SURFACE GRAVITY WAVES PROPAGATING ON WATER OF
FINITE, VARIABLE DEPTH

A Thesis in
Mathematics
by
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Abstract

The evolution of amplitudes of gravity waves propagating in one horizontal dimension on water of variable depth may be governed by a modified nonlinear Schrödinger (MNLS) equation, in which the coefficients are functions of local depth and conservation of wave action flux adds a linear term, which implies growth or decay, depending on the direction of propagation. This work examines the propagation of gravity waves on water of variable depth, with and without viscous dissipation. With a combination of stability, asymptotic and numerical analyses, we study the variable-depth Stokes wavetrain, waves with periodic cnoidal envelopes, and a soliton wave, as they propagate from shallower water to deeper water, and vice versa. We find that both bathymetry and viscous dissipation stabilize the Benjamin-Feir instability of a variable-depth Stokes wavetrain when the wavetrain propagates into shallower water. When waves propagate into deeper water, the effect due to bathymetry is reversed and tends to destabilize the waves. In both cases, however, viscous dissipation eventually causes stabilization, except for a special case in which the growth due to bathymetry and decay due to viscous dissipation balance. In that case, the dynamics are governed by the inviscid uniform-depth nonlinear Schrödinger (NLS) equation, and the Stokes wavetrain solution is unstable. Our numerical results on the stability of the soliton solution and the instability of the cnoidal solutions of the inviscid MNLS are consistent with analytical results in literature. Mathematical analysis and numerical results on cnoidal waves also reveal that the asymptotic solutions can reasonably predict the evolution of the waves for oceanic scales, where the wavelength of the wave is much smaller than the characteristic scale for the depth variation, but fails to do the same for laboratory scales, where the wavelength is not as small compared to the length scale representing the depth variation.
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Dedication

To the victims of the tsunami that hit the Indian coast in 2004.
Chapter 1

The Nonlinear Schrödinger Equation as a Model for Water Waves

1.1 Introduction

The theory of nonlinear water waves has been a subject of intense study over the last several decades. An ubiquitous naturally occurring phenomenon, water waves are important in ocean and coastal engineering, geophysics and related fields, and affect ship safety and scheduling, harbor design, tsunami prediction and warning, and beach maintenance. Other wave phenomena that affect societies include tidal bores, rogue waves, and harbor seiches. The mathematics describing water waves is extremely rich and is of interest to mathematicians for its own sake. In addition, many of the phenomena observed in water waves are common to other fields such as nonlinear optics, nonlinear acoustics, quantum condensates etc., and much of the mathematical development is also common among these fields.

1.2 Modeling Nonlinear Wave Propagation

The nonlinear Schrödinger (NLS) equation is an example of a model that describes many physical nonlinear systems. Wave propagation on deep water is governed by the NLS and its equivalents [105, 106]. It may also be used to model ocean swell.
propagating onto a beach if the waves are narrow-banded surface gravity waves evolving primarily in one horizontal direction. In this case, when in infinite depth, before reaching the beach, the complex amplitudes of these wavetrains may be modeled by the one-dimensional cubic NLS equation

\[ i\psi_x + \vartheta_1 \psi_{tt} + \vartheta_2 |\psi|^2 \psi = 0, \]

(1.1)

where \( \psi(x,t) \) is the complex amplitude of the envelope of the modulated wave and \((\vartheta_1, \vartheta_2)\) are real coefficients. The NLS equation here is written such that the evolution is in space rather than time, which is appropriate for waves measured at fixed locations as they travel across the ocean or down a wave tank in the laboratory. Several researchers including Zakharov [106], Benney & Roskes [16], Hasimoto & Ono [52], Davey [23], Davey & Stewartson [24], Freeman & Davey [45], Yuen & Lake [103], Djordjević & Redekopp [33], Yuen & Lake [104], and Staissnie [91] have derived the NLS equation in different ways and discussed it in various contexts showing that it models the evolution of the complex amplitude of the envelope of a slowly modulated wavetrain propagating on deep water.

Perhaps the most important of all discoveries is concerning the stability of a uniform wave train propagating on deep water, which was discovered independently and in different ways by Benjamin & Feir [12], Whitham [96], Lighthill [66], Ostrovsky [82], Benney & Newell [15], and Zakharov [105, 106] and discussed by several others including Benjamin & Hasselman [13] and Hasselman [53]. The historic result, often referred to as the Benjamin-Feir instability, is that a uniform Stokes wavetrain in deep water is unstable to small, long-wave disturbances in the form of a pair of sideband modes of frequencies slightly different than the underlying wave. More precisely, a uniform Stokes wavetrain of wavenumber \( k \) is unstable to perturbations if \( kh > 1.363 \), where \( h \) is the depth of the undisturbed water surface. The results of Benjamin & Feir and Whitham are extremely important because they established a demarcation, \( kh \approx 1.363 \), between the stable shallow water and unstable deep water regimes. While Benjamin and Feir [12] discuss the instability of the progressive wavetrain based on a subtle kind of resonance between the primary wave with a fundamental frequency and two small modulations at adjacent sideband frequencies, Whitham [96] observed that the differential equations governing the evolution of the slowly varying wavetrain are hyperbolic for \( kh < 1.363 \),
and elliptic otherwise. The elliptic nature of the differential equations implied that the eigenvalues associated with the perturbation are complex, and hence the modulations in the wavetrain grow exponentially. This exponential increase in the rate of energy-transfer between the primary wave and the sidebands, which are coupled by the nonlinear boundary conditions, causes the wavetrain to become irregular, the intensity of which becomes profound with increased interaction between the primary wave and the sidebands, eventually leading to the disintegration of the wavetrain. Thus, a periodic wavetrain is unstable in deep water, characterized by $kh > 1.363$. This instability is also identified in other fields including optics, Bose-Einstein condensates, plasma physics and spin physics, where it is known as "modulational" or "sideband" instability.

The disintegration of a uniform Stokes wavetrain of fundamental wavelength 7.2 ft is shown in figure 1.1, a picture by J. E. Feir obtained from Benjamin & Hasselmann [13] and M. Van Dyke [36]. The top image in the figure shows the wavetrain near the wave-maker in a large towing basin with water of depth approximately 25 ft; the bottom one shows the disintegration due to instability, some 200 ft farther along the basin.

### 1.3 Equations for Uniform Depth

Though the Whitham [96] theory dealt with slowly varying wavetrains, it did not involve a systematic expansion in a small parameter. In addition, the ill-posed differential equations in the theory made it difficult to use the solutions as the leading term of an asymptotic expansion. Hence, Davey & Stewartson [24] used the method of multiple scales to derive two coupled nonlinear partial differential equations for the evolution of the envelope of a three-dimensional wave packet on water of uniform depth, and the resulting mean flow. They are

\[ i\dot{\Psi} + \kappa_1 \Psi_{\xi\xi} + \kappa_2 \Psi_{\eta\eta} = \kappa_3 |\Psi|^2 \Psi + \kappa_4 \Psi \Phi \]  \hspace{1cm} (1.2)

and

\[ \kappa_5 \Phi_{\xi\xi} + \kappa_6 \Phi_{\eta\eta} = \kappa_7 (|\Psi|^2)_{\eta\eta} , \]  \hspace{1cm} (1.3)
where $\Psi(\xi, \eta, \tau) \in \mathbb{C}$ is the envelope of the velocity potential of the progressive wave and $\Phi(\xi, \eta, \tau) \in \mathbb{R}$ is a potential that represents the mean flow set up by the presence of waves in finite depth. They vary in a slow, traveling reference frame with slow space and time scales $\xi = \epsilon(x - c_g t), \eta = \epsilon y, \tau = \epsilon^2 t$; where $\epsilon \ll 1$ is an ordering parameter, $(x, y)$ and $t$ are physical space and time, and $c_g$ is the group velocity [67, 62] of the primary progressive wave. For other versions of the uniform depth equations, see Benney & Roskes [16], Hasimoto & Ono [52] and Ablowitz & Segur [2].
The Davey-Stewartson (DS) equations are written in terms of the velocity potential of the primary wave, which is related to the height of the free surface, \( \zeta \), as

\[
g\zeta = i \epsilon \omega \Psi \exp[i(kx - \omega t)] + \text{c.c.} + O(\epsilon^2). \tag{1.4}
\]

Since these equations are derived for a uniform depth, the coefficients in (1.2) and (1.3) are real constants and are given by

\[
\kappa_1 = \frac{1}{2} \omega_{kk} \leq 0, \tag{1.5}
\]

\[
\kappa_2 = \frac{\omega_k}{2k} = \frac{c_g}{2k} \geq 0, \tag{1.6}
\]

\[
\kappa_3 = \frac{k^4}{4\omega \sigma^2} \left[ 9 - 10\sigma^2 + 9\sigma^4 - \frac{2\sigma^2}{gh - c_g^2} \left( 4c_p^2 + 4c_p c_g (1 - \sigma^2) + gh (1 - \sigma^2)^2 \right) \right], \tag{1.7}
\]

\[
\kappa_4 = \frac{k^4}{c_g^2} \left[ 2c_p + c_g (1 - \sigma^2) \right], \tag{1.8}
\]

\[
\kappa_5 = gh - c_g^2 \geq 0, \tag{1.9}
\]

\[
\kappa_6 = gh, \tag{1.10}
\]

\[
\kappa_7 = gh c_g \left[ \frac{2c_p + c_g (1 - \sigma^2)}{gh - c_g^2} \right], \tag{1.11}
\]

where \( \sigma = \tanh(kh) \), \( c_g = \frac{\partial \omega}{\partial k} \) is the group velocity, \( c_p = \frac{\omega}{k} \) is the phase velocity of the primary progressive wave, \( g \) is the acceleration due to gravity, and \( h \) is the height of the undisturbed free surface from the bottom. These evolution equations are derived from the Stokes [93] boundary value problem and are discussed in Dean & Dalrymple [26].

### 1.4 Equations for Slowly Varying Depth

While studying the evolution of gravity waves, Djordjević and Redekopp [34] allowed for a slowly varying depth in their multiple scale, asymptotic expansion analysis and derived the variable depth version of the DS equations in (1+1) di-
dimensions. They are

\[
\begin{align*}
  i \left[ \Psi_{\xi} + \mu(\xi) \Psi \right] &+ \kappa_8(\xi) \Psi_{\tau\tau} + \kappa_9(\xi) |\Psi|^2 \Psi + \kappa_{10}(\xi) \Phi_{\tau} \Psi = 0 \\
  \Phi_{\tau\tau} - \kappa_{11}(\xi) \left( |A|^2 \right)_{\tau} &= 0, 
\end{align*}
\]  

(1.12)

where \( \Psi(\xi, \tau) \in \mathbb{C} \) is the envelope of the velocity potential of the primary wave and \( \Phi \in \mathbb{R} \) is the potential representing the mean flow. In (1.12), the evolution is in space relative to a slow space scale, \( \xi := \epsilon^2 x \), and the slow, time-like variable is \( \tau := \epsilon \int x \frac{ds}{c_g(s)} - t \), where \( \epsilon \ll 1 \) and \( c_g \) is the group velocity. The bathymetry varies in space, but does not vary in time, and hence, the frequency of the primary wave, \( \omega \), is fixed, while the wavenumber, \( k \), varies with \( \xi \), and may be determined from the dispersion relation

\[
\omega^2 = g k(\xi) \tanh \left[ k(\xi) h(\xi) \right],
\]  

(1.13)

where \( g \) is the acceleration due to gravity, and \( h \) is the local fluid depth. Accordingly, all of the coefficients in (1.12) are slowly varying in space.

For the case of propagation in one horizontal dimension, the envelope of the primary progressive wave and the mean flow decouple, and these equations reduce to a modified NLS equation with variable coefficients,

\[
\begin{align*}
  i \left[ \Psi_{\xi} + \mu(\xi) \Psi \right] &+ \kappa_8(\xi) \Psi_{\tau\tau} + \kappa_{12}(\xi) |\Psi|^2 \Psi + Q_0(\xi) \Psi = 0, \\
\end{align*}
\]  

(1.14)

where \( Q_0 \) appears as a result of integration and acts as a phase shift in the primary wave, and the \( i\mu\Psi \) term comes from conservation of wave action flux [97, 5, 8]. To see this, consider the action \( E/\omega \) being transported by the group velocity, \( c_g \), in a slowly varying medium, so that

\[
\frac{\partial}{\partial t} \left( \frac{E}{\omega} \right) + \frac{\partial}{\partial x} \left( c_g(x) \frac{E}{\omega} \right) = 0,
\]  

(1.15)

with \( E \sim \Psi^2 \) and \( \omega \) fixed. Then

\[
\frac{1}{c_g} \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x} + \frac{1}{2c_g} \frac{dc_g}{dx} \Psi = 0,
\]  

(1.16)
and with respect to the traveling reference frame we are using, (1.16) becomes

$$\frac{\partial \Psi}{\partial \xi} + \frac{1}{2c_g} \frac{d c_g}{d \xi} \Psi = 0.$$ 

So, with

$$\mu := \frac{1}{2c_g} \frac{d c_g}{d \xi},$$  

we get the first two linear terms of (1.14), where the “dissipative” $\mu$ term is due to a non-homogeneous medium. This term appears as a linear damping term such that $\delta \to \delta(\xi) + \mu(\xi)$ (See §1.4.4 for discussion of the boundary-layer induced damping). This $\mu$ term is generic to any envelope-evolution system governed by NLS-type equations for which the medium is variable. The effect of this term on the wave amplitude, and the sign of $\mu$ itself depend on the variation in the medium and on the variation of the group velocity due to the variable medium. We discuss this further in Chapter 2.

As a result of variable depth, the group velocity of the wave, as shown in figure 1.2, is a function of $\xi$ and is given by

$$c_g(\xi) = \frac{g}{2\omega} \left[ \tanh(kh) + kh \text{sech}^2(kh) \right].$$  

(1.18)

The velocity is maximum at $kh \approx 1.2$, and one can show that waves with $kh$ on either side of this maximum have different behaviors resulting from $\mu \neq 0$ (See §1.4.1 for more discussion). Using (1.18) in (1.17), we obtain

$$\mu(\xi) = \frac{(1 - \sigma^2)(1 - kh\sigma)}{\sigma + kh(1 - \sigma^2)} \frac{d(kh)}{d\xi}.$$  

(1.19)

Observe that $\mu \equiv 0$ for uniform depth. The coefficients $\kappa_8$ and $\kappa_{12}$ in (1.14) are given by

$$\kappa_8(\xi) = -\frac{1}{2\omega c_g} \left[ 1 - \frac{gh}{c_g^2} (1 - \sigma^2) (1 - kh\sigma) \right]$$  

(1.20)

and

$$\kappa_{12}(\xi) = -\frac{k^4}{4\omega^2 \sigma^2 c_g} \left[ 9 - 10\sigma^2 + 9\sigma^4 - \frac{2\sigma^2}{gh - c_g^2} \left( 4c_p^2 + 4c_p c_g (1 - \sigma^2) + gh(1 - \sigma^2)^2 \right) \right],$$  

(1.21)
where \( \sigma = \tanh(kh) \). See Djordjević and Redekopp [34] for a comprehensive derivation of these equations.

We observe that (1.14) is written in terms of the velocity potential of the primary wave. To write it in terms of the surface displacement, \( \zeta \), we use

\[
\zeta \sim \frac{\omega}{g} \Psi.
\]  (1.22)

Correspondingly, the coefficient of the nonlinear term in (1.14) gets scaled by a factor of \( \frac{g^2}{c_g^2} \). Then, the equation for the surface displacement of the primary wave is

\[
i \left[ \zeta_\xi + \mu(\xi) \zeta + \alpha(\xi) \zeta_{\tau\tau} + \beta(\xi) |\zeta|^2 \zeta + Q_0(\xi) \zeta \right] = 0,
\]  (1.23)

and the coefficients in the equation are

\[
\alpha(\xi) = \frac{-1}{2\omega c_g} \left[ 1 - \frac{gh}{c_g^2} (1 - \sigma^2) (1 - kh\sigma) \right]
\]  (1.24)

and

\[
\beta(\xi) = \frac{-g^2k^4}{4\omega^3\sigma^2 c_g} \left[ 9 - 10\sigma^2 + 9\sigma^4 - \frac{2\sigma^2}{gh - c_g^2} \left( 4c_p^2 + 4c_p c_g (1 - \sigma^2) + gh (1 - \sigma^2)^2 \right) \right],
\]  (1.25)
where we have corrected a typo in the paper by Djordjević and Redekopp [34].

Following Djordjević and Redekopp [34], we factor out the phase shift due to the mean flow,

$$\zeta(\xi, \tau) = A(\xi, \tau) \exp \left[ -i \int_0^\xi Q_0(s) ds \right], \quad (1.26)$$

to obtain a modified nonlinear Schrödinger (MNLS) equation with variable coefficients for the shifted wave envelope, $A$,

$$i [A_\xi + \mu(\xi) A] + \alpha(\xi) A_{\tau\tau} + \beta(\xi) |A|^2 A = 0, \quad (1.27)$$

and recall that \{\alpha, \beta\} < 0 for the case considered here when $kh > 1.363$. In our work, we avoid this critical value $kh \approx 1.363$ at which $\beta = 0$, where higher-order terms must be accounted for to obtain nonlinear results. However, the MNLS model that we are using is acceptable as long as we do not dwell on the critical point. See Grimshaw & Annenkov [49] for an investigation of waves in this regime.

### 1.4.1 Critical Parameters in Wave Propagation

In this section, we discuss some critical parameters that determine the evolution of a wavepacket as it propagates on water of variable depth. From the dispersion relation (1.13), we obtain

$$\frac{d(kh)}{dh} = \frac{-\sigma}{k (1 - \sigma^2)} \left( \frac{dk}{dh} \right), \quad (1.28)$$

and

$$\frac{dk}{d\xi} = \frac{-g k^2 (1 - \sigma^2)}{2 \omega c_g} \left( \frac{dh}{d\xi} \right). \quad (1.29)$$

These equations imply that the wavenumber, $k$, varies inversely with the depth, $h$, and that the parameter $kh$ varies directly with depth. In general, the wavenumber will increase (wavelength decreases) as the waves (with constant frequency) travel from deeper to shallower water, and vice versa. Similarly, using (1.13), (1.17) and (1.18), the group velocity may be related to the slope of the beach by

$$2 \mu c_g = \frac{dc_g}{d\xi} = \frac{(1 - \sigma^2)(1 - kh\sigma)}{\sigma + kh (1 - \sigma^2)} \left( \frac{g k \sigma}{\omega} \right) \frac{dh}{d\xi}. \quad (1.30)$$
From (1.30), we see that $\mu$ and $\frac{dc_g}{d\xi}$ change sign at $kh\sigma = 1$, which occurs for $kh \approx 1.2$. Accordingly, as the waves travel from deeper to shallower water, the group velocity increases until $kh \approx 1.2$, and then decreases. The reverse is true for waves propagating from shallower to deeper water. However, $kh \approx 1.2$ does not fall in our region of interest, which is $kh \gtrsim 1.363$. The variable depth parameters are discussed in detail in §2.2. Figure 1.3 illustrates the two different regimes in water waves and the critical points. While water of infinite depth typically has a $kh \gg 1.363$, the finite depth region has a $kh$ not much greater than 1.363. In this work, we define the finite depth region to be $1.363 \leq kh \lesssim 4$.

Figure 1.3. Regimes in water wave dynamics.

1.4.2 Conserved Quantities

The NLS equation with constant coefficients is Hamiltonian and has many conserved quantities. The two most important of these are energy, $M$, and momentum, $P$. For the NLS equation (1.1), the conservation equations are $\frac{dM}{d\xi} = 0$ and $\frac{dP}{d\xi} = 0$ where $M = \int_D |\psi|^2 d\tau$ and $P = \int_D [\psi \psi^* \tau - \psi^* \psi \tau] d\tau$, * denotes the complex conjugate and $D$ is a time period. Similarly, energy and momentum are conserved for the MNLS equation with variable coefficients as well. However, in (1.27) the coefficient of the dissipative term, $\mu$, is not constant and may be determined for a given depth variation. Consequently, the conserved quantities corresponding to energy and momentum in the slowly-varying system depend on the variable group
velocity, \( c_g \). They are \( \frac{dM_A}{d\xi} = 0 \) and \( \frac{dP_A}{d\xi} = 0 \), where

\[
M_A = \int_D c_g(\xi)|A|^2 \, d\tau; \quad (1.31a)
\]
\[
P_A = \int_D c_g(\xi) [AA^* - A^*A] \, d\tau. \quad (1.31b)
\]

These conservation equations are derived in Appendix A.

### 1.4.3 Solutions of the MNLS Equation

The MNLS equation with variable coefficients has several types of traveling wave solutions. We are interested in three types of solutions to (1.27), namely,

i. a uniform wave amplitude solution, analogous to the Stokes wave in a uniform medium, of the form

\[
A(\xi, \tau) \sim A_0(\xi, \tau) \exp\left[ \imath \omega_0 \tau \right], \quad (1.32)
\]

where \( A_0 \) is the Fourier amplitude of the carrier wave at frequency \( \omega_0 \). We examine its stability to sideband modes with Fourier amplitudes \( (A_j, A_{-j}) \) at frequencies \( (\omega_0 + \omega_j, \omega_0 - \omega_{-j}) \),

ii. a wave packet solution, analogous to an envelope soliton solution in a uniform medium, of the form

\[
A(\xi, \tau) \sim A_0(\xi) \sech \left[ \lambda(\xi) \tau \right] \exp\left[ \imath f_1(\xi) \tau + \imath f_2(\xi) \right], \quad (1.33)
\]

and

iii. a periodic envelope or a cnoidal wave, analogous to the Jacobi-elliptic function solutions in a uniform medium, of the form

\[
A(\xi, \tau) \sim A_0(\xi) J[\lambda(\xi) \tau; m] \exp\left[ \imath f_3(\xi) \tau + \imath f_4(\xi) \right], \quad (1.34)
\]

where \( J \) is an appropriate Jacobi-elliptic function. A description of the Jacobi-elliptic functions is included in Appendix B.
We discuss these three solutions in detail in Chapter 2, Appendix C and Chapter 3 respectively.

1.4.4 The Dissipative MNLS Equation

Fluid flow that is modeled as inviscid is affected by viscous dissipation primarily near boundaries. This is due to the development of a boundary layer at the rigid boundary and the resulting large gradients in velocity field. Therefore, a wave propagating on water of finite depth will be subjected to damping due to viscous dissipation at the bottom boundary. The effect of damping is inversely proportional to the water depth and so is relatively smaller for larger depths. The resulting decay rates are small in terms of an overall energy budget and disappear in infinitely deep water. However, viscous dissipation also occurs at the moving surface boundary layer, and though the energy decay may be small, the effect on dynamics could be significant.

Several researchers have analyzed the effect of damping on nonlinear systems. Miles [77] reviews several models for decay coefficients. He studied the damping of waves due to viscous dissipation in consequence of surface contamination and that due to capillary hysteresis, in addition to the effect of viscous dissipation at the rigid boundaries. Others that introduced boundary-layer induced damping in their analysis of nonlinear waves include Lake et al. [63], Renouard et al. [87], Helfrich & Melville [54], Henderson & Lee [56], Segur et al. [89], Henderson et al. [57], Henderson & Segur [55].

The variable-coefficient MNLS equation with the inclusion of viscous dissipation is

$$i[A_\xi + \mu(\xi)A + \delta(\xi)A + \alpha(\xi)A_{\tau\tau} + \beta(\xi)|A|^2A = 0, \quad (1.35)$$

where $\delta$ is the coefficient of damping. Henderson & Lee [56] provide the decay rate for the bottom boundary and for a “fully contaminated” or immobile free surface. The combined decay rate is

$$\delta(\xi) = \frac{C + C \cosh^2(\omega h)}{\sinh^2(\omega h) \left[\tanh(\omega h) + \omega \sinh^2(\omega h)\right]}, \quad (1.36)$$

where $C = \sqrt{\frac{\pi}{2}} \left(\frac{\omega^3}{\gamma^2}\right)$ and $\nu$ is the kinematic viscosity of water. The damping
coefficient is a function of local depth, and hence a variable for an uneven bottom. Note that there are two dissipative terms in (1.35), the $\mu$ term and the $\delta$ term. While the $\mu$ term is due to a non-homogeneous medium and is absent for uniform or infinite depth, the $\delta$ term is due to boundary layer effects and disappears only for infinite depth. We will discuss the competition between these two terms in Chapter 2.

1.5 Outline

There is a large literature on nonlinear waves that includes theoretical, experimental and numerical results. Quite a few investigators have studied some solutions of the NLS equation with respect to stability, variable depth, and damping but, not all together. In this work, we include all of stability, variable depth and damping effects. With a combination of numerical and asymptotic analyses, this thesis attempts to study the three solutions-of-interest to the variable coefficient MNLS equation in the context of an uneven bottom and boundary-layer induced damping. We compare the results of our simulations with damping to those without damping and the results with a uniform bottom to those with a non-uniform bottom. In addition, we also study the stability of one of the solutions, the “uniform” amplitude solution.

Benilov [9] showed that envelope solitons propagating from deeper to shallower water spread and undergo a reduction in amplitude. They did not address the converse, which we investigate analytically and numerically and generalize to other wave systems.

The organization of this thesis is as follows. Chapter 2 includes a stability analysis of the “uniform” amplitude solution to the MNLS equation for waves on variable depth. We discuss the stability of the “uniform” amplitude solution to small perturbations and the effect of an uneven bottom and that of the boundary-layer induced damping on the stability. This chapter also includes a proof of linear stability of the solution. Chapter 3 includes an asymptotic analysis of the Jacobi-elliptic function solutions to the MNLS equation for waves propagating on water of variable depth. We compare the asymptotic solution to the numerical solution and discuss the validity of the asymptotic solution. We take into consid-
eration wave propagation in both directions, from shallower to deeper water and vice versa. Chapter 4 includes the numerical analysis of the variable-coefficient MNLS equation with damping. The periodicity of the solutions allows us to use a pseudo-spectral method for the derivatives in space, which is efficient for capturing nonlinear effects. The time derivatives are discretized using a fourth order explicit Runge-Kutta method. Numerical results are produced for the three solutions-of-interest for propagation over an uneven bottom, with and without damping and for propagation from shallower to deeper water and vice versa. The results are discussed and compared with available experimental data. The thesis ends with a comprehensive summary of the results, some discussion and conclusions included in Chapter 5.
Chapter 2

The Stability of a “Uniform” Amplitude Wavetrain Propagating on Water of Variable Depth

2.1 Introduction

In this chapter, we analyze the stability of a “uniform” amplitude wavetrain propagating on water of variable depth, in the presence of dissipation. The stability of the uniform amplitude solution of the constant-coefficient NLS equation (1.1), also referred to as the Stokes solution [93], has been studied extensively in the last few decades. Zakharov [105, 106] was among the first few to examine the stability of nonlinear waves on the surface of an infinitely deep fluid and discover the modulational instability. At about the same time, Benjamin & Feir [12, 13] established that the Stokes wave train in deep water is unstable to small disturbances in the form of a pair of side-band modes, whose frequencies are slightly different from that of the primary wave. Whitham’s [96] results also supported this claim. Hasselmann [13] argued that the resonant nonlinear interactions in any wave spectrum have an irreversible tendency to spread wave energy to all wave numbers and hence a narrow spectral peak may be expected to broaden, which is equivalent to saying that an almost periodic Stokes wave is unstable to its side-
bands. He verified this for gravity waves in deep water with numerical simulations [53]. Other pioneers that studied the stability of nonlinear waves include Hasimoto & Ono [52] and Davey & Stewartson [24], who examined the stability of a uniform Stokes wavetrain on uniform depth, Djordjević & Redekopp [33], who discussed the stability of the Stokes capillary-gravity wavetrain, Ablowitz & Segur [1], who studied the stability of long-wave solitons, Yuen & Lake [104], who studied the stability of deep-water waves, and Dhar & Dhas [30], who examined the stability of a uniform Stokes wave train in the presence of another wavetrain in deep water.

While studying the stability problem in water waves, a few researchers have also identified instabilities other than the Benjamin-Feir type. The stability analysis of Benjamin & Feir was experimentally investigated by Benjamin & Feir [12], Lake et al. [63] and Melville [76]. The experiments of Lake et al. suggested that the end-state of evolution of a nonlinear wavetrain in the absence of dissipation would be a continuing series of modulation-demodulation recurring cycles similar to a Fermi-Pasta-Ulam (FPU) recurrence. In other words, there would be alternating cycles of continuous growth of modulations accompanied by spread of energy over spectral components and return of energy to original spectral components of the initial wavetrain. Likewise, McLean [75] examined the stability of finite amplitude waves and concluded that the dominant instability is two-dimensional (Benjamin-Feir type) for small amplitudes, while it is three-dimensional for large amplitudes. Melville’s measurements [76] agreed with McLean’s predictions [75]. McLean [74] also extended their deep-water results to instability of gravity waves on water of finite depth, showing that the unstable perturbations are three-dimensional for finite depth (in comparison to two-dimensional perturbations for deep water). Additionally, Zhu et al. [86] also affirmed the two-dimensional sideband-like instability for plane standing waves, and showed with the help of direct long time simulations of the nonlinear evolution that the three-dimensional instability is of the FPU recurrence type.

### 2.2 Viscous Dissipation and Variable Depth

Instabilities other than the Benjamin-Feir type are found in waves propagating over variable depth as well. Benney & Roskes [16] were among the first few researchers
that examined the stability of nonlinear waves for variable depth, without viscous dissipation. They showed that results similar to the Whitham theory (the hyperbolic nature of shallow-water equations and elliptic nature of deep-water equations) do not apply for three-dimensional problems. Similarly, while studying the effects of depth and current on a Stokes wavepacket, Turpin [95] extended the criterion of spatial instability to small sidebands, concluding that the fissioning or flattening of such waves is determined by an instability parameter, which depends strongly on the depth variation. Furthermore, Xiao & Lo [100] extended the third order NLS equation to include the next higher order variable depth effects beyond shoaling, and examined the stability for the extended NLS equation. They identified that higher order depth terms introduce a higher order instability for all small wavelength disturbances beyond the Benjamin-Feir type.

These stability results did not take viscous dissipation into consideration. The Benjamin-Feir instability states that in the absence of dissipation, a uniform wave-train of moderate amplitude propagating in deep water is unstable to small perturbations from other waves with nearby frequencies traveling in the same direction. Recently Segur et al. [89] and Henderson & Segur [55] studied the dissipative version of the NLS equation (1.1),

\[ i\psi_x + \delta\psi + \vartheta_1 \psi_{tt} + \vartheta_2 |\psi|^2 \psi = 0, \quad (2.1) \]

which models waves propagating on water of infinite depth with dissipation. They found that dissipation with decay rate \( \delta \), however small, stabilizes this instability and that the growth of these perturbations are bounded within a finite distance, by dissipation, before the nonlinear interactions can become important. Their results encourage us to examine the stability of nonlinear waves propagating on variable depth in the context of the dissipative \( \mu \) term in (1.35) that has the same mathematical role as boundary-layer induced dissipation in uniform depth. It is of interest to see if the dissipation due to the non-homogeneous medium, by itself, without the addition of the boundary-layer induced damping, performs the same role in stabilizing the Benjamin-Feir instability. Consequently, it is also of interest to take into consideration the dissipation effects due to the boundary layer (represented by \( \delta \)), and examine the competition with the dissipation due to the
non-homogeneous medium (represented by \( \mu \)).

To this end, we are interested in the variable-depth Stokes solution of the Djordjevic’c-Redekopp equation with viscous dissipation included (see §1.4.4),

\[
i [A_\xi + \mu(\xi)A + \delta(\xi)A] + \alpha(\xi)A_{\tau\tau} + \beta(\xi)|A|^2A = 0.
\] (2.2)

Following Segur et al. [89], we can factor our the viscous dissipation \( \delta(\xi) \) and the dissipation/growth due to the variable medium, \( \mu(\xi) \), with

\[
A(\xi, \tau) = B(\xi, \tau) \exp \left[ -i \int^{\xi} \{ \mu(s) + \delta(s) \} ds \right],
\] (2.3a)

\[
= \sqrt{\frac{c_{g0}}{c_g(\xi)}} B(\xi, \tau) \exp \left[ -i \int^{\xi} \delta(s) ds \right],
\] (2.3b)

where \( c_{g0} = c_g(0) \), and we have used (1.17) to obtain (2.3b). The envelope amplitude, \( B \), satisfies

\[
i B_\xi + \alpha(\xi)B_{\tau\tau} + \beta(\xi) \exp \left[ -2 \int^{\xi} \{ \mu(s) + \delta(s) \} ds \right] |B|^2B = 0.
\] (2.4)

When viscous dissipation is absent (\( \delta = 0 \)), (2.2) admits conserved quantities (See Appendix A for derivation)

\[
M_A = \int_D c_g(\xi) |A|^2 d\tau,
\] (2.5a)

\[
P_A = \int_D c_g(\xi) [AA^\ast_{\tau} - A_{\tau}A^\ast] d\tau,
\] (2.5b)

for which \( \frac{dM_A}{d\xi} = 0 \), \( \frac{dP_A}{d\xi} = 0 \), the \( * \) indicates complex conjugate, and \( D \) is the temporal domain.

In the presence of viscous dissipation (\( \delta > 0 \)), the integrals in (2.5a) and (2.5b) decay so that

\[
(M_A, P_A) = \left( M_A(0), P_A(0) \right) \exp \left[ -2 \int^\xi \delta(s) ds \right].
\] (2.6)

Then, we can find constants of the motion (following Segur et al. [89]) described
by (2.4) in the reference frame that factors out dissipation so that in the dissipative reference frame,

\[ M_B = \int_D |B|^2 d\tau; \quad (2.7a) \]

\[ P_B = \int_D \left[ B B^*_\tau - B^*_\tau B^* \right] d\tau, \quad (2.7b) \]

for which \( \frac{dM_B}{d\xi} = 0 \) and \( \frac{dP_B}{d\xi} = 0 \).

From (2.5a) we see that \( \frac{d|A|^2}{d\xi} = \frac{dc_g}{c_g} \left( \frac{dc_g}{d\xi} \right) \), so that the spatial change in \( |A|^2 \) is inversely proportional to \( c_g \). Consequently, when \( \frac{dc_g}{d\xi} < 0 \), then necessarily the carrier wave amplitude, \( |A| \), is increasing/decreasing. In §1.4.1, we showed from the dispersion relation (1.13), using \( c_g = \frac{d\omega}{dk} \) (for a fixed \( h \)) and the fact that \( \omega \) is fixed, that \( dc_g/dh \) changes sign when \( kh \tanh kh = 1 \). This happens at \( kh \approx 1.2 \), which is below our boundary of \( kh \gtrsim 1.363 \). For \( kh \gtrsim 1.363 \), \( 1 - kh\sigma < 0 \), and hence (1.30) implies that \( \frac{dc_g}{d\xi} \sim -\frac{dh}{d\xi} \). Therefore, for waves entering a beach, or the continental slope, from deep water, we consider the cases

\[ \frac{dh}{d\xi} < 0 \implies \frac{dk}{d\xi} > 0 \implies \mu \sim \frac{dc_g}{d\xi} > 0 \implies \frac{d|A|}{d\xi} < 0, \quad (2.8a) \]

\[ \frac{dh}{d\xi} > 0 \implies \frac{dk}{d\xi} < 0 \implies \mu \sim \frac{dc_g}{d\xi} < 0 \implies \frac{d|A|}{d\xi} > 0, \quad (2.8b) \]

where we have used (1.17) and (1.29). Relation (2.8a) implies that waves entering shallower water have a decreasing amplitude and decreasing wavelength (increasing wavenumber). Relation (2.8b) implies that the wave amplitudes will increase (as does the wavelength) on regions of the beach (or continental slope) for which the depth is locally increasing (on a scale long compared to the carrier wave). Benilov et al. \cite{9} observed this general result in a numerical simulation using an envelope soliton for initial data.

Figure 2.1 shows the variation of parameters in laboratory scales, for a 3.33 Hz wavepacket propagating up a beach. We have considered a specific beach here for the illustration. The linear beach considered here has been designed to include the critical points \( kh \approx 1.2 \) and \( kh \approx 1.363 \). However, our focus is on \( kh > 1.363 \), and all of the analytical results in this thesis hold for arbitrary depth variation.

In subsequent sections, we address the Benjamin-Feir type instability for the
Figure 2.1. Variation of parameters for a linear beach appropriate for the laboratory. The red/blue dots indicate $kh = 1.363/1.2$.

“uniform” amplitude wavetrain propagating over an uneven bottom and discuss the effects of dissipation due to the boundary layer and that due to the analogous term that necessarily arises for a non-uniform bottom. To this end, following Segur et al. [89], we find a “uniform” amplitude solution to the Djordjević-Redekopp equations and examine the impact of the two dissipative effects on the stability of
the solution and the competition between them.

2.3 The “Uniform” Amplitude Solution

We seek a solution of the model equation in the physical reference frame, (2.2) of the form

\[ A = A_0 \exp \left[ i \int_{\xi}^{\tau} r(s) ds \right]. \quad (2.9) \]

Substituting (2.9) in (2.2) we find that \( r = r_1(\xi) + ir_2(\xi) \) is necessarily complex. The real and imaginary parts are

\[
\begin{align*}
    r_1 &= \beta(\xi) |A_0|^2 \exp \left[ -2 \int_{\xi}^{\tau} r_2(s) ds \right] \\
    &= \beta(\xi) |A_0|^2 \frac{c_{g0}}{c_g(\xi)} \exp \left[ -2 \int_{\xi}^{\tau} \delta(s) ds \right] \\
    r_2 &= \mu(\xi) + \delta(\xi), \quad (2.10)
\end{align*}
\]

using (1.17) and \( c_{g0} = c_g(0) \). It follows that an exact solution of (2.2) is

\[
A(\xi, \tau) = A_0 E(\xi) \sqrt{\frac{c_{g0}}{c_g(\xi)}} \exp \left[ i |A_0|^2 \int_{\xi}^{\tau} \frac{c_{g0}}{c_g(s)} \beta(s) \{E(s)\}^2 ds \right], \quad (2.11)
\]

where \( E(z) = \exp \left[ -\int_{\xi}^{z} \delta(\rho) d\rho \right] \). For a uniform bathymetry \( c_g(\xi) = c_{g0} \), and (2.11) corresponds to the viscous Stokes-like solution of (2.1). For variable bathymetry, the amplitude of the wave will either increase or decrease, depending on the variation of the group velocity in the medium as discussed in §2.2, equations (2.8).

The corresponding solution to (2.4) is

\[
B = B_0 \exp \left[ i |B_0|^2 \int_{\xi}^{\tau} \beta(s) \frac{c_{g0}}{c_g(s)} \{E(s)\}^2 ds \right]. \quad (2.12)
\]

Thus, the amplitude of the solution is not uniform, but instead depends on the local group velocity and dissipation rate.
2.4 Stability Analysis

To examine the stability of the solution, we add a perturbation to (2.9), so that

\[
A(\xi, \tau) = \text{Exp}\left[i\int r(s)ds + i\text{Arg}(A_0)\right]\left(|A_0| + \zeta u + i\zeta v + O(\zeta^2)\right), \tag{2.13}
\]

where \(\zeta \ll 1\) is an ordering parameter and \(r(\xi)\) is given in (2.10). Substituting (2.13) into (2.2), keeping terms of \(O(\zeta)\), we obtain

\[
\begin{align*}
&\left[-r(\xi) + i\mu(\xi) + i\delta(\xi)\right]\left(|A_0| + \zeta u + i\zeta v\right) + i\left[\zeta u_\xi + i\zeta v_\xi\right] + \alpha\left[\zeta u_{\tau\tau} + i\zeta v_{\tau\tau}\right] \\
&+ \beta\text{Exp}\left[-2\int^{\xi} r_2(s)ds\right]\left[|A_0|^3 + 3\zeta u|A_0|^2 + i\zeta v|A_0|^2\right] = 0. \tag{2.14}
\end{align*}
\]

Now, grouping terms with \(\zeta^0\), we recover (2.10),

\[
-r(\xi) + i\left[\mu(\xi) + \delta(\xi)\right] + \beta|A_0|^2\text{Exp}\left[-2\int^{\xi} r_2(s)ds\right] = 0, \tag{2.15}
\]

and grouping terms with \(\zeta^1\), we obtain

\[
iu_\xi - v_\xi + \alpha u_{\tau\tau} + i\alpha v_{\tau\tau} + 2u\beta|A_0|^2\text{Exp}\left[-2\int^{\xi} r_2(s)ds\right] = 0, \tag{2.16}
\]

where we have used (2.10). Separating (2.16) into real and imaginary parts, we obtain

\[
\begin{align*}
v_\xi - \alpha u_{\tau\tau} - 2u\beta|A_0|^2\text{Exp}\left[-2\int^{\xi} r_2(s)ds\right] &= 0, \tag{2.17a} \\
u_\xi + \alpha v_{\tau\tau} &= 0. \tag{2.17b}
\end{align*}
\]

Now, decomposing the perturbation into Fourier modes,

\[
\begin{align*}
u(\xi, \tau) &= U(\xi; \rho)e^{i\rho\tau} + U^*(\xi; \rho)e^{-i\rho\tau}; \\
v(\xi, \tau) &= V(\xi; \rho)e^{i\rho\tau} + V^*(\xi; \rho)e^{-i\rho\tau}, \tag{2.18}
\end{align*}
\]
we obtain a system of ODEs for the Fourier amplitudes,

\[\frac{dV}{d\xi} + \left[ \alpha p^2 - 2\beta|A_0|^2\text{Exp}\left( - 2\int_{\xi}^{e} r_2(s)\text{d}s \right) \right] U = 0, \quad (2.19a)\]

\[\frac{dU}{d\xi} + \alpha p^2 V = 0. \quad (2.19b)\]

This ODE system (2.19) may be rearranged to obtain

\[\frac{d}{d\xi} \left( \frac{1}{|\alpha(\xi)|} \frac{dU}{d\xi} \right) + p^2 \gamma(\xi) U = 0, \quad (2.20a)\]

\[\frac{d}{d\xi} \left( \frac{1}{\gamma(\xi)} \frac{dV}{d\xi} \right) + p^2 |\alpha(\xi)| V = 0, \quad (2.20b)\]

where we have used that \(\{\alpha, \beta\} < 0\) for gravity waves and \(kh > 1.363\), and

\[\gamma(\xi) = |\alpha(\xi)|p^2 - 2|\beta(\xi)||A_0|^2 c_{g0} c_{g0}(\xi)\text{Exp}\left[ - 2\int_{\xi}^{\xi} \delta(s)\text{d}s \right]. \quad (2.21)\]

For analysis using Sturmian theory (cf. Ince [58], Ch 10), [79] we rewrite (2.20) as

\[\frac{d}{d\xi} \left( \frac{K(\xi)}{G(\xi)} \frac{dU}{d\xi} \right) - G(\xi) U = 0, \quad (2.22a)\]

\[\frac{d}{d\xi} \left( \frac{1}{G(\xi)} \frac{dV}{d\xi} \right) - \frac{1}{K(\xi)} V = 0, \quad (2.22b)\]

using

\[K(\xi) = \frac{1}{|\alpha(\xi)|} > 0, \quad (2.23a)\]

\[G(\xi) = -p^2 \gamma(\xi)\]

\[= |A_0|^2 |\alpha(\xi)|p^2 \left[ H(\xi) - \frac{p^2}{|A_0|^2} \right], \quad (2.23b)\]

\[H(\xi; \delta(\xi)) = W(\xi)\text{Exp}\left[ - 2\int_{\xi}^{\xi} \delta(s)\text{d}s \right], \quad (2.23c)\]
\[ W(\xi) = \frac{2|\beta(\xi)|}{|\alpha(\xi)|} \frac{c_g}{c_g(\xi)}. \]  

(2.23d)

The stability of solutions to (2.22) depend on the signs and relative sizes of \( K \) and \( G \). To examine the different possibilities, let

\[
W(\xi) := \text{inf}_{a \leq \xi \leq b} \left[ W(\xi) \right], \\
M_W := \text{sup}_{a \leq \xi \leq b} \left[ W(\xi) \right], \\
(2.24)
\]

\[
m_H := m_W \text{Exp} \left[ -2 \int_\xi \delta(s)ds \right], \\
M_H := M_W \text{Exp} \left[ -2 \int_\xi \delta(s)ds \right], \\
(2.25)
\]

\[
m_G := \text{inf}_{a \leq \xi \leq b} \left[ G(\xi) \right], \\
M_G := \text{sup}_{a \leq \xi \leq b} \left[ G(\xi) \right], \\
(2.26)
\]

where \([a, b]\) is the interval over which the depth is nonuniform. Then we have the following four cases.

**2.4.1 Case 1: \( 0 < \frac{p^2}{|A_0|^2} < m_H \Rightarrow G > 0 \).**

For this case, with no viscous dissipation \((\delta = 0)\), there are no zeros in \([a, b]\); \( U \) and \( V \) will grow. However, viscous dissipation \((\delta > 0)\) will cause \( m_H \) to decrease with increasing \( \xi \) so that there will be a \( \xi \) for which the value of \( \frac{p^2}{|A_0|^2} \) will cease to be less than \( m_H \); every perturbation with fixed \( \frac{p^2}{|A_0|^2} \) that starts out in this regime will eventually leave it and enter the regime of Case 2.

**2.4.2 Case 2: \( m_H \leq \frac{p^2}{|A_0|^2} \leq M_H \).**

For this case, \( m_G = p^2|\alpha||A_0|^2[m_H - M_H] < 0 \), and \( M_G = p^2|\alpha||A_0|^2[M_H - m_H] > 0 \); therefore, \( G \) changes sign during propagation. Solutions of (2.22) will be
oscillatory in the region with $G < 0$, (depending on the boundary size) and will be growing in the region with $G > 0$. However, viscous dissipation, $\delta > 0$, will cause both $m_H$ and $M_H$ to decrease with increasing $\xi$ (regardless of the increasing or decreasing depth), so that eventually, the perturbation with fixed $p^2/|A_0|^2$ will leave this regime, and enter that of Case 3.

2.4.3 Case 3: $\dfrac{p^2}{|A_0|^2} > M_H \Rightarrow G < 0$.

For this case the solutions will be oscillatory if they fit into the boundaries, and the period of the oscillations will depend on the bathymetry as well as viscous dissipation. In summary, Cases 1–3 show that for wavetrains on variable depth, even in the inviscid approximation, there will be regions where all perturbation amplitudes oscillate. Solutions that start out growing will become oscillatory because of viscous dissipation if the distance for evolution is long enough. In §2.5 we discuss the bound on these solutions.

The regions corresponding to the three cases discussed above are shown in figures 2.2, for an arbitrary depth variation. $G = 0$ on the curve $\dfrac{p^2}{|A_0|^2} = H(\xi)$. Similarly, in the region above the curve, $G < 0$, while the region below the curve corresponds to $G > 0$. The figure at the top shows the variation of $H(\xi)$ for a fixed damping coefficient, $\delta(a)$. The maximum and minimum of this function are denoted in the figure as $M_H(a)$ and $m_H(a)$ respectively. However, if we include a variable damping coefficient (due to an uneven bottom), the maximum and minimum of the function decrease exponentially as per (2.25). The figure at the bottom shows the variation of $M_H$ and $m_H$ for an arbitrary depth variation. Thus, as $M_H$ and $m_H$ decrease, the regions with $G > 0$ shrink, and a perturbation with value $p^2/|A_0|^2$ that was initially in the region with $G > 0$ gradually moves into the region with $G < 0$, if the distance of propagation is long enough.

2.4.4 Case 4: $\mu + \delta = 0$

The sign of $\mu(\xi)$ depends on the sign of $dh/d\xi$ and $dc_g/d\xi$ as shown in (2.8). Since $\delta(\xi) > 0$, there is the possibility that $\mu(\xi) + \delta(\xi) = 0$ when the carrier wave is entering deeper water. For this case, the growth/dissipation term in (2.2) would disappear and the amplitude envelope would evolve according to an inviscid
NLS; the solution would be uniform and would be unstable to perturbations with frequencies, $p < 2|\beta(\xi)|A_0|^2/|\alpha(\xi)|$. To find a potential bathymetry to satisfy this condition, one needs to specify the form of viscous dissipation. There are several models in the literature. Here we consider the one given by Lamb [64] that assumes the air-water interface is “inextensible”; the tangential velocity at the interface vanishes. This model was used, for example, by Miles [77] to describe an air-water interface that is “fully contaminated”, that is, it is saturated by a
surfactant. Henderson & Segur [55] showed that it agrees with measured values of dissipation of ocean swell with periods of about 16 s by a factor of about 2. Here we also include dissipation due to a boundary layer at the bottom [22], which makes $\delta$ depend on $\xi$ (See §1.4.4). Then, as recorded by Henderson & Lee [56],

$$
\delta = \sqrt{\frac{\nu \omega}{2}} \left[ \frac{k(\xi) (1 + \cosh^2[k(\xi)h(\xi)])}{c_g(\xi) \sinh[2k(\xi)h(\xi)]} \right],
$$

(2.27)

where $\nu$ is the kinematic viscosity of water. Substituting $c_g = d\omega/dk$ from the dispersion relation (1.13) for fixed $h$, into (2.27), we obtain the viscous decay rate in terms of $kh$ so that

$$
\delta = \frac{C}{\omega^3} \left[ \frac{3 + \cosh(2kh)}{2kh + \sinh(2kh) \tanh^2(kh)} \right],
$$

(2.28)

$$
C = \frac{\omega^3}{g^2} \sqrt{\frac{\nu \omega}{2}}.
$$

The minimum of $\delta/C$ occurs at $kh \sim 3.05$ and then increases asymptotically to

![Figure 2.3. Variation of the dimensionless damping coefficient with $kh$.](image)

its constant, deep-water ($kh \to \infty$) limit, 1, as shown in figure 2.3. From (1.19), we obtain

$$
\mu = \left[ \frac{1 - kh \tanh(kh)}{kh + \cosh(kh) \sinh(kh)} \right] \frac{d(kh)}{d\xi}.
$$

(2.29)
For carrier waves on a depth such that $1.363 < kh < 4$, the numerator of the term in square brackets is negative (see §1.4.1 for discussion). Since $\delta > 0$, we need $\frac{d(kh)}{d\xi} > 0$, using (1.13). (We consider this interval for $kh$ because $\tanh(4) \approx 1$ to three decimal places, so for about that value and larger, the waves behave as “deep-water” waves that do not feel the bottom.) So this case can occur for waves entering deeper water only; it will not occur for waves propagating into shallower water. Setting $\mu = -\delta$ results in a first-order differential equation for $kh$,

$$\frac{d(kh)}{d\xi} = -C f(kh),$$

$$f(kh) = \left[ \frac{\coth^2(kh) \{1 + \cosh^2(kh)\}}{1 - kh \tanh(kh)} \right]. \quad (2.30)$$

Figure 2.4(a) shows the numerical solution of (2.30) with $kh(0) = 1.363$.

Table 2.1. Approximate bounds on the right-hand-side of (2.30) for waves in finite depth for typical carrier waves in the ocean and laboratory.

<table>
<thead>
<tr>
<th>Period (sec)</th>
<th>C (1/cm)</th>
<th>$kh$</th>
<th>$f(kh)$</th>
<th>$\frac{d(kh)}{d\xi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>$2.7 \times 10^{-9}$</td>
<td>1.363</td>
<td>$\sim -35$</td>
<td>$\sim -9.9 \times 10^{-8}$</td>
</tr>
<tr>
<td>6</td>
<td>$\sim -8100$</td>
<td>$\sim -35$</td>
<td>$\sim -2.2 \times 10^{-5}$</td>
<td></td>
</tr>
<tr>
<td>0.3</td>
<td>$3.1 \times 10^{-3}$</td>
<td>1.363</td>
<td>$\sim -35$</td>
<td>$\sim -0.11$</td>
</tr>
<tr>
<td>6</td>
<td>$\sim -8000$</td>
<td>$\sim -35$</td>
<td>$\sim -25$</td>
<td></td>
</tr>
</tbody>
</table>

In table 2.1, we consider values of the right-hand-side of (2.30) for waves in finite depth for typical carrier waves in the ocean and laboratory when $1.363 < kh < 4$. For $kh > 1.363$, $f(kh) < 0$; and $C > 0$ for all $kh$. Therefore, there is no constant value of $kh$ that would satisfy $\mu + \delta = 0$. However, table 2.1 shows that for typical ocean swell $\frac{d(kh)}{d\xi}$ is very close to zero. So for ocean swell, this case would be satisfied for $kh$ = almost constant. Then, to satisfy that constraint and (1.13) the beach slope (with waves propagating in the direction of increasing depth) would have to be very mild. So for the ocean case, there is a balance between two small effects, mild beach slope and viscous dissipation, so that swell propagating on finite depth that is slightly increasing would be unstable to long-wave perturbations. Henderson & Segur [55] showed that for ocean swell in water of infinite depth, dissipation,
even though small in terms of the overall energy balance, can be strong enough to stabilize the Benjamin-Feir instability.

![Graph](image)

**Figure 2.4.** Bathymetry corresponding to Case 4. *(a)* Solution of (2.30) using $kh(0) = 1.363$. *(b)* Solution of (2.31) for a typical laboratory case with a carrier wave of period 0.3 s and $h(0) = 1.363/0.41$ cm.

We note that dissipation plays disproportionate roles in oceanic phenomena and small-scale laboratory experiments. Viscous effects are stronger in laboratory experiments as shown in table 2.1, which gives values of the right-hand-side of (2.30) for a typical laboratory experiment. For that situation, it is useful to use
(1.13) and (2.30), recalling that \( \omega \) is constant, to obtain

\[
\frac{dh}{d\xi} = -\hat{C} f(kh) \left[ \tanh(kh) + kh \ \text{sech}^2(kh) \right],
\]

\[
\hat{C} = \frac{\omega}{g} \sqrt{\frac{\nu \omega}{2}}.
\]

Then one can compute the required depth profile for the case of balanced \( \mu \) and \( \delta \) by specifying the frequency and the value of \( k \) corresponding to that frequency at the initial value of \( kh \) when \( \xi = 0 \). Figure 2.4(b) shows the depth profile obtained by numerically solving (2.30) and using the result to numerically solve (2.31) for a carrier wave of about 3 Hz. The initial conditions are \( kh(0) = 1.363 \) and \( h(0) = 1.363/k(0) \) cm, where \( k(0) = 0.41/\text{cm} \), corresponds to the wavenumber of a wave with period 3 s on a uniform beach with \( kh(0) = 1.363 \).

### 2.5 Linear Stability

In §2.4.1–2.4.4 we found parameter regimes for oscillating and growing solutions of the equations, (2.22), governing the growth of perturbations to (2.9). Except for Case 4, for which \( \mu + \delta = 0 \), we found that with viscous dissipation, \((\delta > 0)\), all perturbations that fit into a case with growing solutions, eventually get kicked into Case 3, which has oscillating solutions only. Here we prove the linear stability of the oscillating solutions. In other words, we show that the oscillating solutions are bounded, so that the solution, (2.9), is linearly stable in the Lyapunov sense (cf. Nemytskii & Stepanov [80]): every perturbation that is bounded at \( \xi = 0 \) stays bounded for all \( \xi > 0 \).

First we note that regardless of the signs of \( K'(\xi) \) and \( G'(\xi) \), the decay due to \( G < 0 \) dominates growth when \( K'(\xi) < 0 \) and \( G'(\xi) > 0 \). Then, following Ince [58], Ch 3, the solution of (2.22) satisfies:

\[
|U(\xi; p)| \leq \sqrt{|U(0; p)|^2 + |V(0; p)|^2} \ \text{Exp} \left[ 2 \int_0^{f(\xi)} \frac{|r_1(s)|}{|\alpha(s)|} \, ds \right],
\]

\[
|V(\xi; p)| \leq \sqrt{|U(0; p)|^2 + |V(0; p)|^2} \ \text{Exp} \left[ 2 \int_0^{f(\xi)} \frac{|r_1(s)|}{|\alpha(s)|} \, ds \right],
\]

(2.32)
where $r_1$ is given in (2.10), and $f(\xi) = \int_0^\xi |\alpha(s)|ds$. Then

$$|U(\xi; p)|^2 + |V(\xi; p)|^2 \leq 2 \left( |U(0; p)|^2 + |V(0; p)|^2 \right) \text{Exp}\left[ 4 \int_0^\xi \frac{|r_1(s)|}{|\alpha(s)|} ds \right]$$

(2.33)

for all perturbations with frequency $p$, since $\delta \geq 0$ and $\xi > 0$. Since $\frac{c_g}{c_g(0)} \sim O(1)$ and the exponential decay due to viscosity eventually beats possible growth due to bathymetry, $r_1(s)$ is bounded, and there exists a constant, $\Sigma > 0$, for which

$$\text{Exp}\left[ 4 \int_0^\xi \frac{|r_1(s)|}{|\alpha(s)|} ds \right] < \Sigma.$$

Then

$$|U(\xi)|^2 + |V(\xi)|^2 \leq 2 \Sigma \left( |U(0)|^2 + |V(0)|^2 \right).$$

(2.34)

We may solve (2.2) with periodic boundary conditions, choosing initial conditions that satisfy

$$\int_T \left( |u(0, \tau)|^2 + |v(0, \tau)|^2 \right) d\tau \leq \Delta$$

(2.35)

for some constant $\Delta > 0$ in the period $T$, and using Parseval's relation (cf. Guenther & Lee [50]) and (2.34), we obtain

$$\int_T \left( |u(\xi, \tau)|^2 + |v(\xi, \tau)|^2 \right) d\tau \leq 2\Sigma\Delta$$

(2.36)

for all $\xi > 0$. The choice of $\Sigma$ and $\Delta$ depends on the depth variation. However, (2.36) implies that for a given bathymetry, we can find a $\Delta$ so that $u$ and $v$ are bounded for all $\xi > 0$, and (2.12) is a linearly stable solution of (2.2).

### 2.6 Summary

In this chapter, we examined the evolution of surface gravity waves with narrow banded spectra that propagate primarily in one direction propagating and that are influenced by slowly varying bathymetry. We found a variable-depth Stokes solution for which bathymetry causes the amplitude to vary like $[c_g(\xi)]^{-1/2}$. Thus, the
amplitude increases/decreases when the wavetrain enters deeper/shallower water. We examined the stability of this solution: a growing amplitude due to bathymetry may enhance modulational instability. However, the presence of viscous damping will cause all perturbations to be have a bounded growth, regardless of the effects of bathymetry. We considered a particular model for viscous dissipation and found a case for which growth due to bathymetry and stabilization due to viscosity balance. For that special case, the solution is unstable.

Numerical simulations of the “uniform” amplitude solution propagating on water of variable depth are included and discussed in Chapter 4.
Chapter 3

Asymptotic Analysis of Cnoidal Wave Solutions

3.1 Introduction

The propagation of gravity waves on water of variable depth is an interesting problem in applied mathematics. It was first studied by Carrier [19], who analyzed the nonlinear beach-climbing problem up to wave breaking. He analyzed the problem for both one-dimensional and two-dimensional bathymetries. The variable depth problem was further explored by Grimshaw [47, 48] who studied the deformation of a solitary wave due to slow variation of the bottom topography. He predicted the inverse variation of wave amplitudes with depth, for small amplitudes. He also derived the differential equations which determine the slow variation of parameters and solved them for one-dimensional bottom topography. Other contributors of the results in wave propagation on a variable bottom include Bampi & Morro [7], who discussed gravity waves propagating on variable depth, Turpin [95] who studied the effects of depth on a Stokes wavepacket, Iusim [59], who studied shoaling of wavegroups on water of variable depth, Liu & Dingemans [68], who worked on multiple scale perturbation methods for waves over an uneven bottom, Madson & Sorenson [70], who predicted a new form of the Boussinesq equation applicable to irregular wave propagation on a slowly varying bathymetry, Matsuno [73], who studied and derived the evolution equations for surface gravity waves propagation on variable depth, and confirmed existing theories, Artiles [6], who examined finite-
amplitude surface gravity waves on a highly variable finite depth, and Magne et al. [71], who studied the propagation of surface gravity waves over steep topography. Dingemans & Otto [31] and Dingemans [32] also discuss waves over uneven bottoms in their books. In addition, Wu et al. [108], Guyenne [51], Benilov et al. [9, 10] and Grimshaw & Annenkov [49] analyzed the problem numerically.

In this chapter, we look for amplitude envelopes that are periodic solutions of the model equation. Exact solutions exist for particular coefficient dependences (See Zhang [107]). However, the required dependences do not apply to water waves. So, we look for asymptotic solutions. We also obtain the limiting case of a soliton solution.

### 3.2 The Variable Depth Problem

The inviscid MNLS equation governing the evolution of a surface gravity wave propagating on water of variable depth in a stationary reference frame, is

\[
i \left[ A_x + \frac{1}{c_g(x)} A_t + \mu(x) A \right] + \alpha(x)A_{tt} + \beta(x)|A|^2A = 0, \tag{3.1}
\]

where

\[
c_g(x) = \frac{g}{2\omega} \left[ \tanh(kh) + kh \operatorname{sech}^2(kh) \right], \tag{3.2}
\]

\[
\mu(x) = \frac{(1 - \sigma^2)(1 - k\sigma)}{\sigma + k h (1 - \sigma^2)} \frac{d(kh)}{dx}, \tag{3.3}
\]

\[
\alpha(x) = \frac{-1}{2\omega c_g} \left[ 1 - \frac{gh}{c_g^2} (1 - \sigma^2)^2 (1 - k\sigma) \right], \tag{3.4}
\]

\[
\beta(x) = \frac{-g^2 k^4}{4\omega^3 \sigma^2 c_g} \left[ 9 - 10\sigma^2 + 9\sigma^4 - \frac{2\sigma^2}{gh - c_g} \left( 4c_p^2 + 4c_p c_g (1 - \sigma^2) + gh (1 - \sigma^2)^2 \right) \right], \tag{3.5}
\]

where \(\sigma = \tanh(kh)\), \(A(x, t)\) is a slowly varying function representing the surface displacement of the envelope of the carrier wave, and \((x, t)\) represent the physical space and time. These equations are introduced in §1.4 in a slightly different form. We restrict our analysis to one dimension, where the depth, \(h\), depends on a single horizontal variable, \(x\), as is the case in waves propagating in an ocean. Unlike the
formulation in §1.4, which is with respect to a traveling reference frame, (3.1) is in a stationary reference frame, and the coefficients are slowly varying functions of the space coordinate, \( x \). Since there is no temporal variation in the bathymetry, the wavenumber, \( k \), may be determined implicitly from the dispersion relation

\[
\omega^2 = gk(x) \tanh \left[ k(x) h(x) \right].
\]  

(3.6)

Our attention is for the case where the spatial length scale of the depth variation is much greater than the wavelength of the wave. Hence, to make the depth a slowly varying function of the spatial variable, we make the transformation

\[
\xi = \epsilon x; \quad \tau = t - \int \left[ \frac{1}{c(x)} + \nu(x) \right] dx,
\]  

(3.7)

where \( \epsilon \ll 1 \) and \( \nu \) is an arbitrary function. Using (3.7) in (3.1), we obtain

\[
\left[ \epsilon A_{\xi} - \nu(\xi) A_{\tau} + \epsilon \mu(\xi) A \right] + \alpha(\xi) A_{\tau \tau} + \beta(\xi) |A|^2 A = 0.
\]  

(3.8)

We now look for a uniform depth solution (\( \alpha, \beta, \nu \) are constants and \( \mu \equiv 0 \)) of the form

\[
A(\xi, \tau) = A_0 f(\lambda \tau) \exp \left[ i (c_1 \tau + c_2 \xi) \right].
\]  

(3.9)

Substituting (3.9) in (3.8), we obtain

\[
-\epsilon f c_2 - i \nu \lambda f' + \nu c_1 f + \alpha \lambda^2 f'' + 2i c_1 \alpha \lambda f' - \alpha c_1^2 f + \beta |A_0|^2 f^3 = 0.
\]  

(3.10)

Comparing the real and imaginary parts of (3.10), we obtain

\[
c_1 = \frac{\nu}{2\alpha},
\]  

(3.11)

and

\[
-\epsilon f c_2 + \nu c_1 f + \alpha \lambda^2 f'' - \alpha c_1^2 f + \beta |A_0|^2 f^3 = 0.
\]  

(3.12)

Rearranging (3.12), we obtain

\[
f'' = \left[ \frac{\epsilon c_2}{\alpha \lambda^2} - \frac{\nu^2}{4 \alpha^2 \lambda^2} \right] f - \frac{\beta |A_0|^2}{\alpha \lambda^2} f^3.
\]  

(3.13)
Integrating (3.13) once, we obtain

$$(f')^2 = \left[ \frac{cc_2}{\alpha \lambda^2} - \frac{\nu^2}{4\alpha^2 \lambda^2} \right] f^2 - \frac{\beta |A_0|^2}{2\alpha \lambda^2} f^4 + c_3. \quad (3.14)$$

The constants $c_2$ and $|A_0|$ in (3.13) are listed in table 3.2. In §3.3, we find solutions to (3.14).

### 3.3 Jacobi-Elliptic Functions as Solutions to the Nonlinear Wave Problem

Jacobi-Elliptic functions are widely used as solutions to the nonlinear equations in water wave theory such as the NLS, the KdV [60] and the Benjamin-Bona-Mahony [11] equations. Several researchers including Fenton [39, 40], Agnon & Pelinovsky and Benjamin & Lighthill [14] have identified the cnoidal structure of water waves described by the NLS and other nonlinear equations. Using analytical and numerical techniques, Agnon & Pelinovsky [4] discuss the disintegration of cnoidal waves described by the variable coefficient KdV equation, when they propagate over a smooth topography. Additionally, Liu et al. [69], Fan [37], Fan & Zheng [38] and Parkes & Duffy [83] propose and discuss some Jacobi-elliptic function methods to construct exact periodic solutions of nonlinear wave equations. Likewise, Yan [101] illustrates the Jacobi-elliptic function method for the coupled DS equation in (2+1) dimensions, and Zhou et al. [109] applies the same for coupled variable-coefficient KdV equations. Jacobi-elliptic functions are used to construct exact solutions to other nonlinear equations such as the combined KdV-mKdV equation as well [61]. The definition of commonly used Jacobi-elliptic functions is provided in Appendix B.

Stability results on cnoidal wave solutions include those of Carter & Segur [21], who showed that every one-dimensional traveling wave solution of the NLS equation with trivial phase is unstable with respect to some infinitesimal perturbation with two dimensional structure, and Carter & Deconinck [20] who establish asymptotically that these one-dimensional, periodic, traveling wave solutions with a trivial phase are unstable with respect to two dimensional perturbations with
long wavelengths, and that they are unstable to perturbations with arbitrarily short wavelengths as well, if the coefficients of the linear dispersive terms in the two-dimensional cubic NLS equation (analogous to (1.1)) have opposite signs. Deconinck and Lovit [27] prove the completeness of Jacobi-elliptic functions and use these functions to decompose a time series of data, instead of using Fourier modes.

Equation (3.14) can be compared to the nonlinear equations for various Jacobi-elliptic functions listed in table 3.1. We find that there are four possible Jacobi-elliptic function solutions for equation (3.14), namely the cn, dn, nd and sd functions. The nature of equation (3.13) suggests that there might also be a solution of the form \( f \sim a(m) \text{nd}(\lambda \tau; m) + b(m) \text{dn}(\lambda \tau; m) \), where \( a \) and \( b \) are appropriate functions of \( m \). Consequently, it turns out that \( Y = \sqrt{1 - m^2} \text{nd}(\lambda \tau; m) + \text{dn}(\lambda \tau; m) \) satisfies the nonlinear equation

\[
Y'' = \left[ 2 + 6\sqrt{1 - m^2} - m^2 \right] Y - 2Y^3. \tag{3.15}
\]

Therefore, there are five exact solutions to (3.13), namely,

i. \( f = \text{cn}(\lambda \tau; m) \),

ii. \( f = \text{dn}(\lambda \tau; m) \),

### Table 3.1. Differential equations for various Jacobi-elliptic functions.

<table>
<thead>
<tr>
<th>Function</th>
<th>Differential Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Y = \text{sn}, \text{cd} )</td>
<td>((Y')^2 = 1 - (1 + m^2)Y^2 + m^2Y^4)</td>
</tr>
<tr>
<td>( Y = \text{cn} )</td>
<td>((Y')^2 = (1 - m^2) + (2m^2 - 1)Y^2 - m^2Y^4)</td>
</tr>
<tr>
<td>( Y = \text{dn} )</td>
<td>((Y')^2 = (m^2 - 1) + (2 - m^2)Y^2 - Y^4)</td>
</tr>
<tr>
<td>( Y = \text{ns}, \text{dc} )</td>
<td>((Y')^2 = m^2 - (1 + m^2)Y^2 + Y^4)</td>
</tr>
<tr>
<td>( Y = \text{nc} )</td>
<td>((Y')^2 = -m^2 + (2m^2 - 1)Y^2 + (1 - m^2)Y^4)</td>
</tr>
<tr>
<td>( Y = \text{nd} )</td>
<td>((Y')^2 = 1 + (2 - m^2)Y^2 - (1 - m^2)Y^4)</td>
</tr>
<tr>
<td>( Y = \text{sc} )</td>
<td>((Y')^2 = 1 - 1 + (2 - m^2)Y^2 + (1 - m^2)Y^4)</td>
</tr>
<tr>
<td>( Y = \text{sd} )</td>
<td>((Y')^2 = 1 + (2m^2 - 1)Y^2 - m^2(1 - m^2)Y^4)</td>
</tr>
<tr>
<td>( Y = \text{cs} )</td>
<td>((Y')^2 = (1 - m^2) + (2 - m^2)Y^2 + Y^4)</td>
</tr>
<tr>
<td>( Y = \text{ds} )</td>
<td>((Y')^2 = -m^2(1 - m^2) + (2m^2 - 1)Y^2 + Y^4)</td>
</tr>
</tbody>
</table>
iii. \( f = \text{nd}(\lambda \tau; m) \),

iv. \( f = \text{sd}(\lambda \tau; m) \), and

v. \( f = \sqrt{1 - m^2} \text{nd}(\lambda \tau; m) + \text{dn}(\lambda \tau; m) \).

Comparing (3.13) and (3.14) with the nonlinear differential equations satisfied by these five functions, we obtain the values of the constants \( c_2 \) and \( |A_0| \); they are listed in table 3.2.

**Table 3.2.** The constants \( c_2 \) and \( |A_0| \) in (3.13).

| \( f, J \)     | \( c_2 \)                                                                 | \( |A_0| \)                              |
|---------------|---------------------------------------------------------------------------|------------------------------------------|
| \( \text{cn}(\lambda \tau; m) \) | \( \frac{1}{\epsilon} \left[ (2m^2 - 1)\alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right] \) | \( m\lambda \sqrt{\frac{2\alpha}{\beta}} \) |
| \( \text{dn}(\lambda \tau; m) \) | \( \frac{1}{\epsilon} \left[ (2 - m^2)\alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right] \) | \( \lambda \sqrt{\frac{2\alpha}{\beta}} \) |
| \( \text{nd}(\lambda \tau; m) \) | \( \frac{1}{\epsilon} \left[ (2 - m^2)\alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right] \) | \( \lambda \sqrt{\frac{2\alpha(1-m^2)}{\beta}} \) |
| \( \text{sd}(\lambda \tau; m) \) | \( \frac{1}{\epsilon} \left[ (2m^2 - 1)\alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right] \) | \( m\lambda \sqrt{\frac{2\alpha(1-m^2)}{\beta}} \) |
| \( \sqrt{1 - m^2} \text{nd}(\lambda \tau; m) + \text{dn}(\lambda \tau; m) \) | \( \frac{1}{\epsilon} \left[ (2 + 6\sqrt{1 - m^2} - m^2)\alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right] \) | \( \lambda \sqrt{\frac{2\alpha}{\beta}} \) |

Using \( A_0 = |A_0|\exp(i\varphi) \), the uniform depth solution may be written as

\[
A(x, \tau) = |A_0|J(\lambda \tau; m)\exp\left[i\left(\frac{\nu}{2\alpha} \tau + c_2 x\right)\right]\exp(i\varphi),
\]

where \( J \) is one of the five Jacobi-elliptic function solutions and \( \varphi \) is the phase of the amplitude \( A_0 \). For \( J = \text{cn}(\lambda \tau; m), \text{dn}(\lambda \tau; m) \) and \( \sqrt{1 - m^2} \text{nd}(\lambda \tau; m) + \text{dn}(\lambda \tau; m) \), we obtain the soliton (sech) solution in the limit \( m \to 1 \),

\[
A(x, \tau) = \lambda \sqrt{\frac{2\alpha}{\beta}} \text{sech}(\lambda \tau)\exp\left[i\left(\frac{\nu}{2\alpha} \tau + \left[ \alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right] x\right)\right]\exp(i\varphi).
\]
3.4 Asymptotic Solution

In this section, we find an asymptotic solution for one of Jacobi-elliptic functions, the cn function, following the procedure outlined by Benilov et al. [9]. Equation (3.16) is the solution of the NLS equation for uniform depth. Building on this, consider a solution in which the amplitude varies slowly, of the form

\[ A(\xi, \tau) = B(\xi, \tau) \exp \left[ i \left( \frac{\nu}{2\alpha} \tau + \frac{1}{\epsilon} \int \left( 2m^2 - 1 \right) \frac{\alpha}{\epsilon} + \frac{\nu^2}{4\alpha} d\xi \right) \right]. \]  

(3.18)

Using (3.18) in (3.8), we obtain

\[ -\alpha \lambda^2 (2m^2 - 1) B + \alpha B_{\tau\tau} + \beta |B|^2 B = \epsilon \left[ -iB_{\xi} + \tau B \frac{d}{d\xi} \left( \frac{\nu}{2\alpha} \right) - i\mu B \right]. \]  

(3.19)

Letting \( m \to 1 \), we obtain the equation corresponding to the soliton solution,

\[ -\alpha \lambda^2 B + \alpha B_{\tau\tau} + \beta |B|^2 B = \epsilon \left[ -iB_{\xi} + \tau B \frac{d}{d\xi} \left( \frac{\nu}{2\alpha} \right) - i\mu B \right]. \]  

(3.20)

Expanding \( B \) in a series, and retaining terms linear in \( \epsilon \), we obtain

\[ B(\xi, \tau) = B_0(\xi, \tau) + \epsilon B_1(\xi, \tau) + O(\epsilon^2). \]  

(3.21)

Substituting (3.21) in (3.19), we obtain

\[ -\alpha \lambda^2 (2m^2 - 1) B_0 - \epsilon \alpha \lambda^2 (2m^2 - 1) B_1 + \alpha (B_0)_{\tau\tau} + \epsilon \alpha (B_1)_{\tau\tau} + \beta |B_0|^2 B_0 \]

\[ + \epsilon \beta \left[ 2B_1 |B_0|^2 + B_1^* B_0^2 \right] = \]

\[ \epsilon \left[ -i(B_0)_{\xi} + \tau B_0 \frac{d}{d\xi} \left( \frac{\nu}{2\alpha} \right) - i\mu B_0 \right] + O(\epsilon^2). \]  

(3.22)

Comparing \( \epsilon^0 \) and \( \epsilon^1 \) terms, we obtain

\[ -\alpha \lambda^2 (2m^2 - 1) B_0 + \alpha (B_0)_{\tau\tau} + \beta |B_0|^2 B_0 = 0, \]  

(3.23)
and
\[-\alpha \lambda^2 (2m^2 - 1) B_1 + \alpha (B_1)_{\tau\tau} + \beta \left[ 2B_1 |B_0|^2 + B_1^* B_0^2 \right] = \left[ -i (B_0)\xi + \tau B_0 \frac{d}{d\xi} \left( \frac{\nu}{2\alpha} \right) - \nu B_0 \right]. \tag{3.24}\]

A solution to (3.23) is
\[B_0(\xi, \tau) = m\lambda \sqrt{\frac{2\alpha}{\beta}} \ cn(\lambda \tau; m) \in \mathbb{R}, \tag{3.25}\]
which corresponds to a periodic envelope solution. Note that the period of the Jacobi cn function is \(D = \frac{4K(m)}{\lambda}\), where \(K(m)\) is the complete elliptic integral of the first kind \([65, 98, 18, 3]\), described in Appendix B. To allow both the amplitude and phase of \(B_1\) to vary slowly with \(X\), let \(B_1(\xi, \tau) = B_{1r}(\xi, \tau) + iB_{1i}(\xi, \tau)\). We separate the real and imaginary parts of (3.24),
\[-\alpha \lambda^2 (2m^2 - 1) B_{1r} + \alpha (B_{1r})_{\tau\tau} + 3\beta |B_0|^2 B_{1r} = \tau B_0 \frac{d}{d\xi} \left( \frac{\nu}{2\alpha} \right), \tag{3.26}\]
and
\[-\alpha \lambda^2 (2m^2 - 1) B_{1i} + \alpha (B_{1i})_{\tau\tau} + \beta |B_0|^2 B_{1i} = -\left[ (B_0)\xi + \nu B_0 \right]. \tag{3.27}\]

Consider the homogeneous versions of (3.26) and (3.27),
\[-\alpha \lambda^2 (2m^2 - 1) B_{1r} + \alpha (B_{1r})_{\tau\tau} + 3\beta |B_0|^2 B_{1r} = 0, \tag{3.28}\]
and
\[-\alpha \lambda^2 (2m^2 - 1) B_{1i} + \alpha (B_{1i})_{\tau\tau} + \beta |B_0|^2 B_{1i} = 0. \tag{3.29}\]
Differentiating (3.23) with respect to \(\tau\), and using \(|B_0|^2 = (B_0)^2\), we obtain
\[-\alpha \lambda^2 (2m^2 - 1) (B_0)_\tau + \alpha (B_0)_{\tau\tau\tau} + 3\beta (B_0)^2 (B_0)_\tau = 0. \tag{3.30}\]
Comparing (3.28) with (3.30) and (3.23) with (3.29), we obtain

\[ B_{1r} = (B_0)_r \quad \text{and} \quad B_{1i} = B_0, \]  

(3.31)
since they satisfy the same linear operator. The operators in the linear non-homogeneous ordinary differential equations (3.26) and (3.27) are self-adjoint. Therefore, using Fredholm Alternative theorem, we could say that the equations (3.26) and (3.27) have bounded solutions if and only if their right-hand sides are orthogonal to the solutions of their homogeneous versions, i.e. (3.28) and (3.29) respectively. Applying the theorem, it follows that

\[ \int_{-D/2}^{D/2} \tau B_0 \frac{d}{d\xi} \left( \frac{\nu}{2\alpha} \right) (B_0)_r d\tau = 0, \]  

(3.32)

and

\[ \int_{-D/2}^{D/2} \left[ (B_0)_\xi + \mu B_0 \right] B_0 d\tau = 0, \]  

(3.33)

where \( D \) is a period. Equation (3.32) implies that

\[ \frac{d}{d\xi} \left( \frac{\nu}{2\alpha} \right) \int_{-D/2}^{D/2} \tau \left[ (B_0)^2 \right]_\tau d\tau = 0, \]  

(3.34)

which yields

\[ \frac{d}{d\xi} \left( \frac{\nu}{2\alpha} \right) = 0, \]  

(3.35)

since \( \int_{-D/2}^{D/2} \tau \left[ (B_0)^2 \right]_\tau d\tau \neq 0 \). It follows from (3.35) that

\[ \frac{\nu}{\alpha} = C_1, \]  

(3.36)

where \( C_1 \) is a constant.
Equation (3.33) may be rewritten as

\[
\frac{d}{d\xi} \left[ \int_{-D/2}^{D/2} (B_0)^2 \, d\tau \right] + 2\mu(\xi) \int_{-D/2}^{D/2} (B_0)^2 \, d\tau = 0. \tag{3.37}
\]

Integrating (3.37), we obtain

\[
\text{Exp} \left[ \int 2\mu(\xi) \, d\xi \right] \int_{-D/2}^{D/2} (B_0)^2 \, d\tau = \text{constant}. \tag{3.38}
\]

Using (3.3) for \(\mu\) and (3.25) for \(B_0\) in (3.37), and integrating with Mathematica [99], we obtain

\[
\frac{m\lambda\alpha}{\beta} \left[ E(m) + (m - 1)K(m) \right] \left[ \tanh(kh) + kh \text{sech}^2(kh) \right] = C_2, \tag{3.39}
\]

where all parameters are functions of \(\xi\), \(C_2\) is a constant, and \(E(.)\) is the complete elliptic integral of the second kind [65, 98, 18, 3], described in Appendix B. Equation (3.39) is the asymptotic solution for the cn function. It predicts the evolution of the parameters \(\lambda\) and \(m\), which determine the amplitude of the cnoidal wave. We have two parameters here. So, to find a unique solution, we need another equation that relates \(\lambda\) and \(m\). To close the system, we assume that the period of the cnoidal wave stays constant throughout the propagation. Based on this assumption, we have two coupled equations in \(\lambda\) and \(m\), namely, (3.39) and the relation between the period of the wavepacket, \(D\), and the parameters \(\lambda\) and \(m\), which is \(D = \frac{4K(m)}{\lambda}\) for the Jacobi-cn function. Therefore, the parameters \(\lambda\) and \(m\) are uniquely determined from these two equations. The agreement between the asymptotic solution and the numerically computed solution is included in Chapter 4.

Equation (3.36) predicts the variation of the parameter \(\nu\), which affects the phase and velocity of the cnoidal wave. The asymptotic solution can also be obtained from the conservation equations, as outlined in §3.5. Corresponding results for the other solutions of the cnoidal wave can be obtained in a similar way. They are

\[
\frac{\lambda\alpha}{\beta} E(m) \left[ \tanh(kh) + kh \text{sech}^2(kh) \right] = C_3, \tag{3.40}
\]
for the Jacobi-dn function, $B_0(\xi, \tau) = \lambda \sqrt{\frac{2\alpha}{\beta}} \, \text{dn} (\lambda \tau; m)$, with $D = \frac{2K(m)}{\lambda}$,

$$
\frac{\lambda \alpha}{\beta} (1 + m) E(m) \left[ \tanh(kh) + kh \text{sech}^2(kh) \right] = C_4,
$$

(3.41)

for the Jacobi-nd function, $B_0(\xi, \tau) = \lambda \sqrt{\frac{2\alpha(1-m^2)}{\beta}} \, \text{nd} (\lambda \tau; m)$ with $D = \frac{2K(m)}{\lambda}$,

$$
m(1 + m) \lambda \alpha \left[ (1 - m) K(m) - E(m) \right] \left[ \tanh(kh) + kh \text{sech}^2(kh) \right] = C_5,
$$

(3.42)

for the Jacobi-sd function, $B_0(\xi, \tau) = m \lambda \sqrt{\frac{2\alpha(1-m^2)}{\beta}} \, \text{sd} (\lambda \tau; m)$ with $D = \frac{4K(m)}{\lambda}$, and

$$
\frac{\lambda \alpha}{\beta} \left[ (m + 2) E(m) + 2 \left( \sqrt{1 - m^2} \right) K(m) \right] \left[ \tanh(kh) + kh \text{sech}^2(kh) \right] = C_6,
$$

(3.43)

for the function $B_0(\xi, \tau) = \lambda \sqrt{\frac{2\alpha}{\beta}} \left[ \sqrt{1 - m^2} \, \text{nd}(\lambda \tau; m) + \text{dn}(\lambda \tau; m) \right]$ with $D = \frac{2K(m)}{\lambda}$, where $C_3, C_4, C_5,$ and $C_6$ are constants. In all cases, the parameters $\lambda$ and $m$ are uniquely determined from the two coupled equations, the asymptotic equation and the corresponding equation relating the period of the wave, $D$, and the parameters $\lambda$ and $m$.

Equations (3.39-3.43) are the asymptotic results for the five Jacobi-elliptic function solutions of the variable-coefficient nonlinear wave equation. Results for the soliton solution $B_0(\xi, \tau) = \lambda \sqrt{\frac{2\alpha}{\beta}} \, \text{sech} (\lambda \tau)$ may be obtained by letting $m \to 1$ in (3.39), (3.40) and (3.43). Letting $m \to 1$, we obtain

$$
\frac{\lambda \alpha}{\beta} \left[ \tanh(kh) + kh \text{sech}^2(kh) \right] = C_7,
$$

(3.44)

where $C_7$ is a constant. This result agrees with that of Benilov et al. [9]. The soliton solution is revisited in Appendix C, where we also fix some errors in the paper by Benilov et al. [9]. Figures showing the agreement between the asymptotic solution and the numerical solution for a solitary wave propagating on water of variable depth are included in Appendix C, for propagation from deeper to shallower water and vice versa.
3.5 Obtaining the Asymptotic Solution from the Conserved Quantities of the MNLS

In this section, we derive the asymptotic solution for the cn function from the conservation equation. The variable-coefficient MNLS equation (3.3) has two conservation equations (see Appendix A for derivation), namely,

\[ \frac{d}{d\xi} \left[ \int_{-D/2}^{D/2} c_g(\xi)|A|^2 d\tau \right] = 0 \]  \hspace{1cm} (3.45)

and

\[ \frac{d}{d\xi} \left[ \int_{-D/2}^{D/2} c_g(\xi)(AA^*-A^*A)d\tau \right] = 0. \]  \hspace{1cm} (3.46)

Assuming that the major contribution to the momentum and energy comes from the leading-order solution, the Jacobi-cn function satisfying (3.1) is

\[ A(\xi,\tau) = m\lambda \sqrt{\frac{2\alpha}{\beta}} \text{cn}(\lambda\tau;m) \text{Exp} \left[ i \left( \frac{\nu}{2\alpha} \tau + \frac{1}{\epsilon} \int \left( 2m^2 - 1 \right) \alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right) d\xi \right]. \]  \hspace{1cm} (3.47)

From (3.47) and (3.45), it follows that

\[ \frac{d}{d\xi} \left[ m^2\lambda^2\nu c_g \int_{-D/2}^{D/2} \text{cn}^2(\lambda\tau;m)d\tau \right] = 0. \]  \hspace{1cm} (3.48)

Integrating with \( D = \frac{4K(m)}{\lambda} \), we obtain

\[ \frac{m\lambda\nu c_g}{\beta} \left[ E(m) + 2(m-1)K(m) \right] = \text{constant}. \]  \hspace{1cm} (3.49)

Then, using (3.2) for the group velocity \( c_g \), we obtain

\[ \frac{m\lambda\nu}{\beta} \left[ E(m) + 2(m-1)K(m) \right] \left[ \tanh(kh) + kh \text{sech}^2(kh) \right] = C_8. \]  \hspace{1cm} (3.50)
Similarly, (3.47) and (3.48) yield

$$\frac{m^2\lambda^2\alpha_c^2}{\beta} \left[ E(m) + 2(m - 1)K(m) \right] = C_9, \quad (3.51)$$

which may be combined with (3.2) to obtain

$$\frac{m\lambda\alpha}{\beta} \left[ E(m) + 2(m - 1)K(m) \right] \left[ \tanh(kh) + kh \text{sech}^2(kh) \right] = C_{10}. \quad (3.52)$$

Equation (3.52) is the same as (3.39). Now, (3.50) and (3.52) yield

$$\frac{\nu}{\alpha} = C_{11}, \quad (3.53)$$

which agrees with (3.36). In the equations above, \((C_8, C_9, C_{10}, C_{11})\) are all real constants. Similar results are applicable for the other four Jacobi-elliptic function solutions of the MNLS equation.
Chapter 4

Numerical Analysis of the Modified Nonlinear Schrödinger Equation

4.1 Introduction

Analytical solutions to differential equations are not always possible. Quite often, it is necessary to rely on other sources such as experiments and numerics. The numerical analysis of differential equations involves deriving equations in discrete space, and solving them by numerical techniques, such as iterative methods, time marching, finite difference methods, finite volume methods and pseudo-spectral methods. Solving a differential equation numerically involves issues such as stability and convergence, as well as aliasing [88], if it uses pseudo-spectral methods. Occurring when a continuous function is discretized, aliasing is the indistinguishability of high frequency components from lower ones on a discrete mesh. Hence, high frequency components which result due to the interaction of the nonlinear terms, will be interpreted as lower frequency components at later times, and hence lead to erroneous solutions. These errors result because the discretized function has a finite spectral bandwidth \((-\pi/\Delta\tau, \pi/\Delta\tau)\), while the continuous function has an infinite spectral bandwidth. In such cases, the solution algorithm should involve dealiasing. Numerical techniques for nonlinear equations and issues such as stability and aliasing are discussed in Roger & Peyret [85], Bengt & Fornberg [41].
and Trefethen [94] and Yang [102].

The numerical solution of nonlinear differential equations is a very popular area of research. The differential equations for wave propagation need special attention because of the nonlinear term and traditional methods may fail to give accurate results for such problems. It was this need that propelled people to look for alternative methods for nonlinear equations. Orszag [81] was one of the pioneers that laid the groundwork for such alternative methods that involves the use of the fast Fourier transform. He is credited with the development of spectral and pseudo-spectral methods, which he used for the numerical simulation of the nonlinear Navier-Stokes equations applicable to turbulence in fluid flow. The spectral and pseudo-spectral methods have since been predominantly used and developed by researchers. Fornberg and Whitham [43] used the pseudo-spectral method to study some nonlinear wave phenomena. They did not observe any aliasing errors in their solution unlike Schamel & Elsässer [88], who observed catastrophic effects due to aliasing interactions in the application of pseudo-spectral methods to numerically simulate the KdV-Burgers equation. Other applications and uses of the pseudo-spectral method may be found in Pathria & Morris [84], Sharmadan [90], Fornberg & Driscoll [42], Milewski et al. [78], Benilov et al. [9, 10], Wu et al. [108], Deghan & Taleei [28], Guyenne [51] and Grimshaw & Amennkov [49]. Furthermore, several others have also analyzed the NLS equation and other nonlinear water wave equations using other numerical methods, including Driscoll [35], who proposed a new composite Runge-Kutta method for semi-linear PDEs, Hox & Vadillo [25], who used the exponential time differencing method for NLS, Dereli & Dağ [29], who examined soliton solutions for the NLS equation using radial basis functions, Segur et al. [89] and Henderson et al. [57] who used the operator splitting algorithm to simulate the NLS equation numerically.

4.2 The Discretized Equations

In this section, we discretize the MNLS equation and write it in a form suitable for numerical evolution. The MNLS equation with damping is

\[
i \left[ A_\xi + \mu(\xi)A + \delta(\xi)A \right] + \alpha(\xi)A_{\tau\tau} + \beta(\xi)|A|^2A = 0.
\]
We define the Fourier transform of $A$ as

$$\mathcal{F}[A] = \hat{A} = \int_{-T}^{T} A(\xi, \tau) \exp(\imath l \tau) \, d\tau,$$  \hspace{1cm} (4.2)

where $T$ is a half-period. Now, taking the Fourier transform of (4.1), we obtain

$$\imath \left[ \hat{A}_\xi + \left\{ (\mu(\xi) + \delta(\xi)) \hat{A} \right\} - l^2 \alpha(\xi) \hat{A} + \beta(\xi) \mathcal{F}[|A|^2 A] \right] = 0.$$ \hspace{1cm} (4.3)

Rearranging, we obtain

$$\hat{A}_\xi = - \left[ (\mu(\xi) + \delta(\xi)) \hat{A} + l^2 \alpha(\xi) \hat{A} - \imath \beta(\xi) \mathcal{F}[|A|^2 A] \right] = 0,$$ \hspace{1cm} (4.4)

which may be re-written in terms of the inverse Fourier transform,

$$\hat{A}_\xi = \left[ (\mu(\xi) + \delta(\xi)) \hat{A} + l^2 \alpha(\xi) \hat{A} - \imath \beta(\xi) \mathcal{F}[|A|^2 A] \right] = 0.$$ \hspace{1cm} (4.5)

Equation (4.5) describes the evolution of the complex Fourier amplitude of the envelope of the wave. When in discrete form, the variable $A$ represents the complex amplitude of the envelope of the wave at discrete nodal points in the (time) grid. Our solutions-of-interest to the nonlinear wave equations are all periodic in time. This periodicity condition allows us to use pseudo-spectral method for the time derivatives. The discovery of the fast Fourier transform (FFT) algorithms has made the use of the pseudo-spectral method practical. Pseudo-spectral methods involve swapping between real space (grid representation) and the Fourier space (transform representation) to simplify the treatment of the nonlinear terms, which might otherwise involve costly convolutions. When evolving in space, every step in (4.5) involves an FFT, an inverse FFT and three point-wise multiplications. For a one-dimensional wave solution approximated by $N$ points, the point-wise multiplications require $O(N)$ effort and both the FFT and inverse FFT require $O(N \log N)$ effort for each step. Thus, the total computational cost is largely determined by the FFT and inverse FFT, so it is important that we use an efficient and accurate implementation of the FFT. In all of our programs, we use the FFTW algorithm
developed by Frigo & Johnson [46], incorporated in MATLAB [72].

We note that (4.5) is a stiff system due to the presence of linear terms. The stiffness of the system and stability issues in the explicit stepping methods limited the usage of large steps in space. For evolution in space, we used the standard fourth order Runge-Kutta method. The space steps could then be decided using the stability criterion for the standard fourth order Runge-Kutta method, illustrated below.

The stability criterion dictates that the eigenvalues of the linearized time discretization operator, scaled by $\Delta \xi$, must lie in the stability region of the (space) stepping scheme [94]. For (4.5), the critical eigenvalues of the linearized time discretization operator (analogous to the $i(\tau)$ term) are $\pm i\pi^2/(2\Delta \tau^2)$. The extrema on the imaginary axis for the fourth order Runge-Kutta scheme are $\pm i2\sqrt{2}$ [92, 17]. Therefore, the critical eigenvalues of the linearized time discretization operator, scaled by $\Delta \xi$, should lie within these extrema for the Runge-Kutta method. Then,

$$\frac{\Delta \xi}{(2\Delta \tau^2)} \leq \frac{2\sqrt{2}}{\pi^2}. \quad (4.6)$$

Using (4.6) and based on our trials with a few test cases, we used $\Delta \xi = 0.2(\Delta \tau)^2$ for all of our simulations. The number of points in the (time) grid, $N$, was chosen to be between $2^{10}$ and $2^{13}$. Fox and Orszag [44] and Fornberg & Whitham [43] did not see errors due to aliasing in their numerical solution. Similarly, we did not observe any qualitative errors in our numerical solutions due to aliasing. Hence, our numerical solution procedure did not require dealiasing. While simulating the inviscid MNLS equation, we also monitored the conserved quantities, and ensured that our numerical solutions satisfy (2.5a) and (2.5b).

### 4.3 Numerical Considerations

We show the results of numerical simulation for four solutions to the MNLS, namely

i. the “uniform” wave amplitude solution,

$$A(\xi, \tau) = A_0(\xi, \tau) + \sum_{j=-n}^{n} A_j(\xi, \tau) \exp [i (\omega_j - \omega_0)] , \quad (4.7)$$
where $A_0$ is the Fourier amplitude of the carrier wave at frequency $\omega_0$ and $(A_j, A_{-j})$ are the Fourier amplitudes of the $j$th sideband at frequencies $(\omega_0 + \omega_j, \omega_0 - \omega_j)$.

**ii. the soliton solution,**

$$A(X, \tau) = \lambda \sqrt{\frac{2\alpha}{\beta}} \text{sech}(\lambda \tau; m) \exp \left[ i \left( \frac{\nu}{2\alpha} \tau + \frac{1}{\epsilon} \left[ \alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right] X \right) \right], \quad (4.8)$$

**iii. the Jacobi-dn function solution**

$$A(X, \tau) = \lambda \sqrt{\frac{2\alpha}{\beta}} \text{dn}(\lambda \tau; m) \exp \left[ i \left( \frac{\nu}{2\alpha} \tau + \frac{1}{\epsilon} \left[ (2-m^2)\alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right] X \right) \right], \quad (4.9)$$

and

**iv. the Jacobi-dn-nd function solution**

$$A(X, \tau) = \lambda \sqrt{\frac{2\alpha}{\beta}} \left( \sqrt{1-m^2} \text{nd}(\lambda \tau; m) + \text{dn}(\lambda \tau; m) \right) \exp \left[ i \left( \frac{\nu}{2\alpha} \tau + \frac{1}{\epsilon} \left[ \left( 2 + 6\sqrt{1-m^2} - m^2 \right) \alpha \lambda^2 + \frac{\nu^2}{4\alpha} \right] X \right) \right]. \quad (4.10)$$

The simulations were performed for the following cases

**i.** wave propagation in infinite depth with dissipation,

**ii.** wave propagation in the forward direction (deeper water to shallower water) on a non-uniform “linear” beach with viscous dissipation,

**iii.** wave propagation in the forward direction on a non-uniform “linear” beach without viscous dissipation,

**iv.** wave propagation in the reverse direction (shallower water to deeper water) on a non-uniform “linear” beach with viscous dissipation,

**v.** wave propagation in the reverse direction on a non-uniform “linear” beach without viscous dissipation,
vi. wave propagation in the forward direction on a non-uniform nonlinear beach with viscous dissipation,

vii. wave propagation in the forward direction on a non-uniform nonlinear beach without viscous dissipation,

viii. wave propagation in the reverse direction on a non-uniform nonlinear beach with viscous dissipation, and

ix. wave propagation in the reverse direction on a non-uniform nonlinear beach without viscous dissipation.

The profiles of the “linear” and nonlinear beaches are described in §4.4. Results are produced for all these cases for each of the “uniform” amplitude solution, the Jacobi-dn solution and the Jacobi-dn-nd solution. Results for the soliton solution are in dimensionless coordinates for ease of comparison with results of Benilov et al. [9], and are hence placed in Appendix C. For the Jacobi-dn and Jacobi-dn-nd solutions, we considered two cases, with \( m_0 = 0.99; \nu_0 = 0 \) and \( m_0 = 0.999; \nu_0 = 0.0001 \text{ s cm}^{-1} \), where \( m_0 = m(\xi = 0) \) and \( \nu_0 = \nu(\xi = 0) \). The initial conditions for the simulations are

i. 

\[
A(\tau) = \sum_{j=-3}^{3} A_j \text{Exp}[i b_j \tau],
\]

(4.11)

for the “uniform” amplitude solution, with (values obtained from Segur et al. [89]) \( A_0 = -0.0578 + 0.0916i, A_1 = -0.0093 - 0.0133i, A_{-1} = -0.0035 - 0.0139i, A_2 = 0.0028 - 0.0011i, A_{-2} = -0.0003 + 0.0024i, A_3 = (2.53 + 3.77i) \times 10^{-4}, A_{-3} = (-3.69 + 2.09i) \times 10^{-4} \text{ cm}; b_0 = 0, b_{\pm 1} = \pm 1.0738, b_{\pm 2} = \pm 2.1476 \text{ and } b_{\pm 3} = \pm 3.2214 /s,

ii. 

\[
A(\tau) = \lambda_0 \sqrt{\frac{2\alpha_0}{\beta_0}} \text{dn}(\lambda_0 \tau; m_0) \text{Exp} \left[ i \frac{\nu_0}{2\alpha_0} \tau \right],
\]

(4.12)

for the Jacobi-dn function solution, where \( \lambda_0 = 2E(m_0)/D_0, \alpha_0 = \alpha(\xi = 0), \beta_0 = \beta(\xi = 0) \) and \( D_0 \) is the fundamental period of the Jacobi-dn solution at \( \xi = 0 \), with
a. \( m_0 = 0.99 \) (smaller modulation) and \( \nu_0 = 0 \),

b. \( m_0 = 0.999 \) (larger modulation) and \( \nu_0 = 0.0001 \) s.cm\(^{-1} \)

and

\[ A(\tau) = \lambda_0 \sqrt{\frac{2\alpha_0}{\beta_0}} \left( \sqrt{1 - (m_0)^2} \operatorname{nd}(\lambda_0 \tau; m_0) + \operatorname{dn}(\lambda_0 \tau; m_0) \right) \exp \left[ i \frac{\nu_0}{2\alpha_0} \tau \right], \]  

for the Jacobi-dn-nd solution, where \( \lambda_0 = 2E(m_0)/D_0 \), \( \alpha_0 = \alpha(\xi = 0) \), \( \beta_0 = \beta(\xi = 0) \) and \( D_0 \) is the fundamental period of the Jacobi-dn-nd solution at \( \xi = 0 \), with \( m_0 = 0.99 \) and \( \nu_0 = 0 \).

We also compare the results of the numerical solution to the asymptotic results derived in Chapter 3.

### 4.4 Beach Profiles

The results obtained in chapters 2 and 3 hold for arbitrary depth with \( kh > 1.363 \). For purposes of numerical simulation, we consider a “linear” and a nonlinear beach with respective profiles

\[ h(\xi) = 6 - \frac{1}{60} \log \left\{ \frac{\cosh [0.3(\xi - 100)]}{\cosh [0.3(\xi - 700)]} \right\}, \]  

\[ h(\xi) = 6 - 3 \tanh \left[ 0.01(\xi - 400) \right], \]

both for \( 0 \leq \xi \leq 900 \) cm. These beaches have been designed for use in a wave tank in the laboratory. The beach profiles are shown in figures 4.1 and 4.2 and the corresponding variation of the coefficients in the MNLS equation (4.1) are plotted in figures 4.3 and 4.4.
4.5 Discussion

With the beach profiles described in §4.4, we computed the numerical solutions for various cases, presented in figures 4.5 - 4.29. Numerical solutions have been computed for propagation with and without the effects of viscous dissipation. The numerical results are compared to the asymptotic solutions derived in Chapter 3 for the Jacobi-dn function. We discuss these results in §4.5.1 - §4.5.3.

4.5.1 The “Uniform” Amplitude Wave

In Chapter 2, we discussed the stability of the “uniform” amplitude solution for propagation on variable depth. We concluded that viscous dissipation limits the growth of all perturbations, irrespective of the effects of bathymetry. Results for the “uniform” amplitude solution are shown in figures 4.5 - 4.9. Figure 4.5 shows the variation of the sideband amplitudes with distance, with and without viscous dissipation. As expected, the viscous NLS model predicts the evolution of the Fourier amplitudes more accurately than the inviscid model. The results with viscous dissipation are compared to experimental data from Segur et al. [89] (See their figure 8). We observe that our numerical results agree with theirs, and as they showed, the numerical results with dissipation agree well with their experimental data. We used their numerical and experimental results as a benchmark to validate our numerics.

The results in figure 4.5 are for a uniform bottom, and hence, the propagating wavetrain is necessarily a Stokes wavetrain with a uniform amplitude and constant group velocity (unlike its nonuniform-bottom counterpart)

\(^1\). The results for the uniform bottom confirm that the small perturbations in the form of sideband modes of the Stokes wavetrain grow until viscous dissipation bounds them in a finite distance before nonlinear interactions can become important. Here, we are interested in seeing how the “uniform” amplitude wavetrain behaves to disturbances in the form of sideband modes when it propagates on water of variable depth. In this case, as described in Chapter 2, the amplitude of the wavetrain is not uniform, and varies as the inverse root of the local group velocity. As shown in figures 4.3

\(^1\)Segur et al. [89] assume infinite depth. Here we use the actual depth of \( h = 20 \text{ cm} \) that gives \( k h \approx 9 \), so that our coefficients in the NLS equation agree with their infinite depth limits.
and 4.4, the group velocity increases as the waves propagate into shallow water. Similarly, the group velocity decreases for reverse propagation. Figures 4.6 - 4.9 show the sideband amplitudes of the shoaling wavetrain for propagation on the “linear” beach and the nonlinear beach, entering deep water from shallow water and vice versa. In all cases, the Fourier amplitudes are greater in the case without viscous dissipation than in the case with it. For forward propagation, since $\mu > 0$, the dissipation due to the variable depth adds to the viscous dissipation. This implies that the sideband amplitudes have to eventually decrease (even without viscous dissipation) and stay bounded throughout the propagation. Alternatively, for reverse propagation, $\mu < 0$, and opposes the effects of viscous dissipation. So, for waves propagating into deeper water, bathymetry acts to destabilize the variable-depth Stokes solution. This explains the difference in behavior of the Fourier amplitudes for forward and reverse propagation. However, eventually, viscous dissipation succeeds in overcoming the growth due to bathymetry, and the sideband amplitudes remain small as shown in figures 4.8 and 4.9. For the special case $\mu + \delta = 0$, the behavior is dictated by the inviscid infinite depth NLS equation (1.1). The differences in the sideband amplitudes for the nonuniform bottom (figures 4.6 - 4.9) and those for the infinite depth (figure 4.5) should be measurable in the laboratory.

These figures show that the inviscid MNLS model predicts that the Fourier amplitudes of the unstable sidebands grow up to a point, reach a maximum, and then decrease. Those maximum amplitudes are when the nonlinear interactions become dominant and start playing a role, thereby causing the amplitudes to decrease. With non-zero viscosity, the MNLS model predicts that viscous dissipation (irrespective of the effects due to variable depth) retards the growth of the unstable sidebands and bounds them, not allowing them to grow large enough for the nonlinear interactions to become important. Therefore, viscous dissipation eventually bounds the growth of the unstable sidebands for all cases except one, in which the growth due to variable depth exactly cancels out the decay due to viscous dissipation, thereby leading to instability. This balance is however difficult to maintain in real-life situations, and small departures affect the instability.
4.5.2 The Jacobi-dn Cnoidal Wave

Results for the Jacobi-dn function are shown in figures 4.10 - 4.23. The Jacobi-dn function is a periodic solution of the MNLS equation, and is unstable, like the “uniform” amplitude solution. From table 3.2, we note that the maximum amplitude of the Jacobi-dn function is \(|A_0| = \lambda \sqrt{2\alpha/\beta}\). This maximum amplitude is plotted as a function of distance in figures 4.10 - 4.14, for both forward and reverse propagation on the “linear” beach and the nonlinear beach, with and without the effects of dissipation. Further, we also consider the effect of a smaller initial modulation \((m_0 = 0.99)\) and a larger initial modulation \((m_0 = 0.999)\). As shown in figure 4.10, the maximum amplitude is a function of \(m\) (since \(\lambda\) is a function of \(m\)) and is greater for a larger \(m\) (larger modulation). The results confirm that for these cases, the amplitude of the wavepacket decays in the presence of viscous dissipation, for a uniform bottom.

Figures 4.11 - 4.14 show the amplitudes of the cnoidal wave propagating on water of variable depth. As observed before, the dissipative effect due to the variable depth adds to the viscous dissipation while the wavepacket travels from deeper water to shallower water and the growth due to variable depth opposes the decay due to viscous dissipation as the wavepacket travels into deeper water. So, similar to the “uniform” amplitude wavetrain, the amplitudes of the cnoidal waves decrease (with or without viscous dissipation) as the wave packet moves into shallower water, as shown in figures 4.11 and 4.12. As expected, the decay in amplitude is greater in the presence of viscous dissipation, than that predicted by inviscid models. For reverse propagation, the variable depth aids in amplitude growth, but since viscous dissipation eventually conquers this growth effect, the amplitudes decrease eventually and stay bounded. However, the inviscid model predicts that the amplitude growth is initially monotonic, and then switches to an oscillatory behavior, with an effective increase in the mean, as shown in figures 4.13 and 4.14. This oscillatory behavior marks the onset of instability, and the cnoidal waves become unstable as they propagate into deep water. Note that the oscillations are also present in the case with viscous dissipation, but are very mild. These oscillations eventually disappear due to damping by viscous dissipation. For analytical results on the cnoidal waves, see Carter & Segur [21].

The parameters \(\lambda\) and \(m\) may be obtained as functions of distance from the
coupled system including the asymptotic solution (3.40) and the equation for the period, \( D = 2K(m)/\lambda \). The parameter \( \lambda \) may also be obtained numerically by \( \lambda = |A_0|\sqrt{\beta/2\alpha} \), where \( |A_0| \) is the maximum amplitude for the Jacobi-dn function, plotted in figures 4.10 - 4.14. The constant \( C_3 \) in (3.40) may be obtained from initial conditions and is listed in table 4.1 for the different cases considered. Figures 4.16 - 4.23 show the asymptotic and numerical solutions for the cases listed in table 4.1. The figures show that the asymptotic solutions do not agree with the numerical solutions beyond a point, which indicates that the asymptotic solutions are not valid for laboratory scales. The range of validity of the asymptotic solution is determined by the ordering parameter \( \epsilon \), in (3.7), which in turn depends on the dimensions of the bathymetry and of the wave packet itself. The ordering parameter \( \epsilon \) is a measure of the ratio of the length scales of the wave packet and the depth variation, and varies as \( \epsilon \sim \frac{2\pi k^{\frac{1}{3}}}{L_h} \), where \( L_h \) is the characteristic length scale for the depth variation. The “linear” and nonlinear beaches in (4.14) and (4.15) are designed to fit in a wavetank in a laboratory. For the beaches we considered, \( \epsilon \sim O(10^{-1}) \). This relatively higher magnitude of \( \epsilon \) is possibly the reason for the difference between the asymptotic and numerical solutions. We expect the asymptotic solution to agree with the numerical solution for lesser values of the ordering parameter \( \epsilon \). The beach described in Appendix C is appropriate for the ocean, and in this case, \( \epsilon \sim O(10^{-3}) \). Hence, as observed in figures C.1 - C.4, the asymptotic and numerical solutions agree to a better extent for the solitary wave propagating on this beach.

4.5.3 The Jacobi-dn-nd Cnoidal Wave

The results for the Jacobi-dn-nd cnoidal wave shown in figures 4.24 - 4.28 are similar to the corresponding results for the Jacobi-dn cnoidal wave. In figures 4.24 - 4.26, we see that the amplitude of the cnoidal wave decays as it propagates forward (irrespective of viscous effects) and exhibits an oscillatory behavior as it propagates backward into deep water. The oscillations for the Jacobi-dn-nd waves are more predominant than those for the corresponding Jacobi-dn wave counterparts. However, the mean in the oscillations increase for the case without dissipation, suggesting a transition to instability as the shoaling waves enter deeper
water. When viscous dissipation effects are included, the oscillations decay and the instability is bounded, so the waves are stable in the presence of viscous dissipation, however small, irrespective of the decay/growth due to variable depth. Figure 4.29 shows the evolution of the Jacobi-dn-nd cnoidal wave as it propagates into deep water, without the inclusion of viscous effects. As seen in the figure, the amplitudes of the periodic wave increase rapidly from an initial amplitude close to 0.4 cm to final amplitudes thrice as high. Beyond a point, envelope fission occurs, and the wave loses its cnoidal structure and becomes more narrow and spiky in shape. As the wave still moves into deeper water, the amplitudes oscillate more and the wave becomes unstable.

4.6 Summary

In this chapter, we have numerically simulated the evolution of the three different solutions of the variable coefficient MNLS equation. The “uniform” amplitude wave and two different cnoidal waves were examined as they propagate on water of variable depth with two types of depth variations, a “linear” beach and a non-linear beach. We considered cases with and without viscous dissipation. For all the three wave solutions, the effect of variable depth aids in amplitude decay, and stabilizes the unstable modes (for the “uniform” amplitude solution), when the waves propagate into shallower water. In this case, the inclusion of viscous dissipation simply increases the rate of decay. For propagation into deeper water, the variable depth by itself is unable to stabilize the waves, as its effect is now reversed and helps in the growth of the waves. In this case, the waves become unstable in the absence of viscous dissipation, with the periodic cnoidal waves exhibiting an oscillatory behavior. Viscous dissipation, if included for propagation into deeper water, will eventually succeed in controlling the growth of these waves, and keep them bounded. Thus, viscous dissipation, however small, stabilizes the waves as they propagate into deeper water.
Figure 4.1. The “linear” beach.

Figure 4.2. The nonlinear beach.
Figure 4.3. Variation of parameters for the “linear” beach.
Figure 4.4. Variation of parameters for the nonlinear beach.
Figure 4.5. The sideband amplitudes on a uniform beach as functions of distance, amplified by $\text{Exp}[\delta \xi]$ to filter out the overall decay, with viscous dissipation (solid curves), and without viscous dissipation (dotted curves). Experimental data (dots) taken from Segur et al. [89].
Figure 4.6. The sideband amplitudes for propagation in the forward direction on the “linear” beach, as functions of distance, with viscous dissipation (solid curves) and without viscous dissipation (dotted curves).
Figure 4.7. The sideband amplitudes for propagation in the forward direction on the nonlinear beach, as functions of distance, with viscous dissipation (solid curves) and without viscous dissipation (dotted curves).
Figure 4.8. The sideband amplitudes for propagation in the reverse direction on the “linear” beach, as functions of distance, with viscous dissipation (solid curves) and without viscous dissipation (dotted curves).
Figure 4.9. The sideband amplitudes for propagation in the reverse direction on the nonlinear beach, as functions of distance, with viscous dissipation (solid curves) and without viscous dissipation (dotted curves).
Figure 4.10. The amplitude of the Jacobi-dn function as a function of distance, for propagation on a uniform beach, with viscous dissipation. (a) $m_0 = 0.99$ and $\nu_0 = 0$. (b) $m_0 = 0.999$ and $\nu_0 = 0.0001 \text{ s.cm}^{-1}$.
Figure 4.11. The amplitude of the Jacobi-dn function as a function of distance, for forward propagation on the “linear” beach, with viscous dissipation (solid curve) and without viscous dissipation (dotted curve). (a) $m_0 = 0.99$ and $\nu_0 = 0$. (b) $m_0 = 0.999$ and $\nu_0 = 0.0001$ s.cm$^{-1}$. 
Figure 4.12. The amplitude of the Jacobi-dn function as a function of distance, for forward propagation on the nonlinear beach, with viscous dissipation (solid curve) and without viscous dissipation (dotted curve). (a) $m_0 = 0.99$ and $\nu_0 = 0$. (b) $m_0 = 0.999$ and $\nu_0 = 0.0001 \text{s.cm}^{-1}$. 
Figure 4.13. The amplitude of the Jacobi-dn function as a function of distance, for reverse propagation on the “linear” beach, with viscous dissipation (solid curve) and without viscous dissipation (dotted curve). (a) $m_0 = 0.99$ and $\nu_0 = 0$. (b) $m_0 = 0.999$ and $\nu_0 = 0.0001$ s.cm$^{-1}$. 
Figure 4.14. The amplitude of the Jacobi-dn function as a function of distance, for reverse propagation on the nonlinear beach, with viscous dissipation (solid curve) and without viscous dissipation (dotted curve). (a) $m_0 = 0.99$ and $\nu_0 = 0$. (b) $m_0 = 0.999$ and $\nu_0 = 0.0001 \text{ s.cm}^{-1}$. 
**Figure 4.15.** The evolution of the Jacobi-dn function in the forward direction on the “linear” beach, without viscous dissipation for $m_0 = 0.99$ and $n_0 = 0$.

**Table 4.1.** Cases considered for the asymptotic solution of the Jacobi-dn function.

<table>
<thead>
<tr>
<th>Case</th>
<th>Beach</th>
<th>Direction</th>
<th>$m_0$</th>
<th>$\lambda_0$</th>
<th>$\alpha_0$</th>
<th>$\beta_0$</th>
<th>$C_3$ (3.40)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>“linear”</td>
<td>forward</td>
<td>0.99</td>
<td>1.176</td>
<td>$-0.001078$</td>
<td>$-0.2628$</td>
<td>0.00492</td>
</tr>
<tr>
<td>2</td>
<td>“linear”</td>
<td>forward</td>
<td>0.999</td>
<td>1.541</td>
<td>$-0.001078$</td>
<td>$-0.2628$</td>
<td>0.00636</td>
</tr>
<tr>
<td>3</td>
<td>nonlinear</td>
<td>forward</td>
<td>0.99</td>
<td>1.176</td>
<td>$-0.001078$</td>
<td>$-0.2628$</td>
<td>0.00492</td>
</tr>
<tr>
<td>4</td>
<td>nonlinear</td>
<td>forward</td>
<td>0.999</td>
<td>1.541</td>
<td>$-0.001078$</td>
<td>$-0.2628$</td>
<td>0.00636</td>
</tr>
<tr>
<td>5</td>
<td>“linear”</td>
<td>reverse</td>
<td>0.99</td>
<td>1.176</td>
<td>$-0.001079$</td>
<td>$-0.0541$</td>
<td>0.02808</td>
</tr>
<tr>
<td>6</td>
<td>“linear”</td>
<td>reverse</td>
<td>0.999</td>
<td>1.541</td>
<td>$-0.001079$</td>
<td>$-0.0541$</td>
<td>0.03628</td>
</tr>
<tr>
<td>7</td>
<td>nonlinear</td>
<td>reverse</td>
<td>0.99</td>
<td>1.176</td>
<td>$-0.001079$</td>
<td>$-0.0541$</td>
<td>0.02808</td>
</tr>
<tr>
<td>8</td>
<td>nonlinear</td>
<td>reverse</td>
<td>0.999</td>
<td>1.541</td>
<td>$-0.001079$</td>
<td>$-0.0541$</td>
<td>0.03628</td>
</tr>
</tbody>
</table>
Figure 4.16. Asymptotic solution (dotted curve) and numerical solution (solid curve) for the Jacobi-dn function, case #1 of table 4.1.

Figure 4.17. Asymptotic solution (dotted curve) and numerical solution (solid curve) for the Jacobi-dn function, case #2 of table 4.1.
Figure 4.18. Asymptotic solution (dotted curve) and numerical solution (solid curve) for the Jacobi-dn function, case #3 of table 4.1.

Figure 4.19. Asymptotic solution (dotted curve) and numerical solution (solid curve) for the Jacobi-dn function, case #4 of table 4.1.
Figure 4.20. Asymptotic solution (dotted curve) and numerical solution (solid curve) for the Jacobi-dn function, case #5 of table 4.1.

Figure 4.21. Asymptotic solution (dotted curve) and numerical solution (solid curve) for the Jacobi-dn function, case #6 of table 4.1.
Figure 4.22. Asymptotic solution (dotted curve) and numerical solution (solid curve) for the Jacobi-dn function, case #7 of table 4.1.

Figure 4.23. Asymptotic solution (dotted curve) and numerical solution (solid curve) for the Jacobi-dn function, case #8 of table 4.1.
Figure 4.24. The amplitude of the Jacobi-dn-nd function as a function of distance, for forward propagation on the “linear” beach, with viscous dissipation (solid curve) and without viscous dissipation (dotted curve), for $m_0 = 0.99$ and $\nu_0 = 0$.

Figure 4.25. The amplitude of the Jacobi-dn-nd function as a function of distance, for forward propagation on the nonlinear beach, with viscous dissipation (solid curve) and without viscous dissipation (dotted curve), for $m_0 = 0.99$ and $\nu_0 = 0$. 
Figure 4.26. The amplitude of the Jacobi-dn-nd function as a function of distance, for reverse propagation on the “linear” beach, with viscous dissipation (solid curve) and without viscous dissipation (dotted curve), for $m_0 = 0.99$ and $n_0 = 0$.

Figure 4.27. The amplitude of the Jacobi-dn-nd function as a function of distance, for reverse propagation on the nonlinear beach, with viscous dissipation (solid curve) and without viscous dissipation (dotted curve), for $m_0 = 0.99$ and $n_0 = 0$. 
Figure 4.28. The amplitude of the Jacobi-dn-nd function (for $m_0 = 0.99$ and $\nu_0 = 0$) as a function of distance, for propagation on a uniform beach, with viscous dissipation.

Figure 4.29. The evolution of the Jacobi-dn-nd function in the reverse direction on the nonlinear beach, without viscous dissipation for $m_0 = 0.99$ and $\nu_0 = 0$. 
Chapter 5

Closure

In this work, we have examined the propagation of nonlinear waves on water of variable depth through a combination of stability, asymptotic and numerical analyses. The two major factors we considered are viscous dissipation and direction of propagation. We considered four different solutions of the MNLS equation, namely, a “uniform” amplitude solution, a soliton solution, and two different cnoidal wave solutions. As a first approximation, we considered a simple linear depth variation, corresponding to a model that is applicable in experiments in a laboratory wavetank, followed by a more complex, nonlinear bathymetry.

The major motivation for this work comes from previous studies on stabilizing the Benjamin-Feir instability by Segur et al. [89] and on the propagation of a solitary wave over an uneven bottom by Benilov et al. [9]. Here, we have extended the results of these two studies to include the stability of a “uniform” amplitude wavetrain (analogous to the Stokes wavetrain on a uniform bottom) as it propagates on water of variable depth (Chapter 2), the asymptotic solution for periodic cnoidal waves (Chapter 3), the effect of viscous dissipation and the direction of propagation, and numerical simulations of the four different solutions (Chapter 4).

The results from all our studies assert that viscous dissipation always plays a major role in the stability of waves propagating in deep water. For propagation into shallower water, the variable depth effects take up the role of viscous dissipation in stabilizing the waves. So, for a variable depth, waves traveling into shallower water are always stable. The converse applies only in the presence of viscous dissipation, since the variable depth effects are now reversed. Thus, vis-
cous dissipation, however small, limits the growth of the unstable waves before nonlinear interactions become important. However, in the presence of dissipation, if the growth due to the variable depth effects are exactly equal and opposite to the decay due to viscous dissipation \((\mu + \delta = 0)\), then these waves are unstable. This case can only occur for waves entering deeper water. Therefore, unless these two effects are equal, the presence of dissipation always ensures stability in a finite distance of propagation. In the absence of viscous dissipation for waves entering deeper water, they are unstable.

The solitary wave is distinctive and a special case of the solutions of the MNLS equation, because of its stable nature unlike the periodic cnoidal waves and the uniform amplitude wave. The solitary wave propagating into shallower water with an uneven bottom was analyzed by Benilov et al. \[9\]. We extend their results to wave propagation into deeper water, to confirm if the soliton is stable in the absence of dissipation. As they described, the amplitude of the solitary wave decreases when it travels into shallower water and identically increases as it travels into deeper water, unlike shallow water solitons. The results for the soliton are included in Appendix C. The solitary wave is stable when it propagates into deeper water, even in the absence of viscous dissipation.

We have attempted to examine issues such as the effect of dissipation combined with variable depth on wave propagation, the stability of waves propagating on variable depth, asymptotic analysis for different solutions to the MNLS equation, and the range of validity of an asymptotic solution. As a continuation of this work, it would be worthwhile to compare the numerical results with experimental data for wave propagation on water of variable depth. These results have made initial predictions that could be directly tested in experiments. It might be of interest to obtain data corresponding to a real beach in the ocean and mimic it in the laboratory, and interpret the small-scale results in an oceanic context, with prospects of understanding real water waves in nature.
Appendix A

Derivation of the Conserved Quantities of the Modified Nonlinear Schrödinger Equation

A.1 Energy Integral

The modified nonlinear Schrödinger (MNLS) equation is

\[ i \left[ A_x + \frac{1}{c_g(x)} A_t + \mu(x) A \right] + \alpha(x) A_{tt} + \beta(x) |A|^2 A = 0. \quad (A.1) \]

Multiplying (A.1) by the complex conjugate of the amplitude, \( A^* \), we obtain

\[ i \left[ A^* A_x + \frac{1}{c_g(x)} A^* A_t + \mu(x) A^* A \right] + \alpha(x) A^* A_{tt} + \beta(x) A^* |A|^2 A = 0. \quad (A.2) \]

Taking the complex conjugate of (A.2), we obtain

\[ -i \left[ A(A^*)_x + \frac{1}{c_g(x)} A(A^*)_t + \mu(x) AA^* \right] + \alpha(x) A(A^*)_{tt} + \beta(x) A|A|^2 A^* = 0. \quad (A.3) \]
Subtracting (A.3) from (A.2), we obtain
\[ i \left[ A^* A_x + A(A^*)_x + \frac{1}{c_g(x)} (A^* A_t + A(A^*)_t) + 2\mu(x) A^* A \right] + \alpha(x) \left[ A^* A_{tt} - A(A^*)_t \right] = 0, \]

which may be re-written as
\[ i \left[ (AA^*)_x + \frac{1}{c_g(x)} (AA^*)_t + 2\mu(x) A^* A \right] + \alpha(x) \left[ A^* A_t - A(A^*)_t \right] = 0. \] (A.5)

Using the fact that \( AA^* = |A|^2 \), we obtain
\[ i \left[ (|A|^2)_x + \frac{1}{c_g(x)} (|A|^2)_t + 2\mu(x)|A|^2 \right] + \alpha(x) \left[ A^* A_t - A(A^*)_t \right] = 0. \] (A.6)

Equation (A.6) may be integrated in \( t \), so that
\[ i \int_{-T}^{T} (|A|^2)_x \, dt + \frac{i}{c_g(x)} \int_{-T}^{T} (|A|^2)_t \, dt + 2i\mu(x) \int_{-T}^{T} |A|^2 \, dt + \alpha(x) \int_{-T}^{T} \left[ A^* A_t - A(A^*)_t \right] \, dt = 0, \] (A.7)

where \( T \) is a half-period. For a periodic wave, \{\( A, A^* \}(x, T) = \{\( A, A^* \}(x, -T) \) and \( \{\( A, A^* \}_t(x, T) = \{\( A, A^* \}_t(x, -T) \). Using these conditions, we obtain
\[ \int_{-T}^{T} \left[ \frac{\partial |A|^2}{\partial x} + 2\mu |A|^2 \right] dt = 0. \] (A.8)

Using \( \mu = \frac{1}{2c_g} \left( \frac{dc_g}{dx} \right) \) from (1.17), we obtain
\[ \int_{-T}^{T} \left[ \frac{\partial |A|^2}{\partial x} + \frac{1}{c_g} \frac{dc_g}{dx} |A|^2 \right] dt = 0, \] (A.9)
which may be rearranged to obtain

\[
\int_{-T}^{T} \left[ c_g \frac{\partial |A|^2}{\partial x} + \frac{dc_g}{dx} |A|^2 \right] dt = 0. \tag{A.10}
\]

Equation (A.10) may be rewritten as

\[
\frac{d}{dx} \left[ \int_{-T}^{T} c_g |A|^2 \, dt \right] = 0, \tag{A.11}
\]

which is the conservation equation involving the energy integral for the MNLS Equation. In the presence of viscous dissipation, (A.1) is modified to

\[
i \left[ A_x + \frac{1}{c_g(x)} A_t + \left( \mu(x) + \delta(x) \right) A \right] + \alpha(x) A_{tt} + \beta(x) |A|^2 A = 0. \tag{A.12}
\]

Then in that case, the conservation equation modifies to

\[
\frac{d}{dx} \left[ \int_{-T}^{T} c_g |A|^2 \, dt \right] = -2 \delta(x) \int_{-T}^{T} c_g |A|^2 \, dt, \tag{A.13}
\]

which may be integrated to obtain

\[
M_A(x) = M_A(0) \exp \left[ -2 \int_x^x \delta(s) \, ds \right], \tag{A.14}
\]

where \( M_A(x) = \int_{-T}^{T} c_g(x) |A(x,t)|^2 \, dt \).
A.2 Momentum Integral

To derive the momentum integral equation, we first take the complex conjugate of (A.1) to obtain

\[-i \left( (A^*)_x + \frac{1}{c_g(x)} (A^*)_t + \mu(x)(A^*) \right) + \alpha(x)(A^*)_tt + \beta(x)|A|^2 A^* = 0. \quad (A.15)\]

Multiplying (A.1) by \((A^*)_t\), we obtain

\[i \left[ A_x (A^*)_t + \frac{1}{c_g(x)} A_t (A^*)_t + \mu(x)A(A^*)_t \right] + \alpha(x)A_{tt}(A^*)_t + \beta(x)|A|^2 A(A^*)_t = 0. \quad (A.16)\]

Multiplying (A.15) by \(A_t\), we obtain

\[-i \left( (A^*)_x A_t + \frac{1}{c_g(x)} (A^*)_t A_t + \mu(x)(A^*)A_t \right) + \alpha(x)(A^*)_{tt}A_t + \beta(x)|A|^2 A^*A_t = 0. \quad (A.17)\]

Adding (A.16) and (A.17), we obtain

\[i \left[ A_x (A^*)_t - (A^*)_x A_t \right] + i\mu(x) \left[ A(A^*)_t - (A^*)_t A_t \right] + \alpha(x)A_{tt}(A^*)_t + \beta(x) \left[ A(A^*)_t + (A^*)_t A_t \right] |A|^2 = 0. \quad (A.18)\]

Differentiating (A.1) and (A.15) with respect to \(t\), we obtain

\[i \left[ A_{xt} + \frac{1}{c_g(x)} A_{tt} + \mu(x)A_t \right] + \alpha(x)A_{ttt} + \beta(x) \left( |A|^2 A \right)_t = 0, \quad (A.19)\]

and

\[-i \left[ (A^*)_{xt} + \frac{1}{c_g(x)} (A^*)_{tt} + \mu(x)(A^*)_t \right] + \alpha(x)(A^*)_{ttt} + \beta(x) \left( |A|^2 A^* \right)_t = 0. \quad (A.20)\]

Multiplying (A.19) by \(A^*\) and (A.20) by \(A\), we obtain

\[i \left[ A_{xt} A^* + \frac{1}{c_g(x)} A_{tt} A^* + \mu(x)A_tA^* \right] + \alpha(x)A_{ttt} A^* + \beta(x) \left( |A|^2 A \right)_t A^* = 0, \quad (A.21)\]
and

\[ -i \left[ (A^*)_{xt} A + \frac{1}{c_g(x)} (A^*)_{tt} A + \mu(x) (A^*)_t A \right] + \alpha(x) (A^*)_{ttt} A + \beta(x) \left( |A|^2 A^* \right)_t A = 0. \]  

(A.22)

Adding (A.21) and (A.22), we obtain

\[ i \left[ A_{xt} A^* - (A^*)_{xt} A \right] + \frac{i}{c_g(x)} \left[ A_{tt} A^* - (A^*)_{tt} A \right] + i \mu(x) \left[ A_t A^* - (A^*)_t A \right] \]

\[ + \alpha(x) \left[ A_{ttt} A^* + (A^*)_{ttt} A \right] + \beta(x) \left[ \left( |A|^2 A \right)_t A^* + \left( |A|^2 A^* \right)_t A \right] = 0 \]

(A.23)

Subtracting (A.23) from (A.18), we obtain

\[ i \left[ A_x (A^*)_t - (A^*)_x A_t - A^* A_t - A(A^*)_{xt} \right] - \frac{i}{c_g(x)} \left[ A^* A_{tt} - (A^*)_t A \right] \]

\[ + i \mu(x) \left[ A (A^*)_t - A^* A_t - A^* A_t + A(A^*)_t \right] \]

\[ + \alpha(x) \left[ A_{tt} (A^*)_t + (A^*)_{tt} A_t - A^* A_{tt} - (A^*)_{ttt} \right] \]

\[ + \beta(x) \left[ A(A^*)_t |A|^2 + A^* A_t |A|^2 - A^* \left( |A|^2 \right)_t A - A \left( |A|^2 \right)_t A^* \right] = 0, \]

(A.24)

which may be rearranged to obtain

\[ i \left[ A(A^*)_t - A^* A_t \right]_x - \frac{i}{c_g(x)} \left[ A^* A_t - (A^*)_t A \right]_t + 2i \mu(x) \left[ (A^*)_t - A^* A_t \right]_t \]

\[ + 2 \alpha(x) \left[ A_t (A^*)_t \right]_t - \alpha(x) \left[ A(A^*)_{tt} + A^* A_{tt} \right]_t - \beta(x) \left( |A|^4 \right)_t = 0. \]

(A.25)

Integrating (A.25) over a period \(2T\), we obtain

\[ i \int_{-T}^{T} \left[ A(A^*)_t - A^* A_t \right]_x dt - \frac{i}{c_g(x)} \int_{-T}^{T} \left[ A^* A_t - (A^*)_t A \right]_t dt \]
Using the fact that the wave amplitude, $A$, its complex conjugate, $A^*$, and their time derivatives are all periodic, we obtain

$$
\int_{-T}^{T} \left( [A(A^*)_t - A^* A_t] + 2\mu(x) [A(A^*)_t - A^* A_t] \right) \, dt = 0,
$$

which when combined with the relation $\mu = \frac{1}{2c_g} \left( \frac{dc_g}{dx} \right)$, becomes

$$
\int_{-T}^{T} \left\{ \frac{c_g}{\partial x} \left[ A(A^*)_t - A^* A_t \right] + \left( \frac{dc_g}{dx} \right) \left[ A(A^*)_t - A^* A_t \right] \right\} \, dt = 0.
$$

Rearranging (A.28), we obtain

$$
\frac{d}{dx} \left[ \int_{-T}^{T} c_g \left( A(A^*)_t - A^* A_t \right) \, dt \right] = 0,
$$

which is the momentum integral equation for the MNLS equation (A.1). When viscous dissipation is added to the MNLS equation to obtain (A.12), the momentum integral equation (A.29) modifies to

$$
\frac{d}{dx} \left[ \int_{-T}^{T} c_g \left( A(A^*)_t - A^* A_t \right) \, dt \right] = -2\delta(x) \int_{-T}^{T} c_g \left( A(A^*)_t - A^* A_t \right) \, dt,
$$
which may be integrated to obtain

\[ P_A(x) = P_A(0)\text{Exp} \left[ -2 \int_0^x \delta(s) \, ds \right], \quad (A.31) \]

where \( P_A(x) = \int_{-T}^{T} c_a(x) \left( A(A^*)_t - A^* A_t \right) \, dt. \)
Appendix B

The Elliptic Integrals and the Jacobi Elliptic Functions

B.1 Definition

In this appendix, we briefly describe the commonly used Jacobi-elliptic functions and the elliptic integrals and list some of their properties. For a complete and detailed description of these functions, see any of Abramowitz & Stegun [3], Byrd & Friedman [18], Lawden [65] or Whittaker & Watson [98]. Let

\[ z = F(\phi, m) = \int_0^\phi \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}}, \]  

(B.1)

where \( m \) is the elliptic modulus, \( \phi \) is the Jacobi amplitude and \( F(\phi, m) \) is the incomplete elliptic integral of the first kind. Then, the jacobi-elliptic functions are defined as

\[ \text{sn}(z; m) = \sin \phi, \]  

(B.2)

\[ \text{cn}(z; m) = \cos \phi, \]  

(B.3)

\[ \text{dn}(z; m) = \sqrt{1 - m^2 \sin^2 \phi}. \]  

(B.4)
The Jacobi-elliptic functions have the following integral forms for their inverses

\[
\int_{0}^{z} \frac{d\rho}{\sqrt{(1-\rho^2)(1-m^2\rho^2)}} = \text{sn}^{-1}(z; m),
\]

(B.5)

\[
\int_{z}^{1} \frac{d\rho}{\sqrt{(1-\rho^2)(m'^2+m^2\rho^2)}} = \text{cn}^{-1}(z; m),
\]

(B.6)

\[
\int_{z}^{1} \frac{d\rho}{\sqrt{(1-\rho^2)(\rho^2-m'^2)}} = \text{dn}^{-1}(z; m),
\]

(B.7)

where \(m'\) is the complementary elliptic modulus defined by \(m'^2 = 1 - m^2\). The elliptic modulus may theoretically be any real or complex number, but here we restrict it to \(0 \leq m \leq 1\). The complete elliptic integral of the first kind is defined by

\[
K(m) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}}.
\]

(B.8)

Note that the incomplete elliptic integral of the first kind, \(F(\phi, m)\), defined in (B.1) becomes the complete elliptic integral of the first kind when \(\phi = \pi/2\). So, \(K(m) = F(\pi/2, m)\). The incomplete elliptic integral of the second kind, \(E(\phi, m)\), is defined by

\[
E(\phi, m) = \int_{0}^{\phi} \sqrt{1 - m^2 \sin^2 \theta} \, d\theta,
\]

(B.9)

and the complete elliptic integral of the second kind is obtained from \(E(\phi, m)\) when \(\phi = \pi/2\). Therefore, the complete elliptic integral of the second kind is

\[
E(m) = \int_{0}^{\pi/2} \sqrt{1 - m^2 \sin^2 \theta} \, d\theta,
\]

(B.10)

The complete elliptic integrals of the first and second kind are graphed in figure B.1. The periods of the Jacobi-elliptic functions are integral multiples of the complete elliptic integral of the first kind. The fundamental period of both the functions \(\text{sn}(z; m)\) and \(\text{cn}(z; m)\) is \(4K(m)\), while that of the function \(\text{dn}(z; m)\) is \(2K(m)\). The
other jacobi-elliptic functions derived from the three basic ones, and their periods are listed in table B.2.

![Graph](image.png)

**Figure B.1.** The complete elliptic integrals of the first and second kind.

### B.2 Specific Cases

The limiting values of the Jacobi-elliptic functions and the complete elliptic integrals are listed in tables B.2 and B.1 respectively. In the limit $m \to 0$, the Jacobi-elliptic functions tend to a constant value or a trigonometric function, while they tend to a constant value or a hyperbolic function in the limit $m \to 1$. Similarly, the complete integrals of the first and second kinds both tend to $\pi/2$ as $m \to 0$ and the former tends to $\infty$ and the latter to 1 in the limit $m \to 1$. Some specific values of the Jacobi-elliptic functions and the elliptic integrals are listed in table B.3. The jacobi elliptic functions for a few values of the parameter $m$ are shown in figure B.2.
Table B.1. The limiting values of the complete elliptic integrals

<table>
<thead>
<tr>
<th>Complete Elliptic Integral of</th>
<th>Function</th>
<th>$m = 0$</th>
<th>$m = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>the first kind</td>
<td>$K(m)$</td>
<td>$\pi/2$</td>
<td>$\infty$</td>
</tr>
<tr>
<td>the second kind</td>
<td>$E(m)$</td>
<td>$\pi/2$</td>
<td>1</td>
</tr>
</tbody>
</table>

Table B.2. Jacobi-elliptic functions, their periods and limiting values

<table>
<thead>
<tr>
<th>Function</th>
<th>Definition</th>
<th>Period</th>
<th>$m = 0$</th>
<th>$m = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$sn(z; m)$</td>
<td>(B.2)</td>
<td>$4K(m)$</td>
<td>$\sin(z)$</td>
<td>$\tanh(z)$</td>
</tr>
<tr>
<td>$cn(z; m)$</td>
<td>(B.3)</td>
<td>$4K(m)$</td>
<td>$\cos(z)$</td>
<td>$\sech(z)$</td>
</tr>
<tr>
<td>$dn(z; m)$</td>
<td>(B.4)</td>
<td>$2K(m)$</td>
<td>1</td>
<td>$\sech(z)$</td>
</tr>
<tr>
<td>$ns(z; m)$</td>
<td>$\frac{1}{sn(z; m)}$</td>
<td>$4K(m)$</td>
<td>$\cosec(z)$</td>
<td>$\coth(z)$</td>
</tr>
<tr>
<td>$nc(z; m)$</td>
<td>$\frac{1}{cn(z; m)}$</td>
<td>$4K(m)$</td>
<td>$\sec(z)$</td>
<td>$\cosh(z)$</td>
</tr>
<tr>
<td>$nd(z; m)$</td>
<td>$\frac{1}{dn(z; m)}$</td>
<td>$2K(m)$</td>
<td>1</td>
<td>$\cosh(z)$</td>
</tr>
<tr>
<td>$sc(z; m)$</td>
<td>$\frac{sn(z; m)}{cn(z; m)}$</td>
<td>$2K(m)$</td>
<td>$\tan(z)$</td>
<td>$\sinh(z)$</td>
</tr>
<tr>
<td>$cs(z; m)$</td>
<td>$\frac{cn(z; m)}{sn(z; m)}$</td>
<td>$2K(m)$</td>
<td>$\cot(z)$</td>
<td>$\cosec(z)$</td>
</tr>
<tr>
<td>$sd(z; m)$</td>
<td>$\frac{sn(z; m)}{dn(z; m)}$</td>
<td>$4K(m)$</td>
<td>$\sin(z)$</td>
<td>$\sinh(z)$</td>
</tr>
<tr>
<td>$ds(z; m)$</td>
<td>$\frac{dn(z; m)}{sn(z; m)}$</td>
<td>$4K(m)$</td>
<td>cosec(z)</td>
<td>cosech(z)</td>
</tr>
<tr>
<td>$cd(z; m)$</td>
<td>$\frac{cn(z; m)}{dn(z; m)}$</td>
<td>$4K(m)$</td>
<td>$\cos(z)$</td>
<td>1</td>
</tr>
<tr>
<td>$dc(z; m)$</td>
<td>$\frac{dn(z; m)}{cn(z; m)}$</td>
<td>$4K(m)$</td>
<td>$\sec(z)$</td>
<td>1</td>
</tr>
</tbody>
</table>
Table B.3. Specific values of the elliptic functions for $n \in \mathbb{Z}$

<table>
<thead>
<tr>
<th>Function</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{sn} \left[ 2nK(m); m \right]$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{sn} \left[ (2n + 1)K(m); m \right]$</td>
<td>$(-1)^n$</td>
</tr>
<tr>
<td>$\text{cn} \left[ 4nK(m); m \right]$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{cn} \left[ (4n + 2)K(m); m \right]$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$\text{cn} \left[ (2n + 1)K(m); m \right]$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{dn} \left[ 4nK(m); m \right]$</td>
<td>0</td>
</tr>
<tr>
<td>$\text{dn} \left[ (4n + 2)K(m); m \right]$</td>
<td>1</td>
</tr>
<tr>
<td>$\text{dn} \left[ (2n + 1)K(m); m \right]$</td>
<td>$\sqrt{1 - m}$</td>
</tr>
<tr>
<td>$\phi[nK(m); m]$</td>
<td>$n\pi/2$</td>
</tr>
<tr>
<td>$E[n\pi/2; m]$</td>
<td>$nE(m)$</td>
</tr>
</tbody>
</table>

Figure B.2. The Jacobi elliptic functions for specific values of $m$. 
Appendix C

The Solitary Wave Revisited

C.1 Introduction

Here we consider the evolution of a soliton on water of variable depth, with and without dissipation. The propagation of the solitary wave on a nonlinear beach, from deeper to shallower water was discussed by Benilov et al. [9]. Their results confirm that the amplitude of the shoaling packet decays as it propagates onto shallow water, which is contrary to the observed phenomenon in monochromatic waves and shallow-water solitons. We extrapolate Benilov’s results here, to include dissipation, and to explore if the solitary wave behaves exactly opposite when it travels in the opposite direction, i.e. from shallow water to deep water.

C.2 Dimensionless Equations

With the initial water depth, $h_0$ as a length scale and $\sqrt{g/h_0}$ as a time scale, we define a dimensionless system, in which the variable coefficient MNLS equation becomes

$$i\left[A_\tilde{x} + \frac{1}{c_g(\tilde{x})}A_t + \tilde{\mu}(\tilde{x})\tilde{A}\right] - \tilde{\alpha}(\tilde{x})\tilde{A}_{\tilde{t}\tilde{t}} - \tilde{\beta}(\tilde{x})|\tilde{A}|^2\tilde{A} = 0,$$  \hspace{1cm} (C.1)

with $(\tilde{\alpha}, \tilde{\beta}) > 0$. In (C.1), $(\cdot)$ represents corresponding dimensionless parameters, and the coefficients have been non-dimensionalized appropriately and listed in equation (2.4) of the paper by Benilov et al. [9]. However, we claim that the
dimensionless nonlinear coefficient, $\tilde{\beta}$ should be

$$\tilde{\beta}(\tilde{x}) = \frac{1}{2\tilde{\omega}^3 \tilde{c}_g} \left\{ \frac{9}{2} \left( \frac{\tilde{k}^6}{\tilde{\omega}^4} \right) - 6\tilde{k}^4 - \tilde{\omega}^8 + \frac{13}{2} \tilde{k}^2 \tilde{\omega}^4 - \left[ \frac{2\tilde{\omega}\tilde{k} - \tilde{c}_g(\tilde{\omega}^4 - \tilde{k}^2)}{\tilde{h} - \tilde{c}_g^2} \right]^2 \right\}, \quad (C.2)$$

rather than the one defined in equation (2.4) of Benilov’s paper [9]. We use (C.2) in our asymptotic analysis and numerical simulations. The uniform depth solution of (C.1), where the coefficients are all constants, is

$$\tilde{A}(\tilde{x}, \tilde{\tau}) = \tilde{\lambda} \sqrt{\frac{2\tilde{\alpha}}{\tilde{\beta}}} \text{sech} \left( \tilde{\lambda} \tilde{\tau} \right) \text{Exp} \left[ -i \left( \tilde{\nu} \tilde{\tau} + \left[ \tilde{\alpha} \tilde{\lambda}^2 + \tilde{\beta}^2 \right] \tilde{x} \right) \right], \quad (C.3)$$

instead of equation (2.6) of the paper. We note that (C.3) is the corresponding dimensionless version of (3.17), with $(\tilde{\alpha}, \tilde{\beta}) > 0$ and $\tilde{\tau}$ is the transformation corresponding to $\tau$, defined in (3.7). The two conserved quantities for (C.1), the energy and momentum integral, are derived in Appendix A. With the inclusion of viscous dissipation, (C.1) becomes

$$i \left[ \tilde{A}_\xi + \frac{1}{\tilde{c}_g(\tilde{\xi})} \tilde{A}_t + \left( \tilde{\mu}(\tilde{\xi}) + \tilde{\delta}(\tilde{\xi}) \right) \tilde{A} \right] - \tilde{\alpha}(\tilde{\xi}) \tilde{A}_{\tilde{h}} - \tilde{\beta}(\tilde{\xi}) |\tilde{A}|^2 \tilde{A} = 0, \quad (C.4)$$

where $\tilde{\xi} = \epsilon \tilde{x}$, and $\epsilon \ll 1$ is a small ordering parameter representing the ratio of the length scales of the solitary wave and the depth variation. We show results for the solitary wave propagating on a “linear” beach and a nonlinear beach, the profiles of which are given by

$$\tilde{h}(\tilde{\xi}) = 1 - \frac{1}{1200} \log \left\{ \frac{\cosh \left[ 0.08(\tilde{\xi} - 500) \right]}{\cosh \left[ 0.08(\tilde{\xi} - 3500) \right]} \right\}, \quad (C.5)$$

and

$$\tilde{h}(\tilde{\xi}) = 1 - 0.2 \tanh \left[ 0.002(\tilde{\xi} - 2000) \right], \quad (C.6)$$

for $0 < \tilde{\xi} < 6000$. These numbers are appropriate for propagation in an ocean, where dissipation has much lesser effect than in propagation in a wave tank in a laboratory as shown in table 2.1. Thus, for a characteristic depth, $h_0 \sim 100$ m, the
dimensionless damping coefficient is of the order of $10^{-5}$. Therefore, we assume the viscous dissipation to vary as

$$
\delta(\tilde{\xi}) = C_\delta \left[ \frac{3 + \cosh(2\tilde{k}\tilde{h})}{\{2\tilde{k}\tilde{h} + \sinh(2\tilde{k}\tilde{h})\} \tanh^2(\tilde{k}\tilde{h})} \right], \quad (C.7)
$$

where $C_\delta \sim 3 \times 10^{-5}$.

### C.3 Results

In this section, we show the results of the solitary wave propagating over the “linear” beach and the nonlinear beach in both forward (deep to shallow) and reverse (shallow to deep) directions. We compare the amplitudes of the wave with and without numerical dissipation. In addition, we also compare the numerical and asymptotic solutions for the case without viscous dissipation. The solutions were assumed periodic in time. The period of the solitary wave was chosen to be large enough to eliminate the interaction of two successive solitons. The initial condition for the solitons, taken from Benilov et al. [9], is

$$
\tilde{A}(\tilde{\tau}) = \tilde{\lambda}_0 \sqrt{\frac{2\tilde{a}_0}{\beta_0}} \text{sech} \left( \tilde{\lambda}_0 \tilde{\tau} \right) \text{Exp} \left( i \frac{\tilde{v}_0 \tilde{\tau}}{2\tilde{a}_0} \right), \quad (C.8)
$$

with $\tilde{\lambda}_0 = 0.1$, $\tilde{v}_0 = 0.1$ and $\tilde{k}_0 = 2$. The periodicity condition allows us to use the pseudo-spectral method for the time derivatives. We used standard fourth order Runge-Kutta method for marching in space. The figures C.1 - C.4 show the maximum amplitude of the solitary wave, $\tilde{\lambda} \sqrt{\frac{2\tilde{a}}{\beta}}$ as a function of distance, $\tilde{\xi}$, scaled by the initial amplitude (for forward propagation) and by the final amplitude (for reverse propagation).

### C.4 Comments

We have examined the evolution of a solitary wave propagating on water of variable depth from deep to shallow water and vice versa. Our results for the forward
propagation on the nonlinear beach are similar to that of Benilov et al. [9]. For propagation on both the beaches, the amplitude of the solitary wave decays as it propagates into shallow water, contrary to the increase in amplitude for shallow water solitons. The wavelength of the soliton decreases and the group velocity increases as it moves into shallow water, and the soliton spreads out, eventually dying out. The reverse is also true, i.e., the amplitude of the shoaling wave packet increases as it enters deep water, as seen in figures C.3 and C.4. The soliton is a stable wave, and hence, dissipation does not play an important role as it does in stabilizing the unstable “uniform” amplitude wave and the periodic cnoidal waves. However, the inclusion of viscous dissipation implies amplitude damping, be it forward propagation or reverse propagation. In this case, irrespective of the direction of propagation, the amplitude of the soliton has to eventually decrease. In the case of forward propagation, the viscous dissipation is with the depth effect, causing additional decay in the amplitude of the wave. In the case of reverse propagation, the viscous dissipation acts opposite to the depth effect, and hence there are two forces in opposite directions trying to increase and decrease the amplitude of the wave. However, it turns out that dissipation succeeds in making the net effect negative, and hence the amplitude of the packet cannot grow forever, and has to decrease beyond a finite distance. The asymptotic solutions agree well with the numerically computed solution for all the cases, unlike for the periodic cnoidal solutions, the limiting case of which (the Jacobi-cn, Jacobi-dn and Jacobi-dnd functions) is the soliton solution. This is because of the smaller magnitude of the ordering parameter $\epsilon$, which, in this case, happens to be of the order of $10^{-3}$. Therefore, for the solitary wave, the asymptotic solution can be used as a good approximation for predicting the evolution of the wave and the variation of the wave parameters with depth.
Figure C.1. The solitary wave propagating on the “linear” beach in the forward direction. Amplitudes computed by numerical solution with viscous dissipation (solid black curve), without viscous dissipation (dotted curve) and asymptotic solution (solid red curve).

Figure C.2. The solitary wave propagating on the nonlinear beach in the forward direction. Amplitudes computed by numerical solution with viscous dissipation (solid black curve), without viscous dissipation (dotted curve) and asymptotic solution (solid red curve).
Figure C.3. The solitary wave propagating on the “linear” beach in the reverse direction. Amplitudes computed by numerical solution with viscous dissipation (solid black curve), without viscous dissipation (dotted curve) and asymptotic solution (solid red curve).

Figure C.4. The solitary wave propagating on the nonlinear beach in the reverse direction. Amplitudes computed by numerical solution with viscous dissipation (solid black curve), without viscous dissipation (dotted curve) and asymptotic solution (solid red curve).
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