A THERMAL HYDRODYNAMIC LUBRICATION MODEL OF
PIVOTED PLANE-PAD THRUST BEARINGS

A Thesis in
Mechanical Engineering
by
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Abstract

This thesis develops an analytical model for the study of oil-lubricated plane-pad slider bearings with thermal effect. The effects of viscosity changes in the lubricant with temperature are taken into account. The performance of the slider bearing system can be determined by reference to three dimensionless design parameters: length-to-width ratio \((B/L)\), bearing load parameter \((w_{th})\) and the pivot location \((X_C)\) of the bearing pad.

The relationships between the performance variables and the design parameters are analyzed. The analysis suggests that centrally pivoted bearings can develop significant hydrodynamic lubrication load capacity when the bearing pad is sufficiently wide such as a length-to-width ratio \((B/L)\) of 1 or less. The results show that the load capacity is reduced as the bearing pad narrows and the reduction becomes steep for bearings of \(B/L > 1\). At \(B/L = 2\), the reduction has become pronounced and the load capacity of the bearing is diminished. In addition, the analysis reveals that the lubricant film thickness in the bearing is relatively insensitive to the applied load when the bearing pad is sufficiently wide such as \(B/L < 1\). As the bearing pad narrows, however, the film thickness becomes increasingly sensitive and the sensitivity becomes large at \(B/L = 2\), making it very difficult to design centrally pivoted narrow bearings.

The model is then used to study off-centrally pivoted bearings, and design charts are given for such bearings. The relationships between film thickness, friction coefficient, and pivot location are studied under various combinations of load parameters and length-to-width ratios.

The results obtained from the model are compared to numerical results in the published literature. The validity and accuracy of the model may be evaluated by
numerical analysis and by performing experiments on actual slider bearings in future work.
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Nomenclature

\[ B = \text{length of the bearing pad} \]
\[ c = \text{specific heat of the lubricant} \]
\[ C_{th} = \text{thermal coefficient of the bearing} \]
\[ f = \text{friction coefficient} \]
\[ h = \text{bearing film thickness} \]
\[ \bar{h} = \text{dimensionless film thickness} \]
\[ h_i = \text{bearing inlet film thickness} \]
\[ h_o = \text{bearing outlet film thickness} \]
\[ h_0^* = \text{dimensionless minimum film thickness} \]
\[ h_{0,max} = \text{maximum film thickness in isothermal model} \]
\[ H = \text{inclination of the bearing} \]
\[ k = \text{thermal conductivity of the lubricant} \]
\[ L = \text{width of the bearing pad} \]
\[ N = \text{length-to-width ratio of the bearing} \]
\[ p = \text{bearing pressure} \]
\[ \bar{p} = \text{dimensionless pressure} \]
\[ \bar{p}_{2D} = \text{dimensionless pressure for the 2D problem} \]
\[ T = \text{bearing temperature} \]
\[ \bar{T} = \text{dimensionless temperature} \]
\[ T_0 = \text{inlet temperature of the lubricant} \]
\[ U = \text{bearing sliding velocity} \]
\[ u = \text{lubricant velocity in the x direction} \]
\[ \bar{w} = \text{dimensionless bearing load} \]
\[ w = \text{bearing load} \]
\[ w_{th} = \text{bearing load parameter} \]
\[ \bar{w}_{2D} = \text{dimensionless load for the 2D problem} \]
\[ w_{2D} = \text{bearing load for the 2D problem} \]
\[ w_{th,2D} = \text{bearing load parameter for the 2D problem} \]
\( x \) = coordinate along the length direction
\( \chi \) = dimensionless x coordinate
\( X_c \) = pivot location in x coordinate
\( y \) = coordinate along the width direction
\( \gamma \) = dimensionless y coordinate
\( z \) = coordinate across the film direction
\( Z \) = dimensionless z coordinate
\( \beta \) = viscosity-temperature coefficient of the lubricant
\( \eta \) = viscosity of the lubricant
\( \eta_0 \) = ambient viscosity of the lubricant
\( \dot{\gamma} \) = shear strain rate in the lubricant
\( \rho \) = density of the lubricant
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Chapter 1: Introduction

Hydrodynamic thrust bearings are commonly used in rotating machinery. An oil-lubricated film between two sliding surfaces is essential to limit wear and to enhance bearing performance. Hydrodynamic lubrication relies on the convergent wedge and the relative motion of the two surfaces together with the viscous nature of the lubricating oil [1]. Pressure is generated inside the lubricating film and, as a result, a normal load can be supported by the bearings [2]. Figure 1.1 shows the assembly of a typical thrust bearing system.

![Figure 1.1 Schematic of a pivoted plane-pad thrust bearing system [1]](image)

Slider bearings can be categorized according to two types: fixed-incline slider bearings and pivoted slider bearings as schematically shown in Figure 1.2.
The operation of a fixed-incline bearing depends on drawing the lubricant into the convergent wedge and producing pressure inside the lubricating film. For fixed-incline bearings, the operating speed must correspond to the inclination to generate the needed hydrodynamic lubrication—a requirement that limits the ways in which such bearings can be used. In contrast, a pivoted bearing is supported at a single point so that the inclination of the bearing pad becomes a variable. Under a given set of operating conditions, a pivoted bearing performs in equilibrium. Any change in the operating conditions alters the pressure distribution, and this alternation temporarily shifts the center of pressure to create a moment that causes the pad to change its inclination until a new equilibrium is obtained. As a result, pivoted bearings have better applicability than do fixed-incline bearings and can be used in either direction of motion.

Classical hydrodynamic lubrication theory for pivoted bearings assumes that the lubricant is ideal; that is, the lubricant’s properties—viscosity, density, and specific heat—are constant. As a result, optimal behavior is obtained by arranging the pad that is
to be pivoted at a certain distance downstream from the pad’s own center. A number of
design guides have been established and documented based on the classical theory [2, 3].

According to the classical isothermal theory [2], no pressure is developed inside the
lubricating film for bearings that are pivoted at the center because no convergent wedge
is formed between the two sliding surfaces. This sharply contradicts with the practical
findings [2, 3]. As a matter of fact, centrally pivoted bearings are widely and successfully
used because of their flexibility of operating in either direction. The classical
hydrodynamic lubrication theory fails to explain this effect; therefore, design guides
cannot be written for such bearings based on its precepts.

A number of textbooks provide design rules for centrally pivoted bearings despite a
lack of theoretical explanations. Ref [1] provides design guidance based on experimental
data for specific lubricant oils under specific operating conditions, which means that, it
does not provide much guidance that is generally applicable. Ref [4] provides design
rules for centrally pivoted bearings, suggesting that they can be treated as equivalent to
bearings with an inlet-outlet film ratio of 2. However, the lack of theoretical support
means that this rule cannot be relied on.

The mechanisms of pressure development for centrally-pivoted bearings have been
studied by a number of researchers. Two mechanisms are thought to play a major role in
generating pressure for centrally pivoted bearings. One is the deflection of the pad.
Raimondi and Boyd [5] made the fundamental discovery that a small curvature that
occurs on the plane-pad slider bearing completely changes the load-supporting capability.
The deflection of the pad can either be induced by thermal distortion or by hydrodynamic
pressure. The effect of thermal distortion for specific cases based on experimental data
has been studied by Ettles [6, 7]. Thermal distortion in large thrust pads is considered a major effect and needs to be controlled; however, for smaller pads, it is considered minor compared to the effect of elastic deflection due to hydrodynamics pressure. Greenwood and Wu [8] developed an elasto-hydrodynamic model for infinitely wide slider bearings. In their model, a single dimensionless parameter was obtained to represent the elastic deflection effect. However, it is difficult to generalize this model to bearings with finite width. Because the deflection occurs in both the length and width directions, it is unlikely that the deflection effect can be described by a single dimensionless parameter for finite-width bearings.

The other major mechanism whereby pressure is generated for centrally pivoted bearings is variations in oil viscosity due to temperature increases in the lubricant. The variation in lubricant viscosity enables the bearing pad to tilt because of the continuity of fluid. As a result, pressure is developed inside the lubricant to support normal load. Chang [9] developed an analytical model to study the thermal effect on viscosity change for centrally-pivoted bearings with infinite width. A single dimensionless parameter was derived to describe the thermal hydrodynamic behavior of the system. The model shows that a significant hydrodynamic load capacity can be developed for centrally pivoted bearings.

Infinitely wide bearings simplify the governing equations so that analytical solutions are relatively easy to obtain. However, the solution for slider bearings with finite width has always been beset with difficulties. Muskat, Morgan, and Meres [10] published their solution for finite-width bearings as a series of graphs. Boegli [11] developed an analytical model to simplify the problem of isothermal bearings with finite width. The
model was based on a simple assumption. For a wide variety of bearings, the model obtained results within a range that is close to the existing ones. Several numerical studies have been performed on such bearings as well [12, 13]. They provide good references and comparisons for the analytical models.

In this thesis, the thermal model of infinitely wide bearings by Chang [9] and the isothermal model for finite bearings by Boegli [11] are integrated to form a new analytical model to study the system of finite-width bearings with thermal effect. The model is first used in reference to the centrally pivoted bearings. Then the off-centrally pivoted bearings are analyzed using the same model. In Chapter 2, the two models used to build the new analytical model are described. In Chapter 3, a new analytical model for finite-width bearings with thermal effect is developed. In Chapter 4, results are presented for both centrally pivoted and off-centrally pivoted bearings. Design charts are also given for different combinations of lubricant thermal properties, loading pressure, and the length-to-width ratios of the bearings. Finally, Chapter 5 summarizes the thesis and suggests future work that may help to verify or improve this model.
Chapter 2: Description of Existing Models

This chapter gives a brief summary of two existing models that provide the building blocks of the analytical model developed in this thesis. The first model is an analytical model developed by Chang [9] for the study of infinitely wide slider bearings with thermal effect. The second model is an analytical model developed by Boegli [11] for analyzing isothermal bearings with finite width.

Figure 2.1  Schematic of a pivoted plane-pad bearing

_Figure 2.1_ illustrates a typical pivoted slider bearing system. Symbols \( B \) and \( L \) define the length and width of the bearing, respectively. Symbols \( h_i \) and \( h_o \) define the film thickness at the inlet and outlet of the bearing pad. The sliding velocity of the bearing is \( U \). The bearing pad is anchored at the pivot point and can alter its inclination.
With a proper design, sufficient hydrodynamic pressure is generated in the lubricating film between the two sliding bearing surfaces. A load can then be supported.

The governing equations for this system are the Reynolds equation and the energy equation:

\[
\frac{\partial}{\partial x} \left( \frac{h^3}{\eta} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{h^3}{\eta} \frac{\partial p}{\partial y} \right) = 6U \frac{\partial h}{\partial x} \tag{2.1}
\]

\[
k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) - \rho c u \frac{\partial T}{\partial x} + \eta \dot{\gamma}^2 = 0 \tag{2.2}
\]

The two primary unknown variables are the lubricant pressure \( p \) and the temperature \( T \) distributions in the bearing. The distribution of these two field variables can be solved from the two governing equations with appropriate boundary conditions. After obtaining the pressure and the temperature distributions, the performance variables, such as lubricant film thickness and friction coefficient, can be calculated and analyzed.

2.1 A 2D thermal model

This section is devoted to describing the analytical thermal model developed by Chang [9] for the centrally pivoted plane-pad infinitely wide (2D) slider bearings. Figure 2.2 shows a schematic of such a bearing, where the dimension \( L \) of the pad into the page is infinite.
For this bearing, the y direction is eliminated from the Reynolds equation (2.1) to give

$$\frac{d}{dx} \left( \frac{h^3}{\eta} \frac{dp}{dx} \right) = 6U \frac{dh}{dx} \quad (2.3)$$

Three assumptions regarding the energy equation (2.2) are made so that an analytical solution of the temperature distribution can be obtained:

1. Conduction heat transfer is negligible compared to convection heat transfer;
2. All the convection heat is carried at the cross-film average velocity of the lubricant; and
3. The shear strain rate is taken to be uniform and equal to the average velocity-induced strain rate.

The validity of the assumptions as well as an order of magnitude analysis to support them can be found in [9].

With the above assumptions, the energy equation (2.2) reduces to
\[
\rho c \frac{U}{2} \frac{dT}{dx} - \eta \left( \frac{U}{(h_i + h_o)/2} \right)^2 = 0 \quad (2.4)
\]

The change of oil viscosity with temperature is assumed to be [4]:

\[
\eta = \eta_0 e^{-\beta(T - T_0)} \quad (2.5)
\]

The following dimensionless variables are defined

\[
\tilde{p} = \frac{h_o^2}{U\eta_0 B} p \quad (2.6)
\]

\[
X = \frac{x}{B} \quad (2.7)
\]

\[
H = \frac{h_i}{h_o} \quad (2.8)
\]

\[
\tilde{h} = \frac{h}{h_o} \quad (2.9)
\]

\[
\tilde{T} = \beta T \quad (2.10)
\]

In addition, for the plane-pad bearings,

\[
\tilde{h} = [H - (H - 1)X] \quad (2.11)
\]

The energy equation is nondimensionalized as

\[
\frac{d\tilde{T}}{dX} - \frac{8C_{th}}{(1+H)^2} e^{-\tilde{T}} = 0 \quad (2.12)
\]

where

\[
C_{th} = \frac{U\eta_0 B\beta}{h_o^2 \rho c} \quad (2.13)
\]

which is defined as the thermal parameter of the system. Solving the temperature distribution and substituting it into equation (2.5) for oil viscosity yields

\[
\eta = \eta_0 \left( 1 + \frac{8C_{th}}{(1+H)^2} X \right)^{-1} \quad (2.14)
\]

Substituting the viscosity in the Reynolds equation (2.3) and integrating yields the pressure distribution.
\[ \bar{p}_{2D}(X) = A(X) + c_1 B(X) + c_2 \]  \hspace{1cm} (2.15)

where

\[ A(X) = 6 \int \frac{dX}{\left[1 + \frac{8C_{th}}{(1+H)^2X} \right][H-(H-1)X]^2} \]  \hspace{1cm} (2.16)

\[ B(X) = \int \frac{dX}{\left[1 + \frac{8C_{th}}{(1+H)^2X} \right][H-(H-1)X]^3} \]  \hspace{1cm} (2.17)

and \( c_1 \) and \( c_2 \) are the integration constants and are determined by applying the boundary conditions \( \bar{p}(0) = 0 \) and \( \bar{p}(1) = 0 \) to equation (2.15).

The pressure center coincides with that of the pivot. For a centrally pivoted bearing, this leads to

\[ \int_0^1 \bar{p}_{2D} X dX = 0.5 \int_0^1 \bar{p}_{2D} dX \]  \hspace{1cm} (2.18)

After the pressure distribution is obtained, performance factors, such as bearing load and lubricant film thickness, are calculated. A dimensionless load is given by

\[ \bar{w}_{2D} = \int_0^1 \bar{p}_{2D} dX = \int_0^B \left( \frac{h_0^2}{U\eta_0 B} \bar{p} \right) d \left( \frac{x}{B} \right) = \frac{h_0^2 w_{2D}}{U\eta_0 B^2} \]  \hspace{1cm} (2.19)

Multiplying \( \bar{w}_{2D} \) and \( C_{th} \) yields an intrinsic parameter for the system

\[ w_{th} = C_{th} \bar{w}_{2D} = \left( \frac{U\eta_0 B\beta}{h_0^2 \rho c} \right) \left( \frac{h_0^2 w_{2D}}{U\eta_0 B^2} \right) = \frac{\beta}{\rho c} \frac{w_{2D}}{B} \]  \hspace{1cm} (2.20)

It is defined as the bearing load parameter. The value of \( w_{th} \) is around 0.1 for typical slider bearings [9].

This model reveals that the thermal effect of the lubricant is a major mechanism enabling hydrodynamic actions for centrally pivoted plane-pad slider bearings. With \( w_{th} = 0.1 \), the model predicts a load capacity of about 42\% of the maximum load.
capacity under isothermal, optimal pivoted conditions, or a film thickness of 65% of the maximum isothermal film thickness [9]. The validity of the model is discussed in detail in [9].

2.2 A 3D isothermal model

In the previous section, variables were assumed to be a function of $x$ and $z$ only. For a finite-width bearing, they are functions of $x$, $y$, and $z$ and the problem is three-dimensional. This section summarizes an analytical model developed by Boegli [11] to study 3D isothermal slider bearing systems.

Figure 2.3 Separation of the pressure function in 3D slider bearing

Figure 2.3 shows a schematic of a 3D slider bearing. Two primary assumptions are made in the model developed by Boegli [11] to simplify the problem. The first assumption is that the pressure distribution can be separated into two independent
functions along the length and width directions. Based on this assumption, the pressure distribution can be expressed as the product of two independent functions

\[ p(x, y) = A f_x(x) f_y(y) \]  \hspace{1cm} (2.21)

where \( A = \frac{u \eta B}{h^2} \) contains all the dimensional terms.

Define one dimensionless variable in addition to those defined by equations (2.6) to (2.10)

\[ Y = \frac{y}{L} \]  \hspace{1cm} (2.22)

Substituting the pressure distribution in equation (2.21) into the 3D Reynolds equation (2.1) and with the nondimensionalization yields

\[ \frac{\partial}{\partial x} \left[ h^3 f_y(Y) \frac{d f_x(x)}{dx} \right] + N^2 h^3 f_x(X) \frac{d^2 f_y(y)}{dy^2} = 6 \frac{d h}{dx} \]  \hspace{1cm} (2.23)

where \( N = \frac{B}{L} \) is the length-to-width ratio of the bearing pad.

The second assumption is that the pressure distribution function along the length direction is the same as that for a 2D model. According to this assumption, the form of the pressure distribution in the \( x \) direction is known from a 2D model. Thus, only \( f_y \) needs to be solved from equation (2.23). Evaluating equation (2.23) at the position where \( f_x \) reaches maximum, or \( df_x/dX = 0 \), enables equation (2.23) to be simplified to

\[ \frac{d^2 f_y(Y)}{dy^2} \Bigg|_{Y=X_m} + \frac{1}{f_x(X)} \frac{d^2 f_x(X)}{dx^2} \frac{1}{N^2} \left( \frac{1}{h^3 f_x(X)} \right) \frac{d h}{dx} \Bigg|_{X=X_m} = 6 \frac{d h}{dx} \]  \hspace{1cm} (2.24)

where \( X_m \) is the value of \( X \) with \( df_x/dX = 0 \).

Remember that \( f_x \) satisfies the 2D Reynolds equation, namely,

\[ \frac{\partial}{\partial X} \left[ h^3 \frac{d f_x(X)}{dX} \right] = 6 \frac{d h}{dX} \]  \hspace{1cm} (2.25)
Evaluating equation (2.25) at $X_m$ yields

$$
\left[ \frac{\partial^2 f_X(X)}{\partial X^2} \right] \bigg|_{X=X_m} = 6 \frac{d\bar{h}}{dX} \bigg|_{X=X_m} \tag{2.26}
$$

Then equation (2.24) can be written as

$$
\frac{d^2 f_Y}{dY^2} + \frac{M}{N^2} f_Y = \frac{M}{N^2} \tag{2.27}
$$

where

$$
M = \left[ \frac{1}{\bar{h}^3 f_X(X)} \right] \left[ 6 \frac{d\bar{h}}{dX} \right] \bigg|_{X=X_m} = \left[ \frac{1}{f_X(X)} \frac{d^2 f_X(X)}{dX^2} \right] \bigg|_{X=X_m} \tag{2.28}
$$

The value of $M$ is obtained from the distribution of $f_X$ in a 2D model. In practice, the first term of equation (2.28) is used as no second derivative of $f_X$ is needed. The solution for $f_Y$ of equation (2.27) is given by [11]

$$
f_Y(Y) = 1 - \left( \frac{e^{n-1}}{e^n - e^{-n}} \right) \left( e^{-nY} + e^{-n(1-Y)} \right) \tag{2.29}
$$

where

$$
n^2 = \frac{-M}{N^2} \tag{2.30}
$$

The total bearing load can then be calculated by

$$
w = ABL\bar{w} \tag{2.31}
$$

where the dimensionless load is

$$
\bar{w} = \int_0^1 f_X(X)dX \int_0^1 f_Y(Y)dY \tag{2.32}
$$

The results show that the error of total bearing load predicted by this model is less than 5% for bearings with a $B/L$ ratio smaller than 1. For narrower bearings of $B/L = 2$, the error from this model is about 10%. The estimate of error and other analysis are discussed in [11].
Chapter 3: Development of a 3D Thermal Model

This chapter describes the development of an analytical model for thermal hydrodynamic lubrication of a 3D pivoted slider bearing system. The two models described in Chapter 2 serve as foundations for this model. Section 3.1 integrates the two models thereby formulating a complete thermal hydrodynamic model that may be used to analyze the 3D pivoted plane-pad slider bearing system. Section 3.2 provides the methods of calculation as well as a program flow chart.

3.1 A 3D thermal model

Based on the ideas from the previous chapter, a 3D thermal model is built in this section. The same approach as in Section 2.2 is used to solve the 3D pressure distribution; namely, the pressure is assumed to be the product of the independent functions along the $x$ and $y$ directions

$$p(x, y) = A f_x(x) f_y(y)$$

(2.22)

Similar to Section 2.2, the pressure distribution along the $x$ direction ($f_x(x)$) is solved first.

Unlike in Section 2.2, the thermal effects of viscosity variation are now taken into consideration. The temperature is assumed to be a function of $x$ only. This assumption is reasonable as heat in the lubricant is mainly generated by the surface motion and is carried by the forward flow of lubricant. This assumption is supported by the numerical results of Raimondi [12] that show, for a square pad ($N = 1$), the temperature variation is small in the width direction of the bearing pad. With this assumption, the resulting viscosity distribution has the same expression as that from the 2D thermal model:

$$\eta = \eta_0 \left(1 + \frac{B C_{th}}{(1+H)^2} X \right)^{-1}$$

(2.14)
The distribution of $f_x(x)$ is obtained by solving the dimensionless 2D Reynolds equation with varying lubricant viscosity:

$$\frac{\partial}{\partial X} \left[ (1 + \frac{8C_{th}}{(1+H)^2} X) \bar{h}^3 \frac{d f_x(X)}{dX} \right] = 6 \frac{d \bar{h}}{dX} \quad (3.1)$$

It can be seen that $f_x$ has the same expression as the 2D pressure distribution $\bar{p}_{2D}$ in Section 2.1, as they both satisfy the same equation (3.1). After the distribution of $f_x$ is obtained, $f_y$ can be solved from the 3D Reynolds equation with variable viscosity:

$$\frac{\partial}{\partial X} \left[ (1 + \frac{8C_{th}}{(1+H)^2} X) \bar{h}^3 f_y(Y) \frac{d f_y(Y)}{dX} \right] + N^2 (1 + \frac{8C_{th}}{(1+H)^2} X) \bar{h}^3 f_x(X) \frac{d^2 f_y(Y)}{dY^2} = 6 \frac{d \bar{h}}{dX} \quad (3.2)$$

Equation (3.2) is evaluated at $X_m$ (where $df_x/dX = 0$). Expanding equation (3.2) and eliminating all the terms in $df_x/dX$ yields

$$\frac{d^2 f_y(Y)}{dY^2} + \left[ \frac{1}{f_x(X)} \frac{d^2 f_x(X)}{dX^2} \right]_{X=X_m} f_y(Y) = \left[ \frac{1}{f_x(X)} \frac{d h}{dX} \right]_{X=X_m} \quad (3.3)$$

In the same time, equation (3.1) is evaluated at $X_m$ to give

$$\left[ (1 + \frac{8C_{th}}{(1+H)^2} X) \bar{h}^3 \frac{d^2 f_x(X)}{dX^2} \right]_{X=X_m} = 6 \frac{d \bar{h}}{dX} \bigg|_{X=X_m} \quad (3.4)$$

Define variable $M$ as

$$M = \left[ \frac{1}{1 + \frac{8C_{th}}{(1+H)^2} X} \bar{h}^3 f_x(X) \frac{d f_x(X)}{dX} \right]_{X=X_m} = \left[ \frac{1}{f_x(X)} \frac{d^2 f_x(X)}{dX^2} \right]_{X=X_m} \quad (3.5)$$

The value of $M$ is calculated from the first term in equation (3.5) in order to avoid the second derivative of $f_x$. As a result, equation (3.3) reduces to

$$\frac{d^2 f_y}{dY^2} + \frac{M}{N^2} f_y = \frac{M}{N^2} \quad (3.6)$$
It can be seen that the reduced form in equation (3.6) has the same expression as equation (2.27). However, the definition of $M$ is different due to the thermal effect on the viscosity variation. Thus, the solution for $f_Y$ is of the same form as that for equation (2.29):

$$f_Y(Y) = 1 - \left(\frac{e^{n-1}}{e^{n} - e^{-n}}\right) (e^{-nY} + e^{-(1-n)Y}) \quad (2.29)$$

where

$$n^2 = \frac{-M}{N^2} \quad (2.30)$$

Then, the entire pressure distribution is obtained and the dimensionless pressure is

$$\bar{p}(X, Y) = \frac{p(x, y)}{A} = f_X(X) f_Y(Y) \quad (3.6)$$

The pressure center in the $x$ direction coincides with the location of the pivot. For a bearing pivoted at $X_C$, this leads to

$$\int_0^1 f_X X dX = X_C \int_0^1 f_X dX \quad (3.7)$$

For centrally pivoted bearings, $X_C = 0.5$. The total load can be calculated by

$$w = \int_0^B \int_0^L p(x, y) dx dy = \int_0^1 \int_0^1 A\bar{p}(X, Y) (BdX) (LdY) = \frac{U\eta_0 B^2 L}{h_0^2} \bar{w} \quad (3.8)$$

where the dimensionless bearing load is

$$\bar{w} = \int_0^1 f_X(X) dX \int_0^1 f_Y(Y) dY \quad (2.32)$$

A bearing load parameter $w_{th}$ can be similarly defined for the 3D problem

$$w_{th} = C_{th} \bar{w} = \left(\frac{U\eta_0 B\beta}{h_0^2 \rho_c}\right) \left(\frac{h_0^2 w}{U\eta_0 B^2 L}\right) = \frac{\beta}{\rho_c} \frac{w}{BL} \quad (3.9)$$

It is an intrinsic system parameter that is determined by the thermal properties of the lubricant ($\frac{\beta}{\rho_c}$) and the average loading pressure ($\frac{w}{BL}$). In fact, the 3D bearing load
parameter is identical to the definition of $w_{th,2D}$ in equation (2.21), as $\frac{W_2D}{B}$ is essentially the average loading pressure for the 2D problem.

The dimensionless minimum film thickness $h_o^*$ is commonly used as a performance factor in the design process. It is defined by [2]

$$h_o^* = \frac{h_0}{B\sqrt{k}} \quad (3.10)$$

where

$$k = \frac{u\eta_o L}{w} \quad (3.11)$$

Comparing equation (3.10) to (3.8), it can be seen that $h_o^*$ has a simple relationship with $\bar{w}$

$$h_o^* = \sqrt{\bar{w}} \quad (3.12)$$

Another important performance measurement is the friction coefficient of the bearing. Friction is the traction force asserted by the lubricating film to the slider bearing surfaces. After the pressure distribution is obtained, the friction coefficient can be calculated from the shear stress in the lubricant:

$$f = \frac{1}{w} \int_0^B \tau_x \, dx \quad (3.13)$$

Carrying out the integral and converting the variables to dimensionless ones yields the expression for the friction coefficient. The detail derivations are given in Appendix A and the resulting expression for the friction coefficient is given by

$$f = \left[ \ln \left( 1 + \frac{8C_{th}H}{(1+H)^2} \right) + \ln H + \frac{(H-1)\bar{w}}{2} \right] / h_o^* \quad (3.14)$$
3.2 Methods of calculation

This section summarizes the key equations comprising the 3D thermal model:

Equations for the pressure distribution:

\[
\bar{p}(X, Y) = f_X(X) f_Y(Y)
\]  
(3.6)

\[
f_X(X) = A(X) + c_1 B(X) + c_2
\]  
(2.16)

\[
A(X) = 6 \int \frac{dX}{\left[1 + \frac{8c_{th}}{(1+H)^2} X \right] \left[H - (H-1)X\right]^2}
\]  
(2.17)

\[
B(X) = \int \frac{dX}{\left[1 + \frac{8c_{th}X}{(1+H)^2} \right] \left[H - (H-1)X\right]^3}
\]  
(2.18)

\[
f_Y(Y) = 1 - \left(\frac{e^{n-1}}{e^n - e^{-n}}\right) (e^{-nY} + e^{-n(1-Y)})
\]  
(2.29)

\[
n^2 = \frac{-M}{N^2}
\]  
(2.30)

\[
M = \left[ \frac{1}{1 + \frac{8c_{th}}{(1+H)^2} X} \frac{1}{f_X(X)} \frac{dH}{dx} \right] \bigg|_{X=X_m} = \left[ \frac{1}{f_X(X)} \frac{d^2 f_X(X)}{dx^2} \right] \bigg|_{X=X_m}
\]  
(3.5)
Equations for the performance variables:

\[ \bar{w} = \int_0^1 f_X(X)\,dX \int_0^1 f_Y(Y)\,dY \tag{2.32} \]

\[ h_o = \sqrt{\bar{w}} \tag{3.12} \]

\[ f = \left[ \ln\left(1+\frac{8C_{th}}{(1+H)^2}\right) + \ln H + \frac{(H-1)\bar{w}}{2} \right]/\bar{h}_o \tag{3.14} \]

In the calculation, the pressure distribution needs to be obtained first followed by the calculation of the performance variables. The pressure distribution depends on the value of the thermal parameter \( C_{th} \) and the bearing inclination \( H \). However, in the design of a pivot bearing, neither \( C_{th} \) nor \( H \) is usually given as the design parameter. Instead, the intrinsic parameters of the pivoted bearing system, such as the length-to-width ratio \( B/L \), the load parameter \( w_{th} \) (defined by equation (3.9)), and the pivot location \( X_c \), are given or selected in the bearing design. Thus, for a given set of \( w_{th}, B/L \) and \( X_c \), the corresponding \( C_{th} \) and \( H \) need to be determined so that the pressure distribution and the performance variables can be calculated. In the calculation, the one-to-one relationships of \( C_{th} \) versus \( w_{th} \) and of \( H \) versus \( w_{th} \) are first created for the given \( B/L \) ratio and \( X_c \): a range of \( C_{th} \) is chosen, and the pressure distribution is computed with the \( H \) that establishes the pressure center in the same location as that of the given pivot. Then, the resulting \( w_{th} \) values are calculated. The corresponding \( C_{th} \) and \( H \) values to the given \( w_{th} \) are obtained from these calculations.

A flow chart of the program is provided in Figure 3.1. The equations directly used in the calculation are included in the flow chart as well.
The Matlab code for the calculation is given in Appendix B. The calculation of the $w_{th}$ versus $C_{th}$ relationship is quite time-consuming. The initial estimate of $C_{th}$ can be chosen according to experience to shorten the calculation time. Optimization methods may be explored in the future to improve the calculation efficiency.

*Figure 3.1* **Flow chart of calculation**

The Matlab code for the calculation is given in Appendix B. The calculation of the $w_{th}$ versus $C_{th}$ relationship is quite time-consuming. The initial estimate of $C_{th}$ can be chosen according to experience to shorten the calculation time. Optimization methods may be explored in the future to improve the calculation efficiency.

*Figure 3.2* shows typical relationships between $w_{th}$ and $C_{th}$ for some different combinations of $X_c$ and $B/L$ ratio. A one-to-one relationship is established between $C_{th}$ and $w_{th}$ for given $X_c$ and $B/L$. For a given design load parameter $w_{th}$, the corresponding $C_{th}$ value is obtained using these relationships.
Figure 3.2  **Relationship between \( C_{th} \) and \( w_{th} \)**

Figure 3.3 shows typical relationships between \( H \) and \( w_{th} \) for some different combination of \( X_c \) and the \( B/L \) ratio. A one-to-one relationship also exists between \( H \) and \( w_{th} \). Thus, for a set of \( w_{th}, X_c \) and \( B/L \), there corresponds a unique set of \( C_{th} \) and \( H \). Once the three design parameters are chosen, the slider bearing system is defined and the performance variable can be calculated.
The procedure for the calculation has been established. The isothermal results for minimum film thickness and the friction coefficient at various pivot locations are well documented in a number of papers and textbooks [2, 3]. However, obtaining those results is computationally intensive. The results from the simple analytical model in this thesis can be compared with these numerical results to examine the validity and accuracy of the model. Figure 3.4 shows the comparison of the film thickness results. It can be seen that, for length-to-width ratios ($B/L$) of 0, 0.5, and 1, the percentage error is within 5%, and for $B/L = 2$, the error is about 10%.
Figure 3.4 Model verification (film thickness) with numerical results

Figure 3.5 shows the comparison of friction coefficient results. The percentage error for the $B/L$ ratios of 0, 0.5, and 1 is less than 10%. For the ratio of $B/L = 2$, the error is relatively large. Nevertheless, the overall results still suggest that the simple analytical model is a reasonable approximation as the accuracy of the friction results is not as critical as that of the film thickness results.
Figure 3.5 Model verification (friction coefficient) with numerical model
Chapter 4: Results and Analysis

In this chapter, the model developed in Chapter 3 is used to analyze the performance of plane-pad slider bearings with thermal effect. In practice, the majority of slider bearings are centrally pivoted because of their flexibility of alternating operating direction. Section 4.1 analyzes the performance of such bearings first. Section 4.2 studies the performance of off-centrally pivoted bearings and generates design charts.

4.1 Centrally pivoted bearings

Recall the schematic of a pivoted plane-pad bearing in Figure 4.1. If the pivot of the bearing is at the center of the pad in the direction of forward lubricant flow (the $x$-direction), it is called centrally pivoted bearing. For this kind of bearing, the pivot location is $x_c = B/2$ or $X_c = 0.5$.

![Schematic of a pivoted plane-pad bearing](image)
According to classical lubrication theory, hydrodynamic actions cannot be developed for such bearings as no convergent wedge can be formed between the two sliding bearing surfaces. Figure 4.2 shows the dimensionless film thickness calculated using equations (3.10) and (3.11) for the isothermal model. The thickness of the lubricating film reflects the protection by the lubrication. The slider bearings need to operate with a lubricating film of sufficient thickness to avoid damage and reduce wear. In Figure 4.2, the x-axis is the dimensionless pivot position. The isothermal model predicts a zero film thickness for centrally pivoted bearings.

![Dimensionless film thickness in isothermal model](image)

**Figure 4.2 Dimensionless film thickness in isothermal model**

The model developed in this thesis takes the thermal effect of the lubricant into consideration. As a result, the variations in lubricant viscosity are introduced, which enables the hydrodynamic actions to be developed for centrally pivoted bearings. Chang [9] studied the centrally pivoted bearings of infinite width with thermal effect. He found
that a lubricating film of 65% of the maximum isothermal film thickness is developed for centrally pivoted bearings under the typical operating condition of $w_{th} = 0.1$. The maximum isothermal film thickness, marked in Figure 4.2, is obtained at different pivot locations for different $B/L$ ratios. It represents the maximum possible load capacity for the bearing at the given $B/L$ ratio. For an infinite wide bearing ($B/L = 0$), the maximum isothermal film thickness is $h_{o,max}^* = 0.4$.

Using the model developed in this thesis, the bearing performance can be studied for any given length-to-width ratio. Figure 4.3 presents the calculated film thickness for different $B/L$ ratios with $w_{th} = 0.1$. The results are presented in the ratio of the film thickness to its corresponding maximum isothermal film thickness. In other words, it measures relative reduction of load capacity from its corresponding maximum possible isothermal load capacity.

Figure 4.3 Relative film thickness under different $B/L$ ratios ($w_{th} = 0.1$)
The film thickness ratio decreases significantly when bearing narrows. For the infinitely wide bearings, a film thickness of 65% of its maximum isothermal value is obtained. For bearings with $B/L = 2$, on the other hand, it is only 12%. It should be noticed that the maximum isothermal film thickness has already decreased significantly when bearing narrows as shown in Figure 4.2. This decrease is mainly due to the side leakage effect in the bearing. In addition to it, the ratio of the film thickness to its maximum further decreases, which suggests the load capacity of the centrally pivoted bearings may be drastically reduced as the bearing narrows.

As shown in Figure 4.3, the reduction of the relative load capacity is somewhat modest for bearing of $B/L$ ratio smaller than 1, which corresponds to bearings of square or wider pads. For $B/L > 1$, the reduction becomes steeper, leading to diminishing relative load capacity. This drastic reduction needs to be noticed as it will reduce the effect of lubrication and may cause serious damage of the machine parts because of the lack of protection. Therefore, the designer may need to be cautious with narrow bearings, especially for bearings of $B/L$ ratios greater than 1.

The mechanism of reduction in the film thickness is analyzed next. As the side leakage effect is already taken into consideration in isothermal models, it should not appear in the relative film thickness results in Figure 4.3. Thus, the thermal effect in the lubricant is likely to cause this reduction. According to the thermal model developed in Chapter 3, the major influence of the thermal effect is to introduce variations in lubricant viscosity, which is assumed to be constant throughout the lubricating film in the isothermal model. The effect on lubricant viscosity is influenced by the thermal parameter $C_{th}$ defined by equation (2.13):
For a given value of load parameter \( w_{th} \), the corresponding \( C_{th} \) needs to be found first using the \( C_{th} \) vs \( w_{th} \) relationship to calculate the pressure distribution. Under the given load parameter of \( w_{th} = 0.1 \), the corresponding \( C_{th} \) value is 153.6 for \( B/L = 2 \); on the other hand, it is only 1.48 for bearings with \( B/L = 0 \). The thermal parameter \( C_{th} \) influences the pressure distribution by altering the viscosity distribution in equation (2.14):

\[
\eta = \eta_0 \left( 1 + \frac{8C_{th}}{(1+H)^2} X \right)^{-1}
\]

The viscosity distributions for bearings with \( B/L \) ratio of 0 and 2 are plotted in Figure 4.4. Both bearings presented in the figure are centrally pivoted. Therefore, the bearing inclination \( H \) and thermal parameter \( C_{th} \) are different for the two bearings. As a result, the viscosity distribution has significant difference between the two bearings.

**Figure 4.4 Viscosity Distribution for \( w_{th} = 0.1 \)**
The viscosity is scaled by its value at the ambient temperature ($\eta_0$). It can be seen that, for the majority of the bearing area, the viscosity is less than 10% of $\eta_0$ for bearings with $B/L = 2$, which vastly reduces the capability of pressure generation. In the case of $B/L = 0$, on the other hand, the viscosity reduction is not very large. As a result, while the variation of the viscosity helps develop a convergent wedge of the bearing to enable the hydrodynamic action, the still significant level of viscosity enables sufficient film thickness or load capacity to be developed.

Figure 4.5 compares the pressure distribution in the $x$-direction obtained using the thermal model and the isothermal model for bearings of $B/L = 2$ under the same bearing inclination $H = 3.28$. The difference in the two pressures is very dramatic. The lack of pressure generation in the thermal model leads to insufficient film thickness, and thus much reduced load capacity.

![Figure 4.5 Pressure Distribution in the x-direction for $w_{th} = 0.1$ ($B/L = 2$)](image-url)
The pressure distribution along the $y$-direction also contributes to the load capacity and film thickness of the bearing. The load of a 3D bearing is related to the load per unit width of a 2D bearing of equal length by

$$\frac{\bar{w}}{\bar{w}_{2D}} = \frac{\int_0^1 \int_0^1 f_x(X) f_y(Y)dXdY}{\int_0^1 f_x(X)dX} = \int_0^1 f_Y(Y)dY = 1 - \frac{2}{n} \frac{(1-e^{-n})^2}{n(1-e^{-2n})} \tag{4.1}$$

Equation (4.1) provides a measure of load reduction due to lubricant side leakage that exists in the finite-width bearings. Because the pressure distribution along the $y$-direction depends on the maximum value of $f_x$ (equations (3.5) and (3.6)), the $f_Y$ distribution is different between the thermal model and the isothermal model. Figure 4.6 shows the pressure distributions along the $y$-direction from these two models for $B/L = 2$ with the same bearing inclination $H = 3.28$. The results suggest that the thermal effect intensifies lubricant side leakage due to reduced viscosity. As a result, the load capacity is furthermore reduced.

Figure 4.6  Pressure Distribution in the $y$-direction for $w_{th} = 0.1 \ (B/L = 2)$
The reduction of the relative load capacity as the bearing narrows is also suggested in the numerical analysis of Raimondi [12]. He performed a numerical study of square-pad bearing \((B/L = 1)\) and provided comparison with the results in this thesis. In Raimondi’s work, the lubricant properties can be converted into \(C_{th}\) by applying equation (2.13), and then converted into \(w_{th}\) by using Figure 3.2. Figure 11 in [12] shows that the bearing inclination for \(w_{th} = 0.06\) is about \(H = 1.8\), and for \(w_{th} = 0.12\), the inclination is about \(H = 1.95\). Using the model in this thesis, the inclination for square-pad bearings is 1.87 for \(w_{th} = 0.1\), which is in good agreement with Raimondi’s numerical study. In Figure 6 of [12], Raimondi obtained the film thickness for square-pad bearings. The relative film thickness for square-pad bearings under \(w_{th} = 0.06\) is 60%, and for \(w_{th} = 0.12\), the film thickness is about 50% of its maximum isothermal thickness. The model in this thesis predicts a film thickness 47% of the isothermal maximum for \(w_{th} = 0.1\), which is lower than Raimondi’s numerical result, but still within fairly close range. More importantly, both Raimondi’s work and the model in this thesis predict that the relative film thickness for the square pad bearings is significantly reduced from the value for infinitely wide bearings.

Up to this point, the study of film thickness for centrally pivoted bearings is based on the operating condition of load parameter \(w_{th} = 0.1\). In practice, the designer can prescribe a different \(w_{th}\) for the bearing. Recall the definition of \(w_{th}\) in equation (3.9): \[ \begin{equation} W_{th} = \frac{\beta \ w}{\rho c B L} \tag{3.9} \end{equation} \]
After the lubricant properties \( \frac{B}{\rho C} \) are chosen, the load parameter \( w_{th} \) is proportional to the average loading pressure \( \frac{w}{BL} \) that is asserted to the bearing. It is of interest to study the behavior of relative film thickness versus load parameter at various length-to-width ratios. Figure 4.7 presents the dimensionless film thickness results for bearings with \( B/L \) ratios equal to 0, 0.5, 1, and 2:

![Figure 4.7 Relative film thickness under different load parameters](image)

The results show that the optimal \( w_{th} \) value, which is defined as the \( w_{th} \) value where the maximum relative film thickness is reached, decreases as the bearing narrows. For \( B/L = 2 \), the optimal \( w_{th} \) value is about 0.01, which is only about 11% of the value for infinitely wide bearings (optimal \( w_{th} = 0.09 \)). Therefore, when bearings are selected, the operating load parameter should be designed with respect to the shape of the bearing. The average loading pressure should be reduced accordingly when a narrower bearing is used so that a lubricating film of sufficient thickness can be generated.
The results in Figure 4.7 also show that the range in which sufficient film thickness can be generated diminishes as the bearing becomes narrower. For infinitely wide bearings, the relative film thickness is not very sensitive to changes in $w_{th}$. A film thickness of more than 60% of its isothermal maximum (which is 5% less than its optimal value of 65%) can be obtained for $w_{th}$ ranging from 0.03 to 0.2. As the bearing narrows, the film thickness becomes more sensitive to $w_{th}$. When $B/L = 0.5$, a film thickness of more than 55% of its maximum can be obtained for $w_{th}$ from 0.02 to 0.15. For $B/L = 2$, $w_{th}$ needs to be in a very narrow range to develop a lubricant film with sufficient thickness. Thus, according to the model’s prediction, when a narrower bearing is designed, the thermal properties and operating pressure must be carefully evaluated so that the precise value of $w_{th}$ can be obtained, or extra safety factors should be considered regarding this effect.

Figure 4.8 shows the optimal $w_{th}$ value as a function of $B/L$ ratio. As the bearing narrows, the optimal load parameter decreases. Because the relative film thickness and load capacity is more sensitive for narrow bearings, designers need to pay more attention when a narrow bearing is chosen, although the reduction in optimal load parameter appears to be steeper for relatively wide bearings ($B/L > 1$). In other words, a $w_{th}$ value that is 10% off its optimal may still be able to provide sufficient film thickness for wide bearings; the same variation, on the other hand, may not generate enough pressure to support a sufficient film thickness.
Figure 4.8 Optimal load parameter for different $B/L$ ratios

Figure 4.9 shows the relative film thickness at the corresponding optimal $w_{th}$ values in Figure 4.8. Although it is still decreasing as the bearing narrows, a relative film thickness of more than 50% of its isothermal maximum can be obtained for all $B/L$ ratios ranging from 0 to 2 at the optimal load parameter.

Figure 4.9 Relative film thickness at optimal load parameter
Another key performance variable is the friction coefficient (equation (3.14)). *Figure 4.10* shows the friction coefficient at various $w_{th}$ values for different $B/L$ ratios. Similar to the film thickness, the friction coefficient is scaled by its isothermal value when the film thickness reaches the maximum. It can be seen that, unlike relative film thickness, the trend for the relative friction coefficient is quite consistent for different $B/L$ ratios: the relative friction coefficient decreases as the load parameter $w_{th}$ increases.

![Graph showing relative friction coefficient under different load parameters](image)

**Figure 4.10** Relative friction coefficient under different load parameters

*Figure 4.11* shows the value of relative friction coefficient at the optimal load parameter in *Figure 4.8*. It represents the value of friction coefficient when the maximum load capacity is obtained. At the optimal load parameter, nearly all the bearings will have a friction coefficient larger than that from the isothermal model.
Figure 4.1 Relative friction coefficient at optimal load parameter

The performance of centrally pivoted bearings has been analyzed in this section. In bearing design, one may make use of the previous results with the following procedure:

1. Determine the length-to-width ratio \( (B/L) \) of the bearing pad;
2. Find the corresponding optimal load parameter \( (w_{th}) \) in Figure 4.8;
3. Find the relative dimensionless film thickness in Figure 4.9, and relative friction coefficient in Figure 4.11;
4. Calculate the dimensionless film thickness by multiplying the relative film thickness by the maximum isothermal film thickness found in Figure 4.2 or other design handbooks, and calculate the friction coefficient by multiplying the relative friction coefficient by its value at the maximum isothermal location;
5. Determine the oil properties using the results from the previous steps to ensure the film thickness satisfies the design requirement.
4.2 Off-centrally pivoted bearings

The centrally pivoted bearings have been studied in the previous section. This section will analyze the effect of pivot location. When the model developed in Chapter 3 is used, once the bearing inclination $H$ and thermal parameter $C_{th}$ are given, the pressure center can be obtained by:

$$\int_0^1 f_X X dX = X_c \int_0^1 f_X dx$$  \hfill (3.7)

For moment equilibrium of a pivoted bearing, the pressure center coincides with the pivot location. Figures 4.12 and 4.13 present the dimensionless minimum film thickness for varying pivot locations and $B/L$ ratios. Both the thermal (with $w_{th} = 0.1$) and the isothermal results are presented so that the thermal effect can be easily observed.

![Graph showing film thickness under different pivot locations](image)

*Figure 4.12  Film thickness under different pivot locations ($B/L = 0$ and 1)*
It can be seen that, in contrast to the isothermal model, a non-zero film thickness is now generated at the centrally pivoted location, which was discussed in the previous section. Also, supporting the previous observation is that for bearings with $B/L = 2$, a very small value of $h_o$ is obtained at the centrally pivoted position, thus, sufficient load capability is difficult to develop under this load parameter for such narrow bearings.

Meanwhile, the maximum value of $h_o^*$ is reduced. Although a non-zero load capacity is generated at the centrally pivoted location, the maximum load capacity when the bearing is pivoted at its optimal pivot position is reduced. Therefore, in designing an off-centrally pivoted bearing, the load capacity needs to be reduced if the bearing is designed to pivot at the optimal position.

In addition, the optimal position where the maximum $h_o^*$ is achieved moves slightly. However, this effect is not quite as significant. Therefore, it is relatively safe to state that the optimal position still stays at the value for isothermal models.
When the pivot location $X_c$ exceeds 0.75, the difference in film thickness between the isothermal and the thermal model becomes much smaller. This is because when $X_c$ becomes larger, the convergent wedge is induced predominantly by the downstream pivot location. Thus, the thermal effect becomes less significant.

*Figures 4.14 to 4.17* show the relationships between pivot location and dimensionless film thickness for various combinations of load parameters and length-to-width ratios. The results may be used as design charts for the thermal effect. One general trend that can be observed is that, as $w_{th}$ increases, the maximum film thickness at the optimal pivot location decreases.

*Figure 4.14  Film thickness under different pivot locations with various $w_{th}$  
\[
(B/L = 0)
\]
Figure 4.15 Film thickness under different pivot locations with various $w_{th}$

($B/L = 0.5$)

Figure 4.16 Film thickness under different pivot locations with various $w_{th}$

($B/L = 1$)
Figure 4.17  **Film thickness under different pivot locations with various $w_{th}$**

($B/L = 2$)

*Figures 4.18 to 4.21* give charts for the friction coefficient at various pivot locations for various $B/L$ ratios of the bearing at various $w_{th}$ values. The results may once again be used as the design charts of the bearing. It should be noted that in *Figure 4.21*, when the $B/L$ ratio is 2, the minimum friction coefficient for $w_{th} = 0.1, 0.075,$ and $0.05$ occurs at the centrally pivoted position. However, as no sufficient load capacity is developed for such load parameter values (as shown in *Figures 4.3*), the friction coefficient does not have practical value; it should, therefore, not be used in the bearing design.
Figure 4.18  Friction coefficient under different pivot locations with various $w_{th}$

$(B/L = 0)$

Figure 4.19  Friction coefficient under different pivot locations with various $w_{th}$

$(B/L = 0.5)$
Figure 4.20 Friction coefficient under different pivot locations with various $w_{th}$

$(B/L = 1)$

Figure 4.21 Friction coefficient under different pivot locations with various $w_{th}$

$(B/L = 2)$
The program used to study the pivoted bearing behavior can be found in Appendix B. It is capable of obtaining the two performance variables (dimensionless film thickness and friction coefficient) once the design parameters (pivot location $X_c$, load parameter $w_{th}$, and length-to-width ratio $B/L$) are given. The program can be used to analyze the relationship between one performance variable and one varying design parameter, and it can be used to create plots that can be used as design charts for bearing designers.
Chapter 5: Conclusion

An analytical model is developed in this thesis to study hydrodynamic lubrication actions in pivoted slider bearings. The viscosity of the lubricant becomes a variable as the temperature variation inside the lubricating film is taken into consideration. Results that contradict with classical isothermal lubrication theory are obtained.

For centrally pivoted bearings, sufficient load capacity can be developed for relatively wide bearing, such as those with length-to-width $B/L < 1$. As the length-to-width ratio $B/L$ of the bearings increases, the pressure generated inside the lubricant film reduces due to the viscosity reduction, which decreases the load capacity for narrow bearings. For narrow bearings such as $B/L = 2$, the reduction becomes so significant that the load capacity of the bearing is diminished. In addition, the relationship between film thickness $h_o$ and load parameter $w_{th}$ depends on the $B/L$ ratio of the bearing. A significant load capacity can be obtained in a broader range of $w_{th}$ for wide bearings. In contrast, the load capacity is much more sensitive to the change of $w_{th}$ for narrow bearings, which makes it very difficult to develop a lubricant film with sufficient load capacity. The friction coefficient, on the other hand, is fairly stable regardless of $B/L$ ratio.

Design charts are presented for a number of combinations of $B/L$ ratios and $w_{th}$ values. The effects of different combinations of these ratios and values on the film thickness and the friction coefficient for various pivot locations can be observed from the design charts.

Numerical analysis can be done to validate the model in this thesis in the future. Experiments on actual pivoted slider bearings can be performed to evaluate the model.
The given design rules in this thesis can serve as a basic conservative design guidance.

Other mechanisms can serve as design safety factors in addition to the mechanism discussed in this thesis, and such mechanisms could be integrated into and thus refine the present model.
Bibliography


Appendix A: Derivation of Friction Coefficient

Friction coefficient is defined by the traction force asserted by the lubricating film to the sliding surface divided by the total bearing load:

\[ f = \frac{1}{w} \int_0^B \tau_x \, dx \]  

(3.13)

where \( \tau_x \) is the shear stress in the \( x \) direction. According to [2], the velocity distribution of forward lubricant flow in the lubrication film is

\[ u(x, z) = \left( 1 - \frac{z}{h} \right) U + \frac{1}{2\eta} \frac{dp}{dx} (z - h)z \]  

(A.1)

Thus, the shear strain rate in the \( x \) direction is

\[ \dot{\gamma}_x = \frac{\partial u}{\partial z} = -\frac{U}{h} + \frac{1}{2\eta} \frac{dp}{dx} (2z - h) \]  

(A.2)

Since

\[ \tau_x = \eta \dot{\gamma}_x \]  

(A.3)

evaluating the shear strain stress at the sliding surface \( (z = 0) \) yields

\[ \tau_x = -\frac{U\eta}{h} - \frac{h \frac{dp}{dx}}{2} \]  

(A.4)

The negative sign indicates the traction force is in the opposite direction of the \( x \)-coordinate. Substituting equation (A.4) into (3.13) and taking the absolute value of traction force yields

\[ f = \frac{1}{w} \int_0^B \left( \frac{U \eta}{h} + \frac{h \frac{dp}{dx}}{2} \right) dx = \frac{1}{w} \int_0^B \frac{U \eta}{h} \, dx + \frac{1}{w} \int_0^B \frac{h \frac{dp}{dx}}{2} \, dx \]  

(A.5)

The first term in equation (A.5) is calculated first: substituting dimensionless variables \( X \) (equation (2.7), \( \tilde{h} \) (equation (2.9)), and viscosity distribution (equation (2.14)) into equation (A.5) yields
\[ \frac{1}{w} \int_0^B \frac{h}{h} \cdot U_\eta \, dx = \frac{UB\eta_0}{wH_o} \int_0^1 \frac{1}{1+\frac{8C_{th}}{1+H^2}X} \, dX \] \quad (A.6)

In addition to dimensionless variables \( X \) and \( \tilde{h} \), substituting dimensionless pressure \( \tilde{\rho} \) (equation (2.6)) into equation (A.5) enables the second term of equation (A.5) to be calculated

\[ \frac{1}{w} \int_0^B \frac{h}{2} \frac{dp}{dx} \, dx = \frac{1}{2w} \int_0^1 \frac{h}{h_0} \, \tilde{h} \frac{UB\eta_0}{h_0^2} \frac{d\tilde{\rho}}{Bdx} \, BdX = \frac{UB\eta_0}{2wh_o} \int_0^1 \frac{1}{dx} \tilde{h} dX \] \quad (A.7)

Equation (A.7) can be further simplified as the following relationship exists

\[ \int_0^1 \frac{d\tilde{\rho}}{dx} \tilde{h} dX = (\tilde{h}\tilde{\rho}|_{x=1} - \tilde{h}\tilde{\rho}|_{x=0}) - \int_0^1 \tilde{\rho} \frac{d\tilde{h}}{dx} dx = - \int_0^1 \tilde{\rho} \frac{d\tilde{h}}{dx} dx \] \quad (A.8)

Substituting equations (A.6), (A.7), and (A.8) into equation (A.5) yields

\[ f = \frac{BU\eta_0}{wh_o} \left[ \int_0^1 \frac{dx}{\tilde{h}(1+\frac{8C_{th}}{1+H^2}X)} - \frac{1}{2} \int_0^1 \frac{d\tilde{h}}{dx} \tilde{\rho} dX \right] \] \quad (A.9)

Recall that, for plane-pad bearings

\[ \tilde{h} = [H - (H - 1)X] \] \quad (2.11)

Substituting equation (2.11) into equation (A.9) yields

\[ f = \frac{BU\eta_0}{wh_o} \left[ \int_0^1 \frac{dx}{[H-(H-1)X](1+\frac{8C_{th}}{1+H^2}X)} + \frac{H-1}{2} \int_0^1 \tilde{\rho} dX \right] \] \quad (A.10)

Carrying out the integrals, the expression for friction coefficient is obtained

\[ f = \left[ \ln\left(1+\frac{8C_{th}}{1+H^2}\right)+\ln(H) + \frac{(H-1)\bar{w}}{2} \right] / \tilde{h}_o \] \quad (3.14)
Appendix B: Matlab Code

Function description:

[xc, load, hmin] = pressdistr(Cth, H, N) obtains pressure distribution for the bearing, and returns the pressure center along with film thickness and load.

y = pressA(x, h, C) returns the value of A(x) in equation (2.17).

y = pressB(x, h, C) returns the value of B(x) in equation (2.18).

C = wfindc(w, xc, n, cint) returns the corresponding Cth value to the given w.

[xc, load, hmin, f] = WNfix(xmin, xmax, num, Wth, N, err) provides the load, film thickness, and friction coefficient against varying pivot positions for the fixed Wth and B/L ratio.

[N, load, hmin, f] = WXfix(Nmin, Nmax, num, Wth, xc, err) provides the load, film thickness, and friction coefficient against varying B/L ratios for the fixed Wth and pivot position.

[C, W, load, hmin, f] = XNfix(Cmin, Cmax, num, xc, N, err) provides the load, film thickness, and friction coefficient against varying Cth or Wth values for the fixed B/L ratio and pivot location.
function [xc,load,hmin] = pressdistr(Cth,H,N)

% this function creates the pressure distribution inside the lubricating
% film. Cth is the thermal parameter in equation (2.14). H is the
% inclination of the bearing. N is the B/L ratio.
% this function returns the pressure center, load, and film thickness.
% Most of the time, the pressure center is read by other subroutines to
% determine whether it fits the requirement.

a0 = 6*pressA(0,H,Cth);  \% calculate A(x) in equation (2.17)
a1 = 6*pressA(1,H,Cth);
b0 = pressB(0,H,Cth);  \% calculate B(x) in equation (2.18)
b1 = pressB(1,H,Cth);

c1 = (a1-a0)/(b0-b1);
c2 = -a0-c1*b0;

pavg = 0;
pxtot = 0;
loadx = 0;
loady = 0;
load = 0;

for i = 1:101
    x(i) = (i-1)/100;
    px(i) = 6*pressA(x(i),H,Cth)+c1*pressB(x(i),H,Cth)+c2; \% generate 2D pressure in
equation (2.16)

    if i>1
        pavg = pavg+(px(i)+px(i-1))/2;
        pxtot = pxtot+(px(i)*x(i)+px(i-1)*x(i-1))*(x(i)-x(i-1))/2;
    end
end
loadx = loadx+(px(i)+px(i-1))*(x(i)-x(i-1))/2;  % calculate 2D load in equation (2.20)
end
end

xc = pxtot/pavg*(i-1);  % obtain pressure center

[pm im] = max(px);
xm = x(im);
m = -6*(H-1)/(H-(H-1)*xm)^3/pm/(1+8*Cth*xm/(1+H)^2);  % obtain M in equation (3.5)

if (N==0)
    loady = 1;
else
    n = sqrt(-m/N^2);
    loady = 1-2/n*(1-exp(-n))^2/(1-exp(-2*n));  % load reduction factor in equation (2.33)
end

load = loadx*loady;  % obtain the total load

hmin = sqrt(load);

clear a0 a1 b0 b1 c1 c2 pavg pxtot i
function y = pressA(x,h,C)

% this function calculated the A(x) value in equation (2.17). Cth is the
% thermal parameter, H is the inclination of the bearing.

a = h;
b = h-1;
c = 8*C/(1+h)^2;

y = 1/(a*c+b)/(a-b*x)-c*log(a-b*x)/(a*c+b)^2+c*log(c*x+1)/(a*c+b)^2;

clear a b c
function y = pressB(x,h,C)

% this function calculated the B(x) value in equation (2.18). Cth is the thermal parameter, H is the inclination of the bearing.

a = h;
b = h-1;
c = 8*C/(1+h)^2;

y = c^2*log((a-b*x)/(1+c*x))/(-b-a*c)^3+c/(a*c+b)^2/(a-b*x)+1/2/(a*c+b)/(a-b*x)^2;

clear a b c
function C = wfindc(w,xc,n,cint)

% this function searches for the corresponding value of Cth to the given Wth.
% w is the given value of Wth, xc is the pivot location, n is the
% length-to-width ratio, and cint is the initial estimate of Cth

% this part is the most time-consuming in the calculation. Here instead of
% creating the entire Cth VS Wth curve, the program creates 5 points of Cth
% and calculates the corresponding Wth. If the given Wth does not lie in
% between the result, 5 new points are created.

% optimization method can be applied to this function and the speed may be
% vastly improved

ifind = 0;
cmin = cint;

while ifind == 0

if w == 0
  %Cth = 0 for isothermal model
  C = 0;
  return
end

if (cmin < 5)  %for Cth<5, the increment of each Cth step is 0.1
  cmax = cmin+0.5;
elseif (cmin >= 5) & (cmin < 50)  %for 5<Cth<50, the increment for each Cth step is 0.5
  cmax = cmin+2.5;
elseif (cmin >= 50)  %for Cth>50, the increment for each Cth step is 5
  cmax = cmin+5;

end

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cmax = cmin + 25;
end

[c0 w0 hmin f] = XNfix(cmin,cmax,5,xc,n,1e-4); % create 5 points of Cth VS Wth

i = 1;
j = 1;
while (j>0)&&(w > w0(i))
i = i+1;
if i > length(w0) % Wth is larger than the max of the 5 Wth's obtained
c = cmin; % increase the Cth range
cmin = cmax;
cmax = cmax+cmax-c;
j = 0;
end
end

if i == 1 % Wth is smaller than the min of the 5 Wth's obtained
c = cmax; % decrease the Cth range
cmax = cmin;
cmin = cmin-c+cmin;
elseif i < (length(w0)+0.5) % Wth lie in between the 5 Wth's obtained
ifind = 1;
end
end

C = c0(i-1)+(c0(i)-c0(i-1))*(w-w0(i-1))/(w0(i)-w0(i-1)); % interpolate to find the Cth value
function [xc, load, hmin, f] = WNfix(xmin, xmax, num, Wth, N, err)

% this function creates the dimensionless load, film thickness, and % friction coefficient against varying xc values at fixed load parameter % and B/L ratio.

% the range of xc is given between xmin and xmax. Num determines the % number of points that performance variables being calculated. Wth and N % are the given load parameter and length-to-width ratio. err is the % allowable error. To create a smooth plot, err is suggested to be 1e-3.

Cth = 0.1;
for i = 1:(num+1)
    xc0 = xmin+(xmax-xmin)/num*(i-1) % current value of xc
    ifind = 1;
    H0 = 1.005; % initial estimate of H
    Cth = wfindc(Wth, xc0, N, Cth); % find the corresponding Cth
    while ifind > 0
        [xcen, load_, h_min] = pressdistr(Cth, H0, N); % generate pressure distribution
        if (abs(xc0-xcen) < err) % correct H is found
            ifind = 0;
            xc(i) = xcen;
            H(i) = H0;
            hmin(i) = h_min;
            load(i) = load_;
            f(i) = ((log(1+8*Cth/(1+H0)^2)+log(H0))/(8*Cth*H0/(1+H0)^2+(H0-1))+(H0-1)*load_/2)/hmin(i);
        end
        H0 = H0+0.01; % supply new estimate of H if the error is too big
    end
end
function [N,load,hmin,f,H] = WXfix(Nmin,Nmax,num,Wth,xc,err)

% this function creates the dimensionless load, film thickness, and
% friction coefficient against varying B/L ratios at fixed pivot
% location and load parameter.

% the range of B/L is given between Nmin and Nmax. Num determines the
% number of points that performance variables being calculated. Xc and Wth
% are the given pivot location and load parameter. err is the
% allowable error. To create a smooth plot, err is suggested to be 1e-3.

Cth = 1.2;
for i = 1:(num+1)
    N0 = Nmin+(Nmax-Nmin)/num*(i-1)  % current value of B/L
    ifind = 1;
    H0 = 1.01;                        % initial estimate of H
    Cth = wfindc(Wth,xc,N0,Cth);     % find the corresponding Cth
    while ifind > 0
        [xcen,load_,h_min] = pressdistr(Cth,H0,N0); % generate pressure distribution
        if (abs(xc-xcen) < err)       % correct H is found
            ifind = 0;
            N(i) = N0;
            hmin(i) = h_min;
            load(i) = load_;
            f(i) = ((log(1+8*Cth/(1+H0)^2)+log(H0))/(8*Cth*H0/(1+H0)^2+(H0-1))+(H0-1)*load_/2)/hmin(i);
            H(i) = H0;
        end
        H0 = H0+0.002; % supply new estimate of H if the error is too big
    end
end
function [C,W,load,hmin,f] = XNfix(Cmin,Cmax,num,xc,N,err)

% this function creates the dimensionless load, film thickness, and
% friction coefficient against varying Cth and Wth values at fixed pivot
% location and B/L ratio.

% the range of Cth is given between Cmin and Cmax. Num determines the
% number of points that performance variables being calculated. Xc and N
% are the given pivot location and length-to-width ratio. err is the
% allowable error. To create a smooth plot, err is suggested to be 1e-4.

for i = 1:(num+1)
    C0 = Cmin+(Cmax-Cmin)/num*(i-1)  % current value of Cth
    ifind = 1;                    % flag
    H0 = 1.01;                  % initial estimate of H
    while ifind > 0
        [xcen,load_,h_min] = pressdistr(C0,H0,N);  % generated pressure distribution
        if (abs(xc-xcen) < err)  % correct H is found
            ifind = 0;
            C(i) = C0;
            hmin(i) = h_min;
            load(i) = load_;
            f(i) = ((log(1+8*C0/(1+H0)^2)+log(H0))/(8*C0*H0/(1+H0)^2+(H0-1))+(H0-1)*load_)/hmin(i);
            W(i) = C(i)*load_;
        end
        H0 = H0+0.002;  % supply new estimate of H if the error is too big
    end
end