$.

Proof. Add $\sum_{l=j}^{j'} h_l$ to the entries off the diagonal of H_j, we have $\hat{H}_j$. Denote $h'_l = h_l + \frac{1}{j-2} \sum_{l=j}^{j'} h_l > 0$, then $\hat{h}_{j_1,j_2} = \sum_{l=1}^{j} h'_l - h'_{j_1} - h'_{j_2}$ and the previous claim implies that $\det \hat{H}_j$ *has sign* $(-1)^{j-1}$. ■

Let us introduce some notations before we move to the next claim.

For a given interval, let $\mathcal{A}$ be the active players and $\mathcal{P}$ be the participating players, $\mathcal{A} \subset \mathcal{P}$. Then for any s in this interval, equilibrium strategies $(G_i)_{i \in \mathcal{A}}$ solve

$$W(\mathbf{G}_{-i}, \mathbf{v}) - c_i s = u_i \qquad \text{for } i \in \mathcal{A} \tag{A.5}$$

where $\mathbf{G} = (G_i)_{i\in\mathcal{P}}$ and $\mathbf{v} = (v^k)_{k\in\mathcal{P}}$.[3]

Suppose $\mathcal{P} = \{1, 2, ..., j'\}$. If $\mathcal{P} \neq \{1, 2, ..., i'\}$, we can order the elements and rename them, and the argument below would be the same. If $\mathbf{v} \neq \left(v^k\right)_{k\in\mathcal{P}}$, we can also rename the prizes similarly, and the analysis below applies as well.

For any $i' \in \mathcal{A}\backslash\{i\}$, take derivatives of both sides of (A.5) with respect to $G_{i'}$, we have

$$\sum_{l\in\mathcal{A}\backslash\{i,i'\}} \frac{dW\left(\mathbf{G}_{-i},\mathbf{v}\right)}{dG_l}\frac{dG_l}{dG_{i'}} = -\frac{dW\left(\mathbf{G}_{-i},\mathbf{v}\right)}{dG_{i'}} \quad \text{for } i \neq i' \tag{A.6}$$

$$\sum_{l\in\mathcal{A}\backslash\{i'\}} \frac{dW\left(\mathbf{G}_{-i'},\mathbf{v}\right)}{dG_l}\frac{dG_l}{dG_{i'}} = 0 \tag{A.7}$$

We can write (A.6) and (A.7) into matrix form

$$\begin{aligned} \hat{\mathbf{W}}_{j-1}\boldsymbol{\delta} &= -\mathbf{d} \\ \mathbf{d}'\boldsymbol{\delta} &> \mathbf{0} \end{aligned} \tag{A.8}$$

where $j = \#\mathcal{A}$ and $\hat{\mathbf{W}}_{j-1}$ is a $(j-1)\times(j-1)$ matrix, $\boldsymbol{\delta}$ and $\mathbf{d}$ are vectors of $j-1$ rows. The diagonal entries of $\hat{\mathbf{W}}$ are zero and the entry at (j_1, j_2) is $dW\left(\mathbf{G}_{-j_1},\mathbf{v}\right)/dG_{j_2}$; the element in row j_1 of $\boldsymbol{\delta}$ is $dG_{j_1}/dG_{i'}$ for $j_1 \neq i'$;the element in row j_1 of $\mathbf{d}$ is $dW\left(\mathbf{G}_{-j_1},\mathbf{v}\right)/dG_{i'}$ for $j_1 \neq i'$. Define $W_j = \begin{pmatrix} \hat{\mathbf{W}}_{j-1} & \mathbf{w} \\ \mathbf{w} & 0 \end{pmatrix}$ and w_{j_1,j_2} as the entry in row j_1 and column j_2 of $\mathbf{W}_j$.

Define first order difference as $\Delta_k^{(1)} = v^{k-1} - v^k$, for $k = 2, ..., j'$. The lth order difference is $\Delta_k^{(l)} = \Delta_{k-1}^{(l-1)} - \Delta_k^{(l-1)}$ for $l = 1, ..., j'-1$, and $k = l+1, ..., j'$.

Claim A.10

$$w_{j_1,j_2} = \Delta_j^{(1)} + \sum_{l'=2}^{j'-1}\Delta_{j'}^{(l')}\left(\sum_{\left\{i_1,...,i_{l'-1}\right\}\subset\Gamma_{j_1,j_2}}\left(\prod_{l=1}^{l'-1}G_{i_l}\right)\right)$$

where $j_1 \neq j_2$ *and* $\Gamma_{j_1,j_2} = \{G_1, ..., G_{j'}\} / \{G_{j_1}, G_{j_2}\}$.

[3]Since the indices of $\mathbf{G}$ and $\mathbf{v}$ in $W\left(\mathbf{G}_{-i},\mathbf{v}\right)$ are the same below, we sometime only mention the indices for $\mathbf{G}$.

Proof. We are going to prove by induction. First, it is easy to verify the statement is true for $j' = 3$. Suppose the statement is true for $j' = j'_I - 1$, we are going to show that it is also true for $j' = j'_I$.

For the purpose of cleaner exhibition, the following proof only focuses on w_{12}.

We know that

$$\begin{aligned} w_{12} &= W\left(G_3, ..., G_{j'_I}, \Delta_2^{(1)}, ..., \Delta_{j'_I}^{(1)}\right) \\ &= G_{j'_I} W\left(G_3, ..., G_{j'_I-1}, \Delta_2^{(1)}, ..., \Delta_{j'_I-1}^{(1)}\right) + \left(1 - G_{j'_I}\right) W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right) \\ &= W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right) \\ &\quad + G_{j'_I}\left(W\left(G_3, ..., G_{j'_I-1}, \Delta_2^{(1)}, ..., \Delta_{j'_I-1}^{(1)}\right) - W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right)\right) \\ &= W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right) \\ &\quad + G_{j'_I} W\left(G_3, ..., G_{j'_I-1}, \Delta_2^{(1)} - \Delta_3^{(1)}, \Delta_3^{(1)} - \Delta_4^{(1)}, ..., \Delta_{j'_I-1}^{(1)} - \Delta_{j'_I}^{(1)}\right) \\ &= W\left(G_3, ..., G_{k'_I-1}, \Delta_3^{(1)}, ..., \Delta_{k'_I}^{(1)}\right) + G_{k'_I} W\left(G_3, ..., G_{k'_I-1}, \Delta_3^{(2)}, ..., \Delta_{k'_I}^{(2)}\right) \end{aligned} \tag{A.9}$$

Since the statement is true for $k' = j'_I - 1$, we have

$$\begin{aligned} & W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(1)}, ..., \Delta_{j'_I}^{(1)}\right) \\ = \; & \Delta_{j'_I}^{(1)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l')} \left(\sum_{\left\{i_1,...,i_{l'-1}\right\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \end{aligned} \tag{A.10}$$

$$\begin{aligned} & W\left(G_3, ..., G_{j'_I-1}, \Delta_3^{(2)}, , ..., \Delta_{j'_I}^{(2)}\right) \\ = \; & \Delta_{j'_I}^{(2)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l'+1)} \left(\sum_{\left\{i_1,...,i_{l'-1}\right\} \subset \Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \end{aligned} \tag{A.11}$$

where $\Gamma'_{12} = \left\{G_1, ..., G_{j'_I-1}\right\} / \left\{G_1, G_2\right\}$ and

Substitute (A.10) and (A.11) into (A.9), then we have

$$\begin{aligned} w_{12} &= \Delta_{j'_I}^{(1)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l')} \left(\sum_{\left\{i_1,\ldots,i_{l'-1}\right\}\subset\Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \\ &\quad + G_{j'_I} \left(\Delta_{j'_I}^{(2)} + \sum_{l'=2}^{j'_I-2} \Delta_{j'_I}^{(l'+1)} \left(\sum_{\left\{i_1,\ldots,i_{l'-1}\right\}\subset\Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \right) \end{aligned}$$

therefore the coefficient of $\Delta_{j'_I}^{(j')}$ is

$$\begin{aligned} &\left(\sum_{\left\{i_1,\ldots,i_{l'-1}\right\}\subset\Gamma'_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) + G_{j'_I} \left(\sum_{\left\{i_1,\ldots,i_{l'-2}\right\}\subset\Gamma'_{12}} \left(\prod_{l=1}^{l'-2} G_{i_l} \right) \right) \\ = &\left(\sum_{\{i_1,\ldots,i_{i'-1}\}\subset\Gamma'_{12}\cup\left(G_{j'_I}\right)} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right) \end{aligned}$$

As a result, $w_{12} = \Delta_{j'}^{(1)} + \sum_{l'=2}^{j'-1} \Delta_{j'}^{(l')} \left(\sum_{\left\{i_1,\ldots,i_{l'-1}\right\}\subset\Gamma_{12}} \left(\prod_{l=1}^{l'-1} G_{i_l} \right) \right)$.

Similarly, we can extend the analysis above to w_{j_1,j_2} for $j_1 \neq j_2$. Hence, the statement is also true for j'_I. ■

Under the assumption of QPS, $\Delta_{j'}^{(l)} = 0$ for $l > 2$. Therefore, both W_j and $\hat{\mathbf{W}}_j$ are simplified, and

$$\begin{aligned} w_{j_1,j_2} &= \left(\sum_{l=1}^{j'} G_l - G_{j_1} - G_{j_2} \right) \left(v^{j'-2} - 2v^{j'-1} \right) + \left(v^{j'-1} - v^{j'} \right) \\ &= \left(\sum_{l=1}^{j'} G_l - G_{j_1} - G_{j_2} \right) \Delta_{j'}^{(2)} + \Delta_{j'}^{(1)} \end{aligned}$$

if $j_1 \neq j_2$.

Claim A.11 $\det W_j$ *and* $\det \hat{\mathbf{W}}_j$ *have sign* $(-1)^{j-1}$ *if the prizes satisfy QPS and* $G_l > 0$ *for* $l \in \mathcal{P}$.[4]

Proof. First, suppose $\mathcal{A} = \mathcal{P}$, so $j = j'$.

$\det W_j = \left(\Delta_{j'}^{(1)}\right)^j \det \mathbf{Z}_j$, where $z_{j_1,j_2} = \sum_{l=1}^{j} h_l - h_{j_1} - h_{j_2}, h_l = G_l \Delta_{j'}^{(2)} / \Delta_{j'}^{(1)} + \frac{1}{j-2}$.

Assume any $G_{i'}$ equals G_i for $i' \in \mathcal{P} \backslash \{i, j\}$, we have $dW_i/dG_j = (j-2)\,\Delta_{j'}^{(2)} G_i + \Delta_{j'}^{(1)} > 0$, where the inequality comes from Claim A.1. As a result, $h_i = G_i \Delta_{j'}^{(2)} / \Delta_{j'}^{(1)} + \frac{1}{j-2} > 0$.

Claim A.8 implies $\det \mathbf{Z}_j$ is of the sign $(-1)^{j-1}$, and so it is $\det W_j$.

Second, suppose $\mathcal{A} \subsetneq \mathcal{P}$, then we have $j < j'$.

$$w_{j_1,j_2} = \left[\Delta_{j'}^{(2)} \left(\sum_{l \in \mathcal{A}} G_l - G_{j_1} - G_{j_2}\right) + \Delta_{j'}^{(1)}\right] + \Delta_{j'}^{(2)} \sum_{l \in \mathcal{P} \backslash \mathcal{A}} G_l$$

if $j_1 \neq j_2$ and $j_1, j_2 \in \mathcal{A}$.

$\det W_j = \left(\Delta_{j'}^{(1)}\right)^j \det \mathbf{Z}_j$, where $z_{j_1,j_2} = \sum_{l \in \mathcal{A}} h_l - h_{j_1} - h_{j_2}$, $h_l = G_l \Delta_{j'}^{(2)} / \Delta_{j'}^{(1)} + \frac{y}{j-2}$, $y = 1 + \left(\Delta_{j'}^{(2)} / \Delta_{j'}^{(1)}\right) \sum_{l \in \mathcal{P} \backslash \mathcal{A}} G_l$. $\det \mathbf{Z}_j$ has sign $(-1)^{j-1}$ according to Claim A.8.

Let us consider $\det \hat{\mathbf{W}}_j$. Suppose $\mathcal{A}' = \mathcal{A} \backslash \{i'\}$ and consider (A.1) with $\mathcal{A}'$ and $\mathcal{P}$. The corresponding W_{j-1} of the new system is just $\hat{\mathbf{W}}_{j-1}$ in the original system. Therefore, $\det \hat{\mathbf{W}}_{j-1}$ has sign $(-1)^{j-2}$ according to Claim A.8. ■

Claim A.12 *Suppose the prizes satisfies QPS. For any* $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$ *and* $i \in \mathcal{A}$*, LHS of (A.5) for* i *decreases if* G_i *increases in other equations of (A.5).*

Proof. Suppose i is the weakest player in $\mathcal{A}$. Claim A.11 shows that $\det W_j$ has sign $(-1)^{j-1}$ and $\det \hat{W}_{j-1}$ has sign $(-1)^{j-2}$, then $\hat{W}_{j-1}$ is invertible and

$$\mathbf{d}'\mathbf{g} = -\mathbf{d}' \hat{\mathbf{W}}_{j-1}^{-1} \mathbf{d} = \det \mathbf{W}_j / \det \hat{\mathbf{W}}_{j-1} < 0$$

[4] This claim may fail if the prizes are not QPS or GPS. Consider a four-player contest with prizes $v_1 = 7, v_2 = 2, v_3 = 1$ and $v_4 = 0$. When G_1 and G_2 are close to 0, G_3 and G_4 are close to 1, $\det \mathbf{F}_4$ is close to 5.

Therefore, we have $\hat{W}_{j-1}\mathbf{g}= -\mathbf{d}$ and $\mathbf{d}'\mathbf{g}> \mathbf{0}$, so the claim is true for $i = a'$.

Since players in $\mathcal{A}$ are symmetric in this problem, the claim is also true for other players in $\mathcal{A}$. ■

Now consider geometric prize sequences.

Claim A.13 *$\det H_j$ has sign $(-1)^{j-1}$, where H_j is a $j \times j$ matrix with zeros diagonal entries and $h_{j_1,j_2} = \left(\prod_{l=1}^{j'} h_l\right) / (h_{j_1} h_{j_2})$ with $h_l > 0$ for any l and $j' \geq j$.*

Proof. Multiply row $j_1 > 1$ by h_{j_1}.

Divided column j_2 by $\left(\prod_{l=1}^{j'} h_l\right) / h_{j_2}$. Let us describe the resulting matrix. First, the entries in the first row are $1/h_1$ except a zero at the first column. Second, the diagonal entries are zero. Third, $h_{j_1,j_2} = 1$ for $j_1 > 1$ and $j_1 \neq j_2$.

Multiply the first row by h_1, we get $\det \mathbf{D}_j = (j-1)(-1)^{j-1}$ by Claim A.2. ■

Claim A.14 *$\det W_j$ and $\det\hat{W}_j$ have sign $(-1)^{j-1}$ if the prizes satisfies GPS and $G_l > 0$ for $l \in \mathcal{P}$.*

Proof. We can verify that $\Delta_{j'}^{(l)} = (\alpha-1)^l v^j$, so $w_{j_1,j_2} = \prod_{l\in\mathcal{P}\setminus\{j_1,j_2\}} ((\alpha-1) G_l + 1)$. Denote $h_l = (a-1) G_l + 1$, and $h_l > 0$ since $\alpha > 1$ and $0 < G_l$. Therefore, Claim A.13 implies $\det W_j$ has sign of $(-1)^{j-1}$.

Similar to the case of QPS, $\det\hat{W}_{j-1} = \det W_{j-1}$ for $\mathcal{A}'= \mathcal{A}\setminus\{i'\}$ and $\mathcal{P}$, so $\det\hat{W}_{j-1}$ has sign $(-1)^{j-2}$. ■

Claim A.15 *Suppose the prizes satisfies GPS, then for any subset $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$ and $i \in \mathcal{A}$, LHS of (A.5) for i decreases if G_i increases in other equations of (A.5).*

Proof. Given the previous claim, the proof is the same as Claim A.12. ■

Claim A.16 *(A.5) has at most one solution in $[0,1]^{\#\mathcal{A}}$.*

Proof. Suppose there are two sets of solutions, $\left(\tilde{G}_i\right)_{i\in\mathcal{A}}$ and $(\hat{G}_i)_{i\in\mathcal{A}}$. Since the solutions are different, suppose $\tilde{G}_i(s_0) > \hat{G}_i(s_0)$ without loss of generality. Therefore Claims A.12 and A.15 imply $W(\tilde{\mathbf{G}}_{-i}, \mathbf{v}) < W(\hat{\mathbf{G}}_{-i}, \mathbf{v})$, so they cannot both equal $u_i + c_i s$, which contradicts the definition of $\tilde{G}_i$ and $\hat{G}_i$. ■

A.2 Equilibrium Properties

This appendix provides proofs for the results in Section 1.3.

Claim A.17 (No Atom) *No score $s > 0$ is chosen with positive probability, and only player $i > m$ may choose 0 with positive probability.*

Proof. Suppose i chooses $s > 0$ with positive probability. If no player chooses score immediately below s, then i could benefit by moving the probability on s to a score slightly below s. If there is a sequence of scores $\{s_l\}$ that are chosen by some players and they converge to s, there exists player j who chooses infinite many scores in this sequence. Then, j's payoff at s is strictly more than at the scores in the sequence, contradiction. Hence, there is no score that is chosen with positive probability.

Suppose $i < m + 1$ chooses 0 with positive probability. Consider two cases. First, suppose no other player chooses 0 with positive probability. Then, i's payoff is 0 at 0, so $u_i^* = 0$. Since i can guarantee himself a payoff no less than u_j^* by choosing slightly above the highest score chosen by j for $j > i$, we have $u_i^* \geq u_j^*$. Therefore, $u_j^* = 0$ for $j > i$. However, $i + 1$ can get a positive payoff by choosing 0. Contradiction. Second, suppose there is another player besides i choosing 0 with positive probability, say player j. If $u_i^* = 0$, there is a contradiction as in the first case, so $u_i^* > 0$. Hence, i and j have positive probability to win a prize at $s = 0$. Then, j would deviate to a score slightly above 0 because the cost is almost the same but he does not have to split the prizes with i. In sum, the two cases imply that i does not choose 0 with positive probability. ■

Claim A.18 (Participation) *Players weaker than $m + 1$ choose 0 with probability one.*

Proof. Zero Lemma by Siegel (2009) is also true here. If we replace the probability of winning (one of the homogeneous prizes) with the probability of winning at least one prize (one of the heterogenous prizes), his proofs also work in this context.

Zero Lemma implies that at least $n-m$ players have zero expected payoff. Recall that $u_i^* \geq u_j^*$ for $j > i$ as in the proof of Claim A.17, $u_i^* = 0$ for $i \geq m+1$. Suppose player $i > m+1$ assigns positive probability on a set of positive scores. Suppose s is any score from that set, consider two cases. First, if $m+1$ does not choose above s, player $m+1$'s expected winnings at s are the same as i's. Second, consider the case in which $m+1$ chooses above s. $m+1$'s expected winnings at s are $W(\mathbf{G}^*_{-(m+1)}(s), \mathbf{v})$, where $\mathbf{G}^* = (G_j^*)_{j\in\mathcal{P}(s)}$ and $\mathbf{v} = (v^k)_{k\in\mathcal{P}(s)}$. Similarly, i's expected winnings at s are

$$W(\mathbf{G}^*_{-(m+1)}(s), G^*_{m+1}(s), \mathbf{v}) < W(\mathbf{G}^*_{-(m+1)}(s), G^*_i(s), \mathbf{v}) = W(\mathbf{G}^*_{-(m+1)}(s), \mathbf{v})$$

where $G_i^*(s) = 1$ and the inequality comes from the monotonicity of W. Therefore, $m+1$'s expected winnings at s are more than i's in the second case. In sum, $m+1$'s expected winnings at s are no less than i's, and $m+1$'s cost is lower at s than i's, so $m+1$ gets a higher payoff than i does at s. Contradiction. ■

Claim A.19 *In an equilibrium, the highest score that stronger player chooses is no less than the highest score that a weaker player chooses. That is,* $\bar{s}^*_{i+1} \leq \bar{s}^*_i$.

Proof. Denote the upper support of player i's strategy as $\bar{s}_i^*$, and the upper support of $i+1$'s equilibrium strategy as $\bar{s}^*_{i+1}$. Suppose $\bar{s}_i^* < \bar{s}^*_{i+1}$. Similar to Claim A.18, $i+1$'s expected winnings are the same as i's at $\bar{s}^*_{i+1}$, and $i+1$'s expected winnings are strictly higher than i's at $\bar{s}_i^*$. Namely, $P_i\left(\bar{s}^*_{i+1}\right) = P_{i+1}\left(\bar{s}^*_{i+1}\right)$ and $P_{i+1}\left(\bar{s}^*_i\right) > P_i\left(\bar{s}^*_i\right)$, where $P_j(s)$ denotes player j's expected winnings at s.

$u_i^* = P_i\left(\bar{s}_i^*\right) - c_i\bar{s}_i^*$ and $u^*_{i+1} = P_{i+1}\left(\bar{s}^*_{i+1}\right) - c_{i+1}\bar{s}^*_{i+1}$.

i's payoff at $\bar{s}^*_{i+1}$ should not be more than u_i^*, so we have

$$P_i\left(\bar{s}^*_{i+1}\right) - c_i\bar{s}^*_{i+1} \leq P_i\left(\bar{s}^*_i\right) - c_i\bar{s}^*_i$$

Note that $P_i\left(\bar{s}_{i+1}^*\right)=P_{i+1}\left(\bar{s}_{i+1}^*\right)$, so the equation above implies

$$\begin{aligned} P_{i+1}\left(\bar{s}_{i+1}^*\right)-c_i\bar{s}_{i+1}^* &\leq P_i\left(\bar{s}_i^*\right)-c_i\bar{s}_i^* \\ P_{i+1}\left(\bar{s}_{i+1}^*\right)-P_i\left(\bar{s}_i^*\right) &\leq c_i\left(\bar{s}_{i+1}^*-\bar{s}_i^*\right) \end{aligned} \tag{A.12}$$

$i+1$'s payoff at $\bar{s}_i^*$ should not be more than u_{i+1}^*, so we have

$$P_{i+1}\left(\bar{s}_i^*\right)-c_{i+1}\bar{s}_i^* \leq P_{i+1}\left(\bar{s}_{i+1}^*\right)-c_{i+1}\bar{s}_{i+1}^*$$

Note that $P_{i+1}\left(\bar{s}_i^*\right)>P_i\left(\bar{s}_i^*\right)$, so we have

$$\begin{aligned} P_i\left(\bar{s}_i^*\right)-c_{i+1}\bar{s}_i^* &< P_{i+1}\left(\bar{s}_i^*\right)-c_{i+1}\bar{s}_i^* \leq P_{i+1}\left(\bar{s}_{i+1}^*\right)-c_{i+1}\bar{s}_{i+1}^* \\ P_{i+1}\left(\bar{s}_{i+1}^*\right)-P_i\left(\bar{s}_i^*\right) &> c_{i+1}\left(\bar{s}_{i+1}^*-\bar{s}_i^*\right) \end{aligned} \tag{A.13}$$

(A.12) and (A.13) contradict with each other. As a result, $\bar{s}_{i+1}^* \leq \bar{s}_i^*$. ■

Claim A.20 *At any score lower than the maximum one, the difference between two players' equilibrium payoffs is no more than the difference in their costs. That is,* $u_i^*+c_is \leq u_{i+1}^*+c_{i+1}s$ *if* $s \leq \bar{s}_{i+1}^*$*; and* $u_i^*+c_is < u_{i+1}^*+c_{i+1}s$ *if* $s < \bar{s}_{i+1}^*$*.*

Proof. Since $\bar{s}_{i+1}^* \leq \bar{s}_i^*$, i's expected winnings at $\bar{s}_{i+1}^*$ are no less than that of $i+1$:

$$u_{i+1}^*+c_{i+1}\bar{s}_{i+1}^* \leq u_i^*+c_i\bar{s}_{i+1}^*$$

$$u_i^*+c_i\bar{s}_{i+1}^*-\left(u_{i+1}^*+c_{i+1}\bar{s}_{i+1}^*\right) \geq 0 \tag{A.14}$$

Therefore we have

$$\begin{aligned} & u_i^*+c_is-\left(u_{i+1}^*+c_{i+1}s\right) \\ = & \; u_i^*-u_{i+1}^*-\left(c_{i+1}-c_i\right)s \\ > & \; u_i^*-u_{i+1}^*-\left(c_{i+1}-c_i\right)\bar{s}_{i+1}^* \\ \geq & \; 0 \end{aligned}$$

where the last inequality comes from (A.14). ■

Claim A.21 *Suppose the prize sequence is either a QPS or GPS, $i, j \in \mathcal{P}(s)$, and $i < j$. If $j \in \mathcal{A}(s)$, $i \in \mathcal{A}(s)$.*

The claim is true at the highest score chosen by the players, $\bar{s}_2^*$. Suppose that this claim is true from $\tilde{s}$ to $\bar{s}_2^*$, we are going to show the two claims below. Then, we will find a contradiction at the supremum of scores that violate this claim, hence the claim is true for all s and so are the two claims below.

Claim A.22 *Suppose Claim A.21 is true for s in $(\tilde{s}, \bar{s}_2^*]$. For any s in $(\tilde{s}, \bar{s}_2^*]$ and any i such that i, $i+1 \in \mathcal{N}$, $G_i^*(s) \leq G_{i+1}^*(s)$; if $i, i+1 \in \mathcal{P}(s)$, $G_i^*(s) < G_{i+1}^*(s)$.*

Proof. Since the upper support of a weaker player is no less than that of a stronger player, $G_i^*(\bar{s}_2^*) \leq G_{i+1}^*(\bar{s}_2^*)$. We are going to consider the case if $s < \bar{s}_2^*$. If i is not in $\mathcal{P}(s)$, $G_i^*(s)$ is 0 or 1, the claim is true. Similarly, the claim is true if i is not in $\mathcal{P}(s)$. Therefore, it is sufficient to examine the case with $i, i+1 \in \mathcal{P}(s)$.

Consider three possibilities. First, suppose both i and $i+1$ are active at s, so $i, i+1 \in \mathcal{A}(s)$. Then, $(G_l^*(s))_{l \in \mathcal{A}(s)}$ is the solution of (A.5) for $\mathcal{A}(s)$ and $\mathcal{P}(s)$. Let us compare the equation for i and $i+1$. Claim A.20 implies that the $u_i^* + c_i s \geq u_{i+1}^* + c_{i+1} s$, so $W(\mathbf{G}_{-(i+1)}^*, \mathbf{v}) \leq W\left(\mathbf{G}_{-i}^*, \mathbf{v}\right)$. Since $W(\mathbf{G}_{-i}, \mathbf{v})$ is increasing in $\mathbf{G}_{-i}$, $G_i^*(s) \leq G_{i+1}^*(s)$.

Second, suppose one of i and $i+1$ are active at s, so $i, i+1 \in \mathcal{P}(s)$, $i \in \mathcal{A}(s)$ and $i+1 \notin \mathcal{A}(s)$. Then, there exists s_{i+1}'' such that $i+1$ is active above it. Since i and $i+1$ are active at s_{i+1}'', $G_i^*\left(s_{i+1}''\right) \leq G_{i+1}^*\left(s_{i+1}''\right)$. $i+1$ is not active over $\left(s, s_{i+1}''\right)$, so $G_{i+1}^*(s) = G_{i+1}^*\left(s_{i+1}''\right) \geq G_i^*\left(s_{i+1}''\right) > G_i^*(s)$.

Third, suppose neither i nor $i+1$ is active at s, so $i, i+1 \notin \mathcal{A}(s)$, but $i, i+1 \in \mathcal{P}(s)$. Then, let i is active at s_i'' and $i+1$ is active at s_{i+1}'' and $s_i'' \leq s_{i+1}''$. Therefore $G_{i+1}^*(s) = G_{i+1}^*\left(s_{i+1}''\right) \geq G_i^*\left(s_{i+1}''\right) = G_i^*(s)$, where the inequality comes from the first two cases.

Now we are going to prove the second part of the claim. If $i, i+1 \in \mathcal{P}(s)$, Claim A.20 implies that $u_i^* + c_i s < u_{i+1}^* + c_{i+1} s$. Similar to the analysis above, we have $G_i^*(s) < G_{i+1}^*(s)$ if $i, i+1 \in \mathcal{P}(s)$. ■

Claim A.23 (Ordered Densities) *Suppose the prize sequence is either a QPS or GPS, and Claim A.21 is true for s in $(\tilde{s}, \bar{s}_2]$. For any s in $(\tilde{s}, \bar{s}_2]$ and any i such that $i, i+1 \in \mathcal{A}(s)$, if s is an interior point of i and $i+1$'s supports, $g_i^*(s) > g_{i+1}^*(s)$.*

Proof. Without loss of generality, suppose $\mathcal{A}(s) = \{1, 2, ..., a\}$.[5]

First, consider players 1 and 2. Suppose s is an interior point of the supports of G_1^* and G_2^*. Consider the equations in (A.5) for $i = 1$ and 2,

$$\begin{aligned} W\left(\mathbf{G}_{-1}^*, \mathbf{v}\right) &= u_1^* + c_1 s \\ W\left(\mathbf{G}_{-2}^*, \mathbf{v}\right) &= u_2^* + c_2 s \end{aligned}$$

Take derivatives w.r.t. s for both sides of the equations, we have

$$\frac{dW\left(\mathbf{G}_{-1}^*, \mathbf{v}\right)}{dG_2} g_2^* + \frac{dW\left(\mathbf{G}_{-1}^*, \mathbf{v}\right)}{dG_3} g_3^* + ... + \frac{dW\left(\mathbf{G}_{-1}^*, \mathbf{v}\right)}{dG_a} g_a^* = c_1 \tag{A.15}$$

$$\frac{dW\left(\mathbf{G}_{-2}^*, \mathbf{v}\right)}{dG_1} g_1^* + \frac{dW\left(\mathbf{G}_{-2}^*, \mathbf{v}\right)}{dG_3} g_3^* + ... + \frac{dW\left(\mathbf{G}_{-2}^*, \mathbf{v}\right)}{dG_a} g_a^* = c_2 \tag{A.16}$$

Since $W(\mathbf{G}_{-1}, \mathbf{v})$ is linear in G_2 for $i = 2, ..., a$, $dW\left(\mathbf{G}_{-1}^*, \mathbf{v}\right)/dG_2$ is independent of G_2^*, therefore

$$\frac{dW\left(\mathbf{G}_{-1}^*, \mathbf{v}\right)}{dG_2} = \frac{dW\left(\mathbf{G}_{-2}^*, \mathbf{v}\right)}{dG_1} \tag{A.17}$$

If the prize sequence satisfies QPS, $dW(\mathbf{G}_{-1}, \mathbf{v})/dG_j = \beta\left(\sum_{j' \in \mathcal{P}(s)} G_{j'} - G_1 - G_j\right) + v^{\max \mathcal{P}(s)}$ as in Claims A.10 and A.11. Therefore, $dW(\mathbf{G}_{-1}, \mathbf{v})/dG_j$ is increasing in G_i for $i \neq 1, j$, and Lemma 1.1 implies

$$\frac{dW\left(\mathbf{G}_{-1}^*, \mathbf{v}\right)}{dG_j} \geq \frac{dW\left(\mathbf{G}_{-2}^*, \mathbf{v}\right)}{dG_j} \tag{A.18}$$

[5] If $\mathcal{A}(s) \neq \{1, ..., a\}$, we can rank the players from the strongest to the weakest, and rename them to $1, 2, ..., a$. Then, the analysis would be the same.

Similarly, $dW(\mathbf{G}_{-1}, \mathbf{v})/dG_j$ is increasing in G_i for $i \neq 1, j$ if the prize sequence satisfies GPS, therefore (A.18) is also true.

Let us compare (A.15) and (A.16). The terms except the first one are bigger in the LHS of (A.15), and the RHS is smaller in (A.15), therefore the first term on the LHS must be smaller in (A.15):

$$\frac{dW\left(\mathbf{G}_{-1}^{*}, \mathbf{v}\right)}{dG_2} g_2^{*} < \frac{dW\left(\mathbf{G}_{-2}^{*}, \mathbf{v}\right)}{dG_1} g_1^{*} \tag{A.19}$$

Then, (A.17) implies $g_2^{*} < g_1^{*}$.

Similarly, $g_{i+1}^{*} < g_i^{*}$. ■

Claim A.24 (Local Solution) *Consider system (A.5) for $\mathcal{A}(s)$ and $\mathcal{P}(s)$ and $u_i = u_i^{*}$. This system has a unique local solution $(\hat{G}_i(s))_{i \in \mathcal{A}(s)}$, and $(\hat{G}_i(s))_{i \in \mathcal{A}(s)}$ is differentiable at s.*

Proof. Take derivatives w.r.t. to s of the system, we have

$$\mathbf{W}_{\#\mathcal{A}(s)} \mathbf{g} = \mathbf{c}$$

where $\mathbf{g} = (g_i)_{i \in \mathcal{A}(s)}$ and $\mathbf{c} = (c_i)_{i \in \mathcal{A}(s)}$. Since $G_i^{*}(s) \in [0,1]$ for $i \in \mathcal{A}(s)$, Claims A.11 and A.14 imply $W_{\#\mathcal{A}(s)}$ is invertible. Therefore, we have an ordinary differential equation system

$$\mathbf{g} = \mathbf{W}_{\#\mathcal{A}(s)}^{-1} \mathbf{c} \tag{A.20}$$

with initial condition

$$\mathbf{G}(s) = \mathbf{G}^{*}(s)$$

Theorem of 20.7 of Olver (2007) implies that there is a local solution $\hat{\mathbf{G}}$ to (A.20), and this solution extends to s' as long as $\hat{G}_l(s'') \geq 0$ for $s'' \in (s, s')$ and all $l \in \mathcal{A}(s)$. It is obvious that $\hat{\mathbf{G}}$ is differentiable at s and it solves (A.5) for $\mathcal{A}(s)$ and $\mathcal{P}(s)$ and $u_i = u_i^{*}$. ■

Proof of Claim A.21 (Nested Gaps) . Suppose s_i'' is the supremum of scores that violate this claim. Therefore, there is a player j weaker than i such that i is not active immediately below s_i'' but j is. Since s_i'' is not the lower support of G_i^* by definition, there are scores below s_i'' such that i is active or j is inactive, and let s_j'' be the supremum of these scores. There are two possible situations at s_j'': i is active or i is inactive.

Consider Case 1: i is active at s_j''. Since j is also active at s_j'' by definition, both i and j are active at s_j''. Since G_i^* and G_j^* satisfy $W(\mathbf{G}_{-i}^*(s_j''), \mathbf{v}) = u_i^* + c_i s_j''$ and $W(\mathbf{G}_{-j}^*(s_j''), \mathbf{v}) = u_j^* + c_j s_j''$, so $G_i^*(s_j'') < G_j^*(s_j'')$, hence $g_i^*(s_j'') > g_j^*(s_j'')$. Since $g_j^*(s_j''+) \geq 0$, $g_i^*(s_j''-) > 0$. Claim A.24 shows that there is a differentiable local solution $(\hat{G}_{j'})_{j' \in \mathcal{A}(s_i)}$ at s_j'' to (A.5) for $\mathcal{A}(s_j'')$ and $\mathcal{P}(s_j'')$. Moreover, $\hat{g}_i(s_j'') > 0$ where $\hat{g}_i(s_j'')$ is the derivative of $\hat{G}_i(s_j'')$. Therefore, i would deviate to slightly above s_j'' according to Claim A.12. Contradiction. Figure A.1 illustrates $\hat{G}_i$ and $\hat{G}_j$ in this case, where the horizontal lines demonstrate the supports.

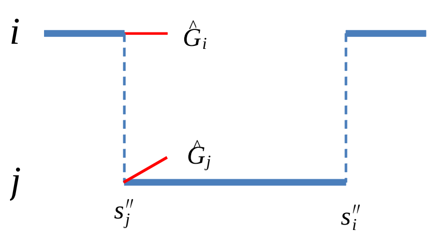

Figure A.1. Case 1

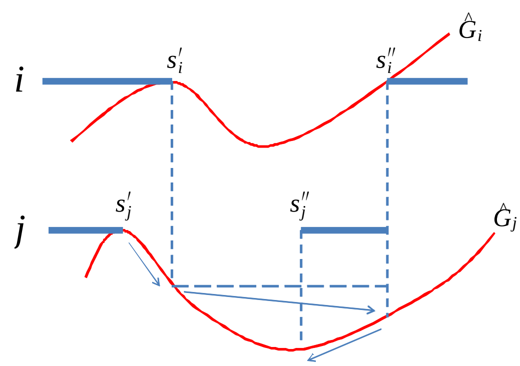

Figure A.2. Case 2

Consider Case 2: i is inactive at s_j''. Let s_i' be the supremum of scores below s_j'' such that i is active. If j is active at s_i', we would have a similar contradiction as in Case 1 above. Let s_j' be the supremum of scores below s_i' such that j is active, therefore $s_j' < s_i' < s_j'' < s_i''$, where (s_i', s_i'') is a gap for i. We proceed the analysis in two steps.

In the first step, for $s \in (s_i', s_i'')$, consider two equations

$$W(\mathbf{G}_{-i,j}^*, G_j, \mathbf{v}) - c_i s = u_i^* \quad \text{(A.21)}$$

$$W(\mathbf{G}_{-i,j}^*, G_i, \mathbf{v}) - c_j s = u_j^* \quad \text{(A.22)}$$

where $\mathbf{G}^*_{-i,j} = (G^*_l)_{l\in\mathcal{N}\setminus\{i,j\}}$ and $\mathbf{v}$ is the prizes available at s. Case 1 implies that the players in $\mathcal{P}(s'_i)$ who are stronger than i are active at s'_i, therefore $(G^*_i(s'_i), G^*_j(s'_i))$ solves (A.21) and (A.22) for $s = s'_i$. According to Claim A.16, this is also the unique solution. Since s''_i is the first violation, Claim A.23 implies that players in $\mathcal{P}(s''_i)$ who are stronger than j are active at s''_i, so $(G^*_i(s''_i), G^*_j(s''_i))$ is the unique solution to (A.21) and (A.22) for $s = s''_i$ similarly.

Take derivatives of both sides of (A.21) and (A.22) w.r.t. s, we have

$$\begin{aligned}
\sum_{j'\in\mathcal{N}\setminus\{i,j\}}\left(\frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_{j'}}g^*_{j'}\right) + \frac{dW(\mathbf{G}^*_{-i,j}, G_j, \mathbf{v})}{dG_j}g_j &= c_i \\
\sum_{j'\in\mathcal{N}\setminus\{i,j\}}\left(\frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_{j'}}g^*_{j'}\right) + \frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_i}g_i &= c_j
\end{aligned}$$

$$g_j = \left[c_i - \sum_{j'\in\mathcal{N}\setminus\{i,j\}}\left(\frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_{j'}}g^*_{j'}\right)\right] \Big/ \frac{dW(\mathbf{G}^*_{-i,j}, G_j, \mathbf{v})}{dG_j} \quad \text{(A.23)}$$

$$g_i = \left[c_j - \sum_{j'\in\mathcal{N}\setminus\{i,j\}}\left(\frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_{j'}}g^*_{j'}\right)\right] \Big/ \frac{dW(\mathbf{G}^*_{-i,j}, G_i, \mathbf{v})}{dG_i} \quad \text{(A.24)}$$

With initial conditions $G_i(s'_i) = G^*_i(s'_i)$ and $G_j(s'_i) = G^*_j(s'_i)$, (A.23) and (A.24) have a unique local solution $\left(\hat{G}_i(s), \hat{G}_j(s)\right)$ at s'_i as in Claim A.24. Moreover, the solution can be extended to s''_i as long as $\hat{G}_i(s), \hat{G}_j(s)$ are finite. Notice that $\hat{G}_j(s)$ also satisfies (A.21) and W in (A.21) is increasing in G_j, so $\hat{G}_j(s)$ must be finite for s in (s'_i, s''_i). Similarly $\hat{G}_i(s)$ is also finite for s in (s'_i, s''_i). Therefore, $\hat{G}_i(s)$ and $\hat{G}_j(s)$ are well defined for s in (s'_i, s''_i). Since $\hat{G}_i(s)$ and $\hat{G}_j(s)$ are differentiable, denote $\hat{g}_i(s)$ and $\hat{g}_j(s)$ as their derivatives.

In the second step, consider the interval (s'_i, s''_i), (A.23) and (A.24) imply $\hat{g}_i \geq \hat{g}_j$ over this interval, therefore $\hat{G}_i(s''_i) - \hat{G}_j(s''_i) \geq \hat{G}_i(s'_i) - \hat{G}_j(s'_i)$. Note that $\hat{G}_i(s''_i) = \hat{G}_i(s'_i)$, so

$$\hat{G}_j(s''_i) \leq \hat{G}_j(s'_i) \quad \text{(A.25)}$$

Recall that (G_i^*, G_j^*) solves (A.21) and (A.22) for $s = s_i'$ and s_i'', so we have

$$\begin{aligned}\hat{G}_j(s_i'') &= G_j^*(s_i'')\\ \hat{G}_j(s_i') &= G_j^*(s_i')\end{aligned}$$

Therefore,

$$G_j^*(s_j'') < G_j^*(s_i'') = \hat{G}_j(s_i'') \leq \hat{G}_j(s_i') = G_j^*(s_i') = G_j^*(s_j') \tag{A.26}$$

where the first inequality comes from G_j^* increases over (s_j'', s_i''), the second inequality comes from (A.25). Figure A.2 illustrates $\hat{G}_i$ and $\hat{G}_j$ in Case 2, and the arrows represent the steps in (A.26). However, mixed strategy G_j^* should be non-decreasing over (s_j', s_j''), which contradicts (A.26). ■

Claim A.21 is true for all $s \in [0, \bar{s}_2^*]$, therefore, Claims A.22 and A.23 are also true for all $s \in [0, \bar{s}_2^*]$. Hence, we have Lemmas 1.1 and 1.2.

A.3 Algorithm Properties

This appendix provides proofs for the results in Section 1.4. Because there always exists an equilibrium, suppose $\bar{s}_1^*$ is the maximum score chosen by player 1 in an equilibrium. In this appendix, we first discuss the properties of Step 1 in the algorithm if it starts with $\bar{s}_1^*$. Then, we discuss the general case if the algorithm starts with any value $\bar{s}$.

Claim A.25 *Suppose the algorithm starts with $\bar{s}_1^*$, and $\bar{s}_j = \bar{s}_j^*$ for $j = 1, ..., i$ after Step 1.i. There exists a unique solution to (1.1) in $[0, \infty]^i$.*

Proof. The proof is similar to the first step in Case 2 in the proof to Claim A.21.

Let $\mathcal{P} = \{i', i'+1, ..., i\}$ be the set of players such that $G_j(\bar{s}_i) > 0$. For any $j < i'$, $G_j(\bar{s}_i) = 0$, so we define $G_j(s) = 0$ and substitute it into (1.1). As a result, (1.1) becomes

$$W(\mathbf{G}_{-j}, \mathbf{v}) - c_j s = u_j \text{ for } j \in \mathcal{P} \tag{A.27}$$

where $\mathbf{G} = (G_j)_{j\in\mathcal{P}}$ and $\mathbf{v} = (v^k)_{k\in\mathcal{P}}$. Then, for $s < \bar{s}_i$, $(G_j)_{j\in\mathcal{P}}$ is the solution to (A.27). Hence $(G_j)_{j\in\mathcal{P}}$ also satisfies the differential equation system

$$\mathbf{W}\mathbf{G}' = \mathbf{c} \tag{A.28}$$

where $w_{j,j'}$ is the derivative of $W(\mathbf{G}_{-j}, \mathbf{v})$ w.r.t. $G_{j'}$ for $j' \in \mathcal{P}\setminus\{j\}$, and $\mathbf{c} = (c_j)_{j\in\mathcal{P}}$. Since W is invertible according to Claims A.11 and A.14, the differential equation can be rewritten as

$$\mathbf{G}' = \mathbf{W}^{-1}\mathbf{c} \tag{A.29}$$

By definition, we already know $G_j(\bar{s}_i)$ for $j = i', ..., i-1$ and $G_i(\bar{s}_i) = 1$, which are the initial conditions of the different equation above. Theorem 20.7 of Olver (2007) implies that (A.29) has a local solution $\mathbf{G}$ around $\bar{s}_i$, and this solution can be extended to $s < \bar{s}_i$ as long as $G_j > 0$ for all $j \in \mathcal{P}$.[6]

By similar analysis for Lemma 1.1, $G_{i'}$ is the smallest in $\mathbf{G}$. It is easy to see in (A.27) that the solution cannot extend to $-\infty$. Therefore, there must exist a score $\underline{s}_{i'}$ such that $G_{i'}(\underline{s}_{i'}) = 0$. Moreover, $G_{i'}$ is strictly increasing. To see why, suppose otherwise and $g_{i'}(s_0) \leq 0$. By similar analysis to Lemma 1.2, $g_j(s_0) < g_{i'}(s_0) \leq 0$ for all $j \in \mathcal{P}$. Therefore, (A.28) is violated. Since $G_{i'}$ is strictly increasing, $\underline{s}_{i'}$ is the only score such that $G_{i'}(\underline{s}_{i'}) = 0$.

The uniqueness comes from Clam A.16. ■

Claim A.26 *Suppose the prize sequence is geometric. For any subset $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$, G_a decreases if G_j increases in (A.5), where j is the weakest player in $\mathcal{A}$ and a is the second weakest player in $\mathcal{A}$.*

Proof. Suppose $\mathcal{P} = \{1, 2, ..., j\}$. The other cases can be proved similarly.

The solution to (A.8) is

$$\delta_i = -\frac{\det \tilde{\mathbf{W}}_i}{\det \hat{\mathbf{W}}_{j-1}}$$

where $\tilde{W}_i$ is $\hat{W}_{j-1}$ with the ith column replaced with $\mathbf{d}$. We want to show that δ_{j-1} is negative. Notice that we already have $\det\hat{W}_{j-1}$ has sign of $(-1)^j$, it is sufficient to

[6] See Olver (2007), pp. 1102-1103.

show that $\det \tilde{W}_{j-1}$ also has sign $(-1)^j$.

Denote $h_l = (\alpha - 1) G_l + 1$, similar to Claim A.15, we can define a $j \times j$ matrix H_j, and $[\hat{\mathbf{W}}_{j-1}, \mathbf{d}] = H_j$. Switch the last two columns, then and drop the last column and last row, we have a $(j-1) \times (j-1)$ matrix $\hat{H}_{j-1}$, and $\det \hat{W}_{j-1} = \det \hat{H}_{j-1}$.

Now we are going to use induction to show that $\det \hat{H}_{j-1}$ has sign of $(-1)^j$.

First, when $j = 3$, we have $\det \hat{H}_2 = \det \begin{pmatrix} 0 & h_2 \\ h_3 & h_1 \end{pmatrix} < 0$.

Suppose $\det \hat{H}_{j'-1}$ has sign of $(-1)^{j'}$. Consider $\det \hat{H}_{j'}$. First, divide all columns except the last one by $h_{j'+1}$, then times column $j' - 1$ by $h_{j'-1}$ and deduct it from the last column. The last column has zeros except in row $j' - 1$. Expand the determinant according to the last column, and we have (-1) times a $(j' - 1)$-dimensional determinant. Take the transpose of the matrix, we get $\det \hat{H}_{j'-1}$ which has the sign of $(-1)^{j'}$. As a result, $\det \hat{H}_{j'}$ has sign of $(-1)^{j'-1}$. ■

Claim A.27 *Suppose the prize sequence is quadratic. For any subset $\mathcal{A} \subset \mathcal{P} \subset \mathcal{N}$, G_a decreases if G_j increases in (A.5), where j is the weakest player in $\mathcal{A}$ and a is the second weakest player in $\mathcal{A}$.*

Proof. Suppose $\mathcal{A} = \mathcal{P} = \{1, 2, ..., j\}$. The other cases can be shown similarly.

Let $\boldsymbol{\delta}$ satisfies (A.8). The rest of the proof has two steps. First, we are going to show that $\delta_1 \geq \delta_2 \geq ... \geq \delta_{j-1}$, second, we are going to show that $\delta_{j-1} < 0$.

Step 1. Similar to the previous claim, the corresponding matrix H_j has entry $h_{ii'} = \sum_{l=1}^{j} h_l - h_i - h_{i'}$ for $i \neq i'$ and zero diagonal elements. Drop the last row of H_j and the resulting matrix is $[\hat{\mathbf{W}}_{j-1}, \mathbf{d}]$. h_i denote the ith column of H for $i = 1, ..., j-1$. Switch the ith column and the last column of H and drop the last column, we get a $(j-1) \times (j-1)$ matrix $\hat{H}_{j-1}$. Drop the last column of H_j, we get another $(j-1) \times (j-1)$ matrix H_{j-1}. As in the previous claim, the solution is

$$\delta_i = -\frac{\det \tilde{\mathbf{H}}_i}{\det \mathbf{H}_{j-1}} \text{ for } i = 1, ..., j-1$$

Since $\det H_{j-1}$ has sign $(-1)^j$ according to Claim A.11, it is sufficient to show that $\det\tilde{H}_i - \det\tilde{H}_{i-1}$ also has sign $(-1)^j$ for $i = 2, ..., j-1$.

Consider $\det\tilde{H}_i - \det\tilde{H}_{i-1}$, and we are going to show $\det\tilde{H}_i - \det\tilde{H}_{i-1}$ has sign of $(-1)^j$ after a series of elementary operations. If we compare $\tilde{H}_i$ and $\tilde{H}_{i-1}$, they are the same except the $(i-1)$th and ith columns. The $(i-1)$th and ith columns in $\tilde{H}_i$ are h_{i-1} and $\mathbf{d}$, and the $(i-1)$th and ith columns in $\tilde{H}_{i-1}$ are $\mathbf{d}$ and h_i. Therefore

$$\begin{aligned} & \det \tilde{\mathbf{H}}_i - \det \tilde{\mathbf{H}}_{i-1} \\ = \; & \det \left(\mathbf{h}_1, ..., \mathbf{h}_{i-1}, \mathbf{d}, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}\right) - \det \left(\mathbf{h}_1, ..., \mathbf{d}, \mathbf{h}_i, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}\right) \\ = \; & \det \left(\mathbf{h}_1, ...\mathbf{h}_{i-2}, \mathbf{h}_{i-1}, \mathbf{d}, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}\right) + \det \left(\mathbf{h}_1, ...\mathbf{h}_{i-2}, \mathbf{h}_i, \mathbf{d}, \mathbf{h}_{i+1}..., \mathbf{h}_{j-1}\right) \\ = \; & \det \left(\mathbf{h}_1, ...\mathbf{h}_{i-2}, \mathbf{h}_{i-1} + \mathbf{h}_i, \mathbf{d}, \mathbf{h}_{i+1}, ..., \mathbf{h}_{j-1}\right) \end{aligned} \tag{A.30}$$

where the second equality comes from switching the $(i-1)$th and ith columns in $\det\hat{H}_{i-1}$.

Deduct ith row from $(i-1)$th row, the $(i-1)$th row in the resulting determinant is $h_i - h_{i-1}$ except 0 in the $(i-1)$th column. Divide the $(i-1)$th row by $h_i - h_{i-1}$. Since $h_i - h_{i-1} > 0$, the resulting determinant has the same sign.

Deduct column i from all other columns except the $(i-1)$th column. The $(i-1)$th row of the resulting determinant has zeros except 1 at column i. The other rows are the same as in (A.30).

Expand the determinant according to the $(i-1)$th row, the result is $-\det \mathbf{Y}_{j-2}$ where $\mathbf{Y}_{j-2}$ is a $(j-2)$-dimensional determinant. Let us describe $\mathbf{Y}_{j-2}$. The $(i-1)$th column of $\mathbf{Y}_{j-2}$ is $h_{i-1}+h_i$ excluding the $(i-1)$th row; Column $i' > i-1$ of $\mathbf{Y}_{j-2}$ has $h_j - h_{i'}$ except $-\sum_{l=1}^{j-1} h_l + h_{i'}$ in column i'; Column $i' < i-1$ of $\mathbf{Y}_{j-2}$ has $h_j - h_{i'+1}$ except $-\sum_{l=1}^{j-1} h_l + h_{i'+1}$ in column i'.

Add all other columns to column $(i-1)$ of $\mathbf{Y}_{j-2}$, the $(i-1)$th column has only $(j-2)\, h_{j-1}$. Since $h_{j-1} > 0$, we can normalize column $i-1$ to ones without change the sign of the determinant.

Multiply column $i-1$ with $h_j - h_{i'}$ and deduct it from column $i' \neq i-1$. Then the i'th column has only zeros except $-\sum_{l=1}^{j-1} h_l + h_{i'+1} - (h_j - h_{i'}) = h_{i'} - \sum_{l\in\mathcal{A}\backslash\{i'\}} h_l < 0$; row $i-1$ has only zeros except 1 at column $i-1$, therefore we can set column $i-1$ to zeros except in row $i-1$ and not affect the sign of the determinant.

The resulting determinant is a diagonal matrix, therefore the determinant equals

$$-\prod_{i'\in\mathcal{A}\backslash\{i-1,j\}} \left(h_{i'} + \sum_{l\in\mathcal{A}\backslash\{i'\}} h_l \right)$$

which is a product of $1+(j-3)$ negative numbers, therefore $\det\hat{H}_i - \det\hat{H}_{i-1}$ has sign of $(-1)^j$.

Step 2. Suppose $\delta_{j-1} \geq 0$, then $\delta_1 > \delta_2 > ... > \delta_{j-1} \geq 0$. Therefore $W_{j-1}\boldsymbol{\delta} \gg 0$. Contradiction.

As a result, $\delta_{j-1} < 0$.

If $\mathcal{A} = \mathcal{P} \neq \{1, 2, ..., j\}$, we can rename the players in $\mathcal{A}$ and the proof is the same. If $\mathcal{A} \subsetneq \mathcal{P}$, the corresponding $\hat{W}_{j-1}$ has $(G_i)_{i\in\mathcal{P}\backslash\mathcal{A}}$ in each entry, then we can define h_i similarly as in Claim A.12. The rest of the proof is the same. ■

Claim A.28 *There exists a unique solution $(\hat{G}_i)_{i\in\mathcal{P}}$ in $[0,1]^{\#\mathcal{P}}$ to (A.5) for $\mathcal{A} = \mathcal{P} = \mathcal{P}(s)$ and $s \in [\underline{s}^*_{p'}, \bar{s}^*_{p''}]$, where p' and p'' are the weakest and strongest players in $\mathcal{P}(s)$. Moreover, $\hat{G}_{p'}(s) \leq G^*_{p'}(s)$ for $s \in [\underline{s}^*_{p'}, \bar{s}^*_{p''}]$.*

Proof. If $g^*_{p'}(s) > 0$, Claim A.21 implies that $g^*_i(s) > 0$ for any $i \in \mathcal{P}$. Therefore, $(G^*_i(s))_{i\in\mathcal{P}}$ is the solution to A.5 and it is unique according to Claim A.16.

Suppose there is a gap (s', s'') in the support of the weakest player's strategy, $G^*_{p'}$. Claim A.25 implies that it is sufficient to show $\hat{G}_i(s) \leq 1$ for $s \in (s', s'')$. Since $\hat{G}_i(s) \leq \hat{G}_{p'}(s)$ by similar analysis to Lemma 1.1, it is sufficient to show that $\hat{G}_{p'}(s) \leq G^*_{p'}(s)$ for $s \in (s', s'')$.

The rest of the proof has three steps.

Step 1. Suppose $\#\mathcal{A}(s) = \#\mathcal{P}(s) - 1$ and $\hat{G}_{p'}(s) > G^*_{p'}(s)$. If we decrease $\hat{G}_{p'}(s)$ to $G^*_{p'}(s)$, Claims A.12 and A.15 imply that $W(\mathbf{G}^*_{-p'}, \mathbf{v}) > u^*_{p'} + c_{p'}s$. Contradiction.

Therefore, $\hat{G}_{p'}(s) \leq G^*_{p'}(s)$.

Suppose $\#\mathcal{A}(s) < \#\mathcal{P}(s) - 1$ and $\hat{G}_{p'}(s) > G^*_{p'}(s)$ for some s in (s', s''). Denote the lower bound of $p'-1$'s gap as $s_d^{p'-1}$. Then, we have $\hat{G}_{p'}(s_d^{p'-1}) < G^*_{p'}(s')$, otherwise, $p'-1$ would deviate to slightly above $s_d^{p'-1}$ as in Case 1 of Claim A.21. Then, intermediate value theorem implies $\hat{G}_{p'}(s_0) = G^*_{p'}(s_0)$ for some $s_0 \in (s_d^{p'-1}, s)$. See Figure A.3. We are going to find a contradiction in the next two steps.

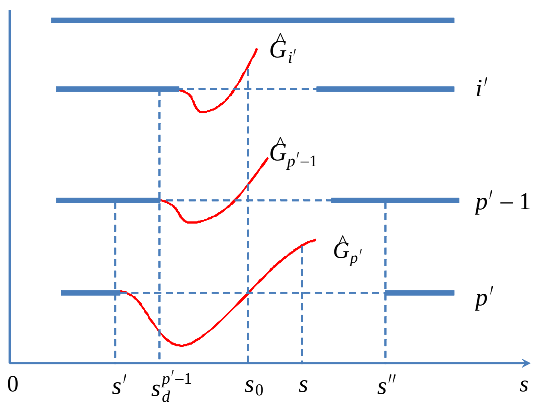

Figure A.3. Contradiction by Step 1 and 2

Step 2. We claim that $\hat{G}_i(s_0) > G^*_i(s_d^i)$, where i is any player in $\mathcal{P}(s_0) \setminus \mathcal{A}(s_0)$ and s_d^i is the lower bound of i's gap.

Since $\hat{g}_{p'} \leq \hat{g}_{p'-1}$ by similar analysis in Claim A.23, $\hat{G}_{p'-1}$ increases faster than $\hat{G}_{p'}$ does. Notice that $\hat{G}_{p'}\left(s_d^{p'-1}\right) < G^*_{p'}(s_0)$, so $\hat{G}_{p'-1}\left(s_d^{p'-1}\right) < \hat{G}_{p'-1}(s_0)$. Moreover, $G^*_{p'-1}\left(s_d^{p'-1}\right) = \hat{G}_{p'-1}\left(s_d^{p'-1}\right)$, then, we have $G^*_{p'-1}\left(s_d^{p'-1}\right) < \hat{G}_{p'-1}(s_0)$. Similarly, $G^*_{p'-1}\left(s_d^{p'-1}\right) < \hat{G}_{p'-1}(s_0)$ implies $G^*_{p'-2}\left(s_d^{p'-2}\right) < \hat{G}_{p'-2}(s_0)$, and so on.

Step 3. We claim that i' would deviate to s_0, where i' is the strongest player in $\mathcal{P}(s_0) \setminus \mathcal{A}(s_0)$.

Decrease $\hat{G}_j(s_0)$ to $G^*_j(s_0)$ for $j = p', p'-1$. Denote $(\tilde{G}_i)_{i \in \mathcal{P}(s) \setminus \{p', p'-1\}}$ as the solution to (A.5) for $\mathcal{A} = \mathcal{P}(s) \setminus \{p', p'-1\}$ and $\mathcal{P} = \mathcal{P}(s)$. Claims A.26 and A.27 imply that $\tilde{G}_{p'-2}(s_0) > \hat{G}_{p'-2}(s_0)$. Similar to Step 1, $\tilde{G}_i(s_0) > G^*_i(s_d^i)$ for $i \in \mathcal{P}(s) \setminus \{p', p'-1\}$. Repeat this process until $\tilde{G}_i(s_0) > G^*_i(s_d^i)$ for $i \in \mathcal{A}(s) \cup \{i'\}$ where i' is the strongest player in $\mathcal{P}(s) \backslash \mathcal{A}(s)$. This would contradict with Step 1. ■

Claim A.29 *If $\bar{s}_j = \bar{s}_j^*$ for $j = 1, ..., i$, there exists a unique solution to (1.1) in* $[0,1]^i$.

Proof. Because of Claim A.25, it is sufficient to show that $G_i \leq 1$ for $j \leq i$.

Claim A.28 shows that this claim is true for $i = m + 1$, and denote the solution as $(G_i^1)_{1 \leq i \leq m+1}$. Now, let $i = m$ and denote the solution as $(G_i^2)_{1 \leq i \leq m}$. Claims A.26 and A.27 imply that $G_m^2(s) < G_m^1(s) \leq 1$. Therefore, the claim is also true for $i = m$. Similarly, we can always exclude the weakest remaining player and show that the claim is true for a smaller i, therefore the claim is true for any $i = 3, ..., m + 1$. ■

Similarly, unique solution can also be proved for the other parts of Step 1.i.

Claim A.30 *The upper support of $i+1$'s equilibrium strategy is the infimum of $i+1$'s best responses in $\left[\underline{s}_{i-1}, \bar{s}_i^*\right]$ against the pseudo strategies yielding $u_1^*, ..., u_i^*$.*

Proof. If we exclude the players weaker than $i + 1$, there are pseudo strategies G_i yielding the equilibrium payoffs for the remaining player according to Claim A.29. Suppose $\mathbf{G}$ is the pseudo strategies yielding $u_1^*, ..., u_i^*$, Claims A.12 and A.15 imply $W(\mathbf{G}, \mathbf{v}) < u_{k+1}^* + c_{k+1}s$ for s between the lower support of G_{i-1} and $\bar{s}_{i+1}^*$.

Claims A.27 and A.26 imply that $G_i(s) \leq G_i^*(s)$, so the lower support of G_i is bigger than G_i^*. This is not a problem because $\bar{s}_{i+1}^*$ cannot be less than the lower support of G_i.

Therefore, $\bar{s}_{i+1}^*$ is the infimum of the best responses to the pseudo strategies $G_1, ..., G_i$ yielding $u_1^*, ..., u_i^*$. ■

Lemma 1.6 implies $\bar{s}_2^* \leq \bar{s}_1^*$. Since there is no aggregate gap, $\bar{s}_2^* = \bar{s}_1^*$. Recall that we let the algorithm starts with $\bar{s}_1^*$, so G_1 and G_2 yields u_1^*, u_2^* in Step 1.2. Therefore, Claim A.30 implies that $\bar{s}_3 = \bar{s}_3^*$ in Step 1.3, then Claim A.29 implies the existence of G_1, G_2, G_3 yielding u_1^*, u_2^*, u_3^*. Similarly, Claim A.30 implies that $\bar{s}_4 = \bar{s}_4^*$ and Claim A.29 implies the existence of $G_1, ..., G_4$ yielding $u_1^*, ..., u_4^*$, and so on. As a result, if the algorithm starts with $\bar{s} = \bar{s}_1^*$, we have $\bar{s}_i = \bar{s}_i^*$ for $i = 1, ..., m+1$. Since G_i^* is increasing slightly below its upper support $\bar{s}_i^*$, Lemma 1.2 implies all participating players are active slight below $\bar{s}_i^*$. Therefore, $u_i = u_i^*$ for $i = 1, ..., m+1$.

Note that $G_m^* (0) = 0$. Consider $G_1, ..., G_{m+1}$ yielding $u_1^*, ..., u_{m+1}^*$, only m and $m+1$ are participating below $\underline{s}_{m-1}^*$, therefore, $G_i (s) = G_i^* (s)$ for $i = m, m+1$ and $s < \underline{s}_{m-1}^*$. As a result, there is no shift in Step 2 of the algorithm, and $\hat{G}_1, ..., \hat{G}_{m+1}$ are the same as $G_1, ..., G_{m+1}$ yielding $u_1^*, ..., u_{m+1}^*$.

Similar to (A.29), there exists a unique solution to (1.2) in $[0,1]^i$ for each $i = m+1, ..., 3$ in Step 3 of the algorithm.

Claim A.31 *If the algorithm starts with $\bar{s}_1^*$, the following two statements are equivalent:*

i) There is a gap (s_i', s_i'') in the support of i's equilibrium strategy.

ii) $\hat{G}_i$ has a dent over (s_i', s_i''), where $\hat{G}_i$ is player i's pseudo strategy after fixing $\hat{G}_{i+1}$'s non-monotonicity.

Proof. First, consider the following statement: $\hat{G}_i$ is strictly increasing if and only if G_i^* does not has a gap in the equilibrium.

"$\Rightarrow$": Suppose $\hat{G}_i$ is strictly increasing and G_i^* has a gap (s', s'') in its support. One the one hand, $G_i^* = \hat{G}_i$ at s' and s'', i's payoff at the boundaries of the gap should be u_i^* for both $\hat{G}_i$ or G_i^*. On the other hand, recall that $\hat{G}_3$ is strictly increasing, but G_i^* is constant over (s', s''), therefore $\hat{G}_i (s') < G_i^* (s')$. Therefore, if we replace $\hat{G}_i$ with G_i^* at s', Claims A.12 and A.15 imply i's payoff at s' is less than u_i^*. Contradiction.

"$\Leftarrow$": Suppose G_i^* has no gap and $\hat{G}_i$ has a dent (s', s''). By definition, G_i^* and $\hat{G}_i$ are different, so there are multiple solutions to (A.5) for $\mathcal{A} = \mathcal{P} = \{a'', a''+1, ..., i\}$. Contradiction to Claim A.28.

Second, consider the following statement: $\hat{G}_i$ has a dent over (s', s'') if and only if G_i^* has gap (s', s'').

"$\Leftarrow$" Suppose G_i^* has gap (s', s'').

It is easy to see that $G_i^* = \hat{G}_i$ at s' and s''. Claim A.28 implies $\hat{G}_i$ has a dent over s' and s''.

"$\Rightarrow$": Suppose $\hat{G}_i$ has a dent over (s', s''), but G_i^* does not have a gap. There are two solutions to (1.2) for $j = 1, ..., i$. Contradiction. Suppose $\hat{G}_i$ has a dent over (s', s'')

and G_i^* has a gap $\left(s'_g, s''_g\right)$ but the gap is different from (s', s''). Let us discuss in four different cases.

Consider the first case: $s' < s'_g$. By definition, $\hat{G}_i$ is not increasing on $\left(s', s'_g\right)$, but G_i^* is. Therefore, we have two different solutions for (A.5) for $\mathcal{A} = \mathcal{P} = \{a'', a''+1, ..., i\}$. Contradiction.

Consider the second case: $s'_g < s'$. Therefore, $\hat{G}_i(s') > G_i^*(s')$, which contradicts Claim A.28.

Consider the third case: $s''_g > s''$. Since $s'_g = s'$, $G_i^*\left(s'_g\right) = \hat{G}_i(s')$. By definition, $G_i^*(s') = G_i^*(s'')$ and $\hat{G}_i\left(s'_g\right) = \hat{G}_i\left(s''_g\right)$. Because $s''_g > s''$, $\hat{G}_i\left(s''_g\right) > \hat{G}_i(s'') = G_i^*(s'') = G_i^*(s') = \hat{G}_i(s')$. Contradiction. (start at the same value but did not end at the same value.)

Consider the fourth case: $s''_g < s''$. We have a contradiction as in the third case. ■

Claim A.32 *If the algorithm starts with $\bar{s}_1^*$, the algorithm ends in a finite number of steps.*

Proof. Consider equation system $W(\mathbf{G}_{-i}, \mathbf{v}) - c_i s = u_i$ for $i \in \mathcal{A}$. Suppose solution $\mathbf{G}$ exists in a neighborhood of s_0. Take derivatives of both hand w.r.t. s, we have

$$\mathbf{W}_j \mathbf{g} = \mathbf{c}$$

where $j = \#\mathcal{A}$, W_j is the $j \times j$ matrix defined above, $\mathbf{g} = (G'_i)_{i\in\mathcal{A}}$ and $\mathbf{c} = (c_i)_{i\in\mathcal{A}}$. According to Claims A.11 and A.14, $\det W_j$ is not zero, hence we have an ordinary differential equation system

$$\mathbf{g} = \mathbf{W}_j^{-1}\mathbf{c}$$

with the initial condition that $\mathbf{G}$'s value at s_0 is $\mathbf{G}(s_0)$. Since $\mathbf{W}_j^{-1}\mathbf{c}$ is an analytic function in $\mathbf{G}$, Theorem 20.10 of Olver (2007) implies that the solution to this system is analytic in a small neighborhood of s_0. Then, $\mathbf{g} = \mathbf{W}_j^{-1}\mathbf{c}$ is a composition of analytic functions, hence is also an analytic function in the neighborhood.

Recall that g_i is defined over an bounded interval in Step 1.i. Since g_i is analytic, Identity Theorem[7] implies that g_i either has a finite number of roots in its domain or

[7] See Chapman (2002), pp. 256.

$g_i = 0$. Either case implies that G_i has only a finite number of dents and the algorithm ends in finite steps. ■

Lemma 1.7 (Nested Gaps): Suppose i, j *both choose above and below* s *and* $i < j$ in the equilibrium. *If the support of* G_i^* *has a gap* (s_i', s_i'') *containing* s, *the support of* G_j^* *also has a gap* (s_j', s_j'') *such that* $s_j' < s_i'$ *and* $s_j'' > s_i''$.

Proof. Suppose i has a gap with lower bound s_i', and j has a gap with lower bound s_j'. Claim A.21 implies that $s_i' \geq s_j'$. Suppose $s_i' = s_j'$, therefore at $g_i^*(s_i') > g_j^*(s_i')$. Lemma 1.6 implies that $g_i^*(s_i') = 0$, therefore $g_j^*(s_i') < 0$ contradiction. Hence, $s_i' > s_j'$.

Suppose i has a gap with upper bound s_i'', and j has a gap with upper bound s_j''. Claim A.21 implies that $s_i'' \leq s_j''$. Suppose $s_i'' = s_j''$. Since $s_i' > s_j'$,

$$G_i^*(s_i') < \hat{G}_j(s_i') < G_j^*(s_j') \tag{A.31}$$

Since $\hat{g}_i(s) > \hat{g}_j(s)$ for $s \in (s_i', s_i'')$, $\hat{G}_i$ increases faster than $\hat{G}_j$, so $\hat{G}_j(s_i'') < \hat{G}_i(s_i'') = G_i^*(s_i'') = G_i^*(s_i') < G_j^*(s_j') = G_j^*(s_i'')$, where the last inequality comes from (A.31). Contradiction. ■

The above lemma is a stronger version of Claim A.21. This lemma ensures that we only need to fix monotonicity of $\hat{G}_i$ in Step 3.i in the gaps of $\hat{G}_{i+1}$. Therefore, we only need to update $\hat{G}_i$ over the gaps of $\hat{G}_{i+1}$ in Step 3.$i+1$.

Claims A.29 to A.31 and Lemma 1.7 imply that the algorithm *constructs* the equilibrium strategies if it starts with $\bar{s}_1^*$, the highest score in the equilibrium. Now let us consider the case if the algorithm starts with an arbitrary score $\bar{s}$.

Lemma 1.4 (Determinateness): *The algorithm uniquely determines* $(G_i^*)_{i \in \mathcal{N}}$, *and* $(G_i^*)_{i \in \mathcal{N}}$ *is independent of the initial value* $\bar{s}$.

Proof. It is sufficient to show that G_i is a function of $\bar{s} - s$.

Substitute $u_i = v^1 - c_i s$ into $W(\mathbf{G}_{-i}, \mathbf{v})$ for $i \in \{1, 2\}$, we have

$$\begin{aligned} W\left(G_2, v^1, v^2\right) &= u_1 + c_1 s = v^1 - c_1(\bar{s} - s) \\ W\left(G_1, v^1, v^2\right) &= u_2 + c_2 s = v^1 - c_2(\bar{s} - s) \end{aligned}$$

therefore G_1 and G_2 in Step I are functions of $\bar{s} - s$.

Suppose $G_1, ..., G_{i-1}$ are the pseudo strategies yielding $u_1, ..., u_{i-1}$, and they are functions of $\bar{s} - s$. Since $\bar{s}_i$ is i's best response in $\left[\underline{s}_{i-2}, \bar{s}_{i-1}\right]$ to $G_1, ..., G_k$, $\bar{s}_1 - \bar{s}_i$ is constant and $u_i + c_i s$ is a function of $\bar{s} - s$. The right hand sides of system (1.1) are functions of $\bar{s} - s$. As a result, the pseudo strategies $G_1, ..., G_i$ are also functions of $\bar{s} - s$. Similarly, at the end of Step 1.$(m+1)$, pseudo strategies $G_1, ..., G_{m+1}$ are also functions of $\bar{s} - s$. Therefore, $(G_i^*)_{i \in \mathcal{N}}$ is independent of the initial value $\bar{s}$, and the algorithm uniquely determines $(G_i^*)_{i \in \mathcal{N}}$. ■

The lemma above shows that if we start with an arbitrary score $\bar{s}$, the algorithm would construct the same pseudo strategies with a shift. Therefore, Claim A.29 to A.31 are also true if the algorithm starts with arbitrary $\bar{s}$ instead of $\bar{s}_1^*$. Hence, we have Lemmas 1.3, 1.5, 1.6 and 1.7.

Theorem 1.1: *The algorithm constructs the unique Nash equilibrium for every all-pay contest with a quadratic or a geometric prize sequence and distinct linear costs.*

Proof. Suppose there are two equilibria, and the corresponding maximum scores in these equilibria is $\bar{s}_1^*$ and $\bar{s}_1^{**}$. If $\bar{s}_1^* = \bar{s}_1^{**}$, Lemma 1.3 to 1.7 imply that the two equilibria must be the same. If $\bar{s}_1^* \neq \bar{s}_1^{**}$, Lemma 1.4 would be violated. Therefore, we have a unique equilibrium and it is constructed by the algorithm. ■

Corollary 1.1: *If $c_i - c_j$ converges to 0 for players $i, j < m+2$, $u_i^* - u_j^*$ also converges to zero and $G_i^* - G_j^*$ pointwise converges to zero.*

Proof. Suppose c_i and c_{i+1} converge. That is, the difference between c_i and c_{i+1} converges to 0. Let us consider the limit. Lemma 1.1 implies $\bar{s}_i^* \geq \bar{s}_{i+1}^*$. Suppose $\bar{s}_i^* > \bar{s}_{i+1}^*$, therefore i's expected winnings at $\bar{s}_{i+1}^*$ are more than $i+1$'s, therefore $i+1$ would deviate to $\bar{s}_i^*$ for a higher payoff. Therefore, $\bar{s}_i^* = \bar{s}_{i+1}^*$, and the payoffs of i and $i+1$ are also the same.

From the way we construct the strategies for i and $i+1$, their strategies $G_i^*(s)$ and $G_{i+1}^*(s)$ must also converge at any s in the common supports. ■

Appendix B

Proof for Chapter 2

Proof of Lemma 2.4: Suppose the high type students are allocated asymmetrically into the two classes. First, consider case 1 in which one L-type student is allocated into the low type class. It is easy to see that the total score is less than $\Pi_{seperating}$.

Proof. Consider case 2 in which one class has 2 L-type students and the other class has $n' = 2n - 2$ L-type students. n' is the number of L-type students in a L-majority class. $u_L \geq (2n - n')\,\varepsilon = 2\varepsilon$ in L-majority class; $u_L \geq n'\varepsilon = (2n-2)\,\varepsilon$ in H-majority class; and

$$
\begin{aligned}
W_H &\geq \varepsilon \sum_{i=1}^{2n-n'} (i-1) + \varepsilon \sum_{i=1}^{n'} (i-1) \\
&= \left(2n^2 - 2nn' - n + (n')^2\right) \varepsilon
\end{aligned}
$$

Therefore, the total score is

$$
\begin{aligned}
\Pi_{2n-2,2} &\leq \frac{V - W_H - n'\,(2n-n')\,\varepsilon - (2n-n')\,n'\varepsilon}{c_L} + \frac{W_H}{c_H} \\
&= \frac{V - (2n^2 - 2nn' - n + (n')^2)\,\varepsilon - n'\,(2n-n')\,\varepsilon - (2n-n')\,n'\varepsilon}{c_L} \\
&\quad + \frac{(2n^2 - 2nn' - n + n'^2)\varepsilon}{c_H}
\end{aligned}
$$

$$= \frac{1}{c_L c_H}\left(4c_L + 4c_H + Vc_H - 5nc_L - 3nc_H + 2n^2 c_L - 2n^2 c_H\right)\varepsilon$$

$$\begin{aligned} & \Pi_{seperating} - \Pi_{2n-2,2} \\ \geq \; & \frac{V - n(2n-1)\varepsilon}{c_L} + \frac{n(2n-1)\varepsilon}{c_H} \\ & - \frac{1}{c_L c_H}\left(4c_L + 4c_H + Vc_H - 5nc_L - 3nc_H + 2n^2 c_L - 2n^2 c_H\right)\varepsilon \\ = \; & \frac{4\varepsilon}{c_L c_H}(c_L + c_H)(n-1) > 0 \end{aligned}$$

Consider case 3 in which one more L-type student from L-majority class to H-majority class, and we are going to show that the total score decreases.

Before the move, $u_L \geq (2n - n')\varepsilon$ in L-majority class; $u_L \geq n'$ in H-majority class; and

$$\begin{aligned} W_H & \geq \varepsilon \sum_{i=1}^{2n-n'} (i-1) + \varepsilon \sum_{i=1}^{n'} (i-1) \\ & = \left(2n^2 - 2nn' - n + (n')^2\right)\varepsilon \end{aligned}$$

Therefore, the total score is

$$\begin{aligned} & \Pi_{n',2n-n'} \\ \leq \; & \bar{\Pi}_{n',2n-n'} \equiv \frac{V - W_H - n'(2n-n')\varepsilon - (2n-n')n'\varepsilon}{c_L} + \frac{W_H}{c_H} \\ = \; & \frac{V - (2n^2 - 2nn' - n + (n')^2)\varepsilon - n'(2n-n')\varepsilon - (2n-n')n'\varepsilon}{c_L} \\ & + \frac{(2n^2 - 2nn' - n + (n')^2)\varepsilon}{c_H} \end{aligned}$$

After the move, $u_L \geq (2n - n' + 1)\varepsilon$ in L-majority class; $u_L \geq (n' - 1)\varepsilon$ in H-majority class; and

$$W_H \geq \varepsilon \sum_{i=1}^{2n-n'+1} (i-1) + \varepsilon \sum_{i=1}^{n'-1} (i-1)$$

$$= \left(2n^2 - 2nn' + n + (n')^2 - 2n' + 1\right)\varepsilon$$

Therefore, the total score is

$$\begin{aligned}
& \Pi_{n'-1,2n-n'+1} \\
\leq \quad & \bar{\Pi}_{n'-1,2n-n'+1} \equiv \frac{V - W_H - (n'-1)(2n-n'+1)\varepsilon - (2n-n'+1)(n'-1)\varepsilon}{c_L} + \frac{W_H}{c_H} \\
= \quad & \frac{\begin{array}{c} V - (2n^2 - 2nn' + n + (n')^2 - 2n' + 1)\varepsilon \\ -(n'-1)(2n-n'+1)\varepsilon - (2n-n'+1)(n'-1)\varepsilon \end{array}}{c_L} \\
& + \frac{2n^2 - 2nn' + n + (n')^2 - 2n' + 1}{c_H}\varepsilon
\end{aligned}$$

$$\begin{aligned}
& \bar{\Pi}_{n',2n-n'} - \bar{\Pi}_{n'-1,2n-n'+1} \\
= \quad & \frac{V - (2n^2 - 2nn' - n + (n')^2)\varepsilon - n'(2n-n')\varepsilon - (2n-n')n'\varepsilon}{c_L} \\
& + \frac{(2n^2 - 2nn' - n + (n')^2)\varepsilon}{c_H} \\
& - \frac{\begin{array}{c} V - (2n^2 - 2nn' + n + (n')^2 - 2n' + 1)\varepsilon \\ -(n'-1)(2n-n'+1)\varepsilon - (2n-n'+1)(n'-1)\varepsilon \end{array}}{c_L} \\
& - \frac{2n^2 - 2nn' + n + (n')^2 - 2n' + 1}{c_H}\varepsilon \\
= \quad & \frac{1}{c_L c_H}(c_L + c_H)(2n' - 2n - 1)\varepsilon > 0
\end{aligned}$$

where the last inequality comes from $n' > n$.

Therefore, the upper bound decreases if we move one L-type students to the H-majority class, so

$$\Pi_{seperating} > \bar{\Pi}_{n',2n-n'} > \Pi_{n',2n-n'}$$

so separating is better than any mixing. ■

Appendix C

Proofs for Chapter 3

C.1 Proof of Proposition 3.1

Let $\bar{\pi}_B$ be the supremum of the buyer's equilibrium payoffs. The proof consists of five claims.

Claim 3.1. *If the buyer's payoff is π_B in an equilibrium, the buyer's payoff is at least π_B in another equilibrium with an agreement in the first period.*

Proof: Suppose the buyer's payoff is π_B in the equilibrium E, where the bargaining order is $i_1, i_2, ...$, and the first agreement is reached in period $t > 1$. Then, E induces an equilibrium, E_t for the subgame $\Gamma\left(B, (i_t, i_{t+1}, ...)\right)$.

If there is no agreement in E, the buyer gets zero in both E and E_t. If there is an agreement in E, the sellers sell at the same prices in E_t as in E but $t-1$ periods earlier, therefore the buyer receives a higher payoff than in E. Hence the buyer's payoff in E_t is at least π_B. As a result, it is another equilibrium where the buyer chooses order $i_t, i_{t+1}, ...$ and every player follows the strategies in E_t. In this equilibrium, the buyer's payoff is at least π_B and an agreement is reached in period 1. ■

Claim 3.2. *$\bar{\pi}_B$ can be approached by buyer's equilibrium payoffs with either order*

$1, 1, \ldots$ *or order* $2, 2, \ldots$.

Proof: By the definition of supremum $\bar{\pi}_B$, there exists a sequence of buyer's equilibrium payoffs $\{\pi_B^k\}_{k=1}^{\infty}$ that converges to $\bar{\pi}_B$. Pick any equilibrium payoff π_B^k from this sequence, denote an associated equilibrium as E^k and the associated order as $i_1^k, i_2^k, i_3^k, \ldots$. Note that there could be multiple equilibria yielding π_B^k. Given equilibrium E^k, denote $\hat{\pi}_B^k$ as the buyer's payoff in the subgame $\Gamma(B, (i_2, i_3, \ldots))$. There must be a subsequence $\{\pi_B^{k_m}\}_{m=1}^{\infty}$, where $\pi_B^{k_m} \geq \hat{\pi}_B^k$ for all m. There are could be two cases.

In the first case, suppose there is a payoff in $\{\pi_B^{k_m}\}_{m=1}^{\infty}$, $\pi_B^{k_j}$, whose equilibrium order starts with i_1. Suppose $\pi_B^{k_j}$ is the buyer's payoff in equilibrium E^{k_j} with order $i_1, i_2^{k_j}, i_3^{k_j}, \ldots$. We are going to construct an equilibrium E^{*k} with the bargaining order $i_1, i_1, i_2^{k_j}, i_3^{k_j}, \ldots$, where the buyer gets no less than in E^k, π_B^k.

Because of Claim 3.1, we can assume that the first agreement is reached in the first period in both E^k and E^{k_j} without loss of generality.

Let us consider equilibrium E^{k_j} with order $\left(i_1, i_2^{k_j}, i_3^{k_j}, \ldots\right)$ first. Because i_1's price is no less than v_{i_1} in E^{k_j}, we have

$$\delta\left(1 - p^1\left(v_{-\tilde{i}_1}\right)\right) - v_{i_1} \geq \pi_B^{k_j} \tag{C.1}$$

where $\tilde{i}_1$ is the player besides i_1. In the first period of $\Gamma\left(i_1, \left(i_1, i_2^{k_j}, i_3^{k_j}, \ldots\right)\right)$, seller i_1 offers $q_{i_1}^{k_j}$ such that

$$\delta\left(1 - p^1\left(v_{\tilde{i}_1}\right)\right) - q_{i_1}^{k_j} = \delta \pi_B^{k_j} \tag{C.2}$$

where $\pi_B^{k_j}$ is the buyer's payoff in $\Gamma\left(B, \left(i_2^{k_j}, i_3^{k_j}, \ldots\right)\right)$ according to E^{k_j}, and $q_{i_1}^{k_j}$ is also accepted. In the first period of $\Gamma\left(B, \left(i_1, i_2^{k_j}, i_3^{k_j}, \ldots\right)\right)$, the buyer offers such that seller i_1 is indifferent between accepting and rejecting and the seller accepts it,

$$p_{i_1}^{k_j} = H_{i_1,1} + \delta q_{i_1}^{k_j} \tag{C.3}$$

and $p_{i_1}^{k_j}$ is also accepted. $q_{i_1}^{k_j} \geq v_{i_1}$ and $p_{i_1}^{k_j} \geq v_{i_1}$ is guaranteed by (C.1).

An equilibrium, E^{*k}, with the order $i_1, i_1, i_2^{k_j}, i_3^{k_j}, \ldots$ is given as follows. The strategies in the subgame $\Gamma\left(B,\left(i_1,i_2^{k_j},i_3^{k_j},....\right)\right)$ are the same as in E^{k_j}. In the subgame $\Gamma\left(i_1,\left(i_1,i_1,i_2^{k_j},i_3^{k_j},...\right)\right)$, seller i_1 offers $q_{i_1}^{*k}$ and it is accepted. In the subgame $\Gamma\left(B,\left(i_1,i_1,i_2^{k_j},i_3^{k_j},...\right)\right)$, the buyer offers $p_{i_1}^{*k}$ and it is also accepted.

In particular, in the first period of $\Gamma\left(i_1,\left(i_1,i_1,i_2^{k_j},i_3^{k_j},...\right)\right)$, seller i_1 offers $q_{i_1}^{*k}$ such that

$$\delta\left(1-p^1\left(v_{\tilde{\imath}_1}\right)\right)-q_{i_1}^{*k}=\delta\pi_B^{k_j} \tag{C.4}$$

where $\pi_B^{k_j}$ is the buyer's payoff in E^{k_j}, and $q_{i_1}^{*k}\geq v_{i_1}$ is guaranteed by (C.1). In the first period of $\Gamma\left(B,\left(i_1,i_1,i_2^{k_j},i_3^{k_j},...\right)\right)$, the buyer offers such that seller i_1 is indifferent between accepting and rejecting and the seller accepts it, so

$$p_{i_1}^{*k}=H_{i_1,1}+\delta q_{i_1}^{*k} \tag{C.5}$$

Now let us compare E^{k_j} and E^{*k}. By definition of $\pi_B^{k_j}$, we have $\pi_B^{k_j}\geq\hat{\pi}_B^k$, therefore (C.2) and (C.4) imply $q_1^{*k}\leq q_1^k$, which, combined with (C.3) and (C.5), gives $p_1^{*k}\leq p_1^k$. Notice that both p_1^{*k} and p_1^k are accepted in E^k and E^{*k} respectively, so the buyer's payoff in E^{*k} is no less than in E^k, therefore E^{*k} achieves a payoff for the buyer, which is higher than π_B^k.

In the second case, suppose there is no payoff in $\left\{\pi_B^{k_m}\right\}_{m=1}^{\infty}$ such that its equilibrium order starts with i_1. Then, there exists a $K\in\mathbb{N}$ such that, for any $k\geq K$, π_B^k is associated with a bargaining order starting with seller $\tilde{\imath}_1$. Pick any element in $\left\{\pi_B^k\right\}_{k=K}^{\infty}$, $\pi_B'^k$. Let the corresponding equilibrium order is $\tilde{\imath}_1,i_2^{k_m},...$, there also exists $\pi_B'^{k_m}$ with order $\bar{\imath}_1,i_2'^{k_m},...$ such that the equilibrium payoff for the buyer is higher than $i_2^{k_m},i_2^{k_m},....$ Similarly as in the first case, there is an equilibrium with order $\tilde{\imath}_1,\tilde{\imath}_1,i_2^{k_m},i_3^{k_m},...$ giving the buyer at least $\pi_B'^k$.

In sum, we have shown that for any sequence of equilibrium payoffs converging to $\bar{\pi}_B$, there is another sequence also converge to $\bar{\pi}_B$, and each equilibrium payoff in this sequence has the same first two elements in its bargaining order.

By induction, if there exists a sequence of equilibria with first t elements identical whose buyer's payoffs converge to $\bar{\pi}_B$, there is another sequence of equilibria with identical first $t+1$ elements, where the buyer's payoffs converge to $\bar{\pi}_B$. Hence, $\bar{\pi}_B$ can be approached by equilibria with either sequence $1, 1, ...$ or sequence $2, 2, ...$ ■

Claim 3.3. $\Gamma(B, (2, 2, ...))$ *has a unique equilibrium with an agreement in the first period,* $\Gamma(B, (1, 1, ...))$ *either has no agreement or has a unique equilibrium with an agreement in the first period.*

Proof: The proof is similar to the analysis for the Rubinstein bargaining game.[1] In particular, the supremum of p_2 and the supremum of q_2 in the equilibria of $\Gamma(B, (2, 2, ...))$ satisfy (3.6) and (3.7), so do the infimum of p_2 and the infimum of q_2. Therefore the supremum and infimum of p_2 coincide, which implies that the selling price for seller 2 is unique. As a result, the unique outcome is $(p_1^1, p_2^2, 2, 1)$ where p_2^2 is given in (3.9).

Given the bargaining order, seller 1 sells first or there is no agreement in $\Gamma(B, (1, 1, ...))$. Suppose seller 1 sells first, his price is at least v_1, then the buyer's payoff is at most $-v_1 - \delta p_2^1 + \delta$, which is negative when $\delta v_2 + (1+\delta) v_1 > \delta$. As a result, $\Gamma(B, (1, 1, ...))$ has no agreement if $\delta v_2 + (1+\delta) v_1 > \delta$. On the other hand, if $\delta v_2 + (1+\delta) v_1 \leq \delta$, similarly as for $\Gamma(B, (2, 2, ...))$, $\Gamma(B, (1, 1, ...))$ also has a unique equilibrium with an agreement in the first period. ■

Claim 3.4. *The buyer's equilibrium payoff in* $\Gamma(B, (1, 1, ...))$ *is less than* $\bar{\pi}_B$.

Proof: $\Gamma(B, (2, 2, ...))$ has a unique equilibrium according to Claim 3.3, and the equilibrium is given in Lemma 3.2. Therefore, the buyer's payoff is given in (3.10), which is positive because of (3.8).

Claim 3.3 implies that $\Gamma(B, (1, 1, ...))$ has no agreement or a unique equilibrium with an agreement in the first period. If $\Gamma(B, (1, 1, ...))$ has no agreement, the buyer's payoff is zero. If $\Gamma(B, (1, 1, ...))$ has a unique equilibrium with an agreement in the first period, an analogue of (3.6) and (3.7) gives the unique selling price for seller 1, so the buyer's payoff is

$$\pi'_B = \frac{1}{1+\delta}\left[\delta \frac{1}{1+\delta}(1 - v_2) - v_1\right]$$

[1]See Fudenberg and Tirole (1991), pp 115-116.

which is also less than π_B^*. Hence, the buyer's equilibrium payoff in $\Gamma(B,(1,1,...))$ is less than π_B^*, so it is also less than $\bar{\pi}_B$. ■

Claim 3.5. *The buyer's equilibrium payoff is $\bar{\pi}_B$ only when the bargaining order is $2,2,...$.*

Proof: Claim 3.2 to 3.4 imply that the buyer's equilibrium payoff in $\Gamma(B,(2,2,...))$ is $\bar{\pi}_B$, so it is sufficient to show that the buyer's equilibrium payoff given any other order is less than $\bar{\pi}_B$. Suppose there is an equilibrium, E_2 with another order $i_1, i_2, i_3, ...$ where the buyer's payoff is also $\bar{\pi}_B$, we are going to show that this assumption leads to a contradiction.

First consider the case with $i_1 = 2$. Since the order $i_1, i_2, ...$ is different from $2, 2, ...$, assume $i_2 = 1$ without loss of generality.[2] By the same reason in Claim 3.1, E_2 has an agreement in period 1. Since seller 2's price is no less than v_2, we have

$$\delta\left(1 - p_1^1\right) - v_2 \geq \bar{\pi}_B \tag{C.6}$$

In the first period of $\Gamma(2,(2,1,i_3,...))$, seller 2 offers q_2'' such that

$$\delta\left(1 - p_1^1\right) - q_2'' = \delta\pi_B'' \tag{C.7}$$

where π_B'' is the payoff of $\Gamma(B,(1,i_3,...))$ in E_2, and $q_2'' \geq v_2$ is guaranteed by

$$\delta\left(1 - p_1^1\right) - v_2 \geq \bar{\pi}_B \geq \pi_B''$$

where the first inequality is given by (C.6) and the second inequality comes from the definition of $\bar{\pi}_B$. Similarly, the buyer offers p_2'' such that seller 2 is indifferent between

[2] To see why it is without loss of generality to assume i_2 is the first 1 in the order, consider, for example, the case where i_3 is the first 1 in the bargaining order. If the buyer's payoff is less than $\bar{\pi}_B$ in the subgame with order $i_2, i_3, ...$, then by backward induction her payoff is also less than $\bar{\pi}_B$ in the game with order $i_1, i_2, ...$

accepting and rejecting in the first period of $\Gamma(B,(2,i_2,i_3,...))$, so

$$p''_2 = H_{2,1} + \delta q''_2 \tag{C.8}$$

In the unique equilibrium given the order $2,2,...$, the buyer offers p_2^2 and seller 2 offers q_2^2 according to (3.6) and (3.7) that can be rewritten as

$$\delta\left(1 - p_1^1\right) - q_2^2 = \delta\bar{\pi}_B \tag{C.9}$$

Claim 3.4 implies that $\pi''_B < \bar{\pi}_B$, so we have $q_2^2 < q''_2$ by comparing (C.9) and (C.9). Similarly, by comparing (3.6) and (C.8), we have $p_2^2 < p''_2$, hence the buyer's payoff in E_2 is less than $\bar{\pi}_B$. This is a contradiction to the definition of E_2.

Similarly, there is also a contradiction when $i_1 = 1$. Hence, if the bargaining order is different from $2,2,...$, the buyer's equilibrium payoff is less than $\bar{\pi}_B$. ■

Claims 3.1 to 3.5 prove Proposition 3.1. ■

C.2 Proof of Proposition 3.2

Lemma C.1. *Perpetual disagreement cannot be an equilibrium outcome.*

Proof: It is easy to see $\pi_B = 0$ under perpetual disagreement.

If the buyer first purchases from seller 1, she would have negative payoff, so seller 2 is the first seller and sells in period t_2 for a price of p_2^2, then seller 1 sells in the next period at price p_1^1. In order to have a balanced budget, the total payments to all the players should equal the total value of the mall,

$$p_2^2 + \delta p_1^1 + \delta\pi_B = \delta,$$

which implies

$$p_2^2 = \delta - \delta p_1^1 - \delta\pi_B. \tag{C.10}$$

So seller 2's payoff is

$$\begin{aligned}\pi_2 &= H_{2,t_2-1} + \delta^{t_2-1} p_2^2 \\ &= H_{2,t_2-1} + \delta^{t_2-1}\left(\delta - \delta p_1^1 - \delta \pi_B\right)\end{aligned}$$

where the second inequality comes from (C.10). π_2 is a function of t_2 and π_B, and it has an upper bound:

$$\begin{aligned}\overline{\pi}_2 &\equiv \max_{t_2 \geq 2, \pi_B \geq 0} \left[H_{2,t_2-1} + \delta^{t_2-1}\left(\delta - \delta p_1^1 - \delta \pi_B\right)\right] \\ &= \delta - \delta p_1^1\end{aligned}$$

When seller 2 faces an offer, his payoff also has the same upper bound. As a result, if seller 2 rejects the buyer's offer, he gets one period of harvest in the current period and at most $\overline{\pi}_2$ in the next period. Because $H_{2,1} + \delta \overline{\pi}_2 < \overline{\pi}_2$, there exits $\varepsilon > 0$ such that

$$H_{2,1} + \delta \overline{\pi}_2 + \varepsilon < \overline{\pi}_2.$$

If the buyer offers to seller 2 the price $H_{2,1} + \delta \overline{\pi}_2 + \varepsilon$ in period 1, seller 2 would accept it because he receives at most $H_{2,1} + \delta \overline{\pi}_2$ otherwise. Then the buyer's payoff π_B satisfies the following budget balance condition:

$$\left(H_{2,1} + \delta \overline{\pi}_2 + \varepsilon\right) + \delta p_1^1 + \delta \pi_B = \delta$$

so

$$\begin{aligned}\pi_B &= -\left(H_{2,1} + \delta \overline{\pi}_2 + \varepsilon\right) - p_1^1 + 1 \\ &> -\left(H_{2,1} + \delta \overline{\pi}_2\right)/\delta - p_1^1 + 1 \\ &= 0\end{aligned}$$

Hence the buyer could always guarantee himself positive payoff by offering $H_{2,1} + \delta \overline{\pi}_2 + \varepsilon$ to seller 2 in the first period. So permanent disagreement cannot be an equilibrium outcome. ■

Proof of Lemma 3.4: Lemma C.1 implies that there is always a first seller. Suppose the first seller sells in period t of $\hat{\Gamma}(B,2)$ and $t > 1$. As discussed after Lemma 3.3, the buyer would have a negative payoff if seller 1 sells first, so the first seller must be seller 2. Let m_B and M_B be the infimum and supremum of equilibrium payoffs of the buyer, and we have $m_B, M_B > 0$.

Let (π_1, π_2, π_B) be a payoff vector, and $U(j,j')$ denotes the set of all equilibrium payoff vectors of subgame $\hat{\Gamma}(j,j')$.

Suppose seller 2 sells immediately at price p_2 in subgame $\hat{\Gamma}(B,2)$, then seller 2 and the buyer's payoffs are

$$\begin{aligned} \pi_B &= \delta - \delta p_1^1 - p_2 \\ \pi_2 &= p_2 \end{aligned}$$

which imply

$$\pi_2 + \pi_B = \delta - \delta p_1^1 \equiv \pi_{-1}(B,2)$$

Similarly, if seller 2 sells in the first period of subgame $\hat{\Gamma}(2,B)$, we have

$$\pi_2 + \pi_B = \pi_{-1}(B,2)$$

If seller 2 sells at price p_2 in the third period of $\hat{\Gamma}(B,1)$, the payoffs for seller 2 and the buyer are

$$\begin{aligned} \pi_B &= \delta^3 - \delta^3 p_1^1 - \delta^2 p_2 \\ \pi_2 &= H_{2,2} + \delta^2 p_2 \end{aligned}$$

which imply

$$\pi_2 + \pi_B = \delta^3 - \delta^3 p_1^1 + H_{2,2} \equiv \pi_{-1}(B,1)$$

Similarly, if seller 2 sells in the second period of $\hat{\Gamma}(1,B)$, we have

$$\pi_2 + \pi_B = \delta^2 - \delta^2 p_1^1 + H_{2,1} \equiv \pi_{-1}(1,B)$$

Let m_{B2} and M_{B2} be the infimum and supremum of π_2 in $U(B,2)$ and m_{2B} and M_{2B} be the infimum and supremum of π_2 in $U(2,B)$. In $\hat{\Gamma}(B,2)$, we have

$$M_{B2} = H_{2,1} + \delta M_{2B} \quad \text{(C.11)}$$

$$m_{B2} = H_{2,1} + \delta m_{2B} \quad \text{(C.12)}$$

In $\hat{\Gamma}(2,B)$, we have

$$\pi_{-1}(2,B) - M_{2B} = \delta M_B \quad \text{(C.13)}$$

$$\pi_{-1}(2,B) - m_{2B} = \delta m_B \quad \text{(C.14)}$$

Since there could not be agreement in the first two periods of subgame $\hat{\Gamma}(B,1)$, the infimum and supremum of π_B in the subgame is $\delta^2 M_B$ and $\delta^2 m_B$.

Since $\delta < 1$ and $m_B, M_B > 0$, infimum and supremum of π_B of the whole game must satisfies

$$M_B = \pi_{-1}(B,2) - m_{B2} \quad \text{(C.15)}$$

$$m_B = \pi_{-1}(B,2) - M_{B2} \quad \text{(C.16)}$$

Since there is a unique solution to (C.11)-(C.16), we have

$$M_{B2} = m_{B2}$$

$$M_{2B} = m_{2B}$$

$$M_B = m_B$$

Suppose that there is an equilibrium with delay, then the equilibrium payoff of the buyer must be lower than that in an equilibrium without delay. Thus we have a contradiction because we would have $m_B < M_B$. Hence, there is unique element in $U(B,2)$, $U(2,B)$, $U(B,1)$ and $U(1,B)$. ■

C.3 Proof of Proposition 3.3

Proof: The arguments for Lemma 3.2 and Proposition 3.1 also apply to the N-seller game with commitment. As a result, only a sketch of the proof is provided below.

Let π_B^{N-1*} be the unique equilibrium payoff in $(N-1)$-seller game. As in Lemma 3.2, let $\left(p_N^N, q_N^N\right)$ be the solution to the equations

$$p_N = H_{N,1} + \delta q_N \tag{C.17}$$
$$\delta \pi_B^{N-1*} - q_N = \delta\left(\delta \pi_B^{N-1*} - p_N\right) \tag{C.18}$$

There is an equilibrium where the buyer suggests price p_N^N to seller N and accepts price no more than q_N^N from seller N; and seller N suggests price q_N^N and accepts price no less than p_N^N. The corresponding equilibrium payoff for the buyer is

$$\pi_B^{N*} = \delta \pi_B^{N-1*} - p_N^N$$

and it is easy to show by induction that

$$\pi_B^{N*} = \frac{1}{1+\delta}\left(\left(\frac{\delta}{\delta+1}\right)^{N-1} - \sum_{i=1}^{N}\left(\frac{\delta}{1+\delta}\right)^{N-i} v_i\right) \tag{C.19}$$

Equilibrium price for seller N can be solved from (C.17), (C.18) and (C.19),

$$p_N^N = \frac{\delta}{1+\delta}\left(\left(\frac{\delta}{\delta+1}\right)^{N-1} - \sum_{i=1}^{N}\left(\frac{\delta}{1+\delta}\right)^{N-i} v_i\right) \tag{C.20}$$

As a result, (3.16) ensures that the mall is profitable for the buyer and seller N. Moreover, it is easy to verify that (3.16) also implies that any other seller's equilibrium price is also higher than the value of his land. (C.20) gives the equilibrium price for seller N in the N-seller game, and it also gives any seller n's equilibrium price p_n^n in the N-seller game if N is replaced with n in the equation. Therefore, the unique equilibrium outcome is $\left(p_1^1, p_2^2, ..., p_N^N, N, N-1, ..., 1\right)$.

Following the same reasoning in the proof of Proposition 3.1, we can show that there is a unique equilibrium outcome with the unique equilibrium bargaining order $N, N, ...$ ■

C.4 Proof of Proposition 3.4

Consider the N-seller game without commitment. Lemma C.4 and Proposition 3.4 extends the analysis in Section 3.3.1 to the N-seller case. (C.19) also defines a function $\pi_B^{N*}(v_1, ..., v_N)$ for any N, so $\pi_B^{N-1*}(\mathbf{v}_{-i})$ denotes the equilibrium payoff in the $(N-1)$-seller game without seller i where $\mathbf{v}_{-i} = (v_1, ..., v_{i-1}, v_{i+1}, ..., v_N)$. It is easy to see from (C.19) that π_B^{n*} is a linear function of $v_1, ..., v_n$, and v_i's coefficient is smaller than v_{i+1}.

Lemma C.4. *For every $i \in \{1, ..., N-1\}$, let (p_i^N, q_{Bi}^N) be the solution to the equations*

$$p_i = H_{i,N+1} + \delta^{N+1} p_i^i \tag{C.21}$$

$$\delta \pi_B^{N-1^*}(\mathbf{v}_{-i}) - q_{Bi} = \delta \pi_B^{N*} \tag{C.22}$$

If (3.17) for $n = 2, ..., N$ and (3.16) are satisfied, given any (q_i^N, p_{Bi}^N) such that $q_i^N > q_{Bi}^N$ and $p_{Bi}^N < q_i^N$, the following strategies constitute an equilibrium for the game $\hat{\Gamma}(B, N)$:

i) seller N suggests price q_N^N and accepts price no less than p_N^N,

ii) seller i suggests price q_i^N and accepts price no less than p_i^N for $i = 1, \cdots, N-1$,

iii) the buyer bargains with seller N before the first agreement; suggests price p_N^N to seller N and price p_{Bi}^i to seller $i = 1, \cdots, N-1$; and accepts price no more than q_N^N from seller N and price no more than q_{Bi}^N from seller $i = 1, \cdots, N-1$.

Proof: Induction on the number of sellers is used to prove. Suppose the lemma is true for $N = k$. By the same backward induction analysis in Lemma 3.3, the above lemma is also true for $N = k+1$. Therefore, only the arguments related with the interpretations are given below.

The conditions in this lemma also have similar interpretations as in Lemma 3.3. For any $i < N$, (C.21) means seller i is indifferent between accepting and rejecting the buyer's offer p_i^N, and (C.22) ensures that the buyer is indifferent between accepting and rejecting seller i's offer q_{Bi}^N.

The sellers sell in the order of increasing size in the first N periods. If seller $N-1$ and seller N exchange their selling time, the buyer's payoff is $\pi_B^{N*}(v_1, ..., v_{N-2}, v_N, v_{N-1})$, and (3.17) for $n = N$ is equivalent to

$$\pi_B^{N*}(v_1, ..., v_{N-2}, v_N, v_{N-1}) < 0.$$

Since v_i's coefficient is smaller than v_{i+1}'s in π_B^{N*} for all i, (3.17) also implies that the buyer receives a negative payoff if any seller other than N sells first. Moreover, (3.17) holds for $n = 2, ..., N$, so the smallest remaining seller has to be the first to sell otherwise the buyer receives a negative payoff. ■

Proof of Proposition 3.4: Similarly as in Lemma 3.4, the equilibrium outcome implied by Lemma C.4 is also unique in $\hat{\Gamma}(B, N)$. As in Section 3.3, if the buyer chooses a seller other than N to bargain with first, the resulting subgame is a proper subgame of $\hat{\Gamma}(B, N)$, so there is no agreement in the first two periods and the buyer chooses seller N to bargain with in the third period and an agreement is reached immediately. As a result, if the buyer bargains with any other seller first, all the selling prices remain the same but there would be two periods of delay, hence the buyer bargains with the smallest remaining seller. ■

Appendix D

A Supplement to Chapter 3

D.1 Two-Seller Game with $2T+1$ Periods

This note studies the two-seller game with a finite horizon of $2T+1$ periods and no commitment.[1] "No commitment" is omitted hereafter since the game with commitment is not discussed here. The two-seller game with $2T+1$ periods is the same as the two-seller game defined in Section 3.3 except that the game ends after $2T+1$ periods.

$\bar{G}(j, j', t)$ denotes the two-seller game that has a horizon of t periods, and player j offering to j' in the first period. $\bar{G}(i, t)$ denotes the one-seller game that has a horizon of t periods and B making the first offer. Recall that $G(j, j')$ denotes the two-seller game with infinite horizon where player j makes the first offer to j', and $G(i)$ denotes the one-seller game with infinite horizon where i is the only seller and the buyer makes the first offer. The buyer offers $p_{i,t}^{n}$ to i in the first period of the n-seller game with t periods, and seller i offers $q_{i,t}^{n}$ in the second period of the n-seller game with t periods.

Lemma D.1: *In the two-seller game with 3 periods, if the mall is profitable as in*

$$(1-\delta)\, v_2 + \delta\,(1-\delta)\, v_1 \leq \delta\,(1-2\delta)\,, \tag{S1}$$

[1]The argument in this note can also be extended to the N-seller game with a horizon of $2T+N-1$ periods and no commitment.

the buyer bargains with seller 2 first until an agreement is reached.

Proof: Backward induction is used. Let us first examine the last period. If neither seller has agreed in the last period, every player gets 0. Suppose that only seller i has not agreed in the last period. If the buyer offers in the last period, she suggests

$$p_{i,1}^{1} = 0.$$

If seller i offers in the last period, he suggests

$$q_{i,1}^{1} = 1.$$

where the second subscript indicates in which period the prices are considered.

Let us move to the second period. On the one hand, suppose that neither seller has agreed in period 2. If seller i offers in period 2, he suggests $q_{i,2}^{2}$ such that the buyer is indifferent between accepting and rejecting:

$$\delta\left(1 - p_{j,1}^{1}\right) - q_{i,2}^{2} = 0$$

On the other hand, suppose only seller i has not agreed. If the buyer offers in period 2, she offers $p_{i,2}^{1}$ such that the seller is indifferent between accepting and rejecting:

$$p_{i,2}^{1} = H_{i,1} + \delta q_{i,1}^{1}$$

where the right land side is the harvest of period 2 and seller i's price in period 3.

Finally, consider the first period. If the buyer bargains with seller i in the first period, she offers $p_{i,3}^{2}$ such that seller i is indifferent between accepting and rejecting:

$$p_{i,3}^{2} = H_{i,1} + \delta q_{i,2}^{2}$$

As a result, if the buyer bargains with seller 2 in the first period, her payoff is

$$\pi_{B,3} = \delta\left(1 - p_{1,2}^{1}\right) - p_{2,3}^{2}$$

In contrast, if the buyer bargains with seller 1 in the first period, her payoff is

$$\pi'_{B,3} = \delta\left(1 - p^1_{2,2}\right) - p^2_{1,3}$$

(S1) ensures that $\pi_{B,3}$ is non-negative. If $\pi'_{B,3}$ is negative, then there is no agreement in the first two periods and the mall is not built, so the buyer prefers bargaining with seller 2 in the first period. If $\pi'_{B,3}$ is positive, it can be verified that $\pi_{B,3} > \pi'_{B,3}$, therefore the buyer chooses seller 2 in the first period. ■

Since the buyer offers first in a one-seller game with a finite horizon, seller 1 offers in the last period of $\bar{G}(1, 2t+2)$, therefore we have

$$\begin{aligned} p^1_{1,2t+2} &= H_{2,1} + \delta q^1_{1,2t+1} \\ 1 - q^1_{1,2t+1} &= \delta\left(1 - p^1_{1,2t}\right) \end{aligned}$$

for all t. Hence, we have

$$p^1_{1,2t+2} = v_2 \frac{1-\delta^{2t+2}}{1+\delta} + \delta\frac{1-\delta^{2t}}{1+\delta} + \delta^{2t+1} \tag{S2}$$

Similarly, the buyer offers in the last period of $\bar{G}(1, 2t+1)$, therefore

$$\begin{aligned} p^1_{1,2t+1} &= H_{2,1} + \delta q^1_{1,2t} \\ 1 - q^1_{1,2t} &= \delta\left(1 - p^1_{1,2t-1}\right) \end{aligned}$$

for all t and

$$p^1_{1,2t+1} = v_2 \frac{1-\delta^{2t}}{1+\delta} + \delta\frac{1-\delta^{2t}}{1+\delta} + \delta^{2t} \tag{S3}$$

Consider the two-seller game with $2t+1$ periods. Suppose that the buyer always bargains with the smallest seller until an agreement is reached, then $p^2_{2,2t+1}$ for $t =$

$1, 2, ...$ can be solved recursively from the equations below

$$p^2_{2,2t+1} = H_{2,1} + \delta q^2_{2,2t} \tag{S4}$$
$$\delta\left(1 - p^1_{1,2t-1}\right) - q^2_{2,2t} = \delta\left[\delta\left(1 - p^1_{1,2t-2}\right) - p^2_{2,2t-1}\right] \tag{S5}$$

Lemma D.2: *Suppose that the buyer chooses to bargain with seller 2 until an agreement is reached in the two-seller game with $2k-1$ periods if the mall is profitable given the horizon $2k-1$ as in*

$$\delta\left(1 - p^1_{1,2k-2}\right) - p^2_{2,2k-1} \geq 0, \tag{S6}$$

then the buyer also chooses to bargain with seller 2 until an agreement is reached in the two-seller game with a horizon of $2k+1$ periods if the mall is profitable given the horizon $2k+1$ as in

$$\delta\left(1 - p^1_{1,2k+2}\right) - p^2_{2,2k+1} \geq 0. \tag{S7}$$

Proof: First, I claim that (S7) implies (S6), which means that if the mall is profitable given a horizon of $2k+1$ periods, it is also profitable given a shorter horizon of $2k-1$ periods. (S4) and (S5) imply that

$$p^2_{2,2k+1} = H_{2,1} + \delta^2\left(1 - p^1_{1,2k-1}\right) - \delta^3\left(1 - p^1_{1,2k-2}\right) + \delta^2 p^2_{2,2k-1}$$

Therefore

$$\begin{aligned} & \pi_{B,2k+1} - \pi_{B,2k-1} \\ = \; & \delta\left(1 - p^1_{1,2k}\right) - \delta^2\left(1 - p^1_{1,2k-1}\right) - H_{2,1} - \left(1 - \delta^2\right)\pi_{B,2k-1} \\ \leq \; & \delta\left(1 - p^1_{1,2k}\right) - \delta^2\left(1 - p^1_{1,2k-1}\right) - H_{2,1} \end{aligned}$$

where $\pi_{B,2k+1} \equiv \delta\left(1 - p^1_{1,2k+2}\right) - p^2_{2,2k+1}$ and $\pi_{B,2k-1} \equiv \delta\left(1 - p^1_{1,2k-2}\right) - p^2_{2,2k-1}$. Substituting (S2) and (S3) into the equation above, we can verify that the right hand side of the above equation is negative. Hence, $\pi_{B,2k+1} < \pi_{B,2k-1}$, therefore (S7) implies (S6).

Suppose the buyer chooses seller 1 in the first period. If no agreement is reached in the first two periods, then in period 3 the buyer choose seller 2 to bargain with by assumption. It is easy to see that the buyer gets higher payoff by choosing seller 2 in the first period.

If an agreement is reached in the first two periods, seller 1 is indifferent between accepting and rejecting in period $t = 1$ and the buyer is indifferent between accepting and rejecting in period $t = 2$. It is easy to see that the buyer gets a higher payoff by choosing seller 2 in the first period. ■

Lemma D.1 and D.2 prove the proposition below.

Proposition D.1: *In the two-seller game with* $2T + 1$ periods, *if the mall is profitable as in*

$$\delta\left(1 - p^1_{1,2T}\right) - p^2_{2,2T+1} \geq 0,$$

there is a unique equilibrium outcome where the buyer bargains with the smaller seller until an agreement is reached.

D.2 Finite horizon vs. Infinite horizon

Lemma D.3: *The equilibrium price in* $\bar{G}(i,t)$ *converges to the equilibrium price in* $G(i)$ *as* t *approaches infinity.*[2]

Lemma D.4: *If the mall is profitable as in (3.8), seller 2's equilibrium price in* $G(B, 2, 2T + 1)$ *converges to the equilibrium price* p^2_2 *in* $G(B, 2)$ *as* T *approaches infinity.*

[2]See in Osborne and Rubinstein (1990), pp 54.

Proof: According to Lemma D.2, the mall is profitable given any finite horizon if (8) is satisfied. Therefore, Proposition D.1 implies that the buyer bargaining with the smaller seller until an agreement is reached and the equilibrium strategies can be solved from (S4) and (S5).

If T approaches infinity, (S4) and (S5) becomes

$$\begin{aligned}
\lim_{t\to\infty} p^2_{2,2t+1} &= H_{2,1} + \delta \lim_{t\to\infty} q^2_{2,2t} \\
\delta\left(1 - \lim_{t\to\infty} p^1_{1,2t-1}\right) - \lim_{t\to\infty} q^2_{2,2t} &= \delta\left[\delta\left(1 - \lim_{t\to\infty} p^1_{1,2t-2}\right) - \lim_{t\to\infty} p^2_{2,2t-1}\right]
\end{aligned}$$

Lemma D.3 implies that $\lim_{t\to\infty} p^1_{1,2t-1} = \lim_{t\to\infty} p^1_{1,2t-2} = p^1_1$, so

$$\begin{aligned}
\lim_{t\to\infty} p^2_{2,2t+1} &= H_{2,1} + \delta \lim_{t\to\infty} q^2_{2,2t} \\
\delta\left(1 - p^1_1\right) - \lim_{t\to\infty} q^2_{2,2t} &= \delta\left[\delta\left(1 - p^1_1\right) - \lim_{t\to\infty} p^2_{2,2t-1}\right]
\end{aligned}$$

Therefore, it can be verified that the equations above imply $\lim_{t\to\infty} p^2_{2,2t+1} = \lim_{t\to\infty} p^2_{2,2t-1} = p^2_2$ and $\lim_{t\to\infty} q^2_{2,2t} = q^2_2$, where p^2_2 and q^2_2 are equilibrium offers in the two-seller game with infinite horizon. As a result, the lemma above is proved. ■

Lemma D.3 and D.4 prove the proposition below.

Proposition D.2: *If the mall is profitable as in (8), the equilibrium outcome of* $\bar{G}(B, 2, 2T+1)$ *converges to the equilibrium outcome* $(2, 1, p^1_1, p^2_2)$ *of* $G(B, 2)$ *as* T *goes to infinity.*

Bibliography

[1] Admati, A. R. and Perry, M. (1987), "Strategic Delay in Bargaining," *Review of Economic Studies,* 54, pp 345-364.

[2] Amann, E., and Leininger, W. (1996), "Asymmetric All-pay Auctions with Incomplete Information: the Two-Player Case", *Game and Economic Behavior*, 14, pp. 1-18.

[3] Blanchard, O. and Kremer, M. (1997), "Disorganization," *Quarterly Journal of Economics,* 112, pp 1091-1126.

[4] Barut, Y., and Kovenock, D. (1998), "The Symmetric Multiple Prize All-Pay Auction with Complete Information", *European Journal of Political Economy*, 14, pp. 627–644.

[5] Baye, M., R., Kovenock, D., and Vries, C. G. (1996), "The All-Pay Auction with Complete Information", *Economic Theory*, 8, pp. 291-305.

[6] Bedrey, Ö. (2009) "Vertical Coordination through Renegotiation and Slotting Fee," Working paper.

[7] Brown, J. (2011), "Quitters Never Win: The (Adverse) Incentive Effects of Competing with Superstars", *Journal of Political Economy*, forthcoming.

[8] Bulow, J., and Levin, J. (2006), "Matching and Price Competition", *American Economic Review,* 96, pp. 652-668.

[9] Cai, H. (2000), "Delay in Multilateral Bargaining under Complete Information," *Journal of Economic Theory,* 93, pp 260-276.

[10] Cai, H. (2003), "Inefficient Markov Perfect Equilibria in Multilateral Bargaining," *Economic Theory,* 22, pp 583-606.

[11] Chae, S. and Yang, J. (1988), "The Unique Perfect Equilibrium of an N-Person Bargaining Game," *Economics Letters,* 28, pp 221-223.

[12] Chapman, C. (2002), *Real Mathematical Analysis*, New York: Springer-Verlag.

[13] Chatterjee, K. and Kim, N. (2005), "Strategic Choice of Sequencing of Negotiations: A Model with One-Sided Incomplete Information," Working paper.

[14] Chowdhury, P. and Sengupta, K. (2008), "Multi-Person Bargaining with Complementarity: Is There Holdout?" MPRA Paper No. 11517.

[15] Chung, J. (2009), "Reaction To Court's Rejection Of Manhattanville Eminent Domain," *News,* December 4, 2009.

[16] Clark, D. J., and Riis, C. (1998), "Competition Over More than One Prize", *American Economic Review*, 88, pp. 276-289.

[17] Coase, R. (1960), "The Problem of Social Cost," *Journal of Law and Economics,* 3, pp 1-44.

[18] Cohen, C., and Sela, A. (2008), "Allocation of Prizes in Asymmetric All-pay Auctions", *European Journal of Political Economy*, 24, pp. 123-132.

[19] Fernandez, R. and Glazer, J. (1991), "Striking for a Bargaining Between Two Completely informed Agents," *American Economic Review,* 81, pp 240-252.

[20] Fershtman, C. (1990), "The Importance of the Agenda in Bargaining," *Games and Economic Behavior,* 2, pp 220-238.

[21] Flamini, F. (2007), "First Things First? The Agenda Formation Problem for Multi-issue Committees," *Journal of Economic Behavior & Organization,* 63, pp 138-157.

[22] French, H. (2007), "In China, Fight Over Development Creates a Star," *New York Times,* March 26, 2007.

[23] González-Díaz, J. and Siegel, R. (2010), "Matching and Price Competition: Beyond Symmetric Linear Costs", working paper, Northwestern University.

[24] Graham, Bryan S.,Imbens, Guido W., and Ridder (2010), "Measuring the Effects of Segregation in the Presence of Social Spillovers: a Nonparametric Approach", NBER working paper.

[25] Cooley, J. (2009), "Desegregation and the Achievement Gap: Do Diverse Peers Help?", working paper, University of Wisconsin-Madison.

[26] Haller, H. and Holden, S. (1990), "A Letter to the Editor on Wage Bargaining," *Journal of Economic Theory,* 52, pp 232-236.

[27] Herrero, M. (1984), "Bargaining and Involuntary Unemployment," unpublished paper, London School of Economics.

[28] Jackson, C. K. (2009), "Ability-Grouping and Academic Inequality: Evidence from Rule-Based Student Assignments", working paper, Cornell University.

[29] Jun, B. H. (1987), "*A Structural Consideration on 3-person Bargaining,*" *(Chapter III in Essays on Topics in Economic Theory,* Ph. D. Thesis, Department of Economics, University of Pennsylvania*).*

[30] Krishna, V. and Serrano, R. (1996), "Multilateral Bargaining," *Review of Economic Studies,* 63, pp 61-80.

[31] Konrad, K. A. (2009), *Strategy and Dynamics in Contests*, Oxford University Press.

[32] Lyle, D. (2008), "The Effects of Peer Heterogeneity and Substitutability on the Production of Human Capital at West Point", West Point working paper.

[33] Moldovanu, B., and Sela, A (2006), "Contest Architecture", *Journal of Economic Theory,* 126, pp. 70-96.

[34] Li, D. (2010), "One-to-Many Bargaining with Endogenous Protocol," Working paper.

[35] Mailath, S. D. and Postelwaite, A. (1990), "Asymmetric Information Bargaining Problems with Many agents," *Review of Economics Studies*, 57, pp 351-367.

[36] Menezes and Pitchford (2004), "A Model of Seller Holdout," *Economic Theory*, 24, pp 231-253.

[37] Muthoo, A. (1999), *Bargaining Theory with Applications,* Cambridge University Press.

[38] "Nail House in Chongqing Demolished," *China Daily*, April 3, 2007.

[39] Noe, T. H. and Wang, J. (2004), "Fooling All of the People Some of the Time: A Theory of Endogenous Sequencing in Confidential Negotiations," *Review of Economic Studies*, 71, 855-881.

[40] Lizzeri, A., and Persico, N. (2000), "Uniqueness and Existence of Equilibrium in Auctions with a Reserve Price", *Game and Economic Behavior*, 30, pp. 83-114.

[41] Lockwood, J. H., and Cleveland, E. F. (1998), "The Challenge of Detracking: Finding the Balance Between Excellence and Equity", U.S. Department of Education, Educational Resources Information Center (ED 422 436).

[42] Moldovanu, B., and Sela, A (2001), "The Optimal Allocation of Prizes in Contests", *American Economic Review*, 91, pp. 542-558.

[43] Olver, P. (2007), *Applied Mathematics Lecture Notes,* Schools of Mathematics, University of Minnesota.

[44] Osborne, M. J. and Rubinstein, A. (1990), *Bargaining and Markets,* Section 3.13, pp 63-65, Academic Press.

[45] Perry, M. and Reny, P. (1993), "A Non-Cooperative Bargaining Model with Strategically Timed Offers," *Journal of Economic Theory*, 59, pp 50-77.

[46] Reinhard, J. and Matthias, G. (2001), "Optimizing Multi-stage Negotiations," *Journal of Economic Behavior & Organization,* 45, pp 155-173.

[47] Rubinstein, A. (1982), "Perfect Equilibrium in a Bargaining Model," *Econometrica,* 50, pp 97-109.

[48] Salmans, S. (1998), "The Tracking Controversy", *New York Times*, April 10, 1988.

[49] Shaked, A., and J. Sutton (1984), "Involuntary Unemployment as a Perfect Equilibrium in a Bargaining Model", *Econometrica*, 52, pp 1351–1364.

[50] Shaw, R. P. (2000), "The Tracking Controversy", http://www.newfoundations.com/ PracEthics/Shaw.html

[51] Sieff, A. (2010), "The Clever Capitalism of Nicholas Sprayregen," *CU Columbia Spectator,* September 16, 2010.

[52] Siegel, R. (2009), "All-Pay Contests", *Econometrica,* 77, pp. 71-92.

[53] Siegel, R. (2010), "Asymmetric Contests with Conditional Investments", *American Economic Review*, 100, pp. 2230–60.

[54] Stole, L. and Zwiebel, J. (1996), "Intra-Firm Bargaining under Non-Binding Contracts," *Review of Economic Studies,* 63, pp 375-410.

[55] Suh, S. and Wen, Q. (2009), "A Multi-Agent Bilateral Bargaining Model with Endogenous Protocol," *Economic Theory,* 40, pp 203-226.

[56] Vanderhart, P. G. (2006), "Why do Some Schools Group by Ability?", *American Journal of Economics and Sociology*, 65, pp. 435-462.

[57] Williams, T. (2008), "2 Gas Stations, and a Family's Resolve, Confront Columbia Expansion Plan," *New York Times,* September 20, 2008.

[58] Winter, E. (1997), "Negotiation in Multi-Issue Committees," *Journal of Public Economics,* 65, pp 323-342.

Jun Xiao

Address: Department of Economics
606 Kern Graduate Building
The Pennsylvania State University
University Park, PA 16802

Phone: (814) 777 1825
E-mail: jux103@psu.edu
Website: http://www.econ.psu.edu/~jux103

Curriculum Vitae

CITIZENSHIP:
- P. R. China (F1 Visa)

EDUCATION:
- Ph.D. Economics, Penn State University, June 2012 (expected)
- M.Phil. Economics, Hong Kong Univ. of Science and Tech., 2007
- B.A. Finance, B.S. Mathematics, Wuhan University, 2004

Ph.D. THESIS:
- *Essays on Contests and Bargaining*
- **Dissertation Advisor:** Prof. Vijay Krishna

FIELDS:
- **Primary:** Microeconomics, Game Theory
- **Secondary:** Industrial Organization

WORKING PAPERS:
- "*Asymmetric All-Pay Contests with Heterogeneous Prizes,"* 2011.
- "*Grouping Players in All-Pay Contests,"* 2011.
- "*Bargaining Order in a Multi-Person Bargaining Game*", 2011.

OTHER CONTRIBUTION:
- "Solution Manual for Vijay Krishna's '*Auction Theory*' (2nd ed.)", with Alexey Kushnir, 2009.

TEACHING EXPERIENCE:
- Teaching Asst., Microeconomic Analysis (Grad.), 2010
- Teaching Asst., Intermediate Microeconomic Analysis, 2008
- Teaching Asst., Intermediate Macroeconomic Analysis, 2007-08

RESEARCH EXPERIENCE:
- Research Assistant for Prof. Vijay Krishna, 2009-2010, 2011-
- Research Assistant for Prof. Timofiy Mylovanov, 2010-2011

PRESENTATION:
- "*Bargaining Order in a Multi-Person Bargaining Game,*" 10th Econometric Society World Congress, 2010