SINGLE AND MULTI-STAGE BLADED ROTORS WITH GEOMETRIC MISTUNING:
REDUCED ORDER MODELING AND MISTUNING IDENTIFICATION

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ABSTRACT

Modified Modal Domain Analysis (MMDA) is a novel method for the development of a reduced-order model of a bladed rotor with geometric mistuning. This method utilizes proper orthogonal decomposition (POD) of Coordinate Measurement Machine (CMM) data on blades’ geometries, and sector analyses using ANSYS and solid modeling. First the algorithm to compute reduced-order mass and stiffness matrices from ANSYS sector analyses is provided. It is also shown that the algorithm can be efficiently used to perform Monte Carlo simulations of mistuning patterns arising out of arrangements of a given set of mistuned blades. Numerical examples dealing with variations in natural frequencies and forced responses are presented for different patterns of geometric mistuning. MMDA is then expanded to use 2nd order Taylor series approximations of perturbations in mass and stiffness matrices (δM and δK) instead of exact δM and δK. It is shown that the reduced order model based on 2nd order approximation is accurate enough to provide results comparable to exact MMDA. It is also shown that with the use of Taylor series expansion the calculation of reduced order matrices is just a block assembling exercise which is computationally inexpensive hence this method is ideally suited for Monte Carlo simulations. As a numerical example 1000 mistuning patterns generated by random values of mistuning parameters are analyzed to support the fact that the technique can be efficiently used for Monte Carlo simulations. It is also shown that although a large number of POD features are present in a mistuned bladed disk assembly, only a few are dominant and inclusion of only the dominant POD features in the bases vectors is sufficient to get accurate results. The algorithm for MMDA is then modified to use the mode shapes from cyclic analysis of actual sectors instead of mode shapes from sectors modified along POD features, hence avoiding the step of creating artificial sector geometries perturbed along POD features. The idea is further extended to be applicable in the case of extremely large mistuning where normal POD approach breaks down.
and it is shown that explicit inclusion of mode shapes from cyclic analysis of blade with extremely large mistuning in the bases rectifies the problem and provides accurate results.

A reduced order model of a multi-stage rotor in which each stage has a different number of blades is developed. It is shown that a reduced order model can be developed on the basis of tuned modes of certain bladed disks which can be easily obtained via sector analyses. Further, it is shown that reduced order model can also be obtained when blades are geometrically mistuned. This algorithm is similar to the modified modal domain analysis. The validity of this algorithm is shown for the finite element model of a two-stage bladed rotor. In addition, the statistical distributions of peak maximum amplitudes and natural frequencies of a two-stage rotor are generated via Monte Carlo simulations for different patterns of geometric mistuning.

The Taylor series expansions of deviations in mass and stiffness matrices due to geometric mistuning gives a direct approach for generating the reduced order model from the components of POD features of spatial variations in blades’ geometries. Reversing the process, algorithms for mistuning identification based on MMDA are presented to calculate the geometric mistuning parameters. Two types of the algorithm, one based on modal analysis and the other on the forced responses are presented. The validity of the method is then verified through a mistuned academic rotor.
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Chapter 1

Introduction

Mistuning is a term adopted to designate the small blade-to-blade variations in geometric and material properties, which are unavoidable in all practical bladed disks due to manufacturing and assembly tolerances and non-uniform wear during service. A great deal of interest to understand mistuning and develop advanced models to study vibration characteristics of bladed rotors can be attributed to its fundamental role in aero engine dynamics. Mistuning phenomenon has a detrimental influence on bladed disk high-cycle fatigue (HCF) life and, consequently a negative impact on durability and reliability of the aero engine itself. HCF failures result from excessive blade vibration cycles, exacerbated by mistuning and aeromechanical sources, as thousands of cycles accumulate rapidly due to high rotation speeds of the engines. HCF has historically led to the premature failure of major aero engine components, and in rare cases has resulted in the loss of the engine and the entire aircraft.

The fundamental blade mistuning problem stems from the fact that unavoidable (but generally small) blade-to-blade variations cause simultaneous and dependent perturbations in mass and stiffness matrices of each blade, which has a dramatic effect on the vibration behavior of a bladed disk system. To mention a few, it could potentially lead to spatial localization of vibration energy around one or a limited number of blades, increase the amplitudes and stresses of blades compared to perfectly tuned system. In extreme cases, this very large uncertainty in the forced response levels of bladed disks can lead to a catastrophic failure. For this reason, the blade mistuning problem has been the focus of attention for many researchers for almost forty years, and indisputably, considerable progress has been made on the subject in that time. Numerous studies have contributed to encapsulate the physics of the mistuning phenomenon and its impact on the vibration characteristics of bladed disks. Computationally efficient algorithms that analyze
perfectly periodic structures using the theory of cyclic symmetry have been developed to study the vibration characteristics of bladed rotors. But in presence of mistuning, the cyclic symmetry property is lost and these algorithms fail to accurately predict the dynamics of a disordered periodic system. Under such circumstances one has to revert back to full 360 degree model of the disk to study the vibration characteristics of the system. In this age of advanced computing, it is quite possible to model the full bladed rotor in ANSYS or NASTRAN. However, because of the stochastic nature of the problem, one must conduct Monte Carlo simulations to determine the statistics of response. The cost of conducting Monte Carlo simulations with finite-element models of full rotors will be prohibitive.

Blade mistuning has been extensively researched since Tobias and Arnold’s [1], Whitehead’s [2] and Ewins’ [3] pioneering works in the 1960s, which shed light on the underlying physics of the problem. Since then, a considerable effort has been directed toward enriching the knowledge of the vibration characteristics of mistuned bladed disks and developing new perspectives and formulations to circumvent the mistuning problem (several comprehensive surveys, such as [4 - 6] summarize the existing mistuning literature). In the early years, simple few degrees-of-freedom lumped parameter mass-spring models (LPMs) were used to study the effects of interblade coupling and mistuning on forced response in a deterministic framework. Later studies have relaxed the assumptions of deterministic framework and looked at the statistics of response. One approach to study statistics is to perform Monte Carlo simulation. First, given a value for the standard deviation of random mistuning, the mistuned blade stiffnesses for one realization of a mistuned bladed disk are assigned by a pseudo-random-number generator to study the vibration characteristics of the particular disk. This process is repeated for many realizations of mistuned rotors to generate the statistics of response. In the days of early computing, even the use of lumped-parameter model for such a statistical analysis was computationally expensive. Therefore, researchers have developed various analytical methods for predicting the distribution
of the mistuned forced response without using Monte Carlo simulations (Sinha et. al. [7]-[13]). Although these analytical techniques were applied to lumped parameter models, the insights gained from such simple modes can be applied to more general finite element models. These techniques also included nonlinearities caused by presence of frictional dampers. Nikolic [14] has looked into the effects of Coriolis forces into bladed disk analysis.

Recent advances in computer hardware has increased the computational power available to researchers many folds and it has become quite easy for them to perform Monte Carlo simulations using simple lumped parameter models. Although these simple models are able to capture various physical phenomenon qualitatively, the use of simple spring-mass models is understood to be inadequate for quantitative evaluation in industrial applications and advanced finite element based models are required. Due to mistuning leading to breakdown in cyclic symmetry, modeling just one sector is not sufficient; a full bladed disk model is needed. Modern industrial finite element models of a full bladed disk can be on the order of millions of degrees of freedom. Even in today’s date of advanced computing, the use of full 360 degree models to perform Monte Carlo simulations is infeasible. Therefore, reduction techniques are used to generate reduced-order models (ROM) from the tuned finite element model and geometric mistuning definitions for a frequency range of interest. A brief study of existing reduced order models and their limitations is performed in the next section. In addition, research topics presented in the proposal are also discussed.
Recent Mistuning Studies

Reduced order modeling of single stage rotors

Throughout the last decade, FE methods have been employed to serve as numerical tools to model real mistuned bladed disks and to examine their quantitative behavior. The use of simple spring-mass models was understood to be inadequate for cases when large numbers of degrees-of-freedom are involved, even though many researchers agreed that a simple model was capable of capturing the essential features of the physical mechanisms associated with mistuning. The analysis of detailed FE structural models of bladed disks is computationally intensive and prohibitively expensive, and, hence, the need for model reduction tools emerged, which would reduce the computational models to manageable size, while retaining the model accuracy. A foundation for reduced-order modeling work that came decades later was established by Craig and Bampton in almost the same year that the initial analytical mistuning studies were beginning [15]. Their article defined an approach for sub structuring based on constraint modes at substructure interfaces and the fixed-interface normal modes of the interior DOF. The defined method was the Craig-Bampton (C-B) Component Mode Synthesis (CMS) approach which was computationally efficient, easily implemented, and is widely adopted by commercial finite element modeling codes. A modified CMS technique, applicable to modern day design practice, was shown by Castanier, et al. [16] for a mistuned bladed disk. The approach used cyclic-sector finite element predictions of modal quantities, stiffness, and mass matrices for use in a modified CMS approach that used disk-induced constraint modes. Use of disk-induced constraint modes limits interface DOF and reduces the assembled substructure size and solution time. The modified CMS technique further assumes that mistuned response can be approximated by linear combination of nominal, i.e. tuned, modes. This limits the method to perturbation of blade
frequencies, and ignores geometric mistuning effects on constraint modes and cantilevered substructure mode shapes. The analytical case studies used a full FEM that applied blade Young’s modulus mistuning to represent mistuning. As such, the blade substructures maintained their nominal mode shapes and the full models themselves were only an approximation of geometric mistuning. In the original validation study, accuracy was degraded by excessive interface stiffness caused by the disk-induced constraint mode assumption, and an iterative approach to artificially adjust mistuned modal stiffnesses was employed for solution improvement. The approach has been used by the turbomachinery industry, and software development led to mistuning prediction software, known as REDUCE. The approach is also referred to as a Component Mode Mistuning (CMM) approach. Bladh et al. [17, 18] developed a C-B CMS approach that used a Secondary Modal Analysis Technique (SMART) to further reduce assembled substructure matrices size. Because the approach is based on C-B CMS, the interface stiffness between substructures is more accurately represented. The resulting model has a larger set of DOF compared to REDUCE, but the SMART approach conducts an eigenanalysis of the C-B reduced matrices to create a second reduced basis. The SMART results are used in a mistuning projection technique to accurately predict mistuning with fewer DOF because of the more accurate constraint mode stiffness modeling. As with REDUCE, validation was conducted with a full FEM that applied various Young’s modulus values for each blade to represent mistuning.

During the development of the CMS simulation approaches at the University of Michigan, researchers at Carnegie Mellon were developing alternate approaches for reduced-order mistuning models from tuned FEM results. Yang and Griffin developed an approach based on the receptance method [19]. The receptance method is similar to that of CMS in that the response of a set of connected substructures is predicted by how it reacts at the interface. The non-interface DOFs are expressed in some manner at the interface DOFs. Yang and Griffin’s effort represented the response of the rotor as rigid-body interface displacements and tuned
clamped-free blade modes. The approach was demonstrated on a two-dimensional FEM representing a rotor, and mistuning in the model was represented as elastic modulus changes to the blades. This approach is similar to the REDUCE approach, though the REDUCE uses disk-induced displacements to couple the mode shapes. Use of rigid-body modes at the interface led to overly-stiff response and development of the receptance approach was not continued. Soon after this, Yang and Griffin developed a new reduced-order modeling technique [20, 21]. Their approach created the reduced model through transformation of the full system model to a modal basis of tuned modes. In this case the rotor is treated as a single structure. The number of reduced DOF is determined by how many tuned modes are retained. The response of the mistuned system is approximated by a weighted sum of the nominal modes that are determined by solving the modal eigenvalue problem. Mistuning again is introduced by perturbing the stiffness matrix resulting from Young’s moduli of elasticity of blades. This model is attractive because it can be constructed from a cyclic sector finite element model, results in a small reduced order model, and accurately accounts for blade-disk interface stiffness. The method was demonstrated on a rotor model with mistuning represented by changing the elastic modulus of the blades. Again, as with previous efforts, the validation problem itself was only an approximation of geometric mistuning. The developed method was implemented in software called Subset of Nominal Modes (SNM) and has been widely used in the turbomachinery industry. Feiner and Griffin introduced a modification to the subset of nominal modes approach [22]. In their article, the authors formulated a method to approximate the mistuning matrix, which requires the stiffness and mass matrix from a FEM, with frequency deviations. They proposed a more sophisticated measure of frequency deviations, a “sector frequency approach”, which effectively uses a frequency quantity that describes the mistuning of an entire sector, rather than considering mistuning to be confined to the blades, thus providing a more accurate mistuning representation. The method was demonstrated on a two-dimensional rotor example where mistuning was represented by changing
the blade length, density, and elastic modulus. The assumptions in the model are that a single, isolated family of modes is involved in the response, the strain energy is stored primarily in the blades, the frequencies are closely spaced, and the blade mode shapes are equivalent. The authors conclude that these assumptions are valid for the fundamental, i.e., low order modes. Examples were shown where the method proved inaccurate at higher modes. The developed method was implemented in software called Fundamental Mistuning Model (FMM) and has been widely used in the turbomachinery industry.

One common observation from the works of Castanier, et al. [16], Bladh ([17, 18]), Yang and Griffin ([19 – 21]), is that typically, blade mistuning has been simulated by changes in blades’ Young’s moduli of elasticity or equivalently specific changes in the stiffness matrix. A consequence of simulating a mistuning by changes in blades’ Young’s moduli of elasticity is that such a mistuning only represents the changes in the frequency of the blade without much emphasis on the blades’ mode shapes. Hence such a mistuning is usually referred as “frequency mistuning”. This is not a very precise definition of realistic mistuning parameters, since the variations in frequencies are consequences of variations in physical properties of blades, so that the consideration of all the factors which cause the blade responses to differ from one blade to another, like blade geometry tolerances, becomes an important concern. Recent work by Lim, Bladh, Castanier, and Pierre researched the impact of an aspect geometric mistuning [23]. It specifically addresses the impact of large geometric deformations caused by Foreign Object Damage (FOD) and how the deformed shape of a single blade impacts forced response. The solution approach uses a CMS approach and divides the rotor into two substructures, a tuned bladed disk and the set of mistuned blades. In the formulation, all the DOF in the mistuned blade component are treated as interface DOF. This can lead to large reduced-order models when all the blades are mistuned. By considering the case of a single FOD deformed blade, the model order is greatly reduced. The two subcomponents in the model are coupled through attachment modes
created by applying unit forces to the interface region of the tuned bladed disk. The authors noted that these attachment modes can lead to matrix ill-conditioning and numerical instability due to the fact that displacement values of the attachment modes are much less than those of normal modes. Also, the attachment and normal modes may not be independent. To overcome the numerical challenges caused by the attachment modes, Lim, Castanier, and Pierre developed a reduced-order modeling approach based on the mode acceleration method based on static mode condensation [24]. In this approach the mistuned system is transformed to a reduced basis space of the tuned system modes and a set of static deflection shapes that account for mistuning. These static deflection shapes can be obtained without the need to conduct a more expensive modal analysis. Again, in this case a single blade deformed by a FOD impact was addressed, which reduced the number of static analyses to conduct. Both papers were constrained to the impact of large geometric effects on a single blade and conducted a deterministic analysis. They also did not consider the general case of all blades having geometry mistuning. Petrov, Sanliturk, and Ewins developed an alternate reduced-order mistuning model [25] to compute the forced response of the mistuned bladed disk, but they did not deal with the computation of mistuned frequencies and mode shapes. This model represents mistuned rotors as a summation of the dynamic stiffness matrix of the tuned matrix plus a mistuning matrix. The matrix inversion operation required in the solution is efficiently solved using Sherman-Morrison-Woodbury identity that simplifies the inverse computations for two added matrices. The mistuning matrix is reduced by considering a subset of the full model that include DOF where mistuning is applied and where response quantities are known. With this approach, if blade geometries are varied then the entire blade surface DOF are required, which could become computationally expensive. To avoid this, the authors proposed using a few active DOF per blade and using lumped mass, stiffness, and damping elements to act as representative mistuning elements. It was also noted by the authors
that the accuracy of the model was reliant on the tuned system modes and it therefore shares the nominal-mode assumption of the REDUCE and SNM approaches.

Sinha [26] expanded the SNM approach for geometric mistuning and termed the approach Multiple Modal Domain Analysis (MMDA). This represents a breakthrough in mistuning research as many researchers in industry and universities were unable to develop an effective method to obtain a reduced order model of a bladed rotor with geometric mistuning. The MMDA approach uses nominal system tuned modes and tuned modes of rotors having perturbed geometry based on the Proper Orthogonal Decomposition (POD) of Coordinate Measuring Machine (CMM) data. The spatial statistical analysis produces a set of POD modes that define the geometric deviations with a reduced basis [27]. The perturbed geometries consist of the nominal geometry with the addition of each retained POD mode. The ROM dimension is the number of tuned modes retained multiplied by the number of retained POD modes. The approach was demonstrated on a geometrically-perturbed academic rotor and showed excellent accuracy for a single mode. The approach uses results from a cyclic sector analysis and sector DOF are transformed to the new basis through pre- and post-multiplying matrices. This is significant for the reason that no analysis is required for the full rotor hence the computational efficiency and minimal memory requirements of the sector analysis are retained. Another salient feature of the method is that it can be easily used to analyze the mistuned bladed disk obtained after random permutations of the original mistuning pattern without requiring any additional Finite element analyses. This is significant because it relates to the problem of finding optimal arrangement of blades to reduce vibration amplitudes. The algorithm ([28]) for the use of sector analysis in MMDA and Monte Carlo simulations of random permutations of the original mistuning pattern form the basis of Chapter 2. Due to stochastic nature of the mistuning problem it is necessary to study not only the effects of random permutations of the mistuned blades, but the also the effects of random values of the mistuning parameters themselves. While MMDA is effective in
analyzing the effect of permutations of mistuned blades, it still requires a finite element analysis for each set of mistuning parameter while studying the effects of random variations in mistuning parameters. This limitation can be overcome by expanding the mass and stiffness matrices of the mistuned sectors using Taylor Series expansion and forms the basis for Chapter 3. Recently Brown [29] has used CMS approach in conjunction with nominal and non-nominal mode approximation as suggested by Sinha [26] to model geometric mistuning. The use of nominal mode approximation captures the qualitative behavior but significant errors have been reported by the author when compared with the full 360 degree model. The addition of non-nominal modes has resulted in improvement of accuracy. But this model again suffers from the loss of efficiency due to large number of constraint modes associated with the interface degrees of freedom.

**Reduced order modeling of multi stage rotors**

Although a lot of emphasis has been given to the research on the modeling of bladed disks, only a few studies are devoted to modeling the flexible bladed disk on a flexible shaft system to assess the coupling effects. Studies by Bladh et al. [30] and Rzadkowski et al. [31] have shown that influence of the interstage coupling are important, as the addition of a shaft can change the spectrum of system’s natural frequencies considerably and the dynamics of the multi-stage assembly can differ significantly from the single stage predictions. Traditionally analysis has been done on the single stages without accounting for the interstage coupling or the interstage coupling has been modeled by applying suitable boundary conditions to the single stage analysis. It is obvious that these approximate boundary conditions will in general not describe the disk flexibility locality at the interstage boundaries. This will lead to inconsistent representations of the interaction between the families of blade and disk dominated modes. Modeling of single stage rotors is done by employing cyclic symmetry but the vibration analysis of a multistage rotor is
complicated by the fact that the number of blades on each rotor can be different. In this case, cyclic symmetry is lost even when all blades are identical (perfectly tuned) on each rotor stage, and sector analysis cannot be performed. The complication of vibration analysis is further enhanced in the presence of mistuning of blades.

The use of component mode synthesis methods (with multilevel reductions) has been proposed by Bladh et al. [32]. Song et al. [33] have also used component mode based reduced order model of a multistage rotor. Laxalde et al. [34] have used sector analysis to find mode shapes of a multistage rotor in a subspace generated by the modes of individual disks. But this solution methodology looks for mono-harmonic (harmonic index same for both the disks) solution only and does not find multi-harmonic solutions which are present in the exact solution. Another restriction on the usage of this method arises from the fact that a compatible mesh is needed at the disk interfaces which is not always possible especially for models of complex industrial rotors. Sternchuss et al. [35] have tried to overcome the necessity of compatible mesh at the disk interfaces by extending classical sub structuring technique in cyclic symmetry to compute mono-harmonic eigenvectors. It should be noted that papers by Song et al. [33], Laxalde et al. [34] and Sternchuss et al. [35] do not deal with mistuning. Since these deviations are random variables, it is important to determine the statistics of response via Monte Carlo simulations. As the cyclic symmetry is lost, the response cannot be obtained by sector analysis and using a full 360 degree finite element analysis will be computationally prohibitive, particularly for a multi-stage rotor. Consequently, it is important to develop a technique to obtain an accurate reduced order of a mistuned multi-stage rotor.

Sinha [36] presented the idea that the mode shapes of a multistage system can be represented as a linear combination of modes of individual stages. The idea is attractive because cyclic symmetry can be utilized to generate the mode shapes of individual tuned rotor stages. Further even in case of mistuning, it is possible to get mode shapes of individual stages using
MMDA. The idea was implemented on a simple spring-mass model of a multi-stage rotor in which the interstage coupling was modeled in an adhoc manner. Excellent agreement was observed between results obtained from full order model and the reduced order model. Although the results are encouraging, there is a need to validate the model under finite element setting for a realistic multistage model. Weak coupling between the stages has been considered for the simple model. It would be interesting to observe the accuracy of the model under strong coupling because strong coupling can significantly alter the mode shapes of the individual stages. Few issues like incompatible interface mesh, which do not arise for simple spring mass system, also need further attention. The validation of the model under finite element setting and these questions about interstage coupling and incompatible mesh interfaces would be the subject of discussion in chapter 6.

**Mistuning Identification**

Most of the mistuning analyses have assumed the blade properties to vary randomly, or in some specified fashion, thus, generally avoiding the characterization of mistuning parameters from experimental data. However, in order to have a realistic mistuning representation, it is necessary to model the mistuning accurately. A precise identification of the degree of mistuning has become an important area of investigation, particularly in a practical implementation of forced response reduction strategies in bladed disks. The standard methods for disks with detachable blades consist of removing the individual blades for measurements of their natural frequency. This is not an accurate procedure, since it ignores the effects of blade attachment to the disk, and the problem is especially pronounced in integrally bladed rotors (blisks), where the blades and the disk form one integral piece. Therefore, in order to accurately identify mistuning in bladed disks, it is essential to develop methods that can produce measurements of the entire
bladed disk assembly. Mignolet et al. [37, 38] have first developed two distinct approaches from the measurements of the natural frequencies of the blades – random modal stiffnesses (RMS) and the maximum likelihood (ML) strategy – for an estimation of dynamic properties of the bladed disk to be used in accurate prediction of the forced response. Recently, Lim et al. [39] have developed a new identification method, in which both free and forced response data can be used to gather blade mistuning data. A new method based on the ROM FMM, called FMM ID has been generated by Feiner and Griffin [40, 41], which relies on the measurements of the vibratory response of the system as a whole, and hence, is well suited for integrally bladed rotors or blisks. Feiner and Griffin [42] extended the approach to perform identification of not only blade mistuning, but also damping variations.

The mistuning techniques developed by Mignolet et al. [37, 38], Lim et al. [39], Feiner and Griffin [40, 41] are based on reduced order models, which themselves are based on frequency mistuning. As mentioned earlier, frequency mistuning is not a very precise definition of realistic mistuning parameters. Since the actual mistuning occurs due to perturbations in blade geometry, the identification techniques based on frequency mistuning leads to slightly erroneous estimates of the bladed disk modal characteristics, which in turn provoke large discrepancies in the prediction of the forced response of the entire system near or at resonance in the presence of a small damping ratio. It is essential to account for geometric mistuning to accurately identify the mistuning parameters. Identification of independent mistuning parameters will help to identify the effect of different mistuning parameters, like thickness of blades, blade inclination etc., on the modal characteristics of the system, which in turn can help to devise better strategies to avoid the detrimental effects of mistuning. Since MMDA is based on geometric mistuning, the effects of perturbation in geometry are transparent. The effect of each factor, like thickness of blades, blade inclination etc., can be taken into account separately using this model. The algorithm based on
MMDA for identification of mistuning parameters using the mode shapes and natural frequencies of the mistuned bladed disk assembly is discussed in the chapter 7.

Summary of Research

Based on the review of the existing research and its limitations, a need for reduced order models based on geometric mistuning is identified. The main requirements from such reduced order modeling techniques are:

- should be able to account for simultaneous perturbations in mass and stiffness matrices, i.e. geometric mistuning.
- should be efficient so that Monte Carlo simulations of disks with permutations of a set of mistuned blades can be easily performed to find optimal arrangement of blades.
- should be efficient so that the Monte Carlo simulations of random geometric mistuning parameters can be easily run without any need for new finite element analysis.

In addition a need for reduced order model applicable to multistage systems even in presence of geometric mistuning is identified. There is also a need to develop techniques to identify geometric mistuning parameters from experimental data. In order to satisfy these requirements, the following research has been conducted:

- Algorithm for MMDA based on sector analysis is developed. It is shown that the algorithm based on sector analysis can easily be used to perform Monte Carlo simulations of disks created by permutations of a set of mistuned blades.
- Taylor series expansion of perturbations in mass and stiffness matrices is used to extend MMDA. It is shown that with this extension, the computation of reduced order matrices
is just a block assembling exercise which is computationally inexpensive, hence ideally suited for Monte Carlo simulations.

- Reduced order modeling technique similar to MMDA is developed for application to multistage systems, even in case of geometric mistuning.
- Algorithm for an identification technique based on MMDA is presented which is able to identify geometric mistuning parameters from experimental natural frequencies and mode shapes, or the harmonic response.
Chapter 2

Modified Modal Domain Analysis (MMDA)

Mistuning has traditionally been modeled through the change in the Young’s modulus of blades, or equivalently through perturbation in the stiffness matrices associated with blades’ degrees of freedom. Such a mistuning is an approximation of actual mistuning because it does not capture the simultaneous perturbations in mass and stiffness matrices due to perturbations in geometry. Such a mistuning is commonly referred to as “Frequency Mistuning”. A consequence of Frequency mistuning is that it does not alter the blades’ mode shapes, but only the blade alone frequencies. For Frequency mistuning, reduced order models \([16 - 21]\), have been developed which represent the solution as weighted sum of the modes of the nominal system. Such an assumption works because Frequency mistuning does not alter the mode shapes associated with the blades. But actual (geometric) mistuning leads to simultaneous perturbations in mass and stiffness matrices, which alter the mode shapes associated with the blades, hence the subset of nominal modes assumption is no longer valid and the accuracy of these models is reduced. In case of lightly damped structures like integrated blade rotors or blisks, this inaccuracy can lead to large errors in the predicted forced response of the mistuned system. Sinha \([26]\) has developed a breakthrough approach for modeling mistuned bladed disks in presence of simultaneous perturbation in mass and stiffness matrices, i.e. geometric mistuning, and termed it Modified Modal Domain Analysis (MMDA). It has been shown by Sinha \([26]\) that MMDA is able to capture the effects of geometric mistuning even in case of large mistuning. In the original paper, the results were presented for simulations run through full 360 degree tuned models, but use of sector analysis for reduced order modeling was mentioned. This chapter provides a detailed algorithm for the use of sector analysis in reduced order modeling. Studies \([44]\) have shown that it possible to minimize forced response levels for certain excitation orders through rearrangement
of blades. Such an analysis would require forced response analysis for a large number of blades’ arrangement to find the optimal arrangement. A consequence of using sector analysis in MMDA is that the modal and hence the forced response analysis of bladed disks generated through permutations of original mistuning pattern can be easily carried out without any need for expensive finite element computations, hence this method is ideal for such computations. Monte Carlo simulations for 1000 permutations of an original mistuning pattern have been carried out to study the statistics of response due to blade rearrangement.

**MMDA**

A mistuned bladed-disk assembly or a bladed rotor can be described by

\[
(M_t + \delta M)\ddot{x} + (K_t + \delta K)x = 0
\]

(2.1)

where \(M_t\) and \(K_t\) are mass and stiffness matrices of the perfectly tuned bladed rotor with each blade having the average geometry, respectively. Matrices \(\delta M\) and \(\delta K\) are deviations in mass and stiffness matrices due to mistuning.

Let

\[
x = \Phi y
\]

(2.2)

where

\[
\Phi = [\Phi_1 \quad \Phi_2 \quad \ldots \quad \Phi_{np+1}]
\]

(2.3)

\(\Phi_1\) : \(r\) tuned modes of the system with blades having the mean geometry.

\(\Phi_l\) : \(r\) tuned modes of the system with blades having perturbed geometry along \((l-1)\)th POD feature, \(l = 2, \ldots, np+1\)
Substituting (2.2) into (2.1), and pre-multiplying by $\Phi^H$, the reduced-order model is obtained as follows:

$$M_r \ddot{y} + K_r y = 0$$  \hspace{1cm} (2.4)

where

$$K_r = \Phi^H K_i \Phi + \Phi^H \delta K \Phi$$  \hspace{1cm} (2.5)

$$M_r = \Phi^H M_i \Phi + \Phi^H \delta M \Phi$$  \hspace{1cm} (2.6)

To obtain reduced-order mass and stiffness matrices, $\Phi_i^H K_i \Phi_j$, $\Phi_i^H M_i \Phi_j$, $\Phi_i^H \delta K \Phi_j$ and $\Phi_i^H \delta M \Phi_j$ are to be calculated for $i = 1, 2, \ldots, np+1$, and $j = i, i+1, \ldots, np+1$. Using (2.3), Hermitian matrices on the right hand sides of (2.5) and (2.6) are expressed as follows:

$$\Phi^H K_i \Phi = \begin{bmatrix}
\Phi_1^H K_i \Phi_1 & \Phi_1^H K_i \Phi_2 & \cdots & \Phi_1^H K_i \Phi_{np+1} \\
\Phi_2^H K_i \Phi_1 & \Phi_2^H K_i \Phi_2 & \cdots & \Phi_2^H K_i \Phi_{np+1} \\
& & \ddots & \\
& & & \Phi_{np+1}^H K_i \Phi_{np+1}
\end{bmatrix}$$ \hspace{1cm} (2.7)

$$\Phi^H M_i \Phi = \begin{bmatrix}
\Phi_1^H M_i \Phi_1 & \Phi_1^H M_i \Phi_2 & \cdots & \Phi_1^H M_i \Phi_{np+1} \\
\Phi_2^H M_i \Phi_1 & \Phi_2^H M_i \Phi_2 & \cdots & \Phi_2^H M_i \Phi_{np+1} \\
& & \ddots & \\
& & & \Phi_{np+1}^H M_i \Phi_{np+1}
\end{bmatrix}$$ \hspace{1cm} (2.8)

$$\Phi^H \delta K \Phi = \begin{bmatrix}
\Phi_1^H \delta K \Phi_1 & \Phi_1^H \delta K \Phi_2 & \cdots & \Phi_1^H \delta K \Phi_{np+1} \\
\Phi_2^H \delta K \Phi_1 & \Phi_2^H \delta K \Phi_2 & \cdots & \Phi_2^H \delta K \Phi_{np+1} \\
& & \ddots & \\
& & & \Phi_{np+1}^H \delta K \Phi_{np+1}
\end{bmatrix}$$ \hspace{1cm} (2.9)
**Computation of reduced order matrices via sector analysis**

The computation of reduced-order stiffness and mass matrices (2.5) and (2.6) are straightforward whenever the matrices $K, M$ and $\Phi$ are known for the full bladed disk. However, it is a common practice to use sector analysis for a perfectly tuned system. Therefore, an algorithm has been developed to compute reduced order matrices $K_r$ and $M_r$ on the basis of sector analyses; i.e. without requiring any full rotor analyses. Here, only algorithms to compute $\Phi_i^H M_i \Phi_j$ and $\Phi_i^H \partial M \Phi_j$ are presented. The procedures to compute $\Phi_i^H K_i \Phi_j$ and $\Phi_i^H \partial K \Phi_j$ are similar.

**Computation of $\Phi_i^H M_i \Phi_j$ via ANSYS sector analyses**

Let

$$
\Phi_i = \begin{bmatrix}
\varphi_{i,1,0} & \varphi_{i,1,1} & \cdots & \varphi_{i,1,n-1} \\
\varphi_{i,2,0} & \varphi_{i,2,1} & \cdots & \varphi_{i,2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{i,n,0} & \varphi_{i,n,1} & \cdots & \varphi_{i,n,n-1}
\end{bmatrix}
$$

(2.11)
where $\phi_{i,j,l}$ represents the nodal displacements for sector $j$ and interblade phase angle 

$$\frac{2\pi l}{n},$$
and $n_r$ is the number of interblade phase angles. Note that

$$\phi_{i,j,p} = e^{i(j-1)\psi_p} \phi_{i,1,p}$$

(2.12)

where

$$\psi_p = \frac{2\pi p}{n}$$

(2.13)

and $t = \sqrt{-1}$. Describe the tuned mass matrix of the full rotor as follows:

$$M_t = \begin{bmatrix}
M_{t,1,1} & M_{t,1,2} & \cdots & M_{t,1,n} \\
M_{t,2,1} & M_{t,2,2} & \cdots & M_{t,2,n} \\
\vdots & \vdots & \ddots & \vdots \\
M_{t,n,1} & M_{t,n,2} & \cdots & M_{t,n,n}
\end{bmatrix}$$

(2.14)

It should be remembered that $M_t$ will be a circulant matrix; i.e., the second block row will be obtained by shifting the first block row by one location, the third block row will be obtained by shifting the second block row by one location, and so on.

$$M_t \Phi_j = \begin{bmatrix}
M_{t,1,1} & M_{t,1,2} & \cdots & M_{t,1,n} \\
M_{t,2,1} & M_{t,2,2} & \cdots & M_{t,2,n} \\
\vdots & \vdots & \ddots & \vdots \\
M_{t,n,1} & M_{t,n,2} & \cdots & M_{t,n,n}
\end{bmatrix} \begin{bmatrix}
\phi_{j,1,0} & \phi_{j,1,1} & \cdots & \phi_{j,1,n-1} \\
\phi_{j,2,0} & \phi_{j,2,1} & \cdots & \phi_{j,2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{j,n,0} & \phi_{j,n,1} & \cdots & \phi_{j,n,n-1}
\end{bmatrix}$$

(2.15)

The $(l, \rho)$ element of $M_t \Phi_j$ is written as

$$M_{t,l,1} \phi_{j,1,\rho} + M_{t,l,2} \phi_{j,2,\rho} + \cdots + M_{t,l,n} \phi_{j,n,\rho} = M_{t,\rho}^c \phi_{j,l,\rho}$$

(2.16)

where

$$M_{t,\rho}^c = M_{t,1,1} + e^{i\psi_p} M_{t,1,2} + \cdots + e^{i(n-1)\psi_p} M_{t,1,n}$$

(2.17)

Note that $M_{t,\rho}^c$ is the complex mass matrix for the interblade phase angle $= \psi_p$. 


Therefore

\[
M_j \Phi_j = \begin{bmatrix}
M^c_{10} \phi_{j,1,0} & M^c_{11} \phi_{j,1,1} & \cdots & M^c_{1n-1} \phi_{j,1,n-1} \\
M^c_{20} \phi_{j,2,0} & M^c_{21} \phi_{j,2,1} & \cdots & M^c_{2n-1} \phi_{j,2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
M^c_{n0} \phi_{j,n,0} & M^c_{n1} \phi_{j,n,1} & \cdots & M^c_{n,n-1} \phi_{j,n,n-1}
\end{bmatrix}
\] (2.18)

Pre-multiplying equation (2.18) by \( \Phi_j^H \), which is the complex conjugate transpose of \( \Phi_j \):

\[
\Phi_j^H M_j \Phi_j = \begin{bmatrix}
\phi_{i,1,0}^H & \phi_{i,2,0}^H & \cdots & \phi_{i,n,0}^H \\
\phi_{i,1,1}^H & \phi_{i,2,1}^H & \cdots & \phi_{i,n,1}^H \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{i,1,n-1}^H & \phi_{i,2,n-1}^H & \cdots & \phi_{i,n,n-1}^H
\end{bmatrix} \times \begin{bmatrix}
M^c_{10} \phi_{j,1,0} & M^c_{11} \phi_{j,1,1} & \cdots & M^c_{1n-1} \phi_{j,1,n-1} \\
M^c_{20} \phi_{j,2,0} & M^c_{21} \phi_{j,2,1} & \cdots & M^c_{2n-1} \phi_{j,2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
M^c_{n0} \phi_{j,n,0} & M^c_{n1} \phi_{j,n,1} & \cdots & M^c_{n,n-1} \phi_{j,n,n-1}
\end{bmatrix}
\]

(2.19)

The \((\ell+1, \rho+1)\) element of \( \Phi_j^H M_j \Phi_j \) is as follows:

\[
\phi_{i,1,\ell}^H M^c_{i\rho} \phi_{j,1,\rho} + \phi_{i,2,\ell}^H M^c_{i\rho} \phi_{j,2,\rho} + \cdots + \phi_{i,n,\ell}^H M^c_{i\rho} \phi_{j,n,\rho} \\
= (1 + e^{i(\psi - \psi_\ell)}) + \cdots + e^{i(n-1)(\psi - \psi_\ell)}) \phi_{i,1,\ell}^H M^c_{i\rho} \phi_{j,1,\rho}
\]

(2.20)

where

\[
\chi = \frac{1 - e^{i(\psi - \psi_\ell)}}{1 - e^{i\psi - \psi_\ell}}
\]

(2.21)

A closer look of equation (2.21) shows that \( e^{i(\psi - \psi_\ell)} = e^{i2\pi(\rho - \ell)} \).

Therefore \( \chi = n \) if \( \rho = \ell \) and zero otherwise. Hence the matrix \( \Phi_j^H M_j \Phi_j \) is a block diagonal matrix.

Let \( \chi_R \) and \( \chi_I \) be real and imaginary parts of \( \chi \), respectively. With \( M^c_{i\rho} \) and \( M^e_{i\rho} \) being real and imaginary parts of \( M^c_{i\rho} \),

\[
\chi M_{i\rho}^c = M_{i\rho}^{mR} + i M_{i\rho}^{mI}
\]

(2.22)
where

\[ M_{ip}^{mcR} = (\chi_R M_{ip}^{cR} - \chi_I M_{ip}^{cI}) \quad \text{and} \quad M_{ip}^{mcI} = (\chi_I M_{ip}^{cR} + \chi_R M_{ip}^{cI}) \]  

(2.23)

Let

\[ \varphi_{j,1,\rho} = \varphi_{j,1,\rho}^R + \nu \varphi_{j,1,\rho}^I \]  

(2.24)

From equation (2.20), the \((l + 1, \rho + 1)\) element of \(\Phi_i^HM_i\Phi_j\) is further expressed as follows:

\[ \chi_{i,j}^HM_{ip}^c \varphi_{j,1,\rho} = \varphi_{i,l,\rho}^R M_{ip}^{mc} \varphi_{j,1,\rho}^I \]

\[ = \left[ \begin{array}{cc} M_{ip}^{mcR} & -M_{ip}^{mcI} \\ M_{ip}^{mcI} & M_{ip}^{mcR} \end{array} \right] \left[ \begin{array}{c} \varphi_{j,1,\rho}^R \\ \varphi_{j,1,\rho}^I \end{array} \right] + \left[ \begin{array}{c} -\varphi_{i,l,\rho}^R \\ -\varphi_{i,l,\rho}^I \end{array} \right] \left[ \begin{array}{ccc} M_{ip}^{mcR} & -M_{ip}^{mcI} \\ M_{ip}^{mcI} & M_{ip}^{mcR} \end{array} \right] \left[ \begin{array}{c} \varphi_{j,1,\rho}^R \\ \varphi_{j,1,\rho}^I \end{array} \right] \]  

(2.25)

From equation (2.23),

\[ \left[ \begin{array}{cc} M_{ip}^{mcR} & -M_{ip}^{mcI} \\ M_{ip}^{mcI} & M_{ip}^{mcR} \end{array} \right] = \chi_R \left[ \begin{array}{cc} M_{ip}^{cR} & -M_{ip}^{cI} \\ M_{ip}^{cI} & M_{ip}^{cR} \end{array} \right] + \chi_I \left[ \begin{array}{cc} -M_{ip}^{cI} & -M_{ip}^{cR} \\ M_{ip}^{cR} & -M_{ip}^{cI} \end{array} \right] \]

(2.26)

The blocks of elements of \(\Phi_i^HM_i\Phi_j\) can be calculated using the above procedure and then assembled to generate \(\Phi_i^HM_i\Phi_j\).

**Computation of \(\Phi_i^HdM\Phi_j\) via ANSYS sector analyses**

The perturbation in the mass matrix, \(dM\), has the block diagonal form:

\[ dM = \begin{bmatrix} dM_1 & 0 & \cdots & 0 \\ 0 & dM_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & dM_n \end{bmatrix} \]  

(2.27)

where \(dM_j\) is the perturbation in the mass matrix of sector\#\(j\) due to geometric mistuning.

Therefore,
\[ \Phi_i^H \delta M \Phi_j = \sum_{\ell=1}^{n} \Phi_i^H \delta M^b_{\ell} \Phi_j \]  

(2.28)

where

\[ \delta M^b_{\ell} = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \cdots & 0 & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \]  

(2.29)

Now,

\[ \Phi_i^H \delta M^b_{\ell} \Phi_j = \begin{bmatrix} \varphi_{i,0}^H & \varphi_{i,1}^H & \cdots & \varphi_{i,n-1}^H \\ \varphi_{i,1,0}^H & \varphi_{i,1,1}^H & \cdots & \varphi_{i,1,n-1}^H \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{i,1,n-1}^H & \varphi_{i,2,n-1}^H & \cdots & \varphi_{i,n,n-1}^H \end{bmatrix} \times \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \vdots & \cdots & 0 & \cdots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \]  

(2.30)

or

\[ \Phi_i^H \delta M^b_{\ell} \Phi_j = \begin{bmatrix} \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,0} & \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,1} & \cdots & \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,n-1} \\ \varphi_{i,1,0}^H \delta M_{\ell} \varphi_{j,0} & \varphi_{i,1,1}^H \delta M_{\ell} \varphi_{j,1} & \cdots & \varphi_{i,1,n-1}^H \delta M_{\ell} \varphi_{j,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{i,1,n-1}^H \delta M_{\ell} \varphi_{j,0} & \varphi_{i,2,n-1}^H \delta M_{\ell} \varphi_{j,1} & \cdots & \varphi_{i,n,n-1}^H \delta M_{\ell} \varphi_{j,n-1} \\ \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,0} & \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,1} & \cdots & \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,n-1} \\ \varphi_{i,1,0}^H \delta M_{\ell} \varphi_{j,0} & \varphi_{i,1,1}^H \delta M_{\ell} \varphi_{j,1} & \cdots & \varphi_{i,1,n-1}^H \delta M_{\ell} \varphi_{j,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{i,1,n-1}^H \delta M_{\ell} \varphi_{j,0} & \varphi_{i,2,n-1}^H \delta M_{\ell} \varphi_{j,1} & \cdots & \varphi_{i,n,n-1}^H \delta M_{\ell} \varphi_{j,n-1} \\ \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,0} & \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,1} & \cdots & \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,n-1} \\ \varphi_{i,1,0}^H \delta M_{\ell} \varphi_{j,0} & \varphi_{i,1,1}^H \delta M_{\ell} \varphi_{j,1} & \cdots & \varphi_{i,1,n-1}^H \delta M_{\ell} \varphi_{j,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{i,1,n-1}^H \delta M_{\ell} \varphi_{j,0} & \varphi_{i,2,n-1}^H \delta M_{\ell} \varphi_{j,1} & \cdots & \varphi_{i,n,n-1}^H \delta M_{\ell} \varphi_{j,n-1} \end{bmatrix} \]  

(2.31)

The \((v + 1, \rho + 1)\) element of \(\Phi_i^H \delta M^b_{\ell} \Phi_j\) in equation (2.31) is as follows:

\[ \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,0} = e^{i(\psi_{i,v} - \psi_{j,v})} \varphi_{i,0}^H \delta M_{\ell} \varphi_{j,0} = \begin{bmatrix} \varphi_{i,0}^R & \varphi_{i,0}^I \\ \varphi_{i,0}^I & \varphi_{i,0}^R \end{bmatrix} \begin{bmatrix} \delta M^R_{i,0} - \delta M^I_{i,0} \\ \delta M^I_{i,0} + \delta M^R_{i,0} \end{bmatrix} + [1 - \varphi_{i,0}^I \varphi_{i,0}^R] \begin{bmatrix} \delta M^R_{i,0} - \delta M^I_{i,0} \\ \delta M^I_{i,0} + \delta M^R_{i,0} \end{bmatrix} \]  

(2.32)

where
\( \delta M_{i\ell} = \delta M_i \cos(\ell(\psi_\rho - \psi_v)) \) \hfill \quad (2.33a)

and \( \delta M_{i\ell} = \delta M_i \sin(\ell(\psi_\rho - \psi_v)) \) \hfill \quad (2.33b)

The blocks of elements of \( \Phi_i^H \delta M_i^H \Phi_j \) can be calculated using the above procedure. These blocks can be summed over all sectors and assembled to generate \( \Phi_i^H \delta M \Phi_j \).

**Connection with ANSYS Sector Analysis**

For a perfectly tuned system, the number of repeated eigenvalues sets equals \((n - 1)/2\) and \((n - 2)/2\) for odd and even number of blades \(n\). This also implies that the number of eigenvalue sets equals 1 and 2 for odd and even number of blades \(n\), respectively. For each eigenvalue, the eigenvector corresponds to a constant interblade phase angle \(\beta\) as follows:

\[
\beta = \frac{2\pi i}{n}; \quad i = 0, 1, 2, \cdots (n - 1)
\]

For odd \(n\), the eigenvector corresponding to the unrepeated eigenvalue represents 0 degree interblade phase angle tuned mode. For even \(n\), the eigenvectors corresponding to unrepeated eigenvalues represent 0 and 180 degrees interblade phase angle tuned modes.

**Figure 2-1: Sector of a Bladed Rotor**
Let \( u_L \) and \( u_R \) be the displacement vectors for the nodes on the left and right edges (Figure 2-1) of a finite element sector, respectively. The condition of a constant interblade phase angle \( \beta \) implies that

\[
u_L = e^{i\beta} u_R \tag{2.35}\]

where \( i = \sqrt{-1} \). Let

\[
u_L = u_L^R + u_L^I \tag{2.36}\]
\[
u_R = u_R^R + u_R^I \tag{2.37}\]

Substituting (2.36) and (2.37) into (2.35),

\[
u_L^R + u_L^I = (u_R^R \cos \beta - u_R^I \sin \beta) + i(u_R^R \sin \beta + u_R^I \cos \beta) \tag{2.38}\]

Then, the equation (2.38) leads to the following relation between the real and the imaginary parts of \( u_L \) and \( u_R \):

\[
\begin{bmatrix}
\nu_L^R \\
\nu_L^I \\
\nu_L^R \\
\nu_L^I \\
\end{bmatrix} =
\begin{bmatrix}
\cos \beta & -\sin \beta \\
\sin \beta & \cos \beta \\
\end{bmatrix}
\begin{bmatrix}
\nu_R^R \\
\nu_R^I \\
\end{bmatrix} \tag{2.39}
\]

Figure 2-2: Two Sectors for Cyclic Symmetry Analysis with ANSYS
The ANSYS code only works with real numbers. It constructs two identical sectors, one for real part and another for imaginary part, Figure 2-2. As a result, only one interblade phase angle for a repeated eigenvalue is considered and the interblade phase angles for odd and even number of blades (or sectors) are as follows:

For odd \( n \),

\[
\beta = \frac{2\pi i}{n} ; \ i = 0,1,2,\cdots, \frac{n-1}{2}
\]  

(2.40)

For even \( n \),

\[
\beta = \frac{2\pi i}{n} ; \ i = 0,1,2,\cdots, \frac{n}{2}
\]  

(2.41)

The mass and stiffness matrices for a double sector are as follows:

\[
\begin{bmatrix}
M^{cR}_{tp} & -M^{cl}_{tp} \\
M^{cl}_{tp} & M^{cR}_{tp}
\end{bmatrix}
\] and

\[
\begin{bmatrix}
K^{cR}_{tp} & -K^{cl}_{tp} \\
K^{cl}_{tp} & K^{cR}_{tp}
\end{bmatrix}
\]  

(2.42)

Therefore, \( M^{cR}_{tp} \), \( M^{cl}_{tp} \) in equation (2.26) can be obtained directly from ANSYS mass matrices.

Furthermore, eigenvectors of the double sector have exactly the form \( \begin{bmatrix} \phi^{R}_{i,1,\ell} & \phi^{I}_{i,1,\ell} \end{bmatrix} \) and \( \begin{bmatrix} -\phi^{I}_{i,1,\ell} & \phi^{R}_{i,1,\ell} \end{bmatrix} \) which appear in equation (2.25) and equation (2.32).

**Workflow, Computational Time and Memory Requirements**

MMDA analysis workflow can be divided into two parts. First part corresponds to the input data generation, in which the inputs for MMDA analysis are generated from the blade geometry data obtained from Coordinate Measuring Machine (CMM). The generation of input data includes following steps:

1. Proper Orthogonal Decomposition of CMM data to obtain average geometry and POD features.
2. Cyclic analysis of average geometry sector to obtain mass and stiffness matrices, and mode shapes for each harmonic index.

3. Cyclic analysis of sector with geometry perturbed along a POD feature to obtain mode shapes for each harmonic index and POD feature.

4. Cyclic analysis of actual blade sectors to obtain mass and stiffness matrices for harmonic index zero.

Figure 2-3: Inputs Generation for MMDA Analysis

The steps for input data generation are summarized in figure 2-3.
The second step consists for executing the MMDA algorithm, i.e. assembling the reduced order matrices and solution of the eigenvalue problem for the reduced order matrices. The execution of MMDA analysis involves the following steps:

1. Reading inputs (Mass and Stiffness Matrices and Mode shapes) from the FEM data.
2. Generating reduced order matrices from input data.
3. Solution of the eigenvalue problem to obtain modal coefficients of the actual mode shapes.
4. Recombining the tuned mode shapes in the basis to obtain actual mode shapes.

Figure 2-4 summarizes the steps for executing, the MMDA Analysis. MMDA analysis block in the figure combines the first 3 steps of MMDA analysis execution.

![Diagram](image)

**Figure 2-4: MMDA Analysis**

In order to estimate the computational time and memory requirements for MMDA analysis, it is essential to understand the work flow of generation of reduced order matrices and the dimensions of the reduced order matrices because the computational time for solving the eigenvalue problem is dependent on the size of the reduced order matrices. Figure 2-5 summarizes the steps involved in the calculation of reduced order matrices.
Figure 2-5: MMDA Analysis (Calculation of Reduced Order Matrices and Solution of Eigenvalue Problem)
**Nomenclature**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>Number of sectors</td>
</tr>
<tr>
<td>$nr$</td>
<td>Number of harmonic indices</td>
</tr>
<tr>
<td>$\rho$</td>
<td>Harmonic index $0 \leq \rho \leq nr$</td>
</tr>
<tr>
<td>$l$</td>
<td>Sector index</td>
</tr>
<tr>
<td>$nd$</td>
<td>Number of degrees of freedom per sector</td>
</tr>
<tr>
<td>$np$</td>
<td>Number of POD features</td>
</tr>
<tr>
<td>$T_*$</td>
<td>Total time for an operation (Read/Calculate matrices/mode shapes)</td>
</tr>
<tr>
<td>$t_*$</td>
<td>Time for calculating a single element of matrix/mode shape</td>
</tr>
</tbody>
</table>

The major steps in MMDA computations, their computational time and memory requirement can be summarized as:

1. Read Mass and Stiffness matrices $M_{\rho}^l$ and $K_{\rho}^l$ of the nominal sector for each harmonic index $0 \leq \rho \leq nr$. The mass and stiffness matrices are sparse matrices for which the number of non-zero elements ($nnzero$) is proportional to the total number of degrees of freedom, i.e. $nnzero = \alpha \times nd$. Here $\alpha (1 \leq \alpha \leq nd)$ is constant dependent on the sparsity pattern of the matrix. For a dense matrix, $\alpha = nd$, whereas $\alpha = 1$ represents a diagonal matrix. The memory ($mem$) for storing a matrix is proportional to the number of non-zero elements in the matrix, i.e. $mem = \beta \times nnzero$, where $\beta$ is a constant amount of memory required to store a matrix element in memory. The time ($t_*$) for reading a matrix is also proportional to the number of non-zero elements in the matrix, i.e. $t_{M_{\rho}^l + K_{\rho}^l} = \mu \times nnzero$. 

Here $\mu$ is a constant time associated with reading one element of the matrix. Hence for matrices for all harmonic indices $\rho$, the total memory and computational time required can be written as:

a) Memory = $\text{mem} \times (1 + nr) = \beta \times \alpha \times \text{nd} \times (1 + nr)$

b) Total Time = $T_{M^c_{\rho},K^c_{\rho}} = \mu \times \text{nnzero} \times (1 + nr) = \mu \times \alpha \times \text{nd} \times (1 + nr)$

2. Read the mode shapes for all harmonic indices $(0 \leq \rho \leq nr)$ and POD features $0 \leq i \leq np$.

   The number of elements of a mode shape is equal to the number of degrees of freedom ($nd$). Therefore the memory ($\text{mem}$) and time ($t_\Phi$) required to read a mode shape are proportional to the number of degrees of freedom respectively, i.e. $\text{mem} = \beta \times \text{nd}$ and $t_\Phi = \mu \times \text{nd}$. $\mu$ and $\beta$ are again the constant time and memory associated with reading and storing a single element of mode shape vector. If $m$ modes are used per harmonic index, the total number of modes ($\text{nmodes}$) required is $\text{nmodes} = m \times (1 + np) \times (1 + nr)$.

   Therefore for mode shapes for all harmonic indices ($\rho$) and POD features ($np$), the total time and memory requirement can be written as:

a) Memory = $\text{mem} \times \text{nmodes} = \beta \times \text{nd} \times m \times (1 + np) \times (1 + nr)$

b) Total Time = $T_\Phi = t_\Phi \times \text{n modes} = \mu \times \text{nd} \times m \times (1 + nr) \times (np + 1)$

3. Calculate $\Phi^HM_i\Phi$ and $\Phi^HK_i\Phi$. The structure of matrices $\Phi^HM_i\Phi$ and $\Phi^HK_i\Phi$ is shown in Figure 2-6a. As suggested from equations 2.20 and 2.21, the matrices are block diagonal with block elements given by $n\phi_{i,\rho}^H M_{ij}\phi_{j,\rho}$. For $m$ modes per harmonic index ($\rho$), the size of a block is $m^2$. The calculation of blocks $n\phi_{i,\rho}^H M_{ij}\phi_{j,\rho}$ or $n\phi_{i,\rho}^H K_{ij}\phi_{j,\rho}$ involves 2 steps:
a) Sparse-dense matrix multiplication ($M_{ij}^C, \varphi_{j,1,\rho}$). For $\varphi_{j,1,\rho}$ having a single mode shape, the time for matrix multiplication is proportional to number of non-zero elements in the matrix. Since the number of non-zero elements (nnzero) of the matrix is itself proportional to the number of degrees of freedom ($nd$), the time for matrix multiplication is proportional to $nd$.

b) Dense-dense matrix multiplication $n\varphi_{i,1,\rho}^H(M_{ij}^C, \varphi_{j,1,\rho})$. For $\varphi_{j,1,\rho}$ having a single mode shape, $M_{ij}^C, \varphi_{j,1,\rho}$ is a vector of length $nd$. Hence the time for calculation of $n\varphi_{i,1,\rho}^H(M_{ij}^C, \varphi_{j,1,\rho})$ (dense-dense matrix multiplication) is proportional to number of degrees of freedom ($nd$).

Hence time ($t_{\phi^{\nu}M\phi^{\nu}K\varphi}$) for computation of one element of matrices $n\varphi_{i,1,\rho}^H M_{ij}^C, \varphi_{j,1,\rho}$ and $n\varphi_{i,1,\rho}^H K_{ij}^C, \varphi_{j,1,\rho}$ is proportional to $nd$, i.e. $t_{\phi^{\nu}M\phi^{\nu}K\varphi} = \tau \times (\alpha + 2) \times nd$. Here $\tau$ is constant time taken for multiply-add step in matrix-matrix multiplication. Hence for number of POD features $= np$, number of sector $n$, and modes $m$ per harmonic index, the number of non-zero elements in matrices $\Phi^H M_i \Phi$ and $\Phi^H K_i \Phi$ is $\text{NNZERO}_{\Phi^H M_i \Phi + \Phi^H K_i \Phi} = n \times (np + 1)^2 \times m^2$. Hence the total memory requirement and computational time can be written as:

a) Memory $= 2\beta \times \text{NNZERO}_{\Phi^H M_F \Phi + \Phi^H K_F}$ $= 2\beta \times n \times (np + 1)^2 \times m^2$

b) $T_{\Phi^H M_F \Phi + \Phi^H K_F} = \tau \times nd \times \text{NNZERO}_{\Phi^H M_F \Phi + \Phi^H K_F}$ $= \tau \times (\alpha + 2) \times nd \times (np + 1)^2 \times m^2$

4. Calculate $\delta M_l$ and $\delta K_l$ for each sector $1 \leq l \leq n$. The calculation involves two steps. Read the mass ($M_{10}^C$) and stiffness ($K_{10}^C$) matrices for each sector and then compute
\[ \delta M_l = M_{t0}^c - M_{t0}^r \quad \text{and} \quad \delta K_l = K_{t0}^c - K_{t0}^r \] for each sector. The structure of the matrices \( M_{t0}^c \) and \( K_{t0}^c \) is similar to the structure of matrices \( M_{t0}^r \) and \( K_{t0}^r \), hence the number of non-zero elements in \((\text{nnzero})\) is proportional to the number of degrees of freedom \((nd)\). Hence for each \( l \), the time for computation of \( \delta M_l \) and \( \delta K_l \) can be written as
\[ t_{\delta M_l+\delta K_l} = \kappa \times \alpha \times nd. \] \( \kappa \) is a constant time required to calculate (read + subtract) 1 element of \( \delta M_l \) and \( \delta K_l \). The number of non-zero \((\text{nnzero} \ \delta M_l+\delta K_l)\) elements of matrix \( \delta M_l \) and \( \delta K_l \) is again proportional to the number of degrees of freedom \((nd)\), i.e. \( \text{nnzero} \ \delta M_l+\delta K_l = \gamma \times nd \). \( \gamma \) is constant dependent sparsity on the pattern of the matrices.

Hence the total memory and computational time can be written as:

a) Memory = \( \beta \times \text{nnzero} \ \delta M_l+\delta K_l \times n = \beta \times \gamma \times nd \times n \)

b) \( T_{\delta M_l+\delta K_l} = t_{\delta M_l+\delta K_l} \times n = \kappa \times \alpha \times nd \times n \)

5. Calculate \( \phi^H_{i,1,v} \delta M_l \phi_{j,1,\rho} \) and \( \phi^H_{i,1,v} \delta K_l \phi_{j,1,\rho} \) for each sector \( 1 \leq l \leq n \), POD \( 0 \leq i, j \leq np \) and harmonic indices \( 0 \leq \rho, v \leq nr \). Figure 2-6b shows the structure of the calculated matrices. The steps for the computation of \( \phi^H_{i,1,v} \delta M_l \phi_{j,1,\rho} \) and \( \phi^H_{i,1,v} \delta K_l \phi_{j,1,\rho} \) are same as discussed for step 3. Hence the time for computation of an element of \( \phi^H_{i,1,v} \delta M_l \phi_{j,1,\rho} \) and \( \phi^H_{i,1,v} \delta K_l \phi_{j,1,\rho} \) can be written as \( t_{\phi^H_{i,1,v} \delta M_l \phi_{j,1,\rho} + \phi^H_{i,1,v} \delta K_l \phi_{j,1,\rho}} = \tau \times (\gamma + 2) \times nd \). But as seen from figure 2-6b, the matrices \( \Phi^H \delta M_l \Phi \) and \( \Phi^H \delta K_l \Phi \) are dense matrices for size \( mn(1+np) \times mn(1+np) \). Hence the total time for computation of matrices \( \Phi^H \delta M_l \Phi \) and \( \Phi^H \delta K_l \Phi \) for each \( l \) can be written as:

a) Memory = \( 2 \beta \times (m \times n \times (1+np))^2 \times n \)
6. Assemble $\Phi^H \delta M \Phi$ & $\Phi^H \delta K \Phi$. This step involves assembling the matrices calculated in the previous step to calculate the reduced order matrices. The dimensions of matrices $\Phi^H \delta M \Phi$ & $\Phi^H \delta K \Phi$ are $mn(1+np) \times mn(1+np)$. Hence the total memory and time requirements are:

- a) Memory = $2\beta \times (m \times n \times (1+np))^2$
- b) $T_{a,\delta M+\Phi^H \delta K \Phi} = t_{\delta M+\Phi^H \delta K \Phi} \times n^3 \times (np+1)^2 \times m^2$
- c) $t_{\delta M+\Phi^H \delta K \Phi}$ is the time in assembling one element of $\Phi^H \delta M \Phi$ & $\Phi^H \delta K \Phi$

7. Solve the eigenvalue problem. The solution of the eigenvalue problem provides the modal coefficients and natural frequencies of the mistuned bladed disk assembly. The computational complexity of the solution of the eigenvalue problem for a dense Hermitian matrix is $O(k^3)$ [45] where $k = \beta \times (m \times n \times (1+np))$ is the number of rows (or columns) of the matrix. Hence the total time and memory requirement can be written as:

- a) Memory = $\beta \times (m \times n \times (1+np))^2 + \beta \times (m \times n \times (1+np))$
- b) $T_{eigs} = O\left((m \times n \times (1+np))^3\right)$
(a) Structure of $\Phi^H M, \Phi$ and $\Phi^H K, \Phi$.

(b) Structure of $\Phi^H \delta M, \Phi$ and $\Phi^H \delta K, \Phi$

Figure 2-6: Structure of the intermediate matrices calculated in MMDA
The memory and time requirements of the different steps in MMDA are summarized in table 2-1.

Table 2-1: Memory and computational time requirements of MMDA

<table>
<thead>
<tr>
<th>Step</th>
<th>Time</th>
<th>Memory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Read Mass and Stiffness Matrices</td>
<td>$\mu \times \alpha \times nd \times (1 + nr')$</td>
<td>$\beta \times \alpha \times nd \times (1 + nr)$</td>
</tr>
<tr>
<td>( $M_{i{\rho}}^c$ and $K_{i{\rho}}^c$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Read Mode Shapes ( $\Phi$ )</td>
<td>$\mu \times nd \times m \times (1 + nr) \times (np + 1)$</td>
<td>$\beta \times nd \times m \times (1 + np) \times (1 + nr)$</td>
</tr>
<tr>
<td>Calculate $\Phi^H M_{i{\rho}} \Phi$ &amp; $\Phi^H K_{i{\rho}} \Phi$</td>
<td>$\tau \times (\alpha + 2) \times nd \times n \times (np + 1)^2 \times m^2$</td>
<td>$2\beta \times n \times (np + 1)^2 \times m^2$</td>
</tr>
<tr>
<td>Calculate $\delta M_I = M_I^c - M_d^c$ &amp; $\delta K_I = K_I^c - K_d^c$</td>
<td>$\kappa \times \alpha \times nd \times n$</td>
<td>$\beta \times \gamma \times nd \times n$</td>
</tr>
<tr>
<td>Calculate $\phi_{i{\rho}1,v}^H \delta M_I \phi_{j{\rho}1,v} &amp; \phi_{i{\rho}1,v}^H \delta K_I \phi_{j{\rho}1,v}$</td>
<td>$\tau \times (\gamma + 2) \times nd \times n^3 \times (np + 1)^2 \times m^2$</td>
<td>$2\beta \times (m \times n \times (1 + np))^2 \times n$</td>
</tr>
<tr>
<td>Assemble $\Phi^H \delta M \Phi$ &amp; $\Phi^H \delta K \Phi$</td>
<td>$\tau a_{\phi^{\rho}}^n \delta M_{\phi^{\rho}} + \delta K_{\phi^{\rho}} \times \cdots$</td>
<td>$\beta \times (m \times n \times (1 + np))^2 + \cdots$</td>
</tr>
<tr>
<td>$\text{eig}(M_r, K_r)$</td>
<td>$O((m \times n \times (1 + np))^3)$</td>
<td>$\beta \times (m \times n \times (1 + np))^2$</td>
</tr>
</tbody>
</table>

where:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Constant dependent on the sparsity pattern of the matrices $M_{i{\rho}}^c$ and $K_{i{\rho}}^c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>Memory required to store a matrix element (complex number)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>Constant time for reading one element of a matrix (ANSYS input)</td>
</tr>
<tr>
<td>$\tau$</td>
<td>Constant time taken for multiply-add step in matrix-matrix multiplication</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>Constant time required to calculate (read + subtract) 1 element of $\delta M_I$ and $\delta K_I$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Constant dependent on the sparsity pattern of the matrices $\delta M_I$ and $\delta K_I$</td>
</tr>
<tr>
<td>$\tau a_{\phi^{\rho}}^n \delta M_{\phi^{\rho}} + \delta K_{\phi^{\rho}}$</td>
<td>Time in assembling one element of $\Phi^H \delta M \Phi$ &amp; $\Phi^H \delta K \Phi$</td>
</tr>
</tbody>
</table>
Numerical Examples

The bladed disk considered by Sinha [26] is considered again. Figure 2-7 represents the finite element model of the disk. The disk consists of 24 blades.

![Finite Element model of a bladed disk](image)

Figure 2-7: Finite Element model of a bladed disk

The mistuning has been introduced in the blades along 2 POD features shown in Figure 2-8.

1. The first POD feature represents each blade having uniform but different thickness. To achieve this the thickness \( b_l \) for blade \#l \( \text{is obtained by multiplying the thickness of the nominal blade } b_0 \text{ by a factor } (1 + \xi_{ll}) \), where \( \xi_{ll} \text{ is the mistuning parameter value for } 1^{st} \text{ POD feature for sector } \#l \), i.e.

\[
b_l = b_0 (1 + \xi_{ll})
\]

2. The second POD feature represents a linear variation of blade thickness along its length.

The slope of this variation in Figure 2-4 is given by
\[
\tan^{-1}\left(\frac{b_0\xi_{2l}}{L}\right)
\]

(2.44)

where \( L \) is the length of the blade and \( \xi_{2l} \) is the mistuning parameter value for 2\textsuperscript{nd} POD feature for sector #1. Specifications of \( \xi_{1l} \) and \( \xi_{2l} \) uniquely determine blade thicknesses \( b_{dl} \), at locations \( p = 1, 2, 3, 4, 5 \) and 6 in Figure 2-4 as follows:

\[
b_{dl} = b_0(\bar{u}_1 + \bar{u}_1 \xi_{1l} + \bar{u}_2 \xi_{2l})
\]

(2.45)

where

\[
b_{dpl} = [b_{d1l} \ b_{d2l} \ b_{d3l} \ b_{d4l} \ b_{d5l} \ b_{d6l}]^T
\]

(2.46)

\[
\bar{u}_1 = [1 \ 1 \ 1 \ 1 \ 1]^T
\]

(2.47)

\[
\bar{u}_2 = [-1 \ -0.6 \ -0.2 \ 0.2 \ 0.6 \ 1]^T
\]

(2.48)

For this numerical example mistuning is introduced only due to POD#1 (representing changes in uniform blades’ thicknesses). Random values of mistuning parameters are generated using ‘\texttt{randn}’ function in MATLAB [46] and are presented in Figure 2-9. The mean of mistuning parameter values is -0.0019 and the standard deviation is 3.42\%. Note that mean (\( \mu \)) is not exactly zero because of a finite number of random variables. The maximum value of deviation in blade thickness is 5.28\% of the nominal blade thickness.
Sector analysis is performed with each blade thickness, and mass and stiffness matrices are stored. Subtracting mass and stiffness matrices of a sector having blade with the average thickness, $\delta K_j$ and $\delta M_j$ ($j = 1, 2, \ldots, 24$) in equation (2.29), are obtained for the implementation of MMDA.

With blade thicknesses represented in Figure 2-9 and Table A-1 in Appendix, 1000 random permutations of the mistuning pattern are created. Using MMDA, all of these mistuned bladed disks are analyzed without needing any new finite element analysis. Using equation (2.32), equation (2.28) is viewed as $\phi^H_{i,1,\rho} \delta M f \phi_{j,1,\rho}$ being multiplied with a weighing factor $e^{it(\psi^d_{\rho}-\psi_{\nu})}$ and then added. Therefore, $\phi^H_{i,1,\rho} \delta M f \phi_{j,1,\rho}$ is calculated only once for each sector and then used for all permutations of the mistuning pattern. Note that the weighing factors $e^{it(\psi^d_{\rho}-\psi_{\nu})}$ change for different permutations. The validity of the MMDA based analysis is shown by comparison of MMDA results with full rotor results for one of the random permutations of original blade thicknesses, Table A-2, and Figure 2-10.
Figure 2-11 shows the deviations in natural frequencies of a mistuned disk as calculated using MMDA and also from the full rotor analysis in ANSYS. The results shown in figure 2-11 suggest that the natural frequencies obtained from MMDA are exact match to the natural frequencies as obtained from ANSYS and MMDA is able to capture the effects of geometric mistuning even in case of large geometry changes. In figure 2-11, the first 24 frequencies represent the family of first bending modes of vibration of blades. Similarly the higher modes represent the lateral bending, torsional bending and other families. The variations in these frequencies are shown in figure 2-12 for different permutations of blade thicknesses. It should be noted that the nodal diameter on the abscissa of the plot refers to the perfectly tuned system only. The histogram representing the distribution of natural frequencies over all mistuning patterns is plotted for the 18th natural frequency (9th Nodal Diameter) in figure 2-13, where NOC stands for number of occurrences in a frequency interval. The normalized frequency is calculated by dividing the frequency with the corresponding frequency of the tuned system. The standard deviation of these frequencies is 0.0023.
Figure 2-11: Deviations in frequencies estimated via MMDA and full rotor FEM (ANSYS) for a random permutation of Original Blade Thicknesses (POD #1)

Figure 2-12: Variations in first 24 natural frequencies for 1000 permutations of mistuning pattern
Figure 2-13: Distribution of 18th natural frequency for 1000 permutations of mistuning pattern

Non-dimensional peak maximum amplitude \( npma \) is defined as the ratio of the maximum amplitude of a blade among all blades in a mistuned bladed disk assembly and across all frequencies to the maximum amplitude of a blade in tuned bladed disk assembly across all frequencies. Specifically, \( npma \) is computed as follows:

\[
npma = \frac{\max_{\omega} \|a\|_{\infty}}{\max_{\omega} \|a_t\|_{\infty}}
\]  

(2.49)

where \( a \) and \( a_t \) are amplitude vectors for mistuned and tuned rotors, respectively. Next the distribution of \( npma \) is plotted for the mistuned disks in figure 2-14. Again, the spatial distribution of force corresponds to the 9\textsuperscript{th} Engine Order. The frequency range is chosen to be 0.98 to 1.02 times the natural frequency corresponding to 9\textsuperscript{th} Nodal Diameter of tuned single stage Disk. Modal damping ratio is again chosen to be 0.001. The standard deviation of these amplitudes is 0.0749.
Verification of Computational Time and Memory Estimates

In order to verify the computational time and memory requirements as listed in table 2-1, numerical experiments were run with different number of sectors, POD features, modes and degrees of freedom per sector and the results are presented in figures 2-15 – 2-18. As observed from the figures the equations listed in table 1 are able to accurately predict the effect of the different parameters (number of sectors, POD features, modes and number of degrees of freedom per sector) on computational time. It is also observed that the dominant time in MMDA computations is for the computations of $\phi_{i,1,v}^{H} \delta M_{j} \phi_{j,1,\rho}$ and $\phi_{i,1,v}^{H} \delta K_{j} \phi_{j,1,\rho}$ and the solution of the eigenvalue problem. As expected, it is observed that the assembly time for calculation of the reduced order matrices is small as compared to computations of $\phi_{i,1,v}^{H} \delta M_{j} \phi_{j,1,\rho}$ and $\phi_{i,1,v}^{H} \delta K_{j} \phi_{j,1,\rho}$.
and the solution of the eigenvalue problem. As also observed from the set of equations, the computational time for $\phi_{i,1,\nu}^H, D_{i,j,k} \phi_{j,1,\rho}$ and $\phi_{i,1,\nu}^H, D_{i,j,k} \phi_{j,1,\rho}$ is proportional to the number of degrees of freedom per sector, whereas the computational time for the assembly step as well as the solution of the eigenvalue problem are independent of the number of degrees of freedom, hence the fraction of computational time in assembly and eigenvalue solution step decreases as the number of degrees of freedom increases. Since the finite element models of the industrial rotors run into millions of degrees of freedom the computational time for the assembly as well as the eigenvalue solution step is negligible. Because of stochastic nature of mistuning, it is essential to run Monte Carlo simulation to study the statistics of response. As $\phi_{i,1,\nu}^H, D_{i,j,k} \phi_{j,1,\rho}$ and $\phi_{i,1,\nu}^H, D_{i,j,k} \phi_{j,1,\rho}$ have to be calculated only once for a set of mistuned blades and the generation of the reduced order matrices for the permutations of the mistuned blades involves only the assembly step, the effective time for MMDA analysis of different permutations of the set of mistuned blades is only that of the solution of the eigenvalue problem which is a small fraction of the total time if the number of degrees of freedom is large, which is often the case with finite element models of the industrial rotors. Hence MMDA is ideally suited for Monte Carlo simulations of the mistuned bladed disk assemblies.
Figure 2-15: Effect of number of modes per harmonic index on computational time

\[ n = 24, \; np = 4, \; nd = 1980 \]

Figure 2-16: Effect of number of POD features on computational time \((n = 24, \; m = 10, \; nd = 1980)\)
Figure 2-17: Effect of number of sectors on computational time \( (m = 10, np = 2, nd = 1980) \)

Figure 2-18: Effect of number of degrees of freedom per sector on computational time

\( (n = 24, m = 10, np = 2) \)
In order to verify the effect of different parameters (number of sectors, POD features, modes and degrees of freedom per sector) on the memory requirements, numerical simulations similar to the previous cases are run and the memory usage is recorded.

Figure 2-19: Effect of number of modes per harmonic index on memory

\( (n = 24, \, np = 4, \, nd = 1980) \)

Figure 2-20: Effect of number of POD features on memory \( (n = 24, \, m = 10, \, nd = 1980) \)
As discussed earlier the MMDA analysis involves 7 steps. There are two plots in each of figures 2-19 – 2-22. The bar chart in each plot is the amount of the memory that is added in a particular step whereas the scatter plot in the figures is the total amount of memory consumption at any step.
in MMDA analysis. Since the memory consumed for storing the mass and stiffness matrices \( M_{ip}^c \) and \( K_{ip}^c \) is released after the computation of \( \Phi^H M_i \Phi \) & \( \Phi^H K_i \Phi \), there is a drop in the total memory usage followed by a rise which is due to the memory required to store the results in the final step. Since the memory required for storing the mass and stiffness matrices, and the mode shapes is dependent on the number of degrees of freedom, whereas the memory required to store the reduced order matrices is independent of the number of degrees of freedom, the fraction of memory required to store mass and stiffness matrices, and the mode shapes increases as the number of degrees of freedom increases, and becomes the dominant fraction for large number of degrees of freedom as shown in figure 2-22. The memory shown in figures 2-19 – 2-22 is the total memory required to store mass and stiffness matrices, and the mode shapes for all the harmonic indices. Since the MMDA computations can be performed by sequentially loading the matrices for a particular harmonic index, performing the computations and then clearing the memory, the memory requirement for loading the mass and stiffness matrices can be reduced by a factor equal to the number of harmonic indices, hence making it more suitable for handling cases with large number of degrees of freedom.

Summary

The algorithm for Modified Modal Domain Analysis (MMDA) has been presented. In particular, it is shown how data (mass matrices, stiffness matrices and modal vectors) from sector analyses can be used to efficiently implement MMDA. Numerical results are provided to support the fact that MMDA can be easily used to analyze the mistuned bladed disk obtained after random permutations of the original mistuning pattern.
Finally the models for computational times and memory requirements for MMDA analysis have been presented and verified using numerical simulations. As suggested by equations and verified by the experiments, the computational time for calculation of $\phi_{l,1,v}^H \partial M_i \phi_{1,1,\rho}$ and $\phi_{l,1,v}^H \partial K_i \phi_{1,1,\rho}$ blocks is the dominant fraction of the computational time, especially for large number of degrees of freedom.
Chapter 3

Modified Modal Domain Analysis with Approximate Deviations in Mass and Stiffness Matrices

In the previous chapter, deviations in mass and stiffness matrices ($\delta M$ & $\delta K$) due to mistuning are obtained by subtracting mass and stiffness matrices of a sector with each blade from those of a sector with a blade with ‘average’ geometry. The advantage of this approach is that $\delta M$ & $\delta K$ are exact. But, this approach is not suitable for Monte Carlo simulations as sector analysis has to be conducted for each blade in a new rotor. The focus of the current work is to extend MMDA so as to avoid running a new FEM sector analysis for every new bladed rotor. The approach is based on the estimation $\delta M$ & $\delta K$ via Taylor series expansion in terms of POD variables representing geometry variations of blades. The calculation of $\Phi^H \delta M \Phi$ & $\Phi^H \delta K \Phi$ in equations (2.5) and (2.6) gets modified with these estimated $\delta M$ & $\delta K$ and the details are provided in this chapter.

Approximation of Deviations in Mass and Stiffness Matrices ($\delta M$ & $\delta K$) by Taylor Series expansion

Let $M_i^c$ be the mass matrix of the mistuned sector and $M_i^t$ the mass matrix of the nominal (tuned) sector. Let $\xi_{s,l}$ be the mistuning parameter associated with POD # $s$ and Sector# $l$. Using Taylor series expansion, the mass matrix of the actual sector can be expanded about the nominal sector matrix as:

$$
\delta M_i = \sum_{s=1}^{np} \frac{\partial M_i^c}{\partial \xi_{s,l}} \xi_{s,l} + \sum_{s=1}^{np} \sum_{l=1}^{np} \frac{\partial^2 M_i^c}{\partial \xi_{s,l} \partial \xi_{s,l}} \frac{\xi_{s,l} \xi_{s,l}}{2} + \Theta(\xi^3)
$$

(3.1)
where $\delta M_l = M^c_l - M^t_l$ is the perturbation in mass matrix for sector $# l$ and $np$ is the number of POD features present. Since $M^c_l$ is the mass matrix of the nominal sector, it is independent of the mistuning parameter $\xi_{s,l}$, i.e. $\frac{\partial M_l}{\partial \xi_{s,l}} = 0$, equation (3.1) can be written as:

$$
\delta M_l = \sum_{s=1}^{np} \frac{\partial M_l}{\partial \xi_{s,l}} \xi_{s,l} + \sum_{s=1}^{np} \sum_{l=1}^{np} \frac{\partial^2 M_l}{\partial \xi_{s,l} \partial \xi_{t,l}} \xi_{s,l} \xi_{t,l} + \Theta(\partial \xi^3)
$$

(3.2a)

Further, $\frac{\partial \delta M_l}{\partial \xi_{s,l}}$ & $\frac{\partial^2 \delta M_l}{\partial \xi_{s,l} \partial \xi_{t,l}}$ are independent of sector $# l$ and will be written as $\frac{\partial \delta M}{\partial \xi_s}$ & $\frac{\partial^2 \delta M}{\partial \xi_s \partial \xi_t}$ respectively. As a result, equation (3.2a) gets modified as

$$
\delta M_l = \sum_{s=1}^{np} \frac{\partial \delta M}{\partial \xi_s} \xi_{s,l} + \sum_{s=1}^{np} \sum_{l=1}^{np} \frac{\partial^2 \delta M}{\partial \xi_s \partial \xi_{t,l}} \xi_{s,l} \xi_{t,l} + \Theta(\partial \xi^3)
$$

(3.2b)

**Calculation of $\frac{\partial \delta M}{\partial \xi_s}$ & $\frac{\partial^2 \delta M}{\partial \xi_s \partial \xi_t}$**

As observed from equation (8b), we need to calculate $np$ first order derivatives and $\frac{np(np+1)}{2}$ second order derivatives. These derivatives are independent of sector $# l$, therefore, subscript $l$ in $\xi_{s,l}$ is not used.

**First and Second order partial derivatives**

Let $\xi_s = \xi_k$ if $s = k$ and $\xi_s = 0$ otherwise, i.e. only the $k^{th}$ POD mistuning parameter is non-zero then:
\( \delta M(\xi_k) = \frac{\partial \delta M}{\partial \xi_k} \xi_k + \frac{\partial^2 \delta M}{\partial \xi_k^2} \frac{\xi_k^2}{2} \) and

\( \delta M(-\xi_k) = -\frac{\partial \delta M}{\partial \xi_k} \xi_k + \frac{\partial^2 \delta M}{\partial \xi_k^2} \frac{\xi_k^2}{2} \) (3.4)

where \( \delta M(\xi_k) \) is the perturbation in mass matrix for mistuning parameter \( \xi_k \) and \( \delta M(-\xi_k) \) for mistuning parameter \( -\xi_k \). Subtracting (3.4) from (3.3)

\[ \frac{\partial \delta M}{\partial \xi_k} = \frac{\delta M(\xi_k) - \delta M(-\xi_k)}{2\xi_k} \] (3.5)

And adding equations (3.4) and (3.3),

\[ \frac{\partial^2 \delta M}{\partial \xi_k^2} = \frac{\delta M(\xi_k) + \delta M(-\xi_k)}{\xi_k^2} \] (3.6)

**Mixed second order partial derivatives**

If \( \xi_s = \xi_j, \xi_k \) if \( s = j, k \) and \( \xi_s = 0 \) otherwise. Then,

\[ \delta M(\xi_j, \xi_k) = \frac{\partial \delta M}{\partial \xi_j} \xi_j + \frac{\partial^2 \delta M}{\partial \xi_j^2} \frac{\xi_j^2}{2} + 1 \left( \frac{\partial \delta M}{\partial \xi_k} \xi_k + \frac{\partial^2 \delta M}{\partial \xi_k^2} \frac{\xi_k^2}{2} + \frac{\partial^2 \delta M}{\partial \xi_j \partial \xi_k} \xi_j \xi_k \right) \] (3.7)

From (3.7),

\[ \frac{\partial^2 \delta M}{\partial \xi_j \partial \xi_k} = \frac{2 \left( \delta M(\xi_j, \xi_k) - \left( \frac{\partial \delta M}{\partial \xi_j} \xi_j + \frac{\partial \delta M}{\partial \xi_k} \xi_k \right) \right) - \left( \frac{\partial^2 \delta M}{\partial \xi_j^2} \xi_j^2 + \frac{\partial^2 \delta M}{\partial \xi_k^2} \xi_k^2 \right)}{2\xi_j \xi_k} \] (3.8)

where the first and second order partial derivatives in equation (3.8) are calculated using equation (3.5) and (3.6).
Estimation of $\Phi^H \delta \Phi & \Phi^H \delta K \Phi$ using Taylor series approximation of $\delta M & \delta K$

Equation (2.10) gives the expression of $\Phi^H \delta \Phi$ as:

$$\Phi^H \delta \Phi = \begin{bmatrix}
\Phi_0^H \delta \Phi_0 & \Phi_1^H \delta \Phi_0 & \cdots & \Phi_{np}^H \delta \Phi_0 \\
\Phi_0^H \delta \Phi_1 & \Phi_1^H \delta \Phi_1 & \cdots & \Phi_{np}^H \delta \Phi_1 \\
\vdots & \vdots & \ddots & \vdots \\
\Phi_0^H \delta \Phi_{n-1} & \Phi_1^H \delta \Phi_{n-1} & \cdots & \Phi_{np}^H \delta \Phi_{n-1}
\end{bmatrix}$$  \hspace{1cm} (2.10)

Note that $\Phi^H \delta \Phi$ is a Hermitian matrix consisting of $(np+1) \times (np+1)$ sub-blocks $\Phi_i^H \delta \Phi_j$.

The perturbation in mass matrix $\delta M$ has a block diagonal form given by equation (2.27):

$$\delta M = \begin{bmatrix}
\delta M_0 & 0 & \cdots & 0 \\
0 & \delta M_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \delta M_{n-1}
\end{bmatrix}$$  \hspace{1cm} (2.27)

and the matrix assembly of mode shapes for $i^{th}$ POD is given by equation (2.11)

$$\Phi_i = \begin{bmatrix}
\varphi_{i,1,0} & \varphi_{i,1,1} & \cdots & \varphi_{i,1,n-1} \\
\varphi_{i,2,0} & \varphi_{i,2,1} & \cdots & \varphi_{i,2,n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_{i,n,0} & \varphi_{i,n,1} & \cdots & \varphi_{i,n,n-1}
\end{bmatrix}$$  \hspace{1cm} (2.11)

where

$$\varphi_{i,k,p} = e^{i(k-1)\psi_p} \varphi_{i,1,p}$$  \hspace{1cm} (2.12)

is the mode shape component of $k^{th}$ sector for $i^{th}$ POD feature and $p^{th}$ harmonic index. And, $n$ is the number of sector. The expression for $\psi_p$ is given by

$$\psi_p = \frac{2\pi p}{n}$$  \hspace{1cm} (2.13)

The mode shape component $\varphi_{i,1,p}$ is calculated by running sector analysis in any standard FEM package like ANSYS or NASTRAN.
Combining equations (2.27) and (2.11), the sub-block \( \Phi_i^H \Delta \Phi_j \) is given by

\[
\Phi_i^H \Delta \Phi_j = \sum_{l=0}^{n-1} \begin{bmatrix}
\Phi_{l,0}^H \Delta M_i \varphi_{j,l,0} & \cdots & \Phi_{l,0}^H \Delta M_i \varphi_{j,l,n-1} \\
\Phi_{l,1}^H \Delta M_i \varphi_{j,l,0} & \cdots & \Phi_{l,1}^H \Delta M_i \varphi_{j,l,n-1} \\
\vdots & \ddots & \vdots \\
\Phi_{l,n-1}^H \Delta M_i \varphi_{j,l,0} & \cdots & \Phi_{l,n-1}^H \Delta M_i \varphi_{j,l,n-1}
\end{bmatrix}
\] (3.9)

Hence the \((\nu+1, \rho+1)\) element of \( \Phi_i^H \Delta \Phi_j \) is

\[
\sum_{l=0}^{n-1} \Phi_{l,1}^H \Delta M_i \varphi_{j,l,\rho} = \sum_{l=0}^{n-1} e^{\frac{2\pi l (\nu-\rho)}{n}} \Phi_{l,1,\nu}^H \Delta M_i \varphi_{j,l,\rho}
\] (3.10)

where \( \Delta M_i \) can be written as a Taylor series expansion given by equation (3.2b). Substituting the expression of \( \Delta M_i \) in equation (3.10), the \((\nu+1, \rho+1)\) element of \( \Phi_i^H \Delta \Phi_j \) can be written as:

\[
\sum_{s=1}^{np} \frac{\partial \Delta M}{\partial \xi_s} \varphi_{j,1,\rho} \sum_{l=0}^{n-1} e^{\frac{2\pi l (\nu-\rho)}{n}} \xi_{s,l} + \sum_{s=1}^{np} \sum_{s'=1}^{np} \frac{\partial^2 \Delta M}{\partial \xi_s \partial \xi_{s'}} \varphi_{j,1,\rho} \sum_{l=0}^{n-1} e^{\frac{2\pi l (\nu-\rho)}{n}} \xi_{s,l} \xi_{s',l}
\]

But

\[
\xi_s(k) = \sum_{l=0}^{n-1} e^{\frac{2\pi l (\nu-\rho)}{n}} \xi_{s,l} \quad \text{and} \quad k = \nu - \rho
\] (3.12)

where \( \xi_s(k) \) is the \( k^{th} \) Discrete Fourier transform [47] of \( \xi_{s,l} \). Further,

\[
\xi_s \bar{\xi}_l(k) = \sum_{l=0}^{n-1} e^{\frac{2\pi l (\nu-\rho)}{n}} \xi_{s,l} \bar{\xi}_{l,l}
\]

(3.13)

where \( \bar{\xi}_s \bar{\xi}_l(k) \) is the \( k^{th} \) Discrete Fourier transform of \( \xi_s \xi_l \). Using the circular convolution property of Discrete Fourier Transform [47], \( \bar{\xi}_s \bar{\xi}_l(k) \) can be written as:
\[
\vec{\xi}_s \vec{\xi}_t (k) = \sum_{i=0}^{n-1} e^{-\frac{2\pi i k}{n}} \xi_{s,i} \xi_{t,i} = \frac{1}{n} \sum_{l=0}^{n-1} \vec{\xi}_s (l) \vec{\xi}_s (k-l) \\
= \frac{1}{n} (\vec{\xi}_s (0) \vec{\xi}_s (k) + \vec{\xi}_s (1) \vec{\xi}_s (k-1) + \ldots + \vec{\xi}_s (k) \vec{\xi}_s (0) + \ldots \\
\ldots \vec{\xi}_s (k+1) \vec{\xi}_s (-1) + \ldots + \vec{\xi}_s (n-1) \vec{\xi}_s (k+1-n)) \\
\] (3.14)

Using the periodicity property of Discrete Fourier transform [47],

\[
\vec{\xi}_s (-r) = \vec{\xi}_s (n-r) \\
\] (3.15)

Combining equations (3.14) and (3.15),

\[
\vec{\xi}_s \vec{\xi}_t (k) = \frac{1}{n} (\vec{\xi}_s (0) \ldots \vec{\xi}_s (k) \ldots \vec{\xi}_s (n-1)) \begin{bmatrix} \vec{\xi}_s (k) \\ \vdots \\ \vec{\xi}_s (0) \\ \vdots \\ \vec{\xi}_s (n-1) \end{bmatrix} \quad \text{or} \\
\begin{bmatrix} \vec{\xi}_s (0) \\ \vdots \\ \vec{\xi}_s (n-1) \end{bmatrix}
\] (3.16)

where

\[
Z_s = \begin{bmatrix} \vec{\xi}_s (0) & \ldots & \vec{\xi}_s (n-1) \end{bmatrix}^T \\
\] (3.17)

and

\[
P(k) = \begin{bmatrix} 0 & \ldots & 1 & \ldots & 0 \\
\vdots \\
1 & 0 & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 1 \\
\vdots \\
0 & \ldots & 0 & 1 & \ldots & 0 \\
\end{bmatrix} \\
\] (3.18)

\(P(k)\) is the permutation matrix used to reorder the rows of \(Z_s\). The 1st element of the \((k+1)^{th}\) row and 1st element of the \((k+1)^{th}\) column of \(P(k)\) is 1. Hence from equation (3.11), the

\((v+1, \rho+1)\) element of \(\Phi_i^H \delta \mathcal{M} \Phi_j\) can be written as:

\[
\frac{np}{\sum_{s=1}^{n} \phi_{i,1,v}^H \frac{\partial \mathcal{M}}{\partial \xi_s} \phi_{j,1,\rho} \vec{\xi}_s (k)} + \frac{1}{n} \sum_{s=1}^{n} \frac{np}{\sum_{s=1}^{n} \phi_{i,1,v}^H} \frac{1}{2} \frac{\partial^2 \mathcal{M}}{\partial \xi_s \partial \xi_t} \phi_{j,1,\rho} Z_i^T P(k) Z_s \\
\] (3.19)
Equation (3.19) can be used to calculate the sub-blocks of $\Phi_i^H \partial M \Phi_j$. Notice from equation (3.19) the expressions $\Phi_{i,1,v}^H \frac{\partial \partial M}{\partial \xi_s} \Phi_{j,1,\rho}$ and $\Phi_{i,1,v}^H \frac{\partial^2 \partial M}{\partial \xi_s \partial \xi_t} \Phi_{j,1,\rho}$ are independent of the mistuning parameter value; hence can be calculated offline and the calculation of $\Phi_i^H \partial M \Phi_j$ at run time requires just the summation step. Equation similar to equation (3.19) can be used to calculate $\Phi_i^H \partial K \Phi_j$. Hence the approximation of $\partial M \& \partial K$ by Taylor Series expansion provides an efficient way to run Monte Carlo simulation on mistuned bladed disk for random values of mistuning parameters.

**Computational Time and Memory Requirements**

The workflow for MMDA based on Taylor series expansion of perturbation in Mass and Stiffness matrices is similar to the MMDA analysis based on exact $\partial M \& \partial K$. The minor difference arises for steps 4, 5 and 6, where instead of calculating $\partial M_l \& \partial K_l$ and $\Phi_l^H \partial M_l \Phi \& \Phi_l^H \partial K_l \Phi$ for each sector $#l$, calculations are made for $\frac{\partial \partial M}{\partial s}, \frac{\partial \partial K}{\partial s}, \frac{\partial^2 \partial M}{\partial s \partial \xi_t}$ and $\frac{\partial^2 \partial K}{\partial s \partial \xi_t}$

and $\Phi_l^H \frac{\partial \partial M}{\partial s} \Phi, \Phi_l^H \frac{\partial \partial K}{\partial s} \Phi, \Phi_l^H \frac{\partial^2 \partial M}{\partial s \partial \xi_t} \Phi \& \Phi_l^H \frac{\partial^2 \partial K}{\partial s \partial \xi_t} \Phi$. Hence the time and memory for steps 4, 5 and 6 can be summarized as:

**Step 4:** Calculation of $\frac{\partial \partial M}{\partial s}, \frac{\partial \partial K}{\partial s}, \frac{\partial^2 \partial M}{\partial s \partial \xi_t}$ and $\frac{\partial^2 \partial K}{\partial s \partial \xi_t}$

a) Memory = $\beta \times \gamma \times nd \times \frac{np(np+3)}{2}$

b) $T_{\partial M_l+\partial K_l} = \kappa \times \alpha \times nd \times \frac{np(np+3)}{2}$
Step 5: Calculation of $\Phi^H \frac{\partial \delta M}{\partial s} \Phi, \Phi^H \frac{\partial \delta K}{\partial s} \Phi, \Phi^H \frac{\partial^2 \delta M}{\partial s \partial t} \Phi$ & $\Phi^H \frac{\partial^2 \delta K}{\partial s \partial t} \Phi$

a) Memory = $2\beta \times (m \times n \times (1 + np))^2 \times \frac{np(np + 3)}{2}$

b) $T_{\phi^h, \partial M \phi + \phi^h, \partial K \phi} = t_{\phi^h, \partial M \phi + \phi^h, \partial K \phi} \times n^2 \times (np + 1)^2 \times m^2 \times \frac{np(np + 3)}{2}$

Step 6: Assembly of $\Phi^H \partial M \Phi$ & $\Phi^H \partial K \Phi$

a) Memory = $2\beta \times (m \times n \times (1 + np))^2$

b) $T_{a\phi^h, \partial M \phi + \phi^h, \partial K \phi} = t_{a\phi^h, \partial M \phi + \phi^h, \partial K \phi} \times n^2 \times (np + 1)^2 \times m^2 \times \frac{np(np + 3)}{2}$

It should be noted that although steps 4 and 5 are more expensive (if $\frac{np(np + 3)}{2} > n$) for MMDA based on approximate analysis, they have to be calculated only once for a reference system, and are not executed for every mistuned rotor.

Numerical Examples

The bladed disk considered in the previous chapter is considered again. Figure 2-7 represents the finite element model of the disk. The mistuning can be introduced in the blades along 2 POD features as discussed in the previous chapter. In the first example the mistuning is introduced only due to POD#1. Random values of mistuning parameters are generated using ‘randn’ function in MATLAB [46] and are presented in Figure 3-1 and Table A-3. The mean of mistuning parameter values is -0.0024 and the standard deviation is 1.7%. The mean ($\mu$) is again not exactly zero because of a finite number of random variables. The maximum value of deviation in blade thickness is 3% of the nominal blade thickness. MMDA analysis based on interpolation of $\partial M$ & $\partial K$ using Taylor series expansion is run for two cases:
1. Mistuning is approximated only by the first order terms in the Taylor series expansion.

2. Mistuning is approximated using both first and second order terms.

Figure 3-1: Mistuning pattern for POD #1 (blade thickness)

Figure 3-2: Deviation in Natural Frequencies estimated via full rotor FEM and MMDA (Exact, 1\textsuperscript{st} and 2\textsuperscript{nd} order approximation of $\delta M$ and $\delta K$)
Figure 3-2 shows deviations in natural frequencies as estimated from ANSYS, MMDA using exact $\delta M$ & $\delta K$, MMDA using $\delta M$ & $\delta K$ approximated from the 1st order terms (case 1) and MMDA approximated from $\delta M$ & $\delta K$ both 1st and 2nd terms (case 2). As observed from the figure, the results from case 2 match exactly with the ANSYS analysis and MMDA based on second order approximation of $\delta M$ & $\delta K$ successfully estimates the natural frequencies of the mistuned system. The results also show that although the values of mistuning parameters is small, the first order approximation is not sufficient due to high sensitivity of mass and stiffness matrices and hence of the eigenvalues to the mistuning.

The % error in the deviation from a tuned bladed disk frequency ($F_{\text{nd}}$) is calculated as the ratio of the error in deviation calculated from MMDA ($\text{Dev}_{\text{MMDA}} - \text{Dev}_{\text{Act}}$) to the actual deviation ($\text{Dev}_{\text{Act}}$), i.e. $\text{ErrorD} (%) = \frac{\text{Dev}_{\text{MMDA}} - \text{Dev}_{\text{Act}}}{\text{Dev}_{\text{Act}}} \times 100$ \hspace{1cm} (3.20)

where $\text{Dev}_{\text{MMDA}} = F_{\text{nd}}\text{MMDA} - F_{\text{nd}}$ and $\text{Dev}_{\text{Act}} = F_{\text{nd}}\text{Act} - F_{\text{nd}}$ \hspace{1cm} (3.21)

\[
\begin{align*}
\text{ErrorD} &\text{ (%) in Deviation} \\
\text{Mode #} &
\end{align*}
\]

Figure 3-3: Comparison of Errors (%) in deviations of natural frequencies estimated via MMDA (exact), MMDA (1st order) and MMDA (2nd order)
The % error in estimates of natural frequency is presented in figure 3-3. Again a behavior similar to figure 3-2 is observed where results from case 2 (2nd order approximation) match closely with the results from the exact analysis whereas the results from case 1 (1st order approximation) fail to capture the small deviations in natural frequencies. But very high value of % errors for case 1 are not only because of its inability to capture the small deviations but also the fact that it is possible to get high values of % error even for small values of $Dev_{ROM} - Dev_{Act}$ if $Dev_{Act}$ is small. Therefore, high values of % errors in figure 3-3 are not necessarily reflected in figure 3-2. In spite of a strict criterion applied for the calculation of % error, small % error in deviation for case 2 suggests that the method is able to capture small deviations.

Note that in the literature [16 - 21] dealing with reduced order modeling (ROM) of mistuned bladed rotor, errors are calculated as follows:

$$Error(\%) = \frac{Freq_{ROM} - Freq_{Act}}{Freq_{Act}} \times 100$$

(3.23)

From this criteria, the maximum value of the Error (%) are 0.98 and 0.05 for $\delta M$ & $\delta K$ based on Taylor series expansion up to first and second order terms respectively. In other words, if criterion (3.23) is applied the first order approximation of $\delta M$ & $\delta K$ will also be acceptable. But, it is believed that criterion (3.20) should be used in evaluating the accuracy of ROM for mistuning, as the goal is to accurately capture small deviations in frequencies caused by mistuning.

Next, the differences between mode shapes from the MMDA and full 360 degree FEM are also examined for case 1 and case 2. The modal vectors are scaled so that the maximum value of an element in a modal vector is 1. As observed from figure 3-4, the error in mode shape vector predicted by 2nd order approximation is much smaller as compared to the mode shape vector by 1st order approximation. The index $i$ of a modal vector represents its $i$th element in figure 3-4. The small absolute value of maximum error for case 2 further confirms that the results obtained on the
basis of 2nd order approximation are almost exact. This result has also been verified by Modal Assurance Criterion [48].

Figure 3-4: Error in mode shapes estimated via MMDA (1st order) and MMDA (2nd order)

In order to verify the applicability of 2nd order approx. technique for multiple POD features, a mistuning pattern consisting of both POD#1 and POD#2 is created. The mistuning
parameter values for POD#1 are the same as given in figure 3-1. The mistuning parameter values for POD#2 ($\xi_{2i}$) are presented in figure 3-5. The mean and standard deviation of the mistuning parameter values is -4.0e-5 and 1.5% respectively.

Figure 3-5: Mistuning pattern for POD #2 (blade surface inclination)

Figure 3-6: Deviation in natural frequencies from MMDA ($2^{nd}$ order) for 2 POD mistuning
Figure 3-6 shows the deviations in natural frequencies of the mistuned system. As observed from the figure, the deviations in natural frequencies estimated via MMDA (up to 2\textsuperscript{nd} order approximation) is almost an exact match to the deviations calculated via full 360\degree analysis.

![Figure 3-6: Deviations in natural frequencies](image)

Figure 3-7: Error in mode shape for 2 POD mistuning using MMDA (2\textsuperscript{nd} order)

Mode shapes from MMDA and full 360 degree FEM are compared in figure 3-7. Figure 3-7 shows that the maximum error in the mode shape vector is 1\% of the maximum value of an index of the mode shape vector; hence MMDA based upon 2\textsuperscript{nd} order approximation is accurate even for mistuning along multiple POD features. This result has also been verified by Modal Assurance Criterion [48].

**Forced Response**

The differential equation of motion is

\[ M\ddot{x} + C\dot{x} + Kx = f(t) \]  

(3.24)

where \( M, K, C \) and \( f(t) \) are mass matrix, stiffness matrix, damping matrix and forcing vector, respectively. Using transformation (2.2),
\[ M \ddot{z} \pm \Phi^H C \Phi z + K z = \Phi^H f(t) \]  

(3.25)

The system having mistuning along single POD feature (Figure 3-1) is excited by a harmonic force corresponding to engine order 9. Amplitudes of steady state responses are obtained from MMDA (equation 3.25) for case 1 (1st order approximation) and case 2 (2nd order approximation) via modal superposition method for excitation frequencies within ±5 percent of the natural frequency corresponding to 9th Nodal diameter of the tuned bladed disk. The damping ratio in each mistuned mode is taken to be 0.001. Figure 3-8 shows the blade tip amplitudes for blade#1 estimated via ANSYS analysis of full (360 degree) model and from MMDA using 1st order approx. (case 1) and MMDA using 2nd order approx. (case 2). The results based on 2nd order approx. match exactly with the results from ANSYS.

Figure 3-8: Comparison of blade tip amplitude (blade #1) from ANSYS and MMDA using 1st order approx. (Case 1) and 2nd order approx. (Case 2), POD #1 only

A similar comparison in forced response is made for the system mistuned along two POD features (Figures 3-1 and 3-5) and the results are presented in figure 3-9. The excitation and damping are same as those considered for the previous case. Here again the blade tip amplitudes
for blade# 1 estimated via ANSYS analysis of full (360 degree) model and from MMDA using 2nd order approx. match exactly with the results from ANSYS. The results from figures 3-8 and 3-9 suggest that MMDA based on 2nd order expansion is able to capture the mistuning effects even in case of mistuning along multiple POD features.

![Excitation Engine Order: 9 Mean Forcing Frequency: 4378.7467 (Hz.)](image)

Figure 3-9: Comparison of blade tip amplitude (blade #1) from ANSYS and MMDA using 1st order approx. (Case 1) and 2nd order approx. (Case 2), both POD features

**Monte Carlo Simulation of Natural Frequencies and Forced Responses via Sector Analyses**

Equation (3.19) suggests that once the blocks $\frac{\partial M}{\partial \xi_j} \frac{\partial \phi_{j,l,\rho}}{\partial \xi_i}$ and $\frac{1}{2} \frac{\partial^2 M}{\partial \xi_i \partial \xi_j} \frac{\partial \phi_{j,l,\rho}}{\partial \xi_i}$ are calculated; generation of reduced order model via Taylor series expansion is just a block assembling exercise. To perform the Monte Carlo simulation on the bladed disk 1000 random mistuning patterns are created using ‘randn’ function in MATLAB. The standard deviation of mistuning parameter for each mistuning pattern is 1.7% and the range of mistuning pattern is ±3%. The natural frequencies of the mistuned bladed assembly are calculated for each mistuning
pattern. The histogram representing the distribution of natural frequency over all mistuning patterns is plotted for the 1st, 13th and 24th natural frequency (Figures 3-10 — 3-12), where NOC stands for number of occurrences in a frequency interval. The normalized frequency is calculated by dividing the frequency with the corresponding frequency of the tuned system. These figures suggest that the average frequency of the lower modes reduces slightly due to mistuning whereas that of the higher modes increases slightly.

Figure 3-10: Distribution of 1st natural frequency for 1000 mistuning patterns

Figure 3-11: Distribution of 13th natural frequency for 1000 mistuning patterns
Next the distribution of non-dimensional peak maximum amplitude \((npma)\) given by equation (2.49) is plotted in figure 3-12. Again, the spatial distribution of force corresponds to the 9\(^{th}\) Nodal Diameter. The frequency range is chosen to be 0.98 to 1.02 times the natural frequency corresponding to 9\(^{th}\) Nodal Diameter of tuned single stage Disk. Modal damping ratio is again chosen to be 0.001.

![Figure 3-12: Distribution of 24\(^{th}\) natural frequency for 1000 mistuning patterns](image)

![Figure 3-13: Distribution of Peak Maximum Amplitude for 1000 mistuning patterns under 9\(^{th}\) EO excitation](image)
Verification of Computational Time and Memory Estimates

Figure 3-14: Effect of number of POD features for MMDA based on approximate $\delta M$ and $\delta K$

\[(n = 24, \ m = 10, \ nd = 1980)\]

For MMDA based on the approximation of $\delta M$ & $\delta K$ using Taylor series approximation, the

\[\varphi_{i,l,v} \frac{\partial^2 \delta M}{\partial \xi_s \partial \xi_t} \varphi_{j,l,1,\rho}, \varphi_{i,l,v} \frac{\partial^2 \delta M}{\partial \xi_s \partial \xi_t} \varphi_{j,l,1,\rho}, \varphi_{i,l,v} \frac{\partial^2 \delta \Phi}{\partial \xi_s \partial \xi_t} \varphi_{j,l,1,\rho}, \text{and} \ \varphi_{i,l,v} \frac{\partial^2 \delta K}{\partial \xi_s \partial \xi_t} \varphi_{j,l,1,\rho}\]

are calculated offline and the calculation of the reduced order matrices is just an assembly step. The computational times for different steps involved in MMDA based on approximate $\delta M$ & $\delta K$ are again given by the equations in table 1. In order to get the estimate of relative times for MMDA analysis based on approximate $\delta M$ & $\delta K$, MMDA analysis is run for the mistuned bladed disk assembly with $n = 24, \ m = 10, \ nd = 1980$ and different number of POD features and the results are presented in figure 3-14. Since the blocks of $\varphi_{i,l,v} \frac{\partial^2 \delta M}{\partial \xi_s \partial \xi_t} \varphi_{j,l,1,\rho}$,
\[ \Phi_i^H \frac{\partial \partial K}{\partial \xi_s} \varphi_{j,1,\rho} , \quad \Phi_i^H \frac{\partial^2 \partial M}{\partial \xi_s \partial \xi_t} \varphi_{j,1,\rho} \text{ and } \Phi_i^H \frac{\partial^2 \partial K}{\partial \xi_s \partial \xi_t} \varphi_{j,1,\rho} \] are calculated offline, the computational times for these blocks are not included but the loading times are listed in results. As observed from the figure, with MMDA based on approximate \( \partial M \) & \( \partial K \), the effective computational time in MMDA analysis is that of the solution of the eigenvalue problem, which depends upon the size of the reduced order model, this method is very efficient for the mistuning analysis of the mistuned bladed disk assemblies.

**Effect of neglecting minor POD features**

Modified Modal Domain Approach is based on projecting the solution of a mistuned bladed disk assembly on the subspace defined by the modes of the nominal system and the modes of the tuned system with geometry perturbed along the POD features. Hence the dimension of the reduced order model is \( m(np+1) \times m(np+1) \). Here \( m \) is the number of modes per POD feature and \( np \) is the number of POD features considered for the reduced order model. In general a complete characterization of the blade geometry will yield a large number of independent mistuning parameters. This has two consequences. First since a sector analysis is required for each POD feature considered, a large \( np \) would require a number of sector analyses to be run. Secondly the number of POD features considered has a direct impact on the dimension of the reduced order model hence the small \( np \) is desirable. Fortunately, amongst the large number of possible independent POD features, only a few would be dominant and the rest could be neglected. But neglecting these minor mistuning parameters may introduce some error. In this part of study, the effect of neglecting minor POD features on the accuracy of the solution has been considered. Studies have been run on geometrically mistuned rotors for solution subspaces having different number of POD features and the accuracy of the solution is compared. Another effect of neglecting the POD features arises from the approximation of geometry and hence the
perturbations in mass and stiffness matrices. As discussed in earlier, the perturbations in mass and stiffness matrices are approximated by Taylor series. Hence neglecting minor POD features in approximating mass and stiffness matrices will also introduce some error.

Mistuned Geometry

Mistuned bladed disk assembly considered earlier in the chapter is considered again. Mistuning is introduced along the two POD features, i.e. the thickness of the blade and the inclination of blade surface from the vertical, as shown in figure 2-8. The dominant mistuning is along the POD#1 with the mistuning pattern given by figure 3-1. The 2nd POD is the minor POD feature with mistuning parameter values as shown in figure 3-15 and table A-4. The mistuning parameter values for POD#2 are taken to be 10% of the mistuning parameter values represented in figure 3-5. The mean and standard deviation of the mistuning parameter values are -4.59e-6 and .15% respectively. The maximum value of a mistuning parameter is 0.003. The range and standard deviation of POD #2 (minor POD feature) is approximately 10% of the range and standard deviation of POD #1 (dominant POD feature).

Figure 3-15: Mistuning pattern for POD #2 (Blade Surface Inclination, Minor POD)
As discussed earlier, two cases can be studied to understand the effect of neglecting minor POD features:

1) Neglecting minor POD features in the solution space, i.e. the mass and stiffness matrices of the mistuned blades are exact but only the tuned and the 1st POD (dominant POD) modes are used for $\Phi$

2) Neglecting minor POD features in mistuning approximation, i.e. only 1st POD (dominant POD) feature is considered for generating mass and stiffness matrices but tuned and all POD feature modes are considered for $\Phi$

Results

![Graph showing error (%) vs mode number for Case 1 and Case 2](image)

Figure 3-16: Errors (%) in Deviations of Natural Frequencies for Case1 (Neglecting minor POD features in the solution space) and Case 2 (Neglecting minor POD features in mistuning approximation)
The modal analysis using MMDA is run for the two cases discussed above and the percentage error in estimates of frequency deviations are plotted for the two cases in figure 3-16. The percentage errors in deviations of natural frequencies are given by equation 3.20. As we can observe from figure 3-16, the % error for case 1 (exact $\delta M$ & $\delta K$) is much less than the % error for case 2. The mean of the absolute values of % error for case 1 and case 2 are 2.4% and 12.1% respectively. Some high values of % errors for case 2 are present because it is possible to get high values of % error even for small values of $Dev_{ROM} - Dev_{Act}$ if $Dev_{Act}$ is small, hence the high values of % error do not necessarily suggest the inability of case 2 to follow mistuning but the fact that the MMDA analysis is more sensitive to the perturbations in mass and stiffness matrices than to the solution subspace.

![Blade Tip Amplitude, Forcing = 9th Nodal Diameter](image)

Figure 3-17: Comparison of blade tip amplitude (blade #1) from ANSYS and MMDA for case 1 and case 2

Another comparison for the two cases can be carried out for the harmonic response estimation of the mistuned bladed disk assembly. The system is excited with harmonic forcing
corresponding to 9th nodal diameter of the tuned bladed disk assembly. The excitation frequencies range within ±3 percent of the mean excitation frequency. The modal damping ratio is again 0.001. Blade tip amplitude for blade #1 of the bladed disk assembly is plotted in figure 3-17 for case1 and case2. It is observed from figure 3-17, the harmonic response estimated for case 1 (exact $\delta M$ and $\delta K$) matches exactly with the harmonic response from full rotor ANSYS analysis. It is also observed that although not exact, the harmonic response for case 2 (approx. $\delta M$ and $\delta K$) matches closely with the full rotor ANSYS analysis. The observations from figures 3-16 and 3-17 suggest that MMDA analysis is more sensitive to perturbation in mass and stiffness matrices as compared to the set of POD modes considered for MMDA. It is also observed that the errors are small for both the cases of neglecting minor POD features in the solution space and neglecting minor POD features in mistuning approximation, hence it can be inferred that MMDA analysis can be performed by neglecting minor POD features in the solution space as well as mistuning approximation without significant loss of accuracy.

Summary

Modified Modal Domain Approach (MMDA) has been extended using Taylor series expansion of perturbations in mass and stiffness matrices due to mistuning. It has been shown that the reduced order model based on 2nd order approximation is accurate enough to capture effects arising out of a geometric mistuning which is inherent to any manufacturing process and provides results comparable to exact MMDA. It has also been shown that with Taylor series approximation, given the blocks $\phi^H_{i,l,v} \frac{\partial \delta M}{\partial \xi_s} \phi_{j,l,\rho}$ and $\phi^H_{i,l,v} \frac{1}{2} \frac{\partial^2 \delta M}{\partial \xi_s \partial \xi_t} \phi_{j,l,\rho}$, generation of reduced order model is just a block assembling exercise which is computationally inexpensive.
A study has also been carried to study the effects of neglecting minor POD features, both in mistuning approximation as well as solution basis. It has been observed that although the mistuning approximation is more sensitive to neglecting the effect of minor POD features as compared to neglecting mode shapes of minor POD features in the solution bases, neglecting minor POD features both in mistuning approximation as well as solution bases does not greatly affect the accuracy of the solution, and the minor POD features can easily be neglected.
Chapter 4

MMDA: Alternative Basis and Large Mistuning

In the previous chapter, the algorithm for MMDA has been developed and the efficacy of the technique has been shown through numerical simulations for the academic rotor. In the previous chapters, the basis for MMDA has been formed using the mode shapes of cyclic sectors perturbed along the POD features. The use of mode shapes from modal analysis of cyclic sectors perturbed along the POD features adds an additional step of creating the finite element models of artificially perturbed geometries. In this chapter an alternative formulation of MMDA is presented which avoids the use of mode shapes from geometries perturbed along POD features to form the basis.

Alternative Basis

In MMDA the true mode shapes of a mistuned bladed disk assembly are approximated by a linear combination of mode shapes of nominal tuned geometry and tuned geometries of sectors with blades perturbed along the POD features as given by equation 2.2. The modes for the geometries from the POD analysis are used because POD analysis provides independent vectors for perturbation in geometry, and by using only dominant POD features to form the basis of geometric perturbation; a minimal set of mode shapes is obtained to form the solution basis. The idea behind POD analysis is to obtain independent perturbation vectors and since the perturbation in actual sector can be represented as a linear combination of the POD vectors using KL expansion [43], it is proposed that alternatively the actual mistuned sectors may themselves be used form a suitable basis, i.e.

\[ x = \Phi y \]  

(4.1)
where

\[
\Phi = [\Phi_0, \Phi_1, \Phi_2, \ldots, \Phi_n]
\]  \hspace{1cm} (4.2)

Equation 4.2 is similar to equation 2.3 but the mode shapes from actual sectors are used to form the basis. Here \( \Phi_0 \) are mode shapes obtained from the modal analysis of the nominal tuned bladed disk assembly and \( \Phi_l \) are the mode shapes from the modal analysis of the tuned bladed disk assembly with each sector represented by the sector \( #l \) of the actual mistuned bladed disk. Since only a few POD features are dominant, the sectors with geometry most aligned along the dominant POD features can be used to form a basis. Hence using the alternative basis, the steps for the MMDA analysis can enumerated as follows:

1. Get the actual mistuned bladed CMM data.
2. Identify independent geometric perturbation vectors using POD analysis.
3. Identify blades/sector most aligned with the dominant POD features.
4. Use mode shapes from cyclic analysis of identified sectors to form the basis of solution.
5. Generate and solve reduced order model using alternative basis.

In the next section MMDA analysis is run with alternative basis and the results are presented.

**Numerical Example**

In this section MMDA analysis is run for the academic rotor used in previous chapters and the finite element model is given by figure 2-7. The sector has 24 blades, with 1980 independent and unconstrained degrees of freedom per sector. The dominant mistuning is again introduced along the two POD features as discussed in chapter 2 and represented by figure 2.8, i.e. the thickness of the blade and the surface inclination of the blade with the vertical. The same mistuning parameter values as used in chapter 3 are again used for generating the mistuning.
Hence the mistuning parameter values for POD features 1 and 2 are given by figures 3-1 and 3-5 (table A.3) respectively.

A look at the values of mistuning parameter values for each blade shows that for sectors 12 ($\xi_1 = 0.0061302$, $\xi_2 = -0.0186800$) and 15 ($\xi_1 = -0.029931$, $\xi_2 = 0.0045726$), mistuning is dominated by POD 2 and 1 respectively, with the mistuning along the other POD feature being very small, i.e. the mistuning values in sectors 12 an 15 are closely aligned to the directions of POD 2 and 1 respectively, hence the mode shapes from cyclic analysis of sectors 12 and 15 can be used to form the basis of solution. Cyclic analysis is run for the two sectors and mode shapes from the first five families are used in MMDA analysis.

![Basis Mode (Sector 12 and Sector 15)](image)

**Figure 4-1: Deviation in Natural Frequencies estimated via full rotor FEM and MMDA**

(Alternative Basis, Sector 12 and 15)
Figure 4-2: Errors (%) in deviations of natural frequencies estimated via MMDA (Alternative Basis, Sector 12 and 15)

Figure 4-1 plots deviations in frequencies as estimated by MMDA and exact full rotor analysis as given by equation 3.21 and 3.22. Figure 4-2 plots the % error in estimation as given by equation 3.20. As observed from the plots the estimates of natural frequencies from MMDA based on mode shapes from actual sectors as bases are very close to the actual values, which suggests that the use of alternative basis is feasible for MMDA analysis.

Since in practical situations the objective behind reduced-order model is to use it as a step in the harmonic analysis to estimate the amplitude magnification, it is essential that the reduced order model not only provides accurate estimates of the natural frequencies but also of the blade tip amplitude under forced response or harmonic excitation. In order to verify the accuracy of the harmonic response, the system is excited by 3rd engine order excitation in ±3% range of the mean forcing frequency of 4157.6 Hz, which is the natural frequency of the nominal tuned disk for the 3 nodal diameter solution of the first family. The modal damping ratio is taken to be 0.1%. The harmonic response, which is the solution of equation 3.24, is obtained using modal superposition.
For MMDA based analysis the mode shapes from the analysis discussed earlier are used whereas for exact analysis (ANSYS) the mode shapes from full rotor analysis are used for modal superposition and the results are presented in figure 4-3.

Figure 4-3: Blade tip amplitude estimated via full rotor FEM and MMDA (Alternative Basis, Sector 12 and 15) for 3EO excitation

Figure 4-3 shows the blade tip amplitude for blades 1 and 13. As observed from the figure, MMDA accurately estimates the harmonic response of the bladed disk assembly.

Figure 4-4: POD features in mistuned bladed disk assembly
As discussed in chapter 3, in a mistuned bladed disk assembly many POD features are present but only a few are dominant and have to be included in MMDA analysis. In order to simulate this, perturbation along an additional POD feature is introduced as shown in figure 4-4. POD features 1 ($\tilde{u}_1$) and 2 ($\tilde{u}_2$) are same as the ones discussed in chapter 2 and are given by equations 2.47 and 2.48 respectively. POD feature 3 ($\tilde{u}_3$) is created by taking the component of vector $v$ (equation 4.3) which is orthogonal to both $\tilde{u}_1$ and $\tilde{u}_2$ via Gram-Schmidt ortho-normalization [49], i.e.

$$\tilde{v} = [1 \ -1 \ 1 \ -1 \ 1]^T$$

$$\tilde{w} = \tilde{v} - proj_{\tilde{u}_1}(\tilde{v}) - proj_{\tilde{u}_2}(\tilde{v})$$

$$\tilde{u}_3 = \tilde{w} / \|\tilde{w}\|$$

Here $\|\tilde{w}\| = \sqrt{\tilde{w}\tilde{w}^T}$ is the norm of $\tilde{w}$ and $proj_{\tilde{u}_n}(v)$ is the projection of vector $v$ on vector $\tilde{u}_n$ and is given by:

$$proj_{\tilde{u}_n}(v) = \frac{\tilde{v}\tilde{u}_n^T}{\|\tilde{u}_n\|^2} \tilde{u}_n$$

The mistuning values along POD features 1 and 2 are the same as those taken earlier for the disk. Mistuning along POD 3 is intended to be minor, whose values are generated in Matlab [46] using randn function and are given in figure 4-5 and table A-5. The mean and standard deviation of the mistuning parameter ($\xi_3$) distribution are 0.0017 and 0.004 respectively. The standard deviation of $\xi_3$ suggests that it is about 25% of the dominant POD features, i.e. POD features 1 and 2.
Due to the random nature of perturbations and hence the mistuning coefficients, it is possible that the blades most aligned in the direction of POD features may have a very small value of mistuning. In this case, the mode shapes obtained from the cyclic analysis of these blades will be very similar to the mode shapes of nominal assembly and no additional information will be introduced by including these mode shapes in the solution basis. Hence an alternative approach for finding the blades most suitable for forming the basis is discussed. When selecting the blades the following conditions can be considered:

The selected blades should:

1. Have perturbation close to the average perturbation of the blades’ geometry. This condition is required to prevent under or over stiffening of mode shapes in the basis.

2. Represent independent directions of perturbations.

In order to satisfy condition 1, we can calculate the norm of perturbation (eq. 4.8) for each blade and then choose the blade whose norm is closest to the median.
\[ \| \delta v_l \| = \sqrt{\delta v_l \delta v_l^T} \quad (4.8) \]

\( \delta v_l \) : Geometry perturbation vector for sector \( #l, 1 \leq l \leq n \)

Once the first blade is selected, the other blades satisfying condition 2 can be successively selected by taking the projection of perturbation in the direction perpendicular to the hyper-plane defined by the previously selected blades and selecting the blade with maximum projection.

The algorithm discussed above is employed in selecting the blades for calculating the basis vectors for this example. The norm of perturbation for each blade is calculated and presented in table A-6. The median of the norm of perturbation of the blades is 0.0031. Blade 4 with a norm of 0.0031 is selected as the first blade. The second blade is selected by taking the projection of perturbation of blades \([1-3, 5-24]\) in the direction perpendicular to blade#4 and choosing the blade with maximum projection. The magnitudes of projections are presented in table A-7 and blade #23 is selected as the second blade to form the basis. Once the blades for forming the basis have been selected, cyclic analysis is run for the two sectors and mode shapes from the first five families are used in MMDA analysis.

![Figure 4-6: Deviation in Natural Frequencies estimated via full rotor FEM and MMDA](alt)
Figure 4-6 plots deviations in frequencies as estimated by MMDA and exact full rotor analysis as given by equation 3.21 and 3.22. As observed from the plots the estimates of natural frequency from MMDA based on mode shapes from actual sectors as basis are very close to the actual values.

As discussed earlier, since the final objective of reduced order analysis is to estimate the forced or the harmonic response of the blades, the accuracy of the results from the reduced order model is verified for the harmonic response as well. The system is excited under the same conditions as for the previous example, i.e. 3rd engine order excitation in ±3% range of the mean forcing frequency of 4157.6 Hz and modal damping ratio of 0.1%. The harmonic response is obtained using modal superposition and the results are compared with the harmonic response from full rotor ANSYS analysis in figure 4-7. As observed from the figure, the harmonic response estimated via MMDA is very close to the exact response from ANSYS analysis.

![Figure 4-7: Blade tip amplitude estimated via full rotor FEM and MMDA (Alternative Basis, Sector 4 and 23) for 3EO excitation](image-url)
A comparison of figures 4-3 and 4-7 shows that the harmonic response is very similar in nature, which is expected because, the mistuned bladed disks in the two examples only differ in mistuning along POD3 (bladed disk for figure 4-3 has no mistuning along POD3), which is the minor POD feature and does not significantly alter the mode shapes or harmonic response. It is also observed that although the harmonic response estimate in figure 4-7 is very close to the true response, it is not as accurate as the estimate in figure 4-3; which is also expected because although minor, some mistuning is present along POD3 and only 2 dominant POD features, and hence 2 sectors are used to form the basis.

**Large mistuning**

Another problem that arises in mistuning is that of the rogue blade, i.e. blade having geometry significantly different from the nominal or average geometry, which can be caused by physical impact like bird hit or blade tip blending to remove blade corrosion. In such cases the mode shape of the rogue blade is significantly different from the nominal mode shape and the techniques discussed so far will fail to provide accurate results. But as it has been observed from the results of alternative basis, the mode shapes of actual blades can be used to form the basis. Same idea can be extended in case of large mistuning and the mode shapes from the cyclic analysis of the rogue blade, \( \Phi_{rogue} \), can also be included in basis to consider the impact of large mistuning. Hence in presence of large mistuning the following modification to the basis in MMDA algorithm is suggested:

\[
\Phi = \begin{bmatrix}
\Phi_0 & \Phi_1 & \Phi_1 & \ldots & \Phi_{np} & \Phi_{rogue}
\end{bmatrix}
\]  

(4.9)

Equation 4.9 is similar to equation 2.3 where \( \Phi_0 \) are tuned modes of the system with blades having the mean geometry and \( \Phi_1 \) are tuned modes of the system with blades having perturbed
geometry along \(^{l}\)th POD feature (the rogue blade is not included in the POD analysis). \(\Phi_{\text{rogue}}\) are the tuned modes from the cyclic analysis of the rogue blade. The explicit inclusion of the mode shapes from the rogue blade in the basis is done to account for the large changes in the mode shapes due to the rogue blade.

**Numerical Example**

MMDA with modified basis is applied to bladed disk with large mistuning in this example and the results are presented. The bladed disk considered earlier in previous examples is considered again. Mistuning is applied along the POD 1, i.e. along the thickness of the blades for all blades. Mistuning parameter values along POD 1 are again given by figure 3.1 and table A.3. Blade #23 is the rogue blade which has additional large mistuning along POD features 2 and 3, represented by figure 4-4. The values of mistuning parameters for the rogue blade are given in table 4-1.

<table>
<thead>
<tr>
<th>(\xi_1)</th>
<th>(\xi_2)</th>
<th>(\xi_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.027854</td>
<td>0.0667100</td>
<td>0.0608370</td>
</tr>
</tbody>
</table>

The mean value (excluding the rogue blade) of norm of the perturbation vectors (eq. 4.8) is 0.0025, whereas the norm of perturbation vector for the rogue blade is 0.0145, i.e. the perturbation in the rogue blade geometry is 5.8 times the average perturbation value.

MMDA analyses are run for the first family of the modes. In order to consider the impact of large mistuning, first analysis is run without including the mode shapes of the rogue blade, i.e. only the nominal mode shapes and the mode shapes from geometry perturbed along POD feature
1 are included in the basis. Then the mode shapes from the rogue blade, $\Phi_{\text{rogue}}$, are also included in the MMDA analysis. Full rotor ANSYS analysis is also run to compare the estimated natural frequency values with the true values. The deviation in frequency and % error in deviation, as given by equations 3.21, and 3.20, are presented in figures 4-8 and 4-9 respectively.

Figure 4-8: Deviation in Natural Frequencies estimated via full rotor FEM, MMDA (POD 1 only)
and MMDA (POD 1 and rogue blade)

Figure 4-9: Errors (%) in deviations of natural frequencies estimated via MMDA (POD 1 only)
and MMDA (POD 1 and rogue blade)
As discussed earlier, in presence of large mistuning, the inclusion of just the nominal modes and the modes from POD analysis is not sufficient to form a suitable basis, as observed from the results of MMDA analysis with just the nominal and POD1 modes in figures 4-8 and 4-9. We can see large errors in the natural frequency estimates from the MMDA analysis without the rogue blade mode shapes. It can also be observed from figures 4-8 and 4-9 that inclusion of mode shapes from the rogue blade rectifies this problem, and the natural frequency estimates from the MMDA analysis with rogue blade are very close to the true values.

Since the objective of the reduced order analysis is not only to get accurate estimates of the natural frequencies but also the blade tip amplitude of the harmonic response, the mistuned bladed disk is subjected to 3rd engine order harmonic excitation in ±3% range of the mean forcing frequency of 4157.6 Hz and modal damping ratio of 0.1%. The harmonic response is obtained using modal superposition and the results are compared with the harmonic response from full rotor ANSYS analysis in figure 4-10.

![Figure 4-10: Blade tip amplitude estimated via full rotor FEM, MMDA (POD 1 only) and MMDA (POD 1 and rogue blade) for 3EO excitation](image-url)
As expected since the mode shapes from just the nominal and POD analysis are not sufficient to form the suitable basis, the blade tip amplitude estimated from MMDA analysis without the rogue blade (Figure 4.10a) differ significantly from the true values. On the other hand the blade tip amplitude estimated via MMDA with rogue blade match exactly with the true response. Hence on the basis of results in this section, it can be said with the explicit inclusion of mode shapes from the rogue blade in the basis, MMDA analysis can be applied even in case of large mistuning.

Summary

Modified Modal Domain Approach (MMDA) has been modified to use mode shapes from the cyclic analysis of the actual sectors to form the basis. The advantage of using the alternative bases is that this avoids the step of creating POD perturbed geometries to calculate the mode shapes in the basis. It has been shown that MMDA with alternative bases is able to accurately estimate the mistuned natural frequencies and harmonic response even when minor PODs are not considered in selection of blades to form the basis. Hence it preserves the accuracy and computational efficiency of the standard MMDA analysis, while reducing a step in the inputs generation for MMDA analysis.

It has also been shown that the idea behind alternative basis can be extended to cases of large mistuning, and the mode shapes from the rogue blade (blade with large mistuning) can be explicitly included in the basis vectors. The results from MMDA analysis on an academic rotor with large mistuning in one of the blades show that although MMDA without the explicit inclusion of mode shapes from the rogue blade does not provide accurate results, the inclusion of mode shapes from the rogue blade rectifies this situation and the results from MMDA analysis with such modified basis are accurate.
Chapter 5

Comparison between Modified Modal Domain Analysis and Frequency Mistuning Approach

Mistuning has traditionally been modeled through the changes in Young’s moduli of blades, or equivalently through perturbations in the stiffness matrices associated with blades’ degrees of freedom. Such a mistuning is an approximation of actual mistuning because it does not capture the simultaneous perturbations in mass and stiffness matrices due to perturbations in geometry. Such a mistuning is commonly referred to as “Frequency Mistuning”. As discussed earlier, a consequence of Frequency Mistuning is that it does not alter the blades’ mode shapes, but only the blade alone frequencies. For Frequency Mistuning, reduced order models [16 - 25], have been developed which represent the solution as a weighted sum of the modes of the nominal tuned system. Such an assumption works because Frequency Mistuning does not alter the mode shapes associated with the blades. But actual (geometric) mistuning leads to simultaneous perturbations in mass and stiffness matrices, which alter the mode shapes associated with the blades, hence the subset of nominal modes assumption [20] is no longer valid and the accuracy of these models is reduced. In case of lightly damped structures like integrated blade rotors or blisks, this inaccuracy can lead to large errors in the predicted forced response of the mistuned system.

Subset of Nominal Modes (SNM) [20] is a reduced order model based on frequency mistuning. SNM and MMDA are similar in the sense that both use nominal modes to form the bases. The difference arises in the use of non-nominal modes in MMDA. Due to the use of additional set of modes, the cost of conducting MMDA analysis is higher as compared to SNM. Due to relatively higher cost of MMDA as compared to SNM it is preferable to identify scenarios where nominal mode solutions may work and where they do not. Hence a comparison study has been undertaken in this chapter, where SNM and MMDA both have been applied to a
geometrically mistuned system and the results have been presented to recognize the advantages of using the non-nominal modes over just the nominal mode approximation.

**SNM and MMDA: Comparison**

SNM algorithm has been summarized in the Appendix D. In this section, both the SNM and MMDA techniques are applied to a bladed disk with geometric mistuning. The disk considered in chapter 2. The number of sectors or blades \( n \) is 24. Mistuning has been introduced by varying the thicknesses of the blades. The thickness of the blade \( l \) is given by equation 2.43. Values of \( \xi_l \) are the same as considered in chapter 3 and are given by figure 3-1. Mean and standard deviation of this random mistuning pattern are \(-0.0024 (\approx 0)\) and 0.017, respectively.

A sector analysis of the nominal tuned sector is run and the natural frequencies (Figure 4-1) and mode shapes are calculated. The first step for frequency mistuning analysis is to calculate the Young’s moduli of the equivalent blades so that the natural frequency of the equivalent blade matches the natural frequency of the actual mistuned blade. The Young’s moduli for the blades can be calculated using equation D.1. Since for same geometric mistuning (in this case, change of thickness), the ratio of the mistuned and tuned natural frequencies depends upon the modes of excitation (the ratio for bending modes would be different from that of torsional modes), it is essential to identify the range of frequency for modal analysis and then identify the blade mode shapes dominant in that range of interest. Once the dominant blade mode shapes are identified then the equivalent Young’s moduli of the blades can be calculated using equation D.1.
Most of the papers dealing with mistuning have employed Young’s modulus mistuning in the reference full order finite models and the results from the reduced-order models have matched exactly with the full order models. Feiner and Griffin [22] applied their reduced order model (FMM technique) to a geometrically mistuned system and showed excellent accuracy of the reduced order model, but they dealt with a very small value of geometric mistuning (0.2% standard deviation in the changes in lengths of blades) and an isolated family of modes. Here SNM technique (on which FMM technique is based) is applied to a geometrically mistuned disk with comparatively larger value of geometric mistuning (1.7% standard deviation in thickness changes of the blades) in order to study its accuracy under larger geometric mistuning values.

Figure 5-1 shows the natural frequencies of the nominal bladed disk assembly for different families of modes. As observed from the figure, two types of regions exist; (i) regions
with isolated family of modes in a narrow frequency band where the primary energy is stored in the blades; for example, family 1. (ii) regions with overlapping families spanning a larger frequency bands where the primary energy is stored in the disk; for example, families 4, 5 and 6. From frequency mistuning point of view these two regions are different in the sense that for isolated families only a single blade mode shape is present in the region and natural frequency of the mistuned blades for that mode can be used to calculate the equivalent Young’s moduli of the blades for frequency mistuning. But in the other region where multiple families overlap, multiple definitions of equivalent frequency mistuning exist depending upon the family of modes used to calculate the equivalent Young’s moduli of the blades. Both the cases for the frequency mistuning have been considered in this study.

As shown in Figure 5-1, the first family of modes is an isolated family of modes. Figure 5-2a shows the 0 harmonic index sector mode shape of the first family. As observed from the figure, the energy in the sector is primarily stored in first bending (FB) mode of the blade, hence the frequency mistuning based on the first bending mode of the blade clamped at the base is generated to apply SNM analysis in the frequency band associated with the first family of the sector modes. Figure 5-3a shows that the 1st mode of the blade clamped at the base is the first bending mode. Figure 5-1 also shows that in frequency band near 22 kHz, families 4, 5 and 6 overlap. The 0 harmonic index sector mode shapes for families 4, 5 and 6 are shown in figure 5-2 (b), (c) and (d) respectively. The mode shapes for the 4th, 5th and 6th families show that the dominant blade mode shapes for the three families are lateral bending (LB), torsion (T) and elongation (E) respectively.

As shown in Figure 5-1, the first family of modes is an isolated family of modes. Figure 5-2a shows the mode shape of the first family of sector modes. As observed from the figure, the energy in the sector is primarily stored in first bending mode of the blade, hence the frequency mistuning based on the first bending mode of the blade clamped at the base is generated to apply SNM analysis in the frequency band associated with the first family of the sector modes. Figure 5-3a shows that the 1st mode of the blade clamped at the base is the first bending mode. Figure 5-
also shows that in frequency band near 22 kHz, families 4, 5 and 6 overlap. The sector mode shapes for families 4, 5 and 6 are shown in figure 4 (b), (c) and (d) respectively. The mode shapes for the 4th, 5th and 6th families show that the dominant blade mode shapes for the three families are lateral bending, torsion and elongation respectively.

![Figure 5-2: Sector mode shapes of the 1st, 4th, 5th and 6th Families](image)

The modal analysis of tuned blade clamped at base shows that the 2nd, 3rd and 5th modes are the lateral bending, torsion and elongation modes respectively (Figure 5-3).
Since the 3 families of blade mode shapes are present in the frequency band around 22 kHz., frequency mistuning is created for each of the 3 cases, i.e. equivalent Young’s moduli for lateral bending, torsion and elongation and the results are presented.
Figure 5-4 shows the equivalent Young’s moduli of the blades calculated for first bending, lateral bending, torsion and elongation modes of the blades. As observed from Figure 5-4, the equivalent Young’s moduli of the blades as calculated for the first bending and torsional modes follows the pattern of actual geometric mistuning parameter as plotted in Figure 3-1. This suggests that the first bending and torsional modes are sensitive to the changes in thicknesses of blades and significant changes in bladed disk assembly mode shapes are expected for the modes dominated by first bending and torsional blade mode shape components. On the other hand, the equivalent Young’s moduli of the blades estimated for the lateral bending and elongation modes of the clamped blades have a constant mean value of $3 \times 10^7$ psi, which suggests that lateral bending and elongation modes are insensitive to changes in thickness of blades and the assembled bladed disk modes dominated by lateral bending or elongation blade mode component are not
expected to change. It should also be noted that since no perturbation in the Young’s moduli of blades is observed for frequency mistuning based on lateral bending or elongation modes, such frequency mistuning will not capture any mistuning effects and the natural frequencies and mode shapes estimated from the SNM analysis will match with the natural frequencies and mode shapes of the nominal system.

Finite element models for these cases of frequency mistuning are generated and SNM analyses are performed on the basis of first 240 tuned modes. MMDA analysis is also performed for the mistuned bladed disk assembly for which number of POD features, \( n_p = 1 \) and 240 modes are used for both \( \Phi_0 \) and \( \Phi_1 \) in equation (D.9). Natural frequencies and mode shapes of the mistuned bladed disk assembly are also generated from finite element analysis of full rotor in ANSYS to compare the accuracy of the two reduced order models.

Figure 5-5 shows deviations in the first 24 natural frequencies estimated via MMDA, SNM and ANSYS analysis. As observed from the Figure, SNM is unable to capture the deviations in natural frequencies due to geometric mistuning with standard deviation equal to 1.7%.

![Figure 5-5: Deviations in frequencies estimated via reduced order models (MMDA and SNM) for the first bending family](image)
Next the mode shapes from the reduced order models (MMDA and SNM) are compared with the mode shapes from the full rotor ANSYS analysis using Modal Assurance Criterion (MAC) [48]. MAC values for the mode shapes estimated via reduced order models are plotted in Figure 5-6. The values closer to 1 on the diagonal suggest that the mode shapes estimated from the reduced order model are identical to the reference mode shapes (mode shapes from full rotor ANSYS analysis), whereas the values closer to 0 on the diagonal suggest that the estimated mode shapes from the reduced order model are orthogonal to the reference mode shapes.
The observation of MAC values for mode shapes estimated via MMDA suggests that MMDA is able to capture the mode shapes exactly. On the other hand MAC values for the modes estimated via SNM suggest that the technique is able to capture mode shapes for modes 1-12, but shows large errors in estimated mode shapes for modes 13-24. A closer look at the mode shapes of the bladed disk assembly shows that the first 12 modes do not show significant mode localization (for example mode #5 in figure 5-7a) and are hence similar to the modes of the nominal tuned bladed disk assembly. For this reason, nominal mode approximation is sufficient to estimate the first 12 mode shapes of the bladed disk assembly. On the other hand, modes 13-24 show significant mode localization (for example mode #19 in figure 5-7b) and are different from the mode shapes of the nominal tuned bladed disk assembly. In this case, the nominal mode
approximation of the mistuned modes is not sufficient and an additional set of non-nominal modes is required to form a suitable basis for the mistuned mode shapes.

Figure 5-7: Mode shapes of mistuned bladed disk assembly

Similar analysis for comparison between SNM and MMDA is also performed for frequency band near 22 kHz. Figure 5-8 shows the deviations in frequencies estimated via MMDA, SNM and full rotor ANSYS analysis. As observed from the figure, MMDA is able to capture the effects of geometric mistuning exactly whereas errors are observed in the frequency deviation estimates from SNM analyses. The observation of deviations in frequencies in Figure 5-8 shows large values of frequency deviation for modes 73-89, whereas small deviations for modes 90-110 and then large and small frequency deviations inter-mixed for modes 111 to 120.
A closer look at the mode shapes of the mistuned bladed disk assembly shows that modes 73-89 are blade dominated torsional mode shapes with significant mode localization. Since the torsional mode shapes are sensitive to the changes in thicknesses of the blades, the mode shapes 73-89 of the mistuned bladed disk assembly are significantly different from the mode shapes of the nominal tuned bladed disk assembly, which results in large frequency deviations. On the other hand, for modes 90-110, lateral bending, torsion and elongation modes are all present. A closer look at these mode shapes shows that the torsional mode shapes present in the range are disk dominated with small or no mode localization, hence they are not significantly altered by the mistuning. The elongation and lateral bending mode shapes present in the range are not disk dominated, but since the lateral bending and elongation mode shapes are not sensitive to the changes in the thicknesses of the blades, these mode shapes are also not altered due to mistuning. This results in small or no deviations in frequencies for modes 90-110. For modes 111 to 120, blade dominated 2nd bending modes (7th family) are also present along with disk dominated torsional and elongation modes, all of which do not show significant mode localization, hence are similar to the nominal tuned bladed disk assembly. Therefore for torsional and elongation modes in the range, significant
deviation in frequency is not observed. For modes corresponding to 2\textsuperscript{nd} bending in the range (modes 111, 115 and 117), although the mode shapes are not localized, they are blade dominated and since the natural frequency of the bending modes is sensitive to the thickness of the blades, significant shift in natural frequency is observed for modes corresponding to 2\textsuperscript{nd} bending modes. This phenomenon is similar to what is observed for the 1\textsuperscript{st} bending family in figures 5-5 and 5-6, where modes 1-12 are not localized but significant deviation in natural frequencies is observed. This analysis is also confirmed by the MAC values plotted for modes 73-120 for MMDA and frequency mistuning based on lateral bending, torsion and elongation blade modes (Figure 5-9). As discussed earlier, frequency mistuning based on lateral bending or elongation modes does not capture geometric mistuning and the mode shapes estimated via SNM analysis match with those of the nominal tuned system. Hence MAC values closer to 1 in Figures 5-8b and 5-8d suggest that the mistuned mode shapes are similar to the mode shapes of the nominal system, whereas MAC values closer to 0 suggest that the mode shapes are significantly altered from the mode shapes of the nominal system. MAC values in Figure 5-9a show that MMDA is able to estimate the mistuned modes accurately.

![MAC Values (MMDA)](image)

(a) MMDA
(b) SNM (Lateral Bending)

(c) SNM (Torsion)
Figure 5-9: MAC values for modes 73-120 calculated via MMDA, SNM (Lateral Bending), SNM (Torsion) and SNM (Elongation)

**Forced Response**

Next the harmonic response of the bladed disk assembly is estimated via different reduced order models and compared with full rotor ANSYS analysis. The differential equation of motion for the bladed disk assembly is given by equation 3.24. Using reduced order modeling transformation (equation D.3 for SNM, and equation 2.2 for MMDA), the reduced-order equations of motion can be written as:

\[ M_r \ddot{z} + \Phi^H C \Phi \dot{z} + K_r z = \Phi^H f(t) \]  \hspace{1cm} (5.1)

Equation 5.1 can be solved by first performing the modal analysis, and then using mode superposition technique to get the harmonic response.
The mistuned blade disk assembly (Figure 2-7) is excited by a harmonic force corresponding to engine order 6. Since the harmonic forcing function corresponds to 6th engine order, in order to study the accuracy of harmonic response for the first family of modes, excitation frequencies are chosen to be within ±3 percent of the mean excitation frequency of 4386.3 Hz (4386.3 Hz is the tuned natural frequency corresponding to 6th harmonic index for 1st family). The damping ratio in each mistuned mode is taken to be 0.001. Figure 5-10 shows the normalized maximum amplitude \( nma \) of the bladed disk assembly, which is defined as the ratio of the maximum amplitude in the bladed disk assembly at a given frequency to the maximum amplitude of the nominal tuned assembly at resonance, i.e.

\[
nma = \frac{\|a\|_\infty}{\max_\omega \|a_t\|_\omega}
\]  

(5.2)

where \( a \) and \( a_t \) are amplitudes vectors for mistuned and nominal tuned bladed disk, respectively. An \( nma \) value greater than 1 indicates that a blade’s response is higher than that of the nominal system at resonance. \( nma \) is also calculated for ANSYS analysis of full (360 degree) model to compare the accuracy of the reduced order models. As observed from Figure 5-10, MMDA is able to capture the effects of mistuning accurately. This is expected because the mode shapes and natural frequencies estimated via MMDA are exact as shown in figures 5-5 and 5-6. On the other hand, mean excitation frequency of 4386.3 Hz falls near the 15th mode of the mistuned bladed disk assembly. As shown in figure 5-6, the mode shapes 13-24 estimated via SNM are not accurate, hence harmonic response estimates based on SNM analysis is not expected to be accurate in the frequency band where these modes are excited. This behavior is also observed in Figure 5-10.
To get the estimates of the worst case scenario, it is important to compute the normalized peak maximum amplitude (npma) defined as:

\[
npma = \frac{\max_{\omega} \|p\|_{\infty}}{\max_{\omega} \|a_{i}\|_{\infty}} = \frac{\max_{\omega} nma}{\max_{\omega} nma}
\]  

(5.3)

The error in npma is defined as the difference in npma values as estimated via reduced order model and the actual npma values estimated via full rotor ANSYS analysis. The error in npma values estimated via MMDA is -1.08e-3%, whereas for npma values estimated via SNM is -22.42%, which suggests that SNM is not suitable for npma estimates for the first bending family.

The frequency deviation analysis in figure 5-8 and MAC values in figure 5-9 show that the natural frequencies and mode shapes for modes 73-89 are significantly different from the
mode shapes of the nominal system, whereas for modes 90-120, deviations in mode shapes are small. In order to study the accuracy of the reduced order models for both the regions of high and low deviations, the system is excited by a harmonic forcing function corresponding to 6th engine order excitation, within ±3 percent of the mean forcing frequencies of 17001.5 Hz and 26788.4 Hz, natural frequencies corresponding to 6th harmonic index of the 4th (lateral bending) and 5th (torsion) family in figure 5-2 respectively. The frequency band of 16491.4 Hz to 17511.5 Hz (mean excitation frequency 17001.5 Hz) excites modes between 73 and 89, whereas frequency band between 25984.8 Hz and 27592.1 Hz excites modes in the range 95-124, which include elongation family modes. The damping ratio in each mistuned mode is again taken to be 0.001.

Figures 5-10 and 5-11 show the normalized maximum amplitudes (nma) estimated via ANSYS analysis of full (360 degree) model and from the different reduced order models. SNM results are only presented for torsional modes of blade vibration. Since the lateral bending and elongation modes are not sensitive to the changes in thicknesses of the blades, SNM results based on these modes are same as responses of the nominal tuned system. Here again it is observed that MMDA estimates match exactly with the full order model ANSYS estimates, whereas nma estimates based on SNM analysis differ from the nma estimates from the ANSYS analysis. In Figure 5-11, peak value of nma predicted by SNM (torsion) is close to its actual value; however, frequency spectrum is quite different. In Figure 5-12, actual response is quite close to that of a nominal tuned system because modes (95 – 124) in the frequency band are similar to the mode shapes of the nominal tuned system.
Figure 5-11: Normalized maximum amplitudes estimated via MMDA and SNM (Torsion).

Engine order = 6, Mean excitation frequency = 17001.5 Hz

Figure 5-12: Normalized maximum amplitudes estimated via MMDA and SNM (Torsion).

Engine order = 6, Mean excitation frequency = 26788.4 Hz

The results in the previous section suggest that frequency mistuning fails to capture the effects of geometric mistuning, especially in the frequency bands of interest where the mode
shapes are significantly altered by geometric mistuning. Figures 5-9, 5-10 and 5-11 show that harmonic response estimates based on SNM analysis do not match with the actual harmonic response in the frequency spectrum, but a look at the normalized peak maximum amplitude ($npma$) from SNM analysis suggests that the error in $nmpa$ estimates are small. This observation could be misleading, as it may suggest that SNM analysis can be employed for calculating $nmpa$ values. Here, the amplitude magnifications obtained for this mistuning pattern are small with a maximum amplitude amplification of 1.4. In order to verify SNM’s ability to accurately estimate $npma$ values even in cases of large amplitude magnification, a worst case mistuning pattern is obtained using the constrained minimization of a nonlinear objective function in MATLAB ($fmincon$) [46]. The inverse of the peak maximum amplitude in the frequency band of interest is the nonlinear function of the mistuning parameters that is minimized. As it has been shown in chapter 3, MMDA analysis based on 2nd order approximation of the perturbations in mass and stiffness matrix provides an accurate reduced order model, which can be used to quickly generate reduced order matrices without any expensive computations; it is used to perform modal analysis of the mistuned system generated for each new set of mistuning parameters. The natural frequencies and mode shapes thus obtained are then used to calculate the peak maximum amplitude at each iteration during the maximization. The worst case mistuning parameters values are calculated to maximize $npma$ for 6th engine order excitation of the first bending family. Excitation frequencies are again chosen to be within ±3 percent of the mean excitation frequency of 4386.3 Hz (4386.3 Hz is the tuned natural frequency corresponding to 6th harmonic index for 1st family). The damping ratio in each mistuned mode is again taken to be 0.001. The mistuning parameters values for the worst case mistuning and the normalized maximum amplitude ($nma$) are plotted in figures 5-13 and 5-14. SNM analysis is also performed for the system and $nma$ values based on SNM analysis are also plotted in figure 5-14.
Figure 5-13: Mistuning pattern for npma maximization for first bending family and 6th EO excitation

Figure 5-14: Normalized maximum amplitude estimated via reduced order models (MMDA and SNM) for the optimum mistuning pattern. Engine order = 6, Mean excitation frequency = 4386.3 Hz
Figure 5-14 shows that \( npma \) value of 2.23 is obtained for the mistuning pattern. It also shows that MMDA provides exact estimates of the maximum amplitudes even this case of large amplitude magnification, whereas the \( npma \) value estimated via SNM analysis is 1.29 which differs significantly from the true \( npma \) value. Figure 5-14 clearly shows that it is possible to get large errors in \( npma \) values estimated via SNM analysis as well; hence SNM analysis cannot be used to reliably estimate \( npma \) values.

**Summary**

A study has been performed to compare the results from MMDA [26] and Frequency Mistuning analysis SNM [20] for a case of geometric mistuning, for both the (i) regions of isolated families of modes, and (ii) regions of multiple family overlap. It has been clearly shown that MMDA provides accurate results for all cases where as Frequency Mistuning is unable to provide accurate results for geometric mistuning in general. The ability of frequency mistuning to capture the effects of geometric mistuning depends on the region of interest. For frequency bands where the mode shapes are either disk dominated or blade dominated with no significant mode localization, the mistuned mode shapes are similar to mode shapes of the nominal tuned bladed disk assembly and as a result, errors in SNM results may not be large. In cases where amplitude amplification due to geometric mistuning is not high, peak maximum amplitude predicted by SNM is comparable to its actual value. However, the SNM frequency spectrum of the response is inaccurate. For a geometric mistuning pattern for which the peak maximum amplitude is high (2.23), the peak maximum amplitude predicted by SNM is 1.29. This result clearly suggests that SNM can miss the cases of “worst” geometric mistuning pattern.
Chapter 6

Reduced Order Model of a Multistage Bladed Rotor with Geometric Mistuning via Modal Analyses of Finite Element Sectors

The vibration of a multi-stage rotor has recently become an important area of research [30-36]. The issue here is the impact of interstage coupling on the vibratory response. It is obvious that a single stage analysis ignores this effect. The vibration analysis of a multistage rotor is complicated by the fact that the number of blades on each rotor can be different. In this case, cyclic symmetry is lost even when all blades are identical (perfectly tuned) on each rotor stage, and sector analysis cannot be performed. The complication of vibration analysis is further enhanced in the presence of mistuning of blades which refers to inevitable variation in blades’ modal properties due to manufacturing tolerances. Since these deviations are random variables, it is important to determine the statistics of response via Monte Carlo simulations. As the cyclic symmetry is lost, the response cannot be obtained by sector analysis and using a full 360 degree finite element analysis will be computationally prohibitive. Consequently, it is important to develop a technique to obtain an accurate reduced order of a mistuned multi-stage rotor.

Song et al. [33] have used component mode based reduced order model of a multistage rotor. Laxalde et al. [34] have used sector analysis to find mode shapes of a multistage rotor in a subspace generated by the modes of individual disks. But this solution methodology looks for mono-harmonic (harmonic index same for both the disks) solution only and does not find multi-harmonic solutions which are present in the exact solution. Another restriction on the usage of this method arises from the fact that a compatible mesh is needed at the disk interfaces which is not always possible especially for models of complex industrial rotors. Sternchuss et al. [35] have tried to overcome the necessity of compatible mesh at the disk interfaces by extending classical sub structuring technique in cyclic symmetry to compute mono-harmonic eigenvectors. It should be noted that papers by Song et al. [33], Laxalde et al. [34] and Sternchuss et al. [35] do not deal
with mistuning. Bladh et. al. [30] have examined the vibration of a two-stage mistuned rotor via a full (360 degree) finite element model.

This chapter develops a reduced order model of a mistuned multistage rotor on the basis of various finite element sector analyses which can be easily performed using a commercial package. This approach is similar to the modified modal domain analysis (MMDA), which has been successfully developed for a single-stage bladed rotor with geometric mistuning [26]. The main idea behind this approach was presented by Sinha [36], but was implemented on a simple spring-mass model of a multi-stage rotor in which the interstage coupling was modeled in an adhoc manner. The objective of this chapter is to show that the idea proposed by Sinha [36] is also valid in a finite element setting.

This chapter is organized as follows. The first section of the paper presents a reduced order model for a tuned two stage rotor (all blades on a stage are identical), figure 6-1. The second section introduces geometric mistuning and discusses how the reduced order model can be extended to analyze the response of a mistuned multistage system. The third section discusses how sector analysis can be applied efficiently for a Monte Carlo simulation of the natural frequencies and forced responses of the mistuned system. Although MMDA for multistage has been applied to an assembly with compatible mesh at the interface, it is only for ease of generation of mesh in ANSYS and the method is applicable for incompatible interface mesh in general.
Figure 6-1: Finite element model of a two-stage rotor

**Tuned Two-stage Rotor**

The free undamped vibration of a tuned two-stage rotor can be represented as

$$M_t \mathbf{x} + K_t \mathbf{x} = 0$$  \hspace{1cm} (6.1)

where $M_t$ and $K_t$ are the mass and stiffness matrix of the tuned system with each blade having the average geometry and

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_L \\ \mathbf{x}_R \end{bmatrix}$$  \hspace{1cm} (6.2)

where $\mathbf{x}_L$ and $\mathbf{x}_R$ are displacement vectors associated with left and right rotor stages, respectively. Let

$$\mathbf{x} = \Phi \mathbf{y}$$  \hspace{1cm} (6.3)

where
\[ \Phi = \begin{bmatrix} \phi_1 & \phi_{1f} & 0 & 0 \\ 0 & 0 & \phi_2 & \phi_{2f} \end{bmatrix} \]  \hspace{1cm} (6.4)

\( \phi_1 \): \( r \) tuned modes of Disk1 with the connecting ring having free end.

\( \phi_{1f} \): \( r \) tuned modes of Disk1 with the connecting ring having fixed end.

\( \phi_2 \): \( q \) tuned modes of Disk2 with the connecting ring having free end.

and

\( \phi_{2f} \): \( q \) tuned modes of Disk2 with the connecting ring having fixed end.

Tuned modes of each disk are natural basis vectors for model reduction. However, since the connecting ring affects mode shapes of each disk, it is attached to each disk and both fixed and free ends of connecting ring are considered for the construction of basis vectors.

Figure 6-2: Two-stage rotor with attached connection ring

Substituting (6.3) into (6.1) and pre-multiplying by \( \Phi^H \), the reduced order model is obtained as follows:

\[ M_r \ddot{y} + K_r y = 0 \]  \hspace{1cm} (6.5)

where
Here, $K_r$ and $M_r$ are obtained by generating the mesh of the full 360 degree finite element model (FEM) and the modal vectors in matrix $\Phi$ are obtained from modal analyses of finite element sectors. It should be noted that modal analysis of the full 360 degree FEM is not required.

**Numerical Example**

The finite element model (FEM) of a two stage rotor is constructed. Numbers of blades on left and right disks are taken to be 24 and 20, respectively. The natural frequencies and mode shapes are computed from the reduced order model with $r = q = 120$; i.e., the reduced order model has the order of 480.

Natural frequencies and mode shapes are also computed from the full (360 degree) finite element model of the two stage rotor.

![Figure 6-3: Error (%) in frequency of Tuned Two Stage Rotor estimated via Reduced Order Model ($r = q = 120$)](image)

Figure 6-3: Error (%) in frequency of Tuned Two Stage Rotor estimated via Reduced Order Model ($r = q = 120$)
Figure 6-3 shows the error in the natural frequency \( (Freq_{ROM}) \) predicted by the reduced order model as a percentage of the frequency of the tuned system \( (Freq_{Tnd}) \) with average geometry estimated via 360 degree FEM analysis, i.e. \( Error(\%) = \frac{Freq_{ROM} - Freq_{Tnd}}{Freq_{Tnd}} \times 100 \), for the first 120 modes. As observed from the Figure, the maximum error is 0.25%. The mean error is 0.0217% whereas the standard deviation of error is 0.0426%. Hence it can clearly be said that the reduced order model provides almost exact natural frequencies of the system.

Next, the differences between mode shapes from the reduced order model and full 360 degree FEM are also examined; for example figure 6-4 in which an index \( i \) of a modal vector represents its \( i^{th} \) element. The modal vectors are scaled so that the maximum value of an element in a modal vector is 1. As observed from figure 6-4, errors in the mode shapes predicted by the reduced order model are very small (maximum value of error is 0.1%) and it can be again said that the mode shapes are almost exact. This conclusion has also been verified by Modal Assurance Criterion (MAC) [48]. The peaks in Figure occur at the indices of the mode shape associated with the blades which have much higher amplitudes compared to disks.
Examining the natural frequencies and mode shapes, it has been observed that there are many families of modes; e.g., the first 44 modes represent first bending modes of vibration of blades, and very little disk vibration. Similarly there exist higher families of modes representing axial (edgewise) bending and torsional modes. For higher frequencies however different families of modes overlap over a frequency range and the modes from different families are close in frequencies.

A reduced order model based upon just the first bending families of Disk1 and Disk2, i.e. \( r = 24 \) and \( q = 20 \), is created and the results are presented in figure 6-5. As observed from the figure, the error is small for mode 10 and above but not for the lower modes. Hence a reduced order model based upon \( r = 24 \) and \( q = 20 \) is probably applicable except for few lower modes.
Figure 6-5: Error (%) in frequency estimated via Reduced Order Model (Tuned Two Stage Rotor, \( r = 24 \) and \( q = 20 \))

Next a reduced order model is created for \( r = 48 \) and \( q = 40 \) and the results are presented in figure 6-6.

Figure 6-6: Error (%) in frequency estimated via Reduced Order Model (Tuned Two Stage Rotor, \( r = 48 \) and \( q = 40 \))
As expected, the observations from figure 6-6 show that the errors in the results from the reduced order model with numbers of tuned modes \( r = 48 \) and \( q = 40 \) are higher in comparison to those from the reduced order model with \( r = q = 120 \) but lower in comparison to those from the reduced order model with \( r = 24 \) and \( q = 20 \). The absolute values of Error (%) in figure 6-6 are also quite small. A comparison of figures 6-5 and 6-6 also suggests that error from the reduced order model falls off rapidly with increasing number of modes. This result implies that an acceptable reduced order model can be created by considering modes of frequency range of interest only. Figure 6-7 shows the differences between mode shapes from the reduced order model and full 360 degree FEM. It should be recalled that the modal vectors are scaled so that the maximum value of an element in a modal vector is 1.

![Graph](image.png)

a) Mode#1
Mistuned Two-stage Rotor

The reduced order model for a two-stage rotor can be extended to include mistuning effects by including the modes of the system with blades having perturbed geometry along a POD feature [26, 27], i.e.

$$\Phi = \begin{bmatrix} \phi_1 & \phi_{1f} & \phi_{11} & \cdots & \phi_{1p} & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & \phi_2 & \phi_{2f} & \phi_{21} & \cdots & \phi_{2p} \end{bmatrix}$$  \hspace{1cm} (6.7)

where $\phi_{ij}$ are tuned modes of $i^{th}$ disk with blades having perturbed geometry along $j^{th}$ POD feature [27]; $j=1,2,\cdots,p$. 

Figure 6-7: Difference between Modal Vector from Reduced Order Model and ANSYS analysis, $(r = 48$ and $q = 40)$
**Numerical Example**

The two-stage rotor considered in the previous section is used here again. The mistuning is introduced by varying the thickness of each blade. The thickness of the $i^{th}$ blade can be represented as:

$$ b_i = b_0(1 + \xi_i) $$

where $b_0$ is the thickness of the tuned blade on each disk. And, $\xi$ is the random variable with zero mean representing the only POD feature and $\xi_i$ is its value for the blade #i. Based upon a set of random values of $\xi$ from the Matlab routine ‘randn’, a mistuning pattern is created for each disk, Figure 6-8 and Table A-8 in Appendix. The standard deviation ($\sigma$) of $\xi$ is about 1.7%. Note again that mean ($\mu$) is not exactly zero because of a finite number of random variables.
b) Disk #2

Figure 6-8: Mistuning Pattern for Disks of a multistage system

Figure 6-9 shows deviations in natural frequencies from their tuned values, which are calculated by the reduced order model and the full (360 degree) ANSYS analysis as well.

Figure 6-9: Deviations in Natural Frequencies of a Mistuned Two Stage Rotor ($r = q = 120$)
As observed in the figure 6-9, the natural frequencies predicted by the reduced-order model are almost an exact match to those calculated by ANSYS. Some disagreements in the higher mode frequencies are expected because only 120 modes are used as a basis vectors for each $\phi$ in equation (6.7).

The % error in the deviation from a tuned two stage frequency ($Freq_{Tnd}$) is calculated as the ratio of the error in deviation calculated from the reduced order model ($Dev_{ROM}$ - $Dev_{Act}$) to the actual deviation ($Dev_{Act}$), i.e.

$$ErrorD(\%) = \frac{Dev_{ROM} - Dev_{Act}}{Dev_{Act}} \times 100$$

where

$$Dev_{ROM} = Freq_{ROM} - Freq_{Tnd}$$

and

$$Dev_{Act} = Freq_{Act} - Freq_{Tnd}.$$

As observed from figure 6-10, the % error in deviation is very small in spite of a strict criterion applied for the calculation of % error. Some high values of %errors in figure 6-10 are reflection of the fact that they are calculated as a percentage of actual frequency deviation. It is possible to get high values of % error even for small values of $Dev_{ROM} - Dev_{Act}$ if $Dev_{Act}$ is small. Therefore, high values of %errors in figure 6-10 are not reflected in figure 6-9.

Figure 6-10: Errors (%) in deviations of natural frequencies of tuned two stage rotor ($r = q = 120$)
Figure 6-11 shows errors in modal vectors corresponding to mode#1 and mode#3 of the mistuned system respectively. It can be seen that the difference between the modal vectors from the reduced order model and ANSYS is almost zero. This conclusion has also been verified by Modal Assurance Criterion (MAC) [48].

Figure 6-11: Difference between Mistuned Modal Vectors (Mode #1 and Mode #3) from Reduced Order Model and Full ANSYS analysis (r = q =120)
Forced Response

The steady state harmonic response of the system can be calculated via mode superposition method. The differential equation of motion is given by equation (3.25). The system is excited by a harmonic force applied to Disk 2. The spatial distribution of the force corresponds to the 2nd Nodal diameter. No force is applied to Disk 1. Amplitudes of steady state responses are obtained from the reduced order model via modal superposition method for excitation frequencies within 5 percent of the natural frequency corresponding to 2nd Nodal diameter of tuned single stage Disk2. The damping ratio in each mistuned mode is taken to be 0.001. Figure 6-12 shows the tip amplitudes for Disk 1 (blade1 and blade13) and Disk 2 (blade1 and blade11). These results match exactly with those from ANSYS analysis of full (360 degree) model.

![Blade Tip Amplitude of Disk 1(top) and Disk 2 (bottom) as a function of excitation frequencies (r = q =120)](image-url)
Figure 6-13 represents the comparison between the forced response of single-stage and multiple stage rotors. The maximum amplitude of a mistuned two-stage rotor is higher than that of a mistuned single-stage rotor in this case.

![Figure 6-13: Forced Responses of Single Stage and Multi-Stage Rotors (Engine Order 2, Mean Forcing Frequency = 4102 Hz.)](image)

**Monte Carlo Simulation of Natural Frequencies and Forced Responses via Sector Analyses**

The mass and stiffness matrices of a mistuned multistage rotor can be represented as:

\[ M = M_I + \delta M \]  \hspace{2cm} (6.8)

\[ K = K_I + \delta K \]  \hspace{2cm} (6.9)

where matrices \( \delta M \) and \( \delta K \) are deviations in mass and stiffness matrices due to geometric mistuning, respectively. Similar to equation (6), reduced order mass and stiffness matrices are

\[ M_r = \Phi^H M_I \Phi + \Phi^H \delta M \Phi \]  \hspace{2cm} (6.10)

and
\[
K_r = \Phi^H K_r \Phi + \Phi^H \delta K \Phi
\]  
(6.11)

Here, \( \Phi^H M_r \Phi \) and \( \Phi^H K_r \Phi \) only need to be calculated once and stored. However, \( \Phi^H \delta M \Phi \) and \( \Phi^H \delta K \Phi \) will change for each case of mistuning pattern of blades’ geometries.

The perturbations in mass and stiffness matrices \( \delta M \) and \( \delta K \) have block diagonal forms:

\[
\delta M = \begin{bmatrix} \delta M_1 & 0 \\ 0 & \delta M_2 \end{bmatrix}, \quad \delta K = \begin{bmatrix} \delta K_1 & 0 \\ 0 & \delta K_2 \end{bmatrix}
\]

(6.12a, b)

where \( \delta M_i \) and \( \delta K_i \) are the perturbations in mass and stiffness matrices due to \( i^{th} \) disk, respectively.

Therefore,

\[
\Phi^H \delta M \Phi = \begin{bmatrix} \Phi_{s1}^H \delta M_1 \Phi_{s1} & 0 \\ 0 & \Phi_{s2}^H \delta M_2 \Phi_{s2} \end{bmatrix}
\]

(6.13)

and

\[
\Phi^H \delta K \Phi = \begin{bmatrix} \Phi_{s1}^H \delta K_1 \Phi_{s1} & 0 \\ 0 & \Phi_{s2}^H \delta K_2 \Phi_{s2} \end{bmatrix}
\]

(6.14)

where

\[
\Phi = \begin{bmatrix} \Phi_{s1} & \Phi_{s2} \end{bmatrix}
\]

(6.15)

The procedures for calculating \( \Phi_{s1}^H \delta M_i \Phi_{s1} \) and \( \Phi_{s1}^H \delta K_i \Phi_{s1} \) are similar to those for a single stage rotor as given by equations (3.26) and (3.31). Further, these computations are done so that \( \Phi_{s1}^H \delta M_i \Phi_{s1} \) and \( \Phi_{s1}^H \delta K_i \Phi_{s1} \) are easily obtained for any permutation of mistuning pattern without running any new finite element analysis and without repeating most of computations.
Numerical Example

The two-stage rotor considered in previous sections is used again. With fractional changes in blade thicknesses shown in Table A-8, 1000 random permutations of the mistuning pattern are created. For each permutation of the mistuning pattern, the reduced order model with $r = 48$ and $q = 40$ is used. The validity of the reduced-order analysis is shown by comparison with full rotor results for one of the random permutations of original blade thicknesses, Table A-9, figures 6-14a, b and figure 6-15.

![Mistuning pattern of a random permutation of blades for Disk#1 and Disk#2](image-url)

Figure 6-14: Mistuning pattern of a random permutation of blades for Disk#1 and Disk#2
The histogram representing the distribution of natural frequency over all permutations of mistuning patterns is plotted for the 1st, 21st and 41st natural frequency (Figures 6-16 – 6-18), where NOC stands for number of occurrences in a frequency interval. These figures suggest that the average frequency of the lower modes reduces slightly due to mistuning whereas that of the higher modes increases slightly.
Figure 6-17: Distribution of 21st natural frequency \( r = 48 \) and \( q = 40 \)

Figure 6-18: Distribution of 41st natural frequency \( r = 48 \) and \( q = 40 \)

Next the distribution of non-dimensional peak maximum amplitude \( npma \) given by equation (2.49) is plotted in figure 6-19. Again, the spatial distribution of force corresponds to the
9th Nodal Diameter. No force is applied to Disk2. The frequency range is chosen to be 0.98 to 1.02 times the natural frequency corresponding to 9th Nodal Diameter of tuned single stage Disk1. Modal damping ratio is again chosen to be 0.001.

![Histogram](image)

**Figure 6-19:** Peak maximum amplitude distribution for harmonic force applied to Disk1 (r = 48 and q = 40)

**Summary**

A reduced order model using the tuned modes from finite element sector analyses have been developed for application to a multistage rotor with geometric variations in blades. It has been shown that the results provided by the reduced order model are almost exact and they are accurate enough to capture effects arising out of a small geometric mistuning which is inherent to any manufacturing process. It has also been shown that the Monte Carlo simulation of natural frequencies and forced responses of a multi-stage rotor can be easily performed for various permutations of a mistuning pattern in a computationally efficient manner.
Chapter 7

Mistuning Identification

Most of the mistuning analyses have assumed the blade properties to vary randomly, or in some specified fashion, thus, generally avoiding the characterization of mistuning parameters from experimental data. However, in order to have a realistic mistuning representation, it is necessary to model the mistuning accurately. A precise identification of the degree of mistuning has become an important area of investigation, particularly in a practical implementation of forced response reduction strategies in bladed disk. The standard methods for disk with detachable blades consist of removing the individual blades for measurements of their natural frequencies. This is not an accurate procedure, since it ignores the effects of blade attachment to the disk, and the problem is especially pronounced in integrally bladed rotors (blisk), where the blades and the disk form one integral piece. Therefore, in order to accurately identify mistuning in bladed disks, it is essential to develop methods that can produce measurements of the entire bladed disk assembly. Mignolet et al. [37, 38], Lim et al. [39], Feiner and Griffin [40, 41] have developed mistuning identification techniques but these identification techniques are based on reduced order models, which themselves are based on frequency mistuning. This frequency mistuning is not a very precise definition of realistic mistuning parameters. Since the actual mistuning occurs due to perturbations in blade geometries, the identification techniques based on frequency mistuning can lead to erroneous estimates of the bladed disk’s modal characteristics, which in turn provoke large discrepancies in the prediction of the forced response of the entire system near or at resonance in the presence of a small damping ratio. It is essential to account for geometric mistuning to accurately identify mistuning parameters. Identification of independent mistuning parameters will help to identify effects of different mistuning parameters, like
thickness of blades, blade inclination etc., on the modal characteristics of the system, which in
turn can help to devise better strategies to avoid the detrimental effects of mistuning.

The reduced order model developed by Sinha [26] is based on geometric mistuning hence
the effect of perturbations in geometries are transparent. The effect of each factor, like thickness
of blades, blade inclination etc., can be taken into account separately using this model. In his
work, Sinha [27] has used Proper Orthogonal Decomposition of CMM data to identify the
independent vibratory parameters (mistuning parameters). The identification of these independent
parameters is crucial for MMDA because the subset of modes of tuned system with blade’s
geometry perturbed along an independent parameter is used to form the bases of solution. The
algorithm for identification of mistuning parameters using the mode shapes and natural
frequencies of the mistuned bladed disk assembly has been discussed in the next section.

Algorithm for mistuning identification

Identification using Modal Analysis

Equation (3.19) provides an expression for the \((\nu + 1, \rho + 1)\) element of \(\Phi^H_i \partial M \Phi_j\). Based
on the expression \(\Phi^H_i \partial M \Phi_j\) can be written in matrix form as:

\[
\Phi^H_i \partial M \Phi_j = \sum_{s=1}^{np} \sum_{l=1}^{np} \frac{1}{N} \begin{pmatrix}
A(s)_{0,0}^i \bar{z}_s(0) & A(s)_{0,1}^i \bar{z}_s(n-1) & \cdots & A(s)_{0,n-1}^i \bar{z}_s(1) \\
A(s)_{1,0}^i \bar{z}_s(1) & A(s)_{1,1}^i \bar{z}_s(0) & \cdots & A(s)_{1,n-1}^i \bar{z}_s(2) \\
\vdots & \vdots & \ddots & \vdots \\
A(s)_{n-1,0}^i \bar{z}_s(n-1) & A(s)_{n-1,1}^i \bar{z}_s(n-2) & \cdots & A(s)_{n-1,n-1}^i \bar{z}_s(0)
\end{pmatrix} + \cdots
\]

\[
\sum_{s=1}^{np} \sum_{l=1}^{np} \frac{1}{N} \begin{pmatrix}
B(s,l)_{0,0}^i Z_l^T P(0) Z_s & B(s,l)_{0,1}^i Z_l^T P(1) Z_s & \cdots & B(s,l)_{0,n-1}^i Z_l^T P(1) Z_s \\
B(s,l)_{1,0}^i Z_l^T P(1) Z_s & B(s,l)_{1,1}^i Z_l^T P(0) Z_s & \cdots & B(s,l)_{1,n-1}^i Z_l^T P(2) Z_s \\
\vdots & \vdots & \ddots & \vdots \\
B(s,l)_{n-1,0}^i Z_l^T P(n-1) Z_s & B(s,l)_{n-1,1}^i Z_l^T P(n-2) Z_s & \cdots & B(s,l)_{n-1,n-1}^i Z_l^T P(0) Z_s
\end{pmatrix}
\]
where

\[
A(s)_{i,j}^{v,\rho} = \Phi_{i,1,v}^H \frac{\partial M}{\partial \xi_s} \Phi_{j,1,\rho} \quad \text{and} \quad (7.2a)
\]

\[
B(s,t)_{i,j}^{v,\rho} = \Phi_{i,1,v}^H \frac{1}{2} \frac{\partial^2 M}{\partial \xi_s \partial \xi_t} \Phi_{j,1,\rho} \quad \text{and} \quad (7.2b)
\]

\[\bar{\xi}_s(k)\] is the \(k\)th Discrete Fourier transform \([47]\) of \(\xi_{s,f}\) defined by equation (3.12). The expressions for \(Z_s\) and \(P(k)\) are provided by equations (3.17) and (3.18) respectively. As discussed earlier, equation (7.1) suggests that the mistuning parameters present themselves as Discrete Fourier transforms in the reduced order matrices. It should be noted that calculation of \(\Phi_i^H \partial M \Phi_j\) is block assembling exercise where the blocks of \(\Phi_{i,1,v}^H \frac{\partial M}{\partial \xi_s} \Phi_{j,1,\rho}\) and \(\Phi_{i,1,v}^H \frac{1}{2} \frac{\partial^2 M}{\partial \xi_s \partial \xi_t} \Phi_{j,1,\rho}\) are multiplied with the \(\bar{\xi}_s(k)\) and \(Z_s\) to assemble the reduced order matrix. This structure of the reduced order matrices suggests that given a mode shape and eigenvalue of the system, we should be able to reverse the process and can calculate the Discrete Fourier transforms and hence the mistuning parameters.

Let \(\beta_p\) be the vector of modal coefficients of the \(p\)th mode shape. Then \(\beta_p\) can be represented as:

\[
\beta_p = \begin{bmatrix} \beta_{p,0} \\ \vdots \\ \beta_{p,np} \end{bmatrix} \quad (7.3)
\]

where \(\beta_{p,j}\) is the coefficient of \(p\)th mode corresponding to \(j\)th POD feature. The eigenvalue problem for the reduced order system for the \(p\)th mode can be written as:

\[
(\Phi^H K \Phi - \omega_p^2 \Phi^H M \Phi) \beta_p + (\Phi^H \partial K \Phi - \omega_p^2 \Phi^H \partial M \Phi) \beta_p = 0 \quad (7.4)
\]
Since $\Phi, K_t, M_t$ are defined by the mean geometry of the system, they are known. The natural
frequencies and mode shapes are measured hence $\omega_p$ and $\beta_p$ are also known. Therefore the
expression $(\Phi^H K_t \Phi - \omega_p^2 \Phi^H M_t \Phi) \beta_p$ can be directly calculated from the known values.

Combining equations (2.19) and (7.3) we can write:

$$
\Phi^H \delta \Phi \beta_p = \left( \sum_{j=0}^{np} \Phi_0^H \delta \Phi_j \beta_{p,j} \right)
$$

(7.5)

Similar expression can be written for $\Phi^H \delta K \beta_p$. Using (5.1) the expression for $\Phi_i^H \delta \Phi \beta_{p,j}$
can be written as:

$$
\Phi_i^H \delta \Phi \beta_{p,j} = \sum_{s=1}^{np} \begin{pmatrix}
A(s)_{i,j}^{0,0} \bar{\xi}_s(0) & A(s)_{i,j}^{0,1} \bar{\xi}_s(n-1) & \cdots & A(s)_{i,j}^{n-1,n-1} \bar{\xi}_s(1) \\
A(s)_{i,j}^{1,0} \bar{\xi}_s(1) & A(s)_{i,j}^{1,1} \bar{\xi}_s(0) & \cdots & A(s)_{i,j}^{n-1,n-1} \bar{\xi}_s(2) \\
\vdots & \vdots & \ddots & \vdots \\
A(s)_{i,j}^{N-1,0} \bar{\xi}_s(n-1) & A(s)_{i,j}^{N-1,1} \bar{\xi}_s(n-2) & \cdots & A(s)_{i,j}^{n-1,n-1} \bar{\xi}_s(0)
\end{pmatrix} \begin{pmatrix}
\beta_{p,j,0} \\
\beta_{p,j,1} \\
\vdots \\
\beta_{p,j,n-1}
\end{pmatrix}
$$

(7.6)

Here

$$
\beta_{p,j} = \begin{pmatrix}
\beta_{p,j,0} \\
\beta_{p,j,1} \\
\vdots \\
\beta_{p,j,n-1}
\end{pmatrix}
$$

(7.7)

where $\beta_{p,j,l}$ is the coefficient of $p^{th}$ mode corresponding to $j^{th}$ POD feature and sector #l.

After a little arrangement, equation (7.6) can be written as:
\[
\Phi_i^H \partial \Phi_j \beta_{p,j} = \sum_{s=1}^{np} \left( A(s)_{0,0} \beta_{p,j,0} \quad A(s)_{0,n-1} \beta_{p,j,n-1} \quad \cdots \quad A(s)_{0,1} \beta_{p,j,1} \right) Z_s + \sum_{s=1}^{np} \left( B(s,t)_{i,j} \beta_{p,j} \right) Z_s
\]

(7.8)

Or

\[
\Phi_i^H \partial \Phi_j \beta_{p,j} = \sum_{s=1}^{np} \hat{A}_{i,j}^p (s) Z_s + \sum_{s=1}^{np} \hat{B}_{i,j}^p (s) Z_s
\]

(7.9)

where

\[
\hat{A}_{i,j}^p (s) = \begin{bmatrix}
A(s)_{0,0} \beta_{p,j,0} & A(s)_{i,j} & \cdots & A(s)_{0,1} \\
A(s)_{1,1} \beta_{p,j,1} & A(s)_{1,j} & \cdots & A(s)_{1,2} \\
\vdots & \vdots & \ddots & \vdots \\
A(s)_{n-1,n-1} \beta_{p,j,n-1} & A(s)_{n-1,j} & \cdots & A(s)_{n-1,0}
\end{bmatrix}
\]

(7.10a)

and

\[
\hat{B}_{i,j}^p (s,t) = \frac{1}{n} \begin{bmatrix}
B(s,t)_{0,0} \beta_{p,j,0} & B(s,t)_{0,n-1} \beta_{p,j,n-1} & \cdots & B(s,t)_{0,1} \\
B(s,t)_{1,0} \beta_{p,j,0} & B(s,t)_{1,n-1} \beta_{p,j,n-1} & \cdots & B(s,t)_{1,1} \\
\vdots & \vdots & \ddots & \vdots \\
B(s,t)_{n-1,0} \beta_{p,j,0} & B(s,t)_{n-1,n-1} \beta_{p,j,n-1} & \cdots & B(s,t)_{n-1,1}
\end{bmatrix}
\]

(7.10b)

Therefore

\[
\sum_{j=0}^{np} \Phi_i^H \partial \Phi_j \beta_{p,j} = \sum_{s=1}^{np} \sum_{j=0}^{np} A_{i,j}^p (s) Z_s + \sum_{s=1}^{np} \sum_{t=1}^{np} \sum_{j=0}^{np} \hat{B}_{i,j}^p (s,t) Z_s
\]

(7.11)
or

\[ \sum_{j=0}^{np} \Phi_i^H \delta \Phi_j \beta_{p,j} = \sum_{s=1}^{np} \hat{A}_i^p(s)Z_s + \sum_{s=1}^{np} \sum_{l=1}^{np} \hat{B}_i^p(s,t) \begin{pmatrix} \frac{Z_t^T P(0)}{} \\ \frac{Z_t^T P(1)}{} \\ \vdots \\ \frac{Z_t^T P(n-1)}{} \end{pmatrix} Z_s \]  

(7.12)

where

\[ \hat{A}_i^p(s) = \sum_{j=0}^{n-1} \hat{A}_{i,j}^p(s) \quad \text{and} \quad \hat{B}_i^p(s,t) = \sum_{j=0}^{np} \hat{B}_{i,j}^p(s,t) \]  

(7.13a, 7.13b)

Equation (7.12) can be written in matrix form as:

\[ \sum_{j=0}^{np} \Phi_i^H \delta \Phi_j \beta_{p,j} = \Psi_{i,p}^M \begin{pmatrix} Z_1 \\ \vdots \\ Z_{np} \end{pmatrix} \]  

(7.14)

where

\[ \Psi_{i,p}^M = \begin{pmatrix} \psi_{i,p,1}^M & \cdots & \psi_{i,p,np}^M \end{pmatrix} \quad \text{and} \quad \Psi_{i,p}^M = \hat{A}_i^p(s) + \sum_{l=1}^{np} \hat{B}_i^p(s,t) \begin{pmatrix} \frac{Z_t^T P(0)}{} \\ \frac{Z_t^T P(1)}{} \\ \vdots \\ \frac{Z_t^T P(n-1)}{} \end{pmatrix} \]  

(7.15, 7.16)

An equation similar to equation (7.14) can be generated for \( \sum_{j=0}^{r} \Phi_i^H \delta \Phi_j \beta_{p,j} \):

\[ \sum_{j=0}^{np} \Phi_i^H \delta \Phi_j \beta_{p,j} = \Psi_{i,p}^K \begin{pmatrix} Z_1 \\ \vdots \\ Z_{np} \end{pmatrix} \]  

(7.17)

From equations (7.14) and (7.17)

\[ \sum_{j=0}^{np} (\Phi_i^H \delta \Phi_j - \omega_i^2 \Phi_i^H \delta \Phi_j) \beta_{p,j} = \begin{pmatrix} \psi_{i,p}^K - \omega_i^2 \psi_{i,p}^M \\ \vdots \\ \psi_{i,p}^r \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_r \end{pmatrix} \]  

(7.18)
Hence the eigenvalue problem from equation (7.4) can be written as:

\[
\Psi_p \begin{pmatrix} Z_1 \\ \vdots \\ Z_{np} \end{pmatrix} = \Gamma_p
\]  

(7.20)

where

\[
(\omega_p^2 \Phi^H M \Phi - \Phi^H K \Phi) \beta_p = \omega_p^2 \Gamma_p^M - \Gamma_p^K = \Gamma_p
\]  

(7.21)

\[
\Psi_p = \begin{pmatrix} \Psi_{0,p}^K \\ \vdots \\ \Psi_{np,p}^K \end{pmatrix} - \omega_p^2 \begin{pmatrix} \Psi_{0,p}^M \\ \vdots \\ \Psi_{np,p}^M \end{pmatrix}
\]  

(7.22)

If \( k \) modes are used then equation (7.20) for each mode can be stacked and the complete set of equations can be written as:

\[
\begin{pmatrix} \Psi_p \\ \vdots \\ \Psi_{p+k} \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ Z_{np} \end{pmatrix} = \begin{pmatrix} \Gamma_p \\ \vdots \\ \Gamma_{p+k} \end{pmatrix}
\]  

(7.23)

Equation (7.23) can be used to calculate the values of mistuning parameters. Since equation (7.23) is nonlinear (\( \psi_q \) is function of

\[
\begin{pmatrix} Z_1^T \\ \vdots \\ Z_{np}^T \end{pmatrix}
\]  

an iterative least square solution can be used to solve the equation. The initial solution for the iterative procedure can be obtained by taking only the constant term in equation (7.16), i.e. \( \Psi^M_{i,p,s} = \hat{A}_i^p(s) \).

**Identification using Harmonic Analysis**

The differential equation of motion governing the forced response of a system is:

\[
M \ddot{x} + C \dot{x} + K x = \mathbf{f}(t)
\]  

(7.24)
where $M$, $K$, $C$ and $f(t)$ are mass matrix, stiffness matrix, damping matrix and the forcing functions respectively. Under harmonic excitation, the excitation force and $f(t)$ and the response $x(t)$ can be represented as:

$$f(t) = F e^{i\omega t} \quad \text{and} \quad x(t) = X(\omega) e^{i\omega t} \quad (7.25a)$$

Substituting equations 7.25a and b in equation (7.24), the equation governing the harmonic response amplitude can be written as:

$$(K + i\omega C - \omega^2 M)X(\omega) = F \quad (7.26)$$

Assuming Raleigh damping, the damping matrix $C$ can be represented as:

$$C = \alpha M + \gamma K \quad (7.27)$$

Substituting equation (7.27) in (7.26), the equation governing the harmonic response, can be written as:

$$[K(1 + i\omega \gamma) - (\omega^2 - i\omega \alpha)M]X(\omega) = F \quad (7.28)$$

Using modal superposition principle, the amplitude of the harmonic response can be represented as a linear combination of the modes of the mistuned system, and using MMDA, the modes of the mistuned system can be represented as a linear combination of the modes of the nominal system and the modes of the system perturbed along specific POD features (equations 2.2 and 2.3), hence the harmonic response amplitude of the mistuned system can be represented as a linear combination for mode shape vectors forming the basis for MMDA analysis, i.e.

$$X = \Phi \beta \quad (7.29)$$

where $\Phi$ is given by equation (2.3). Substituting equation (7.29) in equation (7.26) and pre-multiplying with $\Phi^H$ the equation governing the harmonic response can be written as:

$$(1 + i\omega \gamma)\Phi^H K \Phi \beta - (\omega^2 - i\omega \alpha)\Phi^H M \Phi \beta = \Phi^H F \quad (7.30)$$
As discussed earlier, the mass and stiffness matrices of the mistuned system can be represented as sum of the nominal matrix and the perturbation in the matrix due to mistuning, i.e.

\[
M = M_f + \delta M \quad \text{and} \quad K = K_f + \delta K
\]  

(7.31a, 7.31b)

Substituting these values in equation (7.30), the equation for harmonic amplitude can be written as:

\[
(1 + i \omega \gamma) \Phi^H K_f \Phi \beta - (\omega^2 - i \omega \alpha) \Phi^H M_f \Phi \beta + (1 + i \omega \gamma) \Phi^H \delta K \Phi \beta - (\omega^2 - i \omega \alpha) \Phi^H \delta M \Phi \beta = \Phi^H F
\]  

(7.32)

Let the \( p \)th harmonic excitation frequency be \( \omega_p \) and the vector of modal coefficients for the amplitude be \( \beta_p \). The form of \( \beta_p \) is exactly similar to the one given by the equation (7.7). Since the form of \( \beta_p \) is exactly same as the one discussed in previous section, the analysis in previous section applies and using equations (7.5), (7.18) and (7.22), equation (7.32) can be written as:

\[
\Psi_p \begin{bmatrix} Z_1 \\ \vdots \\ Z_{np} \end{bmatrix} = \Gamma_p + \Phi^H F
\]  

(7.33)

where

\[
\Psi_p = (1 + i \omega_p \gamma) \begin{bmatrix} \Psi_{0,p}^K \\ \vdots \\ \Psi_{np,p}^K \end{bmatrix} - (\omega_p^2 - i \omega_p \alpha) \begin{bmatrix} \Psi_{0,p}^M \\ \vdots \\ \Psi_{np,p}^M \end{bmatrix}
\]  

(7.34)

and

\[
[(\omega_p^2 - i \omega_p \alpha) \Phi^H M_f \Phi - (1 + i \omega_p \gamma) \Phi^H K_f \Phi] \beta_p = (\omega_p^2 - i \omega_p \alpha) \Gamma_p^M - (1 + i \omega_p \gamma) \Gamma_p^K = \Gamma_p
\]  

(7.35)
The definition of $\Psi_{i,p}^{M}$, $\Psi_{i,p}^{K}$, $\Gamma_{p}^{M}$ and $\Gamma_{p}^{K}$ are the same as given by the equations (7.18) and (7.22). For multiple harmonic excitations, the equations represented by equation (7.35) can be stacked in a matrix form and represented in form similar to equation (7.23).

\[
\begin{pmatrix}
\Psi_{p} & Z_{1} \\
\vdots & \vdots \\
\Psi_{p+k} & Z_{np}
\end{pmatrix} = \begin{pmatrix}
\Gamma_{p} \\
\vdots \\
\Gamma_{p+k}
\end{pmatrix} + \begin{pmatrix}
\Phi^{H} F_{p} \\
\vdots \\
\Phi^{H} F_{p+k}
\end{pmatrix}
\]

(7.36)

Equation (7.36) again is a nonlinear equation which can be solved using the non-linear least square method as discussed in previous section.

**Numerical Example**

The bladed disk considered in chapter 2 is considered again. The disk consists of 24 blades. The blades are mistuned by changing the thickness and surface inclination of the blades as discussed in chapter 2 (Figure 2-8). As discussed earlier, the thicknesses of the blades are modified by scaling the thickness of the blades with nominal geometry by a scaling factor, i.e.

\[b_{l} = b_{0}(1 + \xi_{l})\]

(7.37)

Here $\xi_{l}$ is the actual mistuning parameter value for 1st POD feature for sector #l. The surface of the blades is inclined by rotating the surface of the blade by an angle of $\tan^{-1}\left(\frac{b_{0}\xi_{2l}}{L}\right)$ from the vertical. Here $L$ is the length of the blade and $\xi_{2l}$ is the mistuning parameter value for 2nd POD feature for sector #l. Note that positive value of the inclination angle means the leading edge of the blade moves outwards, whereas a negative inclination angle represents inward movement of the leading edge. The actual mistuned bladed disk is created by picking the values of mistuning parameters for the blade thickness and surface inclination from a normal distribution with zero mean and standard deviations of 1.7% and 1.5% respectively.
Identification using Modal Analysis

Modal analysis is performed for the first 48 modes using full rotor analysis in ANSYS. The mode shapes and natural frequencies thus generated are used as input for the identification analysis. For MMDA analysis and identification, the modes from the nominal geometry, geometry perturbed along thickness (POD1) and geometry perturbed by changing the surface inclination (POD2) are used to form the basis vector space. The perturbation in a blade’s geometry is also represented by a linear combination of thickness and surface inclination perturbations so that estimated mistuning parameters can be directly compared to the actual mistuning parameters. Since the actual mistuning parameters are known, the nominal tuned geometry is created such that the mean of the mistuning parameters is zero so as to remove any bias. This constraint is later relaxed. Modal analysis is also performed for cyclic sectors with nominal geometry and geometry perturbed along POD features to generate the basis vectors ($\Phi_0$, $\Phi_1$ and $\Phi_2$), mass and stiffness matrices ($M_t$ and $K_t$), and the gradients of mass and stiffness matrices along POD features ($\frac{\partial^2 M}{\partial \xi_s \partial \xi_t}, \frac{\partial^2 K}{\partial \xi_s \partial \xi_t}$ and $\frac{\partial^2 M}{\partial \xi_s \partial \xi_t}, s, t = 1, 2$). Mistuning identification is performed using the analysis discussed for mistuning identification using modal analysis and the results are presented in figures 7-1 (mistuning parameters for thickness) and 7-2 (mistuning parameters for blade surface inclination). As observed from the figures, the identification technique is able to estimate the values of mistuning parameters accurately, with mean and standard deviation of error ($\mu, \sigma$) for the 1st and 2nd POD feature being (-0.3689e-3, 0.0012) and (0.2140, 0.0007) respectively.
Figure 7-1: Mistuning parameter $\xi_1$ estimated using modal analysis (True Mean Geometry)

Figure 7-2: Mistuning parameter $\xi_2$ estimated using modal analysis (True Mean Geometry)
In the previous analysis, the nominal tuned geometry was generated such that the mean of the mistuning parameters was zero. In practical situations, the mistuning parameters are unknown, therefore the true mean geometry is also unknown, hence the ideal true mean geometry can only be approximated by a present tuned nominal geometry. Obviously under such approximation the mean of the mistuning parameters will not be equal to zero. To simulate this situation, the nominal tuned geometry is created with the blade thickness increased by 1% of the ideal blade thickness, i.e. $b_{\text{present}} = 1.01b_{\text{ideal}}$. Note the difference in thickness of approximate nominal geometry and ideal nominal geometry (1%) is comparable to the perturbations in thicknesses of the actual blades (standard deviation 1.7%), hence this case captures the accuracy of the estimation technique under practical conditions. Modal analysis similar to previous case is performed and the input mode shapes, natural frequencies and nominal and gradient mass and stiffness matrices are calculated. Then the mistuning identification is performed and the results are presented in figures 7-3 and 7-4.
Figure 7-3: Mistuning parameter $\xi_1$ estimated using modal analysis (Approx. Mean Geometry)

Figure 7-4: Mistuning parameter $\xi_2$ estimated using modal analysis (Approx. Mean Geometry)
As observed from figures 7-3 and 7-4, the mistuning parameters estimated with approximate mean geometry are close to their true value, with the mean and standard deviations $(\mu, \sigma)$ of errors for POD1 and POD2 being (-0.0016, 0.0033) and (0.0009, 0.0018) respectively. This is expected because the mode shapes from the nominal and POD perturbed geometries are used to form the bases of MMDA, hence any error arising from the approximation of the nominal tuned geometry is compensated by the use of mode shapes from additional geometries.

**Identification using Harmonic Analysis**

The mistuned bladed disk considered in previous section is considered again. The bladed disk is excited by a 9th Engine Order harmonic excitation within ±3% of mean excitation frequency of 12445Hz. The stiffness damping coefficient for the Rayleigh damping is taken to be 2.5577e-007. Harmonic analysis is performed for 101 excitation frequencies within the excitation range using ANSYS and the amplitude vector for each excitation frequency is obtained. Similar to previous section, since the actual mistuning parameters are known, the nominal tuned geometry is created such that the mean of the mistuning parameters is zero so as to remove any bias. Modal analysis as discussed in previous section is performed to get the mode shapes ($\Phi_0$, $\Phi_1$ and $\Phi_2$), nominal matrices ($M_j$ and $K_j$) and the gradients of matrices ($\frac{\partial \delta M}{\partial \xi_s}$, $\frac{\partial \delta K}{\partial \xi_s}$, $\frac{\partial^2 \delta M}{\partial \xi_s \partial \xi_t}$ and $\frac{\partial^2 \delta K}{\partial \xi_s \partial \xi_t}$, $s,t = 1,2$), which are required inputs for mistuning identifications based on MMDA analysis. The constraint of zero mean for the mistuning parameters is again relaxed later. The identification based on harmonic analysis is performed and the results are presented in figures 7-5 and 7-6. The mean and standard deviation $(\mu, \sigma)$ of errors for POD1 and POD2 in figures 7-5 and 7-6 are (-0.1065e-3, 0.0013) and (0.4245e-3, 0.0017) respectively. The results in figures 7-5
and 7-6 show that the identification based on harmonic analysis is able to estimate the mistuning parameters accurately.

Figure 7-5: Mistuning parameter $\xi_1$ estimated using Harmonic Analysis (True mean geometry and True Damping Coefficient)

Figure 7-6: Mistuning parameter $\xi_2$ estimated using Harmonic Analysis (True mean geometry and True Damping Coefficient)
As discussed earlier, in practical situations the true nominal geometry of the bladed disk is unknown, hence it can best be approximated by the present nominal geometry. Also since the mechanism of damping is not very clear, large errors in the damping coefficients are also possible. As in case of identification using modal analysis, to simulate this behavior, the nominal tuned geometry is created with the blade thickness increased by 1% of the ideal blade thickness, i.e. $b_{\text{present}} = 1.01 b_{\text{ideal}}$. Also the stiffness damping coefficient for Rayleigh damping ($\gamma$) is taken to be 20% of the true value of the damping coefficient. With inputs generated based on the present nominal tuned geometry and present value of damping coefficient, identification analysis based on harmonic analysis is performed and the results are presented in figures 7-7 and 7-8.

Figure 7-7: Mistuning parameter $\xi_1$ estimated using Harmonic Analysis (Approx. Mean Geometry and 80% error in damping coefficient)
Figure 7-8: Mistuning parameter $\xi_2$ estimated using Harmonic Analysis (Approx. Mean Geometry and 80% error in damping coefficient)

The mean and standard deviations ($\mu, \sigma$) of error for POD1 and POD2 in figures 7-7 and 7-8 are (-0.2974e-3, 0.0016) and (0.7902e-3, 0.0022) respectively. The results in figures 7-7 and 7-8 show that the identification technique is able to accurately estimate the mistuning parameters even in presence of shift from the true mean geometry and large errors in damping ratio, conditions both of which will be normally encountered in practice. Hence based on the results it can be said that the mistuning identification technique based on harmonic analysis and MMDA highly robust.
Summary

The forced response amplitude of a bladed rotor is very sensitive to the mistuning and the amplitude magnification can vary significantly depending upon the mistuning present in the structure. Hence it is important to accurately model and predict mistuning. Mistuning has traditionally been modeled through variations in Young’s moduli of the blades. This is called Frequency Mistuning. Frequency mistuning is not a very precise definition of realistic mistuning parameters as it ignores the effects of perturbations in geometries on the mass matrix of the system. In this chapter a more accurate definition of geometric mistuning and techniques to identify mistuning has been presented. The identification technique is based on the reduced order model, called MMDA, developed by Sinha [26]. Two types of the identification technique, one based on modal analysis and another based on forced response have been presented. The techniques have been applied to an academic rotor and it is observed that both methods are able to accurately estimate the mistuning parameters. It has been shown that the methods provide accurate results even when the present (assumed) mean geometry is different from the true mean geometry and significant errors in damping estimates are present, conditions which are normally encountered in practical identification situations. Hence the techniques are suited for practical identification strategies.
Chapter 8

Conclusion

Turbomachinery components (bladed disks) are cyclic by design, i.e. a bladed disk consists of \( n \) identical sectors. Under the assumption of cyclic symmetry, sector analysis can be performed and the results for a sector can be expanded to obtain natural frequencies and mode shapes of full 360 degree rotor. But due to manufacturing tolerances and normal wear and tear during service, cyclic symmetry is lost. This is referred to as mistuning. Mistuning can significantly modify the modal characteristics of a bladed disk and the results (natural frequencies, mode shapes and forced response) from cyclic analysis of the nominal tuned disk can differ significantly from the true values. In this case the finite element analysis of full 360 degree rotor has to be employed, which significantly increases the computational time and memory requirements for the analysis, hence constraining the design space that can be explored in a given period of time. In order to overcome this limitation, many reduced order models have been developed which employ cyclic symmetry even in presence of mistuning. In these models the mistuning has been modeled as changes in Young’s moduli of blades, which is referred to as Frequency Mistuning. Since the actual mistuning happens due to changes in the geometries of the blades, Frequency Mistuning is only an approximation of actual mistuning, and can lead to significant errors. In order to overcome the limitations of Frequency Mistuning, this research focuses on Geometric Mistuning and three areas related to modeling of single and multi-stage rotors are investigated.

The first area focused on the reduced order modeling of single stage rotors with geometric mistuning. An algorithm for modeling of bladed rotors with Geometric Mistuning, termed Modified Modal Domain Analysis (MMDA), has been presented. It has been shown that
mass and stiffness matrices, and mode shapes from sector analyses can be used to efficiently implement MMDA. It has been shown that both the exact perturbations in mass and stiffness matrices, and $2^{nd}$ order Taylor series expansion of the perturbations in mass and stiffness matrices using the semi-analytical sensitivities can be used to develop the reduced order model. It has also been shown that the mode shapes from the sector analyses of actual mistuned blades can be used to form the bases and this technique can be extended to the cases of large mistuning (rogue blades) by explicitly including the mode shapes from the rogue blades in the bases. It has been shown that that MMDA can be used to efficiently perform Monte Carlo simulation of bladed disks for both random permutations of mistuned blades, and random values of mistuning parameters. Several areas of future research are possible:

1. Development of reduced order model for bladed disk with multiple rogue blades.
2. Comparison of the distribution of mistuned response from MMDA based on 1$^{st}$ order Taylor series expansion with the actual distribution to determine if MMDA based on 1$^{st}$ order Taylor series expansion is sufficient for Monte Carlo simulations.
3. Development of reduced order model coupling geometric mistuning and aerodynamic forces.

Second area of research focused on reduced order modeling of multistage rotors, with and without geometric mistuning. A reduced order model using the tuned modes from finite element sector analyses of individual rotor stages has been developed for application to a multistage rotor. The reduced order model has been extended using POD analysis to include the cases of geometric mistuning in multistage rotors. It has been shown that the results provided by the reduced order model are accurate enough to capture effects arising out of a small geometric mistuning which is
inherent to any manufacturing process. It has also been shown that the Monte Carlo simulation of natural frequencies and forced responses of a multi-stage rotor can be easily performed for various permutations of a mistuning pattern in a computationally efficient manner. Several areas of future research are possible:

1. Further development of the presented algorithm to use mass and stiffness matrices from sector analysis for computation of reduced order model.
2. Development of reduced order model based on sensitivities of mass and stiffness matrices.

The third area of research focused on mistuning identification of geometrically mistuned bladed disks. A more accurate definition of mistuning and techniques to identify mistuning parameters for bladed disk with geometric mistuning has been developed. The identification technique is based on MMDA. Two types of the identification techniques, one based on modal analysis and another based on forced response have been presented. The techniques have been applied to an academic rotor and it has been shown that both methods are able to accurately estimate the mistuning parameters, even when the present (assumed) mean geometry is different from the true mean geometry and significant errors in damping estimates are present, conditions which are normally encountered in practical identification situations. Hence the techniques are suited for practical identification strategies. Several areas of future research are possible:

1. Development of mistuning identification techniques for multistage rotors.
2. Development of identification techniques for damping in bladed disks.
References


Appendix A

Mistuning Parameters for perturbation in blade geometry

Table A-1: Factors $\xi$ for Original Pattern of Blade Thicknesses (Geometric Mistuning)

<table>
<thead>
<tr>
<th>Blade 1</th>
<th>Blade 2</th>
<th>Blade 3</th>
<th>Blade 4</th>
<th>Blade 5</th>
<th>Blade 6</th>
<th>Blade 7</th>
<th>Blade 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0051</td>
<td>0.0263</td>
<td>-0.0467</td>
<td>-0.0184</td>
<td>-0.0105</td>
<td>-0.0485</td>
<td>-0.0528</td>
<td>0.0372</td>
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<td>0.0322</td>
<td>0.0093</td>
<td>-0.0396</td>
<td>0.05</td>
<td>0.0116</td>
<td>0.0492</td>
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<td>-0.0517</td>
<td>-0.0029</td>
<td>-0.0132</td>
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<td>0.0199</td>
<td>0.0395</td>
</tr>
</tbody>
</table>

Table A-2: Factors $\xi$ for a Random Permutation of Original Blade Thicknesses

<table>
<thead>
<tr>
<th>Blade 1</th>
<th>Blade 2</th>
<th>Blade 3</th>
<th>Blade 4</th>
<th>Blade 5</th>
<th>Blade 6</th>
<th>Blade 7</th>
<th>Blade 8</th>
</tr>
</thead>
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<tr>
<td>0.0492</td>
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<td>-0.0517</td>
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</table>
Table A-4: Mistuning parameter values for minor POD feature (POD #2)

<table>
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<th>Blade 1</th>
<th>Blade 2</th>
<th>Blade 3</th>
<th>Blade 4</th>
<th>Blade 5</th>
<th>Blade 6</th>
<th>Blade 7</th>
<th>Blade 8</th>
</tr>
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<tbody>
<tr>
<td>0.0013</td>
<td>-0.0002</td>
<td>-0.0021</td>
<td>-0.001</td>
<td>0.0006</td>
<td>-0.0018</td>
<td>0.0014</td>
<td>-0.0015</td>
</tr>
<tr>
<td>Blade 9</td>
<td>Blade 10</td>
<td>Blade 11</td>
<td>Blade 12</td>
<td>Blade 13</td>
<td>Blade 14</td>
<td>Blade 15</td>
<td>Blade 16</td>
</tr>
<tr>
<td>0.0025</td>
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<td>0.0016</td>
<td>-0.0019</td>
<td>-0.0013</td>
<td>-0.0025</td>
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<td>0.0011</td>
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<td>0.0009</td>
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<td>0.0008</td>
<td>0.0027</td>
<td>-0.0017</td>
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Table A-5: Mistuning parameter values for minor POD feature (POD #3)

<table>
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<th>Blade 3</th>
<th>Blade 4</th>
<th>Blade 5</th>
<th>Blade 6</th>
<th>Blade 7</th>
<th>Blade 8</th>
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<td>0.0006</td>
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<td>Blade 11</td>
<td>Blade 12</td>
<td>Blade 13</td>
<td>Blade 14</td>
<td>Blade 15</td>
<td>Blade 16</td>
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<td>-0.0056</td>
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<td>0.0052</td>
</tr>
<tr>
<td>Blade 1</td>
<td>Blade 2</td>
<td>Blade 3</td>
<td>Blade 4</td>
<td>Blade 5</td>
<td>Blade 6</td>
<td>Blade 7</td>
<td>Blade 8</td>
</tr>
<tr>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
<td>---------</td>
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<tr>
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<td>0.0055</td>
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<td>0.0031</td>
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</tr>
<tr>
<td>Blade 9</td>
<td>Blade 10</td>
<td>Blade 11</td>
<td>Blade 12</td>
<td>Blade 13</td>
<td>Blade 14</td>
<td>Blade 15</td>
<td>Blade 16</td>
</tr>
<tr>
<td>0.0045</td>
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<td>0.0034</td>
<td>0.0015</td>
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</table>

Table A-7: Component of perturbation orthogonal to blade#4 for mistuned bladed disk

<table>
<thead>
<tr>
<th>Blade 1</th>
<th>Blade 2</th>
<th>Blade 3</th>
<th>Blade 4</th>
<th>Blade 5</th>
<th>Blade 6</th>
<th>Blade 7</th>
<th>Blade 8</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0015</td>
<td>0.0017</td>
<td>0.0023</td>
<td>0.0000</td>
<td>0.0020</td>
<td>0.0029</td>
<td>0.0007</td>
<td>0.0012</td>
</tr>
<tr>
<td>Blade 9</td>
<td>Blade 10</td>
<td>Blade 11</td>
<td>Blade 12</td>
<td>Blade 13</td>
<td>Blade 14</td>
<td>Blade 15</td>
<td>Blade 16</td>
</tr>
<tr>
<td>0.0026</td>
<td>0.0027</td>
<td>0.0044</td>
<td>0.0028</td>
<td>0.0010</td>
<td>0.0047</td>
<td>0.0029</td>
<td>0.0016</td>
</tr>
<tr>
<td>0.0007</td>
<td>0.0007</td>
<td>0.0015</td>
<td>0.0019</td>
<td>0.0013</td>
<td>0.0014</td>
<td>0.0051</td>
<td>0.0006</td>
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</table>
Table A-8: A Mistuning pattern for Disk1 and Disk2

<table>
<thead>
<tr>
<th>Blade No. ( i )</th>
<th>( \xi_i ) for Disk1</th>
<th>( \xi_i ) for Disk2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.010308</td>
<td>0.010308</td>
</tr>
<tr>
<td>2</td>
<td>0.011708</td>
<td>0.011708</td>
</tr>
<tr>
<td>3</td>
<td>-0.025920</td>
<td>-0.025920</td>
</tr>
<tr>
<td>4</td>
<td>-0.014713</td>
<td>-0.014713</td>
</tr>
<tr>
<td>5</td>
<td>-0.016558</td>
<td>-0.016558</td>
</tr>
<tr>
<td>6</td>
<td>0.010070</td>
<td>0.010070</td>
</tr>
<tr>
<td>7</td>
<td>0.020664</td>
<td>0.020664</td>
</tr>
<tr>
<td>8</td>
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<td>-0.009332</td>
</tr>
<tr>
<td>9</td>
<td>0.016831</td>
<td>0.016831</td>
</tr>
<tr>
<td>10</td>
<td>0.010520</td>
<td>0.010520</td>
</tr>
<tr>
<td>11</td>
<td>-0.029597</td>
<td>-0.029597</td>
</tr>
<tr>
<td>12</td>
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</tr>
<tr>
<td>13</td>
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<td>-0.006794</td>
</tr>
<tr>
<td>14</td>
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<td>0.024959</td>
</tr>
<tr>
<td>15</td>
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<td>-0.029931</td>
</tr>
<tr>
<td>16</td>
<td>-0.002253</td>
<td>-0.002253</td>
</tr>
<tr>
<td>17</td>
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<td>-0.004539</td>
</tr>
<tr>
<td>18</td>
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<tr>
<td>19</td>
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<tr>
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</tr>
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</table>
Table A-9: Mistuning pattern of a random permutation of blades

<table>
<thead>
<tr>
<th>Blade No. $i$</th>
<th>$\xi_i$ for Disk1</th>
<th>$\xi_i$ for Disk2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-0.004539</td>
<td>0.020664</td>
</tr>
<tr>
<td>2</td>
<td>0.010308</td>
<td>-0.025920</td>
</tr>
<tr>
<td>3</td>
<td>-0.029597</td>
<td>-0.029597</td>
</tr>
<tr>
<td>4</td>
<td>0.010070</td>
<td>-0.016558</td>
</tr>
<tr>
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<td>0.010070</td>
</tr>
<tr>
<td>6</td>
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<td>0.010308</td>
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<tr>
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<tr>
<td>8</td>
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<td>-0.014713</td>
</tr>
<tr>
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<td>0.016831</td>
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<tr>
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<td>-0.029931</td>
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<tr>
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<tr>
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<td>0.020664</td>
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</table>
Appendix B

Vibratory Parameters of Blades from Coordinate Measurement Machine Data [27]

Modified Modal Domain Approach (MMDA) is based on the decomposition of the blade geometry along a set of principal component modes that define the geometric deviations with a reduced basis using spatial statistical analysis (Proper Orthogonal Decomposition). The perturbed geometries consist of the nominal geometry with the addition of each retained POD mode. The vibratory parameters of the blades (mistuning parameters) are the coordinates in the spatial vector space that quantify the perturbation in geometry along a principal direction. A subset of nominal mode shapes and mode shapes of tuned system with the blade geometry perturbed in a principal direction forms the basis of reduced space. The procedure of identifying the spatial mode shapes and vibratory parameters of a blade is discussed in the section. The detailed procedure and numerical examples have been discussed by Sinha et al. [27].

Proper Orthogonal Decomposition Expansion of Coordinate Measurement Machine Data on Integrally Bladed Rotors

Coordinate Measurement Machine is used measure the geometry of the blade. At \(nm\) distinct points \(p_i, 1 \leq i \leq nm\) on the surface of the blade, the blade surface coordinates \(v_j(p_i)\) are measured. Each blade surface coordinate is characterized by three components:

\[
v_j(p_i) = \begin{bmatrix} v^x_{ij} \\ v^y_{ij} \\ v^z_{ij} \end{bmatrix}
\]

(B.1)

The CMM data for blade \#j is then defined as:

\[
w_j = [v^x_j(p_1) \cdots v^z_j(p_{nm})] \\
   j = 1, 2, \ldots, nb
\]

(B.2)
where $nb$ is the number of blades. The dimension of the vector is $nd = 3 \times nm$. The mean of $w_j$ is computed as

$$E(w) = \frac{1}{nb} \sum_{j=1}^{nb} w_j$$

(B.3)

and the covariance matrix $C$ is computed as follows:

$$C = \frac{1}{nb} \sum_{j=1}^{nb} \delta w_j \delta w_j^T$$

(B.4)

where

$$\delta w_j = w_j - E(w)$$

(B.5)

Equation (B.4) shows that dimension of the covariance matrix $C$ is $nd \times nd$.

Then, the KL expansion (Appendix C) of the CMM data suggests that the perturbation in the blade geometry can be written as linear combination of spatial modes identifying the principal directions of perturbation, i.e,

$$\delta w_j = \sum_{l=0}^{nd} \xi_l \sqrt{\lambda_l} u_l$$

(B.6)

where $\lambda_l$ and $u_l$ are the $l^{th}$ eigenvalue and normalized eigenvector of the covariance matrix. $\xi_l$ is the coordinate in the spatial vector space quantifying the perturbation in geometry along $l^{th}$ principal direction. $\xi_l$ are uncorrelated random variables with zero mean and unity standard deviation., i.e.,

$$E(\xi_l) = 0$$

(B.7a)

$$E(\xi_l^2) = 1 \quad \text{and} \quad E(\xi_l^2) = 1$$

(B.7b)

$$E(\xi_l \xi_p) = 0 \quad \text{if} \ l \neq p$$

(B.7c)

$nd$ in general would be a large quantity hence the order the matrix $C$ is large and the MATLAB routine `eig` requires a large amount of computation time to determine eigenvalues and
eigenvectors. For an efficient computation of these eigenvalues and eigenvectors, equation (B.4) is rewritten as

\[ C = \frac{1}{nb} \partial W \partial W^T \]  

(B.8)

where

\[ \partial W = [\delta v_1 \cdots \delta v_{nb}] \]  

(B.9)

Since the dimension of \( \partial W \) is \( nd \times nb \), and \( nb < nd \), the rank of the matrix \( C \) cannot exceed \( nb \). An efficient computational algorithm is developed by realizing the connection between eigenvalues of \( C \) and singular values of \( \partial W \). The well-known singular value decomposition is

\[ \partial W = UV^T \]  

(B.10)

where

\[ U = [u_1 \ u_2 \cdots u_{nb}] \]  

(B.11)

\[ V = [v_1 \ v_2 \cdots v_{nb}] \]  

(B.12)

\[ \Sigma = \text{diag}[\sigma_1 \ \sigma_2 \cdots \sigma_{nb}] \]  

(B.13)

\( \sigma_i \) is the \( i \text{th} \) singular value of \( \partial W \). Singular values, vectors \( u_i \) and \( v_i \), are related to eigenvalues and eigenvectors of the covariance matrix as follows:

\[ \partial W^T \partial W \ v_i = \sigma^2 v_i \quad \text{and} \]

\[ \partial W \ \partial W^T u_i = \sigma^2 u_i \]  

(B.14)

(B.15)

Therefore, from equations (B.4), (B.6), (B.9) and (B.15)

\[ \lambda_i = \frac{\sigma_i^2}{nb} \]  

(B.16)

Premultiplying equation (B.6) by \( u_p^T \),

\[ u_p^T \delta v_j = \xi_p \sqrt{\lambda_{jp}} \]
or

\[ \xi_p = \frac{u_p \delta w_j}{\sqrt{\lambda_p}} \]  \hspace{1cm} (B.17)

Equation (B.17) can be used to compute the values of mistuning parameters for actual blades.
Appendix C

Karhunen-Loeve Expansion [43]

Consider a random process \( w(x) \), which is a function of the position vector \( x \) defined over the domain \( D \), with \( \theta \) belonging to the space of random events \( \Omega \). Let \( \bar{w}(x) \) and \( C(x_1, x_2) \) be the expected value and the covariance function of \( w(x) \), respectively. Then,

\[
w(x) = \bar{w}(x) + \alpha(x, \theta)
\]  

(C.1)

where

\[
\alpha(x, \theta) = \sum_{n=0}^{\infty} \xi_n(\theta) \sqrt{\lambda_n} f_n(x)
\]  

(C.2)

Here, \( \xi_n(\theta) \) are random variables satisfying \( E[\xi_n(\theta)] = 0 \) and \( E[\xi_n(\theta) \xi_m(\theta)] = \delta_{nm} \) where \( \delta_{nm} \) are the dirac-delta function. Furthermore \( \lambda_n \) and \( f_n(x) \) are eigenvalues and eigenvectors of the covariance function \( C(x_1, x_2) \) satisfying the following properties:

\[
\int_D C(x_1, x_2) f_n(x_1) dx_1 = \lambda_n f_n(x_2) \quad \text{and}
\]  

(C.3)

\[
\int_D f_n(x) f_m(x) dx = \delta_{nm}
\]  

(C.4)
Appendix D

Mistuning Modeling and SNM [20]

For frequency mistuning the mistuning is simulated by the change in the Young’s moduli of the blades. The representation of actual mistuning in terms of frequency mistuning (equivalent changes in Young’s moduli of blades) involves the following steps:

1. Determination of natural frequency ($f_t$) of the blades with average geometry and Young’s modulus $E_0$ clamped at base (Figure D.1a).

2. Determination of natural frequency ($f_m$) of mistuned blades with Young’s modulus $E_0$ clamped at base (Figure D.1b).

3. Calculation of equivalent Young’s modulus for a blade with average geometry such that the natural frequency ($f_{teq}$) of the blade is same as the natural frequency of the mistuned blade ($f_m$), i.e. $f_{teq} = f_m$ (Figure D.1c). The equivalent Young’s modulus can be calculated as:

$$E_m = E_0 \left( \frac{f_m}{f_t} \right)^2$$  \hfill (D.1)

The finite element model of the bladed disk with the blades modeled as represented in Figure D.1c is used to generate the mass and stiffness matrices of the mistuned bladed disk assembly.

![Blade](image)

Thickness = $b_0$

Natural Frequency = $f_t$

Young’s Modulus = $E_0$

Figure D.1a: Tuned blade
Thickness = $b_0(1 + \xi_1)$

Natural Frequency = $f_m$

Young’s Modulus = $E_0$

Figure D.2b: Mistuned blade

Thickness = $b_0$

Natural Frequency = $f_m$

Young’s Modulus = $E_m = E_0\left(\frac{f_m}{f_1}\right)^2$

Figure D.3c: Blade for Frequency Mistuning

Let $M^{freq} = M_t$ and $K^{freq}$ be the mass and stiffness matrices of the equivalent frequency mistuned bladed disk assembly (Note that the mass matrix of the bladed disk assembly is same as the mass matrix of the assembly with mean geometry and Young’s modulus $E_0$ because the geometry and density of the blades represented in figures D.1a and D.1c are same).

Then the equations of motion for the system can be written as:

$$M_t\ddot{x} + K^{freq}x = 0$$  \hspace{1cm} (D.2)

The idea behind the SNM technique is that the solution $x$ can be represented as a weighted sum of the modes of the nominal assembly, i.e.
\( x = \Phi_0 y \) \hspace{1cm} (D.3)

where \( \Phi_0 \) is the set of modes for the nominal tuned assembly.

Substituting equation (D.3) in equation (D.2) and pre-multiplying with \( \Phi_0^T \) (transpose of \( \Phi_0 \)), the equation of motion can be written as:

\[
M_{r}^{SNM} \ddot{y} + K_{r}^{SNM} y = 0
\]

where

\[
M_{r}^{SNM} = \Phi_0^T M_f \Phi_0 \hspace{1cm} (D.5)
\]

and

\[
K_{r}^{SNM} = \Phi_0^T K^{freq} \Phi_0 \hspace{1cm} (D.6)
\]

The eigenvalue problem associated with equation (D.4) can be solved to get the mode shapes and natural frequencies of the mistuned bladed disk assembly.
Vita

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Yasharth Bhartiya was born October 14, 1984 in Unnao, India, to Kiran and Gulshan Malhotra. After graduating from Vivekanand Inter College in Unnao, India, in 2001, he obtained a Bachelor of Technology in mechanical engineering from the Indian Institute of Technology, Kharagpur, India, in 2006. He then received his M.S. from The Pennsylvania State University in 2010. This dissertation fulfills the requirements of Ph.D. in mechanical engineering from The Pennsylvania State University, which he obtained in 2011 under the advisement of Dr. Alok Sinha. His research interests include Reduced Order Modeling of Single and Multi-stage rotors, Cyclic Symmetry, Mistuning, and Forced Response.