The Pennsylvania State University

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HIGHER HARMONIC GUIDED WAVES IN ISOTROPIC WEAKLY NON-LINEAR ELASTIC PLATES

A Thesis in

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by

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ABSTRACT

Use of non-linear ultrasonic waves for material characterization is a topic of significant interest in many applications. Use of guided waves for this purpose is a promising option but appropriate selection of guided wave modes for the generation of cumulative higher harmonics is of critical importance due to the multi-mode nature of ultrasonic guided waves.

This thesis deals with the problem of generating non-linear guided waves in plates from a theoretical perspective. A theoretical framework to predict the higher harmonic guided wave generation in plates has been developed. Geometric and material non-linearities are incorporated by using the Lagrangian strain (non-linearized strain) and higher order terms in the strain energy function. A new formulation in terms of the displacement gradient has been developed for the present problem. A perturbation technique and normal mode expansion have been used to solve the problem and arrive at the conditions of "internal resonance" which are sufficient for the generation of cumulative second harmonics. A comprehensive analysis as to which guided wave modes have the capability to generate cumulative second harmonic guided waves has been performed. This is extended to predict the non-linear interaction of guided waves in plates that can be used to predict any higher harmonic generation in plate.

The analysis led to the conclusion that Shear Horizontal and Rayleigh Lamb modes can generate only Rayleigh Lamb symmetric modes as second harmonics with single primary mode excitation. Specific modes that satisfy the condition of internal resonance have been identified. The analysis on generalized interaction of guided wave modes led to a more complete understanding of the higher harmonic guided waves in plates.

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I would like to dedicate this work to the invisible yet omnipresent force that has been guiding me all through my life.

Chapter 1

INTRODUCTION

In this chapter we introduce the preliminaries for non-linear ultrasonics and guided waves in general and those in plate in particular. This Chapter is organized as follows. Section 1.1 presents the basics of guided wave mechanics and the advantages it offers as a tool for NDE (Non Destructive Evaluation) and SHM (Structural Health Monitoring). Section 1.2 presents the historical development of the field of non-linear ultrasonics and the advantages it offers as a technique to monitor the micro-structure of the material as suggested by various researchers.

1.1 GUIDED WAVE MECHANICS

Guided waves are waves that travel in bodies with constrained boundaries. The interactions of waves with the boundaries guide the wave through the body. Guided wave inspection has proven to be a valuable tool as it offers the following advantages when compared to methods using bulk-waves:

- Can travel long distances with less attenuation.
- Inspection can be carried out from a single location unlike the bulk wave which requires point by point inspection.
- Sensitivity to a variety of defects can be improved by appropriate mode selection.

Guided waves have been used to inspect pipelines, plate-like structures (air craft wings) and multi layered structures and were found to be promising for various other applications.

1.1.1 Guided waves in plates

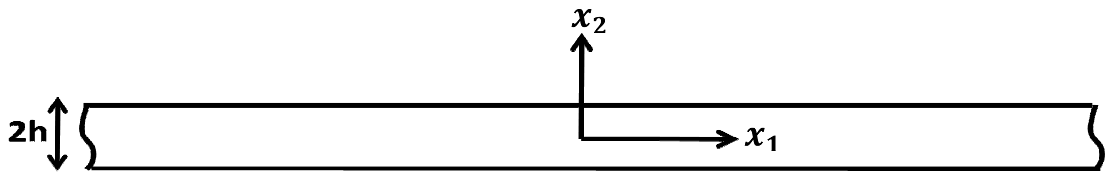

Figure 1.1 Schematic showing an infinite plate with a thickness 2h and the coordinate system used

Consider an infinite traction free plate of thickness 2h shown in Figure 1.1. We use the coordinate system indicated to describe the waves in the plate. The balance of linear momentum for the plate in index notation is given by

$$\mathrm{T}_{ij,j} + \rho b_i = \rho \frac{dv_i}{dt} \qquad \textbf{1.1}$$

with traction free boundary conditions on top and bottom surfaces

$$\mathrm{T}_{ij} n_j = 0 \qquad \textbf{1.2}$$

Where $\mathbf{T}$ is the Cauchy stress tensor, ρ is the mass density of the material, $\boldsymbol{b}$ denotes the body force per unit mass acting on the material and $\boldsymbol{v}$ denotes the particle velocity.

Assuming the material to be an isotropic elastic solid the stress strain relationship is given by classical linearized theory as follows

$$T_{ij} = \lambda\epsilon_{kk}\delta_{ij} + 2\mu\epsilon_{ij} \quad \textbf{1.3}$$

where λ, μ denote the Lame‘s constants, δ_{ij} is the Kronecker delta and ϵ_{ij} is the linearized strain tensor and is related to the displacements as follows

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \textbf{1.4}$$

Using the above relations leads to Navier's equation in terms of displacement as follows:

$$(\lambda + \mu)u_{j,ji} + \mu u_{i,jj} + \rho b_i = \rho\frac{\partial^2 u_i}{\partial t^2} \quad \textbf{1.5}$$

Assuming plane strain (not general), one can use Helmholtz decomposition (Rose,1999) to express the above equation in terms of potentials as follows:

$$u = grad(\phi) + curl(\psi) \quad \textbf{1.6}$$

where ϕ is a scalar and $\boldsymbol{\psi}$ is a vector with components (0,0, $\psi(x_1, x_2)$)

Equation 1.5 is satisfied provided

$$\frac{\partial^2\phi}{\partial {x_1}^2} + \frac{\partial^2\phi}{\partial {x_2}^2} = \frac{1}{{c_l}^2}\frac{\partial^2\phi}{\partial t^2} \quad \textbf{1.7}$$

$$\frac{\partial^2\psi}{\partial {x_1}^2} + \frac{\partial^2\psi}{\partial {x_2}^2} = \frac{1}{{c_t}^2}\frac{\partial^2\psi}{\partial t^2} \quad \textbf{1.8}$$

where c_l, c_t are the longitudinal and transverse wave speeds in the material.

Assuming time harmonic dependence of ϕ, ψ of the form $\phi = \phi(x_2)e^{i(kx_1 - \omega t)}$and $\psi = \psi(x_2)e^{i(kx_1 - \omega t)}$ one can then rewrite the above equations as

$$\frac{\partial^2\phi}{\partial {x_2}^2} + \left(\frac{\omega^2}{{c_l}^2} - k^2\right)\phi = 0 \quad \textbf{1.9}$$

$$\frac{\partial^2\psi}{\partial {x_2}^2} + \left(\frac{\omega^2}{{c_t}^2} - k^2\right)\psi = 0 \qquad \textbf{1.10}$$

The general solutions for the above set of equations are as follows

$$\phi(x_2) = Acos(px_2) + Bsin(px_2) \qquad \textbf{1.11}$$

$$\psi(x_2) = Ccos(qx_2) + Dsin(qx_2) \qquad \textbf{1.12}$$

where $p = \sqrt{\left(\frac{\omega}{c_l}\right)^2 - k^2}$ and q= $\sqrt{\left(\frac{\omega}{c_t}\right)^2 - k^2}$ and A,B,C,D are unknown constants to be determined.

Considering the boundary conditions on the top and bottom surfaces one gets the following relations for the stress components

$$\mathrm{T}_{12}(\mathrm{X}_2 = -h) = \mathrm{T}_{12}(\mathrm{X}_2 = h) = 0 \qquad \textbf{1.13}$$

and

$$\mathrm{T}_{22}(\mathrm{X}_2 = -h) = \mathrm{T}_{22}(\mathrm{X}_2 = h) = 0. \qquad \textbf{1.14}$$

Using these boundary conditions and Equations 1.3,1.4,1.6,1.11,1.12 leads to the system of 4 equations (1.15-1.18) in A,B,C,D which has a non-trivial solution if and only if the determinant of the coefficient matrix vanishes. Guided waves in plates can be classified into two groups depending on the through-thickness displacement profiles. The modes that have a symmetric through thickness u_1 displacement profiles are called symmetric modes (equations 1.15,1.16) and those that have antisymmetric displacement profiles are called antisymmetric modes (Equations 1.17,1.18). Symmetric modes have B=C=0 and antisymmetric modes have A=D=0.

$$\mu\left(-2ikpAsin(ph) + (k^2 - q^2)Dsin(qh)\right) = 0 \qquad \textbf{1.15}$$

$$-\lambda(k^2 + p^2)Acos(ph) - 2\mu[p^2Dcos(ph) + ikDcos(qh)] = 0 \qquad \textbf{1.16}$$

$$\mu\left(2ikpBcos(ph) + (k^2 - q^2)Ccos(qh)\right) = 0 \qquad \textbf{1.17}$$

$$-\lambda(k^2 + p^2)Bsin(ph) - 2\mu[p^2Bcos(ph) - ikCcos(qh)] = 0 \qquad \textbf{1.18}$$

Equating the determinant of the coefficient matrix in each of the systems 1.13&1.14, 1.15&1.16 gives the following dispersion relations

$$\frac{\tan(qh)}{\tan(ph)} = \frac{-4k^2pq}{(q^2-k^2)^2}; \qquad \text{Symmetric modes} \qquad \textbf{1.19}$$

$$\frac{\tan\,(qh)}{\tan\,(ph)} = \frac{-(q^2-k^2)^2}{4k^2pq}; \qquad \text{Antisymmetric modes} \qquad \textbf{1.20}$$

The above relations are the Rayleigh-Lamb (RL) dispersion relations and give those (ω, k) combinations at which guided wave modes exist in the plate. These guided wave modes are named Rayleigh-Lamb (RL) modes and are polarized in the x_1-x_2 plane.

The case of plane strain considered above is not the most general of the problems as it does not consider all possible solutions. There are other modes called the Shear Horizontal (SH) modes polarized in the x_3 direction that propagate in the x_1 direction.

One can use a similar procedure (Rose,1999) to obtain a time harmonic displacement of the form $u_3 = u_3(x_2)e^{i(kx_1-\omega t)}$ and by satisfying the traction free boundary conditions $T_{13}(X_2=\pm h)=0$ we get the following dispersion relation for the shear horizontal modes

$$qh = \frac{n\pi}{2} \qquad \textbf{1.21}$$

where n=0,1,2.. is any arbitrary integer.

1.1.2 Dispersion Curves

This section shows the phase velocity and group velocity dispersion curves for both RL and SH modes.

Phase velocity: The speed with which a given mode i.e, a (ω, k) combination propagates in the material is termed the phase velocity and is denoted by c_p.

$$c_p = \frac{\omega}{k} \qquad \textbf{1.22}$$

Group velocity: The speed with which a wave packet consisting of (ω, k) combinations in a close neighborhood of a given mode propagates is termed as the group velocity and denoted by c_g. This is more important from a practical point of view as in the experiments we generally end up exciting more than one (ω, k) due to the finiteness of the source and the frequency bandwidth of the transducer. The group velocity can be related to the phase velocity by using the relation $c_g = \frac{\partial\omega}{\partial k}$ as follows

$$c_g = \frac{{c_p}^2}{c_p - (fd)\frac{\partial c_p}{\partial (fd)}} \qquad \textbf{1.23}$$

where f= $\frac{\omega}{2\pi}$ and d=2h.

Figures (1.2-1.5) present the phase and group velocity dispersion curves for RL and SH modes. The material chosen for the plate is aluminum whose properties are presented in Table 1-1.

Aluminum		
λ	μ	ρ
$58.5\ GPa$	$26\ GPa$	$2700 kg\ /m^3$

Table 1-1 Material properties of Aluminium

Figure 1.2 shows phase velocity dispersion curves for RL modes in aluminum plate. Different modes have been indicated in the figure. S denotes a symmetric mode and A denotes an antisymmetric mode. Every mode except the fundamental A_0&S_0 have a cut off frequency i.e, a frequency below which they do not exist. Also, all the modes except the fundamental modes converge to the phase velocity equal to the transverse wave speed c_t in the material at high fd products. The modes A_0&S_0 converge to the Rayleigh wave speed (c_R) in the material at high fd products. The flat portions of phase velocity dispersion curves correspond to the modes that have same phase and group velocity.

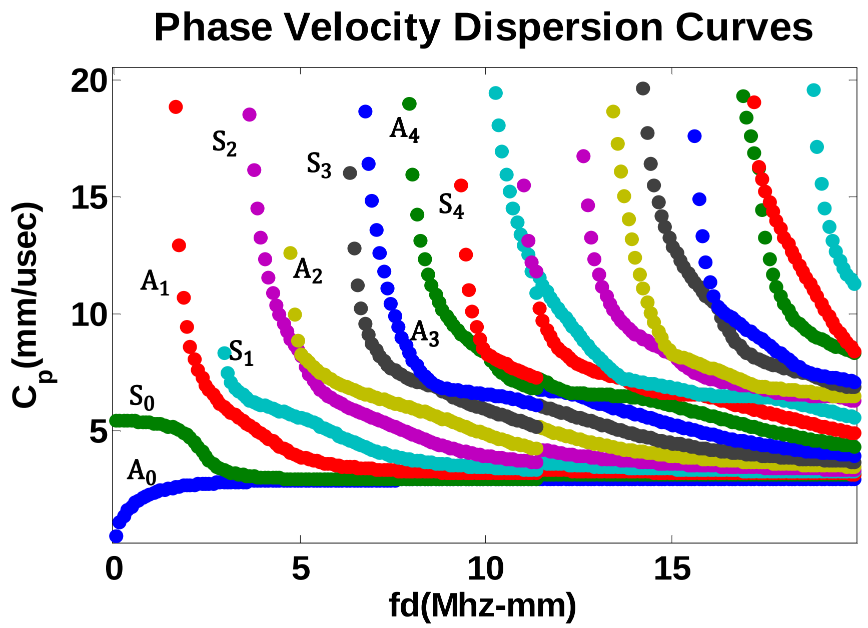

Figure 1.2 Phase velocity dispersion curves for RL mode in Al plate

Figure 1.3 shows group velocity dispersion curves for RL modes in aluminum plate. The peaks in group velocity dispersion curves correspond to the phase velocity and group velocity being equal at those particular modes.

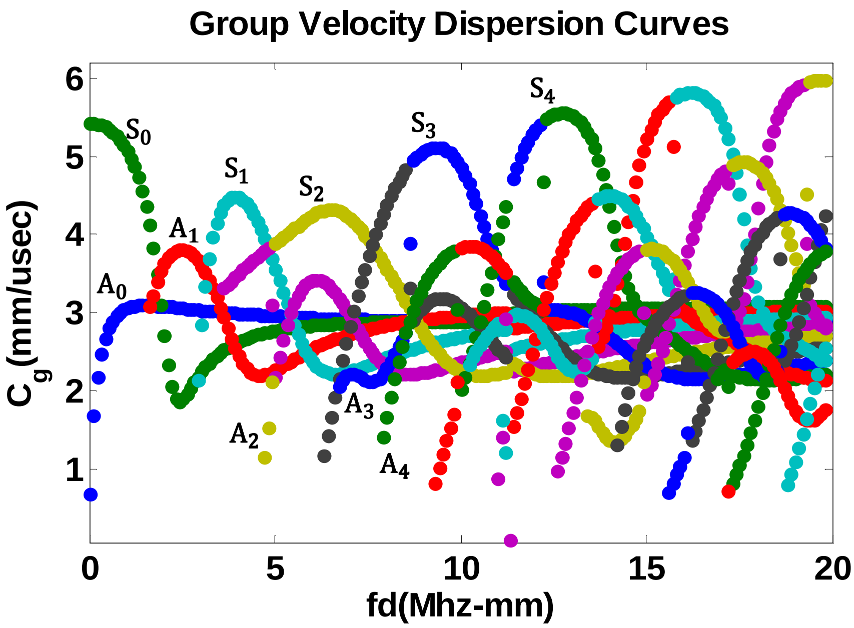

Figure 1.3 Group velocity dispersion curves for RL modes in Al plate

Figure 1.4 shows the phase velocity dispersion curves for SH modes. The fundamental n=0 SH mode is non-dispersive with a phase velocity equal to the shear wave speed (c_t) in the material. All the higher modes approach the fundamental mode at higher fd products. Except for the fundamental mode, all other modes have cut-off frequencies below which that particular mode does not propagate in the plate.

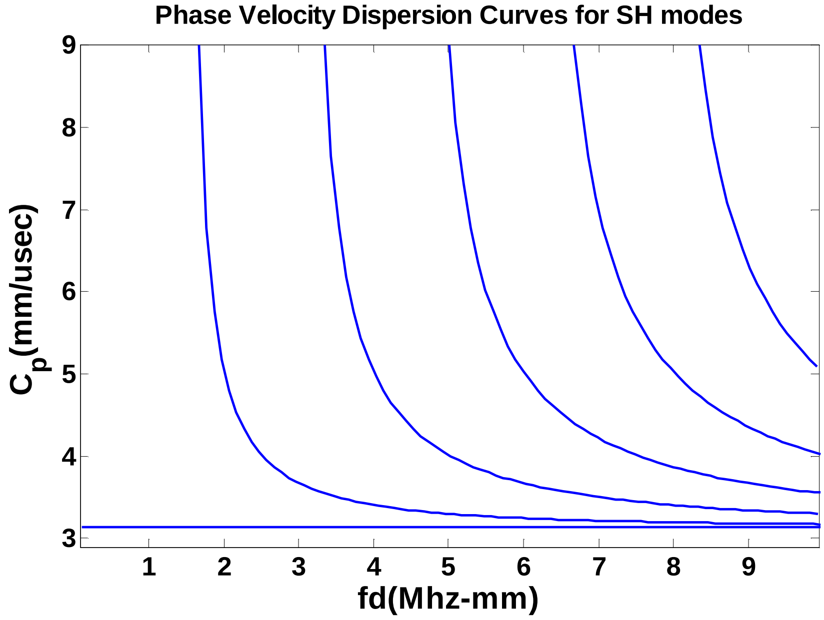

Figure 1.4 Phase velocity dispersion curves for SH modes in Al plate

Figure 1.5 shows group velocity dispersion curves for SH modes in aluminum plate. The primary mode has a group velocity of c_t. All other higher modes have group velocities less than c_t and approach it at higher fd products.

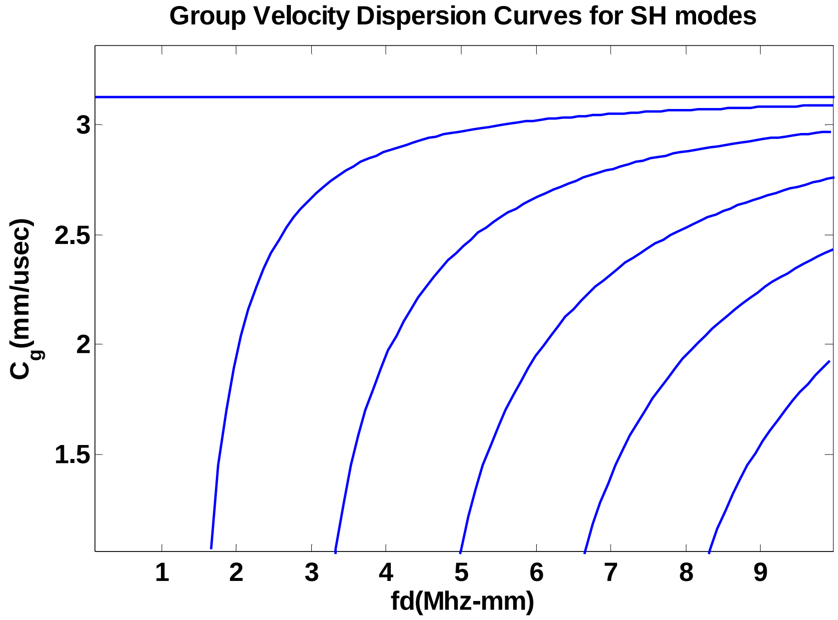

Figure 1.5 Group velocity dispersion curves for SH modes in Al plate

1.1.3 Wave Structures

The through- thickness displacement profiles are referred to as the “wave-structures” and some of the sample wave-structures for some modes are presented in this section. The following figures (1.6-1.9) show the wave structures for a few RL modes in the plate.

Figure 1.6 shows the wave structure for S_0 mode at 0.1 MHz. As can be seen, the through-thickness profile for the displacement u_1 is symmetric about the mid-plane and that of u_2 is antisymmetric about the mid-plane.

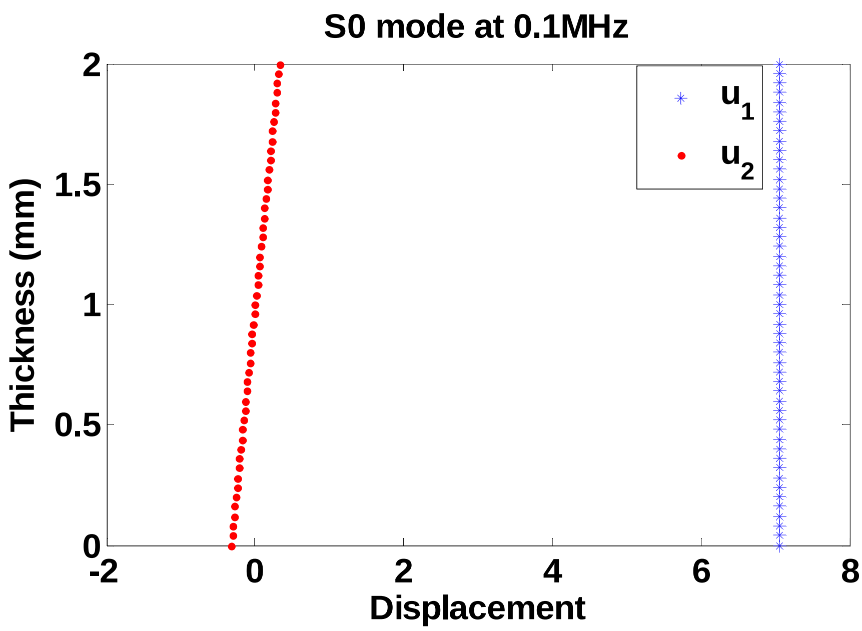

Figure 1.6 Wavestructure for the S0 mode at 0.1 Mhz

Figure 1.7 shows the wave structure for the A_0 mode at 0.1 MHz. As can be seen, the through-thickness profile for the displacement u_1 is antisymmetric about the mid-plane and that of u_2 is symmetric about the mid-plane.

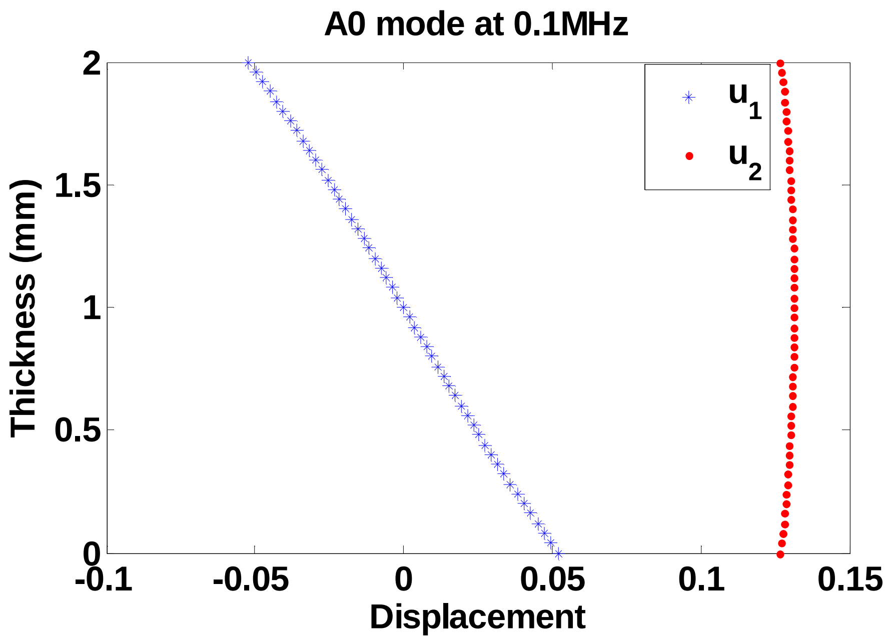

Figure 1.7 Wavestructure for the A0 mode at 0.1 Mhz

Figure 1.8 shows the wave structure for the S_1 mode at 2MHz.

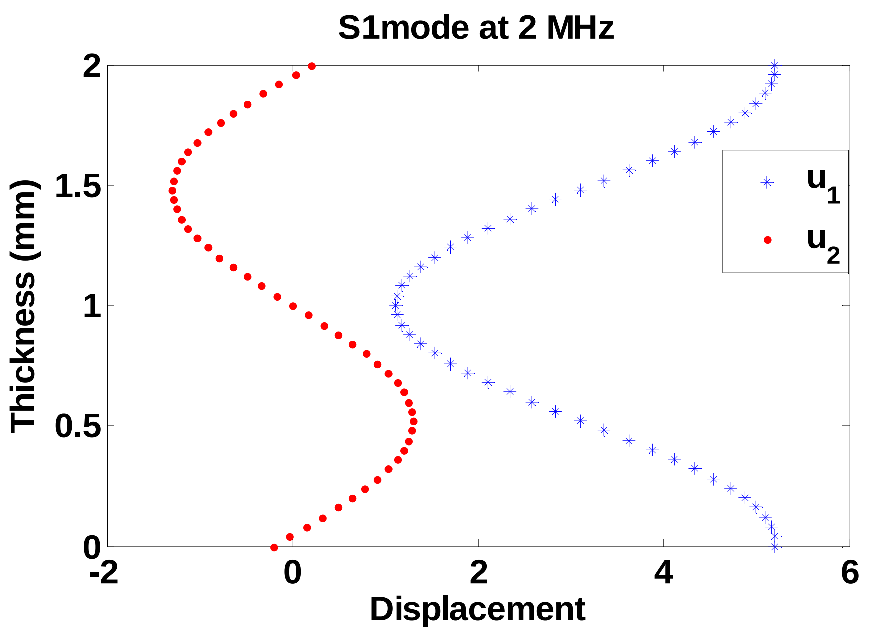

Figure 1.8 Wave structure for the S1 mode at 2Mhz

Figure 1.9 shows the wave structure for the S_1 mode at 2MHz.

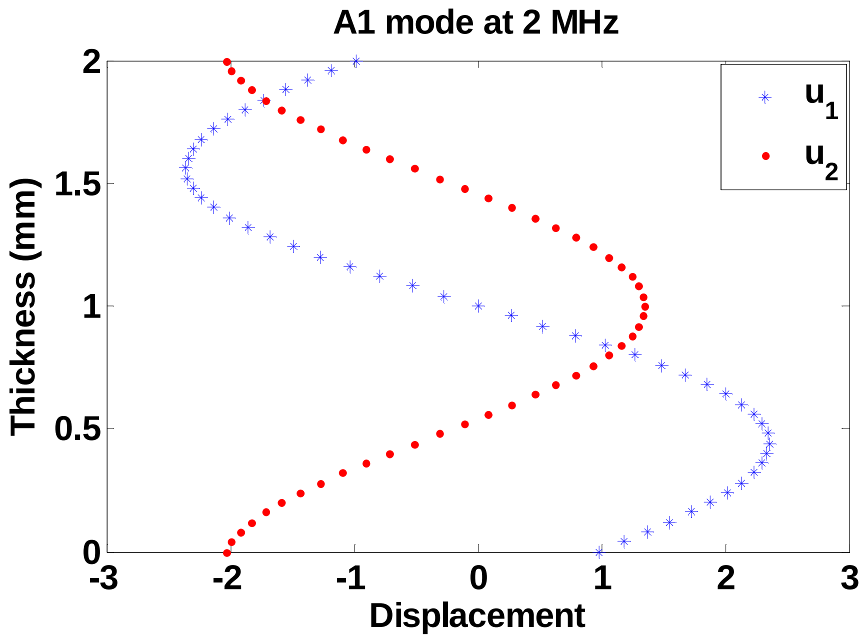

Figure 1.9 Wave structure for the A0 mode at 2 Mhz

The following figures (1.10-1.11) show the wave structures for the SH modes. One important feature concerning the wave structures of the SH modes is that, unlike the wave structures of the RL modes they do not change along a given mode in the dispersion curve. The odd values of 'n' give the antisymmetric modes and even values of 'n' gives the symmetric modes.

Figure 1.10 shows the wave structure for n=1 SH mode. As stated earlier, the wave structure is antisymmetric with respect to the mid-plane.

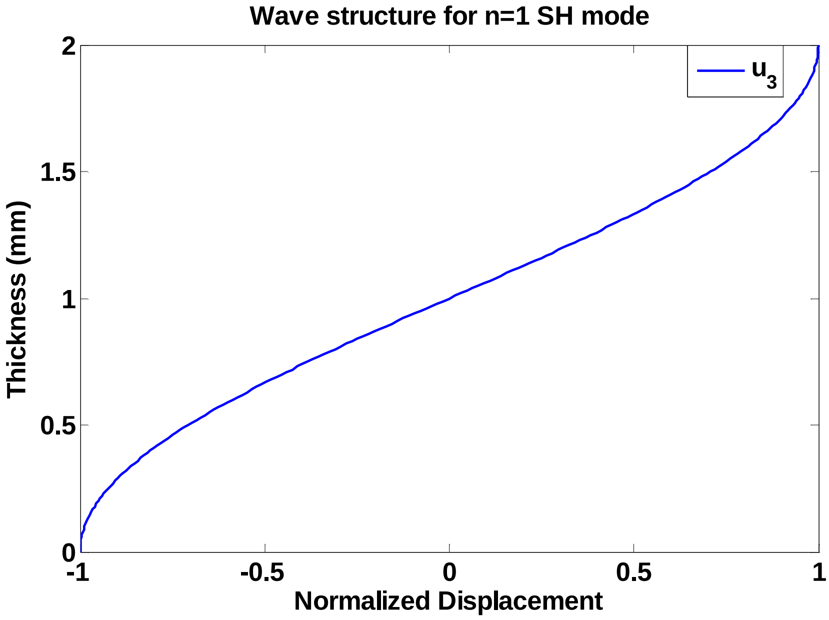

Figure 1.10 Wavestructure for n=1 SH mode

Figure 1.11 shows the wave structure for n=2 SH mode. As stated earlier, the wave structure is symmetric with respect to the mid-plane.

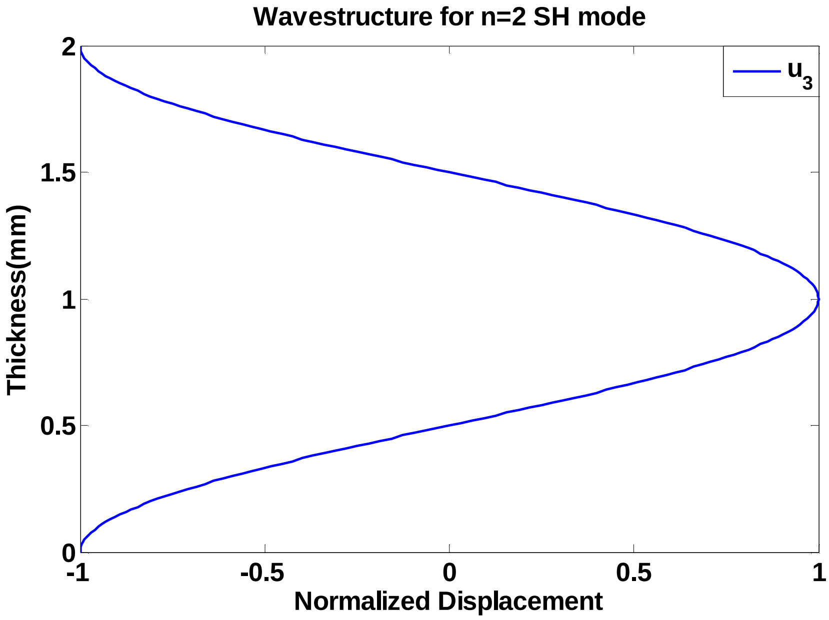

Figure 1.11 Wave structure for n=2 SH mode

1.2 NON-LINEAR ULTRASONICS

1.2.1 History

The theoretical development of the field of non-linear ultrasonics started with researchers examining the effect of introducing non-linear displacement terms in the wave equation and studying the behavior of the solutions. The earliest account of it can be found in Landau and Lifschitz [1956,1970] in a section titled “Anharmonic vibrations”. Later, Goldberg [1960] studied the non-linear interaction of longitudinal and transverse elastic waves from a theoretical standpoint and proved that these waves cannot propagate independently in order to satisfy the wave equation with nonlinear terms. Hikata and Elbaum [1965,1966a,1966b] presented an analysis for the generation of second and third harmonics due to dislocations. It took some time until the higher harmonics were found to be sensitive to the microstructure of material. Cantrell [1994] used a non-linearity parameter β to quantify the degree of non-linearity and examined the effect of crystal structure on β. Cantrell and Yost [2001] used this technique to characterize fatigue damage. Cantrell [2006] used acoustic harmonic generation to quantify fatigue damage accumulation in metals. Cantrell [2009] used ultrasonic harmonic generation for the assessment of fatigue-life and came up with a correlation between the acoustic non-linearity parameter β and the percent remaining life of the material. The work by Cantrell and others employed bulk-waves for higher harmonic generation and used the acoustic non-linearity parameter to quantify the material damage states. Guided waves offer superior inspection capability when compared to bulk-waves and the use of higher harmonic guided waves to monitor the microstructure appears to be a more attractive option. The first step in this regard was taken by Deng [1998] with formulation of the problem of the interaction between Shear Horizontal guided wave modes. Deng [1999] extended the above work for Lamb-wave propagation in plates. de Lima and Hamilton [2003] developed a procedure to qualify the cumulative propagation of second harmonic guided waves in plates. de Lima and Hamilton [2005] later extended this approach to waveguides of arbitrary but constant cross section. Srivastava and Lanza di Scalea used the approach by de Lima and Hamilton to predict the existence of higher harmonics in plates [2009] and rods [2010].

The present work aims to provide a more complete understanding of the problem of higher harmonic generated guided waves in plates. It provides a new approach to predict the nature of guided wave mode interaction which addresses the theoretical inconsistencies that appear in the works of previous researchers.

1.2.1 Preliminaries

In this section we introduce the problem of 1D wave propagation in an elastic material with weak nonlinearities. Consider an elastic material with the following stress-strain relation

$$\sigma = E\varepsilon(1 + \frac{\beta}{2}\varepsilon) \qquad \mathbf{1.24}$$

where the non-linearity considered is of second order in strain ε and E is the Young's modulus, while β quantifies the extent of the non-linearity.

Consider the 1D version of the balance of linear momentum

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \quad \textbf{1.25}$$

$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \quad \textbf{1.26}$$

Using the stress-strain relation (equation 1.24), one can write the equation of motion (Cantrell,1994) as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}(1 + \beta \frac{\partial u}{\partial x}) \quad \textbf{1.27}$$

If we consider a primary wave of the form $u_1 = A_1 \cos(kx - \omega t)$ travelling in the material one can solve the above problem using a perturbation approach by assuming that the second harmonic generated is small in amplitude compared to u_1.

Let $u = u_1 + u_2$. Substituting this in the equation 1.27 one gets two problems

$$\frac{\partial^2 u_1}{\partial t^2} - c^2 \frac{\partial^2 u_1}{\partial x^2} = 0 \quad \textbf{1.28}$$

$$\frac{\partial^2 u_2}{\partial t^2} - c^2 \frac{\partial^2 u_2}{\partial x^2} = -\beta \frac{\partial u_1}{\partial x} \frac{\partial^2 u_1}{\partial x^2} \quad \textbf{1.29}$$

The equation 1.28 is identically satisfied by our assumption that u_1 is a travelling wave in the medium. The second problem can be solved and a particular solution to the second problem is (de Lima and Hamilton, 2003)

$$u_2 = \frac{\beta k^2 {A_1}^2}{16 \rho c^2} x cos(2kx - 2\omega t)\,. \quad \textbf{1.30}$$

This is a wave with amplitude linearly increasing with propagating distance and travelling with the same phase velocity as the primary wave and is termed as a “cumulative second harmonic”. Equation 1.30 is valid provided $u_2 \ll u_1$ as assumed in the perturbation solution.

Chapter 2

SECOND HARMONIC PROBLEM FORMULATION

INTRODUCTION

In this chapter we formulate the ultrasonic guided wave problem of the generation of higher harmonics from the principles of continuum mechanics. This chapter is divided into two sections. Section 2.1 presents the preliminaries of continuum mechanics along with the notation adopted in the remainder of the thesis. Section 2.2 presents the problem formulation for second harmonic guided wave propagation in plates.

2.1 CONTINUUM MECHANICS

2.1.1 Kinematics

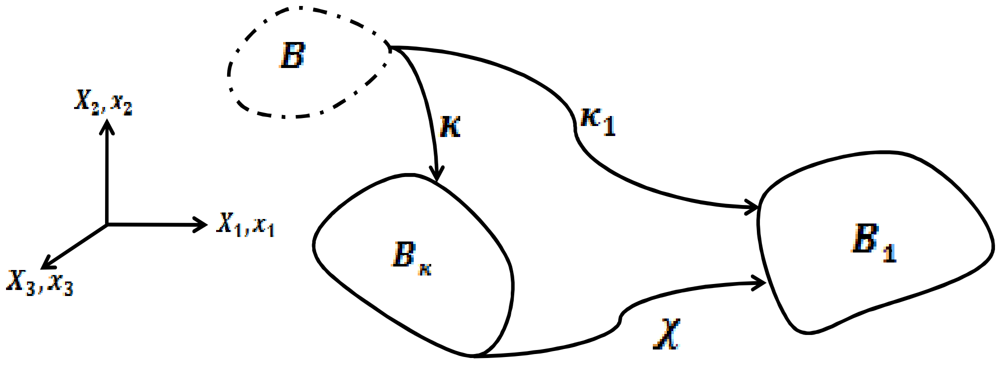

Figure 2.1 Schematic depicting the motion of a body

We denote by $\boldsymbol{B}$ the abstract body consisting of material particles. $\boldsymbol{B_\kappa}$ denotes the reference configuration of the material on which deformation is enforced. We denote by $\boldsymbol{B_1}$ the deformed configuration of the material. By deformation, we mean an invertible mapping $\boldsymbol{\chi} : \boldsymbol{B_\kappa} \rightarrow \boldsymbol{B_1}$.

We use a coordinate system as depicted in Figure 2.1 to describe the deformations. Also, we use letter **X** and $\boldsymbol{x}$ for depicting the position (with reference to the above coordinate system) of a material particle in the reference and deformed configurations respectively. If we consider a continuous sequence of deformations ordered in time we write $\boldsymbol{x} = \boldsymbol{\chi}$ **(X,t)** .We denote by $\boldsymbol{v}(\mathrm{X}, \mathbf{t})$ the velocity of the material particle occupying position **X** in the reference configuration and defined as

$$\boldsymbol{v}(\mathrm{X}, \mathbf{t}) = \frac{\partial \chi}{\partial \mathrm{t}} \text{ .} \qquad \textbf{2.1}$$

We denote by **F** the deformation gradient defined as follows

$$\mathbf{F} = \mathbf{Grad}(\boldsymbol{\chi}\ \mathbf{(X,t)}\ ) = \frac{\partial \chi}{\partial \mathrm{X}} \text{ .} \qquad \textbf{2.2}$$

For sufficiently smooth $\boldsymbol{\chi}$ it can be proved that the determinant of **F** i.e. , $\boldsymbol{det}(\boldsymbol{F}) > 0$.

From the above definitions it is easy to see that the displacement of a material particle is

$$\boldsymbol{u}(\mathrm{X}, \boldsymbol{t}) = \boldsymbol{x} - \mathrm{X} \qquad \textbf{2.3}$$

Also, we have **F=I+Grad(**$\boldsymbol{u}(\mathrm{X}, \boldsymbol{t})$) where **I** is the identity tensor and Grad has the same meaning as defined previously. The quantity $\mathbf{Grad}(\boldsymbol{u}(\mathrm{X}, \boldsymbol{t}))$ is called the Lagrangian displacement gradient and is denoted by **H.**

So, we have

$$\mathrm{F} = \mathrm{I} + \mathrm{H} \text{ .} \qquad \textbf{2.4}$$

We use a Lagrangian measure of strain defined as

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathbf{T}}\mathbf{F} - \mathbf{I}) \qquad \textbf{2.5}$$

where $\mathbf{F}^{\mathbf{T}}$ is the transpose of deformation gradient **F**.

In terms of the displacement gradient **H**, we get

$$\mathbf{E}= \frac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathbf{T}} + \mathbf{H}^{\mathbf{T}}\mathbf{H}). \qquad \textbf{2.6}$$

The above infrastructure will be sufficient to describe the motions we would be dealing with in this work. Any new notation introduced later will be made clear at that point of time. Now, we move on to study the balance laws.

2.1.2 Balance Laws

Balance of mass

Under the assumption that the only cause for a change in mass density is deformation, one can show that

$$\dot{\rho}+\rho div(v) = 0 \qquad \textbf{2.7}$$

where $div(v)$ is the divergence of the velocity field.

Balance of linear and angular momentum

Under Cauchy's assumption, by employing Euler's first and second law to every sub-part $\boldsymbol{P} \subseteq \boldsymbol{B}$ we get

$$\text{div(T)}+\rho b=\rho\frac{dv}{dt} \qquad \text{(Balance of Linear momentum)} \qquad \textbf{2.8}$$

$$\mathbf{T} = \mathbf{T}^{\mathbf{T}} \qquad \text{(Balance of Angular momentum)} \qquad \textbf{2.9}$$

: the local form where T is the Cauchy stress tensor.

The above equations can be expressed in referential form as follows

$$\mathbf{Div}(\boldsymbol{S}) + \boldsymbol{\rho_\kappa b} = \boldsymbol{\rho_\kappa \dot{v}} \quad \text{(Balance of Linear momentum)} \qquad \mathbf{2.10}$$

and

$$\mathbf{SF^T} = \mathbf{FS^T} \quad \text{(Balance of Angular momentum)} \qquad \mathbf{2.11}$$

where **S** is the first Piola-Kirchoff stress tensor and is related to the Cauchy stress **T** as follows

$$\mathbf{S} = \mathbf{det}(\mathbf{F})\mathbf{TF^{-T}}\,. \qquad \mathbf{2.12}$$

2.1.3 Constitutive Theory

Using the Coleman-Noll procedure, which employs Balance of Energy, Second Law of Thermodynamics and material frame indifference one can show that the first Piola-Kirchoff stress for an elastic homogeneous material can be written in terms of the deformation gradient as follows

$$\mathbf{S} = \frac{\partial \mathbf{W}(\mathbf{F})}{\partial \mathbf{F}} \qquad \mathbf{2.13}$$

Where $\mathbf{W}(\mathbf{F})$ is the strain energy function expressed in terms of **F**.

Another expression for the Second Piola-Kirchoff stress $\mathbf{T_{RR}}$ (Appendix A) can be developed in terms of Lagrangian strain **E** as follows

$$\mathbf{T_{RR}} = \frac{\partial \widetilde{\mathbf{W}}(\mathbf{E})}{\partial \mathbf{E}} \qquad \mathbf{2.14}$$

Where $\widetilde{\mathbf{W}}(\mathbf{E})$ is the strain energy function of the material expressed in terms of **E.**

For an isothermal, isotropic, elastic solid one can express $\widetilde{\mathbf{W}}(\mathbf{E})$ in terms of invariants of **E.** Choosing $\mathbf{tr(E)}$,$\mathbf{tr(E^2)}$,$\mathbf{tr(E^3)}$ as the set of invariants of **E** one can write [see Landau and Lifschitz,1970]

$$\widetilde{\mathbf{W}}(\mathbf{E}) = \frac{1}{2}\boldsymbol{\lambda}(\mathbf{tr}(\mathbf{E}))^2 + \boldsymbol{\mu}\mathbf{tr}(\mathbf{E}^2) + \frac{1}{3}\mathbf{C}(\mathbf{tr}(\mathbf{E}))^3 + \mathbf{B}\mathbf{tr}(\mathbf{E})\mathbf{tr}(\mathbf{E}^2) + \frac{1}{3}\mathbf{A}\mathbf{tr}(\mathbf{E}^3) \qquad \mathbf{2.15}$$

up to third order terms in strain multiples where $\boldsymbol{\lambda}, \boldsymbol{\mu}$ are the Lame's constants and $\mathbf{C}, \mathbf{B}, \mathbf{A}$ are third order elastic constants (See Norris [1998]).

Using the above strain energy function one gets the following expression for the second Piola –Kirchoff stress

$$\mathbf{T_{RR}} = \boldsymbol{\lambda}\mathbf{tr}(\mathbf{E})\mathbf{I} + 2\boldsymbol{\mu}\mathbf{E} + \mathbf{C}(\mathbf{tr}(\mathbf{E}))^2\mathbf{I} + \mathbf{B}\mathbf{tr}(\mathbf{E}^2)\mathbf{I} + 2\mathbf{B}\mathbf{tr}(\mathbf{E})\mathbf{E} + \mathbf{A}\mathbf{E}^2 \qquad \mathbf{2.16}$$

In what follows, we use the above expression for the Second Piola-Kirchoff stress in terms of Lagrangian strain **E**.

The first Piola-Kirchoff stress **S** is related to the second Piola-Kirchoff stress tensor as

$$\mathbf{S} = \mathbf{F}\mathbf{T_{RR}} \qquad \mathbf{2.17}$$

where F is the deformation gradient.

For enhancing the clarity, we use the notation $\mathbf{S}(\mathbf{H})/\mathbf{T_{RR}}(\mathbf{H})$ **or** $\mathbf{S(E)}/\,\mathbf{T_{RR}}(\mathbf{E})$ to explicitly describe **S** and $\mathbf{T_{RR}}$ as functions of their arguments.

Using equation 2.6 for **E** in the expression for $\mathbf{T_{RR}}$ **(E)** given in equation 2.16, we have, up to second order in **H**

$$\mathbf{T_{RR}}(\mathbf{H}) = \frac{\boldsymbol{\lambda}}{2}\mathbf{tr}(\mathbf{H} + \mathbf{H^T})\mathbf{I} + \boldsymbol{\mu}(\mathbf{H} + \mathbf{H^T}) + \frac{\boldsymbol{\lambda}}{2}\mathbf{tr}(\mathbf{H^T}\mathbf{H})\mathbf{I} + \mathbf{C}\,(\mathbf{tr}(\mathbf{H}))^2\mathbf{I} + \boldsymbol{\mu}\mathbf{H^T}\mathbf{H} + \mathbf{B}\mathbf{tr}(\mathbf{H})(\mathbf{H} + \mathbf{H^T}) + \frac{\mathbf{B}}{2}\mathbf{tr}(\mathbf{H}^2 + \mathbf{H^T}\mathbf{H})\mathbf{I} + \frac{\mathbf{A}}{4}\left(\mathbf{H}^2 + \mathbf{H^T}^2 + \mathbf{H}\mathbf{H^T} + \mathbf{H^T}\mathbf{H}\right) \qquad \mathbf{2.18}$$

Keeping the future use of the above expression in mind, we break it in to two, namely

$$\mathbf{T_{RR}}^{\mathbf{L}}(\mathbf{H}) = \frac{\boldsymbol{\lambda}}{2}\mathbf{tr}(\mathbf{H} + \mathbf{H^T})\mathbf{I} + \boldsymbol{\mu}(\mathbf{H} + \mathbf{H^T}) \qquad \mathbf{2.19}$$

$$\mathbf{T_{RR}}^{\mathbf{NL}}(\mathbf{H}) = \frac{\lambda}{2}\mathbf{tr}(\mathbf{H^TH})\mathbf{I} + \mathbf{C}\left(\mathbf{tr}(\mathbf{H})\right)^2\mathbf{I} + \mu\mathbf{H^TH} + \mathbf{B}\mathbf{tr}(\mathbf{H})(\mathbf{H} + \mathbf{H^T}) + \frac{\mathbf{B}}{2}\mathbf{tr}(\mathbf{H^2} + \mathbf{H^TH})\mathbf{I} + \frac{\mathbf{A}}{4}\left(\mathbf{H^2} + \mathbf{H^T}^2 + \mathbf{HH^T} + \mathbf{H^TH}\right) \qquad \mathbf{2.20}$$

where $\mathbf{T_{RR}}^{\mathbf{L}}(\mathbf{H})$ and $\mathbf{T_{RR}}^{\mathbf{NL}}(\mathbf{H})$ are the linear and non linear functions of their argument **H** and

$$\mathbf{T_{RR}} = \mathbf{T_{RR}}^{\mathbf{L}}(\mathbf{H}) + \mathbf{T_{RR}}^{\mathbf{NL}}(\mathbf{H}) \qquad \mathbf{2.21}$$

Now we focus on developing similar expressions for $\mathbf{S}(\mathbf{H})$. We know that

$$\mathbf{S} = \mathbf{FT_{RR}} \qquad \text{i.e.,}$$

$$\mathbf{S}(\mathbf{H}) = (\mathbf{I} + \mathbf{H})\left(\mathbf{T_{RR}}^{\mathbf{L}}(\mathbf{H}) + \mathbf{T_{RR}}^{\mathbf{NL}}(\mathbf{H})\right) \qquad \mathbf{2.22}$$

$$\mathbf{S^L}(\mathbf{H}) = \mathbf{T_{RR}}^{\mathbf{L}}(\mathbf{H}) \qquad \mathbf{2.23}$$

$$\mathbf{S^{NL}}(\mathbf{H}) = \mathbf{HT_{RR}}^{\mathbf{L}}(\mathbf{H}) + \mathbf{T_{RR}}^{\mathbf{NL}}(\mathbf{H})\,. \qquad \mathbf{2.24}$$

2.2 SECOND HARMONIC PROBLEM FORMULATION

Consider the equation of balance of linear momentum in referential form for a traction free plate in the absence of body forces $\boldsymbol{b}=\mathbf{0}$:

$$\mathbf{Div}(\boldsymbol{S}) = \boldsymbol{\rho_\kappa}\ddot{\boldsymbol{u}}$$

$$\boldsymbol{S}\boldsymbol{n_\kappa}=\mathbf{0} \qquad \mathbf{2.25}$$

S is the first Piola-Kirchoff stress tensor and $\boldsymbol{n_\kappa}$ is the unit outward normal of the surface of the plate in the reference configuration and $\boldsymbol{u}$ is the displacement.

We use a perturbation method (de Lima and Hamilton [2003]) to decompose the displacement field as

$$\boldsymbol{u} = \boldsymbol{u_1} + \boldsymbol{u_2} \quad \text{with} \qquad |\boldsymbol{u_1}| \gg |\boldsymbol{u_2}|\,,$$

where $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are the primary and secondary displacement fields respectively.

So from the definition of $\mathbf{H}$ we have

$$\mathbf{H} = \mathrm{H}_1 + \mathrm{H}_2 \qquad \mathbf{2.26}$$

where $\mathrm{H}_1 = \boldsymbol{Grad}(\boldsymbol{u_1})$ and $\mathrm{H}_2 = \boldsymbol{Grad}(\boldsymbol{u_2})$.

From the notation used previously

$$\mathbf{S}(\mathbf{H}) = \mathbf{S}^{\mathbf{L}}(\mathbf{H}) + \mathbf{S}^{\mathbf{NL}}(\mathbf{H})$$

$$\Rightarrow \mathbf{S}(\mathbf{H}) = \mathbf{S}^{\mathbf{L}}(\mathbf{H}_1) + \mathbf{S}^{\mathbf{L}}(\mathbf{H}_2) + \mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1 + \mathrm{H}_2). \qquad \mathbf{2.27}$$

In the elaborate expression for $\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1 + \mathrm{H}_2)$ we retain only the terms which involve second order contributions from $\mathbf{H}_1$.We name these as the interaction terms and use the notation $\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1, \mathrm{H}_1, \mathbf{2})$ to designate these. $\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1, \mathrm{H}_1, \mathbf{2})$ can be read as "Non-linear terms which are of order 2 due to self interaction between $\mathbf{H}_1$ and H_1".

Finally we have

$$\mathbf{S}(\mathbf{H}) = \mathbf{S}^{\mathbf{L}}(\mathbf{H}_1) + \mathbf{S}^{\mathbf{L}}(\mathbf{H}_2) + \mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1, \mathrm{H}_1, \mathbf{2}) \qquad \mathbf{2.28}$$

Going back to the equation for balance of linear momentum and substituting the expressions for $\boldsymbol{u} = \boldsymbol{u_1} + \boldsymbol{u_2}$ and $\mathbf{S}(\mathbf{H})$ we have

$$\mathbf{Div}\Big(\mathbf{S}^{\mathbf{L}}(\mathbf{H}_1) + \mathbf{S}^{\mathbf{L}}(\mathbf{H}_2) + \mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1, \mathrm{H}_1, \mathbf{2})\Big) = \boldsymbol{\rho_\kappa}(\ddot{\boldsymbol{u}}_1 + \ddot{\boldsymbol{u}}_2) \qquad \mathbf{2.29}$$

$$\{\mathrm{Div}(\mathbf{S}^{\mathbf{L}}(\mathbf{H}_1)) - \boldsymbol{\rho_\kappa}\ddot{\boldsymbol{u}}_1\} + \{\mathrm{Div}(\mathbf{S}^{\mathbf{L}}(\mathbf{H}_2)) - \boldsymbol{\rho_\kappa}\ddot{\boldsymbol{u}}_2 + \mathrm{Div}(\mathbf{S}^{\mathbf{NL}}(\mathbf{H}_1, \mathbf{H}_1, \mathbf{2}))\} = \mathbf{0} \qquad \mathbf{2.30}$$

with the following boundary condition

$$\mathbf{S}^{\mathbf{L}}(\mathbf{H}_1)\boldsymbol{n_\kappa} + \mathbf{S}^{\mathbf{L}}(\mathbf{H}_2)\boldsymbol{n_\kappa} + \mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1, \mathrm{H}_1, 2)\boldsymbol{n_\kappa} = \mathbf{0} \qquad \mathbf{2.31}$$

Now we decompose the above problem into two problems; one involving $\boldsymbol{u_1}$ and another involving $\boldsymbol{u_2}$,

$$\mathrm{Div}(\mathbf{S}^{\mathbf{L}}(\mathbf{H}_1)) - \boldsymbol{\rho_\kappa}\ddot{\boldsymbol{u}}_1 = \mathbf{0}$$

$$\mathbf{S^L(H_1)}\boldsymbol{n_\kappa} = \mathbf{0} \tag{2.32}$$

and

$$\mathrm{Div}(\mathbf{S^L(H_2)}) - \boldsymbol{\rho_\kappa \ddot{u}_2} = -\mathrm{Div}(\mathbf{S^{NL}(H_1, H_1, 2)})$$

$$\mathbf{S^L(H_2)}\boldsymbol{n_\kappa} = -\mathbf{S^{NL}}(\mathrm{H_1, H_1, 2})\boldsymbol{n_\kappa}\,. \tag{2.33}$$

The first problem is a standard problem for which $\boldsymbol{u_1}$ is a solution.

The second problem is an inhomogeneous version of the first with the forcing term $-\mathrm{Div}(\mathbf{S^{NL}(H_1, H_1, 2)})$. The above formulation is general for any set of displacements $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ such that $|\boldsymbol{u_1}| \gg |\boldsymbol{u_2}|$.

Now, we restrict our attention to guided wave propagation in plates i.e., when both $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ are waves in the plate. Figure 2.2 shows the schematic of the plate and the coordinate system we use for the problem formulation.

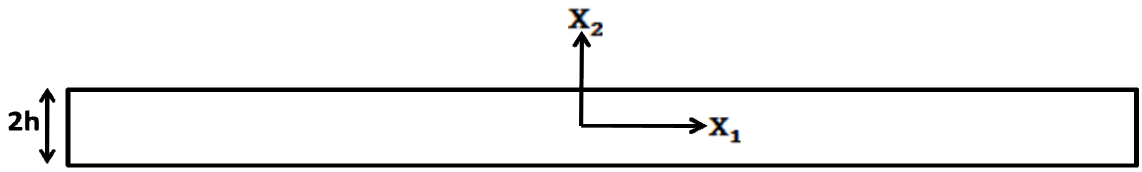

Figure 2.2 Schematic of the plate with the coordinate system used

As said earlier $\boldsymbol{u_1}$ is a guided wave mode and hence can be assumed to be of the form $\boldsymbol{u_1} = \boldsymbol{Re}\{\boldsymbol{u_1}(\mathrm{X_2})\boldsymbol{e}^{i(k\mathrm{X_1}-\omega t)}\}$ where $\boldsymbol{Re}\{\quad\}$ denotes the real part of the argument.

Additionally, $\boldsymbol{u_1}$ is a solution to the first problem and is one among the infinitely many solutions termed as “Normal Modes” to that problem (see Section 1.1).

Now we focus our attention on the second problem. Consider the forcing term $-\mathrm{Div}(\mathbf{S^{NL}(H_1, H_1, 2)})$ on the right hand side of the second problem. Due to the second degree products in $\mathbf{H_1}$ it has a factor of the form $\boldsymbol{e^{-2i\omega t}}$.

Following de Lima and Hamilton [2003] we seek a solution to the second problem in the form of an asymptotic expansion of the normal modes. Also, from the observation made previously it suffices to seek an asymptotic expansion of the solutions at a frequency 2ω, i.e., let

$$\mathbf{S^L}(\mathbf{H_2}) = \sum_{\mathbf{m=1}}^{\infty} \mathrm{A_m}(\mathrm{X_1})\mathbf{S_m} \text{ and}$$

$$\dot{\boldsymbol{u}}_{\mathbf{2}} = \sum_{\boldsymbol{m=1}}^{\boldsymbol{\infty}} \mathrm{A_m}(\mathrm{X_1})\boldsymbol{v_m}. \qquad \mathbf{2.34}$$

Here $\mathbf{S_m}, \boldsymbol{v_m}$ are the stress and velocity variables for m^{th}guided wave modes at a frequency 2ω.

As shown by Auld [1990] , $\mathrm{A_m}(\mathrm{X_1})$ is a solution to the ordinary differential equation for each m and all n for which $P_{mn} \neq 0$.

$$4P_{mn}\left(\frac{d\mathrm{A_m}}{d\mathrm{X_1}} - ik_n{}^*\mathrm{A_m}\right) = \left(f_n{}^{surf} + f_n{}^{vol}\right)e^{2ik\mathrm{X_1}} \qquad \mathbf{2.35}$$

$$P_{mn} = -\frac{\mathbf{1}}{\mathbf{4}}\int_{-\boldsymbol{h}}^{\boldsymbol{h}}\left(\frac{S_m v_n{}^*}{\mathbf{4}} + \frac{S_n{}^* v_m}{\mathbf{4}}\right).\boldsymbol{n_1}\,\mathrm{dX_2} \qquad \mathbf{2.36}$$

$$f_n{}^{surf} = -\frac{\mathbf{1}}{\mathbf{2}}\mathbf{S^{NL}}(\mathrm{H_1}, \mathrm{H_1}, \mathbf{2})\boldsymbol{v_n}{}^*.\boldsymbol{n_2}\,\Big|_{-\boldsymbol{h}}^{\boldsymbol{h}} \qquad \mathbf{2.37}$$

$$f_n{}^{vol} = \frac{\mathbf{1}}{\mathbf{2}}\int_{-\boldsymbol{h}}^{\boldsymbol{h}}\mathbf{Div}\left(\mathbf{S^{NL}}(\mathrm{H_1}, \mathrm{H_1}, \mathbf{2})\right).\boldsymbol{v_n}{}^*\,\mathrm{dX_2} \qquad \mathbf{2.38}$$

For every mode m there is only one mode n such that $P_{mn} \neq 0$. If m is a propagating mode then the mode n is the same as mode m and if m is an evanescent mode then the mode n is such that $k_n = k_m{}^*$.

$\mathrm{A_m}(0) = 0$ as there is no second harmonic propagation initially at $\mathrm{X_1} = 0$.

The general solution to the ordinary differential equation above is

$$A_m(X_1) = \frac{-i(f_n^{surf} + f_n^{vol})}{4P_{mn}(k_n - 2k)}\left(e^{ik_n^* X_1} - e^{i2kX_1}\right) \quad \text{if } k_n^* \neq 2k \qquad \textbf{2.39}$$

$$A_m(X_1) = \frac{(f_n^{surf} + f_n^{vol})}{4P_{mn}} X_1 \qquad \text{if } k_n^* = 2k \qquad \textbf{2.40}$$

The above solutions imply that if the mode n is such that $f_n^{surf} + f_n^{vol} \neq 0$ and $k_n^* = 2k$ then the amplitude of the second harmonic increases linearly with the propagating distance and is termed a cumulative second harmonic.

Thus two conditions are needed for the generation of a cumulative second harmonic

1. A propagating guided wave mode n such that $k_n^* = 2k$ at frequency 2ω. This is termed as the phase matching criterion as the mode n propagates with the same phase velocity as the primary mode. For our future discussions we consider this as a necessary condition.
2. And $f_n^{surf} + f_n^{vol} \neq 0$ for that particular mode n, which is generally termed nonzero power flow.

In the coming chapters we investigate the existence of those primary modes that guarantee cumulative second harmonic generation by employing the above criterion.

Chapter 3

NECESSARY CONDITION FOR THE EXISTENCE OF CUMULATIVE SECOND HARMONIC GUIDED WAVES IN PLATES

INTRODUCTION

This chapter presents an analytical formulation of the necessary condition for the existence of a cumulative second harmonic by employing the phase matching criterion. We do this by considering the dispersion relations for Rayleigh-Lamb and Shear-Horizontal mode propagation in plates. The chapter is organized into three sections. Section 3.1 presents the necessary conditions on the existence of cumulative second harmonic Rayleigh-Lamb modes when the primary mode is also a Rayleigh-Lamb mode. Section 3.2 presents the necessary conditions for the existence of cumulative second harmonic Rayleigh-Lamb modes when the primary mode is a Shear-Horizontal mode. Section 3.3 presents some special cases and discusses the conditions which are sufficient for non-existence of cumulative second harmonic Rayleigh Lamb or Shear Horizontal modes.

3.1 THEORY

The dispersion relations for Rayleigh-Lamb modes (equations 3.1 & 3.2) were developed as equations 1.19 & 1.20 and are presented here for convenience

$$\frac{\tan(qh)}{\tan(ph)} = \frac{-4k^2pq}{(q^2-k^2)^2} \quad \text{symmetric modes} \qquad \mathbf{3.1}$$

$$\frac{\tan\,(qh)}{\tan\,(ph)} = \frac{-(q^2-k^2)^2}{4k^2pq} \quad \text{antisymmetric modes} \qquad \mathbf{3.2}$$

Here 'h' is the half-thickness of the plate, $q = \sqrt{(\frac{\omega}{c_t})^2 - k^2}$, $p = \sqrt{(\frac{\omega}{c_l})^2 - k^2}$.

Where ω=2πf is the angular frequency, k is the wave number, c_l is longitudinal wave speed and c_t is shear wave speed.

We obtain the necessary condition for the existence of a cumulative second harmonic by considering the possible ordered pairs (ω, k) for which both (ω, k) and $(2\omega, 2k)$ are guided wave modes in a plate. This procedure is general in the sense that one could use (ω, k) and $(n\omega, nk)$ for studying the existence of the n^{th} harmonic.

We begin the analysis by observing the following facts about the dispersion relations presented above. As (ω, k) is replaced by $(2\omega, 2k)$ we have

i. $q \to 2q, p \to 2p$

ii. Given the transformation $(\omega, k) \to (2\omega, 2k)$, the right hand sides of Equations 3.1 and 3.2 remain unchanged

3.1.1 Phase matching criterion when both primary and secondary modes are Rayleigh-Lamb modes

We present the analysis in two cases. Case a is when both the primary and secondary modes are of the same nature and Case b is when they are of different nature.

Case a: Both (ω, k) and $(2\omega, 2k)$ are either symmetric or antisymmetric modes.

From Equation 3.1

$$\frac{\tan(qh)}{\tan(ph)} = \frac{-4k^2pq}{(q^2-k^2)^2} \text{ and}$$

$$\frac{\tan(2qh)}{\tan(2ph)} = \frac{-4k^2pq}{(q^2-k^2)^2} \text{ when both are symmetric modes} \qquad \textbf{3.3}$$

Or

$$\frac{\tan\,(qh)}{\tan\,(ph)} = \frac{-(q^2-k^2)^2}{4k^2pq} \text{ and}$$

$$\frac{\tan\,(2qh)}{\tan\,(2ph)} = \frac{-(q^2-k^2)^2}{4k^2pq} \text{ when both are antisymmetric modes} \qquad \textbf{3.4}$$

In writing the above set of equations we exploited the fact that the right hand side of Equation set 3.1 is unaltered by the transformation $(\omega, k) \rightarrow (2\omega, 2k)$

From equations 3.3&3.4 we get;

$$\frac{\tan(2qh)}{\tan(2ph)} = \frac{\tan\,(qh)}{\tan\,(ph)}$$

$$\Leftrightarrow \sin(2qh)\cos(2ph)\sin(ph)\cos(qh) - \sin(2ph)\cos(2qh)\sin(qh)\cos(ph) = 0$$

$\Leftrightarrow 2\sin(qh)\sin(ph)\{\cos^2(qh)\cos(2ph) - \cos^2(ph)\cos(2qh)\} = 0$

$\Leftrightarrow \sin(qh)\sin(ph)\{(1+\cos(2qh))\cos(2ph) - (1+\cos(2\mathrm{ph}))\cos(2qh)\} = 0$

$\Leftrightarrow 2\sin(qh)\sin(ph)(\cos(2\mathrm{ph}) - \cos(2qh)) = 0$

$\Leftrightarrow \mathrm{qh} = \mathrm{n}\pi$ or $\mathrm{ph} = \mathrm{n}\pi$ or $(\mathrm{qh} - \mathrm{ph}) = \mathrm{n}\pi$ or $(\mathrm{qh} + \mathrm{ph}) = \mathrm{n}\pi$

where $q = \sqrt{(\frac{\omega}{c_t})^2 - k^2}$, $p = \sqrt{(\frac{\omega}{c_l})^2 - k^2}$ as mentioned before and n is an arbitrary whole number.

The above analysis holds true even when one or both p and q are imaginary. The only change one has to make to the above derivation is that cos(ph) must be changed into cosh(-iph) and sin(ph) must be changed into sinh(-iph). Of course p can be replaced by q in the previous sentence. Also, care is taken to ensure that we do not miss any solutions falling in these categories while writing the following subcases.

Sub-case a1: $\mathrm{qh} = \mathrm{n}\pi$ but $\mathrm{ph} \neq \mathrm{n}\pi$ but p may be 0.

From the above condition, for symmetric modes we get from Equation (3.1) $-4k^2pq = 0$

$$\Rightarrow (k = 0 \implies (\omega, k) is\ a\ cut\ off\ mode)\ or\ \left(p = 0 \Rightarrow \frac{\omega}{k} = c_l\right) or\ (q = 0 \Rightarrow \frac{\omega}{k} = c_t)$$

$$p = 0 \Rightarrow \frac{\omega}{k} = c_l\ and\ qh = \mathrm{n}\pi \Rightarrow \omega = \frac{n\pi c_l c_t}{h\sqrt{{c_l}^2 - {c_t}^2}}$$

The case q=0 should be handled with limit dispersion relations which are obtained by considering the limit as q goes to 0 in the dispersion relations for symmetric modes (equation 3.1).

The dispersion relation for symmetric modes is

$$\frac{\tan(qh)}{\tan(ph)} = \frac{-4k^2pq}{(q^2-k^2)^2}$$

$$\Rightarrow \frac{(\frac{\tan(qh)}{q})}{\tan(ph)} = \frac{-4k^2p}{(q^2-k^2)^2}$$

Taking limit as q goes to zero on the left hand side we get [As $\lim_{q\to 0}\frac{\tan(qh)}{q} = h$]

$$\frac{h}{\tan(ph)} = \frac{-4k^2p}{(q^2-k^2)^2}$$

Multiplying by p on both sides we get the following limit dispersion relation

$$\frac{ph}{\tan(ph)} = \frac{-4p^2}{k^2} \qquad \textbf{3.5}$$

From equation 3.5 we have

$$\frac{\tan(ph)}{\mathrm{ph}} = -4\frac{\mathrm{k}^2}{\mathrm{p}^2}$$

$\Rightarrow \frac{\tan(ph)}{\mathrm{ph}} = 4\frac{\mathrm{c_l}^2}{\mathrm{c_l}^2-\mathrm{c_t}^2}$ which is a transcendental equation and should be solved for purely imaginary values of p to get the modes at longitudinal velocity c_l on the dispersion curves.

Now the phase matching criterion takes the form

$$\frac{\tan(ph)}{\mathrm{ph}} = \frac{\tan(2ph)}{2\mathrm{ph}}$$

The above equation does not have any solution in the present case.

For antisymmetric modes, from Equation (3.2) we have $(q^2-k^2)^2 = 0$

$$\Rightarrow q^2 = k^2 \Rightarrow \frac{\omega}{k} = \sqrt{2}\,c_t$$

Since qh = nπ we have $\omega = \frac{\sqrt{2}n\pi c_t}{h}$

Conclusions from this sub-case a1:

1. All primary symmetric modes at $\omega = \frac{n\pi c_l c_t}{h\sqrt{c_l^2 - c_t^2}}$ $and\ c_p = \frac{\omega}{k} = c_l$ can give secondary symmetric modes.

2. All primary antisymmetric modes at $\omega = \frac{\sqrt{2}n\pi c_t}{h}$ and $c_p = \frac{\omega}{k} = \sqrt{2}\, c_t$ can give secondary antisymmetric modes.

Sub-case a2: ph = nπ but qh ≠ nπ

From the above condition, for symmetric modes, from equation 3.1 we get $(q^2 - k^2)^2 = 0$

$$\Rightarrow q^2 = k^2 \Rightarrow \frac{\omega}{k} = \sqrt{2}\, c_t$$

If this condition has to be satisfied we need $\sqrt{2}c_t \geq c_l$, as ph = nπ is a real number.

under this assumption, we get $\omega = \frac{\sqrt{2}n\pi c_l c_t}{h\sqrt{2c_t^2 - c_l^2}}$

For antisymmetric modes, from equation 3.2 we get $-4k^2pq = 0$

$$\Rightarrow k = 0\ \big((\omega, k) is\ a\ cut\ off\ mode\big)\ or\ \left(p = 0 \Rightarrow \frac{\omega}{k} = c_l\right) or\ \left(q = 0 \Rightarrow \frac{\omega}{k} = c_t\right)$$

The condition p=0 should be handled using limit dispersion relation for anti symmetric modes. This can be obtained in the similar way we obtained equation 3.5 by considering the limit as p goes to zero for the dispersion relation in equation 3.2.

The limit dispersion relation is

$$\frac{\tan(qh)}{qh} = \frac{-\left(q^2 - k^2\right)^2}{4k^2q^2}\ .$$

The phase matching criterion takes the form

$$\frac{\tan(qh)}{qh} = \frac{\tan(2qh)}{2qh}$$

$\Leftrightarrow qh = n\pi$ for some whole number n (which is against the condition for the present sub-case and also the limit dispersion relation is not satisfied)

($q = 0 \Rightarrow \frac{\omega}{k} = c_t$ and $ph = n\pi$ does not give any modes since $ph \geq 0 \Rightarrow c_p \geq c_l$)

Conclusions from this sub-case a2:

1. If $\sqrt{2}c_t \geq c_l$ then primary symmetric modes at $\omega = \frac{\sqrt{2}n\pi c_l c_t}{h\sqrt{2{c_t}^2 - {c_l}^2}}$ and $c_p = \frac{\omega}{k} = \sqrt{2}\, c_t$ can generate secondary symmetric modes.
2. No primary antisymmetric modes satisying the conditions of this subcase can generate secondary antisymmetric modes.

Sub-case a3: $(\mathrm{qh} - \mathrm{ph}) = \mathrm{n}\pi$ or $(\mathrm{qh} + \mathrm{ph}) = \mathrm{n}\pi$ and $(\mathrm{ph} \neq \mathrm{n}\pi$ and $\mathrm{qh} \neq \mathrm{n}\pi)$

$(\mathrm{qh} - \mathrm{ph}) = \mathrm{n}\pi \Rightarrow \tan(\mathrm{qh}) = \tan(\mathrm{ph})$

$$\Rightarrow \frac{\tan(qh)}{\tan(ph)} = 1 = \frac{-4k^2pq}{(q^2 - k^2)^2} \Rightarrow \frac{-(q^2 - k^2)^2}{4k^2pq} = 1 = \frac{\tan(qh)}{\tan(ph)}$$

$\Longrightarrow$ conditions for both symmetric and anti symmetric modes are satisfied

$\Longrightarrow (\omega, k)$ is a point of intersection of symmetric and antisymmetric modes

$(\mathrm{qh} + \mathrm{ph}) = \mathrm{n}\pi \Rightarrow \tan(\mathrm{qh}) = -\tan(\mathrm{ph})$

$$\Rightarrow \frac{\tan(qh)}{\tan(ph)} = -1 = \frac{-4k^2pq}{(q^2-k^2)^2} \Rightarrow \frac{-(q^2-k^2)^2}{4k^2pq} = -1 = \frac{\tan(qh)}{\tan(ph)}$$

$\Longrightarrow$ conditions for both symmetric and anti symmetric modes are satisfied

$\Longrightarrow$ (ω, k) is a point of intersection of symmetric and antisymmetric modes

Conclusions from this sub-case a3:

1. All primary modes which are intersections of a symmetric and antisymmetric mode can generate a secondary mode which is also an intersection of symmetric and a antisymmetric mode.

Sub-case a4: $ph = n\pi$ and $qh = m\pi$ for some natural numbers m,n.

It is easy to see that in this case also the primary mode is a mode which is an intersection of symmetric and antisymmetric modes.

Case b: (ω, k) is a symmetric/antisymmetric mode and $(2\omega, 2k)$ is an antisymmetric/symmetric mode. (one primary mode generates secondary mode of opposite nature)

$$\frac{\tan(qh)}{\tan(ph)} = \frac{-4k^2pq}{(q^2-k^2)^2} \quad and$$

$$\frac{\tan(2qh)}{\tan(2ph)} = \frac{-(q^2-k^2)^2}{4k^2pq};$$

Or

$$\frac{\tan(qh)}{\tan(ph)} = \frac{-(q^2-k^2)^2}{4k^2pq} \; and$$

$$\frac{\tan(2qh)}{\tan(2ph)} = \frac{-4k^2pq}{(q^2-k^2)^2};$$

From the above cases we get

$$\frac{\tan(qh)}{\tan(ph)}\frac{\tan(2qh)}{\tan(2ph)} = 1$$

$$\Leftrightarrow \sin(2qh)\sin(qh)\cos(ph)\cos(2ph) - \sin(2ph)\cos(2qh)\sin(ph)\cos(qh) = 0$$

$$\Leftrightarrow 2\cos(qh)\cos(ph)\{\cos(2ph)\,sin^2(qh) - \cos(2qh)\,sin^2(ph)\} = 0$$

$$\Leftrightarrow 2\cos(qh)\cos(ph)\{\cos(2ph)\,(1-\cos(2qh)) - \cos(2qh)\,(1-\cos(2ph))\} = 0$$

$$\Leftrightarrow 2\cos(qh)\cos(ph)\{\cos(2ph) - \cos(2qh))\} = 0$$

$$\Leftrightarrow \mathrm{qh} = (2\mathrm{n}+1)\frac{\pi}{2} \text{ or } \mathrm{ph} = (2\mathrm{n}+1)\frac{\pi}{2} \text{ or } (\mathrm{qh}-\mathrm{ph}) = \mathrm{n}\pi \text{ or } (\mathrm{qh}+\mathrm{ph}) = \mathrm{n}\pi$$

where n is an arbitrary whole number.

Sub-case b1: $\mathrm{qh} = (2\mathrm{n}+1)\frac{\pi}{2}$ but $\mathrm{ph} \neq (2\mathrm{n}+1)\frac{\pi}{2}$

For symmetric modes, from equation 3.1, we get $(q^2-k^2)^2 = 0$

$$\Rightarrow q^2 = k^2 \Rightarrow c_p = \frac{\omega}{k} = \sqrt{2}\,c_t$$

As $\mathrm{qh} = (2\mathrm{n}+1)\frac{\pi}{2}$ we have $\omega = \frac{\sqrt{2}(2n+1)\pi c_t}{2h}$

For antisymmetric modes, from equation 1, we get $-4k^2pq = 0$

$$\Rightarrow k = 0\ \left((\omega,k) is\ a\ cut\ off\ mode\right) or\ \left(p = 0 \Rightarrow \frac{\omega}{k} = c_l\right) or\ (q = 0 \Rightarrow \frac{\omega}{k} = c_t)$$

The case p=0 should be dealt with using the limit dispersion relation i.e,

$$\frac{\tan(qh)}{qh} = \frac{-\left(q^2-k^2\right)^2}{4k^2q^2} \text{ and } \mathrm{qh} = (2\mathrm{n}+1)\frac{\pi}{2}$$

$\Leftrightarrow k = 0 \Longrightarrow$ no mode that satisfies this condition exists (no cut off modes exist at c_l).

Conclusions from this Sub-case b1:

1. Primary symmetric modes at $\omega = \frac{\sqrt{2}(2n+1)\pi c_t}{2h}$ and $c_p = \frac{\omega}{k} = \sqrt{2}\, c_t$ can generate secondary antisymmetric modes.

Sub-case b2: $\mathrm{ph} = (2\mathrm{n}+1)\frac{\pi}{2}$ but $\mathrm{qh} \neq (2\mathrm{n}+1)\frac{\pi}{2}$

For Symmetric modes from Equation 3.1, we get $-4k^2pq = 0$

$$\Rightarrow (k = 0 \implies (\omega, k) is\ a\ cut\ off\ mode)\ or\ \left(p = 0 \Rightarrow \frac{\omega}{k} = c_l\right) or\ (q = 0 \Rightarrow \frac{\omega}{k} = c_t)$$

$\left(p = 0 \Rightarrow \frac{\omega}{k} = c_l\right)$ is not a possible case since $\mathrm{ph} = (2\mathrm{n}+1)\frac{\pi}{2}$

The case q=0 should be dealt with the limit dispersion relation i.e,

$\frac{\tan(ph)}{\mathrm{ph}} = -4\frac{\mathrm{k}^2}{\mathrm{p}^2}$ and $\mathrm{ph} = (2\mathrm{n}+1)\frac{\pi}{2}$

$\implies$ No modes that satisfy the above conditions exist.

For Antisymmetric modes, from Equation 3.2, we get $(q^2 - k^2)^2 = 0$

$$\Rightarrow q^2 = k^2 \Rightarrow \frac{\omega}{k} = \sqrt{2}\, c_t$$

If this condition has to be satisfied we need $\sqrt{2}c_t \geq c_l$ as $\mathrm{ph} = (2\mathrm{n}+1)\frac{\pi}{2}$ is a real number.

Under this assumption we get $\omega = \frac{\sqrt{2}(2n+1)\pi c_l c_t}{2h\sqrt{2{c_t}^2 - {c_l}^2}}$

Conclusions from this Sub-case b2:

1. If $\sqrt{2}c_t \geq c_l$ then primary antisymmetric modes at $\omega = \frac{\sqrt{2}(2n+1)\pi c_l c_t}{2h\sqrt{2{c_t}^2 - {c_l}^2}}$ and $c_p = \frac{\omega}{k} = \sqrt{2}\, c_t$ can generate secondary symmetric modes.

Sub-Case b3: $(\mathrm{qh} - \mathrm{ph}) = \mathrm{n}\pi$ or $(\mathrm{qh} + \mathrm{ph}) = \mathrm{n}\pi$

This sub-case has been clearly discussed as part of sub-case a3 and it was observed that both the primary and secondary modes correspond to the intersections of symmetric and antisymmetric modes on the dispersion curves.

Sub-Case b4: $\mathrm{ph} = (2\mathrm{n} + 1)\frac{\pi}{2}$ and $\mathrm{qh} = (2\mathrm{m} + 1)\frac{\pi}{2}$

This Sub-Case would result in the Sub-Case b3 as $(\mathrm{qh} - \mathrm{ph}) = \mathrm{r}\pi$ and $(\mathrm{qh} + \mathrm{ph}) = s\pi$ for arbitrary integers r=m-n,s=m+n+1.

These also correspond to the primary modes being intersections of symmetric and antisymmetric modes in the dispersion curves.

3.1.2 Phase matching criterion when the primary mode is a Shear-Horizontal mode and the secondary mode is a Rayleigh-Lamb mode

In this section we discuss the cases when (ω, k) is a Shear-Horizontal mode and $(2\omega, 2k)$ is a Rayleigh Lamb mode.

Case a: (ω, k) is a Shear-Horizontal mode and $(2\omega, 2k)$ is a Symmetric mode.

$\Rightarrow \mathrm{qh} = \frac{\mathrm{n}\pi}{2}$ and $\frac{\tan(2qh)}{\tan(2ph)} = \frac{-4k^2pq}{(q^2-k^2)^2}$; but $\tan(2qh) = 0\ as\ \mathrm{qh} = \frac{\mathrm{n}\pi}{2}$

$\Longrightarrow \tan(2ph) = 0 \; or \; k = 0\big((\omega, k) is \; a \; cutoff \; mode\big) or \; \left(p = 0 \Rightarrow \frac{\omega}{k} = c_l \right) or$

$(q = 0 \Rightarrow \frac{\omega}{k} = c_t)$

$\Rightarrow 2ph = m\pi \Longrightarrow ph = \frac{m\pi}{2}$; m is an arbitrary whole number or $(q = 0 \Rightarrow \frac{\omega}{k} = c_t)$ or

$(p = 0 \Longrightarrow \frac{\omega}{k} = c_t)$

If $m \neq 0 and \; n \neq 0$ then the secondary symmetric mode is a mode at the intersection of a symmetric and antisymmetric mode at $c_p > c_l$.

If $(p = 0 \Leftrightarrow m = 0) \; and \; n \neq 0$ then the secondary symmetric mode is a mode at $c_p = \frac{\omega}{k} = c_l$ and frequency $2\omega = \frac{n\pi c_l c_t}{h\sqrt{{c_l}^2 - {c_t}^2}}$

The case $(q = 0 \Leftrightarrow n = 0)$ should be dealt with limit dispersion relation i.e,

$\frac{\tan(2ph)}{2\text{ph}} = -4\frac{\text{k}^2}{\text{p}^2}$ and $c_p = c_t$.

The above equation has to be solved numerically and has solutions only at very high frequencies where the dispersion RL curves converge to transverse wave speed.

m=0 and n=0 is not possible because both p and q cannot be zero simultaneously.

$m \neq 0$ and $n \neq 0$ corresponds to primary mode being a mode whose phase velocity satisfies the equation

$$\left(\frac{1}{{c_l}^2} - \frac{1}{{c_p}^2}\right) = \frac{m^2}{n^2}\left(\frac{1}{{c_t}^2} - \frac{1}{{c_p}^2}\right)$$

The secondary mode is an intersection of symmetric and anti symmetric modes in the dispersion curve.

Case b: (ω, k) is a Shear-Horizontal mode and $(2\omega, 2k)$ is an Antisymmetric mode.

$\Rightarrow$ qh $= \frac{n\pi}{2}$ and $\frac{\tan(2qh)}{\tan(2ph)} = \frac{-(q^2-k^2)^2}{4k^2pq}$; but $\tan(2qh) = 0$ as qh $= \frac{n\pi}{2}$

$\Longrightarrow \tan(2ph) = 0$ or $-(q^2 - k^2)^2 = 0;$

$$\Rightarrow ph = \frac{m\pi}{2} \; or \; q^2 = k^2$$

Sub-case b1: qh $= \frac{n\pi}{2}$ and ph $= \frac{m\pi}{2}$

m=0 and n=0 is not possible because one cannot have p=0 and q=0 simultaneously.

m=0 and $n \neq 0$ then this case has to be dealt with the limit dispersion relation i.e,

$\frac{\tan(2qh)}{2qh} = \frac{-\left(q^2-k^2\right)^2}{4k^2q^2}$ and qh $= \frac{n\pi}{2}$

$\Longrightarrow q^2 - k^2$=0 $\Longrightarrow c_p = \frac{\omega}{k} = \sqrt{2}\, c_t$ (which is not possible as p=0 $\Longrightarrow \frac{\omega}{k} = c_l$)

$\Longrightarrow$ there is no solution in this case

$m \neq 0 and\ n = 0 \Longrightarrow$ There is no secondary mode.

$m \neq 0 and\ n \neq 0 \Longrightarrow$ The secondary mode is an intersection of symmetric and antisymmetric modes at $c_p > c_l$. c_p satisfies the following equation as in case a

$$\left(\frac{1}{{c_l}^2} - \frac{1}{{c_p}^2}\right) = \frac{m^2}{n^2}\left(\frac{1}{{c_t}^2} - \frac{1}{{c_p}^2}\right)$$

Sub-case b2 qh $= \frac{n\pi}{2}$ and $q^2 = k^2$

$$\Rightarrow c_p = \frac{\omega}{k} = \sqrt{2}\, c_t \; and \; \omega = \frac{n\pi\sqrt{2}\, c_t}{2h}$$

Primary Shear Horizontal mode at $\omega = \frac{n\pi\sqrt{2}\, c_t}{2h}$ and $c_p = \frac{\omega}{k} = \sqrt{2}\, c_t$ can generate secondary antisymmetric modes at $2\omega = \frac{n\pi\sqrt{2}\, c_t}{h}$ and $c_p = \frac{\omega}{k} = \sqrt{2}\, c_t$.

If n=0 no mode that satisfies the conditions of this sub-case exists.

3.2 CONCLUSIONS

In this section we presented some conditions that are sufficient for non existence of cumulative second harmonics.

1. Shear Horizontal modes cannot be generated as secondary modes when the primary modes are Rayleigh-Lamb modes. Hence, this analysis has not been performed.
2. Any primary mode that is capable of generating a cumulative secondary mode will fall into at least one of the following categories
    a. It has $c_p = c_l$.
    b. It has $c_p = \sqrt{2}c_t$.(Lame modes)
    c. It is an intersection of a symmetric and antisymmetric mode on the dispersion curves.
    d. It is a cut-off mode
3. So, if a mode does not satisfy any of the above conditions then it can be concluded that it cannot be used to generate a cumulative second harmonic in the plate.

Chapter 4

INTERACTION OF GUIDED WAVE MODES IN PLATE

INTRODUCTION

In this chapter we examine the power flow criterion to determine which of the guided wave modes obtained from the phase matching criterion can be used to generate a cumulative second harmonic. We also formulate a generalized problem that will be able to predict the non-linear interaction of two arbitrary guided wave modes propagating in the plate. The content of this chapter is organized as follows. Section 4.1 examines the cumulative second harmonic guided wave problem from a power-flow perspective. Section 4.2 formulates the generalized problem for guided wave mode interaction. Section 4.3 presents the conclusions.

4.1 Power flow analysis for cumulative second harmonic generation with primary Rayleigh-Lamb modes

In Chapter 2 we obtained the non-zero power flow criterion (Equation 2.40) for cumulative second harmonic generation, i.e, existence of a guided wave mode ‘n’ having $\mathrm{k_n} = 2\mathrm{k}$ at a frequency 2ω such that

$$f_n{}^{surf} + f_n{}^{vol} \neq 0 \qquad \textbf{4.1}$$

where (equations 2.37, 2.38)

$$f_n^{\,surf} = -\frac{1}{2}\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1, \mathrm{H}_1, \mathbf{2})\boldsymbol{v}_{\boldsymbol{n}}^{\,*}.\boldsymbol{n}_{\mathbf{2}}\,|_{-\boldsymbol{h}}^{\boldsymbol{h}} \qquad \mathbf{4.2}$$

$$f_n^{\,vol} = \frac{1}{2}\int_{-\boldsymbol{h}}^{\boldsymbol{h}} \mathbf{Div}\left(\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1, \mathrm{H}_1, \mathbf{2})\right).\boldsymbol{v}_{\boldsymbol{n}}^{\,*}\,\mathrm{dX}_2. \qquad \mathbf{4.3}$$

Now we examine under what conditions or which primary modes $\boldsymbol{u}_{\mathbf{1}}$ are capable of satisfying the above criterion. For this we introduce the following terminology.

'Sym' matrix/vector: A matrix of the following structure where E denotes an even function and *O* denotes an odd function is called a 'Sym' matrix. (Note the distinction between O and 0)

$$\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}$$

A vector of the following structure is called a 'Sym' vector

$$\begin{Bmatrix} E \\ O \\ 0 \end{Bmatrix}$$

'Anti' matrix/vector: A matrix of the following structure where E denotes an even function and *O* denotes and odd function is called an 'Anti' matrix.

$$\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}$$

A vector of the following nature is called the 'Anti' vector

$$\begin{Bmatrix} O \\ E \\ 0 \end{Bmatrix}$$

Now we list some lemmas and their proofs, which will help in generalizing the cumulative harmonic generation via mode-interaction.

Lemma 1 If u is a symmetric/antisymmetric Rayleigh Lamb mode then **H=Grad(*u*)** when represented in Cartesian co-ordinate system has 'Sym'/ 'Anti' nature.

Proof Let

$$\boldsymbol{u} = \boldsymbol{Re}\left\{\begin{bmatrix} \boldsymbol{u_1(X_2)} \\ \boldsymbol{u_2(X_2)} \\ \boldsymbol{0} \end{bmatrix} \boldsymbol{e^{i(k\mathrm{X}_1-\omega t)}}\right\}$$

where $u_1$is an even/odd function and u_2 is an odd/even function with respect to $\mathbf{X_2}$ for a symmetric/antisymmetric mode. Since the derivative of an even function is odd and vice-versa, **H** when represented as a matrix looks like $\boldsymbol{Re}\left\{\begin{bmatrix} \boldsymbol{iku_1} & \frac{\boldsymbol{\partial u_1}}{\boldsymbol{\partial X_2}} & \boldsymbol{0} \\ \boldsymbol{iku_2} & \frac{\boldsymbol{\partial u_2}}{\boldsymbol{\partial X_2}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \boldsymbol{e^{i(k\mathrm{X}_1-\omega t)}}\right\}$ **,** which has a 'Sym'/'Anti' structure depending on whether u is a symmetric/antisymmetric mode.

Lemma 2 The product of matrices of identical nature results in a matrix of 'Sym' nature and the product of matrices of opposite nature results in a matrix of 'Anti' nature.

$$\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}$$

$$\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}$$

$$\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}$$

$$\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}$$

Proof The proof is a consequence of the fact that the product of two functions of the same parity is even and that of opposite parity is odd.

$$\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} E*E+O*O & E*O+O*E & 0 \\ O*E+E*O & O*O+E*E & 0 \\ 0 & 0 & E*E \end{bmatrix} = \begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}$$

$$\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} E*O+O*E & E*E+O*O & 0 \\ O*O+E*E & O*E+E*O & 0 \\ 0 & 0 & E*E \end{bmatrix} = \begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}$$

$$\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} E*O+O*E & E*E+O*O & 0 \\ O*O+E*E & O*E+E*O & 0 \\ 0 & 0 & E*E \end{bmatrix} = \begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}$$

$$\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} O*O+E*E & O*E+E*O & 0 \\ E*O+O*E & E*E+O*O & 0 \\ 0 & 0 & E*E \end{bmatrix} = \begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}$$

Lemma 3 The product of a matrix and vector of identical nature (whenever the product is meaningful) results in a 'sym' vector and a product between those of opposite nature results in a vector of 'Anti' nature.

Proof Follows along the same lines as above

$$\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}\begin{Bmatrix} E \\ O \\ 0 \end{Bmatrix} = \begin{Bmatrix} E*E+O*O \\ O*E+E*O \\ 0 \end{Bmatrix} = \begin{Bmatrix} E \\ O \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix}\begin{Bmatrix} O \\ E \\ 0 \end{Bmatrix} = \begin{Bmatrix} O*O+E*E \\ E*O+O*E \\ 0 \end{Bmatrix} = \begin{Bmatrix} E \\ O \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix}\begin{Bmatrix} O \\ E \\ 0 \end{Bmatrix} = \begin{Bmatrix} E*O+O*E \\ O*O+E*E \\ 0 \end{Bmatrix} = \begin{Bmatrix} O \\ E \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix} \begin{Bmatrix} E \\ O \\ 0 \end{Bmatrix} = \begin{Bmatrix} O*E+E*O \\ E*E+O*O \\ 0 \end{Bmatrix} = \begin{Bmatrix} O \\ E \\ 0 \end{Bmatrix}$$

Lemma 4 For any primary Rayleigh Lamb mode u the matrix representation for $\mathbf{S^{NL}(H,H,2)}$ is a 'Sym' matrix.

Proof

From equation 2.24

$$\mathbf{S^{NL}(H,H,2) = HT_{RR}{}^{L}(H) + T_{RR}{}^{NL}(H,H,2)} \qquad \mathbf{4.4}$$

where

$$\mathbf{T_{RR}{}^{L}(H) = \frac{\lambda}{2} tr(H + H^T)I + \mu(H + H^T)} \qquad \mathbf{4.5}$$

$$\mathbf{T_{RR}{}^{NL}(H,H,2) = \frac{\lambda}{2} tr(H^T H)I + C\,(tr(H))^2 I + \mu H^T H + Btr(H)(H + H^T) +}$$

$$\mathbf{\frac{B}{2} tr(H^2 + H^T H)I + \frac{A}{4}(H^2 + H^{T^2} + HH^T + H^T H)} \qquad \mathbf{4.6}$$

$$\mathbf{HT_{RR}{}^{L}(H) = \frac{\lambda}{2} tr(H + H^T)H + \mu(H^2 + HH^T)} \qquad \mathbf{4.7}$$

If **H** is of 'Sym'/'Anti' nature then $\mathrm{tr}(\mathrm{H} + \mathrm{H}^{\mathrm{T}})$ is an even(E)/odd(O) function being linear in the diagonal terms.

So the first term $\mathbf{\frac{\lambda}{2} tr(H + H^T)H}$ in Equation 4.7 has one of the following forms

$$E \begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} E*E & E*O & 0 \\ E*O & E*E & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$O\begin{bmatrix} O & E & 0 \\ E & O & 0 \\ 0 & 0 & E \end{bmatrix} = \begin{bmatrix} O*O & O*E & 0 \\ O*E & O*O & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using lemma's 1 and 2 it is easy to show that $\boldsymbol{\mu}(\mathbf{H^2} + \mathbf{HH^T})$ has a Sym nature. So, we can conclude $\mathbf{HT_{RR}}^{\mathbf{L}}(\mathbf{H})$ has a 'Sym' nature.

Using the consequences of Lemma 2 we can conclude that each of the terms of $\mathbf{T_{RR}}^{\mathbf{NL}}(\mathbf{H})$ has 'Sym' nature and hence $\mathbf{T_{RR}}^{\mathbf{NL}}(\mathbf{H}, \mathbf{H}, \mathbf{2})$ has 'Sym' nature.

So, $\mathbf{S^{NL}}(\mathbf{H}) = \mathbf{HT_{RR}}^{\mathbf{L}}(\mathbf{H}) + \mathbf{T_{RR}}^{\mathbf{NL}}(\mathbf{H}, \mathbf{H}, \mathbf{2})$ has a 'Sym' nature.

Corollary: From the above lemma, it can be seen that the matrix representation of $\mathbf{S^{NL}}(\mathbf{H})$ has the following structure $\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & 0 \end{bmatrix}$ i.e., the third rows are all zero⟹the non linear terms corresponding to the displacement in X_3 direction are zero⟹ Primary Rayleigh Lamb modes cannot generate secondary Shear Horizontal modes (SH modes) as SH modes have only u_3 component of displacement.

Primary RL modes cannot generate secondary SH modes.

Lemma 5 The vector representing $\mathbf{Div}\left(\mathbf{S^{NL}}(\mathbf{H}, \mathbf{H}, \mathbf{2})\right)$ has a 'Sym' nature.

Proof

In lemma 4 we proved that the matrix representation of $\mathbf{S^{NL}}(\mathbf{H}, \mathbf{H}, \mathbf{2})$ has a sym nature.

So, $\mathbf{Div}\left(\mathbf{S}^{\mathbf{NL}}(\mathbf{H},\mathbf{H},\mathbf{2})\right)$ has the following nature $\begin{Bmatrix} \frac{\partial E}{\partial X_1}+\frac{\partial O}{\partial X_2} \\ \frac{\partial O}{\partial X_1}+\frac{\partial E}{\partial X_2} \\ 0 \end{Bmatrix}=\begin{Bmatrix} E \\ O \\ 0 \end{Bmatrix}$ (Since the derivative with respect to X_1 does not change the nature of the function, but a derivative with respect to X_2 flips it).

Lemma 6 The power flow $f_n{}^{surf}+f_n{}^{vol}\neq 0 \Leftrightarrow \boldsymbol{v_n}$ is a symmetric mode.

In the context of the above discussion, $f_n{}^{surf}=-\frac{1}{2}\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})\boldsymbol{v_n}{}^{*}.\boldsymbol{n_2}\ \Big|_{-\boldsymbol{h}}^{\boldsymbol{h}}$ is non-zero $\Leftrightarrow \mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})\boldsymbol{v_n}{}^{*}.\boldsymbol{n_2}$ is an odd function $\Leftrightarrow \mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})\boldsymbol{v_n}{}^{*}$ is a 'Sym' vector $\Leftrightarrow \boldsymbol{v_n}$ is of 'Sym' nature as $\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})$ is a 'Sym' matrix $\Leftrightarrow \boldsymbol{v_n}$ is symmetric mode.

$f_n{}^{vol}=\frac{1}{2}\int_{-h}^{h}\mathbf{Div}\left(\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})\right).\boldsymbol{v_n}{}^{*}\,d\mathrm{X}_2$ is non-zero $\Leftrightarrow \mathbf{Div}\left(\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})\right).\boldsymbol{v_n}{}^{*}$ is an even function $\Leftrightarrow \boldsymbol{v_n}$ is symmetric mode as $\mathbf{Div}\left(\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})\right)$ is 'Sym' vector from lemma 5.

Corollary: From the above lemma, it can be concluded that the power flow is non-zero only to the symmetric modes.

So, for a single primary Rayleigh Lamb mode excitation, cumulative second harmonics exist only as symmetric modes.

4.2 Power flow analysis for generation of second harmonic with primary SH mode excitation

In this section we examine the case when the primary SH modes generate a secondary RL mode.

Let $u = Re\left\{\begin{bmatrix} 0 \\ 0 \\ u_3(X_2) \end{bmatrix} e^{i(kX_1-\omega t)}\right\}$ be the fundamental SH Mode.

H=Grad(*u*) has the following matrix representation

$$\boldsymbol{Re}\left\{\begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{iku_3} & \boldsymbol{\frac{\partial u_3}{\partial X_2}} & \mathbf{0} \end{bmatrix} \boldsymbol{e^{i(k\mathrm{X}_1-\omega t)}}\right\}$$

Following the same procedure as outlined in the preceding section we make the following observations

1. tr(**H**)=tr($\mathbf{H^T}$)=0.
2. $\mathbf{H^2}$=$\mathbf{0}$, $\mathbf{H^{T^2}} = \mathbf{0}$.
3. Matrix representation of $\mathbf{HH^T}$ has the following structure $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E \end{bmatrix}$ where E is an even function.
4. Matrix representation of $\mathbf{H^TH}$ has the following structure $\begin{bmatrix} E & O & 0 \\ O & E & 0 \\ 0 & 0 & 0 \end{bmatrix}$ i.e the structure of 'sym' matrix.
5. The matrix representation of $\mathbf{S^{NL}}(\mathrm{H},\mathrm{H},\mathbf{2})$ (equation 4.4) has sym structure whether u is a symmetric/antisymmetric mode.

6. $\mathbf{Div}\left(\mathbf{S}^{\mathbf{NL}}(\mathbf{H},\mathbf{H},\mathbf{2})\right)$ has the structure of a sym/anti vector depending on whether u is a symmetric/antisymmetric mode.

7. $f_n^{\ surf} = -\frac{1}{2}\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})\boldsymbol{v_n}^*.\boldsymbol{n_2}\ \Big|_{-\boldsymbol{h}}^{\boldsymbol{h}}$ is zero for every SH mode $\boldsymbol{v_n}$

8. $f_n^{\ surf} = -\frac{1}{2}\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})\boldsymbol{v_n}^*.\boldsymbol{n_2}\ \ \Big|_{-\boldsymbol{h}}^{\boldsymbol{h}}$ is non-zero only for symmetric RL modes $\boldsymbol{v_n}$

9. $f_n^{\ vol} = \frac{1}{2}\int_{-h}^{h}\mathbf{Div}\left(\mathbf{S}^{\mathbf{NL}}(\mathrm{H}_1,\mathrm{H}_1,\mathbf{2})\right).\boldsymbol{v_n}^*\, d\mathrm{X}_2$ is non zero only for symmetric RL modes $\boldsymbol{v_n}$.

We can conclude that single primary SH mode excitation can only generate RL symmetric modes as cumulative second harmonics.

4.3 Interaction of Rayleigh-Lamb Guided wave modes

In this section we formulate a generalized problem which helps us to predict the guided wave mode interaction. This is important for the following reasons:

- In most of the experiments involving guided waves, one excites more than one mode owing to the finiteness of source and the frequency bandwidth of the transducer. So, it is important that one studies the problem of guided wave mode interaction.
- It is also important for the study of higher harmonic guided wave generation in plates i.e., harmonics above the second. This is because, one can think of higher harmonics as interactions between primary mode and

the other lower harmonics and it is essential that one has a theoretical framework for studying these interactions.

We consider the interaction of two guided wave modes $\boldsymbol{u_a}, \boldsymbol{u_b}$ propagating in the plate.

Following the same procedure as in Chapter 2 the total displacement field in the plate (up to second order interactions) is

$$\boldsymbol{u} = \boldsymbol{u_a} + \boldsymbol{u_b} + \boldsymbol{u_{aa}} + \boldsymbol{u_{ab}} + \boldsymbol{u_{bb}} \tag{4.8}$$

where $\boldsymbol{u_{aa}}, \boldsymbol{u_{bb}}$ are displacement fields due to the self-interaction of mode a and b respectively, and $\boldsymbol{u_{ab}}$ is the displacement field due to the mutual interaction between modes a and b.

The displacement gradient is

$$\mathbf{H} = \mathbf{H}_a + \mathbf{H}_b + \mathbf{H}_{aa} + \mathbf{H}_{ab} + \mathbf{H}_{bb}. \tag{4.9}$$

The first Piola-Kirchoff stress tensor is given by

$$\mathbf{S}(\mathbf{H}) = \mathbf{S}^{\mathbf{L}}(\mathbf{H}_a) + \mathbf{S}^{\mathbf{L}}(\mathbf{H}_b) + \mathbf{S}^{\mathbf{L}}(\mathbf{H}_{aa}) + \mathbf{S}^{\mathbf{L}}(\mathbf{H}_{ab}) + \mathbf{S}^{\mathbf{L}}(\mathbf{H}_{bb}) + \mathbf{S}^{\mathbf{NL}}(\mathbf{H}_\mathrm{a} + \mathbf{H}_\mathrm{b}). \tag{4.10}$$

We note that

$$\mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a + \mathbf{H}_b) = \mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a, \mathbf{H}_a, \mathbf{2}) + \mathbf{S}^{\mathbf{NL}}(\mathbf{H}_b, \mathbf{H}_b, \mathbf{2}) + \mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a, \mathbf{H}_b, \mathbf{2}) \tag{4.11}$$

where $\mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a, \mathbf{H}_a, \mathbf{2}), \mathbf{S}^{\mathbf{NL}}(\mathbf{H}_b, \mathbf{H}_b, \mathbf{2})$ are the self-interaction terms as defined in Chapter 2 and $\mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a, \mathbf{H}_b, \mathbf{2})$ denotes the other second order interaction terms in $\mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a + \mathbf{H}_b)$ (equation 2.24)

$$\mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a,\mathbf{H}_b,\mathbf{2})$$

$$= \frac{\boldsymbol{\lambda}}{\mathbf{2}}\mathbf{tr}\left(\mathbf{H}_b+\mathbf{H}_b{}^{\mathbf{T}}\right)\mathbf{H}_a+\boldsymbol{\mu}\mathbf{H}_a\left(\mathbf{H}_b+\mathbf{H}_b{}^{\mathbf{T}}\right)+\frac{\boldsymbol{\lambda}}{\mathbf{2}}\mathbf{tr}\left(\mathbf{H}_a+\mathbf{H}_a{}^{\mathbf{T}}\right)\mathbf{H}_b$$

$$+\boldsymbol{\mu}\mathbf{H}_b\left(\mathbf{H}_a+\mathbf{H}_a{}^{\mathbf{T}}\right)+\frac{\boldsymbol{\lambda}}{\mathbf{2}}\mathbf{tr}\left(\mathbf{H}_a{}^{\mathbf{T}}\mathbf{H}_b+\mathbf{H}_b{}^{\mathbf{T}}\mathbf{H}_a\right)\mathbf{I}+\mathbf{2Ctr}(\mathbf{H}_a)\mathbf{tr}(\mathbf{H}_b)\mathbf{I}$$

$$+\boldsymbol{\mu}\left(\mathbf{H}_a{}^{\mathbf{T}}\mathbf{H}_b+\mathbf{H}_b{}^{\mathbf{T}}\mathbf{H}_a\right)+\mathbf{Btr}(\mathbf{H}_a)\left(\mathbf{H}_b+\mathbf{H}_b{}^{\mathbf{T}}\right)+\mathbf{Btr}(\mathbf{H}_b)\left(\mathbf{H}_a+\mathbf{H}_a{}^{\mathbf{T}}\right)$$

$$+\frac{\mathbf{B}}{\mathbf{2}}\mathbf{tr}\left(\mathbf{H}_a\mathbf{H}_b+\mathbf{H}_b\mathbf{H}_a+\mathbf{H}_a{}^{\mathbf{T}}\mathbf{H}_b+\mathbf{H}_b{}^{\mathbf{T}}\mathbf{H}_a\right)\mathbf{I}+\frac{\mathbf{A}}{\mathbf{4}}(\mathbf{H}_a\mathbf{H}_b+\mathbf{H}_b\mathbf{H}_a+$$

$$\mathbf{H}_a{}^{\mathbf{T}}\mathbf{H}_b{}^{\mathbf{T}}+\mathbf{H}_b{}^{\mathbf{T}}\mathbf{H}_a{}^{\mathbf{T}}+\mathbf{H}_a{}^{\mathbf{T}}\mathbf{H}_b+\mathbf{H}_b{}^{\mathbf{T}}\mathbf{H}_a+\mathbf{H}_a\mathbf{H}_b{}^{\mathbf{T}}+\mathbf{H}_b\mathbf{H}_a{}^{\mathbf{T}})$$

4.12

If $\boldsymbol{u}_a=\boldsymbol{Re}\left\{\boldsymbol{u}_a(\mathrm{X}_2)\boldsymbol{e}^{i(k_a\mathrm{X}_1-\omega_a t)}\right\}$, $\boldsymbol{u}_b=\boldsymbol{Re}\left\{\boldsymbol{u}_b(\mathrm{X}_2)\boldsymbol{e}^{i(k_b\mathrm{X}_1-\omega_b t)}\right\}$ then $\mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a,\mathbf{H}_b,\mathbf{2})$ **c**ontains terms with the following exponentials

$e^{i((k_a+k_b)X_1-(\omega_a+\omega_b)t)}$, $e^{i((k_a-k_b)X_1-(\omega_a-\omega_b)t)}$, $e^{-i((k_a+k_b)X_1-(\omega_a+\omega_b)t)}$, $e^{-i((k_a-k_b)X_1-(\omega_a-\omega_b)t)}$

.

These terms correspond to modes at $(\omega_a+\omega_b,k_a+k_b)$ and $(\omega_a-\omega_b,k_a-k_b)$. So, if the phase matching criterion is satisfied i.e., if there exist propagating guided wave modes at any of the above frequency-wave number combinations then there is a possibility of cumulative guided wave mode propagation if the non-zero power flow condition is satisfied for that mode. To comment on this we examine the structure of $\mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a,\mathbf{H}_b,\mathbf{2})$ for various combinations of modes $\boldsymbol{u}_a,\boldsymbol{u}_b$.

We can consider each of the terms of $\mathbf{S}^{\mathbf{NL}}(\mathbf{H}_a,\mathbf{H}_b,\mathbf{2})$ and perform an analysis similar to that in section 4.1 to conclude the following

1. Matrix representation of $\mathbf{S^{NL}}(\mathbf{H}_{\boldsymbol{a}}, \mathbf{H}_{\boldsymbol{b}}, \mathbf{2})$ has a structure of 'Sym' matrix if $\boldsymbol{u_a}, \boldsymbol{u_b}$ are modes of the same nature and has an 'Anti' structure if the modes $\boldsymbol{u_a}, \boldsymbol{u_b}$ are of opposite nature.
2. If $\mathbf{S^{NL}}(\mathbf{H}_{\boldsymbol{a}}, \mathbf{H}_{\boldsymbol{b}}, \mathbf{2})$ has a 'Sym'/'Anti' structure the power flow $f_n^{surf} + f_n^{vol} \neq 0 \Leftrightarrow \boldsymbol{v_n}$ is a symmetric/antisymmetric mode.
3. If $\boldsymbol{u_a}, \boldsymbol{u_b}$ are modes of the same nature then the guided wave mode due to their interaction is a symmetric mode and if they are of opposite nature then the guided wave mode due to their interaction is an antisymmetric mode.

The above observations are consistent with our results for cumulative second harmonic generation using single primary mode excitation. We found that the cumulative second harmonics exist only as symmetric modes and this can be thought of as interaction of the same mode which would lead to a 'Sym' matrix structure for $\mathbf{S^{NL}}(\mathbf{H}_{\boldsymbol{a}}, \mathbf{H}_{\boldsymbol{b}}, \mathbf{2})$ where a=b.

4.4 Conclusions

The following conclusions can be drawn from the analysis performed in this chapter

1. Single primary Rayleigh-Lamb mode excitation can generate only symmetric RL modes as cumulative secondary harmonics.
2. Primary modes consisting of only Rayleigh Lamb modes cannot generate higher harmonic SH modes.

3. Single primary SH mode excitation can generate only symmetric Rayleigh Lamb modes as cumulative secondary harmonics.
4. Interaction of Rayleigh-Lamb modes of same nature can generate symmetric modes as secondary modes, while interaction between those of opposite nature can generate antisymmetric modes as secondary harmonics.

Chapter 5

LITERATURE IN THE CONTEXT OF THESIS FORMULATION

INTRODUCTION

In this chapter we discuss in detail some of the earlier work done in relation to higher harmonic guided wave generation in plates. We discuss the contributions of the earlier work and also adopt a critical viewpoint to examine how the results fit into the theoretical framework developed as part of this thesis. The content of this chapter is organized as follows. Section 5.1 presents the discussion as stated earlier in a chronological manner of the work in higher harmonic guided wave generation in plates.

5.1 Literature in the context of thesis

One of the earlier works on the nonlinear interaction of guided wave modes was by Deng [1998] in which he studied the non-linear interaction of SH modes in plate. He concluded that the cumulative second harmonic generation of SH modes from the self interaction of SH modes is not possible, which is one of the results obtained from the analysis in Chapter 4 in this thesis. The theoretical formalism he adopted was quite different from the one in the present thesis. He expressed the primary wave-field using the partial wave approach and then considered the secondary wave-field as arising out of non-linear interactions among the partial waves. Then the secondary wave field is also made to satisfy the boundary conditions to obtain a set of relations which necessitate the existence of cumulative second harmonics. Deng [1999] extended his approach to study the non-linear interaction of RL modes in a plate. He concluded that the cumulative second harmonics exist only as symmetric modes. This is also one of the conclusions drawn from the analysis presented in chapter 4 of the thesis. Although the work of Deng was able to demonstrate the existence of cumulative second harmonics, it did not present a complete understanding of which modes could be used to generate cumulative second harmonics. The present work in this thesis included a detailed analysis in chapter 3 on how should one go about picking out those modes to generate any higher order harmonic. We were able to come up with a list of guided wave modes which can be used for the generation of cumulative second harmonics.

de Lima and Hamilton [2003] have developed a new formalism for the study of second harmonic guided wave propagation in plates. This is the formalism that we adopted in this thesis. The article by de Lima and Hamilton [2003] used this approach to formulate the generalized problem of guided wave mode interaction. They used a perturbation approach to formulate the second harmonic problem and arrived at the two conditions required for the generation of cumulative second harmonic. The first one is the phase matching criterion and the second one is the non-zero power flux. These two conditions together lead to the "internal resonance" condition. Although this work provided a framework for the analysis of guided wave mode interaction there was no discussion on which modes could be used for the generation of cumulative second harmonics. The present thesis used the formalism by de Lima and Hamilton and went about the analysis to predict which modes generate cumulative second harmonics and also predict the interaction of guided wave modes in plate.

Bermes *et al.* [2007] used the RL modes to characterize material non-linearity in aluminium plates. The primary modes they selected for this purpose is the S_1 mode at the longitudinal velocity, which generates a cumulative second harmonic S_2 mode at the longitudinal velocity. This mode pair was also an outcome of our analysis from chapters 3 and 4. Experiments were performed on two aluminum plates whose material non-linearity parameters were initially estimated using longitudinal waves. The material non-linearity parameters estimated using the Lamb waves were compared to those estimated using longitudinal waves and were found to be in very good agreement. Müller *et al.* [2010] has identified (but not derived) the set of guided wave modes that can be used for the generation of cumulative second harmonics. They considered the group velocity matching criterion in addition to the phase matching and power flow criterion to arrive at those modes. These are exactly the same as the ones obtained going through the analysis presented in chapters 3 and 4 of the present thesis. The criterion of group velocity matching does not arise out of the theoretical solution to the second harmonic problem but is considered necessary from a practical point of view. The rationale behind this is the argument that if the group velocities of the primary and secondary modes differ, then the power flow from the primary to the secondary modes does not take place after a certain propagation distance, then the cumulative increase in amplitude for second harmonic does not occur. Müller *et al.* have not presented a generalized approach for the phase matching criterion as the one presented in chapter 3, which can be extended to study the generation of any higher harmonic. The power flow analysis presented by Müller *et al.* is along the same lines as the one presented in the present thesis except for the fact that the thesis explicitly uses the displacement gradient **H** for carrying out the analysis which greatly simplifies and clarifies it. The usage of $\mathbf{H}$ offers several advantages which are quite evident while studying the generalized problem of RL guided wave mode interaction in chapter 4. Matlack *et al.* [2011] used the RL mode-pairs S_1-S_2 and S_2-S_4 to characterize the efficiency of each of the mode-pairs in estimating the material non-linearity parameter β. It was concluded that S_2-S_4 was more efficient but it resulted in more unwanted modes when compared to S_1-S_2. Hence they used S_1-S_2 for their experiments but made a note stating that S_2-S_4 could be used with more sophisticated experimental methods.

Srivastava and di Scalea [2009] used the theoretical formalism developed by de Lima and Hamilton and tried to predict the existence of symmetric and antisymmetric modes at higher harmonics. To that end, they started with an n^{th} order generalized strain energy function that contains higher order strain multiples up to order *n*. The theoretical formulation was based on linearized strain assumption rather than the full Lagrangian strain. This aspect of their work is plausibly incorrect for reasons listed below:

- The higher order strain term $\mathbf{H}^{\mathbf{T}}\mathbf{H}$ is significant enough in the context of the present work to be able to be neglected in the strain but to be included in the higher order strain multiples.

  For example consider the following strain energy function with third order terms in strain.

$$\widehat{\mathbf{W}}(\mathbf{E}) = \frac{1}{2}\boldsymbol{\lambda}(\mathbf{tr}(\mathbf{E}))^{2} + \boldsymbol{\mu}\mathbf{tr}(\mathbf{E}^{2}) + \frac{1}{3}\mathbf{C}(\mathbf{tr}(\mathbf{E}))^{3} + \mathbf{B}\mathbf{tr}(\mathbf{E})\mathbf{tr}(\mathbf{E}^{2}) + \frac{1}{3}\mathbf{A}\mathbf{tr}(\mathbf{E}^{3}) \quad \mathbf{5.1}$$

  with the resulting second Piola-Kirchoff stress tensor from equation 2.16

$$\mathbf{T_{RR}} = \boldsymbol{\lambda}\mathbf{tr(E)I} + \mathbf{2\mu E} + \mathbf{C(tr(E))^2I} + \mathbf{Btr(E^2)I} + \mathbf{2Btr(E)E} + \mathbf{AE^2} \qquad \mathbf{5.2}$$

We write two expressions for $\mathbf{T_{RR}}$ one using linearized strain ($\mathbf{E_{lin}}$) and the other using the full Lagrangian strain ($\mathbf{E}$).

The linearized strain is $\mathbf{E_{lin}} = \frac{1}{2}(\mathbf{H} + \mathbf{H^T})$

$\mathbf{T_{RR}}$ when linearized strain $\mathbf{E_{lin}}$ is used, is as follows

$$\mathbf{T_{RR}} = \frac{\lambda}{2}\mathbf{tr(H + H^T)I} + \mathbf{\mu(H + H^T)} + \frac{C}{4}\mathbf{tr(H + H^T)^2I} + \frac{B}{4}\mathbf{tr((H + H^T)^2)I} + \frac{B}{2}\mathbf{tr(H + H^T)(H + H^T)} + \frac{A}{4}\mathbf{(H + H^T)^2} \qquad \mathbf{5.3}$$

$\mathbf{T_{RR}}$ when Lagrangian strain $\mathbf{E}$ is used, is as follows

$$\mathbf{T_{RR}(H)} = \frac{\lambda}{2}\mathbf{tr(H + H^T)I} + \mathbf{\mu(H + H^T)} + \frac{\lambda}{2}\mathbf{tr(H^TH)I} + \mathbf{C}\,(tr(H))^2\mathbf{I} + \mathbf{\mu H^TH} + \mathbf{Btr(H)(H + H^T)} + \frac{B}{2}\mathbf{tr(H^2 + H^TH)I} + \frac{A}{4}\mathbf{(H^2 + H^{T^2} + HH^T + H^TH)} \qquad \mathbf{5.4}$$

In the expression with $\mathbf{T_{RR}}$ using linearized strain terms like $\frac{\lambda}{2}\mathbf{tr(H^TH)}I$, $\mathbf{\mu H^TH}$ are dropped but include terms of the same order like $\frac{A}{4}\mathbf{(H + H^T)^2}$, $\frac{C}{4}\mathbf{tr(H + H^T)^2}I$, $\frac{B}{4}\mathbf{tr((H + H^T)^2)I}$ and $\frac{B}{2}\mathbf{tr(H + H^T)(H + H^T)}$. This seems theoretically inconsistent and also has its impact on the higher harmonic guided wave problem formulation as will be illustrated in the following point.

- The work presumed the cause for the generation of higher harmonics is only the primary mode, but in reality once the amplitude of secondary mode becomes comparable to that of the primary mode after a certain propagation distance, the perturbation assumption initially made is incorrect and one has to consider the interaction between the primary and secondary modes too. Thus, if one is formulating the problem for a third harmonic generation one has to include the non-linear stress contribution not only from the primary mode $\mathbf{S^{NL}(H_1, H_1, 3)}$

but also due to the interaction between primary and second harmonic modes $\mathbf{S^{NL}}(\mathbf{H_1}, \mathbf{H_2}, \mathbf{2})$.

We illustrate the above fact by formulating the third harmonic problem. Consider the displacement in the plate up to a third order perturbation.

$$\boldsymbol{u} = \boldsymbol{u_1} + \boldsymbol{u_2} + \boldsymbol{u_3}$$

$$\mathbf{H} = \mathbf{H_1} + \mathbf{H_2} + \mathbf{H_3}$$

The first Piola-Kirchoff stress for the above displacement can be written as (Equation 2.28)

$$\mathbf{S}(\mathbf{H}) = \mathbf{S^L}(\mathbf{H_1}) + \mathbf{S^L}(\mathbf{H_2}) + \mathbf{S^L}(\mathbf{H_3}) + \mathbf{S^{NL}}(\mathbf{H_1}, \mathbf{H_1}, \mathbf{2}) + \mathbf{S^{NL}}(\mathbf{H_1}, \mathbf{H_2}, \mathbf{2}) + \mathbf{S^{NL}}(\mathbf{H_1}, \mathbf{H_1}, \mathbf{3}) \qquad \mathbf{5.5}$$

Following the notation introduced in chapter 2 we can formulate the three problems for $\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}$ as follows

Fundamental wave

$$\mathrm{Div}(\mathbf{S^L}(\mathbf{H_1})) - \boldsymbol{\rho_\kappa}\ddot{\boldsymbol{u}}_\mathbf{1} = \mathbf{0}$$

$$\mathbf{S^L}(\mathbf{H_1})\boldsymbol{n_\kappa} = \mathbf{0} \qquad \mathbf{5.6}$$

Second Harmonic

$$\mathrm{Div}(\mathbf{S^L}(\mathbf{H_2})) - \boldsymbol{\rho_\kappa}\ddot{\boldsymbol{u}}_\mathbf{2} = -\mathrm{Div}(\mathbf{S^{NL}}(\mathbf{H_1}, \mathbf{H_1}, \mathbf{2}))$$

$$\mathbf{S^L}(\mathbf{H_2})\boldsymbol{n_\kappa} = -\mathbf{S^{NL}}(\mathrm{H_1}, \mathrm{H_1}, 2)\boldsymbol{n_\kappa} \qquad \mathbf{5.7}$$

Third Harmonic:

$$\mathrm{Div}(\boldsymbol{S}^{\mathbf{L}}(\mathbf{H_3})) - \boldsymbol{\rho_\kappa}\ddot{\boldsymbol{u}}_{\mathbf{3}} = -\mathrm{Div}(\boldsymbol{S}^{\mathbf{NL}}(\mathbf{H_1}, \mathbf{H_1}, \mathbf{3})) - \mathrm{Div}(\boldsymbol{S}^{\mathbf{NL}}(\mathbf{H_1}, \mathbf{H_2}, \mathbf{2}))$$

$$\mathbf{S}^{\mathbf{L}}(\mathrm{H_3})\boldsymbol{n_\kappa} = -\mathbf{S}^{\mathbf{NL}}(\mathrm{H_1}, \mathrm{H_1}, 3)\boldsymbol{n_\kappa} - \mathbf{S}^{\mathbf{NL}}(\mathrm{H_1}, \mathrm{H_2}, 2)\boldsymbol{n_\kappa} \qquad \mathbf{5.8}$$

Each of the above equations (5.6-5.8) can be solved in a manner analogous to that presented in chapter 2. The behavior of the solution for the third harmonic problem can be inferred from the non-linear stress terms $\mathbf{S}^{\mathbf{NL}}(\mathbf{H_1}, \mathbf{H_1}, \mathbf{3})$ and $\mathbf{S}^{\mathbf{NL}}(\mathrm{H_1}, \mathrm{H_2}, \mathbf{2})$. From the analysis presented in chapter 4 each of the above terms represent the self-interaction between $\mathbf{H_1}$ and mutual interaction between $\mathrm{H_1}, \mathrm{H_2}$. We note that for a given primary mode $\boldsymbol{u_1}$, $\boldsymbol{u_2}$ consists of displacement contributions from all the modes at the second harmonic frequency. So, the non-linear stress term $\mathbf{S}^{\mathbf{NL}}(\mathrm{H_1}, \mathrm{H_2}, \mathbf{2})$ can be thought of as the interaction between the primary mode and the secondary modes which contribute to the displacement $\boldsymbol{u_2}$. If it so happens that the primary mode satisfies the condition of phase matching and non-zero power flow to any of the modes that contribute to displacement $\boldsymbol{u_2}$ then one can expect cumulative third harmonic generation due to their interaction. This exactly is the case that is not considered in the generalized analysis presented in Srivastava and Lanza di Scalea [2009] .

In the light of the above assumptions made, they conclude that antisymmetric modes do not exist at even harmonics but symmetric modes exist at any harmonic. This is one of the conclusions drawn as part of our analysis in chapter 4. These conclusions are true only for a single primary mode excitation. It has been proven as part of this thesis in chapter 4 that the interaction of RL guided wave modes of same nature leads to symmetric modes and that between those of opposite nature can give rise to antisymmetric modes. So, if one starts with an excitation of two modes of opposite nature at the same frequency one could expect their interaction to yield an antisymmetric mode at the second harmonic. This is contradicting the result given by Srivastava and Lanza di Scalea [2009] and arises as a result of not considering the interaction terms like $\mathbf{S^{NL}}(\mathrm{H_1}, \mathrm{H_2}, \mathbf{2})$ that arises in the generalized higher harmonic problem formulation.

Matsuda and Biwa [2011] presented an analysis in a way very similar to that of Müller *et al.* where they identified the set of modes which can be used for cumulative second harmonic generation. These included the Lame modes and extra Rayleigh modes in addition to symmetric modes at longitudinal wave speed and the mode intersection of symmetric and antisymmetric modes. These were the same modes that we obtained as a result of our analysis in chapter 3.

The entire problem formulation and analysis presented in this thesis is carried out independently except for the approach used by de Lima and Hamilton [2003] to solve the non-linear wave equation using normal mode expansion.

Chapter 6

CLOSURE

INTRODUCTION

This chapter presents the summary, conclusions and suggestions for future work with relation to the higher harmonic guided waves. The content of this chapter is organized as follows. Section 1 presents the outcome of the combined analysis of the results obtained in chapters 3 and 4. Section 2 presents a discussion of the results presented in the literature with those discussed herein.

6.1 SUMMARY

The thesis work led to the development of a theoretical framework that can model and predict higher harmonic guided wave generation and propagation in weakly non-linear homogeneous isotropic plates. The framework has been developed from the principles of continuum mechanics. Material non-linearity is taken care of by considering Lagrangian (non-linearized) strain and including higher order terms in the strain energy function. The problem is formulated in the reference configuration using the first Piola-Kirchoff stress. A perturbation approach along with the normal mode expansion technique is used to solve the boundary value problem formulated. This led to the two conditions required for cumulative second harmonic generation in plates. These are the phase matching criterion and the non-zero power flux criterion. These two put together is termed as "internal resonance". The analysis presented in chapters 3 and 4 use the above criterion and this results in the set of guided wave modes that can be used for the generation of cumulative second harmonic guided waves. The generalized problem formulation of guided wave mode interaction in chapter 4 was able to correct the theoretical inconsistencies arising in the problem formulation by previous researchers. The conclusions drawn from this can be used for predicting higher harmonic guided waves in plates.

6.2 CONCLUSIONS

This section presents a summary of the results drawn from the analysis presented in chapters 3 and 4.

6.2.1 Primary modes that are capable of generating cumulative second harmonics

In chapter 4 we concluded that the cumulative second harmonics exist only as symmetric Rayleigh-Lamb modes based on the non-zero power flux requirement for internal resonance. So, it suffices to consider only those modes which are capable of generating secondary symmetric modes for a single fundamental wave. The modes that satisfy the phase matching criterion were listed in chapter 3. We consider three separate cases, i.e., primary symmetric modes giving secondary symmetric modes, primary antisymmetric modes giving secondary symmetric modes and primary Shear Horizontal mode giving secondary Rayleigh-Lamb symmetric mode.

Primary Rayleigh-Lamb symmetric modes that can give secondary symmetric modes

a) **Cut-off modes:** Primary symmetric cutoff modes at $\omega = \frac{n\pi c_t}{h}$ can give secondary symmetric modes.

b) **Symmetric modes at $c_p = c_l$:** Symmetric modes at $\omega = \frac{n\pi c_l c_t}{h\sqrt{c_l^2 - c_t^2}}$ can give secondary symmetric modes.

c) **Mode intersections:** All the modes at the intersection of symmetric and anti symmetric modes can give secondary symmetric modes.

d) **Lame modes with $c_p = \sqrt{2}c_t$ (if $\sqrt{2}c_t > c_l$) :** Primary symmetric modes at $\omega = \frac{\sqrt{2}n\pi c_l c_t}{h\sqrt{2c_t^2 - c_l^2}}$ can generate secondary symmetric modes.

Primary Rayleigh-Lamb antisymmetric modes that can give secondary symmetric modes

a) **Cutoff modes:** Primary antisymmetric cutoff modes at $\omega = \frac{(2n+1)\pi c_t}{2h}$ can give secondary symmetric modes.

b) **Mode intersections:** All the modes at the intersection of symmetric and anti symmetric modes can give secondary symmetric modes.

c) **Lame modes with $c_p = \sqrt{2}c_t$ (if $\sqrt{2}c_t > c_l$) :** Primary antisymmetric modes at $\omega = \frac{\sqrt{2}(2n+1)\pi c_l c_t}{2h\sqrt{2c_t^2 - c_l^2}}$ can generate secondary symmetric modes.

Primary Shear Horizontal modes that can give secondary Rayleigh-Lamb symmetric modes

a) **Cut-off modes:** Primary cut off modes at $\omega = \frac{n\pi c_t}{2h}$ gives secondary symmetric modes.

b) **Modes at $c_p = c_l$:** Primary modes at $\omega = \frac{n\pi c_l c_t}{2h\sqrt{c_l^2 - c_t^2}}$ give secondary symmetric modes.

c) **Special modes:** Primary modes at phase velocities satisfying the equation

$$\left(\frac{1}{c_l^2} - \frac{1}{c_p^2}\right) = \frac{m^2}{n^2}\left(\frac{1}{c_t^2} - \frac{1}{c_p^2}\right)$$

for some integers m,n give secondary modes at the intersection of symmetric and antisymetric modes on the dispersion curves.

6.3 Interaction of RL Guided wave modes - Conclusions

The following conclusions were obtained as result of the analysis of interaction RL guided wave modes. By interaction of two guided wave modes (ω_a, k_a) and (ω_b, k_b) we mean the secondary modes that are generated at $(\omega_a - \omega_b, k_a - k_b)$ or $(\omega_a + \omega_b, k_a + k_b)$ predicted using the approach presented in section 4.3.

1. The interaction of RL guided wave modes of same nature result in symmetric modes and that between those of opposite nature result in antisymmetric modes.
2. With single primary mode excitation, the above result translates as the existence of only symmetric modes as cumulative second harmonics.

6.4 FUTURE WORK SUGGESTIONS

1. Scope for future work lies in performing experiments using the guided wave modes obtained as part of the analysis in the present work for cumulative second harmonic generation in plates.
2. Use the theoretical framework developed to study the problem of higher harmonic guided waves in plates.
3. Develop or extend the theory developed for cumulative second harmonic generation in pipes, shells and other arbitrary cross-sections like rail.
4. Use the cumulative second harmonic guided waves to characterize microstructure evolution of Alloy 617 and other materials.
5. Use the framework developed herein for acoustoelasticity analysis of plates for stress determination.

Appendix

Cauchy and Piola-Kirchoff stress tensors

We present the formal definitions and background of different kinds of stress tensors used in this thesis.

Cauchy stress: This relates the traction in the current configuration to the geometry of the current configuration. If $\boldsymbol{t}$ denoted the traction vector and $\boldsymbol{n}$ denotes the unit normal to the surface of a body in the current configuration, then the Cauchy stress **T** relates them as follows

$$\boldsymbol{t} = \mathrm{T}\boldsymbol{n} \qquad \textbf{A-1}$$

First Piola Kirchoff stress: This relates the traction in the current configuration to the geometry of the reference configuration. If $\boldsymbol{t}$ denoted the traction vector and $\boldsymbol{n_\kappa}$ denotes the unit normal to the surface of a body in the reference configuration, then the first Piola-Kirchoff stress **S** relates them as

$$\boldsymbol{tda} = \mathbf{S}\boldsymbol{n_\kappa dA} \qquad \textbf{A-2}$$

where da, dA are elemental areas in current and reference configurations respectively.

For Hyperelastic materials, if W(F) denotes the strain energy function in terms of the deformation gradient then the first Piola Kirchoff stress is given by

$$\mathbf{S}=\frac{\partial \mathbf{W(F)}}{\partial \mathbf{F}} \qquad \textbf{A-3}$$

Second Piola-Kirchoff stress: It is born out of a need to measure the stress power in the correct configuration with respect to the Lagrangian Strain.

For Hyperelastic materials, if $\widehat{\mathrm{W}}(\mathrm{E})$ denotes the strain energy function in terms of the Lagrangian strain **E**, then the first Piola-Kirchoff stress is given by

$$\mathbf{T_{RR}}=\frac{\partial \widehat{W}(\mathbf{E})}{\partial \mathbf{E}} \qquad \textbf{A-4}$$

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