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Abstracts

Chapter 1

"Stability of Steady States in a Matching Model of Money (Case of Two-Unit Bound)"

We provide a stability analysis of the two monetary steady states in a random matching model of money where money is indivisible, the upper bound on individual money holding is two units, and the trading protocol in the match is a buyer take-it-or-leave-it offer. It is shown that the full-support steady state is locally stable and determinate. The non-full-support steady state is unstable. (JEL classification: C62, C78, E40)
Keywords: random matching model; monetary steady state; local stability; determinacy; instability; Zhu (2003).

Chapter 2

"Why Ten $1's Are Not Treated as a $10? (The Instability of Nonfull-support Steady States in a Matching Model of Money)"

This is extension of the instability result of the non-full-support steady state studied in chapter 1. We study stability of a monetary steady state in a random matching model of money where money is indivisible, the upper bound on individual money holding is finite, and the trading protocol is buyer take-it-or-leave-it offers. The steady state we study has a nonfull-support money-holding distribution. It is shown that there is no equilibrium path with a constant payment rule that converges to this steady state if the initial distribution has a different support. (JEL classification: C62, C78, E40)
Keywords: random matching model; monetary steady state; instability; Zhu (2003).

Chapter 3

A comment on: "Efficient propagation of shocks and the optimal return on money"

Lotteries are introduced into Cavalcanti-Erosa (2008), a version of Trejos-Wright (1995) with aggregate shocks. Lotteries improve welfare and eliminate the two notable features of the optimum with deterministic trades: over-production and history-dependence. Moreover, the optimum can be supported by buyer take-it-or-leave-it offers. (JEL classification: C78; D61; D82; E30; E40; E50)
Keywords: random matching model of money; aggregate shock; optimal allocation; history-dependence; lottery.

Chapter 4
"Distributional effects of hiring through networks"

We present a variant of Galenianos (2011), a version of a random search model with two matching technologies: a standard matching function and worker networks. Our model has two types of workers, networked workers and non-networked workers. A steady state equilibrium exists where networked workers have lower unemployment and higher wages, and it is unique under some conditions. Then we ask a question: how would a policy that bans the use of networks in hiring (e.g., anti-old boy network laws) affect welfare? It is shown that the effects of such a policy on non-networked workers can be either positive or negative, depending on model parameters. In our calibration, such a policy would make non-networked workers slightly worse off and networked workers substantially worse off. (JEL classifications: C78, E24, E60, I3, J20, J30)
Keywords: random search, network, referral, policy analysis, welfare, dynamics.

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Chapter 1

Stability of Steady States in a Matching Model of Money (Case of Two-Unit Bound)

abstract

We provide a stability analysis of the two monetary steady states in a random matching model of money where money is indivisible, the upper bound on individual money holding is two units, and the trading protocol in the match is a buyer take-it-or-leave-it offer. It is shown that the full-support steady state is locally stable and determinate. The non-full-support steady state is unstable.[1] (JEL classification: C62, C78, E40)

Keywords: random matching model; monetary steady state; local stability; determinacy; instability; Zhu (2003).

1.1 Introduction

Trejos and Wright (1995) shows the existence of a monetary steady state in a model where agents' money holding is in $\{0, 1\}$. For the trading mechanism of a buyer take-it-or-leave-it offer, Zhu (2003) extends Trejos-Wright to the case of the arbitrary (finite) bound on money holdings. This extension is not trivial because when the bound is more than one, the money-holding distribution across agents is no longer exogenous. For the general bound, Zhu provides a sufficient condition for the existence of a steady state that has a full-support money-holding distribution and a strictly increasing and strictly concave value function of money holdings.

As Zhu shows, the existence of such a full-support steady state also implies the existence of non-full-support steady states constructed as follows. Consider the full-support steady state in a given economy. Then consider a different economy where both the bound and the total stock of money are some $\ell \in \mathbb{N}$ times as much, relative to the original economy. In this new economy, there is a non-full-support steady state where all the owned/traded units of money are also ℓ times as much, but the quantity of production of goods remains unchanged.

[1]This chapter is joint work with Pidong Huang. We are very grateful to Neil Wallace for his kind and insightful advice.

In other words, the difference between the original full-support steady state and the induced non-full-support steady state is nominal, although the difference between the two economic environments is not. Since the new economy has its own full-support steady state, there is multiplicity of steady states.[2]

We provide the stability analysis of the steady states of the Zhu economy for the smallest bound such that the distribution of money is endogenous and such that there are the above kinds of monetary steady states. This bound is two units.[3] When the bound is two, there are two monetary steady states. One steady state is the full-support one, where some people hold and trade one unit of money. The other steady state is the non-full-support one. It is constructed by starting with the steady state of Trejos-Wright's $\{0,1\}$ economy and setting $\ell = 2$. In this steady state, two units of money resemble 'a pair of socks'. No one holds or trades just one sock as the value of holding one sock is the same as that of holding none. We study both of the steady states.

Although the two-unit bound is restrictive, it is enough to demonstrate a sharp contrast between the full-support and the non-full-support steady states. The full-support steady state is locally stable and also determinate in the sense that the equilibrium path converging to it is uniquely determined, given the initial distribution. In contrast, the non-full-support steady state is shown to be unstable in quite a strong sense; if we start with a nearby distribution, which means one in which a small but positive measure of people hold one unit of money, then there is no equilibrium path that converges to the non-full-support steady state. In this respect, our result is consistent with the preference pertubation studied by Wallace and Zhu (2004). They show that a commodity-money refinement rules out the non-full-support steady states while the full-support steady state survives such a refinement.

1.2 The Zhu (2003) Model

Time is discrete, dated as $t \geq 0$. There is a unit measure of non-atomic agents who are infinitely-lived. Also, there are divisible and non-storable consumption goods at each date. Each agent maximizes the discounted sum of expected utility with discount factor $\beta \in (0,1)$. At each date, if an agent produces an amount $q \geq 0$ of the good, the utility cost is q. If an agent consumes an amount $q \geq 0$ of the good, the period utility he gets is $u(q)$, where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave and continuously differentiable on $\mathbb{R}_+$. Also, we assume $u(0) = 0$, $u'(\infty) = 0$ and that $u'(0)$ is sufficiently large but finite. It is assumed that individuals cannot consume their own production goods.

There exists a fixed stock of indivisible money that is perfectly durable. Let the bound on individual money holding be denoted by $B \in \mathbb{N}$ and the per capita stock of money by BM, where $M \in (0,1)$. Let $\mathbb{B} \equiv \{0, 1, \cdots, B\}$ be the set of possible individual money holdings.

In each period, agents are randomly matched in pairs. With probability $1/N$, where

[2] The multiplicity of steady states bears some resemblance to that in Green-Zhou (2002). However, the models are very different, as are the stability results (see their stability paper.)

[3] Lomeli and Temzelides (2002) provides a stability analysis of Trejos-Wright (1995). There is no such analysis outside the case of $\{0,1\}$ money holdings.

$N \geq 2$, an agent is a consumer (producer) and the partner is a producer (consumer). Such meetings are called single-coincidence meetings. With probability $1 - 2/N$, the match is a no-coincidence meeting. In meetings, agents' money holdings are observable, but any other information about an agent's trading history is private.

Consider a date-t single-coincidence meeting between a consumer (potential buyer) with i units of money (pre-trade) and a producer (potential seller) with j units of money (pre-trade), an (i, j)-meeting. In such a meeting, the consumer gives the producer a take-it-or-leave-it offer consisting of the amount of product the producer should make and the amount the consumer will pay. (There are no lotteries.) The producer accepts or rejects the offer. If the producer rejects it, both sides leave the meeting and go on to the next date. In this context, we use the terms 'consumer/producer' and 'buyer/seller' interchangeably.

For each $k \in \mathbb{B}$, let w_k^t be the expected discounted value of holding k units of money, prior to date-t matching. Using w_k^t's, we can formulate the consumer's problem in an (i, j)-meeting as choosing the payment and quantity of production subject to the producer's participation constraint:

$$\max_{p \in \Gamma(i,j), q \in \mathbb{R}_+} \{u(q) + \beta w_{i-p}^{t+1}\} \tag{1.1}$$

$$\text{s.t.} \ -q + \beta w_{j+p}^{t+1} \geq \beta w_j^{t+1}. \tag{1.2}$$

where $\Gamma(i, j) = \{p \in \mathbb{B} | p \leq \min\{i, B - j\}\}$ is the set of feasible payments. We can further simplify the problem; once the consumer chooses the amount of payment p, the optimal amount of production must be such that it exploits all the benefit the producer gets from trade. That is, the amount of production is $\beta w_{j+p}^{t+1} - \beta w_j^{t+1}$. Therefore, the consumer's problem reduces to

$$f^t(i, j) \equiv \max_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \tag{1.3}$$

$$P^t(i, j) \equiv \underset{p \in \Gamma(i,j)}{\operatorname{argmax}} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \tag{1.4}$$

$P^t(i, j)$ may be multi-valued and Zhu introduces randomization. So denote the set of probability distributions on $P^t(i, j)$ by $\Xi^t(i, j)$.

For each $z \in \mathbb{B}$, let π_z^t denote the fraction of agents holding z units of money at the start of period t, so that π^t is a probability distribution on $\mathbb{B}$ with mean BM. The law of motion for π^{t+1} can be expressed as

$$\begin{aligned} \pi_z^{t+1} &= \frac{N-2}{N}\pi_z^t + \frac{2}{N}\sum_{i=0}^{B}\sum_{j=0}^{B}\pi_i^t\pi_j^t\frac{\xi^t(i-z;i,j) + \xi^t(z-j;i,j)}{2}, \\ & \quad \text{for } \xi^t(\cdot, i, j) \in \Xi^t(i, j). \end{aligned} \tag{1.5}$$

The second term of (1.5) tells who in single-coincidence meetings will end up with z units. It is the buyer who originally had some i units and spent $i - z$ units, and the seller who originally had some j units and acquired $z - j$ units.

The value function w^t satisfies the Bellman equation

$$w_i^t = \frac{N-1}{N}\beta w_i^{t+1} + \frac{1}{N}\sum_{j=0}^{B}\pi_j^t f^t(i,j). \tag{1.6}$$

The first term of the (RHS) corresponds to either entering a no-coincidence meeting or becoming a seller, who is indifferent between trading and not trading. When $i=0$, equation (1.6) reduces to $w_0^t = \beta w_0^{t+1}$, so the only nonexplosive case is $w_0^t = 0, \forall t$. For this reason, we focus on equilibria in which the value from owning no money is always zero, letting $w^t \equiv (w_1^t, \cdots, w_B^t)$. Finally, we allow free disposal of money and consider equilibria in which agents are not willing to throw away money. That is, the value function should be nondecreasing in every period:

$$w_k^t \geq w_{k-1}^t, \quad \text{for } k = 1, \cdots B, \quad \text{and } w_0^t = 0. \tag{1.7}$$

Definition 1. *An* equilibrium path *given π^0 is a sequence $\{(\pi^t, w^t)\}_{t=0}^{\infty}$ that satisfies the law of motion (1.5), the Bellman equation (1.6), and non-disposal of money (1.7). (π, w) is a* monetary steady state *if $(\pi^t, w^t) = (\pi, w)$ $\forall t$ is an equilibrium for $\pi^0 = \pi$ and if $w \neq 0$.*

1.3 Steady states for $\mathbb{B} = \{0, 1, 2\}$

Now we assume $\mathbb{B} = \{0,1,2\}$. At any point in time, the money-holding distribution is represented by π_1 only, because the probabilities must sum to one and the average money holding is constant at $2M$. That is, we have

$$(\pi_0, \pi_1, \pi_2) = (1 - M - \pi_1/2,\ \pi_1,\ M - \pi_1/2)\,, \tag{1.8}$$

$$\text{where} \quad \pi_1 \in \Pi \equiv [0, \min\{2M, 2-2M\}]\,. \tag{1.9}$$

The consumer's *payment rule* maps each $(i,j) \in \mathbb{B} \times \mathbb{B}$ to an element of $\Gamma(i,j)$. Two payment rules are of our particular interest. The first rule p^* is defined as "$p^*(i,j) = 1$ whenever feasible". The second rule p^{**} is defined as "the same as p^* except $p^{**}(2,0) = 2$".

Under these payment rules, the steady-state law of motion and Bellman equation follow from (1.5) and (1.6). (Below, $\gamma = 1$ and $\gamma = 0$ correspond to p^* and p^{**}, respectively.)

$$\pi_1 = \pi_1 - \frac{2(\pi_1)^2}{N} + \gamma\left\{\frac{4M(1-M) - 2\pi_1 + (\pi_1)^2}{2N}\right\}, \tag{1.10}$$

$$\left\{\begin{array}{lcl} w_1 & = & \frac{N-1+\pi_2}{N}\beta w_1 + \frac{\pi_0}{N}u(\beta w_1) + \frac{\pi_1}{N}u(\beta w_2 - \beta w_1) \\ w_2 & = & \frac{N-1+\pi_2}{N}\beta w_2 + \frac{\pi_1}{N}\left[u(\beta w_2 - \beta w_1) + \beta w_1\right] \\ & & +\frac{\pi_0}{N}\left[\gamma\left\{u(\beta w_1) + \beta w_1\right\} + (1-\gamma)u(\beta w_2)\right] \end{array}\right. . \tag{1.11}$$

Lemma 1. *(The full-support steady state (π^*, w^*))*

(I) If a full-support steady state supported by p^* exists, then it satisfies the following.

(i) $\pi_1^* \equiv \left(\sqrt{1+12M(1-M)}-1\right)/3$ *with* $\pi_0^*, \pi_2^* > 0$ *given by (1.8).*

(ii) Let $C \equiv \frac{N(1-\beta)+(1-\pi_2^*)\beta}{\beta}$. *Then* w_1^* *is a positive solution to*

$$C\beta w_1^* = \pi_0^* u(\beta w_1^*) + \pi_1^* u\left(\frac{1-\pi_2^*}{C}\beta w_1^*\right). \tag{1.12}$$

(iii) $w_2^* = \left(\frac{(1-\pi_2^*)\beta}{N(1-\beta)+(1-\pi_2^*)\beta}+1\right) w_1^*.$

(II) Let $\underline{u}_{\text{ful}} \equiv C^2/\left[C\pi_0^* + \pi_1^*(1-\pi_2^*)\right]$. *The full-support steady state exists if and only if* $u'(0) > \underline{u}_{\text{ful}}$ *and*

$$u(\beta w_2^*) - u(\beta w_1^*) \leq \beta w_1^* \leq u(\beta w_2^* - \beta w_1^*). \tag{1.13}$$

Moreover, if the inequalities hold strictly, then p^* is a strictly preferred strategy.

Proof. (Lemma 1) Setting $\gamma = 1$ in (1.10) and solving the resulting quadratic equation proves I-(i). (This is the larger root. The smaller root does not satisfy (1.9).) Also, setting $\gamma = 1$ in (1.11) and subtracting the first equation from the second gives I-(iii):

$$\beta w_2^* - \beta w_1^* = \frac{1-\pi_2^*}{C}\beta w_1^* \; (< \beta w_1^*). \tag{1.14}$$

Substituting this result back into the first equation gives I-(ii). Define function h by the (RHS) of (1.12) or $h(x) = \pi_0^* u(x) + \pi_1^* u\left(\frac{1-\pi_2^*}{C}x\right)$. Then a unique positive solution for βw_1^* to (1.12) exists if and only if $h'(0) > C$. This is equivalent to $u'(0) > \underline{\text{u}}_{\text{ful}}$. Finally, we check whether p^* is actually optimal for the consumer, namely whether p^* satisfies (1.4), given the value function. The condition reduces to the following five inequalities:

$$\begin{aligned}
&(1,0)\text{-match} & u(\beta w_1^*) &\geq \beta w_1^* & (1.15)\\
&(1,1)\text{-match} & u(\beta w_2^* - \beta w_1^*) &\geq \beta w_1^* & (1.16)\\
&(2,1)\text{-match} & u(\beta w_2^* - \beta w_1^*) &\geq \beta w_2^* - \beta w_1^* & (1.17)\\
&(2,0)\text{-match} & u(\beta w_1^*) &\geq \beta w_2^* - \beta w_1^* & (1.18)\\
&(2,0)\text{-match} & \beta w_1^* &\geq u(\beta w_2^*) - u(\beta w_1^*). & (1.19)
\end{aligned}$$

It is not hard to show by using the inequality in (1.14), that (1.15), (1.17) and (1.18) hold with strict inequality. Thus p^* is optimal if and only if (1.13) holds. □

The last statement of Lemma 1-II implies that p^* is generically a strictly preferred strategy, which we assume henceforth. More restrictive but sufficient conditions for the existence in terms of model parameters (N, β, M, u) are found in Zhu (2003) and Camera and Corbae (1999).

Lemma 2. *(The non-full-support steady state (π^{**}, w^{**}))*
Let $\underline{u}_{\text{non}} \equiv \frac{N/\beta - N}{1-M} + 1$. *The non-full-support steady state supported by* p^{**} *exists if and only if* $u'(0) > \underline{u}_{\text{non}}$, *and it satisfies the following:*

(i) The money-holding distribution is $(\pi_0^{**}, \pi_1^{**}, \pi_2^{**}) = (1 - M, 0, M)$.

(ii) $w_1^{**} = 0$.

(iii) $w_2^{**} = \bar{v} > 0$ *is the positive solution to*

$$\left(\frac{N/\beta - N}{1 - M} + 1\right) \beta\bar{v} \quad = \quad u(\beta\bar{v}). \tag{1.20}$$

Moreover, p^{**} *is a strictly preferred strategy in* $(1,1)$-, $(2,1)$- *and* $(2,0)$-*meetings and a weakly preferred strategy in* $(1,0)$-*meetings.*

Proof. (Lemma 2) Equation (1.10) with $\gamma = 0$ implies the invariant distribution $(\pi_0^{**}, \pi_1^{**}, \pi_2^{**}) = (1 - M,\ 0,\ M)$. Because $\pi_1^{**} = 0$, the steady-state Bellman equation is reduced to

$$\left(\frac{N/\beta - N}{1 - M} + 1\right) \beta w_k^{**} \quad = \quad u(\beta w_k^{**}), \quad k = 1, 2. \tag{1.21}$$

This has a unique positive solution if and only if $u'(0) > \underline{u}_{\text{non}}$. Requiring that the value function be nondecreasing, there are two valued-money solutions to (1.21): (i) $0 = w_1^{**} < w_2^{**} = \bar{v}$ and (ii) $0 < w_1^{**} = w_2^{**} = \bar{v}$. Case (ii) does not satisfy the optimality of p^{**}, so that case (i) is the only relevant one. Note also that (1.21) implies $u(\beta w_2^{**}) > \beta w_2^{**} > 0$. Using it, one can easily check the strict optimality of p^{**} in all meetings but the $(1,0)$-meeting. In the $(1,0)$-meeting, because $\beta w_1^{**} = 0$, paying one unit is indifferent to paying zero units, so it is optimal only weakly. □

The non-full-support steady state is isomorphic to the monetary steady state of the Trejos-Wright $\{0, 1\}$ economy. Therefore, the condition for the existence is identical to theirs.

Finally the following gives the relation between the two steady states.

Lemma 3. *If the full-support steady state exists, then the non-full-support steady state exists.*

Proof. (Lemma 3)

$$\underline{u}_{\text{ful}} - \underline{u}_{\text{non}} = \frac{N(1-\beta)}{\beta} \cdot \frac{\pi_1^* N(1-\beta) + \beta\pi_1^*\pi_0^*}{[\pi_0^* N(1-\beta) + \beta(1-\pi_2^*)^2](2-2M)} > 0. \tag{1.22}$$

□

We conclude this section by recalling that there always exists a steady state in which money is valueless.

1.4 Stability and Determinacy

Definition 2. *A steady state* (π, w) *is* locally stable *if there is a neighborhood of* π *such that for any initial distribution in the neighborhood, there is an equilibrium path converging to* (π, w)*.*

Definition 3. *A locally stable steady state is* determinate *if the equilibrium path that converges to that steady state is unique.*

The following are our main results.

Proposition 1. *The full-support steady state is locally stable and determinate.*

Proposition 2. *The non-full-support steady state is unstable.*

Proposition 3. *The valueless-money steady state is stable. It is indeterminate if and only if the non-full-support steady state exists.*

It is interesting to compare our result to the stability of the Trejos-Wright (1995)'s $\{0, 1\}$ model, studied by Lomeli and Temzelides (2002)[4] In the $\{0, 1\}$ economy, the monetary steady state is determinate with a constant value of money. The valueless-money steady state is indeterminate if and only if the monetary steady state exists Our Proposition 1 and 3 imply that these results are somewhat carried over to our environment, where the money-holding distribution endogenously evolves. On the other hand, there is no counterpart of Proposition 2 in the $\{0, 1\}$ economy and it is considered new to our paper.

By Lemma 1 and 2, p^* and p^{**} are strictly preferred payment rules at the full-support and the non-full-support steady states, respectively, so they are optimal along the convergent paths. That is, p^* and p^{**} should be constantly played along the paths.[5] Under these fixed payment rules, the law of motion (1.5) reduces to

$$\pi_1^{t+1} = \pi_1^t - \frac{2(\pi_1^t)^2}{N} + \gamma \left\{ \frac{4M(1-M) - 2\pi_1^t + (\pi_1^t)^2}{2N} \right\}, \tag{1.23}$$

where $\gamma = 1$ under p^* and $\gamma = 0$ under p^{**}. The Bellman equation (1.6) becomes

$$\left\{ \begin{array}{rcl} w_1^t & = & \frac{N-1+\pi_2^t}{N} \beta w_1^{t+1} + \frac{\pi_0^t}{N} u(\beta w_1^{t+1}) + \frac{\pi_1^t}{N} u(\beta w_2^{t+1} - \beta w_1^{t+1}) \\ w_2^t & = & \frac{N-1+\pi_2^t}{N} \beta w_2^{t+1} + \frac{\pi_1^t}{N} \left[u(\beta w_2^{t+1} - \beta w_1^{t+1}) + \beta w_1^{t+1} \right] \\ & & + \frac{\pi_0^t}{N} \left[\gamma \left\{ u(\beta w_1^{t+1}) + \beta w_1^{t+1} \right\} + (1-\gamma) u(\beta w_2^{t+1}) \right] \end{array} \right. , \tag{1.24}$$

where π_0^t and π_2^t are to be eliminated using (1.8). Due to (1.7), this mapping is defined on $\Pi \times W$, where $W \equiv \{(w_1, w_2) | 0 \leq w_1 \leq w_2\}$.

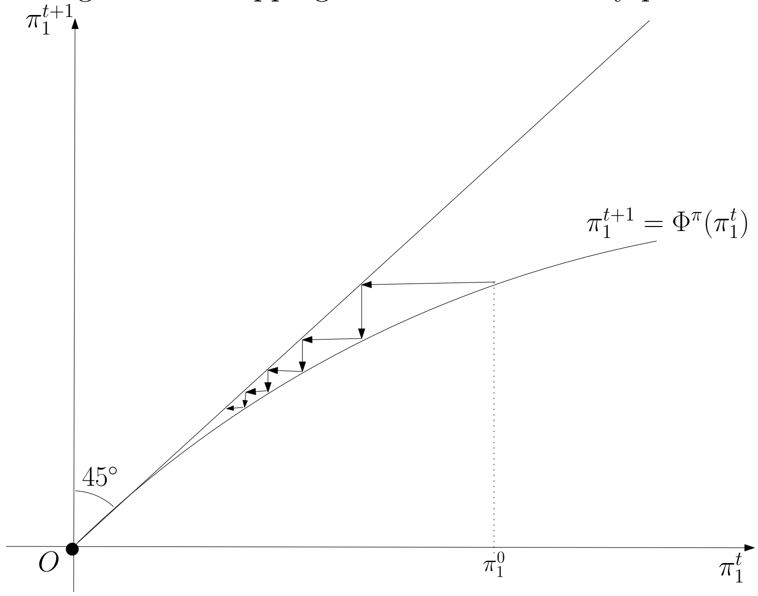
[4] By the above definition of local stability, any steady state in Trejos-Wright is locally stable, because the set of money-holding distributions is a singleton $\{(M, 1-M)\}$.

[5] Paying one unit in $(1, 0)$-meetings is optimal only weakly at the non-full-support steady state, because $w_1^{**} = 0$. However, paying one unit must be strictly preferred along the path converging to that steady state because when w_1^t is close to zero but positive, $u(\beta w_1^t) > \beta w_1^t$ holds.

So far our problem has reduced to analyzing the dynamical system that has three variables π_1^t, w_1^t and w_2^t. Then the proof of Proposition 1 is mostly standard except that the time-direction of the Bellman equation is "backward" so we need to invert it by applying the implicit function theorem. Regarding determinacy, only one variable of our system, π_1^t, has an exogenous initial value. So if the stable manifold of the steady state is one-dimensional, the intersection of the initial condition $\{(\pi_1, w_1, w_2)|\pi_1 = \pi_1^0\}$ and the stable manifold is a single point, resulting in a unique equilibrium path.

As to Proposition 2, we note two prominent features of the dynamics associated with p^{**}. First, the law of motion features unit-root convergence. The law of motion (1.23) under p^{**} is illustrated in Figure 1.1. One can see that $\pi_1^{**} = 0$ is globally stable but the

Figure 1.1: Mapping Φ^π and its stationary point



convergence is extremely slow near that point, the distribution staying almost constant in the end. The reason is that the fewer the people that hold one unit of money, the less frequent are $(1,1)$-meetings, which is the only source of flow from holdings of one unit.

Second, it will be shown that the Bellman equation implies that w_1^t necessarily becomes negative before it converges to zero, and hence violates (1.7).[6] The intuition of this fact is hiden in the first equation of (1.24). If w_1^t converges to zero, always remaining positive, then such convergence should be fast (i.e., exponential), because $u'(0)$ is as high as to satisfy $u'(0) > \underline{\mathrm{u}}_{\mathrm{non}}$. On the other hand, the convergence of π_1^t to zero is extremely slow, remaining almost constant. Therefore, as w_1^t becomes sufficiently small, the third term of the (RHS) becomes too large for the equation to hold (unless w_1^{t+1} is allowed to be negative). To make

[6]Or, if we do not allow w_1^t to take negative values, the value would vanish after some point and there would be no sequence of values that satisfies the Bellman equation.

this argument formal, we explicitly calculate the eigenvector of the Jacobian matrix of the system that approximates the trajectory of convergence.

1.5 Proposition proofs

1.5.1 Proof of Proposition 1

Denote the law of motion derived from equation (1.23) with $\gamma = 1$ as $\Phi^\pi : \Pi \to \Pi$. Denote also the Bellman equation derived from (1.24) with $\gamma = 1$ and (1.8) as $w^t = \phi(\pi_1^t, w^{t+1})$, where $w^t \equiv (w_1^t, w_2^t)$.

Lemma 4. *The full-support steady state satisfies the following inequality:*

$$\pi_0^* u'(\beta w_1^*) + \pi_1^* \frac{1-\pi_2^*}{C} u'(\beta w_2^* - \beta w_1^*) < C. \tag{1.25}$$

Proof. (Lemma 4) Define h as in the proof of Lemma 1. At βw_1^*, the slope of h should necessarily be smaller than C. This together with (1.14) implies the statement. □

Lemma 5.
Generically, $w^t = \phi(\pi_1^t, w^{t+1})$, is invertible around the steady state; that is, there exists a mapping $w^{t+1} = \Phi^w(\pi_1^t, w^t) : U \to \mathbb{R}^2$, where U is an open neighborhood of (π_1^*, w^*), that satisfies

$$w^* = \Phi^w(\pi_1^*, w^*) \tag{1.26}$$

$$w = \phi(\pi_1, \Phi^w(\pi_1, w)), \quad \forall (\pi_1, w) \in U, \textit{ and} \tag{1.27}$$

$$\Phi^w(U) \subset W^o \textit{ (the interior of } W\textit{)}. \tag{1.28}$$

Proof. (Lemma 5) Define a set $S \subset \mathbb{R}^5$ by $S = W^o \times \Pi^o \times W^o$, where Π^o and W^o are the interior of Π and W, and a mapping $F : S \to \mathbb{R}^2$ by $F(w, \pi_1, w') \equiv \phi(\pi_1, w') - w$. By the steady state argument, $F(w^*, \pi_1^*, w^*) = \phi(\pi^*, w^*) - w^* = 0$. $F_{w'}(w^*, \pi^*, w^*)$ is generically invertible. So, applying the implicit function theorem to $F(w, \pi_1, w')$ at (w^*, π_1^*, w^*), we have $w' = \Phi^w(\pi_1, w) : U \to \mathbb{R}^2$, where $U \subset \Pi^o \times W^o$ is a neighborhood of (π_1^*, w^*) that satisfies (1.26) and (1.27). Note that (1.26) and $w^* \in W^o$ implies that if one takes U small enough, then $\Phi^w(U) \subset W^o$, or (1.28). □

Proof. (Proposition 1)
First we show the local stability. The derivative of the law of motion Φ^π evaluated at the steady state is

$$\begin{aligned}\Phi_\pi^\pi(\pi_1^*) &= 1 - \frac{3\pi_1^* + 1}{N} \\ &= 1 - \frac{\sqrt{1 + 12M(1-M)}}{N} \ (< 1),\end{aligned} \tag{1.29}$$

where the second equality uses Lemma 1-I(i). Since it is smaller than one in absolute value, the law of motion is convergent. Moreover, along any convergent path, the no-disposal-of-money condition is ensured by (1.28). Therefore, the full-support steady state is locally stable.

For determinacy, let $\Phi \equiv [\Phi^\pi, \Phi^w]'$ be the joint mapping, where Φ^w exists by Lemma 5. We calculate

$$A \equiv \begin{bmatrix} \Phi^\pi_\pi & O \\ \Phi^w_\pi & \Phi^w_w \end{bmatrix}, \tag{1.30}$$

the 3×3 Jacobian matrix of Φ evaluated at (π_1^*, w^*). The top left matrix of (1.30) is given by (1.29). The two matrices in the second row of (1.30) are obtained by applying the chain rule to equation (1.27). Differentiating (1.27) with respect to π and w and evaluating these derivatives at (π_1^*, w^*), we have

$$\Phi^w_\pi(\pi_1^*, w^*) = -[\phi_2(\pi_1^*, w^*)]^{-1}\phi_1(\pi_1^*, w^*) \tag{1.31}$$
$$\Phi^w_w(\pi_1^*, w^*) = [\phi_2(\pi_1^*, w^*)]^{-1}, \tag{1.32}$$

where ϕ_1 and ϕ_2 denote the partial derivative of ϕ with respect to its first and second argument, respectively. Trivially, one further computes

$$D \equiv \phi_2(\pi_1^*, w^*) = \tag{1.33}$$

$$\begin{bmatrix} \frac{(N-1+\pi_2^*)\beta}{N} + \frac{\pi_0^* u'(\beta w_1^*)\beta}{N} - \frac{\pi_1^* u'(\beta w_2^* - \beta w_1^*)\beta}{N} & \frac{\pi_1^* u'(\beta w_2^* - \beta w_1^*)\beta}{N} \\ \frac{\pi_0^*\{u'(\beta w_1^*)\beta + \beta\}}{N} + \frac{\pi_1^*\{u'(\beta w_2^* - \beta w_1^*)(-\beta) + \beta\}}{N} & \frac{(N-1+\pi_2^*)\beta}{N} + \frac{\pi_1^* u'(\beta w_2^* - \beta w_1^*)\beta}{N} \end{bmatrix}.$$

To study determinacy, we find all of the eigenvalues of A. One eigenvalue is given by (1.29) and is smaller than one. The other two are those of D^{-1}, which are given by the reciprocals of eigenvalues of D. The eigenvalues of D are

$$\frac{\beta}{2N}\left\{2(N-1+\pi_2^*) + \pi_0^* u'(\beta w_1^*) \pm \sqrt{(\pi_0^*)^2 u'(\beta w_1^*)^2 + 4\pi_1^*(1-\pi_2^*)u'(\beta w_2^* - \beta w_1^*)}\right\}. \tag{1.34}$$

Denote the root with plus and minus by λ^+ and λ^-, respectively. If $\lambda^+ < 1$, then λ^- is also smaller than one in absolute value. So we want to show $\lambda^+ < 1$, which is equivalent to

$$2C - \pi_0^* u'(\beta w_1^*) > \sqrt{(\pi_0^*)^2 u'(\beta w_1^*)^2 + 4\pi_1^*(1-\pi_2^*)u'(\beta w_2^* - \beta w_1^*)}.$$

But $(\text{LHS})^2 - (\text{RHS})^2 = 4C^2 - 4C\{\pi_0^* u'(\beta w_1^*) + \pi_1^* \frac{1-\pi_2^*}{C} u'(\beta w_2^* - \beta w_1^*)\} > 0$, where the last inequality follows from Lemma 4. Therefore, both λ^+ and λ^- are always smaller than one in absolute value, so that the eigenvalues of (1.32) are greater than one in absolute value, in which case the full-support steady state has exactly one-dimensional stable manifold and is determinate. □

1.5.2 Proof of Proposition 2

For the names of the mappings, we abuse notation, using the same letters as in Section 1.5.1. Denote the law of motion given p^{**} by $\Phi^\pi : \Pi \to \Pi$. Denote also the Bellman equation derived from (1.24) with $\gamma = 0$ and (1.8) as $w^t = \phi(\pi_1^t, w^{t+1})$.

Lemma 6.
$w^t = \phi(\pi_1^t, w^{t+1})$, is invertible around the steady state; that is, there exists a mapping $w^{t+1} = \Phi^w(\pi_1^t, w^t) : U \to \mathbb{R}^2$, where U is an open neighborhood of (π_1^{**}, w^{**}), that satisfies

$$\begin{aligned} & w^* = \Phi^w(\pi_1^{**}, w^{**}), \ \ \textit{and} \\ & w = \phi(\pi_1, \Phi^w(\pi_1, w)), \ \ \forall(\pi_1, w) \in U. \end{aligned}$$

Proof. (Lemma 6) We allow w^{t+1} to take negative values by extending the domain of u to $\mathbb{R}$, so the function $w^t = \phi(\pi_1^t, w^{t+1})$ is defined on $\mathbb{R}^3$ instead of $\Pi \times \mathbb{R}_+^2$. Applying the implicit function theorem to $F(w, \pi_1, w') \equiv \phi(\pi_1, w') - w$ around $(w^{**}, \pi_1^{**}, w^{**})$, we obtain the "forward-going" Bellman equation $w' = \Phi^w(\pi_1, w) : U \to \mathbb{R}^2$, where $U \subset \mathbb{R}^3$ is a neighborhood of (π_1^{**}, w^{**}). (The invertibility of $F_{w'}(w^{**}, \pi_1^{**}, w^{**}) = \phi_2(\pi_1^{**}, w^{**})$ is confirmed later by (1.37).) □

A major difference from Lemma 5 is that (π^{**}, w^{**}) is at the boundary of $\Pi \times W$. The consequence is that we don't have a counterpart of (1.28), so Φ^w potentially returns a negative value.

Proof. (Proposition 2)
Let $\Phi \equiv [\Phi^\pi, \Phi^w]'$ be the joint mapping, where Φ^w exists by Lemma 6. We take the same steps to obtain the Jacobian matrix of this mapping as we did for the full-support steady state. It is given by differentiating Φ^π and Φ^w, and evaluating the derivatives at $\pi_1^{**} = 0$, $w_1^{**} = 0$ and $w_2^{**} > 0$. That is, we have

$$A \equiv \begin{bmatrix} \Phi_\pi^\pi & O \\ \Phi_\pi^w & \Phi_w^w \end{bmatrix}, \tag{1.35}$$

with

$$\begin{aligned} & \Phi_\pi^\pi(\pi_1^{**}, w^{**}) = 1 \\ & \Phi_\pi^w(\pi_1^{**}, w^{**}) = -[\phi_2(\pi_1^{**}, w^{**})]^{-1}\phi_1(\pi_1^{**}, w^{**}) \\ & \Phi_w^w(\pi_1^{**}, w^{**}) = [\phi_2(\pi_1^{**}, w^{**})]^{-1}, \end{aligned}$$

where

$$\phi_1(\pi_1^{**}, w^{**}) = \begin{bmatrix} \frac{1}{N}u(\beta w_2^{**}) \\ \frac{1}{2N}[u(\beta w_2^{**}) - \beta w_2^{**}] \end{bmatrix} \equiv \begin{bmatrix} p \\ q \end{bmatrix} \tag{1.36}$$

and

$$\phi_2(\pi_1^{**}, w^{**}) \equiv \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = \tag{1.37}$$

$$\begin{bmatrix} \frac{(N-1+M)\beta}{N} + \frac{1-M}{N}\beta u'(0) & 0 \\ 0 & \frac{(N-1+M)\beta}{N} + \frac{1-M}{N} u'(\beta w_2^{**})\beta \end{bmatrix}.$$

It can be shown that $a > 1$ if and only if condition $u'(0) > \underline{\mathrm{u}}_{\mathrm{non}}$ holds. On the other hand, it is always the case that $d \in (0, 1)$.[7] In summary the Jacobian is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -p/a & 1/a & 0 \\ -q/d & 0 & 1/d \end{bmatrix}. \tag{1.38}$$

Since neither a nor d is equal to one, the Jacobian is diagonalized to $A = X\Lambda X^{-1}$, where $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1/d \end{bmatrix}$, $X = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-p}{a-1} & 1 & 0 \\ \frac{q}{1-d} & 0 & 1 \end{bmatrix}$ and $X^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{p}{a-1} & 1 & 0 \\ \frac{-q}{1-d} & 0 & 1 \end{bmatrix}$. In the context of discrete-time dynamical system theory, the unit root is a "boundary" case in which the higher-order terms should be examined. In our case, the unit eigenvalue comes from the law of motion (1.23) with $\gamma = 0$ (i.e., Figure 1.1), so the eigenvector associated with the unit eigenvalue is necessarily a part of the stable manifold. Then the convergent trajectory of $(\pi_1^t, w_1^t, w_2^t - w_2^{**})$ is approximated by the first column of X. This implies that in the limit, π_1^t and w_1^t are of different signs. Since $\pi_1^t > 0$, this means w_1^t necessarily becomes negative along such a path. □

1.5.3 Proof of Proposition 3

A supporting payment rule is the same as p^{**} used in Section 1.5.2, so the law of motion and the Bellman equation are the same. The difference arises when we evaluate Jacobians (1.36) and (1.37). Because the value of money is zero at the valueless-money steady state, (1.36) changes to $\phi_1 = (0, 0)'$ and (1.37) changes to $\phi_2 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ with $a > 1$. Then (1.38) changes to $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/a & 0 \\ 0 & 0 & 1/a \end{bmatrix}$. By the same argument as in Section 1.5.2, the limiting trajectory is represented by the first column of A, meaning that w_1^t and w_2^t go to zero much faster than π_1^t. All the eigenvectors of A are positive vectors so neither w_1^t nor w_2^t become negative along the path. The stable manifold is three dimensional so the intial values of money are fairly arbitrary and hence this steady state is indeterminate.

[7] To see this, note that (1.20) implies that $\beta w_2^{**} > 0$ is given by the intersection of u and a line with slope $(N/\beta - N)/(1-M) + 1$, so that the slope of u at βw_2^{**} is smaller than $(N/\beta - N)/(1-M) + 1$. This fact leads to the result.

1.6 Conclusions

We presented the stability analysis of Zhu (2003) in the case of a two-unit bound, which is the minimum bound such that the money-holding distribution is endogenous. It is shown that the full-support steady state is locally stable and determinate. In contrast, the non-full-support steady state is unstable. Hence our result provides an equilibrium refinement of the multiplicity of monetary steady states in the Trejos-Wright-Zhu model.

When the bound is two, we can identify the candidate payment rule that supports the steady state. For the general bound case, we do not know what the supporting payment rule is. Therefore, a stability argument must be constructed differently and is left to be studied in the future.

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Chapter 2

Why Ten $1's Are Not Treated as a $10? (The Instability of Nonfull-support Steady States in a Matching Model of Money)

abstract

This is extension of the instability result of the non-full-support steady state studied in chapter 1. We study stability of a monetary steady state in a random matching model of money where money is indivisible, the upper bound on individual money holding is finite, and the trading protocol is buyer take-it-or-leave-it offers. The steady state we study has a nonfull-support money-holding distribution. It is shown that there is no equilibrium path with a constant payment rule that converges to this steady state if the initial distribution has a different support. [1] (JEL classification: C62, C78, E40)

Keywords: random matching model; monetary steady state; instability; Zhu (2003).

2.1 Introduction

Zhu (2003) studies existence of steady states in the random matching monetary model of fiat money. In his model, money is indivisible, there is a finite upper bound on individual money holding, and the trading protocol is buyer take-it-or-leave-it offers. Zhu establishes the existence of a monetary steady state that has a strictly increasing and strictly concave value function and a full-support money-holding distribution. A by-product of his result is the existence of nonfull-support steady states constructed in the following way. For any money-holding bound B and per capita stock of money m, consider the full-support steady state with the money-holding distribution $\pi_n,\ n \in \{0, 1, \cdots, B\}$. Then for any integer $x \geq 2$, the economy with bound xB and money stock xm has a nonfull-support steady state with distribution π' with $\pi'_{xn} = \pi_n,\ n \in \{0, 1, \cdots, B\}$. Moreover, if in the economy with bound B, p units of money is traded for q quantity of production when buyer with i units of money meets seller with j units, then in the economy with bound xB, xp units of money is traded

[1]This chapter is joint work with Pidong Huang. We greatly appreciate Neil Wallace.

for q quantity of production, when buyer with xi units meets seller with xj units.[2] Wallace and Zhu (2004) studies these steady states and finds that a commodity-money refinement rules out such nonfull-support steady states while the full-support steady state survives such refinement.

As yet, however, there is no analytical work on convergence to these steady states; i.e., there is no analysis of local stability. This paper gives some instability result of nonfull-support stead state of the above sort. We show that there is no equilibrium path with a constant pure payment rule that converges to the nonfull-support steady state. By 'constant', we mean that the amount of money paid in each type of meeting does not change over time, although the amount of production in return for this payment may change. By 'pure', we mean that randomization over money payment is not allowed.

This instability result is seen as another refinement of the full-support steady state and gives one good reason why in the real world ten one-dollar bills, for example, has more roles than being just a ten-dollar bill.

2.2 Model and Proposition

2.2.1 Model

The model is Zhu (2003) without randomization. Time is discrete, dated as $t \geq 0$. There is a unit measure of non-atomic agents who are infinitely-lived. Also, there is a single kind of divisible and non-storable consumption good. Each agent maximizes the discounted sum of expected utility with discount factor $\beta \in (0, 1)$. At each date, if an agent produces $q \geq 0$ amount of good, the utility cost is q.[3] If an agent consumes $q \geq 0$ amount of good, the period utility he gets is $u(q)$, where $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is differentiable, strictly increasing, strictly concave and satisfies $u(0) = 0$ and $u'(\infty) = 0$. We also assume that $u'(0)$ is sufficiently large but finite.

There exists a fixed stock of indivisible money which is perfectly durable. Let the bound on individual money holding and the per capita stock of money be denoted by $\bar{B} \in \mathbb{N}$ and $\bar{m} \in (0, \bar{B})$, respectively. Let $\bar{\mathbb{B}} = \{0, 1, \cdots, \bar{B}\}$ denote the set of possible individual money holdings. We will later vary the values for these two parameters and use the word "economy with $(\bar{B}, \bar{m})$".

In each period, agents are randomly matched in pairs. With probability $1/N$, where $N \geq 2$, an agent is a consumer (producer) and the partner is a producer (consumer). Such meetings are called single-coincidence meeting. With probability $1 - 2/N$, the match is a no-coincidence meeting. In meetings, agents' money holdings are observable, but any other information about an agent's trading history is private.

Consider a date-t single-coincidence meeting between a consumer (potential buyer) with i units of money (pre-trade) and a producer (potential seller) with j units of money (pre-trade),

[2] This means that if people bundle x units into one bag and treat it as a single unit of money, the economy returns to the original.

[3] The utility cost can be any convex function and linearity assumption is without loss of generality.

an (i, j)-meeting. In such a meeting, the consumer gives the producer a take-it-or-leave-it offer consisting of the amount of production the producer should make and the amount of payment the consumer makes. (There are no lotteries.) The producer accepts or rejects the offer. If the producer rejects, both sides leave the meeting and go on to the next date. In this contex, we use the terms 'consumer/producer' and 'buyer/seller' interchangeably.

Given this trading protocol, we define the value function from holding money. For each $k \in \bar{\mathbb{B}}$, let w_k^t be the expected discounted value of holding k units of money, prior to date-t matching. Using w_k^t's, we can formulate the consumer's problem in (i, j)-meeting as choosing payment and quantity of production subject to producer's participation constraint:

$$\max_{p \in \Gamma(i,j), q \in \mathbb{R}_+} \{u(q) + \beta w_{i-p}^{t+1}\} \tag{2.1}$$

$$\text{s.t.} \; -q + \beta w_{j+p}^{t+1} \geq \beta w_j^{t+1}. \tag{2.2}$$

where $\Gamma(i, j) = \{p \in \bar{\mathbb{B}} | p \leq \min\{i, \bar{B} - j\}\}$. We can further simplify the problem; once the consumer chooses the amount of payment p, the optimal amount of production must be such that it exploits all the benefit the producer gets from trade. That is, the amount of production is $\beta w_{j+p}^{t+1} - \beta w_j^{t+1}$. Therefore, the consumer's problem reduces to

$$f^t(i, j) \equiv \max_{p \in \Gamma(i,j)} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \tag{2.3}$$

Also, let

$$P^t(i, j) \equiv \underset{p \in \Gamma(i,j)}{\text{argmax}} \{u\left(\beta w_{j+p}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p}^{t+1}\}. \tag{2.4}$$

The optimal payment $P^t(i, j)$ is not necessarily single-valued and Zhu introduces randomization over $P^t(i, j)$. However, we focus on pure payment rule throughout this paper.[4] Formally, at each date t, payment rule $p^t : \bar{\mathbb{B}} \times \bar{\mathbb{B}} \to \bar{\mathbb{B}}$ specifies how many units of money should be transfered in each (i, j)-meeting. The payment rule is said to satisfy buyer's optimality condition at date t, if

$$p^t(i, j) \in P^t(i, j), \;\; \forall (i, j) \in \bar{\mathbb{B}} \times \bar{\mathbb{B}}. \tag{2.5}$$

The two key equations of the model are the law of motion of money-holding distribution and the Bellman equation. For each $k \in \bar{\mathbb{B}}$, let π_k^t denote the fraction of agents holding k units of money at the start of period t, so that π^t is a probability distribution on $\bar{\mathbb{B}}$. Given

[4]We will give discussion about this point after we state the main proposition.

payment rule p^t, the law of motion for π^{t+1} can be expressed as

$$\begin{aligned}\pi_k^{t+1} = \pi_k^t \quad &+ \quad \frac{1}{N}\sum_{i,j}\pi_i^t\pi_j^t\mathbb{1}\{p^t(i,j)\neq 0,\ i-p^t(i,j)=k\} \\ &+ \quad \frac{1}{N}\sum_{i,j}\pi_i^t\pi_j^t\mathbb{1}\{p^t(i,j)\neq 0,\ j+p^t(i,j)=k\} \\ &- \quad \frac{1}{N}\sum_{i,j}\pi_i^t\pi_j^t\mathbb{1}\{i=k,\ p^t(i,j)\neq 0\} \\ &- \quad \frac{1}{N}\sum_{i,j}\pi_i^t\pi_j^t\mathbb{1}\{j=k,\ p^t(i,j)\neq 0\},\end{aligned} \tag{2.6}$$

where $\mathbb{1}\{\cdots\}$ is indicator function.[5] Given p^t, the Bellman equation is

$$w_i^t = \frac{N-1}{N}\beta w_i^{t+1} + \frac{1}{N}\sum_{j=0}^{\bar{B}}\pi_j^t\left\{u\left(\beta w_{j+p^t(i,j)}^{t+1} - \beta w_j^{t+1}\right) + \beta w_{i-p^t(i,j)}^{t+1}\right\}. \tag{2.7}$$

The first term of the (RHS) corresponds to getting into no-coincidence meeting or becoming a producer, who is indifferent between trading and not trading. The second term corresponds to becoming a consumer who gets (2.3). When $i=0$, equation (2.7) reduces to $w_0^t = \beta w_0^{t+1}$, so the only nonexplosive case is $w_0^t = 0, \forall t$. For this reason, we hereafter focus on equilibria in which the value from owning no money is always zero. Finally, we allow free disposal of money and consider equilibria in which agents are not willing to throw away money. That is, value function should be nondecreasing every period;

$$w_i^t \geq w_{i-1}^t, \text{ for } i = 1,\cdots \bar{B}, \text{ and } w_0^t = 0. \tag{2.8}$$

Definition 1. *An* equilibrium path *given an initial distribution* π^0 *is a sequence* $\{(p^t,\pi^t,w^t)\}_{t=0}^{\infty}$ *that satisfies (i) law of motion (2.6), (ii) Bellman equation (2.7), (iii) buyer's optimality condition (2.4)-(2.5), and (iv) no-disposal of money (2.8).*

Definition 2. (p,π,w) *is a* monetary steady state *if* $(p^t,\pi^t,w^t)=(p,\pi,w)$ *for all* t *is an equilibrium for* $\pi^0=\pi$ *and* $w\neq 0$*.*

2.2.2 Proposition

Consider an economy with $(\bar{B},\bar{m}) = (B,m)$ for some (B,m), and suppose that in this economy there exists a Zhu (2003) full-support steady state with a unique pure payment rule, say $(\hat{p},\hat{\pi},\hat{w})$. That is, $\hat{\pi}$ has full support, $\hat{w}$ is strictly increasing and strictly concave as is shown in Zhu (2003), and $\hat{p}$ is a unique solution to (2.3). Now take any integer $x \geq 2$

[5]For example, the second term of (2.6) indicates inflows into k-unit state. $\mathbb{1}\{p^t(i,j)\neq 0,\ i-p^t(i,j)=k\}$ is one if consumer originally does not have k units but comes to have k units after payment. $\pi_i^t\pi_j^t$ is the frequency of such meetings.

and consider a different economy with $(\bar{B}, \bar{m}) = (xB, xm)$. In this economy, Zhu constructs a nonfull-support steady state, say (p^*, π^*, w^*), which is inherited from $(\hat{p}, \hat{\pi}, \hat{w})$. Formally, define sets $\mathbb{B} \equiv \{0, 1, \cdots, B\}$ and $\mathbb{X} = \{0, 1, \cdots, x-1\}$. Then the nonfull-support steady state Zhu (2003) constructs satisfies that for all $i, j \in \mathbb{B}$,

$$\pi^*_{ix} = \hat{\pi}_i, \quad \text{and} \quad \pi^*_{i'+ix} = 0, \quad \forall i' \in \mathbb{X} \backslash \{0\}, \tag{2.9}$$

$$w^*_{ix} = \hat{w}_i, \quad \text{and} \quad w^*_{i'+ix} = w^*_{ix}, \quad \forall i' \in \mathbb{X}, \tag{2.10}$$

$$p^*(ix, jx) = \hat{p}(i, j)x. \tag{2.11}$$

In this nonfull-support steady state, amounts of money people hold and trade are always multiples of x units. For this reason, when we study the nonfull-support steady state, it is convenient to call a bundle of x units of money *'bag'*, and any amount less than x *'change'*. In the nonfull-support steady state, people hold and trade only bags, and nobody holds or trades change. Also, w^* has a step-function form. In other words, incremental value from holding change is zero.[6] The trade also follows that of the original full-support steady state. However, note that $\hat{p}$ is silent about off-the-equilibrium components of p^* (i.e., $p^*(i' + ix, j' + jx)$ when i' or j' is not zero), so p^* is not unique, while π^* and w^* are pinned down by (2.9) and (2.10). Throughout this study, we always mean by steady state, this particular nonfull-support steady state characterized above, even if we don't specify.

Now we are ready to state our main proposition. Again, consider Zhu's full-support steady state $(\hat{p}, \hat{\pi}, \hat{w})$ where $\hat{p}$ is unique pure payment rule. Take any $x \in \mathbb{N}$ and consider the nonfull-support steady state (p^*, π^*, w^*) where p^* is pure.

Proposition 1. *Suppose that the initial distribution π^0 has a different support from that of π^*. Then there is no equilibrium path with pure payment rules that satisfies $(p^t, \pi^t, w^t) \to (p^*, \pi^*, w^*)$.*[7]

In other words, if the economy starts with positive-measure of people holding positive amount of change, the economy cannot converge to the nonfull-support steady state.

In the above, we assumed; (i) $\hat{p}$ is a unique pure payment rule; (ii) p^* is pure; and (iii) p^t is pure. For the simplest nonfull-support steady state $B = 1$ and $x = 2$, none of the above three are problem. But in general, these are limitations to our result.

[6] When $B = 1$ and $x \in \mathbb{N}$, it can be shown that w^* necessarily has a step-function form to be nonfull-support steady state, but it may not be the case with general B. We haven't been successful in showing such necessity. However, we focus on nonfull-support steady state with step-function-form w^*, as is constructed in Zhu (2003).

[7] We think it natural to include convergence of payment rule. Payment rule is people's action observed in the economy. If people's action never converged, then the economy wouldn't really seem to be getting 'steady'.

2.3 Proof

2.3.1 Outline of the proof

The proof of Proposition 1 will be by way of contradiction; suppose that there is an initial distribution with positive measure of agents holding change and there is an equilibrium path with a pure strategy that converges to the nonfull-support steady state from that initial distribution. Since payment is discrete choice, convergence of payment rule implies that it will eventually coincide with the steady state payment rule and then stay constant. So we act as if there is a pure payment rule that is fixed all the time. Lemma 1 lists up what we know about the fixed payment rule.

Given this fixed pure payment rule, we convert the law of motion and the Bellman equation to a regular dynamical system whose state variable is (part of) the money-holding distribution and the value function. We divide the money-holding distribution into two parts: bag-state proportions and change-state proportions. Correspondingly, we divide the value function into two parts: values of bags and values of change. Lemma 2(i), 4(i) and 5 show that the pair of 'change-state proportions' and 'values of change' is locally autonomous so we can focus on the dynamics of these two.

We calculate the Jacobian of the system evaluated at the nonfull-support steady state. The Jacobian is seen as a linear approximation of the original system and some of the eigenvectors of the Jacobian are seen as indicating an approximate stable manifold of the original system. In studying convergence, one studies the eigenvector associated with the largest eigenvalue (in absolte value) that does not exceed one, because that characterizes the slowest direction of convergence. Our law of motion has a very peculiar feature that its Jacobian always has a unit eigenvalue and any convergent path of the distribution must involve the eigenvector associated with it. This is formally shown in Lemma 2(ii)-(v) and 3. So, the change-state population goes to zero extremely slowly.

While the distribution converges, how do the values of change behave in the process of convergence? This question is answered by manipulating the Jacobian of the Bellman equation. Lemma 4(ii)(iii) and 6 gives the approximate trajectory for the values of change. Lemma 7 and the main part of the proof then show that the values of change implied by that trajectory will either become negative or violate the optimality of the payment rule, which is contradiction. So there is no such equilibrium that eventually converges to the non-full support steady state. Proofs of lemmas are in Appendix.

2.3.2 Detailed proof

For our goal, it is convenient to divide money holdings into two parts: *change holdings* and *bag holdings*. If one's money holding is $i' + ix$ for some $i' \in \mathbb{X}$ and $i \in \mathbb{B}$, then his change holding is i' and his bag holding is i, so that his money holding can be alternatively expressed as (i', i). There is one-to-one relation between $i' + ix$ and (i', i), so they are conceptually the same. Correspondingly, $(i' + ix, j' + jx)$-meeting is transformed into $((i', i), (j', j))$-meeting, and payment rule such as $p^*(i' + ix, j' + jx)$ is transformed into $p^*((i', i), (j', j))$. It is useful to keep in mind that variables with prime (') are used to mean "change". It is also helpful to keep in mind that in our notation, change always comes first. Although bags are more valuable for agents, change is more important for our analysis.

For p^* to be payment rule of the nonfull-support steady state equilibrium, the only requirement for p^* is (2.11), or with our new notation, $p^*((0,i),(0,j)) = \hat{p}(i,j)x$. However, since payment rule is discrete object, convergence of payment rule, $p^t \to p^*$ implies that $p^t = p^*, \forall t$ from some point. That is, p^* is optimal for the consumer not only at the steady state but also near the steady state. This requires more conditions on p^*.

Lemma 1. *(Necessary Conditions on p^*)*
For each $((i',i),(j',j))$-meeting, $p^*((i,i'),(j,j'))$ is such that the post-trade bag holdings of buyer and seller are as follows:
(i) if $i'+j' < x$, then $i - \hat{p}(i,j)$ for buyer and $j + \hat{p}(i,j)$ for seller;
(ii) if $i'+j' \geq x$ and $\hat{p}(i+1,j) \geq 1$, then $i+1-\hat{p}(i+1,j)$ for buyer and $j+\hat{p}(i+1,j)$ for seller; and
(iii) if $i'+j' \geq x$ and $\hat{p}(i+1,j) = 0$, then i for buyer and $j+1$ for seller.
(iv) Pre-trade change-holdings, i' and j', affect their post-trade bag-holdings only through whether $i'+j' \geq x$ or not.
(v) Comparing total change (the sum of buyer's and seller's change) between pre-trade and post-trade, it remains unchanged in (i)-meetings and decreases in (ii)(iii)-meetings. It never increases.

To analyze Jacobians for stability analysis, we find it convenient to make re-arrangement of our state variables and the dynamical system, although such transformation involves slightly cumbersome notation. So far, our state variables are distribution variables $(\pi_0^t, \pi_1^t, \cdots, \pi_{Bx}^t)$ and value-function variables $(w_1^t, w_2^t, \cdots, w_{Bx}^t)$, and our dynamical system consists of the law of motion (2.6) and the Bellman equation (2.7). After all the transformation is done, we will get the new state variables $(\Pi_c^t, \Pi_b^t, \Delta W^t, W_b^t)$ and the dynamical system $(\Psi^{\Pi_b}, \Psi^{\Pi_b}, \Phi^{\Delta W}, \Phi^{W_b})$, or

$$\begin{cases} \Pi_c^{t+1} & = & \Psi^{\Pi_c}(\Pi_c^t, \Pi_b^t) \\ \Pi_b^{t+1} & = & \Psi^{\Pi_b}(\Pi_c^t, \Pi_b^t) \\ \Delta W^{t+1} & = & \Phi^{\Delta W}(\Pi_c^t, \Pi_b^t, \Delta W^t, W_b^t) \\ W_b^{t+1} & = & \Phi^{W_b}(\Pi_c^t, \Pi_b^t, \Delta W^t, W_b^t) \end{cases} . \tag{2.12}$$

Once we get this form, we will consider the Jacobian of this system evaluated at the steady state and analyze its eigenvalues/eigenvectors.

The following is the transformation process. First, our distribution always satisfies

$$\sum_{i=0}^{Bx} \pi_i^t = 1, \quad \text{and} \quad \sum_{i=0}^{Bx} i\pi_i^t = xm. \tag{2.13}$$

Therefore, π_0^t and π_{Bx}^t can always be recovered from $\pi_1^t, \cdots, \pi_{Bx-1}^t$ through

$$\begin{aligned} g^{Bx}(\pi_1, \cdots, \pi_{Bx-1}) &= \frac{1}{Bx}\left(xm - \sum_{i=1}^{Bx-1} i\pi_i\right), \\ g^0(\pi_1, \cdots, \pi_{Bx-1}) &= 1 - \frac{m}{B} + \sum_{i=1}^{Bx-1}\left(\frac{i}{Bx} - 1\right)\pi_i, \end{aligned} \tag{2.14}$$

so we drop π_0^t and π_{Bx}^t out of the state variables. Secondly, we change the order of the distribution variables as follows. We call states with positive amount of change "change states" and states without change "bag states". Let $\Pi^t \equiv (\Pi_c^t, \Pi_b^t)$, where $\Pi_c^t \equiv (\Pi_c^t(1), \cdots, \Pi_c^t(x-1))$, and $\Pi_c^t(i') \equiv (\pi_{i'+0x}^t, \pi_{i'+1x}^t, \cdots, \pi_{i'+(B-1)x}^t)$ and $\Pi_b^t \equiv (\pi_{1x}^t, \pi_{2x}^t, \cdots, \pi_{(B-1)x}^t)$. Denote the law of motion given payment rule p^* as $\Pi^{t+1} = \Psi(\Pi^t)$.

Thirdly, we transform the Bellman equation by the following linear transformation of variables. Define the 'incremental value of change' as $\Delta w_{i',i}^t \equiv w_{i'+ix}^t - w_{ix}^t$. Then arrange the order so that $\Delta W^t(i') \equiv (\Delta w_{i',0}^t, \cdots, \Delta w_{i',B-1}^t)$ and $\Delta W^t \equiv (\Delta W^t(1), \cdots, \Delta W^t(x-1))$. Then put them together with 'values of bags', $W_b^t \equiv (w_{1x}^t, w_{2x}^t, \cdots, w_{Bx}^t)$. Our new value-function variables are now $W^t \equiv (\Delta W^t, W_b^t)$. The transformation from w^t to W^t is one-to-one and, more importantly, a linear transformation. We denote the new Bellman equation that could be obtained through such transformation as $W^t = \phi(\Pi^t, W^{t+1})$, although there is no need to explicitly calculate it.[8] The mapping ϕ has two parts: $\phi = (\phi^{\Delta W}, \phi^{W_b})$. So, $\Delta W^t = \phi^{\Delta W}(\Pi^t, W^{t+1})$ and $W_b^t = \phi^{W_b}(\Pi^t, W^{t+1})$.

Since time-direction of the mapping ϕ is backward, we apply the implicit function theorem around the steady state, to obtain the forward-looking Bellman equation, denoted Φ: $W^{t+1} = \Phi(\Pi^t, W^t)$.[9] Again, Φ has two parts: $\Phi = (\Phi^{\Delta W}, \Phi^{W_b})$. After all, our dynamical system is (2.12).

The Jacobian of this system is denoted as

$$\begin{bmatrix} \Psi_\Pi & O \\ \Phi_\Pi & \Phi_W \end{bmatrix} = \left[\begin{array}{cc|cc} \Psi_{\Pi_c}^{\Pi_c} & \Psi_{\Pi_b}^{\Pi_c} & O & O \\ \Psi_{\Pi_c}^{\Pi_b} & \Psi_{\Pi_b}^{\Pi_b} & O & O \\ \hline \Phi_{\Pi_c}^{\Delta W} & \Phi_{\Pi_b}^{\Delta W} & \Phi_{\Delta W}^{\Delta W} & \Phi_{W_b}^{\Delta W} \\ \Phi_{\Pi_c}^{W_b} & \Phi_{\Pi_b}^{W_b} & \Phi_{\Delta W}^{W_b} & \Phi_{W_b}^{W_b} \end{array}\right], \tag{2.15}$$

where the subscript indicates the differentiating argument.

Lemma 2. *(The Jacobian of the law of motion)*
Suppose p^* satisfies the conditions of Lemma 1. Then,
(i) $\Psi_{\Pi_b}^{\Pi_c} = O$;

[8] The incremental value of change is zero at the steady state, $\Delta W^* = 0$, so stability implies $\Delta W^t \to 0$.

[9] There is a technical issue here. To apply the implicit function theorem, mapping ϕ must be defined in an open neighborhood of (π^*, w^*). For this reason, we extend the domain of u to negative region and let w^t take negative values. As a matter of course, the value w^t shouldn't take negative value in any equilibrium. The assumption of the implicit function theorem, which in our case is invertibility of ϕ_w, always holds.

and the $B \times B$ matrix,

$$\Psi^{\Pi_c}_{\Pi_c} \equiv \begin{bmatrix} \Psi^{\Pi_c(1)}_{\Pi_c(1)} & & \cdots & & \Psi^{\Pi_c(1)}_{\Pi_c(x-1)} \\ & \ddots & & & \\ \vdots & & \Psi^{\Pi_c(k')}_{\Pi_c(l')} & & \vdots \\ & & & \ddots & \\ \Psi^{\Pi_c(x-1)}_{\Pi_c(1)} & & \cdots & & \Psi^{\Pi_c(x-1)}_{\Pi_c(x-1)} \end{bmatrix}, \tag{2.16}$$

where

$$\Psi^{\Pi_c(k')}_{\Pi_c(l')} \equiv \left(\frac{\partial \pi^{t+1}_{k'+kx}}{\partial \pi^{t}_{l'+lx}} \right)_{k,l=0,\cdots,B-1} \tag{2.17}$$

satisfies that
(ii) if $k' > l'$, then $\Psi^{\Pi_c(k')}_{\Pi_c(l')} = O$;
(iii) if $k' < l'$, then $\Psi^{\Pi_c(k')}_{\Pi_c(l')} \geq 0$, where the inequality for the (k,l)-element is strict if and only if there exists i such that $((0,i),(l',l))$-meeting leaves someone with $k'+kx$ units, or j such that $((l',l),(0,j))$-meeting leaves someone with $k'+kx$ units;[10]
(iv) each diagonal block satisfies $\Psi^{\Pi_c(l')}_{\Pi_c(l')} \geq 0$; Moreover, the sum of each lth column of $\Psi^{\Pi_c(l')}_{\Pi_c(l')}$ is no greater than one, and it is exactly one if and only if for each i, j, $((0,i),(l',l))$-meeting and $((l',l),(0,j))$-meeting leave someone with l' units of change;[11]
(v) The lth column of $\Psi^{\Pi_c(k')}_{\Pi_c(l')}$ for $k' < l'$ is zero vector for all $k' = 1, \cdots, l'-1$ if and only if the sum of the lth column of $\Psi^{\Pi_c(l')}_{\Pi_c(l')}$ is exactly one.

Lemma 2(i) implies that in the linearly approximated system, Π^{t+1}_c does not depend upon Π^t_b. So, Π^t_c is locally autonomous and its behavior is characterized by $\Psi^{\Pi_c}_{\Pi_c}$.

The next lemma is proved by using Lemma 2(ii)-(v) and shows that if under p^* the law of motion converges to the nonfull-support distribution π^*, then $\Psi^{\Pi_c}_{\Pi_c}$ necessarily has a unit eigenvalue and any convergent path must involve this unit eigenvalue. In other words, convergence is necessarily very slow near π^*. The same intuition applies as has been explained in section 3; the fewer people hold change, the less frequent is the meeting in which both sides start with positive amount of change.

Lemma 3. *(Unit-root convergence of the law of motion)*
Suppose that p^* satisfies Lemma 1 condition. Then $\Psi^{\Pi_c}_{\Pi_c}$ has a unit eigenvalue. Moreover, if $\Psi^{\Pi_c}_{\Pi_c}$ has a positive eigenvalue less than one, the associated eigenvector has both positive and negative elements.

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[10] That is, at least one of the following four is equal to $k'+kx$; $l'+lx-p^*((l',l),(0,j))$, $ix-p^*((0,i),(l',l))$, $jx+p^*((l',l),(0,j))$, $l'+lx+p^*((0,i),(l',l))$.

[11] That is, either $jx+p^*((l',l),(0,j))$ or $l'+lx-p((l',l),(0,j))$ is equal to $l'+kx$ for some k, and either $ix-p^*((0,i),(l',l))$ or $l'+lx+p^*((0,i),(l',l))$ is equal to $l'+kx$ for some k.

Next lemma is about the Jacobians of the Bellman equation.

Lemma 4. *(The Jacobian of the Bellman equation)*
(i) Linear approximation of $\phi^{\Delta W}$ *around the steady state has the form* $\Delta W^t = \phi^{\Delta W}_{\Delta W}\Delta W^{t+1} + \phi^{\Delta W}_{\Pi_c}\Pi^t_c$*. That is, it does not depend on* Π^t_b *or* W^{t+1}_b*.*
(ii) $\phi^{\Delta W}_{\Pi_c}$ *consists of* $(x-1)\times(x-1)$ *blocks and has the lower-right triangular form:* $\phi^{\Delta W}_{\Pi_c} = \begin{bmatrix} O & \cdots & O & K \\ \vdots & \iddots & \iddots & \vdots \\ O & \iddots & \iddots & K \\ K & \cdots & K & K \end{bmatrix}$*, where* K *is a* $B\times B$ *matrix whose elements are all strictly positive.*
(iii) $\phi^{\Delta W}_{\Delta W}$ *consists of* $(x-1)\times(x-1)$ *blocks and has the lower-left triangular form:* $\phi^{\Delta W}_{\Delta W} = \begin{bmatrix} \phi^{\Delta W(1)}_{\Delta W(1)} & O & \cdots & O \\ \vdots & \phi^{\Delta W(2)}_{\Delta W(2)} & O & \vdots \\ \vdots & & \ddots & O \\ & \cdots & \cdots & \phi^{\Delta W(x-1)}_{\Delta W(x-1)} \end{bmatrix}$*, where each* $\phi^{\Delta W(i')}_{\Delta W(i')}$ *is a* $B\times B$ *matrix whose elements are all non-negative: for* $i,l = 0,\cdots,B-1$*,*

$$\begin{aligned} \left(\phi^{\Delta W(i')}_{\Delta W(i')}\right)_{i,l} &= \sum_{j\in\mathbb{B}\setminus B} \frac{\pi^*_{jx}}{N} u'(\beta w^*_{i'+lx} - \beta w^*_{jx})\beta\mathbb{1}\{jx + p^*((i',i),(0,j)) = i' + lx\} \\ &+ \sum_{j\in\mathbb{B}\setminus B} \frac{\pi^*_{jx}}{N}\beta\mathbb{1}\{i' + ix - p^*((i',i),(0,j)) = i' + lx\} \\ &+ \frac{\pi^*_{Bx} + N - 1}{N}\beta\mathbb{1}\{i = l\} \qquad (2.18) \\ &\geq 0. \end{aligned}$$

The next lemma is proved by using Lemma 2(i) and 4(iv), and shows that $(\Pi^t_c, \Delta W^t)$ is locally autonomous.

Lemma 5. *(Local autonomy of* $(\Pi^t_c, \Delta W^t)$*)*
The Jacobian of the dynamical system, (2.15), has the form

$$\begin{bmatrix} \Psi^{\Pi_c}_{\Pi_c} & O & O & O \\ \Psi^{\Pi_b}_{\Pi_c} & \Psi^{\Pi_b}_{\Pi_b} & O & O \\ -[\phi^{\Delta W}_{\Delta W}]^{-1}\phi^{\Delta W}_{\Pi_c} & O & [\phi^{\Delta W}_{\Delta W}]^{-1} & O \\ * & * & * & * \end{bmatrix}, \qquad (2.19)$$

where blocks irrelevant to later analysis are expressed by $*$*. That is, the pair* $(\Pi^t_c, \Delta W^t)$ *is locally autonomous.*[12]

[12] In other words, the dynamics of $(\Pi^t_c, \Delta W^t)$ does not depend on that of Π^t_b, W^t_b in the linearized system.

Due to Lemma 5, we have only to focus on the dynamics of $(\Pi_c^t, \Delta W^t)$ for our ultimate goal.[13] The linearized system of $(\Pi_c^t, \Delta W^t)$ is characterized by the specific part of (2.19):

$$A \equiv \begin{bmatrix} \Psi_{\Pi_c}^{\Pi_c} & O \\ \Phi_{\Pi_c}^{\Delta W} & \Phi_{\Delta W}^{\Delta W} \end{bmatrix} = \begin{bmatrix} \Psi_{\Pi_c}^{\Pi_c} & O \\ -[\phi_{\Delta W}^{\Delta W}]^{-1}\phi_{\Pi_c}^{\Delta W} & [\phi_{\Delta W}^{\Delta W}]^{-1} \end{bmatrix} \tag{2.20}$$

As Π_c^t goes to zero, how does ΔW^t behave in the process of convergence? The next lemma gives the trajectory of ΔW^t.

Lemma 6. *(The trajectory of ΔW^t)*
ΔW^t converges to zero along the vector

$$\begin{bmatrix} \omega_{\Delta W(1)} \\ \vdots \\ \omega_{\Delta W(x-1)} \end{bmatrix} \equiv \tag{2.21}$$

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ (I - \phi_{\Delta W(x-1)}^{\Delta W(x-1)})^{-1} K z_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ (I - \phi_{\Delta W(x-2)}^{\Delta W(x-2)})^{-1} K z_2 \\ * \end{bmatrix} + \cdots + \begin{bmatrix} (I - \phi_{\Delta W(1)}^{\Delta W(1)})^{-1} K z_{x-1} \\ * \\ \vdots \\ * \end{bmatrix},$$

where $*$ indicates irrelevant components and $(z_1, \cdots, z_{x-1})'$ is an eigenvector of $\Psi_{\Pi_c}^{\Pi_c}$ associated with its unit eigenvalue.

The final lemma states two properties of the trajectory of ΔW^t given in Lemma 6. This lemma seems technical itself but plays a key role in deriving contradiction in the proof of Proposition 1.

Lemma 7. *(The property of the trajectory)*
Let $\zeta_{\Delta W(i')} \equiv \left(I - \phi_{\Delta W(i')}^{\Delta W(i')}\right)^{-1} K z_{x-i'}$, the $x - i'$th subvector in the i'th term of the (RHS) of (2.21). Suppose $z_{x-i'} \geq 0$ and $z_{x-i'} \neq 0$. Then
(i) if $\zeta_{\Delta W(i')} \geq 0$, then $\zeta_{\Delta W(i')} > 0$;
(ii) if $p^*((i', 0), (0, j)) = i'$ for all j, then $\zeta_{\Delta W(i')}$ has negative elements.

Proof of Proposition 1

Suppose by way of contradiciton that there exists an equilibrium such that $(p^t, \pi^t, w^t) \to (p^*, \pi^*, w^*)$. Transform the variables so that the dynamical sysmem will be given by (2.12). By Lemma 5, the linearized system (the system approximated near the steady state) is given by (2.19). So $(\Pi_c^t, \Delta W^t)$ is locally autonomous, and its dynamics is given by (2.20). By Lemma 3, convergence $\Pi_c^t \to 0$ necessarily involves a unit eigenvalue and such convergence is along an eigenvector, denoted $z \geq 0$, associated with that unit eigenvalue. In the

[13] Not focusing on the locally autonomous variables but dealing with the whole variables would lead to unnecessary complexity with the same conclusion.

following, we will derive contradiction in a particular type of meeting, where buyer has no bag but some change, and seller has some bags but no change.

Case (i): Suppose $z_{x-1} \neq \mathbb{0}$.[14] By (2.21), $\omega_{\Delta W(1)} = \zeta_{\Delta W(1)}$. Since $\Delta W^t(1)$ cannot be negative in the equilibrium and $\Delta W^t(1)$ converges to zero along $\omega_{\Delta(1)}$, it should be the case that $\omega_{\Delta W(1)} \geq 0$. Hence $\zeta_{\Delta W(1)} \geq 0$. By Lemma 7(i), $\zeta_{\Delta W(1)} > 0$. Hence $\omega_{\Delta W(1)} > 0$.[15] Along the equilibrium, $\Delta W^t(1) \to \omega_{\Delta W(1)} > 0$. This implies that near the steady state,

$$u(\beta(w^t_{1+jx} - w^t_{jx})) + \beta w^t_0 > \beta w^t_1, \tag{2.22}$$

for all j, which implies that $p^*((1,0),(0,j)) = 1$ for all j. (We use here the assumption that $u'(0)$ is finite but any large number.) By Lemma 7(ii), $\zeta_{\Delta W(1)}$ has negative components. Contradiction to $\zeta_{\Delta W(1)} > 0$.

Case (ii): Suppose $\hat{i}'(< x-1)$ is such that $z_{\hat{i}'} \neq 0$ and $z_{j'} = 0$ if $j' > \hat{i}'$.[16] By (2.21), $\omega_{\Delta W(x-\hat{i}')} = \zeta_{\Delta W(x-\hat{i}')}$ and $\omega_{\Delta W(j')} = 0$ for $j' < x - \hat{i}'$. Since $\Delta W^t(x-\hat{i}')$ cannot be negative in the equilibrium and $\Delta W^t(x-\hat{i}')$ converges to zero along $\omega_{\Delta(x-\hat{i}')}$, it should be the case that $\omega_{\Delta W(x-\hat{i}')} \geq 0$. Hence $\zeta_{\Delta W(x-\hat{i}')} \geq 0$. By Lemma 7(i), $\zeta_{\Delta W(x-\hat{i}')} > 0$. Hence $\omega_{\Delta W(x-\hat{i}')} > 0$.[17] Along the equilibrium, $((\Delta W^t(j'))_{j'<x-\hat{i}'}, \Delta W^t(x-\hat{i}')) \to (\mathbb{0}, \cdots, \mathbb{0}, \omega_{\Delta W^t(x-\hat{i}')})$. This implies that near the steady state,

$$u(\beta(w^t_{x-\hat{i}'+jx} - w^t_{jx})) + \beta w^t_0 > u(\beta(w^t_{i''+jx} - w^t_{jx})) + \beta w^t_{x-\hat{i}'-i''}, \tag{2.23}$$

for all $i'' < x - \hat{i}'$ and all j, which implies that $p^*((x-\hat{i}',0),(0,j)) = x - \hat{i}'$ for all j. (We use here the assumption that $u'(0)$ is finite but any large number.) By Lemma 7(ii), $\zeta_{\Delta W(x-\hat{i}')}$ has negative components. Contradiction to $\zeta_{\Delta W(x-\hat{i}')} > 0$.

2.4 Conclusion

We have given stability analysis of the nonfull-support steady states of the random matching model of indivisible money in Zhu (2003). It has been shown that there is no equilibrium path with a pure strategy that converges to the nonfull-support steady state if the economy starts with positive measure of people holding change. This instability result is seen as new refinement of the full-support steady state.

There are several possible extensions. In this study, we focused on the steady state and an equilibrium path that have a pure payment rule. So allowing for mixed payment rule is one possibility. Second, the introduction of lotteries in the model may change the result.

[14] That is, along the path, some people hold $x-1$ units of change.

[15] That is, along the path, holding one unit of change is meaningful.

[16] That is, along the path, some people hold $\hat{i}'$ units of change but nobody hold more.

[17] That is, along the path, holding $x - \hat{i}'$ units of change is meaningful but holding less than that is meaningless.

If lotteries are allowed, the buyer's choice set becomes continuous and payment rule can no longer be fixed in the law of motion or Bellman equation. In this case, the inequality conditions for optimality is replaced by the first order conditions, which are now part of the dynamical system.

2.5 Appendices: proofs of lemmas

Proof. (Lemma 1) (a little more revision required)
(i) The unique optimality of $\hat{p}$ implies that in (B, m)-economy,

$$u(\beta \hat{w}_{j+\hat{p}(i,j)} - \beta \hat{w}_j) + \beta \hat{w}_{i-\hat{p}(i,j)} > u(\beta \hat{w}_{j+l} - \beta \hat{w}_j) + \beta \hat{w}_{i-l}, \tag{2.24}$$

for any $l \neq \hat{p}(i,j)$, or equivalently, in (xB, xm)-economy,

$$u(\beta w^*_{jx+\hat{p}(i,j)x} - \beta w^*_{jx}) + \beta w^*_{ix-\hat{p}(i,j)x} > u(\beta w^*_{jx+lx} - \beta w^*_{jx}) + \beta w^*_{ix-lx}. \tag{2.25}$$

Since $w^{t+1} \approx w^*$ near the steady state, we have

$$u(\beta w^{t+1}_{(j+\hat{p}(i,j))x} - \beta w^{t+1}_{jx}) + \beta w^{t+1}_{(i-\hat{p}(i,j))x} > u(\beta w^{t+1}_{(j+l)x} - \beta w^{t+1}_{jx}) + \beta w^{t+1}_{(i-l)x}. \tag{2.26}$$

Here, buyer's pre-trade change and seller's pre-trade change are such that $i' + j' < x$ (i.e., the total change in the meeting does not form one extra bag,) and near the steady state, incremental values of change are negligible. So (2.26) implies that the post-trade bag-holdings implied by the optimal p^* should be $j + \hat{p}(i,j)$ for seller and $i - \hat{p}(i,j)$ for buyer, even if they have positive amount of change pre-trade.
(ii)(iii) Most of the time when $i' + j' \geq x$, buyer can act as if his change were one bag, because his change, which is almost worthless near the steady state, forms one extra bag if added to seller's change and so, for her, buyer's change is as valuable as one bag. In this case, their post-trade bag-holdings are determined by $\hat{p}(i+1, j)$, unless $\hat{p}(i+1, j) = 0$. The exception is when $\hat{p}(i+1, j) = 0$. Buyer's change acts as one extra bag only by being giving to seller so she will form one bag. Therefore when $\hat{p}(i+1, j) = 0$, buyer prefers giving some of his change for seller to form one bag, to following $\hat{p}(i+1, j)$.
(iv) is a consequence of (i)-(iii). □

Proof. (Lemma 2)
First we rewrite the law of motion (2.6) with "change" and "bag" explicit:

$$\begin{aligned}
&\pi^{t+1}_{k'+kx} = \pi^t_{k'+kx} \\
+&\ \frac{1}{N} \sum_{(i',i),(j',j)} \pi^t_{i'+ix} \pi^t_{j'+jx} \mathbb{1}\left\{p^*((i',i),(j',j)) \neq 0, i' + ix - p^*((i',i),(j',j)) = k' + kx\right\} \\
+&\ \frac{1}{N} \sum_{(i',i),(j',j)} \pi^t_{i'+ix} \pi^t_{j'+jx} \mathbb{1}\left\{p^*((i',i),(j',j)) \neq 0, j' + jx + p^*((i',i),(j',j)) = k' + kx\right\} \\
-&\ \frac{1}{N} \sum_{(i',i),(j',j)} \pi^t_{i'+ix} \pi^t_{j'+jx} \mathbb{1}\left\{i' + ix = k' + kx, p^*((i',i),(j',j)) \neq 0\right\} \\
-&\ \frac{1}{N} \sum_{(i',i),(j',j)} \pi^t_{i'+ix} \pi^t_{j'+jx} \mathbb{1}\left\{j' + jx = k' + kx, p^*((i',i),(j',j)) \neq 0\right\},
\end{aligned} \tag{2.27}$$

where π^t_0 and π^t_{Bx} are given by the other probabilities through (2.14). In this lemma, we are interested only in $k' \neq 0$ case. Proof is done by differentiating this law of motion

and evaluating at π^*. Note that most of the terms in the law of motion are quadratic, so differentiating and evaluating them at π^* make many of them disappear.
One can show that for $(l', l) \neq (k', k)$,

$$\begin{aligned}
& \frac{\partial \pi^{t+1}_{k'+kx}}{\partial \pi^{t}_{l'+lx}}(\pi^*) \\
= \; & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{p^*((l',l),(0,j)) \neq 0, l'+lx-p^*((l',l),(0,j)) = k'+kx\} \\
+ \; & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{p^*((0,i),(l',l)) \neq 0, ix-p^*((0,i),(l',l)) = k'+kx\} \\
+ \; & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{p^*((l',l),(0,j)) \neq 0, jx+p^*((l',l),(0,j)) = k'+kx\} \\
+ \; & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{p^*((0,i),(l',l)) \neq 0, l'+lx+p^*((0,i),(l',l)) = k'+kx\} \\
\geq \; & 0,
\end{aligned} \tag{2.28}$$

and for $(l', l) = (k', k)$,

$$\begin{aligned}
& \frac{\partial \pi^{t+1}_{k'+kx}}{\partial \pi^{t}_{k'+kx}}(\pi^*) = 1 + \\
= \; & 1 - \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{p^*((0,i),(k',k)) \neq 0, ix-p^*((0,i),(k',k)) \neq k'+kx\} \\
& -\frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{p^*((k',k),(0,j)) \neq 0, jx+p^*((k',k),(0,j)) \neq k'+kx\} \quad \geq 0,
\end{aligned} \tag{2.29}$$

by certain algebra.[18] Part (i)-(iii) immediately follow from (2.28). (For (i), $k' > 0$ and $l' = 0$. Then the result follows the fact that in a meeting in which both sides are at bag states, nobody ends up with positive amount of change and hence all the terms in (2.28) are zero. For (ii), note that all the terms in (2.28) are associated with meetings where the total

[18] Equation (2.28) and (2.29) is derived as follows. To consider each kind of meeting separately, let $M_1 \equiv \{i' + ix | i' \in \mathbb{X}\backslash\{0\}, i \in \mathbb{B}\backslash\{B\}\}$ (the set of change states), $M_2 \equiv \{ix | i \in \mathbb{B}\backslash\{0, B\}\}$, and $M_3 \equiv \{0, Bx\}$. Depending on which of the three sets the buyer and the seller come from, there are nine kinds of meetings. Meetings where both sides come from $M_2 \cup M_3$ are irrelevant now because such meetings do not produce any outflows from state $k' + kx$, and by Lemma 1(i), such meetings do not produce any inflows into state $k' + kx$ as well. Furthermore, if both sides are from M_1, then differentiation and evaluation lead to zero. Therefore, relevant meetings are such that one side comes from M_1 and the other comes from $M_2 \cup M_3$. Summarized in the following two tables is differentiation and evaluation of terms in (2.27) associated with such meetings.

change is l'. Since $k' > l'$, under Lemma 1(v), nobody ends up with k' units of change, so all the terms are zero. (iii) is exactly what (2.28) states.)

The first part of (iv) follows from the inequalities in (2.28) and (2.29). For the second

Differentiated Term ($i' \neq 0$)	Differentiation w.r.t. $\pi^t_{l'+lx}$		
	$(0,j) = (l',l)$	$(i',i) = (l',l)$	$(i',i) \neq (l',l),\ l' \neq 0$
$\pi^t_{i'+ix}\pi^t_{jx}$	$\pi^t_{i'+ix}(\Rightarrow 0)$	$\pi^t_{jx}(\Rightarrow \pi^*_{jx})$	0
$\pi^t_{i'+ix}\pi^t_{Bx}$	$\pi^t_{i'+ix}\frac{\partial g^{Bx}}{\partial \pi^t_{jx}}(\Rightarrow 0)$	$\pi^t_{Bx} + \pi^t_{i'+ix}\frac{\partial g^{Bx}}{\partial \pi^t_{i'+ix}}(\Rightarrow \pi^*_{Bx})$	$\pi^t_{i'+ix}\frac{\partial g^{Bx}}{\partial \pi^t_{l'+lx}}(\Rightarrow 0)$
$\pi^t_{i'+ix}\pi^t_{0x}$	$\pi^t_{i'+ix}\frac{\partial g^{0x}}{\partial \pi^t_{jx}}(\Rightarrow 0)$	$\pi^t_{0x} + \pi^t_{i'+ix}\frac{\partial g^{0x}}{\partial \pi^t_{i'+ix}}(\Rightarrow \pi^*_{0x})$	$\pi^t_{i'+ix}\frac{\partial g^{0x}}{\partial \pi^t_{l'+lx}}(\Rightarrow 0)$

Differentiated Term ($j' \neq 0$)	Differentiation w.r.t. $\pi^t_{l'+lx}$		
	$(0,i) = (l',l)$	$(j',j) = (l',l)$	$(j',j) \neq (l',l),\ l' \neq 0$
$\pi^t_{ix}\pi^t_{j'+jx}$	$\pi^t_{j'+jx}(\Rightarrow 0)$	$\pi^t_{ix}(\Rightarrow \pi^*_{ix})$	0
$\pi^t_{Bx}\pi^t_{j'+jx}$	$\frac{\partial g^{Bx}}{\partial \pi^t_{ix}}\pi^t_{j'+jx}(\Rightarrow 0)$	$\frac{\partial g^{Bx}}{\partial \pi^t_{j'+jx}}\pi^t_{j'+jx} + \pi^t_{Bx}(\Rightarrow \pi^*_{Bx})$	$\frac{\partial g^{Bx}}{\partial \pi^t_{l'+lx}}\pi^t_{j'+jx}(\Rightarrow 0)$
$\pi^t_{0x}\pi^t_{j'+jx}$	$\frac{\partial g^{0x}}{\partial \pi^t_{ix}}\pi^t_{j'+jx}(\Rightarrow 0)$	$\frac{\partial g^{0x}}{\partial \pi^t_{j'+jx}}\pi^t_{j'+jx} + \pi^t_{0x}(\Rightarrow \pi^*_{0x})$	$\frac{\partial g^{0x}}{\partial \pi^t_{l'+lx}}\pi^t_{j'+jx}(\Rightarrow 0)$

So, differentiating and evaluating (2.27) gives

$$\begin{aligned}
& \frac{\partial \pi^{t+1}_{k'+kx}}{\partial \pi^t_{l'+lx}}(\pi^*) = \mathbb{1}\{l' + lx = k' + kx\} \\
+ \quad & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx}\mathbb{1}\{p^*((l',l),(0,j)) \neq 0, l' + lx - p^*((l',l),(0,j)) = k' + kx\} \\
+ \quad & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix}\mathbb{1}\{p^*((0,i),(l',l)) \neq 0, ix - p^*((0,i),(l',l)) = k' + kx\} \\
+ \quad & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx}\mathbb{1}\{p^*((l',l),(0,j)) \neq 0, jx + p^*((l',l),(0,j)) = k' + kx\} \\
+ \quad & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix}\mathbb{1}\{p^*((0,i),(l',l)) \neq 0, l' + lx + p^*((0,i),(l',l)) = k' + kx\} \\
- \quad & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx}\mathbb{1}\{l' + lx = k' + kx, p^*((l',l),(0,j)) \neq 0\} \\
- \quad & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix}\mathbb{1}\{ix = k' + kx, p^*((0,i),(l',l)) \neq 0\} \\
- \quad & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx}\mathbb{1}\{jx = k' + kx, p^*((l',l),(0,j)) \neq 0\} \\
- \quad & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix}\mathbb{1}\{l' + lx = k' + kx, p^*((0,i),(l',l)) \neq 0\}.
\end{aligned}$$

When $(l',l) \neq (k',k)$, the last four sums are zero, which leads to (2.28). When $(l',l) = (k',k)$, the last equation becomes

$$
\begin{aligned}
& \frac{\partial \pi^{t+1}_{k'+kx}}{\partial \pi^{t}_{k'+kx}}(\pi^*) = 1 + \\
+ \quad & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{p^*((k',k),(0,j)) \neq 0, k'+kx - p^*((k',k),(0,j)) = k'+kx\} \\
+ \quad & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{p^*((0,i),(k',k)) \neq 0, ix - p^*((0,i),(k',k)) = k'+kx\} \\
+ \quad & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{p^*((k',k),(0,j)) \neq 0, jx + p^*((k',k),(0,j)) = k'+kx\} \\
+ \quad & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{p^*((0,i),(k',k)) \neq 0, k'+kx + p^*((0,i),(k',k)) = k'+kx\} \\
- \quad & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{k'+kx = k'+kx, p^*((k',k),(0,j)) \neq 0\} \\
- \quad & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{ix = k'+kx, p^*((0,i),(k',k)) \neq 0\} \\
- \quad & \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{jx = k'+kx, p^*((k',k),(0,j)) \neq 0\} \\
- \quad & \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{k'+kx = k'+kx, p^*((0,i),(k',k)) \neq 0\} \\
= \quad & 1 - \frac{1}{N}\sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{p^*((0,i),(k',k)) \neq 0, ix - p^*((0,i),(k',k)) \neq k'+kx\} \\
& - \frac{1}{N}\sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{p^*((k',k),(0,j)) \neq 0, jx + p^*((k',k),(0,j)) \neq k'+kx\} \quad \geq 0,
\end{aligned}
\tag{2.30}
$$

where the equality follows because the first, fourth, sixth and seventh summations are zero and the second and eighth summations, and the third and fifth summations are combined.

part of (iv), we have

$$
\begin{aligned}
& [\text{The sum of the } l \text{ column of } \Psi^{\Pi_c(l')}_{\Pi_c(l')}] \\
= \; & \sum_{k=0}^{B-1} \frac{\partial \pi^{t+1}_{l'+kx}}{\partial \pi^{t}_{l'+lx}} \\
= \; & \frac{\partial \pi^{t+1}_{l'+lx}}{\partial \pi^{t}_{l'+lx}} + \sum_{k \neq l} \frac{\partial \pi^{t+1}_{l'+kx}}{\partial \pi^{t}_{l'+lx}} \\
= \; & 1 - \frac{1}{N} \sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{p^*((0,i),(l',l)) \neq 0, ix - p^*((0,i),(l',l)) \neq l' + lx\} \\
& \quad - \frac{1}{N} \sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{p^*((l',l),(0,j)) \neq 0, jx + p^*((l',l),(0,j)) \neq l' + lx\} \\
& + \sum_{k \neq l} \Bigg[\frac{1}{N} \sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{p^*((l',l),(0,j)) \neq 0, l' + lx - p^*((l',l),(0,j)) = l' + kx\} \\
& \qquad + \frac{1}{N} \sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{p^*((0,i),(l',l)) \neq 0, l' + lx + p^*((0,i),(l',l)) = l' + kx\} \\
& \qquad + \frac{1}{N} \sum_{i=0}^{B} \pi^*_{ix} \mathbb{1}\{p^*((0,i),(l',l)) \neq 0, ix - p^*((0,i),(l',l)) = l' + kx\} \\
& \qquad + \frac{1}{N} \sum_{j=0}^{B} \pi^*_{jx} \mathbb{1}\{p^*((l',l),(0,j)) \neq 0, jx + p^*((l',l),(0,j)) = l' + kx\} \Bigg] \\
= \; & 1 - \frac{1}{N} \sum_{i=0}^{B} \pi^*_{ix} \Bigg[\mathbb{1}\{p^*((0,i),(l',l)) \neq 0, ix - p^*((0,i),(l',l)) \neq l' + lx\} \\
& \qquad - \sum_{k \neq l} \mathbb{1}\{p^*((0,i),(l',l)) \neq 0, ix - p^*((0,i),(l',l)) = l' + kx\} \\
& \qquad - \sum_{k \neq l} \mathbb{1}\{p^*((0,i),(l',l)) \neq 0, l' + lx + p^*((0,i),(l',l)) = l' + kx\} \Bigg] \\
& - \frac{1}{N} \sum_{j=0}^{B} \pi^*_{jx} \Bigg[\mathbb{1}\{p^*((l',l),(0,j)) \neq 0, jx + p^*((l',l),(0,j)) \neq l' + lx\} \\
& \qquad - \sum_{k \neq l} \mathbb{1}\{p^*((l',l),(0,j)) \neq 0, jx + p^*((l',l),(0,j)) = l' + kx\} \\
& \qquad - \sum_{k \neq l} \mathbb{1}\{p^*((l',l),(0,j)) \neq 0, l' + lx - p^*((l',l),(0,j)) = l' + kx\} \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & 1 - \frac{1}{N}\sum_{i=0}^{B}\pi^*_{ix}[\mathbb{1}\{p^*((0,i),(l',l)) \neq 0, \text{ and } \forall k, ix - p^*((0,i),(l',l)) \neq l' + kx\} \\
& \qquad\qquad -\mathbb{1}\{p^*((0,i),(l',l)) \neq 0, \text{ and } \exists k, l' + lx + p^*((0,i),(l',l)) = l' + kx\}] \\
& -\frac{1}{N}\sum_{j=0}^{B}\pi^*_{jx}[\mathbb{1}\{p^*((l',l),(0,j)) \neq 0, \text{ and } \forall k, jx + p^*((l',l),(0,j)) \neq l' + kx\} \\
& \qquad\qquad -\mathbb{1}\{p^*((l',l),(0,j)) \neq 0, \text{ and } \exists k, l' + lx - p^*((l',l),(0,j)) = l' + kx\}]
\end{aligned}
$$

$$
\begin{aligned}
= 1 \quad & - \quad \frac{1}{N}\sum_{i=0}^{B}\pi^*_{ix}\mathbb{1}\{p^*((0,i),(l',l)) \neq 0, \text{Neither ends up with } l' \text{ units of change}\} \\
& - \quad \frac{1}{N}\sum_{j=0}^{B}\pi^*_{jx}\mathbb{1}\{p^*((l',l),(0,j)) \neq 0, \text{Neither ends up with } l' \text{ units of change}\},
\end{aligned}
$$

which proves (iv).

Part (v) is the implication of part (iii) and (iv). To see this, note that by (iii), the lth column of $\Psi^{\Pi_c(k')}_{\Pi_c(l')}$ for $k' < l'$ is zero vector for all $k' = 1, \cdots, l'-1$ if and only if for (l', l) and $k' = 1, \cdots, l'-1$, there is no i such that $((0,i),(l',l))$-meeting leaves anyone with k' units of change, and there is no j such that $((l',l),(0,j))$-meeting leaves anyone with k' units of change. This is equivalent to the statement that for (l',l), for all i and j, $((0,i),(l',l))$-meeting and $((l',l),(0,j))$-meeting leave one side l' units of change and the other 0 units of change. By (iv), this holds if and only if the sum of the lth column of $\Psi^{\Pi_c(l')}_{\Pi_c(l')}$ is exactly one. □

Proof. (Lemma 3)
By Lemma 2(ii), $\Psi^{\Pi_c}_{\Pi_c}$ is a upper-triangular block matrix, so the eigenvalues of $\Psi^{\Pi_c}_{\Pi_c}$ are those of $\Psi^{\Pi_c(l')}_{\Pi_c(l')}$, $l = 1, \cdots, x-1$. Consider $\Psi^{\Pi_c(1)}_{\Pi_c(1)}$, the block corresponding to one-unit change. Then Lemma 2(iv) implies each lth column of $\Psi^{\Pi_c(1)}_{\Pi_c(1)}$ sums up to one. Therefore, we have $1'_B\Psi^{\Pi_c(1)}_{\Pi_c(1)} = 1'_B$, where $1'_B \equiv (1, 1, \cdots, 1)$ is a B-dimensional row vector with ones. Hence $\Psi^{\Pi_c(1)}_{\Pi_c(1)}$ has a unit eigenvalue, so does $\Psi^{\Pi_c}_{\Pi_c}$.
Now suppose that $\Psi^{\Pi_c}_{\Pi_c}$ also has an eigenvalue that is smaller than one, say $\lambda \in (0,1)$, and that its associated eigenvector has only non-negative elements, so that the law of motion implies exponential convergence to π^* along this eigenvector. Denote that eigenvector $\eta' = (\eta'_1, \cdots, \eta'_{x-1}) \geq 0$, where for each l', $\eta'_{l'} = (\eta_{l',0}, \cdots, \eta_{l',B-1})$ is a B-dimensional vector . We have $B(x-1)$ equations:

$$
0 = \left(\Psi^{\Pi_c}_{\Pi_c} - \lambda I\right)\eta. \tag{2.31}
$$

Sum up the first B equations, the second B equations, etc. This is equivalent to multiplying $I_{x-1} \otimes 1_B$ from the left, where I_{x-1} is an $(x-1)$ by $(x-1)$ identity matrix and $\otimes$ is kroncker's delta.

$$
\begin{aligned}
0_{x-1} &= \begin{bmatrix} 1'_B\Psi^{\Pi_c(1)}_{\Pi_c(1)} - 1'_B\lambda & 1'_B\Psi^{\Pi_c(1)}_{\Pi_c(2)} & \cdots & 1'_B\Psi^{\Pi_c(1)}_{\Pi_c(x-1)} \\ 0'_B & 1'_B\Psi^{\Pi_c(2)}_{\Pi_c(2)} - 1'_B\lambda & \cdots & \\ 0'_B & 0'_B & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1'_B\Psi^{\Pi_c(x-2)}_{\Pi_c(x-1)} \\ 0'_B & 0'_B & \cdots & 1'_B\Psi^{\Pi_c(x-1)}_{\Pi_c(x-1)} - 1'_B\lambda \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \vdots \\ \eta_{x-1} \end{bmatrix} \\
&= \begin{bmatrix} 1'_B\Psi^{\Pi_c(1)}_{\Pi_c(1)} - 1'_B\lambda \\ 0'_B \\ 0'_B \\ \vdots \\ 0'_B \end{bmatrix} \eta_1 + \begin{bmatrix} 1'_B\Psi^{\Pi_c(1)}_{\Pi_c(2)} \\ 1'_B\Psi^{\Pi_c(2)}_{\Pi_c(2)} - 1'_B\lambda \\ 0'_B \\ \vdots \\ 0'_B \end{bmatrix} \eta_2 + \cdots + \begin{bmatrix} 1'_B\Psi^{\Pi_c(1)}_{\Pi_c(x-1)} \\ \vdots \\ \vdots \\ 1'_B\Psi^{\Pi_c(x-2)}_{\Pi_c(x-1)} \\ 1'_B\Psi^{\Pi_c(x-1)}_{\Pi_c(x-1)} - 1'_B\lambda \end{bmatrix} \eta_{x-1} \\
&= \sum_{l\in\mathbb{B}\backslash B} \xi_{1,l}\ \eta_{1,l} + \sum_{l\in\mathbb{B}\backslash B} \xi_{2,l}\ \eta_{2,l} + \cdots + \sum_{l\in\mathbb{B}\backslash B} \xi_{x-1,l}\ \eta_{x-1,l}, \qquad (2.32)
\end{aligned}
$$

where

$$
\xi_{l',l} \equiv \begin{pmatrix} \sum_{k=0}^{B-1} \frac{\partial\pi^{t+1}_{1+kx}}{\partial\pi^t_{l'+lx}} \\ \sum_{k=0}^{B-1} \frac{\partial\pi^{t+1}_{2+kx}}{\partial\pi^t_{l'+lx}} \\ \vdots \\ \sum_{k=0}^{B-1} \frac{\partial\pi^{t+1}_{(l'-1)+kx}}{\partial\pi^t_{l'+lx}} \\ \sum_{k=0}^{B-1} \frac{\partial\pi^{t+1}_{l'+kx}}{\partial\pi^t_{l'+lx}} - \lambda \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad (2.33)
$$

Note that by Lemma 2(iii), $1'_B\Psi^{\Pi_c(k')}_{\Pi_c(l')}$'s are all nonnegative when $k' < l'$. Moreover, the previous argument implies that $1'_B\Psi^{\Pi_c(1)}_{\Pi_c(1)} - \lambda 1'_B = 1'_B - \lambda 1'_B = (1-\lambda)1'_B > 0$. Therefore, $\eta_1 = 0$ is necessary, because the first element of $\xi_{h',l},\ h' > 1, l \in \mathbb{B}\backslash B$ are all nonnegative and $\eta \geq 0$. Now we show that $0 = \eta_1 = \cdots = \eta_{x-1}$. Suppose by way of mathematical induction that $0 = \eta_1 = \cdots = \eta_{l'-1}$, so (2.32) implies $0 = \sum_{l\in\mathbb{B}\backslash B} \xi_{l',l}\ \eta_{l',l} + \cdots + \sum_{l\in\mathbb{B}\backslash B} \xi_{x-1,l}\ \eta_{x-1,l}$. We want to show $\eta_{l'} = 0$. Let $\Xi_{l'} \equiv \{l \in \mathbb{B}\backslash B \mid \text{The first } (l'-1) \text{ elements of } \xi_{l',l} \text{ are zero.}\}$. Then for $l \in \{0, \cdots, B-1\}\backslash\Xi_{l'}$, it is necessary that $\eta_{l',l} = 0$, because the first $l'-1$ elements of $\xi_{h',l},\ h' > l', l \in \mathbb{B}\backslash B$ are all nonnegative and $\eta \geq 0$. Therefore, $\sum_{l\in\mathbb{B}\backslash B} \xi_{l',l}\ \eta_{l',l} = \sum_{l\in\Xi_{l'}} \xi_{l',l}\ \eta_{l',l}$. But in the meantime, by Lemma 2(v), we know

$\Xi_{l'} = \{l \in \mathbb{B}\backslash B \mid \text{The } l'\text{th element of } \xi_{l',l} \text{ is } 1-\lambda > 0.\}$. Hence, the l'th element of $\sum_{l\in\Xi_{l'}} \xi_{l',l}\, \eta_{l',l}$ is strictly positive, unless $\eta_{l',l} = 0$ for all $l \in \Xi_{l'}$. So $\eta_{l',l} = 0$, $\forall l \in \Xi_{l'}$ is necessary, because the l'th element of $\xi_{h',l}$, $h' > l', l \in \mathbb{B}\backslash B$ are all nonnegative and $\eta \geq 0$. Thus $\eta_{l'} = 0$. By induction, we have $\eta = 0$, contradiction. □

Proof. (Lemma 4)
(i) We rewrite the original Bellman equation (2.7) given the payment rule p^*, making the distinction between bag and change explicit:

$$\begin{aligned}
w^t_{i'+ix} &= \sum_{j'+jx} \frac{\pi^t_{j'+jx}}{N}\left[u(\beta w^{t+1}_{j'+jx+p^*((i',i),(j',j))} - \beta w^{t+1}_{j'+jx}) + \beta w^{t+1}_{i'+ix-p^*((i',i),(j',j))}\right] \\
&+ \frac{N-1}{N}\beta w^{t+1}_{i'+ix} \\
&= \sum_{j'+jx\neq 0,Bx} \frac{\pi^t_{j'+jx}}{N}\left[u(\beta w^{t+1}_{j'+jx+p^*((i',i),(j',j))} - \beta w^{t+1}_{j'+jx}) + \beta w^{t+1}_{i'+ix-p^*((i',i),(j',j))}\right] \\
&+ \frac{\pi^t_0}{N}\left[u(\beta w^{t+1}_{p^*((i',i),(0,0))}) + \beta w^{t+1}_{i'+ix-p^*((i',i),(0,0))}\right] + \frac{\pi^t_{Bx}+N-1}{N}\beta w^{t+1}_{i'+ix}, \qquad (2.34)
\end{aligned}$$

where π^t_0 and π^t_{Bx} are given by (2.14). (The second equality is by taking the two terms corresponding to $j'+jx = 0$ and $= Bx$ out of the summation.) We are interested in getting the Jacobians, evaluated at the steady state, of the transformed Bellman equation $\phi^{\Delta W}$ which in principle is obtained by the linear transformation of variables from w^t to W^t. Since the variable transformation is linear, instead of first obtaining the transformed Bellman equation and then linearizing it, we can first linearize the original Bellman equation and then do the variable transformation.

The linear expansion of (2.34) around the steady state gives

$$\begin{aligned}
& w^t_{i'+ix} - w^*_{i'+ix} = \\
& \sum_{j\in\mathbb{B}\backslash B} \frac{\pi^*_{jx}}{N} u'(\beta w^*_{jx+p^*((i',i),(0,j))} - \beta w^*_{jx})\beta\left\{(w^{t+1}_{jx+p^*((i',i),(0,j))} - w^{t+1}_{jx}) - (w^*_{jx+p^*((i',i),(0,j))} - w^*_{jx})\right\} \\
& + \sum_{j\in\mathbb{B}\backslash B} \frac{\pi^*_{jx}}{N}\beta(w^{t+1}_{i'+ix-p^*((i',i),(0,j))} - w^*_{i'+ix-p^*((i',i),(0,j))}) \\
& + \frac{\pi^*_{Bx}+N-1}{N}\beta(w^{t+1}_{i'+ix} - w^*_{i'+ix}) \\
& + \sum_{j'+jx\neq 0,Bx} \frac{1}{N}\left[u(\beta w^*_{j'+jx+p^*((i',i),(j',j))} - \beta w^*_{j'+jx}) + \beta w^*_{i'+ix-p^*((i',i),(j',j))}\right](\pi^t_{j'+jx} - \pi^*_{j'+jx}) \\
& + \frac{1}{N}\left[u(\beta w^*_{p^*((i',i),(0,0))}) + \beta w^*_{i'+ix-p^*((i',i),(0,0))}\right]\nabla g^0(\pi^*)\cdot(\pi^t - \pi^*) \\
& + \frac{\beta w^*_{i'+ix}}{N}\nabla g^{Bx}(\pi^*)\cdot(\pi^t - \pi^*). \qquad (2.35)
\end{aligned}$$

This equation allows to obtain the expression for $\Delta w^t_{i',i} \equiv w^t_{i'+ix} - w^t_{ix}$, the incremental value of change. All the simplification are due to the fact that the steady state values do not depend on the amount of change (i.e., equation (2.10)), and Lemma 1(iv).[19]

$$\begin{aligned} w^t_{i'+ix} - w^t_{ix} &= \sum_{j \in \mathbb{B} \setminus B} \frac{\pi^*_{jx}}{N} u'(\beta w^*_{jx+p^*((0,i),(0,j))} - \beta w^*_{jx}) \beta (w^{t+1}_{jx+p^*((i',i),(0,j))} - w^{t+1}_{jx+p^*((0,i),(0,j))}) \\ &+ \sum_{j \in \mathbb{B} \setminus B} \frac{\pi^*_{jx}}{N} \beta (w^{t+1}_{i'+ix-p^*((i',i),(0,j))} - w^{t+1}_{ix-p^*((0,i),(0,j))}) \\ &+ \frac{\pi^*_{Bx} + N - 1}{N} \beta (w^{t+1}_{i'+ix} - w^{t+1}_{ix}) \\ &+ \sum_{j'+jx \neq 0, Bx} \kappa_{(i',i),(j',j)} (\pi^t_{j'+jx} - \pi^*_{j'+jx}), \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} \kappa_{(i',i),(j',j)} \equiv \frac{1}{N} \Big\{ &\big[u(\beta w^*_{j'+jx+p^*((i',i),(j',j))} - \beta w^*_{j'+jx}) + \beta w^*_{i'+ix-p^*((i',i),(j',j))} \big] \\ &- \big[u(\beta w^*_{j'+jx+p^*((0,i),(j',j))} - \beta w^*_{j'+jx}) + \beta w^*_{ix-p^*((0,i),(j',j))} \big] \Big\} . \end{aligned} \tag{2.37}$$

By Lemma 1(iv), all the three expressions in (2.36), $w^{t+1}_{jx+p^*((i',i),(0,j))} - w^{t+1}_{jx+p^*((0,i),(0,j))}$, $w^{t+1}_{i'+ix-p^*((i',i),(0,j))} - w^{t+1}_{ix-p^*((0,i),(0,j))}$ and $w^{t+1}_{i'+ix} - w^{t+1}_{ix}$, are of the form of $\Delta w^{t+1}_{l',l}$ for some l', l. So the (RHS) depends on ΔW^{t+1}, not W^{t+1}_b. By Lemma 1(iv), $\kappa_{(i',i),(j',j)} = 0$ if $j' = 0$. So the (RHS) does not depend on Π^t_b but Π^t_c only.

(ii) Differentiation of (2.36) with respect to Π^t_c gives $\phi^{\Delta W}_{\Pi_c}$. Again because of (2.10) and Lemma 1(iv), two facts follow; (I) $\kappa_{(i',i),(j',j)} = 0$ for all i', j' such that $i' + j' < x$; and (II) for all i', j' such that $i' + j' \geq x$, $\kappa_{(i',i),(j',j)} = \bar{\kappa}_{i,j}$ for some $\bar{\kappa}_{i,j} > 0$, i.e., it does not depend on i', j'. That is, for given i', j' such that $i' + j' \geq x$, $d(w^t_{i'+ix} - w^t_{ix})/d\pi^t_{j'+jx} = \bar{\kappa}_{i,j}$ regardless of i', j'. So for such i', j', defining $K_{i,j} \equiv \bar{\kappa}_{i,j}$ leads to the statement. The strict positiveness of $\bar{\kappa}_{i,j}$ is due to the following reason. The first term of $\kappa_{(i',i),(j',j)}$ is the buyer's payoff in $((i', i), (j', j))$-meeting, and the second term is that in $((0, i), (j', j))$-meeting. Since $i'+j' \geq x$, the buyer's change i' effectively acts as one bag. To put it in a different way, since paying his change to the seller forms one additional bag, the buyer can get higher payoff either by getting more production good or by saving one bag, compared with in $((0, i), (j', j))$-meeting.

(iii) Differentiating (2.36) with respect to incremental values of change gives $\phi^{\Delta W}_{\Delta W}$. Under Lemma 1(v), the total amount of change in meeting can not increase after trade, so,

$$\frac{d(w^t_{i'+ix} - w^t_{ix})}{d(w^t_{l'+lx} - w^t_{lx})} = 0,$$

[19]For example, $w^*_{jx+p^*((i',i),(0,j))}$ and $w^*_{i'+ix-p^*((i',i),(0,j))}$ in (2.35) do not depend on i'.

if $i' < l'$. Hence all the blocks to the right of the diagonal are zero. For the diagonal blocks, we want

$$\left(\phi^{\Delta W(i')}_{\Delta W(i')}\right)_{i,l} = \frac{d(w^t_{i'+ix} - w^t_{ix})}{d(w^t_{i'+lx} - w^t_{lx})}.$$

The (RHS) of (2.36) includes $w^t_{i'+lx} - w^t_{lx}$ if for some j,

- $jx + p^*((i', i), (0, j)) = i' + lx$;
- $i' + ix - p^*((i', i), (0, j)) = i' + lx$;
- $i' + ix = i' + lx$.

Hence (2.18) follows. □

Proof. (Lemma 5)
Applying the implicit function theorem and the chain rule to $W^t = \phi(\Pi^t, \Phi(\Pi^t, W^t))$, we have

$$\begin{aligned} \Phi_W &= \phi_2^{-1} \\ \Phi_\Pi &= -\phi_2^{-1}\phi_1, \end{aligned}$$

where

$$\phi_1 = \begin{bmatrix} \phi^{\Delta W}_{\Pi_c} & O \\ \phi^{W_b}_{\Pi_c} & \phi^{W_b}_{\Pi_b} \end{bmatrix} \quad \text{and} \quad \phi_2 = \begin{bmatrix} \phi^{\Delta W}_{\Delta W} & O \\ \phi^{W_b}_{\Delta W} & \phi^{W_b}_{W_b} \end{bmatrix}$$

have a zero matrix for the upper-right block due to Lemma 4 (i). Since

$$\phi_2^{-1} = \begin{bmatrix} [\phi^{\Delta W}_{\Delta W}]^{-1} & O \\ * & * \end{bmatrix}$$

and

$$-\phi_2^{-1}\phi_1 = \begin{bmatrix} -[\phi^{\Delta W}_{\Delta W}]^{-1}\phi^{\Delta W}_{\Pi_c} & O \\ * & * \end{bmatrix},$$

(2.15) reduces to (2.19). □

Proof. (Lemma 6)

Claim 1. (General formula for eigenvectors of A)
(i) If μ and z are an eigenvalue and its associated eigenvector of $\Psi^{\Pi_c}_{\Pi_c}$, then A has eigenvalue μ and its associated eigenvector is

$$\begin{bmatrix} z \\ (I - \mu_1\phi^{\Delta W}_{\Delta W})^{-1}\phi^{\Delta W}_{\Pi_c} z \end{bmatrix}. \tag{2.38}$$

(ii) If μ and z are an eigenvalue and its associated eigenvector of $\Phi^{\Delta W}_{\Delta W}$, then A has eigenvalue μ and its associated eigenvector is $(\mathbb{0}, z)$.[20]

[20]Within Claim 1 and its proof, $(\cdot,\cdot)$ indicates vertical concatenation of two column vectors.

Proof. (Claim 1)
Since the upper-right block of A is zero matrix, the eigenvalues of A are those of $\Psi_{\Pi_c}^{\Pi_c}$ and $\Phi_{\Delta W}^{\Delta W}$. Suppose μ and $x = (x_1, x_2)$ are an eigenvalue and its associated eigenvector of A;

$$\begin{bmatrix} \Psi_{\Pi_c}^{\Pi_c} & O \\ \Phi_{\Pi_c}^{\Delta W} & \Phi_{\Delta W}^{\Delta W} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mu \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{2.39}$$

or

$$\begin{cases} \Psi_{\Pi_c}^{\Pi_c} x_1 = \mu x_1 \\ \Phi_{\Pi_c}^{\Delta W} x_1 + \Phi_{\Delta W}^{\Delta W} x_2 = \mu x_2 \end{cases}. \tag{2.40}$$

If μ is an eigenvalue of $\Psi_{\Pi_c}^{\Pi_c}$, then we have $\Psi_{\Pi_c}^{\Pi_c} z = \mu z$ for some $z \neq 0$. In this case, (2.40) implies that $(x_1, x_2) = (z, -(\Phi_{\Delta W}^{\Delta W} - \mu I)^{-1} \Phi_{\Pi_c}^{\Delta W} z)$ acts as eigenvector of A associated with μ. Using (2.20), this eigenvector is explicitly calculated;

$$\begin{aligned} \begin{bmatrix} z \\ -(\Phi_{\Delta W}^{\Delta W} - \mu I)^{-1} \Phi_{\Pi_c}^{\Delta W} z \end{bmatrix} &= \begin{bmatrix} z \\ (\phi_{\Delta W}^{\Delta W^{-1}} - \mu I)^{-1} \phi_{\Delta W}^{\Delta W^{-1}} \phi_{\Pi_c}^{\Delta W} z \end{bmatrix} \\ &= \begin{bmatrix} z \\ (I - \mu \phi_{\Delta W}^{\Delta W})^{-1} \phi_{\Pi_c}^{\Delta W} z \end{bmatrix}. \end{aligned} \tag{2.41}$$

If on the other hand, μ is an eigenvalue of $\Phi_{\Delta W}^{\Delta W}$, then $\Phi_{\Delta W}^{\Delta W} z = \mu z$ for some $z \neq 0$. So equation (2.40) implies that $(x_1, x_2) = (0, z)$ acts as an eigenvector of A associated with μ. □

The eigenvectors have information about the limit behavior of trajectories of the dynamical system. However, one can see that the eivenvectors of our interest are of Claim 1(i), not of (ii). The latter has the form of $(0, z)$, so the change-state probabilities are zero along such an eigenvector.[21]

The common discussion of stability analysis says that any convergent trajectory of a dynamical system is characterized by the eigenvector associated with the largest eigenvalue that does not exceed one in absolute value, because the convergence is slowest in that direction. (See, for example, Luenberger.) Therefore, out of Claim 1(i) eigenvectors, we are interested in those associated with the largest eigenvalues that do not exceed one in absolute value. By Lemma 3, $\Psi_{\Pi_c}^{\Pi_c}$ has a unit eigenvalue. Thus, the eigenvector of our interest corresponds to those of Claim 1(i) with $\mu = 1$, that is,

$$\begin{bmatrix} \omega_{\Delta W(1)} \\ \vdots \\ \omega_{\Delta W(x-1)} \end{bmatrix} \equiv (I - \phi_{\Delta W}^{\Delta W})^{-1} \phi_{\Pi_c}^{\Delta W} z, \tag{2.42}$$

where z is the eigenvector of $\Psi_{\Pi_c}^{\Pi_c}$ associated with a unit eigenvalue. But then, we can explicitly calculate $(I - \phi_{\Delta W}^{\Delta W})^{-1} \phi_{\Pi_c}^{\Delta W}$ by using the following claim. It shows that (2.42) is equal to (2.21).

[21]Recall that we are interested in whether there exists a convergent equlibrium path in which initially there is a positive measure of people who have change but in which eventually nobody has change.

Claim 2. (The product of two triangular block matrices)
Let R and S be triangular block matrices consisting of $n \times n$ pieces of blocks, such that $r_{ij} = O$ (i.e., (i,j)-block is zero matrix) if $i < j$ and $s_{ij} = O$ if $i + j \leq n$. (That is, R is lower triangular block matrix and S is right-lower triangular block matrix.) Suppose also that every block is identically sized square matrix. Then the product RS is also a right-lower triangular block matrix whose (i,j)-block is zero if $i + j \leq n$. Moreover, for i, j such that $i + j = n + 1$, the (i,j)-block (namely, "antidiagonal block") of RS is $r_{ii}s_{ij}$.

Proof. (Claim 2)
$(RS)_{ij} = \sum_{l=1}^{n} r_{il}s_{lj}$, where $r_{il} = O$ if $i < l$ and $s_{lj} = O$ if $l + j \leq n$. If $i + j \leq n$, then either $i < l$ or $l + j \leq n$ is true, as "$i \geq l$ and $l + j > n$" implies $i + j \geq l + j > n$. Hence we have $(i + j \leq n) \Rightarrow (RS)_{ij} = O$. Hence the product is also a right-lower triangular block matrix. Now suppose that $i + j = n + 1$. We have

$$\begin{cases} l \leq i - 1 & \Rightarrow \quad s_{lj} = O \\ l = i & \Rightarrow \quad r_{il} = r_{ii}, s_{lj} = s_{ij} \\ l \geq i + 1 & \Rightarrow \quad r_{il} = O \end{cases} . \tag{2.43}$$

Hence in this case, we have $(RS)_{ij} = r_{ii}s_{ij}$. [22] □

Now we apply Claim 2 to $R = \left(I - \phi_{\Delta W}^{\Delta W}\right)^{-1}$, $S = \phi_{\Pi_c}^{\Delta W}$ and $n = x - 1$. Then Lemma 4(ii)(iii) imply that the product $(I - \phi_{\Delta W}^{\Delta W})^{-1}\phi_{\Pi_c}^{\Delta W}$ is right-lower triangular matrix and has the form of

$$\begin{bmatrix} O & O & \cdots & O & (I - \phi_{\Delta W(1)}^{\Delta W(1)})^{-1}K \\ \vdots & \vdots & & \iddots & * \\ O & O & \iddots & & \vdots \\ O & (I - \phi_{\Delta W(x-2)}^{\Delta W(x-2)})^{-1}K & & * & * \\ (I - \phi_{\Delta W(x-1)}^{\Delta W(x-1)})^{-1}K & * & \cdots & * & * \end{bmatrix},$$

where irrelevant elements are expressed by $*$. □

Proof. (Lemma 7)
(i) Suppose by way of contradiction that $\zeta_{\Delta W(i')} \geq 0$ and $\left(\zeta_{\Delta W(i')}\right)_i = 0$ for some i. We have $\left(I - \phi_{\Delta W(i')}^{\Delta W(i')}\right)\zeta_{\Delta W(i')} = Kz_{x-i'}$. The (RHS) is strictly positive because of $z_{x-i'} \geq 0$ and $z_{x-i'} \neq 0$, and Lemma 4(ii). The i-th element of the (LHS) can not be strictly positive, because by Lemma 4(iii) off-diagonal elements of $\left(I - \phi_{\Delta W(i')}^{\Delta W(i')}\right)$ are all non-positive and $\left(\zeta_{\Delta W(i')}\right)_i = 0$.
(ii) Again we have $\left(I - \phi_{\Delta W(i')}^{\Delta W(i')}\right)\zeta_{\Delta W(i')} = Kz_{x-i'}$. The (RHS) is strictly positive because

[22] Not $r_{jj}s_{ij}$.

of $z_{x-i'} \geq 0$ and $z_{x-i'} \neq 0$, and Lemma 4(ii). By applying $i = 0$ and $p^*((i', 0), (0, j)) = i'$ in (2.18), we have

$$\left(\phi_{\Delta W(i')}^{\Delta W(i')}\right)_{0,l} = \frac{\pi_{lx}^*}{N} u'(0)\beta + \frac{\pi_{Bx}^* + N - 1}{N} \beta \mathbb{1}\{l = 0\}.$$

Therefore, the 0th row of $\left(\phi_{\Delta W(i')}^{\Delta W(i')}\right)$ is

$$\left(\frac{\pi_0^*}{N} u'(0)\beta + \frac{\pi_{Bx}^* + N - 1}{N} \beta, \;\; \frac{\pi_{1x}^*}{N} u'(0)\beta, \;\; \cdots, \frac{\pi_{(B-1)x}^*}{N} u'(0)\beta \right).$$

Therefore, the first element of $\left(I - \phi_{\Delta W(i')}^{\Delta W(i')}\right) \zeta_{\Delta W(i')}$ is

$$\left(1 - \frac{\pi_{Bx}^* + N - 1 + \pi_0^* u'(0)}{N} \beta\right) \left(\zeta_{\Delta W(i')}\right)_0 - \sum_{i=1}^{B-1} \frac{\pi_{ix}^*}{N} u'(0)\beta \left(\zeta_{\Delta W(i')}\right)_i. \tag{2.44}$$

Since $u'(0)$ is assumed to be any large number, at least one element of $\zeta_{\Delta W(i')}$ should be negative for (2.44) to be positive.[23] □

[23] Specifically,

$$u'(0) > \frac{N/\beta - N + (1 - \pi_{Bx}^*)}{\pi_0^*}$$

is sufficient here.

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Chapter 3

A comment on: "Efficient propagation of shocks and the optimal return on money"

abstract

Lotteries are introduced into Cavalcanti-Erosa (2008), a version of Trejos-Wright (1995) with aggregate shocks. Lotteries improve welfare and eliminate the two notable features of the optimum with deterministic trades: over-production and history-dependence. Moreover, the optimum can be supported by buyer take-it-or-leave-it offers.[1] (JEL classification: C78; D61; D82; E30; E40; E50)

Keywords: random matching model of money; aggregate shock; optimal allocation; history-dependence; lottery.

3.1 Introduction

Cavalcanti-Erosa (2008) (CE, hereafter) study optima in a version of Trejos-Wright (1995). They introduce into it *i.i.d.* aggregate shocks to preferences, shocks with a two-point support. They show that for an interval of intermediate magnitudes for the discount factor, the ex ante optimum over all individually rational (IR) and *deterministic* trades displays two properties: output is higher than the first-best when the shock is such that the first-best output is low and there is history dependence—that is, promised utilities play a role. We show that if lotteries are allowed, then higher ex ante utility is achieved and neither property holds at an optimum.[2]

The role of lotteries in the CE setting is easily explained. Consider the situation in which the shock is such that the first best level of output is high and in which the planner would like to weaken the seller IR constraint by making the current acquisition of money more valuable. Absent lotteries, CE achieve that by promising the current seller more output than the first best in the future when he is a buyer and the shock is such that the first

[1]This chapter is joint work with Pidong Huang. We greatly appreciate Neil Wallace.

[2]Berentsen-Molico-Wright [1] are the first to introduce lotteries into matching models of money.

best level of output is low.[3] With lotteries, the current acquisition of money can be made more valuable by having the buyer surrender money with some probability in that future situation.

3.2 Model

The model is Cavalcanti-Erosa (2008) except that lotteries are allowed in trade. Time is discrete, dated as $t \geq 0$, and there is a unit nonatomic measure of agents. At the beginning of every period, the economy is hit by an aggregate shock s with support $\{l, h\}$, low or high, which, as described below, affects preferences. The shock s is iid over time and the probability of state s is $\pi_s (> 0)$.

Each agent maximizes the discounted sum of expected utility with discount factor $\beta \in (0, 1)$. At each date, if an agent produces $y \geq 0$ amount of good, the utility cost is y. If an agent consumes $y \geq 0$ amount of good when the current state is s, the period utility he gets is $u_s(y)$, where $u_s : \mathbb{R}_+ \to \mathbb{R}$ is differentiable, strictly increasing, strictly concave, and satisfies $u_s(0) = 0$, $u_s'(0) = \infty$ and $u_s'(\infty) = 0$. We also assume that u_s is bounded, above by $\bar{u}$, and that $u_l' < u_h'$.[4] Define the first-best output levels by $y_s^* \equiv \arg\max\{u_s(y) - y\}$ or $u_s'(y_s^*) = 1$. That is, the first-best output maximizes the sum of utilities of the consumer and the producer. It follows that $y_l^* < y_h^*$.

In each period, after the aggregate state is observed, agents are randomly matched in pairs. With probability $1/N$, an agent is a consumer, with probability $1/N$, the agent is a producer, and with probability $1 - 2/N$, the match is a no-coincidence meeting., where $N \geq 2$.

There exists a fixed stock of indivisible, perfectly durable money, the per capita amount of which is denoted $m \in (0, 1)$. Individual money holdings are restricted to $\{0, 1\}$. In meetings, agents' money holdings are observable, but any other information about an agent's trading history is private.

3.3 The planner's problem and the solution

We study the mechanism-design problem studied by CE; the planner chooses an allocation to maximize welfare subject to a notion of implementability.

The realization of the date-t aggregate shock is denoted s_t and a history up to date t is denoted $s^t \equiv (s_0, s_1, \cdots, s_t)$. Let $S^t \equiv \{s_0\} \times \{l, h\}^t$ denote the set of possible histories up to date t conditional on the initial state s_0, and let $p(s^t) \equiv \pi_{s_1}\pi_{s_2} \cdots \pi_{s_t}$, the probability of event s^t. It is assumed that the initial state is given and that $p(s_0) = 1$.

[3] This over-production in turn leads to history dependence. See Proposition 10 of their paper for details.

[4] One way to get the linear cost function and the bounded utility function is as follows: suppose that the utility and the cost from consuming and producing z amount are given by a possibly unbounded function $\tilde{u}_s(z)$ and a convex function $\tilde{c}(z)$, respectively. Suppose further that there is a bound $\bar{z}$ on production in a sense that $\lim_{z \to \bar{z}} \tilde{c}(z) = \infty$. Then changing the unit of goods nonlinearly by $y \equiv \tilde{c}(z)$ leads to the bounded utility function $u_s(y) \equiv \tilde{u}_s(\tilde{c}^{-1}(y))$ and the linear cost function $c(y) \equiv \tilde{c}(\tilde{c}^{-1}(y)) = y$ with no bound on y.

An allocation is $\{y(s^t), q(s^t)\}_{s^t}$, where $y(s^t) \in \mathbb{R}_+$ is output (produced by the producer and consumed by the consumer) and $q(s^t) \in [0,1]$ is the probability that the consumer transfers money to the producer.[5] The welfare criterion is

$$\sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t p(s^t) \left[u_{s_t}(y(s^t)) - y(s^t) \right], \tag{3.1}$$

where $u_s(y) - y$ is the social gain, or the sum of period utility of the consumer and the producer.

Because people can exit a meeting without trade and with no further punishment, the planner is subject to IR constraints for the producer and for the consumer. In order to state the IR constraints in a simple way, let $v_j(s^t)$ denote the expected discounted utility of an individual with money holdings $j \in \{0,1\}$ after history s^t and before being matched. These satisfy

$$\begin{aligned} v_1(s^t) = {} & \frac{1-m}{N} \left[u_{s_t}(y(s^t)) + q(s^t)\beta(\pi_l v_0(s^t,l) + \pi_h v_0(s^t,h)) \right] \\ & + \left(1 - \frac{1-m}{N} q(s^t)\right) \beta(\pi_l v_1(s^t,l) + \pi_h v_1(s^t,h)), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} v_0(s^t) = {} & \frac{m}{N} \left[-y(s^t) + q(s^t)\beta(\pi_l v_1(s^t,l) + \pi_h v_1(s^t,h)) \right] \\ & + \left(1 - \frac{m}{N} q(s^t)\right) \beta(\pi_l v_0(s^t,l) + \pi_h v_0(s^t,h)). \end{aligned} \tag{3.3}$$

The IR constraints for the producer and the consumer are expressed as[6]

$$y(s^t) \leq q(s^t)\beta R(s^t) \leq u_{s_t}(y(s^t)), \tag{3.4}$$

where

$$R(s^t) \equiv \pi_l r(s^t,l) + \pi_h r(s^t,h), \tag{3.5}$$

and

$$r(s^t) \equiv v_1(s^t) - v_0(s^t).$$

The planner's problem is as follows.

Definition 1. *An allocation* $\{y(s^t), q(s^t)\}_{s^t}$ *is* implementable *if there exists a sequence* $\{v_0(s^t), v_1(s^t)\}$ *that satisfies conditions (3.2)-(3.4) and* $v_i(s^t) \in [0, \bar{u}/(1-\beta)]$. *An allocation is* optimal *if it maximizes (3.1) among the set of implementable allocations. An allocation is* history-independent *if it depends only on the current state, in which case the allocation is characterized by four numbers:* (y_l, q_l, y_h, q_h).

[5]Because goods are divisible and agents are risk-averse, lotteries over output do not improve welfare. The proof is somewhat analogous to Proposition 3 of Brentsen-Molico-Wright (2002).

[6]Expressing IR constraints by using v_j implicitly relies upon the principle of one-shot deviation. That principle applies because there is discounting and u_s is bounded.

Our result is

Proposition 1. *Let*

$$\frac{1}{\beta_l} \equiv 1 + \frac{1-m}{N} \cdot \frac{\pi_l\,[u_l(y_l^*) - y_l^*] + \pi_h\,[u_h(y_l^*) - y_l^*]}{y_l^*} \tag{3.6}$$

$$\frac{1}{\beta_h} \equiv 1 + \frac{1-m}{N} \cdot \frac{\pi_l\,[u_l(y_l^*) - y_l^*] + \pi_h\,[u_h(y_h^*) - y_h^*]}{y_h^*}. \tag{3.7}$$

There is an optimal allocation and it is history-independent. Moreover, $0 < \beta_l < \beta_h < 1$ *and the optimal allocation is as follows; if* $\beta \le \beta_l$*, then* $y_l = y_h \le y_l^*$ *and* $q_l = q_h = 1$*; if* $\beta \in (\beta_l, \beta_h)$*, then* $y_l = y_l^*$*,* $y_h < y_h^*$*,* $q_l < 1$ *and* $q_h = 1$*; and finally if* $\beta \ge \beta_h$*, then* $y_l = y_l^*$ *and* $y_h = y_h^*$*.*

This differs from CE for intermediate magnitudes for β; state-l output is kept first-best and lotteries are necessary.[7] Moreover, one can see in the proof that the optimum can be supported by buyer take-it-or-leave-it offers. Hence, the trades are not only IR, but also coalition-proof for the pairs in meetings.

3.4 Proof of the proposition

The proof proceeds as folllows. First, an upper bound on $R(s^t)$ (see (3.5)) is established. (The candidate for the upper bound, which depends on β, is provided in Lemma 1. Then, Lemma 2 shows that the candidate is, in fact, an upper bound.) Then, the proposition is proved by constructing the optimum in terms of that upper bound.

Lemma 1. *Let*

$$g(R;\beta) \equiv \beta R + \frac{1-m}{N}[\pi_l \max_{0\le q_l \le 1} H_l(q_l\beta R) + \pi_h \max_{0\le q_h\le 1} H_h(q_h\beta R)],$$

where $H_s(x) \equiv u_s(x) - x$*. The function* $g(\cdot;\beta)$ *has a unique positive fixed point, denoted* $\bar{R}(\beta)$*. Moreover,* $\beta\bar{R}(\beta)$ *is strictly increasing with* $\beta_l\bar{R}(\beta_l) = y_l^*$ *and* $\beta_h\bar{R}(\beta_h) = y_h^*$*, which implies* $\beta_l < \beta_h$*.*

Proof. Note that

$$\arg\max_{q_s\in[0,1]} H_s(q_s\beta R) = \begin{cases} y_s^*/\beta R & \text{if } \beta R \ge y_s^* \\ 1 & \text{otherwise} \end{cases}.$$

It follows that

$$g(R;\beta) = \begin{cases} \beta R + \frac{1-m}{N}[\pi_l H_l(\beta R) + \pi_h H_h(\beta R)] & \text{if} \quad \beta R \le y_l^* \\ \beta R + \frac{1-m}{N}[\pi_l H_l(y_l^*) + \pi_h H_h(\beta R)] & \text{if} \quad \beta R \in [y_l^*, y_h^*] \\ \beta R + \frac{1-m}{N}[\pi_l H_l(y_l^*) + \pi_h H_h(y_h^*)] & \text{if} \quad \beta R \ge y_h^* \end{cases} \tag{3.8}$$

[7] It is not hard to show that in the model without aggregate shocks, optima can be attained without the use of lotteries. In this sense, the CE model is a simple monetary model in which lotteries are necessary.

Therefore, $g(R;\beta)$ is continuous and strictly increasing in R. Moreover, it follows by direct computation that $\partial g(R;\beta)/\partial R$ exists, is infinite at $R = 0$, is weakly decreasing in R, and that $\partial g(R;\beta)/\partial R = \beta$ for $R \geq y_h^*/\beta$. Then, $g(0;\beta) = 0$ implies that there is a unique $R > 0$, denoted $\bar{R}(\beta)$, such that $R = g(R;\beta)$. Also, because $g(R;\beta)$ is strictly increasing in β for $R > 0$, it follows that $\bar{R}(\beta)$ is strictly increasing in β. Finally, continuity of $\bar{R}(\beta)$ follows from the implicit function theorem.

Now consider the equations, $\beta\bar{R}(\beta) = y_s^*$. It follows from the above characterization of g that $\beta\bar{R}(\beta) \to \infty$ as $\beta \to 1$. That, $\beta\bar{R}(\beta) = 0$ at $\beta = 0$, and continuity of $\beta\bar{R}(\beta)$ imply existence of a solution. Also, monotonicity of $\beta\bar{R}(\beta)$ implies that the solution is unique and increasing in y_s^*.

Finally, the closed-form expressions for the β_s are obtained by solving the equations $y_s^*/\beta = g(y_s^*/\beta;\beta)$ for β. In particular, by (3.8), for $s = l$, that equation is

$$y_l^*/\beta = y_l^* + \frac{1-m}{N}[\pi_l H_l(y_l^*) + \pi_h H_h(y_l^*)],$$

while for $s = h$, it is

$$y_h^*/\beta = y_h^* + \frac{1-m}{N}[\pi_l H_l(y_l^*) + \pi_h H_h(y_h^*)].$$

□

Lemma 2. *If $\{y(s^t), q(s^t)\}_{s^t}$ is implementable, then $R(s^t) \leq \bar{R}(\beta)$ for all s^t.*

Proof. For any s^{t-1} and s_t,

$$\begin{aligned}
r(s^{t-1}, s_t) &= \frac{(1-m)u_{s_t}(y(s^t)) + my(s^t)}{N} + \left(1 - \frac{q(s^t)}{N}\right)\beta R(s^t) \\
&\leq \frac{(1-m)u_{s_t}(q(s^t)\beta R(s^t)) + mq(s^t)\beta R(s^t)}{N} + \left(1 - \frac{q(s^t)}{N}\right)\beta R(s^t) \\
&= \beta R(s^t) + \frac{1-m}{N}\left[u_{s_t}(q(s^t)\beta R(s^t)) - q(s^t)\beta R(s^t)\right] \\
&\leq \beta R(s^t) + \frac{1-m}{N}\max_{0 \leq q \leq 1}\left[u_{s_t}(q\beta R(s^t)) - q\beta R(s^t)\right] \\
&\equiv g_s(R(s^t)),
\end{aligned} \tag{3.9}$$

where the first equality follows from the definition of $r(s^t)$ (see (3.3) and (3.2)), and the first inequality from the first inequality in (3.4), the producer IR constraint. Hence, we have

$$\begin{aligned}
R(s^{t-1}) &= \pi_l r(s^{t-1}, l) + \pi_h r(s^{t-1}, h) \\
&\leq \pi_l g_l(R(s^{t-1}, l)) + \pi_h g_h(R(s^{t-1}, h)) \\
&\leq g(\max\{R(s^{t-1}, l), R(s^{t-1}, h)\}),
\end{aligned}$$

where the first inequality follows from (3.9) and the second inequality because g_s is increasing. Therefore,

$$R(s^{t-1}) \leq g(R(s^{t-1}, s_t)) \text{ for either } s_t = l \text{ or } s_t = h. \tag{3.10}$$

Now, suppose, by way of contradiction, that $R(s^t) > \bar{R}(\beta)$ for some s^t. Then, by (3.10), there exists s^{t+1} such that $R(s^{t+1}) \geq f(R(s^t))$, where $f = g^{-1}$. Because f is increasing, by induction there exists a continuation of s^t such that $R(s^{t+n}) \geq f^{(n)}(R(s^t))$ for all n. Moreover, it follows from the properties of g that $f(R(s^t)) > R(s^t)$ and that f is convex. Therefore, the sequence $R(s^{t+n})$ is unbounded, which violates the definition of implementability. □

Proof. (Proposition 1)
We consider, in turn, three exhausitve cases.
Case 1: $\beta \leq \beta_l$
Consider the allocation $(y_s, q_s) = (\beta\bar{R}(\beta), 1), \quad s = l, h$. By construction, this satisfies the first inequality in (3.4). Also, $\beta\bar{R}(\beta) \leq y_l^* < y_h^*$ (see Lemma 1) implies $u_s(y_s) = u_s(\beta\bar{R}(\beta)) \geq \beta\bar{R}(\beta) = \beta\bar{R}(\beta)q_s$. Therefore, the second inquality in (3.4) is also satisfied. Hence, this allocation is implementable.
Now because $\beta\bar{R}(\beta) \leq y_l^*$ and $u_s(y) - y$ is increasing in y for $y \in [0, y_l^*]$, any better allocation must have higher production after some history. However, the bound on $R(s^t)$ and $q_s = 1$ implies that higher production violates the first inquality in (3.4).

Case 2: $\beta_l < \beta < \beta_h$
Consider the allocation $(y_h, q_h) = (\beta\bar{R}(\beta), 1)$ and $(y_l, q_l) = (y_l^*, \frac{y_l^*}{\beta\bar{R}(\beta)})$, where $y_l^* < \beta\bar{R}(\beta) < y_h^*$ (see Lemma 1) guarantees $q_l < 1$. By construction, this satisfies the first inequality of (3.4). Also, $u_l(y_l) = u_l(y_l^*) \geq y_l^* = \beta\bar{R}(\beta)q_l$, and $\beta\bar{R}(\beta) \leq y_h^*$ implies $u_h(y_h) = u_h(\beta\bar{R}(\beta)) \geq \beta\bar{R}(\beta) = \beta\bar{R}(\beta)q_h$. Therefore, the second inequality of (3.4) is also satisfied. Hence, the allocation is implementable.
Now because $\beta\bar{R}(\beta) < y_h^*$ and $u_h(y) - y$ is increasing in y for $y \in [0, y_h^*]$, any better allocation must have higher production after some history s^t with $s_t = h$. (After histories with $s_t = l$, $y(s^t) = y_l^*$, so there is no room for improvement.) However, the bound on $R(s^t)$ and $q_h = 1$ implies that higher production violates the first inequality in (3.4).

Case 3: $\beta_h \leq \beta$
Consider the allocation $(y_s, q_s) = (y_s^*, \frac{y_s^*}{\beta\bar{R}(\beta)}), \ s = l, h$, where $y_l^* < y_h^* \leq \beta\bar{R}(\beta)$ (see Lemma 1) guarantees $q_s \leq 1$. By construction, this satisfies the first inequality of (3.4). Also, $u_s(y_s) = u_s(y_s^*) \geq y_s^* = \beta\bar{R}(\beta)q_s$ implies that the second inequality of (3.4) is satisfied. Hence, the allocation is implementable. It is optimal because it is first-best.[8] □

3.5 The optimal choice of m.

Given the result that the optimal allocation is history-independent, we now consider the optimal choice of m.[9] For that purpose, we now express β_s in (3.6)-(3.7) and $\bar{R}(\beta)$ in Lemma

[8]In this range, the outputs are unique but q's are not. This is similar to what happens in Trejos-Wright for high discount factor. Here, (q_l, q_h) is chosen to maximize $R(s^t)$, which is equivalent to buyer take-it-or-leave-it offers.

[9]Similar discussions are found in previous models without aggregate shock and lotteries: in Trejos-Wright (1995), where consumer and producer have a specific Nash bargaining with equal bargaining powers, and in

1 as $\beta_s(m)$ and $\bar{R}(m,\beta)$, respectively, to make explicit their dependence on m. Suppose that the planner chooses m before the initial shock s_0 is realized. The planner maximizes the product $E(m) \cdot I(m,\beta)$, where $E(m) \equiv m(1-m)/N$, is the frequency of *trade* meetings, and

$$I(m,\beta) \equiv \begin{cases} \frac{1}{1-\beta}\{\pi_l H_l(\beta\bar{R}(m,\beta)) + \pi_h H_h(\beta\bar{R}(m,\beta))\} & \text{if} \quad \beta < \beta_l(m) \\ \frac{1}{1-\beta}\{\pi_l H_l(y_l^*) + \pi_h H_h(\beta\bar{R}(m,\beta))\} & \text{if} \quad \beta_l(m) \le \beta < \beta_h(m) \\ \frac{1}{1-\beta}\{\pi_l H_l(y_l^*) + \pi_h H_h(y_h^*)\} & \text{if} \quad \beta_h(m) \le \beta \end{cases}.$$

$E(m)$ is increasing for $m < 0.5$, a maximum at $m = 0.5$, and decreasing for $m > 0.5$, while $I(m,\beta)$ is decreasing in m, because the cutoff values $\beta_s(m)$ are increasing and the maximum return $\bar{R}(m,\beta)$ is strictly decreasing in m.

One immediate result is that if $\beta \ge \beta_h(0.5)$, then the unique optimal quantity is 0.5. Otherwise, the optimal quantity is less than 0.5, as can be seen from following first-order condition, a necessary condition for the optimal m:

$$\begin{aligned} 0 &= \frac{\partial E}{\partial m} \cdot I(m) + E(m) \cdot \frac{\partial I}{\partial m} \\ &= \frac{1-2m}{N} \cdot I(m,\beta) + \frac{m(1-m)}{N} \cdot \frac{\partial I}{\partial m}. \end{aligned} \tag{3.11}$$

The first term, the 'extensive margin effect,' is zero at $m = 0.5$, while the second term, the 'intensive margin effect,' is negative at $m = 0.5$, because $\partial I/\partial m|_{m=0.5} < 0$ due to $\beta < \beta_h(0.5)$ and $\partial\bar{R}/\partial m < 0$.

3.6 Extension to more than two states

The extension of our results to the case of more than two states is straightforward. Let the support of the preference shock s be $\{1, 2, \cdots, d\}$, where $y_1^* < \cdots < y_d^*$. Then, let

$$\frac{1}{\beta_s} \equiv 1 + \frac{1-m}{N} \cdot \frac{\sum_{i \le s} \pi_i H_i(y_i^*) + \sum_{i \ge s+1} \pi_i H_i(y_s^*)}{y_s^*}$$

for $s = 1, \cdots, d$. The candidate for the optimal allocation is as follows.

$$\begin{array}{llll} \text{If} & \beta \in (0, \beta_1] & \text{then} & (y_i, q_i) = (\beta\bar{R}, 1), \quad i = 1, \cdots, d; \\ \text{if} & \beta \in [\beta_s, \beta_{s+1}] & \text{then} & (y_i, q_i) = \begin{cases} (y_i^*, \frac{y_i^*}{\beta\bar{R}}) & i = 1, \cdots, s \\ (\beta\bar{R}, 1) & i = s+1, \cdots, d \end{cases}; \\ \text{if} & \beta \in [\beta_d, 1) & \text{then} & (y_i, q_i) = (y_i^*, \frac{y_i^*}{\beta\bar{R}}), \quad i = 1, \cdots, d, \end{array}$$

Cavalcanti-Wallace (1999), where the planner chooses the optimal allocation.

where $\bar{R} = \bar{R}(\beta)$ is the unique positive solution to $R = g(R; \beta)$ and

$$g(R; \beta) \equiv \beta R + \frac{1-m}{N} \sum_{s=1}^{d} \pi_s \max_{0 \leq q_s \leq 1} H_s(q_s \beta R).$$

The proof is essentially the same as that for two states.

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Chapter 4

Distributional effects of hiring through networks

abstract

We present a variant of Galenianos (2011), a version of a random search model with two matching technologies: a standard matching function and worker networks. Our model has two types of workers, networked workers and non-networked workers. A steady state equilibrium exists where networked workers have lower unemployment and higher wages, and it is unique under some conditions. Then we ask a question: how would a policy that bans the use of networks in hiring (e.g., anti-old boy network laws) affect welfare? It is shown that the effects of such a policy on non-networked workers can be either positive or negative, depending on model parameters. In our calibration, such a policy would make non-networked workers slightly worse off and networked workers substantially worse off.[1]

Keywords: random search, network, referral, policy analysis, welfare, dynamics.
JEL classifications: C78, E24, E60, I3, J20, J30.

4.1 Introduction

Suppose one is currently unemployed while some of his relatives and friends are employed. Then these people sometimes find him a job; they refer him to their employers. Such social networks also include so-called old-boy networks (i.e., alumni of the same school).[2] As Topa (2010) well summarizes, the use of social networks of workers in the labor market is widespread and seems important.[3] The role of networks is often associated with so-

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[2]In this paper, the word ‘network’ is used in a rather casual sense and does not mean graph-theoretic structures.

[3]A large proportion of workers attempt to use their personal networks in their job search alongside other methods (e.g., Holzer (1987), Elliott (1999)) and many of them actually find their jobs through networks (e.g., Granovetter (1995), Lin et al. (1981)). Also, many surveys report positive effects of the use of personal networks on wages (e.g., Korenman and Turner (1996)).

called ‘search friction’ that causes the co-existence of unemployment and vacancies. After all, there is a lot of unmodeled heterogeneity in workers’ skills and specializations, kinds of jobs, locations, etc., so it takes time for workers and firms to be matched with a right partner. One important role of networks is that one’s network friends know his qualities and hence can tell if he is suitable for the kind of job their employers have. This aspect of networks potentially mitigates the search friction and saves firms hiring cost. For example, some university departments only hire their own Ph.D’s to get a qualified worker without reviewing application materials, having interviews, etc.

On the other hand, the use of social networks in hiring is sometimes restricted. Many govermental/public institutions attempt to avoid hiring through social networks and use more formal channels by posting job openings in public. Also, some policies simply restrict one from hiring a person who belongs to the same social group. (e.g., anti-nepotism laws for public jobs, affirmative actions regarding race, sex, nationality, caste, etc.). In some cases, avoiding informal channels is for the sake of diversity. In other cases, it is because hiring through informal channels is not regarded as fair for those without any such connections.

A natural question arises: what is the welfare implication of hiring through social networks? In particular, are workers without a network hurt because others are hired through networks? In other words, would they benefit from, for example, expanding anti-nepotism laws or affirmative actions to the entire economy? To answer this question, we study a version of the random search model with two permanent types of workers: workers with a social network and workers without one. The model also has two ways of filling jobs, a formal channel (a standard matching function for which firms pay hiring cost) and worker networks, and firms can use both methods. A steady state equilibrium exists where networked workers have lower unemployment and higher wages, and it is unique under some conditions.

Then we ask a question: how would a policy that bans the use of worker networks in hiring affect the welfare of workers? We compare the welfare of the pre-policy economy (the steady state with networks playing a role) and the welfare of the post-policy economy (the transition path to the new steady state). We show that whether the effect of such a policy on non-networked workers is positive or negative depends upon model parameters. In our calibration, such a policy would make non-networked workers slightly worse off and networked workers substantially worse off. This result may seem surprising if one imagines a situation where workers are competing over a fixed number of available posts. In reality, the number of posted vacancies is also determined in equilibrium, affected by the intensity of search friction. Our simple model suggests that the effectiveness of such a policy that increases search friction is ambiguous even for policy-makers who care about people without any networks or connections.

Labor-search models with ‘network’ components are divided into two types. On one hand, there are models with exogenous vacancies. For example, each agent finds a job opening with some exogenous propability and passes that job on to one of his friends if he is employed. Montgomery (1991, 1992, 1994), Calvo-Armengol and Jackson (2004, 2007), and Mayer (2011) belongs to this type. On the other hand, there are models with endogenous vacancies as Mortensen and Pissarides (1994); there is a matching function and the number of vacancies posted by firms is determined by firm free-entry condition. This class includes Calvo-Armengol and Zenou (2005), Fontaine (2008), Kuzubas (2010), and Galenianos (2011). Among them, Galenianos (2011) is different from others. In the first three, a network component is added to the matching function. A firm goes through costly labor search and is matched with a worker through the matching function, but this worker, if he is already employed, passes that job offer to one of his unemployed friends. Galenianos (2011), on the other hand, regards the matching function (i.e., costly search) and network matching as two distinct matching technologies. Firms can search for a worker through formal costly process. But at the same time, they can also count on their current employee to find a qualified person from among his network friends. This paper borrows Galenianos's network matching to address the welfare-related question. The model is however different from Galenianos in other respects.[4]

4.2 Model

The model is a variant of Galenianos (2011). Time is continuous and the horizon is infinite.

Workers and firms

There are continuum-many workers and their total measure is one. Each worker has an exogenously given type $j \in \{0, 1\}$. Type-0 workers have no networks ('non-networked workers') and type-1 workers have networks ('networked workers'). These two types are identical in the other respects. In particular, they have the same productivity. Let n_0 and n_1 be the proportions of non-networked and networked workers, respectively. It follows that $n_0 + n_1 = 1$. There are also continuum-many firms and their total measure is infinite. Both workers and firms are risk-neutral, maximizing expected discounted income with discount factor r.

At any point in time, each worker is either employed or unemployed. If he is unemployed, he gets flow output $b > 0$ from home production. Each firm can employ at most one worker, so we will use the terms *firm* and *job* interchangeably.[5] Each firm/job is either incumbent (i.e., filled with a type-j worker) or unmatched. Incumbent firms produce constant flow output $y > b$. Each unmatched firm is either posting or non-posting. Posting firms pay flow cost k to keep posting its vacancy.[6] Entry is costless so firms currently not posting a vacancy can get posting status simply by starting to pay the flow cost.

Matching technologies

New matches between workers and firms are created in two ways and firms can use both. Existing matches are destroyed at some exogenous rate δ, in which case the worker becomes

[4] In particular, Galenianos does not allow workers to have no networks at all.

[5] One job per firm is a common assumption and it is innocuous under the constant returns to scale production and cost of hiring. We will come back to this point in Section 4.5.

[6] The interpretation of this cost includes cost to post newspaper ads, review application materials, have interviews, etc.

unemployed and the firm becomes unmatched.

One matching technology is described by the following version of the standard matching function. Let u_j be the unemployment rate of type-j workers and $u \equiv n_0 u_0 + n_1 u_1$ be the total unemployment rate. Furthermore, let v be the measure of vacant firms. Then the matching function creates a flow $M(u, v) \equiv \mu u^{1-\eta} v^{\eta}$ of matches between u unemployed workers and v vacant firms, where $\mu > 0$ and $\eta \in (0, 1)$. Therefore, for each unemployed worker, the Poisson arrival rate of a job offer through the matching function is $M(u, v)/u = \mu\theta^{\eta}$, where $\theta \equiv v/u$ is the labor market tightness. For each vacant firm, the arrival rate of a type j-worker is $M(u, v)/v \times (n_j u_j/u) = \mu\theta^{\eta-1} \times (n_j u_j/u)$.

The other matching technology uses in-network referrals. All firms know the same large number of other firms and all type-1 workers know the same large number of other type-1 workers. An existing match between an incumbent firm (firm A) and a type-1 employee (worker A) generates another potential match with Poisson rate $\rho \geq 0$. The timeline is as follows. Firm A knows another unmatched firm (firm B) and worker A knows a friend from his network (worker B), and their finding is that firm B and worker B could potentially be a good match. In such an event, worker A contacts worker B. If worker B is currently unemployed, which will be the case with probability u_1, worker A refers her to his employer (firm A). Then firm A refers worker B to the unmatched firm B and a new match is created.Therefore, from the incumbent firm A's point of view, the arrival rate of network matches is ρu_1. From the point of view of the unemployed type-1 worker B, the arrival rate of referrals is $(1-u_1)\rho$. That is, the arrival of referrals is assumed to depend upon the employment rate of their network friends because only existing matches generate referrals.

An important feature of the network matching is that firms do not have to pay hiring cost k to get a worker referred to them. Any unmatched firm, whether it is currently posting a vacancy or not, could potentially get a referral. Hence each unmatched firm's decision is whether they should just wait for referrals or they should also post a vacancy while waiting for referrals.

Wages and values

A firm that employs a type-j worker pays flow wage w_j to its employee. A type-j worker has lifetime value W_j if he is employed and U_j if he is unemployed. Also, let the lifetime value of a firm be J_j if it is filled by a type-j worker and V if it is posting. When a filled firm refers its employee's friend to an unmatched firm and a new match is created, it receives the whole value $J_1 - V$ created by the new match. This is an implicit one-time payment from the unmatched firm to the incumbent firm.[7]

There is a set of Bellman equations associated with these values. The wage w_j is determined by the traditional Nash bargaining assumption, where the worker's bargaining power is $\beta \in (0, 1)$. We assume the free-entry condition $V = 0$ so a non-posting firm does not gain positive net profit from getting a posting status by starting to pay hiring cost. That is, posting firms and non-posting firms both have value V.

[7] In Section 4.5, we show that one-firm-one-worker model with this referral mechanism is equivalent to the model in which firms can employ multiple workers.

Definition 1. *A* steady state equilibrium *is* $(U_i, W_i, J_i, w_i, u_i)_{i=0,1}$ *and* (V, v) *that satisfy*

$$\begin{aligned}
rU_0 &= b + \mu\theta^\eta (W_0 - U_0) && (4.1)\\
rU_1 &= b + \mu\theta^\eta (W_1 - U_1) + \rho(1-u_1)(W_1 - U_1) && (4.2)\\
rW_0 &= w_0 + \delta(U_0 - W_0) && (4.3)\\
rW_1 &= w_1 + \delta(U_1 - W_1) && (4.4)\\
rJ_0 &= y - w_0 + \delta(V - J_0) && (4.5)\\
rJ_1 &= y - w_1 + \delta(V - J_1) + \rho u_1 (J_1 - V) && (4.6)\\
rV &= -k + \sum_{j=0,1} \mu\theta^{\eta-1} \frac{n_j u_j}{u} (J_j - V) && (4.7)\\
(W_j - U_j)/\beta &= (J_j - V)/(1-\beta), \quad j = 0,1 && (4.8)\\
(1-u_0)\delta &= u_0 \mu\theta^\eta && (4.9)\\
(1-u_1)\delta &= u_1(\mu\theta^\eta + \rho(1-u_1)) && (4.10)\\
V &= 0, && (4.11)
\end{aligned}$$

where $u = n_0 u_0 + n_1 u_1$ *and* $\theta = v/u$.

(4.1)–(4.7) are Bellman equations, (4.8) is wage determination, (4.9)–(4.10) are inflow-equal-outflow conditions, and (4.11) is the free-entry condition. That is, the equilibrium is defined as a solution to the system of 12 equations in 12 unknowns.

What makes this model different from the standard Mortensen-Pissarides model is the three terms involving ρ; the economy has one more matching technology (i.e., the last term of (4.10)), some workers have a chance to be hired through networks (i.e., the last term of (4.2)), and some incumbent firms have a chance to get profits from the use of their employee's networks (i.e., the last term of (4.6)).

Proposition 1. *There exists a steady state equilibrium.*

The proof of the existence of a steady state is much harder than that of the Mortensen-Pissarides model due to the fact that the right hand sides of (4.2) and (4.6) include u_1, not just θ. The proof is one-by-one elimination of variables and found in the appendix. The following features are true in any steady state equilibrium:

Proposition 2. *(Dominance of networked workers)*
$u_0 > u_1$*,* $W_1 > W_0$*,* $U_1 > U_0$*, and* $w_1 > w_0$*.*

The higher wage for type-1 workers arises due to two reasons. First, the firm can potentially benefit from having an employee who sometimes refers a friend. So the firm compensates for that benefit by paying a higher wage to networked workers. Second, the networked worker has higher unemployment value because of the prospect of getting hired through his network. Therefore, he has higher reservation value in Nash bargaining over wage.

On the other hand, depending on the parameters, it is not necessarily the case that $J_1 > J_0$ in equilibrium. Indeed, it is quite possible that $J_0 > J_1$. In such a case, the interpretation is that the equilibrium wage for networked workers is so high that the firm prefers being matched with a non-networked worker. However, in any equilibrium, both J_0 and J_1 are higher than V, so posting firms hire whoever arrives first rather than remaining vacant and waiting for the next arrival of a different type of worker.

Next we give some results on comparative statics and uniqueness of equilibrium. For that purpose, consider the standard Mortensen-Pissarides economy, which coincides with our model when $\rho = 0$. That is, networks do not play a role, thereby rendering the differences between types virtually meaningless. We denote all the endogenous variables of the MP economy with subscript "m". For example, u_m is the (total) unemployment rate of the MP steady state, $\theta_m \equiv v_m/u_m$ is the market tightness of the MP steady state, etc. The unique existence of the MP steady state is easy to show.

Lemma 1. *The MP steady state* $(U_m, W_m, J_m, V_m, w_m, u_m, v_m, \theta_m)$ *exists and is unique.*

Due to Lemma 1, u_m is an implicit function of parameters $(r, \delta, \mu, \eta, y, b, k, \beta)$ that is determined outside our model. In this sense, u_m can be treated as exogenous to our model, although it is endogenous to the MP model.

Proposition 3. *(Dependence of steady states on* ρ*)*
In any equilibrium, u_1 *is decreasing in* ρ *near* $\rho = 0$*. Moreover,*

(i) if $u_m < \beta$*, then* u_0*,* J_0 *are increasing, and* v*,* θ*,* J_1*,* w_0*,* W_0*,* U_0 *are decreasing in* ρ *near* $\rho = 0$*; and*

(ii) if $u_m > \beta$*, then* u_0*,* u*,* J_0 *are decreasing, and* θ*,* J_1*,* w_0*,* W_0*,* U_0*,* w_1*,* W_1*,* U_1 *are increasing in* ρ *near* $\rho = 0$*.*

The effect of positive ρ on the economy is a compound of three effects, corresponding to the three terms involving ρ, each in (4.2), (4.6) and (4.10). When ρ is positive as opposed to zero, the economy has one more matching technology (the last term in (4.10)), so the total unemployment rate u is different. In addition, the term in (4.2) and the term in (4.6) imply two offsetting effects on firms. On one hand, positive ρ gives type-1 workers higher outside value during wage negotiations, which leads to higher wages for them and lower profits for firms. On the other hand positive ρ allows incumbent firms to potentially receive profits from network matchings. In case (i), the former dominates, discouraging firms' entry and decreasing θ. In turn, type-0 workers are disadvantaged by the existence of networked people. In case (ii), the latter dominates, increasing θ. In turn, type-0 workers actually benefit from the existence of networked people. As is shown after the proof of Lemma 1, u_m is increasing in β, k and b, and satisfies $\lim_{k\to\infty} u_m = 1$, $\lim_{\beta\to 0} u_m > 0$, etc. So both $u_m < \beta$ and $u_m > \beta$ are possible.

The following uniqueness property is useful when we compute the equilibrium in the next section.

Proposition 4. *(Uniqueness)*
Suppose $\eta \leq 0.5$*. Then,*

(i) at most one steady state equilibrium satisfies $u_1 \leq 0.5$*;*
(ii) $k \approx 0$ *is sufficient for uniqueness; and*
(iii) $\rho \approx 0$ *and* $u_m \leq 0.5$ *are sufficient for uniqueness.*[8]

4.3 Policy Implications

In this section, we study implications of the policy that bans hiring through networks, imposing $\rho = 0$. After such a policy is imposed, there is no longer a difference between type-0 and type-1 workers so the environment becomes that of Mortensen-Pissarides. (Hereafter, we mean $\rho = 0$ by "the MP model".) As the next proposition states, the MP model has very simple dynamics.

Proposition 5. *(Stability of the MP steady state)*
Consider the dynamics of the MP equilibrium:

$$\dot{U_m} = rU_m(t) - b - \mu\theta_m(t)^\eta[W_m(t) - U_m(t)] \tag{4.12}$$
$$\dot{W_m} = rW_m(t) - w_m(t) - \delta[U_m(t) - W_m(t)] \tag{4.13}$$
$$\dot{J_m} = rJ_m(t) - (y - w_m(t)) - \delta[V_m(t) - J_m(t)] \tag{4.14}$$
$$\dot{V_m} = rV_m(t) + k - \mu\theta_m(t)^{\eta-1}[J_m(t) - V_m(t)] \tag{4.15}$$
$$\dot{u_m} = (1 - u_m(t))\delta - u_m(t)\mu\theta_m(t)^\eta. \tag{4.16}$$

with Nash bargaining condition $(W_m(t) - U_m(t))/\beta = (J_m(t) - V_m(t))/(1 - \beta)$*,* $\forall t$*, and the free-entry* $V_m(t) = \dot{V_m}(t) = 0$*,* $\forall t$*. Given the initial value for* u_m*, an equilibrium path is unique. Moreover,* $u_m(t)$ *converges to the steady state level gradually, while the wage, values, and* $\theta_m(t)$ *are constant.*

When one studies policy effects, it is not legitimate to compare two steady states, one for $\rho > 0$ and the other for $\rho = 0$ (i.e., the MP steady state). The transition should be taken into account; we should compare the former steady state with a path that converges to the latter steady state after such a policy is imposed. The following is our scenario. Suppose that the economy starts with $\rho > 0$ and it is in the steady state with the total unemployment rate $u \equiv n_0u_0 + n_1u_1$. One day, say at time t_0, hiring through networks is banned by law in a permanent and unanticipated manner, so ρ is set to zero. Is such a policy good or bad? By the last proposition, while the economy's unemployment rate, starting with the initial level u, gradually converges to the MP steady-state level u_m, all the values and wages immediately jump to those of the MP steady state. Therefore, for values and hence welfares, the comparison in Proposition 3 is valid even though the whole economy doesn't immediately jump to the MP steady state. The result in Proposition 3, however, is limited to near $\rho = 0$, is only qualitative and still depends on parameters. Hence we perform the calibration below.

In the calibrated model, we first analyze how much each of the four groups of workers (type-0/type-1 and employed/unemployed) becomes better off (or worse off). In addition,

[8] $\eta \leq 0.5$ is not inconsistent with $u_m \leq 0.5$. For instance, Shimer (2005) calibrates η to be 0.28.

Table 4.1: Parameter calibration

t	the unit of time	quarter
r	interest rate	0.012 (Shimer)
δ	job destruction rate	0.1 (Shimer)
ρ	importance of networks	1.2
μ	coefficient of matching function	$0.45/0.63^{\eta}$
η	exponent of matching function	
y	output	1 (normalization)
b	home production	0.4 (Shimer)
k	vacancy cost	7.1
n_1	size of networked workers	0.85
β	worker's bargaining power	0.024

we compute the total resource of the economy: match output plus home production minus vacancy cost. In the pre-policy steady state, the total resource is $[bu + y(1-u) - kv]/r$. In the post-policy economy, it is

$$Y_m(t_0) \equiv \int_{t_0}^{\infty} e^{-r(\tau-t)}[bu_m(\tau) + y(1-u_m(\tau)) - kv_m(\tau)]d\tau, \text{ with } u_m(t_0) = u. \tag{4.17}$$

The following proposition is useful to compute it.

Proposition 6.

$$Y_m(t_0) = uU_m + (1-u)(W_m + J_m).$$

4.3.1 Calibration and numerical exercise

The unit of time is chosen as a quarter of a year. We have 10 parameters, summarized in Table 4.1. Match output y is normalized to one. We inherit from Shimer (2005) $r = 0.012$, $\delta = 0.1$, and $b = 0.4$. Regarding the size of the population of networked workers, Topa (2010) mentions, "Holzer (1987) uses data from the 1981-82 modules of the National Longitudinal Survey of Youth and finds that 87 % of currently employed and 85 % of currently unemployed workers used friends and relatives in their job search, alongside other methods." That is, approximately 85 % of workers have ever attempted to use networks in their job search along with other methods.[9] We interpret it as $n_1 = 0.85$.

There are five parameters left to be calibrated, k, μ, η, ρ and β. The following are our calibration targets.

1. $u = 5.67\%$ (U.S. average of 1951-2003, reported in Shimer (2005))

[9]Other methods include newspaper ads, direct contact, visits to state agencies, private agencies, school placement offices, etc.

Table 4.2: Computed welfare criteria

<table>
<tr><td colspan="2" rowspan="2"></td><td rowspan="2">Pre-policy</td><td colspan="2">Post-policy</td></tr>
<tr><td>$\eta = 0.1$</td><td>$\eta = 0.4$</td></tr>
<tr><td rowspan="2">Unemployment rate</td><td>type-0</td><td>10.7%</td><td rowspan="2">5.67% (converge to 10.9 %)</td><td rowspan="2">5.67% (converge to 11.8%)</td></tr>
<tr><td>type-1</td><td>4.78%</td></tr>
<tr><td rowspan="2">Wage</td><td>type-0</td><td>0.502</td><td rowspan="2">0.501</td><td rowspan="2">0.494</td></tr>
<tr><td>type-1</td><td>0.693</td></tr>
<tr><td rowspan="2">Unemployed workers' value</td><td>type-0</td><td>0.490</td><td rowspan="2">0.489</td><td rowspan="2">0.482</td></tr>
<tr><td>type-1</td><td>0.678</td></tr>
<tr><td rowspan="2">Employed workers' value</td><td>type-0</td><td>0.491</td><td rowspan="2">0.490</td><td rowspan="2">0.483</td></tr>
<tr><td>type-1</td><td>0.679</td></tr>
<tr><td colspan="2">Economy's total output</td><td>0.714</td><td>0.540</td><td>0.534</td></tr>
</table>

$$\left(\text{Wage ratio } = \frac{w_1}{(n_0u_0/u)w_0 + (n_1u_1/u)w_1} = 1.08 \right)$$

2. $\theta = 0.63$ (Hagedorn-Manovskii (2008))

3. average wage $n_0(1-u_0)w_0 + n_1(1-u_1)w_1 = 0.666$ (Two thirds of output)

4. Several surveys report that about 50 % of people find jobs through social contacts. In our model this implies

$$u\mu\theta^\eta \approx n_1u_1\rho(1-u_1)$$

Also Topa (2010) mentions, "Korenman and Turner (1996) also find that the use of social contacts increases wages by about 20 % in a survey of Boston youth, and by 7 % in a sample of young urban males from the 1982 NLSY." Its interpretation in our model is that the ratio of the wage of those hired through network matchings, w_1, to the average wage of those hired through the matching function, $(n_0u_0/u)w_0 + (n_1u_1/u)w_1$, is $1.07 - 1.20$. This can act as a barometer of the model performance.

Target 4, together with 1, pins down $\mu\theta^\eta$, resulting in $u_0 = 10.7\%$, $u_1 = 4.78\%$, and $\rho = 1.2$.[10] Then targets 2 and 3 pin down k and β, resulting in $k = 7.1$ and $\beta = 0.024$. The above targets also imply the relation between μ and η, or $\mu = 0.45/0.63^\eta$, not determining the two parameters separately. These two parameter values are needed to compute the post-policy equilibrium, so we try two extreme values for η, $\eta = 0.1$ and $\eta = 0.4$. In both cases, $u_m > \beta$ results, so we are in case (ii) of Proposition 3 provided that $\rho = 1.2$ is small enough.

Table 4.2 contains the computation results. All the lifetime values are expressed in terms of flow-value equivalence (i.e., lifetime value times r). The pre-policy ratio of the wage of those hired through networks to the average wage of those hired through the matching

[10] To see this, note that adding up (4.9) and (4.10) gives $u\mu\theta^\eta + n_1u_1\rho(1-u_1) = (1-u)\delta$. (The (LHS) is u times the job-finding rate in our model.) Target 4 implies that the first and second terms of the (LHS) are approximately equal, which pins down $\mu\theta^\eta$.

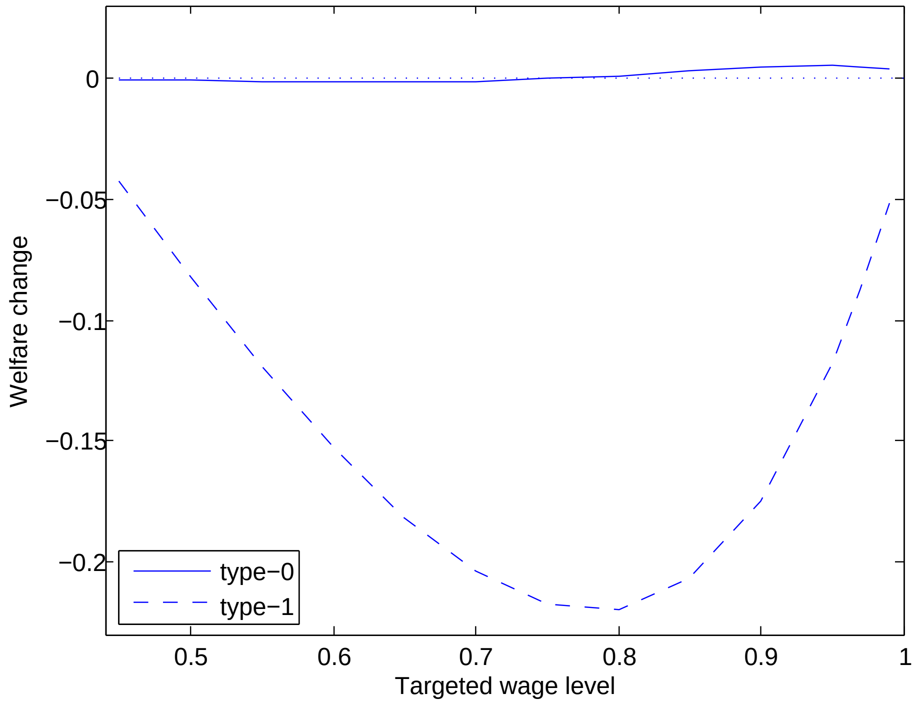

Figure 4.1: Welfare changes for various wage targets
Other calibration targets are maintained. η is set to be 0.3.

function is 1.08 and it matches Korenman and Turner (1996)'s observation well. Not surprisingly, the policy has large negative effects on type-1 workers. The total resource of the economy also falls with such a policy, because the unemployment rate gradually increases in the future. Moreover, the policy slightly lowers wage and values of type-0 workers.

However, in the economy where workers receive higher wages, Proposition 3(i) can also be the case. In such cases, the policy will still largely drop the total resource of the economy, but make type-0 workers slightly better-off. Figure 4.1 shows what happens if the targeted wage levels differ from 0.666. It shows the welfare change (the post-policy lifetime value minus the pre-policy lifetime value) for each type of worker[11] As wage rises above about 0.75 (75 % of output), the change for type-0 workers turns positive, implying that the policy makes them slightly better off, yet still at the cost of the large reduction in type-1 workers' welfare.

Finally, Figure 4.2 shows welfare changes for each type of worker for various n_1. The figure shows that the policy effect on non-networked workers is always negative and small.

[11]The weighted average of the unemployment value and employment value are calculated. Weights are pre-policy steady-state unemployment/employment rates. Note also that as the targeted wage level varies, the calibrated values for β and k also change.

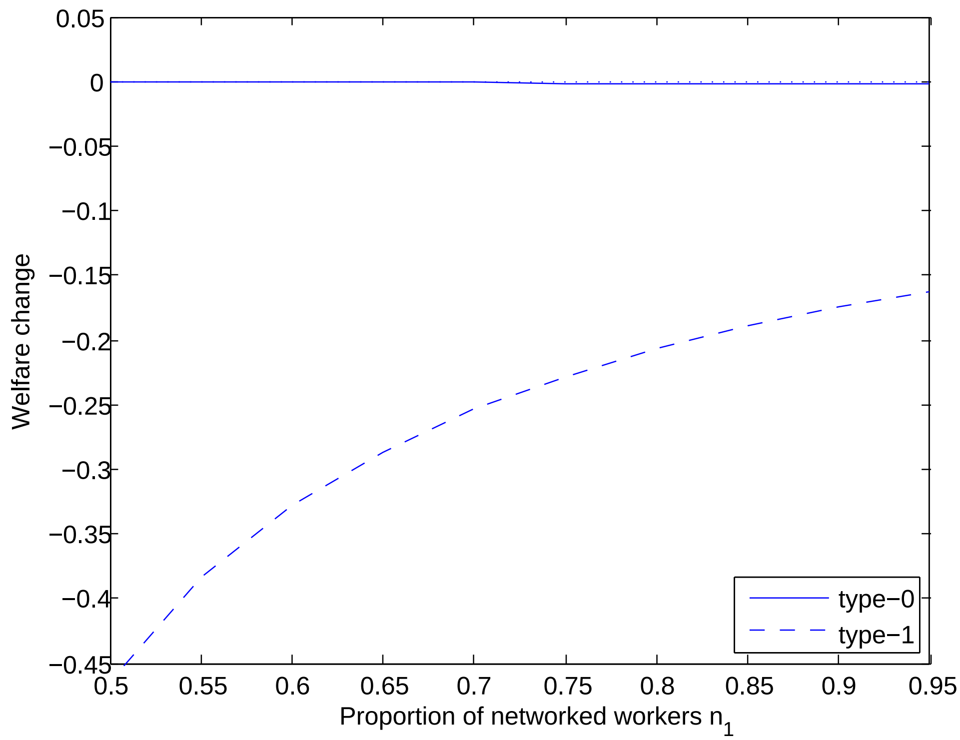

Figure 4.2: Welfare changes for various n_1

4.4 Stability of the steady state for $\rho > 0$

In this section, we numerically check the local stability of the Proposition-1 steady state. The dynamics of the equilibrium with positive ρ are as follows. First, the Bellman equations are[12]

$$\begin{aligned}
\dot{U}_0 &= rU_0(t) - b - \mu\theta(t)^\eta[W_0(t) - U_0(t)] && (4.18)\\
\dot{U}_1 &= rU_1(t) - b - (\mu\theta(t)^\eta + \rho(1 - u_1(t)))[W_1(t) - U_1(t)] && (4.19)\\
\dot{W}_0 &= rW_0(t) - w_0(t) - \delta[U_0(t) - W_0(t)] && (4.20)\\
\dot{W}_1 &= rW_1(t) - w_1(t) - \delta[U_1(t) - W_1(t)] && (4.21)\\
\dot{J}_0 &= rJ_0 - (y - w_0(t)) - \delta[V(t) - J_0(t)] && (4.22)\\
\dot{J}_1 &= rJ_1 - (y - w_1(t)) - \delta[V(t) - J_1(t)] - \rho u_1(t)[J_1(t) - V(t)] && (4.23)\\
\dot{V} &= rV(t) + k - \mu\theta(t)^{\eta-1} \sum_{j=0,1} \frac{n_j u_j(t)}{n_0 u_0(t) + n_1 u_1(t)}[J_j(t) - V(t)]. && (4.24)
\end{aligned}$$

[12]The derivation is given at the end of Appendix.

The laws of motion are

$$\dot{u}_0 = (1 - u_0(t))\delta - u_0\mu\theta(t)^\eta \tag{4.25}$$
$$\dot{u}_1 = (1 - u_1(t))\delta - u_1(\mu\theta(t)^\eta + \rho(1 - u_1(t))). \tag{4.26}$$

The Nash bargaining wage determination $[W_j(t) - U_j(t)]/\beta = [J_j(t) - V(t)]/(1 - \beta)$ and free-entry condition $V(t) = \dot{V}(t) = 0$ hold for all t. We reduce the system to that of the four variables, u_0, u_1, S_0, S_1, where $S_j \equiv W_j - U_j + J_j - V$ is the total surplus of a match with a type-j worker. First, (4.18)-(4.23) are combined to

$$\dot{S}_0 = [r + \delta + \beta\mu\theta(t)^\eta]S_0(t) - (y - b) \tag{4.27}$$
$$\dot{S}_1 = [r + \delta + \beta\mu\theta(t)^\eta + \rho(\beta - u_1(t))]S_1(t) - (y - b). \tag{4.28}$$

Also, (4.24), the Nash bargaining condition and the free-entry condition gives an explicit expression for $\theta(t)$: for all t,

$$\theta(t)^{1-\eta} = \frac{k}{\mu(1-\beta)}\left[\sum_{j=0,1}\frac{n_j u_j(t)}{n_0 u_0(t) + n_1 u_1(t)}S_j(t)\right]^{-1}. \tag{4.29}$$

In summary, (4.25)-(4.28) together with (4.29) gives the dynamical system in the economy with networks. Because u_0 and u_1 are state variables whose initial values are exogenously given while S_0 and S_1 are not such variables, the system is locally stable if the two or more eigenvalues of the 4×4 Jacobian (evaluated at the steady state) are negative. If exactly two are negative and the other two are positive, then the dimension of the stable manifold is two, uniquely determining the initial values for S_0 and S_1 (i.e., the system/path is "determinate"). Although one can obtain the 4×4 Jacobian of the above system analytically, getting its eigenvalues analytically is extremely complex. So we compute the eigenvalues for the model calibrated in the last section. The resulting eigenvalues are $-2.07, -0.907, 0.139$ and 0.120. Thus the fixed point is locally stable and determinate.

4.5 Model in which firms employ multiple workers

In Section 4.2, we assumed that each firm employs at most one worker. In this section, we present a model in which firms can employ multiple workers and show that such a model is reduced to one-firm-one-worker model if both production and cost of hiring satisfy constant returns to scale.

The one-firm-one-worker assumption in Section 4.2 led to the apparently odd referral mechanism that a network friend of a worker currently employed by firm A is referred not to firm A but to some other firm, say firm B. We also assumed that in that case firm B makes payment to firm A to the extent that firm B is indifferent between accepting and not accepting the referred worker (cf. firm A's take-it-or-leave-it offer). One may find these assumptions unnatural. In reality, a firm can employ multiple workers. Therefore, any person referred by a current employee is employed by the same firm, not by a different firm, and hence no firm-to-firm payment results. We claim below that these two environments are equivalent under an assumption that both production and cost of hiring are constant-returns-to-scale; a firm with N employees produces N times as much as a firm with a single employee, and cost to post N vacancies is N times as much as cost to post one vacancy. In other words, our model presented in Section 4.2 is just a fiction that simplifies the model in which firms employ more than one worker.

To see this, let $J^{\ell,m}$ be the value of a firm with ℓ non-networked workers and m networked workers that is not posting another vacancy. Let $\tilde{J}^{\ell,m}$ be the value of a firm with ℓ non-networked workers and m networked workers that is currently posting another vacancy. The free-entry condition is hence $J^{\ell,m} = \tilde{J}^{\ell,m}$ for all ℓ and m. We also make normalization $J^{0,0} = \tilde{J}^{0,0} = 0$. Bellman equations are $\forall \ell, m \geq 1$,

$$\begin{aligned} rJ^{\ell,m} &= \ell(y - w_0) + m(y - w_1) + m\rho u_1(J^{\ell,m+1} - J^{\ell,m}) \\ &+ \ell\delta(J^{\ell-1,m} - J^{\ell,m}) + m\delta(J^{\ell,m-1} - J^{\ell,m}) \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} r\tilde{J}^{\ell,m} &= \ell(y - w_0) + m(y - w_1) - k + m\rho u_1(\tilde{J}^{\ell,m+1} - \tilde{J}^{\ell,m}) \\ &+ \ell\delta(\tilde{J}^{\ell-1,m} - \tilde{J}^{\ell,m}) + m\delta(\tilde{J}^{\ell,m-1} - \tilde{J}^{\ell,m}) \\ &+ \mu\theta^{\eta-1}\frac{u_0}{u}(\tilde{J}^{\ell+1,m} - \tilde{J}^{\ell,m}) + \mu\theta^{\eta-1}\frac{u_1}{u}(\tilde{J}^{\ell,m+1} - \tilde{J}^{\ell,m}), \end{aligned} \tag{4.31}$$

with initial conditions

$$rJ^{1,0} = y - w_0 + \delta(J^{0,0} - J^{1,0}) \tag{4.32}$$

$$rJ^{0,1} = y - w_1 + \delta(J^{0,0} - J^{0,1}) + \rho u_1(J^{0,2} - J^{0,1}) \tag{4.33}$$

$$\begin{aligned} r\tilde{J}^{1,0} = y - w_0 - k + \delta(\tilde{J}^{0,0} - \tilde{J}^{1,0}) &+ \mu\theta^{\eta-1}\frac{u_0}{u}(\tilde{J}^{2,0} - \tilde{J}^{1,0}) \\ &+ \mu\theta^{\eta-1}\frac{u_1}{u}(\tilde{J}^{1,1} - \tilde{J}^{1,0}) \end{aligned} \tag{4.34}$$

and

$$\begin{aligned} r\tilde{J}^{0,1} &= y - w_1 - k + \delta(\tilde{J}^{0,0} - \tilde{J}^{0,1}) + \rho u_1(J^{0,2} - J^{0,1}) \\ &+ \mu\theta^{\eta-1}\frac{u_0}{u}(\tilde{J}^{1,1} - \tilde{J}^{0,1}) + \mu\theta^{\eta-1}\frac{u_1}{u}(\tilde{J}^{0,2} - \tilde{J}^{0,1}). \end{aligned} \tag{4.35}$$

It is easily shown that

$$J^{\ell,m} = \tilde{J}^{\ell,m} = \ell J_0 + m J_1, \tag{4.36}$$

where J_0 and J_1 are from Section 4.2. Other equations remain the same as in the one-firm-one-worker model. So it follows that unemployment rates, wages and vacancy rate remain the same.[13] Thus our one-firm-one-worker model with the incumbent firm's take-it-or-leave-it offer when a referral arises is equivalent to one-firm-multiple-workers model with the constant-returns-to-scale production and cost of hiring.

[13]Equation (4.36) also implies that firms do not get gain from mergers or dissolutions. That is, the size distribution of firms does not matter.

4.6 Conclusion

We consider a version of Mortensen-Pissarides model with two types of workers, workers with networks and workers without networks. The only difference between the two types is that networked workers have chance to get a job through networks in addition to through the standard matching function. A steady state equilibrium exists and under some conditions it is unique. In that steady state, networked workers have a lower unemployment rate, a higher wage and higher lifetime values than non-networked workers. Banning the use of networks in hiring reduces the economy to the Mortensen-Pissarides environment and the economy starts to converge to the MP steady state. First, it is analytically shown that such a policy can have positive or negative effects on non-networked workers, depending on the parameters. In our calibration, such a policy overall discourages firms' entry and hence is bad even for non-networked workers. Moreover, the welfare of networked workers and the total surplus of the economy both drop substantially with such a policy.

In our model, firms can use both hiring methods and do not choose whether to search for workers by means of a formal process or by means of worker networks. If firms make such decisions, one has to allow for two kinds of vacant statuses, one for firms searching via the matching function and the other for firms waiting for network matching. Such a model may have different implications and is left to be studied.

4.7 Appendices

For proofs, we introduce the following notations: for each $j \in \{0,1\}$,

$$S_j \equiv W_j - U_j + J_j - V \tag{4.37}$$

$$\alpha_{F_j} \equiv \mu\theta^{\eta-1}\frac{n_j u_j}{n_0 u_0 + n_1 u_1}. \tag{4.38}$$

To prove Proposition 1, we provide two lemmas.

Lemma 2.
(i) For given v, there is a unique pair (u_0^*, u_1^*) that satisfies (4.9)-(4.10); and
(ii) Such (u_0^*, u_1^*) are strictly decreasing in v.

Proof. (Lemma 2)
(i) First we prove it for the case $\delta \geq \rho$. Define the functions

$$T_0(u_0, u_1, v) \equiv u_0\mu\left(\frac{v}{n_0u_0 + n_1u_1}\right)^\eta - (1-u_0)\delta \tag{4.39}$$

$$T_1(u_0, u_1, v; \rho) \equiv u_1\mu\left(\frac{v}{n_0u_0 + n_1u_1}\right)^\eta + u_1(1-u_1)\rho - (1-u_1)\delta, \tag{4.40}$$

so that the steady-state unemployment rates given v, denoted $u_0^*(v), u_1^*(v)$, are given as the solution to $T_0(u_0^*, u_1^*, v) = 0$ and $T_1(u_0^*, u_1^*, v) = 0$.
First, we have

$$T_1(u_0, 0, v; \rho) = -\delta < 0 \tag{4.41}$$

$$T_1(u_0, 1, v; \rho) = \mu\left(\frac{v}{n_0u_0 + n_11}\right)^\eta > 0 \tag{4.42}$$

$$T_{11} \equiv \frac{\partial T_1}{\partial u_1} = \mu\left(\frac{v}{n_0u_0 + n_1u_1}\right)^\eta\left(1 - \frac{\eta n_1 u_1}{n_0u_0 + n_1u_1}\right) + (1-u_1)\rho + \delta - u_1\rho > 0 \tag{4.43}$$

$$T_{10} \equiv \frac{\partial T_1}{\partial u_0} = \mu\left(\frac{v}{n_0u_0 + n_1u_1}\right)^\eta \frac{-\eta n_0 u_1}{n_0u_0 + n_1u_1} < 0 \tag{4.44}$$

$$T_{1v} \equiv \frac{\partial T_1}{\partial v} = \eta\mu\left(\frac{v}{n_0u_0 + n_1u_1}\right)^{\eta-1}\left(\frac{u_1}{n_0u_0 + n_1u_1}\right) > 0. \tag{4.45}$$

The first three equations imply that for any $v > 0$ and $u_0 \in [0,1]$, there is a unique $u_1 \in (0,1)$ that satisfies $T_1 = 0$, denoted by $u_1^{T_1}(u_0; v)$.[14] Then (4.44) implies that $\partial u_1^{T_1}(u_0; v)/\partial u_0 = -T_{10}/T_{11} > 0$, so that $u_1^{T_1}(u_0; v)$ is increasing in u_0. Moreover, it can be shown that $u_1^{T_1}(u_0; v)$ is bounded away from 0 and 1. That is, we have

$$u_1^{T_1}(u_0\,; v) \text{ is increasing in } u_0, \text{ and } u_1^{T_1}(0; v) > 0, \;\; u_1^{T_1}(1; v) < 1. \tag{4.46}$$

[14]When $u_0 = 0$, (4.40) becomes $T_1(0, u_1, v; \rho) = \mu(v/n_1)^\eta u_1^{1-\eta} + (1-u_1)(u_1\rho - \delta)$. So (4.41) still holds.

Similarly, we have

$$T_0(0, u_1, v) = -\delta < 0 \tag{4.47}$$

$$T_0(1, u_1, v) = \mu \left(\frac{v}{n_0 1 + n_1 u_1} \right)^\eta > 0$$

$$T_{00} \equiv \frac{\partial T_0}{\partial u_0} = \mu \left(\frac{v}{n_0 u_0 + n_1 u_1} \right)^\eta \left(1 - \frac{\eta n_0 u_0}{n_0 u_0 + n_1 u_1} \right) + \delta > 0 \tag{4.48}$$

$$T_{01} \equiv \frac{\partial T_0}{\partial u_1} = \mu \left(\frac{v}{n_0 u_0 + n_1 u_1} \right)^\eta \frac{-\eta n_1 u_0}{n_0 u_0 + n_1 u_1} < 0 \tag{4.49}$$

$$T_{0v} \equiv \frac{\partial T_0}{\partial v} = \eta\mu \left(\frac{v}{n_0 u_0 + n_1 u_1} \right)^{\eta - 1} \left(\frac{u_0}{n_0 u_0 + n_1 u_1} \right) > 0. \tag{4.50}$$

Again, the first three equations imply that for any $v > 0$ and $u_1 \in [0, 1]$, there is a unique $u_0 \in (0, 1)$ that satisfies $T_0 = 0$, denoted by $u_0^{T_0}(u_1; v)$.[15] Then a similar argument leads to

$$u_0^{T_0}(u_1\ ; v) \text{ is increasing in } u_1, \text{ and } u_0^{T_0}(0; v) > 0,\ \ u_0^{T_0}(1; v) < 1. \tag{4.51}$$

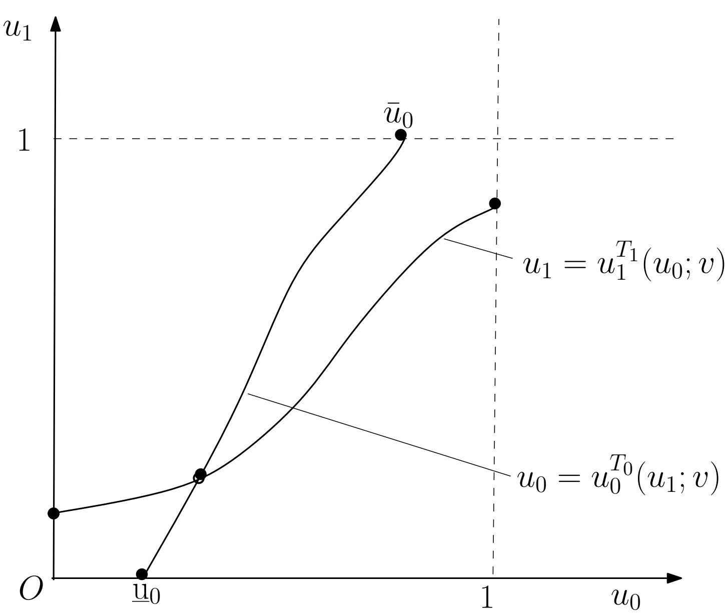

Figure 4.3: Graphical image of the existence proof

So let $\underline{u}_0 \equiv u_0^{T_0}(0; v)$ and $\bar{u}_0 \equiv u_0^{T_0}(1; v)$, as seen in Figure 4.3. Then consider the inverse function of $u_0 = u_0^{T_0}(u_1\ ; v)$ (the inverse in terms of u_1), and denote it as $u_1 = u_1^{T_0}(u_0; v)$, which is defined on $[\underline{u}_0, \bar{u}_0]$. The two functions $u_1^{T_0}(u_0; v)$ and $u_1^{T_1}(u_0; v)$ should intersect at least once because

$$u_1^{T_1}(\underline{u}_0; v) > 0 = u_1^{T_0}(\underline{u}_0; v) \tag{4.52}$$

$$u_1^{T_1}(\bar{u}_0; v) < 1 = u_1^{T_0}(\bar{u}_0; v) \tag{4.53}$$

[15]When $u_1 = 0$, (4.39) becomes $T_0(u_0, 0, v) = \mu(v/n_0)^\eta u_0^{1-\eta} - (1 - u_0)\delta$. So (4.47) still holds.

Now we are ready to show the uniqueness of such an intersection, or

$$\frac{\partial}{\partial u_0}\left\{u_1^{T_0}(u_0;v) - u_1^{T_1}(u_0;v)\right\} > 0. \tag{4.54}$$

Note that

$$\begin{aligned}
& \frac{\partial}{\partial u_0}\left\{u_1^{T_0}(u_0;v) - u_1^{T_1}(u_0;v)\right\} \qquad (4.55)\\
= & \frac{\partial u_1^{T_0}}{\partial u_0} - \frac{\partial u_1^{T_1}}{\partial u_0}\\
= & \frac{T_{00}}{(-T_{01})} - \frac{(-T_{10})}{T_{11}}\\
= & \frac{T_{00}T_{11} - T_{10}T_{01}}{(-T_{01})\,T_{11}},
\end{aligned}$$

and the numerator is proved to be positive because by (4.43)-(4.44), (4.48)-(4.49), and the assumption $\delta > \rho$,

$$\begin{aligned}
& T_{11}T_{00} - T_{10}T_{01}\\
= & \left\{\mu\left(\frac{v}{n_0u_0+n_1u_1}\right)^{\eta}\left(1-\frac{\eta n_1u_1}{n_0u_0+n_1u_1}\right) + (1-u_1)\rho + \delta - u_1\rho\right\}\\
& \times\left\{\mu\left(\frac{v}{n_0u_0+n_1u_1}\right)^{\eta}\left(1-\frac{\eta n_0u_0}{n_0u_0+n_1u_1}\right) + \delta\right\}\\
& -\mu\left(\frac{v}{n_0u_0+n_1u_1}\right)^{\eta}\frac{-\eta n_0u_1}{n_0u_0+n_1u_1}\mu\left(\frac{v}{n_0u_0+n_1u_1}\right)^{\eta}\frac{-\eta n_1u_0}{n_0u_0+n_1u_1} \qquad (4.56)\\
= & (1-\eta)\left\{\mu\left(\frac{v}{n_0u_0+n_1u_1}\right)^{\eta}\right\}^2\\
& +\mu\left(\frac{v}{n_0u_0+n_1u_1}\right)^{\eta}\left(1-\frac{\eta n_1u_1}{n_0u_0+n_1u_1}\right)\delta\\
& +\mu\left(\frac{v}{n_0u_0+n_1u_1}\right)^{\eta}\left(1-\frac{\eta n_0u_0}{n_0u_0+n_1u_1}\right)((1-u_1)\rho + \delta - u_1\rho)\\
& +\delta((1-u_1)\rho + \delta - u_1\rho) \qquad (4.57)\\
> & \ 0, \qquad (4.58)
\end{aligned}$$

which concludes that (4.55) is positive. Therefore, the two functions $u_1^{T_0}(u_0;v)$ and $u_1^{T_1}(u_0;v)$ intersect only once, which proves (i).

Next we prove the lemma for the case $\rho > \delta$. In this case, we can restrict the domain of u_1 to $[0, \delta/\rho] \subset [0,1]$ for the following reason. Consider a hypothetical situation that there is no employment through the matching function. In such a case, the dynamics of u_1 are given by $\dot{u}_1 = (1-u_1)\delta - u_1\rho(1-u_1)$, so its fixed point is $u_1 = \delta/\rho$. The steady state u_1 should be bounded from above by this level.

If we restrict attention to $u_1 \in [0, \delta/\rho]$, (4.43) still holds even if $\rho > \delta$. Also the last part of (4.46) is modified to $u_1^{T_1}(1; v) < \delta/\rho$. Additionally, the definition of $\bar{u}_0$ is modified so that $\bar{u}_0 \equiv u_0^{T_0}(\delta/\rho; v)$ and (4.53) is modified to $u_1^{T_1}(\bar{u}_0; v) < \delta/\rho = u_1^{T_0}(\bar{u}_0; v)$. All the other arguments remain unchanged.
(ii) Note that by the implicit function theorem,

$$\begin{aligned} \begin{bmatrix} \frac{\partial u_0^*}{\partial v} \\ \frac{\partial u_1^*}{\partial v} \end{bmatrix} &= -\begin{bmatrix} T_{00} & T_{01} \\ T_{10} & T_{11} \end{bmatrix}^{-1} \begin{bmatrix} T_{0v} \\ T_{1v} \end{bmatrix} \\ &= \frac{-1}{T_{11}T_{00} - T_{10}T_{01}} \begin{bmatrix} T_{11} & -T_{01} \\ -T_{10} & T_{00} \end{bmatrix} \begin{bmatrix} T_{0v} \\ T_{1v} \end{bmatrix} && (4.59) \\ &< 0, && (4.60) \end{aligned}$$

where the last inequality follows from (4.43)-(4.45), (4.48)-(4.50), and (4.58).

□

Lemma 3.
Given v, there exist unique S_0 and S_1 that satisfy (4.1)-(4.6) and (4.8)-(4.11), denoted $S_0^*(v)$ and $S_1^*(v)$. Both are strictly decreasing in v, satisfying $\lim_{v\to\infty} S_j^*(v) = 0$ and $\lim_{v\to 0} S_j^*(v) < \infty$.

Proof. (Lemma 3)
By (4.1), (4.3), (4.5), (4.8) and (4.11), we have

$$S_0 = \frac{y - b}{r + \delta + \mu\theta^\eta\beta}. \quad (4.61)$$

By (4.2), (4.4), (4.6), (4.8) and (4.11), we have

$$S_1 = \frac{y - b}{r + \delta + \mu\theta^\eta\beta + \rho(\beta - u_1)}. \quad (4.62)$$

By Lemma 2, u_0, u_1 and thus $\mu\theta^\eta$ are uniquely determined, given v. So S_0 and S_1 are unique, given v. By Lemma 2, $\mu\theta^\eta$ is strictly increasing in v, so the denominators of S_0 and S_1 in (4.61)(4.62) are both strictly increasing in v. Therefore, both S_0 and S_1 are strictly decreasing in v. Moreover, both the denominators go to infinity as $v \to \infty$ and go to some positive constants as $v \to 0$. So $\lim_{v\to\infty} S_j^*(v) = 0$ and $\lim_{v\to 0} S_j^*(v) < \infty$. □

Proof. (Proposition 1)
By Lemmas 2 and 3, given v, (4.1)-(4.6) and (4.8)-(4.11) pin down $u_0^*(v)$, $u_1^*(v)$, $S_0^*(v)$ and $S_1^*(v)$, all of which are strictly decreasing in v. The remaining equation, which determines v, is (4.7):

$$k = \alpha_{F_0}^*(v)(1 - \beta)S_0^*(v) + \alpha_{F_1}^*(v)(1 - \beta)S_1^*(v), \quad (4.63)$$

where for $i = 0, 1$,

$$\alpha_{F_i}^*(v) \equiv \mu \left(\frac{n_0 u_0^*(v) + n_1 u_1^*(v)}{v} \right)^{1-\eta} \frac{n_i u_i^*(v)}{n_0 u_0^*(v) + n_1 u_1^*(v)}. \quad (4.64)$$

When $v \to 0$, $\alpha^*_{F_i}(v) \to \infty$. So the (RHS) of (4.63) goes to infinity.
When $v \to \infty$,

$$\alpha^*_{F_i}(v) = \mu \frac{1}{v^{1-\eta}} \left(\frac{n_i u_i^*(v)}{n_0 u_0^*(v) + n_1 u_1^*(v)} \right)^{\eta} (n_i u_i^*(v))^{1-\eta} \to 0,$$

because the expression in the bracket is bounded above by one. So the (RHS) of (4.63) goes to zero. Therefore, by the intermediate value theorem, there exists a v that satisfies (4.63). Once v is determined, all the other equilibrium variables are uniquely pinned down. □

Proof. (Proposition 2)
First, subtracting (4.10) from (4.9) gives

$$(\mu\theta^{\eta} + \delta)(u_0 - u_1) = u_1(1 - u_1)\rho, \tag{4.65}$$

which implies that $u_0 > u_1$. For the rest of the proof, we consider the two cases separately: (i) $S_1 \geq S_0$ and (ii) $S_0 > S_1$. Note that combining (4.1) and (4.2) gives

$$r(U_1 - U_0) = \mu\theta^{\eta}\{(W_1 - U_1) - (W_0 - U_0)\} + (1 - u_1)\rho(W_1 - U_1); \tag{4.66}$$

combining (4.3) and (4.4) gives

$$w_1 - w_0 = r(W_1 - W_0) + \delta\{(W_1 - U_1) - (W_0 - U_0)\}; \tag{4.67}$$

combining (4.5) and (4.6) gives

$$(r + \delta)(J_1 - J_0) = w_0 - w_1 + \rho u_1(J_1 - V); \tag{4.68}$$

and combining (4.66) and (4.67) gives

$$w_1 - w_0 = (r + \delta + \mu\theta^{\eta})\{(W_1 - U_1) - (W_0 - U_0)\} + (1 - u_1)\rho(W_1 - U_1). \tag{4.69}$$

(i) Suppose $S_1 \geq S_0$ in the equilibrium. Then by the Nash bargaining condition (4.8), we have $W_1 - U_1 \geq W_0 - U_0$ and $J_1 \geq J_0$. Then (4.66) implies $U_1 > U_0$. This together with $W_1 - U_1 \geq W_0 - U_0$ implies $W_1 > W_0$. Also, (4.69) implies $w_1 > w_0$.
(ii) Now suppose that $S_0 > S_1$, so that the Nash bargaining condition implies $W_0 - U_0 > W_1 - U_1$ and $J_0 > J_1$. Then by (4.68), $w_1 > w_0$. Hence by (4.67), $W_1 > W_0$. This together with $W_0 - U_0 > W_1 - U_1$ implies $U_1 > U_0$.

The Proposition is proved up to this point. Lastly, we note that $S_0 > S_1$ (hence $J_0 > J_1$ and $W_0 - U_0 > W_1 - U_1$) is not necessarily the case. Subtracting (4.61) from (4.62) gives

$$(S_1 - S_0)(r + \delta + \mu\theta^{\eta}\beta) = S_1\rho(u_1 - \beta). \tag{4.70}$$

So if $u_1 < (>)\beta$, then $S_1 < (>)S_0$. □

Proof. (Lemma 1) The steady state equilibrium of the standard Mortensen-Pissarides model is given as the solution to the following system of equations:

$$\begin{aligned} rU_m &= b + \mu\theta_m^\eta (W_m - U_m) && (4.71) \\ rW_m &= w_m + \delta(U_m - W_m) && (4.72) \\ rJ_m &= y - w_m + \delta(V_m - J_m) && (4.73) \\ rV_m &= -k + \mu\theta_m^{\eta-1}(J_m - V_m) && (4.74) \\ W_m - U_m &= \beta S_m && (4.75) \\ V_m &= 0 && (4.76) \\ u_m \mu\theta_m^\eta &= (1 - u_m)\delta, && (4.77) \end{aligned}$$

where

$$\begin{aligned} S_m &\equiv W_m - U_m + J_m - V_m && (4.78) \\ \theta_m &\equiv v_m/u_m. && (4.79) \end{aligned}$$

Again, (4.71)-(4.73) and (4.75), (4.76) are reduced to

$$S_m = \frac{y - b}{r + \delta + \beta\mu\theta_m^\eta}, \tag{4.80}$$

while (4.74) and (4.75) imply

$$S_m = \frac{k\theta_m^{1-\eta}}{(1 - \beta)\mu}. \tag{4.81}$$

The last two equations pin down (S_m, θ_m). In fact, S_m is given as a unique positive solution to $f_S(S_m) = 0$, where

$$f_S(S_m) \equiv \beta\mu \left(\frac{\mu(1-\beta)}{k}\right)^{\frac{\eta}{1-\eta}} S_m^{\frac{1}{1-\eta}} + (r + \delta)S_m - (y - b). \tag{4.82}$$

Once S_m is determined, so is the v_m-u_m ratio. Meanwhile the steady state condition (4.77) is transformed to

$$v_m = \left[\frac{\delta}{\mu u_m^{1-\eta}} - \frac{\delta}{\mu} u_m^\eta\right]^{\frac{1}{\eta}},$$

which means v_m is decreasing in u_m. Combining it with (4.79) uniquely pins down v_m and u_m.

(Comparative Statics of the MP model.) Eliminating S_m from (4.80) and (4.81) gives

$$\frac{y - b}{k}(1 - \beta) = \frac{r + \delta}{\mu\theta^{\eta-1}} + \beta\theta. \tag{4.83}$$

This implies that increasing β, b, or k leads to lower θ_m, resulting in higher u_m. Eliminating θ_m by using (4.77), one can see that $\lim_{\beta\to 0} u_m > 0$. So for sufficiently low β, $u_m > \beta$ is the case. Also, $\lim_{k\to\infty} u_m = 1$ is easy to check. □

Proof. (Proposition 3)
The equilibrium triple, (u_0, u_1, v), is pinned down by three equation. The first two are $T_0 = 0$ and $T_1 = 0$, where function T_0 and T_1 are defined by (4.39) and (4.40). The third equation, say $T_v = 0$, is defined by (4.7) combined with (4.61)(4.62):

$$\begin{aligned} T_v(u_0, u_1, v; \rho) &\equiv -k + \alpha_{F_0}(1-\beta)\frac{y-b}{r+\delta+\mu\theta^\eta\beta} \\ &+ \alpha_{F_1}(1-\beta)\frac{y-b}{r+\delta+\mu\theta^\eta\beta+\rho(\beta-u_1)}, \end{aligned} \tag{4.84}$$

where α_{F_j}, defined by (4.38), and $\mu\theta^\eta$ are functions of u_0, u_1, v and ρ. By the implicit function theorem,

$$\begin{bmatrix} T_{00} & T_{01} & T_{0v} \\ T_{10} & T_{11} & T_{1v} \\ T_{v0} & T_{v1} & T_{vv} \end{bmatrix} \begin{bmatrix} u_0'(\rho) \\ u_1'(\rho) \\ v'(\rho) \end{bmatrix} = - \begin{bmatrix} T_{0\rho} \\ T_{1\rho} \\ T_{v\rho} \end{bmatrix}.$$

Invoking that $u_0 = u_1 = u_m$ when $\rho = 0$, one can show that

$$u_1'(\rho)|_{\rho=0} = -\frac{u_m\left(\kappa_1 + \mu\left(\frac{v_m}{u_m}\right)^\eta \left(n_1\eta(1-\beta)u_m + \beta(1-n_1\eta)(1-u_m)\right)\right)}{\kappa_2\kappa_3},$$

where

$$\begin{aligned} \kappa_1 &\equiv (r+\delta)(1-\eta)(1-u_m) > 0 \\ \kappa_2 &\equiv \delta + \mu\left(\frac{v_m}{u_m}\right)^\eta > 0 \\ \kappa_3 &\equiv (r+\delta)(1-\eta) + \beta\mu\left(\frac{v_m}{u_m}\right)^\eta > 0 \\ \kappa_4 &\equiv r+\delta+\beta\mu\left(\frac{v_m}{u_m}\right)^\eta > 0, \end{aligned}$$

so u_1 is decreasing in ρ near $\rho = 0$.
(ii) Calculations similar to the above lead to

$$\begin{aligned} u_0'(\rho)|_{\rho=0} &= \frac{n_1\eta\mu\left(\frac{v_m}{u_m}\right)^\eta u_m(\beta-u_m)}{\kappa_2\kappa_3} \\ v'(\rho)|_{\rho=0} &= \frac{n_1 v\{-\kappa_1 - \kappa_2(\beta-u_m) - \mu\left(\frac{v_m}{u_m}\right)^\eta (\eta(1-\beta)u_m + \beta(1-\eta)(1-u_m))\}}{\kappa_2\kappa_3} \end{aligned}$$

Using (2.42) and (4.62),

$$S_0'(\rho)|_{\rho=0} = \frac{n_1\eta\beta S_m\mu\left(\frac{v_m}{u_m}\right)^\eta(\beta-u_m)}{\kappa_3\kappa_4}$$

$$S_1'(\rho)|_{\rho=0} = \frac{(u_m-\beta)S_m((r+\delta)(1-\eta)+\beta\mu\left(\frac{v_m}{u_m}\right)^\eta(1-n_1\eta))}{\kappa_3\kappa_4}.$$

If $u_m < \beta$, we have $u_0'(\rho)|_{\rho=0} > 0$, $v'(\rho)|_{\rho=0} < 0$, $S_0'(\rho)|_{\rho=0} > 0$, and $S_1'(\rho)|_{\rho=0} < 0$. Then (4.8) implies that $W_0 - U_0$ and J_0 are increasing and that $W_1 - U_1$ and J_1 are decreasing in ρ. By (4.5), $w_0 = y - (r+\delta)J_0$, so w_0 is decreasing. Then by (4.3), W_0 is decreasing as well. $W_0 - U_0$ increasing and W_0 decreasing imply U_0 decreasing. Also, u_0 increasing and (4.9) imply that $\mu\theta^\eta$ is decreasing. Hence θ is decreasing.
(iii) If $\beta < u_m$, we have $u_0'(\rho)|_{\rho=0} < 0$, $S_0'(\rho)|_{\rho=0} < 0$, and $S_1'(\rho)|_{\rho=0} > 0$, while the sign of $v'(\rho)|_{\rho=0}$ is indeterminate. The results are then opposite of those of (ii) except for v. Moreover, w_0, W_0 and U_0 being increasing and Proposition 2 imply that w_1, W_1 and U_1 are also increasing near $\rho = 0$. Also, because both u_0 and u_1 are decreasing, u is decreasing. □

Proof. (Proposition 4)
(i)(ii) We want to show that (4.63) in the proof of Lemma 1 has unique solution v. For that, it suffices to show that (4.64) is strictly decreasing. Differentiating (4.64) with respect to v and using (4.59), (4.43)-(4.45) and (4.48)-(4.50), we have

$$\frac{\partial\alpha_{F_0}^*(v)}{\partial v} = -\frac{n_0u_0\mu\theta^\eta\left\{\begin{array}{l} n_0u_0(1-\eta)(\delta+\mu\theta^\eta)(\delta+\mu\theta^\eta+q) \\ +n_1u_1\left[\delta^2(1-\eta)+\delta(1-\eta)(2\mu\theta^\eta+q)+\mu\theta^\eta(\mu\theta^\eta(1-\eta)+q)\right]\end{array}\right\}}{v^2\{n_0u_0(\delta+\mu\theta^\eta(1-\eta))(\delta+\mu\theta^\eta+q)+n_1u_1(\delta+\mu\theta^\eta)(\delta+\mu\theta^\eta(1-\eta)+q)\}},$$

and

$$\frac{\partial\alpha_{F_1}^*(v)}{\partial v} = - \frac{n_1u_1\mu\theta^\eta\left\{\begin{array}{l} n_1u_1(1-\eta)(\delta+\mu\theta^\eta)(\delta+\mu\theta^\eta+q) \\ +n_0u_0\left[\delta^2(1-\eta)+\delta(1-\eta)(2\mu\theta^\eta+q)+\mu\theta^\eta(\mu\theta^\eta(1-\eta)+q(1-2\eta))\right]\end{array}\right\}}{v^2\{n_0u_0(\delta+\mu\theta^\eta(1-\eta))(\delta+\mu\theta^\eta+q)+n_1u_1(\delta+\mu\theta^\eta)(\delta+\mu\theta^\eta(1-\eta)+q)\}},$$

where $q \equiv (1-2u_1^*(v))\rho$. In Lemma 2, we showed that $u_1^*(v)$ is strictly decreasing in v, so the above two equations imply that the $\alpha_{F_i}^*(v)$'s are strictly decreasing in the region such that $q \geq 0$ or $u_1^*(v) \leq 0.5$. So, the (RHS) of (4.63) is also strictly decreasing in v in the region such that $u_1^*(v) \leq 0.5$, as is shown in Figure 4.4. Therefore, there is at most one equilibrium such that $u_1 \leq 0.5$. The (RHS) of (4.63) does not necessarily have humps as is depicted in Figure 4.4. All we know is that it goes to infinity as $v \to 0$, goes to zero as $v \to \infty$, and it is strictly decreasing for sufficiently large v. Parameter k appears only in this equation, so if k

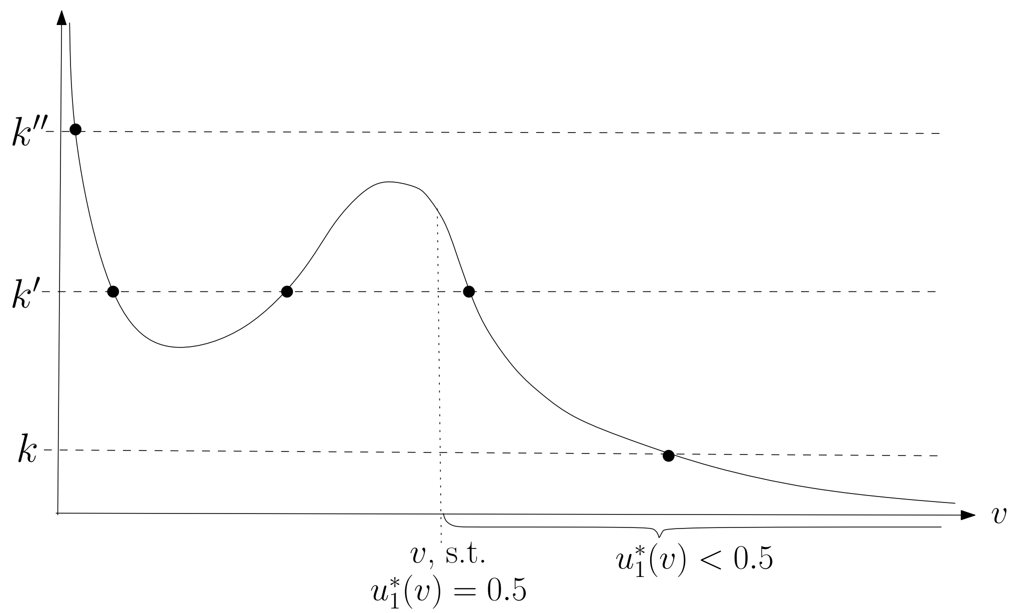

Figure 4.4: The (RHS) of equation (4.63)

is large enough, the equilibrium is unique and $u_1 > 0.5$. If k is small enough, the equilibrium is unique and u_1 can be any small number.
(iii) Proposition 3(i) implies that if $\rho \approx 0$, then $u_1 < u_m$ is the case in any equilibrium. Then part (i) implies the uniqueness of the steady state equilibrium.

□

Proof. (Proposition 5)
Defining the surplus of a match $S_m(t) = W_m(t) - U_m(t) + J_m(t) - V_m(t)$, the Nash bargaining condition and the free-entry condition together with (4.12)-(4.14) imply

$$\dot{S_m} = (r + \delta + \beta\mu\theta_m(t)^\eta)S_m(t) - (y - b). \tag{4.85}$$

In the meantime, (4.15), the Nash bargaining condition and the free-entry condition imply

$$\theta_m(t) = \left(\frac{\mu(1-\beta)}{k}\right)^{\frac{1}{1-\eta}} S_m(t)^{\frac{1}{1-\eta}}, \quad \forall t. \tag{4.86}$$

Substituting the last equation into (4.85), we have

$$\dot{S_m} = \beta\mu\left(\frac{\mu(1-\beta)}{k}\right)^{\frac{\eta}{1-\eta}} S_m(t)^{\frac{1}{1-\eta}} + (r+\delta)S_m(t) - (y-b) \equiv f_S(S_m(t)). \tag{4.87}$$

So the match surplus is autonomous, not depending on other variables such as the unemployment rate. Since $f_s'(S_m) > 0$, the match surplus should be constant at the steady state level for any convergent path. So, $\theta_m(t)$ is also constant. But then, (4.16) is an autonomous system with the (RHS) having negative slope. Therefore, $u_m(t)$ is convergent. By the Nash bargaining condition, $S_m(t)$ being time-invariant implies that $W_m(t) - U_m(t)$, $J_m(t) - V_m(t)$ and $J_m(t)$ are also time-invariant. Then (4.14) implies $w_m(t)$ is time-invariant. So (4.12) and (4.13) become an autonomous system of $U_m(t)$ and $W_m(t)$, respectively. Since the slopes of these systems are positive, $U_m(t)$ and $W_m(t)$ should also be at the steady-state levels from the beginning. □

Proof. (Proposition 6)
Define $Y_m(t) \equiv u_m(t)U_m(t) + (1 - u_m(t))(W_m(t) + J_m(t))$. This is equal to $W_m(t) + J_m(t) - u_m(t)S_m(t)$. Therefore,

$$\begin{aligned} \dot{Y_m} &= \dot{W_m} + \dot{J_m} - \dot{u_m}S_m - u_m\dot{S_m} \\ &= r(W_m + J_m) - y - \delta(U_m - W_m + V_m - J_m) - \dot{u_m}S_m \\ &\quad -u_m[(r + \delta + \beta\mu\theta_m^\eta)S_m - (y - b)] \\ &= -bu_m - y(1 - u_m) + r(W_m + J_m - u_mS_m) - \dot{u_m}S_m + [\delta(1 - u_m) - u_m\mu\theta_m^\eta]S_m \\ &\quad +u_m(1 - \beta)S_m\mu\theta_m^\eta \\ &= -bu_m - y(1 - u_m) + r(W_m + J_m - u_mS_m) + u_m(1 - \beta)S_m\mu\theta_m^\eta \\ &= -bu_m - y(1 - u_m) + kv_m + r(W_m + J_m - u_mS_m) \\ &= -bu_m - y(1 - u_m) + kv_m + rY_m, \end{aligned}$$

where the second equality uses (4.13), (4.14) and (4.85), the fourth equality uses (4.16), and the fifth uses (4.86). The last expression implies

$$Y_m(t) = \int_t^\infty e^{-r(\tau - t)}[bu_m(\tau) + y(1 - u_m(\tau)) - kv_m(\tau)]d\tau,$$

that is, the discounted sum of the economy's total resources. □

Proof. (Derivation of differential equations in section 4.4)
Suppose $\tau \geq 0$ is a random variable that is the time until an event occurs. Let $\Phi(\tau)$ and $\phi(\tau)$ be its distribution function and density function, respectively. That is,

$$\begin{aligned} \Phi(t_2|t_1) &\equiv \mathbb{P}(\tau \leq t_2|\tau > t_1) \\ &= \mathbb{P}(t_1 < \tau \leq t_2|\tau \geq t_1) \\ &= \frac{\Phi(t_2) - \Phi(t_1)}{1 - \Phi(t_1)}. \end{aligned}$$

Differentiating with respect to t_2, we have

$$\phi(t_2|t_1) \equiv \phi(t_2|\tau > t_1) = \frac{\phi(t_2)}{1 - \Phi(t_1)}.$$

Let $\lambda(\tau) \equiv \phi(\tau)/(1 - \Phi(\tau))$, the hazard rate. Then we have

$$\frac{\partial\phi(t_2|t_1)}{\partial t_1} = \lambda(t_1)\phi(t_2|t_1), \tag{4.88}$$

which we often use below. Another formula we often use is as follows. Let

$$D(t,T) \equiv \int_t^T f(s)e^{-r(s-t)}ds,$$

where f is a given function. Then

$$\frac{\partial}{\partial t}D(t,T) = -f(t) + rD(t,T). \tag{4.89}$$

□

The type-0 worker's unemployment value satisfies

$$U_0(t) = \int_t^\infty \phi(T|t)D(t,T)dT + \int_t^\infty \phi(T|t)e^{-r(T-t)}W_0(T)dT,$$

where $f(s) = b$ is a constant function and $\phi(T|t)$ is a density for arrival of employment. (So, $\lambda(t) = \mu\theta(t)^\eta$.) Differentiating it with respect to t and using (4.88)(4.89), we have

$$\dot{U}_0(t) = rU_0(t) - b - \lambda(t)[W_0(t) - U_0(t)]. \tag{4.90}$$

The derivation is similar for the other values.

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