LOSS PHENOMENOLOGY AND THE METHODOLOGY TO DERIVE
LOSS FACTORS IN PIEZOELECTRIC CERAMICS

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by
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ABSTRACT

The key factor for the miniaturization of piezoelectric devices is the power density, which is limited by the heat generation or internal losses. With accelerating the commercialization of piezoelectric actuators and transducers, it becomes more and more significant to clarify the loss phenomenology, reduce the hysteresis losses, and increase the mechanical quality factor to enhance the efficiency and performance.

There are three loss components for piezoelectric vibrators, i.e., dielectric, elastic and piezoelectric losses. These loss factors are related with mechanical quality factors, which are the figure of merit of the material or device in the sense of efficiency. In order to obtain the mechanical quality factor $Q_m$, IEEE standard provided the measurement method using the resonance and antiresonance frequencies. However, this characterization assumes that the $Q$ value at resonance is equal to the one at antiresonance all the time, which is not consistent with experiment results. For practical materials there is sometimes a big difference between the mechanical quality factors of the resonance ($Q_A$) and the antiresonance ($Q_B$), and in most cases higher $Q_B$ can be observed. Therefore, in recent several years we are focusing on a new resonance AC drive methodology to measure the admittance curve accurately around both the resonance and antiresonance peaks for piezoelectric materials, from which $Q_A$ and $Q_B$ can be derived by the 3dB method. Various piezoelectric materials were characterized with this technique.

Further, the equations were derived showing the relations between quality factors and loss factors by the complex analysis of the admittance/impedance expressions for specific piezoelectric vibrators. Using $Q_A$, $Q_B$, and the electromechanical coupling factor $k$, we can obtain three types of loss factors precisely. Among various vibration modes of piezoelectric vibrators, we focus on $k_{31}$, $k_{15}$, $k_{33}$, $k_p$, and $k_{15}$ modes, which cover all the 20 parameters of the ferroelectric material with $\infty mm/6mm$ crystal symmetry, i.e. piezoelectric ceramic. Plus some other
derivations using the fundamental correlations, 20 loss factors can be obtained for all parameters. Using this technique the piezoelectric loss factor is confirmed to be comparable to dielectric and elastic losses and it is the factor that determines whether $Q_B$ is larger than $Q_A$, though it was previously neglected by most researchers. After getting the full loss matrices, the loss anisotropy was accordingly discussed, and the extensive loss factor was verified to be smaller than the intensive one.

This methodology is an essential supplement to the current IEEE standard on piezoelectric characterization. The simplicity and accuracy of this technique are very attractive, and hopefully this proposal will be widely accepted as a standard in the piezoelectric community in the future. The inclusion of three loss factors is important for the admittance analysis and thermal simulation of piezoelectric devices in the “finite element method” software. Furthermore, taking into account the piezoelectric loss in addition to the dielectric and elastic ones, a new domain dynamic model will be established. Our phenomenological solution can be directly applied for the high power characterization methodology to clarify various materials’ loss performances, and a principle for preparing high power density piezoelectric materials will be developed in the future.
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Chapter 1

Introduction

1.1 Background

Piezoelectric devices gain increasing interest for consumer applications where volumetric space is a scarce commodity. Low, cramped space and the desire to pack more functions into smaller/lighter, portable devices forced electromagnetic (EM) technologies out of scope since small scaled EM devices are bound to suffer from increased ohmic losses (the magnetic flux is generated via finite thickness coils). [1] Since piezoelectric devices, such as ultrasonic motors and transformers, demonstrated 1/10th size for equivalent-power-level electromagnetic devices, [2, 3, 4, 5, 6, 7, 8] piezoelectric components have been replacing electromagnetic conjugates. The interest of the industry in ever higher efficiency to achieve a sustainable society while scaling sizes down, forced research towards more power dense piezoelectric technologies. [9, 10, 11] However, the power density in piezoelectric devices is still limited by material’s inherent losses which stem from the microscopic domain dynamics, resulting in the heat generation. [12, 13, 14] Therefore, to advance the device miniaturization it is necessary to clarify the loss phenomenology and mechanism.

Losses in piezoelectrics are considered to have three components: dielectric, elastic, and piezoelectric losses. [15, 16, 17, 18] The dielectric and elastic loss factors are commonly reported by researchers and companies, while so far little attention has been paid to the piezoelectric loss factor. [19] However, relatively large piezoelectric loss factors were reported in our previous study, which also explained the discrepancy between the theoretical expectation and experimental
results of the mechanical quality factors. [20, 21, 22] We further classified each loss into intensive and extensive components, which are briefly summarized in the next section. [20, 23]

Mechanical quality factors play a significant role in the loss study of piezoelectrics. A higher mechanical quality factor leads to the increase of the efficiency by reducing the heat generation, which usually limits the power density of the device. [24, 25, 26] The quality factor is basically related to dielectric, elastic and piezoelectric loss factors. [27, 28] Besides, a higher quality factor at the antiresonance is usually observed in experiments, in comparison with that at the resonance. [29, 30] We provide comprehensive derivations of piezoelectric quality factors for both resonance and antiresonance in various vibration modes.

The orientation-dependence of the loss performance of the piezoelectrics is discussed, especially for materials with 6mm crystal symmetry whose piezoelectric constant matrix is equivalent to the polycrystalline piezo-ceramics ($\infty m$ symmetry), such as lead zirconate titanate (PZT). [31, 32] A methodology to determine the losses in different directions is introduced, employing the impedance measurements.

1.2 Objective

First of all, the discrepancy of resonance quality factor $Q_A$ and antiresonance quality factor $Q_B$ needs to be clarified. As reported before, in experiment $Q_B$ is usually higher than $Q_A$. Accordingly, antiresonance drive possesses some potential advantages over the conventional resonance drive for piezoelectric devices, including high efficiency, high voltage and low current drive, etc. Besides, the high output impedance provides a good match with piezoelectric transformer, which may result in a novel drive technique for piezoelectric devices. However, the previous theory could not explain the deviation of $Q_A$ and $Q_B$ explicitly. IEEE Std. provided the method to derive the mechanical quality factor, based on the equivalent circuit as shown in Figure
1-1. [33] Though the value of quality factor was also given by eq. (1.1), the assumption was made that the resonance quality factor is equal to the antiresonance quality factor. This is a seriously misleading point, which is inconsistent with the experimental results. Therefore, we are working on the derivations of $Q_A$ and $Q_B$ from the impedance expressions for a variety of vibration modes, with regard to loss factors and electromechanical coupling factors. From the analytical results, the difference between $Q_A$ and $Q_B$ can be explained.

Figure 1-1. Equivalent electrical circuit of a piezoelectric vibrator.

$$Q = \sqrt{\frac{L_1}{C_1}} / R_1.$$  \hspace{2cm} (1.1)

Secondly, we would like to propose a methodology to determine the loss factors under different vibration modes via the impedance/admittance measurement, based on the derivations of quality factors. In the previous research, we obtained the loss factors by measuring the hysteresis loops, e.g. strain vs. electric field (stress free). [34] However, this method requires rather sophisticated and bulky instruments, such as tensile and compression testing machine (Instron).
Admittance measurement, on the other hand, is easy and friendly with high accuracy. Therefore, it can replace the previous technique to become a standard of piezoelectric loss measurement. Among various vibration modes of piezoelectric vibrators, we focus on $k_{31}$, $k_p$, $k_i$, $k_{33}$, and $k_{15}$ modes, which will be discussed in detail in the next chapter. By these analyses, all the loss factors can be derived for the material with $\approx$mm/6mm crystal symmetry, e.g. piezoelectric ceramics. Consequently, the loss matrices will be established and the orientation dependence can be studied. Besides, the inclusion of three loss factors is essential for the admittance analysis and thermal simulation of piezoelectric devices in the “finite element method” software. [35, 36] The simulation quality will be enhanced a lot by integrating all the loss factors. Furthermore, taking into account the piezoelectric loss in addition to the dielectric and elastic ones, a new domain dynamic model will be established. Our phenomenological solution can be directly applied for the high power characterization methodology to clarify various materials’ loss performances, and a principle for preparing high power density piezoelectrics will be developed in the future.

1.3 Outline

This thesis is organized as follows.

In Chapter 2, the theoretical derivations of quality factors and loss factors are explained. First, the fundamentals on piezoelectric loss phenomenology will be introduced, including hysteresis losses in piezoelectrics, the definitions of intensive and extensive loss factors, the resonance and antiresonance quality factors, material properties and crystal symmetry, and vibration modes for piezoelectric vibrators. Next, the derivations of quality factors will be given for $k_{31}$, $k_p$, $k_i$, $k_{33}$, and $k_{15}$ modes, using the analytical impedance analysis with first order approximation.
In Chapter 3, the methodology to determine loss factors is illustrated. Initially, some fundamental relations of material properties will be introduced, which are the basis of the methodology. Then the procedures to measure the real parts of material properties will be discussed according to the IEEE Std. After that our technique to measure and derive the imaginary parts (loss factors) will be presented step by step.

In Chapter 4, the results of finite element method (FEM) simulations are introduced. First, a brief introduction is given about the principles of FEM analysis. Then the simulations with ATILA FEM software (ISEN) are made to verify the analytical conclusions. In addition, the significant effect of piezoelectric loss factor is demonstrated.

In Chapter 5, the experiment setup and results will be explained. First, the instruction of accurate impedance measurement is given, including the history of the development of impedance characterization systems. Then the equipment for the characterization, impedance analyzer, will be introduced with its specifications and use conditions. Next, the loss characterization methodology was utilized for lead zirconate titanate (PZT) ceramic APC 850. All the real and imaginary material properties were obtained, and the orientation dependence of the loss factors was accordingly analyzed. The piezoelectric loss factors were again confirmed to be essential, and the intensive loss factors were verified to be larger than the extensive ones.

In Chapter 6, a summary is provided with the conclusions and the proposed future work.
Chapter 2

Derivations of Quality Factors in Different Vibration Modes

There exist three loss components in piezoelectric materials, i.e. dielectric, elastic, and piezoelectric losses. The heat generation of piezoelectric devices is controlled by loss factors, and the mechanical quality factor is the general figure of merit in the sense of efficiency.

In this chapter, the theoretical derivations of quality factors will be explained, which illustrate the relations of quality factors and loss factors. First, the fundamentals on piezoelectric loss phenomenology will be introduced, including hysteresis losses in piezoelectrics, the definitions of intensive and extensive losses, the quality factor definitions for resonance and antiresonance, material properties and crystal symmetry, and the various vibration modes of piezoelectric vibrators. Next, the impedance analysis will be made to derive the quality factors of $k_{31}$, $k_t$, $k_{33}$, $k_p$, and $k_{15}$ vibration modes, using the analytical impedance analysis with first order approximation. Both the resonance quality factor $Q_A$ and antiresonance quality factor $Q_B$ are discussed.

2.1 Fundamentals of Piezoelectric Loss Phenomenology

2.1.1 Hysteresis loss

The loss or heat generation in piezoelectrics originates from the inherent hysteresis of electric and mechanical parameters, which is schematically depicted in Figure 2-1. [23] Here, $D_0$ is the maximum electric displacement, $E_0$ the maximum electric field, $S_0$ the maximum strain, and $T_0$ the maximum stress. The area of the loops indicate the hysteresis, and $w_e$, $w_m$, and $w_{em}$ (or $w_{me}$)
were used to denote dielectric (electrical) loss, elastic (mechanical) loss, and piezoelectric (electromechanical) loss, respectively. Notice that the area of the triangle in each curve stands for the energy of the sample. $U_m$ is the mechanical energy, $U_e$ is the electrical energy, and $U_{em}$ and $U_{me}$ represent the electromechanical energy. Here Figure 2-1(a) shows the hysteresis loops for intensive losses while (b) indicates the extensive losses with different boundary conditions. The definitions of ‘intensive’ and ‘extensive’ losses will be explained in the next part.

The loss for each process can be derived by the hysteresis loop. It should be noted that for the S-E loop the product $S_0E_0$ must be multiplied by the scale factor $(d/s)$ to convert to energy density units $(N/m^2)$, where $d$ is the piezoelectric coefficient $(m/V)$ and $s$ is the elastic compliance $(m^2/N)$, and for the T-D loop the product $D_0T_0$ must be multiplied by $(d/\varepsilon\varepsilon_0)$ with $\varepsilon$ being the dielectric constant. This is the principle for our previous measurements to determine the loss factors.

![Diagram of hysteresis loops](image)
Figure 2-1. Schematic representations of hysteresis curves: (a) intensive losses: S–T (short-circuit condition), D–E (stress-free condition), S–E (stress-free condition), and D–T (short circuit condition); (b) extensive losses: T–S (open-circuit condition), E–D (clamping condition), E–S (clamping condition), and T–D (open circuit condition)

2.1.2 Loss factor

Complex parameters are integrated to express the hysteresis losses in piezoelectrics. We use $\tan \delta'$, $\tan \phi'$ and $\tan \theta'$ to represent “intensive” dielectric, elastic and piezoelectric loss factors, respectively. The “extensive” loss factors are given by corresponding notations without prime. The definitions are given by

$$\varepsilon^{T*} = \varepsilon^T (1 - j \tan \delta') ,$$  \hspace{1cm} (2.1)

$$s^{E*} = s^E (1 - j \tan \phi') ,$$  \hspace{1cm} (2.2)
Here \( j \) is the imaginary notation, \( \varepsilon^T \) the dielectric constant under constant stress, \( \beta^S \) the inverse dielectric constant under constant strain, \( s^E \) the elastic compliance under constant electric field, \( c^D \) the elastic stiffness under constant electric displacement, \( d \) the piezoelectric constant, and \( h \) the inverse piezoelectric charge constant.

Note that the phenomenological equations only hold when the piezoelectric sample works in the linear region and the loss is treated as a perturbation. In practice, the theoretical equations derived in this way are accurate for the cases where the loss factors are less than 0.1.

From the view point of the physical sciences, an intensive property is a physical property of a system that does not depend on the system size or the amount of material in the system. In other words, it is scale invariant. By contrast, an extensive property a system is directly proportional to the system size or the amount of material in the system. Note that state variables are extensive, while field variables and point are intensive. [38, 39]

As for the loss factors, intensive loss corresponds to the boundary conditions of constant stress \( T \) or electric field \( E \) as shown in piezoelectric eqs. (2.7) and (2.8), while extensive loss is attributed to the boundary conditions of constant strain \( S \) or electric displacement \( D \) as illustrated in eqs. (2.9) and (2.10). Here \( T \) and \( E \) are intensive parameters, and \( S \) and \( D \) are extensive properties.

\[
d^* = d(1 - j \tan \theta'), 
\]

\[
\beta^S = \beta^S (1 + j \tan \delta'), 
\]

\[
c^D = c^D (1 + j \tan \phi'), 
\]

\[
h^* = h(1 + j \tan \theta). 
\]
\[ T = c^D S - hD, \]  
(2.9)

\[ E = -hS + \beta^S D. \]  
(2.10)

In 1-dimensional consideration, eqs. (2.4) and (2.5) are equivalent to

\[ \varepsilon^S = \varepsilon^S (1 - j \tan \delta), \]  
(2.11)

\[ s^D = s^D (1 - j \tan \phi). \]  
(2.12)

The electromechanical coupling coefficient is given by

\[ k^2 = \frac{d^2}{s^E (\varepsilon^T / \varepsilon_0)} = \frac{h^2}{c^D (\beta^S / \varepsilon_0)}. \]  
(2.13)

Figure 2-2. Conceptual figures for explaining the relation between \( \varepsilon^T \) and \( \varepsilon^S, s^E \) and \( s^D \).
The relations of $\varepsilon^T$ and $\varepsilon^S$, $s^E$ and $s^D$ are shown conceptually in Figure 2-2 in the sense of energy conversion. [23] When an electric field is applied on a piezoelectric sample, this state will be equivalent to the superposition of the following two steps: first, the sample is completely clamped and the field $E_0$ is applied (pure electrical energy $\varepsilon_0 \varepsilon^S E_0^2 / 2$ is input); second, keeping the field at $E_0$, the mechanical constraint is released (additional mechanical energy ($d^2/s^E)E_0^2 / 2$ is necessary). The total energy corresponds to the total input electrical energy $\varepsilon_0 \varepsilon^T E_0^2 / 2$. Similar energy calculation can be obtained for the case of mechanical input. Therefore, the following two relations are derived as

\[
\varepsilon^S = \varepsilon^T (1 - k^2), \tag{2.14}
\]

\[
s^D = s^E (1 - k^2). \tag{2.15}
\]

Furthermore, extensive and intensive loss factors have the following relationship:

\[
\begin{bmatrix}
\tan \delta' \\
\tan \phi' \\
\tan \theta'
\end{bmatrix}
= K
\begin{bmatrix}
\tan \delta \\
\tan \phi \\
\tan \theta
\end{bmatrix}, \tag{2.16}
\]

\[
K = \frac{1}{1 - k^2}
\begin{bmatrix}
1 & k^2 & -2k^2 \\
k^2 & 1 & -2k^2 \\
1 & 1 & -1 - k^2
\end{bmatrix}. \tag{2.17}
\]

The matrix $K$ is proven to be involutory, i.e. $K^2 = I$, or $K = K^{-1}$, where $I$ is the identity matrix. Hence the conversion relationship between the intensive (prime) and extensive (non-prime) exhibits full symmetry.

Upon the conventional method, intensive loss factors are measured through the analysis of hysteresis loss as illustrated in Figure 2-1, and extensive losses are derived then using eq. (2.16).

It should be noted that the above analysis is based on the 1-dimensional assumption. We should apply matrix operations rather than single values in the 3-dimensional case.
Previously we proposed a model to explain the physical meaning of hysteresis losses. To make the situation simplest, we consider here only the domain wall motion-related losses. Taking into account the fact that the polarization change is primarily attributed to 180° domain wall motion, while the strain is attributed to 90° (or non-180°) domain wall motion, we suppose that the dielectric and mechanical losses are originated from 180° and 90° domain wall motions, respectively, as illustrated in Figure 2-3. [20] The dielectric loss comes from the hysteresis during the 180° polarization reversal under E, while the elastic loss comes from the hysteresis during the 90° polarization reorientation under T. In this model, the piezoelectric loss is explained by the 90° polarization reorientation under E, which can be realized by superimposing the 90° polarization reorientation under E and the 180° polarization reversal under E. [20]

![Figure 2-3. Polarization reversal/reorientation model for dielectric, elastic and piezoelectric losses.](image-url)
2.1.3 Quality factor

Quality factor is the figure of merit for the material in terms of efficiency or heat generation, and it is closely related to loss factors. For piezoelectric materials, resonance and antiresonance are defined as the maximum and minimum point of admittance, as shown in Figure 2-4. $Q_A$ and $Q_B$ are used to represent the quality factor under the resonance (A-type resonance) and antiresonance (B-type resonance), respectively.

Figure 2-4. Admittance spectrum of a piezoelectric sample.

In Figure 2-5, the definitions of the mechanical quality factors at resonance and antiresonance are illustrated schematically. The values of $Q_A$ and $Q_B$ can be calculated as

$$Q_A = \frac{\omega_2}{\omega_{a2} - \omega_{a1}},$$  \hspace{1cm} (2.18)

$$Q_B = \frac{\omega_b}{\omega_{b2} - \omega_{b1}},$$  \hspace{1cm} (2.19)
where $\omega_a$ is the resonance frequency, $\omega_b$ is the antiresonance frequency, and $(\omega_{a2} - \omega_{a1})$ and $(\omega_{b2} - \omega_{b1})$ correspond to the 3dB bandwidth in the admittance curves around the resonance and antiresonance peaks, respectively.

![Admittance vs Frequency](image)

(a) Admittance

(b) Admittance

Figure 2-5. Definitions of quality factors at (a) resonance and (b) antiresonance.

2.1.4 Material properties

The properties of piezoelectric materials can be represented in the form of matrix. Three groups (elastic, dielectric, and piezoelectric characteristics) and four conditions (constant electric field, constant electric displacement, constant strain, and constant stress) are discussed.

Specifically, piezoelectric ceramics (poled in the $x_3$ direction) have the crystal symmetry of $\infty mm$ (equivalent to 6mm). There are 20 material properties shown in the following four matrices.
\[ S^E = \begin{bmatrix} s^E_{11} & s^E_{12} & s^E_{13} & 0 & 0 & 0 \\ s^E_{12} & s^E_{11} & s^E_{13} & 0 & 0 & 0 \\ s^E_{13} & s^E_{13} & s^E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s^E_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & s^E_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & s^E_{66} \end{bmatrix} = (e^E)^{-1}, \] (2.20)

\[ E^D = \begin{bmatrix} e^D_{11} & e^D_{12} & e^D_{13} & 0 & 0 & 0 \\ e^D_{12} & e^D_{11} & e^D_{13} & 0 & 0 & 0 \\ e^D_{13} & e^D_{13} & e^D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^D_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^D_{66} \end{bmatrix} = (s^D)^{-1}; \] (2.21)

\[ E^T = \begin{bmatrix} e^T_{11} & 0 & 0 \\ 0 & e^T_{11} & 0 \\ 0 & 0 & e^T_{33} \end{bmatrix} = (\beta^T)^{-1}, \] (2.22)

\[ \beta^S = \begin{bmatrix} \beta^S_{11} & 0 & 0 \\ 0 & \beta^S_{11} & 0 \\ 0 & 0 & \beta^S_{33} \end{bmatrix} = (e^S)^{-1}; \] (2.23)

\[ d = \begin{bmatrix} 0 & 0 & 0 & 0 & d_{15} & 0 \\ 0 & 0 & 0 & d_{15} & 0 & 0 \\ d_{31} & d_{31} & d_{33} & 0 & 0 & 0 \end{bmatrix}, \] (2.24)

\[ h = \begin{bmatrix} 0 & 0 & 0 & 0 & h_{15} & 0 \\ 0 & 0 & 0 & h_{15} & 0 & 0 \\ h_{31} & h_{31} & h_{33} & 0 & 0 & 0 \end{bmatrix}. \] (2.25)

Here

\[ s_{66} = 2(s_{11} - s_{12}), \] (2.26)

\[ c_{66} = (c_{11} - c_{12})/2. \] (2.27)
Among these parameters, $g^E$, $g^T$, and $d$ are related with intensive losses, while $e^D$, $h^S$, and $h$ are related with extensive ones.

### 2.1.5 Vibration modes

Table 2-1. The characteristics of various piezoelectric resonators with different shapes and sizes.

<table>
<thead>
<tr>
<th>Factor</th>
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<tr>
<td>a</td>
<td>$k_{31}$</td>
<td>$T_1 \neq 0, T_2 = T_3 = 0$</td>
<td>$S_1 \neq 0, S_2 \neq 0, S_3 \neq 0$</td>
</tr>
<tr>
<td>b</td>
<td>$k_{33}$</td>
<td>$T_1 = T_2 = 0, T_3 \neq 0$</td>
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<td>c</td>
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<td>$T_1 = T_2 \neq 0, T_3 = 0$</td>
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<td>d</td>
<td>$k_t$</td>
<td>$T_1 = T_2 \neq 0, T_3 \neq 0$</td>
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<td>$k_{24}=k_{15}$</td>
<td>$T_1 = T_2 = T_3 = 0, T_6 \neq 0$</td>
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</table>

Piezoelectric vibrators have different vibration modes upon the geometry and dimension. Some typical vibration modes are given in Table 2-1. [40] The electromechanical coupling factor has different definition for each mode, and certain parameters are primary in each case. Therefore,
in order to cover all material properties for piezoelectric ceramics several modes must be considered, including $k_{31}$, $k_\nu$, $k_{33}$, $k_p$, and $k_{15}$ modes.

In $k_{31}$ mode, for example, the effective parameters are $d_{31}$, $s_{11}^E$, and $\varepsilon_{33}^T$. In this case,

$$k_{31}^2 = \frac{d_{31}^2}{s_{11}^E (\varepsilon_{33}^T \varepsilon_0)} .$$

(2.28)

Accordingly, by the impedance characterization in $k_{31}$ mode the loss factors can be derived for $d_{31}$, $s_{11}^E$, and $\varepsilon_{33}^T$.

Therefore, different loss factors can be obtained by studying specific vibration modes. However, the relations of quality factors and loss factors must be derived first.

### 2.2 Derivations of Quality Factors under Various Modes

In this section, the theoretical derivations of quality factors will be introduced. We focus on the impedance analysis of $k_{31}$, $k_\nu$, $k_{33}$, $k_p$ and $k_{15}$ modes, and the analytical solutions are derived with the first order approximation. Also the derivations are based on the initial condition that the piezoelectric sample works in the linear region and the loss is treated as a perturbation, practically less than 0.1.

#### 2.2.1 $k_{31}$ mode (transverse extensional mode)

![Figure 2-6. Longitudinal vibration through the transverse piezoelectric effect in a rectangular plate ($k_{31}$ mode).](image-url)
For the $k_{31}$ mode piezoelectric plate, the length $l$, width $w$ and thickness $b$ should satisfy $l \gg w \gg b$, as shown in Figure 2-6.

In this case the electromechanical coupling factor $k_{31}$ is given by eq. (2.28), and the electric admittance for the sample can be represented as [33]

$$Y = j \omega \frac{e_0 e_{33}^T}{b} \left[1 - k_{31}^2 + k_{31}^2 \frac{\tan(\omega l / 2v_{11}^E)}{\omega l / 2v_{11}^E}\right], \quad (2.29)$$

$$v_{11}^E = \frac{1}{\sqrt{\rho_{11}^E}}. \quad (2.30)$$

Here $v_{11}^E$ is the sound velocity, and $\rho$ is the mass density. Then introduce a new parameter for analysis:

$$\Omega = \frac{\omega l}{2v_{11}^E}. \quad (2.31)$$

$\Omega$ is called a “normalized frequency” here, which is a real number and proportional to the angular frequency $\omega$. Then the expression with losses is represented by including the elastic loss factor for $s_{11}^E$.

$$\Omega^* = \frac{\omega l}{2v_{11}^E} = \Omega \sqrt{1 - j \tan \phi_{11}'} = \Omega \left(1 - j \frac{1}{2} \tan \phi_{11}'\right). \quad (2.32)$$

In eq. (2.32) and following derivations, the result is derived using the Taylor expansion with the first-order approximation considering that the loss factor is a small number (typically less than 0.1).

Let’s discuss the resonance first. The admittance expression with loss factors can be written as

$$Y = j \omega C_d (1 - j \tan \delta_{33}'''') + j \omega C_y k_{31}^2 \left[1 - j(2 \tan \theta_{31}' - \tan \phi_{11}')\right] \frac{\tan \Omega^*}{\Omega^*}, \quad (2.33)$$
where $\omega$ is the driving frequency in radians, $C_d$ the damped capacitance, $C_0$ the motional capacitance, and $v_{11}^{E*}$ the sound velocity in the material with loss effect:

$$C_0 = \frac{w l}{b} \varepsilon_0 \varepsilon_{33}^r,$$

$$C_d = (1 - k_{31}^2) C_0,$$

$$\tan \delta_{33}''' = \frac{1}{1 - k_{31}^2} \left[ \tan \delta_{33}' - k_{31}^2 (2 \tan \theta_{31}' - \tan \phi_{11}') \right],$$

$$v_{11}^{E*} = \frac{1}{\sqrt{\rho_{11}E(1 - j \tan \phi_{11}')}} = v_{11}^{E} \left( 1 + j \frac{\tan \phi_{11}'}{2} \right).$$

Here $\tan \delta_{33}'$, $\tan \phi_{11}'$, $\tan \theta_{31}'$ are intensive loss factors for $\varepsilon_{33}^T$, $s_{11}^E$, $d_{31}$, respectively.

The admittance approaches the maximum when $\Omega$ is around $\pi/2$. In this case, the damped capacitance can be neglected considering the motional one is much larger for the resonance.

$$Y \approx j \omega C_0 k_{31}^2 \left[ 1 - j (2 \tan \theta_{31}' - \tan \phi_{11}') \right] \frac{\tan(\omega l / 2 v_{11}^{E*})}{\omega l / 2 v_{11}^{E*}}.$$  \hspace{1cm} (2.38)

Assume the resonance frequency corresponds to

$$\Omega_A = \frac{\pi}{2} + \Delta_A.$$

Here $\Delta_A$ is also a small fluctuation, and therefore the following Taylor expansion around $\pi/2$ with the first order approximation is utilized.

$$\frac{1}{\tan \Omega_A} = \cot \left( \frac{\pi}{2} + \Delta_A - j \frac{\pi}{4} \tan \phi_{11}' \right) = \Delta_A - j \frac{\pi}{4} \tan \phi_{11}'.$$  \hspace{1cm} (2.40)

Integrate the approximation, and the admittance is obtained by

$$Y = j \omega C_0 k_{31}^2 \frac{1 - j (2 \tan \theta_{31}' - \tan \phi_{11}')}{{\left( \frac{\pi}{2} + \Delta_A \right)} \left( \Delta_A - j \frac{\pi}{4} \tan \phi_{11}' \right)}.$$  \hspace{1cm} (2.41)
\[ Y \approx \frac{2}{\pi} \omega C_0 k_{31}^2 \frac{1-j(2 \tan \theta_{31} - \tan \phi_{11}')}{\Delta_A - j \frac{\pi}{4} \tan \phi_{11}'} . \] (2.42)

In order to achieve the maximum absolute value of admittance \( \Delta_A \) should be zero, and the maximum \( |Y| \) is can be represented as

\[ |Y|_{\text{max}} = \frac{8}{\pi^2} \omega C_0 k_{31}^2 \frac{1}{\tan \phi_{11}'} . \] (2.43)

Next the frequency with 3dB admittance down corresponds to the situation when

\[ |Y| = |Y|_{\text{max}} / \sqrt{2} , \] (2.44)

i.e.,

\[ \Delta_A = \frac{\pi}{4} \tan \phi_{11}' . \] (2.45)

Then the quality factor \( Q_{A,31} \) can be calculated by

\[ Q_{A,31} = \frac{\Omega_A}{2 \Delta_A} = \frac{\frac{\pi}{2}}{2 \left( \frac{\pi}{4} \tan \phi_{11}' \right)} = \frac{1}{\tan \phi_{11}'} . \] (2.46)

On the other hand, the antiresonance corresponds to the minimum admittance. Since the expansion series of \( \tan \Omega \) is convergent in this case, we can apply the following expansion.

\[ \tan(\Omega^*) = \tan \left( \Omega - \frac{\Omega \tan \phi_{11}'}{2} \right) = \tan \Omega - j \frac{\Omega \tan \phi_{11}'}{2 \cos^2 \Omega} . \] (2.47)

The admittance expression, eq. (2.29), could be simplified as follows.

\[ Y' = 1 - k_{31}^2 + k_{31}^2 \frac{\tan(\omega l / 2\nu_{11}^E)}{\omega l / 2\nu_{11}^E} . \] (2.48)

Introduce losses for the parameters and we can get
\[
Y' = 1 - k_{31}^2 \left[ 1 - j \left( 2 \tan \theta_{31} - \tan \delta_{33} - \tan \phi_{11} \right) \right] + k_{31}^2 \left[ 1 - j \left( 2 \tan \theta_{31} - \tan \delta_{33} - \tan \phi_{11} \right) \right] \frac{\tan \Omega^*}{\Omega^*}.
\]

(2.49)

By simplifying this expression, we can obtain the conductance \( G \) (real part) and susceptance \( B \) (imaginary part) separately as

\[
G = 1 - k_{31}^2 + k_{31}^2 \frac{\tan \Omega}{\Omega}.
\]

(2.50)

\[
B = k_{31}^2 \left( 2 \tan \theta_{31} - \tan \delta_{33} - \tan \phi_{11} \right) - \frac{k_{31}^2}{\Omega}
\times \left[ \tan \Omega \left( 2 \tan \theta_{31} - \tan \delta_{33} - \tan \phi_{11} \right) + \frac{\Omega}{2 \cos^2 \Omega} \tan \phi_{11} - \tan \phi_{11} \right].
\]

(2.51)

\[
B = \left( k_{31}^2 - \frac{\tan \Omega}{\Omega} \right) \left( 2 \tan \theta_{31} - \tan \delta_{33} - \tan \phi_{11} \right) - \frac{k_{31}^2}{2} \left( \frac{1}{\cos^2 \Omega} - \frac{\tan \Omega}{\Omega} \right) \tan \phi_{11}.
\]

(2.52)

Assume that

\[
\Omega = \Omega_B + \Delta_B,
\]

(2.53)

where

\[
1 - k_{31}^2 + k_{31}^2 \frac{\tan \Omega_B}{\Omega_B} = 0.
\]

(2.54)

Similar to \( \Delta_A \), \( \Delta_B \) is also a small number and the first order approximation can be utilized.

\[
\frac{\tan \Omega}{\Omega} = \frac{\tan \Omega_B}{\Omega_B} + \frac{1}{\Omega_B} \left( \frac{1}{\cos^2 \Omega_B} - \frac{\tan \Omega_B}{\Omega_B} \right) \Delta_B.
\]

(2.55)

Apply the approximation, and neglect high order term which has two or more small factors (loss factor or \( \Delta_B \)).

\[
Y' = G + jB.
\]

(2.56)
\[ G = \frac{k_{31}^2}{\Omega_B} \left( \frac{1}{\cos^2 \Omega_B} - \frac{\tan \Omega_B}{\Omega_B} \right) \Delta_B. \] (2.57)

\[ B = \left( 2 \tan \theta_{31}' - \tan \delta_{33}' - \tan \phi_{11}' \right) - \frac{k_{31}^2}{2} \left( \frac{1}{\cos^2 \Omega_B} - \frac{\tan \Omega_B}{\Omega_B} \right) \tan \phi_{11}'. \] (2.58)

Consequently, the minimum absolute value of admittance can be achieved when \( \Delta_B \) is 0.

Therefore, the antiresonance frequency is determined by \( \Omega_B \).

In order to find the 3dB point, let \( G = B \):

\[ \frac{k_{31}^2}{\Omega_B} \left( \frac{1}{\cos^2 \Omega_B} - \frac{\tan \Omega_B}{\Omega_B} \right) \Delta_B = \left( 2 \tan \theta_{31}' - \tan \delta_{33}' - \tan \phi_{11}' \right) \] (2.59)

\[ - \frac{k_{31}^2}{2} \left( \frac{1}{\cos^2 \Omega_B} - \frac{\tan \Omega_B}{\Omega_B} \right) \tan \phi_{11}'. \]

Further, the antiresonance quality factor is given by

\[ Q_{B,31} = \frac{\Omega_B}{2 | \Delta_B |}. \] (2.60)

So eq. (2.59) can be represented as

\[ \frac{k_{31}^2}{2Q_{B,31}} \left( \frac{1}{\cos^2 \Omega_B} - \frac{\tan \Omega_B}{\Omega_B} \right) = \left( 2 \tan \theta_{31}' - \tan \delta_{33}' - \tan \phi_{11}' \right) \] (2.61)

\[ + \frac{k_{31}^2}{2} \left( \frac{1}{\cos^2 \Omega_B} - \frac{\tan \Omega_B}{\Omega_B} \right) \tan \phi_{11}'. \]

Considering eq. (2.46), we can obtain the result as

\[ \frac{1}{Q_{B,31}} = \frac{1}{Q_{4,31}} - \frac{2}{k_{31}^2} \left( 2 \tan \theta_{31}' - \tan \delta_{33}' - \tan \phi_{11}' \right) \left( \frac{1}{\cos^2 \Omega_B} - \frac{\tan \Omega_B}{\Omega_B} \right). \] (2.62)

According to eq. (2.54),

\[ \frac{\tan \Omega_B}{\Omega_B} = - \frac{1 - k_{31}^2}{k_{31}^2}, \] (2.63)
\[
\frac{1}{\cos^2 \Omega_B} = \frac{(1 - k_{31}^2)^2 \Omega_B^2}{k_{31}^4} + k_{31}^4.
\]  
(2.64)

Therefore,

\[
\frac{1}{\cos^2 \Omega_B} - \tan \Omega_B = \frac{(1 - k_{31}^2)^2 \Omega_B^2}{k_{31}^4} + \frac{1 - k_{31}^2}{k_{31}^2},
\]  
(2.65)

\[
\frac{1}{\cos^2 \Omega_B} - \frac{\tan \Omega_B}{\Omega} = \frac{(1 - k_{31}^2)^2 \Omega_B^2}{k_{31}^4} + k_{31}^2,
\]  
(2.66)

\[
\frac{1}{Q_{B,31}} = \frac{1}{Q_{A,31}} - \frac{2}{k_{31}} \left(2 \tan \theta_{31}' - \tan \delta_{33}' - \tan \phi_{1}'\right) \left(1 - k_{31}^2 \right) \Omega_B^2 \frac{k_{31}^4}{2}.
\]  
(2.67)

Finally, the equation of the antiresonance quality factor is given by

\[
\frac{1}{Q_{B,31}} = \frac{1}{Q_{A,31}} - \frac{2}{k_{31}} \left(2 \tan \theta_{31}' - \tan \delta_{33}' - \tan \phi_{1}'\right) \Omega_B^2 \frac{k_{31}^4}{2}.
\]  
(2.68)

2.2.2 kₜ mode (thickness mode)

![Diagram of thin plate in thickness vibration mode](image)

Figure 2-7. Thin plate in thickness vibration mode (kₜ mode).

The thickness vibration mode is analyzed by characterizing thin plate (square or disk), which is poled and vibrating along the thickness direction as shown in Figure 2-7. The impedance expression of the kₜ mode plate is given by: [33]
\[ Z(\omega) = \frac{1}{j\omega C_d} \left( 1 - k_i^2 \frac{\tan \Omega}{\Omega} \right), \]  

(2.69)

where

\[ C_d = \frac{\pi a^2}{b} e_0 e_{33}, \]  

(2.70)

\[ \Omega = \frac{\omega b}{2\nu_{33}^D} = \frac{\omega b}{2} \frac{\rho}{c_{33}^D}, \]  

(2.71)

\[ k_i^2 = \frac{h_{33}^2}{c_{33}^D (\beta_{33}^S / e_0)} = \frac{h_{33}^2 e_0 e_{33}^{s^*}}{c_{33}^D}. \]  

(2.72)

By introducing the complex parameters,

\[ C_d^* = \frac{\pi a^2}{t} e_0 e_{33}^{s^*} = C_d (1 - j \tan \delta_{33}), \]  

(2.73)

\[ \Omega^* = \frac{\omega b}{2} \frac{\rho}{c_{33}^D (1 + j \tan \phi_{33})} = \Omega \sqrt{1 - j \tan \phi_{33}}, \]  

(2.74)

\[ \left( k_i^2 \right)^* = k_i^2 (1 + j \chi_i), \]  

(2.75)

\[ \chi_i = 2 \tan \theta_{33} - \tan \delta_{33} - \tan \phi_{33}. \]  

(2.76)

Here extensive losses are considered rather than intensive ones in the case of \( k_{31} \) mode, because the boundary conditions are constant strain \( S \) and electric displacement \( D \).

First-order approximation can be applied to \( \Omega^* \):

\[ \Omega^* = \Omega \sqrt{1 - j \tan \phi_{33}} = \Omega \left( 1 - j \frac{\tan \phi_{33}}{2} \right). \]  

(2.77)

Then we can obtain the impedance expression with losses by

\[ Z(\omega) = \frac{1}{j\omega C_d (1 - j \tan \delta_{33})} \left\{ 1 - k_i^2 (1 + j \chi_i) \frac{\tan \Omega \left( 1 - j \frac{\tan \phi_{33}}{2} \right)}{\Omega \left( 1 - j \frac{\tan \phi_{33}}{2} \right)} \right\}. \]  

(2.78)
According to eq. (2.69), the impedance goes to infinity when $\Omega$ approaches $\pi/2$, which corresponds to the antiresonance. Therefore, around the antiresonance:

$$Z(\omega) \approx j \frac{1}{\omega C_d (1 - j \tan \delta_{33})} k^2 r (1 + j \chi_i) \tan \left[ \frac{\Omega \left(1 - j \frac{\tan \phi_{33}}{2}\right)}{\Omega \left(1 - j \frac{\tan \phi_{33}}{2}\right)} \right].$$

(2.79)

Assume that $\Omega = \pi/2 + \Delta_B$, where $\Delta_B$ is considered as a small deviation.

$$\Omega^* = \left(\frac{\pi}{2} + \Delta_B\right) \left(1 - j \frac{\tan \phi_{33}}{2}\right) = \frac{\pi}{2} + \Delta_B - j \frac{\pi \tan \phi_{33}}{4},$$

(2.80)

$$\frac{1}{\tan \Omega^*} = \cot \Omega^* = -\Delta_B + j \frac{\pi \tan \phi_{33}}{4}.$$  

(2.81)

Here, the Taylor expansion is utilized around $\pi/2$ and the higher order terms are neglected considering loss factors and $\Delta_B$ are quite small numbers. This assumption will also be applied in the following derivation. Accordingly, the impedance expression is given by:

$$Z(\omega) = j \frac{1}{\omega C_d (1 - j \tan \delta_{33})} \frac{k^2 r (1 + j \chi_i)}{\left(\frac{\pi}{2} + \Delta_B - j \frac{\pi \tan \phi_{33}}{4}\right) \left(-\Delta_B + j \frac{\pi \tan \phi_{33}}{4}\right)}.$$ 

(2.82)

Simplify this equation by neglecting the high order terms:

$$Z(\omega) = j \frac{1}{\omega C_d} \frac{k^2 r (1 + j \chi_i)}{\left(\frac{\pi}{2}\right) \left(-\Delta_B + j \frac{\pi \tan \phi_{33}}{4}\right)}.$$  

(2.83)

So the maximum impedance is achieved when $\Delta_B$ is zero, i.e. $\Omega = \Omega_B = \pi/2$.

$$|Z|_{\text{max}} = \frac{8}{\pi^2 \omega C_d \tan \phi_{33}} k^2 r.$$  

(2.84)

On the other hand, the 3dB boundary corresponds to the situation when $|\Delta_B| = \pi \tan \phi_{33}/4$.

Finally the quality factor at the antiresonance can be derived as:
\[ Q_{B,t} = \frac{\Omega_B}{2 |\Delta_B|} = \frac{\pi / 2}{2 \times (\pi \tan \phi_{33} / 4)} = \frac{1}{\tan \phi_{33}}. \]  

(2.85)

As for the resonance, the minimum impedance is achieved around

\[ 1 - k_i^2 \tan \frac{\Omega}{\Omega} = 0. \]  

(2.86)

In this case,

\[ \Omega_A = k_i^2 \tan \Omega_A, \]  

(2.87)

\[ \frac{1}{\cos^2 \Omega_A} = 1 + \frac{\Omega_A^2}{k_i^4}. \]  

(2.88)

Then, let us discuss eq. (2.78) again, which can be written as:

\[ Z(\Omega) = \frac{1}{j \omega C_d (1 - j \tan \delta_{33}) \Omega} \cdot \left\{ \Omega - k_i^2 \left[ 1 + j \left( \chi_i + \frac{\tan \phi_{33}}{2} \right) \right] \tan \left[ \Omega \left( 1 - j \frac{\tan \phi_{33}}{2} \right) \right] \right\}. \]  

(2.89)

Then apply the following expansion for the tangent term:

\[ \tan \left[ \Omega \left( 1 - j \frac{\tan \phi_{33}}{2} \right) \right] = \tan \Omega - j \frac{\Omega \tan \phi_{33}}{2 \cos^2 \Omega}. \]  

(2.90)

Therefore, eq. (2.89) becomes

\[ Z(\Omega) = \frac{1}{j \omega C_d (1 - j \tan \delta_{33}) \Omega} \cdot \left\{ \Omega - k_i^2 \left[ 1 + j \left( \chi_i + \frac{\tan \phi_{33}}{2} \right) \right] \tan \left[ \Omega \left( 1 - j \frac{\tan \phi_{33}}{2} \right) \right] \right\}. \]  

(2.91)

\[ Z(\Omega) = \frac{1}{j \omega C_d (1 - j \tan \delta_{33}) \Omega} \cdot \left[ \Omega - k_i^2 \tan \Omega - jk_i^2 \left( \chi_i + \frac{\tan \phi_{33}}{2} \right) \tan \Omega + jk_i^2 \frac{\Omega \tan \phi_{33}}{2 \cos^2 \Omega} \right]. \]  

(2.92)

Let \( \Omega = \Omega_{\lambda} + \Delta_{\lambda} \), where \( \Delta_{\lambda} \) is a small value.
\[
\tan(\Omega_A + \Delta_A) = \tan \Omega_A + \frac{\Delta_A}{\cos^2 \Omega_A}.
\]

(2.93)

\[
\frac{1}{\cos^2(\Omega_A + \Delta_A)} = \left(1 + \frac{\Omega_A^2}{k_i^2}\right) + \frac{2\Omega_A^2}{k_i^2} \Delta_A = \frac{\Omega_A^2 + 2\Omega_A \Delta_A + k_i^4}{k_i^4}.
\]

(2.94)

Here eq. (2.94) is obtained by the first-order approximation of eq. (2.88). Therefore, we can derive the following result by using eqs. (2.87) and (2.88) and neglecting higher order small terms.

\[
Z(\Omega_A + \Delta_A) = \frac{1}{j \omega C_d (1 - j \tan \delta_{33}) \Omega_A} \left\{-\Delta_A \left(\frac{\Omega_A^2}{k_i^2} + k_i^2 - 1\right) - j \Omega_A \left[\chi_t - \frac{\tan \phi_{33}}{2} \left(\frac{\Omega_A^2}{k_i^2} + k_i^2 - 1\right)\right]\right\}.
\]

(2.95)

\[Z\] is minimum when \(\Delta_A = 0\), and the 3dB condition corresponds to

\[
\Delta_A \left(\frac{\Omega_A^2}{k_i^2} + k_i^2 - 1\right) = \Omega_A \left[\chi_t - \frac{\tan \phi_{33}}{2} \left(\frac{\Omega_A^2}{k_i^2} + k_i^2 - 1\right)\right].
\]

(2.96)

The quality factor at the resonance can be represented as

\[
\frac{1}{Q_{A,t}} = \frac{2|\Delta_A|}{\Omega_A} = \tan \phi_{33} - \frac{2}{k_i^2 - 1 + \Omega_A^2 / k_i^2} \chi_t.
\]

(2.97)

Therefore, the final result is given by

\[
\frac{1}{Q_{A,t}} = \frac{1}{Q_{B,t}} + \frac{2}{k_i^2 - 1 + \Omega_A^2 / k_i^2} \left(\tan \delta_{33} + \tan \phi_{33} - 2 \tan \theta_{33}\right).
\]

(2.98)

Note that the sign of the term \((k_i^2 - 1 + \Omega_A^2 / k_i^2)\) should be clarified. According to eq. (2.87),

\[
k_i^2 - 1 + \frac{\Omega_A^2}{k_i^2} = \frac{\Omega_A}{\tan \Omega_A} - 1 + \Omega_A \tan \Omega_A.
\]

(2.99)
Since \( 0 < \Omega_A < \Omega_B = \pi/2 \), \( 2\Omega_A \) is greater than \( \sin(2\Omega_A) \) and therefore \((k_i^2 - 1 + \Omega_A^2 / k_i^2)\) is always positive.

### 2.2.3 \( k_{33} \) mode (length extensional mode)

The length extensional mode is shown in Figure 2-8, where \( l >> w, b \). The impedance expression of the \( k_{33} \) mode bar is very similar to \( k_i \) mode. [14]

\[
Z(\omega) = \frac{1}{j\omega C_d} \left( 1 - k_{33}^2 \frac{\tan \Omega}{\Omega} \right),
\]

where

\[
C_d = \frac{wb}{l} \varepsilon_0 \varepsilon_{33}^T \left( 1 - k_{33}^2 \right),
\]
\[ \Omega = \frac{\omega l}{2} \sqrt{\rho s_{33}^P} = \frac{\omega l}{2} \sqrt{\rho s_{33}^E (1 - k_{33}^2)}, \] (2.103)

\[ k_{33}^2 = \frac{d_{33}^2}{\varepsilon_{33}^{eff} \varepsilon_{33}^{E}}. \] (2.104)

By introducing the complex parameters,

\[ \left( k_{33}^2 \right)^* = k_{33}^2 (1 - j \chi_{33}), \] (2.105)

\[ \chi_{33} = 2 \tan \theta_{33} \tan \delta_{33} \tan \phi_{33}. \] (2.106)

\[ C_d^* = C_d (1 - j \tan \delta_{33}'''), \] (2.107)

\[ \tan \delta_{33}''' = \frac{1}{1 - k_{33}^2} \left[ \tan \delta_{33} - k_{33}^2 (2 \tan \theta_{33} \tan \phi_{33}' \tan \delta_{33}') \right], \] (2.108)

\[ \Omega^* = \Omega \sqrt{1 - j \tan \phi_{33}'''}, \] (2.109)

\[ \tan \phi_{33}''' = \frac{1}{1 - k_{33}^2} \left[ \tan \phi_{33} - k_{33}^2 (2 \tan \theta_{33} \tan \phi_{33} \tan \delta_{33}') \right]. \] (2.110)

Note that the parameters in \( k_{33} \) mode have the same forms as \( k_t \) mode, and the difference is that the loss factors \( \chi_t, \tan \delta_{33}, \tan \phi_{33} \) are replaced by \( -\chi_{33}, \tan \delta_{33}'''', \tan \phi_{33}'''', \) respectively. Therefore, the same derivation process can be applied, and the results are given by

\[ Q_{b,33} = \frac{1}{\tan \phi_{33}''''} = \frac{1}{\tan \phi_{33}' - k_{33}^2 (2 \tan \theta_{33} \tan \phi_{33} \tan \delta_{33}')} \] (2.111)

\[ \frac{1}{Q_{a,33}} = \frac{1}{Q_{b,33}} + \frac{2}{k_{33}^2 - 1 + \Omega_4^2 / k_{33}^2} (2 \tan \theta_{33} \tan \delta_{33} \tan \phi_{33}'). \] (2.112)

Here

\[ \Omega_4 = k_{33}^2 \tan \Omega_4. \] (2.113)
2.2.4 kp mode (radial mode)

The piezoelectric vibrator under kp mode is a thin plate, which is poled along the thickness and vibrating along the radius. Here the radius should be much larger than the thickness, and the admittance expression is given by [33]

\[ Y = j \omega \varepsilon_0 \varepsilon \frac{A}{b} \left( 1 - k_p^2 + k_p^2 \frac{1 + \sigma}{J - 1 + \sigma} \right), \]  \hspace{1cm} (2.114)

where

\[ k_p^2 = \frac{2 k_{31}^2}{1 - \sigma}. \]  \hspace{1cm} (2.115)

\[ \Omega = \frac{\omega a}{\nu^p}, \]  \hspace{1cm} (2.116)

\[ J = \Omega J_0(\Omega) / J_1(\Omega). \]  \hspace{1cm} (2.117)

Here A is the surface area, \( \nu^p \) is the sound velocity, and \( \sigma \) is Poisson’s ratio. \( J_0 \) and \( J_1 \) are Bessel functions of the first kind.

\[ A = \pi a^2, \]  \hspace{1cm} (2.118)

\[ \sigma = -s_{12}^E / s_{11}^E, \]  \hspace{1cm} (2.119)

\[ \nu^p = \sqrt{\frac{c_{11}^p}{\rho}} = \sqrt{\frac{1}{\rho s_{11}^E} \frac{1}{1 - \sigma^2}}. \]  \hspace{1cm} (2.120)

For the analysis of the resonance,
\[ Y \approx j\omega e_0 e_{33}^T \frac{A}{b} k_p^2 \frac{1 + \sigma}{J - 1 + \sigma}. \quad (2.121) \]

Consequently, we focus on the denominator of the last term:

\[ D(\Omega) = J - 1 + \sigma = \Omega J_0(\Omega)/J_1(\Omega) - 1 + \sigma. \quad (2.122) \]

The resonance corresponds to the minimum \( (D_{\min}) \), and the 3dB boundary is achieved when

\[ D = \sqrt{2} D_{\min}. \quad (2.123) \]

Assume \( \Omega = \Omega_{A,p} + \Delta_A \), where \( \Delta_A \) is a small factor and \( \Omega_{A}, J_0(\Omega_A)/J_1(\Omega_A) - 1 + \sigma = 0. \quad (2.124) \)

Bessel functions have the following relation:

\[ \frac{d}{d\Omega} \left[ \Omega^m J_m(\Omega) \right] = \Omega^m J_{m-1}(\Omega). \quad (2.125) \]

Accordingly,

\[ \frac{d}{d\Omega} \left[ J_0(\Omega) \right] = -J_1(\Omega), \quad (2.126) \]

\[ \frac{d}{d\Omega} \left[ \Omega J_1(\Omega) \right] = \Omega J_0(\Omega). \quad (2.127) \]

Then eq. (2.127) can be represented as

\[ \frac{d}{d\Omega} \left[ J_1(\Omega) \right] = J_0(\Omega) - \frac{1}{\Omega} J_1(\Omega). \quad (2.128) \]

Apply the Taylor expansions by

\[ J_0(\Omega_A + \Delta_A) = J_0(\Omega_A) - J_1(\Omega_A) \Delta_A, \quad (2.129) \]

\[ J_1(\Omega_A + \Delta_A) = J_1(\Omega_A) + \left[ J_0(\Omega_A) - \frac{1}{\Omega_A} J_1(\Omega_A) \right] \Delta_A. \quad (2.130) \]

So
\[
D(\Omega) = (\Omega + \Delta) J_0(\Omega + \Delta) / J_1(\Omega + \Delta) - 1 + \sigma. \tag{2.131}
\]
\[
D(\Omega) = \frac{\left(\Omega + \Delta\right)\left[J_0(\Omega + \Delta) - J_1(\Omega + \Delta)\Delta\right]}{J_1(\Omega + \Delta) + \left[J_0(\Omega + \Delta) - \frac{1}{\Omega} J_1(\Omega + \Delta)\right] \Delta} - 1 + \sigma. \tag{2.132}
\]

Next use complex numbers to include losses.

\[
s_{11}^E = s_{11}^E (1 - j \tan \phi_1'). \tag{2.133}
\]
\[
s_{12}^E = s_{12}^E (1 - j \tan \phi_2'). \tag{2.134}
\]
\[
\sigma^* = \sigma [1 - j (\tan \phi_2' - \tan \phi_1')]. \tag{2.135}
\]
\[
v^p = v^p \left[1 - j \left(\frac{\sigma^2 \tan \phi_2'}{1 - \sigma^2} - \frac{1 + \sigma^2}{2(1 - \sigma^2)} \tan \phi_1'\right)\right]. \tag{2.136}
\]
\[
v^p = v^p \left[1 - j \left(\frac{\sigma^2}{1 - \sigma^2} (\tan \phi_2' - \tan \phi_1') - \frac{1}{2} \tan \phi_1'\right)\right]. \tag{2.137}
\]
\[
\Omega^* = \Omega \left[1 - j \left(\frac{1 + \sigma^2}{2(1 - \sigma^2)} \tan \phi_1' - \frac{\sigma^2}{1 - \sigma^2} \tan \phi_2'\right)\right]. \tag{2.138}
\]

For the simplicity of derivation, let

\[
\Omega^* = \Omega (1 - j \gamma) \tag{2.139}
\]
\[
\gamma = \frac{1 + \sigma^2}{2(1 - \sigma^2)} \tan \phi_1' - \frac{\sigma^2}{1 - \sigma^2} \tan \phi_2'. \tag{2.140}
\]

Then apply the Taylor expansions:

\[
J_0(\Omega_A^*) = J_0(\Omega_A) + j \Omega_A J_1(\Omega_A) \gamma, \tag{2.141}
\]
\[
J_1(\Omega_A^*) = J_1(\Omega_A) - j \left[J_0(\Omega_A) - \frac{1}{\Omega_A} J_1(\Omega_A)\right] \Omega_A \gamma. \tag{2.142}
\]

According to eq. (2.124),
\[ J_1(\Omega_A^*) = J_1(\Omega_A) + j\sigma J_1(\Omega_A)\gamma. \quad (2.143) \]

Then discuss the relation with losses:

\[
D^* = \frac{\left(\Omega_A + j\sigma \Delta_A\right)\left[J_0(\Omega_A^*) - J_1(\Omega_A^*)\Delta_A\right]}{J_1(\Omega_A^*) + \left[J_0(\Omega_A^*) - \frac{1}{\Omega_A^*} J_1(\Omega_A^*)\right] \Delta_A} - 1 + \sigma^*. \quad (2.144)
\]

In order to simply the equation, use the first order approximations by neglecting high
order terms which have two or more small factors (loss factor or \(\Delta_A\)). Here eq. (2.124) is also
utilized during the simplification.

\[
D^* = \frac{\left(\Omega_A - j\Omega_A\gamma + \Delta_A\right)\left[J_0(\Omega_A) + j\Omega_A J_1(\Omega_A)\gamma - J_1(\Omega_A)\Delta_A\right]}{J_1(\Omega_A) + j\sigma J_1(\Omega_A)\gamma + \left[J_0(\Omega_A) - \frac{1}{\Omega_A} J_1(\Omega_A)\right] \Delta_A} - 1 + \sigma - j\sigma(\tan\phi_{12}' - \tan\phi_1'). \quad (2.145)
\]

\[
D^* = \frac{\left(\Omega_A + \Delta_A - j\Omega_A\gamma\right)\left[J_0(\Omega_A) - J_1(\Omega_A)\Delta_A + j\Omega_A J_1(\Omega_A)\gamma\right]}{J_1(\Omega_A)\left(1 - \frac{\sigma}{\Omega_A} \Delta_A + j\sigma\gamma\right)} - 1 + \sigma - j\sigma(\tan\phi_{12}' - \tan\phi_1'). \quad (2.146)
\]

\[
D^* = \frac{\left(\Omega_A + \Delta_A - j\Omega_A\gamma\right)\left[J_0(\Omega_A) - J_1(\Omega_A)\Delta_A + j\Omega_A J_1(\Omega_A)\gamma\right]}{J_1(\Omega_A)\left(1 - \frac{\sigma}{\Omega_A} \Delta_A\right)^2} \cdot \left(1 - \frac{\sigma}{\Omega_A} \Delta_A - j\sigma\gamma\right) - 1 + \sigma - j\sigma(\tan\phi_{12}' - \tan\phi_1'). \quad (2.147)
\]

\[
D^* = \frac{\left(\Omega_A + \Delta_A - j\Omega_A\gamma\right)\left[J_0(\Omega_A) - J_1(\Omega_A)\Delta_A + j\Omega_A J_1(\Omega_A)\gamma\right]}{J_1(\Omega_A)\left(1 - \frac{2\sigma}{\Omega_A} \Delta_A\right)} \cdot \left(1 - \frac{\sigma}{\Omega_A} \Delta_A - j\sigma\gamma\right) - 1 + \sigma - j\sigma(\tan\phi_{12}' - \tan\phi_1'). \quad (2.148)
\]
\[ D^* = \frac{(\Omega_A + \Delta_A - j\Omega_A \gamma)\left[J_0(\Omega_A) - J_1(\Omega_A)\Delta_A + j\Omega_A J_1(\Omega_A)\gamma\right]}{J_1(\Omega_A)\left(1 - \frac{2\sigma}{\Omega_A \Delta_A}\right)\left(1 + \frac{2\sigma}{\Omega_A \Delta_A}\right)} \cdot \left(1 - \frac{\sigma}{\Omega_A} \Delta_A - j\sigma\gamma\right) - 1 + \sigma - j\sigma(\tan \phi_{12} - \tan \phi_{11}'). \] (2.149)

\[ D^* = \frac{(\Omega_A + \Delta_A - j\Omega_A \gamma)\left[J_0(\Omega_A) - J_1(\Omega_A)\Delta_A + j\Omega_A J_1(\Omega_A)\gamma\right]}{J_1(\Omega_A)} \cdot \left(1 + \frac{\sigma}{\Omega_A} \Delta_A - j\sigma\gamma\right) - 1 + \sigma - j\sigma(\tan \phi_{12} - \tan \phi_{11}'). \] (2.150)

\[ D^* = \frac{\Omega_A J_0(\Omega_A) + (1 + \sigma)J_0(\Omega_A) - \Omega_A J_1(\Omega_A)}{J_1(\Omega_A)} \Delta_A - j\Omega_A \frac{(1 + \sigma)J_0(\Omega_A) - \Omega_A J_1(\Omega_A)}{J_1(\Omega_A)} \gamma - 1 + \sigma - j\sigma(\tan \phi_{12} - \tan \phi_{11}'). \] (2.151)

Considering again eq. (2.124) \( \Omega_A J_0(\Omega_A)/J_1(\Omega_A) - 1 + \sigma = 0 \),

\[ D^* = \frac{(1 + \sigma)J_0(\Omega_A) - \Omega_A J_1(\Omega_A)}{J_1(\Omega_A)} \Delta_A - j\Omega_A \frac{(1 + \sigma)J_0(\Omega_A) - \Omega_A J_1(\Omega_A)}{J_1(\Omega_A)} \gamma - j\sigma(\tan \phi_{12} - \tan \phi_{11}'). \] (2.152)

\[ D^* = \frac{(1 + \sigma)J_0(\Omega_A) - \Omega_A J_1(\Omega_A)}{J_1(\Omega_A)} \Delta_A - j\left[\Omega_A \frac{(1 + \sigma)J_0(\Omega_A) - \Omega_A J_1(\Omega_A)}{J_1(\Omega_A)} \gamma + \sigma(\tan \phi_{12} - \tan \phi_{11}'). \right] \] (2.153)

To achieve the minimum absolute value of \( D \), \( \Delta_A \) should be zero. Therefore, \( \Omega_A \) corresponds to the resonance of \( k_p \) mode. To find the 3dB point, let the real part equal the imaginary.

\[ \frac{(1 + \sigma)J_0(\Omega_A) - \Omega_A J_1(\Omega_A)}{J_1(\Omega_A)} \Delta_A = \Omega_A \frac{(1 + \sigma)J_0(\Omega_A) - \Omega_A J_1(\Omega_A)}{J_1(\Omega_A)} \gamma \]

\[ + \sigma(\tan \phi_{12} - \tan \phi_{11}'). \] (2.154)
According to eq. (2.124) again,

$$\Omega_A J_0(\Omega_A) = (1 - \sigma) J_1(\Omega_A).$$

(2.155)

Therefore, eq. (2.154) can be represented as

$$\frac{\Delta A}{\Omega_A} \left[ (1 - \sigma^2 - \Omega_A^2) \right] = \left[ (1 - \sigma^2 - \Omega_A^2) \right] \gamma + \sigma (\tan \phi_{12} - \tan \phi_{11'}).$$

(2.156)

Note that

$$Q_{A,p} = \frac{\Omega_A}{2\Delta_A}.$$  

(2.157)

Therefore,

$$\frac{1}{Q_{A,p}} = 2\gamma - \frac{2\sigma}{\Omega_A^2 + \sigma^2 - 1} \left( \tan \phi_{12} - \tan \phi_{11'} \right).$$

(2.158)

Integrate eq. (2.140) \( \gamma = \frac{1+\sigma^2}{2(1-\sigma^2)} \tan \phi_{11} \right), \tan \phi_{12}', \) and simplify the equation.

Then the final result of the resonance quality factor is given by

$$Q_{A,p} = \frac{\Omega_A^2 + \sigma^2 - 1}{\alpha_{11} \tan \phi_{11} - \alpha_{12} \tan \phi_{12}'},$$

(2.159)

or

$$Q_{A,p} = \frac{\alpha_{11} - \alpha_{12}}{\alpha_{11} \tan \phi_{11} - \alpha_{12} \tan \phi_{12}'}.$$ 

(2.160)

where

$$\alpha_{11} = \frac{1 + \sigma^2}{1 - \sigma^2} \Omega_A^2 - (1 - \sigma)^2.$$ 

(2.161)

$$\alpha_{12} = 2\sigma \left[ \frac{\Omega_A^2}{1 - \sigma^2} + (1 - \sigma) \right].$$

(2.162)
The value of $\Omega_A$ can be derived by solving eq. (2.124) \( \Omega_A J_0(\Omega_A) / J_1(\Omega_A) - 1 + \sigma = 0 \).

Computational calculations are applied using MATLAB, and the results are presented in Figure 2-10.

![Figure 2-10. The value of $\Omega_A$ with regard to $\sigma$.](image)

In addition, $Q_{B,p}$ can also be derived and one complicated equation was obtained. However, the result does not affect the methodology of loss characterization. This is because the only material property characterized in $k_p$ mode is $s_{12}^E$, which is already covered by $Q_{A,p}$. Therefore, this portion is neglected here.

### 2.2.5 $k_{15}$ mode (shear mode)

The vibration mode with applied electric field perpendicular to the polarization is called shear mode or $k_{15}$ mode. Two boundary conditions are analyzed for the shear mode, as shown in Figure 2-11. When $L<<t$, it is called length shear mode with the boundary condition of constant field, i.e. the electric field $E$ is constant. When $L>>t$, it is called thickness shear mode with the boundary condition of constant induction, i.e. the electric displacement $D$ is constant. [41]
The electromechanical coupling factor is represented as

\[
k_{15}^2 = \frac{d_{15}^2}{\varepsilon_0 \varepsilon_{11}^E s_{55}^E} = \frac{h_{15}^2}{\left(\beta_{11}^S / \varepsilon_0\right) s_{55}^D}. \tag{2.163}
\]

In the case of constant field condition (length shear mode), the admittance expression is given by:

\[
Y = j \omega \varepsilon_0 \varepsilon_{11}^E L w \frac{T}{t} \left[1 - k_{15}^2 + k_{15}^2 \frac{\tan(\omega L / 2 v_{55}^E)}{\omega L / 2 v_{55}^E}\right], \tag{2.164}
\]

Notice that this equation is similar to the admittance expression of \( k_{31} \) mode. Therefore, the results of quality factors are given in the same way by

\[
Q_{A,15}^E = \frac{1}{\tan \phi_{55}^E}; \tag{2.165}
\]

\[
\frac{1}{Q_{B,15}^E} = \frac{1}{Q_{A,15}^E} - \frac{2}{1 + \left(\frac{1}{k_{15}} - k_{15}\right)^2 \Omega_B^2} \left(2 \tan \delta_{15}' - \tan \delta_{11}' - \tan \phi_{55}'\right). \tag{2.166}
\]

Here,
\[ \Omega_B = \frac{\omega_0 L}{2v_{55}^E} = \frac{\omega_0 L}{2v_{55}^E} \sqrt{\rho_{55}^E}. \]  
(2.167)

\[ 1 - k_{15}^2 + k_{15}^2 \tan \frac{\Omega_B}{\Omega_B} = 0. \]  
(2.168)

In the case of constant induction condition (thickness shear mode), the impedance expression is given by:

\[ Z = \frac{1}{j\omega C_d} \left[ 1 - k_{15}^2 \tan \left( \frac{\omega t}{2v_{55}^D} \right) \right], \]  
(2.169)

\[ C_d = \frac{Lw}{t} \epsilon_0 \epsilon_{11}^s. \]  
(2.170)

The equation has the same form as the impedance expression of \( k_t \) mode. Therefore, the results of quality factors are given by

\[ Q_{B,15}^D = \frac{1}{\tan \phi_{55}}, \]  
(2.171)

\[ \frac{1}{Q_{D,15}} = \frac{1}{Q_{B,15}^D} + \frac{2}{k_{15}^2 - 1 + \Omega_A^2 / k_{15}^2} (\tan \delta_{11} + \tan \phi_{55} - 2 \tan \theta_{15}). \]  
(2.172)

Here,

\[ \Omega_A = \frac{\omega_0 t}{2v_{55}^D} = \frac{\omega_0 t}{2} \sqrt{\frac{\rho_{55}^D}{c_{55}^D}}. \]  
(2.173)

\[ \Omega_A = k_{15}^2 \tan \Omega_A. \]  
(2.174)

2.2.6 Summary of quality factors in different vibration modes

(1) \( k_{31} \) mode:
\[ Q_{A,31} = \frac{1}{\tan \phi_{11}}. \]

\[ \frac{1}{Q_{B,31}} = \frac{1}{Q_{A,31}} - \frac{2}{1 + \left( \frac{1}{k_{31}} - k_{31} \right)^2} \left( 2 \tan \theta_{31} - \tan \delta_{33} - \tan \phi_{11} \right). \]

(2) \( k_t \) mode:

\[ Q_{B,t} = \frac{1}{\tan \phi_{33}}. \]

\[ \frac{1}{Q_{A,t}} = \frac{1}{Q_{B,t}} + \frac{2}{k_t^2 - 1 + \Omega_t^2 / k_t^2} (\tan \delta_{33} + \tan \phi_{33} - 2 \tan \theta_{33}). \]

(3) \( k_{33} \) mode:

\[ Q_{B,33} = \frac{1-k_{33}^2}{\tan \phi_{33} - k_{33}^2} \left( 2 \tan \theta_{33} - \tan \delta_{33}' \right), \]

\[ \frac{1}{Q_{A,33}} = \frac{1}{Q_{B,33}} + \frac{2}{k_{33}^2 - 1 + \Omega_{33}^2 / k_{33}^2} \left( 2 \tan \theta_{33} - \tan \delta_{33} - \tan \phi_{33}' \right). \]

(4) \( k_p \) mode:

\[ Q_{A,p} = \frac{\alpha_{11} - \alpha_{12}}{\alpha_{11} \tan \phi_{11} - \alpha_{12} \tan \phi_{12}} \]

\[ \alpha_{11} = \frac{1+\sigma^2}{1-\sigma^2} \Omega_A^2 - (1-\sigma)^2, \]

\[ \alpha_{12} = 2\sigma \left[ \frac{\sigma}{1-\sigma^2} \Omega_A^2 + (1-\sigma) \right]. \]

(5) \( k_{15} \) mode (constant \( E \) – length shear mode):

\[ Q_{A,15}^E = \frac{1}{\tan \phi_{15}}. \]
\[
\frac{1}{Q_{B,15}^E} = \frac{1}{Q_{A,15}^E} - \frac{2}{1 + \left( \frac{1}{k_{15}} - k_{13} \right)^2 \Omega_B^2} (2 \tan \theta_{15}' - \tan \delta_{11}' - \tan \phi_{35}').
\]

(6) \( k_{15} \) mode (constant \( D \) – thickness shear mode):

\[
\frac{1}{Q_{B,15}^D} = \frac{1}{\tan \phi_{35}},
\]

\[
\frac{1}{Q_{A,15}^D} = \frac{2}{k_{15}^2 - 1 + \Omega_A^2 / k_{15}^2} \left( \tan \delta_{11}' + \tan \phi_{35} - 2 \tan \theta_{15}' \right).
\]

According to the analytical solutions, we can explain the discrepancy between \( Q_A \) and \( Q_B \), which is basically determined by \( (2 \tan \theta' - \tan \delta' - \tan \phi') \) or \( (\tan \delta' + \tan \phi' - 2 \tan \theta') \). In experiment \( Q_B \) is always larger than \( Q_A \), which indicates that \( 2 \tan \theta' > (\tan \delta' + \tan \phi') \), or \( (\tan \delta' + \tan \phi') > 2 \tan \theta' \). Therefore, the piezoelectric loss factor plays a significant role in the loss performance of piezoelectric ceramics. If \( \tan \theta' \) is neglected as some researchers did previously, \( Q_A \) will be larger than \( Q_B \). For the special assumption by IEEE Std. that \( Q_A \) and \( Q_B \) are identical, the piezoelectric loss is exactly equal to the average value of elastic loss and dielectric loss.
Chapter 3

Methodology to Determine Loss Factors for Piezoelectric Ceramics

In this chapter, the methodology to characterize the material properties of piezoelectric ceramics will be introduced. First some fundamental relations will be presented, which are the basics for the material characterization. Then the procedures to measure the real parts of material properties will be discussed, followed by our technique to measure and derive the imaginary parts (loss factors).

3.1 Fundamental Relations

As mentioned before, piezoelectric ceramics (poled in the $x_3$ direction) have the crystal symmetry of $\infty mm$ (equivalent to 6mm). There are 20 material properties, which can be summarized by the matrices below. Accordingly, 20 loss factors will be measured or derived for each number.

\[
\begin{bmatrix}
S_{11}^E & S_{12}^E & S_{13}^E & 0 & 0 & 0 \\
S_{12}^E & S_{11}^E & S_{13}^E & 0 & 0 & 0 \\
S_{13}^E & S_{13}^E & S_{33}^E & 0 & 0 & 0 \\
0 & 0 & 0 & S_{55}^E & 0 & 0 \\
0 & 0 & 0 & 0 & S_{66}^E & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
= (\varepsilon^E)^{-1},
\begin{bmatrix}
C_{11}^D & C_{12}^D & C_{13}^D & 0 & 0 & 0 \\
C_{12}^D & C_{11}^D & C_{13}^D & 0 & 0 & 0 \\
C_{13}^D & C_{13}^D & C_{33}^D & 0 & 0 & 0 \\
0 & 0 & 0 & C_{55}^D & 0 & 0 \\
0 & 0 & 0 & 0 & C_{66}^D & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
= (\varepsilon^D)^{-1};
\]

\[s_{66} = 2(s_{11} - s_{12}),
\]

\[c_{66} = (c_{11} - c_{12})/2 .\]

\[
\begin{bmatrix}
\varepsilon_{11}^T & 0 & 0 \\
0 & \varepsilon_{11}^T & 0 \\
0 & 0 & \varepsilon_{33}^T \\
\end{bmatrix}
= (\beta^T)^{-1},
\begin{bmatrix}
\beta_{11}^S & 0 & 0 \\
0 & \beta_{11}^S & 0 \\
0 & 0 & \beta_{33}^S \\
\end{bmatrix}
= (\varepsilon^S)^{-1};
\]
From the inverse matrix relations, some equations are given for the dielectric and elastic properties. Here the superscripts are not included, and parameters in one equation are assumed under the same condition.

\[ \varepsilon_{11} = 1 / \beta_{11}. \]  
\[ \varepsilon_{33} = 1 / \beta_{33}. \]  
\[ c_{11} = \frac{s_{11} s_{33} - (s_{13})^2}{(s_{11} - s_{12}) s_{33} (s_{11} + s_{12}) - 2(s_{13})^2}; \quad s_{11} = \frac{c_{11} c_{33} - (c_{13})^2}{(c_{11} - c_{12}) c_{33} (c_{11} + c_{12}) - 2(c_{13})^2}. \]  
\[ c_{12} = \frac{(s_{13})^2 - s_{12} s_{33}}{(s_{11} - s_{12}) s_{33} (s_{11} + s_{12}) - 2(s_{13})^2}; \quad s_{12} = \frac{(c_{13})^2 - c_{12} c_{33}}{(c_{11} - c_{12}) c_{33} (c_{11} + c_{12}) - 2(c_{13})^2}. \]  
\[ c_{13} = \frac{-s_{13}}{s_{33} (s_{11} + s_{12}) - 2(s_{13})^2}; \quad s_{13} = \frac{-c_{13}}{c_{33} (c_{11} + c_{12}) - 2(c_{13})^2}. \]  
\[ c_{33} = \frac{s_{11} + s_{12}}{s_{33} (s_{11} + s_{12}) - 2(s_{13})^2}; \quad s_{33} = \frac{c_{11} + c_{12}}{c_{33} (c_{11} + c_{12}) - 2(c_{13})^2}. \]  
\[ c_{55} = 1 / s_{55}. \]  

Then let us discuss several constitutive piezoelectric equations in different forms. [33]

\[ S_{ij} = S_{ijkl} T_{kl} + d_{ijkl} E_{k}, \]  
\[ D_i = d_{ikl} T_{kl} + e_{ikl} E_{k}. \]  
\[ S_{ij} = S_{ijkl} T_{kl} + g_{ijkl} D_{k}, \]  
\[ E_i = -g_{ikl} T_{kl} + h_{ikl} D_{k}. \]  
\[ T_{ij} = c_{ijkl} S_{kl} - h_{ijkl} D_{k}. \]
\[ E_i = -h_{id} S_{di} + \beta_{ik}^S D_k. \]  

(3.13)

There are four piezoelectric constants depending on different conditions. The conversion equations are represented as [33]

\[ e_{ip} = d_{iq} c_{qp} = \varepsilon_{ij}^S h_{ij}, \]

(3.14)

\[ d_{ip} = \varepsilon_{ik}^T g_{kp}, \]

(3.15)

\[ g_{ip} = \beta_{ik}^T d_{kp}, \]

(3.16)

\[ h_{ip} = g_{iq} c_{qp}^D. \]

(3.17)

Specifically, for the crystal symmetry of \( \infty \)mm or 6mm the following equations are given.

\[ e_{31} = d_{31} (c_{11}^E + c_{12}^E) + d_{33} c_{13}^E = \varepsilon_{33}^S h_{31}, \]

(3.18)

\[ e_{33} = 2d_{31} c_{13}^E + d_{33} c_{33}^E = \varepsilon_{33}^S h_{33}, \]

(3.19)

\[ e_{15} = d_{15} c_{55}^E = \varepsilon_{11}^S h_{15}. \]

(3.20)

\[ d_{31} = \varepsilon_{13}^T g_{31} = g_{31} / \beta_{13}^T, \]

(3.21)

\[ d_{33} = \varepsilon_{13}^T g_{33} = g_{33} / \beta_{33}^T, \]

(3.22)

\[ d_{15} = \varepsilon_{11}^T g_{15} = g_{15} / \beta_{11}^T. \]

(3.23)

\[ h_{31} = g_{31} (c_{11}^D + c_{12}^D) + g_{33} c_{13}^D, \]

(3.24)

\[ h_{33} = 2g_{31} c_{13}^D + g_{33} c_{33}^D, \]

(3.25)

\[ h_{15} = g_{15} c_{55}^D. \]

(3.26)

Furthermore, the elastic and dielectric properties under different conditions can be converted using the following relations.

\[ c_{pq}^D = c_{pq}^E + e_{qp} h_{pq}, \]

(3.27)

\[ s_{pq}^D = s_{pq}^E - d_{qp} g_{pq}. \]

(3.28)
\[ \varepsilon_{ij}^T = \varepsilon_{ij}^S + d_{ij} \varepsilon_{pq}. \] (3.29)

\[ \beta_{ij}^T = \beta_{ij}^S - g_{ij} h_{pq}. \] (3.30)

Also these relations can be specified for the \( \infty \text{mm} \) or \( 6 \text{mm} \) symmetry as

\[ c_{11}^D = c_{11}^E + e_{31} h_{31}, \] (3.31)

\[ c_{12}^D = c_{12}^E + e_{31} h_{31}, \] (3.32)

\[ c_{13}^D = c_{13}^E + e_{31} h_{33}, \] (3.33)

\[ c_{33}^D = c_{33}^E + e_{33} h_{33} = c_{33}^E / (1 - k_i^2), \] (3.34)

\[ s_{11}^D = s_{11}^E - d_{31} g_{31} = s_{11}^E / (1 - k_{31}^2), \] (3.36)

\[ s_{12}^D = s_{12}^E - d_{31} g_{31} = s_{12}^E - k_{31}^2 s_{11}^E, \] (3.37)

\[ s_{13}^D = s_{13}^E - d_{31} g_{33}, \] (3.38)

\[ s_{33}^D = s_{33}^E - d_{33} g_{33} = s_{33}^E / (1 - k_{33}^2), \] (3.39)

\[ s_{55}^D = s_{55}^E - d_{15} g_{15} = s_{55}^E / (1 - k_{15}^2), \] (3.40)

\[ \varepsilon_{11}^T = \varepsilon_{11}^S + e_{15} e_{15} = \varepsilon_{11}^S / (1 - k_{15}^2), \] (3.41)

\[ \varepsilon_{33}^T = \varepsilon_{33}^S + 2 d_{31} e_{31} + d_{33} e_{33}, \] (3.42)

\[ \beta_{11}^T = \beta_{11}^S - g_{15} h_{15} = \beta_{11}^S / (1 - k_{15}^2), \] (3.43)

\[ \beta_{33}^T = \beta_{33}^S - 2 g_{31} h_{31} - g_{33} h_{33}, \] (3.44)

The equations above are essential in the methodology of material characterization for piezoelectric ceramics.
3.2 Characterization of Real Material Properties

The characterization of real material properties have been standardized for many years. [33, 42, 43] In this portion, a brief introduction is made based on the IEEE Std. The admittance/impedance characterizations of \( k_{31} \), \( k_p \), \( k_t \), \( k_{33} \), and \( k_{15} \) vibration modes are involved, as shown in Figure 3-1.

![Figure 3-1. Sketches of different vibration modes.](image)

3.2.1 \( k_{31} \) mode characterization

The geometry requirements for this mode is \( l >> w \), and \( l >> b \). Practically \( w/l < 0.3 \) and \( w/b > 2 \) should be satisfied.
First the dielectric constant (permittivity) $\varepsilon_{33}^T$ can be obtained by the capacitance $C$ measurement under low frequency. The measurements are best made at a frequency $1/100^{th}$ or less than the lowest resonance frequency of the plate.

$$\varepsilon_{33}^T = \left( \frac{b}{l_w} \right) \frac{C}{\varepsilon_0}.$$  \hspace{1cm} (3.45)

Then make the impedance measurements to get the resonance frequency $f_A$ and antiresonance frequency $f_B$, and derive the following parameters: $k_{31}$, $s_{11}^E$, $s_{11}^D$, and $d_{31}$.

$$\frac{k_{31}^2}{1-k_{31}^2} = \frac{\pi f_B}{2 f_A} \tan \left( \frac{\pi f_B - f_A}{2 f_A} \right).$$ \hspace{1cm} (3.46)

$$s_{11}^E = \frac{1}{4 \rho f_A^2 l^2}.$$ \hspace{1cm} (3.47)

$$s_{11}^D = s_{11}^E (1-k_{31}^2).$$ \hspace{1cm} (3.48)

$$d_{31}^2 = k_{31}^2 s_{11}^E \varepsilon_{33}^T \varepsilon_0.$$ \hspace{1cm} (3.49)

**3.2.2 $k_p$ mode characterization**

The geometry requirements for the thin disk is $a >> b$, and in practice $2a/b > 16$ should be satisfied.

Do the impedance measurements to get the first and second resonance frequencies $f_{A1}$ and $f_{A2}$, and find the Poisson’s ratio $\sigma$ in Table 3-1. [44]

Next the electromechanical coupling factor $k_p$ can be obtained by

$$k_p^2 = \left( \frac{2}{1-\sigma} \right) k_{31}^2.$$ \hspace{1cm} (3.50)

Two elastic compliances can be derived by

$$s_{12}^E = -\sigma s_{11}^E,$$ \hspace{1cm} (3.51)
\[ S_{12}^D = S_{12}^E - S_{12}^E k_{31}^2. \]  

(3.52)

Table 3-1. Relation between \( \sigma \) and ratio of first overtone to fundamental resonant frequencies.

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>0.15</th>
<th>0.16</th>
<th>0.17</th>
<th>0.18</th>
<th>0.19</th>
<th>0.20</th>
<th>0.21</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{A2}/f_{A1} )</td>
<td>2.7482</td>
<td>2.7396</td>
<td>2.7310</td>
<td>2.7225</td>
<td>2.7143</td>
<td>2.7061</td>
<td>2.6981</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.22</td>
<td>0.23</td>
<td>0.24</td>
<td>0.25</td>
<td>0.26</td>
<td>0.27</td>
<td>0.28</td>
</tr>
<tr>
<td>( f_{A2}/f_{A1} )</td>
<td>2.6901</td>
<td>2.6823</td>
<td>2.6746</td>
<td>2.6670</td>
<td>2.6595</td>
<td>2.6521</td>
<td>2.6447</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.29</td>
<td>0.30</td>
<td>0.31</td>
<td>0.32</td>
<td>0.33</td>
<td>0.34</td>
<td>0.35</td>
</tr>
<tr>
<td>( f_{A2}/f_{A1} )</td>
<td>2.6375</td>
<td>2.6304</td>
<td>2.6234</td>
<td>2.6165</td>
<td>2.6095</td>
<td>2.6029</td>
<td>2.5962</td>
</tr>
</tbody>
</table>

3.2.3 \( k_t \) mode characterization

The same size of disk plate could be utilized for the thickness mode characterization. By the impedance measurements to get the resonance frequency \( f_A \) and antiresonance frequency \( f_B \), the following parameters could be derived: \( k_t \), \( c_{33}^D \), and \( c_{33}^E \).

\[
k_t^2 = \frac{\pi}{2} \frac{f_A}{f_B} \tan \left( \frac{\pi}{2} \frac{f_B - f_A}{f_B} \right),
\]

(3.53)

\[
c_{33}^D = 4 \rho f_B^2 b^2.
\]

(3.54)

\[
c_{33}^E = c_{33}^D \left( 1 - k_t^2 \right).
\]

(3.55)
3.2.4 \( k_{33} \) mode characterization

For the \( k_{33} \) mode bar, \((l/w)\) or \((l/b)\) must be larger than 3 in practice. By the impedance measurements, derive the following parameters using the resonance frequency \( f_A \) and antiresonance frequency \( f_B \).

\[
k_{33}^2 = \frac{\pi}{2} \frac{f_A}{f_B} \tan\left( \frac{\pi}{2} \frac{f_B - f_A}{f_B} \right).
\]  
(3.56)

\[
s_{33}^D = \frac{1}{4 \rho f_B^2 l^2}.
\]  
(3.57)

\[
s_{33}^E = \frac{s_{33}^D}{1 - k_{33}^2}.
\]  
(3.58)

\[
d_{33}^2 = k_{33}^2 \left( s_{33}^E \varepsilon_{33}^T \varepsilon_0 \right).
\]  
(3.59)

3.2.5 Thickness shear mode characterization

The geometry requirement for thickness shear mode is \( L, w > 10t \). The dielectric constant (permittivity) \( \varepsilon_{11}^T \) can be obtained by the capacitance \( C \) measurement under low frequency (1/100th or less than the lowest resonance frequency of the plate).

\[
\varepsilon_{11}^T = \left( \frac{t}{Lw} \right) \frac{C}{\varepsilon_0}.
\]  
(3.60)

Then make the impedance measurements to get the resonance frequency \( f_A \) and antiresonance frequency \( f_B \), and derive the following parameters: \( k_{15}, \varepsilon_{11}^S, \varepsilon_{55}^D, s_{55}^D, \) and \( h_{15} \).

\[
k_{15}^2 = \frac{\pi}{2} \frac{f_A}{f_B} \tan\left( \frac{\pi}{2} \frac{f_B - f_A}{f_B} \right).
\]  
(3.61)

\[
\varepsilon_{11}^S = \varepsilon_{11}^T \left( 1 - k_{15}^2 \right).
\]  
(3.62)
\[ c_{ss}^D = \frac{1}{s_{ss}^D} = 4 \rho f_B^2 t^2. \]  
(3.63)

\[ h_{15}^2 = k_{15}^2 \left( \frac{c_{ss}^D}{\varepsilon_{11}^0} \right). \]  
(3.64)

### 3.2.6 Length shear mode characterization

The geometry requirement for length shear mode is \( t, w > 10L \). Measure the resonance frequency \( f_A \), and derive \( s_{ss}^E, c_{ss}^E \), and \( d_{15} \).

\[ s_{ss}^E = \frac{1}{c_{ss}^E} = \frac{1}{4 \rho f_A^2 L^2}. \]  
(3.65)

\[ d_{15}^2 = k_{15}^2 \left( s_{ss}^E \varepsilon_{11}^0 \right). \]  
(3.66)

### 3.2.7 Derive other parameters

First, apply eqs. (3.3) – (3.6) to derive the other eight elastic properties.

\[ c_{33}^E = \frac{s_{11}^E + s_{12}^E}{s_{33}^E (s_{11}^E + s_{12}^E) - 2(s_{13}^E)^2}. \]

Use \( c_{33}^E, s_{11}^E, s_{12}^E \) and \( s_{33}^E \) to calculate \( s_{13}^E \). Then \( c_{11}^E, c_{12}^E \), and \( c_{13}^E \) could be derived by \( s_{11}^E, s_{12}^E, s_{13}^E \), and \( s_{33}^E \).

Similarly,

\[ c_{33}^D = \frac{s_{11}^D + s_{12}^D}{s_{33}^D (s_{11}^D + s_{12}^D) - 2(s_{13}^D)^2}. \]

Using \( c_{33}^D, s_{11}^D, s_{12}^D \) and \( s_{33}^D \), we can obtain \( s_{13}^D \). Then \( c_{11}^D, c_{12}^D \), and \( c_{13}^D \) could be derived by \( s_{11}^D, s_{12}^D, s_{13}^D \), and \( s_{33}^D \).
Next, $\varepsilon_{33}^S$ is calculated by the equation below.

$$
\varepsilon_{33}^S = \varepsilon_{33}^T \left( 1 - \frac{k_{\rho}^2 + k_{33}^2 + 2Ak_{\rho}k_{33}}{1 - A^2} \right), \quad (3.67)
$$

where

$$
A = \sqrt{\frac{2s_{31}^E}{s_{33}^E}} \sqrt{\frac{s_{33}^E}{s_{11}^E + s_{12}^E}}. \quad (3.68)
$$

Notice that $\varepsilon_{33}^S$ can also be measured on a thin $k_1$ mode plate at high frequency, which is well below any ionic resonances but sufficiently removed from high overtone resonance frequencies. However, the accuracy of this measurement is not so good, and therefore the derivation provided above is preferred.

Then $h_{33}$ is given by

$$
h_{33} = k_1 \sqrt{\frac{\varepsilon_{33}^T}{\varepsilon_{33}^S}}. \quad (3.69)
$$

At last, $h_{31}$ can be derived based on eq. (3.42).

$$
h_{31} = \frac{\varepsilon_{33}^T / \varepsilon_{33}^S - 1 - d_{33}h_{33}}{2d_{31}}. \quad (3.70)
$$

### 3.3 Characterization of Imaginary Material Properties – Loss Factor Derivations

As defined in Chapter 2, the intensive and extensive loss factors are represented as

$$
\varepsilon^T = \varepsilon^T (1 - j \tan \delta^*), \quad (3.71)
$$

$$
s^E = s^E (1 - j \tan \phi^*), \quad (3.72)
$$

$$
d^* = d (1 - j \tan \theta^*), \quad (3.73)
$$

$$
\beta^S = \beta^S (1 + j \tan \delta), \quad (3.74)
$$
\[ c^{D*} = c^D (1 + j \tan \phi), \quad (3.75) \]

\[ h^* = h (1 + j \tan \theta). \quad (3.76) \]

For piezoelectric ceramics, 20 loss factors need to be derived for the properties:
\[ \varepsilon_{11}^T, \varepsilon_{33}^T, \beta_{11}^{S*}, \beta_{33}^{S*}, \beta_{11}^{E*}, \beta_{33}^{E*}, \beta_{11}^{D*}, \beta_{33}^{D*}, \varepsilon_{11}^{D*}, \varepsilon_{33}^{D*}, \varepsilon_{55}^{D*}, d_{31}^{*}, d_{33}^{*}, d_{15}^{*}, h_{31}^{*}, h_{33}^{*}, h_{15}^{*}. \]

This methodology is based on the results of quality factor derivations in the previous chapter, and the relations of piezoelectric parameters. Like the characterization of real properties, the impedance measurement of \( k_{31}, k_p, k_t, k_{33}, \) and \( k_{15} \) vibration modes are involved.

### 3.3.1 \( k_{31} \) mode analysis

First, measure the dissipation factor of the capacitance of a \( k_{31} \) mode plate to obtain \( \tan \delta_{33}' \) for \( \varepsilon_{33}^T \). The measurement should be made by a capacitance meter or impedance analyzer at a low frequency, 1/100th or less than the lowest resonance frequency of the plate.

Measure the resonance quality factor \( Q_A \) in \( k_{31} \) mode to get \( \tan \phi_{11}' \) for \( s_{11}^{E*} \).

\[ \tan \phi_{11}' = \frac{1}{Q_{4,31}}. \quad (3.77) \]

Measure the antiresonance quality factor \( Q_B \) in \( k_{31} \) mode to obtain \( \tan \theta_{31}' \) for \( d_{31}^{*} \).

\[ \tan \theta_{31}' = \frac{\tan \delta_{33}' + \tan \phi_{11}'}{2} + \frac{1}{4} \left( \frac{1}{Q_{4,31}} - \frac{1}{Q_{B,31}} \right) \left[ 1 + \left( \frac{1}{k_{31} - k_{31}} \right)^2 \Omega_B^2 \right]. \quad (3.78) \]

### 3.3.2 \( k_p \) mode analysis

Measure the resonance quality factor \( Q_A \) in \( k_p \) mode to derive \( \tan \phi_{12}' \), which is the loss factor of \( s_{12}^{E*} \).
\[ \tan \phi_{12} = \frac{\alpha_{11}}{\alpha_{12}} \tan \phi_{11} \frac{\alpha_{11} - \alpha_{12}}{\alpha_{12} Q_{A,p}}. \]  

Here,

\[ \alpha_{11} = \frac{1 + \sigma^2}{1 - \sigma^2} \Omega_A^2 - (1 - \sigma)^2. \]  

\[ \alpha_{12} = 2\sigma \left[ \frac{\sigma}{1 - \sigma^2} \Omega_A^2 + (1 - \sigma) \right]. \]

\[ \Omega_A J_0(\Omega_A)/J_1(\Omega_A) - 1 + \sigma = 0. \]

### 3.3.3 k_{l} mode analysis

Measure the antiresonance quality factor \( Q_B \) in \( k_l \) mode to get \( \tan \phi_{33} \) for \( c_{33}^{D^*} \).

\[ \tan \phi_{33} = \frac{1}{Q_{B,33}}. \]  

### 3.3.4 k_{33} mode analysis

Measure \( Q_A \) and \( Q_B \) in \( k_{33} \) mode, and solve the following equations to derive \( \tan \phi_{33} \) and \( \tan \theta_{33} \), which are the loss factors for \( s_{33}^{E^*} \) and \( d_{33}^{*} \), respectively.

\[ \tan \phi_{33} - 2k_{33}^2 \tan \theta_{33} = \left( 1 - k_{33}^2 \right) \frac{1}{Q_{B,33}} - k_{33}^2 \tan \delta_{33} \]  

\[ - \tan \phi_{33} + 2 \tan \theta_{33} = \left( \frac{1}{Q_{A,33}} - \frac{1}{Q_{B,33}} \right) k_{33}^2 - 1 + \frac{\Omega_A^2}{k_{33}^2} + \tan \delta_{33}. \]
3.3.5 $k_{15}$ shear mode analysis

Here both thickness shear mode (constant E condition) and length shear mode (constant D condition) need to be characterized.

First, use the capacitance meter or impedance analyzer to measure the dissipation factor of a thickness shear mode plate as the dielectric loss $\tan \delta_{11}'$ for $\epsilon_{11}^T\epsilon^*$. A low frequency, 1/100th or less than the lowest resonance frequency of the plate, should be utilized.

Then measure $Q_A$ in length shear mode (constant E condition) to get $\tan \phi_{55}'$ for $s_{55}E^*$.  

$$\tan \phi_{55}' = \frac{1}{Q_{A,15}^E}.$$  \hspace{1cm} (3.86)

Next measure $Q_B$ in length shear mode (constant E condition) and calculate $\tan \theta_{15}'$ for $d_{15}^*$.  

$$\tan \theta_{15}' = \frac{\tan \delta_{11}' + \tan \phi_{55}'}{2} + \frac{1}{4} \left( \frac{1}{Q_{A,15}^E} - \frac{1}{Q_{B,15}^E} \right) \left[ 1 + \left( \frac{1}{k_{15}} - k_{15} \right)^2 \right].$$  \hspace{1cm} (3.87)

After that the extensive dielectric loss $\tan \delta_{11}$ could be derived, which is the loss factor for $\epsilon_{11}^S\epsilon^*$.  

$$\tan \delta_{11} = \frac{1}{1 - k_{15}^2} \left[ \tan \delta_{11}' - k_{15}^2 (2 \tan \theta_{15}' - \tan \phi_{55}') \right].$$  \hspace{1cm} (3.88)

Measure $Q_B$ in thickness shear mode (constant D condition) to get $\tan \phi_{55}$ for $c_{55}D^*$.  

$$\tan \phi_{55} = \frac{1}{Q_{B,15}^D}.$$  \hspace{1cm} (3.89)

Measure $Q_A$ in thickness shear mode (constant D condition) to derive $\tan \theta_{15}$ for $h_{15}^*$.  

$$\tan \theta_{15} = \frac{\tan \delta_{11} + \tan \phi_{55}}{2} - \frac{1}{4} \left( \frac{1}{Q_{A,15}^D} - \frac{1}{Q_{B,15}^D} \right) \left( k_{15}^2 - 1 + \Omega_\alpha^2 / k_{15}^2 \right).$$  \hspace{1cm} (3.90)
3.3.6 Other derivations

After the loss characterizations of five vibration modes, seven loss factors are still missing: \( \tan \delta_{33} \), \( \tan \theta_{33} \), \( \tan \theta_{31} \), \( \tan \phi_{13} \), \( \tan \phi_{12} \). They are the imaginary parts for \( \beta_{33}^S \) (or \( \varepsilon_{33}^S \)), \( h_{33}^* \), \( h_{31}^* \), \( e_{13}^D \), \( s_{13}^E \), \( c_{11}^D \), \( c_{12}^D \), respectively.

Some relations of the material properties, mentioned in the first portion of this chapter, can be applied to derive these loss factors.

i) First, \( \tan \phi_{13} \) can be derived for \( s_{13}^E \) based on eqs. (3.5) and (3.34):

\[
2 \tan \phi_{13}' = \frac{c_{33}^E s_{33}^E - \tan \phi_{33} - \tan \phi_{33}'}{c_{33}^E s_{33}^E - 1} + \frac{s_{11}^E \tan \phi_{11}' + s_{12}^E \tan \phi_{12}'}{s_{11}^E + s_{12}^E}
\]

\[
- \frac{1}{c_{33}^E s_{33}^E - 1} \left( \tan \delta_{33} + \tan \phi_{33} - 2 \tan \theta_{33} \right) \frac{k_i^2}{1 - k_i^2}.
\]

Consider the quality factor derivation of \( k_i \) mode, eq. (2.98):

\[
\frac{1}{Q_{A,i}} = \frac{1}{Q_{B,i}} + \frac{2}{k_i^2 - 1 + \omega_i^2 / k_i^2} \left( \tan \delta_{33} + \tan \phi_{33} - 2 \tan \theta_{33} \right).
\]

So the result is given for \( \tan \phi_{13} \) by

\[
2 \tan \phi_{13}' = \frac{c_{33}^E s_{33}^E - \tan \phi_{33} - \tan \phi_{33}'}{c_{33}^E s_{33}^E - 1} + \frac{s_{11}^E \tan \phi_{11}' + s_{12}^E \tan \phi_{12}'}{s_{11}^E + s_{12}^E}
\]

\[
- \frac{1}{2 \left( c_{33}^E s_{33}^E - 1 \right)} \left( \frac{1}{Q_{A,i}} - \frac{1}{Q_{B,i}} \right) \frac{k_i^4 - k_i^2 + \omega_i^2}{1 - k_i^2}.
\]

ii) Solve the equations with regard to \( \tan \delta_{33} \), \( \tan \theta_{33} \), \( \tan \theta_{31} \). Three equations are needed for these three unknowns.
Consider first the quality factor result eq. (2.98):

\[
\frac{1}{Q_{A,t}} = \frac{1}{Q_{B,t}} + \frac{2}{k_i^2 - 1 + \Omega_{A}^2 / k_i^2} \left( \tan \delta_{33} + \tan \phi_{33} - 2 \tan \theta_{33} \right).
\]

So the equation below is obtained with regard to \( \tan \delta_{33} \) and \( \tan \theta_{33} \).

\[
- \tan \delta_{33} + 2 \tan \theta_{33} = \tan \phi_{33} - \frac{1}{2} \left( \frac{1}{Q_{A,t}} - \frac{1}{Q_{B,t}} \right) \left( k_i^2 - 1 + \Omega_{A}^2 / k_i^2 \right). \tag{3.93}
\]

Then use eqs. (3.18), (3.19), and (3.42):

\[
e_{31} = e_{33}^S h_{31},
\]

\[
e_{33} = e_{33}^S h_{33},
\]

\[
e_{33}^T = e_{33}^S + 2d_{31}e_{31} + d_{33}e_{33}.
\]

Therefore,

\[
\frac{e_{33}^T}{e_{33}^S} = 1 + 2d_{31}h_{31} + d_{33}h_{33}. \tag{3.94}
\]

Similarly introduce loss factor for each parameter, and compare the imaginary parts. The following equation with regard to \( \tan \delta_{33}, \tan \theta_{33}, \) and \( \tan \theta_{31} \) is obtained.

\[
- \frac{e_{33}^T}{e_{33}^S} \tan \delta_{33} + d_{33} h_{33} \tan \theta_{33} + 2d_{31} h_{31} \tan \theta_{31} =
\]

\[
- \frac{e_{33}^T}{e_{33}^S} \tan \delta_{33} + 2d_{31} h_{31} \tan \theta_{31} + d_{33} h_{33} \tan \theta_{33}. \tag{3.95}
\]

Next we take into account the relation:

\[
s_{13}^E = \frac{d_{33} - e_{33}^S e_{33}^E}{2e_{31}} = \frac{d_{33} - e_{33}^S h_{33} s_{33}^E}{2e_{33}^S h_{31}}, \tag{3.96}
\]

which can be verified by eqs. (3.5), (3.6), (3.18), and (3.19):

\[
s_{13}^E = \frac{-c_{13}^E}{c_{33}^E (c_{11}^E + c_{12}^E) - 2(c_{13}^E)^2},
\]
\[
S_{33}^E = \frac{c_{11}^E + c_{12}^E}{c_{33}^E (c_{11}^E + c_{12}^E) - 2(c_{13}^E)^2},
\]
\[
e_{31} = d_{31} (c_{11}^E + c_{12}^E) + d_{33} c_{33}^E = \varepsilon_{31}^S h_{31},
\]
\[
e_{33} = 2d_{31} c_{13}^E + d_{33} c_{33}^E = \varepsilon_{33}^S h_{33}.
\]

Then by including losses and considering the imaginary parts of eq. (3.96), another equation with regard to \(\tan \delta_{33}, \tan \theta_{33},\) and \(\tan \theta_{31}\) is represented by
\[
\begin{align*}
-d_{33}^E & \tan \delta_{33} + S_{33}^E h_{33} \tan \theta_{33} + 2S_{13}^E h_{31} \tan \theta_{31} = \\
& S_{33}^E h_{33} \tan \phi_{33} + 2S_{13}^E h_{31} \tan \phi_{31} - \frac{d_{33}}{\varepsilon_{33}^S} \tan \theta_{33}.
\end{align*}
\]

\[(3.97)\]

Therefore, we can calculate \(\tan \delta_{33}, \tan \theta_{33},\) and \(\tan \theta_{31},\) using eqs. (3.93), (3.95), and (3.97).

iii) Derive \(\tan \phi_{13}\) using eq. (3.25):
\[
h_{33} = 2g_{31} c_{13}^D + g_{33} c_{33}^D,
\]
i. e.,
\[
h_{33} e_{33}^T = 2d_{31} c_{13}^D + d_{33} c_{33}^D.
\]

Apply the same method to introduce loss factor for each parameter and compare the imaginary parts of two sides. The result is given by
\[
\begin{align*}
-2d_{31} c_{13}^D \tan \phi_{13} &= h_{33} e_{33}^T \tan \delta_{33} - 2d_{31} c_{13}^D \tan \theta_{31} \ \\
+d_{33} c_{33}^D \tan \phi_{33} - d_{33} c_{33}^D \tan \theta_{33} - h_{33} e_{33}^T \tan \theta_{33}.
\end{align*}
\]

\[(3.98)\]

iv) At last two loss factors, \(\tan \phi_{11}\) and \(\tan \phi_{12}\), will be derived. First consider eq. (3.24):
\[
h_{31} - g_{31} (c_{11}^D + c_{12}^D) + g_{33} c_{33}^D = \frac{d_{31} (c_{11}^D + c_{12}^D) + d_{33} c_{33}^D}{\varepsilon_{33}^T}.
\]

Therefore,
\[ c_{11}^D + c_{12}^D = \frac{h_{31}e_{33} - d_{33}e_{13}^D}{d_{31}}. \]  

On the other hand according to eqs. (3.3) and (3.4):

\[ c_{11}^D = \frac{s_{11}^D s_{33}^D - (s_{11}^D)^2}{(s_{11}^D - s_{12}^D)(s_{33}^D(s_{11} + s_{12}^D) - 2(s_{13}^D)^2)}. \]

\[ c_{12}^D = \frac{(s_{13}^D)^2 - s_{12}^D s_{33}^D}{(s_{11}^D - s_{12}^D)(s_{33}^D(s_{11} + s_{12}^D) - 2(s_{13}^D)^2)}. \]

the following relation is given:

\[ c_{11}^D - c_{12}^D = \frac{1}{s_{11}^D - s_{12}^D}. \]

Also taking into account eqs. (3.36) and (3.37),

\[ s_{11}^D = s_{11}^E(1 - k_{31}^2), \]

\[ s_{12}^D = s_{12}^E - k_{31}^2 s_{11}^E, \]

then

\[ s_{11}^D - s_{12}^D = s_{11}^E - s_{12}^E. \]

Therefore,

\[ c_{11}^D - c_{12}^D = \frac{1}{s_{11}^E - s_{12}^E}. \]  

Next integrate losses into eqs. (3.99) and (3.100) and equate the imaginary parts on the left and right sides.

\[ c_{11}^D \tan \phi_{11} + c_{12}^D \tan \phi_{12} = \left( c_{11}^D + c_{12}^D \right) \tan \theta_3 \]

\[ + \frac{h_{31}e_{33}^T (\tan \theta_3 - \tan \delta_{33}) - d_{33}e_{13}^D (\tan \phi_3 - \tan \delta_3)}{d_{31}}. \]  

\[ c_{11}^D \tan \phi_{11} - c_{12}^D \tan \phi_{12} = \frac{s_{11}^E \tan \phi_{11} - s_{12}^E \tan \phi_{12}}{s_{11}^E - s_{12}^E}. \]  

\[ (3.101) \]

\[ (3.102) \]
Finally by solving the equations (3.101) and (3.102), $\tan \phi_{11}$ and $\tan \phi_{12}$ can be obtained, which are the imaginary parts of $c_{11}^{D*}$ and $c_{12}^{D*}$, respectively.

Note that this methodology is effective for materials with loss factors less than 0.1, which can be treated as perturbations in the theory. Most piezoelectric ceramics, even soft ones, satisfy this prerequisite. Besides, the sample must work in the linear region without significant nonlinear effects.
Chapter 4

Finite Element Method (FEM) Simulations

In this chapter, the results of finite element method (FEM) simulations are introduced. The simulations with ATILA FEM software (ISEN) are made to verify the analytical conclusions. In addition, the significant effect of piezoelectric loss factor is confirmed.

Initially, the FEM principles will be briefly introduced. Then the software is used in the loss analysis for piezoelectric materials, and the results will be discussed.

4.1 Principles of FEM

For most practical material or device structures, the problem complexity is too significant to be handled by analytical methods. In order to find the operation performance, like strain and stress level, FEM simulations can be applied. Here the principles will be discussed briefly as below.

Figure 4-1. Schematic representation of the problem domain $\Omega$ with boundary $\Gamma$. 
Consider the piezoelectric domain $\Omega$ pictured in Figure 4-1, [40] within which the displacement field $u$, and electric potential field $\varphi$, are to be determined. The $u$ and $\varphi$ fields satisfy a set of differential equations that represent the physics of the continuum problem considered. Boundary conditions are usually imposed on the domain’s boundary $\Gamma$, to complete the definition of the problem.

The finite element method is an approximation technique for finding solution functions. The method consists of subdividing the domain $\Omega$ into sub-domains, or finite elements, as illustrated in Figure 4-2. [40] These finite elements are interconnected at a finite number of points, or nodes, along their peripheries. The ensemble of finite elements defines the problem mesh. Note that because the subdivision of $\Omega$ into finite elements is arbitrary, there is not a unique mesh for a given problem.

![Figure 4-2. Discretization of the domain $\Omega$.](image)

Within each finite element, the displacement and electric potential fields are uniquely defined by the values they assume at the element nodes. This is achieved by a process of interpolation or weighing in which shape functions are associated with the element. By
combining or assembling these local definitions throughout the whole mesh, we can obtain a trial function for $\Omega$ that depends only on the nodal values of $u$ and $\varphi$, and that is “piecewise” defined over all the interconnected elementary domains. Unlike the domain $\Omega$, these elementary domains may have a simple geometric shape and homogeneous composition, and therefore the analytical solutions are available.

### 4.2 FEM Simulations for Loss Analysis

To verify the analytical solutions, we employed the finite element method (FEM) software ATILA (ver. 5.2.4) commercialized by ISEN (Institute Superieure de l’Electronique et du Numerique, Lille, France) and distributed by Micromechatronics (US), which has the capability to apply three intensive loss factors (dielectric, elastic and piezoelectric losses). Currently the software does not include the loss anisotropy, and the loss factors are therefore same for all directions.

![Figure 4-3. Mesh conditions of FEM simulations for (a) $k_{31}$ mode and (b) $k_{33}$ mode.](image)

In the simulation, a typical hard lead zirconate titanate ceramic (PZT-8) was utilized. For this material, $\tan\varphi' = 0.001$, $\tan\delta' = 0.004$, and different $\tan\theta'$ values were selected. The $k_{31}$ and $k_{33}$ vibration modes are applied in the simulations, which are two of the most practical modes for
piezoelectric device design. The dimensions for $k_{31}$ and $k_{33}$ analysis are $20 \times 3 \times 1$ mm and $2 \times 2 \times 20$ mm, respectively. The mesh conditions are shown in Figure 4-3.

![Figure 4-3](image)

**Figure 4-4.** Quality factors derived by simulation and analytical calculation for (a) $k_{31}$ mode and (b) $k_{33}$ mode.

In simulations the quality factor $Q_A$ or $Q_B$ can be obtained around the resonance or antiresonance peak of the admittance curve, and the theoretical solutions are determined by equations given in Chapter 2. As shown in Figure 4-4, the simulation results match the calculations quite well.
Further, in both cases when $2\tan\theta' > \tan\phi' + \tan\delta'$, $Q_A < Q_B$; when $2\tan\theta' < \tan\phi' + \tan\delta'$, $Q_A > Q_B$; when $2\tan\theta' = \tan\phi' + \tan\delta'$, $Q_A = Q_B$. In experiments, $Q_B$ usually has larger value than $Q_A$ in either $k_{31}$ or $k_{33}$ mode for piezoelectric ceramics, which indicates a relatively high intensive piezoelectric loss factor.

Therefore, the piezoelectric loss factor $\tan\theta'$ is essential in the loss analysis for piezoelectric materials, though it is neglected by most researchers before. Figure 4-5 shows clearly the effect of piezoelectric loss factor $\tan\theta'$ on the admittance curve of the $k_{31}$ sample. As we can see, the antiresonance peak becomes lower and sharper when the piezoelectric loss $\tan\theta'$ increases.

![Figure 4-5. Effect of piezoelectric loss factor on the admittance curve.](image)

Consequently, in order to obtain the admittance simulation results more accurately the inclusion of the piezoelectric loss factor is necessary. Therefore, the characterization of loss factors must be complete to obtain the data.
Chapter 5

Experiment Setup and Results

In this chapter, the experiment setup and results will be explained. First, the principles of accurate impedance measurement are given, including the history of the development of impedance characterization systems. Then the equipment, impedance analyzer, will be introduced with its specifications and use conditions. Next, the loss characterization methodology was utilized for lead zirconate titanate (PZT) based ceramic APC 850. All the real and imaginary material properties were obtained, and orientation-dependence of loss factors was accordingly analyzed. The piezoelectric loss factors were again confirmed to be essential, and the intensive loss factors were verified to be larger than the extensive ones.

5.1 Experiment Setup

5.1.1 Impedance characterization principles and techniques

Basically the impedance characterization is the frequency sweep for the electric properties of the sample, including the voltage, current, and phase information. For piezoelectric materials, how to drive the sample through the sweep is the most essential issue.

In the beginning the most straightforward drive, constant voltage drive, was utilized for the impedance measurement, i.e., the drive voltage is kept constant while the frequency changes step by step. This idea is still popularly used in many research and practical applications. However, constant voltage drive method may lead to serious problem around the resonance frequency, especially when the drive power is high. The significant distortion of the admittance
spectrum may happen when the sample is driven by a constant voltage, due to the nonlinear behavior of elastic compliance at high vibration amplitude. Figure 5-1(a) exemplifies the problem, where the admittance spectrum is skewed with a jump around the maximum admittance point. Thus, we cannot determine the resonance frequency or the mechanical quality factor precisely from these skewed spectra.

Figure 5-1. Admittance curves around the resonance frequency for a rectangular plate of PZT 8. (a) Constant voltage drive. (b) Constant current drive.

In order to eliminate the problem with a constant voltage measurement, we proposed a constant current measurement technique. [45] Since the vibration amplitude is stable by keeping
the drive current constant, the elastic property of the sample will not suffer from the dramatic change when the frequency approaches the resonance. As demonstrated in Figure 5-1(b), the spectra exhibit symmetric curves, from which we can determine the resonance frequency and the mechanical quality factor $Q_A$ precisely.

The constant current measurement method improved the quality of impedance characterization around the resonance, but it was not suitable for the antiresonance. To the contrary, in order to obtain good impedance curves the constant voltage drive should be utilized.

![Plot of Voltage and Current Characteristics](image)

**Figure 5-2.** Voltage and current characteristics in the frequency sweep from the resonance to the antiresonance while keeping the vibration amplitude constant.

Therefore, the key point to obtain good impedance curves is to keep the sample’s mechanical property steady. To clarify this conclusion, we did the analytical analysis for a $k_{31}$ mode piezoelectric plate. In the analysis, the tip vibration velocity was kept as a constant, and then the voltage and current on the sample were calculated from resonance to antiresonance. The results were plotted in Figure 5-2. Here the voltage approaches the minimum point at the
resonance while the current curve is relatively flat throughout the resonance. On the other hand, the current approaches the minimum point at the antiresonance while the voltage is stable around the antiresonance frequency. The analysis confirmed the specific drive method for resonance and antiresonance.

Furthermore, if we want to obtain the full impedance spectrum including both resonance and antiresonance simultaneously, an advanced measurement technique should be applied. The vibration amplitude of the vibrator should be measured as the feedback and kept as a constant. The constant vibration velocity measurement guarantees the stability of mechanical energy of the sample, and therefore can provide the perfect impedance characterization results for any frequency as shown in Figure 5-3. [21]

![Figure 5-3. Impedance curves of a k31 mode plate under a constant vibration velocity sweep conducted across the resonance and antiresonance frequencies.](image)

However, for the simplicity the constant vibration velocity drive was not utilized for the loss characterization due to some limitations of the current setup. For example, the vibration velocity of the shear mode is difficult to detect, and the frequency for several modes are above the limit of the amplifier. Therefore, the impedance analyzer was applied in our measurements, which will be introduced as follows.
5.1.2 Impedance analyzer

Typically, the capacitance and impedance measurements for piezoelectric materials are realized by the commercialized impedance analyzer. Specifically, in our experiments Agilent 4294A Precision Impedance Analyzer was utilized as shown in Figure 5-4.

![Figure 5-4. Agilent 4294A Precision Impedance Analyzer.](image)

With the impedance analyzer, the damped capacitance with dissipation factor (intensive dielectric loss factor) could be measured for a $k_{31}$ mode or thickness shear mode plate under low frequency ($1/100^{th}$ or less than the lowest resonance frequency of the plate). Considering the frequency range of the equipment is 40Hz to 110MHz, it could handle the impedance characterizations for the samples of all different vibration modes. For the measurements of impedance curves, constant voltage drive and constant current drive are available. As illustrated before, constant current is the ideal condition around the resonance peak, while for the antiresonance characterization constant voltage is the better method. The maximum driving voltage this equipment could provide is $1V_{\text{rms}}$, and the maximum current is $20mA_{\text{rms}}$. Therefore, all the measurements were taken under small signals, and the nonlinearity of the material is not pronounced. Besides, for each individual sample the drive current at the resonance and the drive voltage at the antiresonance were adjusted upon the impedance values of resonance and antiresonance, so that the electric input power is equivalent. Accordingly, the vibration amplitude is also approximately equal for the resonance and antiresonance.
5.2 Experiment Results

The loss characterization methodology was applied on the soft PZT based piezoelectric ceramic APC 850 (American Piezo Ceramic International Ltd., Mackeyville, PA, USA), following the procedures explained in Chapter 3. The admittance/impedance characterizations of $k_{31}$, $k_p$, $k_t$, $k_{33}$, and $k_{15}$ vibration modes were carried out, as shown in Figure 5-5. The dimensions of different modes are given by: $k_{31}$ mode – $20 \times 4 \times 1$ mm; $k_{33}$ mode – $20 \times 4 \times 4$ mm; $k_t$ and $k_p$ mode disk – diameter 25 mm, thickness 1 mm; thickness shear mode – $18 \times 18 \times 1$ mm; length shear mode – $1 \times 18 \times 18$ mm.

First, the real material properties were measured upon the IEEE Std. The full property matrices were obtained, which are listed in Table 5-1. Then the loss characterization methodology was utilized, and all the 20 loss factors were measured and derived. Table 5-2 shows the quality...
factor values for all the vibration modes, and in Table 5-3 loss factors are listed. Besides, the impedance curves for different vibration modes are also shown in Figure 5-6.

According to the data all the loss factors obtained here are much less than 0.1, which satisfies the requirement that the loss could be treated as a perturbation. Therefore, the methodology is effective and accurate enough from the viewpoint of the theory.

Note that the error analysis is also included in the results. The result of each parameter is the mean value of several samples with the same condition. The experimental measurement errors were obtained by the standards deviations. For the derived parameters, the error propagation was made using the software GUM Workbench Professional Version 2.4 (Metrodata GmbH, Germany).

For real material properties, the errors are small and very high accuracy is guaranteed. On the other hand, some of the loss factors have quite large uncertainties due to the error propagations during the derivations, especially the extensive piezoelectric losses $\tan \theta_{31}$, $\tan \theta_{33}$, $\tan \theta_{13}$. The methodology will be optimized in the future using some alternative measurements or derivations to reduce these errors. Though the deviations for extensive piezoelectric losses here are very large, the data can still be used for the following analysis on the loss performance of piezoelectric ceramics.

Table 5-1. The real material properties of APC 850 with experimental uncertainties.

<table>
<thead>
<tr>
<th></th>
<th>$s_{11}^E$</th>
<th>$s_{12}^E$</th>
<th>$s_{13}^E$</th>
<th>$s_{33}^E$</th>
<th>$s_{55}^E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Value</td>
<td>16.46</td>
<td>-5.27</td>
<td>-7.58</td>
<td>18.48</td>
<td>47.57</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>0.05</td>
<td>0.02</td>
<td>0.08</td>
<td>0.20</td>
<td>0.18</td>
</tr>
<tr>
<td>Relative Uncertainty</td>
<td>0.3%</td>
<td>0.3%</td>
<td>1.0%</td>
<td>1.1%</td>
<td>0.4%</td>
</tr>
</tbody>
</table>
### c (10^{-10} \text{ N/m}^2)

<table>
<thead>
<tr>
<th></th>
<th>$c_{11}^D$</th>
<th>$c_{12}^D$</th>
<th>$c_{13}^D$</th>
<th>$c_{33}^D$</th>
<th>$c_{55}^D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Value</td>
<td>12.83</td>
<td>8.24</td>
<td>7.51</td>
<td>16.37</td>
<td>4.62</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>0.26</td>
<td>0.26</td>
<td>0.29</td>
<td>0.08</td>
<td>0.05</td>
</tr>
<tr>
<td>Relative Uncertainty</td>
<td>2.0%</td>
<td>3.2%</td>
<td>3.8%</td>
<td>0.5%</td>
<td>1.0%</td>
</tr>
</tbody>
</table>

### $\varepsilon$

<table>
<thead>
<tr>
<th></th>
<th>$\varepsilon_{33}^T$</th>
<th>$\varepsilon_{11}^T$</th>
<th>$\varepsilon_{33}^S$</th>
<th>$\varepsilon_{11}^S$</th>
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<td>2075</td>
<td>1834</td>
<td>988</td>
<td>835</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>15</td>
<td>29</td>
<td>11</td>
<td>16</td>
</tr>
<tr>
<td>Relative Uncertainty</td>
<td>0.7%</td>
<td>1.6%</td>
<td>1.1%</td>
<td>1.9%</td>
</tr>
</tbody>
</table>

### d (10^{-12} \text{ C/N or m/V})

<table>
<thead>
<tr>
<th></th>
<th>d$_{31}$</th>
<th>d$_{33}$</th>
<th>d$_{15}$</th>
<th>h$_{31}$</th>
<th>h$_{33}$</th>
<th>h$_{15}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Value</td>
<td>-196</td>
<td>416</td>
<td>649</td>
<td>Mean Value</td>
<td>-5.10</td>
<td>21.94</td>
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<tr>
<td>Uncertainty</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>Uncertainty</td>
<td>0.43</td>
<td>0.16</td>
</tr>
<tr>
<td>Relative Uncertainty</td>
<td>1.0%</td>
<td>0.7%</td>
<td>0.9%</td>
<td>Relative Uncertainty</td>
<td>8.4%</td>
<td>0.7%</td>
</tr>
</tbody>
</table>
Table 5-2. The quality factors of APC 850 with experimental uncertainties.

<table>
<thead>
<tr>
<th></th>
<th>$Q_{A,31}$</th>
<th>$Q_{B,31}$</th>
<th>$Q_{A,p}$</th>
<th>$Q_{B,p}$</th>
<th>$Q_{A,t}$</th>
<th>$Q_{B,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Value</td>
<td>91.27</td>
<td>103.8</td>
<td>86.44</td>
<td>156.08</td>
<td>130.2</td>
<td>231.0</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>0.58</td>
<td>1.1</td>
<td>0.61</td>
<td>1.49</td>
<td>6.7</td>
<td>4.0</td>
</tr>
<tr>
<td>Relative Uncertainty</td>
<td>0.6%</td>
<td>1.1%</td>
<td>0.7%</td>
<td>1.0%</td>
<td>5.2%</td>
<td>1.7%</td>
</tr>
<tr>
<td></td>
<td>$Q_{A,33}$</td>
<td>$Q_{B,33}$</td>
<td>$Q_{A,15}^D$</td>
<td>$Q_{B,15}^D$</td>
<td>$Q_{A,15}^E$</td>
<td>$Q_{B15}^E$</td>
</tr>
<tr>
<td>Mean Value</td>
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<td>27.73</td>
<td>67.1</td>
<td>43.0</td>
<td>106.1</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>1.9</td>
<td>13.0</td>
<td>0.96</td>
<td>1.4</td>
<td>4.1</td>
<td>1.5</td>
</tr>
<tr>
<td>Relative Uncertainty</td>
<td>2.3%</td>
<td>6.3%</td>
<td>3.5%</td>
<td>2.1%</td>
<td>9.5%</td>
<td>1.5%</td>
</tr>
</tbody>
</table>

Table 5-3. The loss factors of APC 850 with experimental uncertainties.

<table>
<thead>
<tr>
<th></th>
<th>$\tan \phi_{11}'$</th>
<th>$\tan \phi_{12}'$</th>
<th>$\tan \phi_{13}'$</th>
<th>$\tan \phi_{33}'$</th>
<th>$\tan \phi_{55}'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.01096</td>
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<td>0.01507</td>
<td>0.01325</td>
<td>0.0233</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>0.00007</td>
<td>0.0003</td>
<td>0.00034</td>
<td>0.00033</td>
<td>0.0022</td>
</tr>
<tr>
<td>Relative</td>
<td>0.6%</td>
<td>3.2%</td>
<td>2.2%</td>
<td>2.5%</td>
<td>9.6%</td>
</tr>
<tr>
<td></td>
<td>$\tan \phi_{11}$</td>
<td>$\tan \phi_{12}$</td>
<td>$\tan \phi_{13}$</td>
<td>$\tan \phi_{33}$</td>
<td>$\tan \phi_{55}$</td>
</tr>
<tr>
<td>Mean</td>
<td>0.0105</td>
<td>0.0104</td>
<td>0.0076</td>
<td>0.00433</td>
<td>0.0149</td>
</tr>
<tr>
<td>Uncertainty</td>
<td>0.0018</td>
<td>0.0028</td>
<td>0.0013</td>
<td>0.00008</td>
<td>0.0003</td>
</tr>
<tr>
<td>Relative</td>
<td>17%</td>
<td>28%</td>
<td>17%</td>
<td>1.7%</td>
<td>2.1%</td>
</tr>
<tr>
<td></td>
<td>$\tan\delta_{33}'$</td>
<td>$\tan\delta_{11}'$</td>
<td>$\tan\delta_{33}$</td>
<td>$\tan\delta_{11}$</td>
<td></td>
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<td>----------------</td>
<td>---------------------</td>
<td>---------------------</td>
<td>---------------------</td>
<td>---------------------</td>
<td></td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>0.0143</td>
<td>0.0176</td>
<td>0.0058</td>
<td>0.0092</td>
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</tr>
<tr>
<td><strong>Uncertainty</strong></td>
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<td>0.0004</td>
<td>0.0011</td>
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<tr>
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<td>25%</td>
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</tr>
<tr>
<td></td>
<td>$\tan\theta_{11}'$</td>
<td>$\tan\theta_{33}'$</td>
<td>$\tan\theta_{15}'$</td>
<td>$\tan\theta_{31}$</td>
<td>$\tan\theta_{13}$</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td>0.0184</td>
<td>0.0178</td>
<td>0.0296</td>
<td>0.0133</td>
<td>0.0004</td>
</tr>
<tr>
<td><strong>Uncertainty</strong></td>
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<td>0.0004</td>
<td>0.0026</td>
<td>0.0081</td>
<td>0.0004</td>
</tr>
<tr>
<td><strong>Relative</strong></td>
<td>3.2%</td>
<td>2.1%</td>
<td>8.8%</td>
<td>61%</td>
<td>100%</td>
</tr>
</tbody>
</table>

According to the data, some conclusions are made as follows.

First, the piezoelectric loss factor $\tan\theta'$ is essential and comparable with elastic and dielectric losses. However, this factor was neglected by most previous researchers, and IEEE Std. assumed this loss to be the average value of elastic and dielectric losses so that the resonance quality factor and antiresonance quality factor are identical, which is a serious problem that needs to be corrected.

Next all the 20 loss factors have been derived from the impedance characterizations of $k_{31}$, $k_r$, $k_{33}$, $k_p$, and $k_{15}$ modes, and losses are orientation dependent. Therefore, the characterizations of all the six vibration modes are necessary in order to cover all the orientations. For other crystal symmetries, some other modes should be involved.

What’s more, the data shows that the intensive losses are generally larger than the extensive ones. This result is expected because constant strain condition and constant electric displacement condition usually provide less loss, comparing with constant stress condition and constant electric field condition.
Figure 5-6. Experimental impedance curves of different vibration modes.
Chapter 6

Summary

In summary, a comprehensive study on the loss phenomenology of piezoelectrics was made theoretically and experimentally, and one novel methodology was proposed to derive the loss factors for piezoelectric ceramics. In this chapter, some conclusions will be given and the future work will be discussed.

6.1 Conclusions

We introduced a new loss phenomenology and innovative measuring method based on the theory derived at first. Quality factors at resonance and antiresonance for the $k_{31}$, $k_p$, $k$, $k_{33}$, and $k_{15}$ vibration modes are derived theoretically from the impedance expressions with first order approximations, and the methodology for determining loss factors in various orientations is proposed, which is accurate and simple comparing with the conventional measurement technique. We focus on materials with $\infty$mm (equivalent to 6mm) crystal symmetry for deriving the loss factors of a polycrystalline piezoelectric ceramic, and all the 20 losses could be obtained from the measurements and derivations. The theoretical requirement for the methodology is that the material has small loss factors, practically less than 0.1, which is satisfied by most piezoelectric ceramics. Also the sample must work in the linear region without significant nonlinear behavior.

In addition, we proposed the experimental methods for measuring mechanical quality factors $Q_A$ and $Q_B$ at the resonance and antiresonance modes accurately. The constant current drive should be utilized for the resonance measurement, while the constant voltage drive is a better option when characterizing the antiresonance peak. Furthermore, in order to obtain a
continuous and stable admittance/impedance spectrum the constant vibration velocity measurement should be applied.

The resonance quality factor $Q_A$ and the antiresonance quality factor $Q_B$ were compared in experiment. Results showed that under all vibration modes the piezoelectric sample driven in the antiresonance mode has significantly lower losses and higher quality factors than those in the resonance mode. This conclusion indicates that the antiresonance drive may be a promising alternative technique for the conventional resonance drive method. Besides, the piezoelectric sample or device possesses high voltage and low current while running at the antiresonance, and this drive condition is easier to be realized. Therefore, applying the antiresonance drive may simplify the power supply and reduce the cost for piezoelectric devices.

The FEM simulations with ATILA software were made, and the simulation results verified the analytical solutions quite well. It was shown that the key factor to determine whether $Q_B$ is larger than $Q_A$ is the term $(2\tan\theta' - \tan\phi' - \tan\delta')$ or $(\tan\phi + \tan\delta - 2\tan\theta)$. This term should be positive, leading to the higher $Q_B$ which matches the experimental results. Therefore, the significant impact of the piezoelectric loss factor $\tan\theta'$ was confirmed. Also proved in the later experiments, the piezoelectric loss factor $\tan\theta'$ is essential and comparable with elastic and dielectric losses. However, this factor was neglected by most previous researchers, and IEEE Std. assumed this loss to be the average value of elastic and dielectric losses so that the resonance quality factor and antiresonance quality factor are identical, which is a serious problem that needs to be corrected.

The methodology was applied on soft PZT ceramic APC 850, and all the real and imaginary material properties were obtained. The orientation-dependence of loss factors was confirmed, and the intensive losses are verified to be larger than the extensive ones. The complete loss characterization is very meaningful for the future FEM simulations, since the accurate imaginary part can be added into the software for each parameter. Therefore, a better accuracy
will be achieved for the simulation, especially in the impedance and thermal analysis. Moreover, this technique could be applied for the high power characterization, which may clarify various materials’ loss/heat generation performances. Also based on the methodology a principle for preparing high power density piezoelectric material may be developed.

6.2 Future Work

First, the quality factor measurement technique still needs to be improved. The details of the characterization will be optimized, taking into account the electrode effects, drive conditions, etc. The quality factor measurements of some critical modes, like $k_t$ and thickness shear modes, are difficult to proceed due to a lot of noises around the peak, especially for the hard ceramics. Specific electrode shape design may suppress the noise and provide better data.

After accomplishing the methodology, it will be submitted to IEEE as an important supplement to the material characterization techniques. We will also collaborate with the FEM software company to integrate the imaginary parts for the material properties, in order to improve the accuracy of simulation results.

The methodology will be extended to other materials, e.g. ferroelectric single crystals. Piezoelectric single crystals, such as Pb(Mg$_{1/3}$Nb$_{2/3}$)O$_3$-PbTiO$_3$, exhibit very large electromechanical coupling coefficient, high piezoelectric constant, and high operational temperature. They are the materials of the next generation for the applications with large bandwidth and high sensitivity. [46, 47, 48, 49, 50] The loss characterization for single crystals is essential to clarify the performance of heat generation. In this case the crystal symmetry is different from piezoelectric ceramic, and therefore the derivations need to be modified and more vibration modes will be included. In addition, the loss performance of piezoelectric materials at high temperature is also interesting, which may be analyzed by the characterization of quartz.
Furthermore, the loss mechanisms for piezoelectric materials need to be clarified, and the physical origins of elastic, dielectric, and piezoelectric loss should be found. By collecting enough loss data for different materials, the microscopic analysis will be made in the future. Also this technique will be applied for the high power characterizations of piezoelectrics, and the development of high power density piezoelectric materials.
Appendix

MATLAB Code for Loss Factor Derivations

function l = piezo_losses ()

% APC 850: loss factors obtained from the quality factor characterizations.
% Loss factors: p means ‘prime’ (intensive losses); fi – elastic loss, dt – dielectric losses, th–
piezoelectric loss.
fi11p=0.01096; fi12p=0.0095; fi33p=0.01325; fi33=0.00433; fi13p=0.0076;

% Real properties of APC 850.
% Elastic compliance under constant electric field, and elastic stiffness under constant electric
% displacement.
s11E=16.46e-12; s12E=-5.27e-12; s13E=-7.58e-12; s33E=18.48e-12;
c11D=12.83e+10; c12D=8.24e+10; c13D=7.51e+10; c33D=16.37e+10;

% Solve the following 3 equations to get tanδ33, tanθ33, tanθ31.
% A*X=B. Unknowns: X=[x1; x2; x3]. x1—dt33, x2—th33, x3—th31.

% Eq 1: a11*x1 +a12*x2+a13*x3 = b1.
\[a11=-1; \, a12=2; \, a13=0;\]

% Data of kt mode: electromechanical coupling factor, quality factors.
\[kt=0.507; \, Qa=130.2; \, Qb=231;\]
\[
str1=['\tan(x)*',num2str(kt^2),'-x'];
\]
\[fun1=inline(str1);\]
\[Omega = \text{fsolve} (\text{fun1}, 1.1, \text{optimset}('\text{Display}', 'off'));\]
\[b1 = 1/Qb - 1/2*(1/Qa-1/Qb)*(kt^2-1+Omega^2/kt^2);\]

% Eq 2: a21*x1 +a22*x2+a23*x3 = b2.
\[a21=-e33T/e33S; \, a22=d33*h33; \, a23=2*d31*h31;\]
\[b2=2*d31*h31*th31p - e33T/e33S*dt33p + d33*h33*th33p;\]

% Eq 3: a31*x1 +a32*x2+a33*x3 = b3.
\[fi13p=0.01446;\]
\[a31=-d33/e33S; \, a32=h33*s33E; \, a33=2*s13E*h31;\]
\[b3= h33*s33E*fi33p+2*s13E*h31*fi13p-d33/e33S*th33p;\]

% Matrix calculation.
\[A= [a11,a12,a13;a21,a22,a23; a31,a32,a33];\]
\[B=[b1;b2;b3];\]
\[X=A^{-1}*B;\]
% Calculate tanφ11 and tanφ12.

C1 = (c11D+c12D)*th31p + ( h31*e33T*(th31-dt33p) - d33*c13D*(-th33p+fi13) ) / d31;

C2 = (s11E*fi11p-s12E*fi12p) / (s11E-s12E)^2;

fi11 = (C1+C2) / (2*c11D);

fi12 = (C1-C2) / (2*c12D);

% Show all the results.

l = [X; fi11; fi12];
Bibliography


Curriculum Vitae

Yuan Zhuang

Education

Research
• Loss characterization of piezoelectric materials.
• High power characterization of piezoelectric materials.
• Design and control of piezoelectric ultrasonic motor and transducer design and control.
• Piezoelectric single crystal transformer design and application.

Awards
• Melvin P. Bloom Memorial Outstanding Doctoral Research Award in Electrical Engineering, The Pennsylvania State University, April 2011.
• Student Poster Competition Award Winner, 2011 International Workshop on Acoustic Transduction Materials and Devices, May 2011.

Publications