OPTIMAL CONTROL PROBLEMS ON STRATIFIED DOMAINS:
APPLICATION TO SINGLE-STATION MULTICLASS QUEUEING
SYSTEMS WITH FINITE BUFFERS AND OVERFLOW COSTS

A Thesis in
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Abstract

We study a class of optimal control problems on stratified domains (OCPSD) and apply the results to an optimal control problem for a single-station multiclass queueing system with finite buffers and overflow costs.

In the optimal control problems on stratified domains, we assume that the state space $\mathbb{R}^N$ admits a stratification as a disjoint union of finitely many embedded submanifolds $M_i$. The dynamics of the system and the cost function are Lipschitz continuous restricted to each submanifold. We provide conditions, which guarantee the existence of an optimal solution, and study sufficient conditions for optimality. These are obtained by proving a uniqueness result for solutions to a corresponding Hamilton-Jacobi equation with discontinuous coefficients, describing the value function. Our results are motivated by various applications, such as minimum time problems with discontinuous dynamics, and optimization problems constrained to a bounded domain, in the presence of an additional overflow cost at the boundary.

The multiclass queueing system is modeled to have stochastic fluid flows. We model the problem as an optimal control problem with a closed piecewise smooth reflecting boundary and reflecting cost. For this, we extend the results of OCPSD into the stochastic fluid model. Then, we characterize the value function of the queueing control problem by the unique viscosity solution for a set of Hamilton-Jacobi equations. Furthermore, we validate the Markov chain approximation method in our problem. We illustrate the results and provide a numerical solution for an example problem.
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Chapter 1

Introduction

1.1 Single-Station Multiclass Queueing Systems with Finite Buffers and Overflow Costs

We study a class of stochastic optimal control problems for a single-station multiclass queueing system with finite buffers and overflow costs. The system is modeled to have stochastic fluid flows (alternatively, Markov modulated fluid flows) [44]. By class, customers (or fluid flow) have different processing rates and costs. The system has one or more finite buffers. We assume that the server has a finite service rate and can allocate fractions of the service rate to the buffers. The environment state, which typically indicates a certain combination of arrival rates or service capacity at a time, is time varying by a finite state continuous time Markov chain (CTMC). Depending on the environment state, even the cost function could be different. Our concern is in the optimal dynamic allocation of the shared service rate onto each class in terms of a given cost functional.

We approach the optimal control problem by the Hamilton-Jacobi-Bellman (HJB) equation. The overflow cost makes our study different from others. Due to the overflow cost, the HJB equation has discontinuities on the boundary. We cannot use the conventional viscosity solution concept, which were developed for continuous HJB equations. In order to resolve this issue, we define a class of optimal control problems, which have a special structure in problem data. In detail, the state space is a stratified domain, which consists of submanifolds with
a certain rule (cf. (3.1)). On each submanifold, different cost and dynamics are defined. Thus, cost and dynamics are discontinuous functions. However, when we restrict them to any one submanifold in the stratification, they are sufficiently regular (cf. (H1) in Chapter 3). We call this problem “the optimal control problems on stratified domains (OCPSD)”.

Then, we provide a new form of viscosity solution for the corresponding discontinuous HJB equations of OCPSD. As a main result, we characterize the value function of OCPSD as a unique viscosity solution of the HJB equation.

Now we discuss the queueing problem in detail. Suppose there are $N$ classes of customers (or fluid). In our canonical problem, one finite buffer is prepared for one class for simplicity. The buffer sizes are $B_i$, $i = 1, \ldots, N$. Let $x : [0, \infty) \mapsto \mathbb{R}^N$ be an absolute continuous function, which denotes queue lengths as a function of time. It is required that $x(t) \in \bar{M} = [0, B_i]^N$ for all $t \geq 0$. The interior of $\bar{M}$ and its boundary are denoted by $M = (0, B_i)^N$ and $\partial M$ respectively. Let $\{z(t), t \geq 0\}$ be a CTMC with a finite countable state space, $Z = \{1, 2, \ldots, I\}$. Each state of $Z$ indicates a certain environment state, according to which dynamics or cost function would be different. We assume $z(t)$ does not depend on queue length and control. The distribution of $z(\cdot)$ is determined by an infinitesimal generator, which is $I \times I$ matrix

$$Q = \begin{pmatrix}
-q_1 & q_{12} & \cdots & q_{1I} \\
q_{21} & -q_2 & \cdots & q_{2I} \\
\vdots & \vdots & \ddots & \vdots \\
q_{I1} & q_{I2} & \cdots & -q_I
\end{pmatrix}, \quad q_i = \sum_{j \in Z \setminus \{i\}} q_{ij} > 0, \ \forall i \in Z. \quad (1.1)$$

and the initial distribution, $P(z(0) \in \Gamma)$, where $\Gamma$ is an element of $\sigma$-algebra of $Z$. Here, we consider a unit-mass distribution, $P(z(0) = i) = 1$ for a certain $i \in Z$.

Now we consider a family of dynamics for $z \in Z$

$$\dot{x} = g^z(x, a), \quad x \in \bar{M}, \ a \in A^z, \quad (1.2)$$

where $A^z$, for each $z \in Z$, is a compact set of $\mathbb{R}^M$. Because buffers are finite,
the system has a reflecting boundary \cite{51}, that is, whenever the system cross the boundary, it will be pushed back immediately to stay in \(\hat{M}\). For a given \(z \in Z\), the dynamics, \(g^z(x, a)\), is discontinuous on the boundary. However, we assume that \(g^z(x, a)\) is sufficiently regular on each of well-defined submanifolds of \(\hat{M}\). This will be discussed in detail in section \ref{1.2}. With a certain regularity condition of \(g^z\), the existence and uniqueness of the solution of \eqref{1.2} has been studied through Skorokhod problem \cite{51} or differential inclusion \cite{4} (cf. chapter \ref{2}).

The family of dynamics in \eqref{1.2} is used to describe the controlled stochastic fluid model. Let \(T_n, X_n\) and \(Z_n\) be the random variables representing the \(n\)-th jump time of environment variable \(z(t)\), the corresponding system state and environment state at the \(n\)-th jump time for \(n = 0, 1, \cdots\). For any \(Z_n = i \in Z\) and \(t \geq 0\),

\[
P(T_{n+1} - T_n > t \mid T_1, X_1, Z_1, \cdots, T_n, X_n, Z_n) = P(T_{n+1} - T_n > t \mid Z_n = i) = e^{-qi t}.
\]

After the \(n\)-th jump occurs, an admissible control is a function of \((X_n, Z_n)\) and the elapsed time since \(T_n\). The system dynamics is

\[
\dot{x}(t) = g^Z_n(x(t), \hat{\alpha}(X_n, Z_n, t - T_n)), \quad t \in [T_n, T_{n+1}), \tag{1.3}
\]

where \(\hat{\alpha} \in A\) is the control function. The set \(A\) denotes a set of admissible controls,

\[
A \doteq \{ \hat{\alpha}(x, z, \cdot) : [0, \infty) \rightarrow A^z, \text{ measurable, } \forall (x, z) \in \hat{M} \times Z \}. \tag{1.4}
\]

The system follows the dynamics \eqref{1.3} until the next jump. At the next jump time, with a new initial state, \((X_{n+1}, Z_{n+1})\), the system starts again following \eqref{1.3}. This process will repeat for the consecutive jumps. This stochastic process, \(\{x(t), z(t)\}_{t \geq 0}\), is a type of piecewise deterministic Markov process (PDMP) and is a strong Markov process \cite{24}.

For simplicity, we use another notation for the control in the rest of this thesis,

\[
\alpha(t) \doteq \hat{\alpha}(X_n, Z_n, t - T_n) \quad \text{if } t \in [T_n, T_{n+1}). \tag{1.5}
\]

We want to find an optimal control, which minimizes the cost function \(J : \)}
\[ \tilde{M} \times Z \times A \mapsto \mathbb{R}, \]

\[ J(\bar{x}, \bar{z}, \alpha) = E \left\{ \int_0^\infty e^{-\beta t} \ell_i^\varepsilon(x(t), \alpha(t)) \, dt \right\} = E \left\{ \sum_{n=0}^{\infty} \int_{T_n}^{T_{n+1}} e^{-\beta t} \ell_z^\varepsilon(x(t), \hat{\alpha}(X_n, Z_n, t - T_n)) \, dt \right\}, \quad (1.6) \]

where \( x(0) = \bar{x}, \, z(0) = \bar{z} \) and

\[ \ell_z^\varepsilon(x(t), \alpha(t)) = c_z^\varepsilon(x(t), \alpha(t)) + b_z^\varepsilon(x(t), n(x(t), \alpha(t))). \]

The lagrangian \( \ell^i(\cdot, \cdot) \) consists of two parts: (1) holding / service related cost, \( c^i(\cdot, \cdot) \) and (2) the overflow cost (or reflecting cost) \( b^i(\cdot, \cdot) \) for \( i \in Z \). The function, \( n(x(t), \alpha(t)) \), denote the reflecting force, which pushes back the system into \( \tilde{M} \) whenever it crosses over \( \partial \tilde{M} \). The lagrangian function is also discontinuous on the boundary. However, it is sufficiently regular only on each well-defined submanifold of \( \tilde{M} \). The cost functional is discounted by a constant \( \beta > 0 \). Then, the value function, \( V : \tilde{M} \times Z \mapsto \mathbb{R} \), is

\[ V(\bar{x}, \bar{z}) = \inf_{\alpha \in A} J(\bar{x}, \bar{z}, \alpha). \quad (1.7) \]

The abstract model above is a controlled stochastic fluid model. The deterministic dynamics in a certain environment state would be a model for the dynamics of real fluid like water or a deterministic continuous time approximation of the discrete time stochastic queueing system \[16, 18\]. The fluid approximation is based on the observation about the asymptotic behavior of the queues when the processing time of each customer is relatively smaller than the waiting time. This has been observed in an ATM communication network \[32\]. A packet is very small chunk of signal, which has very small processing time. However, packets arrive in a bulk from various sources. In other words, arrival process is characterized by, so called, burstiness. The random environment variable indicates the slow changes on the system parameters such as the arrival rates or processing rate. Such changes as machine breakdowns are sudden but the times between events are relatively long compared to the processing time of a single customer \[63\].
In order to give a clear idea about the problem, we present an example control problem with simple cost and dynamics function. However, the mathematical model for our general optimal control problem (cf. chapter 3), can have nonlinear cost and dynamics with some assumptions in section 1.2 or chapter 3.

In figure 1.1 we depict a queueing system as an example application.

![Queueing System Diagram](image)

**Figure 1.1.** A multiclass queueing system with finite buffers

**Example 1.** We consider an extension of Martinelli and Valigi’s problem [53] to a stochastic fluid model. The stochastic factor is not taken into account in their model. The optimal control problem is

\[
\min_{\alpha \in \mathcal{A}} \mathbb{E} \left\{ \int_0^\infty e^{-\beta t} \left[ c(x(t), \alpha(t)) + b z(t)(x(t), \alpha(t)) \right] dt \right\}
\]  

(1.8)

subject to

\[
\dot{x}_k(t) = \begin{cases} 
  r_k^z(t) - \alpha_k(t) & \text{if } 0 < x_k(t) < B_k \\
  \min\{0, r_k^z(t) - \alpha_k(t)\} & \text{if } x_k(t) = B_k \\
  \max\{0, r_k^z(t) - \alpha_k(t)\} & \text{if } x_k(t) = 0 
\end{cases}, \quad k = 1, 2, \cdots, N, \quad \text{a.e. } t \geq 0.
\]

(1.9a)

\[
x(0) = \bar{x}, \quad z(0) = i.
\]

(1.9b)

The environment variable \(z(t)\) is a CTMC as above. The symbol \(r_k^z(t)\) denotes the arrival rate to queue \(k\) for given \(z(t)\). The set of control value, \(A^z\), represents the set of processing rates to be assigned to each queue, i.e.

\[
A^z = \left\{ a \in \mathbb{R}^N; \sum_{k=1}^N a_k \leq D^z, \; a_k \geq 0, \; k = 1, \cdots, N \right\},
\]

where a scalar \(D^z\) is the total processing rate, which depends on \(z \in Z\). The control parameter \(\alpha_k(t)\) denotes the processing rate allocation to queue \(k\) at time \(t\).
Martinelli and Valigi [53] consider a linear holding and overflow cost function, which are similar to
\[
c(x, a) = \sum_{k=1}^{N} c_k x_k, \quad 0 \leq c_k < \infty,
\]
\[
b^z(x, a) = \sum_{k=1}^{N} b_k \max(r^z_k - a, 0) 1_{\{x_k = B_k\}},
\]
where \( x \in \tilde{M}, \ z \in Z, \) and \( a \in A^z. \) We will use this linear cost function for demonstration purpose in chapter 6 even though we can use more general cost function. For simplicity, we consider the same cost function over all environment states. This control problem is an instance of the reflecting boundary-cost (RBC) problem on convex domain in section 1.2.

We can recognize that some other problems with different shape of domain also can be considered such as the multiclass queueing system with a single shared buffer or with service rate control (leaking fluid). In a single shared buffer case, the state space is, for \( B > 0, \)
\[
\tilde{M} = \left\{ x \in \mathbb{R}^N; \sum_{i=1}^{N} x_i \leq B, \ x_i \geq 0 \ i = 1, \cdots, N \right\}.
\]
By the service rate control, we mean that unit processing rate can be increased at the expense of some additional cost. This can be seen as a leaking fluid in communication models [58]. In a communication switching device, leaky bucket is used to intentionally discard some packets of certain traffics, which is relatively insensitive to the loss of data, in order to reduce congestion. For example, we may be able to increase the processing speed by leaking some packets of audio or video stream within a limit that quality of voice or image is acceptable. Power control
in wireless communication is another example [3]. Then, the set of control is

\[ A^z = \left\{ (a, l) \in \mathbb{R}^{2N} : \sum_{k=1}^{N} a_k \leq D^z, \ a_k \geq 0, \ k = 1, \ldots, N, \right\}, \]

where \( P^z_k, k = 1, \ldots, N, \) is the upper limit of speed-up rate. The set of admissible controls is

\[ A = \{ (\hat{\alpha}(x, z, \cdot), \hat{\lambda}(x, z, \cdot)) : [0, \infty) \rightarrow A^z \text{ measurable } \forall (x, z) \in \bar{M} \times Z \}. \] (1.12)

As in (1.5), we consider \((\alpha, \lambda)\) instead of \((\hat{\alpha}, \hat{\lambda})\). Then, in example 1, (1.9a) becomes

\[
\dot{x}_k(t) = \begin{cases} 
  r_k^{z(t)} - \alpha_k(t) \lambda_k(t) & \text{if } 0 < x_k(t) < B_k \\
  \min \{0, r_k^{z(t)} - \alpha_k(t) \lambda_k(t)\} & \text{if } x_k(t) = B_k \\
  \max \{0, r_k^{z(t)} - \alpha_k(t) \lambda_k(t)\} & \text{if } x_k(t) = 0 \\
\end{cases}, \quad k = 1, 2, \ldots, N, \quad \text{a.e. } t \geq 0.
\] (1.13)

### 1.2 Reflecting Boundary-Cost Problems

We can regard the overflow cost as one of the costs, which arise on the boundary. We will call such costs as a whole, the reflecting cost. The typical reflecting boundary problem does not have the reflecting cost. In order to distinguish our problem from other reflecting boundary problems, we name our problem ‘reflecting boundary-cost (RBC) problem’.

For \( z \in Z \), we assume (H0) for the problem data.
1. Let $f^z$ be a continuous function from $\mathbb{R}^N \times A^z$ into $\mathbb{R}^N$ such that

$$|f^z(x, a) - f^z(y, a)| \leq L_{f^z}|x - y| \quad \forall x, y \in \mathbb{R}^N, \ a \in A^z, \ 0 < L_{f^z} < \infty.$$  

(1.14)

2. The running cost $c^z$ is in $C(\mathbb{R}^N \times \mathbb{R}^M; [0, \infty))$ such that

$$|c^z(x, a) - c^z(y, a)| \leq L_{c^z}|x - y| \quad \forall x, y \in \mathbb{R}^N, \ a \in A^z, \ 0 < L_{c^z} < \infty.$$  

(1.15)

3. We consider two forms of reflecting (or overflow) cost: linear and general form.

(i) The linear form is $b^z(x) \cdot n_x$, where

$$b^z(x) = (b^z_1(x), \ldots, b^z_N(x)) \in C(\mathbb{R}^N; [0, \infty)^N)$$

is a cost vector and is Lipschitz continuous such that

$$|b^z(x) - b^z(y)| \leq L_{b^z} |x - y|, \ x, y \in \partial M, \ 0 < L_{b^z} < \infty.$$  

(1.16)

(ii) The general form, $b^z(x, n_x) \in C (\mathbb{R}^N \times \mathbb{R}^N; [0, \infty))$, which is convex with respect to $n_x$, satisfies

$$|b^z(x, n_x) - b^z(y, n_y)| \leq L_{b^z_x} |x - y| + L_{b^z_n} |n_x - n_y|, \ x, y \in \partial M, \ 0 < L_{b^z_x} < \infty \text{ and } 0 < L_{b^z_n} < \infty.$$  

(1.17)

The both reflecting costs are nondecreasing with respect to $|n|$ and has zero values when $x \in M$. For unbounded $\partial M$, we assume that the reflecting costs is bounded.

Let’s fix the environment variable by $z$ for explanation purpose. For a given initial condition $x(0) = \bar{x} \in \bar{M}$, the system dynamics (1.2) satisfies (1.18) and (1.19).

$$\dot{x}(t) = \pi_{T_{\bar{x}}(x(t))}f^z(x(t), \alpha(t)) = f^z(x(t), \alpha(t)) - n_x(t) = g^z(x(t), \alpha(t)), \quad x(t) \in \bar{M}, \quad \forall t > 0.$$  

(1.18) (1.19)

The dynamics (1.18) is a projected ordinary differential equation. We denote the
projection by $\pi_S(x)$, i.e.

$$\pi_S(x) \doteq \left\{ s \in S; \|s - x\| = d_S(x) \right\},$$

where

$$d_S(x) \doteq \inf \left\{ \|s - x\|; s \in S \right\}.$$

We recall the definition of normal cone to $\bar{M}$ at $x$,

$$N_{\bar{M}}(x) \doteq \left\{ p \in \mathbb{R}^N; \langle p, v \rangle \leq 0, \, \forall v \in T_{\bar{M}}(x) \right\}, \tag{1.20}$$

where $T_{\bar{M}}(x)$ is the tangent (or adjacent) cone to $\bar{M}$ at $x$ [5] such that

$$T_{\bar{M}}(x) \doteq \left\{ v \in \mathbb{R}^N; \lim_{h \to 0^+} \frac{d_{\bar{M}}(x + hv)}{h} = 0 \right\}.$$

For simplicity, we use the notation $n_x(t)$ for the reflecting force instead of

$$n(x(t), \alpha(\bar{x}, z, t)) = \pi_{N_{\bar{M}}(x(t))} f^\bar{z}(x(t), \alpha(\bar{x}, z, t)).$$

### 1.3 Significance of Research

Control problems for the multiclass queueing system with finite buffers and overflow costs are found in various applications such as manufacturing systems [53], communication switching devices [3, 46]. Typically, efficient resource allocation, and good QoS in terms of holding cost (or waiting time) and customer loss have been concerns on these systems. Optimal control of these systems cannot be overemphasized because of their broad applicability and economic importance.

In general, physical queueing systems have finite buffers. However, the amount of studies on the finite buffer systems is much smaller than the one about infinite buffers. We may be able to approximate the finite buffer model by the infinite buffer if the buffer is sufficiently large and overflow is regarded as a rare-event [1]. In such studies, stability condition is considered to be more important and the estimation of the probability of the rare event (large queue) is a significant research topic. However, if we take into account the overflow cost (in the case of finite
buffers), the optimal control policy has a more complicated form than the infinite case [53]. They are not simple threshold-type control policies. In addition, the finite buffer is used to take into account some business policy. For example, the finite buffer would be introduced on purpose in order to make sure a certain level of minimum waiting time to the customer [3] as a service policy. Thus, the finite buffer models have its own importance to study.

There are few studies on the optimal control of continuous-time dynamic model with the overflow costs. Martinelli and Valigi [53] study a make-to-order, two-part-type manufacturing system, which has a single server and two finite buffers. They consider a deterministic optimal control problem with a stability assumption, that is, the total arrival rate is strictly less than the total service rate. Because of their assumed stability condition, the system always reaches the origin in the state space. They find the optimal control policy for all combinations of problem data. However, Martinelli and Valigi model cannot be easily extended to include the service rate control and stochastic version and more than two queues. It is not about developing a general control theory such as HJB equation approach for this type of problems.

Ata et al. [3] is another related work, which takes into account the overflow cost even though it is a single queue problem. The disturbances of arrival rate is described by the Brownian motion. Service rate is controlled to minimize a long run average cost, which is the sum of service cost and overflow cost. Even though Ata et al. [3] derive a closed form of control policy from a second order HJB equation, they can not justify the use of the HJB equation rigorously.

In usual queueing studies, main interest is to find the steady-state distribution of the system state for a certain type of parameterized control policy such as threshold-type. Then, an analytical performance index is found from the distribution [44]. This analytical model can be used to find the best one in the set of parameterized control policy, e.g. finding a threshold value. Different from the typical queueing studies, through control theory, we want to find an optimal control policy, which is not restricted to a certain type of control policy, directly from a controlled stochastic fluid model without the steady-state performance analysis. In this thesis, our purpose is to develop an optimal control theory for the controlled stochastic fluid model. Specifically, we approach the problem by the HJB equation.
Other than the theoretical interest, the advantage of the HJB equation is: Once we solve the HJB equation, we can synthesize the feedback optimal control from its solution, i.e. the value function.

There have been a lot of studies on the solvability of HJB equation through the viscosity solution since 1980’s [6]. Most of them are for the continuous HJB equations. However, we can not use the existing viscosity solution concept and proving method for our problem because the Hamiltonian is discontinuous in our problem. We observe that discontinuity in our problem has a certain level of regularity depending on regions in the state space. Based on this observation, we define a class of general optimal control problem so called ‘optimal control problems on a stratified domain (OCPSD)’ [11]. Then, we define the viscosity solution and prove the uniqueness of viscosity solution for the deterministic OCPSD. We feel that in many practical applications outside the queueing theory, discontinuous HJB equations have the form we suggested rather than only being Borel measurable.

To the best of our knowledge, the solvability of HJB equation, which is related with the overflow cost, has not been studied. In this thesis, we investigate the solvability, that is, characterizing the value function as a unique viscosity solution of the HJB equation for the controlled stochastic fluid problem. Furthermore, we can deal with the nonsmooth boundary, which is piecewise-smooth with some corners.

Because the OCPSD could be found in other applications (cf. Example 1 in chapter 3) even outside the queueing theory, the result would provide a theoretical foundation to other areas also. The queueing problem is an instance of OCPSD. Specifically, it would be helpful to the hybrid system, in which a dynamics can be defined differently depending on submanifold. In queueing theory, we can study various systems with finite buffers, which has not been approached by optimal control theory. For example, we can apply or extend the theory to the problem of service differentiation with pricing [50], which needs to solve dynamic resource allocation and pricing at the same time. As long as the cost function and dynamics satisfy some regularity assumption, which was presented in section 1.2, we can deal with diverse problem situations.
1.4 Organization

In chapter 2, we will discuss the related studies such as the controlled piecewise deterministic Markov process (PDMP) and Skorokhod problem and differential inclusion approach to the reflecting boundary problem. In chapters 3 and 4, we deal with the deterministic control problem. In chapter 3, we characterize the value function of the deterministic problem as the viscosity solution of HJ equation for optimal control problems on stratified domains. Toward this goal, we introduce suitable notions of upper and lower solutions to the corresponding H-J equation with discontinuous coefficients (3.12)-(3.13), which are valid in connection with the given stratification. We then prove that the value function $V$ of the deterministic problem is an admissible solution of the HJ equation. Its uniqueness, within the class of admissible solutions, is proved by showing that

$$w(x) \leq V(x) \leq v(x)$$

for all $x \in \mathbb{R}^N$, where $w$ and $v$ denote respectively a lower and an upper solution. In Chapter 4, we characterize the continuity of the value function and provide explicit conditions for the Lipschitz continuity. Then, we show that the theory developed in Chapter 3 can be applied to the deterministic RBC problem. In Chapter 5, we extend the deterministic optimal control result to the PDMP control problem. In Chapter 6, we apply the Markov chain approximation method [48] to our problem to get a numerical solution. We will show local consistency of the approximated chains and tightness of the space of the approximated chains so that we can use the Markov chain approximation method without modification.
Related Studies

In this chapter, we discuss some related research with respect to modeling and problem solving approach. We will review the stochastic fluid model and its application to multiclass queueing systems. The viscosity solution and the Hamilton-Jacobi equation will be discussed in chapter 3.

2.1 The Stochastic Fluid Model and Piecewise Deterministic Markov Process

The stochastic fluid model can be considered as a special case of piecewise deterministic Markov process (PDMP). The PDMP could have boundary hitting jumps additionally. When the system hits the boundary (deterministically), the system is located randomly at a point in the state space. On the other hand, in the stochastic fluid model, there is no boundary hitting jump and only the environment variable has random jumps. The PDMP equipped with control theory has been used in various areas as one of main stochastic modeling approaches [25] [7]. Especially, it has been studied extensively in manufacturing area [63] [37]. However, it can be applied to other applications also such as communication networks [54] [55].

The PDMP is a time-homogeneous strong Markov process with right continuous and left hand limit (RCLL) sample paths [25]. It has a deterministic dynamics between random jumps. Davis [24] argues that PDMP can cover almost all non-diffusion stochastic models. In order to give some idea about PDMP, we give a
brief introduction to PDMP without rigorous treatment.

Let \( E = E^0 \cup \partial E \subset \mathbb{R}^N \) be the state space, where \( E^0 \) is the interior and \( \partial E \) is the boundary. Admissible control functions in \( E^0 \) are different from the ones in \( \partial E \) such that \( u_0 : [0, \infty) \times E^0 \mapsto U_0 \subset \mathbb{R}^N \) on \( E^0 \) and \( u_\partial : \partial E \mapsto U_\partial \subset \mathbb{R}^M \) on \( \partial E \), where \( u_0 \) and \( u_\partial \) are measurable functions. As usual, \( U_0, U_\partial \) are assumed to be compact.

The probability distribution of PDMP is determined by three local characteristics:

(i) A vector field, \( f : E \times U_0 \mapsto \mathbb{R}^N \). For \( t \geq 0 \)

\[
\dot{\xi}(t) = f(\xi(t), u_0(t, \xi)), \quad \xi(0) = \xi \quad \forall \xi \in E^0. \tag{2.1}
\]

(ii) A jump rate, \( \lambda : E^0 \times U_0 \mapsto [0, \infty) \). For each \( \xi \in E^0 \), there is an \( \epsilon(x) > 0 \) such that

\[
\int_0^{\epsilon(x)} \lambda(\xi(t), u_0(t, \xi))dt < \infty. \tag{2.2}
\]

(iii) A transition measure, \( Q = (Q_0, Q_\partial) \) for \( E^0 \) and \( \partial E \).

\[
Q_0 : E^0 \times U_0 \mapsto \mathcal{P}(E^0), \quad \text{and} \quad Q_\partial : \partial E \times U_\partial \mapsto \mathcal{P}(E^0),
\]

where \( \mathcal{P}(E^0) \) is the set of probability measures on \( E^0 \).

Let a PDMP process \( \{\xi(t); \ t > 0\} \) starts at \( \xi \in E^0 \). Let \( T_1 \) be the first jump time and

\[
P[T_1 > t|\xi(0) = \xi] = \begin{cases} 
\exp[-\int_0^t \lambda(\xi(s), u_0(t, \xi))ds], & \text{if } t < t_*(\xi), \\
0, & \text{if } t \geq t_*(\xi), 
\end{cases}
\]

where

\[
t_*(\xi) = \inf\{ t > 0; \xi(t) \in \partial E \}. \tag{2.3}
\]

By \( T_1 \), the system trajectory is determined by \( (2.1) \). The post jump state \( x(T_1) \) is

\[\footnote{On the boundary, the system jumps immediately. The control on the boundary, \( u_\partial \), does not have the time variable as an argument.}\]
determined by the distribution
\begin{equation}
\text{Prob}[\xi(T_1) \in A | \xi(T_1^-), v] = Q(A; \xi(T_1^-), v).
\end{equation}
for the Borel sets $A$ of $E^0$. The control $v = u_0(T_1^-, \xi)$ if $T_1 \neq t_*(\xi)$. Otherwise, $v = u_\partial(\xi(T_1^-))$. Note $Q(\{\xi\}; \xi, v) = 0$ for any $\xi \in E^0$. After the jump, the above process repeats with the new starting point $\xi(T_1)$.

We can consider two types of costs: the running cost $\ell: E^0 \times U_0 \mapsto [0, \infty)$ and the boundary jump cost $\ell_\partial: \partial E \times U_\partial \mapsto [0, \infty)$. Let $T_1, T_2, \cdots$ are jump times and $\beta > 0$ is a discount rate. Let $T_0 = 0$. The expected discounted total cost for a given control policy $u = (u_0, u_\partial)$ and an initial point $\xi(0) = \bar{\xi}$ is
\begin{equation}
J(\bar{\xi}, u) = E \left[ \sum_{i=0}^{\infty} \left( \int_{T_i}^{T_{i+1}} e^{-\beta t} \ell(\xi(t), u_0(t - T_i, \xi(T_i))) dt + e^{-\beta T_i} \ell_\partial(\xi(T_i^-), u_\partial(\xi(T_i^-))) 1_{\{\xi(T_i^-) \in \partial E\}} \right) \bigg| u, \xi(0) = \bar{\xi} \right].
\end{equation}
Then, the value function is defined as
\[
V(\bar{\xi}) = \inf_u J(\bar{\xi}; u), \quad \bar{\xi} \in E^0.
\]

For the well-posedness of the solution, additional appropriate regularity conditions would be assumed on $f$, $\lambda$, $\ell$, $\ell_\partial$, $Q_0$ and $Q_\partial$. For example, we might assume Lipschitz continuity in the first variable for $f$, $\lambda$, $\ell$, $\ell_\partial$ or Lipschitz continuity w.r.t. the weak* topology on $\mathcal{P}(E^0)$ in the first variable for $Q_0$ and $Q_\partial$.

\section{The Optimal Control Problem with Reflecting Boundary}

The fluid model is used to approximate a discrete time queueing process, e.g., $G/G/1$. The approximation is based on the functional strong law of large numbers (FSLLN) \cite{12}, which is obtained by time scaling. For example, let \{ $X(t), \ t \geq 0$ \}^2v\theta \in C(E^0), \ x \mapsto \int_{E^0} \theta(y)Q_0(dy; x, v)$ is Lipschitz for $v \in U_0$. So does $Q_\partial$.\footnote{The fluid model is used to approximate a discrete time queueing process, e.g., $G/G/1$. The approximation is based on the functional strong law of large numbers (FSLLN) \cite{12}, which is obtained by time scaling. For example, let \{ $X(t), \ t \geq 0$ \} such that $\int_{E^0} \theta(y)Q_0(dy; x, v)$ is Lipschitz for $v \in U_0$. So does $Q_\partial$.}
be a summation process such that

$$X(t) = \sum_{i=1}^{\lfloor t \rfloor} \xi_i, \quad t \geq 0,$$

where \( \{ \xi_i, i = 1, 2, \cdots \} \) is a sequence of nonnegative i.i.d. random variable in \( \mathbb{R} \) with a finite mean \( m > 0 \). Then, by FSLLN, as \( n \to \infty \), a time scaling of the process \( X(t) \) converges to a simple fluid approximation,

$$\tilde{X}^n \xrightarrow{a.s.} \tilde{X}, \quad (2.5)$$

where \( \tilde{X}^n(t) = \frac{1}{n} X(nt) \) and \( \tilde{X}(t) = mt \). The resulting process has a simple linear dynamics, which is a special case of our general model.

When the dynamics of queueing systems are approximated by fluid models or Brownian motions, the dynamics can be described by a (stochastic) differential equation with reflecting boundary condition \([30, 31, 38, 29]\). The reflecting boundary is introduced usually for nonnegativity constraints.

The dynamics on the domain with reflecting boundary is expressed through Skorokhod problems \([27, 51]\) or differential inclusions \([62]\). The existence and uniqueness of the solution for each approach has been studied for various domains or reflecting directions. Moreover, the solvability of HJB equations also has been studied from each approach \([62]\). The overflow cost is a function of the reflecting force on the boundary. Thus, it may be considered to be enough as a slight modification of the previous HJB equation result. However, as we discuss in chapter \([1]\), it causes discontinuities of the HJB equation on the boundary, which is a violation of the assumptions of the existing results.

### 2.2.1 Differential inclusion

The reflecting boundary was modeled by differential inclusion in \([62, 4, 22]\). Basic idea of using differential inclusion is almost same. In the first step, the dynamics
(1.18) can be reformulated into a projected differential inclusion [4]:

\[
\begin{cases}
\dot{x}(t) \in \pi_{T\bar{M}(x(t))} F^z(x(t)) & \text{for almost all } t \geq 0, \\
x(t) \in \mathcal{M} & \forall t \geq 0,
\end{cases}
\]

(2.6)

where the map, \( x \rightarrow F^z(x) \), is a set-valued function such that

\[
F^z(x) = \{ f^z(x, a); \ a \in A \}, \ x \in \bar{M}.
\]

(2.7)

We assume \( \mathcal{M} \) is convex. By assumption (H0), \( F^z(x) \) is Lipschitz continuous on \( \mathbb{R}^N \).

The differential inclusion in the above formulation might not be convex and upper semicontinuous. Thus, we cannot use the result of existence and uniqueness of the solution (or trajectory) [4]. In order to make sure the existence of solution of (2.6), we can regularize it into another differential inclusion

\[
\begin{cases}
\dot{x}(t) \in F^z(x(t)) - \tilde{N}_{\bar{M}}(x(t)) & \text{for almost all } t \geq 0, \\
x(t) \in \bar{M} & \forall t \geq 0.
\end{cases}
\]

(2.8)

We consider a bounded set of outer normal vectors to \( \bar{M} \) at \( x \),

\[
\tilde{N}_{\bar{M}}(x) = \{ p \in \mathbb{R}^N; \langle p, v \rangle \leq 0, \ \forall v \in T_{\bar{M}}(x), \ |p| \leq M \},
\]

(2.9)

where \( M = \max \{ |y|; \ y \in F^z(x), \ x \in \bar{M} \} < \infty \). It has been proved that the projection form of differential inclusion (2.6) and the extended form (2.8) provide same locally Lipschitz continuous solution [22].

If we can make the deparameterized form of the cost also become lower semicontinuous, with an appropriate assumption of cost function, then, we can prove the existence of optimal control by the direct method in Calculus of Variation [13, 23]. This basic principle is used in the next chapter for our problem also in a different form of differential inclusion.
Another approach to the reflecting boundary problem is to formulate the problem as a Skorokhod problem. For each point \( x \in \partial M \), reflection directions are defined as a set, \( r(x) \subset \{ y \in \mathbb{R}^N; |y| = 1 \} \). The system trajectory \( x(\cdot) \) consists of two components, \( \psi, \eta \in C([0,\infty); \mathbb{R}^N) \). Suppose \( \psi(0) \in M \) and \( \eta \) has bounded variation for a finite interval \([0, t], t < \infty\). Let \( |\eta|(t) \) denote the total variation of \( \eta \) over \([0, t]\).³

**Definition 2.2.1** (Skorokhod problem \([47]\)). We call \((x, \eta)\) a solution of the Skorokhod problem for \( \psi \) (w.r.t. \( M \) and \( r \)) if

(1) \( x(t) = \psi(t) - \eta(t), \ x(0) = \psi(0) \),
(2) \( x(t) \in M \) for \( t \in [0, \infty) \),
(3) \( |\eta|(t) < \infty \) for all \( t < \infty \),
(4) \( |\eta|(t) = \int_0^t I_{\{x(s)\in \partial M\}} d|\eta|(s) \), and
(5) there exists a smooth vector field \( r \) on \( \partial M \) pointing outward, i.e.

\[ \exists \nu > 0, \ \forall x \in \partial M, \ (n(x), r(x)) \geq \nu \] (2.10)

and \( \eta(t) = \int_0^t r(x(s)) d|\eta|(s) \).

The trajectory \( x(t) \) satisfies the following integral equation

\[ x(t) = x(0) + \int_0^t f(x(s), \alpha(s))ds - \eta(t) \]

with \( x(t) \in \bar{M} \), for all \( t \geq 0 \).

Lions\([50]\) uses the Skorokhod problem approach to study a deterministic optimal control problem, which has the reflecting boundary on a domain of smooth closed boundary. He showed that the value function is characterized as a viscosity solution of a HJB equation with a Neumann (or oblique derivative) boundary condition. If the reflecting direction is normal, the boundary conditions are

³The total variation of \( \eta \) over \([0, t]\) is defined by \( \sup \sum_{i=1}^N |\eta(t_i) - \eta(t_{i-1})| \) for all integers \( N \) and any sequence \( \{t_i\} \) such that \( 0 < t_0 < t_1 < \cdots < t_N < t \) [14].
Neumann-type, that is,
\[
\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial M
\]
(2.11)
where \( u \) is the unknown of HJB equation and \( n \) is the unit outward normal to \( \partial M \).
We can consider more general (or oblique) reflecting directions by
\[
\frac{\partial u}{\partial r} = 0 \quad \text{on } \partial M.
\]
(2.12)

Dupuis and Ishii [28] take into account nonsmooth domains with corners for second order nonlinear elliptic PDEs. They consider a state space, which is represented as an intersection of several finite number of bounded open domains \( \Omega_i, \ i \in I = \{1, 2, \cdots, M\} \), each of which has relatively regular boundary, such that
\[
\Omega = \bigcap_{i \in I} \Omega_i,
\]
(2.13)
where the closure of \( \Omega \), i.e., \( \bar{\Omega} \) is the domain on which the PDEs are defined.

Different from ours, they do not characterize dynamics and cost function by \( \Omega_i \).
In addition, we consider a state space, which is a union of submanifolds, i.e. the stratification [9] (cf. (3.1)). In [28], \( \bar{\Omega} \) is not represented by such a stratification.
In our problem, a submanifold might be used to represent even a border, at which the cost or dynamics are discontinuous even inside the state space.

They show that the value function of the optimal control is a viscosity solution of a HJB equation with the oblique derivative boundary condition,
\[
\frac{\partial u}{\partial \gamma_i} + f_i(x, u) = 0 \quad \text{for } x \in \partial \Omega \text{ and } i \in I(x),
\]
where \( I(x) = \{i \in I; \ x \in \Omega_i\} \) and \( \{f_i\}_{i \in I} \) is a set of real functions on \( \partial \Omega \times \mathbb{R} \). A set of vector fields \( \{\gamma_i\}_{i \in I} \) on \( \mathbb{R}^N \) is the oblique reflection direction, \( \langle \gamma_i(x), n_i(x) \rangle > 0 \) for \( x \in \partial \Omega_i \), where \( n_i(x) \) is the outward normal vector of \( \Omega_i \) at \( x \).
2.3 Single-server Multiclass Queueing Systems in Applications

2.3.1 Control of the Queueing Systems

The single-server multiclass queueing models have been used to study various systems such as multiplexers, flexible manufacturing systems. The models describe a phenomenon that multiple classes (or types) of customers compete for a bounded and shared processing (or service) rate. Finding a good queueing policy (e.g. dynamic priority, dropping some designated packets) for efficient processing rate allocation to minimize cost has been an issue. We see that the problem of finding an optimal queueing policy is basically an optimal control problem. We discuss how the dynamic control can be used in queueing problems through example 3-1 and 3-2.

Example 3-1 (A dynamic priority queue). Switching devices in ATM communication network can be modeled by a fluid queueing system\[43\]. Because the communication signals have a form of streams, which consists of many small chunks, called packets, of very small processing time, it is natural to approximate the dynamics of queueing system by the fluid model. There are several different types of traffics. Their service requirements and processing rates are different from each other. For example, we can see largely two types of communication traffics: Data traffic and Real time traffic. Data traffic such as FTP service or mail is loss-sensitive but delay-insensitive. Real time traffic such as voice or video is delay-sensitive but loss-insensitive.

In the usual priority queueing system\[43\], the real time data has the high priority over the non-real time data. A static priority policy gives good QoS (Quality of Service) to the real time data while the non-real time data experience worse QoS. Excessively unbalanced service quality could lead to loss of profits to the service provider. Thus, by a dynamic priority policy, people want to balance the service qualities among the classes. Further, we may be able to think of trade-off between service rate and quality on the real time traffic. For example, because the real-time traffic is loss-insensitive, we can degrade the quality of streaming by dropping some packets in order to speed up the processing\[32\]. The packets to be
dropped could be designated by traffic sources [46]. The degraded quality would be reflected in the cost function. This kind of trade-off could be implemented in different ways depending on applications such as trade-off between power consumption and processing speed [3]. The best trade-offs for the above diverse circumstances in terms of a given cost function could be achieved by the dynamic control of the system parameters.

Optimal control approach could be also one of ways to pricing differentiated services [52, 56]. Because of the rapid growth of internet, efficient control of heterogeneous communication traffics in a server with differentiated service requirement becomes an important issue. A difference from existing approach [52, 56] is that in optimal control approach, we do not explicitly take into account waiting time distribution in the cost function. Instead, we can indirectly reflect it through holding cost.

**Example 3-2 (A Resource Allocation Problem in a Software Multiagent System)**

Multiagent systems have significant advantages in the development of complex distributed software systems [42]. Components in a complex application system and interactions among them can be naturally implemented as agents and the interactions of agents. Due to the modularity and autonomy of agents, applications could be composed by assembling agents. Thus, multiagent systems are flexible in design such that partial changes in the system could be localized on a few agents without affecting the design of the rest of the system. Hence, constructing or altering a large software system could become easier with agent technology.

In the multiagent system, functions in software are distributed into agents. The software runs through interactions of agents [19]. Each agent has a role of providing a service to other agents. Each agent might be a service customer and a service provider at the same time. A service provider agent receives service request messages and store them temporarily stored in a buffer before processing if it is processing another service request. The traffic of service requests has been described as queueing models [11].

When we focus on a resource allocation problem in a single machine (or single station), the multiagent system can be considered as a single-station multiclass queueing system. Several agents, which have different service functions, are located
in the single machine and compete for computing resources, CPU time. Each agent has a preassigned finite memory space, which has a role of a buffer. The assigned memories may have different sizes.\textsuperscript{4} We also assume the existence of a software component, called node, which has a capability to control the allocation of the processing rate (or CPU time) to agents in the multiagent architecture.

The work load level is not constant in time (even bursty \textsuperscript{1}) and not equally distributed among agents in usual agent systems \textsuperscript{19}. Thus, we feel that a dynamic resource control could give a better performance. The problem is to find an optimal control policy for the node such that the expected total discount cost is minimized while the system is subject to stochastically time varying workload and CPU availability. We consider an infinite time horizon problem. The cost function consists of holding cost of service requests, and cost of service request message loss due to the lack of buffer space. The cost function is evaluated for each agent independently. The cost of the overall system is assumed to be the sum of individual agents’ cost.

The problem can be modeled as a single-station multiclass queueing system with finite buffers and overflow costs. The overflow cost is introduced to penalize the failure of providing service of agents. In addition to the bursty workload traffics, even the importance of agent service could be different depending on situation.

2.3.2 Controlled Fluid Models in Applications

Optimal control approach using HJB equation has been studied extensively in manufacturing systems. The problem has a form of determining production rates in an unreliable flexible manufacturing system to minimize a convex cost functional. Stochastic disturbances are caused by failures of machines and time varying demand \textsuperscript{63}, which are described by a finite state CTMC. The value function of the optimal control problem is characterized as the unique viscosity solution of a HJB equation. By investigating the Hamiltonian dynamics, we can find that the optimal control would have a hedging point or turnpike property in some general conditions \textsuperscript{39, 40}. This turnpike property is found relatively easily because the value function is convex when we consider the convex cost function, which consists

\textsuperscript{4}They may have a shared memory. This can be covered by our approach.
of backlog and inventory cost. Based on these characterizations, an analytical or approximate solution has been sought.

Akella and Kumar \[2\] find an analytical form of the optimal control policy for a single machine and single part type problem. It is a rare closed-form analytical result. It is hard to expect such a closed-form control policy for more complicated problems. Thus, studies on this type of problems are largely classified into the study on the characterization of the optimal control policy or finding numerical solutions or approximation. A single-unreliable-machine two-part type system \[9,44\] has a similar characteristics as our problem in that the decisions on the production rate of two parts are interdependent because they share a single machine. Srivatsan and Dallery \[44\] provide a partial characterization of the control policy. According to them, the optimality of hedging point policies in two-part-type systems has not been characterized completely. Shaoxiang \[41\] proves the optimality of hedging point policies but for a similar discrete time model. Those models are for the make-to-stock production system.

The above studies in manufacturing system assume infinite buffers. We will see single-server multiclass queueing system problems with reflecting boundary in the next section. We want to see how the nonnegativity constraints are dealt in the literature. The flowshop or jobshop problem have the nonnegativity constraints. In \[63\], the nonnegativity constraints is treated as a state constraints as in \[65\] rather than reflecting boundary. In this case, the HJBDD (HJB directional derivative) equation is used to characterize the value function. Idea is to replace the term $Dv \cdot f$ in HJB equation by directional derivative of the value function $v'_f$ (or $D_f v$), where $f$ is a direction, which does not make the system violate the constraints and $v$ is the unknown. The direction $f$ is the velocity $f(x,a)$, which is a function of state and control. The HJB equation has the following form in our notation,

$$
\beta v(x, i) = \inf_{a \in A(x)} \left\{ v'_f(x, i) + \ell^i(x, a) \right\} + \sum_{j \neq i} q_{ij} [v(x, j) - v(x, i)],
$$

(2.14)

where $v'_f$ is a directional derivative,

$$
\lim_{\delta \to 0} \frac{v(x + \delta f) - v(x)}{\delta} = v'_f(x)
$$

(2.15)
and $f$ depends on $a$. The control set, $A(x)$, is prepared to be a set of admissible controls at $x$, i.e. the set of controls, which make the system satisfy the constraints.

The solution of the HJBDD is a viscosity solution. However, only a verification theorem is provided using the convexity and Lipschitz continuity of the value function \[63\].

In communication network area, the stochastic fluid model has been popularly used in analyzing the performance of communication networks [20,27] rather than optimal control.
Chapter 3

Optimal Control Problems on Stratified Domains

3.1 Introduction

In this chapter, we deal with a deterministic ‘optimal control problems on stratified domains’. The result in this chapter will be adopted for the deterministic and stochastic RBC problem, which we introduced in chapter 1.

The theory of viscosity solutions was initially developed in connection with continuous solutions of Hamilton-Jacobi equations, whose coefficients are also continuous.

Various authors have then extended the theory in cases where the value function is discontinuous [6, 69]. In particular, upper or lower solutions to a HJ equation can now be defined within a more general class of semicontinuous functions. In a different direction, motivated by problems in optimal control, sufficient conditions for the optimality of a feedback synthesis have been established in [60], under assumptions that do not require the continuity of the value function.

A further line of investigation, more recently pursued in [67, 15], is the case where the coefficients of the HJ equation are themselves discontinuous. This chapter represents a contribution in this direction, in a specific case. Namely, we study the value function for an infinite-horizon optimal control problem, on a structured domain. The space $\mathbb{R}^N$ is decomposed as the disjoint union of finitely many sub-
manifolds of different dimensions, and we assume that the dynamics of the system as well as the running cost are sufficiently regular when restricted to each given manifold, but may well differ from one manifold to the other.

More precisely, we assume that there exists a decomposition

\[ \mathbb{R}^N = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_M \]  

(3.1)

with the following properties. Each \( \mathcal{M}_j \subset \mathbb{R}^N \) is an embedded submanifold. If \( j \neq k \), then \( \mathcal{M}_j \cap \mathcal{M}_k = \emptyset \). In addition, if \( \mathcal{M}_j \cap \overline{\mathcal{M}}_k \neq \emptyset \), then \( \mathcal{M}_j \subset \overline{\mathcal{M}}_k \), where the upper bar denotes closure.

We call \( d_k = \dim(\mathcal{M}_k) \), so that \( d_k = 0 \) if \( \mathcal{M}_k \) consists of a single point and \( d_k = N \) if \( \mathcal{M}_k \) is an open subset of \( \mathbb{R}^N \). For example, in figure 3.1 we have

\[ d_1 = d_2 = 2, \quad d_3 = d_4 = d_5 = d_6 = 1, \quad d_7 = d_8 = d_9 = d_{10} = 0. \]

We now consider an optimal control problem with infinite horizon and exponentially discounted cost with \( \beta > 0 \):

\[ \text{minimize} \quad J(\bar{x}, \alpha) = \int_0^\infty e^{-\beta t} \ell(x(t), \alpha(t)) \, dt \]  

(3.2)

for a system with dynamics

\[ \dot{x}(t) = f(x(t), \alpha(t)), \quad x(0) = \bar{x} \in \mathbb{R}^N. \]  

(3.3)

Here \( t \mapsto \alpha(t) \) denotes the control function.
The value function is defined as

\[ V(\bar{x}) := \inf_{\alpha \in \mathcal{A}} J(\bar{x}, \alpha), \]  

(3.4)

where \( \mathcal{A} \) is the set of all admissible control functions.

Our key assumption is that both the field \( f \) and the cost \( \ell \) are sufficiently regular when restricted to each of the manifolds \( \mathcal{M}_j \). More precisely

**\( \mathbf{H1} \)** For each \( i = 1, \ldots , M \) there exists a compact set of controls \( A_i \subset \mathbb{R}^m \), a continuous map \( f_i : \mathcal{M}_i \times A_i \mapsto \mathbb{R}^N \), and a cost function \( \ell_i \) with the following properties

(a) At each point \( x \in \mathcal{M}_i \), all vectors \( f_i(x, a) \), \( a \in A_i \) are tangent to the manifold \( \mathcal{M}_i \).

(b) \( |f_i(x, a) - f_i(y, a)| \leq \text{Lip}(f_i) |x - y| \), for all \( x, y \in \mathcal{M}_i \), \( a \in A_i \).

(c) Each cost function \( \ell_i(x, a) \) is non-negative and continuous.

(d) We have \( f(x, a) = f_i(x, a) \) and \( \ell(x, a) = \ell_i(x, a) \) whenever \( x \in \mathcal{M}_i \), \( i = 1, \ldots , M \).

By \( \text{Lip}(f_i) \) we denote here a Lipschitz constant for the function \( f_i \) w.r.t. the first variable. In the following, for any \( x \in \mathbb{R}^N \), the index \( i(x) \in \{1, \ldots , M\} \) identifies the manifold which contains the point \( x \). In other words,

\[ i(x) \doteq k \quad \text{if} \quad x \in \mathcal{M}_k. \]

The assumption (d) can now be written as

\[ f(x, a) = f_i(x, a), \quad \ell(x, a) = \ell_i(x, a), \quad a \in A_{i(x)}. \]

The tangency condition in \( \mathbf{H1}-\text{a} \) can thus be restated as

\[ f_i(x, a) \in T_{\mathcal{M}_i}(x) \quad \forall x \in \mathcal{M}_i, \quad a \in A_i. \]
Since the functions \(f_i\) are Lipschitz continuous w.r.t. \(x\) and the sets of controls \(A_i\) are assumed to be compact, it follows that trajectories of the control system cannot approach infinity in finite time. Indeed, all solutions of (3.3) satisfy the a-priori bounds
\[
|\dot{x}(t)| \leq C(1 + |x(t)|), \tag{3.6}
\]
\[
|x(t)| \leq e^{Ct}(1 + |x(0)|), \tag{3.7}
\]
for some constant \(C\) (Corollary 3.1 [12]).

In the above setting, our main interest is in the existence of optimal controls, and in the characterization of the value function as the unique solution to the corresponding HJ equation, in an appropriate sense. In Section 3 we discuss a simple example (Example 3), showing that the standard definition of viscosity solution is not adequate in the case of discontinuous dynamics and cost functions. Indeed, in addition to the value function, one can now have infinitely many other Lipschitz continuous admissible solutions to the HJ equation. We then show how to modify the definition of solution, in connection with the stratification (3.1), in order to uniquely characterize the value function.

In the case of an upper solution \(v\), the comparison result relies on an invariance property of the epigraph of \(v\), as in [69]. To analyze lower solutions, our techniques resemble those used in [36] and [60] to prove the optimality of a regular feedback synthesis. The main technical difficulty encountered here is due to the stratification (3.1). In particular, the case of an optimal trajectory that enters and exits infinitely many times from the same manifold \(\mathcal{M}_i\) cannot be ruled out a priori, and requires a more careful study.

Here, we give an example of OCPSD, which is not a queueing problem.

**Example 1 (Minimum time problem with discontinuous coefficients.)**

Consider a minimum time problem on \(\mathbb{R}^2\), assuming that the speed can be much higher along “highways”, described by a finite number of curves in the plane. As admissible velocity sets one can then take, for example

\[
F_0(x) = \{y \in \mathbb{R}^2; \ |y| \leq c_0(x)\}
\]

\(^1\text{Consider a Cauchy problem } \dot{r} = C(1 + r) \text{ with } r(0) = |x(0)|.\)
outside the highways, and

\[ F_i(x) = \{ y \in \mathbb{R}^2 ; \; y \in T_{\mathcal{M}_i(x)} , \; |y| \leq c_i(x) \} \]

along the highway \( \mathcal{M}_i \), for some speeds \( 0 < c_0(x) \ll c_i(x) \). Given a target point \( x^\dagger \in \mathbb{R}^2 \), consider the cost function \( \ell(x, \cdot) = 1 \) if \( x \neq x^\dagger \) while \( \ell(x^\dagger, \cdot) = 0 \). Then value function for the problem

\[
\min_{x(\cdot)} \int_0^\infty e^{-\beta t} \ell(x(t), \dot{x}(t)) \, dt
\]

subject to

\[
x(0) = \bar{x}, \quad \dot{x}(t) \in F(x(t))
\]

is given by

\[
V(\bar{x}) = \frac{1 - e^{-\beta T(\bar{x})}}{\beta}.
\]

Here \( T(\bar{x}) \) is the minimum time needed to steer the system from \( \bar{x} \) to the target point \( x^\dagger \). This provides a simple example of a minimum time problem with discontinuous velocities, which can be recast in the form (3.1).

**A related differential inclusion**

To study certain aspects of the optimization problem, it is convenient to reformulate it as a differential inclusion, leaving aside the parametrization of the velocity sets in terms of the control values.

For each \( x \in \mathbb{R}^N \), define the set of admissible velocities

\[
F(x) = \left\{ f_{i(x)}(x, a) ; \; a \in A_{i(x)} \right\} \subset \mathbb{R}^N.
\]

(3.8)

Sometimes, we will use \( f_{i(x)}(x, a) \) for \( F(x) \) in order to show explicitly \( i(x) \). Moreover, define the extended multifunction

\[
\hat{F}(x) = \left\{ (y, \eta) ; \; y = f_{i(x)}(x, a) , \; \eta \geq \ell_{i(x)}(x, a) , \; a \in A_{i(x)} \right\} \subset \mathbb{R}^{N+1}.
\]

(3.9)

Denoting by \( \overline{\text{co}} S \) the closed convex hull of a set \( S \), we shall also consider the upper
semicontinuous, convex-valued regularization

\[ G(x) = \bigcap_{\varepsilon > 0} \overline{\co \{(y, \eta) \in \widehat{F}(x') : |x' - x| < \varepsilon \}} \subset \mathbb{R}^{N+1}. \] (3.10)

To achieve the existence of an optimal control for the problem (3.2)-(3.3), we shall use the following assumption.

(H2) For every \( x \in \mathbb{R}^N \) one has

\[ \left\{ (y, \eta) \in G(x) : y \in T_{M_{i(x)}}(x) \right\} = \widehat{F}(x). \] (3.11)

In particular, (H2) implies

(H2') For each fixed \( x \in \mathbb{R}^N \), the set \( F(x) \subset \mathbb{R}^N \) is convex. Moreover, the function

\[ p \mapsto L(x, p) \doteq \min \left\{ \ell_{i(x)}(x, a) : f_{i(x)}(x, a) = p, \ a \in A_{i(x)} \right\}, \]

defined for \( p \in F(x) \), is convex.

**The Hamilton-Jacobi equation**

Besides proving the existence of an optimal control, we wish to characterize the value function as the unique solution of the corresponding Hamilton-Jacobi (HJ) equation

\[ \beta u(x) + H(x, Du(x)) = 0. \] (3.12)

Here the Hamiltonian function is defined as

\[ H(x, p) \doteq \sup_{(f, \eta) \in G(x)} \left\{ -f \cdot p - \eta \right\}. \] (3.13)

We mention here some relations of the present work with earlier literature. The type of stratified control system which we consider in (H1) is reminiscent of the definition of regular synthesis by Boltyanskii [8] and by Brunovský [13]. However,
in their case the stratification referred to the structure of the value function, while here the stratification is a property of the control system. In certain ways, our framework is similar to a hybrid control system [59, 71], where the state can jump within a finite set of manifolds. The main differences here are that (i) the times $t_i$ where the state moves from one manifold to another are determined by the position of the system itself, and not directly by the controller, and (ii) there is no cost associated to the transition from one manifold to another. As a result, an optimal trajectory may well leave and re-enter a given manifold $M_i$ infinitely many times.

Various studies on HJ equations with discontinuous coefficients have appeared in recent years, due to a growing recognition of the importance of these equations. Newcomb and Su [57] introduced the Monge solution for an equation of eikonal type

$$H(Du) = n(x),$$

where $H$ is assumed to be convex and $n(x)$ is a lower semicontinuous function. In [67], Soravia studied a class of optimal control problems with discontinuous dynamics. The HJ equations take the special form

$$\beta u(x) + \sup_{a \in A} \{-f(x, a) \cdot Du(x) - h(x, a)\} = g(x), \quad (3.14)$$

where $f$ and $h$ are locally Lipschitz continuous and $g$ is a Borel measurable function.

In a paper by Camilli and Siconolfi, a general class of HJ equations with measurable coefficients is considered. In [15], they propose a definition of solution which disregards sets of measure zero. This is very different from our approach, where the form of the control system on submanifolds $M_j$ of dimension $d_j < N$ (hence of measure zero) plays a key role in the optimization problem.

In Section 3.2 we prove a theorem on the existence of optimal controls. The main ingredients of the proof are the same as in the standard case, with continuous dynamic and cost functionals. The convexity assumption (H2) here provides the key tool for passing to the limit in a minimizing sequence.

In the remaining sections we seek conditions which imply the optimality of a given trajectory. Toward this goal, in Section 3.3, we introduce suitable notions of upper and lower solutions to the corresponding HJ equation with discontinuous coefficients (3.12)-(3.13), valid in connection with the given stratification. We then
prove that the value function $V$ in (3.4) is an admissible solution. In Section 3.3, its uniqueness, within the class of admissible solutions, is proved by showing that

$$u(x) \leq V(x) \leq v(x) \quad \text{for all } x \in \mathbb{R}^N.$$ 

where $u$ and $v$ denote respectively a lower and an upper solution. These comparison results require some minimum regularity assumptions. Namely, the value function $V$ should be globally Hölder continuous of exponent $1/2$, and its restriction to each submanifold $M_k$ should be a.e. differentiable (almost everywhere w.r.t. the $d_k$-dimensional measure). By Rademacher’s theorem, this last condition certainly holds if $V$ is locally Lipschitz continuous in a neighborhood of a.e. point $x \in M_k$. For example, the function $V(x, y) = \sqrt{|x|} + \sqrt{|y|}$ satisfies the above requirements, for a stratification with $M_1 = \{(x, y) ; y = 0\}, M_2 = \mathbb{R}^2 \setminus M_1$.

### 3.2 Existence of an optimal control

Aim of this section is to prove a theorem on the existence of optimal controls. This will be achieved by a suitable modification of Filippov’s argument [35], to account for the discontinuities in the dynamics and in the cost functions.

**Theorem 3.2.1.** Consider the optimization problem (3.2), for the control system (3.3) on a stratified domain. Let the assumptions (H1), (H2) hold. If there exists at least one trajectory having finite cost, then the minimization problem admits an optimal solution.

**Proof.** The proof will be given in several steps.

1. **(Existence of a minimizing sequence).** By assumption, there exists a sequence of admissible controls $\alpha_k(\cdot)$ with corresponding trajectories $x_k(\cdot)$ such that

$$\dot{x}_k(t) = f(x_k(t), \alpha_k(t)), \quad x_k(0) = \bar{x}$$

$$\lim_{k \to \infty} \int_0^\infty e^{-\beta t} \ell(x_k(t), \alpha_k(t)) \, dt = \inf_{\alpha \in A} J(\bar{x}, \alpha) = m < +\infty. \quad (3.15)$$

2. **(Compactness $\implies$ existence of a convergent subsequence).** By the
continuity assumption in (H1), the cost function $\ell$ is locally bounded. We can thus find a continuous function $x \mapsto K^\dagger(x)$ such that

$$\ell(x, a) < K^\dagger(x) \quad \text{for all } x \in \mathbb{R}^N, \ a \in A_i(x). \quad (3.16)$$

Recalling (3.10), we define the truncated, time dependent multifunction

$$G^\dagger(t, x) \doteq \left\{ (y, e^{-\beta t}\eta); \ (y, \eta) \in G(x), \ \eta \leq K^\dagger(x) \right\} \subset \mathbb{R}^{N+1}. \quad (3.17)$$

We observe that $G^\dagger$ is upper semicontinuous with convex, compact values. Define

$$\gamma_k(t) \doteq \int_0^t e^{-\beta s}\ell(x_k(s), \alpha_k(s)) \, ds.$$  

Then for each $k \geq 1$ the map

$$t \mapsto (x_k(t), \gamma_k(t))$$

provides a solution to the differential inclusion

$$\frac{d}{dt} (x(t), \gamma(t)) \in G^\dagger(t, x), \quad (x(0), \gamma(0)) = (\bar{x}, 0). \quad (3.18)$$

The Lipschitz continuity of the functions $f_i$, and the compactness of the sets of controls $A_i$, imply that all solutions of (3.3) satisfy the a-priori bounds (3.6), (3.7). In particular, on any given time interval $[0, T]$, all values $|x_k(t)|$ as well as all derivatives $|\dot{x}_k(t)|$ remain uniformly bounded. Because of (3.16), the cost functions $\ell(x_k, \alpha_k)$ are also uniformly bounded. By the Ascoli-Arzelà compactness theorem, by possibly taking a subsequence, we can assume the convergence

$$x_k(t) \to x^*(t), \quad \gamma_k(t) \to \gamma^*(t)$$

for some limit functions $x^*(\cdot), \gamma^*(\cdot)$, uniformly for $t$ in bounded sets.

3. (The limit trajectory is admissible). By the theory of differential inclusions [4], the upper semicontinuity and convexity properties of the multifunction $G^\dagger$
imply that the limit trajectory satisfies
\[\frac{d}{dt}(x^*(t), \gamma^*(t)) \in G\dot{t}(t, x^*(t)), \quad (x^*(0), \gamma^*(0)) = (\bar{x}, 0).\]

For \(i = 1, \ldots, M\), consider the set of times
\[J_i = \{t \geq 0; \quad x^*(t) \in M_i\}.\] (3.19)

Each \(J_i\) is a Borel measurable subset of the real line. Moreover,
\[\dot{x}^*(t) \in T_{M_i}(x^*(t)) \quad \text{for a.e. } t \in J_i.\]

We can thus use the assumption (H2') and deduce that, for a.e. time \(t \geq 0,\)
\[\dot{x}^*(t) \in F(x^*(t)), \quad \dot{\gamma}^*(t) \geq \min \left\{e^{-\beta t} \ell_i(x^*(t), a); \quad f_i(x^*(t), a) = \dot{x}^*(t), \quad a \in A_i(x^*(t))\right\}.\]

4. (The limit trajectory is optimal). By the previous step, and by Filippov's measurable selection theorem [35], we can select control functions \(\alpha_i^* : J_i \mapsto A_i\) such that
\[\ell_i(x^*(t), \alpha_i^*(t)) = \min \left\{\ell_i(x^*(t), a); \quad a \in A_i, \quad f_i(x^*(t), a) = \dot{x}^*(t)\right\}\]
for a.e. \(t \in J_i\). Defining
\[\alpha^*(t) = \alpha_i^*(t) \quad \text{for } t \in J_i,\]
we obtain
\[\dot{x}^*(t) = f_i(x^*(t), \alpha^*(t)).\] (3.20)

Moreover, for every fixed \(T > 0,\)
\[\int_0^T e^{-\beta t} \ell(x^*(t), \alpha^*(t)) \, dt \leq \gamma^*(T) = \lim_{k \to \infty} \int_0^T e^{-\beta t} \ell(x_k(t), \alpha_k(t)) \, dt \leq m.\]
Letting $T \to \infty$ we obtain
\[
\int_{0}^{\infty} e^{-\beta t} \ell(x^*(t), \alpha^*(t)) \, dt = \sup_{T > 0} \int_{0}^{T} e^{-\beta t} \ell(x^*(t), \alpha^*(t)) \, dt \leq m .
\] (3.21)

Together, (3.20) and (3.21) yield the result.

3.3 Viscosity solutions of the Hamilton-Jacobi equation

For optimal control problems with continuous dynamics, it is well known that the value function provides a solution of a corresponding Hamilton-Jacobi equation, in a viscosity sense [6, 49]. In the remainder of this chapter, we show how the definition of viscosity solution can be adapted to the case of stratified control system, and extend the well known comparison and uniqueness results to this case.

3.3.1 Upper and lower solutions

We now introduce the definitions of upper and lower solution for (3.12)-(3.13), relative to the stratified domain (3.1).

**Definition 3.3.1.** We say that a continuous function $w$ is an upper solution of (3.12)-(3.13) relative to the stratification (3.1) if the following holds. If $w - \varphi$ has a local minimum at $\bar{x}$ for some $\varphi \in C^1$, then
\[
\beta w(\bar{x}) + \sup_{(y, \eta) \in G(\bar{x})} \left\{ - y \cdot D\varphi(\bar{x}) - \eta \right\} \geq 0.
\] (3.22)

**Definition 3.3.2.** We say that a continuous function $w$ is a lower solution of (3.12)-(3.13) relative to the stratification (3.1) if the following condition holds.

If $\bar{x} \in M_i$ and the restriction of $w - \varphi$ to $M_i$ has a local maximum at $\bar{x}$ for some $\varphi \in C^1$, then
\[
\beta w(\bar{x}) + \sup_{(y, \eta) \in G(\bar{x})} \left\{ - y \cdot D\varphi(\bar{x}) - \eta \right\} \leq 0.
\] (3.23)
Definition 3.3.3. A continuous function, which is at the same time an upper and a lower solution relative to the stratification (3.1), will be called a viscosity solution.

Notice that, in the definition of lower solution, we restrict the analysis to the manifold $\mathcal{M}_{l(x)}$. This is motivated by the following example.

Example 3. Consider the problem of reaching the origin in minimum time, for the system of $\mathbb{R}^2$ described by

$$\frac{d}{dt} (x_1, x_2) \in F(x_1, x_2) \triangleq \begin{cases} 
\{(y_1, 0) ; \, |y_1| \leq 3\} & \text{if } x_2 = 0, \\
\{(y_1, y_2) ; \, |y_1| + |y_2| \leq 1\} & \text{if } x_2 \neq 0.
\end{cases}$$

In this case, the optimal trajectories are easy to describe: To reach the origin starting from $(\bar{x}_1, \bar{x}_2)$ we first move vertically toward the point $(\bar{x}_1, 0)$ with speed 1, then move horizontally to the origin, with speed 3. The minimum time function is thus

$$V(x_1, x_2) = \frac{|x_1|}{3} + |x_2|. $$

This provides a viscosity solution on $\mathbb{R}^2 \setminus \{0\}$ to the corresponding HJ equation

$$\sup_{y \in F(x)} \{ - y \cdot \nabla v(x) \} - 1 = 0. \quad (3.24)$$

However, if we use the standard notion of viscosity solution, then also the function

$$U(x_1, x_2) = \frac{|x_1|}{2} + |x_2|. $$

provides a solution. Indeed, at any point $\bar{x} = (\bar{x}_1, 0)$ there is no $C^1$ function $\varphi$ such that $u - \varphi$ has a local maximum at $\bar{x}$. Therefore, the usual definition of viscosity lower solution does not pose any requirement at these points.

Recalling Example 1, one checks that the functions

$$\tilde{V}(x) \equiv 1 - e^{-V(x)}, \quad \tilde{U}(x) \equiv 1 - e^{-U(x)}$$


provide two distinct viscosity solutions (in the standard sense) to the same equation

\[ u(x) + \sup_{y \in F(x)} \left\{ -y \cdot \nabla u(x) \right\} - 1 = 0 \quad (3.25) \]

on \( \mathbb{R}^2 \setminus \{0\} \). Notice however that \( \tilde{U} \) does not satisfy our present definition of lower solution.

### 3.3.2 The value function as a viscosity solution

**Proposition 3.3.4.** Consider the optimal control problem \((3.2)\), for the control system \((3.3)\) on a stratified domain. Let the assumptions \((H1), (H2)\) hold and assume that the value function \(V\) is continuous. Then, \(V\) is a viscosity solution according to Definition \[3.3.3\].

**Proof.** The argument naturally consists of two parts.

1. **\(V\) is an upper solution.** Let \(\varphi \in C^1\) and let \(\bar{x}\) be a point where \(V - \varphi\) attains a local minimum. We can assume that, for some \(r > 0\),

\[ \varphi(\bar{x}) = V(\bar{x}), \quad \varphi(x') \leq V(x') \quad \forall x' \in B(\bar{x}, r). \quad (3.26) \]

Let \(t \mapsto \alpha^*(t)\) and \(t \mapsto x^*(t)\) be respectively an optimal control and a corresponding optimal trajectory, starting from the point \(\bar{x}\). Their existence was proved in Theorem 1. For all \(T \geq 0\) we now have

\[ V(\bar{x}) = \int_0^T e^{-\beta t} \ell(x^*(t), \alpha^*(t)) dt + e^{-\beta T} V(x^*(T)), \quad (3.27) \]

If we had \(x^*(t) \in M_j\) for a fixed index \(j \in \{1, \ldots, M\}\) and all \(t \in ]0, \delta]\), \(\delta > 0\), it would now be easy to conclude. However, we must consider the possibility that the optimal trajectory \(x^*(\cdot)\) switches infinitely many times between different manifolds \(M_i\). To handle this more general situation, we consider the minimum dimension among these manifolds:

\[ d^- \doteq \lim \inf_{t \to 0} d_i(x^*(t)). \]
We then choose a manifold $M_k$ of minimum dimension $d^-$ such that

$$x^*(T_n) \in M_k$$

for a sequence of times $T_n \to 0$.

![Figure 3.2. The convergent subsequence $T_n$](image)

By possibly taking a subsequence, as $T_n \to 0$ we can assume that

$$\lim_{n \to \infty} \frac{x^*(T_n) - \bar{x}}{T_n} = \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} f(x^*(t), \alpha^*(t)) dt = \tilde{f}, \quad (3.28)$$

$$\lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} e^{-\beta t} \ell(x^*(t), \alpha^*(t)) dt = \bar{\eta}, \quad (3.29)$$

for some vector $\tilde{f} \in T_{\bar{x}}M_k(\bar{x})$ and some $\bar{\eta} \geq 0$. Observing that

$$\frac{d}{dt} \left( x^*(t), -V(x^*(t)) \right) = \left( f_i(x^*(t))(x^*(t), \alpha^*(t)), -\beta V(x^*(t)) + \ell_i(x^*(t))(x^*(t), \alpha^*(t)) \right),$$

we have

$$\frac{d}{dt} \left( x^*(t), -V(x^*(t)) \right) + \left( 0, \beta V(x^*(t)) \right) \in G(x^*(t)). \quad (3.30)$$

By (3.27) and (3.29) it follows

$$\lim_{n \to \infty} \frac{V(\bar{x}) - V(x^*(T_n))}{T_n} + \beta V(\bar{x}) = \lim_{n \to \infty} \frac{1}{T_n} \int_0^{T_n} e^{-\beta t} \ell(x^*(t), \alpha^*(t)) dt = \bar{\eta}.$$  

(3.31)
The upper semicontinuity and the convexity of the multifunction $G$ implies $(\bar{f}, \bar{\eta}) \in G(\bar{x})$.

To prove that $V$ is a supersolution, we need to show that

$$
\beta \varphi(\bar{x}) + \sup_{(y, \eta) \in G(\bar{x})} \left\{ - y \cdot \nabla \varphi(\bar{x}) - \eta \right\} \geq \beta \varphi(\bar{x}) - \bar{f} \cdot \nabla \varphi(\bar{x}) - \bar{\eta} \geq 0. \tag{3.32}
$$

From (3.31) and (3.26) it now follows

$$
\bar{\eta} - \beta V(\bar{x}) = \lim_{n \to \infty} \frac{V(\bar{x}) - V(x^*(T_n))}{T_n} \leq \lim_{n \to \infty} \frac{\varphi(\bar{x}) - \varphi(x^*(T_n))}{T_n} = - \bar{f} \cdot \nabla \varphi(\bar{x}),
$$
proving (3.32).

2. **$V$ is a lower solution.** Assume that $\varphi \in C^1$ and that the function $V - \varphi$, restricted to $\mathcal{M}_i$, attains a strict local maximum at $\bar{x} \in \mathcal{M}_i$. We can assume that, for some $r > 0$,

$$
\varphi(\bar{x}) = V(\bar{x}), \quad \varphi(x') \geq V(x'), \quad \forall x' \in B(\bar{x}, r) \cap \mathcal{M}_i. \tag{3.33}
$$

Fix any $(y, \eta) \in G(\bar{x})$. We need to show that

$$
\beta V(\bar{x}) - y \cdot D \varphi(\bar{x}) - \eta \leq 0. \tag{3.34}
$$

By the assumption (H2), there exists a control value $a \in A_i$ such that

$$
y = f_i(\bar{x}, a), \quad \eta \geq \ell_i(\bar{x}, a). \tag{3.35}
$$

Consider the trajectory $t \mapsto x(t)$ corresponding to the constant control $a(t) \equiv a$. Our assumptions imply

$$
x(t) \in \mathcal{M}_i \cap B(\bar{x}, r),
$$
at least for a short time interval, say $t \in [0, T]$. Since

$$
V(\bar{x}) \leq \int_0^t e^{-\beta s} \ell_i(x(s), a) \, ds + e^{-\beta t} V(x(t)),
$$
we compute
\[
\lim_{t \to 0} \frac{\phi(\bar{x}) - e^{-\beta t} \varphi(x(t))}{t} \leq \lim_{t \to 0} \frac{V(\bar{x}) - e^{-\beta t} V(x(t))}{t} \leq \lim_{t \to 0} \frac{1}{t} \int_0^t e^{-\beta s} \ell_i(x(s), a) \, ds = \ell_i(\bar{x}, a).
\]

Therefore, by (3.35),
\[
\eta \geq \ell_i(\bar{x}, a) \geq \lim_{t \to 0} \frac{\varphi(\bar{x}) - e^{-\beta t} \varphi(x(t))}{t} = \beta \varphi(\bar{x}) - f_i(\bar{x}, a) \cdot \nabla \varphi(\bar{x}) = \beta V(\bar{x}) - y \cdot \nabla \varphi(\bar{x}).
\]
This establishes (3.34), completing the proof.

3.4 Uniqueness of the viscosity solution

Aim of this section is to characterize the value function $V$ as the unique solution to the Hamilton-Jacobi equation (3.12)-(3.13). Toward this goal, we shall establish comparison results stating that
\[
u(x) \leq V(x) \leq u(x)
\]
for all $x \in \mathbb{R}^N$, (3.36)

where $V$ is the value function for the optimal control problem (3.2)-(3.3), while $v$ and $u$ are respectively an upper and a lower solution relative to the stratification (3.1), according to Definitions 3.3.1 and 3.3.2.

For an upper solution $v \geq 0$, a continuity assumption already suffices to achieve the comparison result. For lower solutions, a comparison theorem is valid under stronger assumptions, such as the Lipschitz continuity of the value function. An alternative set of assumptions, somewhat less restrictive than Lipschitz continuity, is the following.

(H3) The function $u$ is Hölder continuous of exponent $1/2$. Moreover, the restriction of $u$ to each manifold $\mathcal{M}_i$ is locally Lipschitz continuous outside a countable union of $C^1$ sub-manifolds of strictly smaller dimension.
Still in connection with lower solutions, we shall need a bound on the growth of $u$ as $|x| \to \infty$.

(H4) Either $u$ is globally bounded or

\[ |u(x)| \leq C_0(1 + |x|), \quad |f_i(x, a)| \leq C_1(1 + |x|), \]

where $C_0$ and $C_1$ are some positive constants, with $C_1 < \beta$.\footnote{Consider (3.7) and that the cost $\ell$ also has a linear growth rate such that $\ell \leq C_\ell(1 + |x|)$, for some $C_\ell > 0$.}

### 3.4.1 The upper solution and the value function

**Theorem 3.4.1.** Consider the optimal control problem (3.2), for the control system (3.3) on a stratified domain. Let the assumptions (H1), (H2) hold. Let $V$ be the value function and let $v$ be a non-negative, continuous upper solution to the HJ equation (3.12)-(3.13). Then

\[ V(x) \leq v(x) \quad x \in \mathbb{R}^N. \quad (3.37) \]

**Proof.** Recalling (3.10), we introduce a new multifunction $\Gamma$ on $\mathbb{R}^{N+1}$, defined as

\[ \Gamma(x, z) = \{ (y, \xi) ; (y, \beta z - \xi) \in G(x), \beta z - \xi \leq K^1(x) \}, \quad (x, z) \in \mathbb{R}^N \times \mathbb{R}. \quad (3.38) \]

By the properties of $G$ it follows that $\Gamma$ is upper semicontinuous, with compact convex, nonempty values. We then consider the differential inclusion

\[ (\dot{x}, \dot{z}) \in \Gamma(x, z). \quad (3.39) \]

Assuming that $v : \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous upper solution of (3.12)-(3.13), we claim that its epigraph

\[ \text{epi}\{v\} = \{ (x, z) \in \mathbb{R}^N \times \mathbb{R} ; \quad z \geq v(x) \} \]

is weakly invariant w.r.t. the differential inclusion (3.39). By a basic viability theorem \cite{4}, to prove this invariance it suffices to check that, at each point $(x, z) \in$
epi\{v\}, one has
\[ \Gamma(x, z) \cap T_{\text{epi}\{v\}}(x, z) \neq \emptyset. \] (3.40)

By \( T_S(p) \) we denote here the Bouligand contingent cone to a set \( S \) at a point \( p \), namely
\[ T_S(p) = \left\{ y \in \mathbb{R}^n ; \liminf_{h \to 0} \frac{d(p + hy; S)}{h} = 0 \right\}. \]

We recall here that the set \( D^-(v(x)) \) of lower differentials to a function \( v \) at a point \( x \) is
\[ D^-(v(x)) = \left\{ p \in \mathbb{R}^n ; \liminf_{y \to 0} \frac{v(x + y) - v(x) - p \cdot y}{|y|} \geq 0 \right\}. \] (3.41)

According to Lemma 3.4.2 below, the nonempty intersection property (3.40) holds at every point \((x, z) \in \text{epi}\{v\}\) if and only if
\[ \beta v(x) + \sup_{(y, \eta) \in G(x)} \left\{ - y \cdot p - \eta \right\} \geq 0 \] (3.42)
for every \( x \in \mathbb{R}^n \) and \( p \in D^-(v(x)) \). This condition holds if \( v \) is an upper solution.

We can thus construct a trajectory \( t \mapsto (x^*(t), z^*(t)) \) of the differential inclusion (3.39), with initial data
\[ (x^*(0), z^*(0)) = (\bar{x}, v(\bar{x})). \]

Consider the set of times
\[ J_i \doteq \left\{ t \geq 0 ; \ x^*(t) \in M_i \right\}. \]

We then have
\[ \dot{x}^*(t) \in T_{M_i}(x^*(t)) \]
for a.e. \( t \in J_i \). By the property (H2), and using Filippov’s measurable selection theorem, we can find measurable control functions \( \alpha_i : J_i \mapsto A_i \) such that
\[ \dot{x}^*(t) = f_i(x^*(t), \alpha_i(t)) , \quad \ell_i(x^*(t), \alpha_i(t)) \leq \beta z^*(t) - \dot{z}^*(t) \]
for a.e. \( t \in J_i \). Setting \( \alpha(t) = \alpha_i(t) \) for \( t \in J_i \), the above implies

\[
\frac{d}{dt} \left[ \int_0^t e^{-\beta_\ell(x^*(s), \alpha(s))} ds + \left[ e^{-\beta_\ell} v(x^*(s)) \right]_0^t \right] \leq 0
\]

for a.e. \( t \geq 0 \). Assuming that \( v(x) \geq 0 \ \forall x \in \mathbb{R}^N \) and letting \( t \to \infty \), we conclude

\[
v(\bar{x}) \geq \lim_{t \to \infty} \int_0^t e^{-\beta_\ell(x^*(s), \alpha(s))} ds \geq V(\bar{x}).
\]

as desired.

---

**Nonsmooth analysis results.** We want to briefly introduce some results and definition in nonsmooth analysis, which will be used in Lemma 3.4.2. The lower Dini semiderivatives \( d^-v(x; f) \) of \( v \) at a point \( x \in \mathbb{R}^N \) in direction \( f \) is

\[
d^-v(x; f) = \liminf_{h \to 0} \frac{v(x + th) - v(x)}{t}, \tag{3.43}
\]

and has relations with tangent space of epi \( \{v\} \) and subdifferential as follows \([21]\).

\[
d^-v(x; f) = \inf\{g \in \mathbb{R} : (f, g) \in T_{\text{epi}\{v\}}(x, v(x))\}. \tag{3.44}
\]

\[
D^-v(x) = \{p \in \mathbb{R}^N : f \cdot p \leq d^-v(x; f), \ \forall f \in \mathbb{R}^N\}. \tag{3.45}
\]

**Lemma 3.4.2.** A solution \( v \) satisfies

\[
T_{\text{epi}\{v\}}(\bar{x}, \bar{z}) \cap \Gamma(\bar{x}, \bar{z}) \neq \emptyset, \ \forall (\bar{x}, \bar{z}) \in \partial\text{epi}\{v\}. \tag{3.46}
\]

if and only if \( v \) satisfies (3.42) at \( \bar{x} \in \mathbb{R}^N \).

**Proof.** Let \( (\bar{x}, \bar{z}) = (\bar{x}, v(\bar{x})) \). Note \( (\bar{x}, v(\bar{x})) \in \partial\text{epi}\{v\}\).

1. (3.46) \( \Rightarrow \) (3.42). Let \( (y, \xi) \in T_{\text{epi}\{v\}}(\bar{x}, \bar{z}) \cap \Gamma(\bar{x}, \bar{z}) \). By (3.44) and (3.38),

\[
\xi \geq d^-v(\bar{x}; y) \ \text{and} \ \xi \leq \beta \bar{z} - \ell_{i(\bar{x})}(\bar{x}, a),
\]
where \( a \in \{ \hat{a} \in A_i(\bar{x}) : f_i(\bar{x}, \hat{a}) = y \} \). Then,

\[
d^-v(\bar{x};y) - \beta v(\bar{x}) + \ell_i(\bar{x}, a) \leq 0.
\] (3.47)

By (3.45), we know \( \langle y, p \rangle \leq d^-v(\bar{x};y) \) for all \( p \in D^-v(\bar{x}) \) and \( y \in \mathbb{R}^N \). Then,

\[
\langle y, p \rangle - \beta v(\bar{x}) + \ell_i(\bar{x}, a) \leq d^-v(\bar{x};y) - \beta v(\bar{x}) + \ell_i(\bar{x}, a) \leq 0.
\] (3.48)

Thus, we get (3.42).

2. (3.46) \( \iff \) (3.42). By Lemma 3.4.3 below, (3.42) is equivalent to

\[
\beta v(\bar{x}) + \sup_{(y, \eta) \in \mathcal{G}(\bar{x})} \left\{ -d^-v(\bar{x};y) - \eta \right\} \geq 0.
\] (3.49)

for a continuous upper solution \( v \).

We can show that (3.49) means (3.46) using Theorem 6.1 in [70].

**CASE 1:** \( d^-v(\bar{x};f) > -\infty \) for all \( f \in F(\bar{x}) \). By (3.44), we have

\[
(f, d^-v(\bar{x};f)) \in T_{epi\{v\}}(\bar{x}, \bar{z}).
\]

From (3.49), we find that

\[
(\hat{f}, d^-v(\bar{x};\hat{f})) \in \Gamma(\bar{x}, \bar{z}),
\]

where \( \hat{f} \) is the velocity, which gives sup in (3.49). Consequently, \( (\hat{f}, d^-v(\bar{x};\hat{f})) \in T_{epi\{v\}}(\bar{x}, \bar{z}) \cap \Gamma(\bar{x}, \bar{z}) \neq \emptyset \).

**CASE 2:** \( d^-v(\bar{x};f) = -\infty \) for some \( f \) such that \( (f, \xi) \in \Gamma(\bar{x}, \bar{z}) \). From (3.44),

\[
(f, g) \in T_{epi\{v\}}(\bar{x}, \bar{z}) \quad \forall \ g \in \mathbb{R}.
\]

Thus, \( T_{epi\{v\}}(\bar{x}, \bar{z}) \cap \Gamma(\bar{x}, \bar{z}) \neq \emptyset \).

\[ \square \]

**Lemma 3.4.3.** (3.42) is equivalent to (3.49) for a continuous upper solution \( v \).

**Proof.** It is obvious that (3.49) means (3.42). We can use the result of Theorem...
Theorem A.1.5 says that if $u$ is an upper solution in the viscosity sense, then (3.42) is satisfied. Here, we want to show that (3.51) is same with (3.49) in our problem. Thus, (3.42) means (3.49). In addition, we point out that the proof of Theorem A.1.5 is still valid even though $F$ is discontinuous because $G$ is upper semicontinuous.

1. Dini subderivate form. We can map the formulations in Theorem A.1.5 to the current context. The Hamilton-Jacobi equation in Theorem A.1.5

$$F(x, u(x), p) = -\beta u(x) - \sup_{(y, \eta) \in G(x)} \left\{ - y \cdot p - \eta \right\}$$  (3.50)

satisfies the main assumptions of (A.2). The alternative condition for the minimax upper solution

$$\inf \left\{ d^- u(x; f) - \langle p, f \rangle + F(x, u(x), p) ; \|f\| \leq \rho(x) \right\} \leq 0, \quad \forall x, p \in \mathbb{R}^N \quad (3.51)$$

corresponds to

$$- \beta u(x) - \sup_{(y, \eta) \in G(x)} \left\{ - d^- u(x; y) - \eta \right\} \leq 0. \quad (3.52)$$

We would like to discuss the relation between (3.51) and (3.52). Here, we narrow the velocity set $\{ f \in \mathbb{R}^N; \|f\| \leq \rho(x) \}$ to $Y(x) \doteq \{ y \in \mathbb{R}^N; (y, \eta) \in G(x) \}$.

$$\inf \left\{ d^- u(x; f) - \langle p, f \rangle + F(x, u(x), p); \|f\| \leq \rho(x) \right\} \quad \Rightarrow \quad \inf_{f \in Y(x)} \left\{ d^- u(x; f) - \langle p, f \rangle + F(x, u(x), p) \right\}.$$  

Then,

$$\inf_{f \in Y(x)} \left\{ d^- u(x; f) - \langle p, f \rangle + F(x, u(x), p) \right\} \quad \left\{ \begin{array}{l}
\leq - \beta u(x) - \sup_{(y, \eta) \in G(x)} \left\{ - d^- u(x; f) - \eta \right\} \\
= \inf_{f \in Y(x)} \left\{ d^- u(x; f) - \langle p, f \rangle \right\} - \beta u(x) - \sup_{(y, \eta) \in G(x)} \left\{ - \langle p, y \rangle - \eta \right\} \\
= - \beta u(x) - \sup_{(y, \eta) \in G(x)} \left\{ - d^- u(x; y) - \eta \right\} \\
\end{array} \right.$$
2. Discontinuity of $F$. In Theorem [A.1.5] a bound estimation of $F(x_\varepsilon, u(x_\varepsilon), q)$ is given by (A.22). We repeat the first inequality of (A.22) here for discussion.

$$F(x_\varepsilon, u(x_\varepsilon), q) \geq F(x_0, u(x_\varepsilon), q) - \gamma^{(2)}_\varepsilon,$$  \hspace{1cm} (3.53)

where $\gamma^{(2)}_\varepsilon \downarrow 0$ as $\varepsilon \downarrow 0$. Subbotin assumes that $F$ is continuous.

In our problem, $F$ is not continuous w.r.t. $x$. This estimation seems not natural. However, by the weak lower semicontinuous assumption (H2) and (3.8),

$$F(x_\varepsilon, z_\varepsilon, p) \geq F(x_0, z_\varepsilon, p),$$  \hspace{1cm} (3.54)

because $\lim_{\varepsilon \to 0} G(x_\varepsilon) \subset G(x_0)$, i.e.

$$- \sup_{(y, \eta) \in G(x_\varepsilon)} \{- y \cdot p - \eta\} \geq - \sup_{(y, \eta) \in G(x_0)} \{- y \cdot p - \eta\}.$$  

Depending on discontinuity, $\geq$ may be $>$. Obviously, it is not the opposite direction, $\leq$ or $<$. Then, we can use the inequality (3.53) even though $F$ is discontinuous w.r.t. $x$. Thus, we can use Theorem [A.1.5] to get the result, i.e. (3.42) means (3.49).

3.4.2 The lower solution and the value function

Theorem 3.4.4. Consider the optimal control problem (3.2), for the control system (3.3) on a stratified domain. Let the assumptions (H1), (H2) hold. Let $V$ be the value function and let $u$ be a lower solution to the HJ equation (3.12)-(3.13). Let the cost functions $\ell_i$ be Lipschitz continuous w.r.t. $x$, so that

$$\left| \ell_i(x, a) - \ell_i(y, a) \right| \leq \text{Lip}(\ell_i) |x - y|, \quad \forall \ x, y \in \mathcal{M}_i, \ a \in A_i,$$ \hspace{1cm} (3.55)

for some Lipschitz constants $\text{Lip}(\ell_i)$. If $u$ satisfies the assumptions (H3) and (H4), then

$$u(x) \leq V(x), \quad x \in \mathbb{R}^N.$$ \hspace{1cm} (3.56)

Proof. For clarity of exposition, we first give a proof assuming that $u, V$ are both locally Lipschitz. Then we mention the minor changes needed in the more general
case where the assumptions (H3) hold.

Fix any point $\bar{x}$ and let $t \mapsto x^*(t)$ be an optimal trajectory, corresponding to the optimal control $t \mapsto \alpha^*(t)$. This will achieve the minimum cost

$$V(\bar{x}) = \int_0^\infty e^{-\beta t} \ell(x^*(t), \alpha^*(t)) \, dt.$$  \hfill (3.57)

In order to show that

$$V(\bar{x}) \geq u(\bar{x}),$$  \hfill (3.58)

for any fixed time interval $[a, b]$ we shall first establish the following basic estimate:

$$e^{-\beta a} u(x^*(a)) - e^{-\beta b} u(x^*(b)) \leq \int_a^b e^{-\beta t} \ell(x^*(t), \alpha^*(t)) \, dt$$  \hfill (3.59)

$$= e^{-\beta a} V(x^*(a)) - e^{-\beta b} V(x^*(b)).$$

To prove (3.59), we consider various cases.

**CASE 1**: For all $t \in ]a, b[$, the trajectory $x^*(\cdot)$ remains inside one single manifold $M_j$ of maximal dimension $N$.

In this case, the estimate (3.59) follows by standard argument. Assume first that $x^*(t) \in M_j$ for all $t \in [a, b]$, i.e., including the end-points of the interval. By our assumptions, the O.D.E.

$$\dot{x}(t) = f_j(x(t), \alpha^*(t))$$  \hfill (3.60)

is Lipschitz continuous w.r.t. $x$ and measurable w.r.t. $t$. Therefore, for each initial condition

$$x(a) = y \in M_j,$$  \hfill (3.61)

the Cauchy problem (3.60)-(3.61) admits a unique solution $t \mapsto x(t, y)$. Moreover, for a suitable Lipschitz constant $L = Lip(f_j)$, the solutions corresponding to different initial data $y, \tilde{y}$ satisfy

$$e^{-L(t-a)} |y - \tilde{y}| \leq |x(t, y) - x(t, \tilde{y})| \leq e^{L(t-a)} |y - \tilde{y}| \quad t \geq a.$$  \hfill (3.62)
Since the function \( u \) is differentiable a.e. on the open set \( \mathcal{M}_j \subseteq \mathbb{R}^N \), we can find a sequence of initial points \( y_n \) and trajectories \( t \mapsto x_n(t) = x(t, y_n) \) such that:

(i) \( y_n \to x^*(a) \), and hence \( x_n(t) \to x^*(t) \) uniformly for \( t \in [a, b] \).

(ii) For each \( n \geq 1 \), the function \( u \) is differentiable at the point \( x_n(t) \), for a.e. \( t \in [a, b] \).

We now compute

\[
e^{-\beta a} u(x_n(a)) - e^{-\beta b} u(x_n(b)) = - \int_a^b \left( \frac{d}{dt} e^{-\beta t} u(x_n(t)) \right) dt
= \int_a^b e^{-\beta t} \left[ \beta u(x_n(t)) - \nabla u(x_n(t)) \cdot f_j(x_n(t), \alpha^*(t)) \right] dt
\leq \int_a^b e^{-\beta t} \ell_j(x_n(t), \alpha^*(t)) dt,
\]

(3.63)
because of the definition of lower solution. Letting \( n \to \infty \) in (3.63) we obtain the desired inequality (3.59).

If now \( x^*(t) \in \mathcal{M}_j \) only for \( t \in ]a, b[ \), we can still apply the above result to the smaller closed interval \([a + \varepsilon, b - \varepsilon]\). This yields

\[
e^{-\beta(a + \varepsilon)} u(x^*(a + \varepsilon)) - e^{-\beta(b - \varepsilon)} u(x^*(b - \varepsilon)) \leq \int_{a+\varepsilon}^{b-\varepsilon} e^{-\beta t} \ell(x^*(t), \alpha^*(t)) dt.
\]

Letting \( \varepsilon \to 0 \) we recover again (3.59).

**CASE 2:** We assume now that \( x^*(a), x^*(b) \in \mathcal{M}_j \), the dimension of \( \mathcal{M}_j \) is \( d_j = N - 1 \), and moreover the trajectory \( t \mapsto x^*(t) \) remains either inside \( \mathcal{M}_j \) or inside other manifolds of dimension \( N \), for all \( t \in [a, b] \).

Using a local chart, we can assume that

\[
\mathcal{M}_j = \{ (x_1, \ldots, x_N) \in \mathbb{R}^N ; \ x_N = 0 \}.
\]

(3.64)

By continuity, \( x^*(\cdot) \) leaves \( \mathcal{M}_j \) and enters some other \( N \)-dimensional manifold \( \mathcal{M}_k \).
on an open set of times, say

$$\{ t \in [a, b] ; \ x^*(t) \not\in \mathcal{M}_j \} = \bigcup_{i \in I} [a_i, b_i].$$

Here $I$ is a finite or countable set of indices.

For every $i \in I$, by the analysis in Case 1 we already know that

$$e^{-\beta a_i} u(x^*(a_i)) - e^{-\beta b_i} u(x^*(b_i)) \leq e^{-\beta a_i} V(x^*(a_i)) - e^{-\beta b_i} V(x^*(b_i)) \quad (3.65)$$

A further estimate will be needed. For each $i \in I$, by the assumption (H1) of Lipschitz continuity of the functions $f_j$, $f_k$, and by the assumption (H2') of upper semicontinuity of the velocity sets, we have

$$\overline{\co} \left( \bigcup_{t \in [a_i, b_i]} F_k(x^*(t)) \right) \cap T_{\mathcal{M}_j} \subseteq B \left( F(x^*(a_i)), L(b_i - a_i) \right), \quad (3.66)$$

for some Lipschitz constant $L$. Here $B(S, r)$ denotes the closed neighborhood of radius $r$ around the set $S$.

By (3.66) we can choose a constant control $\alpha_{j,i} \in A_j$ such that

$$\left| f_j(x^*(a_i), \alpha_{j,i}) - \frac{1}{b_i - a_i} \int_{a_i}^{b_i} f_k(x^*(t), \alpha^*(t)) \, dt \right| \leq L(b_i - a_i) \quad (3.67)$$

observing that

$$f_j(x^*(a_i), \alpha_{j,i}) \in F(x^*(a_i)),$$

$$\frac{x^*(b_i) - x^*(a_i)}{b_i - a_i} = \frac{1}{b_i - a_i} \int_{a_i}^{b_i} f_k(x^*(t), \alpha^*(t)) \, dt \in \overline{\co} \left( \bigcup_{t \in [a_i, b_i]} F_k(x^*(t)) \right) \cap T_{\mathcal{M}_j}.$$

Moreover, by the Jensen’s inequality and the Lipschitz continuity of the cost function $\ell_k$, we can also achieve

$$\ell_j(x^*(a_i), \alpha_{j,i}) \leq \frac{1}{b_i - a_i} \int_{a_i}^{b_i} \ell_k(x^*(t), \alpha^*(t)) \, dt + C(b_i - a_i). \quad (3.68)$$
Using the constant control $\alpha_{j,i}$ on the whole interval $[a_i, b_i]$, the solution of 
\[
\dot{x}(t) = f_j(x(t), \alpha_{j,i}) \quad \quad x(a_i) = x^*(a_i),
\]
satisfies 
\[
y_i \doteq x(b_i) = x^*(a_i) + f_j(x^*(a_i), \alpha_{j,i})(b_i - a_i) + \mathcal{O}(1)(b_i - a_i)^2,
\]
and $y_i \in \mathcal{M}_j$. Hence, setting 
\[
v_i \doteq x^*(b_i) - y_i
\]
we have 
\[
|v_i| \leq \kappa (b_i - a_i)^2. \quad (3.69)
\]
for some constant $\kappa$, uniformly valid for all $i \in I$.

We are now ready to define a family of perturbed trajectories. Define the control function 
\[
\alpha^\dagger(t) \doteq \begin{cases} 
\alpha^*(t) & \text{if } t \notin \bigcup_i [a_i, b_i[, \\
\alpha_{j,i} & \text{if } t \in ]a_i, b_i[ \text{ for some } i \in I.
\end{cases} \quad (3.70)
\]

For each initial point $y \in \mathcal{M}_j$ close to $x^*(a)$, let $t \mapsto x(t, y)$ be the solution of the impulsive Cauchy problem 
\[
\dot{x}(t) = f_j(x(t), \alpha^\dagger(t)) \\
x(0) = y, \quad x(b_i) = x(b_i-) + v_i.
\]

The figure 3.3 illustrates the solution. Notice that this trajectory is unique, because it corresponds to the unique fixed point of the integral transformation $x(\cdot) \mapsto \hat{T}x(\cdot)$, defined as 
\[
\hat{T}x(t) = y + \int_0^t f_j(x(s), \alpha^\dagger(s)) \, ds + \sum_{b_i \leq t} v_i.
\]
As in Case 1, we can select a sequence of initial points \( y_n \), with corresponding trajectories \( x(\cdot, y_n) \), such that

(i) As \( n \to \infty \), one has \( y_n \to x^*(a) \), and hence \( x_n(t) \to x^*(t) \) uniformly for \( t \in [a, b] \).

(ii) For each \( n \geq 1 \), the restriction of \( u \) to \( M_j \) is differentiable at the point \( x_n(t) \), for a.e. \( t \in [a, b] \).

Using (3.68) and (3.69) we now compute

\[
e^{-\beta a}u(x_n(a)) - e^{-\beta b}u(x_n(b)) = - \int_a^b \left[ \frac{d}{dt} e^{-\beta t}u(x_n(t)) \right] dt - \sum_{i \in I} e^{-\beta b_i} \left[ u(x_n(b_i)) - u(x_n(b_i -)) \right] \\
\leq \int_a^b e^{-\beta t} \left[ \beta u(x_n(t)) - \nabla u(x_n(t)) \cdot f_j(x_n(t), \alpha^\dagger(t)) \right] dt \\
+ \sum_{i \in I} e^{-\beta b_i} L_u \kappa(b_i - a_i)^2 \\
\leq \int_a^b e^{-\beta t} \ell_j(x_n(t), \alpha^\dagger(t)) dt + \sum_{i \in I} e^{-\beta b_i} L_u \kappa(b_i - a_i)^2 \\
\leq \int_a^b e^{-\beta t} \ell(x_n(t), \alpha^*(t)) dt + \sum_{i \in I} e^{-\beta a_i} C(b_i - a_i)^2 \\
+ \sum_{i \in I} e^{-\beta b_i} L_u \kappa(b_i - a_i)^2 ,
\]

(3.71)

because \( u \) is a lower solution. The Lipschitz constant of \( u \) is denoted by \( L_u \).

Let \( \varepsilon > 0 \) be given. Choose a finite subset of indices \( I' \subset I \) such that

\[
\sum_{i \in I \setminus I'} e^{-\beta a_i} C(b_i - a_i)^2 + \sum_{i \in I \setminus I'} e^{-\beta b_i} L_u \kappa(b_i - a_i)^2 < \varepsilon .
\]

To fix the ideas, let \( I' = \{1, \ldots, \nu\} \), with

\[
a \leq a_1 < b_1 < a_2 < b_2 < \ldots < a_\nu < b_\nu \leq b.
\]
We can now use the estimate \((3.65)\) on each of the subintervals \([a_k, b_k] , k = 1, \ldots, \nu,\) and an estimate of the form \((3.70)\) on the remaining finitely many intervals 
\[[a, a_1], [b_1, a_2], \ldots, [b_{\nu}, b] .\]

Setting for convenience \(b_0 = a, a_{\nu+1} = b,\) we thus obtain
\[
e^{-\beta a} u(x_n(a)) - e^{-\beta b} u(x_n(b)) = \sum_{k=1}^{\nu+1} \left( e^{-\beta b_{k-1}} u(x_n(b_{k-1})) - e^{-\beta a_k} u(x_n(a_k)) \right) + \sum_{k=1}^{\nu} \left( e^{-\beta a_k} u(x_n(a_k)) - e^{-\beta b_k} u(x_n(b_k)) \right).
\]

(3.72)

\[
\leq \int_a^b e^{-\beta t} \ell(x_n(t), \alpha^*(t)) \, dt + \sum_{i \in I \setminus I'} e^{-\beta a_i} C(b_i - a_i)^2 + \sum_{i \in I \setminus I'} e^{-\beta b_i} L_u \kappa(b_i - a_i)^2.
\]

Letting \(n \to \infty,\) from \((3.72)\) it follows
\[
e^{-\beta a} u(x^*(a)) - e^{-\beta b} u(x^*(b)) \leq \int_a^b \ell(x^*(t), \alpha^*(t)) \, dt + \sum_{i \in I \setminus I'} e^{-\beta a_i} C(b_i - a_i)^2 + \sum_{i \in I \setminus I'} e^{-\beta b_i} L_u \kappa(b_i - a_i)^2
\]
\[
\leq e^{-\beta a} V(x^*(a)) - e^{-\beta b} V(x^*(b)) + \varepsilon .
\]

(3.73)

Since \(\varepsilon > 0\) was arbitrary, once again we obtain the basic inequality \((3.59)\).

\section*{Case 3:}
During the interval \([a, b]\) the optimal trajectory \(x^*(\cdot)\) remains inside manifolds of dimension \(N\) or \(N - 1.\)

This is a slight generalization of the previous case. The validity of \((3.59)\) is clear, observing that we can find finitely many times \(a = t_0 < t_1 < \cdots < t_n = b\) such that the restriction of \(x^*\) to each subinterval \([t_{i-1}, t_i]\) satisfies the conditions in Case 2.

\section*{Case 4:}
We now assume that the estimate \((3.59)\) holds whenever the optimal
trajectory \( x^*(\cdot) \) remains on manifolds of dimension \( \geq m + 1 \), and prove that it still holds when \( x^*(\cdot) \) stays on manifolds of dimension \( \geq m \). By induction, this will establish (3.59) in the general case.

The proof of this inductive step relies on the same ideas used in Case 2. We thus only sketch the main lines.

Assume that \( x^*(\cdot) \) remains inside a manifold \( M_j \) of dimension \( d_j = m \), or other manifolds of strictly higher dimension. Using a local chart, we can assume that

\[
M_j = \{ (x_1, \ldots, x_N) \in \mathbb{R}^N ; \quad x_i = 0 \quad i = m + 1, \ldots, N \}.
\]

By continuity, we again have

\[
\{ t \in [a, b] ; \quad x^*(t) \notin M_j \} = \bigcup_{i \in I} [a_i, b_i],
\]

where \( I \) is a finite or countable set of indices. For every \( i \in I \), by the inductive assumption we still have (3.65). Furthermore, for each subinterval \([a_i, b_i]\) we can find a control \( \alpha_{j,i} \) such that (3.67) and (3.68) hold. We thus define the control \( \alpha \) as in (3.70), choose a sequence of trajectories \( x_n = x(y_n, \alpha) \) and retrace all steps (3.71)–(3.73). This concludes the proof of (3.59).

We now conclude the proof of (3.58). For any given initial condition \( \bar{x} \), let \( \alpha^*(\cdot) \) and \( x^*(\cdot) \) be a corresponding optimal control and optimal trajectory. For every \( T > 0 \), using (3.59) on the interval \([0, T]\) we find

\[
u(\bar{x}) \leq V(\bar{x}) + e^{-\beta T} u(x^*(T)). \tag{3.74}
\]

Letting \( T \to \infty \), by (H4) we have

\[
e^{-\beta T} u(x^*(T)) \to 0.
\]

Finally, we observe that the above result remains valid if the assumption of Lipschitz continuity of the lower solution \( u \) is replaced by the assumption (H3). The proof would go through as before, except that the last term in (3.71) would
Figure 3.3. The solution of the impulsive Cauchy problem

be replaced by

$$\sum_{i \in I} e^{-\beta b_i} L_u \kappa(b_i - a_i).$$

Now $L_u$ denotes the Hölder constant of $u$. This estimate suffices to complete the remainder of the proof.

From the above comparison theorems one immediately obtains a uniqueness result:

**Corollary 3.4.5.** Consider the optimal control problem (3.2), for the control system (3.3) on a stratified domain (3.1). Let the assumptions (H1), (H2), (H4) and (3.55) hold. Let’s assume that the value function $V$ satisfies the regularity assumptions (H3). Then $V$ is the unique non-negative solution to the HJ equation (3.12)-(3.13) with such regularity properties.

**Remark 3.4.1**

All the results in this chapter remain valid in the more general case where we allow the control set $A_i$ to be empty, i.e. $A_i = \emptyset$, on some manifold $M_i$ of dimension $d_i < N$.

Notice that, in this case, there is no control which keeps the system inside $M_i$. The assumption (H2) now implies that

$$\left\{ (y, \eta) \in G(x) : y \in T_{M_i(x)}(x) \right\} = \emptyset,$$

for all $x \in M_i$. In particular, this means that all trajectories cross the manifold $M_i$ transversally, spending a zero amount of time inside $M_i$. 
Viscosity Solution for the Deterministic Reflecting Boundary-Cost Problems

If we fix the environmental variable to an environmental state, the stochastic reflecting boundary-cost (RBC) problem in section 1.2 is an instance of the OCPSD in chapter 3. In this chapter, we discuss the application of the result in chapter 3 to the deterministic RBC problem.

4.1 Existence of Optimal Control

We drop the environmental variable, $z$, in the RBC problem (1.7)-(1.18) for simplicity. This corresponds to a system with single environmental state.

Now, we check whether the problem condition satisfies the assumptions in chapter 3. In the case of convex domains with piecewise smooth boundary, a problem, whose data satisfy (H0) in section 1.2 with the following additional condition (H0-1), fits naturally into OCPSD.

(H0-1) We can form a stratification, $\mathcal{M} = \mathcal{M}_1$ and $\partial \mathcal{M} = \mathcal{M}_2 \cup \cdots \cup \mathcal{M}_M$, such that the following conditions are satisfied:
1. \( F_i(x)(x) \) is convex for all \( x \in \bar{M} \).

\[
F_i(x)(x) = \{ f(x, a) - n(x, a) ; a \in A_{i(x)} \},
\]

where \( A_{i(x)} = \{ a \in A ; f(x, a) - n(x, a) \in T_{\mathcal{M}_i(x)}, \forall x \in \mathcal{M}_i(x) \} \). We note that \( n(x, a) = 0 \) if \( x \in \mathcal{M}_1 \) and \( a \in A_1 = A \).

2. The deparameterized cost function, \( L(x, p) \), is convex in \( p \in F_i(x)(x) \) for all \( x \in \bar{M} \),

\[
L(x, p) = \min \{ c(x, a) + b(x, n(x, a)) ; f_i(x)(x, a) = p, a \in A_{i(x)} \}.
\]

We note that \( b(x, 0) = 0 \) and \( a \mapsto n(x, a) \) is continuous because \( N_{\bar{M}}(x) \) is a closed convex set. For all \( x \in \bar{M}, c(x, a) \) and \( b(x, n) \) is convex in \( a \in A \) and \( n \in N_{\bar{M}}(x) \) respectively.

In order to retain the whole space \( \mathbb{R}^N \) as domain for the control system, it suffices to choose a cost \( c(x, a) \) very large when \( x \notin \bar{M} \). This will force the solution of the optimization problem to remain inside \( \bar{M} \) at all times. Thus, (H1) and (H2) in chapter 3 are satisfied.

The case where boundary reflection occurs at no additional cost, i.e. \( b \equiv 0 \), has been studied in the literature as the Skorokhod problem [51, 62].

By Theorem 3.2.1, the optimal control exists for the deterministic RBC problem. We summarize this as the following lemma.

**Lemma 4.1.1.** Suppose \( f \) and \( \ell \) satisfies the assumption (H0) in section 1.2 and (H0-1). Then, the deterministic RBC satisfies (H1) and (H2). In addition, the optimal control exists for the deterministic RBC problem.

### 4.2 Continuity of the Value Function

In this chapter, we characterize continuity of the value function. Firstly, we present the result without a special local controllability assumption (S3), which will be introduced in Section 4.2.3. For the simple presentation, we use \( x_t \) \((y_t, \alpha_t)\) instead of \( x(t) \) \((y(t), \alpha(t))\) in this section.
In this section, we prove that the value function is Hölder continuous with exponent $\gamma$ such that

$$
\gamma =
\begin{cases} 1 & \text{if } \beta > L_f, \\ 
\text{any } \gamma < 1 & \text{if } \beta = L_f, \\ 
\beta / L_f & \text{if } \beta < L_f.
\end{cases}
$$

(4.1)

We get (4.1) for the following two cases (cf. (H0))

CASE 1. A general convex boundary and a linear overflow cost,

CASE 2. A rectangular boundary and a general overflow cost.

For the proof of the above two cases, we use a growth estimation

$$
|y_t - x_t| \leq e^{Lt} |\bar{y} - \bar{x}|,
$$

(4.2)

where $x$ and $y$ are Lipschitz mappings with different starting points, $x_0 = \bar{x}$ and $y_0 = \bar{y}$ and $\bar{x}, \bar{y} \in \bar{M}$. This estimation can be obtained from Gronwall inequality (cf. Theorem 5.5 [6]).

Section 4.2.1 and 4.2.2 deal with CASE 1 and CASE 2 respectively.

### 4.2.1 A General Boundary and a Linear Overflow Cost

Bardi and Capuzzo-Dolcetta [6] made similar results when $\mathcal{M} = \mathbb{R}^N$ and there is no overflow cost.

**Theorem 4.2.1** (CASE 1). In the case of a linear overflow cost and general piecewise smooth boundary, the value function $v(x)$ is Hölder continuous with the exponent $\gamma$ in (4.1).

**Proof.** Let $x$ and $y$ be the trajectories generated by a control $\alpha$ with $x_0 = \bar{x}$ and $y_0 = \bar{y}$ respectively. Let $n^x_t$ and $n^y_t$ be the reflecting forces for $x_t$ and $y_t$. We determine $\alpha$ such that $\epsilon > 0$ satisfies

$$
v(\bar{y}) \geq \int_{0}^{\infty} e^{-\beta t} \left\{ c(y_t, \alpha_t) + b(y_t) \cdot n^y_t \right\} dt - \epsilon, \quad \text{and}
$$

$$
v(\bar{x}) \leq \int_{0}^{\infty} e^{-\beta t} \left\{ c(x_t, \alpha_t) + b(x_t) \cdot n^x_t \right\} dt.
$$
Then, for an arbitrary $T > 0$,\
\[
|v(\bar{x}) - v(\bar{y})| \leq \left| \int_0^T e^{-\beta t} \{ c(x_t, \alpha_t) + b(x_t) \cdot n_t^x - c(y_t, \alpha_t) - b(y_t) \cdot n_t^y \} \, dt \right| + \int_T^\infty e^{-\beta t} (L_c + L_b L_f) \, dt + \epsilon
\]

(4.3)\

The second term in the right hand side of (4.3) is obtained from (1.14), (1.16) and\
\[
|n_t^x| = |\pi_{N_{x_t}} f(x_t, \alpha_t)| \leq |f(x_t, \alpha_t)| \leq L_f.
\]

(4.4)\

For simplicity, we introduce a map, $u \mapsto T(u)$ for $u \in L^1([0,\infty))$, which is defined as\
\[
T(u_t) = \int_0^T e^{-\beta \tau} u_\tau \, d\tau.
\]

(4.5)\

We observe that\
\[
\left| T\left( c(x_t, \alpha_t) - c(y_t, \alpha_t) \right) \right| \leq T\left( L_c |x_t - y_t| \right) \leq L_c T(e^{L_f t}) |\bar{x} - \bar{y}|,
\]

(4.6)\

and\
\[
\left| T\left( b(x_t) \cdot n_t^x - b(y_t) \cdot n_t^y \right) \right| = \left| T\left( [b(x_t) - b(y_t)] \cdot n_t^x + b(y_t) \cdot [n_t^x - n_t^y] \right) \right| \\
\leq T\left( L_b |x_t - y_t| |n_t^x| \right) + \left| T\left( b(y_t) \cdot \{ f(x_t, \alpha_t) - f(y_t, \alpha_t) - \dot{x}_t + \dot{y}_t \} \right) \right| \\
\leq 2L_f L_b T(e^{L_f t}) |\bar{x} - \bar{y}| + \left| T\left( b(y_t) \cdot (\dot{y}_t - \dot{x}_t) \right) \right| .
\]

(4.7)\

Let us consider integration-by-part of the second term of the last equation in (4.7). Let $u = e^{-\beta t} b(y_t)$ and $v' = \dot{y}_t - \dot{x}_t$.\
\[
\left| T\left( b(y_t) \cdot (\dot{y}_t - \dot{x}_t) \right) \right| = \left| \int_0^T e^{-\beta \tau} b(y_\tau) \cdot (\dot{y}_\tau - \dot{x}_\tau) \, d\tau \right| \\
= \left| e^{-\beta T} L_b e^{L_f T} + L_b \right| |\bar{x} - \bar{y}| + \int_0^T e^{-\beta \tau} (L_b L_f + \beta L_b) e^{L_f \tau} |\bar{x} - \bar{y}| \, d\tau \\
\leq \left\{ L_b (e^{(L_f - \beta) T} + 1) + (L_b L_f + \beta L_b) T(e^{L_f t}) \right\} |\bar{x} - \bar{y}|,
\]

(4.8)
We note that
\[ T(e^{L_f T}) = \begin{cases} \frac{1}{L_f - \beta} (e^{(L_f - \beta)T} - 1), & \text{if } \beta \neq L_f, \\ T, & \text{if } \beta = L_f. \end{cases} \]  
(4.9)

Then, from (4.6), (4.7) and (4.8)
\[ |v(x) - v(y)| \leq \left[ L_b \left( e^{(L_f - \beta)T} + 1 \right) + AT \left( e^{L_f T} \right) \right] |x - y| + M e^{-\beta T} + \epsilon, \]  
(4.10)
where \( A = L_c + 3L_f L_b + \beta L_b \) and \( M = 2(L_c + L_b L_f)/\beta \).

We can consider three different cases from (4.10) as in Proposition 2.1 p99 [6].

1. If \( \beta > L_f \), letting \( T \to \infty \) and \( \epsilon \to 0 \), we get
\[ |v(x) - v(y)| \leq \left[ L_b + \frac{A}{\beta - L_f} \right] |x - y|. \]  
(4.11)

The value function \( v(x) \) is Lipschitz continuous, i.e., \( \gamma = 1 \).

2. If \( \beta < L_f \), let \( |x - y| < 1 \) and \( T = -\frac{1}{L_f} \ln |x - y| \). Then,
\[ T(e^{L_f t}) = \frac{1}{L_f - \beta} \left( |x - y|^{\frac{\beta}{L_f}} - 1 \right), \]  
and
\[ |v(x) - v(y)| \leq \left( \frac{A}{L_f - \beta} + L_b + M \right) |x - y|^{\frac{\beta}{L_f}} + \left( L_b - \frac{A}{L_f - \beta} \right) |x - y| + \epsilon \]  
(4.12)
\[ \leq C_3 |x - y|^{\frac{\beta}{L_f}}, \quad \text{as } \epsilon \to 0, \]
where \( C_3 \) is a positive constant. Thus, The value function \( v(x) \) is Hölder continuous with \( \gamma = \beta/L_f \).

3. If \( \beta = L_f \), we consider \( |x - y| < 1 \) without loss of generality. Let \( T = -\frac{1}{\beta} \ln |x - y| \). Then,
\[ |v(x) - v(y)| \leq \frac{-A}{\beta} |x - y| \ln |y - x| + (2L_b + M) |x - y| + \epsilon \]  
\[ \leq C_4 |x - y| |\ln |x - y||, \quad \text{as } \epsilon \to 0, \]  
(4.13)
where $C_4$ is a positive constant. We note that when $0 < z < 1$,

$$\left| \ln z \right| \leq \frac{1}{z^\varepsilon} \quad \forall \varepsilon > 0.$$ 

Therefore, the value function $v(x)$ is Hölder continuous with $\gamma < 1$. 

\[ \square \]

### 4.2.2 A Rectangular State Space and a General Overflow Cost

Generally, we cannot easily expect strong convergence of derivatives in $L^1$ space. However, in rectangular state spaces, we can find a bound estimation on the derivatives by the initial data.

We consider a rectangular state space such that

$$\tilde{\mathcal{M}} \doteq \{ x \in \mathbb{R}^N; 0 \leq x_i \leq B_i, \quad B_i > 0, \quad i = 1, \cdots, N \}.$$  \hfill (4.14)

In order to prove the strong convergence of $\dot{x}_t$ to $\dot{y}_t$ as $r \to 0$, $|\bar{x} - \bar{y}| < r$, we introduce a function,

$$\psi_t = \sum_{i=1}^N |x_{t,i} - y_{t,i}|,$$  \hfill (4.15)

where $x_t = (x_{t,1}, \cdots, x_{t,N})$ and $y_t = (y_{t,1}, \cdots, y_{t,N})$.

**Lemma 4.2.2.** Suppose (H0) and (H0-1). If the state space, $\tilde{\mathcal{M}}$, is rectangular as in (4.14) and the overflow cost is general, then,

$$\frac{d}{dt} \psi_t \leq NL_f |x_t - y_t| - |n^x_t - n^y_t|$$  \hfill (4.16)

**Proof.** Let $\Delta_i = \frac{x_{t,i} - y_{t,i}}{|x_{t,i} - y_{t,i}|}$, $i = 1, \cdots, N$. $\Delta_i$ is -1, 0, or 1.

$$\frac{d}{dt} \psi_t = \sum_{i=1}^N \frac{(x_{t,i} - y_{t,i})}{|x_{t,i} - y_{t,i}|} (\dot{x}_{t,i} - \dot{y}_{t,i}) = \Delta \cdot (\dot{x}_t - \dot{y}_t)$$

$$= \Delta \cdot (f(x_t, \alpha_t) - f(y_t, \alpha_t) - (n^x_t - n^y_t))$$

$$\leq NL_f |x_t - y_t| - \sum_{i=1}^N \Delta_i (n^x_{t,i} - n^y_{t,i})$$  \hfill (4.17)
We argue that
\[
\sum_{i=1}^{N} \Delta_i (n_{t,i}^x - n_{t,i}^y) = \sum_{i=1}^{N} |n_{t,i}^x - n_{t,i}^y| \geq |n_t^x - n_t^y|.
\] (4.18)

We observe that \( n_t^x \) (or \( n_t^y \)) has nonzero value only when \( x \in \partial M \) (or \( y \in \partial M \)). When \( x_{t,i} = B_i \) (or \( y_{t,i} = B_i \)), \( n_{t,i}^x \geq 0 \) (or \( n_{t,i}^y \geq 0 \)). When \( x_{t,i} = 0 \) (or \( y_{t,i} = 0 \)), \( n_{t,i}^x \leq 0 \) (or \( n_{t,i}^y \leq 0 \)).

Let \( \delta_i = \Delta_i (n_{t,i}^x - n_{t,i}^y) \). We observe the followings:

- If \( x_{t,i} = y_{t,i} = B_i \) or 0, then \( \Delta_i = 0 \). \( \delta_i = 0 \).
- If \( x_{t,i} = B_i \) and \( y_{t,i} \leq B_i \), then \( \Delta_i = 1 \) and \( n_{t,i}^x \geq 0 \). \( \delta_i \geq 0 \).
- If \( x_{t,i} \leq B_i \) and \( y_{t,i} = B_i \), then \( \Delta_i = -1 \) and \( -n_{t,i}^y \leq 0 \). \( \delta_i \geq 0 \).
- If \( x_{t,i} = 0 \) and \( y_{t,i} > 0 \), then \( \Delta_i = -1 \) and \( -n_{t,i}^x \leq 0 \). \( \delta_i \geq 0 \).
- If \( x_{t,i} > 0 \) and \( y_{t,i} = 0 \), then \( \Delta_i = 1 \) and \( -n_{t,i}^y \geq 0 \). \( \delta_i \geq 0 \).

We do not consider the case \( x_{t,i} = B_i \) and \( y_{t,i} = 0 \) or \( x_{t,i} = 0 \) and \( y_{t,i} = B_i \) because the two trajectories depend on the distance between two starting points and it’s enough to observe two trajectories sufficiently close to each other. From all the above cases, \( \delta_i \geq 0 \), and \( \delta_i = |\Delta_i (n_{t,i}^x - n_{t,i}^y) | \). Thus, (4.18) is proved. From (4.17),
\[
\frac{d}{dt} \psi_t \leq N L f |x_t - y_t| - |n_t^x - n_t^y|.
\] (4.19)

\[\square\]

**Theorem 4.2.3** (Strong convergence of derivatives). Suppose (H0) and (H0-1). If the state space, \( \bar{M} \), is rectangular as in (4.14) and the overflow cost is general, then,
\[
|\dot{x}_t - \dot{y}_t| \leq C |x_t - y_t| - \frac{d}{dt} \psi_t.
\] (4.20)

In addition,
\[
\int_{0}^{t} |\dot{x}_s - \dot{y}_s| \, ds \to 0 \quad \text{as} \quad |\bar{x} - \bar{y}| \to 0.
\] (4.21)
Proof.
\[
|\dot{x}_t - \dot{y}_t| = \left| f(x_t, \alpha_t) - f(y_t, \alpha_t) + n^x_t - n^y_t \right| \leq L_f |x_t - y_t| + |n^x_t - n^y_t|
\]
\[
\leq L_f |x_t - y_t| + NL_f |x_t - y_t| - \frac{d}{dt} \psi_t \quad \text{from (4.16)} \tag{4.22}
\]
\[
\leq (1 + N)L_f e^{L_f t} |\bar{x} - \bar{y}| - \frac{d}{dt} \psi_t.
\]

Then,
\[
\int_0^t |\dot{x}_s - \dot{y}_s| ds \leq \int_0^t (1 + N)L_f e^{L_f s} |\bar{x} - \bar{y}| ds - \int_0^t \frac{d}{ds} \psi(s) ds
\]
\[
\leq (1 + N) (e^{L_f t} - 1) |\bar{x} - \bar{y}| + \sum_{i=1}^N |\bar{x}_i - \bar{y}_i| \tag{4.23}
\]
\[
\leq \left\{ (1 + N) (e^{L_f t} - 1) + \sqrt{N} \right\} |\bar{x} - \bar{y}|.
\]

Thus, we can get (4.21).

Now, we get the result of the problem with a rectangular state space and a general Lipschitz continuous cost.

**Theorem 4.2.4 (CASE 2).** In the case of a general Lipschitz continuous overflow cost and a rectangular state space, the value function \( v(x) \) is Hölder continuous with the exponent \( \gamma \) in (4.1).

**Proof.** We can fix \( \varepsilon \) and \( \alpha \) as in Theorem 4.2.1. Then, by (1.15) and (1.17),
\[
|v(\bar{x}) - v(\bar{y})| \leq \left| \int_0^T e^{-\beta t} \{ c(x_t, \alpha_t) + b(x_t, n^x_t) - c(y_t, \alpha_t) - b(y_t, n^y_t) \} dt \right|
\]
\[
+ \int_T^\infty e^{-\beta t} (L_c + K_b) dt + \varepsilon \tag{4.24}
\]

An example of \( b(x_t, n^x_t) \) is \( b(x_t) \cdot g(n^x_t) \), where \( g \) is locally Lipschitz continuous.
\[
\left| b(x_t) \cdot g(n^x_t) - b(y_t) \cdot g(n^y_t) \right| \leq \left| b(x_t) - b(y_t) \right| \cdot \left| g(n^x_t) \right| + \left| b(y_t) \cdot [g(n^x_t) - g(n^y_t)] \right|
\]
We observe that
\[
T \left( |b(x_t, n_t^x) - b(y_t, n_t^y)| \right) \leq T \left( L_{bx} |x_t - y_t| + L_{bn} |n_t^x - n_t^y| \right) \\
\leq T \left( L_{bx} |x_t - y_t| \right) + T \left( L_{bn} \left[ NL_f |x_t - y_t| - \frac{d}{dt} \psi_t \right] \right) \quad \text{by (4.16)}
\]
\[
\leq L_{bx} T \left( e^{L_f t} |\bar{x} - \bar{y}| \right) + L_{bn} N L_f T \left( e^{L_f t} \right) |\bar{x} - \bar{y}| - L_{bn} T \left( \frac{d}{dt} \psi_t \right) \\
\leq \left\{ (L_{bx} + L_{bn} N L_f) T \left( e^{L_f t} \right) + L_{bn} \sqrt{N} \right\} |\bar{x} - \bar{y}|.
\]

We note that
\[
- T \left( \frac{d}{dt} \psi_t \right) = - \int_0^T e^{-\beta t} \frac{d}{dt} \psi_t dt = - [e^{-\beta t} \psi_t]_0^T - \int_0^T e^{-\beta t} \psi_t dt \\
\leq \psi_0 = \sum_{i=1}^N |\bar{x}_i - \bar{y}_i| \leq \sqrt{N} |\bar{x} - \bar{y}|.
\]

Then, we get from (4.6) and (4.25)
\[
|v(\bar{x}) - v(\bar{y})| \leq \left\{ A T \left( e^{L_f t} \right) + L_{bn} \sqrt{N} \right\} |\bar{x} - \bar{y}| + M e^{-\beta T} + \epsilon,
\]
where \(A = L_c + L_{bx} + L_{bn} N L_f\) and \(M = 2(L_c + K_b)/\beta\).

We can consider three different cases also as in Theorem 4.2.1:

1. if \(\beta > L_f\), letting \(T \to \infty\) and \(\epsilon \to 0\), we get
\[
|v(\bar{x}) - v(\bar{y})| \leq \left\{ \frac{A}{\beta - L_f} + L_{bn} \sqrt{N} \right\} |\bar{x} - \bar{y}|.
\]

The value function \(v(x)\) is Lipschitz continuous.

2. if \(\beta < L_f\), let \(|\bar{x} - \bar{y}| < 1\) and \(T = -\frac{1}{L_f} \ln |\bar{x} - \bar{y}|\). Then,
\[
|v(\bar{x}) - v(\bar{y})| \leq \left( \frac{A}{L_f - \beta} + M \right) |\bar{x} - \bar{y}|^{\frac{\beta}{L_f}} \\
+ \left( L_{bn} \sqrt{N} - \frac{A}{L_f - \beta} \right) |\bar{x} - \bar{y}| + \epsilon \\
\leq C_3 |\bar{x} - \bar{y}|^{\frac{\beta}{L_f}}, \quad \text{as} \quad \epsilon \to 0,
\]
where $C_3$ depends on $\beta$, $L_f$ and $A$. Thus, the value function $v(x)$ is Hölder continuous with $\gamma = \beta/L_f$.

3. if $\beta = L_f$,

$$|v(\bar{x}) - v(\bar{y})| \leq AT |\bar{x} - \bar{y}| + L_{bn} \sqrt{N} |\bar{x} - \bar{y}| + Me^{-\beta T} + \epsilon. \quad (4.30)$$

We consider $|\bar{x} - \bar{y}| < 1$ without loss of generality. Let $T = -\frac{1}{\beta} \ln |\bar{x} - \bar{y}|$ and $\epsilon \to 0$. Then,

$$|v(\bar{x}) - v(\bar{y})| \leq \frac{A}{\beta} |\bar{x} - \bar{y}| \ln |\bar{x} - \bar{y}| + \left( L_{bn} \sqrt{N} + M \right) |\bar{x} - \bar{y}|$$

$$\leq \frac{A}{\beta} |\bar{x} - \bar{y}|^{1-\epsilon} + \left( L_{bn} \sqrt{N} + M \right) |\bar{x} - \bar{y}|^{1-\epsilon}, \quad \epsilon > 0$$

$$\leq C_2 |\bar{x} - \bar{y}|^{\gamma}, \quad \gamma = 1 - \epsilon < 1. \quad (4.31)$$

Therefore, the value function $v(x)$ is Hölder continuous with $\gamma < 1$.

\[\square\]

### 4.2.3 Locally Lipschitz Continuous Value Function

In the previous two sections, we gave characterizations of uniform continuity of value function without local controllability assumption. Now we want to show that the value function is locally Lipschitz continuous with a stronger geometric assumption (S3) in [68].

Let $\Sigma$ be a closed target set in $\bar{M}$ and $B$ be an open unit ball in $\mathbb{R}^N$.

**S3** There exist $\delta_1 > 0$, $\delta_2 > 0$ such that for each $x \in \bar{M} \cap \Sigma^c \cap \{\Sigma + \delta_1 B\}$, there exists a control $u(x) \in U$ for which

$$\langle n, f(x, u(x)) \rangle \leq 0 \quad \forall n \in N_{\bar{M}}(x) \quad (4.32)$$

\[\square\] and the “proximal aiming” condition

$$\langle x - y, f(x, u(x)) \rangle \leq -\delta_2 \|x - y\| \quad (4.33)$$

\[1\] In [68], the proximal normal cone, $N_{\bar{M}}^p(x)$, is used.
for some \( y \in \pi_{\Sigma}(x) \).\(^2\)

Let \( T(x) \) be the minimal time to reach \( \Sigma \) from \( x \). By the “proximal aiming” technique, one can show that for some \( \gamma_1 > 0, \gamma_2 > 0 \),\(^2\)

\[
T(\bar{x}) \leq \gamma_2 \| \bar{x} - \pi_{\Sigma}(\bar{x}) \| \quad \forall \bar{x} \in \bar{\mathcal{M}} \cap \{ \Sigma + \gamma_1 B \}. \quad (4.34)
\]

**Lemma 4.2.5.** We assume (4.34) for all \( x \in \mathcal{M} \). Then, the value function \( v \) is locally Lipschitz continuous in \( \bar{\mathcal{M}} \).

**Proof.** Let \( x_1 \in \mathcal{M} \) and \( T \) be the minimal time function to \( x_1 \). Because of (4.34), \( T \) is locally Lipschitz continuous in \( \mathcal{R} \), which is the reachable set to \( x_1 \). Let \( \mathcal{R}_x \) be the reachable set to \( x \). Thus,

\[
T(x_2) \leq C \| x_2 - x_1 \|, \quad \forall x_2 \in \mathcal{R}_x, \quad (4.35)
\]

where \( C \) is the Lipschitz constant for \( T \).

Let \( \bar{\alpha} \) be the control to make the system move from \( x_2 \) to \( x_1 \) with the minimal time \( T(x_2) \). The cost \( K(x_2) \) from \( x_2 \) to \( x_1 \) is bounded: \( x(0) = x_2 \),

\[
K(x_2) = \int_0^{T(x_2)} e^{-\beta t} \left\{ c(x, \bar{\alpha}_t) + b(x_t, n_t^x) \right\} dt \leq \int_0^{T(x_2)} e^{-\beta t} \left\{ L_c + L_b \right\} dt
\]

\[
= \frac{L_c + L_b}{\beta} \left( 1 - e^{-\beta T(x_2)} \right) \leq \frac{L_c + L_b}{\beta} \beta T(x_2) \quad \therefore \frac{d}{dt} (1 - e^{-\beta t}) \leq \beta
\]

\[
\leq \frac{L_c + L_b}{\beta} \beta C \| x_2 - x_1 \| \leq C_1 \| x_2 - x_1 \|, \quad C_1 = \frac{L_c + L_b}{\beta} \beta C.
\]

By the DP principle

\[
v(x_2) \leq K(x_2) + e^{-\beta T(x_2)} v(x_1) \leq C_1 \| x_2 - x_1 \| + u(x_1) \quad \therefore e^{-\beta T(x_2)} \leq 1
\]

We can get this result for all \( x \) in \( \bar{\mathcal{M}} \). Then,

\[
|v(x_2) - v(x_1)| \leq C_1 \| x_2 - x_1 \|, \quad x_2 \in \bar{\mathcal{M}} \cap B(x_1, \epsilon), \forall x_1 \in \bar{\mathcal{M}}, \text{ some } \epsilon > 0.
\]

\(2\)In [68], \( \bar{\mathcal{M}} \) is \( S \). \( S \) is assumed to be compact and wedged at each \( x \in \partial S \). \( \bar{\mathcal{M}} \) is also wedged because \( \mathcal{M} \) is a convex set.
Therefore, \( v \) is locally Lipschitz continuous in \( \tilde{M} \). 

\[ \]

### 4.3 Viscosity Solution and Uniqueness

We show that the problem data of RBC satisfies conditions for OCPSD in section [4.1]. Thus, we can define the viscosity solution by Definition [3.3.3] and we know that when there is no random jump, the value function (1.7) is the viscosity solution of HJ equation (3.12)-(3.13) by Proposition [3.3.4]. Now we want to check the conditions for the uniqueness of the viscosity solution.

\[ (H3') \quad L_f < \beta \]

**Corollary 4.3.1.** Consider a deterministic reflecting boundary problem with value function (1.7), for the control system (1.18) on a convex domain with piecewise smooth boundary. Let the assumptions (H0) and (H0-1) hold. Let us assume (H3’) or (S3). Then \( V \) is characterized as the unique viscosity solution to the H-J equation (3.12)-(3.13).

**Proof.** We showed that (H2) is satisfied in Section [4.1]. We know Lipschitz continuity of \( \ell \) from (H0). (H4) is satisfied because \( \ell \) is Lipschitz continuous and \( \tilde{M} \) is compact. By Corollary [3.4.5] we can conclude. \qed
Chapter 5

Viscosity Solution for the Stochastic Reflecting Boundary-Cost Problems

We extend the viscosity solution result of the deterministic problem developed in the previous chapters to a stochastic model, so called, piecewise deterministic Markov process (PDMP). We discussed the modeling aspects of PDMP for the optimal queueing control problem, which is based on stochastic fluid model, in chapter 1. The PDMP has been studied in a very general form, which includes the control and state dependent time-varying jump rate or boundary hitting jumps [24, 25]. Here, we have interests only in queueing application. Thus, we consider a minimum PDMP model for the stochastic fluid model as in [7, 63].

5.1 Existence of an optimal control process and the continuity of the value function

In controlled PDMP studies, even though the details of problems are different, we can see largely two different ways for the proof of the existence of optimal control and characterization of continuity of value function. Depster and Ye [26] reduce the PDMP problem to the discrete time dynamic programming problem by converting the dynamics and cost during the inter-jump time into the transition probability and cost over a single stage. On the other hand, Soner [66] utilizes the recursive relation in the DP equation between the value functions. We adopt
Soner’s approach.

Let $V^0$ be the value function of the deterministic problem \((1.7)\)–\((1.18)\) in chapter 1.2, i.e. $T_0 = 0$ and $T_1 = \infty$,

\[
V^0(\bar{x}, i) = \inf_{\hat{\alpha} \in A} J(\bar{x}, i, \hat{\alpha}),
\]

where

\[
J(\bar{x}, i, \hat{\alpha}) = \int_0^\infty e^{-\beta t} \ell^i(x(t), \hat{\alpha}(\bar{x}, i, t)) dt
\]

and the dynamics is

\[
f^i_{\bar{x}(x), a}(x, a) \in \bar{M}, \quad a \in A_i(x).
\]

We know the existence of optimal control for $V^0$ from Theorem 3.2.1 and boundedness and uniform continuity of $V^0$ from Theorem 4.2.1–4.2.4 in chapter 4. We define

\[
V^N(\bar{x}, i) = \inf_{\hat{\alpha} \in A} J^N(\bar{x}, i, \hat{\alpha})
\]

where

\[
J^N(\bar{x}, i, \hat{\alpha}) = E \left[ \int_0^{T_1} e^{-\beta t} \ell^i(x(t), \hat{\alpha}(\bar{x}, i, t)) dt + e^{-\beta T_1} V^{N-1}(x(T_1), z_1) \right].
\]

Here, $T_1$ is the first jump time and $z_1$ is the post-jump environment state. $E^i$ is the conditional expectation for given $z(0) = i$. We want to show that the existence of optimal control for $V^N$ and $V^N \in BUC(\bar{M})$ by induction.

(H6) $\inf_{z \in Z} (q^z + \beta - L_j^z) \geq 0$.

Lemma 5.1.1. We assume (H6) or local controllability assumption in chapter 4. Suppose $V^{N-1}(\cdot, i)$ is locally Lipschitz continuous for each $i \in Z$ given $N \geq 1$. Then, there exists an optimal control for $V^N$ and $V^N$ is bounded and Lipschitz continuous.

Proof. 1. Existence of an optimal control. From (5.4) and (1.1), as in
Let 

\[ V^N(\bar{x}, i) = \inf_{\hat{\alpha} \in \Theta} \mathbb{E}^i \left[ \int_0^{T_1} e^{-\beta t} \ell^i(x(t), \hat{\alpha}(\bar{x}, i, t)) dt + e^{-\beta T_1} V^{N-1}(x(T_1), z_1) \right] \]

\[ = \inf_{\hat{\alpha} \in \Theta} \int_0^\infty q_i e^{-q_i t} \left\{ \int_0^t e^{-\beta s} \ell^i(x(s), \hat{\alpha}(\bar{x}, i, s)) ds + e^{-\beta t} \sum_{j \in Z \setminus \{i\}} q_{ij} V^{N-1}(x(t), j) \right\} dt. \] (5.6)

Let \( U(t) = \int_0^t e^{-\beta s} \ell^i(x(s), \hat{\alpha}(\bar{x}, i, s)) ds \) and \( V(t) = -e^{-q_i t} \) for integrating by parts. We can see

\[ \lim_{t \to \infty} U(t) V(t) = 0 \] (5.7)

because \( |U(t)| < \infty \) and \( \lim_{t \to \infty} V(t) = 0 \). Then, we get

\[ V^N(\bar{x}, i) = \inf_{\hat{\alpha} \in \Theta} \int_0^\infty e^{-(\beta + q_i) t} \left\{ \ell^i(x(t), \hat{\alpha}(\bar{x}, i, t)) + \sum_{j \in Z \setminus \{i\}} q_{ij} V^{N-1}(x(t), j) \right\} dt. \] (5.8)

We can see that (5.4) is another deterministic optimal control problem such that the lagrangian is

\[ \mathcal{L}^i(x(t), \hat{\alpha}(\bar{x}, i, t)) = \ell^i(x(t), \hat{\alpha}(\bar{x}, i, t)) + \sum_{j \in Z \setminus \{i\}} q_{ij} V^{N-1}(x(t), j), \] (5.9)

and discount rate is \( \beta + q_i > 0 \). The dynamics is (5.3). The value function \( V^{N-1} \) is uniformly continuous and independent of control. Thus, the new lagrangian satisfies assumption (H1). As we saw in chapter 4, there exists an optimal control for all \( \bar{x} \in \bar{M} \) and \( i \in Z \).

2. Lipschitz continuity of \( V^N \). Let \( x(\cdot) \) be the trajectory of (5.3) with an initial condition \( x(0) = \bar{x} \) and a control \( \hat{\alpha} \). Let \( y(\cdot) \) be another trajectory of (5.3) with \( y(0) = \bar{y} \) and the same control \( \hat{\alpha} \). Let \( n_x(t) \) and \( n_y(t) \) be the reflecting forces...
for \( x(t) \) and \( y(t) \) as we defined in chapter 1.2. We determine \( \hat{\alpha} \) and \( \epsilon > 0 \) such that

\[
V^N(\bar{y}, i) \geq E^i \left[ \int_0^{T_1} e^{-\beta t} \ell^i(y(t), \hat{\alpha}(\bar{y}, i, t)) dt + e^{-\beta T_1} V^{N-1}(y(T_1), z_1) \right] - \epsilon,
\]

and

\[
V^N(\bar{x}, i) \leq E^i \left[ \int_0^{T_1} e^{-\beta t} \ell^i(x(t), \hat{\alpha}(\bar{x}, i, t)) dt + e^{-\beta T_1} V^{N-1}(x(T_1), z_1) \right].
\]

Then,

\[
\begin{align*}
|V^N(\bar{x}, i) - V^N(\bar{y}, i)| & \leq E^i \left[ \int_0^{T_1} e^{-\beta t} \left\{ \ell^i(x(t), \hat{\alpha}(\bar{x}, i, t)) - \ell^i(y(t), \hat{\alpha}(\bar{y}, i, t)) \right\} dt \right] \\
& + E^i \left[ e^{-\beta T_1} \left\{ V^{N-1}(x(T_1), z_1) - V^{N-1}(y(T_1), z_1) \right\} \right] + \epsilon = I_1 + I_2 + \epsilon \\
\end{align*}
\]

(5.10)

We observe that as in chapter 4.

(1) Piecewise smooth boundary and linear overflow cost

\[
I_1 \leq E^i \left\{ \left[ L_b \left( e^{(L_f - \beta)T} + 1 \right) + \frac{A}{L_f - \beta} \left( e^{(L_f - \beta)T} - 1 \right) \right] |\bar{x} - \bar{y}| \right\}
\]

\[
\leq \left[ L_b \left( \frac{q_i}{L_f - \beta - q_i} + 1 \right) + \frac{A}{L_f - \beta} \left( \frac{q_i}{L_f - \beta - q_i} - 1 \right) \right] |\bar{x} - \bar{y}| \\
\]

(5.11)

(2) Rectangular boundary and general overflow cost

\[
I_1 \leq E^i \left\{ \left[ \frac{A}{L_f - \beta} \left( e^{(L_f - \beta)T} - 1 \right) + L_{bn} \sqrt{N} \right] |\bar{x} - \bar{y}| \right\}
\]

\[
\leq \left[ \frac{A}{L_f - \beta} \left( \frac{q_i}{L_f - \beta - q_i} - 1 \right) + L_{bn} \sqrt{N} \right] |\bar{x} - \bar{y}| \\
\]

(5.12)

We can represent for both (5.11) and (5.12),

\[
I_1 \leq R_i |\bar{y} - \bar{x}|, \quad 0 < R_i < \infty. \\
\]

(5.13)
By assumption, $V^N(\cdot, i)$ is Lipschitz continuous, and by (H6)

$$I_2 \leq E^i \left[ e^{-\beta T_i} L_{V^{N-1}} \left( e^{L_i/T_i} |\bar{y} - \bar{x}| \right) \right] = S_i L_{V^{N-1}} |\bar{y} - \bar{x}|, \quad 0 < S_i L_{V^{N-1}} < \infty,$$

(5.14)

where $S_i = |\frac{q_i}{L_i - \beta \cdot q_i}|$ and $L_{V^{N-1}} > 0$ is the Lipschitz constant of $V^{N-1}$. Suppose $L_{V^{N-1}} < \infty$ for a while. We will discuss this below.

From (5.13) and (5.14), we get

$$|V^N(\bar{x}, i) - V^N(\bar{y}, i)| \leq R_i |\bar{y} - \bar{x}| + S_i L_{V^{N-1}} |\bar{y} - \bar{x}| + \epsilon \quad (5.15)$$

Thus, as $\epsilon \to 0$ by choosing optimal control for $y(\cdot)$, we conclude $V^N(\bar{x}, i)$ is locally Lipschitz continuous if $V^{N-1}(\cdot, i)$ is locally Lipschitz continuous for all $i \in \mathbb{Z}$.

Now we want to check whether this is valid when $N \to \infty$. By the same way,

$$|V^1(\bar{x}, i) - V^1(\bar{y}, i)| \leq R_i |\bar{y} - \bar{x}| + S_i L_{V^0} |\bar{y} - \bar{x}|,$$

where $L_{V^0}$ is the Lipschitz constant of $V^0$, which is given in one of (4.11) or (4.28) or (4.36) depending on the assumption for the deterministic problem. Let $i^k = Z_k$, $k = 1, 2, \cdots$. Then, we can get

$$|V^1(\bar{x}, i^1) - V^1(\bar{y}, i^1)| \leq R_{i^1} |\bar{y} - \bar{x}| + S_{i^1} L_{V^0} |\bar{y} - \bar{x}| = L_{V^1} |\bar{y} - \bar{x}|$$

$$|V^2(\bar{x}, i^2) - V^2(\bar{y}, i^2)| \leq R_{i^2} |\bar{y} - \bar{x}| + S_{i^2} L_{V^1} |\bar{y} - \bar{x}|$$

$$= \left( R_{i^2} + S_{i^2} R_{i^1} + S_{i^2} S_{i^1} L_{V^0} \right) |\bar{y} - \bar{x}|$$

$$|V^3(\bar{x}, i^3) - V^3(\bar{y}, i^3)| \leq R_{i^3} |\bar{y} - \bar{x}| + S_{i^3} L_{V^2} |\bar{y} - \bar{x}|$$

$$= \left( R_{i^3} + S_{i^3} R_{i^2} + S_{i^3} S_{i^2} R_{i^1} + S_{i^3} S_{i^2} S_{i^1} L_{V^0} \right) |\bar{y} - \bar{x}|$$

$$\vdots$$

$$|V^N(\bar{x}, i^N) - V^N(\bar{y}, i^N)| \leq R_{i^N} |\bar{y} - \bar{x}| + S_{i^N} L_{V^{N-1}} |\bar{y} - \bar{x}|$$

$$= \left( R_{i^N} + \sum_{k=1}^{N-1} R_{i^k} \prod_{j=k+1}^{N} S_{i^j} + \prod_{j=1}^{N} S_{i^j} L_{V^0} \right) |\bar{y} - \bar{x}|.$$
Let $S = \max_{i \in \mathbb{Z}} S_i$ and $R = \max_{i \in \mathbb{Z}} R_i$. Then,

$$\lim_{N \to \infty} |V^N(\bar{x}, i^N) - V^N(\bar{y}, i^N)| \leq \lim_{N \to \infty} \left( R \sum_{k=0}^{N-1} S^k + S^N L_v \right) |\bar{y} - \bar{x}|$$

$$= \frac{R}{1 - S} |\bar{y} - \bar{x}|$$

because $0 < S < 1$. The value function, $V^\infty(x, i) = \lim_{N \to \infty} V^N(x, i)$, is still Lipschitz continuous.

The next theorem shows that $V^\infty(x, i) = V(x, i)$ and $V(x, i)$ satisfies the result for $V^N(x, i)$.

**Theorem 5.1.2 (t:3.4 [66]).** By the assumption (H6), the value function $V$ is locally Lipschitz continuous.

**Proof.** By the recursive relation of (5.5)

$$V^N(\bar{x}, i) = \inf_{\alpha \in \mathcal{A}} E \left\{ \sum_{n=0}^{N-1} \int_{T_n}^{T_{n+1}} e^{-\beta t} \ell(x_n(t), \check{\alpha}(X_n, z_n, t - T_n)) dt + \int_{T_N}^{\infty} e^{-\beta t} \ell(x_N(t), \check{\alpha}(X_N, z_N, t - T_N)) dt \right\}$$

(5.16)

Let $x_n(t)$ be the trajectory of $\check{\alpha}(X_n, z_n, t - T_n)$, $t \in [T_n, T_{n+1})$ and $x_n(T_n) = X_n$ and $x_0(0) = \bar{x}$, $z_0 = i$.

We recall

$$V(\bar{x}, i) = \inf_{\check{\alpha} \in \mathcal{A}} E \left\{ \sum_{n=0}^{\infty} \int_{T_n}^{T_{n+1}} e^{-\beta t} \ell(x_n(t), \check{\alpha}(X_n, z_n, t - T_n)) dt \right\}.$$  

(5.17)

We want to prove $V^N$ converges to $V$ as $N \to \infty$. Firstly we can observe that for a given $i \in \mathbb{Z}$

$$\sup_{x \in \mathcal{X}} |V^N(\bar{x}, i) - V(\bar{x}, i)| \leq \sup_{\check{\alpha} \in \mathcal{A}} E \left\{ \int_{T_N}^{\infty} 2L^* e^{-\beta t} dt \right\}$$

$$\leq \frac{2L^*}{\beta} \sup_{\check{\alpha} \in \mathcal{A}} E \left\{ e^{-\beta T_N} \right\},$$

(5.18)
where
\[ L^* = \max_{x \in \hat{M}, a \in A, i \in Z} \ell^i(x, a) < \infty. \]

Let us assume that the optimal controls for \( V_N \) and \( V \) are different. Let \( \tilde{J}^N(\bar{x}, i) \) be the cost by the control of \( V \) up to stage \( N \). Obviously, (5.18) is the bound on the difference between \( \tilde{J}^N(\bar{x}, i) \) and \( V(\bar{x}, i) \). Because \( V(\bar{x}, i) \leq V_N(\bar{x}, i) \leq \tilde{J}^N(\bar{x}, i) \), (5.18) can be the bound on difference between \( V_N \) and \( V \) still.

We can show that \( E \left( e^{-\beta T_N} \right) \) is decreasing to zero independent of control. The environment variable \( z(t) \) is a regular Markov process because \( q_i < \infty \) for all \( i \in Z \).

Thus, \( \sup_{N} T_N \to \infty \) as \( N \to \infty \) [20] (p251).

Therefore, we can conclude
\[ \sup_{x \in \hat{M}} \left| V_N(\bar{x}, i) - V(\bar{x}, i) \right| \leq 0 \] (5.19)
and \( V \in C^{0,1}(\hat{M}) \) because \( V_N \in C^{0,1}(\hat{M}) \).

\[ \square \]

### 5.2 Viscosity solution and uniqueness

By the same way of obtaining (5.4), we can get the similar form of deterministic formula for \( V \) as follows because \( V_N \to V \) as \( N \to \infty \).

\[ V(\bar{x}, i) = \inf_{\hat{\alpha} \in \hat{A}} \int_0^\infty e^{-(\beta+q_i)t} \left\{ \ell^i(x(t), \hat{\alpha}(\bar{x}, i, t)) + \sum_{j \in Z \setminus \{i\}} q_{ij} V(x(t), j) \right\} dt \]
\[ = \inf_{\hat{\alpha} \in \hat{A}} \int_0^\infty e^{-(\beta+q_i)t} \mathcal{L}^i(x(t), \hat{\alpha}(\bar{x}, i, t)) dt \] (5.20)
for all \((\bar{x}, i) \in \hat{M} \times Z\).

We assume that \( \ell^z(x, a) \) and \( f^z(x, a) \) satisfy assumption (H1) in chapter 3 and
\[ f^z_{i(x)}(x, a) = f^z_{i(x)}(x, a), \quad \ell^z(x, a) = \ell^z_{i(x)}(x, a), \quad a \in A_{i(x)}. \]

Because \( V \) is Lipschitz continuous, the lagrangian \( \mathcal{L}^z(x, a) \) also satisfies (H1) and
\[ \mathcal{L}^z(x, a) = \mathcal{L}^z_{i(x)}(x, a). \]
As in the deterministic case, we can define the extended multifunction for each 
\( z \in \mathbb{Z} \),

\[
\hat{F}^z(x) = \left\{ (y, \eta) \; ; \; y = f^z_i(x, a), \; \eta \geq \mathcal{L}^z_i(x(t), a), \; a \in \mathcal{A}^i(x) \right\} \subset \mathbb{R}^{N+1}.
\] (5.21)

We shall also consider the upper semicontinuous, convex-valued regularization

\[
G^z(x) = \bigcap_{\varepsilon > 0} \text{co} \left\{ (y, \eta) \in \hat{F}^z(x') ; \; |x' - x| < \varepsilon, \; x' \in \bar{M} \right\} \subset \mathbb{R}^{N+1}.
\] (5.22)

We also assume that (H2) is satisfied for each deterministic problem, which use the data in each environmental state. Then, we can see easily that (5.22) also satisfies (H2) also.

We can consider a HJ equation for the deterministic optimal control problem for \( Z = i \),

\[
(\beta + q^z)w(x, z) + H(x, z, Dw(x, z)) = 0,
\] (5.23)

where,

\[
H(x, z, p) = \sup_{(f, \eta) \in G^z(x)} \left\{ -f \cdot p - \eta \right\},
\] (5.24)

where \( w \) is the Lipschitz continuous function. Because the new cost function contains the value functions of other environmental variables, the HJ equations are interdependent. Thus, we have a set of HJ equations for the controlled PDMP. However, we know that the value function is Lipschitz continuous. Thus, the value function is unique by Corollary 3.4.5.

We summarize the result for the controlled PDMP as a Corollary.

**Corollary 5.2.1.** Consider the optimal control problem (1.6), for the control system (1.3) on a stratified domain (3.1). Let the assumptions (H1), (H2), (H4), (3.55) and (H6) hold for the problem data of each environmental state. Let us assume that the value function \( V \) satisfies the regularity assumptions (H3). Then \( V \) is the unique non-negative solution to the H-J equation (3.12)-(3.13) with such regularity properties.

**Proof.** By Lemma 5.1.1, we know that the value function, \( V(\cdot, i) \), for the problem (1.6)-(1.3) be locally Lipschitz continuous on \( \bar{M} \) for each \( i \in \mathbb{Z} \). Because the controlled PDMP problem can be transformed into the deterministic control
problem, (5.20), for a fixed $i \in Z$. Because problem data at each $i \in Z$ satisfies the assumptions (H1), (H2), (H4), (3.55) and (H6). The new deterministic control problem also satisfies the assumptions (H1), (H2), (H4) and (3.55). Thus, $V(\cdot, i)$ is a viscosity solution of (5.23)-(5.24) according to Definition 3.3.3 by Proposition 3.3.4.

By Corollary 3.4.5, the viscosity solution of (5.23)-(5.24) is unique. Thus, we can conclude. \qed
Chapter 6

Numerical solution

6.1 Introduction

Now, we want to get the value function by solving the HJB equations numerically. Using the numerical result, we can find an optimal control and corresponding system trajectory. We use the Markov chain approximation method, which has been studied to solve various stochastic optimal control problems [48, 6]. Because it is an iterative method, the computation time tends to long. However, it is easy to apply to diverse optimal control problems. We may consider this method as a convenient approach. On the other hand, we have used different approach to deal with the reflecting boundary from the literature on the Markov chain approximation. We need to check the validity of the method for our problem in this chapter. We also show how the Markov chain approximation is derived from the HJB equation, which has also a different form from the ones used in literature [48].

In numerical methods, the main concern is to assure that approximated solutions approach the real solution as an approximation parameter, let say, \( h \) gets close to zero. Conventionally the smaller \( h \) means the finer grain approximation. This is shown by proving the weak convergence of the approximated solutions in an appropriate function space.

In the literature, weak convergence in a problem with the reflecting boundary has been proved in the framework of Skorokhod problem [38]. However, in this thesis, we made the result of solvability not using Skorokhod problem approach.
Naturally, weak convergence proof also does not deal with Skorokhod problem formulation.

There are several different ways of convergence proof for the Markov chain approximations. We use the probabilistic method of Kushner and Dupuis [48]. We will give a brief explanation about the principle of the approximate Markov chain and weak convergence result to our problem. Then, we will give how we can use the HJB equation to derive the transition probability using the finite difference quotient. Then, we will present a numerical result for an example.

6.2 Controlled Markov Chain Approximation

The main idea is that the system trajectory can be approximated by a continuous time interpolation of a controlled Markov chain. The optimal trajectory and value function of the Markov chain approximation converges weakly to the optimal trajectory and value function, which we want to get from the corresponding HJB equation.

Let a $N$-dimensional vector $h > 0$ be the approximation parameter, which is the grid size in this thesis. It represents the level of approximation. Let $R^N_h \subset \mathbb{R}^N$ be a discrete state space such that if $e_i$ is the $i$th coordinate unit vector,

$$R^N_h = \left\{ x \in \mathbb{R}^N; \quad x_k = mh_ke_k, \quad m = 0, \pm 1, \pm 2, \cdots, k = 1, 2, \cdots, N \right\}.$$ 

Let $\hat{\mathcal{M}}_h = \mathcal{M} \cap R^N_h$ be the discretized domain. We assume that $B_k/h_k$, $k = 1, \cdots, N$ are integer valued without loss of generality so that the points in $\partial \mathcal{M} \cap R^N_h$ belongs to $\hat{\mathcal{M}}_h$. For a given $h$, we consider a $R^N_h$-valued Markov chain $\{\xi_h^n, n = 0, 1, \cdots\}$ whose interpolation is an approximation of the trajectory of the system. Its one-step transition probabilities $p^h(y|x, a)$ from $x$ to $y$ is dependent on a control $a \in A$ for $x, y \in \hat{\mathcal{M}}_h$.

Let $a^n_h$ denote the control used at step $n$, and suppose that an interpolation interval $\Delta t^h(x, a)$ is given and $\Delta t^h(x, a)$ is continuous in $x$ and $a$. Define $\Delta t^n_h = \Delta t^h(\xi^n_h, a^n_h)$ and $\delta \xi^n_h = \xi^n_{h+1} - \xi^n_h$. Let $E^n_h$ denote the expectation given $\{\xi^n_i, a^n_i, 0 \leq i \leq n\}$.

We use same notations and assumptions for cost and dynamics as in the pre-
vious chapters. For simplicity, let’s fix \( z \in \mathbb{Z} \).

**Definition 6.2.1** (Local consistency condition \([48]\)). For some \( \alpha > 0 \) and \( \xi_{n}^{h} = x \),

\[
E_{n}^{h} \delta \xi_{n}^{h} = \Delta^{h}(x, a)f^{\ast}(x, a) + O(h^{\alpha}\Delta^{h}(x, a)),
\]

\[
|\xi_{n+1}^{h} - \xi_{n}^{h}| = O(h). \tag{6.1}
\]

We will show this condition is satisfied in our problem later. We define the interpolated chain \( \xi^{h}(\cdot) \) and control \( a^{h}(\cdot) \) by

\[
\xi^{h}(t) \equiv \xi_{n}^{h} \quad \text{and} \quad a^{h}(t) \equiv a_{n}^{h} \quad \text{on} \quad \left[ t_{n}^{h}, t_{n+1}^{h} \right], \quad t_{n}^{h} = \sum_{i=0}^{n-1} \Delta t_{i}^{h}. \tag{6.2}
\]

The numerical method is based on the weak convergence of \( \{ \xi^{h}(\cdot) \} \) to \( x(\cdot) \) as \( h \to 0 \). This means that the probability measures \( \mathcal{P}^{h} \) induced by the Markov chain \( \{ \xi_{n}^{h}, n = 0, 1, \ldots \} \) converges to a probability measure \( \mathcal{P} \) in the space of probability measure and the sample path of \( \{ \xi^{h}(\cdot) \} \) to \( x(\cdot) \) in an appropriate functional space of certain metrics respectively. We gives some necessary definition and results briefly.

**Definition 6.2.2** (Weak convergence). Let \( \{X_{n}\} \) and \( X \) be random variables with values in a metric space \( S \), and with induced probabilities \( \{\mathcal{P}_{n}\} \) and \( \mathcal{P} \) on the \( \sigma \)-algebra \( \mathcal{S} \) of \( S \). If \( Ef(X_{n}) \to Ef(X) \) for every bounded real-valued continuous function \( f(\cdot) \), then we say that \( \{X_{n}\} \) (or \( \{\mathcal{P}_{n}\} \)) converges weakly to \( X \) (or \( \mathcal{P} \)), and write \( X_{n} \Rightarrow X \) (or \( \mathcal{P}_{n} \Rightarrow \mathcal{P} \)).

For weak convergence, the concept of tightness is used. Here, we briefly introduced.

**Definition 6.2.3** (Tightness \([48]\)). The sequence \( \{X_{n}\} \) (or \( \{\mathcal{P}_{n}\} \)) is said to be tight (relatively compact) if for each \( \varepsilon > 0 \), there is a compact set \( K_{\varepsilon} \subset S \) such that

\[
\inf_{n} \mathcal{P}\{X_{n} \in K_{\varepsilon}\} \geq 1 - \varepsilon.
\]

If the \( X_{n} \) are vector-valued, then tightness is equivalent to

\[
\lim_{N \to \infty} \sup_{n} \mathcal{P}\{|X_{n}| \geq N\} = 0.
\]
Theorem 6.2.4 (Prohorov’s theorem (Theorem 4.2 in [45])). Let \( \{X_n\} \) be tight in \((S, \mathcal{S})\). Then for each subsequence, there is a further subsequence \( \{X_{n_k}'\} \) and an \( X \) such that \( X_{n_k}' \Rightarrow X \).

We can apply definition 6.2.2 to our situation. We note that \( \{X_n\} \) and \( X \) corresponds to \( \{\xi^h\} \) and \( x \). Now we want the metric space \( S \) to be a space of \( \xi^h \). Let \( D^N[0, \infty) \) denote the space of \( \mathbb{R}^N \)-valued functions on \([0, \infty)\) that are right continuous and have left-hand limits. By being topolgized with the Skorokhod metric, \( D^N[0, \infty) \) becomes a complete and separable metric space [33]. Let \( \Xi \) be the set of the interpolation, \( \xi^h \), of Markov chains for all \( h > 0 \). It is easy to see \( \Xi \subset D^N[0, \infty) \). Then, as we saw above, if \( \Xi \) is tight, then a sequence \( \{\xi^h\} \) has the convergent subsequence \( \{\xi^h'\} \) in \( D^N[0, \infty) \). In the following sections, we will show that the condition for the above theorem and definition are satisfied and \( x \), the limiting process of \( \xi^h \) is in \( C^N[0, \infty) \).

6.3 Construction of Markov Chain Approximation and Convergence

In this section, we want to show that the HJB equation can be transformed into Markov Decision Process (MDP) in discretized domain through the Markov chain approximation. Basic idea is to replace the gradient term \( Dv \) in the HJB equation with finite difference quotients and to get the MDP form. This approach was suggested by Kushner [48] originally for controlled diffusion models. However, it can be applied to deterministic or PDMP models [10].

The HJ equation. By replacing the finite-difference quotient with the gradient of the value function, we can get the transition probability for the approximate Markov chain [48]. The regularized HJ equation in chapter 3 has a upper semi-continuous regularized and deparameterized form. Thus, it is not appropriate to use it directly for our purpose here. We will use an equivalent HJ equation with the projected dynamics (6.3). For \( z \in Z \),

\[
\beta v(x, z) + \sup_{a \in U} \left\{ - \pi_M f^z(x, a) \nabla v(x, z) - \ell^z(x, a) \right\} = 0. \tag{6.3}
\]
Transition probability. In this subsection, we assume $x \in \bar{\mathcal{M}}_h$. As in Boukas and Haurie [10], we define

$$f_k^i(x, a) = \max(0, \pi_k \bar{f}_k^i(x, a)) \quad \text{and} \quad f_k^{-i}(x, a) = \max(0, -\pi_k \bar{f}_k^i(x, a)).$$

As in finite difference method, we replace the first order derivative $\frac{\partial v}{\partial x_k}(x, i)$ by the difference quotient,

$$\begin{cases} 
\frac{v^h(x + h_k e_k, i) - v^h(x, i)}{h_k} & \text{if } f_k^i(x, a) \geq 0, \\
\frac{v^h(x, i) - v^h(x - h_k e_k, i)}{h_k} & \text{if } f_k^i(x, a) < 0,
\end{cases} \quad k = 1, 2, \ldots, N.$$

Then, the HJB equations become

$$\beta v^h(x, i) = \inf_{a \in A^i} \left\{ \ell(x, a) + \sum_{k=1}^N |f_k^i(x, a)| \frac{v^h(x \pm h_k e_k, i) - v^h(x, i)}{h_k} \right. \\
+ \sum_{j \neq i} q_{ij} (v^h(x, j) - v^h(x, i)) \right\},$$

where we simply represent the sign by $\pm$ because we know the sign by $f_k^i(x, a)$.

$$0 = \inf_{a \in A^i} \left\{ \ell(x, a) + \sum_{k=1}^N |f_k^i(x, a)| \frac{v^h(x \pm h_k e_k, i)}{h_k} \\
+ \sum_{j \neq i} q_{ij} v^h(x, j) - \left( \beta + \sum_{j \neq i} q_{ij} + \sum_{k=1}^N |f_k^i(x, a)| \frac{1}{h_k} \right) v(x, i) \right\},$$

Let

$$Q^h_i(x, a) = \sum_{j \neq i} q_{ij} + \sum_{k=1}^N |f_k^i(x, a)| \frac{1}{h_k}. \quad (6.4)$$
Then, we get

\[
 v^h(x,i) = \inf_{a \in A} \frac{1}{\beta + Q^h(x,a)} \left\{ \ell(x,a) + \sum_{k=1}^{N} |f_k^i(x,a)| \frac{v^h(x \pm h_ek, i)}{h_k} 
+ \sum_{j \neq i} q_{ij} v^h(x, j) \right\}.
\] (6.5)

Define a set of points, \( Y(x,i) \), by

\[
 Y^h(x,i) = \{ (x \pm h_ek, i), \ k = 1, 2, \cdots, N, \text{and} \ (x,j), \ \forall j \in Z \}.
\] (6.6)

We can define new transition probability,

\[
 p^h[(x \pm h_ek, i)](x,i), a] = \frac{f_k^i(x,a)}{h_k Q^h(x,a)},
 p^h[(x,j)](x,i), a] = \frac{q_{ij} v^h(x, j)}{Q^h(x,a)}, \quad \text{for} \ i \neq j
 p^h[(y,j)](x,i), a] = 0, \quad \forall (y,j) \in (\bar{\mathcal{M}}^h \times Z) \setminus Y^h(x,i),
\] (6.7)

discount rate and cost,

\[
 \hat{\beta} = \frac{1}{1 + \beta Q^h(x,a)}, \quad \text{and} \quad \hat{\ell}(x,a) = \frac{1}{\beta + Q^h(x,a)} \ell(x,a).
\] (6.8)

Then, we get a discrete dynamic programming equation,

\[
 v^h(x,i) = \inf_{a \in A^i} \left\{ \hat{\ell}(x,a) + \hat{\beta} \left\{ \sum_{k=1}^{N} p^h[(x \pm h_ek, i)](x,i), a] v^h(x \pm h_ek, i) 
+ \sum_{j \neq i} p^h[(x,j)](x,i), a] v^h(x, j) \right\} \right\}
\] (6.9)

\[
 = \inf_{a \in A^i} \left\{ \hat{\ell}(x,a) + \hat{\beta} \sum_{(y,j) \in Y^h(x,i)} p^h[(y,j)](x,i), a] v^h(y, j) \right\}.
\]
We can see easily that the local consistency (6.1) is satisfied with
\[ \Delta t^h(x, a) \doteq \frac{1}{\beta + Q_i^h(x, a)}. \]
An actual next jump point \( x' \) by control, \( a \), from point \( x \) with \( \Delta t^h(x, a) \) is within
the grid. In addition, because the domain is rectangular, it will not happen that
\( x' \notin \mathcal{M}^h \). Thus, the local consistency is maintained at all points in \( \mathcal{M}^h \).

**Weak Convergence in** \( D^N[0, \infty) \). Because \( f \) has a linear growth and \( \mathcal{M} \) is
bounded, the approximated Markov chain is tight for all \( h > 0 \) by Theorem 4.5
in [45]. Actually, the system trajectory is contained in the compact space \( \mathcal{M} \),
this is obvious. We know also that \( D^N[0, \infty) \), a complete separable metric space
with Skorokhod topology, contains \( \Xi \), the set of all \( \xi^h, h > 0 \). Thus, by the
Prohorov’s theorem (Theorem 4.2 in [45]), there exists a convergent subsequence
\( \{\xi^h(\cdot)\} \) of \( \{\xi^i(\cdot)\} \) in \( \Xi \) for the weak convergence. Because the limiting process
is deterministic in each environment, we can get the same result by the simple
probabilistic approach in section 4.5 p80 [48].

### 6.4 Numerical Results

#### 6.4.1 Discrete Infinite-Horizon Discounted-Cost Problems and Algorithm.

The discrete DP formulation (6.12) is related with the infinite-horizon stochastic
optimal control problem of the expected total discounted cost [41],

\[ v^h_*(\bar{x}, i) \doteq v^h_\pi(\bar{x}, i) = \inf_{\pi \in \Pi} v^h_\pi(\bar{x}, i), \tag{6.10} \]

where

\[ v^h_\pi(\bar{x}, i) \doteq \lim_{N \to \infty} E^{\pi}_{(\bar{x}, i)} \left\{ \sum_{n=0}^{N} \beta^n \ell(x_n, a_n) \right\}, \tag{6.11} \]

and \( x_0 = \bar{x} \in \mathcal{M}^h \) and \( \Pi \) is the set of (randomized) control policies.

We summarize the related result as a lemma.
Lemma 6.4.1 (61). By our problem condition, the value function $v^h$ satisfies the optimality equation

$$v^h(x, i) = \min_{a \in A^i} \left[ \hat{\ell}(x, a) + \hat{\beta} \sum_{(y, j) \in Y^h(x, i)} p^h[(y, j)|(x, i), a]v^h(y, j) \right]$$  \hspace{1cm} (6.12)

and the unique solution of (6.12) with a deterministic stationary optimal control policy, is the value function $v^*_h$.

Proof. Because $0 < \hat{\beta} < 1$, the state space $\hat{\mathcal{M}}^h \times Z$ is finite and discrete, and $\hat{\ell}$ is bounded below, (6.12) has a unique $v^*_h(x, i)$ (Thm 6.2.5). The value function $v^*_\pi(x, i)$ is a solution of (6.12) iff $\pi^* \in \Pi$ is an optimal policy (Thm 6.2.6). Because $A^i$ is compact, $\hat{\ell}$ and $p^h()$ is continuous in $a \in A^i$, there exists an optimal deterministic stationary policy (Thm 6.2.10), which gives the infimum of (6.9). Then, $\inf$ is replaced by $\min$ as in (6.12).

We can solve the controlled Markov chain (6.10) by the value iteration algorithm, which is based on (6.12) in principle. In practice, we need to stop the algorithm with a certain bound because we cannot run the value iteration infinitely long time. We define $\varepsilon$-optimal policy, $\pi^*_\varepsilon$ for $\varepsilon > 0$, for all $(x, i) \in \hat{\mathcal{M}}^h \times Z$,

$$v^h_{\pi^*_\varepsilon}(x, i) \geq v^*_h(x, i) - \varepsilon.$$  \hspace{1cm} (6.13)

The following is the value iteration algorithm, which gives $\varepsilon$-optimal policy. It has been proved that $\|v^h_n - v^*_h\| \to 0$ as $\varepsilon \to 0$ in Theorem 6.3.1. \[61\].

Value iteration algorithm for $\varepsilon$-optimal policy \[61\]

1. Select $v^h_0 \in B(\hat{\mathcal{M}}^h \times Z)$, specify $\varepsilon > 0$, and $n = 0$.

2. For each $(x, i) \in \hat{\mathcal{M}}^h \times Z$, compute $v^h_{n+1}(x, i)$ by

$$v^h_{n+1}(x, i) = \min_{a \in A^i} \left[ \hat{\ell}(x, a) + \hat{\beta} \sum_{(y, j) \in Y^h(x, i)} p^h[(y, j)|(x, i), a]v^h_n(y, j) \right],$$  \hspace{1cm} (6.14)

for $n = 1, 2, \cdots$. 


3. If
\[ \| v^h_{n+1} - v^h_n \|_2 < \varepsilon \frac{1 - \hat{\beta}}{2\hat{\beta}}, \]  
(6.15)
go to step 4. Otherwise increment \( n \) by 1 and return to step 2.

4. For each \( (x, i) \in \mathcal{M}^h \times Z \), choose
\[ d_\varepsilon(x, i) \in \arg \min_{a \in A} \left[ \ell(x, a) + \hat{\beta} \sum_{(y, j) \in Y^h(x, i)} p^h[(y, j)|(x, i), a] v^h_{n+1}(y, j) \right]. \]  
(6.16)
and stop.

### 6.4.2 An Example

We want to solve Example 1 in Chapter \[�1\] Martinelli and Valigi \[53\] consider only the case that stability condition is satisfied, i.e., \( \sum_{k=1}^{N} r_k < A \). Taking a step forward, we consider the case of \( \sum_{k=1}^{N} r_k^z(t) \geq A_z(t) \) for some \( z \in Z \) for PDMP model.

#### 6.4.2.1 Deterministic control problem.

In contrast to PDMP, it might happen that \( Q^h_k(x, a) = 0 \) in deterministic control problem. In such a case, (6.12) becomes
\[ v^h(x, i) = \frac{\ell(x, a)}{\beta} \quad \text{for } a \in A \text{ such that } Q^h_k(x, a) = 0. \]  
(6.17)

#### 6.4.2.2 PDMP control problem

We consider the following problem data. The environmental variable, \( Z \), has the values on the following table.

<table>
<thead>
<tr>
<th>( Z )</th>
<th>( r_1 )</th>
<th>( r_2 )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.3</td>
<td>0.3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.9</td>
<td>1</td>
</tr>
</tbody>
</table>

We use \( \varepsilon = 0.001 \), \( \beta = 0.3 \), \( c_1 = c_2 = 1 \), \( b_1 = b_2 = 1 \), \( B_1 = B_2 = 10 \) and \( h = 0.1 \).
We consider normalized processing rate $d = 1$. However, we can also consider $d < 1$ if there are situations in which processing rate is degraded compared to normal situation. The infinitesimal generator for $Z$ is

$$Q = \begin{pmatrix} -1 & 1 \\ 3 & -3 \end{pmatrix}.$$ 

We solved the problem using the VI in section 6.4.1. We use an incremental search for $d$, which minimize right hand side of (6.14) with incremental size 0.01. We show the value function for the state $Z = 1$ in figure 6.1.

![Figure 6.1. the value function for the state Z = 1](image)

We can get the similar value function with the smaller set of controls, that is bang-bang style control. The control set is

$$\left\{ (d_1, d_2) \in \left( \{ 0, r_1, d - r_2, d \} \times \{ 0, r_2, d - r_1, d \} \right) : d_1 \geq 0, d_2 \geq 0, d_1 + d_2 \leq d \right\}. \quad (6.18)$$

The difference between two value function is $\|v^{\text{inc}}(x) - v^{\text{bang}}(x)\|_\infty = 0.071889$, where $v^{\text{inc}}$ and $v^{\text{bang}}$ are the value function obtained by incremental search and bang-bang control respectively. As expected, $v^{\text{inc}}$ gives smaller value than $v^{\text{bang}}$. However, the difference is not significant. Computationally is is required less time.
We conjecture that when the holding cost is linear and dynamics is simply a linear function of control except boundary. In $\mathcal{M}$, the control does not change cost. With the given problem data, we can reduce the holding cost at the expense of overflow. Thus, the optimal control is to move the system to the boundary and make it overflow. Thus, the control value is the one to move the system to boundary as possible as it can, that is, allocating processing rate only on one queue - bang-bang control. If we have nonlinear holding cost and overflow cost function, the phenomenon would be more difficult to explain.

The following figure shows the velocity of the optimal trajectory.

![Figure 6.2. the velocity of the optimal trajectory](image)
Conclusion and Discussion

7.1 Summary

We study an optimal control problem in a single station, multiclass queueing system with finite buffers and overflow costs. The system dynamics is modeled by a controlled stochastic fluid model, which depicts random burstiness of demand or degradation of processing rate. We take into account overflow costs in addition to other usual costs such as holding cost and service rate cost. Various control parameters can be considered. However, we discussed our result in terms of dynamic processing rate allocation.

Because the overflow costs arise only on the boundary and state space is nonsmooth, the corresponding HJB equation is discontinuous. We found that the general theory for HJB equations has not been paid enough attention. Thus, we had intended to develop a theory for this type of problems. However, because we found that the control problem could be regarded as a special case of a larger set of control problem, i.e. the optimal control problem on a stratified domain (OCPSD), we studied the HJB equation for OCPD.

We describe the problem using the differential inclusion, which has been used for general reflecting boundary in chapter 1. We add the overflow cost to the differential inclusion. The existence of optimal control is proved by the direct method in calculus of variation through regularizing a differential inclusion for the system dynamics and cost functions. Then, we show that the regularized differential inclusion gives the same optimal control to the original problem. We characterize
the uniform continuity of the value function. We show that the value function is (locally) Lipschitz continuous when \( L_f < \beta \) under fairly general assumption on \( f \) or when local controllability assumption is satisfied in chapter 1.

We define the optimal control problem on a stratified domain (OCPSD) to prove the uniqueness of the viscosity solution to a certain type of discontinuous HJ equations. The OCPSD is general enough to cover various practical optimal control problems, whose dynamics and cost have good regularity locally on a submanifold in a stratified domain (see (H1) in chapter 3). Even the domain is not necessary to be compact.

We introduce a new concept of viscosity solution for the OCPSD and prove the uniqueness of the viscosity solution for the corresponding HJ equation in chapter 3. Then, we extend the theory of OCPSD to the stochastic fluid problem. We characterize the uniform continuity and prove the existence of optimal control and the uniqueness of the viscosity solution based on the proof in chapter 5. The multiple classes queueing system in chapter 1.2 is a special case of the optimal control problem on a stratified domain.

In chapter 6, we show how to get the numerical solution through the Markov chain approximation 48. Typically, reflecting boundary problem has been modeled as a Skorokhod problem, which has been also a basis for numerical methods in literature 48. We show briefly that without relying on the Skorokhod problem, we can solve the reflecting boundary problem. The discontinuities of cost and dynamics on the boundary in our problem violate the basic assumption of 48 and are not the type of discontinuities discussed in 48. We show that the local consistency is valid still and the weak convergence of the probability measures and the Markov chain approximation make sense without modification of proof of the existing result.

7.2 Contribution

There are many studies on deterministic or stochastic reflecting boundary problem 27, 28, 50, 62. However, reflecting cost (or overflow cost) has not been paid attention sufficiently (see chapter 2) in control theory. Especially, there has not been a rigorous study on uniqueness of solution of corresponding HJB equation for
the overflow cost problem, to our best knowledge.

The contributions of this thesis are:

1. we prove the existence of minimizer of the optimal control problem on the stratified domain.

2. we prove the uniform continuity of the value function for linear cost function with arbitrary piecewise smooth boundary and general cost function with rectangular boundary respectively.

3. we define a new concept of viscosity solution and prove the uniqueness of the viscosity solution to the HJB equation when the system has the deterministic dynamics or the stochastic fluid dynamics respectively.

4. we show that the numerical method based on the Markov chain approximation is still valid even though it has the discontinuity, which has not been taken into account.

The differential inclusion model for the problem of reflecting boundary and reflecting cost is not restricted to the queueing problem. The uniqueness result is also so general that it can be applied to the other control problems such as minimal time problem (see example 1 in chapter 3) and any convex reflecting boundary and reflecting cost problem, which satisfies the assumption (H1).

### 7.3 Future work

We can extend the result into the several different directions.

1. Asymptotic stability and switching curves

   The numerical example in chapter 6 has the origin as equilibrium state. However, depending on the cost or dynamics, there might be other equilibrium states. Finding such equilibrium (or equilibria in the case of stochastic fluid model) would be an interesting topic to develop a efficient algorithm. Typically such equilibrium is searched or characterized by asymptotic stability study using Hamiltonian dynamics [34]. Because of the discontinuous dynamics, we need adapt the conventional approach to our problem.
From the numerical solution of example in chapter 6, we can see that the state space is divided into several regions by switching curves. Finding or characterizing the switching curves is important because it would be helpful to find analytical solution or improve the numerical method. In our problem, it might be more difficult because the value function is not convex. In the other studies, characterizing hedging points relies on the convexity of the value function [39] but still it is an elusive goal.

2. Extending to the general PDMP model

The stochastic fluid model, where the environment variable is independent of control, is interpreted as a special case of PDMP model. In the general PDMP [24, 26], we need to consider the general jump rate and transition probability, which are dependent on the control. In addition, controlled boundary hitting jumps could be considered. We expect that there could be applications, which have those additional modeling factors and the discontinuities, which we addressed in this thesis. Extending our theory to the general PDMP model would require the adaptation of the proofs.

3. More accurate numerical solution for large size problems

Reducing the errors caused by the approximation is always a major concern in the numerical solution. Here, rather than providing a general methodology, we may pursue a way tailored to this problem for improving accuracy and at the same time reducing the computation time by concentrating on computational efforts on the essential part. This would be done accompanying the analytical study including asymptotic stability and switching curves.

It would be desirable to develop a application-oriented single-pass numerical method similar to the ordered upwind method [64] instead of using iterative methods [48].

4. The theory could be extended to hybrid optimal control problem with discontinuous dynamics and cost functions.

5. Our theory would be applied to the differentiated service design and pricing models [56] if we use a holding cost instead of a utility function of waiting
time. Those models basically are multiclass queueing systems. The existing models determine price based on a heuristic scheduling policy. We may be able to apply our theory to a pricing control model, which determines pricing and optimal scheduling for more general dynamics and cost functions (including overflow cost). In addition, the optimal control solution would be a basement from which other approximation or heuristic algorithms are developed.

6. We can extend the theory to other (stochastic fluid) queueing problems such as multiclass queueing networks in logistics and supply chain management or production and service systems.
Appendix A

Preliminaries

A.1 Viscosity Solution and Minimax solution

We introduce the viscosity solution and the minimax solution based on [70]. The more details are in [6, 70, 69].

We denote the Hamilton-Jacobi equation of an unknown $u : \bar{\mathcal{M}} \mapsto \mathbb{R}$ by

$$F(x, u(x), Du(x)) = 0,$$  \hskip1cm (HJ)

where $\mathcal{M}$ an open domain of $\mathbb{R}^N$ and $F(x, r, p)$ is a continuous real valued function on $\mathcal{M} \times \mathbb{R} \times \mathbb{R}^N$.

The main assumptions on $F$ are

$$F(x, z_1, p) \geq F(x, z_2, p) \quad \forall (x, z_1, z_2, p) \in \mathcal{M} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N, \; z_1 \leq z_2; \quad \text{(A.1)}$$

$$|F(x, z, 0)| \leq (1 + \|x\| + |z|)\mu,$$

$$|F(x, z, p) - F(x, z, q)| \leq \|p - q\| \rho(x) \quad \text{(A.2)}$$

$$\forall (x, z, p, q) \in \mathcal{M} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N,$$

where $\rho(x) \doteq (1 + \|x\|)\mu$ and $\mu$ is a positive number.

Even though we provide the definition for the lower and upper semicontinuous function $u$, the definition can be applied to continuous functions.
A.1.1 Viscosity solution

Definition A.1.1 ([70] Definition 6.1). An upper semicontinuous function $u : \bar{M} \mapsto \mathbb{R}$ is a viscosity subsolution (or lower solution) of (HJ) if, for any $\varphi \in C^1(M)$,

$$F(x_0, u(x_0), D\varphi(x_0)) \leq 0$$  \hspace{1cm} (A.3)

at any local maximum point $x_0 \in M$ of $u - \varphi$. Similarly, a lower semicontinuous function $u : \bar{M} \mapsto \mathbb{R}$ is a viscosity supersolution (or upper solution) of (HJ) if, for any $\varphi \in C^1(M)$,

$$F(x_0, u(x_0), D\varphi(x_0)) \geq 0$$  \hspace{1cm} (A.4)

at any local minimum point $x_1 \in M$ of $u - \varphi$. Finally, $u$ is a viscosity solution of (HJ) if it is simultaneously a viscosity sub- and supersolution. □

Note that $F$ is equivalent to being multiplied by -1 to the Hamilton-Jacobi equation in [6]. Thus, the inequality of definition A.1.1 has opposite inequalities from definition 2.1.1 in [6].

The above definition is equivalent to the definition of using sub- and superdifferential. The sub- and superdifferential of $v$ at $x$ are respectively

$$D^-v(x) = \left\{ q \in \mathbb{R}^N; \liminf_{y \to x} \frac{v(y) - v(x) - q \cdot (y-x)}{|y-x|} \geq 0 \right\},$$

$$D^+v(x) = \left\{ p \in \mathbb{R}^N; \limsup_{y \to x} \frac{v(y) - v(x) - p \cdot (y-x)}{|y-x|} \leq 0 \right\}.$$

Then, a continuous function $u$ is a viscosity solution of (HJ) if the following conditions are satisfied:

$$F(x, u(x), q) \leq 0 \quad \forall x \in \mathbb{R}^N, \forall q \in D^-u(x)$$  \hspace{1cm} (A.5)

$$F(x, u(x), p) \geq 0 \quad \forall x \in \mathbb{R}^N, \forall p \in D^+u(x)$$  \hspace{1cm} (A.6)

If (i) holds, $u$ is a viscosity subsolution of (HJ). If (ii) holds, $u$ is a viscosity supersolution of (HJ).
A.1.2 Minimax solution

For the definition of the minimax solution, Subbotin uses the differential (characteristic) inclusion, i.e.

\[(\dot{x}(t), \dot{z}(t)) \in E(x(t), z(t), p)\]  

where

\[E(x, z, p) = \{ (f, g) \in \mathbb{R}^N \times \mathbb{R}^N : \|f\| \leq \rho(x), \; g = \langle f, p \rangle - F(x, z, p) \} \],  

(A.7)

Subbotin denotes the family of closed sets \(W \subset \bar{M} \times \mathbb{R}^N\), which is weakly invariant w.r.t. (3.6) for a fixed \(p \in \mathbb{R}^N\) by \(\text{Inv}_{(3.6)}(p)\).

Definition A.1.2 ([70] Definition 3.1). A minimax solution of (HJ) is a continuous function \(u : \bar{M} \mapsto \mathbb{R}\) such that

\[\text{gr} \; u \in \text{Inv}_{(3.6)}(p), \quad \forall p \in \mathbb{R}^N.\]  

(A.8)

\[\text{gr} \; u\] is the graph of \(u\).

Note that

\[E(x, z, p) \cap E(x, z, q) \neq \emptyset \; \forall (x, z, p, q) \in \mathcal{M} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N.\]  

(A.9)

Recalling (A.2), we can see that there exists \((f_*, g_*)\) such that

\[f_* = \frac{[F(x, z, p) - F(x, z, q)](p - q)}{\|p - q\|^2},\]

\[g_* = \langle f_*, p \rangle - F(x, z, p) = \langle f_*, q \rangle - F(x, z, q).\]

Definition A.1.3. An upper (a lower) minimax solution of equation (HJ) is a lower semicontinuous (a upper semicontinuous) function \(u : \bar{M} \mapsto \mathbb{R}\), which satisfy (A.10) (A.11).

\[\text{epi} u \in \text{Inv}_{(3.6)}(p), \quad \forall p \in \mathbb{R}^N,\]  

(A.10)

\[\text{hypo} u \in \text{Inv}_{(3.6)}(p), \quad \forall p \in \mathbb{R}^N,\]  

(A.11)
where

\[ \text{epi} u \doteq \{ (x, z) \in \hat{M} \times \mathbb{R} : z \geq u(x) \} \quad \text{the epigraph of } u, \text{ and} \]

\[ \text{hypo } u \doteq \{ (x, z) \in \hat{M} \times \mathbb{R} : z \leq u(x) \} \quad \text{the hypograph of } u. \]

**Proposition A.1.4** ([70] Proposition 4.1). For a continuous function \( u : \hat{M} \mapsto \mathbb{R} \), the following equivalence holds

\[ ([A.10], [A.11]) \Leftrightarrow (A.8). \]

The minimax solution is proved to be equivalent to the viscosity solution [70]. There are many various equivalent forms. Here, we introduce a proof related with the comparison result of our upper solution in Chapter 3. We provide more details on it than [70].

**Theorem A.1.5** ([70] Theorem 6.1). Let a lower semicontinuous function \( u \) satisfies (A.5) for all \( x \in \mathcal{M} \). Then, \( u \) satisfies also

\[
\inf \{ d^- u(x; f) - \langle p, f \rangle + F(x, u(x), p) ; \| f \| \leq \rho(x) \} \leq 0, \quad \forall x \in \mathcal{M}, \forall p \in \mathbb{R}^N. \tag{A.12}
\]

*Proof.* Assume the contrary: there exist \( x_0 \in \mathcal{M} \) and \( q \in \mathbb{R}^N \) such that

\[
\min \{ d^- u(x_0; f) - \langle q, f \rangle + F(x_0, u(x_0), q) ; \| f \| \leq \rho(x_0) \} > 0. \tag{A.13}
\]

Since \( f \mapsto d^- u(x_0; f) \) is lower semicontinuous (cf. p136 [21]), there exists a \( \varrho > 0 \) such that

\[
\min \{ d^- u(x_0; f) - \langle q, f \rangle + F(x_0, u(x_0), q) : \| f \| \leq \rho(x_0) + \varrho \} > \varrho. \tag{A.14}
\]

Fix \( r > 0 \) such that \( B(x_0, r) \subseteq \mathcal{M} \). Let

\[ z_0 \doteq u(x_0), \quad a \doteq \inf_x u(x), \quad b \doteq \sup_x u(x), \quad \text{for } x \in B(x_0, r). \]
Because \( z \mapsto F(x_0, z, q) \) is continuous, there exists \( \delta > 0 \) such that
\[
|F(x_0, z, q) - F(x_0, z_0, q)| \leq \frac{\rho}{3} \quad \forall z \in [z_0 - \delta, z_0 + \delta].
\]

Let
\[
\lambda = \sup \left\{ \frac{|F(x_0, z, q) - F(x_0, z_0, q)|}{|z - z_0|} : z \in [a, b], |z - z_0| \geq \delta \right\}.
\]
Then, we can have an estimate
\[
|F(x_0, z, q) - F(x_0, z_0, q)| \leq \lambda|z - z_0| + \frac{\rho}{3} \quad \forall z \in [a, b]. \quad (A.15)
\]

Let
\[
c = F(x_0, u(x_0), q) + \lambda u(x_0) - \rho,
\]
\[
Y = \left\{ (t, x) \in \mathbb{R} \times \mathbb{R}^N ; |t| < 1/\lambda, \|x - x_0\| < r \right\}.
\]
We consider a function \( y \mapsto v(y) : Y \mapsto \mathbb{R} \) such that
\[
v(y) = v(t, x) = (1 - \lambda t)u(x) - \langle q, x \rangle + ct. \quad (A.16)
\]
Note that \( v \) is lower semicontinuous on \( Y \) (\( \vdash u \) lower semicontinuous),
\[
d^- v(y; h) = (1 - \lambda t)d^- u(x; f) - \langle q, f \rangle + \alpha(c - \lambda u(x)), \quad (A.18)
\]
holds, where \( h = (\alpha, f) \in \mathbb{R} \times \mathbb{R}^N \). According to \((3.45)\), if \((\sigma, p) \in D^- v(t, x), \)
\[
\sigma = c - \lambda u(x) \quad \text{and} \quad s = (p + q)(1 - \lambda t)^{-1} \in D^- u(x). \quad (A.19)
\]
Using Lemma \(A.1.6\) and \((A.19)\), we obtain that for every \( \varepsilon > 0 \), there exist a pair \( y_\varepsilon = (t_\varepsilon, x_\varepsilon) \in Y \) and subgradient \( s_\varepsilon \in D^- u(x_\varepsilon) \) such that
\[
\|x_\varepsilon - x_0\| < \varepsilon, \quad |t_\varepsilon| < \varepsilon,
\]
\[
\min_f [\langle s_\varepsilon, f \rangle - \nu_\varepsilon \langle q, f \rangle + \nu_\varepsilon (c - \lambda u(x_\varepsilon))] > 0 \quad \text{for} \quad \|f\| \leq \rho(x_0) + \rho, \quad (A.20)
\]

where \( \nu_\varepsilon = (1 - \lambda t_\varepsilon)^{-1} \). Note that \( \nu_\varepsilon \to 1 \) as \( \varepsilon \downarrow 0 \). In detail, from \((A.18)\) and
\[(A.16)\]

\[d^- v(y_0; h) = d^- u(x_0; f) - \langle q, f \rangle + \alpha(c - \lambda u(x_0)) \quad \because y_0 = (0, x_0).\]

\[= d^- u(x_0; f) - \langle q, f \rangle + F(x_0, u(x_0), q) - q.\]

By \[(A.14)\]

\[d^- v(y_0; h) > 0 \quad \forall h \in H \doteq \{(1, f) \in \mathbb{R} \times \mathbb{R}^N ; \|f\| \leq \rho(x_0) + \varrho \}.$

By Lemma \[A.1.6\] there exists \((\sigma_\varepsilon, p_\varepsilon)\) such that from \[(A.26)\]

\[\langle (\sigma_\varepsilon, p_\varepsilon), (1, f) \rangle = (c - \lambda u(x_\varepsilon)) + (1 - \lambda t_\varepsilon) \langle s_\varepsilon, f \rangle - \langle q, f \rangle \geq 0 \quad \forall (1, f) \in H,\]

where \((\sigma_\varepsilon, p_\varepsilon) = (c - \lambda u(x_\varepsilon), (1 - \lambda t_\varepsilon)s_\varepsilon - q) \in D^- v(t_\varepsilon, x_\varepsilon)\) and

\[D^- v(t_\varepsilon, x_\varepsilon) = (c - \lambda u(x_\varepsilon), (1 - \lambda t_\varepsilon)D^- u(x_\varepsilon) - q).\]

By multiplying \(\nu_\varepsilon\), we get \[(A.20).\]

The lower semicontinuity of \(u\), continuity of \(F\), and \[(A.15)\] imply

\[\lambda(u(x_\varepsilon) - u(x_0)) \geq \lambda|u(x_\varepsilon) - u(x_0)| - \gamma_\varepsilon^{(1)}, \quad (A.21)\]

\[F(x_\varepsilon, u(x_\varepsilon), q) \geq F(x_0, u(x_\varepsilon), q) - \gamma_\varepsilon^{(2)} \geq F(x_0, u(x_0), q) - \lambda(u(x_\varepsilon) - u(x_0)) - \varrho/3 - \gamma_\varepsilon, \quad (A.22)\]

where \(\gamma_\varepsilon = \gamma_\varepsilon^{(1)} + \gamma_\varepsilon^{(2)} \to 0\) as \(\varepsilon \downarrow 0\). We note that from \[(A.15)\]

\[-\lambda|z - z_0| - \varrho/3 \leq F(x_0, z, q) - F(x_0, z_0, q) \leq \lambda|z - z_0| + \varrho/3 \]

\[\Rightarrow \quad F(x_0, z, q) \geq F(x_0, z_0, q) - \lambda|z - z_0| - \varrho/3.\]

Then, we get \[(A.22).\]

For a sufficiently small \(\varepsilon\), according to \[(A.16), (A.22)\) and , we obtain

\[F(x_\varepsilon, u(x_\varepsilon), \nu_\varepsilon q) - \varrho/3 \geq \nu_\varepsilon[F(x_\varepsilon, u(x_\varepsilon), q) + \gamma_\varepsilon - 2\varrho/3] \geq \nu_\varepsilon(c - \lambda u(x_\varepsilon)). \quad (A.23)\]
In detail, in (A.16) subtract $\lambda u(x_{\varepsilon})$ both sides,
\[ c - \lambda u(x_{\varepsilon}) = F(x_0, u(x_0), q) + \lambda u(x_0) - \lambda u(x_{\varepsilon}) - \varrho. \]

Using this relation, from (A.22), we can get the second inequality of (A.23). Now we consider the first inequality. We observe, as $\varepsilon \downarrow 0$, the first inequality of (A.23) becomes
\[ F(x_0, u(x_0), q) - \varrho/3 \geq F(x_0, u(x_0), q) - 2\varrho/3. \]
Thus, we can find a sufficiently small $\varepsilon > 0$ such that the first inequality holds.

By (A.20) and (A.23),
\[ \langle s_{\varepsilon}, f \rangle - \nu_{\varepsilon} \langle q, f \rangle + F(x_{\varepsilon}, u(x_{\varepsilon}), \nu_{\varepsilon} q) - \varrho/3 > 0, \quad \text{for } \|f\| \leq \rho(x_{\varepsilon}). \quad (A.24) \]

On the other hand, (A.9) implies that there exists a vector $f \in \mathbb{R}^N$ such that
\[ \langle s_{\varepsilon}, f \rangle - \nu_{\varepsilon} \langle q, f \rangle + F(x_{\varepsilon}, u(x_{\varepsilon}), \nu_{\varepsilon} q) = F(x_{\varepsilon}, u(x_{\varepsilon}), s_{\varepsilon}), \quad \|f\| \leq \rho(x_{\varepsilon}). \]

We note that
\[ E(x_{\varepsilon}, u(x_{\varepsilon}), \nu_{\varepsilon} q) = \{ \ldots, g = \langle \nu_{\varepsilon} q, f \rangle - F(x_{\varepsilon}, u(x_{\varepsilon}), \nu_{\varepsilon} q) \} \]
\[ E(x_{\varepsilon}, u(x_{\varepsilon}), s_{\varepsilon}) = \{ \ldots, g = \langle s_{\varepsilon}, f \rangle - F(x_{\varepsilon}, u(x_{\varepsilon}), s_{\varepsilon}) \} \]

Thus, for a sufficiently small $\varepsilon$, $F(x_{\varepsilon}, u(x_{\varepsilon}), s_{\varepsilon}) > \varrho/3 > 0$, where $x_{\varepsilon} \in \mathcal{M}$, $s_{\varepsilon} \in D^-u(x_{\varepsilon})$. This contradicts (A.5). Therefore, (A.5) $\Rightarrow$ (A.12).

\[ \square \]

**Lemma A.1.6.** Let $v$ be a lower semicontinuous function defined on an open set $Y \subset \mathbb{R}^m$. Let $H$ be a convex compact set in $\mathbb{R}^m$, and let $y_0 \in Y$. Assume that
\[ d^-v(y_0; h) > 0, \quad \forall h \in H. \quad (A.25) \]

Then for any $\varepsilon > 0$, there exist a pt $y_{\varepsilon} \in Y$ and a subgradient $s_{\varepsilon} \in D^-v(y_{\varepsilon})$ such that
\[ \|y_0 - y_{\varepsilon}\| < \varepsilon, \quad \langle s_{\varepsilon}, h \rangle > 0, \quad \forall h \in H. \quad (A.26) \]
\[ \|v(y_0) - v(y_{\varepsilon})\| < \varepsilon \quad \text{added in [69], p276} \]
We can find the proof of this lemma in [69 21].
Bibliography


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