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LOOP QUANTUM COSMOLOGY:

A WINDOW INTO THE PATH INTEGRAL REPRESENTATION

OF QUANTUM GRAVITY

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Abstract

In this dissertation we present sum of histories formulations of loop quantum cosmology. These are considered for exactly soluble LQC expressed in a Schrodinger form with time provided by a scalar field where we are interested in transition amplitudes and as a constrained system where we are interested in the physical inner product defined by group averaging. As loop quantum cosmology is based on the new quantum mechanics motivated from the representation of loop quantum gravity the sum over histories have many distinct features from standard Feynman path integrals. The sum over histories is given by an *discrete* sum over histories which change configuration only *finitely* many times. The resulting sum over histories have strong similarities to the current spin foam models while being derived from a well-controlled canonical quantization. Thus we are able to probe many of the open questions of spin foam models as well as provide new insights and avenues of research.

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Dedication

To those who fostered my love of math and physics at each stage of my life: Doris Maclaughlin, Fred Campisano, Uriel Nauenburg, and Abhay Ashtekar. Most especially to my parents, whose love and support made this possible.

Chapter 1

Introduction

The beginning of the 20th century saw a great leap in our understanding of both the large scale and the small structure of the universe through the development of general relativity and quantum mechanics. The pressures of theoretical inconsistencies between established theories and their conflict with new experimental data required a new understanding of our universe. General relativity and quantum mechanics have each been astonishingly successful in explaining the physics within their own realms of validity. Despite a lack of direct conflict with experiment, it quickly became apparent that while the two theories should be combined within a single framework, commonly called quantum gravity, the physical and mathematical structures upon which they are based are radically different. Perhaps the most striking difference is that quantum mechanics and its relativistic extension, quantum field theory (QFT), make heavy use of fixed background geometries while on the other hand general relativity indicates that the force of gravity is encoded in the geometry of space-time which is itself a dynamic object.

There have been many attempts and are currently many approaches to obtain a theory of quantum gravity most of which derive from one of two paths from a classical system to a corresponding quantum theory. The first is the canonical approach as pioneered by Dirac for the quantization of constrained systems [1], which is rooted in the Hamiltonian formulation of classical mechanics. Second, is path integral or sum over histories quantization as introduced by Feynman [2] with closer ties to Lagrangian mechanics. In well understood systems the two approaches are known to be equivalent [3]. It is then natural to expect the same

to hold for quantum gravity; given a canonical description of quantum gravity there should exist an equivalent sum over histories description and vice versa. In this dissertation we analyze the sum over histories representations of canonical loop quantum gravity (LQG) by constructing them in the simplified model of loop quantum cosmology (LQC). Further we use the sum over histories representations of LQC as a window into the open problems of spin foam models (SFM) which are conjectured to be related to canonical LQG.

1.1   Canonical Quantum Gravity

Canonical quantum gravity is a long standing program to quantize gravity following the procedures for the quantization of systems with constraints as pioneered by Bergmann and Dirac. It is rooted in the development of a Hamiltonian framework for general relativity where the basic canonical variables are a 3-metric on a spatial slice and its conjugate momentum. Thus the Einstein field equations for the full metric, which determines the geometry of space-time, are reinterpreted as evolution equations for the 3-metric. The dynamics are encoded in the Hamiltonian for general relativity, which due to the diffeomorphism invariance of the theory is a sum of constraints. The constraints serve two purposes: one to restrict the choice of 3-metric and its conjugate momentum, and second to generate the symmetries of the theory through the Poisson brackets of the canonical variables with the constraints.

The basic content of the Dirac procedure for such a constrained theory is to first quantize the theory ignoring the constraints obtaining a kinematical Hilbert space and afterwards to impose the constraints as operators on the kinematic Hilbert space to obtain the physical theory. The first step is to choose a set of elementary variables that will be represented in the quantum theory such as the configuration and momentum variables of Schrodinger quantum mechanics. These form an algebra generated by products and sums of variables and most importantly the Poisson brackets between them. The second step is to obtain a representation of this algebra by operators on a Hilbert space where the Poisson bracket between elementary variables is replaced by the commutator between the corresponding operators. The resulting space is referred to as the kinematic Hilbert space and

serves as the foundation for the quantization of the constrained theory. The third step is to write the constraints in terms of the elementary variables and represent them as operators on the kinematic Hilbert space. The fourth and final step is to find the physical states which are annihilated by the constraint operators and to choose an inner product on the space of these physical states. The result is the physical Hilbert space consisting of states that are annihilated by each constraint.

Loop quantum gravity is the latest development of the canonical quantization program. [4, 5, 6]. Beginning with the introduction of a new set of variables [7, 8] rephrasing general relativity as a theory of connections, the theory is expressed in the same language as the gauge theories which describe the other fundamental forces. The phase space of general relativity is then given by pairs of SU(2) connections, A_a^i, and its conjugate momenta, E_i^a, from which the metric can be reconstructed. Drawing intuition from lattice gauge theory the elementary configuration variables are taken to be the holonomy, $h_e(A)$, of the connection, A_a^i, which determine how a spinor is parallel transported along the edge, e. The elementary momentum variables are the fluxes of the gravitational 'electric' field, E_i^a through surfaces S, $E(S, f) = \int_S E_i^a f^i d^2 S_a$. Together these define the holonomy/flux algebra that is represented in the quantum theory. Through the rigorous construction of a measure on the space of connections, it was possible to construct the Hilbert space of square integrable functions of (generalized) connections, $L_2(A, d\mu_{AL})$, on which the holonomy/flux algebra can be represented [9, 10, 11, 12, 13] . Further it has been shown that up to technical assumptions this representation is *unique* due to the requirement of diffeomorphism invariance [14, 15]. Therefore unlike the the original approach to canonical quantum gravity in terms of metric variables, in LQG the kinematic Hilbert space can be rigorously constructed thereby providing a solid foundation upon which to build the physical theory.

Already at the kinematic level there are profound implication for quantum gravity. First although the full kinematic Hilbert space is non-separable and seemingly too large to allow practical computations, it can be decomposed into orthogonal Hilbert spaces $\mathcal{H}_{\gamma,j}$ which are each finite dimensional. Each Hilbert space, $\mathcal{H}_{\gamma,j}$ is labelled by a graph, γ and spins j (non-zero half integers) associated to each edge of the graph. For each choice of graph, γ, and spins the Hilbert space $\mathcal{H}_{\gamma,j}$ is the Hilbert space of a simple spin system. Each element of the Hilbert spaces $\mathcal{H}_{\gamma,j}$

is referred to as a spin-network state and can be specified by the assignment of an intertwiner (an object from representation theory) to each vertex of the graph, γ. These spin-network states provide a basis for the full kinematic Hilbert space called the spin-network basis. Second, these spaces are invariant under the action of geometric operators constructed solely from the fluxes, including most importantly the operators associated to the area of a surface and to the volume of a region. Further these geometric operators constructed from the fluxes have a discrete spectrum rovelli:1995, qtg1, qtg2, thiemann:1998. This has the surprising implication that both area and volume are quantized and are allowed to take only certain discrete values. In this sense the spin-network states provide the eigenstates of *quantum geometry.*

1.2 The Problem of Dynamics

As much is known about the kinematical structure of the LQG, there is a large outstanding problem for the field: Dynamics. The rigorously developed kinematical Hilbert space, $\mathcal{H}_{\text{kin}}$, provides the foundation upon which to construct the physical Hilbert space by incorporating the constraints. The dynamics of general relativity written as theory of connections is given by three constraints: the Gauss and diffeomorphism constraints which generate local gauge transformations and spatial diffeomorphisms and the Hamiltonian constraint which generates the dynamics or equivalently time reparametrizations. The problem of dynamics is three-fold: first, to find a viable representation of the constraints (or their exponentiation) on the kinematic Hilbert space [1] by writing the classical constraints in terms of the elementary operators; second, given a particular candidate for the quantum constraints to find the states annihilated by them (or equivalently the states invariant under the group generated by their exponentiation) and a suitable inner product between these physical states ; and third, to extract from this physical Hilbert space observable physics.

For both the Gauss and diffeomorphism constraints this construction has been carried out. The construction of the full physical Hilbert space composed of solutions to all the constraints including the Hamiltonian constraint remains the major

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[1]Or a suitable space of distributional states

open question of the theory. The main obstruction to obtaining the complete dynamics has been the myriad of ambiguities present in writing the Hamiltonian constraint operator in terms of the elementary operators [16, 17, 18] and further the lack of simple means to determine the 'viability' of a particular choice. Given a particular choice of the quantum Hamiltonian constraint the second difficulty is in obtaining physical Hilbert space. The Gauss and diffeomorphism constraints each generate a simple action on the kinematic Hilbert space making it possible to construct the physical states and the physical inner product for example through the group averaging procedure. On the other hand, even classically the Hamiltonian constraint does not generate a simple geometric action, so naturally the corresponding operator is equally complex. The problem of dynamics has been a point of bifurcation leading researchers in two primary directions. First, working in simpler systems such as symmetry reduced models, gravity in lower dimensions, and recently parametrized field theory which has provided considerable insight and key directions for future progress. The other primary direction has been to try to gain insight into the canonical theory by developing an analogous theory within the path integral paradigm, which has led to a very enticing picture for quantum gravity.

1.3   Spin Foam Models

One approach to understand the dynamics of LQG has been spin foam models (SFM). These models side-step the problem of defining the Hamiltonian constraint operator and obtaining the physical Hilbert space by working within the path integral approach to quantization. Recall that in non-relativistic quantum mechanics the probability amplitude for a particle initially in the configuration q_i at time $t = t_i$ to evolve to a final configuration q_f at time $t = t_f$, as defined by the canonical theory, can be equivalently expressed as an integral over all histories of the configuration variable $q(t)$ interpolating between these configurations ($q(t_i) = q_i$ and $q(t_f) = q_f$) each weighted by the exponential of the classical action, e^{iS}, evaluated on $q(t)$. The path integral representation can be constructed from the canonical theory following the method introduced by Feynman [2], or can be used to directly define the quantum theory. The latter route is often used in quantum

field theory where a path integral is used to define the theory and to derive the Feynman diagrams used to compute scattering amplitudes.

The path integral paradigm gives a distinct route to quantum gravity as defined by a sum over all possible histories of the 3-geometries of a spatial slice, or equivalently a sum over all 4-geometries each weighted by an exponential of the classical Einstein-Hilbert action. In ordinary quantum mechanics and quantum field theories on a fixed background geometry, such a path integral provides the transition amplitude for the first state, specified at initial time, to evolve to the second state at the final time. In the background independent context of quantum gravity, one does not have access to a time variable and dynamics is encoded in constraints. Therefore the notion of a transition in a pre-specified time interval is not a priori meaningful. Rather, the sum over histories would have to provide the physical inner product between solutions to the quantum constraints, extracted from the initial and final states.

Such an approach was introduced formally by Misner [19]. Defining quantum gravity by a sum over all possible metrics or 4-geometries is exceedingly difficult to implement beyond the formal level due to the diffeomorphims invariance of the theory. The presence of this symmetry implies that there are many 4-geometries which are physically equivalent only one of which should contribute to the path integral. In path integrals for gauge theories such as QED where there are similarly many physically equivalent histories it is necessary to introduce a gauge fixing or some method to parametrize the space of physically distinct histories. For gravity the exceptional difficulty is in parametrizing the space of diffeomorphism equivalence classes of geometries. A more recent approach is to introduce a discretization of gravity using the ideas of Regge calculus [20, 21, 22, 23]. In this approach the space of physically distinct geometries can be parametrized allowing for the definition of a regularized sum over geometries. Approaches of this type have a distinct difficulty which is the need to eventually take the continuum limit where the regularization is removed.

New insight coming from both the kinematic structure of LQG as well as results from lower dimensional gravity has led the an enticing picture for quantum gravity expressed as a sum over *quantum* 4-geometries. The basic concept is that in constructing a path integral from the canonical theory the precise form of the histories

of the configuration variables is specified by the eigenvalues of the corresponding operators. In Schrodinger quantum mechanics the spectrum of the position operator is identical to the classical configuration space, so the histories summed over are simply those $q(t)$ taking values in the classical configuration space. In LQG the eigenvalues of geometrical operators such as area and volume are discrete as opposed to the continuous classical geometry. As such we would expect a path integral derived from LQG to be a sum over histories of quantum 3-geometries, or since the spin-network basis states effectively provide the eigenstates of geometry the path integral is expected to consist of sums over histories of spin-network states.

In the language of SFM this heuristic idea is expressed as follows. A general SFM can be defined by considering a general 2-dimensional piecewise linear cell complex Δ^*, which is roughly a finite collection of polygons glued together along their edges with two boundaries that are graphs. Depending on the model this 2-cell complex is either considered to be embedded in a 4-manifold M where the its boundaries lie on the two 3-surfaces, S_1 and S_2, bounding the manifold M or it is considered to be simply an abstract object. The cell complex Δ^* plays the role of a discretization of the 4-manifold M; for certain choices of the 2-dimensional cell complex there is an associated discretization of the manifold into the 4-dimensional analogue of tetrahedra, a simplicial decomposition Δ where each vertex of Δ^* lies inside a 4-dim tetrahedra, each edge of Δ^* is related to a tetrahedra of Δ, and so on. For this reason Δ^* is often referred to as the dual triangulation. The 2-cell complex is often referred to as the dual triangulation as One can think of S_1 as an 'initial' surface and S_2 as a 'final' surface. On each surface one can fix a basis state of the kinematic Hilbert space of LQG, a spin network, to specify an 'initial' and a 'final' state of the quantum 3-geometry. A quantum 4-geometry interpolating between the initial and final states is given by assigning to each face (polygon) of the 2-cell complex a spin j and to each edge of the 2-cell complex an intertwiner. The dual triangulation colored with spin labels and intertwiners of SU(2) can be seen as a history of spin-network states and therefore a history of quantum 3-geometries interpolating between the initial and final states. To each such coloring of Δ^* by spins and intertwiners there is an associated amplitude. The spin foam amplitude associated to the 2-cell complex is then given by a sum over all coloring

of the 2-cell complex. The conjecture is that the physical inner product can then be obtained by summing these spin foam amplitudes over a suitable family of the dual triangulations Δ^*. The sum over all dual triangulations is often referred to as the *vertex expansion* because the $M - th$ term in the series corresponds to a dual triangulation with M vertices.

Four different avenues to quantum gravity point towards the framework which underlies the spin-foam models (SFMs). The first avenue is the heuristic idea from LQG as above. The other routes are each provided by quantum gravity in lower dimensions. The first comes from the equivalence of general relativity in 3 space-time dimensions with BF theory [24], a topological gauge theory having no local degrees of freedom. For BF theory and thus gravity in 3 space-time dimensions there is a sum over histories of the SFM type. Although as the theory is topological the path integral is defined with a fixed but arbitrary discretization and is independent of the choice of discretization. In higher dimension general relativity can be regarded as a constrained BF theory [25]. The path integral formulation of the BF theory provided the second avenue and led to the SFM of Barret and Crane [26] as defined by constraining the SFM of BF theory. The third route comes from the Ponzano-Regge model of 3-dimensional gravity [27] that inspired Regge calculus in higher dimensions [20, 21, 22]. Here one begins with a simplicial decomposition of the space-time manifold, describes its discrete Riemannian geometry using edge lengths and deficit angles and constructs a path integral in terms of them. If one uses holonomies and discrete areas of loop quantum gravity in place of edge lengths, one is again led to a spin foam. The fourth avenue starts from approaches to quantum gravity in which gravity is to emerge from a more fundamental theory based on abstract structures that, to begin with, have nothing to do with space-time geometry. Examples are matrix models for 2-dimensional gravity and their extension to 3-dimensions —the Boulatov model [28]— where the basic object is a field on a group manifold rather than a matrix. The Boulatov model was further generalized to a group field theory (GFT) tailored to 4-dimensional gravity [5, 29, 30]. The perturbative expansion of this GFT turned out be very closely related to the vertex expansions in SFMs. Thus the SFMs lie at a junction where four apparently distinct paths to quantum gravity meet.

While the heuristic idea is well motivated by the convergence of the kinematic

structure of LQG and quantum gravity in 3-dimensions, there are numerous open questions about the detailed structure of SFMs for 4-dimensional gravity First, SFMs in 4-dimension are developed entirely within the framework of discretized gravity, so the relationship to continuous gravity is not clear. As the theory does have local degrees of freedom it is not independent of the choice of discretization and this dependence must be removed to obtain a quantum theory of continuous gravity. The candidates for recovering continuous gravity are either a sum over all triangulations or a refinement of a fixed triangulation, but not a continuum limit as expected from non-topological gauge theories or the standard Regge approach [31]. Hints in this direction come from the perturbative expansion of GFT which gives a sum over triangulations as in the vertex expansion of SFMs. Second the SFMs in 4-dimensions have not not been systemically derived following established quantization procedures, either those coming from path integral or canonical quantization. The derivations are based on new methods of imposing second class constraints in the quantum theory, which have yet to be tested sufficiently in known systems. Where they have been analyzed they have been shown to fail by not giving the correct physical states and dynamics of the quantum theory [32]. We are thus led to ask if the SFM picture survives beyond topological models.

1.4 Simpler Systems as Probe of Dynamics

The second main program to understand the dynamics of LQG has been the application of the quantization techniques of LQG to simpler systems such as symmetry reduced gravitational systems, gravity in lower dimensions , and parametrized field theory. Each provides a simpler arena in which to probe the problems mentioned above by reducing the degrees of freedom and the number of constraints, or by allowing comparison with systems whose quantization is known through other methods. First, Loop quantum cosmology (LQC) provides a technically simpler yet physically interesting context to explore the problem of dynamics. In LQC the principles of LQG are applied to cosmological models having a high degree of symmetry where this symmetry is imposed at the classical level. Thanks to this symmetry it has been possible to construct and analyze in detail the physical Hilbert space for these models in a number of cases

[33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. LQC shares many of the conceptual problems and mathematical structures of LQG and thus provides a fertile ground to test various and ideas of full LQG. LQC thus far has successfully provided insight into the dynamics of LQG including how to extract physical information and most importantly hints into the correct form of the Hamiltonian constraint. Similarly recent work by Varadarajan and Laddha [47, 48, 49] and later by Thiemann [50] used insights gained from LQC to carry out a detailed analysis of the dynamics of polymer parametrized field theory. Further the combined insight from this work and LQC has led to the remarkable construction of a diffeomorphism constraint operator for full LQG [51]. Thus insights from simplified systems have had dramatic impact on the full theory of LQG

As LQC has provided a successful test bed for many of the open questions of LQG we are led to ask if LQC can additionally provide insight into the structure of possible path integral representations of LQG including the SFMs. In this dissertation we start from the well understood and exactly soluble model of LQC, which is the canonical quantization of homogeneous and isotropic cosmology with a massless scalar using the tools of LQG. This model has the benefit of two equivalent descriptions. First, as a system where the scalar field is treated as a clock and the quantum dynamics are defined by a Schrodinger like equation. This framework allows for contact with the standard quantum mechanics and allows for simpler extraction of physical results form the model. Second, the system can be analyzed within the timeless framework where instead of the quantum states evolving in 'time' according to a Schrodinger equation, the quantum dynamics are given by the requirement that the physical states are annihilated by the quantum constraint. For each picture we will derive a path integral for soluble LQC by closely following the Feynman construction from the canonical theory. The resulting path integrals though are quite distinct from those of Schrodinger quantum mechanics due to the different representation used in LQC which is motivated from the unique representation used in LQG. We show that the path integrals derived mimic the structure of the vertex expansion of the SFMs. In particular the path integrals are given by *discrete* sums over histories where the configuration variable changes value only a *finite* number of times.

The vertex expansion for LQC has the benefit of being constructed from the

well-understood canonical quantization. As such it is possible to avoid formal manipulations and to pinpoint the single assumption required to obtain the vertex expansion. Further we can compare the resulting expansion to the exact physical inner product. We can then use the similarity between the path integrals of LQC and the SFM to shed light on the open questions of the current SFMs. Most importantly the vertex expansion of LQC provides a proof of concept that a vertex expansion as in the SFMs can be constructed starting from the canonical theory. We show that this construction exists for a large class of models with representations similar to LQG. Within the context of LQC we are able to study how the vertex expansion solves the quantum constraint and further how well it provides an approximation to the exact physical inner product. We are able to probe long-standing issues such as the role of orientation in the spin foam histories and the physical interpretation of the coupling constant λ appearing in the perturbative expansion of GFT. We also introduce new tools for the study of vertex expansions of SFM such as a notion of coarse graining or renormalization.

1.5 Overview

The outline of the dissertation is as follows. In chapter two we first overview the salient features of LQC and introduce the deparametrized form of LQC where the scalar field is used as a clock variable. We then follow the Feynman construction of sum over histories representation for the deparametrized form leading to the vertex expansion for LQC, and conclude with a study of the properties of the expansion and its relation to SFMs. Chapter three introduces the group averaging procedure as a way to solve the constraints of quantum gravity and extends the construction of chapter two to obtain a distinct vertex expansion as derived from the timeless framework. Chapter four then extends the construction of the vertex expansion beyond LQC to more complex gravitational systems coupled to a scalar field. Finally chapter five considers the vertex expansion of vacuum models of LQC which require a regularization to remain well defined.

Chapter 2

Deparametrized LQC

Loop quantum cosmology provides a physically interesting yet technically simpler context in which explore the open issues of loop quantum gravity [52, 53]. In this chapter we will use LQC to probe the connection between loop quantum gravity (LQG) and the spin foam models (SFM). The goal of this chapter is to show that a detailed analysis of this example provides support for the general paradigm underlying SFMs and also sharpens the discussion of many open questions. Further this chapter serves to introduce the techniques and ideas used in the later chapters in a technically and conceptually simpler setting.

In this chapter we will first provide an overview of the salient aspects of LQC. Then starting from the deparametrized form of LQC where the scalar field is treated as a clock variable we construct a sum over histories expansion by following the Feynman construction as closely as possible. The result is a sum over discrete histories that closely mimics the vertex expansion in the spin foam paradigm. We then test the vertex expansion to see how well it provides a perturbative approximation to the exact transition amplitudes of LQC. We finally conclude with discussion of the relation to SFM and insights gained.

2.1   Soluble Loop Quantum Cosmology

In this chapter and the next we will focus on the simplest LQC model that has been analyzed in detail [33, 34, 35, 38]: the k=0, Λ=0 Friedmann model with a massless scalar field as a source. However, since the derivations do not depend

on the detailed structure of the model, it is rather straightforward to extend this analysis to allow for more complex models such as those with non-zero cosmological constant [39, 40], the spatially compact k=1 case [36], anisotropic models [41, 42], and inhomogeneous models [44, 45, 46]. In this section then we provide an overview of the important aspects of soluble LQC.

As mentioned in the introductory chapter LQC is the quantization of symmetric systems in general relativity following as closely as possible the quantization of LQG. The canonical quantization procedure begins with the Hamiltonian formulation of general relativity in terms of connection variables [7, 8] where the phase space consists of an SU(2) valued connection, A_a^i and the conjugate electric field , E_i^a. Based on this there is a natural algebra of functions on this classical phase space: the holonomies of the connection along edges, $h(A, e) = \mathcal{P} \exp \int_e A^i \tau_i$, and the fluxes of the conjugate field through surfaces, $E(S, f) = \int_S E_i f^i$. This 'holonomy-flux' algebra is what is represented in the quantum theory in such a way that preserves the background independence of the theory.

For the homogeneous and isotropic FRW models, classically one begins by fixing a (spatial) manifold S, topologically $\mathbb{R}^3$, cartesian coordinates x^i thereon, and a fiducial metric q_{ab}^o given by $q_{ab}^o \mathrm{d}x^a \mathrm{d}x^b = \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2$. Since the model considers homogeneous and isotropic metrics, the physical 3-metric q_{ab} is then determined solely by a scale factor a; $q_{ab} = a^2 q_{ab}^o$. For the Hamiltonian analysis one further fixes a cubical fiducial cell $\mathcal{V}$ whose volume with respect to q_{ab} is given by $V = a^3 V_o$. This cell is necessary to obtain finite results in the Hamiltonian theory, but it is ensured that any physical quantity is independent of the cell.

Using homogeneity and isotropy the triad and connection can be fixed such that they depend on the spatially constant variables c, p.

$$A_a^i = cV_o^{-1/3o}\omega_a^i \quad E_i^a = pV_o^{-2/3}\sqrt{{}^o q}{}^o e_i^a \tag{2.1}$$

From these we classically construct the holonomies and fluxes along the edges and surfaces that are tailored to the symmetries of the system. These are entirely determined by the quantities p and $e^{i\mu c}$, which form algebra that we seek a representation for. Following input from the full theory the resulting representation is one where the operator $e^{i\mu c}$ is well-defined, while there is no operator corre-

sponding to the connetion, c. From the dynamics of LQC it is more convenient to express the theory in terms of the conjugate variables ν and b where the quantity ν is related to the volume by $V = 2\pi\gamma\ell_{\text{Pl}}^2 |\nu|$ [38].[1]

The kinematical Hilbert space is a tensor product $\mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{kin}}^{\text{grav}} \otimes \mathcal{H}_{\text{kin}}^{\text{matt}}$ of the gravitational and matter Hilbert spaces. Elements $\Psi(\nu)$ of $\mathcal{H}_{\text{kin}}^{\text{grav}}$ are functions of ν with finite norm

$$||\Psi||^2 := \sum_\nu |\Psi(\nu)|^2 \tag{2.2}$$

where the sum runs over all real values ν. Elements of $\mathcal{H}_{\text{kin}}^{\text{grav}}$ must then have support on only a countable number of points. On this space ν acts simply by multiplication and $e^{i\lambda b}$ acts by translations. The matter Hilbert space on the other hand is the standard one: $\mathcal{H}_{\text{kin}}^{\text{matt}} = L^2(\mathbb{R}, d\phi)$. Thus, the kinematic quantum states of the model are functions $\Psi(\nu, \phi)$ with finite norm

$$||\Psi||^2 := \sum_\nu \int d\phi \, |\Psi(\nu, \phi)|^2. \tag{2.3}$$

A (generalized) orthonormal basis in $\mathcal{H}_{\text{kin}}$ is given by $|\nu, \phi\rangle$ with

$$\langle \nu', \phi' \,|\, \nu, \phi \rangle = \delta_{\nu'\nu} \, \delta(\phi', \phi). \tag{2.4}$$

Notice that here the volume eigenstates have an inner product given by the Kronecker delta function as opposed to the Dirac delta function of matter Hilbert space. This is a key similarity to LQG where the kinematic Hilbert space is non-separable and the spin network basis, the eigenstates of quantum geometry, has a similar inner product that can be normalized to a Kronecker delta. This property is a result by the diffeomorphism invariance of the theory and leads to many of the interesting aspects including the non-existence of an operator corresponding to the connection. Additionally having a basis normalizable to the Kronecker delta is *the* essential element for the derivation of the vertex expansion in the following sections and any system sharing this feature will admit a similar expansion. Namely the

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[1]In LQG the basic geometric variable is an orthonormal triad and the physical metric q_{ab} is constructed from it. If the triad has the same orientation as the fiducial one, given by the coordinates x^i, the configuration variable ν is positive and if the orientations are opposite, ν is negative. Physics of the model is insensitive to the triad orientation and hence to the sign of ν. In particular quantum states satisfy $\Psi(\nu, \phi) = \Psi(-v, \phi)$.

vertex expansion presented here can be extended to full loop quantum gravity. We will refer to such systems as having a discrete kinematical Hilbert space or a discrete inner product. Where here discrete refers to the discrete topology of the Hilbert space coming from the Kronecker delta inner product.

Now to obtain the physical Hilbert space one must impose a quantum version of the constraint equation. One first notes that the quantum constraint can be written in terms of the basic operators as

$$C\Psi(\nu,\phi) \equiv p_\phi^2/\hbar^2 \Psi(\nu,\phi) - \Theta\Psi(\nu,\phi) = 0 \tag{2.5}$$

where Θ is a positive and self-adjoint operator on $\mathcal{H}_{\text{kin}}^{\text{grav}}$ [54]. More explicitly, Θ is a second order difference operator [41]

$$\left(\Theta\Psi\right)(\nu) := -\frac{3\pi G}{4\ell_o^2}\Big[\quad \sqrt{|\nu(\nu+4\ell_o)|}\,(\nu+2\ell_o)\,\Psi(\nu+4\ell_o) \; - \; 2\nu^2\Psi(\nu)$$
$$+ \quad \sqrt{|\nu(\nu-4\ell_o)|}\,(\nu-2\ell_o)\,\Psi(\nu-4\ell_o)\Big], \tag{2.6}$$

where ℓ_o is related to the minimum area eigenvalue of LQG, the 'area gap', $\Delta = 4\sqrt{3}\pi\gamma\,\ell_{\text{Pl}}^2$ via $\ell_o^2 = \Delta$. The form of Θ shows that the space of solutions to the quantum constraint can be naturally decomposed into sectors in which the wave functions have support on specific 'ν-lattices' [34] given by $\{\nu_o + 4n\ell_o \mid \nu_o \; \epsilon \; [0, 4\ell_o)$ and n $\epsilon \; \mathbb{Z}\}$. For definiteness, we will restrict ourselves to the lattice $\{4n\ell_o\}$ where n is an integer. Details of the expression of Θ will not be needed in the following analysis.

2.2   Deparameterized Framework

To obtain the physical states we first notice that the scalar field ϕ is monotonic on all classical solutions (also in the cases when k=1, and $\Lambda \neq 0$) and therefore serves as a relational time variable, a la Leibnitz, in the classical theory. This interpretation carries over to the quantum theory. Where, the form of the quantum constraint (2.5) is similar to that of the Klein-Gordon equation with ϕ playing the role of time and $-\Theta$ of the spatial Laplacian (or, the elliptic operator generalizing the Laplacian if we are in a general static space-time). Therefore, in LQC, one can use

ϕ as an internal time variable with respect to which physical quantities such as the density, scalar curvature, anisotropies and other geometric quantities evolve.

A second conceptually important observation is that, as in the Klein-Gordon case, there is a superselection. A complete set of physical observables (Dirac observables), which are those that commute with the Hamiltonian constraint, is given by the scalar field momentum $p_\phi = -i\partial_\phi$ and the volume $V|_{\phi_o}$ (or, equivalently, the energy density operator $\rho|_{\phi_o}$) at the value $\phi = \phi_o$ of the internal time [34, 38, 35]. The action of these Dirac observables as well as time evolution leaves the space of positive and negative frequency solutions invariant. Therefore, as in the Klein-Gordon theory, we are led to work with either set. In LQC, one generally works with the positive frequency ones. The similarity of the form of the quantum constraint to the Klein-Gordon equation suggests that we use ϕ as relational time to deparameterize the quantum theory. As in the Klein-Gordon theory, one can perform a group averaging procedure to arrive at the physical Hilbert space $\mathcal{H}_{\mathrm{phy}}$ [33]. Then the physical Hilbert space $\mathcal{H}_{\mathrm{phy}}$ of LQC consists of *positive frequency* solutions $\Psi_+(\nu, \phi)$ to the quantum constraint (2.5), i.e. solutions satisfying

$$-i\partial_\phi\,\Psi_+(\nu, \phi) = \sqrt{\Theta}\Psi_+(\nu, \phi) \equiv H\Psi_+(\nu, \phi) \qquad (2.7)$$

with the inner-product

$$(\Psi_+,\, \Phi_+)_{\mathrm{phy}} = \sum_{\nu=4n\ell_o} \bar{\Psi}_+(\nu, \phi_o)\, \Phi_+(\nu, \phi_o)\,. \qquad (2.8)$$

which is independent of the value ϕ_o of ϕ at which the right side is evaluated. Note that, because of deparametrization, the quantum constraint can be regarded as 'evolving the state in relational time ϕ'.

While this construction of $\mathcal{H}_{\mathrm{phy}}$ does not *require* us to think of ϕ as internal time in quantum theory, this interpretation is natural in the light of final Eqs (2.7) and (2.8). For, these equations suggest that we can think of ν as the sole configuration variable and introduce 'Schrödinger states' $\Psi(\nu)$ through the physical inner product (2.8). These 'evolve' via (2.7). In this picture, the restriction to positive frequency states has direct interpretation: $p_\phi \equiv \sqrt{\Theta}$ is now a positive operator on $\mathcal{H}_{\mathrm{phy}}$ just as p_0 is a positive operator on the traditional Klein-Gordon Hilbert space.

2.3 Sum Over Histories

Given LQC expressed in the Schrodinger picture where we regard ν as a configuration variable and ϕ as time we will now construct a configuration path integral for this model. As the physical states evolve via a Schrödinger like equation:

$$-i\partial_\phi \Psi(\nu, \phi) = \sqrt{\Theta}\Psi(\nu, \phi) \equiv H\Psi(\nu, \phi), \qquad (2.9)$$

the primary object of interest will be the transition amplitude

$$A(\nu_f, \phi_f; \nu_i, \phi_i) = \langle \nu_f| e^{iH(\phi_f - \phi_i)}|\nu_i\rangle \qquad (2.10)$$

for the initial configuration $|\nu_i\rangle$ at time ϕ_i to evolve to $|\nu_f\rangle$ at time ϕ_f. Because of the close similarity of (2.9) with the Schrödinger equation, we can now pass to a sum over histories formulation of quantum dynamics using the procedure introduced by Feynman [2]. It is immediate from the form of (2.9) that this amplitude depends only on the difference $\phi_f - \phi_i$. Therefore, without loss of generality we will set $\phi_i = 0$ and $\phi_f = \varphi$ and refer to this time interval as $\mathcal{I}$. Let us divide $\mathcal{I}$ into N parts each of length $\epsilon = \varphi/N$ and write the transition amplitude as

$$A(\nu_f, \varphi; \nu_i, 0) := \langle \nu_f|e^{i\varphi H}|\nu_i\rangle = \langle \nu_f| \prod_{n=1}^{N} e^{i\epsilon H}|\nu_i\rangle \qquad (2.11)$$

we then introduce a decomposition of the identity operator at each intermediate time $\phi = n\epsilon$, $(n = 1, 2, \ldots, N-1)$, $I = \sum_{\bar{\nu}} |\bar{\nu}\rangle\langle\bar{\nu}|$, where the sum runs over the invariant lattice. [2]

$$
\begin{aligned}
A(\nu_f, \varphi; \nu_i, 0) &= \sum_{\bar{\nu}_{N-1}, \ldots, \bar{\nu}_1} \langle \bar{\nu}_N|e^{i\epsilon H}|\bar{\nu}_{N-1}\rangle\langle \bar{\nu}_{N-1}|e^{i\epsilon H}|\bar{\nu}_{N-2}\rangle \ldots \langle \bar{\nu}_1|e^{i\epsilon H}|\bar{\nu}_0\rangle \\
&= \sum_{\bar{\nu}_{N-1}, \ldots, \bar{\nu}_1} U_{\bar{\nu}_N\bar{\nu}_{N-1}} U_{\bar{\nu}_{N-1}\bar{\nu}_{N-2}} \ldots U_{\bar{\nu}_1\bar{\nu}_0}, \qquad (2.12)
\end{aligned}
$$

For notational simplicity we set $\nu_f = \bar{\nu}_N$ and $\nu_i = \bar{\nu}_0$ and to further simplify denoted the matrix element $\langle \bar{\nu}_n| \exp i\epsilon H|\bar{\nu}_{n-1}\rangle$ by $U_{\bar{\nu}_n\bar{\nu}_{n-1}}$. The division of $\mathcal{I}$ provides

---

[2]Here the complete basis can be just the lattice $\nu = 4n\ell_o$ or equivalently all real values of ν. The dynamics will automatically select the invariant lattice as the matrix elements $U_{\bar{\nu}_n\bar{\nu}_{n-1}}$ are non-zero only if both ν_n and ν_{n-1} lie on the same lattice.

a skeletonization of the time interval. An assignment $\sigma_N = (\bar\nu_N, \ldots, \bar\nu_0)$ of volumes to the $N+1$ time instants $\phi = \epsilon n$ can be regarded as a history of the configuration variable associated with this skeletonization since one can envision the system as evolving from each $\bar\nu_{n-1}$ at time $\phi = \epsilon(n-1)$ to $\bar\nu_n$ at time $\phi_\epsilon n$. The transition amplitude is thus given by a sum of amplitudes for each of these histories, a la Feynman:

$$A(\nu_f, \varphi; \nu_i, 0) = \sum_{\bar\sigma_N} A(\bar\sigma_N) \qquad \text{with} \quad A(\bar\sigma_N) = U_{\bar\nu_N \bar\nu_{N-1}} U_{\bar\nu_{N-1} \bar\nu_{N-2}} \ldots U_{\bar\nu_2 \bar\nu_1} U_{\bar\nu_1 \bar\nu_0}.$$
(2.13)

Recall, that the Hamiltonian theory implies $A(\nu_f, \varphi; \nu_i, 0) = \langle \nu_f | e^{iH\varphi} | \nu_i \rangle$. Hence the value of the amplitude (2.13) does not depend on N; the skeletonization was introduced just to express this well-defined amplitude as a sum over histories. In non-relativistic quantum mechanics, the amplitude for each history would be simplified in the limit of infinitely refining the skeletonization ($N \to \infty$) where the amplitude for each history can be expanded in epsilon giving,

$$A(\bar\sigma_N) = \langle \bar\nu_N | 1 + i\epsilon H | \bar\nu_{N-1} \rangle \langle \bar\nu_N - 1 | 1 + i\epsilon H | \bar\nu_{N-2} \rangle \ldots \qquad (2.14)$$
$$\times \langle \bar\nu_2 | 1 + i\epsilon H | \bar\nu_1 \rangle \langle \bar\nu_1 | 1 + i\epsilon H | \bar\nu_0 \rangle + \mathcal{O}(\epsilon)$$

Here, if the gravitational kinematic Hilbert space was the standard $L^2(\mathbb{R}, \mathrm{d}\nu)$ each matrix element in the above amplitude is distributional. The amplitude would then be evaluated by inserting a complete basis in the conjugate momenta b for each matrix element which results in the following,

$$A(\bar\sigma_N) \approx \prod_{m=0}^{N-1} \left(\int \mathrm{d}b_m \right) \exp i \sum_{n=0}^{N-1} \epsilon \left(b_n \frac{(\bar\nu_{n+1} - \bar\nu_n)}{\epsilon} + H(\bar\nu, b) \right) \qquad (2.15)$$

where $H(\bar\nu, b)$ is some discretization of the Hamiltonian (depending on the factor ordering chosen). The full amplitude in the limit $N \to \infty$ would be formally written as a phase space path integral as in non-relativistic quantum mechanics where if possible one would integrate out the momenta to arrive at a configuration space path integral.

Where in the standard representation each term in the product is distributional, for LQC each matrix element in (2.14) is simply a complex number since the volume

eigenstates are normalizable in the kinematic Hilbert space. That the eigenstates of the configuration variable are normalizable leads to sum over histories very distinct from that of Schrodinger quantum mechanics. First it allows for a more rigorous analysis of the limit $N \to \infty$ of removing the skeletonization. Further since the amplitudes already depend only on histories in the configuration variable it is not necessary to pass to the phase space path integral. [3].

We find that while the continuum limit of the amplitude for a single configuration history (2.15) in the Schrodinger representation gives to a non-trivial contribution to the overall amplitude, for LQC the continuum limit of each individual history (2.14) is zero. This can be seen by looking at the family of amplitudes of a generic path for each refinement of the skeletonizationi, N, as defined by $\nu_n = \nu(\epsilon n)$ for some history $\nu(\phi)$. Up to order ϵ the amplitude is a product of the following matrix elements

$$\langle \bar{\nu}_n | 1 + i\epsilon H | \bar{\nu}_{n-1} \rangle = \delta_{\bar{\nu}_n, \bar{\nu}_{n-1}} + i\epsilon H_{\bar{\nu}_n \bar{\nu}_{n-1}} \qquad (2.16)$$

where $H_{\bar{\nu}'\bar{\nu}}$ denote the matrix elements of the physical Hamiltonian. Notice that due to the Kronecker delta, unless the two volumes are equal the matrix elements is of order ϵ. Thus if along a history the volume changes values M times the amplitude for that history will be of order ϵ^M and is zero in the limit that the skeletonization is removed. It is not possible then to use the limit $N \to \infty$ to simplify the amplitude for each individual history

2.4   Reorganizing the Sum Over Histories

In section 2.3 we found that the limit, $N \to \infty$, cannot be taken in a naive manner. In this section we will reorganize the sum (2.13) such that this limit can be taken. The result will be a sum over discrete histories changing values only finitely many times, which can be cast as a 'vertex expansion' in the spirit of SFM. The idea is that while the continuum time limit of individual histories is trivial by grouping together histories the limit can be taken. This is clearly true for the entire set of

---

[3]Although, a phase space path integral can be constructed and has many interesting features [55]

histories as summing over all histories returns the exact amplitude, but we will see this is equally true for subsets of the set of histories. More explicitly the idea is as follows: the total amplitude is written as a sum over amplitudes of histories at a fixed N,

$$A(\nu_f, \varphi; \nu_i, 0) = \sum_{\bar{\sigma}_N} A(\bar{\sigma}_N) \qquad (2.17)$$

We will remove the dependence on the skeletonization N from the sum over histories by summing over all histories of all lengths N,

$$A(\nu_f, \varphi; \nu_i, 0) = \sum_{\bar{\sigma}} A_N(\bar{\sigma}) \qquad (2.18)$$

where the amplitude $A_N(\bar{\sigma})$ is zero if the skeletonization associated to the history σ is not N and returns the amplitude (2.13) otherwise. Now we are interested in finding a partitioning of the space of histories of all lengths $N + 1$, such that for each subset the limit of removing the skeletonization is well defined and commutes with the sum over histories.

$$A(\nu_f, \varphi; \nu_i, 0) = \sum_{\alpha} \sum_{\bar{\sigma}_\alpha} A_N(\sigma_\alpha) \qquad (2.19)$$

where each alpha denotes a subset of the set of histories and

$$\lim_{N\to\infty} \sum_{\alpha} \sum_{\bar{\sigma}_\alpha} A_N(\sigma_\alpha) = \sum_{\alpha} \lim_{N\to\infty} \sum_{\bar{\sigma}_\alpha} A_N(\sigma_\alpha) \qquad (2.20)$$

In the following we construct a given partition such that the limit is well defined and the commutitivity above holds.

Since the configuration space given by the set of $\bar{\nu}$ eigenvalues is discrete, even in the limit that the skeletonization is removed the histories are not continuous functions $\nu(\phi)$. The paths then simply consist of those that are constant in $\bar{\nu}$ aside from discontinuous jumps to new values of the volume. At fixed N we first note that along a path σ_N, the volume $\bar{\nu}$ is allowed to remain constant along a number of time steps, then transition to another value, where it could again remain constant for a certain number of time steps, and so on. The first key idea is to group paths according to the number of *volume transitions* rather than time steps. Let us then

consider a path $\bar{\sigma}_N^M$ which involves M volume transitions (clearly, $M \leq N$):

$$\bar{\sigma}_N^M = (\nu_M, \ldots, \nu_M;\, \nu_{M-1}, \ldots, \nu_{M-1};\, \ldots \ldots, \nu_2; \overbrace{\nu_1, \ldots, \nu_1}^{N_2}; \underbrace{\nu_0, \ldots, \nu_0}_{N_1}). \qquad (2.21)$$

Thus, the volume changes from ν_{m-1} to ν_m at time $\phi = N_m\epsilon$ and remains ν_m till time $\phi = N_{m+1}\,\epsilon$. Clearly $N_1 \geq 1$ and $N_M \leq N$. (Note that ν_m is distinct from $\bar{\nu}_m$ used in section 2.3: while ν_m is the volume after the m-th *volume transition* along the given discrete path, $\bar{\nu}_m$ is the volume at the end of the m-th *time interval*, i.e., at $\phi = m\epsilon$.) It is clear that any history is of this type.

These histories can be labelled more transparently by two ordered sequences

$$\bar{\sigma}_N^M = \{\, (\nu_M, \nu_{M-1}, \ldots, \nu_1, \nu_0);\; (N_M, N_{M-1}, \ldots, N_2, N_1) \,\}, \quad \nu_m \neq \nu_{m-1}, N_m > N_{m-1}. \blacksquare$$
$$(2.22)$$

Note that while no two *consecutive* volume values can be equal, a given volume value can repeat in the sequence; ν_m can equal some ν_n if $n \neq m \pm 1$. The probability amplitude for such a history σ_N^M is given by:

$$A(\bar{\sigma}_N^M) = [U_{\nu_M\nu_M}]^{N-N_M}\, U_{\nu_M\nu_{M-1}} \,\cdots\, [U_{\nu_1\nu_1}]^{N_2-N_1-1}\, U_{\nu_1\nu_0}\, [U_{\nu_0\nu_0}]^{N_1-1} \qquad (2.23)$$

The second key idea is to carry out the sum over all these amplitudes in three steps. First we keep the ordered set of volumes $(\nu_M, \ldots, \nu_0)$ fixed, but allow the volume transitions to occur at *any* value $\phi = n\epsilon$ in the interval $\mathcal{I}$, subject only to the constraint that the m-th transition occurs before the $(m+1)$-th for all m. The sum of amplitudes over this group of histories is given by

$$A_N(\nu_M, \ldots, \nu_0) = \sum_{N_M=M}^{N} \sum_{N_{M-1}=M-1}^{N_M-1} \cdots \sum_{N_1=1}^{N_2-1} A(\sigma_N^M). \qquad (2.24)$$

Next we sum over all possible intermediate values of ν_m such that $\nu_m \neq \nu_{m-1}$, keeping $\nu_0 = \nu_i$, $\nu_M = \nu_f$ to obtain the amplitude $A(M)$ associated with the set of all paths in which there are precisely M volume transitions:

$$A_N(M, \nu_f, \nu_i, \varphi) = \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \ldots, \nu_0, \varphi) \qquad (2.25)$$

Finally the total amplitude $A(\nu_f, \varphi; \nu_i, 0)$ is obtained by summing over all volume transitions that are permissible within our initially fixed skeletonization with N time steps:

$$A(\nu_f, \varphi; \nu_i, 0) = \sum_{M=0}^{N} A_N(M, \nu_f, \nu_i, \varphi) \equiv \sum_{M=0}^{N} \left[\sum_{\substack{\nu_{M-1},\ldots,\nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \ldots, \nu_0, \varphi) \right].$$

(2.26)

This is clearly a reorganization of the sum over histories given by

$$\sum_{\bar{\sigma}_N} = \sum_{M=0}^{N} \sum_{\sigma_M} \sum_{\bar{\sigma}_N^M}$$

(2.27)

Where the barred $\bar{\sigma}_N$ and $\bar{\sigma}_N^M$ are the histories in the time ϕ and σ_M keeps track of only the values of the volume taken along the history. In the following we will call σ_M a discrete history.

Recall, however, that the value of the amplitude (2.26) does not depend on N at all; the skeletanization was introduced just to express this well-defined amplitude as a sum over histories. Thus, while the range of M in the sum and the amplitude $A_N(M)$ in (2.26) both depend on N, the sum does not. We are now well positioned to get rid of the skeletonization altogether by taking the limit N goes to infinity. Note first that with our fixed skeletonization, the gravitational amplitude is a finite sum of terms,

$$\begin{aligned} A_G(\nu_f, \nu_i, \varphi) =& A_N(0; \nu_f, \nu_i, \varphi) + A_N(1; \nu_f, \nu_i, \varphi) + \ldots \\ &+ A_N(M; \nu_f, \nu_i, \varphi) + \ldots + A_N(N; \nu_f, \nu_i, \varphi) \end{aligned}$$

(2.28)

each providing the contribution of all discrete paths that contain a fixed number of volume transitions. We will rewrite this as

$$A_G(\nu_f, \nu_i; \alpha) = \lim_{N \to \infty} \sum_{M=0}^{\infty} \sum_{\substack{\nu_{M-1},\ldots,\nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \ldots, \nu_0; \varphi)$$

(2.29)

where the partial amplitude $A_N(M; \alpha)$ is defined to be zero if M is larger than N the total number of transitions thus decoupling the limit $N \to \infty$ from the sum

over the amplitudes. Focusing on the Mth term in the sum:

The third key idea is to get rid of the skeletonization altogether by taking the limit as N goes to infinity, to express the total transition amplitude as a vertex expansion in the spirit of spin-foam models. To carry out this step, we first note that a straightforward but non-trivial calculation presented in A.1 shows that with the grouping of paths defined above $\lim_{N \to \infty} A_N(\nu_M, \ldots, \nu_0)$ exists and is given by:

$$
A(\nu_M, \ldots, \nu_0) := \lim_{N \to \infty} A_N(\nu_M, \ldots, \nu_0)
$$
$$
= \int_0^\varphi \mathrm{d}\phi_M \int_0^{\phi_M} \mathrm{d}\phi_{M-1} \ldots \int_0^{\phi_2} \mathrm{d}\phi_1 \; A(\nu_M, \ldots, \nu_0; \phi_M, \ldots, \phi_1), \quad (2.30)
$$

where,

$$
A(\nu_M, \ldots, \nu_0; \phi_M, \ldots, \phi_1) \;:=\; e^{i(\varphi - \phi_M)H_{\nu_M \nu_M}} \left(i H_{\nu_M \nu_{M_1}}\right) e^{i(\phi_M - \phi_{M-1})H_{\nu_{M-1}\nu_{M-1}}} \times
$$
$$
\ldots \; e^{i(\phi_2 - \phi_1)H_{\nu_1 \nu_1}} \left(i H_{\nu_1 \nu_0}\right) e^{i\phi_1 H_{\nu_0 \nu_0}} \quad (2.31)
$$

The structure of these equations can be understood as follows. In the limit $N \to \infty$, the length $\epsilon = \varphi/N$ of the elementary time intervals goes to zero, which made the amplitude due to a single history vanish. Here each ϵ combines with a discrete sum appearing in (2.24) over the time where the transitions occur resulting in continuous integrals over $\phi_1, ..., \phi_M$. Clearly the ordered sums over N_i in (2.24) gives the structure of Eq (2.30) which says that the final, i.e., M-th volume transition can occur anywhere in the interval $\mathcal{I}$, the $(M\text{-}1)$-th transition can occur anywhere before the M-th, and so on. In passing from (2.24) to (3.29), the factors $[U_{\nu_M \nu_M}]^{N_M - N_{M-1}} = [1 + i((N_M - N_{M-1})\epsilon)H_{\nu_M \nu_M} + \mathcal{O}(\epsilon^2)]$ go to $e^{i(\phi_M - \phi_{M-1})H_{\nu_M \nu_M}}$ while the factors $U_{\nu_M \nu_{M-1}} = i\epsilon H_{\nu_M \nu_{M-1}} + O(\epsilon^2)$ simply contribute multiplicative factors, $i H_{\nu_M \nu_{M-1}}$. Thus we find that as expected by a suitable grouping of paths the limit $N \to \infty$ can be taken leading to a non-zero value.

The dependence on the intermediate times in (2.30) can be removed by integrating out the variables (2.24) leading to an expression $A(\nu_M, \ldots, \nu_0)$ just in terms of matrix elements of H. For simplicity, let us first consider the case of a discrete history where the diagonal elements $H_{\nu_m \nu_m}$ are all distinct values, which for this system is equivalent to the requirement that the volumes $(\nu_M, \ldots, \nu_0)$ are

all distinct. Then, we have:

$$A(\nu_M, \ldots, \nu_0, \varphi) = H_{\nu_M \nu_{M-1}} \ldots H_{\nu_1, \nu_0} \sum_{m=0}^{M} \frac{e^{i\varphi H_{\nu_m \nu_m}}}{\prod_{\substack{j=0 \\ j \neq m}}^{M} (H_{\nu_m \nu_m} - H_{\nu_j \nu_j})}. \tag{2.32}$$

More generally if the diagonal elements are not distinct, there are P distinct volumes $\nu_{i(d)}$ taken during the discrete history each repeating n_i times. Clearly then $n_1 + \ldots + n_P = M + 1$. For this general case the integral is given by the following:

$$\begin{aligned} A(\nu_M, \ldots, \nu_0, \varphi) &= H_{\nu_M \nu_{M-1}} H_{\nu_{M-1} \nu_{M-2}} \ldots H_{\nu_2 \nu_1} H_{\nu_1 \nu_0} \times \\ &\prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial H_k} \right)^{n_k - 1} \sum_{i=1}^{P} \frac{e^{iH_i \varphi}}{\prod_{j \neq i}^{P} (H_i - H_j)} \Bigg|_{H_i = H_{\nu_{i(d)} \nu_{i(d)}}} \end{aligned} \tag{2.33}$$

which is computed by first taking the derivatives and then evaluating the resulting function at the distinct values of the diagonal matrix elements $H_{\nu_{i(d)} \nu_{i(d)}}$. Finally, since $\lim_{N \to \infty} A_N(\nu_M, \ldots, \nu_0, \varphi)$ exists, the contribution A_M from paths with precisely M volume changes has a well defined 'continuous time' limit and the total amplitude is given by a discrete sum over M:

$$A(\nu_f, \varphi; \nu_i, 0) = \sum_{M=0}^{\infty} A_M(\nu_f, \varphi; \nu_i, 0) \tag{2.34}$$

where the partial amplitudes A_M are given by

$$\begin{aligned} A_M(\nu_f, \varphi; \nu_i, 0) &= \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_f, \nu_{M-1}, \ldots \nu_1, \nu_i, \varphi) \tag{2.35} \\ &= \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} H_{\nu_M \nu_{M-1}} H_{\nu_{M-1} \nu_{M-2}} \ldots H_{\nu_2 \nu_1} H_{\nu_1 \nu_0} \times \\ &\prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial H_k} \right)^{n_k - 1} \sum_{i=1}^{P} \frac{e^{iH_i \varphi}}{\prod_{j \neq i}^{P} (H_i - H_j)} \Bigg|_{H_i = H_{\nu_{i(d)} \nu_{i(d)}}}. \end{aligned}$$

In this limit the reference to the skeletonization of the time interval disappears and volume changes can now occur at *any* time in the continuous interval ($\phi_i = 0$, $\phi_f = \varphi$). As one might expect, the final expression involves just the matrix

elements of the Hamiltonian $H = \sqrt{\Theta}$. These are calculated in Appendix B.

Eq (2.34) mimics the vertex expansion of SFMs. The key similarity is that the resulting sum over histories extends over the discrete histories characterized by simply a finite number of values the volume takes along the history instead of continuous functions. More precisely, the parallels are as follows. The analog of the manifold M with boundaries Σ_i, Σ_f in SFMs is the manifold $\mathcal{V} \times \mathcal{I}$, where $\mathcal{V}$ is the elementary cell in LQC and $\mathcal{I}$ the closed interval bounded by $\phi = 0$ and $\phi = \varphi$. The analog of the dual-triangulation in SFMs is just a 'vertical' line in $\mathcal{V} \times \mathcal{I}$ with M marked points or 'vertices' (not including the two end-points of $\mathcal{I}$). What matters here is the number M; the precise location of vertices is irrelevant. Coloring of the dual-triangulation in SFMs corresponds to an ordered assignment $(\nu_M, \nu_{M-1}, \ldots \nu_1, \nu_0)$ of volumes to edges bounded by these marked points (subject only to the constraints $\nu_M = \nu_f$, $\nu_0 = \nu_i$ and $\nu_m \neq \nu_{m-1}$). Each vertex signals a change in the physical volume along the quantum history. The probability amplitude associated with the given coloring is $A(\nu_M, \ldots, \nu_0)$ and a sum over colorings yields the amplitude $A(M)$ associated with the triangulation with M 'vertices', which is related to current SFM which are defined on fixed triangulations. The total amplitude $A(\nu_f, \varphi; \nu_i, 0)$ is given by a sum (2.34) over these M-vertex amplitudes, which gives insight into the conjectured sum over triangulations needed to recover continuum general relativity.

Here we also highlight one key difference as well. Current SFM have locality as a key assumption. By locality it is meant that the amplitude corresponding to a fixed triangulation with a given set of labels is a product of amplitudes associated to each element of the dual triangulation (faces, edges, and vertices) which may depend only on those elements in their immediate neighborhood. In this system the amplitude (2.35) associated to each triangulation is not local in this sense. The amplitude depends on the entire history as can be seen by the explicit dependence on the number of times a given volume repeats in a history.

The assumption of locality can be related to the assumption that given two manifolds M and M' and triangulation thereof, the amplitude associated to the triangulation obtained by glueing together M and M' is given by the product of the amplitudes of the two triangulations summed over the labels of the glueing surface. The corresponding statement for the vertex expansion (2.34) would be

that the partial amplitude for $M + M'$ vertices is given by the product of the partial amplitudes for M and M' vertices with a sum over the intermediate value of the volume. It can be clearly seen that this feature is not present at the level of (2.34)

Starting from the full amplitude we can derive composition law for the partial amplitudes corresponding to two triangulations M and M' by simply splitting the transition amplitude by inserting a complete basis in ν and applying the vertex expansion to each resulting transition amplitude.

$$\langle \nu_f | e^{i(\varphi_1 + \varphi_2)H} | \nu_i \rangle = \sum_\nu \langle \nu_f | e^{i\varphi_2 H} | \nu \rangle \langle \nu | e^{i\varphi_1 H} | \nu_i \rangle \tag{2.36}$$

We can then apply the vertex expansion to each amplitude on the right hand side of (2.36).

$$\sum_{M=0} \lambda^M A_M(\nu_f, \varphi_1 + \varphi_2; \nu_i, 0) = \sum_\nu \sum_{M_1=0} \lambda^{M_1} A_{M_1}(\nu_f, \varphi_2; \nu, 0) \sum_{M_2=0} \lambda^{M_2} A_{M_2}(\nu, \varphi_1; \nu_i, 0) \tag{2.37}$$

From which we can find the following composition law for the partial amplitudes,

$$A_M(\nu_f, \varphi_1 + \varphi_2; \nu_i, 0) = \sum_{M'=0}^{M} \sum_\nu A_{M-M'}(\nu_f, \varphi_2; \nu, 0) A_{M'}(\nu, \varphi_1; \nu_i, 0) \tag{2.38}$$

The amplitude associated to a larger triangulation $M + M'$ is then the not product of two smaller triangulations but the sum of all possible decompositions into two triangulations.

One the other hand we can see from the derivation of the amplitudes that the following holds

$$A_M(\nu_f, ..., \nu_i, \varphi) = \int \mathrm{d}\varphi' A_{M_1}(\nu_f, ..., \nu_{M_2+1}, \varphi - \varphi') i H_{\nu_{M_2+1}\nu_{M_2}} A_{M_2}(\nu_{M_2}, ..., \nu_i, \varphi',) \tag{2.39}$$

where $M_1 + M_2 + 1 = M$. Here we do find that the amplitude corresponding to the larger triangulation $M_1 + M_2$ is given by a product of the amplitudes corresponding to M_1 and M_2 connected by an action of the constraint, if additionally we integrate over all possible 'times' at which these amplitudes are combined. In this sense the

expansion is local when one considers both degrees of freedom.

To conclude this section, we emphasize that the resulting vertex expansion was *derived* from a Hamiltonian theory. We did not postulate that the left side of (2.34) is given by a formal path integral. Rather, a rigorously developed Hamiltonian theory guaranteed that the left side is well-defined and provided the expression (2.12) for it. We simply recast this expression as a vertex expansion with one assumption of the exchange of the limit $N \to \infty$ with the sum over the discrete histories.

2.5 Vertex Expansion as a Perturbation Series

We will now show that the expression (2.34) of the transition amplitude can also be obtained using a specific perturbative expansion. This alternative derivation provides three benefits. First it provides a direct justification of the reordering of the sum used in the previous section and the interchange of the sum over discrete histories and the limit $N \to \infty$. The perturbative approach avoids skeletonization altogether and has the advantage that it guarantees a convergent series. Second, structurally this derivation of the vertex expansion is reminiscent of the perturbative strategy used in group field theory (see, e.g., [29, 30]), which presents new insights for group field theory and its connection with quantum gravity. Finally we will see in later chapters that this derivation provides an additional avenue for generalization of the expansion presented in this chapter and the next.

Let us begin by decomposing the 'Hamiltonian' H into the diagonal part D and the remainder, the non-diagonal part K which is responsible for volume change of the in the basis $|\nu = 4n\ell_o\rangle$

$$D_{\nu'\nu} = H_{\nu\nu}\,\delta_{\nu'\nu}, \qquad K_{\nu'\nu} = \begin{cases} H_{\nu'\nu} & \nu' \neq \nu \\ 0 & \nu' = \nu \end{cases} \tag{2.40}$$

Again this decomposition of H is possible due to the discrete inner product of the basis. Clearly $H = D + K$. The idea is to think of D as the main part of H and K as a perturbation. To implement it, introduce a 1-parameter family of operators $H_\lambda = D + \lambda K$ as an intermediate mathematical step. The parameter λ will simply serve as a marker to keep track of powers of K in the perturbative expansion; we

will have to set $\lambda = 1$ at the end of the calculation as there is no true coupling constant in the theory - the only dimensionfull parameters in H appear equally in each term of the constraint.

Following the textbook procedure, let us define the 'interaction Hamiltonian' as

$$H_I(\phi) = e^{-iD\phi} \, \lambda K \, e^{iD\phi}. \tag{2.41}$$

Then the evolution in the interaction picture is dictated by the operator

$$\tilde{U}_\lambda(\phi) = e^{-iD\phi} e^{iH_\lambda \phi}, \qquad \text{satisfying} \quad \frac{d\tilde{U}_\lambda(\phi)}{d\phi} = iH_I(\phi)\tilde{U}_\lambda(\phi), \tag{2.42}$$

whose solution is given by a time-ordered exponential:

$$\begin{aligned}
\tilde{U}_\lambda(\varphi) &= \mathcal{T} \, e^{i \int_0^\varphi H_I(\phi)d\phi} \\
&= \sum_{M=0}^\infty \int_0^\varphi d\phi_M \int_0^{\phi_M} d\phi_{M-1} \dots \int_0^{\phi_2} d\phi_1 \, [iH_I(\phi_M)] \dots [iH_I(\phi_1)]
\end{aligned} \tag{2.43}$$

Next, let us express the evolution operator as $U_\lambda(\varphi) = e^{iD\varphi}\tilde{U}_\lambda(\varphi)$, with $\tilde{U}_\lambda(\varphi)$ given by (2.43), take matrix element between initial and final states, $|\nu_i \equiv \nu_0\rangle$ and $|\nu_f \equiv \nu_M\rangle$, write out explicitly the product of the H_I's, and insert a complete a complete basis between each H_I. The result is

$$\begin{aligned}
A_\lambda(\nu_f, \varphi; \nu_i, 0) = \sum_{M=0}^\infty \int_0^\varphi d\phi_M \dots \int_0^{\phi_2} d\phi_1 \sum_{\nu_{M-1}, \dots, \nu_1} [e^{i(\varphi-\phi_M)D_{\nu_M \nu_M}}] \, (i\lambda K_{\nu_M \nu_{M-1}}) \times \\
[e^{i(\phi_M - \phi_{M-1})D_{\nu_{M-1}\nu_{M-1}}}] \dots (i\lambda K_{\nu_1 \nu_0}) \, [e^{i\phi_1 D_{\nu_0\nu_0}}].
\end{aligned} \tag{2.44}$$

We can now replace D and K by their definition (3.42). Because K has no diagonal matrix elements, only the terms with $\nu_m \neq \nu_{m+1}$ contribute to the sum and the sum becomes

$$A_\lambda(\nu_f, \varphi; \nu_i, 0) = \sum_{M=0}^\infty \lambda^M \left[\sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \dots, \nu_0) \right], \tag{2.45}$$

where $A(\nu_M, \dots, \nu_0)$ is defined in (2.32). If we now set $\lambda = 1$, Eq. (2.45) reduces

to Eq. (2.34) obtained independently in section 2.4.

Thus, by formally regarding the off-diagonal piece of the Hamiltonian as a perturbation of the diagonal piece we have obtained an independent derivation of the vertex expansion of the amplitude $A_\lambda(\nu_f, \varphi; \nu_i, 0)$ as a power series expansion in λ, the power of λ serving as a book-keeping device to keep track of the order in the vertex expansion. In this sense this alternate derivation is analogous to the vertex expansion obtained using group field theory.

2.6   Satisfaction of the Schrodinger Equation

Recall that in the deparametrization scheme, the Schrödinger equation (2.9) incorporates both the quantum constraint and the positive frequency condition. By its very definition, the exact transition amplitude $A(\nu_f, \varphi; v_i, 0)$ satisfies this Schrödinger equation if viewed as a function of ν_f and φ.

$$(i\partial\varphi + H)A(\nu_f, \varphi; v_i, 0) = \langle\nu_f|(i\partial\varphi + H)e^{i\varphi H}|\nu_i\rangle = 0 \tag{2.46}$$

As a non-trivial check of the perturbative expansion (2.34) we are led to ask whether the Schrödinger equation is solved by the vertex expansion and in a controlled approximate sense if we were to truncate the series on the right side of (2.34) at a finite number , $M^\star$, of vertices. We will now show that this is indeed the case.

The action of H on the vertex expansion is given by

$$H_\lambda A(\nu_f, \varphi; v_i, 0) = \sum_M \lambda^M H_\lambda A_M(\nu_f, \varphi; v_i, 0) \tag{2.47}$$

since each term of the vertex expansion can be treated as a function of ν_f and φ. Here we have kept λ which will act as a place-holder to simplify the calculation. Since $H_\lambda = D + \lambda K$, the Schrödinger equation would be solved order by order in λ if for each M we have:

$$(i\partial_\varphi + D_f) A_M(\nu_f, \varphi; v_i, 0) + K_f A_{M-1}(\nu_f, \varphi; v_i, 0) = 0\,. \tag{2.48}$$

Using the expression of the partial amplitudes $A_M(\nu_f, \varphi; \nu_i, 0)$ this is equivalent to

the requirement that

$$\sum_{\substack{\nu_{M-2},\ldots,\nu_1 \\ \nu_m \neq \nu_{m+1}}} \Bigg[\sum_{\substack{\nu_{M-1} \\ \nu_{M-1}\neq\nu_{M-2}}} \left(i\partial_\varphi + D_f\right) A_M(\nu_f, \nu_{M-1}, \ldots, \nu_1, \nu_i; \varphi) \tag{2.49}$$

$$+ K_f A_{M-1}(\nu_f, \nu_{M-2}, \ldots, \nu_1, \nu_i; \varphi) \Bigg] \tag{2.50}$$

vanishes for each M. Where again we view $A_M(\nu_f, \nu_{M-1}, \ldots, \nu_1, \nu_i; \varphi)$ as functions of ν_f and φ for fixed $\nu_{M-1}, \ldots, \nu_i$. It is clearly possible that the above expression could vanish only through cancellations between terms from the entire sum over $\nu_{M-1}, \ldots, \nu_1$. Surprisingly though using the expression (2.35) for $A(\nu_f, v_{M-1}, \ldots \nu_1, \nu_i; \varphi)$, one can verify that the cancellations occur path by path i.e. for each $\nu_{M-1}, \ldots, \nu_i$ the term in brackets is zero.

$$\sum_{\substack{\nu_{M-1} \\ \nu_{M-1}\neq\nu_{M-2}}} \left(i\partial_\varphi + D_f\right) A_M(\nu_f, \nu_{M-1}, \ldots, \nu_1, \nu_i; \varphi) \tag{2.51}$$

$$+ K_f A_{M-1}(\nu_f, \nu_{M-2}, \ldots, \nu_1, \nu_i; \varphi) = 0 \tag{2.52}$$

To see this we compute the action of the diagonal and off-diagonal parts of the Schrödinger equation on the partial amplitudes (2.35). Without loss of generality we assume that $\nu_f = v_{P(d)}$ in (2.35). Then we have that the action of $i\partial_\varphi + D_f$ is

$$(i\partial_\varphi + D_f) A(\nu_f, \nu_{M-1}, \ldots, \nu_1, \nu_i; \varphi) = H_{\nu_f\nu_{M-1}} H_{\nu_{M-1}\nu_{M-2}} \ldots H_{\nu_2\nu_1} H_{\nu_1\nu_0} \times$$

$$\left[\prod_{k=1}^{P} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial H_k}\right)^{n_k-1} \sum_{i=1}^{P} \frac{-H_i e^{iH_i\varphi}}{\prod_{j\neq i}^{P}(H_i - H_j)} \right.$$

$$\left. + H_{v_{p(d)}v_{p(d)}} \prod_{k=1}^{P} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial H_k}\right)^{n_k-1} \sum_{i=1}^{P} \frac{e^{iH_i\varphi}}{\prod_{j\neq i}^{P}(H_i - H_j)} \right]\Bigg|_{H_i = H_{\nu_{i(d)}\nu_{i(d)}}} \tag{2.53}$$

If $H_{\nu_{P(d)}\nu_{P(d)}}$ occurs with multiplicity $n_P = 1$, if ν_f is the only point in the history taking the value $\nu_{P(d)}$ then there are no derivatives in $H_{\nu_{P(d)}\nu_{P(d)}}$ in the above equation and it simplifies to

$$(i\partial_\varphi + D_f) A(\nu_f, \nu_{M-1}, \ldots, \nu_1, \nu_i; \varphi) = H_{\nu_f\nu_{M-1}} H_{\nu_{M-1}\nu_{M-2}} \ldots H_{\nu_2\nu_1} H_{\nu_1\nu_i} \times$$

$$\left[\prod_{k=1}^{P}\frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial H_k}\right)^{n_k-1}\sum_{i=1}^{P}\frac{(H_P-H_i)e^{iH_i\varphi}}{\prod_{j\neq i}^{P}(H_i-H_j)}\right]\Bigg|_{H_i=H_{\nu_{i(d)}\nu_{i(d)}}}$$

$$=-H_{\nu_f\nu_{M-1}}A(\nu_{M-1},\ldots,\nu_1,\nu_i;\varphi)\,. \quad (2.54)$$

Where in the final step we recognized that the term (H_P-H_i) renders the $P-th$ term of the sum zero and removes the H_P dependence from each other term in the series. Together these reduce the structure to that of a history without the final point ν_f. Thus, on simple paths where the final volume occurs only once in the history, the action of $i\partial_\varphi+D_f$ is to give the amplitude of the history without ν_f, times a matrix element of H related to the transition from ν_{M-1} to ν_f. In general, the value of the final volume can be repeated in the discrete history; $n_P\neq 1$. In that case we need to push $H_{\nu_{P(d)}\nu_{P(d)}}$ under the derivatives in (2.53). We push the factor of H_p under the derivatives using the identity that

$$x\frac{\partial^n}{\partial x^n}f(x)=\frac{\partial^n}{\partial x^n}(xf(x))-n\frac{\partial^{n-1}}{\partial x^{n-1}}f(x) \quad (2.55)$$

giving

$$(i\partial_\varphi+D_f)A(\nu_M,\ldots,\nu_0;\varphi)=H_{\nu_f\nu_{M-1}}H_{\nu_{M-1}\nu_{M-2}}\ldots H_{\nu_2\nu_1}H_{\nu_1\nu_0}\times$$

$$\left[\prod_{k=1}^{P}\frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial H_k}\right)^{n_k-1}\sum_{i=1}^{P}\frac{(H_p-H_i)e^{iH_i\varphi}}{\prod_{j\neq i}^{P}(H_i-H_j)}\right. \quad (2.56)$$

$$\left.-\prod_{k=1}^{P-1}\frac{1}{(n_k-1)!}\left(\frac{\partial}{\partial H_k}\right)^{n_k-1}\frac{1}{(n_p-2)!}\left(\frac{\partial}{\partial H_p}\right)^{n_p-2}\sum_{i=1}^{P}\frac{e^{iH_i\varphi}}{\prod_{j\neq i}^{P}(H_i-H_j)}\right]$$

$$=-H_{\nu_f\nu_{M-1}}A(\nu_{M-1},\ldots,\nu_1,\nu_i;\varphi)$$

As above the (H_P-H_i) appearing in the first term removes the H_P dependence, so the derivative with respect to H_P is zero. The result is simply the second term which is again the amplitude of the history without the final point, ν_f, times a matrix element $H_{\nu_f\nu_{M-1}}$. Thus, in all cases we have

$$(i\partial_\varphi+D_f)A(\nu_M,\ldots,\nu_0;\varphi)=-H_{\nu_f\nu_{M-1}}A(\nu_{M-1},\ldots,\nu_1,\nu_i;\varphi)\,. \quad (2.57)$$

Finally, it is straightforward to evaluate the action of the off-diagonal part on

A_{M-1}:

$$K_f \, A(\nu_f, \nu_{M-2}, \ldots, \nu_1, \nu_i; \varphi) = \sum_{\nu_{M-1}} H_{\nu_f \nu_{M-1}} A(\nu_{M-1}, \nu_{M-2}, \ldots, \nu_1, \nu_i; \varphi) \,. \quad (2.58)$$

Putting these two results together we see that (2.51) is satisfied by the partial amplitudes $A(\nu_M, \nu_{M-1}, \ldots, \nu_1, \nu_i; \varphi)$. Thus the Schrödinger equation is solved in a surprisingly simple fashion. It is satisfied *path by path* due to the satisfaction of eqn (2.51) by the partial amplitudes $A(\nu_M, v_{M-1}, \ldots \nu_1, \nu_i; \varphi)$ Thus we have shown that the vertex expansion resulting from the perturbation series satisfies quantum dynamics in a well-controlled fashion: If we were to terminate the sum at $M = M^\star$, we would have

$$(i\partial_\varphi + D_f + \lambda K) \left[\sum_{M=0}^{M^\star} \lambda^M A_M(\nu_f, \varphi; v_i, 0) \right] = \mathcal{O}(\lambda^{M^\star+1}) \quad (2.59)$$

This brings out the precise sense in which a truncation to a finite order of the vertex expansion incorporates the quantum dynamics of the deparameterized theory approximately. If only up to a finite number $M = M^\star$ of vertices are included then the sum provides a solution to the Schrödinger equation up to terms depending on paths with $M = M^\star + 1$ vertices. The difficulty though is that to recover expansion corresponding to the original constraint we must set $\lambda = 1$ so it is not clear that the terms of higher order in λ are necessarily small. It would be necessary to explicitly compute each term or be able to set bounds on them to show that each successive term is smaller. Here we do know that the series converges as it is derived from the expansion of the unitary evolution operator via standard perturbation theory, so there does exist $M^\star$ such that for all $M > M^\star$, the partial amplitudes A_M have the necessary falloff. The terms of the expansion could then be increasing up to say $M = 100$, so there is a no a priori reason why any finite number of terms will provide a good approximation the physical inner product without additional knowledge of the amplitudes.
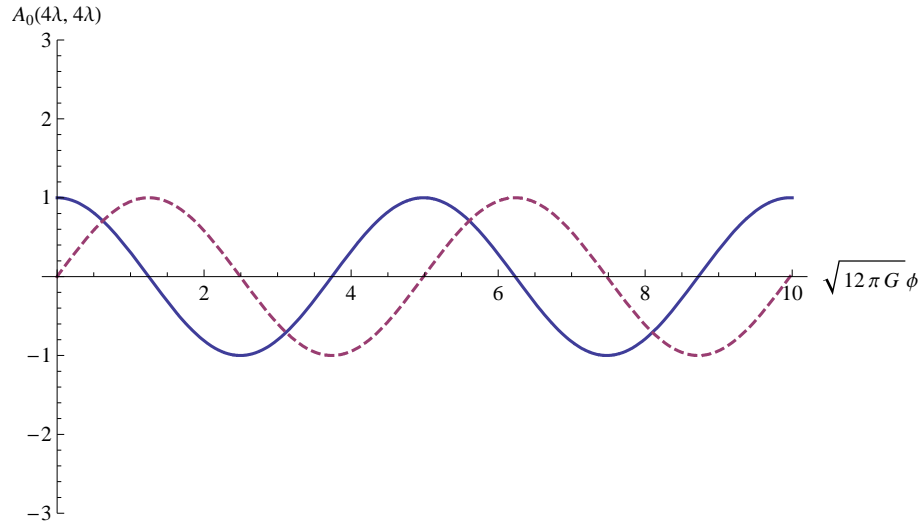
Figure 2.1. Real (solid) and imaginary (dashed) parts of the partial amplitude with zero vertices $A_0(4\lambda, \varphi; 4\lambda, 0)$ plotted as a function of $\sqrt{12\pi G}\varphi$. This provides the first term of the vertex expansion.

2.7 Testing the Expansion

In the previous section we demonstrated that the vertex expansion provides a perturbative solution to the constraint in a very simple manner - i.e. path by path. On the other hand there is at present no a priori way to determine if the concatenation of the series at some finite order will provide a good approximation to the exact physical inner product. In this section then we explicitly test the expansion through comparison with the exact amplitude. It is imperative to test any new approximation methods or quantization technique with well known systems to ensure that results can be trusted in more complex systems where exact calculations are not available.

The strategy for this section is as follows: For the simple choice of initial and final volumes, $\nu_i = 4\lambda$ and $\nu_f = 4\lambda$, we analyze the first few terms of the expansion first individually to check the falloff behavior and second how the sum to compares to the exact amplitude. We will see that the behavior of the expansion exhibits the concern of the previous section that higher order terms can dominate the sum. In particular we see that higher order terms become more important at large values of scalar field.

We see here an explicit realization of the concern that even though the series
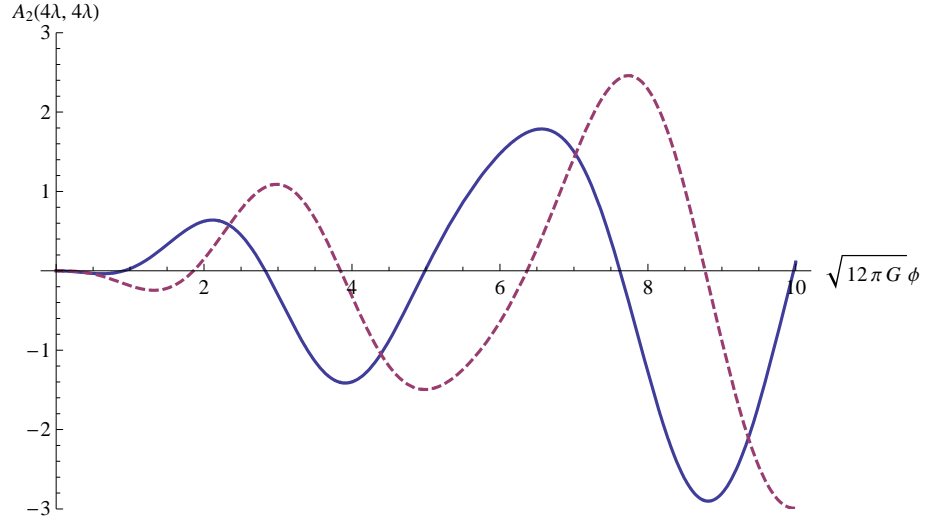
Figure 2.2. Real (solid) and imaginary (dashed) parts of the second term in the vertex expansion with two vertices $(M = 2)$ $A_2(4\lambda, \varphi; 4\lambda, 0)$ plotted as a function of $\sqrt{12\pi G}\varphi$. Comparing with figure 2.1, for $\sqrt{12\pi G}\varphi < 2$ the $M = 2$ term is smaller than the $M = 0$ while clearly the $M = 2$ term of the vertex expansion dominates the $M = 0$ for larger values of φ, so we see for longer evolution in φ terms with more vertices can contribute more than the lower terms in the vertex expansion.
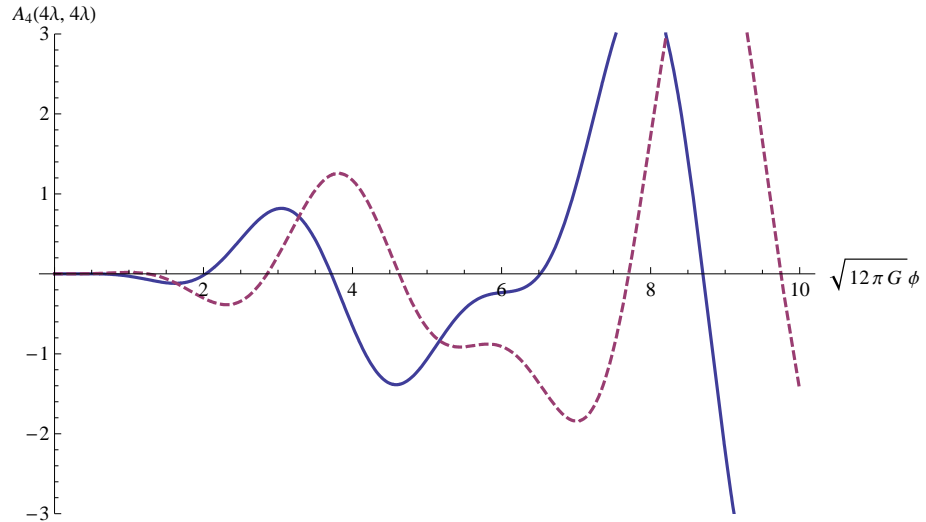
Figure 2.3. Real (solid) and imaginary (dashed) parts of the partial amplitude with two vertices $A_4(4\lambda, \varphi; 4\lambda, 0)$ plotted as a function of $\sqrt{12\pi G}\varphi$. Again comparison with figures 2.1 and 2.2 the $M = 4$ term of the vertex expansion is larger than both $M = 0, M = 2$ terms for larger values of φ. We again clearly see that for longer evolution in φ higher order terms in the vertex expansion will dominate the expansion.
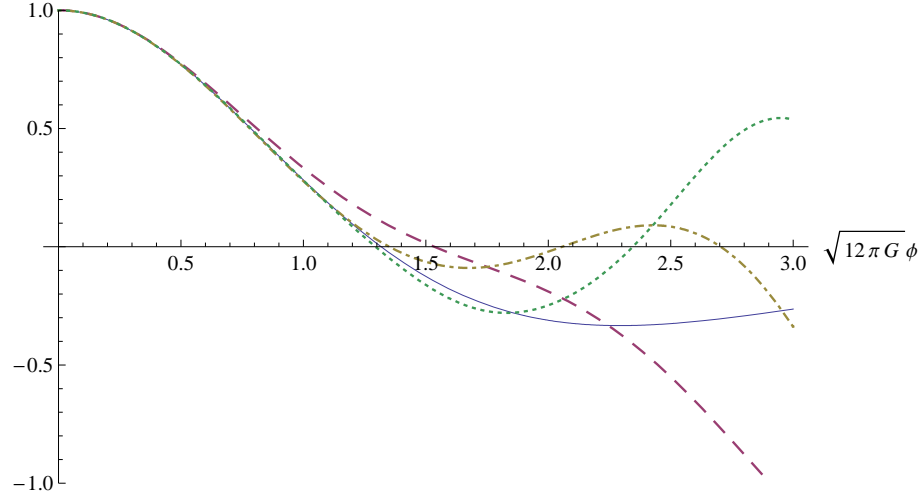
Figure 2.4. The exact transition amplitude $A(4\lambda, 4\lambda, \varphi)$ (solid line) and the vertex expansion up to order $M = 2$ (dashed), $M = 4$ (dot-dashed), and $M = 6$ (dotted) plotted as functions of $\sqrt{12\pi G}\varphi$. With more terms of the expansion included the vertex expansion converges up to a larger value of φ, but there is large deviation from the exact amplitude beyond that value. Thus it is necessary to include higher and higher terms in the vertex expansion to have even a first-order approximation of the exact transition amplitude.

is convergent it does not require that the each subsequent term after the first is smaller - that especially for longer evolution in φ the amplitudes increase with M up to some critical value of M beyond which they must fall-off since the overall series is convergent. It is necessary to then to take many terms in the vertex expansion even to obtain a first order approximation of the exact transition amplitude. This is an important realization for spin-foam models where there is equally no coupling constant and little knowledge about the relative size of different terms of the vertex expansion. Carrying over our intuition from this system we would expect that for 'short time-scale' phenomena the first few terms of the expansion may suffice, but for 'long time-scale' phenomena such as cosmological evolution or scattering amplitudes the first terms would actually provide a negligible contribution.

2.8 Discussion

Let us start with a brief summary. In section 2.3 we began with Hamiltonian LQC, divided the time interval into N segments thereby expressing the transition

amplitude $A(\nu_f, \phi_f; v_i, \phi_i)$ as a sum (2.13) over histories. In section 2.4 we reorganized this sum according to the number of volume transitions allowing us to take the $N \to \infty$ limit to get rid of the skeletonization of the time interval. This led us to the expression (2.45) of the transition amplitude. The M-th term in this expansion corresponds to a sum over all histories in which there are precisely M volume transitions, allowed to occur at *any* time in the interval (ϕ_f, ϕ_i). Therefore as discussed in 2.4 the expansion resembles the vertex expansion of SFMs where the dual-triangulation is simply a line with M vertices and the coloring is simply a labeling of each edge by a volume. In section 2.4 we showed that the same vertex expansion can be arrived at by formally splitting the Hamiltonian H into a main part D and a 'perturbation' λK and expanding the transition amplitude using standard perturbation theory in the interaction picture. (Here the 'coupling constant' λ was introduced just as a mathematical label to keep track of the number of vertices in various terms and we have to set $\lambda = 1$ at the end to recover the physical transition amplitude). This expansion in powers of λ resembles the vertex expansion in group field theory.

While our final result yields vertex expansions in the spirit of SFMs, we did not begin with a particular SFM and arrive at the vertex expansion by a suitable symmetry reduction, e.g., by summing over the degrees of freedom other than the total volume. Rather, since our goal is to analyze spin foam models from the point of view of standard quantization techniques we begin from the canonical theory. A parallel approach would be to study SFM from standard path integral techniques of quantum field theory [56, 57, 58]. While ideally our analysis would begin with LQG, our procedure is much more modest: As is usual in LQC, we carried out the symmetry reduction at the *classical* level by partial gauge fixing, constructed the Hamiltonian quantum theory and used it to obtain vertex expansions. Also, so far, vertex expansions have been discussed in SFMs only for source-free gravity, while the presence of a scalar field played a key role in LQC. Thus the results obtained here are better tailored then to deparametrized LQG based on for example the Brown-Kuchar model [59]. Nonetheless our results provide support for the heuristic framework underlying SFMs. In addition, since we have an exactly soluble, concrete example, we can use it to analyze the status of open issues of the SFM.

First, as hoped in SFMs, the physical inner product can indeed be expressed

as a vertex expansion, a sum over discrete histories each characterized by a finite number of configurations. The inner product is defined independently by the canonical theory and this well-defined quantity is merely expanded out as a convenient series. Second, the expectation that the expansion should be derivable from a suitable Hamiltonian theory has been realized. In addition, as one would expect from [60], each vertex can be thought of as emerging from the action of the Hamiltonian operator. Finally, in our decomposition $H = D + \lambda K$ (or $\Theta = \underline{D} + \underline{\lambda K}$) one can think of D as the free part of the Hamiltonian because it does not change the volume and K as the interaction part because it does. Thus, as in group field theory, the factors of λ are associated with 'interaction' piece of the Hamiltonian (which is responsible for the volume transitions in LQC). Third, the expansion provide a perturbative solution to the Schrodinger equation order by order in the number of vertices, changes in volume.

One key issue of SFMs is the lack of connection to continuum general relativity, since the construction of SFMs is based on a discretization of GR. An issue that is often raised in the literature on SFMs is whether the physical inner product should be given by summing over triangulations, each with a finite number of vertices, or if one should take a "continuum" limit at the end as in, e.g., lattice QCD. If one *defines* the inner product as a discretized path integral there is no clear answer. However, since we began with a well-controlled Hamiltonian theory, in the LQC example, the answer *is* clear. We did not have to take the limit; the physical inner product is given by a discrete sum over amplitudes associated to triangulation with a finite number of vertices or equivalently over discrete histories with a finite number of volume changes.

A related issue is how well a finite number of elements of this sum over triangulations will approximate the exact transition amplitude. As the coupling constant λ here and coming from GFT is a dimensionless parameter that is set to one in the expansion there is a priori no reason for the terms of higher order in λ to be sub-dominant. In the vertex expansion for LQC presented here we explicitly see that the terms of higher order in λ can be larger than the lower order terms. In particular at large values of the scalar field terms with more volume transitions dominate the expansion.

Finally, the vertex expansion of LQC does not fit with one of the key assump-

tions of the SFMs. In the general definition of a SFM the amplitude associated to a given dual triangulation is given by a product of amplitudes associated to each vertex, edge, and face which depends on only the immediate neighborhood of each. The LQC vertex expansion in contrast was extremely non-local in its dependence on the volume labels. This feature will persist in the next chapter where we derive a distinct vertex expansion without using the scalar field as a clock variable to deparametrize the theory. There we will further explore this lack of locality.

Vertex Expansion from Group Averaging

As seen in the previous chapter we derived at a sum over histories expansion of LQC starting from a deparametrization of the canonical theory using the scalar field as a clock. This sum over histories extended over those histories that feature a discrete number of changes in volume mimicking the structure of SFMs, so we refer to the expansion as a vertex expansion of LQC. In general there is no such method for deparametrizing full general relativity. Although, there are proposals for deparametrized models which will be well defined on a suitable sub-space of the full phase space of general relativity coupled to suitable matter fields [59, 61]. The expansion for deparametrized LQC could then be extended to such models, but not to full LQG. In this chapter we then derive a distinct vertex expansion starting from the timeless framework with a Hamiltonian constraint instead of the deparametrized system.

Recall that spin network states are used in LQG to construct a convenient orthonormal basis in the kinematical Hilbert space. A key challenge is to extract physical states from them by imposing constraints. Formally this can be accomplished by the group averaging procedure which also provides the physical inner product between the resulting states [62, 12]. From the LQG perspective, the primary goal of SFMs is to construct a path integral that leads to this physical Hilbert space. Similarly in LQC the physical Hilbert space can be obtained from the group averaging procedure. We then construct a vertex expansion starting

from the group averaging procedure.

We will find that this expansion bears out the ideas and conjectures that drive the spin foam paradigm more thoroughly as it has a stronger connection to the spin foam framework. Specifically, we will show that: i) the physical inner product in the timeless framework equals the transition amplitude in the theory that is deparameterized using the scalar field; ii) the physical inner product admits a vertex expansion a la SFMs distinct from that of the deparametrized system; iii) the exact physical inner product is again obtained by summing over just the discrete geometries; no 'continuum limit' is involved; and, iv) the vertex expansion can be interpreted as a perturbative expansion in the spirit of GFT, where, moreover, the GFT coupling constant λ is closely related to the cosmological constant Λ. Since the Hilbert space theory is fully under control in this example, we will be able to avoid formal manipulations and pin-point the one technical assumption that is necessary to obtain the desired vertex expansion: that one can interchange the group averaging integral and a convergent but infinite sum defining the gravitational contribution to the vertex expansion(see discussion at the end of section 3.2). This analysis will shed light on some long standing issues in SFMs such as the role of orientation in the spin foam histories [63], the somewhat puzzling fact that spin foam amplitudes are real rather than complex [64]. Further it will introduce new insights and new issues for SFM.

The chapter is organized as follows. We begin with an overview of group averaging as a way to obtain the physical Hilbert space of LQC. In section 3.2 we adapt the main results from chapter 2 to the timeless framework by showing that the physical inner product as defined by group averaging can be expressed as a vertex expansion. In section 3.3 we will arrive at the same expansion using perturbation theory in a suitably defined interaction picture. As an important consistency check, in section 3.4 we verify that this perturbative expansion does satisfy the constraint order by order. In section 3.5 we observe that, in this simple example, the coupling constant λ used in the expansion is intimately related to the cosmological constant Λ. Although the precise relation we obtain is tied to LQC, the observation illustrates in a concrete fashion how one may be able to provide a gravitational interpretation to λ in GFTs. Finally, in 3.6 we numerically test the vertex expansion for a simple choice of initial and final volumes. Section 3.7

summarizes the main results and discusses some generalizations and open issues.

3.1 Group Averaging LQC

We began our discussion in the previous chapter by carrying out a deparametriza-tion (using ϕ as the relational time variable) because this is the procedure used in LQC to extract physics from the quantum theory and provides contact with stan-dard non-relativistic quantum mechanics. In the SFM and LQG, by contrast, one does not have access to such a preferred time and therefore one chooses to work with the timeless formalism where the dynamics is encoded by constraints instead of a Schrodinger like equation. Furthermore, a convenient deparametrization is not always available even in cosmology if we allow the scalar field to have general potentials or the gravitational field to admit inhomogeneities or when considering vacuum general relativity. In this case we have to return to the constrained system and construct the physical Hilbert space differently. The standard construction of $\mathcal{H}_{\mathrm{phy}}$ in LQG is done using group averaging which provides both the solutions to the constraint and their inner product. Therefore we first forgo the emphasis on using ϕ as internal time and implement the group averaging procedure which uses the constraint operator as a whole [62, 12]. Then the basic object of interest is not the transition amplitude but the physical inner product.

As the group averaging procedure is the basis of the remaining discussions we will summarize it in some detail. The necessity of the procedure arises since the spectrum of the constraint is continuous and thus the eigenstates of the constraint are not normalizable in the kinematical Hilbert space requiring the construction of a new Hilbert space and inner product which we will denote as the physical Hilbert space and physical inner product. One begins by fixing a dense sub-space $\mathbb{S}$ of $\mathcal{H}_{\mathrm{kin}}$. In LQC, this is generally taken to be the Schwartz space of smooth functions $f(\nu, \phi)$ which fall off to zero at infinity faster than any polynomial. This Schwartz space serves two purposes one to provide a set of kinematic states from which to generate solutions to the constraint and second to give a meaning to the distributional solutions as states in the dual of $\mathbb{S}$. The first step in the group averaging procedure is then to extract a solution $\Psi_f(\nu, \phi)$ to the quantum constraint operator (2.5) from each $f \in \mathbb{S}$. Here the constraint is a self-adjoint operator, so we can define the

physical state as.

$$(\Psi_f| = \left[\int \mathrm{d}\alpha\, e^{i\alpha C}|f\rangle\right]^{\dagger} \tag{3.1}$$

where $C = p^2/\hbar^2 - \Theta$ is the constraint operator and the round bracket on the left side of the equation denotes a 'generalized bra', an element of the algebraic dual (called $\mathrm{Cyl}^\star$ in the literature) of a suitable dense subspace of $\mathcal{H}_{\mathrm{kin}}$, $\mathbb{S}$. Where here the terminology 'group averaging' is clear - as we are averaging over the action of the unitary evolution operator generated from the constraint thus constructing a state that is invariant under this group.

The second step of the group averaging procedure provides an appropriate inner product between solutions Ψ_f. As the solutions are distributional states the physical inner product between Ψ_f and Ψ_g is defined by the action of Ψ_f on any state $|g\rangle$ which gives Ψ_g when group averaged.

$$(\Psi_f|\Psi_g)_{\mathrm{phys}} = (\Psi_f|g\rangle = \overline{(\Psi_g|f\rangle} \tag{3.2}$$

[1] Formally, integration over the 'lapse' α introduces the factor $\delta(C)$ that is necessary to extract physical states from kinematical ones and also yields the physical inner product between the resulting physical states. This procedure can be carried out in detail for LQC as follows [33].

Denote by $e_k(\nu)$, with $k \in (-\infty, \infty)$ a complete set of orthonormal eigenfunctions of Θ on $\mathcal{H}_{\mathrm{kin}}^{\mathrm{grav}}$. We will denote the eigenvalues by ω_k^2 and, without loss of generality, assume that $\omega_k \geq 0$ [34, 35]. (Eigenfunctions and operator functions of Θ are discussed in appendix B.) Any $f(\nu, \phi) \in \mathbb{S}$ can be expanded as

$$f(\nu, \phi) = \int \mathrm{d}k \, \tfrac{1}{2\pi} \int \mathrm{d}p_\phi \, \tilde{f}(k, p_\phi)\, e^{ip_\phi \phi}\, e_k(\nu)\,. \tag{3.3}$$

Here and in what follows the range of integrals will be from $-\infty$ to ∞ unless otherwise stated. Using this expansion, we can group-average any $f(\nu, \phi)$ to obtain a distributional solution (in $\mathbb{S}^\star$) $\Psi_f(\nu, \phi)$ to the quantum constraint:

$$\Psi_f(\nu, \phi) \; := \; \int \mathrm{d}\alpha \, [e^{i\alpha C}\, 2|p_\phi|\, f(\nu, \phi)] \tag{3.4}$$

[1]There is some freedom in the definition of the action of elements of $\mathrm{Cyl}^\star$. In LQC, this freedom is used to simplify the expression of the physical inner product [33] and the subsequent action of Dirac observables on $\mathcal{H}_{\mathrm{phy}}$. We will use the same conventions here.

$$= \int \mathrm{d}k \int \mathrm{d}p_\phi \, \delta(p_\phi^2 - \omega_k^2) \, 2|p_\phi| \tilde{f}(k, p_\phi) \, e^{ip_\phi \phi} \, e_k(\nu) \, ,$$

where, the operator $2|p_\phi|$ has been introduced just for later technical simplification. Had we dropped it, we would have associated with f the solution $(2|p_\phi|)^{-1} \, \Psi_f$ and, in the end, obtained a unitarily equivalent representation of the algebra of Dirac observables.

By carrying out the integral over p_ϕ the expression of Ψ_f can be brought to the desired form:

$$
\begin{aligned}
\Psi_f(\nu, \phi) & = \int \mathrm{d}k \left[\tilde{f}(k, \omega_k) \, e^{i\omega_k \phi} \, e_k(\nu) + \tilde{f}(k, -\omega_k) \, e^{-i\omega_k \phi} \, e_k(\nu) \right] \\
& =: \Psi_f^+(\nu, \phi) + \Psi_f^-(\nu, \phi) \, .
\end{aligned}
\tag{3.5}
$$

By their very definition $\Psi_f^\pm(\nu, \phi)$ satisfy

$$\Psi_f^\pm(\nu, \phi) = e^{\pm i\sqrt{\Theta} \, (\phi - \phi_o)} \, \Psi_f^\pm(\nu, \phi_o) \, , \tag{3.6}$$

whence they can be interpreted as the 'positive and negative frequency solutions' to (2.5) with respect to the relational time ϕ considered in the deparametrized system. Thus the group average of f is a solution Ψ_f to the quantum constraint (2.5) which, furthermore, is naturally decomposed into positive and negative frequency parts. Ψ_f is to be regarded as a distribution in $\mathbb{S}^\star$ which acts on elements $g \in \mathbb{S}$ via the kinematic inner product [62, 12]:

$$
\begin{aligned}
(\Psi_f | g \rangle & := \langle \Psi_f | g \rangle \\
& = \int \mathrm{d}k \int \mathrm{d}p_\phi \, \delta(p_\phi^2 - \omega_k^2) \, 2\omega_k \, \overline{\tilde{f}}(k, p_\phi) \, \tilde{g}(k, p_\phi) \\
& = \int \mathrm{d}k \left[\overline{\tilde{f}}(k, \omega_k) \, \tilde{g}(k, \omega_k) + \overline{\tilde{f}}(k, -\omega_k) \, \tilde{g}(k, -\omega_k) \right] .
\end{aligned}
\tag{3.7}
$$

Finally, the group averaged scalar product on solutions Ψ_f is given just by this action [62, 12]. Thus, given any elements f, g in $\mathbb{S}$, the scalar product between the corresponding group averaged states Ψ_f, Ψ_g is given by

$$(\Psi_f, \, \Psi_g) := (\Psi_f | g \rangle = \overline{(\Psi_g | f \rangle} \, . \tag{3.8}$$

In section 3.2 we will obtain a vertex expansion for this scalar product.

As discussed in the previous chapter there is a superselection allowing us to work with either the set of positive or negative frequency solutions. Therefore, as in the Klein-Gordon theory, we are led to work with either set. In LQC, one generally works with the positive frequency ones. Then the physical Hilbert space $\mathcal{H}_{\text{phy}}$ of LQC consists of *positive frequency* solutions $\Psi_+(\nu, \phi)$ to the quantum constraint (2.5), i.e. solutions satisfying

$$-i\partial_\phi \, \Psi_+(\nu, \phi) = \sqrt{\Theta}\Psi_+(\nu, \phi) \; \equiv \; H\Psi_+(\nu, \phi) \tag{3.9}$$

with inner-product (3.8). This inner product can be re-expressed simply as:

$$(\Psi_+, \, \Phi_+)_{\text{phy}} = \sum_{\nu=4n\ell_o} \bar{\Psi}_+(\nu, \phi_o) \, \Phi_+(\nu, \phi_o) \,. \tag{3.10}$$

and is independent of the value ϕ_o of ϕ at which the right side is evaluated. The physical Hilbert space obtained through group averaging is precisely that discussed in the previous chapter.

3.2 Sum Over Histories

To mimic the general spin foam constructions in LQC, in this section we will largely disregard the fact that the scalar field can be used as relational time and that the final constraint has the form of the Schrödinger equation. Instead, we will use the group averaging procedure for the full constraint

$$C = -\partial_\phi^2 - \Theta \; \equiv \; p_\phi^2 - \Theta \tag{3.11}$$

and incorporate the positive frequency condition in a second step. None of the steps in this analysis refer the evolution in relational time mentioned above. The primary object of interest will be the physical scalar product, rather than the transition amplitude for a Schrodinger state $\Psi(\nu, \phi_i)$ at an initial 'time' ϕ_i to evolve to another state $\Phi(\nu, \phi_f)$ at a final 'time' ϕ_f.

In section 3.1 we considered general kinematic states $f(\nu, \phi)$. Here, in contrast, we will focus on the basis vectors $|\nu, \phi\rangle$ in $\mathcal{H}_{\text{kin}}$ which are the LQC analogs of spin networks that are used to specify the boundary states in SFMs. Following the

setup introduced in section 2.2 let us then fix two kinematic states, $|\nu_i, \phi_i\rangle$ and $|\nu_f, \phi_f\rangle$. For notational simplicity, we will denote the group averaged solutions to (3.11) they define by $|[\nu_i, \phi_i])$ and $|[\nu_f, \phi_f])$. The group averaged inner product between these states is given by

$$([\nu_f, \phi_f], \, [\nu_i, \phi_i]) = 2\textstyle\int \mathrm{d}\alpha \, \langle \nu_f, \phi_f | \, e^{i\alpha C} \, |p_\phi| \, |\nu_i, \phi_i\rangle . \tag{3.12}$$

From our discussion in section 3.1, one would expect this physical scalar product $([\nu_f, \varphi]_+, \, [\nu_i, 0]_+)_{\mathrm{phy}} = G(\nu_f, \varphi; \nu_i, 0)$ to be equivalent to the transition amplitudes considered in section 2.2. This is indeed the case. For, the positive frequency solution $\Psi_{\nu_i, \phi_i} \equiv [\nu_i, \phi_i]_+$ obtained by group averaging the kinematic basis vector $|\nu_i, \phi_i\rangle$ is given by

$$\Psi_{\nu_i, \phi_i}(\nu, \phi) = \textstyle\int \mathrm{d}k \, (\bar{e}_k(\nu_i) \, e^{-i\omega_k \phi_i}) \, e^{i\omega_k(\phi)} \, e_k(\nu) \tag{3.13}$$

(see Eq.(3.5)) so that the physical scalar product between positive frequency solutions $[\nu_i, \phi_i]_+$ and $[\nu_f, \phi_f]_+$ is given by

$$([\nu_f, \phi_f]_+, \, [\nu_i, \phi_i]_+)_{\mathrm{phy}} = \textstyle\int \mathrm{d}k \, e^{i\omega_k(\phi_f - \phi_i)} \, \bar{e}_k(\nu_i) \, e_k(v_f) \tag{3.14}$$

(see Eq (3.8)). The right hand side is precisely the expression of the transition amplitude $\langle \nu_f | \, e^{iH\varphi} | \nu_i \rangle = \int \mathrm{d}k \, \langle \nu_f | \, e^{iH\varphi} | k \rangle \langle k | \nu_i \rangle$. Since $e_k(\nu) = \langle \nu | k \rangle$, we have the equality: $G(\nu_f, \varphi; \nu_i, 0) = A(\nu_f, \varphi; \nu_i, 0)$.

If we restrict ourselves to the positive part of the spectrum of $\hat{p}_\phi$ —or, to 'positive frequency' physical states— as in LQC, *the physical inner product is given precisely by the transition amplitude* $A(\nu_f, \phi_f; \nu_i, \phi_i)$ we focused on in chapter 2. On the 'negative frequency' solutions it is given by the complex conjugate, $[A(\nu_f, \phi_f; \nu_i, \phi_i)]^\star$ (because the matrix elements H_{ν_m, ν_n} and Θ_{ν_m, ν_n} are all real.) If we allow both, then the inner product is always real: $\langle \nu_f, \phi_f | \nu_i, \phi_i \rangle_{\mathrm{phys}} = A(\nu_f, \phi_f; \nu_i, \phi_i) + [A(\nu_f, \phi_f; \nu_i, \phi_i)]^\star$. Thus, the physical inner product, the key object in the timeless framework, can be readily constructed from the transition amplitude, the key object in the deparameterized framework.

Our goal is to express the scalar product given by (3.12) as a vertex expansion

a la SFMs and study its properties. To do so we first focus on the amplitude

$$A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha) = 2 \langle \nu_f, \phi_f| \, e^{i\alpha C} \, |p_\phi| \, |\nu_i, \phi_i\rangle \qquad (3.15)$$

which constitutes the integrand of (3.12). Our strategy is to first expand this amplitude such that the integral α can be carried out term by term in the resulting vertex expansion. We will find that this is a highly non-trivial requirement.

Mathematically one can choose to regard αC as a Hamiltonian operator. Then $A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha)$ can be interpreted as the probability amplitude for an initial kinematic state $|\nu_i, \phi_i\rangle$ to evolve to a final kinematic state $|\nu_f, \phi_f\rangle$ in a unit 'time interval' and we can follow Feynman's procedure [2] to express it as a sum over histories. More precisely the constraint C generates gauge evolution, so we will obtain a sum over gauge histories. Technically, a key simplification comes from the fact that the constraint C is a sum of two commuting pieces that act separately on $\mathcal{H}_{\text{kin}}^{\text{matt}}$ and $\mathcal{H}_{\text{kin}}^{\text{grav}}$. Consequently, the amplitude (3.15) factorizes as

$$A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha) = A_\phi(\phi_f, \phi_i; \alpha) \, A_G(\nu_f, \nu_i; \alpha) \qquad (3.16)$$

with

$$A_\phi(\phi_f, \phi_i; \alpha) = 2 \langle \phi_f| e^{i\alpha p_\phi^2} \, |p_\phi| |\phi_i\rangle, \quad \text{and} \quad A_G(\nu_f, \nu_i; \alpha) = \langle \nu_f| e^{-i\alpha \Theta}|\nu_i\rangle . \quad (3.17)$$

It is easy to cast the first amplitude, A_ϕ, in the desired form using either a standard Feynman expansion or simply evaluating it by inserting a complete eigen-basis of p_ϕ. The result is:

$$A_\phi(\phi_f, \phi_i; \alpha) = 2 \int \mathrm{d}p_\phi \, e^{i\alpha p_\phi^2} \, e^{ip_\phi(\phi_f - \phi_i)} \, |p_\phi| \qquad (3.18)$$

The expansion of the gravitational amplitude A_G is not as simple. We will first express it as a sum over histories following the strategy from chapter 2. In a second step, we will evaluate the total amplitude (3.15) by integrating over α for each history separately. Although it is not a priori obvious, we will find that the amplitude associated to each history is manifestly finite and the total amplitude can be written as a discrete sum that mimics the vertex expansion in SFMs.

3.2.1 The gravitational amplitude A_G

As mentioned above, to apply the standard Feynman procedure we will regard $e^{-i\alpha\Theta}$ as an 'evolution operator' with 'Hamiltonian' $\alpha\Theta$ and a 'time interval' $\Delta\tau = 1$. We emphasize that this 'evolution' is a just a convenient mathematical construct and does not correspond to the physical evolution with respect to a relational time variable. Rather, since it is generated by the constraint C, physically it represents gauge transformations (or time reparameterizations).

As the construction of chapter 2 did not depend on the particular details of the Hamiltonian, H, it is simply extended to this amplitude. We then follow the construction of chapter 2 leaving out details. We first divide the interval $\Delta\tau = 1$ into N parts each of length $\epsilon = 1/N$ and write the gravitational amplitude $A_G(\nu_f, \nu_i; \alpha)$ as

$$\langle\nu_f|e^{-i\alpha\Theta}|\nu_i\rangle = \sum_{\bar\nu_{N-1},...,\bar\nu_1} \langle\nu_f|e^{-i\epsilon\alpha\Theta}|\bar\nu_{N-1}\rangle\langle\bar\nu_{N-1}|e^{-i\epsilon\alpha\Theta}|\bar\nu_{N-2}\rangle ... \langle\bar\nu_1|e^{-i\epsilon\alpha\Theta}|\nu_i\rangle$$

(3.19)

where we have first split the exponential into N identical terms and then introduced a decomposition of the identity operator at each intermediate 'time' $\tau = n\epsilon$, $n = 1, 2, .., N-1$. For notational simplicity, we will denote the matrix element $\langle\bar\nu_n|e^{-i\epsilon\alpha\Theta}|\bar\nu_{n-1}\rangle$ by $U_{\bar\nu_n\bar\nu_{n-1}}$ and set $\nu_f = \bar\nu_N$ and $\nu_i = \bar\nu_0$. We then have

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{\bar\nu_{N-1},...,\bar\nu_1} U_{\bar\nu_N\bar\nu_{N-1}} U_{\bar\nu_{N-1}\bar\nu_{N-2}} \cdots U_{\bar\nu_1\bar\nu_0}.$$

(3.20)

Where it is important to note that the $U_{\bar\nu_{n+1}\bar\nu_n}$ appearing here is distinct from the U introduced in the previous chapter as it depends on $\alpha\Theta$ instead of $\sqrt{\Theta}$. The division of $\Delta\tau$ again provides a skeletonization of this 'time interval'. An assignment $\sigma_N = (\bar\nu_N, \ldots, \bar\nu_0)$ of volumes to the $N+1$ time instants $\tau = \epsilon n$ is now regarded as a discrete (gauge) history associated with this skeletonization. The matrix element is given by a sum of amplitudes over histories with fixed endpoints,

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{\sigma_N} A(\sigma_N) \equiv \sum_{\sigma_N} U_{\bar\nu_N\bar\nu_{N-1}} U_{\bar\nu_{N-1}\bar\nu_{N-2}} \ldots U_{\bar\nu_2\bar\nu_1} U_{\bar\nu_1\bar\nu_0}.$$

(3.21)

The next step is to take the 'continuum' limit, $N \to \infty$, of the skeletonization.

Again since our basis $|\nu_n\rangle$ is discrete, one can make rigorous sense of the $N \to \infty$ limit by reorganizing the well-defined sum (3.21) according to the number of volume transitions.

We first group paths according to the number of volume transitions rather than time steps. Let us then consider a path σ_N^M which involves M volume transitions (clearly, $M \leq N$):

$$\sigma_N^M = (\,\nu_M, \ldots, \nu_M;\ \nu_{M-1}, \ldots, \nu_{M-1};\ \ldots \ldots;\ \overbrace{\nu_1, \ldots, \nu_1;}^{N_2} \underbrace{\nu_0, \ldots, \nu_0}_{N_1}\,). \qquad (3.22)$$

Thus, the volume changes from ν_{m-1} to ν_m at 'time' $\tau = N_m \epsilon$ and remains ν_m till time $\tau = N_{m+1}\,\epsilon$. Where ν_m is the volume after the m-th *volume transition* along the given discrete path, $\bar{\nu}_m$ is the volume at the end of the m-th *time interval*, i.e., at $\tau = m\epsilon$. These discrete histories are labeled more transparently by two ordered sequences

$$\sigma_N^M = \{\,(\nu_M, \nu_{M-1}, \ldots, \nu_1, \nu_0);\ (N_M, N_{M-1}, \ldots, N_2, N_1)\,\}, \quad \nu_m \neq \nu_{m-1},\ N \geq N_m > N_{m-1}. \qquad (3.23)$$

where $\nu_M, \ldots, \nu_0$ denote the volumes that feature in the history σ_N^M and N_k denotes the number of time steps after which the volume changes from ν_{k-1} to ν_k. Note that while no two *consecutive* volume values can be equal, a given volume value can repeat in the sequence; ν_m can equal some ν_n if $n \neq m \pm 1$. The probability amplitude for such a history σ_N^M is given by:

$$A(\sigma_N^M) = [U_{\nu_M \nu_M}]^{N-N_M}\, U_{\nu_M \nu_{M-1}}\ \cdots\ [U_{\nu_1 \nu_1}]^{N_2 - N_1 - 1}\, U_{\nu_1 \nu_0}\, [U_{\nu_0 \nu_0}]^{N_1 - 1}. \qquad (3.24)$$

Second we to perform the sum over all these amplitudes in three steps. First we keep the ordered set of volumes $(\nu_M, \ldots, \nu_0)$ fixed, but allow the volume transitions to occur at *any* value $\tau = n\epsilon$ in the interval $\Delta\tau$, subject only to the constraint that the m-th transition occurs before the $(m+1)$-th for all m. The sum of amplitudes over this group of histories is given by

$$A_N(\nu_M, \ldots, \nu_0; \alpha) = \sum_{N_M = M}^{N-1}\ \sum_{N_{M-1} = M-1}^{N_M - 1} \cdots \sum_{N_1 = 1}^{N_2 - 1} A(\sigma_N^M). \qquad (3.25)$$

This is again the grouping of histories with well defined $N \to \infty$ limit. Next we sum over all possible intermediate values of ν_m such that $\nu_m \neq \nu_{m-1}$, keeping $\nu_0 = \nu_i$, $\nu_M = \nu_f$ to obtain the amplitude $A_N(M)$ associated with the set of all paths in which there are precisely M volume transitions:

$$A_N(M; \alpha) = \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A_N(\nu_M, \ldots, \nu_0; \alpha) \tag{3.26}$$

Finally the total amplitude $A_G(\nu_f; \nu_i, \alpha)$ is obtained by summing over all volume transitions that are permissible within our initially fixed skeletonization with N time steps:

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{M=0}^{N} A_N(M; \alpha) \tag{3.27}$$

This concludes the desired re-arrangement of the sum (3.21). The sum on the right side is manifestly finite. Furthermore, since $A_G(\nu_f, \nu_i; \alpha) = \langle \nu_f | e^{-i\alpha\Theta} | \nu_i \rangle$, the value of the amplitude (3.27) does not depend on N at all; the skeletonization was introduced just to express this well-defined amplitude as a sum over histories. Thus, while the range of M in the sum and the amplitude $A_N(M; \alpha)$ in (3.27) both depend on N, the sum does not.

Now, in 2.4 the limit $\lim_{N\to\infty} A_N(\nu_M, \ldots, \nu_0; \alpha)$ exists for each grouping of histories and is given by

$$\begin{aligned} A(\nu_M, \quad &\ldots, \nu_0; \alpha) := \lim_{N\to\infty} A_N(\nu_M, \ldots, \nu_0; \alpha) \\ &= \int_0^1 d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_2} d\tau_1 \ A(\nu_M, \ldots, \nu_0; \tau_M, \ldots, \tau_1; \alpha) \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} A(\nu_M, \ldots, \nu_0; \tau_M, \ldots, \tau_1; \alpha) := e^{-i(1-\tau_M)\alpha\Theta_{\nu_M\nu_M}} \ (-i\alpha\Theta_{\nu_M\nu_{M-1}}) \times \\ \ldots \ e^{-i(\tau_2-\tau_1)\alpha\Theta_{\nu_1\nu_1}} \ (-i\alpha\Theta_{\nu_1\nu_0}) \ e^{-i\tau_1\alpha\Theta_{\nu_0\nu_0}} . \end{aligned} \tag{3.29}$$

Note that the matrix elements $\Theta_{\nu_m\nu_n} = \langle \nu_m | \Theta | \nu_n \rangle$ of Θ in $\mathcal{H}_{\text{kin}}^{\text{grav}}$ can be calculated easily from (2.6) and vanish if $(\nu_m - \nu_n) \notin \{0, \pm 4\ell_0\}$. Therefore, explicit evaluation of the limit is rather straightforward. We will assume that the limit $N \to \infty$ can be interchanged with the sum over $\nu_{M-1}, \ldots \nu_1$. (This assumption is motivated by the

fact that in the expression of $A(\nu_M, \ldots, \nu_0; \alpha)$ most matrix elements of Θ vanish, and since the initial and final volumes are fixed, the sums over intermediate volumes $\nu_{M-1}, \ldots, \nu_1$ extend over only a finite number of non-zero terms. It is further verified by the perturbation expansion of section 3.3). With this assumption it follows that

$$A_G(M; \alpha) := \lim_{N \to \infty} A_N(M; \alpha) \tag{3.30}$$
$$= \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} \lim_{N \to \infty} A_N(\nu_M, \ldots, \nu_0; \alpha)$$

exists for each finite M. Finally, the total gravitational amplitude can be written as an infinite sum:

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{M=0}^{\infty} A_G(M; \alpha) \tag{3.31}$$

$$A_G(M; \alpha) = \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \ldots, \nu_0; \alpha) \tag{3.32}$$

where the partial amplitude associated to each discrete history $\nu_M, \ldots, \nu_0$ is given by (3.29). Note that the reference to the skeletonization disappears in this limit. Thus, $A_G(M; \alpha)$ is the amplitude obtained by summing over all paths that contain precisely M volume transitions within the 'time interval' $\Delta\tau = 1$, irrespective of precisely when and at what values of volume they occurred.

While each partial amplitude $A_G(M; \alpha)$ is well-defined and finite, it does not ensure that the infinite sum converges. A priori the infinite sum on the right hand side of (3.31) could be, for example, only an asymptotic series to the well-defined left side. Also, our derivation assumed that the limit $N \to \infty$ commutes with the partial sums. Both these limitations will be overcome in section 3.3: We will see that $A_G(\nu_f, \nu_i; \alpha)$ is indeed given by a convergent sum (3.31).

Here the expansion is clearly seen as a sum over all histories of the configuration variable ν which are given by piecewise constant functions each taking a countable number of values ν_m. As compared to the standard path integral the sum over histories is clearly defined as a sum over M, the number of times ν changes along the history, a sum over $(\nu_{M-1}, \ldots, \nu_1)$ the constant values taken, and finally a time ordered integral over the length of each constant part. We have not yet

removed the dependence of τ_m, the time of each transition, yet here this sum can be viewed equivalently as a sum over all triangulations of the interval $\mathcal{I}$ in a distinct sense from the sum over triangulation analogous to that of the spin foam models presented in section 2.4. This brings to light two distinct notions of triangulation. The first being the construction of an abstract simplicial complex composed of an abstract set of simplices, the basic geometric objects, and the information about how they are connected. This we refer to as an abstract triangulation as it is not built upon some pre-existing manifold. The second notion of triangulation arises from geometric realization of these abstract simplicial complexes or the embedding in a topological space. This we refer to as an embedded triangulation as it is built upon the underlying manifold. It is clear that these are two very different notions of triangulation as for a given number of simplices, M, here there is a single abstract triangulation of the interval $\mathcal{I}$ while there is an uncountable number of embedded triangulations given by the set of possible partitions of the interval $\mathcal{I}$ into M subsets. Expressed as a sum over embedded triangulations the amplitude for each triangulation is simply a product of elements associated to each piece while we find in the following the the amplitude can be reduced to a sum over abstract triangulations whereby the local nature is *lost*. This realization raises the natural question as to whether the spinfoam sum over triangulations should extend over simply abstract triangulations or embedded triangulations.

As mentioned the expression (3.28) still contains some integrals of τ relating to the length of each constant portion of the history. These can be performed exactly thus reducing to a sum over abstract triangulations depending only on the number of times the volume changes and not on the time at which it changes. This is akin to modding out by time-reparametrizations. The case when all of $(\nu_M, \ldots, \nu_0)$ are distinct is straightforward and the result as given in [65]. The general case is a little more complicated and is analyzed in Appendix A.2. The final result is:

$$
A(\nu_M, \ldots, \nu_0; \alpha) = \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \ldots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \times
$$

$$
\prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k} \right)^{n_k - 1} \sum_{i=1}^{p} \frac{e^{-i\alpha \Theta_i \Delta \tau}}{\prod_{j \neq i}^{p} (\Theta_i - \Theta_j)} \Bigg|_{\Theta_i = \Theta_{\nu_{i(d)} \nu_{i(d)}}}
$$

$$
(3.33)
$$

where, since the volumes can repeat along the discrete path, $\nu_{i(d)}$ label the p distinct values taken by the volume and n_m the number of times that each value occurs in the sequence. The n_m satisfy $n_1 + \ldots + n_p = M + 1$. In doing so we notice that the amplitude is no longer local as it depends on how many times a given volume reoccurs during the history. It is clear here that locality is not a necessary ingredient for a sum over histories expansion and may disappear at some order of the construction.

To summarize, we have written the gravitational part $A_G(\nu_f, \nu_i; \alpha)$ of the amplitude as a 'sum over histories':

$$A_G(\nu_f, \nu_i; \alpha) = \sum_{M=0}^{\infty} \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \ldots, \nu_0; \alpha) \qquad (3.34)$$

with $A(\nu_M, \ldots, \nu_0; \alpha)$ given by (3.33). This expression consists of a sum over M, the number of volume transitions, and a sum over the (finite number of) sequences of $M-1$ intermediate volumes that are consistent with the boundary conditions and the condition that $\nu_m \neq \nu_{m+1}$. It is important to note that this is only one component of the physical inner product it is necessary to combine this with the scalar field component and integrate out the lapse α. The ability to carry out this integral is a highly non-trivial requirement.

3.2.2   Vertex expansion of the physical inner product

Thus for we have only considered the transition amplitudes for the evolution between an initial and final state as generated by the gauge evolution. Recall that the group-averaged scalar product is expressed in terms of these amplitudes as

$$([\nu_f, \phi_f], [\nu_i, \phi_i]) = 2\int \mathrm{d}\alpha \, A_\phi(\nu_i, \phi_i; \alpha) \, A_G(\nu_f, \nu_i; \alpha) \,. \qquad (3.35)$$

In the previous section we constructed a vertex expansion for the gravitational part of (3.35) which we can insert in the above expression.

$$([\nu_f, \phi_f], [\nu_i, \phi_i]) = 2\int \mathrm{d}\alpha \, A_\phi(\phi_f, \phi_i; \alpha) \sum_{M=0}^{\infty} \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \ldots, \nu_0; \alpha)$$
$$(3.36)$$

If we desire a vertex expansion for the physical inner product it is then necessary to carry out the integral over lapse α for each term of the vertex expansion seperately. If this does not give meaningful results it is then clear that the sum over vertices must be carried out before the group averaging and there does not exist a vertex expansion ala SFMs as constructed in this way. The main assumption then in our derivation —the only one that will be required also in section 3.3— is that one can interchange the integration over α and the (convergent but infinite) sum over M in the expression of $A_G(\nu_f, \nu_i; \alpha)$. Let us then use expressions (3.18) and (3.34) of A_ϕ and A_G, make the interchange and carry out the integral over α. If this can be done, the scalar product (3.35) will then be re-expressed as a sum of amplitudes associated with each discrete path $(\nu_M, \ldots, \nu_0)$:

$$([\nu_f, \phi_f],\ [\nu_i, \phi_i]) = \sum_{M=0}^{\infty} \left[\sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \ldots, \nu_0; \phi_f, \phi_i) \right]. \qquad (3.37)$$

where,

$$A(\nu_M, \ldots, \nu_0; \phi_f, \phi_i) = 2 \int \mathrm{d}\alpha \, A_\phi(\phi_f, \phi_i; \alpha) A(\nu_M, \ldots, \nu_0; \alpha) \qquad (3.38)$$

By replacing the partial amplitude $A(\nu_M, \ldots, \nu_0; \phi_f, \phi_i)$ with the expression (3.33) and carrying out the integral over the lapse α.

$$A(\nu_M, \ldots, \nu_0;\ \phi_f, \phi_i) = \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \ldots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \times$$
$$\prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k} \right)^{n_k - 1} \sum_{i=1}^{P} \int \mathrm{d}p_\phi \, e^{i p_\phi (\phi_f - \phi_i)} \, 2|p_\phi| \left. \frac{\delta(p_\phi^2 - \Theta_i \Delta \tau)}{\prod_{j \neq i}^{P} (\Theta_i - \Theta_j)} \right|_{\Theta_i = \Theta_{\nu_{i(d)} \nu_{i(d)}}}$$
$$(3.39)$$

The right side is a sum of distributions depending on the matrix elements of Θ and p_ϕ, but the result is well defined as it is integrated over p_ϕ. It is straightforward to perform this integral over p_ϕ and express $A(\nu_M, \ldots, \nu_0; \phi_f, \phi_i)$ in terms of the matrix elements of Θ:

$$A(\nu_M, \ldots, \nu_0; \phi_f, \phi_i) = \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \ldots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \times \qquad (3.40)$$

$$\prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k}\right)^{n_k - 1} \sum_{i=1}^{P} \frac{e^{i\sqrt{\Theta_i}\Delta\phi} + e^{-i\sqrt{\Theta_i}\Delta\phi}}{\prod_{j\neq i}^{P}(\Theta_i - \Theta_j)}\Bigg|_{\Theta_i = \Theta_{\nu_{i(d)}\nu_{i(d)}}}$$

where $\Delta\phi = \phi_f - \phi_i$. Since by inspection each amplitude $A(\nu_M, \ldots, \nu_0, \phi_f, \phi_i)$ is real, the group averaged scalar product (3.37) is also real.

Finally, as explained in section 3.1, the group averaging procedure yields a solution which has both positive and negative frequency components while the physical Hilbert space consists only of positive frequency solutions. Let us denote the positive frequency parts of the group averaged ket $|[\nu, \phi]\rangle$ by $|[\nu, \phi]_+\rangle$. Then, the physical scalar product between these states in $\mathcal{H}_{\text{phy}}$ is given by a sum over amplitudes $A(M)$, each associated with a fixed number of volume transitions:

$$
\begin{aligned}
([\nu_f, \phi_f]_+, \ [\nu_i, \phi_i]_+)_{\text{phy}} \ = \ & \sum_{M=0}^{\infty} A(M) \quad\quad\quad\quad\quad\quad\quad\quad (3.41)\\
= \ & \sum_{M=0}^{\infty} \Big[\sum_{\substack{\nu_{M-1},\ldots,\nu_1 \\ \nu_m \neq \nu_{m+1}}} \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1}\nu_{M-2}} \cdots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \\
& \times \ \prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k}\right)^{n_k - 1} \sum_{i=1}^{P} \frac{e^{i\sqrt{\Theta_i}\Delta\phi}}{\prod_{j\neq i}^{P}(\Theta_i - \Theta_j)}\Bigg|_{\Theta_i = \Theta_{\nu_{i(d)}\nu_{i(d)}}} \Big].
\end{aligned}
$$

(Note that the right side is in general complex, a point to which we will return in section 3.3.) This is the vertex expansion of the physical inner product we were seeking. It has two key features. First, the integral over the parameter α was carried out and is not divergent. This is a non-trivial and important result if we are interested in computing the physical inner product perturbatively, i.e., order by order in the number of vertices. Second, the summand involves only the matrix elements of Θ which are easy to compute. As remarked earlier, significant simplification arises because Eq (2.6) implies that $\Theta_{\nu_m \nu_n}$ is zero if $\nu_m - \nu_n \notin \{0, \pm 4\ell_0\}$. This provides a clear benefit over the deparametrized model - as the sum runs over a finite number of paths for each partial amplitudes with M vertices.

Let us summarize. We did not begin by postulating that the physical inner product is given by a formal path integral. Rather, we started with the kinematical Hilbert space and the group averaging procedure and *derived* a vertex expansion of the physical inner product. Because the Hilbert space framework is fully under

control, we could pin-point the one assumption that is needed to arrive at (3.41): the sum over vertices and the integral over α can be interchanged. In the full theory, one often performs formal manipulations which result in divergent individual terms in the series under consideration. (For instance sometimes one starts by expanding the very first amplitude (3.15) in powers of α even though the α integral of each term is then divergent [60, 66]). *In our case, individual terms in the series are all finite.* Nonetheless, at present the interchange of the α-integral and the infinite sum over M has not been justified. If this gap can be filled, we would have a fully rigorous argument that the well-defined physical inner product admits an exact, convergent vertex expansion (3.41). (This assumption is needed only in the timeless framework because the integration over α never appears in the deparameterized framework of section 2.3.)

Thus, the total transition amplitude has been expressed as a vertex expansion (3.41) a la SFMs. We provided several intermediate steps because, although the left hand sides are equal, the final vertex expansions is *different* from that obtained in section 2.3: While (2.34) features matrix elements of $H = \sqrt{\Theta}$, (3.41) features matrix elements of Θ itself. The existence of distinct but equivalent vertex expansions is quite surprising. In each case we emphasized a distinct aspect of dynamics: the timeless framework and group averaging in (3.41), and relational time and deparametrization in (2.34). This fact leads to two important differences. First $\sqrt{\Theta_{\nu_m \nu_m}}$, the square root of the matrix element of Θ, is distinct from $H_{\nu_m \nu_m}$, the matrix element of the square root of Θ. Second, because the off diagonal elements $\Theta_{\nu_m \nu_n}$ in Eq (3.40) are non-zero only if $\nu_m = \nu_n \pm 4\ell_o$, consecutive ν_m in second sum in Eq (3.41) can differ only by $\pm 4\ell_o$. There is no such simplification in Eq (2.34). Because of these differences, although the physical inner product obtained by group averaging is related in a simple manner to the transition amplitude, the vertex expansion obtained in this section is completely different from that obtained in the last two sections. If we were to terminate the sum at any finite order, the results would not be simply related.

3.3 Perturbation Series

We will now show that the expression (3.41) of the transition amplitude can also be obtained using a specific perturbative expansion in the same manner as the deparametrized system. Structurally, this second derivation of the vertex expansion is reminiscent of the perturbative strategy used in group field theory (see, e.g., [29, 30]).

We begin by considering the diagonal and off-diagonal parts D and K of the operator Θ in the basis $|\nu = 4n\ell_o\rangle$. Thus, matrix elements of D and K are given by:

$$D_{\nu'\nu} = \Theta_{\nu\nu}\,\delta_{\nu'\nu}, \qquad K_{\nu'\nu} = \begin{cases} \Theta_{\nu'\nu} & \nu' \neq \nu \\ 0 & \nu' = \nu \end{cases} \tag{3.42}$$

Clearly $C = p_\phi^2 - D - K$. We think of $p_\phi^2 - D$ as the 'main part' of C and K as a 'perturbation'. To implement it, introduce a 1-parameter family of operators

$$C_\lambda = p_\phi^2 - \Theta_\lambda := p_\phi^2 - D - \lambda K \tag{3.43}$$

as an intermediate mathematical step. The parameter λ simply serves as a marker to keep track of powers of K in the perturbative expansion and we will have to set $\lambda = 1$ at the end of the calculation. As opposed to the deparametrized case, here we must be more careful and ensure that the the spectrum of these constraints C_λ is continuous in the parameter λ at the point $\lambda = 0$. At the very least we require that zero remains in the continuous part of the spectrum. This problem is brought to the fore in the vacuum models where when defining the operator C_λ as above the spectrum becomes completely discrete at $\lambda = 0$, at which point expansion becomes ill-defined.

Our starting point is again the decomposition (3.16) of the amplitude $A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha)$ into a scalar field and a gravitational part. The λ dependence appears in the gravitational part:

$$A_G^{(\lambda)}(\nu_f, \nu_i, \alpha) := \langle \nu_f | e^{-i\alpha\Theta_\lambda} | \nu_i \rangle. \tag{3.44}$$

We construct a perturbative expansion of this amplitude. Again we think of $e^{-i\alpha\Theta_\lambda}$ as a mathematical 'evolution operator' defined by the 'Hamiltonian' $\alpha\Theta_\lambda$ and a 'time interval' $\Delta\tau = 1$. The ' unperturbed Hamiltonian' is αD and the 'per-

turbation' is $\lambda \alpha K$. Following the textbook procedure, we define the 'interaction Hamiltonian' as

$$H_I(\tau) = e^{i\alpha D\tau} \; \alpha K \; e^{-i\alpha D\tau}. \tag{3.45}$$

Then the evolution in the interaction picture is dictated by the 1-parameter family of unitary operators on $\mathcal{H}_{\mathrm{kin}}^{\mathrm{grav}}$

$$\tilde{U}_\lambda(\tau) = e^{i\alpha D\tau} e^{-i\alpha\Theta_\lambda \tau}, \qquad \text{satisfying} \quad \frac{\mathrm{d}\tilde{U}_\lambda(\tau)}{\mathrm{d}\tau} = -i\lambda \, H_I(\tau) \tilde{U}_\lambda(\tau). \tag{3.46}$$

The solution of this equation is given by a time-ordered exponential:

$$\begin{aligned}
\tilde{U}_\lambda(\tau) &= \mathcal{T} \, e^{-i\int_0^\tau H_I(\tau)\mathrm{d}\tau} \\
&= \sum_{M=0}^{\infty} \lambda^M \int_0^\tau \mathrm{d}\tau_M \int_0^{\tau_M} \mathrm{d}\tau_{M-1} \ldots \int_0^{\tau_2} \mathrm{d}\tau_1 \; [-iH_I(\tau_M)] \ldots [-iH_I(\tau_1)]
\end{aligned} \tag{3.47}$$

Using use the relation $e^{-i\alpha\Theta_\lambda} = e^{-i\alpha D}\tilde{U}_\lambda(1)$, with $\tilde{U}_\lambda$ given by (3.47), take the matrix element of $e^{i\alpha\Theta_\lambda}$ between initial and final states, $|\nu_i \equiv \nu_0\rangle$ and $|\nu_f \equiv \nu_M\rangle$, and write out explicitly the product of the H_I's. The result is

$$\begin{aligned}
A_G^{(\lambda)}(\nu_f, \nu_i, \alpha) &= \sum_{M=0}^{\infty} \lambda^M \int_0^1 \mathrm{d}\tau_M . \int_0^{\tau_2} \mathrm{d}\tau_1 \sum_{\nu_{M-1}, \ldots, \nu_1} [e^{-i(1-\tau_M)\alpha D_{\nu_M \nu_M}}] \times \\
& \qquad (-i\alpha K_{\nu_M \nu_{M-1}}) \ldots (-i\alpha K_{\nu_1 \nu_0}) \, [e^{-i\tau_1 \alpha D_{\nu_0 \nu_0}}].
\end{aligned} \tag{3.48}$$

We can now replace D and K by their definition (3.42). Because K has no diagonal matrix elements, only the terms with $\nu_m \neq \nu_{m+1}$ contribute and the sum reduces precisely to

$$A_G^{(\lambda)}(\nu_f, \nu_i, \alpha) = \sum_{M=0}^{\infty} \lambda^M \left[\sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \ldots, \nu_0; \alpha) \right], \tag{3.49}$$

where $A(\nu_M, \ldots, \nu_0; \alpha)$ is given by (3.33) as in the sum over histories expansion of section 3.2.1.

We can now construct the total amplitude by including the scalar field factor (3.18) and performing the α integral as in section 3.2.2. Then the group averaged

scalar product is given by

$$([\nu_f, \phi_f], [\nu_i, \phi_i])^{(\lambda)} = \sum_{M=0}^{\infty} \lambda^M \Big[\sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_M, \ldots, \nu_0, \phi_f, \phi_i) \Big] \tag{3.50}$$

where $A(\nu_M, \ldots, \nu_0, \phi_f, \phi_i)$ is given in (3.40). If we now set $\lambda = 1$, (3.50) reduces to (3.41) obtained independently in section 3.2.1. Finally, let us restrict ourselves to the positive frequency parts $|[\nu, \phi]_+\rangle$ of $[\nu, \phi]\rangle$ which provide elements of $\mathcal{H}_{\text{phy}}$. Reasoning of section 3.2.2 tells us that the physical scalar product $([\nu_f, \phi_f]_+, [v_i, \phi_i]_+)_{\text{phy}}$ is given by (3.41).

Thus, by formally regarding the *volume changing*, off-diagonal piece of the constraint as a perturbation we have obtained an independent derivation of the vertex expansion for $([\nu_f, \phi_f]_+, [v_i, \phi_i]_+)_{\text{phy}}$ as a power series expansion in λ, the power of λ serving as a bookmark that keeps track of the number of vertices in each term. In this sense this alternate derivation is analogous to the vertex expansion obtained using group field theory. This derivation has a technical advantage. Since H_I is self-adjoint on $\mathcal{H}_{\text{kin}}^{\text{grav}}$, it follows that the expansion (3.47) of $\tilde{U}_\lambda(\tau)$ is convergent everywhere on $\mathcal{H}_{\text{kin}}^{\text{grav}}$ [67]. This in turn implies that the right hand side of (3.49) converges to the well-defined gravitational amplitude $A_G^{(\lambda)} = \langle \nu_f | e^{-i\alpha\Theta_\lambda} | \nu_i \rangle$. However, to arrive at the final vertex expansion starting from (3.49) we followed the same procedure as in section 3.2.2. Therefore, this second derivation of the vertex amplitude also assumes that one can interchange the integral over α with the (convergent but) infinite sum over M in (3.49). This derivation has a second benefit as it clearly demonstrates a potential breaking point of the theory, which is where the spectrum of the family of operators C_λ is discontinuous around $\lambda = 0$.

3.4   Satisfaction of the constraint

The physical inner product between the basis states defines a 2-point function:

$$G(\nu_f, \phi_f; \nu_i, \phi_i) := ([\nu_f, \phi_f]_+, [\nu_i, \phi_i]_+)_{\text{phy}} \tag{3.51}$$

and it follows from section 3.1 that it satisfies the constraint equation in each argument. Since $G(\nu_f, \phi_f;\, \nu_i, \phi_i) = \bar{G}(\nu_i, \phi_i;\, \nu_f, \phi_f)$, it suffices to focus just on one argument, say the final one. Then we have:

$$[\partial_{\phi_f}^2 - \Theta_f]G(\nu_f, \phi_f;\, \nu_i, \phi_i) = 0 \tag{3.52}$$

where Θ_f acts as in (2.6) but on ν_f in place of ν. If one replaces Θ by Θ_λ, one obtains a 2-point function $G_\lambda(\nu_f, \phi_f;\, \nu_i, \phi_i)$ which, as we saw in section 3.2.2 admits a perturbative expansion:

$$G_\lambda(\nu_f, \phi_f;\, \nu_i, \phi_i) = \sum_{M=0}^{\infty} \lambda^M A_M(\nu_f, \phi_f; \nu_i, \phi_i), \tag{3.53}$$

where A_M is the amplitude defined in (3.41):

$$
\begin{aligned}
A_M(\nu_f, \phi_f; \nu_i, \phi_i) &= \sum_{\substack{\nu_{M-1},\ldots,\nu_1 \\ \nu_m \neq \nu_{m+1}}} A_+(\nu_M, \ldots \nu_0;\, \phi_f, \phi_i) \\
&\equiv \sum_{\substack{\nu_{M-1},\ldots,\nu_1 \\ \nu_m \neq \nu_{m+1}}} \Theta_{\nu_M \nu_{M-1}} \Theta_{\nu_{M-1}\nu_{M-2}} \cdots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_0} \\
&\quad \times \prod_{k=1}^{P} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial \Theta_k}\right)^{n_k-1} \sum_{i=1}^{P} \frac{e^{i\sqrt{\Theta_i}\Delta\phi}}{\prod_{j\neq i}^{P}(\Theta_i - \Theta_j)}\bigg|_{\Theta_i = \Theta_{\nu_{i(d)}\nu_{i(d)}}}
\end{aligned}
\tag{3.54}
$$

The suffix $+$ in $A_+(\nu_M, \ldots, \nu_0;\, \phi_f, \phi_i)$ emphasizes that we have taken the positive frequency part.

As a non-trivial check on this expansion we will now show that G_λ satisfies (3.52) order by order. As the derivation of 3.54 requires the key assumption that the integral over the lapse α can be exchanged with the sum over vertices this provides a check of the validity of the expansion. Since $\Theta_\lambda = D + \lambda K$, our task reduces to showing

$$(\partial_{\phi_f}^2 - D_f)\, A_M(\nu_f, \phi_f;\, \nu_i, \phi_i) - K_f\, A_{M-1}(\nu_f, \phi_f;\, \nu_i, \phi_i) = 0 \,. \tag{3.55}$$

We will show that as with the expansion of Chapter 2, the left hand side is zero path by path in the sense that for every history acted on by the off-diagonal part

there are two histories acted on the diagonal part that cancel it. To see how the constraint acts on the partial amplitude with M vertices we first act on each amplitude $A_+(\nu_M, \dots \nu_0; \phi_f, \phi_i)$ viewed as a free function of ν_f and ϕ_f. Without loss of generality we assume that $\nu_f = \nu_{i(d)}$. Then we have

$$
(\partial^2_{\phi_f} - D_f) A_+(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) = \Theta_{\nu_f \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_i} \times
$$
$$
\left[\prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k} \right)^{n_k - 1} \sum_{i=1}^{P} \frac{\Theta_i e^{i\sqrt{\Theta_i}\Delta\phi}}{\prod_{j \neq i}^{P}(\Theta_i - \Theta_j)} \right.
$$
$$
\left. - \Theta_{v_{p(d)} v_{p(d)}} \prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k} \right)^{n_k - 1} \sum_{i=1}^{P} \frac{e^{i\sqrt{\Theta_i}\Delta\phi}}{\prod_{j \neq i}^{P}(\Theta_i - \Theta_j)} \right] \Bigg|_{\Theta_i = \Theta_{\nu_{i(d)} \nu_{i(d)}}} . \quad (3.56)
$$

If $\nu_{P(d)}$ occurs with multiplicity $n_P = 1$, if ν_f is the only volume to take the value $\nu_{P(d)}$ then there are no derivatives in Θ_P in the above equation and it simplifies to

$$
(\partial^2_{\phi_f} - D_f) A_+(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) = \Theta_{\nu_f \nu_{M-1}} \Theta_{\nu_{M-1} \nu_{M-2}} \dots \Theta_{\nu_2 \nu_1} \Theta_{\nu_1 \nu_i} \times
$$
$$
\left[\prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k} \right)^{n_k - 1} \sum_{i=1}^{P} \frac{(\Theta_i - \Theta_P) e^{i\sqrt{\Theta_i}\Delta\phi}}{\prod_{j \neq i}^{P}(\Theta_i - \Theta_j)} \right] \Bigg|_{\Theta_i = \Theta_{\nu_{i(d)} \nu_{i(d)}}} . \quad (3.57)
$$
$$
= \Theta_{\nu_f \nu_{M-1}} A_+(\nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) .
$$

Where in the final step we recognized that the term $(\Theta_i - \Theta_P)$ renders the $P - th$ term of the sum zero and removes the Θ_P dependence for each other term in the series. Together these reduce the structure to that of a history without the final point ν_f. Thus, on simple histories where the final volume occurs only once in the sequence, the action of $[\partial^2_{\phi_f} - D]$ is to give the amplitude of the history without ν_f, times a matrix element of Θ related to the transition from ν_{M-1} to ν_f. In general, the value of the final volume can be repeated in the discrete history; $n_p \neq 1$. It can be shown using the same strategy as that of section 2.4 that in all cases,

$$
(\partial^2_{\phi_f} - D_f) A_+(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) = \Theta_{\nu_f \nu_{M-1}} A_+(\nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) .
$$
$$
(3.58)
$$

Finally, it is straightforward to evaluate the action of the off-diagonal part on A_{M-1}

(see (3.55)):

$$K\,A_+(\nu_f,\nu_{M-2},\ldots,\nu_1,\nu_i;\phi_f,\phi_i) = \sum_{\nu_{M-1}} \Theta_{\nu_f\,\nu_{M-1}} A_+(\nu_{M-1},\nu_{M-2},\ldots,\nu_1,\nu_i;\phi_f,\phi_i)\,.$$

(3.59)

Combining these results we see that Eq. (3.55) is satisfied as it can be expressed as,

$$\sum_{\substack{\nu_{M-2},\ldots,\nu_1 \\ \nu_m \neq \nu_{m+1}}} \left[\sum_{\nu_{M-1}} (\partial_{\phi_f}^2 - D_f)A_+(\nu_f,\ldots\nu_0;\,\phi_f,\phi_i) + K A_+(\nu_f,\nu_{M-2},\ldots\nu_0;\,\phi_f,\phi_i) \right] = 0$$

(3.60)

Thus the vertex expansion we obtained is a solution to the quantum constraint equation. Further it is a good perturbative solution in the sense that, if we only take histories in which the number of volume transitions is less than some $M^\star$, then the constraint is satisfied to the order $\lambda^{M^\star}$:

$$[\partial_{\phi_f}^2 - (D_f + \lambda K_f)] \sum_{M=0}^{M^\star} \lambda^M A_M(\nu_f,\phi_f;\,\nu_i,\phi_i) = \mathcal{O}(\lambda^{M^\star+1})$$

(3.61)

Also in this calculation the cancelations occur in a simple manner; the off-diagonal part acting on histories with $M-1$ transitions gives a contribution for each history with M transitions that could be obtained by a adding a single additional transition in the original history. These contributions cancel with the action of the diagonal part on the histories with M transitions.

We again stress though that while the series above solves the constraint the solution may be purely formal. It is not a priori clear whether that the series is convergent, asymptotic, or simply a divergent series and secondly if cutting off the series at some finite point provides a good approximation of the exact physical inner product. In a later section we well examine these questions through explicit computation. Nonetheless this section provides an explicit check on our perturbative expansion of the physical inner product. This is a concrete realization, in this simple example, of a central hope of SFMs: to show that the physical inner product between spin networks, expressed as a vertex expansion, does solve the Hamiltonian constraint of LQG order by order.

This further leads to important insights into the structure of the vertex expansion for the full spin foam models. First it is clear in this system that no single term of the vertex expansion provides a solution of the constraint. It is necessary to include the full sum over all terms of the vertex expansion to obtain a solution. Although, since the solution of the constraint happens through the cancellation of closely related triangulations and even closely related colored triangulations this can provide very deep insight into the structure of the sum over triangulations including the relative weighting of the amplitude associated to each triangulation. Further this can be used as a procedure to generate higher order triangulations from lower order terms in the expansion.

3.5 The 'coupling constant' λ and the cosmological constant Λ

So far we have regarded the perturbation theory as purely a calculational tool and the coupling constant λ as a book-keeping device which merely keeps track of the number of vertices in the vertex expansion. We found that by taking only histories with up to M^* vertices that the constraint is solved to $\mathcal{O}(\lambda^{M^*} + 1)$, but from this standpoint values of λ other than $\lambda = 1$ have no physical significance. This leads us to question what the more convergent series with $\lambda < 1$ means. Further, if one regards GFT as fundamental and gravity as an emergent phenomenon, one is forced to change the viewpoint. In this perspective, the coupling constant λ is physical and can, for example, run under a renormalization group flow. The question we raised in section 3 is: What would then be the physical meaning of λ from the *gravitational* perspective? Surprisingly, in the LQC model under consideration, λ can be regarded as (a function of) the cosmological constant Λ.

Let us begin by noting how the quantum constraint changes in presence of a cosmological constant Λ:

$$-C(\Lambda) = \partial_\phi^2 + \Theta(\Lambda) \equiv \partial_\phi^2 + \Theta - \pi G \gamma^2 \Lambda \nu^2 \,. \tag{3.62}$$

Thus, only the diagonal part of Θ is modified and it just acquires an additional term proportional to Λ. In the GFT-like perturbation expansion, then, we are led

to decompose $\Theta_\lambda(\Lambda)$ as

$$\Theta_\lambda(\Lambda) = D(\Lambda) + \lambda K \qquad \text{where} \qquad D(\Lambda) = \pi G \left(\frac{3}{2\ell_o^2} - \gamma^2 \Lambda \right) \nu^2 . \qquad (3.63)$$

It is now easy to check that $\Psi(\nu, \phi)$ satisfies the constraint equation

$$[\partial_\phi^2 + D(\Lambda) + \lambda K] \, \Psi(\nu, \phi) = 0 \qquad (3.64)$$

with cosmological constant Λ if and only if $\tilde{\Psi}(\nu, \tilde{\phi})$ satisfies

$$[\partial_{\tilde{\phi}}^2 + D(\tilde{\Lambda}) + K] \, \tilde{\Psi}(\nu, \tilde{\phi}) = 0 \qquad (3.65)$$

where

$$\tilde{\Lambda} = \frac{\Lambda}{\lambda} + \frac{3}{2\gamma^2 \ell_o^2 \lambda} (\lambda - 1), \qquad \tilde{\phi} = \sqrt{\lambda} \, \phi, \quad \text{and} \quad \tilde{\Psi}(\nu, \tilde{\phi}) = \Psi(\nu, \phi) . \qquad (3.66)$$

Consequently the two theories are isomorphic. Because of this isomorphism, the gravitational meaning of the coupling constant λ is surprisingly simple: it is related to the cosmological constant Λ.

First this gives meaning to the vertex expansion with $\lambda \neq 1$. We find that it is related to the expansion of the normal gravitational constraint with a shifted value of the cosmological constant. Thus if we take the expansion where $\lambda \neq 1$ and $\Lambda = 0$ this is equivalent to the expansion with $\lambda = 1$ and with the following values of the cosmological constant

$$\Lambda = \frac{3}{2\gamma^2 \ell_o^2}(1 - \frac{1}{\lambda}) \qquad (3.67)$$

In particular if we look at the values of λ for which the series is clearly more divergent $\lambda > 1$ we find that this corresponds to a positive cosmological constant taking values $\Lambda \in (0, \frac{3}{2\gamma^2 \ell_o^2})$. On the other hand taking the values of $\lambda < 1$ leading to a more convergent series is equivalent to values of the cosmological constant $\Lambda \in (-\infty, 0]$.

Second a conjectural but interesting point is the following. Suppose we want to consider the Hamiltonian theory (or the SFM) for zero cosmological constant.

Then we are interested in the Hamiltonian constraint (3.65) with $\tilde{\Lambda} = 0$. From the GFT perspective, on the other hand, the cosmological constant is Λ which 'runs with the coupling constant' λ via[2]

$$\Lambda = \frac{3}{2\gamma^2\ell_o^2}(1 - \lambda) \tag{3.68}$$

At $\lambda = 1$, we have $\Lambda = 0$, whence the GFT reproduces the amplitudes of the SFM with zero cosmological constant. The question is: What is the space-time interpretation of GFT for other values of λ? From the perturbation theory perspective, λ will start out being zero in GFT and, under the renormalization group flow, it will hopefully increase to the desired value $\lambda = 1$. In the *weak coupling limit* $\lambda \approx 0$, the SFM will reproduce the amplitudes of the theory which has a positive but Planck scale cosmological constant $\Lambda \approx 3/2\gamma^2\ell_o^2$. This is just what one would expect from the 'vacuum energy' considerations in quantum field theories in Minkowski space-time. As the coupling constant λ increases and approaches the SFM value $\lambda = 1$, the cosmological constant Λ decreases. Now, suppose that the renormalization group flow leads us close to but not all the way to $\lambda = 1$. If we are just slightly away from the fixed point $\lambda = 1$, the cosmological constant Λ would be small and positive. These considerations are only heuristic. But they suggest an avenue by which a fully developed GFT could perhaps account for the smallness of the cosmological constant.

3.6 Testing the Expansion

As done for the deparametrized model in 2.4, we will now directly test the ability of the expansion to approximate the exact physical inner product using only finitely many terms of the vertex expansion. In the following we again analyze the expansion for a simple choice of initial and final volumes, $\nu_i = 4$, $\nu_f = 4$, and a range of values of the scalar field. We compute the first few orders in M of the vertex expansion exactly seeing how they vary with increasing order and compare the sum to the exact expression for the physical inner product.

---

[2]Note incidentally that, contrary to what is often assumed, running of constant under a renormalization group flow is not related to the physical time evolution in cosmology [68].

Figure 3.1. Real (solid) and imaginary (dashed) parts of the partial amplitude with zero vertices $A_0(4\lambda, 4\lambda, \Delta\phi)$ plotted as a function of $\sqrt{12\pi G}\varphi$.

Similar to the deparametrized vertex expansion the initial terms of the expansion do not necessarily provide a good approximation to the exact physical inner product. In particular at larger values of $\Delta\phi$ the sum is dominated not by the first terms of the expansion, but higher orders. This can be seen in figures 3.1-3.3 where for example both the $M = 2$, $M = 6$ partial amplitude dominates the $M = 0$ partial amplitudes for large $\Delta\phi$. We see in 3.4 the comparison of the sum over partial amplitudes up to order $M = 0$, $M = 6$, $M = 10$, and $M = 18$ as compared to the exact physical inner product. As more terms are included the expansion converges up to larger values of $\Delta\phi$, but has drastically large error beyond that. This indicates that either terms of higher order dominate the sum at those values of the scalar field or the series is asymptotic beyond a certain value of $\Delta\phi$.

We again find that a priori it is not clear what a fixed number of terms of the vertex expansion mean. For small values of $\Delta\phi$ they do provide a very good approximation to the exact physical inner product, but for larger values they may not even provide a zeroth order approximation. This expansion then serves as a clear warning for similar computations within more complicated models. Without a coupling constant in the theory that ensures that higher order terms are subdominate as in QED or in large N expansions of Yang-Mills, a finite number of terms in an expansion may not provide a good approximation to the physical

Figure 3.2. Real (solid) and imaginary (dashed) parts of the partial amplitude with two vertices $A_2(4\lambda, 4\lambda; \Delta\phi)$ plotted as a function of $\sqrt{12\pi G}\Delta\phi$. For $\sqrt{12\pi G}\Delta\phi < 2$ the $M = 2$ is smaller than the $M = 0$ while clearly the $M = 2$ term of the vertex expansion dominates the $M = 0$ for larger values of $\Delta\phi$, so we see for larger $\Delta\phi$ terms with more vertices can contribute more than the lower terms in the vertex expansion.
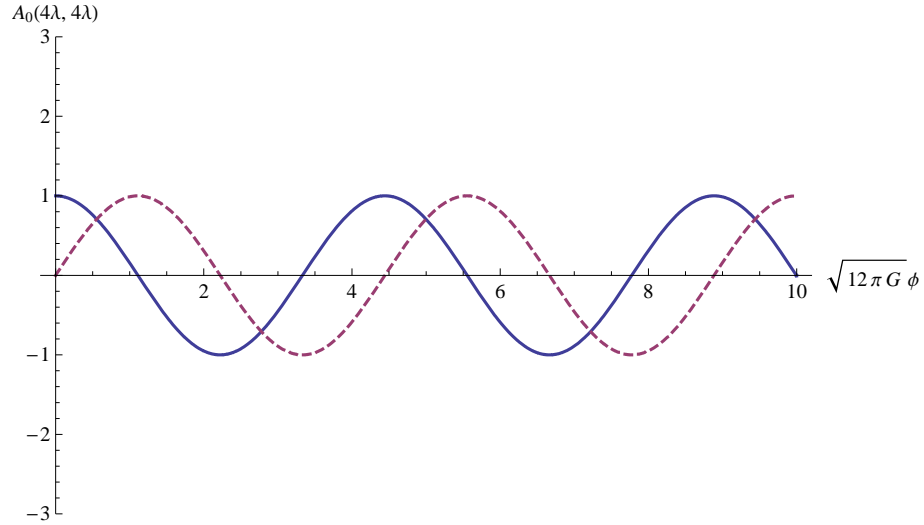

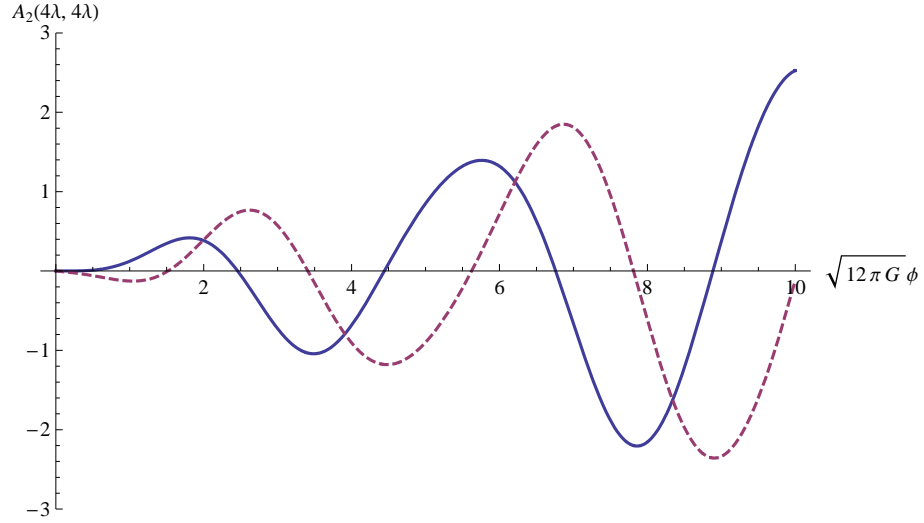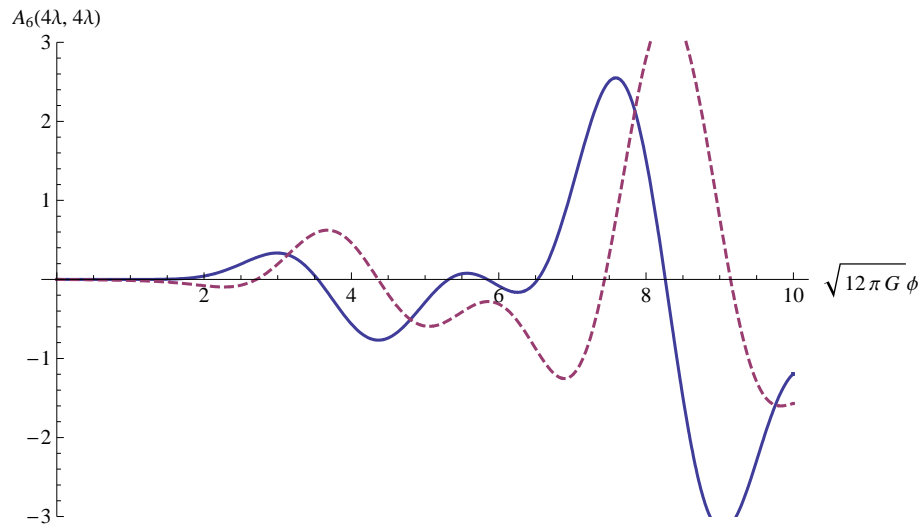**Figure 3.3.** Real (solid) and imaginary (dashed) parts of the partial amplitude with two vertices $A_6(4\lambda, 4\lambda; \Delta\phi)$ plotted as a function of $\sqrt{12\pi G}\Delta\phi$. Again the $M = 6$ term of the vertex expansion is larger than both $M = 0, M = 2$ terms for larger values of $\Delta\phi$. We again clearly see that for larger $\Delta\phi$ terms with more vertices can dominate the expansion.
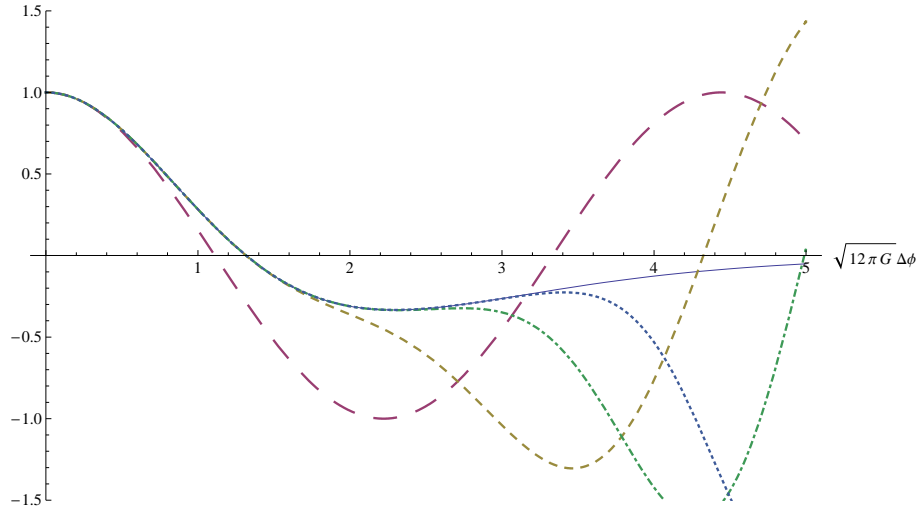
Figure 3.4. The exact transition amplitude $A(4\lambda, 4\lambda, \Delta\phi)$ (solid line) and the vertex expansion up to order $M = 2$ (large dashes), $M = 6$ (dashed), and $M = 10$ (dot-doshed), and $M = 18$ (dotted) plotted as functions of $\sqrt{12\pi G}\Delta\phi$. With more terms of the expansion included it converges at a larger value of $\Delta\phi$, but diverges strongly from the exact amplitude beyond that value. Thus it is necessary to include higher and higher terms in the vertex expansion to have even a first-order approximation of the exact transition amplitude.

inner product. Although, while there is no coupling constant, when computing the physical inner product at a fixed value of $\Delta\phi$ using the vertex expansion the series does converge to the exact result.

3.7 Discussion and Relation to SFM

We begin a brief summary. Group averaging provides a Green's function $G(\nu_f, \phi_f; \nu_i, \phi_i)$ representing the inner product between physical states extracted from the kinematic kets $|v_f, \phi_f\rangle$ and $|\nu_i, \phi_i\rangle$. We saw in section 3.2 that this quantity is equal to the transition amplitude $A(\nu_f, \phi_f; \nu_i, \phi_i)$ for the physical state $|\nu_i\rangle$ at the initial instant ϕ_i to evolve to the state $|\nu_f\rangle$ at the final instant of time ϕ_f as provided by the Schrödinger evolution of the deparameterized theory. Although they are equal, they emphasize different physics. Following the procedure of chapter 2 to pass from a Hamiltonian theory to a sum over discrete histories, we were able to obtain a series expansion, (3.41) for $G(\nu_f, \phi_f; \nu_i, \phi_i)$ - that mimics the vertex expansion of SFMs. In section 3.2, we had to make one assumption in the derivation

of the vertex expansion of $G(\nu_f, \phi_f; \nu_i, \phi_i)$: in the passage from (3.49) to (3.50) we assumed that the integration over α of the group averaging procedure commutes with an infinite sum in (3.49). This is an additional assumption on top of the those required for chapter 2.

In sections 3.3 we were able to obtain the same vertex sum using a perturbative expansion, in a coupling constant λ, that is reminiscent of GFTs. In sections 3.4 we showed that this is a useful expansion in the sense that the Green's function satisfy the constraint equation order by order in λ. Thus, if we were to truncate the expansion to order M, the truncated Green's function would satisfy the constraint equation up to terms of the order $O(\lambda^{M+1})$. In section 3.5 we showed that the coupling constant λ inspired by GFTs is closely related to the cosmological constant. This interpretation opens a possibility that a detailed study of the renormalization group flow in GFT may be able to account for the very small, positive value of the cosmological constant.

There is again a detailed parallel between this construction and the SFMS. The vertex expansion (3.41) can be seen as a sum over amplitudes corresponding to triangulations Δ_M of a given manifold as in the spin foam paradigm. The analog of the manifold M with boundaries Σ_i, Σ_f in SFMs is the manifold $\mathcal{V} \times \mathcal{I}$, where $\mathcal{V}$ is the elementary cell in LQC and $\mathcal{I}$, a closed interval in the real line (corresponding to $\tau \in [0,1]$ in the timeless framework. The analog of a triangulation in spin-foams is just a division of $\mathcal{V} \times \mathcal{I}$ into M parts by introducing $M-1$ time slices. Just as the triangulation in SFMs is determined by the number of 4-simplices, what matters in LQC is the number M; the precise location of slices is irrelevant. The analog of the dual-triangulation in SFMs is just a 'vertical' line in $\mathcal{V} \times \mathcal{I}$ with M marked points or 'vertices' (not including the two end-points of $\mathcal{I}$). Again, what matters is the number M; the precise location of vertices is irrelevant. Coloring of the dual-triangulation in SFMs corresponds to an ordered assignment $(\nu_M, \nu_{M-1}, \ldots \nu_1, \nu_0)$ of volumes to edges bounded by these marked points (subject only to the constraints $\nu_M = \nu_f$, $\nu_0 = \nu_i$ and $\nu_m \neq \nu_{m-1}$). Each vertex signals a change in the physical volume along the quantum history. The probability amplitude associated with the given coloring is given by $A(\nu_f, \ldots, \nu_0; \phi_f, \phi_i)$, (3.40)). A sum over these colorings yields the partial amplitude associated with the triangulation with M 'vertices'. These partial amplitudes then correspond to the spin

foam amplitude corresponding to a fixed triangulation Δ_M. Finally, the Green's function $G(\nu_f, \phi_f; \nu_i, \phi_i)$ is given by a sum over all M-vertex amplitudes.

Thus, the physical inner product of the timeless framework can be expressed as a *discrete sum* of amplitudes associated with triangulations of the interval $\mathcal{I}$ without the need of a 'continuum limit': A countable number of vertices suffices; the number of volume transitions does not have to become continuously infinite. This result supports the view that LQG and SFMs are not quite analogous to quantum field theories on classical space-times. Discrete quantum geometry at the Planck scale makes a key difference. Taken together, these results provide concrete support for the general paradigms that underlie SFM and GFT. By giving a construction from the canonical theory leading to a spin foam like structure we have both a proof of concept that canonical quantum gravity and spin foam models can be related and a more rigorous framework to help pin-down open issues of the spin foam models.

However, we again emphasize that this analysis has a limitation: We did not begin with a SFM in full general relativity and then arrive at the LQC model through a systematic symmetry reduction of the full vertex expansion. Rather, we began with an already symmetry reduced model in the canonical language and recast the results in the spin foam language. Reciprocally, a key strength of these results is that we did not have to start by *postulating* that the physical inner product or the transition amplitude is given by a formal path integral. Rather, a rigorously developed Hamiltonian theory guaranteed that these quantities are well-defined. We simply recast their expressions as vertex expansions. Thus we are not necessarily providing insight about the current spin-foam models from within their own framework rather we provide insight into the open questions of the current spin foam models through the assumption that they are derivable from a given canonical theory. This has the strong benefit of being based on traditional quantum mechanics as well as retaining a connection to continuous gravity.

It is often the case that exactly soluble models not only provide support for or against general paradigms but they can also uncover new issues whose significance had not been realized before. The LQC analysis has brought to forefront five such issues.

First, as the derivations of the current SFM's is based on a discretization of

general relativity on a fixed triangulation the theory does not tell us how to recover the continuum theory. The expected route to recover the continuum theory is to sum over all choices of triangulations, which is supported by the vertex expansion constructed here. The vertex expansion of LQC actually presents two distinct notions of a sum over triangulations. First there are the abstract simplicial complexes that are usually considered in the SFM. These are defined without reference to an underlying manifold, depending only on the set of abstract simplices, the basic geometric objects, and how they are connected together. The second notion of triangulation arises from the geometric realization of these abstract simplicial complexes, from the their embedding in a manifold. At different stages of the derivation each notion of triangulation, abstract and embedded, appears in the vertex expansion of LQC. Prior to the integration over the times at which each vertex appears, the amplitude 3.28 can be seen as an *integral* over the set of all embedded triangulation - here given by all partitions of the interval $\mathcal{I}$ into $M + 1$ intervals. In this case two triangulations with the same number of vertices but different times at which the vertices are located will have different amplitudes. Following the integration over these intermediate times the amplitude naturally reduced to a sum over abstract triangulations where all that matters is how many vertices there are not at what time they occur. This extends the standard question in SFM about what triangulations should be summed over beyond whether one consider triangulations more general than simplicial complexes; in addition one may need to consider not abstract triangulations, but also all possible embeddings.

Second, related to these two notions of triangulation is the presence or lack of 'locality' in the vertex expansions presented here. Recall that locality in SFM is defined as the property where the amplitude corresponding to a coloring of a fixed triangulation is given as the product of amplitudes corresponding to each edge, face, and vertex (each simplex) of the dual triangulation. Further the amplitude corresponding to each simplex depends only on its immediate neighbors. For the vertex expansion of LQC expressed as a sum over embedded triangulations, the gravitational amplitude 3.29 for a given choice of triangulation and of volumes is simply a product of elements associated to each edge and vertex of the triangulation. Following the reduction to a sum over abstract triangulations this simple product structure is *lost* 3.33. This loss of locality is then transferred to the vertex

expansion for the physical inner product. Locality is a key assumption of SFM that must of course be tested. In many systems locality as mentioned above does occur. The examples often cited include standard Feynman path integrals and the amplitudes associated to Feynman diagrams in momentum space. Similarly though there are many systems for which this locality is not present - for example Feynman diagrams in position space.

Third, it has revealed the advantage of adding matter fields. It is widely appreciated that on physical grounds it is important to extend SFMs beyond vacuum general relativity. However what was not realized before is that, rather than complicating the analysis, this generalization can in fact lead to interesting and significant technical simplifications. This point is brought out by a recent analysis of Rovelli and Vidotto [69]. They considered a simple model on a finite dimensional Hilbert space where there is no analog of the scalar field or the possibility of deparametrization. There, individual terms in the vertex expansion turn out to be well defined only after a (natural) regularization. In our example, the presence of the scalar field simplified the analysis (in the transition from (3.37) to (3.40)) and individual terms in the vertex expansion are finite without the need of any regularization. Furthermore, this simplification is not an artefact of our restriction to the simplest cosmological model. For example, in the Bianchi I model the Hamiltonian theory is also well-developed in the *vacuum* case [70]. Work done by Campiglia, Henderson, Nelson and Wilson-Ewing shows that technical problems illustrated in [69] arise also in this case, making it necessary to consider a regularization or other modification. These problems simply disappear if one also includes a scalar field. This issue will be further discussed in the following chapters.

Fourth, it came as a surprise that there are two *distinct* vertex expansions. The standard group averaging procedure implies that (for 'positive frequency' physical states) the physical inner product in the timeless framework equals the transition amplitude in the deparameterized framework used in LQC [33]. But whereas the perturbative expansion (2.34) involves the matrix elements of H, (3.41) involves the matrix elements of Θ (and their square-roots). Thus, while the sum yields the same quantity, if we were to truncate the perturbation series to any finite order one would obtain distinct results. This is not an artefact of using the simplest cosmological model. Indeed, from a Hamiltonian perspective, it would appear that distinct

vertex expansions can arise whenever a well-defined deparametrization is available or when the constraint can be solved explicitly as in $2+1$ gravity. This raises an interesting and more general possibility. Can there exist distinct spin foam models —constructed by using, say, distinct vertex amplitudes— for which the complete vertex expansions yield the same answer? Finite truncations of these expansions could be inequivalent, but each could be tailored to provide an approximation to the full answer for a specific physical question. One may then be able to choose which truncated expansion to use to probe a specific physical effect.

The fifth issue concerns three related questions in the spin foam literature: i) Should the physical inner products between states associated with spin networks be real rather than complex [64]? ii) In the classical limit, should one recover $\cos S$ in place of the usual term e^{iS}, where S is the Einstein Hilbert action [71, 72]? iii) Should the choice of orientation play a role in the sum of histories [63]? In the LQC example we studied in this paper, these three questions are intimately related. The inner product between the physical states $[\nu, \phi]_+$ determined by the kinematic basis vectors —which are the analogs of spin networks in this example— are in general complex (see Eq (3.41)). However, if we had dropped the positive frequency requirement, the group averaged inner products would have been real (see Eq (3.37)). The situation with action is analogous. And, as we show in the next paragraph, the positive frequency condition also selects a time-orientation.

Since this is an important issue, we will discuss it in some detail. Let us begin with the classical theory. The phase space is 4-dimensional and there is a single constraint: $C(\nu, b; \phi, p_\phi) := G\, p_\phi^2 - 3\pi\, (\ell_{\rm Pl}^2 \nu^2)\, b^2 = 0$. Dynamics has two conceptually interesting features. First, given a solution $(\nu(t), \phi(t))$ to the constraint and dynamical equations, $(-\nu(t), \phi(t))$ is also a solution (where t denotes proper time). They define the same space-time metric and scalar field; only the parity of the spatial triad is reversed. Therefore $(\nu(t), \phi(t)) \to (-\nu(t), \phi(t))$ is regarded as a gauge transformation. The second feature arises from the fact that the constraint surface has two 'branches', $p_\phi > 0$ and $p_\phi < 0$, joined at points $p_\phi = 0$ which represent Minkowski space-time. As is usual in quantum cosmology, let us ignore the trivial flat solution. Then each of the two portions $\bar{\Gamma}^\pm$ of the constraint surface defined by the sign of p_ϕ is left invariant by dynamics. Furthermore, there is a symmetry: Given a dynamical trajectory $(\nu(t), \phi(t))$ in $\bar{\Gamma}^+$, there is a trajectory $(\nu(t), -\phi(t))$

which lies in $\bar{\Gamma}^-$. This represents a *redundancy* in the description in the sense that we recover all physical space-time geometries $g_{ab}(t)$ even if we restrict only to one of the two branches $\bar{\Gamma}^\pm$. In particular, the dynamical trajectories on $\bar{\Gamma}^+$, for example, include solutions which start with a big-bang and expand out to infinity *as well as* those which start out with infinite volume and end their lives in a big crunch. The difference is in only in time orientation: If we regard ϕ as an internal or relational time variable and *reconstruct* space-time geometries from phase space trajectories, space-times obtained from a trajectory on $\bar{\Gamma}^+$ defines the same geometry as the one obtained from the corresponding trajectory on $\bar{\Gamma}^-$ but with opposite time orientation. As in the Klein-Gordon theory of a free relativistic particle, this redundancy is removed by restricting oneself either to the $p_\phi > 0$ sector or to the $p_\phi < 0$ sector. In the quantum theory, then, the physical Hilbert space is given by solutions $\Psi(\nu, \phi)$ to the quantum constraint (3.11) which in addition have only positive (or negative) frequency so that the operator p_ϕ is positive (or negative) definite. (They are also invariant under parity, $\Psi(\nu, \phi) = \Psi(-\nu, \phi)$). Thus, the LQC example suggests that in general SFMs one should fix the time-orientation, lending independent support to the new ideas proposed in [63]. Reality of the physical inner products between spin network states [64] and the emergence of $\cos S$ in place of e^{iS} [71, 72] can be traced back to the fact that in most of the SFM literature one sums over both orientations. However, our analysis provides only a hint rather than an iron-clad argument because all our discussion is tied to LQC models where symmetry reduction occurs prior to quantization.

Next, we saw that the inner product between physical states extracted from the kinematic basis vectors $|\nu, \phi\rangle$ are in general complex (as is generally true for constrained systems). However, if we were to enlarge the physical Hilbert space allowing for both 'positive *and* negative frequency' solutions, they become real. In SFMs the obvious analog of the LQC 'positive frequency' restriction is a choice of an orientation. Currently, the sum involves both orientations and the inner products between the physical states extracted from spin network states are all real. The LQC analysis naturally raises a question: Should one also impose a suitable restriction allowing, e.g., only those histories with only one orientation [63]? Or, does correct quantum physics require us to have only real physical scalar products in this basis? If so, why is there a qualitative difference in LQC?

We conclude with a hopeful direction for new insight into the construction of SFM. Coming from the view of the gravitational amplitude expressed as a sum of embedded triangulations 3.28. The vertex amplitude of LQC can be thought of as a sum over all piece-wise constant histories in the configuration variable. This is natural given the discrete topology of the spectrum of the configuration variable coming from the Kronecker delta inner product. This raises the hope that it is possible to derive the vertex expansion of spin foam models not by first discretizing the theory and then writing a path integral expression, but given the insight into the the structure of the kinematic Hilbert space of LQG of quantum geometries consider the space of piecewise constant configurations. The sum over all histories compatible with the quantum geometry would then decompose into a sum over different embedded triangulations and integrals over the values of the configuration variables in each constant patch.

Chapter 4

Generalization of Vertex Expansion

In the previous chapters we introduced two distinct sum over histories expansions for exactly soluble LQC, a quantization of FRW coupled to a massless scalar field. The first was constructed by identifying the scalar field as a clock, deparametrizing the system, and then following the Feynman construction of a sum over histories. The result is a sum over *discrete* histories where the volume changes only a finite number of times, which we call a vertex expansion due to the connection with the expansion of the same name in the spin foam models. The second was constructed without deparametrizing, but instead through constructing a vertex expansion for the group averaging expression for the physical inner product. The vertex expansion of this soluble model provided both a method of perturbative calculation for LQC as well as insight into the current SFMs albeit in a simplified setting. For both reasons we should extend the construction to more complex models such as anisotropic cosmologies [41, 42, 43] and inhomogeneous systems ([44, 45, 46]). For these systems there is little known analytically, so the vertex expansion provides a tool to perturbatively compute the physics of these models. Second, systems with more degrees of freedom capture more features of full quantum gravity and thus provide more precise insight into the structure of SFMs.

Although each vertex expansion was constructed for a particular model, the derivations did not make extensive use of the detailed properties of the model. In this chapter we analyze more general models with a single Hamiltonian constraint: first by extending the construction to general gravitational parts of the constraint while keeping the massless scalar field and then by additionally allowing for more

general matter contribution to the constraint. We show that the vertex expansion directly extends to a large class of systems. We find though that the requirement of being able to carry out the integral over the lapse, α, in the group averaging procedure is the main obstacle to generalization of the vertex expansion of the timeless framework. New input is needed for the integrals over α to be well defined Rather than being an insurmountable problem though, this leads to deeper insights into the vertex expansion and to spin foam models.

4.1   General Sum Over Discrete Histories

We will first return to the discrete sum over histories introduced in section 2.4 that was the basis of the vertex expansion for both the deparametrized and timeless framework. Here we analyze the necessary assumptions of that expansion to see to how specific the construction is to soluble LQC and hence how generally it can be applied. Recall that that the structure of the sum over discrete histories corresponding to the transition amplitude is,

$$\langle \nu_f | \, e^{iHt} | \nu_i \rangle = A(\nu_f, t; \, \nu_i, 0) = \sum_{M=0}^{\infty} A_M(\nu_f, t; \, \nu_i, 0) \tag{4.1}$$

where the partial amplitudes A_M are given by a sum over amplitudes corresponding to each discrete history where the volume changes M times.

$$A_M(\nu_f, t; \, \nu_i, 0) = \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} A(\nu_f, \nu_{M-1}, \ldots \nu_1, \nu_i, t) \tag{4.2}$$

The amplitude for a general discrete history is then given by

$$\begin{aligned}
A(\nu_f, \nu_{M-1}, \ldots \nu_1, \nu_i, t) \;\; = & \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} H_{\nu_M \nu_{M-1}} H_{\nu_{M-1} \nu_{M-2}} \ldots H_{\nu_2 \nu_1} H_{\nu_1 \nu_0} \;\times \\
& \prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial H_k} \right)^{n_k - 1} \sum_{i=1}^{P} \frac{e^{iH_i \varphi}}{\prod_{j \neq i}^{P} (H_i - H_j)} \Bigg|_{H_i = H_{\nu_{i(d)} \nu_{i(d)}}} .
\end{aligned}$$

where even though neighboring volumes are required to be distinct the same volume can repeat many times during the history. The $\nu_{i(d)}$ then denote the P distinct values of the volume taken along the history and n_i denote the degeneracy of each of those values. Here this encompasses both the vertex expansion of the deparametrized system, $H = \sqrt{\Theta}$ and $t = \varphi$, and of the gravitational amplitude of the timeless system, $H = -\alpha\Theta$ and $t = 1$.

The key ingredients going into the construction were the following. We require a kinematical Hilbert space with a basis normalizable to the Kronecker delta, which encompasses every Hilbert space. For the cases above this was given by the basis of volume eigenstates with an inner product given by the Kronecker delta,

$$\langle \nu_1 | \nu_2 \rangle = \delta_{\nu_1,\nu_2} \tag{4.3}$$

It was this structure allowed the continuous time limit $(N \to \infty)$ to be rigorously taken leading to a sum over discrete histories instead a path integral over continuous histories. Similarly the use of such a normalizable basis allowed for the decomposition of the Hamiltonian into diagonal and off-diagonal parts in the derivation of the vertex expansion from perturbation theory. It is preferable if the normalizable basis is provided by the eigenstates of a natural configuration variable. This is one point where LQG and LQC are distinct from standard Schrodinger quantum mechanics where the position and momentum eigenstates used to construct the Feynman path integral are non-normalizable in the Hilbert space, as opposed to the normalizable volume eigenstates of LQC. The second input was a self-adjoint operator H on this kinematical Hilbert space of which no additional assumptions were made.

From here we see that the derivation of $(4.1 - 4.3)$ is completely general. It extends to *any* Hilbert space, with any choice of a normalizable basis and *any* self-adjoint operator on that Hilbert space. The vertex expansion can then be applied to standard Schrodinger quantum mechanics using for example the eigenstates of the simple harmonic oscillator. It most naturally extend though to theories where the normalizable basis consists of the eigenstates of a natural configuration variable. Importantly, the spin network basis of LQG has precisely such a normalizable basis, so the vertex expansion is directly generalizable to LQG. The vertex expan-

sion In the following we will take a general kinematical Hilbert space given by finite linear combinations of basis elements given by states $|s\rangle$ for all s in some index set S. In soluble LQC the states $|s\rangle$ are the eigenstates of the volume and S is isomorphic to the set of integers. For the LQC of Bianchi I the states $|s\rangle$ are the simultaneous eigenstates of the volume ν and two variables describing the anisotropy and in LQG the states $|s\rangle$ are elements of the spin network basis with S given by the set of all piecewise-analytic graphs on the spatial slice Σ colored with $SU(2)$ spins. The inner product between these basis elements is given simply as

$$\langle s_1|s_2\rangle = \delta_{s_1,s_2} \quad \forall s_1, s_2 \,\epsilon\, S \tag{4.4}$$

We have then that given any deparametrizable system or any system with a true Hamiltonian the construction of the vertex expansion presented in chapter two can be simply generalized. The barriers to constructing a useful expansion of these systems lie primarily in the ability to compute the matrix elements of the Hamiltonian. Even in the exactly soluble system this was highly non-trivial.

4.2   Vertex Expansion of Physical Inner Product

In the previous section we saw that the vertex expansion of the amplitude to evolve between two kinematical states can be extended to any constrained system admitting a deparametrization through a choice of clock variable. For a general constrained system such a deparametrization is not available. Additionally, for more complex LQC models such as Bianchi I that do admit a deparametrization it is extremely difficult to calculate the matrix elements appearing in the vertex expansion (4.3). In either case we must return to the timeless framework and obtain the physical states and physical inner product through the group averaging procedure. We will find in this section though that to derive the vertex expansion in the timeless framework there are more stringent requirements to on the choice of basis states and the form of the constraint.

The derivation of the vertex expansion for the physical inner product constructed in chapter 3 required the following inputs. First we required a kinematical Hilbert space that was the tensor product of a gravitational Hilbert space with a

normalizable basis with the standard Hilbert space, $\mathcal{H}_{\text{kin}}^{\text{matt}} = L^2(\mathbb{R}, \mathrm{d}\phi)$, for the scalar field. In the following we denote the gravitational basis states by $|s\rangle$ where s is an element of some index sex S and the states have an inner product given by (4.4. The construction necessarily required the presence of a matter degree of freedom - the extension to purely vacuum models will be presented in the next chapter. Second the constraint is assumed to be given by a gravitational part plus a matter contribution corresponding to a massless scalar field

$$C = p_\phi^2/\hbar^2 - \Theta \tag{4.5}$$

It is this feature that allows the system to be simply deparametrized. Third, the gravitational part of the constraint, Θ has all non-zero matrix elements in the basis, $\langle s|\Theta|s\rangle \neq 0$ for all s. Given the first two assumptions the physical inner product can be written as follows.

$$([s_f, \phi_f], [s_i, \phi_i]) = 2\int \mathrm{d}\alpha \, \langle s_f, \phi_f| \, e^{i\alpha C} \, |p_\phi| \, |s_i, \phi_i\rangle \tag{4.6}$$
$$= \int \mathrm{d}\alpha A_\phi(\phi_f, \phi_i; \alpha) \, A_G(s_f, s_i; \alpha) \tag{4.7}$$

with

$$A_\phi(\phi_f, \phi_i; \alpha) = 2 \, \langle \phi_f| e^{i\alpha p_\phi^2} \, |p_\phi| |\phi_i\rangle, \quad \text{and} \quad A_G(s_f, s_i; \alpha) = \langle s_f| e^{-i\alpha\Theta} |s_i\rangle. \tag{4.8}$$

since the constraint simply breaks into one term acting on the gravitational Hilbert space and one term acting on the scalar field Hilbert space. Here without any additional assumptions the gravitational amplitude $A_G(s_f, s_i; \alpha)$ can be expanded in a vertex expansion using (4.1-4.3).

Inserting this vertex expansion into the expression for the physical inner product above, assuming that the diagonal matrix elements are non-zero, and again interchanging the integration over α with the sum over vertices M we arrive at the following.

$$([s_f, \phi_f], [s_i, \phi_i]) = \sum_{M=0}^{\infty} \left[\sum_{\substack{s_{M-1}, \ldots, s_1 \\ s_m \neq s_{m+1}}} A(s_M, \ldots, s_0; \phi_f, \phi_i) \right], \tag{4.9}$$

where the partial amplitude associated to each number of vertices M is given by

$$
\begin{aligned}
A(s_M, \ldots, s_0; \phi_f, \phi_i) \quad = \quad & \Theta_{s_M s_{M-1}} \Theta_{s_{M-1} s_{M-2}} \cdots \Theta_{s_2 s_1} \Theta_{s_1 s_0} \times \quad (4.10) \\
& \prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k} \right)^{n_k - 1} \sum_{i=1}^{P} \frac{e^{i\sqrt{\Theta_i} \Delta\phi} + e^{-i\sqrt{\Theta_i} \Delta\phi}}{\prod_{j \neq i}^{P} (\Theta_i - \Theta_j)} \bigg|_{\Theta_i = \Theta_{s_{i(d)} s_{i(d)}}}
\end{aligned}
$$

as obtained by first integrating over the lapse α and then over the scalar field momentum p_ϕ. In a discrete history $s_f, s_{M-1}, \ldots, s_1, s_i$ there can be many states s_i that lead to the same diagonal element. In the above expression then P is the number of distinct values of Θ_{ss} taken along the history, $\Theta_{s_{i(d)} s_{i(d)}}$ denotes the i-th such distinct value, and n_i denotes the number of times that value repeats. For SLQC the diagonal elements of $\Theta_{\nu\nu}$ were distinct if the volumes ν were distinct along the discrete history. In that case this construction led to the correct expansion as observed by the fact that it solved the constraint and that it well approximated the physical inner product at least for at least some range of values. Similarly the vertex expansion (4.9) solves the constraint $C = p_\phi^2 - \Theta$ for a generic gravitational part Θ with non-zero diagonal matrix elements.

To reiterate the necessary features for the construction above are the following:

- The presence of a scalar field

- A splitting of the constraint operator into a gravitational part and a massless scalar field part.

- A gravitational part of the constraint Θ having nonzero diagonal elements.

Together these three assumptions allowed us to interchange the integral over α with the sum over M the number of vertices obtaining a well-defined amplitude for each discrete history. In the following we will seek to relax the last two assumptions and finally relax the first assumption in the next chapter.

4.3 Zero Diagonal Elements

The simplest generalization leaves the matter part of the constraint untouched and simply considers the most general possible self-adjoint operator Θ whose diagonal matrix elements can be zero. Even for the Bianchi I model, the simplest

anisotropic cosmology, the gravitational part of the constraint Θ_{BI} has all zero diagonal matrix elements in a natural basis given by eigenstates of geometric operators. The presence of a single zero diagonal matrix leads to divergences in the vertex expansion in the timeless framework. It is then necessary to introduce new ideas to obtain a well defined vertex expansion.

We first recall that the derivation of the vertex expansion for the gravitational part of the amplitude does not depend on the details of the operator Θ.

$$A_G(s_f, s_i; \alpha) = \langle s_f | e^{-i\alpha\Theta} | s_i \rangle = \sum_{M=0}^{\infty} \left[\sum_{\substack{s_{M-1},\dots,s_1 \\ s_m \neq s_{m+1}}} A(s_M, \dots, s_0; \alpha) \right] \tag{4.11}$$

$$A(s_M, \dots, s_0; \alpha) = \Theta_{s_M s_{M-1}} \Theta_{s_{M-1} s_{M-2}} \dots \Theta_{s_2 s_1} \Theta_{s_1 s_0} \times$$
$$\prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k} \right)^{n_k - 1} \sum_{i=1}^{P} \frac{e^{-i\alpha\Theta_i \Delta\tau}}{\prod_{j\neq i}^{P}(\Theta_i - \Theta_j)} \Bigg|_{\Theta_i = \Theta_{s_{i(d)} s_{i(d)}}} \tag{4.12}$$

where the derivates are taken and then the expression is evaluated at values of the diagonal elements associated to the distinct states in the history. This is equally valid when the diagonal elements are zero.

The barrier to obtaining a vertex expansion of the physical inner product is whether or not the integral over α appearing in (4.6) can be carried out for each discrete history separately. Without loss of generality we can assume that that the matrix element $\Theta_{s_{P(d)} s_{P(d)}}$ is zero. To clearly see the breakdown of the expansion we carry out the integrals over α and p_ϕ and then take the limit as Θ_P goes to zero for the amplitude corresponding a single discrete history (4.10).

$$A(s_M, \dots, s_0; \phi_f, \phi_i) = \Theta_{s_M s_{M-1}} \Theta_{s_{M-1} s_{M-2}} \dots \Theta_{s_2 s_1} \Theta_{s_1 s_0} \times$$
$$4.3.2 \prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_k} \right)^{n_k - 1} \sum_{i=1}^{P} \frac{e^{i\sqrt{\Theta_i}\Delta\phi}}{\prod_{j\neq i}^{P}(\Theta_i - \Theta_j)} \Bigg|_{\Theta_i = \Theta_{s_{i(d)} s_{i(d)}}} \tag{4.13}$$

The limit $\Theta_P \to 0$ is well defined for the terms $i = 1$ to $P - 1$ of the sum. Though

the limit is clearly divergent for the P-th term of the sum when $n_P \geq 2$. The dominant term in that case is given by

$$A(s_M, \ldots, s_0; \phi_f, \phi_i) \approx \lim_{\Theta_P \to 0} \frac{1}{\Theta_P{}^{n_P - 3/2}} \tag{4.14}$$

coming from the repeated derivatives of $e^{i\sqrt{\Theta_P}\Delta\phi}$ with respect to Θ_P. The amplitude for a discrete history which contains states whose diagonal matrix elements are zero have divergent amplitudes and further the divergence is proportional to the number of such states appearing in the history repeat. We thus see a clear need to modify the expansion such that the integral over α will be well defined. There are three natural solutions

- Introduce a different grouping of histories resulting in a finite integral over α, which was the strategy to obtain a well-defined limit as the skeletonization was removed, $N \to \infty$.

- Change to a basis where the operator Θ has all non-zero diagonal matrix elements.

- Introduce a different splitting of the Hamiltonian into 'Free' and 'Interaction' terms in the derivation from perturbation theory.

These are each related but provide different intuition for generalizing the vertex expansion. In the following we discuss these three solutions with focus on the first as it has the closest contact to the path integral paradigm.

4.3.1 New Basis/Choice of Free and Interaction Terms

Two natural solutions from the canonical perspective are simply consider change of basis in which the diagonal matrix elements are all non-zero or to consider a different choice of free and interaction terms in the perturbation theory expansion. We can simply see that the choosing a different basis while following the exact derivation of the vertex expansion as in 3.3 is equivalent working with the same basis and using a different choice of free and interaction terms. We can first choose a new normalizable basis $\{|d\rangle\}$ such that $\langle d|d'\rangle = \delta_{dd'}$. Splitting the operator Θ

into a diagonal part D', and off-diagonal part K' in terms of the new basis states and following the derivation of section 3.3 of the vertex expansion where we find,

$$A_G^{(\lambda)}(s_f, s_i, \alpha) = \sum_{M=0}^{\infty} \lambda^M \int_0^1 d\tau_M ... \int_0^{\tau_2} d\tau_1 \sum_{\substack{d_M, ..., d_0 \\ d_m \neq d_{m+1}}} \langle s_f | d_M [e^{-i(1-\tau_M)\alpha D'_{d_M d_M}}] \times$$
$$(-i\alpha K'_{d_M d_{M-1}}) \ ... \ (-i\alpha K'_{d_1 d_0}) [e^{-i\tau_1 \alpha D'_{d_0 d_0}}] \langle s_0 | s_i \rangle. \tag{4.15}$$

The gravitational amplitude is simply expressed as a sum over discrete histories of the new basis states.

Similarly we can consider working with the original basis and choosing a different splitting of the gravitational part of the constraint, $\Theta = D' + K'$, where D' is not diagonal in the original basis and the spectrum of D' is still discrete (the eigenstates of D' are normalizable). Treating D' as the free part of the constraint and K' as the interaction term we can again follow the derivation of section 3.3 of the vertex expansion. Starting with the gravitational part of the amplitude we take the perturbative expansion around the free part D',

$$A_G^{(\lambda)}(s_f, s_i, \alpha) = \sum_{M=0}^{\infty} \lambda^M \int_0^1 d\tau_M ... \int_0^{\tau_2} d\tau_1 \ \langle s_f | [e^{-i(1-\tau_M)\alpha D'}] \ (-i\alpha K') \times$$
$$(-i\alpha K') [e^{-i(\tau_M - \tau_{M-1})\alpha D'}] ... [e^{-i(\tau_2 - \tau_1)\alpha D'}] \ (-i\alpha K') [e^{-i\tau_1 \alpha D'}] | s_i \rangle. \tag{4.16}$$

To evaluate the above expression we insert a decomposition of the identity operator in terms of the eigenstates $|d\rangle$ of free part D' between each interaction term. Since the spectrum of D' was assumed to be discrete the eigenstates can be normalized to an inner product given by the Kronecker delta and the resulting perturbative expansion is a vertex expansion,

$$A_G^{(\lambda)}(s_f, s_i, \alpha) = \sum_{M=0}^{\infty} \lambda^M \int_0^1 d\tau_M ... \int_0^{\tau_2} d\tau_1 \sum_{d_M, ..., d_0} \overline{d_M(s_f)} [e^{-i(1-\tau_M)\alpha D'_{d_M d_M}}] \times$$
$$(-i\alpha K'_{d_M d_{M-1}}) \ ... \ (-i\alpha K'_{d_1 d_0}) [e^{-i\tau_1 \alpha D'_{d_0 d_0}}] \ d_0(s_i). \tag{4.17}$$

which is equivalent to (4.15) if additionally the free and interaction terms are chosen such that for all eigenstates of D' $|d\rangle$ the diagonal matrix elements of the

interaction term, K_{dd} are zero. We then find that either approach produces the same vertex expansion, so in the following we focus on the choice of a different splitting of the constraint.

If the free part, D', is chosen such its eigenvalues are always non-zero the vertex expansion for the physical inner product can be derived leading to

$$([s_f, \phi_f], [s_i, \phi_i]) = \sum_{M=0}^{\infty} \Big[\sum_{\substack{d_M,\ldots,d_0 \\ d_m \neq d_{m+1}}} \langle s_f|d_M\rangle A(d_M,\ldots,d_0; \phi_f, \phi_i)\langle d_0|s_i\rangle \Big], \quad (4.18)$$

with sums over discrete histories of the eigenstates d instead of the original basis states. The partial amplitude associated to each number of vertices M is given by

$$
\begin{aligned}
A(d_M,\ldots,d_0; \phi_f, \phi_i) &= K'_{d_M d_{M-1}} K'_{d_{M-1} d_{M-2}} \cdots K'_{d_2 d_1} K'_{d_1 d_0} \times \qquad (4.19) \\
&\quad \prod_{k=1}^{P} \frac{1}{(n_k-1)!} \Big(\frac{\partial}{\partial D'_k}\Big)^{n_k-1} \sum_{i=1}^{P} \frac{e^{i\sqrt{D'_i}\Delta\phi} + e^{-i\sqrt{D'_i}\Delta\phi}}{\prod_{j\neq i}^{P}(D'_i - D'_j)}
\end{aligned}
$$

where the D'_i the distinct values of the matrix elements $D_{d_m d_m}$ taken along the discrete history and n_i are the number of times each distinct value D'_i is repeated along the history. The problem is then to find a suitable splitting of the constraint that will lead to a free part with no zero eigenvalues and to matrix element of the interaction term, K'_{dd}, that are zero for each eigenstate of D'. Such a splitting will lead to a well defined vertex expansion given.

For LQC and similar systems where the kinematic basis is isomorphic to the integers there is a simple generalization of the splitting of Θ into diagonal and off-diagonal parts satisfying the above properties. In such systems the operator Θ can be written as a matrix. For soluble LQC for example the operator Θ is a tri-diagonal matrix with entries only along the main diagonal and the diagonals above and below it. A natural splitting in these cases is to consider the free part of Θ, D', to be a block-diagonal matrix instead of simply diagonal. The spectrum of this block diagonal free part is then given simply by the eigenvalues of each block, D_i, along the diagonal. The simplest such choice is to take each zero diagonal

element to be in a 2×2 block as in the following example,

$$
\begin{bmatrix}
\Theta_{4\lambda 4\lambda} & \Theta_{4\lambda 8\lambda} & 0 & 0 & 0 \\
\Theta_{8\lambda 4\lambda} & 0 & \Theta_{8\lambda 12\lambda} & 0 & 0 \\
0 & \Theta_{12\lambda 8\lambda} & \Theta_{12\lambda 12\lambda} & \Theta_{12\lambda 16\lambda} & 0 \\
0 & 0 & \Theta_{16\lambda 12\lambda} & 0 & \dots \\
0 & 0 & 0 & \dots & \dots
\end{bmatrix}
=
\begin{bmatrix}
D_1 & K_{12} & 0 \\
 & & 0 \\
K_{21} & D_2 & 0 \\
 & & \dots \\
0 \ 0 & 0 \ \dots & \dots
\end{bmatrix}
\tag{4.20}
$$

where D_1 is the block containing the matrix elements with respect to 4λ and 8λ and D_2 with respect to $12\lambda,16\lambda$. For this simple choice the spectrum of the free part D' will necessarily not contain 0. Each zero diagonal element of Θ is contained in a 2×2 block

$$
D_s = \begin{bmatrix} 0 & \Theta_{ss'} \\ \Theta_{s's} & \Theta_{s's'} \end{bmatrix}
\tag{4.21}
$$

whose eigenvalues are non-zero for all values of $\Theta_{ss'} \neq 0$ and $\Theta_{s's'}$.

More generally the gravitational part of the constraint Θ cannot be written as a simple matrix as above. There remains though a clear notion of a block diagonal decomposition. It is helpful to first introduce the following picture: as Θ is a self-adjoint operator on the kinematic Hilbert space it necessarily maps a single element of the basis to a finite linear combination of states $|s\rangle$. The index set S then inherits from Θ a graph structure where to each $s \epsilon S$ there is a vertex and there is a link between the vertices s and s' iff $\Theta_{ss'} \neq 0$ (Allowing links between s and itself). From this there is a clear notion of distance between two elements of the kinematic basis as the minimum number of links connecting the two points, or equivalently the minimum n such that $\langle s'|\Theta^n|s\rangle$ is non-zero. We say that two points are n-connected if there exists is a path of n-links connecting them.

The block diagonal decomposition is then constructed as follows. We take a partitioning of the set S into disjoint countable subsets S_α such that the induced graph on each S_α is connected graph with some maximum number of distance betweens elements. In other words we partition the graph into connected subgraphs S_α. The decomposition into free and interaction parts is then as follows:

$$
D_{ss'} = \Theta_{ss'} \quad \text{if there exists } \alpha \text{ such that s}, \text{s}' \epsilon \text{ S}_\alpha
\tag{4.22}
$$

$$K_{ss'} = \Theta_{ss'} \quad \text{if there does not exist } \alpha \text{ such that s, s}' \in S_\alpha \tag{4.23}$$

Here each block of the free part of the constraint D is given by the restriction of Θ to a single subset S_α. The off-diagonal part K then connects elements within distinct subsets S_α and S'_α acting as a generalization of the interaction term introduced in section 3.3. The eigenvalues of D are simply given by the eigenvalues of each block D_α, the restriction of D to a single S_α and the eigenstates of D be chosen to be the eigenstates of each D_α. Then it is true that the diagonal matrix elements of the interaction term in the basis of eigenstates of D. The final and most important requirement is to chose the subsets S_α such that the eigenvalues of each block D_α are non-zero. The simplest choice is to take each S_α to be a connected pair of states, which corresponds to the simple example above for LQC. If such a decomposition exists then a vertex expansion exists for the model and is given by (4.18) and (4.19).

The choice of splitting Θ into a free part which is block diagonal and an interaction term provides one way to resolve the divergences that occur in the vertex expansion when the Θ has diagonal matrix elements that are zero. Although, in practice it may be difficult to construct a splitting such that the free part D does not have zero as an eigenvalue.

4.3.2 Coarse Graining

The most natural generalization from the sum over histories derivation of the vertex expansion is to consider further grouping together the histories prior to taking integral over α coming from the group averaging. This was precisely the approach that led to a well-defined limit $N \to \infty$ in sections 2.4 and 3.2. Originally we considered the simplest grouping of histories that allowed for the limit $N \to \infty$ to be taken. This was defined for each finite set of volumes $(\nu_M, \nu_{M-1}, \ldots, \nu_0)$ by summing together all piecewise constant histories where the transitions between each constant value, $\nu_m \to \nu_{m-1}$, is allowed to happen at any time, τ_m along the history. Coincidentally this choice also resulted in a well defined expression when interchanging the integral over α with the sum over discrete histories when the

operator Θ had no zero diagonal matrix elements.

$$([s_f, \phi_f],\ [s_i, \phi_i]) = 2 \int d\alpha \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1},\ldots,s_1 \\ s_m \neq s_{m+1}}} A_\phi(\phi_f, \phi_i; \alpha) A(s_M, \ldots, s_0; \alpha) \quad (4.24)$$

$$= \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1},\ldots,s_1 \\ s_m \neq s_{m+1}}} \int d\alpha\ 2 A_\phi(\phi_f, \phi_i; \alpha) A(s_M, \ldots, s_0; \alpha) \quad (4.25)$$

Denoting the sum over discrete histories as

$$\sum_\sigma := \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1},\ldots,s_1 \\ s_m \neq s_{m+1}}} \quad (4.26)$$

where σ denotes the set of all discrete histories of any length M. The assumption above is that the sum over all discrete histories in σ commutes with the integral over α. Clearly assumption breaks down in the case that Θ has some diagonal matrix elements that are zero where the integral over α is divergent.

Since the integral over α cannot be trivially interchanged with the entire sum over all histories, we are led to ask if there exist a further grouping of paths such that the integral over α is well-defined for sum of amplitudes corresponding to the whole grouping. It is clear that if we sum over *all* discrete histories in (4.9) the gravitational part of the amplitude reduces to its exact value and the resulting integral over α will give the exact physical inner product (assuming everything is convergent). We then seek subsets, σ_β of the set of all discrete histories such that the sum over amplitudes of the histories within a σ_β has a well-defined integral over the lapse and that the integral over the lapse, α, can be interchanged with the sum over the subsets indexed by $\beta \,\epsilon\, B$.

$$\int d\alpha \sum_{\beta\,\epsilon\,B} \sum_{\sigma_\beta} = \sum_{\beta\,\epsilon\,B} \int d\alpha \sum_{\sigma_\beta} \quad (4.27)$$

To obtain a well-defined limit $N \to \infty$ in sections 2.4 and 3.2 we consider the sets of histories with M volume transitions between a fixed set of volumes $\nu_M, \ldots, \nu_0$ where the transitions could occur at arbitrary times. Here we consider a similar construction: As we saw in (4.14) if for some basis state, $|s_o\rangle$ the diagonal matrix

element of $\Theta_{s_o s_o}$ is zero the size of the divergence in the amplitude of a discrete history is given by the number of times that state repeats in the history. As such our intuition is to take for each σ_β those paths which oscillate arbitrarily around states with zero diagonal matrix element to those states are nearby (states, $|s\rangle$ for which $\Theta_{s_o s}$ is non-zero). We can view this as a coarse graining of our notion of 'constant' path to be one that fluctuates to nearby points.

To describe this grouping of paths we can return to the picture of the graph on S. For each element $s \in S$ we take its immediate neighborhood to be $C(s, 1)$ set of states that are a distance one from s.

$$C(s, 1) = \{s' | \Theta_{s\,s'} \neq 0 \ s' \neq s\} \tag{4.28}$$

and $C(s, 2)$ to be the set of states that are a distance two away from s not including s itself.

$$C(s, 2) = \{s' | \Theta^2_{ss'} \neq 0 \ \text{and} \ s' \neq s\} \tag{4.29}$$

We now follow the same idea as that of 2.3 and 3.2 where we re-organized the sum over histories in 'time' emphasizing the number of volume transitions thereby obtaining a well-defined limit as the skeletonization was removed. Here we re-organize the sum over discrete histories emphasizing the 'double-transitions', or transitions from a state s to a state $s_2 \in C(s, 2)$. We first notice that any discrete history with M transitions can be written as a sequence of N_0 (possibly 0) transitions between the initial point s_0 and states of $C(s_0, 1)$, followed by a double transition directly to an element s_1 in $C(s_0, 2)$ followed by N_1 transitions between s_1 and states of $C(s_1, 1)$, etc... Then instead of focusing on the number of single transitions we characterize histories by the number of double transitions which do not return to the original state.

Each discrete history can be characterized by a set of ordered sequences. First a set σ_M of $M + 1$ states along the history each separated by a double transition. A set of $M + 1$ integers N_m denoting the number of transitions between each element of σ_M and it's neighbouring states. Finally $M + 1$ sequences of length $2N_m$ giving the sequence of transitions between s_m and nearby states in $C(s_m, 1)$. A generic

history can then be written as

$$\sigma_{M,N} = \big(s_M, \overbrace{s_1^{(M)}, s_M, s_2^{(M)}, \ldots, s_{N_M}^{(M)}, s_M}^{2N_M}, s^{(M,M-1)}, s_{M-1}, s_1^{(M-1)}, s_{M-1}, \ldots \quad (4.30)$$

$$, s_1, \underbrace{s_1^{(1)}, s_1, \ldots, s_{N_1}^{(1)}, s_1}_{2N_1}, s^{(1,0)}, s_0, \underbrace{s_1^{(0)}, s_0, \ldots s_{N_0}^{(0)}, s_0}_{2N_0}\big) \quad s_{m+1} \in C(s_m, 1)$$

where $s_M, s_{M-1}, \ldots, s_1, s_0$ denote the states separated by double transitions, each $s_m \in C(s_{m-1}, 2)$. $s_1^{(M)}, \ldots, s_{N_M}^{(M)}$ denote the states in $C(s_M, 1)$, the neighboring states of s_M, and similarly each each element $s_n^{(m)}$ is an element of $C(s_m, 1)$. The states $s^{(m,m-1)}$ then are the intermediate states that connect s_m and s_{m+1} separated by two transitions, they are elements of $C(s_{m+1}, 1) \cap C(s_m, 1)$.

The idea is to sum over all histories with the same coarse grained history $s_M, s_{M-1}, \ldots, s_1, s_0$ of states separated by two transitions. To do so we first sum over all possible sequences of transitions between each states of the coarse grained history, s_m, and it's neighboring states in $C(s_m, 1)$ for a fixed coarse grained history and fixed numbers of transitions N_m. Second we sum over all N_m the number of transitions between each s_m and it's neighboring states. The result is an amplitude depending only on the coarse grained amplitude $s_M, s_{M-1}, \ldots, s_1, s_0$. To obtain the full amplitude we must then sum over all such histories of fixed length and finally over the number of double transitions M. This will provide the full amplitude only in the case that the final states s_M can be reached from the initial state s_0 only by an even number of actions of the operator Θ.

For simplicity we will discuss here the case where all diagonal elements of the Θ are zero, which is the case for simplest anisotropic LQC model, Bianchi I, with basis states labelled by the volume and two variables encoding the anisotropy. For such cases the gravitational amplitude for a single discrete history is simply

$$A(s_M, \ldots, s_0; \alpha) = \Theta_{s_M s_{M-1}} \Theta_{s_{M-1} s_{M-2}} \ldots \Theta_{s_2 s_1} \Theta_{s_1 s_0} \frac{(-i\alpha)^M}{M!} \quad (4.31)$$

whereby the resulting integral over α would be strongly divergent as in 4.14. To carry out the coarse graining we first then consider a fixed history of states that are separated by two transitions, $s_M, s_{M-1}, \ldots, s_0$, and a fixed number N_m of oscillations around each state, s_m, in the history. The amplitude corresponding to

this is given by a sum over all possible sets of N_m neighboring states for each s_m and all possible states connecting each s_m to s_{m-1}. This can easily be seen to be the following,

$$A_{(2)}(s_M, \ldots, s_0; N_M, \ldots, N_0) = \frac{(-i\alpha)^{2(N_M+N_{M-1}+\ldots+N_0+M)}}{(2(N_M + N_{M-1} + \ldots + N_0 + M))!} \quad (4.32)$$
$$\times (\Theta^2{}_{s_M s_M})^{N_M}(\Theta^2{}_{s_M s_{M-1}})(\Theta^2{}_{s_{M-1} s_{M-1}})^{N_{M-1}} \ldots (\Theta^2{}_{s_1 s_0})(\Theta^2{}_{s_0 s_0})^{N_0}$$

where $(\Theta^2{}_{ss'})$ simply denote the matrix elements of the operator Θ^2

We next sum over all $N_M, \ldots, N_0$ with a fixed coarse grained history.

$$A_{(2)}(s_M, \ldots, s_0; \alpha) = \sum_{N_M=0}^{\infty} \ldots \sum_{N_0=0}^{\infty} A_{(2)}(s_M, \ldots, s_0; N_M, \ldots, N_0) \quad (4.33)$$

By inserting (4.32) and defining $P_M = N_M + N_{M-1} + \ldots + N_0$, $P_{M-1} = N_{M-1} + N_{M-2} + \ldots + N_0$, $\ldots$, $P_0 = N_0$, (4.33) can be expressed as,

$$A_{(2)}(s_M, \ldots, s_0; \alpha) = (\Theta^2_{s_M s_{M-1}}) \ldots (\Theta^2{}_{s_1 s_0}) \sum_{P_M=0}^{\infty} \sum_{P_{M-1}=0}^{P_M} \ldots \sum_{P_0=0}^{P_1} \quad (4.34)$$
$$\frac{(-i\alpha)^{2(P_M+M)}}{(2(N_M+M))!}(\Theta^2{}_{s_M s_M})^{P_M-P_{M-1}}(\Theta^2{}_{s_{M-1} s_{M-1}})^{P_{M-1}-P_{M-2}} \ldots (\Theta^2{}_{s_0 s_0})^{P_0}$$

From this form the sums can be simply carried out for the case where the diagonal matrix elements of Θ^2 for states in the coarse grained history are all distinct.

$$A_{(2)}(s_M, \ldots, s_0; \alpha) = (\Theta^2_{s_M s_{M-1}}) \ldots (\Theta^2{}_{s_1 s_0}) \sum_{i=0}^{M} \frac{e^{-i\alpha\sqrt{\Theta^2{}_{s_i s_i}}} + e^{+i\alpha\sqrt{\Theta^2{}_{s_i s_i}}}}{\prod_{j \neq i}(\Theta^2{}_{s_i s_i} - \Theta^2{}_{s_j s_j})} \quad (4.35)$$

When the diagonal matrix elements of Θ^2 are not distinct along a history (4.35) is generalized in the same manner as (4.3.2),

$$A_{(2)}(s_M, \ldots, s_0; \alpha) = (\Theta^2_{s_M s_{M-1}}) \ldots (\Theta^2{}_{s_1 s_0}) \quad (4.36)$$
$$\times \prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta^2{}_k}\right)^{n_k-1} \sum_{i=1}^{P} \frac{e^{-i\alpha\sqrt{\Theta^2{}_i}} + e^{+i\alpha\sqrt{\Theta^2{}_i}}}{\prod_{j \neq i}(\Theta^2{}_i - \Theta^2{}_j)}$$

where $\Theta^2{}_i$ now denote the P distinct values of the diagonal matrix elements of

Θ^2 along the history and n_i again label how many times each value $\Theta^2{}_i$ repeats. The overall gravitational amplitude is then given as a sum over these amplitudes corresponding to coarse grained histories.

$$A_G(s_f, s_i; \alpha) = \sum_{M=0}^{\infty} \left[\sum_{\substack{s_{M-1}, \ldots, s_1 \\ s_m \, \epsilon \, C(s_{m+1}, 2)}} A_{(2)}(s_M, \ldots, s_0; \alpha) \right] \qquad (4.37)$$

where since we have introduced a coarse graining this expansion can only give the gravitational amplitude for states s_f and s_i that are connected by an even number of actions of the constraint. In terms of the picture of Θ giving a graph on the space of basis states, s_f and s_i are connected by only even length paths. This can be simply generalized, but will be discussed in the following chapter where it can be given a simpler formulation.

The final expression for the amplitude corresponding to coarse grained histories now does not depend on the diagonal matrix elements of Θ, which if zero rendered the expansion of the physical inner product divergent. Instead the amplitude (4.36) depends on the diagonal matrix elements of Θ^2 which if zero for some state indicate that the state is entirely decoupled from the dynamics. Further the partial amplitude for each coarse grained history has the same form as the amplitude for the original discrete histories (4.10) except it depends on the matrix elements of Θ^2 in place of Θ and it contains exponential terms with both signs. From here this amplitude can be combined with the amplitude for the scalar field as in (4.6) and the group averaging integral can be carried out leading to a result much like (4.9-4.10) but now depending on the matrix elements of Θ^2. The coarse graining has taken a gravitational amplitude that would lead to a divergent vertex expansion for the physical inner product and by a suitable resummation has produced a gravitational amplitude that will allow for a well defined vertex expansion.

To summarize we have found that the construction of the vertex expansion for the physical inner product breaks down when the diagonal matrix elements of Θ are zero. There exists at least two modifications to the construction that produce a well defined vertex expansion. First to choose a new basis such that the diagonal matrix elements of Θ are non-zero or equivalently chose a new splitting of the constraint into free and interaction terms while keeping the original basis.

A candidate for a splitting of the constraint was presented based on choosing a 'block-diagonal' free part as opposed to the diagonal free term in the original construction. In examples such a choice does produce a well defined expansion, but may be difficult to implement in models with non-separable Hilbert spaces. A second way to remove the divergences is to effectively coarse grain the sum over histories prior to carrying out the integral over α. For each M and for each coarse grained history $(s_M, s_{M-1}, \ldots, s_0)$ with each state s_{m+1} connected to s_m now by two transitions, we group together all discrete histories that oscillate arbitrarily many times between each s_m and the nearby states. This process reduces the vertex expansion of the gravitational amplitude to a sum over the coarse grained histories which admits a well defined integral over α and thus a well-defined vertex expansion for the physical inner product. We will return to this coarse graining in the following chapter where it can be analyzed in general.

4.4   General Matter Terms

Up till now we have consistently worked with systems with a trivial matter part, excluding those that are physically interesting such as inflationary models which require a potential term for the scalar field. A very important generalization then is to extend the vertex expansion to more general matter terms including those where the constraint does not split into commuting gravitational and matter parts. We will explore these again in the context of homogeneous cosmology while recalling that the construction is independent of the details of $\mathcal{H}_{\mathrm{kin}}$. While initially one would expect this to be a large hurdle to overcome we find that for a very general class of matter contributions to the Hamiltonian constraint the vertex expansion can be carried out nearly to completion and leads to important insights into the possible structure of spin foams coupled to matter.

We analyze a very general class of constraints given by

$$C = H_{matt} - \Theta \tag{4.38}$$

where Θ is a self-adjoint operator acting only on the gravitational part of $\mathcal{H}_{\mathrm{kin}}$ with non-zero diagonal matrix elements (which can be generalized using the ideas

of the previous section). H_{matt} acts on both the gravitational and matter parts of $\mathcal{H}_{\text{kin}}$, but we assume that it depends only on the volume, ν. This is a natural assumption and is satisfied by most couplings of LQC with matter. Thus the matter Hamiltonian is diagonal on the gravitational part of the kinematic Hilbert space. For such models it is no longer possible to deparametrize the system by using the scalar field as a clock, so we must necessarily work within the timeless framework and use group averaging to extract physics from the model. We will then begin with the argument of the group averaging expression for the physical inner product.

$$A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha) = \langle \nu_f, \phi_f | \, e^{i\alpha C} \, | \nu_i, \phi_i \rangle \tag{4.39}$$

Where since the constraint does not trivially split into one part acting on the matter Hilbert space and one acting on the gravitational part, the amplitude can not be split into a product of scalar field and gravitational amplitudes. We will again follow the Feynman construction as far as possible, where now we treat the entire constraint, αC as our 'Hamiltonian' with a unit time interval. Starting from (4.39 we split the exponential as a product of N terms, and insert a complete kinematical basis between each giving,

$$A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha) = \sum_{\bar{\nu}_{N-1},...,\bar{\nu}_1} \int d\phi_{N-1}...d\phi_1 \, \langle \nu_N, \phi_N | e^{i\epsilon\alpha C} | \bar{\nu}_{N-1}, \phi_{N-1} \rangle \tag{4.40}$$

$$\times \, ... \, \langle \nu_1, \phi_1 | e^{i\epsilon\alpha C} | \bar{\nu}_0, \phi_0 \rangle \tag{4.41}$$

Then as with the standard derivation we make use of the limit $N \to \infty$ to simplify the matrix elements in the above expression. The gravitational matrix elements are simple to calculate, while to evaluate the scalar field matrix elements it is necessary to insert a complete basis of the scalar field momentum, p, resulting in,

$$A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha) = \sum_{\bar{\nu}_{N-1},...,\bar{\nu}_1} \int d\phi_{N-1}...d\phi_1 \int dp_N \, ... \, dp_1 \, e^{i \sum_{n=1}^N p_n(\phi_n - \phi_{n-1})}$$

$$\tag{4.42}$$

$$\times \, U(\bar{\nu}_N, \bar{\nu}_{N-1}, \phi_N, \phi_{N-1}, p_N) ... U(\bar{\nu}_1, \bar{\nu}_0, \phi_1, \phi_0, p_1)$$

where up to order $\epsilon = 1/N$,

$$U(\bar{\nu}_n, \bar{\nu}_{n-1}, \phi_n, \phi_{n-1}, p_n) = \delta_{\bar{\nu}_n, \bar{\nu}_{n-1}} + i\epsilon\, \delta_{\bar{\nu}_n, \bar{\nu}_{n-1}} H(p_n, \phi_n, \phi_{n-1}, \bar{\nu}_n) \qquad (4.43)$$
$$- i\epsilon\, \alpha \Theta_{\bar{\nu}_n \bar{\nu}_{n-1}} + \mathcal{O}(\epsilon)$$

In the limit as the skeletonization is removed, $N \to \infty$ the gravitational part will again take the form of a vertex expansion due to the normalizable volume eigenstates while on the other hand the matter part will take the form of a standard path integral as it is constructed with the distributional eigenstates of ϕ and p. To see this we again group together those histories with a finite number of M transitions in the volume ν to allow for a well defined limit as the triangulation is removed. At fixed N we first focus on the histories with M volume transitions taking the values $(\nu_M, \nu_{M-1}, \ldots, \nu_0)$ and a fixed scalar field history $\phi(\tau), p(\tau)$ from which we obtain $(\phi_N, \ldots, \phi_0)$ and $(p_N, \ldots, p_1)$ by $\phi_n = \phi(n\epsilon)$ and $p_n = p(n\epsilon)$. By grouping together all histories where the volume transitions are allowed to happen at any time during the history according, we find that by a slight extension of the proof in appendix A.1 the limit $N \to \infty$ exists for this sum over histories and is given by

$$A(\nu_M, \nu_{M-1}, \ldots, \nu_0; \phi(\tau), p(\tau); \alpha) = \left[\int_0^1 d\tau_M \ldots \int_0^{\tau_2} d\tau_1 \, \exp(i\int_0^1 d\tau\, p\dot{\phi}) \qquad (4.44) \right.$$
$$e^{-i(1-\tau_M)\alpha\Theta_{\nu_M \nu_M}} \left(-i\alpha\Theta_{\nu_M \nu_{M-1}} \right) \ldots \left(-i\alpha\Theta_{\nu_1 \nu_0} \right) e^{-i(\tau_1)\alpha\Theta_{\nu_0 \nu_0}}$$
$$\left. \exp(i\alpha\int_{\tau_M}^1 d\tau\, H_{matt}(p, \phi, \nu_M)) \times \ldots \exp(i\alpha\int_0^{\tau_1} d\tau\, H_{matt}(p, \phi, \nu_0)) \right]$$

The full amplitude is again given by a sum over all numbers of volume transitions, M, discrete histories of volumes, $(\nu_M, \nu_{M-1}, \ldots, \nu_0)$, and additionally integrals over all continuous scalar field histories.

$$A(\nu_f, \phi_f; \nu_i, \phi_i; \alpha) = \sum_{M=0}^\infty \sum_{\substack{\nu_{M-1}, \ldots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} \int_{\substack{\phi(0)=\phi_i \\ \phi(1)=\phi_f}} \mathcal{D}\phi \int \mathcal{D}p\, A(\nu_f, \nu_{M-1}, \ldots, \nu_0; \phi(\tau), p(\tau); \alpha)$$

$$(4.45)$$

Thus we obtain a sum over histories for the argument of the group averaging integral which is a hybrid of a traditional path integral for the matter contribution with

a sum over piecewise constant paths for the gravitational part. The resulting amplitude for a given history of geometries specified by the volumes $(\nu_M, \nu_{M-1}, \ldots, \nu_0)$ and scalar field histories 4.44 can be written as

$$A(\nu_M, \nu_{M-1}, \ldots, \nu_0; \phi(\tau), p(\tau); \alpha) = \int_0^1 d\tau_M \ldots \int_0^{\tau_2} d\tau_1 e^{-i(1-\tau_M)\alpha\Theta_{\nu_M\nu_M}} \quad (4.46)$$

$$\times \, (-i\alpha\Theta_{\nu_M\nu_{M-1}}) \ldots (-i\alpha\Theta_{\nu_1\nu_0}) \, e^{-i(\tau_1)\alpha\Theta_{\nu_0\nu_0}} \, \exp(iS_{matt}(p, \phi, \nu))$$

Where the matter action $S_{matt}(p, \phi, \nu)$ is the original action [1] where the gravitational dependence in the action is evaluated on simple piecewise constant histories.

$$S_{matt}(p, \phi, \nu) = \int_0^1 d\tau \, p\dot{\phi}) + \int_0^{\tau_1} \alpha d\tau \, H_{matt}(p, \phi, \nu_0) + \ldots \int_{\tau_M}^1 d\tau \, \alpha H_{matt}(p, \phi, \nu_M)$$

$$(4.47)$$

This is not surprising given the appearance of such piecewise histories of the volume appearing in the simpler models, but may be very powerful for analyzing quantum cosmologies with more complicated matter Hamiltonians.

This it opens the possibility of utilizing standard path integral tools of quantum field theory on fixed backgrounds, since the matter path integral simple reduces to a path integral on constant geometries. The vertex expansion (4.45) is still expressed as a sum over embedded triangulations with explicit integrals over τ the times at which the volume transitions occur. An open question is precisely how the vertex expansion (4.45) is reduced to a sum over discrete histories where the dependence on τ is removed.

4.5   Conclusion

In this chapter we found that the vertex expansion introduced for exactly soluble LQC is extremely general. FIrst, for *any* non-separable Hilbert space and *any* self-adjoint operator H there exists a vertex expansion for the transition amplitude toi 'evolve' between two kinematic states via the evolution operator e^{itH}. This vertex expansion is characterized by a discrete sum over histories where the state changes

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[1]Up to modifications depending on precisely how H_{matt} is represented on the kinematic Hilbert space

only finitely many times. This structure will be shared by *all* configuration space path integrals in such models unless the path integral expansion is carried out with a non-normalizable basis in the kinematic Hilbert space. While the vertex expansion can be trivially extended to a wide range of systems, the further step of carrying out the group averaging procedure by integrating over the lapse α for each element of the vertex expansion separately is more difficult to generalize. When extending the vertex expansion to generic gravitational parts of the constraint we found the terms of the sum to be divergent if the operator Θ has any diagonal matrix elements with respect to the basis states that are zero. These divergences can be cured by a variety of closely related techniques: considering a different basis for the gravitational kinematic Hilbert space, considering a different splitting of the constraint into 'free' and 'interacting' terms when performing derivation via perturbation expansion, and finally consider a coarse graining of the sum over histories. The third will be further investigated in the following chapter as a possible cure for both divergences due to a vanishing diagonal terms and divergences due to the entire summation.

We finally considered extensions of the vertex expansion to systems with more general matter terms in the constraint. There again the vertex expansion of the transition amplitude generated by $e^{i\alpha C}$ can be easily generalized. This already leads to an interesting structure. The resulting sum over histories expansion of the above transition amplitude is a hybrid of the vertex expansion and a standard path integral for the matter. The first construction takes the form of a sum over piecewise histories in the volume ν together with a standard path integral over the scalar field and its conjugate momentum. The matter action appearing in this hybrid path integral depends not an continuous geometries as we would expect but instead on piecewise constant geometries. This opens the possibility to use established path integral techniques associated to QFT on a fixed background spacetime even within the context of quantum gravity. This also leads to an intuition for the addition of matter to spin-foam models. We find that instead of the matter having a discrete evolution tailored to the discrete histories in the volume, the matter follows a continuous evolution on piecewise constant configurations. Of course if we additionally consider diffeomorphism invariant measures for the matter Hilbert space we will be similarly led to non-seperable spaces for the matter as well.

Chapter 5

Vacuum Vertex Expansion

We have seen in the previous chapters that the relationship between the canonical and spin-foam languages can be studied in the simplified context of cosmology. For a very general class of models, the standard path integral construction carried out using a normalizable basis leads to a vertex expansion that mimics the structure of the vertex expansion of SFM. The previous discussions though included a massless scalar field which was later generalized to allow for more general matter contributions to the constraint, but current spin-foam models are intended to be models of vacuum quantum gravity only. In this chapter we then extend of the construction of the vertex expansion to vacuum models.

For the vertex expansions presented thus far, matter played two roles. The presence of a massless scalar field allowed for the theory to be deparametrized by using the scalar field as a clock. Further, integration over the matter degrees of freedom makes the otherwise distributional terms of the vertex expansion well defined. In the absence of matter or another suitable variable with continuous spectrum the theory cannot be deparametrized, requiring us to work within the timeless framework where we will show that the resulting vertex expansion will be in general distributional. Although the expansion is distributional we will see that it still provides a formal perturbative solution to the constraint equation.

In this chapter we will introduce a regular δ in the group averaging procedure making each term of the expansion a regular function of the discrete histories. We introduce two choices of regulator, the Gaussian and Feynman regularizations. The second leads to a simple local expansion which makes better contact with

the current SFM, so we will explore it in more detail. The hope is that the vertex expansion for the regulated physical inner product is accurate up to some specifiable small error which vanishes as $\delta \to 0$. In simple examples though, the resulting vertex expansion is divergent in the usual sense even when the exact physical inner product is well defined. We will show that while divergent in the normal sense the vertex expansion for these simple systems is Borel summable and does produce the exact physical inner product.

The chapter is organized as follows. We first carry out the expansion of a generic vacuum model showing how the expansion breaks down and use this to motivate the regularizations. In sections 5.2 and 5.3 we introduce the expansion with each type of regularization. We then focus on the local expansion as it provides the best contact with current SFM. In section 5.4 we derive the conditions under which the local expansion provides a good approximation to the physical inner product. In sections 5.5 and 5.6 we test the local expansion by applying first to exactly soluble LQC and then to a simple vacuum model. As the expansion for the vacuum model is a strongly divergent in 5.7 we analyze ways to improve the series properties through the coarse graining introduced in chapter 4.

5.1   The Vertex Expansion

In this section we illustrate the breakdown of the vertex expansion in the timeless framework when applied to the vacuum models of LQC and similarly LQG. When followed exactly each term of the vertex expansion is a sum of distributions. While each term is distributional we find that in examples the entire series is well defined and does provide a solution to the constraint. Thus much like the divergences of standard quantum field theory there is physical information contained even in these distributional terms.

We start with a vacuum model with a normalizable basis of states $|s\rangle$ with inner product $\langle s'|s\rangle = \delta_{s',s}$ and a single constraint operator, C. We will initially assume that the diagonal matrix elements of the constraint are non-zero in the given basis. Following the introduction of the regularization we will find that the assumption can be relaxed. We will also naturally assume that zero lies within the continuous spectrum of the constraint. As for a general system it is not possible to

deparametrize we will work within the timeless framework and construct a vertex expansion for the physical inner product defined by group averaging.

$$([s_f], [s_i]) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \langle s_f | e^{i\alpha C} | s_i \rangle \tag{5.1}$$

We again follow the strategy of first obtaining a vertex expansion of the argument of the group averaging integral. For generic constraint operator this vertex expansion is given by (4.1-4.3),

$$\langle s_f | e^{i\alpha C} | s_i \rangle = \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1}, \ldots, s_1 \\ s_m \neq s_{m+1}}} A(s_M, s_{M-1}, \ldots, s_1, s_0; \alpha), \tag{5.2}$$

where each partial amplitude is given by

$$A(s_M, s_{M-1}, \ldots, s_1, s_0; \alpha) = C_{s_M s_{M-1}} \ldots C_{s_1 s_0} \times \tag{5.3}$$

$$\left[\prod_{k=1}^{P} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial C_k} \right)^{n_k - 1} \right] \sum_{i=1}^{P} \frac{e^{i\alpha C_i}}{\prod_{j \neq i}^{P} (C_i - C_j)}, \tag{5.4}$$

where C_i are the P distinct values of $C_{s_i s_i}$ taken along the history and n_i is the number of times the each value C_i is repeated. Recall that in the simplest case of all the n_i being 1, this is simply[1]

$$A(s_M, s_{M-1}, \ldots, s_1, s_0; \alpha) = \sum_{i=0}^{M} \frac{e^{i\alpha C_i}}{\prod_{j \neq i} (C_i - C_j)} . \tag{5.5}$$

The last step is to perform the integral over α and express the physical inner product, $([s_f], [s_i])$ by interchanging of the infinite sum M over the number of transitions with the integral over α coming from group averaging. In the simplest case where the diagonal matrix elements along the history are all distinct, $(C_{s_m s_m} \neq C_{s'_m s'_m}$ for all s_m and $s_{m'}$ along the discrete history, , one obtains a sum of Dirac delta distributions. More generally the amplitude for each path is a sum of derivatives of Dirac delta distributions. The total amplitude is then a discrete sum

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[1]Even though expression (5.3) is well defined in the limit that some of the diagonal matrix elements of the constraint are zero, the integral over α in Eq. (5.2) is divergent instead of distributional in this limit.

over distributions outside of any integral and the vertex expansion would strictly fail to be well defined.

Carrying out the integral over α without introducing any sort of regulator results in the following expansion

$$([s_f]|[s_i]) = \sum_{M=0}^{\infty} A_M(s_f, s_i) \tag{5.6}$$

$$= \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1},\ldots,s_1 \\ s_m \neq s_{m+1}}} \bar{A}(s_M, \ldots, s_0),$$

where

$$\bar{A}(s_M, \ldots, s_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \ A(s_M, \ldots, s_0; \alpha). \tag{5.7}$$

This integral can be easily evaluated giving

$$\bar{A}(s_M, \ldots, s_0) = \left[\prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial C_k} \right)^{n_k - 1} \right] \sum_{i=1}^{p} \frac{\delta(C_i)}{\prod_{j\neq i}^{p}(C_i - C_j)}, \tag{5.8}$$

where, as before, C_i are the p distinct values of $C_{s_i s_i}$ taken along the history and n_i is the number of times the value C_i is repeated. When viewed as a distribution depending on $C_1, \ldots, C_P$ (or equivalently some parameter common to each diagonal matrix element) one can verify that this distribution is equivalent to the following simpler one:

$$\bar{A}(s_M, \ldots, s_0) = (-1)^{M+1} \sum_{i=1}^{p} \frac{(-1)^{n_i}}{(n_i - 1)!} \left(\frac{\partial}{\partial C_i} \right)^{n_i - 1} \frac{\delta(C_i)}{\prod_{j\neq i}^{p} C_j^{n_j}}. \tag{5.9}$$

Despite being a totally distributional expression the vertex expansion (5.6) provides formal solution to the constraint. As in sections 2.5 and 3.4 we can introduce a parameter λ as a marker of the number of vertices and write the constraint as diagonal plus off-diagonal part, $C = D + \lambda K$. The constraint equation is then satisfied if for each M,

$$DA_M(s_f, s_i) + KA_{M-1}(s_f, s_i) = 0 \tag{5.10}$$

From the previous chapter we would expect that this is satisfied by cancellations between similar histories,

$$\sum_{s_{M-2}} \cdots \sum_{s_1} \left[\sum_{s_{M-1}} DA(s_f, s_{M-1}, \ldots, s_i) + KA(s_f, s_{M-2}, \ldots, s_i) \right] = 0 \qquad (5.11)$$

meaning that (5.11) vanishes simply because the term in brackets is zero. To show this we again look at the action of D and K on the amplitude corresponding to a discrete history, $\bar{A}(s_M, \ldots, s_0)$. Without loss of generality we may assume that is $C_{s_f s_f}$ is the $P - th$ distinct diagonal matrix element, C_P. The action of the diagonal part of the constraint can be simply evaluated,

$$DA(s_f, \ldots, s_i) = C_{s_f s_f} A(s_f, \ldots, s_i) = C_P \sum_{i=1}^{p} \frac{(-1)^{n_i}}{(n_i - 1)!} \frac{\delta^{(n_i-1)}(C_i)}{\prod_{j \neq i}^{p} C_j^{n_j}}. \qquad (5.12)$$

where $\delta^{(n)}(x)$ denotes the $n - th$ derivative of the Dirac delta function. For $i \neq P$ in the sum the C_P simply reduces the power of C_P in the denominator. While the Pth term in the sum simplifies to

$$(-1)^{M+1} \frac{(-1)^{n_P-1}}{(n_P - 2)!} \frac{\delta^{(n_P-2)}(C_P)}{\prod_{j \neq P}^{P} C_j^{n_j}} \qquad (5.13)$$

where we have used the identity that

$$x\delta^{(n)}(x) = -n\delta^{(n-1)}(x) \qquad (5.14)$$

Overall all then the action of D gives the amplitude for the discrete history with one less configuration having the diagonal matrix element C_P.

$$DA(s_f, s_{M-1}, \ldots, s_i) = -C_{s_f s_{M-1}} A(s_{M-1}, \ldots, s_i) \qquad (5.15)$$

The off-diagonal part of the constraint acts very simply

$$KA(s_f, s_{M-2}, \ldots, s_i) = \sum_{s_f \neq s_{M-1}} C_{s_f s_{M-1}} A(s_{M-1}, s_{M-2}, \ldots, s_i) \qquad (5.16)$$

Together we find that (5.11) is satisfied, so the distributional vertex expansion (5.6) *does* provide a formal solution to the constraint.

As this series provides a formal solution to the constraint one would naturally ask how this relates to the actual physical inner product. Above we have obtained an infinite sequence of distributions. These have a natural notion of convergence obtained from their action on test functions. A sequence of distributions, d_n, converges to a given distribution, d, if the sequence $\int d_n(x)f(x)$ converges to $\int d(x)f(x)$ for all test-functions $f(x)$. For simple sequence of distributions the limit $d(x)$ can be found, but it is intractable in general. Alternatively we consider in the following sections a regularization of the distributional series. We can look at the convergence of the regulated series and if convergent take the limit as the regulator is removed for the sum of the regulated series. We introduce two such regulators parametrized by δ, first a Gaussian suppression of the group averaging integral and then a linear suppression akin to that used in the Feynman propagator.

5.2 Gaussian Regulator

A natural way to regulate the distributional vertex expansion (5.6) is by introducing a gaussian suppression in the integral over α in the group averaging procedure. The approximate physical inner product given by this regulated group averaging is

$$([s_f]|[s_i])_{G,\delta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \, \langle s_F|e^{i\alpha C - \delta^2 \alpha^2}|s_i\rangle_{\text{kin}}. \tag{5.17}$$

While the standard group averaging procedure extracts only the kernel of the constraint, the regulated definition will clearly include eigenstates within some spread δ around the zero eigenstates. The correct physical inner product is recovered in the limit $\delta \to 0$.

Following the same procedure outlined in section 5.1 in order to obtain a vertex expansion for this regulated inner product, we obtain

$$([s_f]|[s_i])_{G,\delta} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1},\dots,s_1 \\ s_m \neq s_{m+1}}} e^{-\delta^2 \alpha^2} A(s_M, s_{M-1}, \dots, s_1, s_0; \alpha), \tag{5.18}$$

where $A(s_M, s_{M-1}, \ldots, s_1, s_0; \alpha)$ is again given by Eq. (5.3). Pulling the sum over M outside of the integral and evaluating the integral over α leads to the following regulated vertex expansion

$$([s_f]|[s_i])_{G,\delta} = \frac{1}{2\pi} \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1},\ldots,s_1 \\ s_m \neq s_{m+1}}} \bar{A}_{G,\delta}(s_M, \ldots, s_0). \tag{5.19}$$

Where the gaussian regulated amplitude for each discrete history is given by

$$\bar{A}_{G,\delta}(s_M, s_{M-1} \ldots, s_1, s_0; \alpha) = \prod_{k=1}^{P} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial C_k}\right)^{n_k-1} \sum_{i=1}^{P} \frac{\sqrt{\pi}}{\delta} \frac{e^{-C_i^2/4\delta^2}}{\prod_{j\neq i}^{P}(C_i - C_j)}. \tag{5.20}$$

Again the C_i label the p distinct values of C_{ss} taken along the history $(s_M, \ldots, s_0)$ and n_i the number of times that value is repeated. Notice that one cannot take the limit of $\delta \to 0$ at this point, the fact that δ is nonzero is absolutely necessary in order to pull the sum over M outside of the integral.

The integral here gives Gaussians and derivatives thereof instead of Dirac delta distributions and derivatives thereof. Each term of the regulated physical inner product is then well defined. A priori though it is not clear how well this expansion will approximate the exact physical inner product. First, since the expansion itself is not clearly convergent or even asymptotic for a given choice of constraint and initial/final states. Second, if we assume that the series is convergent, since it is necessary to leave δ at some finite value, there remains an error term parametrized by δ.

$$\langle s_F|s_0\rangle_{\text{phy}} = \langle s_F|s_0\rangle_{G,\delta} + \mathcal{O}(\delta) \tag{5.21}$$

The properties of this error term are not known as of yet for the Gaussian regularization, but will be discussed in more detail for the Feynman regularization.

5.3 Feynman Regulator

Another possible approach is to introduce a linear suppression in the group averaging integral much as is done in computing the Feynman propagator. This provides the closest contact with the current SFM as it leads to a local expansion where the

amplitude for each discrete history is simply a product. The Feynman regulated physical inner product is as follows,

$$
\begin{aligned}
(s_f|s_i)_{\mathrm{F},\delta} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha \, \langle s_f | e^{i\alpha C - |\alpha|\delta} | s_i \rangle \quad &(5.22)\\
&= \frac{1}{2\pi} \int_{0}^{\infty} d\alpha \, \langle s_f | e^{i\alpha C - \delta\alpha} | s_i \rangle_{\mathrm{kin}} + \frac{1}{2\pi} \int_{-\infty}^{0} d\alpha \, \langle s_f | e^{i\alpha C + \delta\alpha} | s_i \rangle_{\mathrm{kin}}\\
&= (s_f|s_i)_{+,\delta} + (s_f|s_i)_{-,\delta},
\end{aligned}
$$

where $(s_f|s_i)_{+/-,\delta}$ denote the integrals over positive and negative, α. If we construct a phase space path integral for this system as done for LQC [55] we find that α plays the role of the lapse. Each term $(s_f|s_i)_{+/-,\delta}$ then corresponds to fixing one sign for the lapse or fixing a single direction for the time evolution, and will be directly related to the positive and negative frequency solutions to the constraint for deparametrizable systems such as soluble LQC. In what follows we will refer to $(s_f|s_i)_{+/-,\delta}$ as the positive and negative lapse amplitudes. For general systems though it is necessary to include both terms to obtain the correct physical inner product when the regulator is removed.

The negative lapse amplitude can be obtained by simply replacing C by $-C$ in the positive lapse amplitude, so for simplicity we first focus on just one half of this expression.

$$
(s_f|s_i)_{+,\delta} = \frac{1}{2\pi} \int_{0}^{\infty} d\alpha \, \langle s_f | e^{i\alpha C - \delta\alpha} | s_i \rangle, \quad (5.23)
$$

the vertex expansion of $\langle s_f | e^{i\alpha C} | s_i \rangle$ can be carried out as in (5.2) and assuming that the integral over α can be exchanged with the sum over discrete histories the positive lapse expansion is given by

$$
(s_f, s_i)_{+,\delta} = \frac{1}{2\pi} \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1},\ldots,s_1 \\ s_m \neq s_{m+1}}} \bar{A}_{+,\delta}(s_f, s_{M-1} \ldots, s_1, s_i), \quad (5.24)
$$

where the amplitude associated to each discrete history is,

$$
\bar{A}_{+,\delta}(s_M, s_{M-1} \ldots, s_1, s_0) = \int_{0}^{\infty} d\alpha \, e^{-\delta\alpha} A(s_M, s_{M-1}, \ldots, s_0; \alpha), \quad (5.25)
$$

and $A(s_M, s_{M-1}, \ldots, s_1, s_0; \alpha)$ is given by (5.3). Surprisingly, the integral over α

reduces the very nonlocal expression given in (5.3) with a simple local one which is just a product of matrix elements:

$$\bar{A}_{+,\delta}(s_M, s_{M-1}, \ldots, s_1, s_0) = \frac{i(-1)^M C_{s_M s_{M-1}} \ldots C_{s_1 s_0}}{\prod_{m=0}^{M}(C_{s_m s_m} + i\delta)}.$$

(5.26)

Inserting this simple result into the full expression, the resulting positive lapse expansion is

$$(s_f, s_i)_{+\delta} = \frac{i}{2\pi} \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1}, \ldots, s_1 \\ s_m \neq s_{m+1}}} \frac{(-1)^M C_{s_M s_{M-1}} \ldots C_{s_1 s_0}}{(C_{s_M s_M} + i\delta)(C_{s_{M-1} s_{M-1}} + i\delta) \ldots (C_{s_0 s_0} + i\delta)}.$$

(5.27)

The negative lapse expansion can be obtained by interchanging C with $-C$ in 5.28.

$$(s_f | s_i)_{-\delta} = -\frac{i}{2\pi} \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1}, \ldots, s_1 \\ s_m \neq s_{m+1}}} \frac{(-1)^M C_{s_f s_{M-1}} \ldots C_{s_1 s_0}}{(C_{s_M s_M} - i\delta)(C_{s_{M-1} s_{M-1}} - i\delta) \ldots (C_{s_0 s_0} - i\delta)}.$$

(5.28)

The vertex expansion for the regulated physical inner product is then given as the sum of the two local expansions above,

$$(s_f | s_i)_{\mathrm{F},\delta} = (s_f | s_i)_{+\delta} + (s_f | s_i)_{-\delta}$$

(5.29)

As with the Gaussian regulator it is *not* possible to remove δ at a finite order in the expansion. The positive and negative lapse expansions separately have a well-defined limit as $\delta \to 0$ when each C_{ss} is non-zero. Together though they limit to the distributional expansion 5.6. When the matrix elements of the constraint are all real the physical inner product, regulated by the parameter δ, is given simply by the real part of the positive vertex expansion. Equivalently we have that the exact physical inner product in that case is given by

$$(s_f | s_i) = 2 \operatorname{Re}(s_f | s_i)_{+\delta} + \operatorname{err}(\delta),$$

(5.30)

where $\operatorname{err}(\delta)$ is an error term which vanishes in the limit of $\delta \to 0$. Notice that

(5.28) can be rewritten in the form

$$(s_f|s_i)_{+\delta} = \frac{i}{2\pi} \sum_{M=0}^{\infty} \sum_{\substack{s_{M-1},\dots,s_1 \\ s_m \neq s_{m+1}}} \prod_f A_f(s_f) \prod_v A_v(s_f), \qquad (5.31)$$

where $f = 0, \dots, M$; $v = 1, \dots, M$; the "face" amplitude is $A_f = (\Theta_{s_f s_f} + i\delta)^{-1}$, the "vertex" amplitude is $A_v = -\Theta_{s_v s_{v-1}}$. The expression (5.31) is precisely the general expression for (local) SFM, if as in chapter 3 we identify the sum over M with a sum over triangulations, the sum over the s_f with a sum over coloring of each dual triangulation.

The Feynman regulated expansion reproduces a local expression fitting with the current expression for SFM. Where again locality means that the amplitude of a single history is a product of amplitudes of the elements that form the history. This type of locality was not present for the constructions of the vertex expansions in previous chapters. Locality is one of the key assumptions of the current SFM. This assumption is motivated by path integrals of Schrodinger QM and QFT where the amplitude $e^{i\int_M \mathcal{L}}$ can be written as a product of amplitudes each depending on the Lagrangian, $\mathcal{L}$, integrated on a subset of M, and by the amplitude associated to Feynman diagrams which is a product of amplitudes associated to propagators and vertices when written in momentum space. Locality though is *not* a universal feature. For example the simple local form of Feynman diagram amplitudes breaks down when written in position space. Similarly we saw in chapter 3 that the locality present in a path integral can be lost when reducing the vertex expansion to a sum over only abstract triangulations. In the following then we will focus on the Feynman regulated expansion, henceforth referred to as the regulated vertex expansion as it provides a simpler form of the expansion allowing for analytical results and as it provides the best contact with the current SFM.

5.4   Solution to the Constraint

For the vertex expansions in systems with matter we found that the vertex expansion solved the constraint equation, and further in a very simple manner. For the regulated expansions derived above it is critical to test the hope that they solve the

constraint up to a controlled error in the regulator δ. We analyze this here for the Feynman expansion. We show that there are conditions such that the regulated expansion provides a solution to the constraint up to a controlled error in δ, but these do not appear to be satisfied by the physically interesting vacuum models. For this analysis we are again interested in evaluating the action of the constraint operator on the vertex expansion for the regulated physical inner product viewed as a function of the final state. We would expect that the result will be non-zero, but will vanish in the limit $\delta \to 0$.

$$C_f(s_f, s_i)_{F,\delta} = \mathcal{O}(\delta) \tag{5.32}$$

To check this we first find the action of the constraint on the positive lapse expansion by acting on each term of the expansion as follows

$$C_f(s_f, s_i)_{+\delta} = \frac{1}{2\pi} \sum_{M=0}^{\infty} (D + \lambda K) \lambda^M A_M(s_f, s_i) \tag{5.33}$$

where we have included the placeholder λ to simplify calculations. It is again instructive to find the λ^M term of the above expression given by the action of D on the partial amplitudes with M transitions and the action of K on the partial amplitudes with $M - 1$ transtions.

$$D\bar{A}_{+\delta}(s_f, s_{M-1}, \ldots, s_1, s_i) = \frac{(-1)^M C_{s_f s_f} C_{s_f s_{M-1}} \ldots C_{s_1 s_i}}{(C_{s_f s_f} + i\delta) \ldots (C_{s_i s_i} + i\delta)} \tag{5.34}$$

$$K\bar{A}_{+\delta}(s_f, s_{M-2}, \ldots, s_i, s_i) = \sum_{s_{M-1} \neq s_f, s_{M-2}} \frac{(-1)^{M-1} C_{s_f s_{M-1}} \ldots C_{s_1 s_0}}{(C_{s_{M-1} s_{M-1}} + i\delta) \ldots (C_{s_i s_i} + i\delta)} \tag{5.35}$$

Together we find that for all $M \neq 0$

$$DA_M(s_f, s_i) + KA_{M-1}(s_f, s_i) = \sum_{\substack{s_{M-1}, \ldots, s_1 \\ s_m \neq s_{m+1}}} \frac{C_{s_f s_{M-1}} \ldots C_{s_1 s_0}}{(C_{s_f s_f} + i\delta) \ldots (C_{s_i s_i} + i\delta)} \left(\frac{C_{s_f s_f}}{C_{s_f s_f} + i\delta} - 1 \right) \tag{5.36}$$

$$= -i\delta A_M(s_f, s_i)$$

As opposed to the previous expansions where (5.36) was identically 0, here there is a non-zero result that vanishes as $\delta \to 0$. For $M = 0$ the only contribution is the action of the diagonal part D given by

$$DA_0(s_f, s_i) = -i\delta A_0(s_f, s_i) + \delta(s_f, s_i) \tag{5.37}$$

Together we find that the action of the constraint on the positive lapse expansion is,

$$C_f(s_f|s_i)_{+\delta} = -i\delta(s_f|s_i)_{+\delta} + i\delta(s_f, s_i). \tag{5.38}$$

From which the action of the constraint negative lapse expansion of the can be simply obtained by replacing $i \to -i$ in the result for the positive expansion.

$$C_f(s_f|s_i)_{-\delta} = +i\delta(s_f|s_i)_{-\delta} - i\delta(s_f, s_i) \tag{5.39}$$

The vertex expansion for the regulated physical inner product is provided by the sum of the positive and negative expansions, so overall the action of the constraint on the complete vertex expansion is given by

$$C_f(s_f|s_i)_{\delta} = -i\delta[(s_f|s_i)_{+\delta} - (s_f|s_i)_{-\delta}] \tag{5.40}$$

In the case where all matrix elements of the constraint are real the above expression reduces to

$$C_f(s_f|s_i)_{\delta} = -i\delta[2Im((s_f|s_i)_{+\delta})] \tag{5.41}$$

so while the real part gives the approximate physical inner product, the imaginary part specifies the error. Thus we find that there is a controllable error in how well the regulated physical inner product solves the quantum constraint equation *if* the positive and negative lapse expansions are convergent series. If the positive and negative expansions are each convergent then both the regulated vertex expansion for the physical inner product and the error term are well-defined. In practice though each of the positive and negative lapse expansions are actually divergent series.

Note that the above discussion does not specify the error between the regulated vertex expansion and the exact physical inner product, but just how well the

regulated vertex expansion approximates a solution to the quantum constraint. In addition the statement that the series well-approximates the constraint is conditional on the convergence of the vertex expansions corresponding to each the positive and negative halves of the regulated inner product while in the following we will see though that the expansions are far from convergent in the normal sense.

5.5   Local Expansion of Soluble LQC

As an example we can apply the local vertex expansion to exactly soluble LQC in the timeless framework. For this system the local expansion leads to the same result as obtained from the group averaging procedure (3.41). It provides an alternate, more compact expression for the amplitudes associated to each history (3.40). Further for this deparametrizable system the positive and negative lapse expansions correspond to the positive and negative frequency solutions of the constraint.

We begin with the expression, 3.12, for the physical inner product

$$([\nu_f, \phi_f], \ [\nu_i, \phi_i]) = \frac{1}{2\pi} \int d\alpha \, \langle \nu_f, \phi_f| \, e^{i\alpha C} \, 2|p_\phi| \, |\nu_i, \phi_i\rangle. \tag{5.42}$$

with the constraint

$$C = p_\phi^2 - \Theta. \tag{5.43}$$

By inserting a complete basis in the scalar field momentum we obtain

$$([\nu_f, \phi_f], \ [\nu_i, \phi_i]) = \frac{1}{2\pi} \int dp_\phi \, e^{ip_\phi(\phi_f - \phi_i)} \, 2|p_\phi| \int d\alpha \, \langle \nu_f| \, e^{i\alpha(p_\phi^2 - \Theta)} |\nu_i\rangle \tag{5.44}$$

. Now we can focus on the term under the integral over p_ϕ which for each fixed value of p_ϕ is an expression of the form 5.1, so we can construct the positive lapse vertex expansion for this term.

$$\int d\alpha \, \langle \nu_f| \, e^{i\alpha(p_\phi^2 - \Theta)} |\nu_i\rangle_{+,\delta} = \sum_{M=0}^{\infty} \sum_{\substack{\nu_{M-1}, \dots, \nu_1 \\ \nu_m \neq \nu_{m+1}}} \tag{5.45}$$

$$\frac{\Theta_{\nu_M \nu_{M-1}} \dots \Theta_{\nu_1 \nu_0}}{(p_\phi^2 - \Theta_{\nu_M \nu_M} + i\delta)(p_\phi^2 - \Theta_{\nu_{M-1} \nu_{M-1}} + i\delta) \dots (p_\phi^2 - \Theta_{\nu_0 \nu_0} + i\delta)}$$

We can insert this positive lapse expansion into the original expression to obtain

the following positive lapse expansion for the regulated physical inner product.

$$([\nu_f, \phi_f], \ [\nu_i, \phi_i])_{+\delta} = \sum_{M=0}^{\infty} \sum_{\substack{\nu_{M-1},\dots,\nu_1 \\ \nu_m \neq \nu_{m+1}}} A_{+,\delta}(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) \qquad (5.46)$$

where each partial amplitude is given by

$$A_{+,\delta}(\nu_f, \nu_{M-1}, \dots, \nu_1, \nu_i; \phi_f, \phi_i) = \int \mathrm{d}p_\phi \, e^{ip_\phi(\phi_f - \phi_i)} \, 2|p_\phi| \frac{i}{2\pi} \qquad (5.47)$$

$$\frac{\Theta_{\nu_M \nu_{M-1}} \dots \Theta_{\nu_1 \nu_0}}{(p_\phi^2 - \Theta_{\nu_M \nu_M} + i\delta)(p_\phi^2 - \Theta_{\nu_{M-1}\nu_{M-1}} + i\delta)\dots(p_\phi^2 - \Theta_{\nu_0 \nu_0} + i\delta)}$$

The full physical inner product, (5.42), is then real part of (5.46) in the limit $\delta \to 0$, which reproduces exactly the expression (3.39) obtained from the original vertex expansion in the timeless framework. Additionally though each of the positive and negative expansions individually have well defined limits as $\delta \to 0$ and provide the positive and negative *frequency* expansions obtained in chapter 3.

Focusing on the partial amplitudes (5.47) we see that the integrand has poles for each distinct values of $\Theta_{\nu\nu}$ and further that the factor of $i\delta$ selects only one of the positive or negative pole $p_\phi = \pm\sqrt{\Theta_{\nu\nu}}$. The limit $\delta \to 0$ of the partial amplitude (5.47) then reduces by the residue theorem to the negative frequency analogue of the expression (3.41) $\Delta\phi$ is positive and positive frequency is $\Delta\phi$ is negative. [2] This observation was made independently in [73]. We see in this example that the regulated vertex expansion is equivalent to the original vertex expansion constructed for LQC in timeless framework. Further we find that for LQC the regulator *can* be removed at a finite order since the positive and negative lapse expansions separately provide solutions to the constraint so we can work with one or the two expansions where the limit $\delta \to 0$ is well defined. It is unknown if this behavior extends to models that are not deparametrizable.

---

[2]The result is the negative frequency expression due to the choice of sign of the constraint appearing in the group averaging. Choosing $e^{-i\alpha C}$ leads to both the correct action and to the match up of the positive lapse expansion and the positive frequency solutions.

5.6 DIvergence of Local Expansion

Above we tested whether the regulated vertex expansion is well-defined in the sense that it provides a solution to the constraint up to a controlled error depending on δ. We found that the regulated vertex expansion provides an approximate solution to the physical inner product only if each the positive and negative vertex expansions converge independently. When applied to physically interesting systems such as the vacuum Bianchi I LQC it is found that the positive and negative vertex expansions are divergent and thus the regulated vertex expansion does not provide an approximation to the physical inner product. In this section we analyze a simple example which nonetheless captures the features of more complex models that lead to the divergence of the regulated vertex expansion. For this simple example we find that while the regulated expansion is divergent in the usual sense it is *convergent* in a more general sense as the series is Borel summable.

The simple vacuum model we consider is defined as follows: The kinematical Hilbert space is the same as the gravitational Kinematic space of LQC, whose basis states we will denote by $|n\rangle$. The constraint is chosen to generate a very simple difference equation in the 'volume' n,

$$C = \sin^2(b) - \sin^2(\theta) \tag{5.48}$$

where $\theta \, \epsilon \, [0, \pi/2]$ is a constant. The action of the constraint on states $\psi(n)$ is the simple difference equation,

$$C\psi(n) = -\frac{1}{4}\psi(n-2) + (\frac{1}{2} - \sin^2(\theta))\psi(n) - \frac{1}{4}\psi(n+2) \tag{5.49}$$

The matrix elements of this constraint in the n basis are then simply,

$$C_{n\,n\pm2} = -\frac{1}{4} \tag{5.50}$$

$$C_{n\,n} = \frac{1}{2} - \sin^2(\theta) = \frac{1}{2}\cos(2\theta). \tag{5.51}$$

In this model the difference equation satisfied by the physical states, the group averaging procedure, and the vertex expansion can all be analyzed exactly allowing for direct comparison of the exact physical inner product with the regulated

expansions.

First the group averaging can be carried out exactly for this system. Starting with the group averaging expression for the physical inner product we obtain,

$$([n], [m]) = \frac{1}{2\pi} \int d\alpha \, \langle n | e^{i\alpha(\sin^2(b) - \sin^2(\theta))} | m \rangle \tag{5.52}$$

$$= \frac{1}{2\pi} \int d\alpha \, \frac{1}{2\pi} \int_0^{2\pi} db \, e^{ib(n-m)} e^{i\alpha(\sin^2(b) - \sin^2(\theta))} \tag{5.53}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} db \, e^{ib(n-m)} \delta(\sin^2(b) - \sin^2(\theta)) \tag{5.54}$$

. Where we have inserted a complete basis in the conjugate variable b and then carried out the integral over α. Now for $\theta = 0$ and $\theta = \pi/2$ the above expression is divergent, so consider $\theta \in (0, \pi/2)$ and note that the group averaging expression above can be simply modified to ensure convergence. The integral over b can then be carried out exactly giving the following physical inner product between the physical states extracted from the kinematic basis states.

$$([n], [m]) = \begin{cases} 0 & n - m \text{ odd} \\ \frac{2}{\pi} \frac{\cos(\theta(n-m))}{\sin(2\theta)} & n - m \text{ even} \end{cases} \tag{5.55}$$

It is simply checked that treating this physical inner product as a free function of n that it satisfies the constraint.

On the other hand we can also consider the regulated vertex expansion constructed for this system. The positive lapse expansion of the regulated physical inner product is the following,

$$(n, m)_{+\delta} = \frac{i}{2\pi} \sum_{M=0}^{\infty} \sum_{\substack{n_{M-1},\dots,n_1 \\ n_{m+1} = n_m \pm 2}} \frac{(-1)^M (-1/4)^M}{(1/2 \cos(2\theta) + i\delta)^{M+1}} \tag{5.56}$$

Since amplitude is independent of the values n_m taken during each discrete history, the sums over the intermediate states can be evaluated by counting the number of paths between the initial state m and final state n with M transitions. T his is a simple combinatoric problem with the result that for $M = 2M' + (n - m)/2$ transitions there are $2M$ choose M' paths between the initial and final states. The

positive lapse expansion above then reduces to the following

$$(n,m)_{+\delta} = \frac{i}{2\pi} \sum_{M'=0}^{\infty} \binom{2M' + (n-m)/2}{M'} \frac{(4}{(2\cos(2\theta) + 4i\delta)^{2M'+(n-m)/2+1}} \quad (5.57)$$

The resulting series is highly divergent do to the large number of possible histories for each M'. For all θ in $(0, \pi/2)$ the above series is purely divergent. We are led to ask in what sense this expansion is related to the exact inner product. The resulting vertex expansion does not provide a good approximation to the physical inner product even as an asymptotic series, so how can we extract information from it?

We can first see that although this series is divergent it does reproduce the exact physical inner product in a clear way. The divergence appearing here is much like that of traditional quantum field theory where even though each term of the Feynman diagram expansion is finite after renormalization the overall series is diverges. There we typically find that the series is asymptotic up to some order as compared to this example where the series simply diverges. Overall though the perturbation series as defined by Feynman diagrams while divergent according to the usual notion of series convergence, is convergent according to more general notions. In particular the perturbation series in many cases is known to be Borel summable, which says that under certain technical conditions there is a unique function whose power series expansion will reproduce the terms of the series. We find a similar property for the vertex expansion obtained here.

Obtaining the Borel sum of a divergent series $\sum \lambda^n b_n$ requires two steps. First to introduce a new series

$$B(\lambda, t) = \sum_{n=0}^{infty} \frac{b_n \lambda^n t^n}{n!} \quad (5.58)$$

which naturally has better convergence properties than the original due the factorial. If this series converges for all t then we take the Laplace transform of the function $B(\lambda, t)$.

$$A(\lambda) = \int_0^{\infty} dt \, e^{-t} B(\lambda, t) \quad (5.59)$$

$A(\lambda)$ then is the Borel sum of the original series. For series that converge in the normal sense we can check that this process returns the expected sum, since term-

wise the Laplace transform acts to remove the $t^n/n!$ introduced in forming the more convergent series.

We can now carry out this process for the vertex expansion given above. First we define the new series

$$B(t) = \frac{i}{2\pi(1/2\cos(2\theta) + i\delta)} \sum_{M'=0}^{\infty} \frac{t^{2M'+(n-m)/2}}{(M'+(n-m)/2)!M'!} \left(\frac{1}{2\cos(2\theta)+4i\delta}\right)^{2M'+(n-m)/2} \tag{5.60}$$

Recalling that the series expansion of the Bessel function is

$$J_n(x) = \sum_{M=0}^{\infty} \frac{(-1)^M}{M!(M+n)!} \left(\frac{x}{2}\right)^{2m+n} \tag{5.61}$$

We find that $B(t)$ does converge and is given by

$$B(t) = \frac{i}{2\pi}(-i)^{(n-m)/2} 2\omega J_{(n-m)/2}(i\omega t) \tag{5.62}$$

$$\omega = \frac{1}{\cos(2\theta) + 2i\delta} \tag{5.63}$$

Finally we can carry out the Laplace of transform of $B(t)$ to give the Borel sum of the series $(n,m)_{+\delta}$,

$$(n,m)_{+\delta} = \int_0^{\infty} dt\, e^{-t} B(t) = \frac{i}{\pi} \frac{(1+\sqrt{1-\omega^2})^{-(n-m)/2}\, \omega^{(n-m)/2+1}}{\sqrt{1-\omega^2}} \tag{5.64}$$

The limit $\delta \to 0$ of this final result is clearly defined and can be simply evaluated to be

$$(n|m)_+ = \frac{1}{\pi\sin(2\theta)}(e^{-i(n-m)\theta}) \quad (n-m) \quad \text{even} \tag{5.65}$$

Now finally as this is a system whose matrix elements are all real the exact physical inner product is recovered by simply taking twice the real part of 5.65, which reproduces the exact expression obtained from group averaging (5.55). We see then that the local vertex expansion constructed does reproduce the exact result of the group averaging procedure, but not in the desired manner. To recover the group averaging result it was necessary to have knowledge of the full series and to compute the Borel sum. The above technique cannot be carried out at a finite order

of the vertex expansion, so it does not give a well-defined perturbative procedure for computing the physical inner product.

A surprising consequence though is that the positive frequency expansion of the physical inner product returned exactly the positive frequency solution of the constraint equation. Looking at simply breaking the group averaging integral into integrals over positive and negative α it is not clear that the restriction to integration on the positive frequency part should return a solution to constraint. A similar analysis can be considered for the cases $\theta = 0$ and $\theta = \pi$, but there the group averaging procedure as carried out above is divergent. It is necessary to insert a factor of $\sqrt{C}$ in the definition of the physical inner product to recover a finite result.

A further interesting question is to analyze for when the expansion for the above simple system will be convergent. Taking the constraint to be

$$C = \sin^2(b) - \Lambda \tag{5.66}$$

where $\Lambda \geq 1$, the constraint equation does not have even distributional solutions. The vertex expansion though

$$(n, m)_{+\delta} = \frac{i}{2\pi} \sum_{M'=0}^{\infty} \binom{2M' + (n-m)/2}{M'} \frac{4}{(2 - 4\Lambda + 4i\delta)^{2M'+(n-m)/2+1}} \tag{5.67}$$

is *convergent*. If we let $\Lambda = \cosh(\phi)^2$ then the series converges to

$$(n, m)_{+\delta} = \frac{i}{\pi \sinh(2\phi)}(-1)^{(n-m/2)}e^{-(n-m)\phi}. \tag{5.68}$$

Further, much in the way that the Borel sum (5.65) gives the positive frequency solution to the constraint (5.48), (5.68) gives the divergent solutions to (5.66). The full physical inner product though is given by the real part of the (5.68), which is zero. While the positive/negative vertex expansions can individually be non-zero even when the constraint equation does not have solutions, in this case the physical inner product is zero as expected.

In this explicit example we see that the local expansion is a *divergent* series due to the large number of discrete histories. Further it does not give a good

asymptotic series for the physical inner product since the terms in the expansion typically grow, so similarly the error grows at each order of the expansion. Thus in this simple example we have obtained a local expansion similar to the current SFM at the expense of the ability of the series to approximate the physical inner product using a finite number of terms. Additionally this problem is not particular to the simple model considered. If we consider more complex models such as vacuum Bianchi I the sum is divergent in a similar manner since for large volumes the amplitudes approach the simple constant amplitudes of this simple model. Thus this is a very general concern.

5.7 Coarse-Graining/Renormalization

In section 4.3.2 we introduced a notion of coarse graining of the sum over histories as a cure for the divergences arising from the presence zero matrix elements of the constraint. The coarse graining was conceptually simple, just considering the sum over all paths that fluctuate around those states leading to zero diagonal matrix elements, but the implementation was more complicated. Here since the amplitude for a given history is much simpler we can state the exact action of this renormalization. For each basis state s we consider the set of states that are in its immediate neighborhood, $C(s, 1)$ defined by

$$C(s, 1) = \{s'|C_{ss'} \neq 0 \text{ and } s' \neq s\} \tag{5.69}$$

and the set of states that are separated by two actions of the constraint, $C(2, s)$, defined by

$$C(s, 2) = \{s'|C_{ss'}^2 \neq 0 \text{ and } s' \neq s\} \tag{5.70}$$

As in section 4.3.2 we consider coarse grained histories characterized by a sequence of a states connected by 2-transitions

$$\sigma_M = (s_M, s_{M-1}, \ldots, s_1, s_0) \quad s_{m+1} \, \epsilon \, C(2, s_m) \tag{5.71}$$

We will refer to these as 2-histories since they are made up of states separated by two transitions and use $A_{(2)}$ to denote the amplitude associated to them. We can

similarly refer to the original discrete histories as 1-histories.

The amplitude corresponding to such a history is given by the sum over all histories of single transitions that start at s_0 transition arbitrarily many times between states in $C(s_0, 1)$ and s_0 followed by 2-transition leading to s_1, arbitrarily many transitions between states in $C(s_1, 1)$ and s_1 and so on. Since the amplitude corresponding to a 1-history is totally local it can be written as a product amplitudes corresponding to each part of the history,

$$A_{(2)}(s_M, s_{M-1}, \ldots, s_1, s_0) = \sum_{N_M=0}^{\infty} \ldots \sum_{N_0=0}^{\infty} A_{(2)}(s_M, N_M) K_{(2)}(s_M, s_{M-1}) \quad (5.72)$$

$$\times A_{(2)}(s_{M-1}, N_{M-1}) \ldots K_{(2)}(s_1, s_0) A_{(2)}(s_0, N_0)$$

Where $A_{(2)}(s, N)$ denotes the amplitude corresponding to the sum over all histories that transition between s and elements of $C(s, 1)$ a total N times and $K_{(2)}(s, s')$ denotes the amplitude corresponding to the sum over all 2-transitions between s and s'. The amplitude above can be simplified as follows: We first absorb the 'edge' amplitudes into each 'vertex' amplitude to simplify the calculation. Then the amplitude corresponding to a single discrete 1-history can be written as

$$A(s_M, s_{M-1}, \ldots, s_1, s_0) = V_{s_M s_{M-1}} \ldots V_{s_1 s_0} \quad (5.73)$$

where

$$V_{ss'} = \frac{-C_{ss'}}{\sqrt{(C_{ss} + i\delta)(C_{s's'} + i\delta)}} \text{ for s} \neq \text{s}' \quad (5.74)$$

and the overall amplitude for the positive part is then

$$(s_f|s_i)_{+\delta} = \frac{i}{2\pi} \frac{1}{\sqrt{(C_{s_f s_f} + i\delta)(C_{s_i s_i} + i\delta)}} \sum_{M=0}^{\infty} \sum_{s_{M-1}, \ldots, s_1} A(s_M, s_{M-1}, \ldots, s_1, s_0)$$

$$(5.75)$$

The amplitudes corresponding to an arbitrary number of oscillations between a state s and it's neighboring elements can be simply computed,

$$\sum_{N=0}^{\infty} A_{(2)}(s, N) = \sum_{N=0}^{\infty} \sum_{s_N, \ldots, s_1 \, \epsilon \, C(s,1)} V_{ss_N} V_{s_N s} V_{ss_{N-1}} V_{s_{N-1} s} \ldots V_{ss_1} V_{s_1 s} \quad (5.76)$$

$$= \sum_{N=0}^{\infty} \left(\sum_{s' \, \epsilon \, C(s,1)} V_{ss'} V_{s's} \right)^N \tag{5.77}$$

$$= \frac{1}{1 - \sum_{s' \, \epsilon \, C(s,1)} V_{ss'} V_{s's}} \equiv \frac{1}{D_{(2)}(s,s)} \tag{5.78}$$

Similarly the amplitude corresponding to the double transition can be simply evaluated as

$$K_{(2)}(s,s') = \sum_{B(s,1) \cap B(s'',1)} V_{ss''} V_{s''s'} \tag{5.79}$$

Thus the overall amplitude corresponding to a coarse grained history is given by

$$A_{(2)}(s_M, s_{M-1}, \ldots, s_1, s_0) = \frac{K_{(2)}(s_M, s_{M-1})K_{(2)}(s_{M-1}, s_{M-2}) \ldots K_{(2)}(s_1, s_0)}{D_{(2)}(s_M, s_M)D_{(2)}(s_{M-1}, s_{M-1} \ldots D_{(2)}(s_0, s_0))} \tag{5.80}$$

Thus the coarse grained amplitude has an identical form to the original with new vertex and edge amplitudes. We can clearly read off the action of this renormalization group on the vertex and edge amplitudes.

$$K_{(2)}(s',s) = \sum_{C(s,1) \cap C(s',1)} \frac{C_{s's''}C_{s''s}}{\sqrt{C_{ss}C_{s''s''}}\sqrt{C_{s's'}}} \tag{5.81}$$

$$D_{(2)}(s,s) = 1 - \sum_{s' \, \epsilon \, C(s,1)} \frac{C_{ss'}C_{s's}}{C_{ss}C_{s's'}} \tag{5.82}$$

Since the coarse graining reproduced an amplitude of the exact same form this process can be repeated arbitrarily many times generating a renormalization group flow of the edge and vertex amplitudes. The overall amplitude is then given as a sum over all sequences of states separated by double transitions with the above amplitude for each history,

$$(s_f, s_i)_{+\delta} = \frac{i}{2\pi} \frac{1}{\sqrt{(C_{s_f s_f} + i\delta)(C_{s_i s_i} + i\delta)}} \sum_{\substack{s_{M-1}, \ldots, s_1 \\ s_{m+1} \, \epsilon \, C(s_m, 2)}} A_{(2)}(s_M, s_{M-1}, \ldots, s_1, s_0) \tag{5.83}$$

if the states s_f and s_i are connected by an even number of actions of the constraint.

Applied to the simple example considered in section 5.6 this coarse graining does provide improved convergence. In the regime where the vertex expansion

was convergent, which is the non-physical region where there is no solution to the constraint, the convergence is accelerated. For the physically interesting range of values θ the coarse graining does not produce a convergent sequence although it dramatically improves the accuracy of the first terms of the series. If the coarse graining is repeated eventually the first term of the series provides a sufficiently good approximation to the exact physical inner product. Thus the coarse graining discussed here can provide a means to extract physical information from the otherwise often purely divergent series generated by the local vertex expansion.

5.8 Discussion

For the systems without matter considered in this chapter the resulting vertex expansion was a sum over distributions. The resulting expansion while providing a formal solution to the constraint does not provide a viable perturbative scheme for computing the physical inner product. We have introduced two possible regulators in the group averaging procedure which lead to vertex expansions that are well defined term by term. The Feynman regulator further gives vertex expansion with a simple local expression for the amplitude for each discrete history. The Feynman regularization adds an additional similarity to spin foams. The spin foam amplitude for a single triangulation and set of labels, for a single history of spin networks, is given by a product of amplitudes associated to each vertex, face, and edge. This is in contrast to the vertex expansions presented in the previous chapters which were essentially non-local, with the amplitude for a history depending on the properties of the entire history. The Feynman regularization was given by a sum of two vertex expansions, the positive and negative part, each of which are local in the above sense. In the simple case where all matrix elements of the constraint are real, the regulated physical inner product is given simply by the real part of the positive expansion.

This regulator though *cannot* be removed at a finite order of the expansion of the regulated physical inner product as in the limit $\delta \to 0$ each term becomes distributional. The hope was then that the regulator could be left at a small value δ at which the regulated vertex expansion would provide an approximation to the exact physical inner product up to a controlled error. Unfortunately we generally

find that the regulated vertex expansion is divergent and thus does not provide a good approximation to the physical inner product. The regulated vertex expansion is not devoid of physical information though. Similar to the Feynman diagram expansion of QFT the regulated vertex expansion in examples is convergent in a more general sense as it it Borel summable and further the Borel sum does reproduce the exact physical inner product. Thus the regulated local vertex expansion may similarly generate a Borel summable series for physical inner product.

While the local vertex expansion provides better contact with the current SFMs it also leads to a highly divergent series that does not provide a good approximation to the physical inner product by taking finitely many terms.

Proofs and Identities for Vertex Expansion

A.1 Limit in Eq (3.28)

It is convenient to rewrite $A_N(\nu_M, \ldots, \nu_0; \alpha)$ defined in (3.25) in the following way:

$$
A_N(\nu_M, \ldots, \nu_0; \alpha) = U_{\nu_M \nu_{M-1}} \ldots U_{\nu_1 \nu_0} \left[U_{\nu_M \nu_M} \right]^N \left[U_{\nu_M \nu_M} \ldots U_{\nu_0 \nu_0} \right]^{-1} \times
$$

$$
\sum_{N_M = M}^{N-1} \sum_{N_{M-1} = M-1}^{N_M - 1} \cdots \sum_{N_1 = 1}^{N_2 - 1} \left[\frac{U_{\nu_{M-1} \nu_{M-1}}}{U_{\nu_M \nu_M}} \right]^{N_M} \cdots \left[\frac{U_{\nu_0 \nu_0}}{U_{\nu_1 \nu_1}} \right]^{N_1}.
$$

$$(A.1)$$

Our aim is to calculate the limit $N \to \infty$ of (A.1) and show that is given by $A(\nu_M, \ldots, \nu_0; \alpha)$, of Eq (3.28) which we rewrite as

$$
A(\nu_M, \ldots, \nu_0; \alpha) = (-i\alpha)^M \, \Theta_{\nu_M \nu_{M-1}} \ldots \Theta_{\nu_1 \nu_0} \, e^{-i\alpha \Theta_{\nu_M \nu_M}} \times
$$

$$
\int_0^1 \mathrm{d}\tau_M \int_0^{\tau_M} \mathrm{d}\tau_{M-1} \ldots \int_0^{\tau_2} \mathrm{d}\tau_1 \; e^{\tau_M b_M} \ldots e^{\tau_1 b_1}
$$

$$(A.2)$$

where

$$
b_m := -i\alpha (\Theta_{\nu_{m-1} \nu_{m-1}} - \Theta_{\nu_m \nu_m}). \tag{A.3}
$$

We start by calculating the $N \gg 1$ behavior of the terms appearing in (A.1).

These are:

$$U_{\nu_{m+1}\nu_m} = -\frac{i\alpha}{N}\Theta_{\nu_{m+1}\nu_m} + O(N^{-2}), \tag{A.4}$$

$$
\begin{aligned}
[U_{\nu_M\nu_M}]^N &= e^{N\log U_{\nu_M\nu_M}} \\
&= e^{N\left(-i\frac{\alpha}{N}\Theta_{\nu_M\nu_M}+O(N^{-2})\right)} \\
&= e^{-i\alpha\Theta_{\nu_M\nu_M}} + O(N^{-1}),
\end{aligned}
\tag{A.5}
$$

$$[U_{\nu_M\nu_M}\ldots U_{\nu_0\nu_0}]^{-1} = 1 + O(N^{-1}), \tag{A.6}$$

$$
\begin{aligned}
\left[\frac{U_{\nu_{m-1}\nu_{m-1}}}{U_{\nu_m\nu_m}}\right]^{N_m} &= e^{N_m\left(\log U_{\nu_{m-1}\nu_{m-1}}-\log U_{\nu_m\nu_m}\right)} \\
&= e^{N_m\left(b_m/N+O(N^{-2})\right)} \\
&= e^{\frac{N_m}{N}b_m} + O(N_m N^{-2}),
\end{aligned}
\tag{A.7}
$$

with b_m given in (A.3). In (A.5) and (A.7) we have used the fact that the multivalued nature of the log function does not affect the final result: $e^{N(\log x+2\pi ik)} = e^{N\log x}$ where $k \in \mathbb{Z}$ reflects the multiple values that log can take.

We now substitute expressions (A.4) to (A.7) in (A.1) to obtain

$$
\begin{aligned}
A_N(\nu_M,\ldots,\nu_0;\alpha) = &\left[(-i\alpha)^M\Theta_{\nu_M\nu_{M-1}}\ldots\Theta_{\nu_1\nu_0}e^{-i\alpha\Theta_{\nu_M\nu_M}}N^{-M} + O(N^{-M-1})\right] \times \\
&\prod_{m=1}^{M}\left[\sum_{N_m=m}^{N_{m+1}-1} e^{\frac{N_m}{N}b_m} + O(N_m N^{-2})\right]
\end{aligned}
\tag{A.8}
$$

where the product denotes the M nested sums in (A.1). Each sum in (A.8) has two terms. The first one gives a contribution of $\sum_{N_m} e^{\frac{N_m}{N}b_m} \sim O(N)$ while the second one is $\sum_{N_m} O(N_m N^{-2}) \sim O(1)$. The M sums then give a contribution of order $[O(N) + O(1)]^M \sim O(N^M) + O(N^{M-1})$. By combining this with the first factor of (A.8), we find that the non-vanishing contribution comes from the first

terms of the sums:

$$A_N(\nu_M, \ldots, \nu_0; \alpha) = (-i\alpha)^M \, \Theta_{\nu_M \nu_{M-1}} \ldots \Theta_{\nu_1 \nu_0} \, e^{-i\alpha\Theta_{\nu_M \nu_M}} \times$$

$$N^{-M} \prod_{m=1}^{M} \left[\sum_{N_m=m}^{N_{m+1}-1} e^{\frac{N_m}{N} b_m} \right] + O(N^{-1}). \qquad (A.9)$$

Eq (A.9) has all the pre-factors appearing in (A.2). It then remains to show that N^{-M} times the sums in (A.9) limits to the integrals in (A.2). But this is rather obvious, as the sums can be seen as Riemann sums for the integrals. Specifically,

$$\lim_{N\to\infty} N^{-M} \prod_{m=1}^{M} \left[\sum_{N_m=m}^{N_{m+1}-1} e^{\frac{N_m}{N} b_m} \right]$$

$$= \lim_{N\to\infty} N^{-M} \sum_{N_M=0}^{N} \sum_{N_{M-1}=0}^{N_M} \ldots \sum_{N_1=0}^{N_2} e^{\frac{N_M}{N} b_M} \ldots e^{\frac{N_1}{N} b_1}$$

$$= \int_0^1 d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_2} d\tau_1 \, e^{\tau_M b_M} \ldots e^{\tau_1 b_1} \qquad (A.10)$$

where, in the second line, we have slightly changed the limits on the sums, introducing an $O(N^{-1})$-term which vanishes in the limit. This concludes the proof of the limit (3.28).

A.2   General Integrals in Eq (3.28)

The integrals over τ appearing in the amplitude for a single discrete path (3.28) can be evaluated for a general sequence of volumes $(\nu_M, ..., \nu_0)$. In this appendix we will perform these integrals first for the case where all ν_i are distinct and then for the general case. The amplitude for a single discrete path given by (3.28) is

$$A(\nu_M, \ldots, \nu_0, \alpha) = \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_2} d\tau_1 e^{-i(\Delta\tau - \tau_M)\alpha\Theta_{\nu_M \nu_M}} \left(-i\alpha\Theta_{\nu_M \nu_{M_1}} \right) \times$$

$$e^{-i(\tau_M - \tau_{M-1})\alpha\Theta_{\nu_{M-1}\nu_{M-1}}} \ldots e^{-i(\tau_2 - \tau_1)\alpha\Theta_{\nu_1 \nu_1}} \left(-i\alpha\Theta_{\nu_1 \nu_0} \right) e^{i\tau_1 \alpha\Theta_{\nu_0 \nu_0}}$$

$$(A.11)$$

This expression can be written in terms of the following integral.

$$I(x_M, \ldots, x_0, \Delta\tau) = \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_2} d\tau_1 (i)^M e^{i(\Delta\tau - \tau_M)x_M} e^{i(\tau_M - \tau_{M-1})x_{M-1}}$$

$$\text{(A.12)}$$

$$\ldots e^{i(\tau_2 - \tau_1)x_1} e^{i\tau_1 x_0}$$

We will first evaluate this integral for the case where all x_i are distinct. By induction on M —the number of vertices or the number of times that x changes value— we will show that when the x_i are all distinct the integral is given by

$$I(x_M, \ldots, x_0, \Delta\tau) = \sum_{i=0}^{M} \frac{e^{ix_i \Delta\tau}}{\prod_{j \neq i}^{M}(x_i - x_j)} \tag{A.13}$$

This is true by inspection for $M = 0$. If we assume that (A.13) holds for M we can evaluate the integral with $M + 1$ vertices.

$$I(x_{M+1}, x_M, \ldots, x_0, \Delta\tau) = \int_0^{\Delta\tau} d\tau_{M+1}\, ie^{i(\Delta\tau - \tau_{M+1})x_{M+1}} I(x_M, \ldots, x_0, \tau_{M+1})$$

$$\text{(A.14)}$$

$$= \int_0^{\Delta\tau} d\tau_{M+1}\, ie^{i(\Delta\tau - \tau_{M+1})x_{M+1}} \sum_{i=0}^{M} \frac{e^{ix_i \tau_{M+1}}}{\prod_{j \neq i}^{M}(x_i - x_j)}$$

$$= \sum_{i=0}^{M} \frac{e^{ix_i \Delta\tau}}{\prod_{j \neq i}^{M+1}(x_i - x_j)} - e^{i\Delta\tau x_{M+1}} \sum_{i=0}^{M} \frac{1}{\prod_{j \neq i}^{M+1}(x_i - x_j)}$$

In the first step we recognized that the $M + 1$-th integral contains the M-th and then, in the second step, we inserted the assumed result for the $M - th$ integral. In the second step the integral over τ_{M+1} is carried out. Finally using the identity

$$\sum_{i=1}^{M+1} \frac{1}{\prod_{j \neq i}^{M+1}(x_i - x_j)} = 0 \tag{A.15}$$

The integral can be written as

$$I(x_{M+1}, x_M, \ldots, x_0, \Delta\tau) = \sum_{i=0}^{M+1} \frac{e^{ix_i \Delta\tau}}{\prod_{j \neq i}^{M+1}(x_i - x_j)} \tag{A.16}$$

Therefore if (A.13) holds for M it also holds for $M + 1$, thus by induction it holds for all $M \geq 0$.

If the x_i are not distinct, if there exist i, j such that $x_i = x_j$, then the proof follows in a similar way. The key element is that the integral $I(x_M, ..., x_0)$ is independent of the order of the x_i's. This can be seen by rewriting the integral in terms of the time intervals $\Delta\tau_i = \tau_{i+1} - \tau_i$ where $\tau_0 = 0$ and $\tau_{m+1} = \Delta\tau$.

$$I(x_0, x_1, ...x_M, \Delta\tau) = \int_0^{\Delta\tau} d\Delta\tau_M d\Delta\tau_{M-1}..d\Delta\tau_0 \ \delta(\Delta\tau_m + ... + \Delta\tau_0 - \Delta\tau) \quad \text{(A.17)}$$
$$(i)^M e^{i\Delta\tau_M x_M} e^{i\Delta\tau_{M-1} x_{M-1}} ... e^{i\Delta\tau_1 x_1} e^{i\Delta\tau_0 x_0}$$

It is clear that this is symmetric under the interchange of x_i with x_j for all i, j, so the integral is independent of the order of the sequence x_i. Since the integral is independent of the order of the values x_i it should be characterized by the distinct values, labeled by y_i and their multiplicity n_i. Where $n_1 + \ldots + n_p = M + 1$. Given a set of values x_i we will evaluate the integral for the case where they are organized such that any x_i sharing the same value are grouped together. Doing so the integral simplifies to

$$I(y_p, n_p, \ldots, y_1, n_1, \Delta\tau) = \int_0^{\Delta\tau} d\tau_M \int_0^{\tau_M} d\tau_{M-1} \ldots \int_0^{\tau_2} d\tau_1 (i)^M e^{i(\Delta\tau - \tau_{n_1} + ... + n_{p-1})y_p}$$

$$\text{(A.18)}$$

$$e^{i(\tau_{n_1} + ... + n_{p-1} - \tau_{n_1} + ... + n_{p-2})y_{p-1}} ... e^{i(\tau_{n_1} + n_2 - \tau_{n_1})y_2} e^{i\tau_{n_1} y_1}$$

By induction on p, the number of distinct values, we show that this integral is given by

$$I(y_p, n_p, ..., y_1, n_1, \Delta\tau) = \frac{1}{(n_p - 1)!} \left(\frac{\partial}{\partial y_p}\right)^{n_p - 1} \cdots \frac{1}{(n_1 - 1)!} \left(\frac{\partial}{\partial y_1}\right)^{n_1 - 1} \sum_{i=1}^{p} \frac{e^{iy_i \Delta\tau}}{\prod_{j \neq i}^{p}(y_i - y_j)}$$

$$\text{(A.19)}$$

$$= \prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k - 1} \sum_{i=1}^{p} \frac{e^{iy_i \Delta\tau}}{\prod_{j \neq i}^{p}(y_i - y_j)}$$

For $p = 1$ (A.18) can be easily evaluated giving

$$I(y_1, n_1) = \int_0^{\Delta\tau} d\tau_{n_1 - 1} \ldots \int_0^{\tau_2} d\tau_1 (i)^{n_1 - 1} e^{iy_1 \Delta\tau} = \frac{(i\Delta\tau)^{n_1 - 1}}{(n_1 - 1)!} e^{iy_1 \Delta\tau} \quad \text{(A.20)}$$

$$= \left(\frac{\partial}{\partial y_1}\right)^{n_1-1} \frac{1}{(n_1-1)!} e^{iy_i\Delta\tau}$$

And if $y_1 = 0$

$$I(y_1, n_1) = (i\Delta\tau)^{n_1-1}/(n_1-1)! \tag{A.21}$$

If we assume that (A.19) holds for p distinct values then we can evaluate it for $p+1$ distinct values as follows.

$$I(y_{p+1}, n_{p+1}, y_p, n_p \ldots, y_1, n_1, \Delta\tau) = \int_0^{\Delta\tau} d\tau_M \ldots \int_0^{\tau_{M-n_{p+1}+2}} d\tau_{M-n_{p+1}+1} \tag{A.22}$$
$$(i)^{n_{p+1}-1} e^{i(\Delta\tau - \tau_{M-n_{p+1}+1})y_{p+1}} I(y_p, n_p, \ldots, y_1, n_1, \tau_{M-n_{p+1}+1})$$

Plugging in the assumed result for p distinct values and performing the integrals over τ we obtain

$$I(y_{p+1}, n_{p+1}, \ldots, y_1, n_1, \Delta\tau) = \prod_{k=1}^{p} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)} \tag{A.23}$$
$$\left[\frac{e^{iy_i\Delta\tau}}{(y_i-y_{p+1})^{n_{p+1}}} - \sum_{m=0}^{n_{p+1}} \frac{e^{iy_{p+1}\Delta\tau}}{(y_i-y_{p+1})^m} \frac{(i\Delta\tau)^{n_{p+1}-m}}{(n_{p+1}-m)!}\right]$$

If one diagonal element is zero without loss of generality we can assume the $p+1-th$ element is.

$$I(y_{p+1}, n_{p+1}, \ldots, y_1, n_1, \Delta\tau) = \prod_{k=1}^{p} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)} \tag{A.24}$$
$$\left[\frac{e^{iy_i\Delta\tau}}{(y_i)^{n_{p+1}}} - \sum_{m=0}^{n_{p+1}} \frac{1}{(y_i)^m} \frac{(i\Delta\tau)^{n_{p+1}-m}}{(n_{p+1}-m)!}\right]$$

We recognize that the term in brackets can be written as derivatives with respect to y_{p+1} of a simple function.

$$I(y_{p+1}, n_{p+1}, y_p, n_p \ldots, y_1, n_1, \Delta\tau) = \prod_{k=1}^{p} \frac{1}{(n_k-1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k-1} \sum_{i=1}^{p} \frac{1}{\prod_{j\neq i}^{p}(y_i-y_j)}$$
$$\tag{A.25}$$
$$\left[\frac{1}{(n_{p+1}-1)!} \left(\frac{\partial}{\partial y_{p+1}}\right)^{n_{p+1}-1} \left(\frac{e^{iy_i\Delta\tau}}{y_i-y_{p+1}} - \frac{e^{iy_{p+1}\Delta\tau}}{y_i-y_{p+1}}\right)\right]$$

Finally simplifying the expression and using eqn (A.15) we obtain

$$I(y_{p+1}, n_{p+1}, \ldots, y_1, n_1, \Delta\tau) = \prod_{k=1}^{p+1} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial y_k}\right)^{n_k - 1} \sum_{i=1}^{p+1} \frac{e^{iy_i \Delta\tau}}{\prod_{j\neq i}^{p}(y_i - y_j)} \quad \text{(A.26)}$$

Thus if (A.19) holds for p then it also holds for $p + 1$, so it is true for all $p \geq 0$. Using this result we find that the contribution due to each discrete path is

$$A(\nu_M, \ldots, \nu_0, \alpha) = (\Theta_{\nu_M \nu_{M-1}})(\Theta_{\nu_{M-1}\nu_{M-2}}) \ldots (\Theta_{\nu_2 \nu_1})(\Theta_{\nu_1 \nu_0}) \quad \text{(A.27)}$$

$$\prod_{k=1}^{p} \frac{1}{(n_k - 1)!} \left(\frac{\partial}{\partial \Theta_{w_k w_k}}\right)^{n_k - 1} \sum_{i=1}^{p} \frac{e^{-i\alpha\Theta_{w_i w_i}\Delta\tau}}{\prod_{j\neq i}^{p}(\Theta_{w_i w_i} - \Theta_{w_j w_j})}$$

where w_i label the distinct values taken by ν along the path and n_i the multiplicity of each value.

Appendix B

Eigenstates and Matrix Elements of f(Θ)

In the timeless framework of section of chapter 3, the vertex expansion featured matrix elements $\Theta_{\nu_m \nu_n} = \langle \nu_m | \Theta | \nu_n \rangle$. These are easy to evaluate directly from the definition of Θ. In the deparameterized framework of chapter 2, on the other hand, the vertex expansion involves matrix elements of $\sqrt{\Theta}$. To evaluate these one needs the spectral decomposition of Θ. In the first part of this Appendix we construct eigenstates of Θ and discuss their relevant properties. In the second part we use these eigenstates to evaluate the matrix elements functions of Θ, including $\sqrt{\Theta}$.

B.1   Eigenstates of Θ

Recall that Θ is a positive, self-adjoint operator on $\mathcal{H}_{\text{kin}}^{\text{grav}}$. By its definition (2.6), it follows that Θ preserves each of the three sub-spaces in the decomposition $\mathcal{H}_{\text{kin}}^{\text{grav}} = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+$, spanned by functions with support on $\nu < 0$, $\nu = 0$ and $\nu > 0$ respectively. In particular, $|\nu = 0\rangle$ is the unique eigenvector of Θ, with eigenvalue 0; $\mathcal{H}_0$ is 1-dimensional. Our first task is to solve the eigenvalue equation for a general eigenvalue ω_k^2:

$$\Theta \, e_k(\nu) = \omega_k^2 \, e_k(\nu) \,. \tag{B.1}$$

This task becomes simpler in the representation in which states are functions $\chi(b)$ of the variable b conjugate to ν: [1]

$$\chi(b) := \sqrt{\frac{\ell_o}{\pi}} \sum_{\nu = 4n\ell_o} e^{\frac{i}{2}\nu b} \frac{\Psi(\nu)}{\sqrt{|\nu|}} . \tag{B.2}$$

In this representation, the eigenvalue equation (B.1) takes the form of a simple differential equation

$$\left(\Theta\chi_k\right)(b) = -12\pi G \left(\frac{\sin \ell_o b}{\ell_o}\partial_b\right)^2 \chi_k(b) = \omega_k^2\, \chi_k(b), \tag{B.3}$$

whose solutions are

$$\chi_k(b) = A(k)\, e^{ik\log(\tan\frac{\ell_o b}{2})} \qquad \text{with} \quad \omega_k^2 = 12\pi G\, k^2\,, \tag{B.4}$$

where $A(k)$ is a normalization factor and $k \in (-\infty, \infty)$. $k = 0$ yields a discrete eigenvalue $\omega_k = 0$ and in the ν representation the eigenvector can be expressed simply as $e_0(\nu) = \delta_{0,\nu}$. Eigenvectors with non-zero eigenvalues can also be expressed in the ν representation by applying the inverse transformation of (B.2) to (B.4):

$$e_k(\nu) = A(k)\sqrt{\frac{\ell_o|\nu|}{\pi}}\int_0^{\pi/\ell_o}\mathrm{d}b\, e^{-\frac{i}{2}\nu b}\, e^{ik\log(\tan\frac{\ell_o b}{2})} \qquad \text{where} \quad k \neq 0\,. \tag{B.5}$$

Let us note two properties of these eigenvectors. First, e_k and e_{-k} have the same eigenvalue and so the ω_k^2-eigenspace is two-dimensional. Second, the vectors $e_k(\nu)$ we have obtained have support on both $\nu > 0$ and $\nu < 0$. However, since Θ preserves the sub-spaces $\mathcal{H}_\pm$, it is natural to seek linear combinations $e_k^\pm(\nu)$ of $e_k(\nu)$ and $e_{-k}(\nu)$ which lie in these sub-spaces. In particular, this will simplify the problem of normalization of eigenfunctions.

Let us begin by rewriting the integral in (B.5) as a contour integral in the

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[1]Our normalization is different from that in [38]. The wave function $\tilde{\Psi}(\nu)$ in [38] is related to the one here by $\Psi(\nu) = \sqrt{\frac{\ell_o}{\pi|\nu|}}\, \tilde{\Psi}(\nu)$.

complex plane. Recalling that $\nu = 4\ell_o n$ and setting $z = e^{ib\ell_o}$ we obtain

$$\frac{\ell_o}{\pi} \int_0^{\pi/\ell_o} db \, e^{-2ibn} e^{ik \log(\tan\frac{\ell_o b}{2})} = \frac{e^{-\pi k/2}}{\pi i} \int_{\mathcal{C}} z^{-2n-1} \left(\frac{1-z}{1+z}\right)^{ik} dz =: J(k,n), \qquad (B.6)$$

where $\mathcal{C}$ is the unit semicircle in counterclockwise direction in the upper half, $\Im z > 0$, of the complex plane. As remarked earlier, $e_k(\nu) = A(k)\sqrt{\ell_o|\nu|/\pi}\, J(k, \nu/4\ell_o)$ has support on both positive and negative values of $\nu = 4\ell_o n$. Now, the second independent eigenfunction $e_{-k}(\nu)$ with the same eigenvalue ω_k^2 can be represented in a similar fashion by setting $z = -e^{ib\ell_o}$. The result is a contour-integral along the unit semicircle in counterclockwise direction in the lower half, $\Im z < 0$ of the complex plane. By combining the two integrals, we obtain a closed integral along the unit circle:

$$\frac{1}{2\pi i} \oint z^{-2n-1} \left(\frac{1-z}{1+z}\right)^{ik} dz = \frac{1}{2}\left(e^{\pi k/2} J(k,n) + e^{-\pi k/2} J(-k,n)\right) =: I(k,n).$$

$$(B.7)$$

Being a linear combination of $e_k(\nu)$ and $e_{-k}(\nu)$, this $I(k,n)$ gives also an eigenfunction of Θ with eigenvalue ω_k^2. Moreover, using elementary complex analysis, one finds that *it has support only on positive n*:

$$I(k,n) = \begin{cases} \frac{1}{(2n)!} \frac{d^{2n}}{ds^{2n}}\Big|_{s=0} \left(\frac{1-s}{1+s}\right)^{ik} & n \geq 0 \\ 0 & n < 0. \end{cases} \qquad (B.8)$$

Repeating the argument but taking $z = e^{-ib\ell_o}$ and $z = -e^{-ib\ell_o}$ one obtains

$$\frac{1}{2}\left(e^{-\pi k/2} J(k,n) + e^{\pi k/2} J(-k,n)\right) = \frac{1}{2\pi i} \oint z^{2n-1} \left(\frac{1-z}{1+z}\right)^{ik} dz = I(k,-n) \quad (B.9)$$

which has support only on negative n. Thus, the basis we are looking for is given by

$$e_k^{\pm}(\nu) := \frac{1}{2}\left(e^{\pm\pi k/2} e_k(\nu) + e^{\mp\pi k/2} e_{-k}(\nu)\right) = A(k)\sqrt{\frac{\pi|\nu|}{\ell_o}}\, I(k, \pm\frac{\nu}{4\ell_o}). \qquad (B.10)$$

By construction, $e_k^{\pm} \in \mathcal{H}_{\pm}$.

Next, let us calculate the normalization of these vectors. It is convenient to

introduce kets $|k\pm\rangle$ such that $\langle\nu|k\pm\rangle = e_k^{\pm}(\nu)$. Then, it is clear that $\langle k'\pm|k\mp\rangle = 0$. To calculate the nontrivial inner product, $\langle k'\pm|k\pm\rangle$, let us return to the b representation. There, the functions describing the states $|k\pm\rangle$ are

$$\chi_k^{\pm}(b) = \frac{A(k)}{2}\left(e^{\pm\pi k/2}e^{ik\log(\tan\frac{\ell_o b}{2})} + e^{\mp\pi k/2}e^{-ik\log(\tan\frac{\ell_o b}{2})}\right) \tag{B.11}$$

and their inner product is given by [38]

$$\langle k'\pm|k\pm\rangle = \int_0^{\pi/\ell_o}db\,|A(k)|^2\,\overline{\chi^{\pm}}_{k'}(b)\,|2i\partial_b|\,\chi_k^{\pm}(b) \tag{B.12}$$

where $|2i\partial_b|$ is the absolute value of the volume operator $\hat{\nu} = 2i\partial_b$. Simplification occurs because $e_k^{\pm}(\nu)$ have support only on positive/negative ν values. Because of this property, one can replace $|\partial_b|$ in (B.12) by $\pm\partial_b$. The calculation now reduces to a straightforward integration. The result is

$$\langle k'\pm|k\pm\rangle = |A(k)|^2\,2\pi k\,\sinh(\pi k)\,\delta(k',k). \tag{B.13}$$

B.2 Matrix Elements for $f(\Theta)$

We will now use the eigenbasis $|\pm k\rangle$ of Θ to calculate the matrix elements $\langle 4n\ell_o|f(\Theta)|4m\ell_o\rangle$, of the operators of the form $f(\Theta)$, for a measurable function f. Throughout this section, the normalization factor $A(k)$ is chosen to be unity. From the normalization condition (B.13) with $A(k) = 1$, we have the following decomposition of the identity:

$$\mathbf{I} = \int_0^{\infty}\frac{dk}{2\pi k\sinh(\pi k)}\,\left(|k+\rangle\langle k+| + |k-\rangle\langle k-|\right). \tag{B.14}$$

which can be inserted in $\langle 4n\ell_o|f(\widehat{\Theta})|4m\ell_o\rangle$. If m and n have different signs, the result is zero. It suffices to consider the case where both are positive. By writing $\langle 4n\ell_o|k+\rangle$ in terms of derivatives (see equations (B.10) and (B.8)), one obtains

$$\langle 4n\ell_o|f(\widehat{\Theta})|4m\ell_o\rangle = \frac{2\sqrt{mn}}{(2n)!(2m)!}\left.\frac{d^{2m}}{ds^{2m}}\frac{d^{2n}}{dt^{2n}}\right|_{s=t=0}F_{f(\Theta)}\left(\frac{1+s}{1-s}\frac{1-t}{1+t}\right) \tag{B.15}$$

with $F_{f(\Theta)}$ the 'generating function' given by[2]

$$F_{f(\Theta)}(x) = \int_0^\infty dk \; \frac{f(12\pi G k^2) x^{ik}}{k \sinh(\pi k)}.$$ (B.16)

We now give the generating function for $\sqrt{\Theta}$. It is also useful (at least to check normalization factors) to write down the generating functions for operators whose matrix elements are known, namely Θ and the identity I. These generating functions are given by,

$$F_I(x) = -2 \left(\log(1 + x) + \log \Gamma(1/2 + i \frac{\log x}{2\pi}) \right)$$ (B.17)

$$F_{\sqrt{\Theta}}(x) = \sqrt{12\pi G} \left(\frac{2ix}{1 + x} - \frac{1}{\pi} \psi(1/2 + i \frac{\log x}{2\pi}) \right)$$ (B.18)

$$F_\Theta(x) = 12\pi G \left(\frac{2x}{(1 + x)^2} - \frac{1}{2\pi^2} \psi'(1/2 + i \frac{\log x}{2\pi}) \right)$$ (B.19)

where $\Gamma(z)$ is the gamma function, and $\psi(z) = \Gamma'(z)/\Gamma(z)$ the polygamma function.

In obtaining these functions, it is useful to observe the following relations among them:

$$F_{\sqrt{\Theta}}(x) = -i\sqrt{12\pi G}\, x \frac{d}{dx} F_I(x)$$ (B.20)

$$F_\Theta(x) = -i\sqrt{12\pi G}\, x \frac{d}{dx} F_{\sqrt{\Theta}}(x),$$ (B.21)

which can be derived from (B.16).

We will conclude by noting that the matrix elements for the evolution operator $U(\varphi) = e^{i\varphi\sqrt{\Theta}}$ are easy to find: From (B.16) one sees that $F_{U(\varphi)}(x) = F_I(e^{\sqrt{12\pi G}\varphi} x)$.

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[2]For a general f, integral as defined may diverge. However the divergent terms (e.g., those which are x-independent) do not contribute to the expression of the matrix element and can therefore be discarded. This 'finite part extraction' is implicit in going from (B.16) to (B.17), (B.18) and (B.19) .

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