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MANAGING INVENTORY SYSTEMS WITH TECHNOLOGY INNOVATIONS

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Abstract

In the "faster, better, and cheaper" information age, rapid technological breakthroughs create significant risks of obsolescence at the product level or the component level. Consequently, enormous challenges in *jointly* coordinating inventory replenishment and technology selection arise. The purpose of this dissertation is to develop analytical models to study technology selection and inventory replenishment *joint* optimization in inventory systems that face frequent technology innovations.

Chapter 1 describes rapid technological obsolescence in the high-tech industry and documents the incredible turn-around in eMachines to demonstrate the importance of *jointly* optimizing technology selection and inventory control. Several research questions related to *jointly* optimizing technology selection and inventory control are raised. Chapter 2 reviews the related literatures. The major differences between the models in this dissertation and those in the existing literatures are also discussed. Chapter 3 to Chapter 6 study the technology selection and inventory replenishment *joint* optimization in three different production systems.

First, Chapter 3 considers a Make-to-Stock (MTS) system that markets a *single* product in each time period and investigates optimal timing for technology upgrading. Over the infinite planning horizon, the firm introduces the new and improved product depending on the availability of new generation products. The arrival time of next generation product follows an exogenous discrete-time, phase-type (PH) distribution. The cost parameters and demand distributions are phase- and technology-dependent.

In each period, the firm needs to decide which generation product to offer and how much inventory to order. A dynamic programming problem is formulated to determine the technology and inventory joint control policy that maximizes the total expected discounted profit over the infinite horizon. In a special case, where the firm is under the "inventory return" protection, the optimal policy is explicitly derived. However, when the "inventory return" condition is not satisfied and there is only one period remaining, the optimal policy operates as follows: if and only if the initial inventory is old generation and its inventory level is strictly below a state-dependent threshold, upgrading is optimal; meanwhile the optimal inventory policy follows a two-limit policy with control limits independent of initial inventory (i.e., when the initial technology is kept and its inventory level is below the first limit, order up-to the first limit; if the inventory is above the second limit, salvage inventory down to the second limit; otherwise, neither order nor salvage). However, when the analysis is extended to the multi-period model, the optimal policy is unclear because the objective function could be neither concave nor quasi-concave. A sufficient condition that guarantees the objective function to be concave is discussed. When the objective function is concave, the structure of the optimal joint technology and inventory policy resembles to the counterparts for the single-period problem. Based on the structural property of the special case, a sequential optimization heuristic is proposed, in which the product is upgraded as soon as the development phase reaches a threshold. Under this technology plan, the optimal inventory policy is a two-limit policy.

Second, Chapter 4 extends the model in Chapter 3 to allow the firm to offer an *assortment* of products in each time period. At the beginning of each period, the firm learns the cost parameters and phases associated with each available technology and makes the *assortment* and *inventory* decisions. If the firm decides to remove a product from the initial assortment, the initial inventory of that product will be salvaged immediately and will not be re-introduced. On the other hand, after knowing what assortment the firm offers, customers make their choice decision following a multinomial logit (MNL) model. As a result, the actual demand for each product included in the assortment is dependent on the firm's assortment decision as well as the phase and generation of the product. The unmet demand is assumed to be lost and the replenishment lead time is assumed to be negligible. For the final (single) period, the optimal ordering policy follows a two-limit control policy and the optimal assortment policy is a switch-over policy. That is, there exist two increasing functions that partition the plane into (at most) three non-overlapping regions, where each region corresponds to one dominating assortment. When the analysis is extended to the infinite-period model, once again, the objective function could be neither quasi-concave nor concave. If concavity is preserved, then the optimal ordering and assortment policies resemble to their counterparts in the single-period model. To gain more insights into the structural properties of the optimal policy, I analyze a special case where the firm is under the protection of "inventory return". I show that the optimal inventory policy is myopic and always adjusts the inventory level to the newsvender solution. With an additional assumption that the assortment decision does not change the coefficient of variation of product demand, I derive the optimal policy for the unconstrained problem. In addition, I devise a comprehensive measure, the *profit per unit*, which takes into account the profit margin, overage cost, and demand uncertainty, to conveniently determine which assortment to use. Under some conditions on cost parameters, the optimal policy for the unconstrained problem is also optimal for the constrained problem and indeed the optimal assortment evolves in a phase-based pattern. Several heuristics are proposed. Among them, the sequential optimization heuristic (SH) outperforms others. Under the SH heuristic, when the development state is below the first threshold, the firm offers the old generation product only; when the development state is between the first and the second thresholds, the firm offers both generation products; when the development state exceeds the second threshold, the firm discontinues the older product and offers the new generation product only. In the meantime, the inventory ordering policy follows a state-dependent two-limit policy according to the assortment plan. Numerical examples (based on 800 random samples) show that the SH heuristic outperforms the other heuristics, although the computation time for SH heuristic is more intensive. Numerical experiments also suggest that it is more important for the firm with a low profit margin to jointly optimize assortment and inventory control than for a firm with a high profit margin. In particular, the joint optimization becomes more important when demand variability is high or technology innovation is fast.

In practice, the final product is usually assembled from multiple components. As such, technology innovations occur at the *component* level and impact the system at the *product* level. The firm needs to effectively coordinate the technology configurations as well as the inventory decisions across different components. Chapter 5 analyzes an Assemble-to-Order (ATO) system that faces frequent component-based innovations. The innovation process of various components is governed by an exogenous discrete-time, multivariate, phase-type renewal process. I first consider centralized decision making, in which the firm jointly determines the technologies to be used and the inventories to be ordered across different components. Under the "inventory return" condition, the joint optimal technology and inventory policy is myopic and can be solved efficiently by the Interval Partitioning Algorithm (IPA). I then study the dynamic behavior of the optimal myopic policy to gain insights on how the innovation processes affect the firm's technology selection and inventory replenishment decisions. Using the insights from the structural analysis, a heuristic method that makes the *decentralized* technology selection decision and the *centralized* inventory replenishment decision is proposed in Chapter 6. A numerical study using randomly generated data shows that the heuristic is efficient, robust, and yields near-optimal solutions.

Finally, Chapter 7 concludes the dissertation by summarizing the analytical and managerial insights. Several directions for future research are outlined. The appendix contains the definitions of several key terminologies that have been used in the dissertation. In addition the proofs of several important lemmas and theorems are also relegated into the appendix.

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Chapter 1

Introduction

1.1 Technology Innovations

In the "faster, better, and cheaper" information age, rapid technology innovations have significantly shortened the product life-cycle and increased the pace of product obsolescence. For example, since the price war in microprocessors started in 1991, it has frequently occurred that the price of PCs have plummeted after the introduction of new technology. An industrial survey by *PC Today* [34] reports that just over the first year of its life cycle, the retail price of a desktop PC declines by 50% to 58%, on average. This phenomenon created enormous new challenges in coordinating technology selection and inventory control for modern production systems. The most noticeable trend is that the increasing speed of technology innovations forces the firm to review its own technology and inventory more frequently. This pushes technology management closer to the level of operational decisions and calls for effective coordination between technology management and inventory control decisions.

Ineffective coordination between technology management and inventory control is costly. Before June 2001, eMachines, an low-end computer manufacturer, like many others, rarely paid attention to coordinating its technology management and inventory control effectively. Since many of eMachines' products had relatively short life cycles and low profit margins, the lack of coordination between technology and inventory management exacerbated the obsolescence problem. In the fiscal year ending December 2000, eMachines lost $220 million on sales of $684 million (Hesseldahi [19]). In May 2001, after eMachines was de-listed from NASDAQ, the new CEO took office and eMachines never looked back. One of the many important steps to revive the company was to effectively coordinate its technology selection and inventory control. Now, eMachines decides when to introduce new computers and what features to include, based on reams of detailed data collected from each retailer on what features are most popular and how much customers are willing to pay for them. By August of 2003, eMachines reported eight consecutive profitable quarters. eMachines' market share of PCs sold through retail outlets in the US topped 26% and now ranks number three in the US desktop market, behind only Dell and HP, and ahead of Gateway. Now, eMachines is regarded as the most efficient computer manufacturer because it achieves the highest number of computers produced per employee on the lowest administrative cost per computer sold. The miraculous turnaround in eMachines suggests that an effective coordination between technology selection and inventory control has to be effectively implemented in today's ever-changing market.

The existing literatures have not been fully addressed the issues that eMachines was facing.

- The technology replacement literature, which models the innovation process as a Markov chain (Hopp & Nair [21], Nair [31], Rajagopalan, Singh, & Morton [39], among others), analyze the optimal timing for adopting new technology under deterministic demand. Product obsolescence models (see the survey papers by

Nahmias [29], Raafat [38], and Tekin, Gurlar, & Berk [51]) consider obsolescence in a single component (product) at either the supply or the demand side. When obsolescence occurs at the supply side, for example, blood products and perishable foods are discarded after their lifetimes, it effectively reduces inventory on hand (Nahmias [28], Nandasakumar & Morton [32]). When obsolescence occurs at the demand side, demand for the product diminishes over its lifetime (Pierskalla [36], Song & Zipkin [47], among others). However, the afore-mentioned results on product obsolescence or technology adoption often assume that the cost parameters are constant or that demand is deterministic and known, and thus did not fully capture the real problems that eMachines was facing.

- There are several papers addressing the technology acquisition with inventory aspect. Li, Loulou, & Rahman [24] and Wilhem, Damodaran, & Li [55], with some modeling differences, study the strategic decision in technology acquisition over a finite planning horizon with random demand, and apply the scenario approach to solve the stochastic programming problem. Souza, Bayus, & Wagner [49] formulate a dynamic programming problem that determines technology and inventory joint optimization by modelling the demand uncertainty, inventory salvage loss, and the firm's marketing influence. But these papers did not address the interaction between the customer choice and firm's assortment decisions. As in eMachines, the product configuration and inventory replenishment decisions need to be made jointly across different products/components by considering various factors such as customer choice, inventory, and technology status.

- The existing ATO models (see a comprehensive survey paper by Song & Zipkin [48]) focused on determining the optimal or near-optimal inventory control policy, assuming that cost parameters are constant and that components do not become obsolete. Rosling [40], and Hsu, Lee & So [22], with some modelling differences, derived the structure of the optimal ordering policy for the single-product assembly system with multiple components and deterministic lead time. Zhang [56], Agrawal & Cohen [1], and Akcay & Xu [2], focused on determining the independent order-up-to levels under some component allocation schemes for the periodic-review multi-component and multi-product ATO systems. But how price fluctuation and component obsolescence could affect the performance of ATO systems have not been sufficiently studied.

In summary, it is important to develop analytical models to study how to mitigate the negative impact of technology innovations (which lead to component obsolescence and state-dependent cost parameters) on the production systems.

1.2 Research Questions

The purpose of this dissertation is to develop analytical models to study the *joint* technology selection and inventory control optimization problem in production systems with technology innovations. In various types of production systems, the scope of such a coordination is naturally different. For example, in the Make-to-Stock (MTS) system where the final product has been assembled before receiving customer orders, the technology decision is made at the product level and is often related to the assortment decision. On the other hand, in the Assemble-to-Order (ATO) system where the final product is assembled from several components, the technology decision needs to be made at the component level. Because of these operational differences in MTS and ATO systems, this dissertation separately studies the technology selection and inventory control policies in the MTS and ATO systems separately.

I first consider the single-product Make-to-Stock system in which the firm markets a *single* generation product in each time period. The major issues to be addressed are the following:

- What is the optimal technology and inventory *joint* control policy?
- What are the structural properties of the optimal policy as the innovation state changes?
- How important is it to jointly optimize technology management and inventory replenishment?

Second, I consider extending the analysis to allow the firm to offer an assortment of products in each time period. That is, the firm needs to decide the mixture of new and old generations products to offer by anticipating how customers will make their choice from the firm's assortment. I plan to pursue answers to the following questions:

- *Customer Choice.* Each customer is a utility maximizer. After knowing the firm's assortment, a customer will choose the option with the highest utility. However, the customer's utility generally is a random variable and is dependent on the prices and the product generation. What customer choice model is the most appropriate to use? How does customer choice affect the firm's assortment decision?

- *Dynamic Assortment and Inventory Joint Control.* When a new generation product becomes available, the firm needs to decide whether to introduce the new one or discontinue the old one. What is the optimal assortment and inventory joint control policy? Is there any structural property of the optimal assortment when the innovation process evolves? What is the impact of positive salvage loss on the system?

Third, in many circumstances, the final product is assembled from multiple components. It is useful to extend the analysis to the ATO system, where the technology decision is made at the component level and coordination has to be made across different components. The major issues addressed are the following:

- *Technology Selection.* Although the value of dynamically adjusting the product configuration in response to the changing market is clear (as demonstrated in the eMachines' turnaround), how to select an appropriate product configuration is challenging. A new technology may have a lucrative profit margin but also a high overage cost. As such, a tradeoff must be made between the performance and cost of each possible product configuration. Related questions to be considered include the following: What are the general rules of thumb in selecting the desired product configuration? How does the optimal configuration change when its components are aging?

- *Inventory Control.* When some components are facing significant risk of price erosion, reducing inventory levels could mitigate the salvage loss, but may hurt the firm's ability to fulfill customer orders. Given that current components may be upgraded in the future, what is the optimal inventory replenishment policy across different components? How do the ages of existing technologies affect the optimal inventory levels?

- *Coordination between Technology Selection and Inventory Control.* The successful ATO practice depends on the effective coordination between technology selection and inventory management. The related questions are the following: how to jointly optimize inventory replenishment and technology selection decisions? Is there any price protection contract that can simplify the coordination between technology selection and inventory control? Can both firm and supplier benefit from such a price protection contract?

Chapter 2

Literature Review

This literature review examines three avenues: machine replacement, product obsolescence, and ATO literatures.

2.1 Machine Replacement

In the traditional machine replacement literature, the primary motivation for replacing existing machines is the deterioration process which causes either increasing operating cost or increasing probability of failure. In recent years, technological obsolescence has become the motivating factor for many replacement decisions, because technology develops so rapidly. Existing components/products are replaced or discontinued not because they are physically worn out, but because better components/products become available. Replacing the old generation components/products could allow the firm to keep up with the industry clock-speed (Souza, Bayus, & Wagner [49]).

Equipment replacement under technological changes has received increasing attention. Chand & Sethi [7] assumed that technology development and equipment deterioration are deterministic. Their objective is to minimize the sum of operating cost, replacement cost, and salvage cost. They developed a forward algorithm to solve the problem. They also identified the forecast horizon T such that the first machine replacement, which is based on a T-periods long forecast horizon, remains optimal for any longer (than T) horizon. Glodstein, Landany, & Mehrez [17] considered the replacement problem when the debut time of new technology is a geometrically distributed random variable. Hopp & Nair [21] modeled technology improvements and deterioration as a Markov process. In their model, only one new technology may appear in the future in addition to the technology already available in the market at the current time. They develop algorithms to find the optimal "keep" or "replace" decision using a forecast horizon approach. They study the interaction between the deterioration and the technology change. Nair [31] studied the equipment replacement problem with sequential technological change. The timing of the appearance of new technology, costs, revenues, and cost of capital change over time. He presented an efficient algorithm to determine the optimal replacement decision for the infinite planning horizon. Rajagopalan, Singh, & Morton [39] considered a firm facing a sequence of technological breakthroughs with uncertain timing and magnitude. The demand is deterministic. The firm must decide how much capacity of current technology to acquire to meet future demand and whether to upgrade it. A dynamic programming problem was formulated and solved by a regeneration point-based dynamic programming algorithm. They showed that it is optimal to dispose of excess capacity only in periods when a new technology appears and to replace used capacity only when capacity is in any case going to be acquired to meet future demand. The capacity investment model by Eberly and Van Mieghem [11], hereafter EV, is also related to our work. However, the differences between our model and EV's are (i) ours is concerned with the inventory replenishment; thus, the initial inventory is a random variable, whereas EV concerns with capacity adjustment, where initial capacity equals that determined in the previous period; (ii) in our model the change in assortment leads to the change in the demand distribution whereas in EV the type of technology used by the firm does not affect the demand distribution; (iii) our future profit function is conditionally concave, while that in EV is concave.

The existing machine replacement models do not consider demand uncertainty. Since demand uncertainty is not included, the assortment problem for the multiple-product system is trivial, where the multiple-components system can be decomposed into multiple single-component systems. However, when demand is random, the assortment decision for the multiple-item system is challenge; whereas in the multiple-components system, the decomposition approach is no longer applicable to ATO system, and the exact analysis is also challenging.

There are several papers addressing the technology acquisition with inventory aspect. Li, Loulou, & Rahman [24] and Wilhem, Damodaran, & Li [55] studied the strategic decision in technology acquisition over a finite planning horizon with random demand, and applied the scenario approach to solve the stochastic programming problem. Souza, Bayus, & Wagner [49] formulated a dynamic programming problem that determines the technology and inventory joint optimization problem by modelling the demand uncertainty, inventory salvage loss, and firm's marketing influence. Using a large-scale numerical study, Souza, Bayus, & Wagner provided some insights about the timing for adopting new technology. Since these papers assume that at any time the firm only offers one product (which is built upon the previous work by Cohen, Eliashberg, & Ho [9], Bayus, Jain, & Rao [5]), the internal competition among different products in the firm's assortment is not considered.

2.2 Product Obsolescence

Two different approaches have been used to model product obsolescence. For a comprehensive survey on product obsolescence, readers can refer to Nahmias [29], Raafat [38], and Tekin, Gurlar, & Berk [51]. As mentioned before, the first approach considers the obsolescence at the *supply* side, for example, the blood products and the fresh produces are discarded after their lifetimes. As such, obsolescence reduces the salable inventory. The second approach considers the obsolescence at the *demand* side, namely, the age-dependent demand pattern. For example, the demand for fashion goods drops to the lowest level after a random lifetime.

2.2.1 Perishable Product Models with Stochastic Demands

Nahmias & Pierskalla [30] characterized the optimal ordering policy for a single product with a fixed life time of two periods, zero lead time, and stochastic demand. They assume that the stock remaining on hand at the end of the period can be salvaged at the purchase cost and that excess demand can be satisfied by an emergency order. The stationary optimal policy is that when the on-hand inventory is positive, the optimal order quantity is a complicated convex and decreasing function of on-hand inventory; when the on-hand inventory is negative, the optimal order quantity is the backlog plus what would be ordered from zero following an optimal policy. By generalizing the work of Nahmias & Pierskalla [30], Fires [13] & Nahmias [27] independently and simultaneously characterized the structure of the optimal ordering policy for a periodic review inventory system with a single product of fixed life time. Nahmias [28] considered the ordering policy under random demand and random product life time. Nahmias assumed that successive orders outdate in the same sequence that they enter stock, that is First-In-First-Out (FIFO). The expected future outdating of the current order is determined by the current stock and realizations of future demands. Nahmias showed that the optimal policy has the same structure as that of Fries [13] and Nahmias [27]. Nandasakumar & Morton [32] extended the work of Nahmias [27] and provide easy-to-compute upper and lower bounds of the optimal order-up-to level.

2.2.2 State-Dependent Demand Models

Pierskalla [36] studied a discrete-time model, in which the demands are independent and identically distributed until obsolescence occurs. The time until obsolescence is a random variable with increasing failure rate and after obsolescence, there is no demand. Pierskalla showed that the optimal policy is a base-stock policy with non-increasing base-stock levels. Song & Zpkin [47] studied how obsolescence affects the inventory system with exponential lead time and Markovian decreasing demand pattern. They showed the structure of the optimal policy under linear costs and zero lead time: at the lowest state, the myopic solution is optimal, whereas in other states, the myopic policy is an upper bound, and the optimal base-stock levels are monotone with respect to the state of the system. Angelus & Porteus [3] studied a model in which the life cycle of the product is deterministic and the demand increases stochastically at the beginning of the life cycle and decreases thereafter. Capacity can be reduced as well as added, at an exogenously given price. They explicitly derive the optimal simultaneous capacity and production plan for a short-life cycle, produce-to-stock goods under stochastic demand.

There are two major modelling differences between our models and product obsolescence literature. First, product obsolescence models assume constant cost parameters, whereas we consider state-dependent cost parameters and assortment decisions. Second, in ATO systems, the product configuration and inventory replenishment decisions must be made jointly across different components, whereas the product obsolescence models only consider a single-component system.

2.3 Assemble-to-Order

Our classification of the ATO literature are the following: the discrete-time, single-product model; the discrete-time, multiple-product model; and the continuous time model.

2.3.1 Discrete Time, Single-Product Model

Schmidt & Nahmias [42] investigated a finite-horizon periodic review system of two components assembled into a single end product. Production cost, holding cost, and shortage penalty cost are assumed to be linear. The component replenishment lead time is deterministic and could be different for the two components. They derive a complicated optimal ordering policy under general assumptions on initial stock levels. Essentially, the optimal policy they derive is a base-stock policy, but the order quantity for each component has a fairly complex form. Rosling [40] extended the work of Schimidt & Nahmias [42] and considered a single-product assembly system with multiple components and subassemblies in an infinite planning horizon. In his model, a number of components are acquired from suppliers; then, in several stages, the components are assembled into subassemblies and finally into a single end product. The ordered components are available after a fixed lead time, whereas the assemble time is zero. Rosling [40] derived the long-run optimal policy with positive echelon holding costs and penalty costs and shows that under some conditions on the initial inventory, the system will eventually get into the long-run balance. When the system is in long-run balance from the very first period, the assembly system can be interpreted as a series system, and hence, the optimal policy is an echelon base-stock policy. Hsu, Lee & So [22] analyzed a single-period, single-product ATO system where the price of the end product depends on the delivery lead time. The procurement lead time is deterministic and component-specific. They develop an efficient algorithm to find the optimal order quantity for each of the required components.

2.3.2 Discrete-Time, Multiple-Products Model

The earlier works on ATO system focus on documenting the benefits of risk-pooling. Baker, Magazine, & Nuttle [4] studied a single-period, two-product ATO system, in which each product requires a unique component and a common component. The objective is to minimize the total safety stock of the components subject to an aggregated service level constraint. With uniform demand distribution, they show that risk pooling reduces the total inventory level. Besides, they study an alternative problem with the objective to minimize the total inventory level, subject to individual service level constraints. When the stock of the common component is insufficient to meet the demand for both products, an allocation rule that gives priority to the product with the smaller realized demand is proposed. Their analytical results show that under aggregated or individual service level constraints, commonality reduces the total stock but increases the stock level of the unique components. Eynan & Rosenblatt [12] extended the model of Baker et al. [4], but allow the price of the common component to exceed the price of the components that it replaces. They derive the conditions under which introducing commonality would result in lower inventory costs. Hillier [20] showed that when the common component is more expensive than those it replaces, the risk-pooling benefit is soon dominated by the additional purchasing cost in the multiple-periods setting, which implies that to keep total inventory cost and aggregate service level unchanged, the price premium paid for the common component should be lower in the multiple-periods model than that in the single period model. Hiller also derives the lower and upper bounds of the average cost per period. The derived upper bound on the cost can be computed separately by product. Based on this upper bound, a heuristic solution for the base-stock level is derived.

There are several works focusing on finding the optimal inventory control policy. Gerchak, Magazine, & Gamble [15] extended the results of Baker et al [4] to a more general product structure setting and revisit the component allocation problem. The problem is formulated as a two-stage stochastic program. They derive the optimal allocation policy for the two-product model giving priority to the first product with a certain probability. In a similar fashion, Gerchack & Henig [14] considered a multi-period problem with general product configuration and show that under some mild conditions, the optimal solution is a myopic policy. Swaminathan & Tayur [50] formulated a two-stage stochastic linear program to study an ATO system. They treat the second stage (allocation) problem as a standard linear program and apply perturbation analysis to solve the first stage (replenishment) problem. van Mieghem & Rudi [52] analyzed a similar two-stage stochastic programming problem and characterize the optimal inventory policy for the lost sales case.

Because the optimal inventory policy is difficult to find and implement, there are several papers focusing on finding the independent order-up-to levels for each component under some allocation schemes. Hausman, Lee, & Zhang [18] studied a periodic review, multiple-components ATO system with an independent order-up-to policy. They proposed an *equal fractile* heuristic component allocation policy and determine the order-up-to levels. The measure of service level is the fill rate within a time window or the probability of filling a demand within a time interval. Zhang [56] studied a similar system where the objective is to minimize the total inventory cost subject to a service level requirement for each product. The component allocation rule proposed by Zhang is a *fixed priority* rule, under which all the product demands within the same period are assigned a predetermined priority order and the available components are allocated accordingly. Two easy-to-compute lower bounds on the order fill rate are proposed for the multivariate normal distribution. Numerical examples show that neither bound dominates. Agrawal & Cohen [1] investigated the performance of the *fair share* component allocation scheme. Under this scheme, the quantity of the component allocated to a product is determined by the ratio of the realized demand of that product to the total realized demands that require the same component but independent of the availability of the other components. They develop an expression for the order fill rate under this scheme. The objective is to minimize the total inventory cost subject to a product-specific service level constraint. Akcay & Xu [2] considered a multi-components, multi-products, periodic-review ATO system with an independent order-up-to policy for each component. They showed that the component allocation problem is a Multidimensional General Knapsack problem, which is NP-hard. They propose a simple, *order-based* component allocation rule. Extensive numerical examples show that their component allocation rule solves the allocation problem in polynomial time and outperforms other existing allocation heuristics.

2.3.3 Continuous Time Model

For continuous-review, multi-product systems, all research to date assumes independent compound-Poisson demands, which also implies that the demand for each component is a compound-Poisson process. Since ours is a discrete time model, I only review some representative results for the continuous time model. For a more complete review, readers can refer to the survey paper by Song & Zipkin [48].

Song [44] formulated a continuous time ATO system under base-stock control policy. The demand is filled first-come-first-served (FCFS). Demands that cannot be filled immediately are backordered. When a demand arrives and some of its requested components are in stock but some are not, the system either sends the in-stock components or puts them aside as committed inventory. However, a demand is considered backordered until it is satisfied completely. Song provides the formulations to evaluate system performance under the order-based and the component-based controls. Glasserman & Wang [16] modeled the ATO system as a set of M/G/1 queues (G/G/1 queues for the single-product case). Assuming the fill rate remains high, they investigate the trade-off between the delivery time window and the total base-stock levels. Wang [53] extends this model to study a single-product system with an objective to minimize the average inventory cost subject to a fill rate constraint. He develops a closed-form solution for the base-stock level. Song, Xu, & Liu [45] considered a multiple-products, unit-demand model and showed that the outstanding order vector under total order service or partial order service is an irreducible continuous-time Markov chain with finite state spaces. The unique stationary distribution can be obtained through a *quasi-birth-and-death* process. Song & Yao [46] considered a single product ATO system with Poisson demand, iid random lead time for each component. For any given base-stock policy, the problem is reduced to evaluating a set of $M/G/\infty$ queues with a common arrival stream. They showed that lead time variability has a larger impact on the system performance than the demand variability. They also develop heuristics to solve the problem and find that it is better to approximate the lead time uncertainty than to ignore it. Chen, Ettl, Lin & Yao [8] analyzed the inventory-service tradeoff in an ATO system and develop an exact algorithm to solve the special case and a greedy heuristic (to solve the general case). Dayanik, Song, & Xu [10] evaluated several performance bounds for the capacitated ATO systems. Lu, Song, & Yao [26] demonstrated that utilizing advanced information of future demand, the ATO firm can improve performance.

As Song & Zipking [48] mention, “Nearly all the (existing) models assume stationary data,” which implies that the cost parameters are constant and that the components do not become obsolete. To the best of our knowledge, our proposed models would be the first to emphasize some important non-stationary elements in an ATO environment, such as state-dependent cost parameters and component obsolescence due to technology innovations.

Chapter 3

Joint Technology and Inventory Control in a Make-to-Stock System

The Make-to-Stock (MTS) system is commonly used in many industries. The model studied in this Chapter is built upon the previous work by Cohen, Eliashberg, & Ho [9], Bayus, Jain, & Rao [5], and Souza, Bayus, & Wagner [49]. We consider that a firm is planning to introduce new and improved products over the infinite planning horizon. The interarrival time between two consecutive generations of technologies follows an exogenous discrete-time, phase-type distribution, where the phase denotes the current development state of the pipeline technology. At any time, there are two coexisting generations of technologies, labelled as the generation 0 for the latest technology and the generation 1 for the earlier technology. When a technology innovation occurs, the newly released technology is introduced to the market as a generation-0 technology. Subsequently, each previously existing technology becomes one generation older and the generation-2 technology is immediately phased out. The system parameters associated with each technology, such as selling price, procurement cost, and demand distributions are functions of the product generation and the innovation state. At the beginning of each period, the firm learns the cost parameters associated with each available technology. If the firm is offering the generation 0 product, no technology action is taken and the firm only needs to decide how much inventory to order for the current period. If the firm is offering the generation 1 product, then the firm needs to decide whether to upgrade the product or to keep it, and then decide how much inventory to order. If the firm decides to upgrade the product, the initial inventory of the generation 1 technology will be salvaged and that technology will not be re-introduced. We consider the lost sales model and assume that the replenishment lead time is negligible.

The major differences between our model and the aforementioned models are: first, in our model the technology development is exogenous and the introduction of new product depends on its availability, whereas in the aforementioned models, the firm can launch the new product any time; second, in our model the firm is allowed to partially salvage unwanted inventory in each period (the salvage value is less than the current purchase cost) whereas in the aforementioned models, the firm salvages inventory only when the product is discontinued. third, we assume that the demand is generally distributed, whereas in the aforementioned models the demand is treated as discrete random variables. With these differences, the structure of the optimal policy can be explicitly characterized under some conditions.

We formulate a dynamic programming problem that determines the technology and inventory joint control policy that maximizes the total expected discounted profits over the infinite horizon. We characterize the structure of the optimal policy for the single-period problem and then derive the optimal policy explicitly for a special case. When the value loss to slash non-obsolete inventory is significant, we develop a "τ-policy" under which as soon as the development state reaches the pre-specified τ, the upgrading to the new technology is made. In the mean time, the inventory policy follows a two-limit policy.

3.1 Model Formulation

3.1.1 Technology Innovation and Co-existing Technologies

We consider that a firm is planning to introduce new and improved products over the infinite planning horizon. The introduction of the next generation products depends on the availability of new generation technologies. We assume that the time between two consecutive generations of technologies follows an exogenous *discrete phase-type* (PH) distribution. A phase-type distribution is a natural representation of an innovation process, as the development of a new technology usually undergoes several phases with each phase lasting a geometrically distributed amount of time. More specifically, suppose the innovation consists of d development phases. Denote $d+1$ as the completion state, that is, as soon as the innovation process visits state $d+1$, the pipeline technology is released to the market and the firm needs to decide whether to adopt it. Let $p_{tt'}$ be the probability that the development of the pipeline technology moves from phase t to phase t' in one period, $t, t' = 1, 2 \ldots, d$. For expositional simplicity, we let $p_{tt'} = 0$ if $t' < t$, which implies that the phase of a pipeline technology is non-decreasing as time elapses. Let $q_t \equiv 1 - \sum_{t'=t}^{d} p_{tt'}$, which is understood as the probability that an innovation occurs in the next period, given the development phase is in t in the current period. Consistent with the standard definition of the discrete-time PH distribution (Neuts [33]), we treat $d+1$ as an *instantaneous* state, that is, as soon as the process enters state $d+1$, it instantaneously jumps to state t with probability α_t, $\sum_{t=1}^{d} \alpha_t = 1$. In other words, as soon as an innovation occurs, the development of the next pipeline technology immediately re-starts in phase t with probability α_t, $t = 1, \ldots, d$. The interarrival time of new technologies is a discrete phase-type random variable, denoted by Γ, with representation $(\{\alpha_t\}, \{p_{tt'}\})$.

The PH distribution is a powerful tool to characterize a stochastic technology innovation process. It is known that any distribution can be represented by a PH distribution by choosing appropriate values for $(\{\alpha_t\}, \{p_{tt'}\})$ and d. For example, when $d = 1$, Γ is geometrically distributed; if we let $d = 2$, $(\alpha_1, \alpha_2) = (1, 0)$, and $\{p_{tt'}\} = \begin{pmatrix} p_1 & 1-p_1 \\ 0 & p_2 \end{pmatrix}$, then Γ follows a (generalized) negative binomial distribution of order 2 with parameters p_1 and p_2.

Let T^n be the development phase of a pipeline technology at time n, $T^n \in \{1, 2, \ldots, d\}$. The one-step transition probabilities of $\{T^n, n \geq 0\}$ satisfy

$$p(t'|t) = P(T^{n+1} = t'|T^n = t) = p_{t,t'} + \alpha_{t'} q_t, \quad t, t' = 1, 2, \ldots, d, \tag{3.1}$$

where, by assumption, $p_{t,t'} = 0$, if $t' < t$. The above probability means that the innovation process can visit state t' from state t either via incremental phase change, which occurs with probability $p_{t,t'}$, $t' \geq t$, or via innovation, which occurs with probability $\alpha_{t'} q_t$.

Next, consider the existing technologies. We assume that only the latest two generation technologies can co-exist. Of these two generations, we label the latest technology as the generation 0 technology and the earlier technology as the generation 1 technology. When an innovation occurs, the newly released technology is introduced to the market as the generation-0 technology. Subsequently, the product that was labelled as a generation-0 product in the last period will become a generation-1 product, whereas the product that was labelled as a generation-1 product in the last period will become obsolete and will be salvaged immediately.

One can also model the technology breakthrough by including multiple "instantaneous" states. For example, the new technology could achieve either a small or large improvement over the existing technology. We can define two instantaneous states, say, state $d+1$ for small improvement and state $d+2$ for large improvement. Then we need a two-dimensional notation to describe the system state, that is, a phase index to indicate the phase of pipeline technology and an improvement index to indicate the improvement between the generation-0 and generation-1 technologies. When the system enters the instantaneous state $d+1$ $(d+2)$, the innovation process re-starts according to the initialization vector and the improvement index takes a small (large) value.

3.1.2 The Joint Technology and Inventory Control System

We assume that the cost parameters are dependent on the generation and development phase of the product. For convenience, we call a product a type $t-b$ product if it uses a generation-b $(b=0,1)$ technology and the development phase is in phase t. Let $r_b(t)$ and $c_b(t)$ be the selling price and procurement cost of a type $t-b$ product, where $r_b(t) \geq c_b(t)$. Let $s_b(t)$ be the salvage value of a type $t-b$ product. Then $c_b(t) - s_b(t)$ (≥ 0) is the difference between the current procurement cost and the salvage value, which can be understood as the unit cost to reduce the type $t-b$ inventory. A technology will become obsolete as soon as its generation index reaches 2, and its inventory will be salvaged immediately with zero value. We assume that $c_b(t)$ is decreasing in t and $c_0(t) \geq c_1(t) > 0$, for any t.

We use $a \in \{0, 1\}$ to indicate the product generation the firm offers currently and use x to represent its initial inventory. The state of the system at the beginning of the period is denoted by (t, a, x). The lead time is assumed to be zero. We write b as the product generation *after* the technology decision, where $b \in \{0, 1\}$ and $b \leq a$, meaning that technology downgrading is not allowed. Let y be the inventory level after replenishment. Then, the state of the system *after* technology and inventory decisions is represented by (t, b, y). For convenience, we write $(\cdot)^+ = \max(0, \cdot)$. We describe the cost due to the technology and inventory decisions as follows. If $a = b$, the firm does not change the product technology in the current period. Then the firm incurs cost $c_b(t)(y - x)^+ - s_b(t)(x - y)^+$, where the first term is the procurement cost the firm pays if $y \geq x$ and the second term is the salvage revenue the firm receives if $y < x$. If $a = 1$ and $b = 0$, the firm upgrades to the new technology. Then the firm incurs the cost $c_0(t)y - s_1(t)x$, where the first term represents the procurement cost for ordering inventory $y > 0$ and the second term represents the salvage revenue received from the discontinued generation 1 inventory x.

Note that the "partial salvage" can be achieved in many ways. For example, the firm while commanding sufficient market power, might have an agreement with its supplier or a third party to return/sell the unwanted inventory. It is also possible that there exists a secondary market where the firm can trade the inventory with negligible transaction fee. The firm can also temporarily mark down the price at the beginning of the period, i.e., using the "kick-off" sale or "early-bird-special" promotion with limited quantity as discussed in Petruzzi and Dada [35]. When the market size is sufficiently large, the firm can reduce inventory level to a desired level. And sometimes it could recoup current procurement cost for the initial inventory.

The demand of a type $t-b$ product is denoted by $D_b(t)$. For any given b and t, let $F_b(t,\cdot)$ be the cumulative distribution function of $D_b(t)$ and denote $f_b(t,\cdot)$ as its density function. Unmet demand is lost.

3.1.3 Dynamic Programming Formulation

Let $Z_b(t,y) = \min[y, D_b(t)]$ be the sales of the type $t-b$ product in the current period. Given that the system state is adjusted from state (t,a,x) to (t,b,y), we call $G(t,a,x;b,y)$ the single-period expected profit. We have, for $a \geq b$,

$$G(t,a,x;b,y) = \begin{cases} r_b(t)E[Z_b(t,y)] - c_b(t)(y-x)^+ + s_b(t)(x-y)^+, & a=b, \\ r_0(t)E[Z_0(t,y)] - c_0(t)y + s_1(t)x, & a=1,\ b=0. \end{cases} \tag{3.2}$$

Let $V(t,a,x)$ be the attainable maximal expected total discounted profit over the infinite planning horizon with the starting state (t,a,x) and discount factor $0 \leq \theta < 1$. The optimality equation satisfies

$$V(t,a,x) = \max_{\substack{a\geq b\\ y\geq 0}} \left\{ \begin{array}{l} G(t,a,x;b,y) + \theta \sum_{t'\geq t} p_{t,t'} E[V(t',b,(y-D_b(t))^+)] \\ +\theta q_t \sum_{t'} \alpha_{t'} E[V(t',b+1,(y-D_b(t))^+)] \end{array} \right\} \tag{3.3}$$

$$= \max_{a\geq b,\ y\geq 0} \{w(t,a,x;b,y)\}. \tag{3.4}$$

Inside the optimization operator of (3.3), the first term is the expected profit in the current period, and the second and third terms are the future profits conditioning on whether an innovation occurs or not in the next period. For convenience, we define

$$V(t;2,x) = V(t;1,0), \forall x \geq 0,$$

since the initial inventory of the obsolete product has zero salvage value.

3.2 The Single-Period Analysis

This section considers the joint optimal technology and inventory decisions for the single (final) period model. We first obtain the optimal inventory policy for fixed b. We then consider how to determine the optimal technology decision provided that the inventory level will be fixed at the optimal level for each technology decision.

At the end of the planning horizon, we assume that the remaining inventory is salvaged at its terminal value. We compute the expected terminal value of a type $t-b$ inventory as follows. Recall that with probability $p_{t,t'}$, no innovation occurs, and the development process moves to state t' from state t in the next period, with the terminal value of a type $t-b$ product $s_b(t')$. With probability $q_t\alpha_{t'}$, an innovation occurs and the next pipeline technology restarts in phase t'. Then, the current $t-b$ product will become a type $t'-(b+1)$ product and has the terminal value $s_{b+1}(t')$. Thus, the expected discounted terminal value of each unit of the leftover type $t-b$ inventory is

$$h_b(t) = \theta \sum_{t'=t}^{d} p_{t,t'} s_b(t') + \theta q_t \sum_{t'=1}^{d} \alpha_{t'} s_{b+1}(t'), \tag{3.5}$$

where $s_2(t) = 0$ for any t. It can be easily shown that $h_b(t) \leq s_b(t)$.

Evidently, $c_b(t) - h_b(t) \geq 0$ is the overage cost of a type $t-b$ product, which equals the expected value depreciation due to the pipeline technology development. For expositional purpose, our formulation does not include the one-period warehousing cost for the leftover inventory. However, such a cost can be easily incorporated into our model. The final period profit function is

$$g(t, a, x; b, y) = G(t, a, x; b, y) + h_b(t)E(y - Z_b(t, y)), \quad b = 0, 1. \tag{3.6}$$

In the remainder of this section, we suppress t in the notation when convenient; for example, $F_b(t, y)$ is written as $F_b(y)$.

3.2.1 The Optimal Inventory Policy for a Selected Technology

In this subsection we determine the optimal inventory policy for a given technology decision b. If $a = 1$ and $b = 0$, the initial inventory is salvaged and we find that

$$g(1, x; 0, y) = r_0 E[Z_0(y)] - c_0 y + h_0(y - E[Z_0(y)]) + s_1 x, \tag{3.7}$$

which is concave in y with given x, and the optimal inventory is the well-known newsvender solution. We focus on the non-trivial case $a = b$. From (3.6), we get for $a = b$,

$$g(a, x; b, y) = (r_b - h_b)E[Z_b(y)] - c_b(y - x)^+ + s_b(x - y)^+ + h_b y. \tag{3.8}$$

Clearly, each term is concave in y with given (b, x). From (3.8), we find that the first derivative with respect to y satisfies

$$\frac{\partial g(a,x;b,y)}{\partial y} = \begin{cases} (r_b - h_b)(1 - F_b(y)) - c_b + h_b & \text{if } y > x \\ (r_b - h_b)(1 - F_b(y)) - s_b + h_b & \text{if } y < x. \end{cases} \tag{3.9}$$

Define $k_b = \frac{r_b - c_b}{r_b - h_b}$, the newsvender ratio for product b with underage cost $r_b - c_b$ and overage cost $c_b - h_b$. Also let $k_b' = \frac{r_b - s_b}{r_b - h_b}$. We define the following identities:

$$y_b = F^{-1}(k_b), \text{ and } y_b' = F^{-1}(k_b'). \tag{3.10}$$

Because $c_b \geq s_b > h_b$, we have $y_b \leq y_b' < \infty$.

Obviously, if $y_b \geq x$, then y_b maximizes $g(a, x; b, y)$. If $y_b' \leq x$, then y_b' maximizes $g(a, x; b,y)$. If $y_b < x < y_b'$ and $y < x$, which implies $y < y_b'$, then the second expression of (3.9) is positive. On the other hand, if $y_b < x < y_b'$ and $x < y$, which implies $y > y_b$, then the first expression of (3.9) is negative. Therefore, if $y_b < x < y_b'$, then $g(a, x; b, y)$ is maximized at $y = x$. In summary, the optimal procurement level, when $a = b$, denoted by $y_{(1)}(b, x; b)$, is given by

$$y_{(1)}(b,x,b) = \begin{cases} y_b, & \text{if } x \leq y_b, \\ x, & \text{if } y_b < x < y_b', \\ y_b', & \text{if } x \geq y_b'. \end{cases} \tag{3.11}$$

Equation (3.11) means that if the salvage value of an item is greater than its discounted net worth in the next period, the optimal ordering policy follows a two-limit control policy. Specifically, if the initial inventory is below the newsvender level, it is optimal to raise the inventory to the newsvender level; if the initial inventory x is greater than y_b', then the overage cost exceeds the salvage value loss, and, thus, it is optimal to slash the inventory down to y_b'; finally, if the initial inventory is between y_b and y_b', then slashing the inventory is more costly than keeping it; thus, it is optimal to keep the inventory level unchanged. This result is parallel to the Invest-Stay-Disinvest (ISD) policy of Eberly and Van Mieghem [11].

Substituting y_b into (3.8), we can write for $a = b$,

$$\begin{aligned} g^*(a,x;b) &= \max_{y\geq 0} g(a,x;b,y) = g(a,x;b,y_{(1)}(b,x;b)) \\ &= \begin{cases} g(a,0;b,y_b) + c_b x, & \text{if } x \leq y_b, \\ (r_b - h_b)E[Z_b(x)] + h_b x, & \text{if } y_b < x < y_b', \\ g(a,y_b';b,y_b') + s_b(x - y_b'), & \text{if } x \geq y_b'. \end{cases} \end{aligned} \tag{3.12}$$

By checking the derivative at points $x = y_b$ and $x = y_b'$, one can show that $g^*(a,x;b)$ is differentiable at these two points. Then one can further show that $g^*(a,x;b)$ is differentiable and concave for any $x \geq 0$. We will use this property later.

3.2.2 The Optimal Technology Decision

In this section we determine the optimal technology decision, assuming that the optimal inventory policy will be implemented after the technology is selected. We again suppress t for notation simplicity. The optimal technology selection problem can be expressed as

$$V_{(1)}(a,x) = \max_{a \geq b}\{g^*(a,x;b)\}, \tag{3.13}$$

where $V_{(1)}(a,x)$ is the optimal single-period profit attainable with starting state (a,x).

Since technology downgrading is not allowed, when $a = 0$ the firm has to keep the technology unchanged (i.e. $b^* = 0$ when $a = 0$). We focus on the case with $a = 1$, where the firm needs to decide whether to upgrade to the new technology or continue to use the old technology. According to (3.7), if the firm decides to upgrade, the optimal inventory level is $y_0 = F^{-1}(k_0)$ and $g^*(1,x;0) = g^*(1,0;0) + s_1 x$ (We shall derive the closed form for $g^*(1,0;0)$ in Section 3.4 with normally distributed demand distribution).

Theorem 3.1. *Given the current state $(a,x) = (1,x)$, the optimal technology decision is characterized by a threshold policy: there exists a unique number $\delta_{(1)} \geq 0$ such that it is optimal to upgrade to the generation-0 technology if and only if $x < \delta_{(1)}$.*

Proof. Given the current state $(1,x)$, the single-period optimization problem is

$$V_{(1)}(1,x) = \max\{g^*(1,x;0),\ g^*(1,x;1)\} = \max\{g^*(1,0;0) + s_1 x,\ g^*(1,x;1)\},$$

From (3.12), it can be verified that

$$\frac{\partial g^*(1,x;1)}{\partial x} = \begin{cases} c_1, & \text{if } x \le y_1, \\ (r_1 - h_1)(1 - F_1(x)) + h_1, & \text{if } y_1 < x < y_1', \\ s_1, & \text{if } x \ge y_1'. \end{cases} \tag{3.14}$$

Define $\Delta(x) = g^*(1,0;0) + s_1 x - g^*(1,x;1)$. Because $c_1 \ge \frac{\partial g^*(1,x;1)}{\partial x} \ge s_1$, we immediately see that $\Delta(x)$ is non-increasing in x, and it crosses zero at most once. Let $\delta_{(1)}$ be the root of $\Delta(x) = 0$, where $\delta_{(1)} \equiv 0$ if $\Delta(0) < 0$ and $\delta_{(1)} \equiv \infty$ if $\Delta(\infty) \ge 0$.

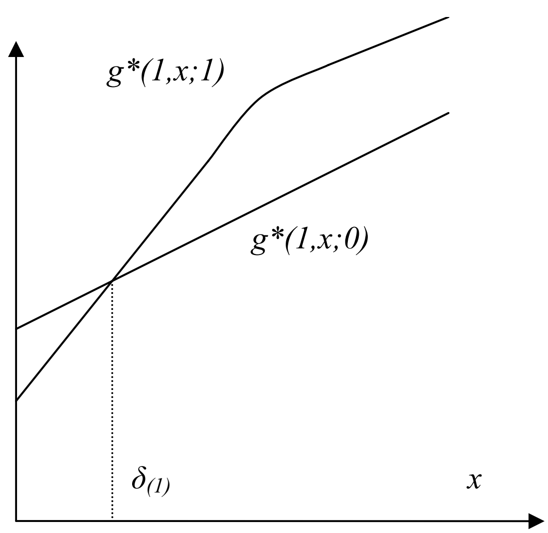

Fig. 3.1. Optimal Technology Decision

Clearly, and also can be seen from Figure 3.1, it is optimal to upgrade to the new technology if and only if x is less than $\delta_{(1)}$. In the special case $\delta_{(1)} = 0$ ($\delta_{(1)} = \infty$), it is never (always) optimal to upgrade to the new technology regardless of the initial inventory x. ■

Combining (3.11) and Theorem 3.1, we see that the *joint* optimal technology selection and inventory replenishment policy operates as follows. The firm upgrades to the generation-0 technology if and only if the initial inventory satisfies $x < \delta_{(1)}$. After the optimal technology decision is made, the inventory ordering policy follows the two-limit threshold policy described by (3.11).

In addition, from Figure 3.1, we can see that $V_{(1)}(a, x)$ is strictly increasing in x. This property of $V_{(1)}(a, x)$ is formalized in the following Lemma.

LEMMA 3.1. *The single period profit function $V_{(1)}(a, x)$ is strictly increasing in x for fixed a and satisfies $c_1 \geq \frac{\partial V_{(1)}(a,x)}{\partial x} \geq s_1$. Furthermore,*

$$\lim_{x \to \infty} \frac{\partial V_{(1)}(a, x)}{\partial x} = s_a.$$

Proof. By definition,

$$V_{(1)}(a, x) = \begin{cases} \max\{g^*(1, 0; 0) + s_1 x, \ g^*(1, x; 1)\}, & \text{if } a = 1, \\ g^*(0, x; 0), & \text{if } a = 0. \end{cases}$$

It is trivial to see that $g^*(t, 1, 0; 0) + s_1 x$ is strictly increasing in x. From (3.12), we note that $g^*(b, x; b)$ is strictly increasing in x for $b = 0, 1$. After taking the maximum, we conclude that $V_{(1)}(a, x)$ a strictly increasing function of x (refer to Figure 3.1). Because

$y_{(1)}(b,x,b)$ is not a function of x when x is sufficiently large (i.e., $x \geq y'_b$, where y'_b is defined in (3.10)), it can be seen from (3.12) that $\lim_{x\to\infty} \frac{\partial V_{(1)}(a,x)}{\partial x} = s_a$. ■

3.3 The Infinite-Period Analysis

In this section, we extend the analysis to the infinite-period model. We will show that the structure of the optimal inventory and technology upgrading policies resemble their counterparts in the single period model under certain condition on the demand distribution. We append t to the notation.

Define $y^*(t,a,x;b)$ as the optimal inventory level that maximizes $w(t,a,x;b,y)$ with fixed (t,a,x,b), where $w(t,a,x;b,y)$ is given in (3.4). We truncate the infinite-period model into the n-period model. Let $V_{(n)}(t,a,x)$ be the optimal value function with state (t,a,x) and n periods remaining. The optimality equation is

$$\begin{aligned} V_{(n+1)}(t,a,x) &= \max_{y\geq 0, b\leq a} \left\{ \begin{array}{l} G(t,a,x;b,y) + \theta \sum_{t'\geq t} p_{t,t'} E[V_{(n)}(t',b,(y-D_b(t))^+)] \\ +\theta q_t \sum_{t'} \alpha_{t'} E[V_{(n)}(t',b+1,(y-D_b(t))^+)] \end{array} \right\} \\ &= \max_{y\geq 0, b\leq a} \{w_{(n+1)}(t,a,x;b,y)\}. \end{aligned}$$

It should be noted that $w_{(n+1)}(t,a,x;b,y)$ is not necessarily quasi-concave or concave in y with fixed (t,a,x,b). For example, consider the 2-period model. By (3.2), $G(t,a,x;b,y)$ is strictly concave in y for fixed $(t,a,x;b)$. In addition, both $E[V_{(1)}(t',b,(y-D_b(t))^+)]$ and $E[V_{(1)}(t',b+1,(y-D_b(t))^+)]$ are strictly increasing and quasi-concave in y, by Lemma 3.1. Unfortunately, $w_{(2)}(t,a,x;b,y)$ which is the sum of three quasi-concave functions, is not necessarily a quasi-concave function (see Silverman [43], page 15).

3.4 Special Case: The Zero Salvage Loss Condition

We analyze the structure of the optimal policy under the condition that the firm can salvage excessive inventory at the current procurement cost, i.e., $c_j(t) = s_j(t)$ for any j and t. We call it the *zero salvage loss* condition. In many cases, the zero salvage loss condition can be achieved. For example, the firm, commanding sufficient market power, might have an agreement with its supplier or a third party to return/sell the unwanted inventory at the current procurement cost. It is also possible that there exists a secondary market for the discounted product, where the firm can trade the inventory with the current procurement cost with the negligible transaction fee.

When $c_j(t) = s_j(t)$ for any j and t, $G(t, a, x; b, y)$ in (3.2) reduces to

$$G(t, a, x; b, y) = r_b(t)E[Z_b(t, y)] - c_b(t)y + c_a(t)x. \tag{3.15}$$

In addition, $V(t, a, x) = V(t, a, 0) + c_a(t)x$. Hence, the optimality equation (3.3) becomes

$$V(t, a, x) = c_a(t)x + \max_{\substack{a \geq b \\ y \geq 0}} \left\{ \begin{array}{l} r_b(t)E[Z_b(t, y)] - c_b(t)y + h_b(t)E[(y - D_b(t))^+] \\ +\theta \sum_{t' \geq t} p_{t,t'} E\left[V(t', b, 0)\right] + \theta q_t \sum_{t'} \alpha_{t'} E\left[V(t', b+1, 0)\right] \end{array} \right\}, \tag{3.16}$$

where $h_b(t)$, defined by (3.5), is the expected net worth of the product in state (t, b) by the end of the period. Let

$$g(t; b, y) = (r_b(t) - c_b(t))E[Z_b(t, y)] - (c_b(t) - h_b(t))y,$$

be the newsvender function with profit margin $r_b(t)-c_b(t)$ and overage cost $c_b(t)-h_b(t)$. So $y_b(t)$, defined (3.10), maximizes $g(t;b,y)$.

3.4.1 Profit Per Unit

We assume that $D_b(t)$ follows normal distribution with mean $\lambda_b(t)$ and standard deviation $\sigma_b(t)$ for any given t and b. For expositional simplicity, we suppress t when convenient. Let $\Phi(\cdot)$ be the cumulative distribution function of a standard normal variable and denote $\phi(\cdot)$ as its density function. Using the facts that for a standard normal random variable Z, $E[\min(z,Z)] = -\phi(z)+z(1-\Phi(z))$, and that $\frac{y_b-\lambda_b}{\sigma_b} = \Phi^{-1}(k_b)$, we have

$$\begin{aligned} E[Z_b(y_b)] = E[\min\{y_b, D_b\}] &= \sigma_b E[\min\{z_b, Z\}] + \lambda_b \\ &= -\sigma_b\phi(z_b) + \frac{y_b(c_b-h_b)}{r_b-h_b} + \frac{\lambda_b(r_b-c_b)}{r_b-h_b}. \end{aligned}$$

Substituting the above expression into (3.8), we can write for any $a=b$ and $x=0$,

$$g^*(b) = \max_{y\geq 0}\{g(b,y)\} = g(b,y_b) = (r_b-c_b)\lambda_b - (r_b-h_b)\sigma_b\phi(z_b) = \lambda_b R_b,$$

where

$$R_b = (r_b-c_b)(1-\frac{\sigma_b\phi(z_b)}{\lambda_b}) - (c_b-h_b)\frac{\sigma_b\phi(z_b)}{\lambda_b},\ \text{b=0,1}, \qquad (3.17)$$

is called the *profit per unit* for the generation-b product, $b=0,1$. Once we know R_b, the expected profit (with zero initial inventory) can be easily computed by taking the product of R_b and the expected demand.

We can see that the *profit per unit* consists of two parts: the first term is the profit margin multiplied by $1 - \frac{\sigma_b \phi(z_b)}{\lambda_b}$, and the second term is the overage cost multiplied by $\frac{\sigma_b \phi(z_b)}{\lambda_b}$. Note that $\frac{\sigma_b}{\lambda_b}$ is the coefficient of variation. As such, when keeping other parameters unchanged, a larger value of $\frac{\sigma_b}{\lambda_b}$ (which corresponds to a higher level of variability) leads to a lower expected profit. As we can see, the *profit per unit* is a comprehensive measure that takes into account the profit margin, service level, overage cost, and demand uncertainty together.

3.4.2 Myopic Policy

Hereafter, we expand the notation on t. Empirical evidences support that the new technology development (i.e., when t increases) often results in a lower profitability for the old technology. Because of the generational difference, it is plausible that the difference $\lambda_0(t)R_0(t) - \lambda_1(t)R_1(t)$ is increasing in t.

LEMMA 3.2. *Under the condition that* $\lambda_0(t)R_0(t) - \lambda_1(t)R_1(t)$ *is increasing in* t*, there exists a unique threshold* t^* *such that for the single-period unconstrained optimization problem*

$$\bar{g}^*(t; b) = \max_{b=0,1} \{\lambda_0(t)R_0(t),\ \lambda_1(t)R_1(t)\},$$

the optimal technology selection is when $t < t^*$, $\bar{b}_t = 1$*; when* $t^* \leq t$, $\bar{b}_t = 0$.

The proof of Lemma 3.2 is straightforward and is omitted. In the special case when $t^* = d + 1$, we always offer generation-1 product (i.e., when on hand inventory is obsolete, upgrade to the second new product); when $t^* = 1$, we always offer the generation-0 product (i.e., adopt the newest technology whenever there is an innovation).

We define the *myopic* policy for the unconstrained problem, represented by $(\bar{b}_t, \bar{y}_t)$, as follows.

$$(\bar{b}_t, \bar{y}_t) = \begin{cases} (1, y_1(t)), & \text{if } t < t^*, \\ (0, y_0(t)), & \text{if } t \geq t^*. \end{cases} \tag{3.18}$$

Under the myopic policy, when the development state is $t < t^*$, we offer the generation-1 product and adjust the inventory level to $y_1(t)$; when the development state is $t \geq t^*$, we offer the generation-0 product and adjust the inventory level to $y_0(t)$.

THEOREM 3.2. *The optimal policy for the constrained problem (3.16) is*

$$(b_t^*, y_t^*) = \begin{cases} (0, y_0(t)), & \textit{if } t < t^* \textit{ and } a = 0, \\ (\bar{b}_t, \bar{y}_t) & \textit{otherwise.} \end{cases} \tag{3.19}$$

Proof: We truncate the planning horizon into n periods. Let $V_{(n)}(t, a)$ be the optimal objective value function with state (t, a) and n periods remaining. First, when $n = 1$, we have

$$V_{(1)}(t, a, x) = c_a(t)x + \max_{a \geq b, y > 0} \{g^*(t, b, y)\}.$$

According to Lemma 3.2, if $a = 1$, or $t \geq t^*$ and $a = 0$, then the myopic policy is feasible for $V_{(1)}(t, a)$, thus, is optimal for $V_{(1)}(t, a)$; if $t < t^*$ and $a = 0$, the only choice the firm has is to keep the generation-0 product, so $b_t^* = 0$ and $y_t^* = y_0(t)$.

Next, we hypothesize that (3.19) is optimal for $V_{(n)}(t,a,x)$ and consider when there are $n+1$ periods remaining. The optimality equation is

$$V_{(n+1)}(t,a) = c_a(t)x + \max_{a \geq b, y > 0} \left\{ \begin{array}{l} g^*(t;b,y) + \theta \sum_{t' \geq t} p_{t,t'} E[V_{(n)}(t',b)] \\ +\theta q_t \sum_{t'} \alpha_{t'} E[V_{(n)}(t',b+1)] \end{array} \right\}.$$

Obviously, if $t < t^*$ and $a = 0$, the firm must continue to offer the generation-0 product, so the optimal policy is $(0, y_0(t))$. If $a = 1$, or $t \geq t^*$ and $a = 0$, then the myopic policy is feasible for $V_{(n+1)}(t,a)$ and maximizes $g^*(t;b,y)$. Now consider the next period. If there is no innovation and the system enters state t', then since $t' \geq t$ (recall $p_{t,t'} = 0$ if $t' < t$), the myopic technology selection decision $\bar{b}_{t'}$ is again feasible for $V_{(n)}(t', b_t)$, since $b_t \geq b_{t'}$. By our hypothesis, the myopic policy is optimal from the next period onward. If there is an innovation in the next period and the system enters state $(t', 1)$, the myopic policy is feasible and optimal for $V_{(n)}(t', 1)$. Clearly, once the myopic policy is feasible, it becomes feasible, and thus optimal, for the rest of the planning horizon. Therefore, (3.19) is optimal for $V_{(n+1)}(t,a,x)$. Finally, letting $n \to \infty$, and using the fact that $\lim_{n\to\infty} V_{(n+1)}(t,a,x) \to V(t,a,x)$, we conclude that (3.19) is optimal for $V(t,a,x)$. ■

We say that the system is in a *balanced* state if $a = 1$, or $t \geq t^*$ and $a = 0$. As noted, once the myopic policy is feasible, it will be feasible forever and keep the system in a balanced state. When the system is in a balanced state, the optimal policy to choose the technology indeed has a threshold structure: when the development state reaches a certain threshold, it is optimal to discontinue the old product and upgrade to the new one. Such a threshold policy has been widely supported by empirical evidences.

One can also show that the zero salvage loss condition improves the firm's profit, since it reduces the overage cost. Furthermore, it also simplifies the joint optimization of technology management and inventory replenishment. On the other hand, the supplier can also benefit from engaging in such a contract, as Webster and Weng [54] and Li, Xu, and Hayya [25] show that the inventory return agreement encourages the firm to place larger order quantities.

REMARK 1. *It should be mentioned that if there is a fixed cost to upgrade, say K, the problem can be reduced to a machine replacement problem as the following.*

$$V(t,1) = \max \left\{ \begin{array}{l} -K + g^*(t;0) + \theta \sum_{t' \geq t} p_{t,t'} V(t',0) + \theta q_t \sum_{t'} \alpha V(t',1), \\ g^*(t;1) + \theta \sum_{t' \geq t} p_{t,t'} V(t',1) + \theta q_t \sum_{t'} \alpha V(t',2) \end{array} \right\}$$

and

$$V(t,2) = \max \left\{ \begin{array}{l} -K + g^*(t;0) + \theta \sum_{t' \geq t} p_{t,t'} V(t',0) + \theta q_t \sum_{t'} \alpha V(t',1), \\ -K + g^*(t;1) + \theta \sum_{t' \geq t} p_{t,t'} V(t',1) + \theta q_t \sum_{t'} \alpha V(t',2) \end{array} \right\}$$

When $\lambda_0(t)R_0(t) - \lambda_1(t)R_1(t)$ is increasing in t (in other words, $g^*(t;1) - g^*(t;1)$ is increasing in t), it is well-known that there exists a threshold $\tilde{t}$ ($\tilde{t} \geq t^*$, where t^* is defined in Lemma 3.2) such that it is optimal to upgrade if and only if $t \geq \tilde{t}$ (Ross [41]).

3.5 A Phase-Based Heuristic

When $c_j(t) > s_j(t)$, at the current stage, the structure of joint optimization is unclear for arbitrary demand distributions. Using the insights from Section 3.4, we propose a "τ-policy", under which as soon as the development state reaches the pre-specified τ, the upgrading to the new technology is made. In the mean time, the inventory policy is optimized by solving the following dynamic programming problem.

$$\begin{aligned} V(t,a,x;\tau) &= \max_{y\geq 0}\left\{\begin{array}{l} G(t,a,x;b(t),y) \\ +\theta\sum_{t'} p_{t,t'\geq t} E[V(t',b(t),(y-D_{b(t)}(t))^{+};\tau)] \\ +\theta q_t \sum_{t'} \alpha_{t'} E[V(t',b_0(t)+1,(y-D_{b(t)}(t))^{+};\tau)] \end{array}\right\} \qquad (3.20) \\ &= \max_{y\geq 0}\{w(t,a,x;b(t),y)\}. \end{aligned}$$

where $b(t) = 1$ if $t \leq \tau - 1$, otherwise $b(t) = 0$. We iterate all possible τ $(1 \leq \tau \leq d)$ and choose τ^* that maximizes the time-average profit $V(\tau) = E[V(t,0,0;\tau)]$, where we let θ approach to 1.

Under a given "τ-policy", since the assortment decision is independent of x, we have

$$V(t',a,x;\tau) = \begin{cases} w^*(t',a,x;1) & \text{if } x < \tau, \\ w^*(t',a,x;0) & \text{if } t \geq \tau. \end{cases} \qquad (3.21)$$

where $w^*(t', a, x; b) = \max_{y \geq 0}\{w(t', a, x; b(t), y)\}$. We can re-write $w(t, a, x; b(t), y)$ as follows.

$$\begin{aligned} w(t, a, x; b(t), y) &= G(t, a, x; b(t), y) + \theta \sum_{t'} p_{t,t'} E[w^*(t', b(t), (y - D_{b(t)}(t))^+; b(t'))] \\ &\quad + + \theta q_t \sum_{t'} \alpha_{t'} E[w^*(t', b(t) + 1, (y - D_{b(t)}(t))^+; b(t'))]. \end{aligned}$$

The next theorem characterizes the structure of the optimal inventory policy for (3.20) with fixed τ.

THEOREM 3.3. *For any given τ, the optimal inventory policy for (3.20) is a two-limit policy. That is, there exit two unique numbers $y_b^L(t, \tau) \leq y_b^H(t, \tau)$ such that when no upgrade is made in the current period, the optimal inventory level is determined by*

$$y^\tau(t, x) = \begin{cases} y_b^L(t, \tau), & \text{if } x \leq y_b^L(t, \tau), \\ x, & \text{if } y_b^L(t, \tau) < x < y_b^H(t, \tau), \\ y_b^H(t, \tau), & \text{if } x \geq y_b^H(t, \tau). \end{cases} \tag{3.22}$$

and $y^\tau(t, x) = y_0^L(t, \tau)$ is independent on x when the current technology is upgraded in the current period.

The proof is presented in the Appendix A.2. In the next chapter, we will develop a PB heuristic that is similar to this "τ-heuristic". We believe that the large scale numerical experiments in the next chapter will reveal the conditions under which the "τ-heuristic" has a good performance. So we did not perform numerical experiments to test the "τ-heuristic" in this chapter.

Chapter 4

Joint Assortment and Inventory Control in a Make-to-Stock System

In previous Chapter, we analyze a Make-to-Stock inventory system in which the firm markets a *single* generation product in each time period. While different generation products may have overlapping life cycles, it is possible that the firm might offer multiple generation products, or an assortment, in each time period. In this chapter, we extend the analysis in Chapter 3 to allow the firm to offer an assortment of products (i.e., with up to two generation products) in each time period. With a given assortment, we model the customer choice process using a multi-nomial logit (MNL) model. At the beginning of each period, the firm learns the cost parameters associated with each available technology. The firm needs to jointly decide which product(s) should be included in the assortment and the inventory level for each variant in the assortment. If the firm decides to remove a product from the initial assortment, the inventory of that product will be salvaged immediately and the discontinued product will not be re-introduced in the future. We consider the lost sales model and assume that the replenishment lead time is negligible. When the firm can offer an assortment of products in each time period, the tradeoff between introducing the new product and discontinuing the old one becomes more delicate. For example, if the firm chooses not to introduce the new generation product, it loses the tech-savvy customers who usually are willing to pay for the newest product. If the firm chooses to discontinue the old generation product, it incurs the salvage value loss particularly when the old product is still popular. If the firm decides to offer multiple generation products in the same period, then the internal competition among different generation products arise and the task of inventory control becomes more complex.

We formulate a discounted dynamic programming problem that determines the joint assortment and inventory control policy to maximize the total expected discounted profits over the infinite horizon. We characterize the structure of the jointly optimal policy for the single-period problem. In a special case where the salvage loss of unwanted inventory is immaterial and the coefficients of variation of demands for both products are independent of assortment, we show that the optimal replenishment order quantity is the myopic and equals the newsvendor solution. We also show that the optimal assortment evolves in a "three-stage" pattern. When the value loss to salvage unwanted inventory is positive, the objective function is not necessarily concave or quasi-concave. Based on the structural property obtained in the special case, we propose several heuristics for the joint assortment and inventory control. Numerical experiments show that the sequential optimization heuristic outperforms others and suggest that for it is important for the firm with a low profit margin to jointly optimize assortment and inventory control than for a firm with a high profit margin. In particular, the joint optimization becomes more important when demand variability is high or technology innovation is fast.

4.1 Model Formulation

4.1.1 The Joint Technology and Inventory Control System

We consider the same Make-to-Stock system studied in Chapter 3 except that we allow the firm to offer one or both generation products. We use $a_j \in \{0, 1, 2\}$ to indicate the initial assortment status of the generation-j product. If the current generation-j product has not been introduced, $a_j = 0$; if it was offered in the previous period, $a_j = 1$; if it has been discontinued in the past, $a_j = 2$. Let x_j ($j = 0, 1$) represent the initial inventory level of the generation-j product *before* technology and inventory decisions. We write $\mathbf{x} = (x_0, x_1)$ as the initial inventory vector and $\mathbf{a} = (a_0, a_1)$ as the initial assortment vector. We denote the state of the system at the beginning of the period by $(t, \mathbf{a}, \mathbf{x})$.

The lead time is assumed to be zero. We write $\mathbf{b} = (b_0, b_1)$ as the assortment vector *after* technology decision, where $b_j \in \{0, 1, 2\}$ and $b_j \geq a_j$. This admits three scenarios: (i) $a_j = 0$: the firm can either introduce the generation-j product (i.e., $b_j = 1$) or not introduce it (i.e., $b_j = 0$); (ii) $a_j = 1$: the firm can either keep the generation-j product in the assortment (i.e., $b_j = 1$) or discontinue it (i.e., $b_j = 2$). In the latter case, the initial inventory x_j will be salvaged immediately); (iii) $a_j = 2$: the firm cannot reintroduce the product that was discontinued in the past, so $b_j = 2$. Based on the assortment decision, we denote $\mathbf{y} = (y_0, y_1)$ as the inventory levels after ordering, where $y_j = 0$ if $b_j \neq 1$. The state of the system *after* technology and inventory decisions is represented by $(t, \mathbf{b}, \mathbf{y})$.

We describe the cost due to the technology and inventory decisions as follows. If $a_j = b_j = 0$, the firm does not offer the generation-j product in the current period. If $a_j = 0$ and $b_j = 1$, the firm adds the generation-j product to the assortment and incurs a procurement cost $c_j(t)y_j$ to bring the inventory level to $y_j(> 0)$. If $a_j = b_j = 1$, the firm continues to offer the generation-j product and incurs a cost $c_j(t)(y_j - x_j)^+ - s_j(t)(x_j - y_j)^+$, where the first term is the procurement cost if $y_j \geq x_j$ and the second terms is the salvage revenue if $y_j < x_j$. If $a_j = 1$ and $b_j = 2$, the firm discontinues the current generation-j product and receives salvage revenue $s_j(t)x_j$.

4.1.2 Demand Process

Customers are expected utility maximizers. A customer's utility for each available choice (including a no-purchase option) is a random variable. Formally, we denote the random utility of a customer associated with a type $t - j$ product by $\tilde{u}_j(t) = \bar{u}_j(t) - r_j(t) + \xi$, where $\bar{u}_j(t) - r_j(t)$ is the expected net surplus (mean utility minus price) of a type $t - j$ product and ξ is an i.i.d., Gumbel random noise term with mean zero and scale parameter one for any j and t. The Gumbel distribution, which is also called the double-exponential distribution, has the distribution $P(\xi \leq v) = \exp(-e^{-(v+\gamma)})$, where $\gamma \simeq 0.5772$ is the Euler's constant. Without loss of generality, we assume that the net surplus associated with the no-purchase option is zero, hence, the random surplus associated with the no-purchase option is ξ. For notational convenience, we define the customer's "preference" for a type $j - t$ product by $q_j(t) = e^{\bar{u}_j(t) - r_j(t)}$. This implies the preference associated with the no-purchase option is normalized to 1. Because e^x is increasing in x, a higher value of $\bar{u}_j(t) - r_j(t)$ implies a higher value of $q_j(t)$. Let

$S(\mathbf{b}) = \{j : b_j = 1, j = 0, 1\}$ denote the set of products being offered in assortment $\mathbf{b}$. Under this utility model, as shown in vin Ryzin and Mahajan (1999), $P_j(t, \mathbf{b})$, the probability that a customer chooses a type $t - j$ product, $j \in S(\mathbf{b})$, is given by

$$P_j(t, \mathbf{b}) = \frac{q_j(t)}{1 + \sum_{j \in S(\mathbf{b})} q_j(t)}, \quad j \in S(\mathbf{b}),$$

where $1 - \sum_{j \in S(\mathbf{b})} P_j(t, \mathbf{b})$ is the probability of no-purchase. The expression shows that the demand for each product depends on the offered products in set $S(\mathbf{b})$.

We model the customer choice process by assuming: (i) customers make their choice decision based on the knowledge of assortment $\mathbf{b}$ and utility function $\tilde{u}_j(t)$, $j \in S(\mathbf{b})$, and they have no knowledge of the inventory status; (ii) if a customer selects a type $t - j$ product in $S(\mathbf{b})$ and the firm runs out of stock, the customer does not undertake a second choice and the sale is lost. The mean number of customers making the choice in each period is λ. Denote the number of customers selecting a type $t - j$ product by $D_j(t, \mathbf{b})$, where $D_j(t, \mathbf{b}) \equiv 0$ if $b_j \neq 1$. Following the independent customer choice model in Ryzin and Mahajan (1999), we see that $E[D_j(t, \mathbf{b})] = \lambda P_j(t, \mathbf{b})$. We assume, for $b_j = 1$, that $D_j(t, \mathbf{b})$ is normally distributed with a standard deviation $\sigma(\lambda P_j(t, \mathbf{b}))^{\beta}$, where $\sigma \geq 0$ and $0 \leq \beta \leq 1$. Note that when $0 \leq \beta < 1$, this demand model implicitly implies that the assortment decision $\mathbf{b}$ changes the coefficient of variation for the product demand; when $\beta = 1$, the coefficient of variation is independent of the assortment decision. In a natural special case where the number of customers making the choice is a Poisson random variable with mean λ, then $D_j(t, \mathbf{b})$ is also a Poisson random variable with mean $\lambda P_j(t, \mathbf{b})$.

Assumption 1. *(1)* $u_j(t) - r_j(t) \geq 0$ *for any* $j = 0, 1$ *and* $t = 1, 2, ...d$. *(2)* $q_0(t)$ *is increasing in* t*,* $q_1(t)$ *is decreasing in* t*.*

Recall that when only the new product is being offered, the expected demand for the product is $\frac{\lambda q_0(t)}{q_0(t)+1}$, which is increasing in t by Assumption 1. As soon as an innovation occurs, the generation-0 product becomes the generation-1 product with the expected demand $\frac{\lambda q_1(t)}{q_1(t)+1}$, which is decreasing in t. We see that the demand for the newly-released product increases steadily until the next innovation occurs. From that moment onward the demand for the product starts to decline. The generation-1 product becomes obsolete as soon as another innovation takes place.

4.1.3 Dynamic Programming Formulation

For convenience, we write $(\cdot)^+ = \max(0, \cdot)$. Let $Z_j(t, \mathbf{b}, y_j) = \min[y_j, D_j(t, \mathbf{b})]$ be the sales of the type $t - j$ product. For expositional simplicity, we write $Z_j(t, \mathbf{b}, y_j)$ as Z_j whenever convenient. Let $\mathbf{Z} = (Z_0, Z_1)$. Given that the system state is adjusted from state $(t, \mathbf{a}, \mathbf{x})$ to $(t, \mathbf{b}, \mathbf{y})$, we define $G_j(t, a_j, x_j; \mathbf{b}, y_j)$ as the single-period expected profit generated by the type $t - j$ product. We require that (i) $x_j = 0$ if $a_j \neq 1$, (ii) $y_j = 0$ and $E(Z_j) = 0$ if $b_j \neq 1$, and (iii) $a_j = 1$ if $x_j > 0$. Then, for $a_j \leq b_j$,

$$G_j(t, a_j, x_j; \mathbf{b}, y_j) = r_j(t)E(Z_j) + s_j(t)(x_j - y_j)^+ - c_j(t)(y_j - x_j)^+. \tag{4.1}$$

For each given a_j and $\mathbf{b}$, the above formula will yield the correct cost function. For example, if product-j is discontinued ($b_j = 2$), then $y_j = 0$ and $Z_j = 0$, and $G_j(t, a_j, x_j; \mathbf{b}, y_j) =$

$s_j(t)x_j$. For expositional purpose, our formulation does not include the one-period warehousing cost for the leftover inventory. However, such a cost can be easily incorporated into our model.

Let $V(t, \mathbf{a}, \mathbf{x})$ be the maximal expected total discounted profit over the infinite planning horizon with the starting state $(t, \mathbf{a}, \mathbf{x})$ and discount factor $0 \leq \theta < 1$. The optimality equation satisfies

$$V(t, \mathbf{a}, \mathbf{x}) = \max_{\substack{\mathbf{a} \leq \mathbf{b}, \\ \mathbf{y} \geq \mathbf{0}}} \left\{ \begin{array}{l} \sum_{j=0}^{1} G_j(t, a_j, x_j; \mathbf{b}, y_j) + \theta \sum_{t'} p_{t,t' \geq t} E[V(t', \mathbf{b}, \mathbf{y} - \mathbf{Z})] \\ +\theta q_t \sum_{t'} \alpha_{t'} E[V(t', (0, b_0), (0, y_0 - Z_0))] \end{array} \right\} \tag{4.2}$$

$$= \max_{\mathbf{a} \leq \mathbf{b},\ \mathbf{y} \geq \mathbf{0}} \{w(t, \mathbf{a}, \mathbf{x}; \mathbf{b}, \mathbf{y})\}.$$

The first summation term in (4.2) is the expected profit in the current period, and the second and third terms are the discounted expected future profits, conditioning whether an innovation occurs or not.

4.2 The Single-Period Analysis

This section considers the joint optimal assortment and inventory decisions for the single (final) period model. We first obtain the optimal inventory policy with fixed assortment $\mathbf{b}$. We then consider how to determine the optimal assortment decision $\mathbf{b}$ provided that the inventory level $\mathbf{y}$ will be fixed at the optimal level for each assortment decision.

We assume that the remaining inventory at the end of the final period is salvaged at its terminal value. We compute the expected terminal value of a type $t - j$ inventory as follows. Recall that with probability $p_{tt'}$, no innovation occurs and the development process moves to state t' from state t in the next period, the terminal value of a type $t-j$ product is $s_j(t')$. With probability $f_t\alpha_{t'}$, an innovation occurs and the next pipeline technology restarts in phase t'. Then, the current $t-j$ product will become a type $t'-(j+1)$ product and has the terminal value $s_{j+1}(t')$. Thus, the expected discounted terminal value of each unit of the leftover type $t-j$ inventory is

$$h_j(t) = \theta \sum_{t'=t}^{d} p_{t,t'} s_j(t') + \theta f_t \sum_{t'=1}^{d} \alpha_{t'} s_{j+1}(t'), \tag{4.3}$$

where $s_2(t) = 0$ for any t. It can be easily shown that $h_j(t) \leq \theta s_j(t)$.

Evidently, $c_j(t) - h_j(t) \geq 0$ is the overage cost of a type $t-j$ product, which equals the expected value depreciation due to innovations. The final period profit function for product j is

$$g_j(t, a_j, x_j; \mathbf{b}, y_j) = G_j(t, a_j, x_j; \mathbf{b}, y_j) + h_j(t)(y_j - Z_j(t, \mathbf{b}, y_j)), \quad j = 0, 1. \tag{4.4}$$

In the remainder of this section, we suppress t in the notation when convenient, for example, $h_j(t)$ will be written as h_j.

4.2.1 The Optimal Inventory Policy for a Fixed Assortment

In this subsection we determine the optimal inventory policy for a given assortment decision $\mathbf{b}$. Clearly, if $b_j \neq 1$, then the type $t-j$ product was either discontinued or has not been introduced, then $y_j = 0$. Therefore, we focus on the case with $b_j = 1$

and $a_j \leq 1$. From (4.4), we get that when $a_j = 0$, $b_j = 1$,

$$g_j(0, 0; \mathbf{b}, y_j) = (r_j - h_j)E[Z_j(\mathbf{b}, y_j)] - c_j y_j + h_j y_j,$$

which is concave in y_j with given x_j and the optimal inventory level is the well-known newsvender solution. When $a_j = b_j = 1$,

$$g_j(1, x_j; \mathbf{b}, y_j) = (r_j - h_j)E[Z_j(\mathbf{b}, y_j)] - c_j(y_j - x_j)^+ + s_j(x_j - y_j)^+ + h_j y_j. \quad (4.5)$$

Clearly, each term is concave in y_j with fixed $(x_j; \mathbf{b})$. Let $\Phi(\cdot)$ be the cumulative distribution function of a standard normal random variable. From (4.5), we find that the first derivative with respect to y_j, $y_j \neq x_j$, satisfies

$$\frac{\partial g_j(a_j, x_j; \mathbf{b}, y_j)}{\partial y_j} = \begin{cases} (r_j - h_j)(1 - \Phi(\frac{y_j - \lambda P_j}{\sigma(\lambda P_j)^\beta})) - c_j + h_j & \text{if } y_j > x_j \\ (r_j - h_j)(1 - \Phi(\frac{y_j - \lambda P_j}{\sigma(\lambda P_j)^\beta})) - s_j + h_j & \text{if } y_j < x_j. \end{cases} \quad (4.6)$$

Define $k_j = \frac{r_j - c_j}{r_j - h_j}$, which is the well-known newsvender ratio for product j with underage cost $r_j - c_j$ and overage cost $c_j - h_j$. Also let $k_j^{'} = \frac{r_j - s_j}{r_j - h_j}$. We define the following.

$$y_j(\mathbf{b}) = \lambda P_j + \Phi^{-1}(k_j)\sigma(\lambda P_j)^\beta, \text{ and } y_j^{'}(\mathbf{b}) = \lambda P_j + \Phi^{-1}(k_j^{'})\sigma(\lambda P_j)^\beta. \quad (4.7)$$

From (4.3), we see that $c_j \geq s_j > h_j$, thus, $y_j(\mathbf{b}) \leq y_j^{'}(\mathbf{b}) < \infty$.

Obviously, if $y_j(t, \mathbf{b}) > x_j$, then $y_j(t, \mathbf{b})$ maximizes (4.5). If $y'_j(t, \mathbf{b}) < x_j$, then $y'_j(t, \mathbf{b})$ maximizes (4.5). Now consider when $y_j(t, \mathbf{b}) < x_j < y'_j(t, \mathbf{b})$. If $y_j < x_j$, which implies $y_j < y'(t, \mathbf{b})$, then the second expression of (4.6) is positive; on the other hand, if $x_j < y_j$, which implies $y_j > y_j(t, \mathbf{b})$, then the first expression of (4.6) is negative. Therefore, if $y_j(t, \mathbf{b}) < x_j < y'_j(t, \mathbf{b})$, then (4.5) is maximized at $y_j = x_j$.

Lemma 4.1. *1. For the single period model, when $a_j = b_j = 1$, the optimal inventory level for product-j is given by*

$$y_j(1, x_j, \mathbf{b}) = \begin{cases} y_j(\mathbf{b}) & \text{if } x_j \leq y_j(\mathbf{b}) \\ x_j & \text{if } y_j(\mathbf{b}) < x_j < y'_j(\mathbf{b}) \\ y'_j(\mathbf{b}) & \text{if } x_j \geq y'_j(\mathbf{b}); \end{cases} \tag{4.8}$$

where $y_j(\mathbf{b})$ and $y'_j(\mathbf{b})$ are independent of $(\mathbf{a}, \mathbf{x})$ and defined in (4.7).

2. When $a_j = 0$ and $b_j = 1$, the optimal inventory level for product-j is $y_j(0, 0, \mathbf{b}) = y_j(\mathbf{b})$.

Equation (4.8) indicates that if $s_j \geq h_j$, that is, the salvage value of an item is greater than its discounted net worth in the next period, then the optimal ordering policy follows a two-limit policy. Specifically, if the initial inventory is below the first limit, it is optimal to raise the inventory to the first limit (in the single period problem, the first limit equals the newsvender solution); if the initial inventory x_j is greater than $y'_j(\mathbf{b})$, then the overage cost exceeds the salvage loss, thus, it is optimal to slash the inventory down to $y'_j(\mathbf{b})$; finally, if the initial inventory is between $y_j(\mathbf{b})$ and $y'_j(\mathbf{b})$, then slashing inventory is more costly than keeping it, thus, it is optimal to keep the inventory level unchanged.

Hereafter, we shall call the above inventory policy described in (4.8) the *two-limit inventory policy*. Substituting $y_j(\mathbf{b}, x_j)$ into (4.5), we can write

$$\begin{aligned} g_j^*(a_j, x_j; \mathbf{b}) &= \max_{y_j \geq 0} g_j(a_j, x_j; \mathbf{b}, y_j) = g_j(a_j, x_j; \mathbf{b}, y_j(\mathbf{b}, x_j)) \\ &= \begin{cases} g_j(a_j, 0; \mathbf{b}, y_j(\mathbf{b})) + c_j x_j, & \text{if } x_j \leq y_j(\mathbf{b}) \\ (r_j - h_j)E[Z_j(\mathbf{b}, x_j)] + h_j x_j, & \text{if } y_j(\mathbf{b}) < x_j < y'_j(\mathbf{b}) \\ g_j(a_j, y'_j(\mathbf{b}); \mathbf{b}, y'_j(\mathbf{b})) + s_j(x_j - y'_j(\mathbf{b})), & \text{if } x_j \geq y'_j(\mathbf{b}). \end{cases} \end{aligned} \tag{4.9}$$

4.2.2 The Optimal Assortment Decision

In this subsection, we determine the optimal assortment decision, provided that the optimal inventory policy will be implemented after the assortment is selected. We again suppress t for notation simplicity. The optimal assortment problem can be expressed as

$$V_{(1)}(\mathbf{a}, \mathbf{x}) = \max_{\mathbf{a} \leq \mathbf{b}} \left\{ \sum_{j=0}^{1} g_j^*(a_j, x_j; \mathbf{b}) \right\} = \max_{\mathbf{a} \leq \mathbf{b}} \left\{ g^*(\mathbf{a}, \mathbf{x}; \mathbf{b}) \right\},$$

where $V_{(1)}(\mathbf{a}, \mathbf{x})$ is the optimal single-period profit attainable with starting state $(\mathbf{a}, \mathbf{x})$. Clearly, for each fixed $(\mathbf{a}, \mathbf{x})$, the optimal assortment can be determined by comparing the profits under at most three feasible assortments satisfying $\mathbf{b} \geq \mathbf{a}$. Numerical evaluation of each assortment option is straightforward. Our objective, however, is to gain insight on how the assortment policy is affected by the initial inventory $\mathbf{x}$.

Without loss of generality, we assume that at any time offering at least one product is better than not offering any product. Therefore, the case $\mathbf{a} = (2,2)$, which means that both products were discontinued, can be excluded from consideration. We have eight possible cases: $\mathbf{a} =(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,1)$, $(2,0)$, and $(0,2)$. When there exists a product that has already been discontinued, for example, $\mathbf{a} =(1,2)$, $(2,1)$, $(2,0)$ or $(0,2)$, obviously the only feasible solution is to offer the product that has not been discontinued. We shall analyze the case $\mathbf{a} = (1,1)$ and $\mathbf{x} \geq 0$. Since $g^*((1,1),(0,x_1),\mathbf{b}) = g^*((0,1),(0,x_1),\mathbf{b})$, the initial state $(\mathbf{a},\mathbf{x}) = ((1,0),(x_0,0))$ can be treated as a special case of $(\mathbf{a},\mathbf{x}) = ((1,1),\mathbf{x})$. Similarly, both $(\mathbf{a},\mathbf{x}) = ((0,1),(0,x_1))$ and $(\mathbf{a},\mathbf{x}) = ((0,0),(0,0))$ are special cases of $(\mathbf{a},\mathbf{x}) = ((1,1),\mathbf{x})$.

From (4.9), we find that

$$V_{(1)}((1,1),\mathbf{x}) = \max \left\{ \begin{array}{l} g^*((1,1),\mathbf{x},(2,1)) = s_0 x_0 + g_1^*(1,x_1;(2,1)), \\ g^*((1,1),\mathbf{x},(1,1)) = g_0^*(1,x_0;(1,1)) + g_1^*(1,x_1;(1,1)), \\ g^*((1,1),\mathbf{x},(1,2)) = g_0^*(1,x_0;(1,2)) + s_1 x_1 \end{array} \right\}, \tag{4.10}$$

where the first, second and third terms inside the optimization operator correspond to the single period profits under the assortment decisions (2,1), (1,1) and (1,2), respectively. Before characterizing the structure of the optimal assortment policy, we first show the following lemma, which states that given x_0 (x_1), the marginal values of inventory x_1 (x_0) is decreasing (increasing) with respect to assortments $(2,1)$, $(1,1)$ and $(1,2)$.

Lemma 4.2. *It holds that*

$$\frac{\partial g^*((1,1),\mathbf{x},(2,1))}{\partial x_1} \geq \frac{\partial g^*((1,1),\mathbf{x},(1,1))}{\partial x_1} \geq \frac{\partial g^*((1,1),\mathbf{x},(1,2))}{\partial x_1} \quad \textit{for all } x_1 \tag{4.11}$$

and

$$\frac{\partial g^*((1,1),\mathbf{x},(2,1))}{\partial x_0} \leq \frac{\partial g^*((1,1),\mathbf{x},(1,1))}{\partial x_0} \leq \frac{\partial g^*((1,1),\mathbf{x},(1,2))}{\partial x_0} \quad \textit{for all } x_1 \tag{4.12}$$

Proof. Because of symmetry, we shall prove (4.11) only. Note that if product-1 is offered, then for $\mathbf{b} = (b_0, 1)$,

$$\begin{aligned} \frac{\partial g^*((1,1),\mathbf{x},\mathbf{b})}{\partial x_1} &= \frac{\partial g_1^*(1,x_1;\mathbf{b})}{\partial x_1} \\ &= \begin{cases} c_1 & \text{if } x_1 \leq y_1(\mathbf{b}) \\ (r_1 - h_1)(1 - \Phi(\frac{x_1 - \lambda P_1(\mathbf{b})}{\sigma_1(\lambda P_1(\mathbf{b}))^\beta})) + h_1 & \text{if } y_1(\mathbf{b}) < x_1 < y_1'(\mathbf{b}) \\ s_1 & \text{if } x_1 \geq y_1'(\mathbf{b}). \end{cases} \end{aligned} \tag{4.13}$$

Clearly, it holds that $c_1 \geq \frac{\partial g^*((1,1),\mathbf{x};\mathbf{b})}{\partial x_1} \geq s_1$ for all x_1 and $\mathbf{b} = (b_0, 1)$. In particular, for $\mathbf{b} = (1,1)$,

$$\frac{\partial g^*((1,1),\mathbf{x},(1,1))}{\partial x_1} \geq s_1 = \frac{\partial g^*((1,1),\mathbf{x},(1,2))}{\partial x_1},$$

which leads to the second inequality of (4.11). To prove the first inequality of (4.11), we consider two cases.

Case 1. $y_1(2,1) < y_1^{'}(1,1)$. Observe that, because $P_1(2,1) = \frac{q_1}{1+q_1} > P_1(1,1) = \frac{q_1}{1+q_0+q_1}$, we see that $y_1(2,1) > y_1(1,1)$, $y_1'(2,1) > y_1'(1,1)$, and $\Phi(\frac{x_1-\lambda P_1(1,1)}{\sigma_1(\lambda P_1(1,1))^\beta}) > \Phi(\frac{x_1-\lambda P_1(2,1)}{\sigma_1(\lambda P_1(2,1))^\beta})$. Therefore, we find from (4.13) that

i. when $x_1 \leq y_1(1,1)$, $\frac{\partial g_1^*(1,x_1;(2,1))}{\partial x_1} = \frac{\partial g_1^*(1,x_1;(1,1))}{\partial x_1} = c_1$;

ii. when $y_1(1,1) < x_1 \leq y_1(2,1)$, $\frac{\partial g_1^*(1,x_1;(2,1))}{\partial x_1} = c_1 \geq \frac{\partial g_1^*(1,x_1;(1,1))}{\partial x_1}$;

iii. when $y_1(2,1) < x_1 \leq y_1^{'}(1,1)$,

$$\frac{\partial g_1^*(1,x_1;(2,1))}{\partial x_1} - \frac{\partial g_1^*(1,x_1;(1,1))}{\partial x_1} = (r_1 - h_1)[\Phi(\frac{x_1 - \lambda P_1(1,1)}{\sigma_1(\lambda P_1(1,1))^\beta}) - \Phi(\frac{x_1 - \lambda P_1(2,1)}{\sigma_1(\lambda P_1(2,1))^\beta})] > 0;$$

iv. when $y_1'(1,1) < x_1$, $\frac{\partial g_1^*(1,x_1;(2,1))}{\partial x_1} \geq s_1 = \frac{\partial g_1^*(1,x_1;(1,1))}{\partial x_1}$.

Thus, the first inequality of (4.11) is true in this case.

Case 2. $y_1^{'}(1,1) < y_1(2,1)$. We find from (4.13) that

i when $x_1 \leq y_1(2,1)$, $\frac{\partial g_1^*(1,x_1;(2,1))}{\partial x_1} = c_1 \geq \frac{\partial g_1^*(1,x_1;(1,1))}{\partial x_1}$;

ii. when $x_1 > y_1(2,1)$, $\frac{\partial g_1^*(1,x_1;(2,1))}{\partial x_1} \geq s_1 = \frac{\partial g_1^*(1,x_1;(1,1))}{\partial x_1}$.

Thus, the first inequality of (4.11) is also true. ■

Lemma 4.2 indicates that for any fixed x_0, $g^*((1,1),\mathbf{x},(1,1)) - g^*((1,1),\mathbf{x},(1,2))$ is non-decreasing in x_1. Let $\beta_1(x_0)$ be the indifference point between using assortments $\mathbf{b} = (1,1)$ and $\mathbf{b} = (1,2)$, defined as

$$\beta_1(x_0) = \min\{x_1 : g^*((1,1),\mathbf{x},(1,1)) - g^*((1,1),\mathbf{x},(1,2)) \geq 0\}, \tag{4.14}$$

where $\beta_1(x_0) = \infty$ if $g^*((1,1),\mathbf{x},(1,1)) - g^*((1,1),\mathbf{x},(1,2)) < 0$ for all x_1. Since, by (4.12), $g^*((1,1),\mathbf{x},(1,1)) - g^*((1,1),\mathbf{x},(1,2))$ is a decreasing function of x_0, $\beta_1(x_0)$ is an increasing function of x_0.

Similarly, we define

$$\beta_2(x_0) = \min\{x_1 : g^*((1,1),\mathbf{x},(2,1)) - g^*((1,1),\mathbf{x},(1,1)) \geq 0\}, \tag{4.15}$$

$$\beta_3(x_0) = \min\{x_1 : g^*((1,1),\mathbf{x},(2,1)) - g^*((1,1),\mathbf{x},(1,2)) \geq 0\}. \tag{4.16}$$

Again, both $\beta_2(x_0)$ and $\beta_3(x_0)$ are increasing functions of x_0. Let us further define

$$\underline{\delta}_{(1)}(x_0) = \min\{\beta_1(x_0), \beta_3(x_0)\}, \quad x_0 \geq 0, \tag{4.17}$$

$$\overline{\delta}_{(1)}(x_0) = \max\{\beta_2(x_0), \beta_3(x_0)\}, \quad x_0 \geq 0. \tag{4.18}$$

It is obvious $\underline{\delta}_{(1)}(x_0) \leq \overline{\delta}_{(1)}(x_0)$. Furthermore, since $\beta_i(x_0)$, $i = 1, 2, 3$, are all increasing functions of x_0, both $\underline{\delta}_{(1)}(x_0)$ and $\overline{\delta}_{(1)}(x_0)$ are increasing functions of x_0.

Now, we are ready to characterize the optimal assortment policy with initial assortment $\mathbf{a} = (1,1)$.

LEMMA 4.3. *With initial state* $(\mathbf{a},\mathbf{x}) = ((1,1),\mathbf{x})$*, the optimal assortment policy is a switch-over policy: the increasing functions* $\underline{\delta}_{(1)}(x_0)$ *and* $\overline{\delta}_{(1)}(x_0)$ *partition the* (x_0, x_1)

plane into (at most) three non-overlapping regions:

$$A_1 = \{\mathbf{x} :\ 0 \le x_1 \le \underline{\delta}_{(1)}(x_0)\}, \tag{4.19}$$

$$A_2 = \{\mathbf{x} :\ \underline{\delta}_{(1)}(x_0) < x_1 \le \overline{\delta}_{(1)}(x_0)\}, \tag{4.20}$$

$$A_3 = \{\mathbf{x} :\ \overline{\delta}_{(1)}(x_0) < x_1\}, \tag{4.21}$$

such that assortment $(1, 2)$ *is optimal if* $\mathbf{x} \in A_1$*, assortment* $(1, 1)$ *is optimal if* $\mathbf{x} \in A_2$*, and assortment* $(2, 1)$ *is optimal if* $\mathbf{x} \in A_3$*.*

Proof. As we have shown, $g^*((1, 1), \mathbf{x}, (1, 1)) - g^*((1, 1), \mathbf{x}; (1, 2))$ is an increasing function of x_1, for any fixed x_0. Therefore, by the definition of $\beta_1(x_0)$, $g^*((1, 1), \mathbf{x}, (1, 1)) < g^*((1, 1), \mathbf{x}; (1, 2))$ if and only if $x < \beta_1(x_0)$, in words, assortment $(1, 2)$ is better than assortment $(1, 1)$ if and only if $x_1 < \beta_1(x_0)$. Along the same line of argument, it can be shown that assortment $(1, 2)$ is better than assortment $(2, 1)$ if and only if $x_1 < \beta_3(x_0)$. Hence, assortment $(1, 2)$ is the best assortment among the three assortments if and only if $x_1 < \min\{\beta_1(x_0), \beta_3(x_0)\} = \underline{\delta}_{(1)}(x_0)$. This proves that assortment (1,2) is optimal if and only if $\mathbf{x} \in A_1$. The proofs for the other two cases are similar and we omit the details. ■

As depicted in Figure 4.1, each of the switching curves $\underline{\delta}_{(1)}(x_0)$ and $\overline{\delta}_{(1)}(x_0)$ consists of two constant segments when x_0 is small and when x_0 is large. This property, in fact, is due to the *two-limit* inventory policy. To understand it, note that $\underline{\delta}_{(1)}(x_0)$ and $\overline{\delta}_{(1)}(x_0)$ inherit the properties of $\beta_i(x_0)$, $i = 1, 2, 3$. We use $\beta_1(x_0)$ as an example to illustrate that its shape resembles that of $\underline{\delta}_{(1)}(x_0)$. When $x_0 \le y_0(1, 1)$, the inventory level of product-0 will be adjusted to $y_0(1, 2)$ and $y_0(1, 1)$, respectively, under assortments

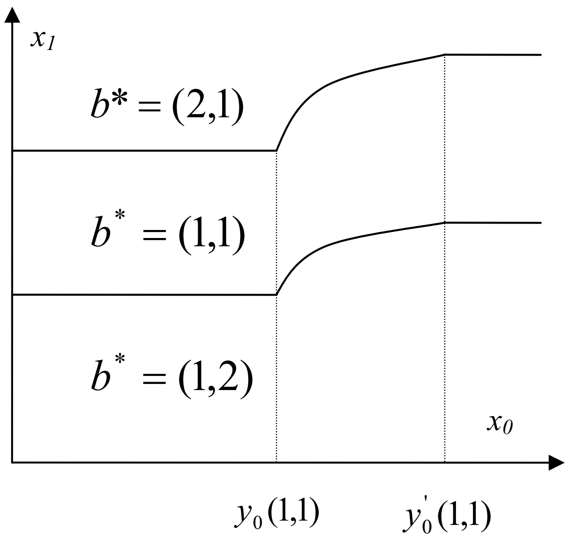

Fig. 4.1. The Optimal Technology Assortment With $\mathbf{a} = (1, 1)$

(1,2) and (1,1), $y_0(1, 2) > y_0(1, 1)$. Therefore,

$$
\begin{aligned}
& g^*((1,1), \mathbf{x}; (1,1)) - g^*((1,1)1, \mathbf{x}; (1,2)) \\
= \; & g_0^*(1, 0; (1,1)) + c_0 x_0 + g_1^*(1, x_1; (1,1)) - g_0^*(1, 0; (1,2)) - c_0 x_0 - s_1 x_1 \\
= \; & g_0^*(1, 0; (1,1)) + g_1^*(1, x_1; (1,1)) - g_0^*(1, 0; (1,2)) - s_1 x_1,
\end{aligned}
$$

which is independent of x_0, implying that the $\beta_1(x_0)$ is a constant for $x_1 \leq y_0(1, 1)$. On the other hand, when $x_0 > y_0'(1, 1)$, the inventory level of product-0 will be reduced to $y_0'(1, 1)$ and $y_0'(1, 2)$, respectively, under assortments (1,1) and (1,2). Similarly as in the previous case, we can show that $g^*((1, 1), \mathbf{x}; (1, 1)) - g^*((1, 1)1, \mathbf{x}; (1, 2))$ is independent of x_0 for $x_0 > y_0'(1, 1)$. Finally, when $y_0(1, 1) < x_0 < y_0'(1, 1)$, as shown in part (2) of Lemma 4.3, the switching curve $\underline{\delta}_{(1)}(x_0)$ is increasing in x_0.

As we mentioned, $(\mathbf{a}, \mathbf{x}) = ((0,1),(0,x_1))$ is a special case of $(\mathbf{a}, \mathbf{x}) = ((1,1), \mathbf{x})$ with $x_0 = 0$. Then Lemma 4.3 implies that the optimal assortment policy in state $\mathbf{a} = (0,1)$ is a threshold policy: if $x_1 < \underline{\delta}_{(1)}(0)$, assortment (1,2) is optimal; if $\underline{\delta}_{(1)}(0) < x_1 \leq \overline{\delta}_{(1)}(0)$, assortment (1,1) is optimal, otherwise assortment (0,1) is optimal. Similarly, it can be shown that the optimal assortment policy for $\mathbf{a} = (1,0)$ is a threshold policy.

Since each term inside the optimization operator of (4.10) is increasing in $\mathbf{x}$ (i.e., $\frac{\partial g^*(\mathbf{a},\mathbf{x};\mathbf{b})}{\partial x_j} \geq s_j$) we see that $V_{(1)}(\mathbf{a}, \mathbf{x})$ is strictly increasing (and hence quasi-concave) in $\mathbf{x}$ for any fixed $\mathbf{a}$. Furthermore, when initial inventory is sufficiently high, i.e., exceeds the second limit for any assortment decision, the firm will always start to salvage inventory and the marginal benefit of having one extra unit of initial inventory equals to its salvage value. These two properties of $V_{(1)}(\mathbf{a}, \mathbf{x})$ are formalized in the following Lemma.

Lemma 4.4. *For $a_j = 1$, $\frac{\partial V_{(1)}(\mathbf{a},\mathbf{x})}{\partial x_j} \geq s_j$, thus, the single-period profit function $V_{(1)}(\mathbf{a}, \mathbf{x})$ is strictly increasing in $\mathbf{x}$ for any fixed $\mathbf{a}$. Furthermore,*

$$\lim_{x_j \to \infty} \frac{\partial V_{(1)}(\mathbf{a}, \mathbf{x})}{\partial x_j} = s_j, \text{ for } a_j = 1.$$

4.2.3 The Infinite-Period Analysis

Unfortunately, when we extend the analysis to the multi-period model, the analysis breaks down for the same reason discussed in Chapter 3, i.e., we cannot guarantee that the objective function $w(t, \mathbf{a}, \mathbf{x}; \mathbf{b}, \mathbf{y})$ is concave in $\mathbf{y}$ for fixed $(t, \mathbf{a}, \mathbf{x}, \mathbf{b})$. In our numerical study using normally distributed demands, we did not find any instance that $w(t, \mathbf{a}, \mathbf{x}; \mathbf{b}, \mathbf{y})$ is not concave. We leave the analysis to find a sufficient condition that guarantee the concavity of $w(t, \mathbf{a}, \mathbf{x}; \mathbf{b}, \mathbf{y})$ in $\mathbf{y}$ for the future research.

If under some conditions, the objective function $w(t, \mathbf{a}, \mathbf{x}; \mathbf{b}, \mathbf{y})$ is either concave or quasi-concave in $\mathbf{y}$ for fixed $(t, \mathbf{a}, \mathbf{x}, \mathbf{b})$, then one should be able to show that the optimal ordering policy is a two-limit policy (the proof directly follows Theorem 2 in Eberly and Van Mieghem [11]). As such, the structure of the optimal policy at the current stage remains unclear.

4.3 The Special Case

In this section, we consider the special case with the following simplifying conditions: 1) no salvage loss for salvaged inventory, that is, $c_j(t) = s_j(t)$ for any j and t, and 2) $\beta = 1$, that is, demand $D_j(t, \mathbf{b})$ follows a normal distribution with mean $\lambda P_j(t, \mathbf{b})$ and standard deviation $\sigma \lambda P_j(t, \mathbf{b})$. There are many ways to achieve Condition 1. For example, the firm, with sufficient bargaining power, might have an agreement with its supplier or a third party to return/sell the unwanted inventory at the current procurement cost. It is also possible that there exists a secondary market where the firm can trade the inventory at the current procurement cost with negligible transaction fee. On the other hand, Condition 2 means that the coefficient of variation for each product demand is a constant σ. Under those simplifying conditions, we show that that, for a fixed assortment $\mathbf{b}$, the optimal inventory policy is myopic, i.e., the policy that maximizes the single-period profit function $g(t, \mathbf{a}, \mathbf{x}; \mathbf{b}, \mathbf{y})$ is optimal for the infinite-period problem. Furthermore, the optimal assortment policy is also myopic as soon as the system enters the long-run balancing states (to be defined more precisely later).

4.3.1 The Optimal Inventory Policy for a Fixed Assortment

Under the condition $c_j(t) = s_j(t)$ for any j and t, we have $V(t, \mathbf{a}, \mathbf{x}) = V(t, \mathbf{a}, \mathbf{0}) + \sum_{j=0}^{1} c_j(t)x_j$. For notation simplicity, we write $V(t, \mathbf{a}, \mathbf{0})$ as $V(t, \mathbf{a})$. Then the optimality equation (4.2) becomes

$$\begin{aligned} V(t,\mathbf{a},\mathbf{x}) &= \sum_{j=0}^{1} c_j(t)x_j + \max_{\substack{\mathbf{a}\le\mathbf{b},\\ \mathbf{y}\ge\mathbf{0}}} \left\{ \begin{array}{l} \sum_{j=0}^{1}\Big(r_j(t)E(Z_j) - c_j(t)y_j + h_j(t)(y_j - E(Z_j))\Big) \\ +\theta\sum_{t'\ge t} p_{t,t'}E[V(t',\mathbf{b})] + \theta f_t \sum_{t'} \alpha_{t'} E[V(t',(0,b_0))] \end{array} \right\} \\ &= \sum_{j=0}^{1} c_j(t)x_j + \max_{\mathbf{a}\le\mathbf{b}} \left\{ \begin{array}{l} \max_{\mathbf{y}\ge\mathbf{0}}\{g(t,\mathbf{b},\mathbf{y})\} + \theta\sum_{t'\ge t} p_{t,t'}E[V(t',\mathbf{b})] \\ +\theta f_t \sum_{t'} \alpha_{t'} E[V(t',(0,b_0))] \end{array} \right\}, \end{aligned} \tag{4.22}$$

where $g(t, \mathbf{b}, \mathbf{y}) = \sum_{j=0}^{1} g_j(t, \mathbf{b}, y_j)$ and

$$g_j(t,\mathbf{b},y_j) = r_j(t)E(Z_j) - c_j(t)y_j + h_j(t)(y_j - E(Z_j)), \quad j = 0, 1. \tag{4.23}$$

From the optimality equation (4.22), we immediately conclude that the optimal inventory policy is myopic. As $g_j(t, \mathbf{b}, y_j)$ is concave in y_j with given $(t, \mathbf{b})$, it is maximized when y_j equals the newsvender solution

$$\bar{y}_j(t,\mathbf{b}) = \lambda P_j(t,\mathbf{b}) + \Phi^{-1}\left(\frac{r_j(t) - c_j(t)}{r_j(t) - h_j(t)}\right)\sigma\lambda P_j(t,\mathbf{b}).$$

4.3.2 The Optimal Assortment Policy

The optimal assortment policy for (4.22) is generally not myopic due to the constraint $\mathbf{b} \ge \mathbf{a}$. Our solution procedure consists of two steps. We first derive the optimal assortment policy by relaxing the assortment constraint $\mathbf{b} \geq \mathbf{a}$. That is, the firm is allowed to offer any product, even the product that has already been discontinued previously. We prove that the optimal assortment policy for the unconstrained problem is myopic and takes a simple form. We then show that this myopic assortment policy can be implemented for the constrained optimization problem (4.22) with slight adjustment.

For expositional simplicity, we suppress t when convenient. Let $\phi(\cdot)$ be the density function of a standard normal random variable. Using the fact that for a standard normal random variable Z, $E[\min(z, Z)] = -\phi(z) + z(1 - \Phi(z))$, and that $\frac{\bar{y}_j(\mathbf{b}) - \lambda P_j(\mathbf{b})}{\sigma(\lambda P_j(\mathbf{b}))} = \Phi^{-1}(k_j)$, we have

$$\begin{aligned} E[Z_j(\mathbf{b}, y_j)] &= E[\min\{\bar{y}_j(\mathbf{b}), D_j(\mathbf{b})\}] = \sigma\lambda P_j(\mathbf{b}) E[\min\{z_j(\mathbf{b}), Z\}] + \lambda P_j(\mathbf{b}) \\ &= -\sigma\lambda P_j(\mathbf{b})\phi_j + \frac{\bar{y}_j(\mathbf{b})(c_j - h_j)}{r_j - h_j} + \frac{\lambda P_j(\mathbf{b})(r_j - c_j)}{r_j - h_j}, \end{aligned}$$

where $\phi_j = \phi(\Phi^{-1}(k_j))$. Substituting the above expression into (4.23), we have

$$g_j(\mathbf{b}, y_j(\mathbf{b})) = \lambda P_j(\mathbf{b})[(r_j - c_j) - (r_j - h_j)\sigma\phi_j],$$

where $P_j(\mathbf{b}) = \frac{\lambda q_j}{1 + \sum_{j \in S(\mathbf{b})} q_j}$. When it does not cause confusion, we write $\sum_{j=0}^{1} g_j(\mathbf{b}, y_j(\mathbf{b})) = \bar{g}^*(\mathbf{b})$. Define

$$R_j = (r_j - c_j) - (r_j - h_j)\sigma\phi_j. \tag{4.24}$$

We call R_j the *profit per unit* for product j, $j = 0, 1$. As it can be seen, the *profit per unit* is a comprehensive measure that takes into account the profit margin, service level, overage cost, and demand uncertainty together. Once we know R_j, the expected profit contributed by product-j can be easily computed by taking the product of R_j and the expected demand $\lambda P_j(\mathbf{b})$. We assume that $R_j > 0$.

After relaxing the constraint $\mathbf{b} \geq \mathbf{a}$, it is obvious that the optimal assortment decision $\mathbf{b}$ in (4.22) becomes myopic which is independent of $\mathbf{a}$. Therefore, we only need to focus on the single-period, unconstrained assortment problem $\max_{\mathbf{b}} \left\{ R_j \lambda P_j(\mathbf{b}) \right\}$.

THEOREM 4.1. *1. The optimal assortment that maximizes* $\max_{\mathbf{b}} \left\{ R_j \lambda P_j(\mathbf{b}) \right\}$ *can be characterized as the following :*

- *When* $\frac{R_0}{R_1} > \frac{q_0+1}{q_0}$*, the optimal assortment is to offer product-*0 *only;*
- *when* $\frac{q_1}{q_1+1} \leq \frac{R_0}{R_1} \leq \frac{q_0+1}{q_0}$*, the optimal assortment is to offer both products;*
- *when* $\frac{R_0}{R_1} < \frac{q_1}{q_1+1}$*, the optimal assortment is to offer product-1 only.*

2. $\bar{g}^*(1,1) \geq \min\{\bar{g}^*(1,0), \bar{g}^*(0,1)\}$.

Proof. 1) It can be verified that

$$\bar{g}^*(1,0) = \frac{\lambda R_0 q_0}{1+q_0},\ \bar{g}^*(0,1) = \frac{\lambda R_1 q_1}{1+q_1},\ \text{and } \bar{g}^*(1,1) = \frac{\lambda(R_0 q_0 + R_1 q_1)}{1+q_0+q_1}.$$

First consider the case $\frac{R_0}{R_1} > \frac{q_0+1}{q_0}$. We obtain

$$\bar{g}^*(1,0) = \frac{\lambda R_0 q_0}{1+q_0} > \lambda R_1 > \frac{\lambda R_1 q_1}{1+q_1} = \bar{g}^*(0,1).$$

Hence offering product-0 is better than offering product-1. On the other hand,

$$\bar{g}^*(1,0) - \bar{g}^*(1,1) = \frac{\lambda R_0 q_0}{1+q_0} - \frac{\lambda(R_0 q_0 + R_1 q_1)}{1+q_0+q_1} = \frac{\lambda q_1 [R_0 q_0 - R_1(1+q_0)]}{(1+q_0)(1+q_0+q_1)}.$$

When $\frac{R_0}{R_1} > \frac{q_0+1}{q_0}$, $\bar{g}^*(1,0) > \bar{g}^*(1,1)$, which means offering product-0 is also better than offering both products. The proofs of the other two cases are similar.

2). This claim follows from

$$\begin{aligned}\bar{g}^*(1,1) &= \frac{1+q_0}{1+q_0+q_1}\bar{g}^*(1,0) + \frac{1+q_1}{1+q_0+q_1}\bar{g}^*(0,1) \\ &\geq \frac{1+q_0+1+q_1}{1+q_0+q_1}\min\{\bar{g}^*(1,0), \bar{g}^*(0,1)\} > \min(\bar{g}^*(1,0), \bar{g}^*(0,1)).\end{aligned}$$

This completes the proof of Theorem 4.1. ■

We call the policy stated in Theorem 4.1 the myopic joint control policy. To facilitate the analysis the optimal assortment policy for the unconstrained problem, we need to utilize the dynamic structure of the myopic policy as a function of t, which we will consider in the next subsection.

4.3.3 Dynamic Property of the Myopic Policy

We expand the notation on t, for example, R_j will be written as $R_j(t)$. When $\frac{R_0(t)}{R_1(t)} \geq (<)1$, it implies that the profit-per-unit of the newer (older) product is higher. The new development of the pipeline technology (i.e., when t increases) often results in a lower profit-per-unit for an existing product. Because of the generational difference, it is reasonable to assume that the profit-per-unit of the new product decreases at a slower rate than that of the old product. Therefore, we make the following assumption on the ratio when the innovation process evolves.

Assumption 2. *The ratio* $\frac{R_0(t)}{R_1(t)}$ *is increasing in* t.

Then, as a direct consequence of Theorem 4.1, we have the following corollary.

Corollary 4.1. *Under Assumptions 1 and 2, there exists two thresholds* t^*_1 *and* t^*_2 *(*$t^*_1 \leq t^*_2$*) such that the optimal assortment for the unconstrained problem is the following: when* $t < t^*_1$, $\bar{\mathbf{b}}(t) = (0, 1)$*; when* $t^*_1 \leq t \leq t^*_2$, $\bar{\mathbf{b}}(t) = (1, 1)$*; and when* $t > t^*_2$, $\bar{\mathbf{b}}(t) = (1, 2)$*.*

Proof. The result follows immediately from Theorem 4.1, and our assumptions that $\frac{R_0(t)}{R_1(t)}$ is increasing in t, and both ratios $\frac{q_0(t)+1}{q_0(t)}$ and $\frac{q_1(t)}{q_1(t)+1}$ are decreasing in t. ■

Essentially, Corollary 4.1 states that the optimal assortment evolves in the following pattern: when the pipeline technology is in the early development phase (i.e., $t < t^*_1$), offer the old product only; when the development of the pipeline technology is in the medium phase (i.e., $t^*_1 \leq t \leq t^*_2$), offer both products; and when the pipeline technology is near completion (i.e., $t > t^*_2$), offer the new product only. There are several special cases. For example,

- when $t^*_1 = t^*_2 - 1$, the optimal assortment contains only one product for any t, i.e., when $t \leq t^*_1$, it is optimal to offer product-1 only; when $t > t^*_1$, it is optimal to offer product-0 only.

- when $t^*_1 = 0$ and $t^*_2 \geq 2$, the optimal assortment will evolve from (1,1) to (1,2) (in other words, the new product should be introduced as soon as it is available; and after the state reaches t^*_2 the old product should be phased out);

- when $t^*_1 = d$ and $t^*_2 = d + 1$, the optimal assortment is to offer the product-1 only at all states;

- when $t_1^* = 0$ and $t_2^* = 1$, the optimal assortment is to offer the product-0 at all state (whenever there is an innovation, upgrade to the newest technology immediately);
- when $t_1^* = 0$ and $t_2^* = d+1$, the optimal assortment is always to offer both products.

4.3.4 The Constrained Assortment Problem

We now return to our original infinite-period problem defined in (4.22). We say the system is in a *balancing* state if the myopic policy given in Corollary 4.1 is feasible. The next theorem states that the myopic policy becomes optimal as soon as the system enters a balanced state.

THEOREM 4.2. *If the myopic policy* $\bar{\mathbf{b}}(t)$ *defined in Corollary 4.1 satisfies* $\mathbf{a} \leq \bar{\mathbf{b}}(t)$*, then the myopic policy is optimal for (4.22).*

Proof: We truncate the planning horizon into n periods. When $n = 1$, we have

$$V_{(1)}(t, \mathbf{a}) = \max_{\mathbf{a} \leq \mathbf{b}} \{g(t, \mathbf{b})\}.$$

If the initial assortment is $\mathbf{a} \leq \bar{\mathbf{b}}(t)$, then $\bar{\mathbf{b}}(t)$ is a feasible, hence optimal, solution of $\max_{\mathbf{a} \leq \mathbf{b}} \{g(t, \mathbf{b})\}$. Next, we hypothesize that the myopic policy is optimal for $V_{(n)}(t, \mathbf{a})$ when $\mathbf{a} \leq \bar{\mathbf{b}}(t)$ is satisfied. Then $V_{(n+1)}(t, \mathbf{a})$ is given by

$$V_{(n+1)}(t, \mathbf{a}) = \max_{\mathbf{a} \leq \mathbf{b}} \left\{ g(t, \mathbf{b}) + \theta \sum_{t' \geq t} p_{t,t'} E[V_n(t', \mathbf{b})] + \theta f_t \sum_{t'} \alpha_{t'} E[V_n(t', (0, b_0))] \right\}. \tag{4.25}$$

Obviously, given $\mathbf{a} \leq \bar{\mathbf{b}}(t)$, it is feasible to implement the myopic policy in the first period, which maximizes $g(t, \mathbf{b})$. If in the next period no innovation occurs and the development phase moves to state $t' \geq t$, then according to Corollary 4.1, it holds that $\bar{\mathbf{b}}(t) \leq \bar{\mathbf{b}}(t')$ if $t \leq t'$, and hence the myopic policy $\bar{\mathbf{b}}(t')$ is again feasible in state $(t', \bar{\mathbf{b}}(t))$. Applying our hypothesis, we conclude that the myopic policy maximizes $V_{(n)}(t', \mathbf{b}(t))$ for any $t' \geq t$. On the other hand, If the innovation occurs in the next period and the development phase moves to t', then the initial assortment state in the beginning of the next period becomes $(0, \bar{b}_0(t))$. Observe that the new product is never discontinued under the myopic policy, that is, $\bar{b}_0(t) \leq 1$. Therefore, $(0, \bar{b}_0(t)) \leq (0, 1)$ for any t. This means that the myopic policy is feasible when the system enters state $(t', (0, \bar{b}_0))$ in the next period for any t'. By our hypothesis, the myopic policy also maximizes $V_{(n)}(t', (0, \bar{b}_0(t))$ for any t'. Finally letting $n \to \infty$, and using the fact that $\lim_{n\to\infty} V_{(n)}(t, \mathbf{a}) \to V(t, \mathbf{a})$, we conclude that the myopic policy is optimal for the infinite-period problem when $\mathbf{a} \leq \bar{\mathbf{b}}(t)$. ■

What if the system starts with a non-balanced state? The following theorem prescribes the optimal assortment policy in a non-balanced state.

THEOREM 4.3. *1. If* $\mathbf{a} = (2, 1)$ *or* $\mathbf{a} = (1, 2)$*, then it is optimal to keep the current assortment unchanged.*

2. If $t < t_1^*$ *and* $\mathbf{a} =(1, 1)$ *or* $t < t_1^*$ *and* $\mathbf{a} =(1, 0)$*, then the optimal assortment for (4.22) is* $\mathbf{b}^* = (1, 1)$.

Proof. The first part of the result is trivial, since keeping the current assortment is the only option available. We focus on the second part of the theorem. According to part 2 of Theorem 4.1 and Corollary 4.1, when $t < t_1^*$, it satisfies that $\bar{g}^*(1, 1) \geq \bar{g}^*(1, 0) =$

$\bar{g}^*(1,2)$. So $(1,1)$ is optimal for the single-period constrained problem with $\mathbf{a}=(1,1)$ or $(0,1)$. We hypothesize that when $t < t_1^*$, the optimal assortment for $V_n(t,(1,1))$ and $V_n(t,(1,0))$ is $\mathbf{b}^* = (1,1)$. From (4.25), we see that if the system enters state $(t',(0,1))$ in the next period, $t' > t_1^*$, the myopic policy is optimal from then on. Suppose the system enters state $(t',(1,1))$. If $t' \geq t_1^*$, then the myopic policy is optimal from then on. If $t' < t_1^*$, then by our induction hypothesis, $\mathbf{b}^* = (1,1)$ is optimal for state $(t',(1,1))$. Hence, we conclude that $\mathbf{b}^* = (1,1)$ is optimal for $V_{(n+1)}(t,(1,1))$ and $V_{(n+1)}(t,(1,0))$. Finally letting $n \to \infty$, and using the fact that $\lim_{n\to\infty} V_{(n)}(t,\mathbf{a}) \to V(t,\mathbf{a})$, we conclude that $\mathbf{b}^* = (1,1)$ is optimal for $V(t,(1,1))$ and $V(t,(1,0))$ when $t < t_1^*$. ■

4.3.5 Optimal Assortment for N-Products

We extend our special case analysis to the single-period model with $N(\geq 2)$ generations of products co-existing. Let

$$R_j = r_j - c_j - (r_j - h_j)\sigma\phi_j, \quad j = 0,1,...,N-1$$

be the profit per unit for product-j. In fact, our following analysis can be generalized to the case where the demand for each product has its own coefficient of variation that is independent of the assortment decision. The unconstrained optimization problem is

$$\max_{\mathbf{b}} \bar{g}^*(\mathbf{b}) = \sum_{j\in S(\mathbf{b})} \frac{\lambda q_j R_j}{1+\sum_{j\in S(\mathbf{b})} q_j}. \tag{4.26}$$

Without loss of generality, assume that $R_0 \geq R_1 \geq ... \geq R_{N-1}$ (if necessary re-index j such that $\{R_j\}$ is a decreasing sequence). Let

$$\mathbf{b}_{(i)} = \{\underbrace{1, 1, ..., 1,}_{\text{first } i+1 \text{ elements}} \quad 0, ..., 0\},$$

where $0 \leq i \leq N-1$.

THEOREM 4.4. *Let n^* be the largest integer such that*

$$\frac{\sum_{j=0}^{n^*} q_j R_j}{1 + \sum_{j=0}^{n^*} q_j} - R_{n^*+1} \geq 0.$$

Then the optimal assortment $\mathbf{b}^* = \mathbf{b}_{(n^*)}$, meaning that it is optimal to offer the first n^* (most profitable) products.

The proof of Theorem 4.4 is presented in Appendix A.3. Theorem 4.4 intuitively indicates that the optimal assortment should include the first few items that have the highest *profit per unit*. However, the extension to multiple-periods is left for future research.

4.4 Heuristics and Numerical Study

With $c_j(t) > s_j(t)$ and $\beta < 1$, since we are unable to prove $w(t, \mathbf{a}, \mathbf{x}; \mathbf{b}, \mathbf{y})$ to be a concave or a quasi concave function, the structural property of the optimal assortment and inventory policy for the infinity-period discounted problem is unclear. In this section, we propose several heuristics based on the insight we gained from the single-period analysis and the special case of the multi-period analysis.

4.4.1 Phase-based Assortment Heuristic

Motivated by the dynamic structure of the optimal assortment shown in Section 4.3, we propose a phase-based heuristic, represented by $\mathbf{b}(t)$, that controls the assortment based on t only. That is, for the assortment decision, we select the thresholds $\tau = (t_1, t_2)$, where $1 \le t_1 \le t_2 \le d$ and let

$$\mathbf{b}(t) = \begin{cases} (0,1), & \text{for } t < t_1 - 1, \\ (1,1), & \text{for } t_1 \le t \le t_2, \\ (1,2), & \text{for } t_2 < t. \end{cases} \tag{4.27}$$

For any given $\mathbf{b}(t)$, the inventory vector $\mathbf{y}(t, \mathbf{b}(t))$ is determined by equation (4.8), the single-period, two-limit inventory policy[1]. Hereafter, we call this heuristic the PB heuristic.

With given thresholds $\tau = (t_1, t_2)$, let $V_{PB}(\tau, \mathbf{y})$ be the time-average profit under a PB heuristic. Essentially, under a PB heuristic with $\tau = (t_1, t_2)$, the assortment decision is controlled as follows. When the development of the pipeline technology is in the early stage ($t < t_1 - 1$), offer product-1 only; when the development of the pipeline technology enters the intermediate stage ($t_1 \le t \le t_2$), offer both products; when the

[1] Since in our numerical study we consider time-average profit, we let $\theta = 1$ in our computation.

development of the pipeline technology is near completion ($t_2 < t$), discontinue product-1 and offer product-0 only. As demonstrated in Section 4.3, once the assortment control rule in (4.27) is feasible, the system will enter the balanced state and (4.27) remains feasible. In our computation, we use $t = 1$, $\mathbf{a} = (0, 1)$, and $\mathbf{x} = \mathbf{0}$ as the initial states.

There are totally $\frac{d(d+1)}{2}$ possible combinations of t_1 and t_2 (since t_1 and t_2 can be equal). For each given combination of t_1 and t_2, we compute $V_{PB}(\tau, \mathbf{y})$ using the standard policy evaluation techniques (Puterman [37]). We iterate all the possible combinations and choose the values $\tau_{PB}=(t_1^{PB}, t_2^{PB})$ that maximize $V_{PB}(\tau, \mathbf{y})$ to determine the assortment control rule in (4.27).

As the structure of the optimal policy is unclear at this stage, we develop an easy-to-compute upper bound for the optimal objective value to benchmark our heuristics. Denote the time-average profit with $s_j(t) = c_j(t)$ by V_u, which serves as an upper bound of its counterpart with $s_j(t) < c_j(t)$. The optimal assortment plan is obtained by solving the dynamic programming problem (4.22) and the inventory policy is determined by

$$y_j^u(t, \mathbf{b}) = \lambda P_j(t, \mathbf{b}) + \Phi^{-1}\left(\frac{r_j(t) - c_j(t)}{r_j(t) - h_j(t)}\right) \sigma(\lambda P_j(t, \mathbf{b}))^{\beta},$$

where $h_j(t)$ is from (4.3) with $s_j(t) = c_j(t)$. Once again, we use $\mathbf{a} = (0, 1)$ as the initial assortment. Note that

$$V_u(t) = \lim_{\theta \to 1} (1 - \theta) V(t, (0, 1)),$$

where $V(t, (0, 1))$ is the total discounted profit (discount factor is θ). And then $V_u = \sum_t \pi(t) V_u(t)$, where $\pi(t)$ is the long-run probability that the system is in state t. We will compare the percentage difference between $V_{PB}(\tau_{PB}, \mathbf{y})$ and V_u. Furthermore, quantifying the difference against V_u can also help us to see how much the firm can benefit from the "zero salvage" condition.

A PB heuristic is easy to compute and implement. However, it has two major weaknesses. First, the two-limit inventory policy for the single period case is not necessarily optimal for assortment control threshold τ_{PB}. In the next section, we discuss how to improve the inventory policy. Second, the single-period analysis indicates that the assortment decision should be dependent on the initial inventory levels. For example, in the period when $t_1 \le t \le t_2$ or $t > t_2$, high inventory level of product-1 may postpone the phase-out of product-1 and/or the introduction of product-0. In the next section, we propose some modifications to improve the PB heuristic in these situations.

4.4.2 Other Heuristics

We develop several heuristics in this section.

1. The Myopic Heuristic. Under the myopic heuristic, for initial state $(t, \mathbf{a}, \mathbf{x})$, we determine the assortment $\mathbf{b}$ following Lemma 4.3 and the discussions afterward in Section 4.2.2. This includes three major scenarios.

- If one of the two generation products has already been discontinued, then keep the initial assortment unchanged.
- If the initial assortment contains both generation products, we determine the assortment according to the switching curves defined in Lemma 4.3.

- If the initial assortment only contains one product and the other one has not been introduced, for example, $a_j = 0$, we treat this as a special case of Lemma 4.3 with $x_j = 0$.

As soon as the assortment is determined, we use equation (4.8), the corresponding single-period two-limit inventory policy, to control inventory. Let $V_{MY}(\mathbf{t}, \mathbf{y})$ be the time-average profit under the myopic heuristic with initial state $(t, \mathbf{0}, \mathbf{0})$. The advantages of the myopic heuristic are: first, it has a closed form solution for the assortment and inventory decisions; second, it is easy to implement; and third, computation effort required to evaluate $V_{MY}(\mathbf{t}, \mathbf{y})$ is not intensive. A weakness of the myopic heuristic is that the myopic inventory policy is based on the assumption that inventory will be salvaged in the next period. This could lead to a non-optimal inventory policy in the multiple-period setting (for example, the firm under-stocks because of an overestimation of overage costs).

2. The Sequential Optimization Heuristic (SH). This heuristic is determined by the following steps. First, we choose two thresholds $\tau = (t_1, t_2)$, then the assortment $\mathbf{b}(t)$ is determined by equation (4.27). Second, we determine the inventory policy by solving the following dynamic programming problem:

$$V^{\tau}(t, \mathbf{a}, \mathbf{x}) = \max_{\mathbf{y} \geq \mathbf{0}} \left\{ \begin{array}{l} \sum_{j=0}^{1} G_j(t, a_j, x_j; \mathbf{b}(t), y_j) \\ +\theta \sum_{t' \geq t} p_{t,t'} E[V^{\tau}(t', \mathbf{b}(t), \mathbf{y} - \mathbf{Z})] \\ +\theta f_t \sum_{t'} \alpha_{t'} E[V^{\tau}(t', (0, b_0(t)), (0, y_0 - Z_0))] \end{array} \right\}. \tag{4.28}$$

It is easy to argue that the above dynamic programming problem can be decomposed as $V^{\tau}(t, \mathbf{a}, \mathbf{x}) = \sum_{j=0}^{1} V_j^{\tau}(t, a_j, x_j)$, where $V_j^{\tau}(t, a_j, x_j)$ is the single-product profit function with state (t, a_j, x_j) and assortment control threshold vector τ. $V_j^{\tau}(t, a_j, x_j)$ satisfies

$$V_j^{\tau}(t, a_j, x_j) = \max_{y_j \geq 0} \left\{ \begin{array}{l} G_j(t, a_j, x_j; \mathbf{b}(t), y_j) \\ +\theta \sum_{t'} p_{t,t' \geq t} E[V_j(t', b_j(t), y_j - Z_j(\mathbf{b}(t)); \mathbf{t})] \\ +\theta f_t \sum_{t'} \alpha_{t'} E[V_{j+1}(t', b_j(t), y_j - Z_j(\mathbf{b}(t)); \mathbf{t})] \end{array} \right\} \quad (4.29)$$

$$= \max_{y_j \geq 0} \{ w_j^{\tau}(t, a_j, x_j; \mathbf{b}(t), y_j) \}. \quad (4.30)$$

for $j = 0, 1$ and $V_2^{\tau}(t, 2, x) = V_1^{\tau}(t, 1, 0)$ (since the generation-2 product will be salvaged with zero salvage value). Because the assortment decision is determined by thresholds $\tau = (t_1, t_2)$ and is independent of x_j, using the similar approach in Theorem 3.3, one can show that $w_j^{\tau}(t, a_j, x_j; \mathbf{b}(t), y_j)$ is concave in y_j and that the optimal inventory policy for (4.29) is determined by two thresholds, with the values of both thresholds only dependent of t and j and not on x_j. Finally, we iterate all possible $\mathbf{t}$ and choose $\mathbf{t}_{SH}$ that maximizes the time-average profit $V^{\tau}(t, \mathbf{0}, \mathbf{0})$ to determine the assortment control rule in (4.27).

4.4.3 Numerical Study

We investigate the performance of various heuristics using randomly generated data. We describe the procedure to generate random data as follows.

1. *Innovation Process:* The number of phases, d, is either 3 or 6. For simplicity, we assume that the transition probability satisfies $p_{t,t'} = 0$ if $t > t'$ and $p_{t,t'} = p$ otherwise. For example, when $d = 3$, the transition matrix is $\begin{pmatrix} p & p & p \\ 0 & p & p \\ 0 & 0 & p \end{pmatrix}$. The

value of p is uniformly distributed between $\frac{1}{2d}$ and $\frac{1}{d}$ (the lower bound on p is to avoid having a very small p, which could lead to an unreasonably high innovation speed). One can show that $1 - \sum p_{t,t'}$ is increasing in t, i.e., the innovation time has an increasing failure rate. We assume that the innovation process will always restart at phase 1. Accordingly, the initialization vector α_t satisfies $\alpha_1 = 1$ and $\alpha_t = 0$ for $t \geq 2$.

2. *Cost Parameters:* The price for product-0 is

$$r_0(t) = r_0(1)t^{-\eta}, \ t \geq 1$$

where η=0.1 or 0.3 and $r_0(1)$ is uniformly distributed between 400 and 500. Souza, Bayus, and Wagner [49] discussed the empirical evidences about the price decay function $r_0(t) = r_0(1)t^{-\eta}$. When η is taking a smaller value, the price decays slower. For product-1, the price is

$$r_1(t) = r_1(1)t^{-\eta}, \ t \geq 1$$

where $r_1(1)$ is uniformly distributed between $0.8r_0(d)$ and $0.9r_0(d)$. This guarantees that $r_j(t)$ is decreasing in t for $j = 0, 1$ and $r_0(d) > r_1(t)$ for any given t. In our experiment, the price range for product-0 is between 234 and 500 and for product-2 is between 109 and 403. The purchasing cost $c_j(t) = C \times r_j(t)$, where C is fixed for each simulation run and is taking one of the two different values 0.6 and 0.8. The salvage value $s_j(t) = S \times c_j(t)$, where S is fixed for each simulation

Table 4.1. Numerical Study Design

Parameters	Symbol	Values
Number of phases	d	3, 6
Price decay factor	η	0.1, 0.3
Purchase cost coefficient	C	0.6, 0.8
Salvage value coefficient	S	0.6, 0.9
Coefficient of variation	σ	1/6, 1/3

run and is either 0.7 or 0.9. For example, when $S = 0.7$, the firm loses 30% of the current unit value when salvaging the unwanted initial inventory.

3. *Demand Distribution:* $\lambda = 10$ and $\beta = 1$, i.e., the total market size has a mean of 10 and the coefficient of variation is not affected by the assortment decision. We consider two levels of coefficient of variation $\sigma = 1/6$ and $1/3$ (this will assure the probability of having negative demand to be negligible). The normalized preference $q_j(t)$ is randomly generated and satisfies that $q_0(t)$ is increasing in t and that $q_1(t)$ is decreasing in t. Overall, the range of $q_j(t)$ is between 1 and 9 and the non-purchase preference is 1. As such, the choice probability $P_j(t, \mathbf{b})$ is between 0.09 and 0.90.

Notice that for the number of phases d, price decay factor η, purchasing cost coefficient C, salvage value coefficient S, and coefficient of variation σ, each of them can take two difference values. This leads to 32 scenarios. For each of these 32 scenarios, we generate 25 problem instances. For each simulation run, we evaluate their time-average profit: V_{PB}, V_{MY}, and V_{SH} (using MatLab and the standard policy evaluation techniques in Puterman [37], as well as the discussions in previous section) and then compute the percentage difference of the profits of a given heuristic against the upper bound profit V_u. Table 4.2 reports the maximum percentage difference and mean percentage difference for various heuristics with $d = 3$, in which we use H/L to indicate the controlled parameter is taking Hign/Low value.

We observe that among the three heuristics, the SH heuristic has the smallest maximum and mean percentage differences against upper bound profit. Table 4.3 reports the computational results with $d = 6$. There are no noticeable differences between $d = 3$ and $d = 6$, which implies that the number of phases has little impact on the gap between the profit of our heuristics and the upper bound profit.

We also found that the SH heuristic indeed outperforms the other two methods except in 5 instances (in totally 800 problem instances) when the myopic heuristic slightly outperforms the SH heuristic. As we discussed before, the inventory policy is optimized in the SH heuristic according to the future assortment plan whereas it is not optimized in the other two heuristics. So we conclude that the SH heuristic is the best among these three.

The maximum difference between the SH heuristic and the upper bound profit is 9%, which implies that the upper bound may not be a very reliable benchmark in some circumstances. Nonetheless, it indicates that the benefit for the firm to engage an agreement with its supplier to return inventory at the current purchase cost is substantial. Certainly the supplier may charge a higher purchase cost. We leave the risk-sharing and transfer pricing issues for future research.

From Tables 4.2 and 4.3, we can also see that the price decay factor η, purchase cost coefficient C, and salvage value coefficient S have larger impact on the performance

Table 4.2. Summary of Numerical Experiment with d=3

Case	Parameters Setting				Maximum percentage difference vs V_u			Mean percentage difference vs V_u		
	η	σ	S	C	PB	MY	SH	PB	MY	SH
1	H	H	H	H	2.4%	2.0%	1.7%	1.7%	1.4%	1.1%
2	L	H	H	H	7.7%	7.5%	6.0%	4.3%	3.9%	2.8%
3	H	L	H	H	1.2%	1%	0.8%	0.7%	0.6%	0.5%
4	L	L	H	H	1.6%	1.3%	1.2%	1%	0.8%	0.7%
5	H	H	L	H	8%	7.2%	6.0%	6.3%	5.7%	4.7%
6	L	H	L	H	12%	11%	9.0%	9.1%	8.2%	7.2%
7	H	L	L	H	3.6%	3.2%	2.2%	2.8%	2.5%	1.4%
8	L	L	L	H	5.5%	4.9%	4.6%	3.5%	3.1%	2.4%
9	H	H	H	L	1.3%	1.1%	1.0%	1%	0.8%	0.7%
10	L	H	H	L	3.1%	2.9%	1.7%	1.6%	1.3%	1.0%
11	H	L	H	L	0.6%	0.5%	0.5%	0.5%	0.4%	0.4%
12	L	L	H	L	0.9%	0.7%	0.8%	0.6%	0.4%	0.4%
13	H	H	L	L	4.7%	4.0%	3.6%	3.3%	2.8%	2.4%
14	L	H	L	L	5.7%	4.8%	4.5%	4.6%	3.9%	3.5%
15	H	L	L	L	2.2%	1.9%	1.5%	1.7%	1.5%	1.2%
16	L	L	L	L	2.9%	2.5%	2.4%	2.0%	1.7%	1.4%

Table 4.3. Summary of Numerical Experiment with d=6

Case	Parameters Setting				Maximum percentage difference vs V_u			Mean percentage difference vs V_u		
	η	σ	S	C	PB	MY	SH	PB	MY	SH
17	H	H	H	H	2.5%	2.1%	1.6%	1.8%	1.5%	1.2%
18	L	H	H	H	7.8%	7%	6.1%	4.2%	3.8%	2.9%
19	H	L	H	H	1.3%	1.1%	0.9%	0.6%	0.5%	0.4%
20	L	L	H	H	1.9%	1.4%	1.2%	1.1%	0.8%	0.7%
21	H	H	L	H	8%	7.3%	6.4%	6.2%	5.4%	4.6%
22	L	H	L	H	11%	9.8%	8.8%	8.6%	7.6%	6.8%
23	H	L	L	H	3.6%	3.2%	2.2%	2.8%	2.5%	1.8%
24	L	L	L	H	5.5%	4.9%	4.6%	3.5%	3.1%	2.6%
25	H	H	H	L	1.3%	1.1%	1.1%	1.1%	0.8%	0.7%
26	L	H	H	L	3.1%	2.8%	2.2%	1.6%	1.3%	1.1%
27	H	L	H	L	0.6%	0.5%	0.6%	0.5%	0.4%	0.4%
28	L	L	H	L	0.9%	0.8%	0.8%	0.6%	0.4%	0.4%
29	H	H	L	L	4.7%	4.1%	3.6%	3.4%	2.9%	2.7%
30	L	H	L	L	5.7%	4.8%	4.5%	4.6%	3.9%	3.5%
31	H	L	L	L	2.2%	1.9%	1.5%	1.8%	1.5%	1.3%
32	L	L	L	L	3.1%	2.7%	2.3%	2.1%	2.8%	1.5%

gaps (note that this does not mean the performance of our heuristic worsen since we are comparing it with the upper bound profit). One can verify that when η takes a low value, the purchase cost is higher than it could be when η takes a high value. The purchase cost in turn will determine the overage cost and salvage value loss. For example, in the scenario 6 of Table 4.2 and the scenario 22 of Table 4.3, where η takes a low value, C takes a high value, and S takes a low value, the overage cost and salvage loss are the highest among all different settings. Coincidentally, the gap between our heuristics and the upper bound profit is the widest. This suggests that when the profit margin is low and salvage loss is severe, it is very important for the firm to have the inventory return condition and to jointly optimize the assortment and inventory controls.

We also did a small scale experiment using those problem instances with a large $\frac{V_u - V_{SH}}{V_u}$. We found that a modification on the assortment plan under the SH heuristic seems to be promising. For example, when the system first enters state $t (> t_2)$, under the SH heuristic, it is to discontinue product-1 immediately. We propose a modification as follows: product-1 is discontinued only if the initial inventory of product-1 is strictly below a threshold (say $x_1 < \beta_1(t, 0)$, where $\beta_1(t, 0)$ is the myopic threshold defined in equation (4.14) with $x_0 = 0$); otherwise, keep offering both products. This one-period delay in the assortment change (we introduce a similar delay when the system first enters state t where $t_1 \leq t \leq t_2$) can improve the time-average profit of the SH heuristic by about 1 to 2%. In our on-going research, we will explore the direction to find a tighter bound or solve the joint optimal control policy.

Chapter 5

The Single-Product, Multiple-Components ATO Model

The ATO system keeps inventory at the component level. After customer orders arrive, the required components are pulled from the inventory and the end products are assembled and delivered to the customers. Because the ATO system can simultaneously achieve lower inventory costs, quicker response to market, and greater product variety as compared with the MTS system, in the past twenty years, it has been broadly adopted in the electronic industry. Even the automobile industry, which now predominantly follows the MTS system, is deliberating upon the adoption of the the ATO system [6].

In the ATO system, because the component obsolescence occurs at the component level but affects the system at the product level, several challenges arise. The first challenge is to select the appropriate technology for each component. The second challenge is to determine inventory replenishment levels across different components while taking price erosions and short component life cycles into account. In the next two Chapters, we study the technology and inventory joint control policy in a single-product, multi-component, periodic-review ATO system.

5.1 Technology Innovation Processes and Co-existing Technologies

We consider an ATO system with a single product assembled from m components. Let $E = \{1, 2, \ldots, m\}$ be the index set of components. Each component is subject to generational change or obsolescence due to the introduction of the next generation technology for that component. We assume that the innovation process for each component is exogenous and follows a *discrete-time phase-type* (PH) renewal process, characterized by the inter-arrival times of new technologies that have a PH distribution. A PH renewal process is a natural representation of an innovation process. The development of a new technology usually undergoes several phases with each phase lasting a geometrically distributed amount of time. More specifically, suppose the innovation for component i, $i \in E$, consists of d_i development phases. Denote $d_i + 1$ as the completion state. As soon as the innovation process visits state $d_i + 1$, the latest technology of that component is released to the market. Let $p^i_{t,t'}$ be the probability that the development of component i moves from phase t to phase t' in one period, $t, t' = 1, 2 \ldots, d_i$. For expositional simplicity, we let $p^i_{tt'} = 0$ if $t' < t$, i.e., the phase of a given pipeline technology is non-decreasing as time elapses (our result is still valid if the next phase is stochastically increasing in t_i). Let $q_i(t) \equiv 1 - \sum_{t'=t}^{d_i} p^i_{tt'}$, which is understood as the probability that the innovation occurs in the next period, given the current phase t. Consistent with the standard definition of the discrete-time PH renewal process (Neuts [33]), we treat $d_i + 1$ as an *instantaneous* state, that is, as soon as the process enters state $d_i + 1$, it instantaneously jumps to state t with probability α^i_t, $\sum_{t=1}^{d_i} \alpha^i_t = 1$. In our problem context, its means that as soon as an innovation occurs, the development of the next pipeline technology for the component immediately starts in phase t with probability α^i_t, $t = 1, \ldots, d_i$. By allowing an arbitrary initial phase of a pipeline technology, we model the scenario of overlapping developments of several future technologies so that when the current pipeline technology is released, the next pipeline technology may have already undergone some early development stages. Following Neuts [33], the innovation time for component i is a discrete-time PH random variable with representation $(\{\alpha^i_t\}, \{p^i_{tt'}\})$, $i \in E$. The PH distribution is a powerful tool to characterize a stochastic technology innovation process. It is known that any distribution can be represented by a PH distribution by choosing appropriate values for $(\{\alpha^i_t\}, \{p^i_{tt'}\})$ and d_i.

Let T^n_i be the state of component i at time n, $T^n_i \in \{1, 2, \ldots, d_i\}$. For $i \in E$, the one-step transition probability of $\{T^n_i, n \geq 0\}$ satisfies

$$p^i(t'|t) = P(T^{n+1}_i = t'|T^n_i = t) = p^i_{t,t'} + \alpha^i_{t'} q_i(t), \quad t, t' = 1, 2, \ldots, d_i, \tag{5.1}$$

where, by assumption, $p^i_{t,t'} = 0$ if $t' < t$. The above probability means that the innovation process can visit state t' from state t either via incremental phase change, which occurs with probability $p^i_{t,t'}$, $t' \geq t$, or via innovation, which occurs with probability $\alpha^i_{t'} q_i(t)$.

Let $\mathbf{T}^n = \{T^n_i\}$ be the m-dimensional state vector of various pipeline technologies at time n. Denote $\mathbf{t} = \{t_i\}$ and $\mathbf{t}' = \{t'_i\}$ as possible realizations of $\mathbf{T}^n$. If the marginal innovation processes $\{T^n_i, n \geq 1\}$, $i \in E$, are independent, then the one-step transition probability of process $\{\mathbf{T}^n, n \geq 1\}$ can be expressed as

$$p(\mathbf{t}'|\mathbf{t}) = P(\mathbf{T}^{n+1} = \mathbf{t}'|\mathbf{T}^n = \mathbf{t}) = \prod_{i \in E} p^i(t'_i|t_i), \tag{5.2}$$

where $p^i(t'_i|t_i)$ is given in (5.1). If the marginal innovation processes $\{T^n_i, n \geq 1\}$, $i \in E$, are PH renewal processes but mutually dependent (which will be the case if the development of one set of technologies may accelerate or decelerate the development of another set of related technologies), then the joint technology innovation process is called a *multivariate discrete-time PH process.* Note that each marginal process possesses the Markovian property, that is, for each component $i \in E$, the next state of component i, T_i^{n+1}, depends only on T_i^n, but not on the states of other components.

Next, consider the existing technologies of various components on the market. We assume that the firm only uses the latest two generations of technologies for various components to assemble the final product. Of the two generations of technologies coexisting in the market, we label the latest technology as generation 0 and the earlier technology as generation 1. We assume that a technology will become obsolete as soon as its generation index reaches 2, and generation 2 inventory must be immediately salvaged. For convenience, we call a type-i component that currently uses a generation-j technology a type i–j component, $i \in E$, $j = 0, 1$. We denote $\mathbf{j} = (j_1, \ldots, j_m)$ as the current product configuration, where $j_i = 1$ if a type $i - 1$ component is adopted and zero otherwise, $i \in E$. Hereafter, the end product that has configuration $\mathbf{j}$ is called Config-$\mathbf{j}$. When an innovation for component i occurs, the newly released technology is introduced to the market as a type $i - 0$ component. Consequently, the component that was labelled as a type $i - j$ component in the last period will become a type $i - (j + 1)$ component, $j = 0, 1$. As noted earlier, a type $i - 2$ component is considered outdated and must be salvaged immediately.

Remark 2. *An alternative approach to model the innovation process is by the age of an existing technology since its introduction to the market. The age-based approach is* *applicable if the innovation process is not directly observable by the firm so a component's age is used as a proxy to the development status of the pipeline technology. The disadvantage of the age-based approach is that if the innovation time is relatively long (usually in the frequency of months) as compared to the inventory review period (usually in the frequency of days), then the firm has to collect the information for a large number of states (ages), even though the innovation process may not experience significant changes over several inventory review periods. In contrast, the phase-based approach assumes that the development status of a pipeline technology is observable. As compared to the age-based approach, the phase-based approach uses a relatively small number of states to register the significant progress in the innovation process, and, hence, requires less data to implement joint technology and inventory control. The characterization of a PH distribution can be obtained from managerial judgment or by estimating the innovation process from historical data.*

5.2 The Joint Technology and Inventory Management System

We consider the lost sales model. We assume that demands in different periods are independent and identically distributed (iid) random variables and denote the generic version of demands by D and its distribution by $F(\cdot)$. The underlying assumption of iid demands is that the firm uses pricing as a tool to foster stable demand for different product configurations over time, where different product offerings may attract consumers from different market segments. To envision this, suppose a consumer values Config-**j** in state **t** by $R(\mathbf{t}, \mathbf{j})$, $j \in \{0,1\}^m$. The firm can set up the price $r(\mathbf{t}, \mathbf{j})$ such that $P(R(\mathbf{t}, \mathbf{j}) \geq r(\mathbf{t}, \mathbf{j})) =$ constant, independent of the configuration offering and its phase vector. Even though consumers may differ in their individual preferences, the aggregate demand across the consumer population for each configuration remains stochastically identical. In Section 6.3, we consider an extension where the demand pattern is dependent on the product offering. We also let the lead time for each component be zero (In Section 6.3 we relax this assumption).

We assume that for each component, its cost parameters are generation and phase-dependent. More specifically, let $r_i(t, j)$, $c_i(t, j)$, and $h_i(t, j)$ be the selling price, purchase cost, and holding cost of a type $i-j$ component in phase t, $1 \leq t \leq d_i$, $j = 0, 1$, $i \in E$. Our assumption conforms with the common practice that R&D progress of future technologies triggers procurement cost discounts by the supplier, which subsequently leads to price cuts by the firm. Recall that a component will be salvaged as soon as its generation index reaches 2. Let $c_i(t, 2)$ be the salvage value of such a component, where $c_i(t, 2) = 0$ if an obsolete component becomes worthless. The following assumption states that the development or pending release of the pipeline technology erodes the value of an existing technology, and that a new technology is worth more than an old technology.

Assumption A0. i) Selling price $r_i(t, j)$ and purchase cost $c_i(t, j)$ are decreasing in t, $i \in E$. ii) $r_i(t, 0) \geq r_i(t', 1)$ and $c_i(t, 0) \geq c_i(t', 1) \geq c_i(t'', 2)$, for any t, t', t'', $i \in E$.

We denote the state of the system at the beginning of a period by $(\mathbf{t}; \mathbf{j}, \mathbf{x})$, where $\mathbf{t}$ is the phase vector of pipeline components, $\mathbf{j}$ is the current end-product configuration, and $\mathbf{x} = (x_1, \ldots, x_m)$ is the initial inventory vector, with x_i the initial inventory of component i, $i \in E$. We denote the state of the system *after* technology selection and inventory decisions by $(\mathbf{t}; \mathbf{k}, \mathbf{y})$, where $\mathbf{k}$ is the new product configuration and $\mathbf{y} = (y_1, \ldots, y_m)$ represents the inventory levels after salvage and replenishment decisions. Observe that, if

$j_i > k_i$, the firm replaces the existing type $i-1$ components with type $i-0$ components, receives salvage revenue $c_i(t_i,1)x_i$ for the type $i-1$ components, and pays purchase cost $c_i(t_i,0)y_i$ for type $i-0$ components. Hereafter, we call such an action *upgrading*. If $j_i = k_i$, the firm retains the current technology and brings its inventory level to y_i, with purchase cost $c_i(t_i,j)(x_i - y_i)$. If $j_i < k_i$, the firm replaces the initial generation-0 components with generation-1 components, receives salvage revenue $c_i(t_i,0)x_i$ and pays purchase cost $c_i(t_i,1)y_i$. We call such an action *downgrading*. We assume that the price (cost) of a configuration equals the sum of the prices (cost) of its constituting components. Denote

$$c(\mathbf{t},\mathbf{j}) = \sum_{i \in E} c_i(t_i, j_i) \quad \text{and} \quad r(\mathbf{t},\mathbf{j}) = \sum_{i \in E} r_i(t_i, j_i) \tag{5.3}$$

as the purchase cost and selling price, respectively, of a single unit of Config-$\mathbf{j}$ in state $\mathbf{t}$.

5.3 Discounted Dynamic Programming Formulation

Let $V(\mathbf{t};\mathbf{j},\mathbf{x})$ be the maximal expected total discounted profit over the infinite planning horizon with the starting state $(\mathbf{t};\mathbf{j},\mathbf{x})$ and discount factor $0 \leq \lambda < 1$. Let $Z(\mathbf{y}) = \min_{i \in E}\{y_i, D\}$ be the product demand that can be satisfied by available inventory $\mathbf{y}$. Suppose the product configuration has been updated from Config-$\mathbf{j}$ to Config-$\mathbf{k}$.

Let $\mathbf{0}$ be the zero vector. Then, the optimality equation satisfies

$$V(\mathbf{t};\mathbf{j},\mathbf{x}) = \max_{\substack{\mathbf{k}\in\{0,1\}^m \\ \mathbf{y}\geq\mathbf{0}}} \left\{ \sum_{i\in E} c_i(t_i, j_i)x_i + W(\mathbf{t},\mathbf{k},\mathbf{y}) \right\} \tag{5.4}$$

$$= \sum_{i\in E} c_i(t_i, j_i)x_i + \max_{\substack{\mathbf{k}\in\{0,1\}^m \\ \mathbf{y}\geq\mathbf{0}}} \{W(\mathbf{t},\mathbf{k},\mathbf{y})\} = \sum_{i\in E} c_i(t_i, j_i)x_i + W(\mathbf{t}), \tag{5.5}$$

where the first term of (5.4) is the procurement cost of on-hand inventory and $W(\mathbf{t},\mathbf{k},\mathbf{y})$ is the expected maximal profit attainable given that Config-$\mathbf{k}$ is selected and inventory is replenished to $\mathbf{y}$. We can factor the first term out since it is a constant with respect to the maximum operator. We also denote $W(\mathbf{t}) \equiv \max_{\mathbf{k}\in\{0,1\}^m,\ \mathbf{y}\geq\mathbf{0}}\{W(\mathbf{t},\mathbf{k},\mathbf{y})\}$ as the maximum expected discounted profit, with zero initial inventory. Next we derive the optimality equation for $W(\mathbf{t})$. Let $M_i(t_i)$ be the binary random variable that is equal to 1 if component i introduces the new technology in the next period (with probability $p^i_{t_i,d_i+1}$) and 0 otherwise, given current phase t_i, $i \in E$. Then

$$W(\mathbf{t}) = \max_{\substack{\mathbf{k}\in\{0,1\}^m \\ \mathbf{y}\geq 0}} \left\{ \begin{array}{l} r(\mathbf{t},\mathbf{k})E[Z(\mathbf{y})] - \sum_{i\in E} c_i(t_i,k_i)y_i - \sum_{i\in E} h_i(t_i,k_i)E[y_i - Z(\mathbf{y})] \\ +\lambda \sum_{i\in E} E[c_i(T'_i, k_i + M_i(t_i)|t_i)]E[y_i - Z(\mathbf{y})] + \lambda \sum_{\mathbf{t'}} \mathbf{p}(\mathbf{t'}|\mathbf{t})\mathbf{W}(\mathbf{t'}) \end{array} \right\}, \tag{5.6}$$

where $r(\mathbf{t},\mathbf{k})$ is defined in (5.3). We interpret the RHS of (5.6) as follows. The first term is the expected sales revenue. The second and third terms are the procurement and holding costs. The fourth term is the expected total discounted market value of the leftover inventory at the beginning of the next period. Finally, the last term is the maximal expected discounted profit from the next period onward, with initial inventory

0. The expected market value of leftover inventory (the fourth term of (5.6)) can be computed as follows. With probability $p^i_{t_i,t'_i}$, no innovation occurs and the innovation process moves to phase $t'_i \geq t_i$ in the next period, and one unit of leftover component i will be worth $c_i(t'_i, k_i)$; with probability $q_i(t_i)a^i_{t'_i}$, the innovation occurs and the next pipeline technology starts in phase t'_i, and one unit of leftover component i will be worth $c_i(t'_i, k_i + 1)$, $i \in E$. Therefore,

$$E[c_i(T'_i, k_i + M_i(t_i)|t_i)] = \sum_{t'_i=t_i}^{d_i} p^i_{t_i,t'_i} c_i(t'_i, k_i) + \sum_{t'_i=1}^{m} q_i(t_i)a^i_{t'_i} c_i(t'_i, k_i + 1), \quad i \in E. \quad (5.7)$$

Define

$$H_i(t_i, j) \equiv h_i(t_i, j) - \lambda[\sum_{t'_i=t_i}^{d_i} p^i_{t_i,t'_i} c_i(t'_i, j) + q_i(t_i) \sum_{t'_i=1}^{d_i} a^i_{t'_i} c_i(t'_i, j + 1)] \qquad (5.8)$$

as the *effective holding cost* of a type $i - j$ component with phase t_i, $i \in E$. Then, $c_i(t_i, j) + H_i(t_i, j)$ is the *overage cost* of a type $i - j$ component with phase t_i, which is the sum of the holding cost and the expected loss of market value of a type $i - j$ component in a single period. From Assumption A0 and our assumption that the phase of a pipeline technology cannot decrease, it can be easily argued that the overage cost $c_i(t_i, j) + H_i(t_i, j)$ is non-negative.

From (5.5) to (5.8), we rewrite (5.6) as

$$
\begin{aligned}
W(\mathbf{t}) &= \max_{\substack{\mathbf{k}\in\{0,1\}^m \\ \mathbf{y}\geq 0}} \left\{ r(\mathbf{t},\mathbf{k})E[Z(\mathbf{y})] - \sum_{i\in E} c_i(t_i,k_i)y_i - \sum_{i\in E} H_i(t_i,k_i)E[y_i - Z(\mathbf{y})] + \lambda \sum_{\mathbf{t}'} p(\mathbf{t}'|\mathbf{t})W(\mathbf{t}') \right\} \\
&= \max_{\substack{\mathbf{k}\in\{0,1\}^m \\ \mathbf{y}\geq 0}} \left\{ g(\mathbf{t},\mathbf{k},\mathbf{y}) + \lambda \sum_{\mathbf{t}'} p(\mathbf{t}'|\mathbf{t})W(\mathbf{t}') \right\} = \max_{\substack{\mathbf{k}\in\{0,1\}^m \\ \mathbf{y}\geq 0}} \{g(\mathbf{t},\mathbf{k},\mathbf{y})\} + \lambda \sum_{\mathbf{t}'} p(\mathbf{t}'|\mathbf{t})W(\mathbf{t}'). \quad (5.9)
\end{aligned}
$$

Here, $g(\mathbf{t},\mathbf{k},\mathbf{y})$ is the *expected single-period profit*, given that the phase vector is $\mathbf{t}$, the product configuration is $\mathbf{k}$, and the inventory vector is $\mathbf{y}$.

5.4 Myopic Policy

It is evident from (5.9) that the myopic policy that maximizes the single-period profit function $g(\mathbf{t};\mathbf{k},\mathbf{y})$ over configuration $\mathbf{k} \in \{0,1\}^m$ and inventory vector $\mathbf{y} \geq \mathbf{0}$ is also globally optimal. The focus of this section is to derive the efficient method to obtain the optimal myopic policy.

5.4.1 Balanced Base-Stock Policy

It can be shown that for each given configuration $\mathbf{k}$ and vector $\mathbf{t}$, the optimal order quantities, denoted by $y_i(\mathbf{t},\mathbf{k})$, $i \in E$, are equal, $y_1(\mathbf{t},\mathbf{k}) = \cdots = y_m(\mathbf{t},\mathbf{k}) = y(\mathbf{t},\mathbf{k})$. Following the convention, we call the policy that orders an equal amount of inventory for different components the *balanced base-stock policy.* With a slight abuse of notation, we write the single-period profit function with the balanced base-stock level y, as $g(\mathbf{t},\mathbf{k},y)$. Also let $H(\mathbf{t},\mathbf{k}) = \sum_{i\in E} H_i(t_i,k_i)$, $k_i = 0,1$, be the effective holding cost of one unit of Config-$\mathbf{k}$ product with phase vector $\mathbf{t}$. Then the single-period cost function under the balanced base-stock level y is simplified to

$$g(\mathbf{t}, \mathbf{k}, \mathbf{y}) = g(\mathbf{t}, \mathbf{k}, y) = [r(\mathbf{t}, \mathbf{k}) - H(\mathbf{t}, \mathbf{k})]E[Z(y)] - [c(\mathbf{t}, \mathbf{k}) + H(\mathbf{t}, \mathbf{k})]y, \tag{5.10}$$

where $E[Z(y)] = E[\min(y, D)]$. Evidently, $g(\mathbf{t}, \mathbf{k}, y)$ is the single-product, single-period, newsvendor profit function, where $r(\mathbf{t}, \mathbf{y}) - c(\mathbf{t}, \mathbf{k})$ is the product profit margin and $c(\mathbf{t}, \mathbf{k}) + H(\mathbf{t}, \mathbf{k})$ the product overage cost. It is well known that, given $\mathbf{t}$ and $\mathbf{k}$, $g(\mathbf{t}, \mathbf{k}, y)$ is concave in y and maximized when y equals the product-level newsvendor solution. Therefore, we have the following theorem.

LEMMA 5.1. *Let* $\mathbf{y}(\mathbf{t}, \mathbf{k})$ *be the optimal order quantity vector that maximizes* $g(\mathbf{t}; \mathbf{k}, \mathbf{y})$ *for fixed Config-*$\mathbf{k}$ *and vector* $\mathbf{t}$ *and let* $y_i(\mathbf{t}, \mathbf{k})$ *be the* i*-th element of* $\mathbf{y}(\mathbf{t}, \mathbf{k})$*. Then* $y_1(\mathbf{t}, \mathbf{k}) = \cdots = y_m(\mathbf{t}, \mathbf{k}) = y(\mathbf{t}, \mathbf{k})$*. In addition,*

$$y(\mathbf{t}, \mathbf{k}) = F^{-1}\left(\frac{r(\mathbf{t}, \mathbf{k}) - c(\mathbf{t}, \mathbf{k})}{r(\mathbf{t}, \mathbf{k}) + H(\mathbf{t}, \mathbf{k})}\right), \tag{5.11}$$

where F^{-1} *is the inverse of the cumulative distribution function of demand* D*.*

5.4.2 Dominant Configurations and Interval Partitioning Algorithm

Now, we turn our attention to determining the optimal configuration $\mathbf{k}^*(\mathbf{t})$, knowing that given $\mathbf{k}^*(\mathbf{t})$, $y(\mathbf{t}, \mathbf{k}^*(\mathbf{t}))$ can be obtained by (5.11). The optimal configuration $\mathbf{k}^*(\mathbf{t})$ can certainly be determined by exhaustively evaluating the profit functions over 2^m configurations, which is obviously inefficient. Next, we develop an algorithm, called the *Interval Partitioning Algorithm* (IPA), that can efficiently identify the optimal configuration.

Recall that $g(\mathbf{t}, \mathbf{k}, y)$ is continuous and concave in y. Therefore, $\max_{\mathbf{k}\in\{0,1\}^m}\{g(\mathbf{t}, \mathbf{k}, y)\}$, the maximum frontier of continuous concave functions, is continuous and *piecewise concave* in y (see Figure 5.1). Each of such a "concave piece" in a subinterval corresponds to the profit function of a frontier configuration (call it a *dominant configuration* hereafter) that outperforms all other configurations in the subinterval. Let us consider how to

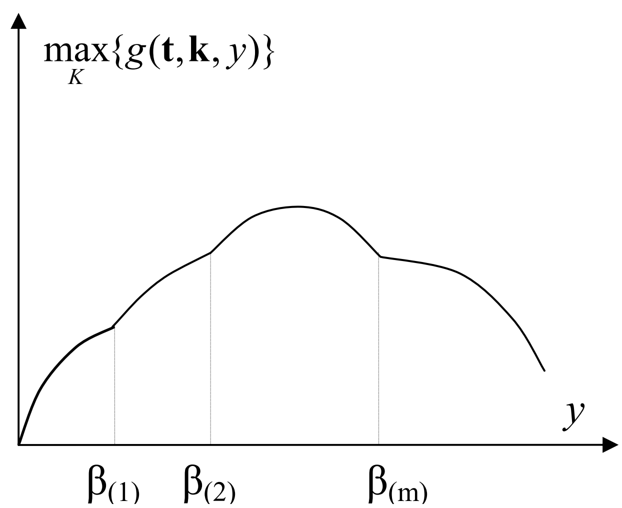

Fig. 5.1. Piecewise Concavity of $\pi(T, y)$ in y

identify those dominant configurations. For this, we decompose the product-level profit function, $g(\mathbf{t}, \mathbf{k}, y)$, into the sum of component-level profit functions,

$$g(\mathbf{t}, \mathbf{k}, y) = \sum_{i=1}^{m} g_i(t_i, k_i, y), \tag{5.12}$$

where

$$g_i(t_i, k_i, y) = [r_i(t_i, k_i) - c_i(t_i, k_i)]E[Z(y)] - [c_i(t_i, k_i) + H_i(t_i, k_i)]E[y - Z(y)]. \tag{5.13}$$

The differences of the profit margins and the differences of the overage costs between the new and old technologies, denoted by a_i and b_i (for simplicity we suppress their dependence on t_i) are, respectively,

$$a_i = [r_i(t_i, 0) - c_i(t_i, 0)] - [r_i(t_i, 1) - c_{i1}(t_i, 1)], \tag{5.14}$$

$$b_i = [c_i(t_i, 0) + H_i(t_i, 0)] - [c_i(t_i, 1) + H_i(t_i, 1)]. \tag{5.15}$$

Then, we can write the difference of the profit functions under the new and old technologies as

$$\begin{aligned} \Delta g_i(y) \equiv g_i(t_i, 0, y) - g_i(t_i, 1, y) &= a_i E[Z(y)] - b_i E[y - Z(y)] \\ &= [a_i + b_i]E[Z(y)] - b_i y. \end{aligned} \tag{5.16}$$

Lemma 5.2. **(a)** *$\Delta g_i(y)$ is a concave function if $a_i + b_i \geq 0$ and a convex function if $a_i + b_i \leq 0$.*

(b) *Let β_i be the largest value of y such that $\Delta g_i(y) = 0$, that is, β_i is the largest solution of*

$$\frac{b_i}{a_i + b_i} = \frac{E[\min(y, D)]}{y}, \tag{5.17}$$

where we let $\beta_i \equiv \infty$, if $\Delta g_i(y) \geq 0$ or $\Delta g_i(y) \leq 0$ for all $y \geq 0$. Then $\Delta g_i(y) \geq 0$ ($\Delta g_i(y) \leq 0$) if and only if $a_i \geq 0$ ($a_i < 0$) and $y \leq \beta_i$.

(c) *If $a_i \geq 0$ ($a_i < 0$), then $\Delta g_i(y) < 0 (> 0)$ and is decreasing (increasing) when $y > \beta_i$.*

Lemma 5.2 states that for each component i, the unique break-even point β_i partitions $[0, \infty)$ into two subintervals $[0, \beta_i]$ and (β_i, ∞) such that one technology generates a larger profit than another in each subinterval. We call the technology that generates a larger profit the dominant technology and the interval over which the technology dominates its *dominant interval.* Also, from (5.14) and (5.15), $a_i \geq 0$ ($a_i < 0$) means the type $i - 0$ ($i - 1$) component has a higher profit margin, and $b_i \geq 0$ ($b_i < 0$) means the type $i - 1$ ($i - 0$) component has a lower overage cost. Thus, Lemma 5.2 indicates that it is optimal to use the technology with the higher profit margin if y is below β_i; otherwise, it is optimal to use the technology with the lower overage cost. Let us rank order β_i such that

$$0 \equiv \beta_{(0)} \leq \beta_{(1)} \leq \beta_{(2)} \leq \cdots \leq \beta_{(m)} \leq \beta_{(m+1)} \equiv \infty. \tag{5.18}$$

Then $\beta_{(\ell)}$, $\ell = 0, 1, \ldots, m + 1$, partition interval $[0, \infty)$ into at most $m + 1$ subintervals. Consider the difference function $\Delta g_i(y)$ for $y \in [\beta_{(\ell)}, \beta_{(\ell+1)})]$, $\ell = 0, 1, \ldots, m$, $i \in E$. Lemma 5.2 (b) states that if $\beta_i \leq \beta_{(\ell)}$ ($\beta_i > \beta_{(\ell)}$), then the old (new) technology dominates the new (old) technology in $[\beta_{(\ell)}, \beta_{(\ell+1)}))$ if and only if $a_i \geq 0$. By selecting the better technology for each component in $[\beta_{(\ell)}, \beta_{(\ell+1)})$ as described above, we obtain a unique configuration, denoted by $\mathbf{k}^{(\ell)}$, that outperforms all other configurations for $y \in [\beta_{(\ell)}, \beta_{(\ell+1)})$, $\ell = 0, 1, \ldots, m$. Since there are at most $(m+1)$ such intervals, there are at most $(m+1)$ such configurations that dominate all others for any $y \geq 0$. Therefore, to identify the optimal configuration, we only need to search among dominant configurations, rather than all possible 2^m configurations. The following theorem shows that for a dominant configuration to be optimal, its optimal order quantity must belong to its dominant interval. This theorem is also the foundation of the Interval Partitioning Algorithm (IPA).

LEMMA 5.3. *Let* $\mathbf{k}^{(\ell)}$ *be the dominant configuration of interval* $[\beta_{(\ell)}, \beta_{(\ell+1)})$, $\ell = 0, 1, 2, \ldots, m$, *and* $y(\mathbf{t}, \mathbf{k}^{(\ell)})$ *the optimal inventory level with Config-*$\mathbf{k}^{(\ell)}$. *Then*

a. $(\mathbf{k}^{(\ell)}, y(\mathbf{t}, \mathbf{k}^{(\ell)}))$ *is a locally optimal solution of* $g(\mathbf{t}, \mathbf{k}, y)$ *if and only if* $y(\mathbf{t}, \mathbf{k}^{(\ell)}) \in [\beta_{(\ell)}, \beta_{(\ell+1)})$.

b. The globally optimal solution $(\mathbf{k}^*(\mathbf{t}), y^*(\mathbf{t}))$ *has the property that for each component i,* $y^*(\mathbf{t})$ *belongs to the dominant interval of technology* $k_i^*(\mathbf{t})$, $i \in E$.

Based on Lemma 5.3, we develop an Interval Partitioning Algorithm to solve the optimal configuration.

ALGORITHM 5.1. *The Interval Partitioning Algorithm.*

Step 1 *[Construct dominant intervals]: For each* $i \in E$, *compute* a_i, b_i *and* β_i, *using (5.14), (5.15), and (5.17), respectively. Rank* β_i, $i \in E$, *according to (5.18).*

Step 2 *[Construct dominant configurations]: For each subinterval* $[\beta_{(\ell)}, \beta_{(\ell+1)})$*, generate its dominant configuration, denoted by* $\mathbf{k}^{(\ell)} = (k_1^{(\ell)}, \ldots, k_m^{(\ell)})$*, as follows. For each* $i \in E$*,*

$$\text{If } \beta_i \le \beta_{(\ell)}, \text{ then } k_i^{(\ell)} = \begin{cases} 1 & a_i \ge 0, \\ 0 & a_i < 0 \end{cases} \tag{5.19}$$

$$\text{If } \beta_i > \beta_{(\ell)}, \text{ then } k_i^{(\ell)} = \begin{cases} 0 & a_i \ge 0 \\ 1 & a_i < 0. \end{cases} \tag{5.20}$$

Step 3 *[Obtain locally optimal solutions]: For each* ℓ*, compute* $y(\mathbf{t}, \mathbf{k}^{(\ell)})$ *using (5.11). If* $y(\mathbf{t}, \mathbf{k}^{(\ell)}) \in [\beta_{(\ell)}, \beta_{(\ell+1)})$*, label* $(\mathbf{k}^{(\ell)}, y(\mathbf{t}, \mathbf{k}^{(\ell)}))$ *as a local maximum and compute* $g(\mathbf{t}, \mathbf{k}^{(\ell)}, y(\mathbf{t}, \mathbf{k}^{(\ell)}))$*.*

Step 4 *[Identify the globally optimal solution]: Among the locally optimal solutions* $(\mathbf{k}^{(\ell)}, y(\mathbf{t}, \mathbf{k}^{(\ell)}))$ *identified in Step 3, let*

$$(\mathbf{k}^*(\mathbf{t}), y^*(\mathbf{t})) = \arg\max_{\ell} \{g(\mathbf{t}, \mathbf{k}^{(\ell)}, y(\mathbf{t}, \mathbf{k}^{(\ell)}))\}.$$

Then $(\mathbf{k}^*(\mathbf{t}), y^*(\mathbf{t}))$ *is the globally optimal solution for* $g(\mathbf{t}, \mathbf{k}, y)$*.*

It is worth noting that the state change of a pipeline technology may have a cascading effect not only on the technological choice for that component but also on other components. Certainly, the optimal order quantity will also be affected. Indeed, the introduction of a pipeline technology may trigger downgrades of several components, including itself. This will happen if, with the new cost parameters, the firm increases the inventory level, and needs to switch to the technologies with low overage costs in order to reduce the risk of overstocking. Similarly, the advance of a pipeline technology may trigger upgrade actions of several components. This will be the case if the firm decreases the inventory level and needs to switch to the technologies with high profit margins to offset the low volume of sales. Therefore, the truly joint optimal technology and inventory management can only be achieved by centralized control that utilizes all the information across different components.

EXAMPLE 1. *We illustrate IPA using the following 5-component system with the cost data showing in Table 5.1. Let demand be an Erlang random variable with mean 4 and variance 8. The discount factor* λ *is set at 0.9. We assume* $d_i = 1$*, so the innovation time for each component is geometrically distributed with innovation rate given in the fifth column of Table 5.1. Since* $\mathbf{t} \equiv \mathbf{1}$*, we suppress* $\mathbf{t}$ *from all expressions. The salvage value of the generation-2 technology is set at* $c_i(2) \equiv 0$.

Table 5.1. A PC Example

Component	*Generation 0*		*Generation 1*		*Innovation Probability*
	Price ($)	*Cost ($)*	*Price ($)*	*Cost ($)*	p^i_{t,d_i+1}
1	250	120	170	50	0.25
2	100	40	82	20	0.25
3	210	163	140	100	0.1
4	80	40	55	30	0.25
5	100	60	73	35	0.2

The result of IPA is reported in Table 5.2, which shows that $\{\beta_i\}$ *partition* $[0, \infty)$ *into 4 subintervals. In each subinterval, we select the technology of each component using (5.19) or (5.20), and determine whether the resultant configuration is locally optimal by Step 3 of IPA. In this example, we have a unique locally, and hence globally, optimal solution* $\mathbf{k}^* = \{1, 1, 0, 0, 1\}$ *and* $y^* = 5.63$*. Here, we evaluate only 4 dominant configurations, rather than* $2^5 = 32$ *alternatives.*

Table 5.2. The Result of IPA for the PC Example

				Subintervals			
Component	a_i	b_i	β_i	[0,5.03)	[5.03,5.52)	[5.52,6.58)	[6.58,∞)
1	10	18.6	5.03	0	1	1	1
2	-2	4	∞	1	1	1	1
3	7.5	9.1	6.58	0	0	0	1
4	15	-2.5	∞	0	0	0	0
5	2	3.2	5.52	0	0	1	1
$r(\mathbf{k}^{(\ell)}) - \mathbf{c}(\mathbf{k}^{(\ell)})$				319.5	309.5	307.5	300
$c(\mathbf{k}^{(\ell)}) + H(\mathbf{k}^{(\ell)})$				113.1	94.6	91.4	82.3
$y(\mathbf{k}^{(\ell)})$				5.27	5.57	5.63	5.80
Local Optimum?				No	No	Yes	No
$g(\mathbf{k}^{(\ell)}, y(\mathbf{k}^{(\ell)}))$				N/A	N/A	850.9	N/A

THEOREM 5.1. *Both the profit margin* $r(\mathbf{t}, \mathbf{k}^{(\ell)}) - c(\mathbf{t}, \mathbf{k}^{(\ell)})$ *and the overage cost* $c(\mathbf{t}, \mathbf{k}^{(\ell)}) + H(\mathbf{t}, \mathbf{k}^{(\ell)})$ *associated with the dominant configuration* $\mathbf{k}^{(\ell)}$ *decrease in* ℓ*.*

Proof of Theorem 5.1.

We suppress t_i or $\mathbf{t}$ whenever convenient. Consider two consecutive subintervals $[\beta_{(\ell)}, \beta_{(\ell+1)})$ and $[\beta_{(\ell+1)}, \beta_{(\ell+2)})$ and their corresponding dominant configurations $\mathbf{k}^{(\ell)}$

and $\mathbf{k}^{(\ell+1)}$, $\ell = 0, 1, ..., m-1$. Suppose $\beta_{(\ell+1)} < \infty$ and $\beta_{(\ell+1)} = \beta_i$. Consider $a_i \geq 0$ first. As suggested in Step 2 of Algorithm 5.1, Config-$\mathbf{k}^{(\ell)}$ and Config-$\mathbf{k}^{(\ell+1)}$ use the same technology for each component except for component i, where Config-$\mathbf{k}^{(\ell)}$ uses technology 0 and Config-$\mathbf{k}^{(\ell+1)}$ uses technology 1. Therefore,

$$\begin{aligned}
[r(\mathbf{k}^{(\ell)}) - c(\mathbf{k}^{(\ell)})] - [r(\mathbf{k}^{(\ell+\mathbf{1})}) - c(\mathbf{k}^{(\ell+\mathbf{1})})] &= a_i \geq 0, \\
[c(\mathbf{k}^{(\ell)}) + H(\mathbf{k}^{(\ell)})] - [c(\mathbf{k}^{(\ell+\mathbf{1})}) + H(\mathbf{k}^{(\ell+\mathbf{1})})] &= b_i \geq 0,
\end{aligned}$$

where the last inequality holds since $\beta_i < \infty$. Thus, both the profit margin and overage cost of $\mathbf{k}^{(\ell)}$ decrease in ℓ. Conversely, if $a_i < 0$, Config-$\mathbf{k}^{(\ell)}$ uses generation 1 and Config-$\mathbf{k}^{(\ell+1)}$ uses generation 0 for component i, but both use the same technology for any other component. Hence, the LHSs of the above expressions are $-a_i > 0$ and $-b_i > 0$, where $-b_i > 0$ since $\beta_i < \infty$ means $b_i < 0$. ■

We can interpret Theorem 5.1 in light of the planned service level. It states that as the planned service level increases (i.e., y increases), both the profit margin and overage cost of the corresponding dominant configuration decrease. This is plausible, since a low service level means that the firm is more likely to sell on-hand inventory and less likely to incur the overage cost. Therefore, the configuration with the high profit margin and high overage cost dominates others. As the planned service level increases, the configuration with the low profit margin and low overage cost moves to the frontier, since the firm is less likely to sell on-hand inventory and more likely to incur the overage cost.

5.5 Dynamic Structure of the Myopic Policy

While the myopic solution $(\mathbf{k}^*(\mathbf{t}), y^*(\mathbf{t}))$ prescribes the best course of action for each given state $\mathbf{t}$, it does not reveal the dynamic properties of $(\mathbf{k}^*(\mathbf{t}), y^*(\mathbf{t}))$ as $\mathbf{t}$ evolves. The following questions naturally arise: What is the impact of the R&D progress of pipeline technologies on product configuration and inventory levels? In this section, we develop several structural properties of the myopic solution $(\mathbf{k}^*(\mathbf{t}), y^*(\mathbf{t}))$ as a function of $\mathbf{t}$. We start with the notion of submodularity of a function $f(x, y)$.

Definition 1. *A function $f(x, y)$ is said to be submodular (supermodular) if for any $x_1 \geq x_2$ and $y_1 \geq y_2$, it holds that*

$$f(x_2, y_2) - f(x_1, y_2) \leq (\geq) f(x_2, y_1) - f(x_1, y_1). \tag{5.21}$$

or, equivalently, $f(x_2, y) - f(x_1, y)$ is increasing (decreasing) in y.

Our monotone properties are derived based on the combination of following assumptions. As $\mathbf{t}$ becomes crucial in our analysis, we append t_i to a_i, b_i and β_i, $i \in E$.

A1 $r_i(t_i, j) - c_i(t_i, j)$ and $c_i(t_i, j) + H_i(t_i, j)$ are respectively *decreasing* and *increasing* in t_i for all $i \in E$, $j = 0, 1$; or

A2 $r_i(t_i, j) - c_i(t_i, j)$ and $c_i(t_i, j) + H_i(t_i, j)$ are respectively *increasing* and *decreasing* in t_i for all $i \in E$, $j = 0, 1$;

and

B1 $a_i(t_i)$ and $b_i(t_i)$ are respectively *increasing* and *decreasing* in t_i, for all $i \in E$; or

B2 $a_i(t_i)$ and $b_i(t_i)$ are respectively *decreasing* and *increasing* in t_i, for all $i \in E$.

Assumption A1 (A2) means that as the innovation moves to a more advanced stage, the selling price erodes at a smaller (greater) rate than that of the purchase cost, whereas the purchase cost is reduced at a smaller (greater) rate than that of the effective holding cost. Assumption B1 (B2) states that, as the development of a future technology advances, the profit margin of the new technology changes at a slower (faster) rate than that of the old technology, whereas the overage cost of the new technology changes at a faster (slower) pace than that of the old technology. The above assumptions yield four possible combinations: A1 and B1, A1 and B2, A2 and B1, and A2 and B2. Since A1 is a mirror image of A2, and B1 is a mirror image of B2, the result associated with each combination is also symmetric in nature. We shall only present the result and proof (in the Appendix) under A1-B1, and summarize the results under other combinations in Table 5.3. The following lemma facilitates the derivation of Theorem 5.2, our major results in this section.

LEMMA 5.4. *a. Under Assumption A1 (A2), $g_i(t_i, j, y)$ is a submodular (supermodular) function of t_i and y, $j = 0, 1$. In other words, for $j = 1, 2$,*

$$g_i(t_i, j, y) - g_i(t_i + 1, j, y) \geq (\leq) g_i(t_i, j, y + 1) - g_i(t_i + 1, j, y + 1). \tag{5.22}$$

b. Under Assumption B1 (B2), $g_i(t_i, j, y)$ is a submodular (supermodular) function of t_i and j. In other words, for any given y,

$$g_i(t_i, 0, y) - g_i(t_i, 1, y) \leq (\geq) g_i(t_i + 1, 0, y) - g_i(t_i + 1, 1, y). \tag{5.23}$$

c. *Let* $y(t_i, j)$ *maximize* $g_i(t_i, j, y)$ *for given* t_i *and* j. *Under Assumption A1 (A2),* $y(t_i, j)$ *is decreasing (increasing) in* t_i *for a given* j.

d. *Under Assumption B1 (B2),* $\beta_i(t_i)$ *is increasing (decreasing) in* t_i, $i \in E$.

Proof of Lemma 5.4.

a. Taking the partial derivative of $g_i(t_i, j, y) - g_i(t_i + 1, j, y)$ with respect to y, we obtain

$$\begin{aligned}
&\frac{\partial}{\partial y}\Big(g_i(t_i, j, y) - g_i(t_i + 1, j, y)\Big) \\
= \;& [(r_i(t_i, j) - c_i(t_i, j)) - (r_i(t_i + 1, j) - c_i(t_i + 1, j))]P(D \geq y) \\
&-[(c_i(t_i, j) + H_i(t_i, j)) - (c_i(t_i + 1, j) + H_i(t_i + 1, j)](1 - P(D \geq y)).
\end{aligned}$$

If A1 (A2) holds, then the above expression is positive (negative), which implies that $g_i(t_i, j, y) - g_i(t_i + 1, j, y)$ is increasing (decreasing) in y, or, equivalently, $g_i(t_i, j, y)$ is a submodular (supermodular) function of t_i and y.

b. We have

$$g_i(t_i, 0, y) - g_i(t_i, 1, y) = a_i(t_i)E[\min(y, D)] - b_i(t_i)(y - E[\min(y, D)]).$$

which is increasing (decreasing) in t_i if Assumption B1 (B2) holds.

c. It is well-known that if a function $f(x, y)$ is submodular (supermodular), and if $y(x)$ maximizes $f(x, y)$ for a given x, then $y(x)$ is decreasing (increasing) in x. This fact, in conjunction with part (a), establishes (c).

d. Recall that $\beta_i(t_i)$ is the largest solution of (5.17). Under B1 (B2), $1 + \frac{a_i(t_i)}{b_i(t_i)}$ is increasing (decreasing) in t_i, or, equivalently, the RHS of (5.17), $\frac{b_i(t_i)}{a_i(t_i)+b_i(t_i)}$, is decreasing (increasing) in t_i. In addition,

$$\begin{aligned} \frac{d}{dy}\left(\frac{E[\min(y,D)]}{y}\right) &= \frac{d}{dy}\left(\frac{\int_0^y P(D \geq u)du}{y}\right) = \frac{P(D \geq y)y - \int_0^y P(D \geq u)du}{y^2} \\ &\leq \frac{P(D \geq y)y - \int_0^y P(D \geq u)du}{y^2} = 0, \end{aligned} \tag{5.24}$$

which implies that $\frac{E[\min(y,D)]}{y}$ is decreasing in y. Therefore, $\beta_i(t_i)$ is increasing (decreasing) in t_i under B1 (B2). ■

The next theorem points out that under some assumptions on the cost parameters, the optimal configuration and inventory level exhibit some monotone properties.

THEOREM 5.2. *Suppose $a_i(1) > 0$. Under Assumptions A0, A1 and B1, then (a) $\mathbf{k}^*(\mathbf{t})$ is a decreasing function of $\mathbf{t}$; (b) $y^*(\mathbf{t})$ is a decreasing function of $\mathbf{t}$.*

Proof of Theorem 5.2.

a. Let 1_i be the vector whose i-th element is 1 and others 0. It is sufficient to show $\mathbf{k}^*(\mathbf{t}) \geq \mathbf{k}^*(\mathbf{t} + 1_i)$. For notational simplicity, denote the optimal solution pair in state $\mathbf{t}$ by $(\mathbf{k}^*, y^*) \equiv (\mathbf{k}^*(\mathbf{t}), y(\mathbf{t}, \mathbf{k}^*(\mathbf{t})))$ and the optimal solution pair in state $\mathbf{t} + 1_i$ by $(\bar{\mathbf{k}}, \bar{y}) \equiv (\mathbf{k}^*(\mathbf{t} + 1_i), y(\mathbf{t} + 1_i, \mathbf{k}^*(\mathbf{t} + 1_i)))$. Without loss of generality, assume that $\beta_i(t_i)$ is increasing in i, and

$$\mathbf{k}^* = (\underbrace{1, 1, ..., 1,}_{\text{first } \ell^* \text{ elements}} \; 0, 0, ..., 0), \tag{5.25}$$

i.e., $\mathbf{k}^*$ uses generation-1 technologies for components $1, 2, ..., \ell^*$, and generation-0 technologies for components $\ell^*+1, ..., m$. We establish our result via contradiction. Suppose $\bar{\mathbf{k}} \not\leq \mathbf{k}^*$. This implies that there exists $\ell > \ell^*$, such that $k^*_\ell = 0$ (i.e., component ℓ uses the new technology in state $\mathbf{t}$) but $\bar{k}_\ell = 1$ (i.e., component ℓ uses the old technology in state $\mathbf{t}+1_i$). According to Lemma 5.3 (b), y^* and $\bar{y}$ belongs to the dominant intervals of k^*_ℓ and $\bar{k}_\ell$. This implies

$$\begin{cases} \beta_{\ell^*}(t_{\ell^*}) \leq y^* < \beta_{\ell^*+1}(t_{\ell^*+1}) \leq \beta_\ell(t_\ell) \leq \bar{y} & \text{if } \ell \neq i, \\ \beta_{\ell^*}(t_{\ell^*}) \leq y^* < \beta_{\ell^*+1}(t_{\ell^*+1}) \leq \beta_\ell(t_\ell) = \beta_i(t_i) \leq \beta_i(t_i+1) \leq \bar{y}, & \text{if } \ell = i. \end{cases}$$

In either case, $y^* < \bar{y}$. We need to consider two cases.

Case 1. $k^*_i \leq \bar{k}_i$. We prove the solution pair $(\bar{\mathbf{k}}, \bar{y})$ outperforms the presumed optimal solution pair $(\mathbf{k}^*, y^*)$ in state $\mathbf{t}$, resulting in a contradiction. We have

$$\begin{aligned} g(\mathbf{t}, \mathbf{k}^*, y^*) - g(\mathbf{t}+1_i, \bar{\mathbf{k}}, \bar{y}) &\leq g(\mathbf{t}, \mathbf{k}^*, y^*) - g(\mathbf{t}+1_i, \mathbf{k}^*, y^*) \\ &= g_i(t_i, k^*_i, y^*) - g_i(t_i+1, k^*_i, y^*) \\ &\leq g_i(t_i, k^*_i, \bar{y}) - g_i(t_i+1, k^*_i, \bar{y}) \\ &\leq g_i(t_i, \bar{k}_i, \bar{y}) - g_i(t_i+1, \bar{k}_i, \bar{y}) \\ &= g(\mathbf{t}, \bar{\mathbf{k}}, \bar{y}) - g(\mathbf{t}+1_i, \bar{\mathbf{k}}, \bar{y}). \end{aligned} \tag{5.26}$$

Here, the first inequality follows, because $(\bar{\mathbf{k}}, \bar{y})$ is the optimal solution pair in state $\mathbf{t}+1_i$. The second inequality is due to Lemma 5.4 (a) under A1. The third inequality uses $k^*_i \leq \bar{k}_i$ and Lemma 5.4 (b) under B1. However, (5.26) yields

$$g(\mathbf{t}, \mathbf{k}^*, y^*) \leq g(\mathbf{t}, \bar{\mathbf{k}}, \bar{y}),$$

which contradicts to the optimality of $(\mathbf{k}^*, y^*)$ in state $\mathbf{t}$.

Case 2. $k^*_i > \bar{k}_i$. This means $k^*_i = 1$ and $\bar{k}_i = 0$, that is, component i, which uses the old technology in state $\mathbf{t}$, is upgraded to the new technology in state $\mathbf{t} + 1_i$. By assumption of $\mathbf{k}^*$ in (5.25), $i \leq \ell^*$. From Lemma 5.4 (d), $\beta_i(t_i) \leq \beta_i(t_i + 1)$ under B1. In addition, since $k^*_i = 1$ and $\bar{k}_i = 0$, by Lemma 5.3 (b), y^* and $\bar{y}$ belong to their respective dominant intervals. Therefore, $\beta_i(t_i) \leq y^* < \bar{y} < \beta_i(t_i + 1)$. Let $\bar{\ell}$ be the largest integer such that $\beta_{\bar{\ell}}(t_{\bar{\ell}}) \leq \bar{y} < \beta_i(t_i + 1)$, $\bar{\ell} \geq \ell > \ell^* \geq i$. This, however, implies that $\bar{\mathbf{k}}$ must take the form

$$\bar{\mathbf{k}} = (\underbrace{1, 1, ..., 1,}_{\text{first } i-1 \text{ elements}} \quad 0, \quad \underbrace{1, 1, \ldots, 1,}_{i+1 \text{ to } \bar{\ell} \text{ elements}} \quad 0, 0, ..., 0).$$

Let us define

$$\mathbf{k}' = \bar{\mathbf{k}} + 1_i \text{ and } \mathbf{k}'' = \mathbf{k}^* - 1_i. \tag{5.27}$$

Then, by our assumption, the solution pair $(\mathbf{k}^*, y^*)$ outperforms the solution pair $(\mathbf{k}', \bar{y})$ in state $\mathbf{t}$, that is,

$$g(\mathbf{t}, \mathbf{k}^*, y^*) \geq g(\mathbf{t}, \mathbf{k}', \bar{y}), \tag{5.28}$$

or equivalently,

$$g_i(t_i, 1, y^*) - g_i(t_i, 1, \bar{y}) \geq \sum_{j \neq i} g_j(t_j, k'_j, \bar{y}) - \sum_{j \neq i} g_j(t_j, k^*_j, y^*). \quad (5.29)$$

On the other hand, the solution pair $(\bar{\mathbf{k}}, \bar{y})$ outperforms the solution pair $(\mathbf{k}'', y^*)$ in state $\mathbf{t} + 1_i$:

$$g(\mathbf{t} + 1_i, \bar{\mathbf{k}}, \bar{y}) \geq g(\mathbf{t} + 1_i, \mathbf{k}'', y^*), \quad (5.30)$$

or equivalently,

$$\sum_{j \neq i} g_j(t_j, \bar{k}_j, \bar{y}) - \sum_{j \neq i} g_j(t_j, k''_j, y^*) \geq g_i(t_i + 1, 0, y^*) - g_i(t_i + 1, 0, \bar{y}). \quad (5.31)$$

However, our construction of $\mathbf{k}'$ and $\mathbf{k}''$ in (5.27) implies that the RHS of (5.29) is equal to the LHS of (5.31), and thus we have, from (5.29) and (5.31),

$$\begin{aligned} g_i(t_i, 1, y^*) - g_i(t_i, 1, \bar{y}) &\geq g_i(t_i + 1, 0, y^*) - g_i(t_i + 1, 0, \bar{y}) \\ &\geq g_i(t_i, 0, y^*) - g_i(t_i, 0, \bar{y}), \quad (5.32) \end{aligned}$$

where the last inequality is due to Lemma 5.4 (a). Rearranging the above inequality, we obtain

$$g_i(t_i, 0, y^*) - g_i(t_i, 1, y^*) \leq g_i(t_i, 0, \bar{y}) - g_i(t_i, 1, \bar{y}).$$

However, this contradicts Lemma 5.2 (c), which states that $g_i(t_i, 0, y) - g_i(t_i, 1, y) < 0$ and is decreasing in y for $y > \beta_i(t_i)$. Therefore, we must have $\bar{\mathbf{k}} \leq \mathbf{k}^*$.

b. First note that, from (5.12), we can write

$$g(\mathbf{t}, \mathbf{k}, y) - g(\mathbf{t} + 1_i, \mathbf{k}, y) = g_i(t_i, k_i, y) - g_i(t_i + 1, k_i, y).$$

Then Lemma 5.4 (a) means that $g(\mathbf{t}, \mathbf{k}, y)$ is a submodular function of t_i and y. Hence $y(\mathbf{t}, \mathbf{k})$ is decreasing in t_i for any fixed $\mathbf{k}$ and t_l, $l \neq i$.

This result means that it is sufficient to show $y' \equiv y(\mathbf{t}, \bar{\mathbf{k}}) \leq y^*$, since $y' \geq \bar{y} \equiv y(\mathbf{t}+1_i, \bar{\mathbf{k}})$. Suppose, on the contrary, $y' > y^*$. Note that $\beta_{\ell^*}(t_{\ell^*}) < y^*$, because of (5.25). Recall that both functions $g(\mathbf{t}, \mathbf{k}^*, y)$ and $g(\mathbf{t}, \bar{\mathbf{k}}, y)$ are concave functions of y. Since $g(\mathbf{t}, \mathbf{k}^*, y)$ reaches the maximum at y^* and $g(\mathbf{t}, \bar{\mathbf{k}}, y)$ reaches the maximum at y', $y^* < y'$, $g(\mathbf{t}, \mathbf{k}^*, y)$ is decreasing in y and $g(\mathbf{t}, \bar{\mathbf{k}}, y)$ is increasing in y in the interval $[y^*, y')$. This implies that the difference function

$$g(\mathbf{t}, \bar{\mathbf{k}}, y) - g(\mathbf{t}, \mathbf{k}^*, y) = \sum_{\bar{k}_j < k_j^*, j \leq \ell^*} [g_j(t_j, 0, y) - g_j(t_j, 1, y)]$$

is increasing in y in the interval $[y^*, y')$, with $y^* \geq \beta_{\ell^*}(t_{\ell^*})$. However, by Lemma 5.2 (c), each term on the RHS of the above expression, $g_j(t_j, 0, y) - g_j(t_j, 1, y)$, is decreasing in y for $y > \beta(t_{\ell^*}) \geq \beta(t_j)$, $j \leq \ell^*$, resulting in a contradiction. Hence we must have $\bar{y} \leq y' \leq y^*$. ■

In words, Theorem 5.2 states that, under A0, A1 and B1, as the pipeline technology development advances ($\mathbf{t}$ increases), the firm should use more new technologies and in the meantime reduce the inventory level. The intuition behind this result is as follows. A larger $\mathbf{t}$ means that, first, the future release of pipeline technologies becomes more imminent, which increases the likelihood that old technologies will become obsolete soon; second, for each component, the profit margin decreases and the overage cost increases (A1), and hence the inventory should be adjusted downward; and finally, the difference in profit margins of the new and old technologies increases and the difference in their overage costs decreases (B1), and hence a new technology becomes more attractive. Consequently, the firm is motivated to upgrade to new technologies to lessen the risk of incurring obsolescence cost and order less inventory to avoid paying higher overage cost.

Besides the results shown in Theorem 5.2, we also derive the monotone structure of the myopic solution under Assumptions A1-B2, A2-B1 and A2-B2. We summarize those results in Table 5.3.

Table 5.3. Monotone Properties of $(\mathbf{k}^*(\mathbf{t}), y^*(\mathbf{t}))$ as $\mathbf{t}$ Increases

Case	Assumptions	$y^*(t)$	$k^*(t)$
1	$a_i(1) \geq 0\ \forall i, A0, A1, B1$	Decrease	Decrease
2	$a_i(1) \leq 0\ \forall i, A0, A1, B2$	Decrease	Increase
3	$a_i(1) \geq 0\ \forall i, A0, A2, B1$	Increase	Decrease
4	$a_i(1) \leq 0\ \forall i, A0, A2, B2$	Increase	Increase

Chapter 6

Decentralized Heuristics and Advantages of ATO System

In this Chapter, first, we propose a Component Phase-Partitioning Heuristic (CPPH) and investigate its effectiveness of by using randomly generated data; second, we demonstrate the competitive advantages of using ATO over MTS system; and third, we discuss several extensions of the model formulated in Chapter 5.

6.1 Heuristic

In this section we answer the question whether decentralized technology management can be effective. Decentralized control has the advantages that it lessens the firm's burden of coordinating technology selection decisions across different components, avoids frequent upgrading and downgrading of a component triggered by the state change of other components, and leads to a policy that is transparent, intuitive, and easy to implement in practice, especially for the ATO system with a large number of components. We first develop a simple, decentralized technology selection mechanism and then use randomly generated data to test its effectiveness.

6.1.1 Component Phase-Partitioning Heuristic (CPPH)

Our Component Phase-Partitioning Heuristic (CPPH) is based on the insight gained from Theorem 5.2. We note that if the product consists of only a single component, then Theorem 5.2 leads to

Corollary 6.1. *Suppose the product consists of only a single component i. Let $(k_i^*(t_i), y_i^*(t_i))$ be the optimal solution in state t_i. Suppose $a_i(1) > 0$, and Assumptions A1-B1 hold for component i. Then (a). $k_i^*(t_i)$ is a decreasing function of t_i; (b). $y_i^*(t_i)$ is a decreasing function of t_i.*

Corollary 6.1 implies that, under the stated assumptions, the optimal policy is of a threshold type: there exists a threshold t_i^* such that the new technology is used if and only if $t_i > t_i^*$. For the multi-component system, we can select the configuration in state $\mathbf{t}$, say $\mathbf{k}^H(\mathbf{t})$, by letting $k_i^H(t_i) = 1$ if $t_i \leq t_i^*$ and 0 otherwise, $i \in E$. Under this component-based technology selection mechanism, we only take one upgrade action for each generation of technology based on a single threshold. Also, the threshold t_i^* depends only on the cost parameters associated with component i; thus, the phase changes of other components will not affect the technology choice of component i.

In a more general setting where no assumptions on cost structures are imposed, we can implement the decentralized technology control by letting

$$k_i^H(t_i) = \begin{cases} 0, & \text{if } g_i(t_i, 0, y(t_i, 0)) \geq g_i(t_i, 1, y(t_i, 1)), \\ 1, & \text{otherwise.} \end{cases} \tag{6.1}$$

In this case, we may have several thresholds that partition state space $\{1, 2, \ldots, d_i\}$ into several subintervals, such that a given technology is adopted in each subinterval. In most situations, the number of phases is small. Our computational results show that even without imposing any cost structure, the above rule often results in a single threshold control. We now state our heuristic.

Algorithm 6.1. *Component Phase-Partitioning Heuristic (CPPH).*

Step 1. *For each* $i \in E$ *and* t_i, *make the technology choice using (6.1). Under the conditions of Corollary 6.1, set* t_i^* *as the smallest* t_i *such that* $g_i(t_i, 0, y(t_i, 0)) \geq g_i(t_i, 1, y(t_i, 1))$, *and let* $k_i^H(t_i) = 1$ *if* $t_i \leq t_i^*$ *and* $k_i^H(t_i) = 0$ *if* $t_i > t_i^*$.

Steps 2. *Compute* $y^H(\mathbf{t}) = y(\mathbf{t}, \mathbf{k}^H(\mathbf{t}))$ *using (5.11). Use* $(\mathbf{k}^H(\mathbf{t}), y^H(\mathbf{t}))$ *as the approximate solution in state* $\mathbf{t}$.

In the special case when $m = 1$, CPPH is optimal. In terms of component selections, CPPH makes decentralized decisions but for the inventory replenishment decision, CPPH makes the centralized decision. We emphasize that the inventory policy must be determined *jointly* after the configuration is determined. The reader can be easily convinced that even a small degree of imbalance in inventories over different components may substantially increase the firm's overage cost.

6.1.2 Numerical Study of CPPH

In this section, we design an experiment to test the effectiveness of CPPH, as follows.

1. *Innovation Process:* The innovation time for each component is independent and follows a negative binomial distribution that has 4 phases ($d_i = 4$), $i \in E$. The probability of advancing from one phase to the next is 0.1. Let $\{\alpha^i_{t_i}\} = (1, 0, 0, 0)$, $i \in E$.

2. *Demand Distribution:* Demand, D, follows a gamma distribution with parameters (θ, μ). The mean and variance of demand are $\theta\mu$ and $\theta\mu^2$, respectively. We consider

two levels of demand variability with the same mean, $(\theta, \mu) = (2, 2)$ and $(\theta, \mu) = (4, 1)$.

3. *Number of Components:* We consider $m = 3$ and $m = 5$.

4. *Cost Parameters:* The salvage value for a generation-2 component is assumed to be 0. The holding cost $h_i(t, j)$ is 10% of the procurement cost $c_i(t, j)$ for any i and $j = 0, 1$. We consider two cases.

Case i. *Cost parameters satisfy Assumption A0 only.* For each component $i \in E$, we generate two samples from a uniform distribution between 100 and 400 of size 8 each, and order both data sets in a decreasing order. One ordered sample is used as purchase costs so that $c_i(1, 0) \geq \cdots \geq c_i(4, 0) \geq c_i(1, 1) \geq \cdots \geq c_i(4, 1)$. Another ordered sample is used as profit margins, which are added to the corresponding procurement costs to generate $r_i(t_i, j)$ for each t_i and j. Clearly, the data generated as such satisfy Assumption A0.

Case ii. *Cost parameters satisfy Assumptions A0, A1 and B1.* We assume purchase costs are generation-dependent only. We generate two random numbers, $c_i(0)$ and $c_i(1)$, from a uniform distribution between 100 and 400 satisfying $c_i(1) > c_i(0)$, and let $c_i(t, j) = c_i(j)$, $j = 0, 1$, $t = 1, 2, 3, 4$. Then $b_i(1) = b_i(2) = b_i(3) > b_i(4)$ and overage cost $c_i(t, j) + H_i(t, j)$ is non-decreasing in t for any i. In addition, for each i, we generate two independent sequences of size 4 each, with $\{a_i(t)\}$ being the (increasing) order statistic from a uniform distribution between 0 and 25 and $\{\pi_i(t)\}$ the decreasing order statistic from a uniformly distribution between 25 and 200. We let

$$\begin{cases} r_i(t,0) = \pi_i(t) + c_i(t,0), \\ r_i(t,1) = \pi_i(t) - a_i(t_i) + c_i(t,1). \end{cases}.$$

Our procedure guarantees that the profit margins for both generations, $\pi_i(t)$ and $\pi_i(t) - a_i(t)$, decrease in t and the difference of the profit margins between two generations, $a_i(t)$, increases in t.

5. *Expected Discounted Reward with Random Initial State.* We treat the initial state $\mathbf{T}^0$ as a random vector, and let it follow the long-run stationary distribution of the innovation processes, denoted by $p(\mathbf{t})$. From our assumption on the innovation process, $P(\mathbf{T}^0 = \mathbf{t}) = p(\mathbf{t}) = \frac{1}{64}$. Also, the innovation process is stationary with $p(\mathbf{T}^n) = p(\mathbf{t})$ for any n and $\mathbf{t}$. Let $g(\mathbf{t}, y^*(\mathbf{t}), \mathbf{k}^*(\mathbf{t})) = g^*(\mathbf{t})$ be the optimal single-period profit with state $\mathbf{t}$. The optimal expected discounted reward with initial random state $\mathbf{T}^0$ will be

$$E[W(\mathbf{T}^0)] = \sum_{n=0}^{\infty} \lambda^n \sum_{\mathbf{t}} p(\mathbf{t}) g^*(\mathbf{t}) = \frac{1}{1-\lambda} \sum_{\mathbf{t}} p(\mathbf{t}) g^*(\mathbf{t}), \tag{6.2}$$

where $\lambda = 0.9$. IPA is used to compute $g^*(\mathbf{t})$ for each $\mathbf{t}$. Let $E[W^H(\mathbf{T})]$ be the expected discounted reward with initial random state $\mathbf{T}^0$, under CPPH. We can compute $E[W^H(\mathbf{T})]$ by substituting $g^H(\mathbf{t})$, computed using CPPH, into the above equation.

Our experiment generates 8 scenarios consisting of two levels of numbers of components, two levels of demand variability, and two cost parameter structures. For each scenario, we generate 100 random problem instances and report the average percentage error (Mean Error %) under the optimal solution and CPPH, defined as $1 - E[W^H(\mathbf{T})]/E[W(\mathbf{T})]$. We also report the maximal percentage error (Max Error %) and the percentage of the problem instances where CPPH attains the optimal solution (Hit %). The results are reported in Table 6.1.

Table 6.1. Numerical Study of CPPH

	# of Comp.	Assumptions	Demand	Mean Error %	Max Error %	Hit %
1	3	A0	Erlang (2,2)	0.20%	3.37%	50%
2	3	A0, A1-B1	Erlang (2,2)	0.39%	6.44%	60%
3	5	A0	Erlang (2,2)	0.13%	3.37%	37%
4	5	A0, A1-B1	Erlang (2,2)	0.18%	3.23%	64%
5	3	A0	Erlang (4,1)	0.11%	1.43%	44%
6	3	A0, A1-B1	Erlang (4,1)	0.02%	0.43%	57%
7	5	A0	Erlang (4,1)	0.01%	0.32%	68%
8	5	A0, A1-B1	Erlang (4,1)	0.01%	0.29%	57%
Average				0.13%		55%

Table 6.1 shows that CPPH performs extremely well against the optimal solution $(\mathbf{k}^*(\mathbf{t}), y^*(\mathbf{t}))$. For over 800 randomly generated problem instances, the mean error is merely 0.13% and the percentage of hits is 55%. CPPH also appears robust, regardless of whether the cost parameters satisfy additional Assumptions A1-B1. Further, Table 6.1 suggests that as demand variability decreases or the number of component increases, CPPH's performance improves.

Observe that the range of the maximum percentage errors of CPPH over 8 scenarios is between 0.29-6.4%. This raises the question of under what conditions CPPH may yield a solution that is substantially below optimal. We retrieve the data of these problem instances that generate the maximum percentage errors and find that if the component-based optimal order quantities, $y_i(t_i, j) = F^{-1}\left(\frac{r_i(t_i,j)-c_i(t_i,j)}{c_i(t_i,j)-H_i(t_i,j)}\right)$, $1 \le t_i \le d_i$, $i \in E$, diverge significantly across different components and phases, then CPPH becomes less effective. To verify this observation, we define the dispersion index of $\{y_i(t_i, j)\}$ by

$$\text{Dispersion Index of } \{y_i(t_i, j)\} = \max_{t_i,j}\{y_i(t_i, j)\} - \min_{t_i,j}\{y_i(t_i, j)\}.$$

For each problem instance we compute its dispersion index. We then divide the dispersion indices of 800 problem instances into three groups, and compute the mean percentage error for each group. We report the results in Table 6.2.

Table 6.2. Mean Error % as Function of Dispersion Index

3 Components			5 Components		
Group	Range of Dispersion Index	Mean Error %	Group	Range of Dispersion Index	Mean Error %
1	0.97 - 1.95	0.04 %	1	1.06 - 2.40	0.02 %
2	1.95 - 2.92	0.10 %	2	2.40 - 3.24	0.02 %
3	2.92 - 3.90	1.0 %	3	3.24 - 4.38	0.14 %

Table 6.2 shows that as the dispersion index increases, the mean percentage error also increases. As expected, the problem instance with the maximum percentage error

6.4% belongs to group 3. Nevertheless, the mean percentage errors over different ranges of dispersion indices are quite small ($\leq 1\%$), especially when the number of components increases. Therefore, we conclude the following:

OBSERVATION 1. *If the critical ratios* $\frac{r_i(t_i,j)-c_i(t_i,j)}{c_i(t_i,j)+H_i(t_i,j)}$ *do not diverge significantly across different components and phases, CPPH is near-optimal. The performance of CPPH deteriorates when the critical ratios deviate from each other significantly. However, the performance gap between CPPH and IPA for most problem instances is insignificant.*

The superior performance of CPPH with similar critical ratios is not surprising, since when the dispersion index equals zero, the policy of decentralized technology and inventory control becomes globally optimal. What might be surprising is that, even when the dispersion index is relatively large, decentralized technology management, supplemented by centralized inventory control, proves to be extremely effective.

6.2 Effects of Postponement: A Comparison of ATO and MTS

Suppose the ATO firm incurs an assembly cost, c_a, for each unit of the *filled* demand, where c_a is independent of the product offering. Then it is evident that, if we adjust the selling price from $r(\mathbf{t}, \mathbf{k})$ to $r(\mathbf{t}, \mathbf{k}) - c_a$, then all our results developed in the previous sections are still valid. Consider the counterpart MTS system, which has the same cost parameters, except that MTS purchases the assembled units of a configuration, with additional cost c_a for each unit acquired. In addition, when MTS salvages a unit of Config-$\mathbf{k}$, it only receives $c(\mathbf{t}, \mathbf{k}) - c_b$ for each salvaged product, that is, the firm cannot recoup the assembly cost and also incurs additional market value discount for a returned product. The cost $c_b > 0$ can be interpreted as the disassembly cost, but more generally, it is meant to reflect the fact that a salvaged product is worth less than its parts. It is often the case that a product equipped with outdated or obsolete components is discounted more heavily than its constituent components. What is the impact of postponed assembly in ATO on the total profit, product offering, and inventory level? Many competitive advantages of ATO, as compared to MTS, have been documented. However, the effect of postponement in ATO to protect the firm from the risk of component obsolescence has not been formally analyzed. To gain insight, we compared two single-period systems operating under the ATO and MTS modes. For a notation used in ATO, say x, its counterpart notation in MTS will be $\tilde{x}$. We again suppress $\mathbf{t}$. The single period profit functions, given $\mathbf{k}$ and y, in the two systems are

$$\begin{aligned} g(\mathbf{k}, y) &= [r(\mathbf{k}) - c_a - H(\mathbf{k})]E[Z(y)] - [c(\mathbf{k}) + H(\mathbf{k})]y, \\ \tilde{g}(\mathbf{k}, y) &= [r(\mathbf{k}) - \lambda c_b - H(\mathbf{k})]E[Z(y)] - [c(\mathbf{k}) + c_a + H(\mathbf{k}) + \lambda c_b]y. \end{aligned}$$

It is easily seen that each $i \in E$, $a_i = \tilde{a}_i$ and $b_i = \tilde{b}_i$. This implies that the break-even points β_i in both systems are the same. Consequently, the dominant intervals and dominant configurations in both systems are also the same. In addition,

$$g(\mathbf{k}, y) - \tilde{g}(\mathbf{k}, y) = (\lambda c_b - c_a)E[Z(y)] + (c_a + \lambda c_b)y, \tag{6.3}$$

which is positive and increasing in y. Using the above expressions, we can prove the following:

Theorem 6.1. *Suppose the optimal solutions in ATO and MTS are* $(\mathbf{k}^{(\ell^*)}, y(\mathbf{k}^{(\ell^*)})$ *and* $(\mathbf{k}^{(\tilde{\ell})}, \tilde{y}(\mathbf{k}^{(\tilde{\ell})})$*, respectively. Then, (a).* $\tilde{\ell} \leq \ell^*$*; (b).* $\tilde{y}(\mathbf{k}^{(\tilde{\ell})}) \leq y^*(\mathbf{k}^{(\ell^*)})$*.*

Proof of Theorem 6.1.

a. If the contrary, $\ell^* < \tilde{\ell}$, is true, then

$$\begin{aligned}
&\tilde{g}(\mathbf{k}^{(\ell^*)}, y(\mathbf{k}^{(\ell^*)}) - \tilde{g}(\mathbf{k}^{(\tilde{\ell})}, \tilde{y}(\mathbf{k}^{(\tilde{\ell})}) \\
&\quad \geq \tilde{g}(\mathbf{k}^{(\ell^*)}, y(\mathbf{k}^{(\ell^*)}) - g(\mathbf{k}^{(\ell^*)}, y(\mathbf{k}^{(\ell^*)}) + g(\mathbf{k}^{(\tilde{\ell})}, \tilde{y}(\mathbf{k}^{(\tilde{\ell})}) - \tilde{g}((\mathbf{k}^{(\tilde{\ell})}, \tilde{y}(\mathbf{k}^{(\tilde{\ell})}) \\
&\quad = -(\lambda c_b - c_a)E[Z(y(\mathbf{k}^{(\ell^*)}))] - (c_a + \lambda c_b)y(\mathbf{k}^{(\ell^*)}) \\
&\qquad +(\lambda c_b - c_a)E[Z(\tilde{y}(\mathbf{k}^{(\tilde{\ell})}))] + (c_a + \lambda c_b)\tilde{y}(\mathbf{k}^{(\tilde{\ell})}) > 0,
\end{aligned}$$

where the last inequality is due to $\tilde{y}(\mathbf{k}^{(\tilde{\ell})}) > y(\mathbf{k}^{(\ell^*)})$ and also the RHS of (6.3) is increasing in y. The above inequality, however, is against the optimality of $(\mathbf{k}^{(\tilde{\ell})}, y(\mathbf{k}^{(\tilde{\ell})})$, and the contradiction implies $\tilde{\ell} \leq \ell^*$.

b. For any fixed Config-$\mathbf{k}$, the optimal order quantity of MTS and ATO are, $\tilde{y}(\mathbf{k}) = F^{-1}(\frac{r(\mathbf{k})-c(\mathbf{k})-c_a}{r(\mathbf{k})+H(\mathbf{k})+\lambda c_b})$ and $y(\mathbf{k}) = F^{-1}(\frac{r(\mathbf{k})-c_a-c(\mathbf{k})}{r(\mathbf{k})-c_a+H(\mathbf{k})})$, implying $\tilde{y}(\mathbf{k}^{(\tilde{\ell})}) \leq y(\mathbf{k}^{(\tilde{\ell})}) \leq y(\mathbf{k}^{(\ell^*)})$. ■

We draw several conclusions from our analysis. Compared with MTS, 1) ATO has a higher profit, as seen from (6.3); 2) Theorems 5.1 and 6.1 (a) together mean that ATO uses the configuration that has a lower profit margin and lower overage cost. In particular, under Assumptions A1-B1, it means that ATO uses an "older" configuration than MTS does; and 3). Theorem 6.1 implies that ATO has a higher base-stock level, and hence, better customer service.

The insight gained from the single-period analysis is not lost under the infinite-period setting, although the detailed comparison of the two systems is beyond the scope of this paper. We note that the optimal policy for ATO remains myopic, which is not true for MTS. This is because MTS suffers salvage loss $c_a + \lambda c_b$ for each returned unit, which causes it to become an "earlier adopter" of new technologies and to be less flexible to change to different configurations, as compared with ATO.

6.3 Extensions

In this section, we consider several variants of our basic model in Chapter 5.

1. Configuration and Phase-Dependent Demand. It is perceivable that the choices of some major components may have significant impact on the demand pattern. Let $G \subset E$ be the subset of components so that the demand distribution depends on the technological choices of components in G but not on that in $\bar{G}$. Let $\mathbf{k}(G) = (k_i, i \in G)$ be the technological choices for components in G. The demand distribution given $\mathbf{k}(G)$ is denoted by $D(\mathbf{k}(G))$. Then IPA can be applied with some slight modifications. For each demand $D(\mathbf{k}(G))$, we use IPA to find the optimal configuration for the components in $\bar{G}$, say $\mathbf{k}(\bar{G})$, subject to the constraint that the components in G use technologies $\mathbf{k}(G)$. Then, we determine the optimal solution pair by

$$(\mathbf{k}^*(\mathbf{t}), y^*(\mathbf{t})) = \arg\max_{\mathbf{k}(G)} \{g(\mathbf{t}, \mathbf{k}(G) \cup \mathbf{k}(\bar{G}), y(\mathbf{t}, \mathbf{k}(G) \cup \mathbf{k}(\bar{G})).$$

If the number of components in set G is k, $0 \leq k \leq m$, then we need to evaluate at most $2^k(m-k+1)$ configurations. If k is small, as in most practical circumstances where only one or two major components could have significant influence on demand, IPA remains effective. The generalization of CPPH to handle the configuration and phase-dependence demand case is similar.

2. Deterministic Lead Times and Deterministic Cost Parameters. Let L_i be the procurement lead time for component i, where L_i is a positive constant, $i \in E$, satisfying $L_1 \leq L_2 \leq \cdots \leq L_m$. We impose two additional conditions: 1) The cost parameters for the next L_m periods are deterministic. This will be the case if the supplier informs the firm about the discount price schedule for each component in the next L_m periods and the firm subsequently determines the selling prices and holding cost for each component. This is realistic since, nowadays, a firm often establishes a long-term relationship with its supplier, who may provide the firm with a "road map" of future price discount plans (for example, Intel publishes the price cut schedule of its processors for next several quarters); and 2) the purchase cost is charged in the period when the order is received. Consider the following modified Interval Partitioning Algorithm: first, in any period n, we use the cost parameters in period $n + L_m$ in IPA to identify the myopic solution $(\mathbf{k}^*, y^*)$; second, in period $n + L_m - L_i$, $i \in E$, order y^* units of type $i - k_i^*$ component, which will arrive in period $n + L_m$; third, at the beginning of period $n + L_m + 1$, return any leftover inventory carried over from period $n + L_m$ with the salvage revenue equal to the discounted market value in period $n + L_m + 1$, and the salvage revenue is charged back to period $n + L_m$ as part of the overage cost. Clearly, the quantities of various components arriving in period $n + L_m$ are balanced and equal to y^*. It can be shown that the modified IPA under Conditions (1) and (2) is optimal.

3. Multiple Coexisting Technologies. When the number of coexisting technologies for each component is $n > 2$, it can be shown that the myopic policy is still optimal and the optimal inventory policy for a given configuration remains a balanced base-stock policy computed using the product-level newsvendor formula. In addition, we can extend our Interval Partitioning Algorithm to find the optimal configuration by evaluating at most $m(n-1)+1$ dominant configurations, rather than n^m possible alternatives.

Chapter 7

Conclusions

Rapid technological innovations pose several challenges to modern production systems. The increasing speed of technology innovations pushes the need to effectively coordinate technology management and inventory control. The miraculous turn-around in eMachines Inc. demonstrated the importance of *jointly* optimizing technology selection and inventory control. This dissertation develops several analytical models to systematically study the joint technology and inventory control policy in the MTS and ATO systems with technology innovations.

7.1 Summary of Managerial Insights

The managerial and analytical insights gained from this dissertation are summarized as follows.

- The joint technology and inventory control is generally complex and requires extensive computation efforts (see the analysis in Chapter 3 to 6). As identified in this dissertation, the positive value loss due to salvaging the non-obsolete inventory is the major source that complicates the analysis. It is useful for the firm to engage in some risk-sharing contract with its supplier to allow the firm to return the unwanted inventory at the current procurement cost. Such an "inventory return" condition simplifies the coordination between technology selection and inventory control, improves the firm's profit, and narrows the time window for the technology forecast. On the other hand, the supplier may benefit from the larger order size placed by the firm. Other than returning the unwanted inventory to its supplier, the firm can create its own "salvage market" by temporarily marking down the price at the beginning of each time period, i.e., using "kick-off" sales or "early-bird-specials". Provided that the market size is sufficiently large, the firm can reduce the inventory to desired level and re-coup the current procurement cost of the non- obsolete inventory. However, the pricing issues are left for future research.

- For different production systems, the scope of coordinating technology and inventory controls is different. In a Make-to-Stock system that markets a single product in each period (see Chapter 3), the technology and inventory decisions are made at the product level. With positive salvage loss, the objective function is not necessarily concave, except in the final period. If the concavity is preserved in the multi-period model, the optimal timing to upgrade depends on a sequence of state-dependent threshold: when the initial inventory is old technology and the inventory level is strictly below the threshold, it is optimal to upgrade. For any given technology decision, the optimal ordering policy follows a two-limit policy. That is, when the initial inventory is below the first limit, raise the inventory up to the first limit; when the inventory is between the first and second limits, neither order nor slash inventory; when the initial inventory is above the second limit, slash the inventory down to the second limit.

- In a MTS system that markets an assortment of products, i.e., with up to two generation products, (see Chapter 4), the technology decision is made at the assortment level whereas the inventory decision is made at the product level. I show that the optimal assortment policy for the final period is determined by a switching policy. That is, for each state, there exists two switching curves that partition the plane into three regions. Each region corresponds to one dominating assortment. When the initial inventory falls in one of these three regions, the optimal assortment is to use the corresponding dominating assortment. With the "zero salvage loss" condition and some plausible conditions on the cost parameters, the optimal assortment policy indeed has a "three-stage" structure: when the development state is in the early stage, it is optimal to offer the old generation product; when the development state is in the middle stage, it is optimal to offer both generation products; and when the development state is in the advanced stage, it is optimal to discontinue the old generation product. Inspired by the structural properties of the special case analysis, a sequential optimization heuristic (SH) is proposed. Numerical experiments show that the SH heuristic generally performs better against the upper-bound profit (i.e., the profit attainable under "inventory return condition") when the percentage profit margin is high. Our numerical experiments also suggest that joint optimization is important for the firm with low profit margin, particularly when the demand variability is high or technology innovation speed is fast.

- In the multi-components ATO system, technology innovations occurs at the component level but impacts the system at the product level (see Chapter 5 and 6). Under the "zero salvage loss" condition, the myopic policy is optimal and in the single-product case, the optimal solution can found by using the Interval Partitioning Algorithm, which searches the optimal configuration along the efficient frontier. A general rule of thumbs for jointly optimizing technology selection and inventory control is the following: when the planned inventory level or demand variability is high, it is better to use the technology that has lower overage cost; when the planned inventory level or demand variability is low, it is better to use the technology that has a higher profit margin. To choose the optimal configuration efficiently, the firm should focus on a set of dominant configurations only. Under several plausible conditions on the cost parameters, the optimal configuration and inventory level exhibit some monotone properties. A decentralized technology management, supplemented by joint inventory optimization, can simplify technology coordination, avoid frequent configuration changeovers, and ease implementation. It proves effective and robust in most problem instances tested.

- When facing technology innovations, ATO outperforms MTS in dealing with rapid technology breakthroughs and dynamically changing prices. It offers better customer service at lower cost and is also more flexible and dynamic in product offerings.

7.2 Future Research

There are several interesting ways to extend the research of this dissertation.

- It would be very useful to extend the analysis in Chapter 5 and 6 to the multi-component and multi-product ATO system with customer choices. In this setting, the firm needs to first announce a technology menu listing the available options for each major components. Then customers choose their favorite configurations following a multi-nomial logit (MNL) process. In the single-period setting, this problem can be naturally formulated as a two-stage stochastic programming problem. In the first stage, the firm determines the technology menu to offer and the inventory levels. In the second stage, the firm receives customer orders and uses the available components to assemble the final product and fulfill the demand. This problem is closely related to the work by Swaminathan and Tayur ([50]). However, it is computationally demanding to solve for the optimal menu and corresponding inventory levels. Extending the analysis to a multi-period model is more challenging. In case the structure of the optimal policy for the multi-period problem is difficult to characterize, it is useful to consider some heuristic approaches to select the technology menu. For instance, will the "three-stage" structure derived in Chapter 4 still hold in the multi-products setting? Research along this direction may help us evaluate how customer choice, risk-pooling, and demand fulfillment could affect the system.

- Our assumption of lead-time in this dissertation is too restrictive. It would be useful to extend the analysis to the case with positive and deterministic lead-time or even random lead-time.

- Empirical evidence shows that the technology upgrading decisions are significantly influenced by the competitor's behavior. Using game theory models, it would be useful to analyze how the competitor's behavior will affect the optimal assortment decision.

- This dissertation accentuates the value of the "inventory return" contract, which could greatly simplify the joint optimization. But it is important to investigate how to devise the optimal risk-sharing contract for the firm and its suppliers.

- It would be interesting to consider the pricing issue. For example, it is possible to extend the analysis in Chapter 3 and 4 by considering the additive or multiplicative demand model as discussed in Petruzzi and Dada [35] with generational differences in the risk-free demand function. To be specific,

$$D_j(t) = (U_j(t) - p_j(t))D \text{ or } D_j(t) = D + (U_j(t) - p_j(t)),$$

  where D is i.i.d. random variable, $U_j(t)$ is the reserve value that the customer is willing to pay for product-j when system is in state t, and $p_j(t)$ is the price that the firm charges.

- The objective value functions in Chapter 3 and 4 could potentially be neither concave nor quasi-concave. Although we can gain some insight on the optimal policy if the objective function preserves concavity, it calls for further investigation on the sufficient conditions that guarantee concavity and the structure of the optimal policy when concavity or quasi-concavity is lost in the multi-period analysis. In addition, it is helpful to explore other directions to develop a method better than the SH heuristic.

In summary, this dissertation research is the first attempt to investigate how state-dependent factors such as price erosion caused by technology innovation affect the performance of production systems. The insight gained from this dissertation could help academic researchers and industry practitioners better understand the value of *jointly* optimizing inventory replenishment and technology selection in modern production systems. I hope that this dissertation can open a new door to research on the issues related to *jointly* optimizing technology selection and inventory replenishment in a dynamic environment.

Appendix A

Key Terms and Proofs

The appendix contains the explanations of some key terminologies frequently used in the dissertation and the proofs of several important lemmas and theorems.

A.1 Key Terms

A.1.1 Concavity and Quasi-concavity

DEFINITION 2. *A function $f\colon X \to R$ is said to be* ***concave*** *at $x \in X$, if $f(\lambda x + (1-\lambda)x') \geq \lambda f(x) + (1-\lambda)f(x')$ for all $x' \in X$ and $\lambda \in [0,1]$. f is said to be concave on X if it is concave at each $x \in X$. If function f is continuous and twice differentiable, then it is strictly concave if $\nabla^2 f(x) \leq 0$ for each $x \in X$.*

DEFINITION 3. *A function $f\colon X \to R$ is said to be* ***strictly concave*** *at $x \in X$, if $f(\lambda x + (1-\lambda)x') > \lambda f(x) + (1-\lambda)f(x')$ for all $x' \in X$ and $\lambda \in (0,1)$. f is said to be strictly concave on X if it is strictly concave at each $x \in X$. If function f is continuous and twice differentiable, then it is strictly concave if $\nabla^2 f(x) < 0$ for each $x \in X$.*

DEFINITION 4. *A function $f\colon X \to R$ is said to be* ***quasi-concave*** *if $f(\lambda x+(1-\lambda)x') \geq \min\{f(x), f(x')\}$ for all $(x', x) \in X \times X$ and all $\lambda \in [0,1]$. A function $f\colon X \to R$ is said to be* ***strictly quasi-concave*** *if $f(\lambda x + (1-\lambda)x') > \min\{f(x), f(x')\}$ for all $(x', x) \in X \times X$ such that $x \neq x'$ and all $\lambda \in [0,1]$.*

There are several important properties related to concavity and quasi-concavity.

1. A function f: $X \to R$ is quasi-concave if it is concave. But the converse is not true.

2. Function f: $X \to R$ is quasi-concave. $h(x)$ is a linear function of x. Then both $g_1 = h(f(x))$ and $g_2 = f(h(x))$ are quasi-concave.

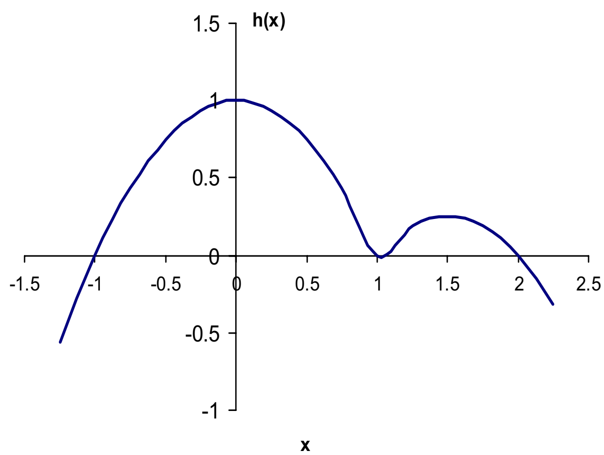

Fig. A.1. A counter example showing that the sum of two quasi-concave functions is not a quasi-concave function

3. Quasi-concavity is **not** preserved under nonnegative summation while concavity is. For example, suppose that f_1 and f_2 are quasi-concave. $f_1 + f_2$ is not necessarily quasi-concave (Silverman [43], page 15). A counter example is sufficient. Let

$$f_1(x) = 1 - x^2$$

and

$$f_2(x) = \begin{cases} 0 & \text{if } x < 1; \\ 3(x-3) & \text{if } x \geq 1. \end{cases}$$

It is easy to see that $f_1(x)$ is quasi-concave in x and so is $f_2(x)$. Then

$$h(x) = f_1(x) + f_2(x) = \begin{cases} 1 - x^2 & \text{if } x < 1; \\ -x^2 + 3x - 2 & \text{if } x \geq 1. \end{cases}$$

It is clear that $h(x)$ has two local optimums at $x = 0$ and $x = 1.5$. Also, $h(x)$ is neither concave nor quasi-concave in x, although it is piecewise concave in x.

4. Suppose that $C \in R^n$ is a convex set and let $X^* = \arg\max_{x \in C}\{f(x)\}$. Then

   (a) X^* is a convex set if f is quasi-concave;

   (b) $X^* = \{x^*\}$ if f is strictly quasi-concave, i.e., the solution if exists is unique.

A.1.2 Leibniz's Rule

The Leibniz integral rule gives a formula for differentiation of a definite integral whose limits are functions of the differential variable,

$$\frac{d}{dy}\int_{a_1(y)}^{a_2(y)} h(x,y)dx = \int_{a_1(y)}^{a_2(y)} \frac{dh(x,y)}{dy}dx + h(a_2(y),y)a_2'(y) - h(a_1(y),y)a_1'(y).$$

It is sometimes known as differentiation under the integral sign (Kaplan [23], page 256-258). It has been frequently used in this dissertation to obtain the first and second derivative of the objective functions.

A.2 Proof of Theorem 3.3 in Chapter 3

Proof of Theorem 3.3: We prove the theorem by establishing that $w(t,a,x;b(t),y,\tau)$ is concave in y using induction. We truncate the planning horizon to n periods. We have

$$\begin{aligned} w_{(n+1)}(t,a,x;b(t),y) &= G(t,a,x;b(t),y) + \theta \sum_{t'} p_{t,t'} E[w^*_{(n)}(t',b(t),(y-D_{b(t)}(t))^+;b(t'))] \\ &\quad + + \theta q_t \sum_{t'} \alpha_{t'} E[w^*_{(n)}(t',b(t)+1,(y-D_{b(t)}(t))^+;b(t'))]. \end{aligned}$$

It can be seen that when $n=1$, $w_{(1)}(t,a,x;b(t),y) = g(t,a,x;b(t),y)$, which is differentiable and is concave in y for any given $(t,a,x,b(t))$. Equation (3.11) indicates that the optimal inventory policy is a two-limit policy. Furthermore, from (3.12), we see that $w^*_{(1)}(t,a,x;b(t)) = g^*(t,a,x;b(t))$, which is also differentiable and concave for any $x \geq 0$ with any given (t,a). This proves the theorem for $n=1$. In addition, from Lemma 3.1, we see that $\nabla_x w^*_{(1)}(t,a,x;b(t)) = \nabla_x g^*(t,a,x;b(t)) \leq c_a(t)$.

We hypothesize that $w_{(n)}(t,a,x;b(t),y)$ is differentiable and concave in y and that $\nabla_x w^*_{(n)}(t,a,x;b(t)) \leq c_a(t)$. We have

$$\begin{aligned} w_{(n+1)}(t,a,x;b(t),y) &= G(t,a,x;b(t),y) + \theta \sum_{t' \geq t} p_{t,t'} E[w^*_{(n)}(t',b(t),(y-D_{b(t)}(t))^+;b(t'))] \\ &\quad + \theta q_t \sum_{t'} \alpha_{t'} E[w^*_{(n)}(t',b(t)+1,(y-D_{b(t)}(t))^+;b(t'))]. \end{aligned}$$

Clearly, $G(t,a,x;b(t),y)$ is strictly concave in y and its second derivative is $-r_{b(t)} f_{b(t)}(t,y)$.

We shall establish that

$$\nabla_y^2 E[w^*_{(n)}(t', b(t), (y - D_{b(t)}(t))^+; b(t'))] < r_{b(t)} f_{b(t)}(t, y), \text{ for any } t \geq t',$$

and that

$$\nabla_y^2 E[w^*_{(n)}(t', b(t), (y - D_{b(t)}(t))^+; b(t'))] < r_{b(t)} f_{b(t)}(t, y), \text{ for any } t'$$

Because of symmetry, we only show the first inequality. Toward this end, we express the term $E[w^*_{(n)}(t', b(t), (y - D_{b(t)}(t))^+; b(t'))]$ by conditioning whether $y > D_{b(t)}(t)$ or not. We find

$$\begin{aligned} & E[w^*_{(n)}(t', b(t), (y - D_{b(t)}(t))^+; b(t'))] \\ = \ & P(D_{b(t)}(t) > y) w^*_{(n)}(t', b(t), 0; b(t')) + \int_0^y w^*_{(n)}(t', b(t), y - u; b(t')) f_{b(t)}(t, u) du \\ = \ & w^*_{(n)}(t', b(t), 0; b(t')) + \int_0^y [w^*_{(n)}(t', b(t), y - u; b(t')) - w^*_{(n)}(t', b(t), 0; b(t'))] f_{b(t)}(t, u) du. \end{aligned}$$

Since $w^*_{(n)}(t', b(t), 0; b(t'))$ is a constant and $w^*_{(n)}(t', b(t), y - u; b(t'))$ is concave in $y - u$, we see that $w^*_{(n)}(t', b(t), y - u; b(t'))$ is also concave in y. We repeatedly use the Leibniz' rule and find that

$$\begin{aligned} & \nabla_y^2 E[w^*_{(n)}(t', b(t), (y - D_{b(t)}(t))^+; b(t'))] \\ = \ & \int_0^y \nabla_y^2 w^*_{(n)}(t', b(t), y - u; b(t')) f_{b(t)}(t, u) du + f_{b(t)}(t, u) \frac{\partial w^*_{(n)}(t', b(t), y - u; b(t'))}{\partial y} |_{u=y}. \end{aligned}$$

Because $\nabla^2_y w^*_{(n)}(t', b(t), y-u; b(t')) \le 0$ and $\frac{\partial w^*_{(n)}(t', b(t), y-u; b(t'))}{\partial y} \le c_{b(t)}(t')$, we see that

$$\nabla^2_y E[w^*_{(n)}(t', b(t), (y - D_{b(t)}(t))^+; b(t'))] \le c_{b(t)}(t') f_{b(t)}(t, y) < r_{b(t)} f_{b(t)}(t, y),$$

where the last inequality follows the assumption on cost parameters that $r_{b(t)}(t) > c_{b(t)}(t) \ge c_{b(t)}(t')$ for $t \ge t'$. As such, we see that

$$\begin{aligned}\nabla^2_y w_{(n+1)}(t, a, x; b(t), y) &< -r_{b(t)} f_{b(t)}(t, y) + \theta \sum_{t' \ge t} p_{t,t'} r_{b(t)} f_{b(t)}(t, y) + \theta q_t \sum_{t'} \alpha_{t'} r_{b(t)} f_{b(t)}(t, y) \\ &\le r_{b(t)} f_{b(t)}(t, y)[1 - \sum_{t' \ge t} p_{t,t'} + q_t \sum_{t'} \alpha_{t'}] = 0,\end{aligned}$$

indicating that $w_{(n+1)}(t, a, x; b(t), y)$ is concave in y.

Suppose no technology upgrade is made. When $y \ge x$, the first derivative of $w_{(n+1)}(t, a, x; a, y)$ with respect to y is

$$\begin{aligned}\nabla_y w_{(n+1)}(t, a, x; a, y) &= -c_a(t) + r_a(t)(1 - F_a(t, y)) + \theta \sum_{t'} p_{t,t'} \nabla_y E[w^*_{(n)}(t', a, y - Z; b(t'))] \\ &\quad + \theta q_t \sum_{t'} \alpha_{t'} \nabla_y E[w^*_{(n)}(t', a+1, y - Z)].\end{aligned}$$

Let the above equation equal to zero and denote the root by $y^L_{(n+1),a}(t,\tau)$. When $y < x$, the first derivative of $w^\tau_{(n+1)}(t,a,x;a,y)$ with respect to y is

$$\begin{aligned}\nabla_y w_{(n+1)}(t,a,x;a,y) &= -s_a(t) + r_a(t)(1 - F_a(t,y)) + \theta \sum_{t'} p_{t,t'} \nabla_y E[w^*_{(n)}(t',a,y-Z;b(t'))] \\ &\quad + \theta q_t \sum_{t'} \alpha_{t'} \nabla_y E[w^*_{(n)}(t',a+1,y-Z)].\end{aligned}$$

Let the above equation equal to zero and denote the root by $y^H_{(n+1),a}(t,\tau)$. Since $s_a(t) \le c_a(t)$, in conjunction of the strictly concavity of $w^\tau_{(n+1)}(t,a,x;a,y)$ in y, we have $y^L_{(n+1),a}(t,\tau) \le y^H_{(n+1),a}(t,\tau)$. Using the similar argument in showing equation (3.11), one can show that the optimal inventory policy is a two-limit policy where the first limit is $y^L_{(n+1),a}(t,\tau)$ and the second limit is $y^H_{(n+1),a}(t,\tau)$. In addition, following the similar approach in (3.12), one can verify that $w^*_{(n+1)}(t',a,x;a)$ is differentiable in x. Furthermore, it satisfies $\nabla_x w^*_{(n+1)}(t',a,x;a) \le c_a(t)$ and

$$\nabla^2_x w^*_{(n+1)}(t',a,x;a) = \begin{cases} 0, & \text{if } x \le y^L_{(n+1),a}(t,\tau), \\ \nabla^2_y w_{(n+1)}(t,a,0;a,y)|_{y=x}, & \text{if } y^L_{(n+1),a}(t,\tau) < x < y^H_{(n+1),a}(t,\tau), \\ 0, & \text{if } x \ge y^H_{(n+1),a}(t,\tau). \end{cases} \tag{A.1}$$

Clearly, $\nabla^2_x w^*_{(n+1)}(t',a,x;a) \le 0$ for all $x \ge 0$.

Finally, let n approach to infinity. $w_{(n)}(t,a,x;b(t),y)$ converges to $w(t,a,x;b(t),y)$. We see that $w(t,a,x;b(t),y)$ is concave in y and the optimal inventory policy is a two-limit policy, i.e., $y^L_{(n),b}(t,\tau)$ and $y^H_{(n),b}(t,\tau)$ converge to $y^L_b(t,\tau)$ and $y^H_b(t,\tau)$. ■

A.3 Proof of Theorem 4.4 in Chapter 4

Proof of Theorem 4.4. Suppose the i-th element of assortment vector $\mathbf{b}$ is not equal to 1, meaning that assortment $\mathbf{b}$ does not include product i. Let 1_i be the N-dimensional vector with its ith element equal to 1 and others zero. We see that

$$\begin{aligned}\bar{g}^*(\mathbf{b}) - \bar{g}^*(\mathbf{b}+1_i) &= \sum_{j\in S(\mathbf{b})} \frac{\lambda q_j R_j}{1+\sum_{j\in S(\mathbf{b})} q_j} - \frac{\lambda q_i R_i + \sum_{j\in S(\mathbf{b})} \lambda q_j R_j}{1+q_i+\sum_{j\in S(\mathbf{b})} q_j} \\ &= \frac{\lambda q_i}{1+q_i+\sum_{j\in S(\mathbf{b})} q_j}\left(\frac{\sum_{j\in S(\mathbf{b})} q_j R_j}{1+\sum_{j\in S(\mathbf{b})} q_j} - R_i\right).\end{aligned}$$

Therefore, if $\frac{\sum_{j\in S(\mathbf{b})} q_j R_j}{1+\sum_{j\in S(\mathbf{b})} q_j} - R_i \geq 0$, then adding product i to the assortment $\mathbf{b}$ decreases the total profits; if $\frac{\sum_{j\in S(\mathbf{b})} q_j R_j}{1+\sum_{j\in S(\mathbf{b})} q_j} - R_i < 0$, then adding product i to the assortment $\mathbf{b}$ increases the total profits. Recall that $R_0 \geq R_1 \geq ... \geq R_{N-1}$, it is clear that

$$\frac{\sum_{j\in S(\mathbf{b})} q_j R_j}{1+\sum_{j\in S(\mathbf{b})} q_j} - R_0 < 0, \text{ for any } \mathbf{b}.$$

Therefore, the optimal assortment $\mathbf{b}^*$ must include product-0.

Next, we prove that if the optimal assortment includes product $i+1$, then it must include product i. Suppose the optimal assortment $\mathbf{b}^*$ includes product $i+1$ but does not include product i. Then it must hold that

$$\frac{\sum_{j\in S(\mathbf{b}^*)} q_j R_j}{1+\sum_{j\in S(\mathbf{b}^*)} q_j} - R_i \geq 0. \tag{A.2}$$

Otherwise, adding product i into assortment $\mathbf{b}^*$ increases the total profits. By rearranging the terms, we have

$$\sum_{j\in S(\mathbf{b}^*)} q_j R_j - R_i(1 + \sum_{j\in S(\mathbf{b}^*)} q_j) \geq 0.$$

While the LHS can be written as

$$\begin{aligned} LHS &= \left(\sum_{j\in S(\mathbf{b}^*-1_{i+1})} q_j R_j\right) + q_{i+1}R_{i+1} - R_i\left(1 + \sum_{j\in S(\mathbf{b}^*-1_{i+1})} q_j\right) - R_i q_{i+1} \\ &= \left(\sum_{j\in S(\mathbf{b}^*-1_{i+1})} q_j R_j\right) - R_i\left(1 + \sum_{j\in S(\mathbf{b}^*-1_{i+1})} q_j\right) + q_{i+1}(R_{i+1} - R_i). \quad \text{(A.3)} \end{aligned}$$

Since $\mathbf{b}^*$ is optimal, adding product $i+1$ to assortment $\mathbf{b}^* - 1_{i+1}$ increases total profits. So it must be the case that

$$\frac{\sum_{j\in S(\mathbf{b}-1_{i+1})} q_j R_j}{1 + \sum_{j\in S(\mathbf{b}-1_{i+1})} q_j} - R_{i+1} < 0,$$

which is equivalent to

$$\sum_{j\in S(\mathbf{b}^*-1_{i+1})} q_j R_j < R_{i+1}(1 + \sum_{j\in S(\mathbf{b}^*-1_{i+1})} q_j) \leq R_i(1 + \sum_{j\in S(\mathbf{b}^*-1_{i+1})} q_j),$$

where second inequality is due to $R_{i+1} < R_i$. Now, we see that both terms in (A.3) are negative. This implies that $\frac{\sum_{j\in S(\mathbf{b}^*)} q_j R_j}{1+\sum_{j\in S(\mathbf{b}^*)} q_j} - R_i < 0$, in other words, $\bar{g}^*(\mathbf{b}^*) < \bar{g}^*(\mathbf{b} + 1_{(i)})$, which contradicts to the optimality of $\mathbf{b}^*$. Therefore, if the optimal assortment includes product $i+1$, then it must include product i. As such, $\mathbf{b}^* \in \{\mathbf{b}_{(0)}, \mathbf{b}_{(1)}, ...\mathbf{b}_{(N-1)}\}$. ■

A.4 Proof of Lemmas in Chapter 5

Proof of Lemma 5.2.

a. From (5.16), the second derivative of $\Delta g_i(y)$ satisfies

$$\frac{\partial^2 \Delta g_i(y)}{\partial y^2} = -(a_i + b_i) f(y) \tag{A.4}$$

which is concave (convex) in y if and only if $a_i + b_i \geq 0 (a_i + b_i \leq 0)$.

b. If $a_i \geq 0$ and $b_i \leq 0$ ($a_i < 0$ and $b_i > 0$), then $\Delta g_i(y) \geq 0$ ($\Delta g_i(y) \leq 0$) for all y. Thus, $\beta_i = \infty$ and our claim holds true trivially. If $a_i \geq 0$ and $b_i \geq 0$ ($a_i < 0$ and $b_i < 0$), then $\Delta g_i(y)$ is concave (convex), as shown in (a). Our claim follows immediately from the definition of β_i.

c. This is a direct consequence of (a) and (b). ■

Proof of Lemma 5.3.

a. Suppose $y(\mathbf{t}, \mathbf{k}^{(\ell)}) \in [\beta_{(\ell)}, \beta_{(\ell+1)})$. Then for $y \in [\beta_{(\ell)}, \beta_{(\ell+1)})$ and $\mathbf{k}$,

$$g(\mathbf{t}, \mathbf{k}, y) \leq g(\mathbf{t}, \mathbf{k}^{(\ell)}, y) \leq g(\mathbf{t}, \mathbf{k}^{(\ell)}, y(\mathbf{t}, \mathbf{k}^{(\ell)})),$$

where the first inequality follows since $\mathbf{k}^{(\ell)}$ is the dominant configuration in subinterval $[\beta_{(\ell)}, \beta_{(\ell+1)})$ and the second inequality is true because $y(\mathbf{t}, \mathbf{k}^{(\ell)})$ is the optimal inventory level for Config-$\mathbf{k}^{(\ell)}$. This shows that $(\mathbf{k}^{(\ell)}, y(\mathbf{t}, \mathbf{k}^{(\ell)}))$ is a locally optimum solution of $g(\mathbf{t}, \mathbf{k}, y)$ if $y(\mathbf{t}, \mathbf{k}^{(\ell)}) \in [\beta_{(\ell)}, \beta_{(\ell+1)})$. On the other hand, if $y(\mathbf{t}, \mathbf{k}^{(\ell)}) \in [\beta_{(r)}, \beta_{(r+1)})$ and $\ell \neq r$, then

$$g(\mathbf{t}, \mathbf{k}^{(\ell)}, y(\mathbf{t}, \mathbf{k}^{(\ell)})) < g(\mathbf{t}, \mathbf{k}^{(r)}, y(\mathbf{t}, \mathbf{k}^{(\ell)})) \leq g(\mathbf{t}, \mathbf{k}^{(r)}, y(\mathbf{t}, \mathbf{k}^{(r)})),$$

which implies that $(\mathbf{k}^{(\ell)}, y(\mathbf{t}, \mathbf{k}^{(\ell)}))$ cannot be a locally optima if $y(\mathbf{t}, \mathbf{k}^{(\ell)}) \notin [\beta_{(\ell)}, \beta_{(\ell+1)})$.

b. Suppose $(\mathbf{k}^*(\mathbf{t}), \mathbf{y}^*(\mathbf{t})) = (\mathbf{k}^{(\ell^*)}, \mathbf{y}(\mathbf{t}, \mathbf{k}^{(\ell^*)})$, where $(\mathbf{k}^{(\ell^*)}, y(\mathbf{t}, \mathbf{k}^{(\ell^*)})$ is a locally optimal solution. By our construction of the dominant configuration in subinterval $[\beta_{(\ell^*)}, \beta_{(\ell^*+1)})$, $k_i^*(\mathbf{t}) = k_i^{(\ell^*)}(\mathbf{t})$ is the dominant technology of component i in $[\beta_{(\ell^*)}, \beta_{(\ell^*+1)})$, $i \in E$. By part (a) of the theorem, $y^*(\mathbf{t}) = y(\mathbf{t}, \mathbf{k}^{(\ell^*)}) \in [\beta_{(\ell^*)}, \beta_{(\ell^*+1)})$. Therefore, $y^*(\mathbf{t})$ belongs to the dominant interval of technology $k_i^*(\mathbf{t})$, for all $i \in E$. ■

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Vita

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