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ZETA FUNCTIONS OF COMPLEXES FROM PGSP(4)

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ABSTRACT

In this thesis we study the zeta functions arising from PGSp(4) over a nonarchimedean local field. In this case, the complexes have dimension two, like PGL(3). However, the vertices are distinguished as special and nonspecial vertices, unlike the case of PGL(3). We define the (edge) zeta function as the counting function of the number of tailless closed geodesics of all type-one or type-two edges, which has a closed form expression in terms of parahoric Hecke operators. The main result shows that the zeta function satisfies a zeta identity involving the Euler characteristic of the complex, the characteristic polynomial of the recurrence relations of the Hecke algebra, the Iwahori-Hecke operator and the number of special and nonspecial vertices. Moreover, we study the operators on nonspecial vertices.

Contents

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Chapter 1

Introduction

In the 1960′s, Ihara [4] first defined a zeta function in group theoretical terms, which was interpreted geometrically as a zeta function of a finite quotient of the Bruhat-Tits tree by Serre [13]. Let F be a non-archimedean local field with q elements in its residue field and π a uniformizer. Let X_Γ be the finite quotient of the Bruhat-Tits tree associated to $\mathrm{PGL}(2, F)$ modulo a discrete co-compact torsion-free subgroup Γ. The zeta function counts the number of closed geodesics in X_Γ, defined as

$$Z(X_\Gamma, u) = \prod_{[C]} (1 - u^{l([C])})^{-1},$$

where the product is over equivalence classes $[C]$ of primitive closed tailless geodesics C, and $l([C])$ is the length of a geodesic in $[C]$. The graph X_Γ is $(q + 1)$-regular. The following is the main result of Ihara's zeta function:

Theorem 1.1. *Suppose X_Γ has the vertex set V and the edge set E. Then* *its zeta function is a rational function of the form*

$$Z(X_\Gamma, u) = \frac{(1-u^2)^{\chi(X_\Gamma)}}{det(I - Au + qu^2 I)},$$

where $\chi(X_\Gamma) = |V| - |E|$ *is the Euler characteristic of* X_Γ *and* A *is the adjacency matrix of* X_Γ.

In recent years, there have been several attempts to define zeta functions on finite complexes of the Bruhat-Tits buildings associated to higher rank groups. However, no closed form expression was obtained until Kang-Li [5] first gave a closed form expression for PGL$(3, F)$ in 2008. Let Γ be a discrete, cocompact and torsion-free subgroup of PGL$(3, F)$. As before, let X_Γ be the finite quotient of the Bruhat-Tits building associated to PGL$(3, F)$ by Γ. There are two types of edges in X_Γ, called type-one edges and type-two edges. The edge zeta function counts the number of closed one-dimensional geodesics of all type-one edges and all type-two edges in X_Γ, defined as

$$Z_1(X_\Gamma, u) = \prod_{[C]} (1 - u^{l([C])})^{-1},$$

where the product is over equivalence classes $[C]$ of primitive closed tailless one-dimensional geodesics using one type of edges, and $l([C])$ is the algebraic length of a geodesic in $[C]$. The following is the main theorem of [5]:

Theorem 1.2. *Assume* $ord_\pi det\, \Gamma \subseteq 3Z$. *Then* $Z_1(X_\Gamma)$ *is a rational func-*

tion and it satisfies the zeta identity

$$Z_1(X_\Gamma, u) = \frac{(1-u^3)^{\chi(X_\Gamma)}}{det(I - A_1 u + qA_2 u^2 - q^3 u^3 I) det(I + L_B u)},$$

where $\chi(X_\Gamma)$ *is the Euler characteristic of* X_Γ*,* A_1 *and* A_2 *are the two Hecke operators which generate the Hecke algebra, and* L_B *is the adjacency operator on chambers.*

This theorem was proved by two different methods: the combinatorial approach in [5] and the representation-theoretic approach in [6].

The purpose of this thesis is to obtain the zeta identity of PGSp$(4, F)$, which also has rank two. In Chapter 2, we introduce the Bruhat-Tits building $\triangle_4$ associated to PGSp$(4, F)$. The group PGSp$(4, F)$ has two maximal compact subgroups $K =$PGSp$(4, O_F)$ and

$$P_{02} = \{g \in G : g \in \begin{pmatrix} O_F & O_F & O_F & \pi^{-1}O_F \\ \pi O_F & O_F & O_F & O_F \\ \pi O_F & O_F & O_F & O_F \\ \pi O_F & \pi O_F & \pi O_F & O_F \end{pmatrix}, det(g) \in F^*\}$$

up to conjugacy [11]. The special vertices of $\triangle_4$ are parameterized by G/K. A special vertex is called primitive if ord_π of the determinant of the coset representative is in $4Z$. There exists an involution $\tau = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \end{pmatrix}$ between the primitive vertices and the nonprimitive special vertices. Let P'_{02} be the group generated by τ and P_{02}. The remaining vertices, called nonspecial, are parameterized by G/P'_{02}. There are two kinds of directed edges. Let E_1 be the stabilizer of the directed type-one edge from vertex

K to τK in G. Let E_2 be the stabilizer of the directed type-two edge from vertex K to P'_{02} in G. Let I be the stabilizer of the directed chamber connecting vertices K, τK and P'_{02}. The directed type-one edges, type-two edges and the directed chambers are parameterized by G/E_1, G/E_2, and G/I respectively.

In Chapter 3, we introduce the operators on special vertices, edges, and chambers of the finite quotient X_Γ of $\triangle_4$. The operators A_1 and A_2 on special vertices are the two Hecke operators which generate the Hecke algebra. The operator L_{E_1} is an adjacency operator which sends a type-one edge to its type-one neighbors. The operator L_{E_2} sends a type-two edge to so called type-two neighbors. The directed chamber operator L_I acting twice is related to L_{E_2}. We define the (edge) zeta function of X_Γ as

$$Z(X_\Gamma, u) = \prod_{[C]} (1 - u^{l([C])})^{-1},$$

where the product is over equivalence classes $[C]$ of primitive closed geodesics of all type-one edges or all type-two edges, and $l([C])$ is the algebraic length of a geodesic in $[C]$. Here

$$\prod_{[C]:type\ i} (1 - u^{l([C])})^{-1} = \frac{1}{det(I - L_{E_i} u^i)}.$$

Therefore, we obtain

$$Z(X_\Gamma, u) = det(I - L_{E_1} u)^{-1} det(I - L_{E_2} u^2)^{-1}.$$

The main theorem of the thesis is the following zeta identity.

Theorem 1.3. [2] *Let Γ be a discrete, co-compact and torsion-free subgroup of G such that $ord_\pi det\Gamma \subseteq 4Z$. Then $Z(X_\Gamma, u)$ is a rational function satisfying the zeta identity*

$$Z(X_\Gamma, u) = \frac{(1-u^2)^{\chi(X_\Gamma)}(1-q^2u^2)^{-(q^2-1)m}}{det(I - A_1 u + qA_2 u^2 - q^3 A_1 u^3 + q^6 I u^4) det(I - L_I u)},$$

where $\chi(X_\Gamma)$ is the Euler characteristic of X_Γ, m is the number of primitive vertices in X_Γ, A_1 and A_2 are the two Hecke operators which generate the Hecke algebra, L_I is the operator on directed chambers.

In Chapter 4, we prove the main theorem by computing the eigenvalues of each operator via each unitary representation of PGSp$(4, F)$, as in the proof of the zeta identity of PGL$(3, F)$.

Ramanujan graphs/complexes from PGL(n, F) are regular graphs/complexes such that the nontrivial eigenvalues of adjacency operators on graphs/complexes fall into the spectrum of those of their universal cover, the Bruhat-Tits building attached to PGL(n, F) [8]. Ramanujan graphs/complexes are spectrally extremal. For the case of PGL(2) and PGL(3), X_Γ is Ramanujan if and only if its zeta function satisfies the Riemann Hypothesis. In the case of PGSp(4), the complex X_Γ is called Ramanujan if the nontrivial irreducible unitary representations occurring in $L^2(\Gamma \setminus G/I)$ are tempered. By observing the eigenvalues of each operator on each representation space, we get equivalent conditions for X_Γ to be Ramanujan.

Corollary 1.1. [2] *The following statements are equivalent:*

(1) X_Γ *is Ramanujan.*

(2) *The nontrivial zeros of* $det(I - L_{E_2}u)$ *do not have absolute values* q^{-3} *or* $q^{-\frac{5}{2}}$. *When the nontrivial zeros of* $det(I - L_{E_2}u)$ *have absolute value* q^{-2}, *the corresponding eigenspace must contain* K*-invariant vectors.*

(3) *The nontrivial zeros of* $det(I - L_I u)$ *do not have absolute values* $q^{-\frac{3}{2}}$ *or* $q^{-\frac{5}{4}}$. *When the nontrivial zeros of* $det(I - L_I u)$ *are equal to* $\pm q^{-1}i$, *the corresponding eigenspace must contain* K*-invariant vectors. When the nontrivial zeros of* $det(I - L_I u)$ *are equal to* $\pm q^{-1}$, *the corresponding eigenspace must contain* E_1*-invariant vectors.*

The operators on vertices appearing in the zeta identity act on special vertices. It is natural to explore an identity involving operators acting on nonspecial vertices. Chapter 5 is devoted to this. We study the irreducible representation spaces which contain $P'_{02}-$invariant vectors. And in a similar way, we define operators on nonspecial vertices, edges and chambers. However, since $P'_{02}-$invariant spaces have lower dimensions, we could achieve an identity with only trivial eigenvalues left after the cancelation with eigenvalues of other operators.

Chapter 2

Building Associated to PSGp

2.1 The group $\mathbf{GSp}(4, F)$

Let F be a nonarchimedean local field with q elements in its residue field k. Let O_F be the ring of integers and π a uniformizer of F. Let V be a $4-$dimensional vector space over F, endowed with the alternating binear form $< \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} >= a_1b_4 + a_2b_3 - a_3b_2 - a_4b_1$. Let $G =$GSp$(4, F)$ be the general symplectic group, defined as $G = \{g \in GL(V) :< gv, gw >= \lambda(g) < v, w >, \forall v, w \in V, \lambda(g) \in F^*\}$. With respect to the standard basis of V, the elements in G can be expressed as 4×4 matrices, satisfying $G = \{g \in GL_4(F) : g^t \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} g = \lambda(g) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \lambda(g) \in F^*\}$. The group G has two maximal compact subgroups $K =$GSp$(4, O_F)$ and $P_{02} = \{g \in G : g \in \begin{pmatrix} O_F & O_F & O_F & \pi^{-1}O_F \\ \pi O_F & O_F & O_F & O_F \\ \pi O_F & O_F & O_F & O_F \\ \pi O_F & \pi O_F & \pi O_F & O_F \end{pmatrix}, det(g) \in F^*\}$ up to conjugacy [11].

2.2 The affine building $\triangle_4$ of $\mathbf{PGSp}(4, F)$

We say $\{e_1, e_2, f_1, f_2\} \subset V$ forms a symplectic basis if they satisfy $< e_1, f_2 >=< e_2, f_1 >= 1$, $< e_i, e_j >=< e_i, f_j >=< f_i, f_j >= 0$ otherwise. Let L be a rank 4 lattice in V, $L = e_1O_F + e_2O_F + f_1O_F + f_2O_F$. Two lattices L and L' are called equivalent if $L' = kL, k \in F^*$. Denote by $[L]$ the equivalence class of L. We say $[L_0]$ is primitive if it contains a L_0, which has a symplectic basis. The equivalence class $[L]$ is called adjacent to a primitive $[L_0]$, if there exist $L \in [L]$, $L_0 \in [L_0]$, such that $\pi L_0 \subset L \subset L_0$ and $< L, L >= \pi O_F$.

Definition 2.1. *The vertices of the building $\triangle_4$ are the primitive equivalence classes of lattices and the equivalence classes of lattices that are adjacent to the primitive ones.*

Example 2.1. *Write $I_4 = (e_1, e_2, f_1, f_2)$.*

Let $L_0 = e_1O_F + e_2O_F + f_1O_F + f_2O_F$, $[L_0]$ is a primitive vertex.

Let $L_1 = e_1O_F + e_2O_F + \pi f_1O_F + \pi f_2O_F$, $[L_1]$ is adjacent to $[L_0]$.

Let $L' = \pi e_1O_F + e_2O_F + f_1O_F + \pi f_2O_F$, $[L']$ is not a primitive vertex since it does not contain a lattice with a symplectic basis. And $[L']$ is not adjacent to $[L_0]$ since $< L', L' >\neq \pi O_F$.

Definition 2.2. *The chambers (hyperedges) of the building $\triangle_4$ are the ver-*

tices $\{[L_0], [L_1], [L_2]\}$ *satisfying:* $[L_0]$ *is primitive,* $[L_1]$, $[L_2]$ *are adjacent to* $[L_0]$. *And there exist* L_0, L_1, L_2, *such that* $\pi L_0 \subset L_2 \subset L_1 \subset L_0$.

Example 2.2. *Let* $L_0 = e_1 O_F + e_2 O_F + f_1 O_F + f_2 O_F$.

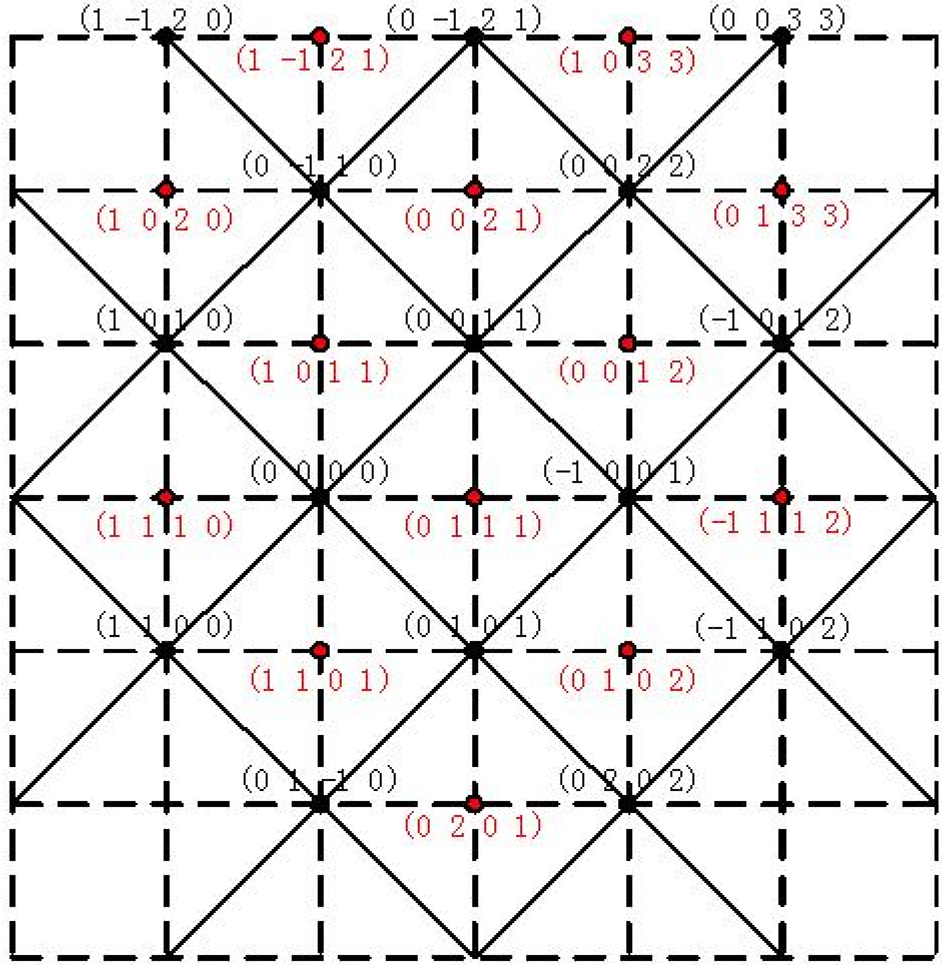
Let $L_1 = e_1 O_F + e_2 O_F + \pi f_1 O_F + \pi f_2 O_F$.

Let $L_2 = e_1 O_F + \pi e_2 O_F + \pi f_1 O_F + \pi f_2 O_F$.

$\{[L_0], [L_1], [L_2]\}$ *forms a chamber.*

An apartment of the building $\triangle_4$ is defined to be the set of all vertices of the form $[\pi^{a_1} e_1 O_F + \pi^{a_2} e_2 O_F + \pi^{b_1} f_1 O_F + \pi^{b_2} f_2 O_F]$, where $\{e_1, e_2, f_1, f_2\}$ is a symplectic basis. Any vertex $[L]$ can be represented by $L = \pi^{a_1} e_1 O_F + \pi^{a_2} e_2 O_F + \pi^{b_1} f_1 O_F + \pi^{b_2} f_2 O_F$, where $\{e_1, e_2, f_1, f_2\}$ is a symplectic basis. If $a_1 + b_2 = a_2 + b_1$, $[L]$ is called a special vertex. Otherwise, it is called a nonspecial vertex.

Example 2.3. *The vertex* $[L]$, *where* $L = \pi^{a_1} e_1 O_F + \pi^{a_2} e_2 O_F + \pi^{b_1} f_1 O_F + \pi^{b_2} f_2 O_F$, *is represented by* (a_1, a_2, b_1, b_2). *The following is an example of an* *apartment*



Proposition 2.1. [14] *The group G acts transitively on special vertices.*

Since the center $Z(G)$ of G acts trivially on the building, we may replace G by $G/Z(G)$ =PGSp$(4, F)$. In Example 2.1, L_0 is a primitive and special vertex whose stabilizer is K. Since G acts transitively on special vertices, the special vertices can be expressed as gK, $g \in G$.

In Example 2.2, L_2 is a nonspecial vertex whose stabilizer is P_{02} [11]. However, G acts on P_{02} and generates some nonvertices. Hence, we modify the definition of nonspecial vertices as following: Let $\tau = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \end{pmatrix}$, which is an involution on special vertices. Let P'_{02} be the group generated by P_{02} and τ, the nonspecial special are parameterized by G/P'_{02}.

Let $E_1 = K \bigcap \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix}^{-1}$. Then E_1 is the stabilizer of the directed type-one edge from the primitive vertex K to the nonprimitive special vertex $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K$. The directed type-one edges are parameterized by G/E_1.

Let $E_2 = K \bigcap P'_{02}$. Then E_2 is the stabilizer of the directed type-two edge from the primitive vertex K to the nonspecial vertex P'_{02}. The directed type-two edges are parameterized by G/E_2.

Let $I = E_1 \bigcap E_2$, called the Iwahori subgroup. Then I is the stabilizer of the chamber consisting of the primitive vertex K, nonspecial vertex P_{02} and the nonprimitive special vertex $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K$. The directed chambers are parameterized by G/I.

Chapter 3

Zeta functions

3.1 Operators on vertices, edges and chambers

Let Γ be a cocompact, discrete, torsion-free subgroup of G. Then Γ intersects any compact subgroup of G trivially. In particular, Γ acts on Δ_4 free of fixed points. We shall assume Γ satisfies $ord_\pi det\Gamma \subset 4Z$ so that Γ identifies vertices of the same type, consequently X_Γ is a finite connected two-dimensional simplicial complex.

Definition 3.1. *Operators on vertices.*

$$\text{Let } A_1 = K \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K = \bigcup_{a,b,c\in F_q} \begin{pmatrix} \pi & 0 & b & a \\ 0 & \pi & c & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} K$$

$$\bigcup_{\alpha,\beta\in F_q} \begin{pmatrix} \pi & -\alpha & 0 & \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} K \bigcup_{\gamma\in F_q} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi & \gamma & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K \bigcup \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K$$

with coset representatives $\{g_i\}_i$. The operator A_1 sends a primitive vertex to its nonprimitive special neighbors and sends a nonprimitive special *vertex to its primitive neighbors. It may also be regarded as an operator acting on continuous functions space $L^2(\Gamma\backslash G/K)$ by sending f to $A_1 f$, where $A_1 f(gK) = \sum_i f(gg_iK)$.*

Let $A_2 = K \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi^2 \end{pmatrix} K \bigcup (q^2+1) \begin{pmatrix} \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K$

$$= \bigcup_{\alpha, b\in F_q, a\in F_{q^2}} \begin{pmatrix} \pi^2 & -\pi\alpha & \pi b & a \\ 0 & \pi & 0 & b \\ 0 & 0 & \pi & \alpha \\ 0 & 0 & 0 & 1 \end{pmatrix} K \bigcup_{b\in F_q, c\in F_{q^2}} \begin{pmatrix} \pi & 0 & b & 0 \\ 0 & \pi^2 & c & \pi b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K$$

$$\bigcup_{\alpha\in F_q} \begin{pmatrix} \pi & -\alpha & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi^2 & \pi\alpha \\ 0 & 0 & 0 & \pi \end{pmatrix} K \bigcup \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi^2 \end{pmatrix} K \bigcup_{a\in F_q^*} \begin{pmatrix} \pi & 0 & 0 & a \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K$$

$$\bigcup_{b\in F_q, a\in F_q^*} \begin{pmatrix} \pi & 0 & b & a \\ 0 & \pi & \frac{b^2}{a} & b \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K \bigcup (q^2+1) \begin{pmatrix} \pi & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K$$

with coset representatives $\{h_i\}_i$. The operator A_2 sends a primitive(nonprimitive special) vertex to the primitive(nonprimitive special) vertices which share a common nonspecial neighbor with it. It may also be regarded as an operator acting on $L^2(\Gamma\backslash G/K)$ by sending f to $A_2 f$, where $A_2 f(gK) = \sum_i f(gh_iK)$.

Definition 3.2. *Operators on edges.*

Let $L_{E_1} = E_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} E_1 = \bigcup_{a,b,c\in F_q} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \pi b & \pi a & \pi & 0 \\ \pi c & \pi b & 0 & \pi \end{pmatrix} E_1$

with coset representatives $\{g_\alpha\}_\alpha$. The operator L_{E_1} sends a type-one edge to its type-one neighbors which share a common vertex with it and the other two distinct vertices share no common nonspecial neighbors. It may also be *regarded as an operator acting on* $L^2(\Gamma\backslash G/E_1)$ *by sending* f *to* $L_{E_1}f$*, where* $L_{E_1}f(gE_1) = \sum_\alpha f(gg_\alpha E_1)$.

Let $L_{E_2} = E_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \pi & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi^2 \end{pmatrix} E_2 = \bigcup_{b,\alpha\in F_q, c\in F_{q^2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\pi\alpha & \pi & 0 & 0 \\ \pi b & 0 & \pi & 0 \\ \pi c & \pi^2 b & \pi^2\alpha & \pi^2 \end{pmatrix} E_2$ *with coset representatives* $\{h_\beta\}_\beta$*. The operator* L_{E_2} *sends a type-two edge to the type-two edges, such that the their primitive vertices share the formal nonspecial vertex as a neighbor and the their nonspecial vertices share no nonprimitve special neighbors. It may also be regarded as an operator acting on* $L^2(\Gamma\backslash G/E_2)$ *by sending* f *to* $L_{E_2}f$*, where* $L_{E_2}f(gE_2) = \sum_\beta f(gh_\beta E_2)$.

Definition 3.3. *Operator on chambers.*

Let $L_I = I \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\pi & 0 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} I = \bigcup_{b,c\in F_q} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \pi b & 0 & \pi & 0 \\ \pi c & \pi b & 0 & \pi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} I$ *with coset representatives* $\{l_\gamma\}_\gamma$*. The operator* L_I *acting twice sends a directed chamber to the directed chambers, such that the action on the type-two edge coincides with the action of* L_{E_2} *and their nonprimitive special vertices share a common nonspecial neighbor. It could also be regarded as an operator acting on* $L^2(\Gamma\backslash G/I)$ *by sending* f *to* $L_I f$*, where* $L_I f(gI) = \sum_\gamma f(gl_\gamma I)$.

3.2 Zeta functions of complexes from PGSp$(4, F)$

Denote by X_Γ the finite quotient of the building $\triangle_4$ by Γ.

Definition 3.4. *The complex zeta function counts tailless closed geodesics* *in X_Γ. More precisely, it is defined as*

$$Z(X_\Gamma, u) = \prod_{[C]} \frac{1}{(1 - u^{l(C)})},$$

where the product runs through the equivalence classes of primitive tailless closed geodesics C consisting of solely type-one edges or solely type-two edges in X_Γ, and $l(C)$ denotes the length of C.

The zeta function can be expressed in terms of the edge operators, namely

$$Z(X_\Gamma, u) = \frac{1}{\det(I - L_{E_1}u)\det(I - L_{E_2}u^2)}.$$

Proof. The number of closed tailless geodesics of type-one and length m is $TrL_{E_1}^m$, the number of closed tailless geodesics of type-two and length $2m$ is $2TrL_{E_2}^m$. Take the derivative log of the zeta function and multiply by u, it counts the number of closed tailless geodesics of type-one and type-two. Take the derivative log of $\frac{1}{\det(I - L_{E_1}u)\det(I - L_{E_2}u^2)}$ and multiply by u, it is equal to $\sum_{m\geq 1} TrL_{E_1}^m u^m + 2TrL_{E_2}^m u^{2m}$. Hence the equality holds.

The zeta function could also be expressed as

Theorem 3.1. *(Main Theorem) In terms of operators A_1, A_2 on vertices, L_{E_1}, L_{E_2} on edges, and L_I on chambers of X_Γ:*

$$Z(X_\Gamma, u) = \frac{(1 - u^2)^{\chi(X_\Gamma)}(1 - q^2u^2)^{-(q^2-1)m}}{det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)det(I - L_Iu)},$$

where $\chi(X_\Gamma)$ *is the Euler characteristic of* X_Γ *and* m *is the number of primitive vertices in* X_Γ*.*

Chapter 4

Proof of the Main Theorem

The group G acts on $L^2(\Gamma\backslash G)$ by right translation. $L^2(\Gamma\backslash G) = \oplus_\rho m_\rho V_\rho$, where V_ρ is an irreducible, unitary representation of G and m_ρ is the multiplicity of the representation V_ρ. The operators A_1, A_2, L_{E_1}, L_{E_2} and L_I act on $C(\Gamma\backslash G/K)$, $C(\Gamma\backslash G/E_1)$, $C(\Gamma\backslash G/E_2)$ and $C(\Gamma\backslash G/I)$ respectively. $C(\Gamma\backslash G/I) = L^2(\Gamma\backslash G/I) = L^2(\Gamma\backslash G)^I = \oplus_\rho m_\rho V_\rho^I$, where V_ρ^I is the space of I-invariant vectors in V_ρ. And similar decomposition holds for the spaces $C(\Gamma\backslash G/K)$,$C(\Gamma\backslash G/E_1)$,$C(\Gamma\backslash G/E_2)$.

To understand the actions of A_1, A_2, L_{E_1}, L_{E_2} and L_I, it then suffices to study the actions of these operators on the subspaces V_ρ^K, $V_\rho^{E_1}$, $V_\rho^{E_2}$ and V_ρ^I, for each irreducible unitary representation ρ with nontrivial I-invariant vectors. [1] shows that an irreducible representation of G that contains an Iwahori fixed vector if and only if it is an irreducible subquotient of an unramified principal series representation.

Theorem 4.1. [10] *The irreducible subquotients of principal series representations of* $GSP_4(F)$ *are:*

Case 1*: Irreducible representations of the form* $\chi_1 \times \chi_2 \rtimes \sigma$.

Case 2*: Constituents of* $\nu^{\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi \rtimes \sigma = \chi St_{GL_2} \rtimes \sigma + \chi 1_{GL_2} \rtimes \sigma$.

Case 3*: Constituents of* $\chi \times \nu \rtimes \nu^{-\frac{1}{2}}\sigma = \chi \rtimes \sigma St_{GSP_2} + \chi \rtimes \sigma 1_{GSP_2}$.

Case 4*: Constituents of* $\nu^2 \times \nu \rtimes \nu^{-\frac{3}{2}}\sigma = \sigma St_{GSP_4} + L(\nu^2, \nu^{-1}\sigma St_{GSP_2}) + L(\nu^{\frac{3}{2}} St_{GL_2}, \nu^{-\frac{3}{2}}\sigma) + \sigma 1_{GSP_4}$.

Case 5*: Constituents of* $\nu\xi_0 \times \xi_0 \rtimes \nu^{-\frac{1}{2}}\sigma = \delta([\xi_0, \nu\xi_0], \nu^{-\frac{1}{2}}\sigma) + L(\nu^{\frac{1}{2}}\xi_0 St_{GL_2}, \nu^{-\frac{1}{2}}\sigma)$ $+ L(\nu^{\frac{1}{2}}\xi_0 St_{GL_2}, \xi_0\nu^{-\frac{1}{2}}\sigma) + L(\nu\xi_0, \xi_0 \rtimes \nu^{-\frac{1}{2}}\sigma)$.

Case 6*: Constituents of* $\nu \times 1_{F^*} \rtimes \nu^{-\frac{1}{2}}\sigma = \tau(S, \nu^{-\frac{1}{2}}\sigma) + \tau(T, \nu^{-\frac{1}{2}}\sigma) + L(\nu^{\frac{1}{2}} St_{GL_2}, \nu^{-\frac{1}{2}}\sigma) + L(\nu, 1_{F^*} \rtimes \nu^{-\frac{1}{2}}\sigma)$

4.1 Eigenvalues of each operator for each representation

We first find the basis of each operator for each representation. With the coset representatives listed in Chapter 3 for each operator, a straightforward computation shows the eigenvalues of each operator for each representation.

Case 1

The irreducible representation $(\rho, V) = Ind(\chi_1 \times \chi_2 \rtimes \sigma)$, where $\chi_1\chi_2\sigma^2 = id$. The space V is a Borel induced representation space, defined as

$V = \{f : G \mapsto C : f(hg) = |a^2bc^{-\frac{3}{2}}|\chi_1(a)\chi_2(b)\sigma(c)f(g), \forall h = \begin{pmatrix} a & * & * & * \\ 0 & b & * & * \\ 0 & 0 & ca^{-1} & * \\ 0 & 0 & 0 & cb^{-1} \end{pmatrix}, g \in G\}$.

Let $W =< s_1, s_2 >$ be the Weyl group of G, consisting of eight elements $id, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2$, where $s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

The Iwahori invariant vector space V^I has dimension eight from the decomposition $G = \bigcup_{w \in W} BwI$. Let $f_\alpha(x)$ be the function in V supported on the coset $B\alpha I$ with $f_\alpha(\alpha I) = 1$. Then V^I has a basis consisting of $f_1 := f_{id}, f_2 := f_{s_1}, f_3 := f_{s_2}, f_4 := f_{s_1s_2}, f_5 := f_{s_2s_1}, f_6 := f_{s_1s_2s_1}, f_7 := f_{s_2s_1s_2}, f_8 := f_{s_1s_2s_1s_2}$.

A straightforward computation using the coset decomposition of L_I gives the following actions of L_I:

$L_I f_1 = q^{\frac{1}{2}}\chi_2\sigma(\pi)f_3,$

$L_I f_2 = q^{\frac{1}{2}}\chi_1\sigma(\pi)f_4 + (q-1)q^{\frac{1}{2}}\chi_2\sigma(\pi)f_3,$

$L_I f_3 = q^{\frac{3}{2}}\sigma(\pi)f_1,$

$L_I f_4 = q^{\frac{3}{2}}\sigma(\pi)f_2,$

$L_I f_5 = q^{\frac{1}{2}}\chi_1\chi_2\sigma(\pi)f_7 + (q-1)q^{\frac{1}{2}}\chi_1\sigma(\pi)f_4 + (q-1)q^{\frac{3}{2}}\sigma(\pi)f_1,$

$L_I f_6 = q^{\frac{1}{2}}\chi_1\chi_2\sigma(\pi)f_8 + (q-1)q^{\frac{3}{2}}\chi_2\sigma(\pi)f_3 + (q-1)q^{\frac{3}{2}}\sigma(\pi)f_2,$

$L_I f_7 = q^{\frac{3}{2}}\chi_2\sigma(\pi)f_5 + (q-1)q^{\frac{3}{2}}\sigma(\pi)f_2,$

$L_I f_8 = q^{\frac{3}{2}}\chi_1\sigma(\pi)f_6 + (q-1)q^{\frac{3}{2}}\chi_2\sigma(\pi)f_5 + (q-1)^2q^{\frac{3}{2}}\sigma(\pi)f_2 + (q-1)q^{\frac{5}{2}}\sigma(\pi)f_1.$

It shows that L_I on V^I has eigenvalues $\pm q\sqrt{\chi_1\sigma^2(\pi)}$, $\pm q\sqrt{\chi_2\sigma^2(\pi)}$, $\pm q\sqrt{\chi_1\chi_2^2\sigma^2(\pi)}$, $\pm q\sqrt{\chi_1^2\chi_2\sigma^2(\pi)}$.

Since $s_1 \in E_1$, the E_1-invariant vector space V^{E_1} has dimension four from the decomposition $G = \bigcup_{w\in W} BwI = \bigcup_{w'\in W'} Bw'E_1$, where $W' = W/ < s_1 >$, represented by $id, s_2, s_1s_2, s_2s_1s_2$. Let $g_\alpha(x)$ be the function in V supported on the coset $B\alpha E_1$ with $g_\alpha(\alpha E_1) = 1$. Then V^{E_1} has a basis consisting of $g_1 := g_{id}, g_2 := g_{s_2}, g_3 := g_{s_1s_2}, g_4 := g_{s_2s_1s_2}$.

Applying the coset decomposition of L_{E_1}, we get the following actions of L_{E_1}:

$L_{E_1}g_1 = q^{\frac{3}{2}}\sigma(\pi)g_1,$

$L_{E_1}g_2 = q^{\frac{3}{2}}\chi_2\sigma(\pi)g_2 + (q-1)q^{\frac{3}{2}}\sigma(\pi)g_1,$

$L_{E_1}g_3 = q^{\frac{3}{2}}\chi_1\sigma(\pi)g_3 + (q-1)q^{\frac{3}{2}}\chi_2\sigma(\pi)g_2 + (q-1)q^{\frac{5}{2}}\sigma(\pi)g_1,$

$L_{E_1}g_4 = q^{\frac{3}{2}}\chi_1\chi_2\sigma(\pi)g_4 + (q-1)q^{\frac{3}{2}}\chi_1\sigma(\pi)g_3 + (q-1)q^{\frac{5}{2}}\chi_2\sigma(\pi)g_2 + (q-1)q^{\frac{7}{2}}\sigma(\pi)g_1.$

Hence L_{E_1} on V^{E_1} has eigenvalues $q^{\frac{3}{2}}\chi_1\sigma(\pi)$, $q^{\frac{3}{2}}\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\chi_1\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\sigma(\pi)$.

Since $s_2 \in E_2$, the E_2- invariant vector space V^{E_2} has dimension four from the decomposition $G = \bigcup_{w \in W} BwI = \bigcup_{w'' \in W''} Bw''E_2$, where $W'' = W/ < s_2 >$, represented by $id, s_1, s_2 s_1, s_1 s_2 s_1$. Let $h_\alpha(x)$ be the function in V supported on the coset $B\alpha E_2$ with $h_\alpha(\alpha E_2) = 1$. Then V^{E_2} has a basis consisting of $h_1 := h_{id}, h_2 := h_{s_1}, h_3 := h_{s_2 s_1}, h_4 := h_{s_1 s_2 s_1}$.

Applying the coset decomposition of L_{E_2}, we get the following actions of L_{E_2}:

$L_{E_2} h_1 = q^2 \chi_2 \sigma^2(\pi) h_1$,

$L_{E_2} h_2 = q^2 \chi_1 \sigma^2(\pi) h_2 + (q-1) q^2 \chi_2 \sigma^2(\pi) h_1$,

$L_{E_2} h_3 = q^2 \chi_1 \chi_2^2 \sigma^2(\pi) h_3 + ((q-1) q^2 \chi_1 \chi_2 \sigma^2(\pi) + (q-1) q^2 \chi_1 \sigma^2(\pi)) h_2 + (q-1) q^3 \chi_2 \sigma^2(\pi) h_1$,

$L_{E_2} h_4 = q^2 \chi_1^2 \chi_2 \sigma^2(\pi) h_4 + (q-1) q^2 \chi_1 \chi_2^2 \sigma^2(\pi) h_3 + ((q-1)^2 q^2 \chi_1 \chi_2 \sigma^2(\pi) + (q-1) q^3 \chi_1 \sigma^2(\pi)) h_2 + ((q-1) q^3 \chi_1 \chi_2 \sigma^2(\pi) + (q-1) q^4 \chi_2 \sigma^2(\pi)) h_1$.

Hence L_{E_2} on V^{E_2} has eigenvalues $q^2 \chi_1 \sigma^2(\pi)$, $q^2 \chi_2 \sigma^2(\pi)$, $q^2 \chi_1 \chi_2^2 \sigma^2(\pi)$, $q^2 \chi_1^2 \chi_2 \sigma^2(\pi)$.

The $K-$invariant vector space V^K has dimension one from $G = BK$. Let $F(x)$ be the function in V supported on the coset BK with $F(K) = 1$. Then $F(x)$ is the basis of V^K.

From the coset decompositions of A_1 and A_2, we get the actions of A_1 and A_2:

$A_1 F = q^{\frac{3}{2}} (\chi_1 \chi_2(\pi) + \chi_1(\pi) + \chi_2(\pi) + 1) \sigma(\pi) F$,

$$A_2F = q^2(\chi_1^2\chi_2(\pi) + \chi_1\chi_2^2(\pi) + \chi_1(\pi) + \chi_2(\pi) + 2\chi_1\chi_2(\pi))\sigma^2(\pi)F.$$

Hence the inverse of the roots of $det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)$ on V^K are $q^{\frac{3}{2}}\chi_1\sigma(\pi)$, $q^{\frac{3}{2}}\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\chi_1\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\sigma(\pi)$.

Observe that the eigenvalues from E_1- and $K-$invariant spaces get canceled, and the eigenvalues from E_2- and $I-$invariant spaces also get canceled, hence the zeta identity for Case 1 is

$$\frac{1}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}.$$

Case $2b$

The irreducible subrepresentation $(\rho, V) = \chi 1_{GL_2} \rtimes \sigma$ of $Ind(\nu^{-\frac{1}{2}}\chi \times \nu^{\frac{1}{2}}\chi \rtimes \sigma)$, where $\chi^2\sigma^2 = id$.

Let P be a Siegel parabolic group, defined as $P = \begin{pmatrix} A & * \\ 0 & cA' \end{pmatrix}$, where A and $*$ are in $GL_2(F)$, $c \in F^*$. Let $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$, then $\Delta = det(A)$ and $A' = \frac{1}{\Delta}\begin{pmatrix} a_1 & -a_2 \\ -a_3 & a_4 \end{pmatrix}$. The space V is a Siegel parabolic group $P-$induced space, defined as

$$V = \{f : G \mapsto C : f(hg) = |det(A)^{\frac{3}{2}}c^{-\frac{3}{2}}|\chi(det(A))\sigma(c)f(g), \forall h = \begin{pmatrix} A & * \\ 0 & cA' \end{pmatrix}\}.$$

Since $s_1 \in P$, the Iwahori invariant vector space V^I has dimension four from $G = \bigcup_{w\in W} BwI = \bigcup_{u\in U} PuI$, where $U =< s_1 > \backslash W$ is represented by $id, s_2, s_2s_1, s_2s_1s_2$. Let $f_\alpha(x)$ be the function in V supported on the coset $P\alpha I$ with $f_\alpha(\alpha I) = 1$. Then V^I has a basis consisting of $f_1 := f_{id}, f_2 := f_{s_2}, f_3 := f_{s_2s_1}, f_4 := f_{s_2s_1s_2}$. The E_1-invariant vector space V^{E_1}

has dimension three from $G = \bigcup_{u \in U} PuI = \bigcup_{u' \in U'} Pu'E_1$, where $U' =< s_1 > \backslash W / < s_1 >$ is represented by $id, s_2, s_2s_1s_2$. Let $g_\alpha(x)$ be the function in V supported on the coset $P\alpha E_1$ with $g_\alpha(\alpha E_1) = 1$. Then V^{E_1} has a basis consisting of $g_1 := g_{id}, g_2 := g_{s_2}, g_3 := g_{s_2s_1s_2}$. The E_2-invariant vector space V^{E_2} has dimension two from $G = \bigcup_{u \in U} PuI = \bigcup_{u'' \in U''} Pu''E_2$, where $U'' =< s_1 > \backslash W / < s_2 >$ is represented by id, s_2s_1. Let $h_\alpha(x)$ be the function in V supported on the coset $P\alpha E_2$ with $h_\alpha(\alpha E_2) = 1$. Then V^{E_2} has a basis consisting of $h_1 := h_{id}, h_2 := h_{s_2s_1}$. The space of K invariant vectors V^K has dimension one from $G = PK$. Let $F(x)$ be the function in V supported on the coset PK with $F(K) = 1$. Then $F(x)$ is a basis of V^K.

With the basis listed above, a straightforward computation shows the actions of L_I, L_{E_1}, L_{E_2} as follows:

$L_I(f_1) = q\chi\sigma(\pi)f_2,$

$L_I(f_2) = q^{\frac{3}{2}}\sigma(\pi)f_1,$

$L_I(f_3) = q^{\frac{1}{2}}\chi^2\sigma(\pi)f_4 + (q-1)q\chi\sigma(\pi)f_2 + (q-1)q^{\frac{3}{2}}\sigma(\pi)f_1,$

$L_I(f_4) = q^2\chi\sigma(\pi)f_3 + (q-1)q^{\frac{5}{2}}\sigma(\pi)f_1.$

$L_{E_1}(g_1) = q^{\frac{3}{2}}\sigma(\pi)g_1,$

$L_{E_1}(g_2) = q^2\chi\sigma(\pi)g_2 + (q^2-1)q^{\frac{3}{2}}\sigma(\pi)g_1,$

$L_{E_1}(g_3) = q^{\frac{3}{2}}\chi^2\sigma(\pi)g_3 + (q-1)q^2\chi\sigma(\pi)g_2 + (q-1)q^{\frac{7}{2}}\sigma(\pi)g_1.$

$L_{E_2}(h_1) = q^{\frac{5}{2}}\chi\sigma^2(\pi)h_1,$

$L_{E_2}(h_2) = q^{\frac{5}{2}}\chi^3\sigma^2(\pi)h_2 + ((q-1)q^3\chi^2\sigma^2(\pi) + (q^2-1)q^{\frac{5}{2}}\chi\sigma^2(\pi))h_1.$

Hence, the eigenvalues of L_I on V^I are $\pm\sqrt{q^{\frac{5}{2}}\chi\sigma^2(\pi)}$, $\pm\sqrt{q^{\frac{5}{2}}\chi^3\sigma^2(\pi)}$. The eigenvalues of L_{E_1} on V^{E_1} are $q^{\frac{3}{2}}\chi^2\sigma(\pi)$, $q^{\frac{3}{2}}\sigma(\pi)$, $q^2\chi\sigma(\pi)$. The eigenvalues of L_{E_2} on V^{E_2} are $q^{\frac{5}{2}}\chi\sigma^2(\pi)$, $q^{\frac{5}{2}}\chi^3\sigma^2(\pi)$.

Replace χ_1 by $\nu^{-\frac{1}{2}}\chi$, χ_2 by $\nu^{\frac{1}{2}}\chi$ and apply results in Case 1; the inverse of the roots of $det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)$ are $q^{\frac{3}{2}}\chi^2\sigma(\pi)$, $q^{\frac{3}{2}}\sigma(\pi)$, $q^2\chi\sigma(\pi), q\chi\sigma(\pi)$.

Observe that the eigenvalues from E_2- and $I-$invariant spaces get canceled, and the eigenvalues from E_1- and $K-$invariant spaces also get canceled except for $q\chi\sigma(\pi)$. Notice that $\chi\sigma(\pi) = \pm 1$, hence the zeta identity for Case $2b$ is

$\frac{(1-q^2u^2)^{\frac{1}{2}m(2b)}}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}$, where $m(2b)$ represents the multiplicity of the representations of Case $2b$ in $L^2(\Gamma\backslash G)$.

Case $2a$

The irreducible subrepresentation $(\rho, V) = \chi St_{GL_2} \rtimes \sigma$ of $Ind(\nu^{\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi \rtimes \sigma)$, where $\chi^2\sigma^2 = id$.

Let $f_\alpha(x)$ be the function in V supported on the coset $B\alpha I$ with $f_\alpha(\alpha I) = 1$. From the intertwining map $T_1 : Ind(\nu^{\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi \rtimes \sigma) \mapsto Ind(\nu^{-\frac{1}{2}}\chi \times \nu^{\frac{1}{2}}\chi \rtimes \sigma)$ in [1] and [16], we get a basis of the Iwahori invariant vector space V^I, consisting of $f_1 := f_{s_1} - qf_{id}$, $f_2 := f_{s_1s_2} - qf_{s_2}$, $f_3 := f_{s_1s_2s_1} - qf_{s_2s_1}$, $f_4 :=$

$f_{s_1s_2s_1s_2} - qf_{s_2s_1s_2}$. Since $s_1 \in E_1$, V^{E_1} has a basis of $g_1 = f_2 + f_3$. Since $s_2 \in E_2$, V^{E_2} has a basis of $h_1 = f_1 + f_2$, $h_2 = f_3 + f_4$. And there is no $K-$invariant vectors [11].

With the basis listed above, we obtain the actions of L_I, L_{E_1}, L_{E_2}:

$L_I(f_1) = \chi\sigma(\pi)f_2$,

$L_I(f_2) = q^{\frac{3}{2}}\sigma(\pi)f_1$,

$L_I(f_3) = q^{\frac{1}{2}}\chi^2\sigma(\pi)f_4 - (q-1)q\chi\sigma(\pi)f_2 + (q-1)q^{\frac{3}{2}}\sigma(\pi)f_1$,

$L_I(f_4) = q\chi\sigma(\pi)f_3 - (q-1)q^{\frac{3}{2}}\sigma(\pi)f_1$.

$L_{E_1}(g_1) = q\chi\sigma(\pi)g_1$.

$L_{E_2}(h_1) = q^{\frac{3}{2}}\chi\sigma^2(\pi)h_1$,

$L_{E_2}(h_2) = q^{\frac{3}{2}}\chi^3\sigma^2(\pi)h_2 - (q-1)q^2\chi^2\sigma^2(\pi)h_1$.

Hence, the eigenvalues of L_I on V^I are $\pm\sqrt{q^{\frac{3}{2}}\chi\sigma^2(\pi)}$, $\pm\sqrt{q^{\frac{3}{2}}\chi^3\sigma^2(\pi)}$. The eigenvalues of L_{E_1} on V^{E_1} is $q\chi\sigma(\pi)$. The eigenvalues of L_{E_2} on V^{E_2} are $q^{\frac{3}{2}}\chi\sigma^2(\pi)$, $q^{\frac{3}{2}}\chi^3\sigma^2(\pi)$.

Observe that the eigenvalues from E_2- and $I-$invariant spaces get canceled, and the eigenvalues from E_1- and $K-$invariant spaces also get canceled except for $q\chi\sigma(\pi)$. Notice that $\chi\sigma(\pi) = \pm 1$, hence the zeta identity for Case $2a$ is

$\frac{(1-q^2u^2)^{-\frac{1}{2}m(2a)}}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}$, where $m(2a)$ represents the multiplicity of the representations of Case $2a$ in $L^2(\Gamma\backslash G)$.

Case $3b$

The irreducible subrepresentation $(\rho, V) = \chi \rtimes \sigma 1_{GSP_2}$ of $Ind(\chi \times \nu^{-1} \rtimes \nu^{\frac{1}{2}}\sigma)$, where $\chi^2\sigma^2 = id$.

Let Q be a Klingen parabolic group, defined as $Q = \begin{pmatrix} b & * & * & * \\ 0 & a_1 & a_2 & * \\ 0 & a_3 & a_4 & * \\ 0 & 0 & 0 & \frac{c}{b} \end{pmatrix}$, where $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in GL_2(F)$, $c = det(A)$ and $b, * \in F^*$. The space V is a Klingen parabolic $Q-$induced space, defined as

$$V = \{f : G \mapsto C : f(hg) = |b^2c^{-1}|\chi(b)\sigma(c)f(g), \forall h = \begin{pmatrix} b & * & * & * \\ 0 & a_1 & a_2 & * \\ 0 & a_3 & a_4 & * \\ 0 & 0 & 0 & \frac{c}{b} \end{pmatrix}\}.$$

Since $s_2 \in Q$, the Iwahori invariant vector space V^I has dimension four from $G = \bigcup_{w \in W} BwI = \bigcup_{u \in U} QuI$, where $U =< s_2 > \backslash W$ is represented by $id, s_1, s_1s_2, s_1s_2s_1$. Let $f_\alpha(x)$ be the function in V supported on the coset $Q\alpha I$ with $f_\alpha(\alpha I) = 1$. Then V^I has a basis consisting of $f_1 := f_{id}, f_2 := f_{s_1}, f_3 := f_{s_1s_2}, f_4 := f_{s_1s_2s_1}$. The E_1-invariant vector space V^{E_1} has dimension two from $G = \bigcup_{u \in U} QuI = \bigcup_{u' \in U'} Qu'E_1$, where $U' =< s_2 > \backslash W / < s_1 >$ is represented by id, s_1s_2. Let $g_\alpha(x)$ be the function in V supported on the coset $Q\alpha E_1$ with $g_\alpha(\alpha E_1) = 1$. Then V^{E_1} has a basis consisting of $g_1 := g_{id}, g_2 := g_{s_1s_2}$. The E_2-invariant vector space V^{E_2} has dimension three from $G = \bigcup_{u \in U} QuI = \bigcup_{u'' \in U''} Qu''E_2$, where $U'' =< s_2 > \backslash W / < s_2 >$ is represented by $id, s_1, s_1s_2s_1$. Let $h_\alpha(x)$ be the function in V supported on the coset $Q\alpha E_2$ with $h_\alpha(\alpha E_2) = 1$. Then V^{E_2}

has a basis consisting of $h_1 := h_{id}, h_2 := h_{s_1}, h_3 = h_{s_1 s_2 s_1}$. The space of K invariant vectors V^K has dimension one from $G = QK$. Let $F(x)$ be the function in V supported on the coset QK with $F(K) = 1$. $F(x)$ is the basis of V^K.

Using the basis listed above, we obtain the actions of L_I, L_{E_1}, L_{E_2}:

$L_I(f_1) = q\sigma(\pi)f_1$,

$L_I(f_2) = q\chi\sigma(\pi)f_3 + (q-1)q\sigma(\pi)f_1$,

$L_I(f_3) = q^2\sigma(\pi)f_2$,

$L_I(f_4) = q\chi\sigma(\pi)f_4 + (q-1)q^2\sigma(\pi)f_2 + (q-1)q^2\sigma(\pi)f_1$.

$L_{E_1}(g_1) = q^2\sigma(\pi)g_1$,

$L_{E_1}(g_2) = q^2\chi\sigma(\pi)g_2 + (q^2-1)q^2\sigma(\pi)g_1$.

$L_{E_2}(h_1) = q^2\sigma^2(\pi)h_1$,

$L_{E_2}(h_2) = q^3\chi\sigma^2(\pi)h_2 + (q^2-1)q^2\sigma^2(\pi)h_1$,

$L_{E_2}(h_3) = q^2\chi^2\sigma^2(\pi)h_3 + (q-1)q^3\chi\sigma^2(\pi)h_2 + ((q-1)q^3\chi\sigma^2(\pi) + (q-1)q^4\sigma^2(\pi))h_1$.

Hence,the eigenvalues of L_I on V^I are $q\chi\sigma(\pi)$, $q\sigma(\pi)$, $\pm\sqrt{q^3\chi\sigma^2(\pi)}$. The eigenvalues of L_{E_1} on V^{E_1} are $q^2\chi\sigma(\pi)$, $q^2\sigma(\pi)$. The eigenvalues of L_{E_2} on V^{E_1} are $q^2\chi^2\sigma^2(\pi)$, $q^2\sigma^2(\pi)$, $q^3\chi\sigma^2(\pi)$.

Replace χ_1 by χ, χ_2 by ν^{-1}, σ by $\nu^{\frac{1}{2}}\sigma$ and apply results in Case 1; the inverse of the roots of $det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)$ are $q^2\sigma(\pi)$,

$q\sigma(\pi)$, $q^2\chi\sigma(\pi), q\chi\sigma(\pi)$.

Observe that the eigenvalues from E_2- and $I-$invariant spaces get canceled, and the eigenvalues from E_1- and $K-$invariant spaces also get canceled, hence the zeta identity for Case $3b$ is

$$\frac{1}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}.$$

Case $3a$

The irreducible subrepresentation $(\rho, V) = \chi \rtimes \sigma St_{GSP_2}$ of $Ind(\chi \times \nu \rtimes \nu^{-\frac{1}{2}}\sigma)$, where $\chi^2\sigma^2 = id$.

Let $f_\alpha(x)$ be the function in V supported on the coset $B\alpha I$ with $f_\alpha(\alpha I) = 1$. From the intertwining map $T_2 : Ind(\chi \times \nu \rtimes \nu^{-\frac{1}{2}}\sigma) \mapsto Ind(\chi \times \nu^{-1} \rtimes \nu^{\frac{1}{2}}\sigma)$ in [1] and [16], we get a basis of the Iwahori invariant vector space V^I, consisting of $f_1 := f_{s_2} - qf_{id}, f_2 := f_{s_2s_1} - qf_{s_1}, f_3 := f_{s_2s_1s_2} - qf_{s_1s_2}, f_4 := f_{s_1s_2s_1s_2} - qf_{s_1s_2s_1}$. Since $s_1 \in E_1$, V^{E_1} has a basis of $g_1 = f_1+f_2$, $g_2 = f_3+f_4$. Since $s_2 \in E_2$, V^{E_2} has a basis of $h_1 = f_2 + f_3$. There is no $K-$invariant vector [11].

Using the basis listed above, we obtain the actions of L_I, L_{E_1}, L_{E_2}:

$L_I(f_1) = -q\sigma(\pi)f_1$, $L_I(f_2) = \chi\sigma(\pi)f_3 - (q-1)q\sigma(\pi)f_1$, $L_I(f_3) = q\sigma(\pi)f_2$, $L_I(f_4) = -q\chi\sigma(\pi)f_4 + (q-1)q\sigma(\pi)f_2 - (q-1)q^2\sigma(\pi)f_1$.

$L_{E_1}(g_1) = q\sigma(\pi)g_1$, $L_{E_1}(g_2) = q\chi\sigma(\pi)g_2$.

$L_{E_2}(h_1) = q\chi\sigma^2(\pi)h_1$.

Hence,the eigenvalues of L_I on V^I are $-q\chi\sigma(\pi)$, $-q\sigma(\pi)$ and $\pm\sqrt{q\chi\sigma^2(\pi)}$. The eigenvalues of L_{E_1} on V^{E_1} are $q\chi\sigma(\pi)$ and $q\sigma(\pi)$. The eigenvalue of L_{E_2} on V^{E_2} is $q\chi\sigma^2(\pi)$.

Observe that the eigenvalues from E_2- and $I-$invariant spaces get canceled, and the eigenvalues from E_1- and $K-$invariant spaces also get canceled, hence the zeta identity for Case $3a$ is

$$\frac{1}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}.$$

Case $4d$

The irreducible subrepresentation $(\rho, V) = \sigma 1_{GSP_4}$ of $Ind(\nu^{-2} \times \nu^{-1} \rtimes \nu^{\frac{3}{2}}\sigma)$, where $\sigma^2 = id$.

Let $f_\alpha(x)$ be the function in V supported on the coset $B\alpha I$ with $f_\alpha(\alpha I) = 1$. $f_1 := f_{id}, f_2 := f_{s_1}, f_3 := f_{s_2}$, $f_4 := f_{s_1s_2}, f_5 := f_{s_2s_1}, f_6 := f_{s_1s_2s_1}, f_7 := f_{s_2s_1s_2}, f_8 := f_{s_1s_2s_1s_2}$. Let $F(x) = \sum_i f_i$, $F(x)$ is a basis of $I-$, E_1-, E_2- and $K-$invariant spaces. Hence, we obtain the eigenvalue of L_I on V^I is $q^2\sigma(\pi)$. The eigenvalue of L_{E_1} on V^{E_1} is $q^3\sigma(\pi)$. The eigenvalue of L_{E_2} on V^{E_2} is $q^4\sigma^2(\pi)$.

Replace χ_1 by ν^{-2}, χ_2 by ν^{-1}, σ by $\nu^{\frac{3}{2}}\sigma$ and apply results in Case 1; the inverse of the roots of $det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)$ are $q^3\sigma(\pi)$, $q^2\sigma(\pi)$, $q\sigma(\pi)$, $\sigma(\pi)$.

After the cancelation of these eigenvalues, the zeta identity for Case $4d$

is

$\frac{(1-u^2)(1-q^2u^2)}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}$.

Case $4a$

The irreducible subrepresentation $(\rho, V) = \sigma St_{GSP_4}$ of $Ind(\nu^2 \times \nu \rtimes \nu^{-\frac{3}{2}}\sigma)$, where $\sigma^2 = id$.

Let $f_\alpha(x)$ be the function in V supported on the coset $B\alpha I$ with $f_\alpha(\alpha I) = 1$. $f_1 := f_{id}, f_2 := f_{s_1}, f_3 := f_{s_2}, f_4 := f_{s_1s_2}, f_5 := f_{s_2s_1}, f_6 := f_{s_1s_2s_1}, f_7 := f_{s_2s_1s_2}, f_8 := f_{s_1s_2s_1s_2}$. From the intertwining maps T_1 and T_2 in [1] and [16], we get a basis of V^I is $F(x) = f_8 - qf_7 - qf_6 + q^2f_5 + q^2f_4 - q^3f_3 - q^3f_2 + q^4f_1$. There are no $E_1-, E_2-, K-$invariant vectors [11]. The eigenvalue of L_I is $-\sigma(\pi)$.

The zeta identity for Case $4a$ is

$\frac{(1-u^2)^{m(4a)}}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}$, where $m(4a)$ represents the multiplicity of each Steinberg representation of Case $4a$ in $L^2(\Gamma\backslash G)$.

Case $5b$

The irreducible subrepresentation $(\rho, V) = L(\nu^{\frac{1}{2}}\xi_0 St_{GL_2}, \nu^{-\frac{1}{2}}\sigma)$ with $\sigma^2 = id$ of $Ind(\nu^{\frac{1}{2}}\xi_0 1_{GL_2} \rtimes \nu^{-\frac{1}{2}}\xi_0\sigma)$, which is a subrepresentation of $Ind(\xi_0 \times \nu\xi_0 \rtimes \nu^{-\frac{1}{2}}\xi_0\sigma)$.

Let $f_\alpha(x)$ be the function in V supported on the coset $P\alpha I$ with $f_\alpha(\alpha I) = 1$. Let $f_1 := f_{id}, f_2 := f_{s_2}, f_3 := f_{s_2s_1}, f_4 := f_{s_2s_1s_2}$. From [12], the Iwahori invariant vector space V^I has a basis as $i_1 = -f_4 - f_3 + q^2 f_2 + q^2 f_1$ and $i_2 = -(q+1)f_4 + (q^2-q)f_3 + (q^2-q)f_2 + (q^3+q^2)f_1$. Since $s_1 \in E_1$, the E_1-invariant space V^{E_1} has i_2 as a basis. Since $s_2 \in E_2$, the E_2-invariant space V^{E_2} has i_1 as a basis. There is no $K-$invariant vector [11].

With the basis listed above, we obtain the actions of L_I, L_{E_1}, L_{E_2}: $L_I(i_1) = q\sigma(\pi)i_1 - \sigma(\pi)i_2$, $L_I(i_2) = 2q^2\sigma(\pi)i_1 - q\sigma(\pi)i_2$. $L_{E_1}(i_2) = -q\sigma(\pi)i_2$. $L_{E_2}(i_1) = -q^2\sigma^2(\pi)i_1$. Hence, the eigenvalues of L_I on V^I are $\pm\sqrt{-q^2\sigma^2(\pi)}$. The eigenvalue of L_{E_1} on V^{E_1} is $-q\sigma(\pi)$. The eigenvalue of L_{E_2} on V^{E_2} is $-q^2\sigma^2(\pi)$.

With a similar discussion to the previous cases, the zeta identity for Case $5b$ is

$\frac{(1-q^2u^2)^{-\frac{1}{2}m(5b)}}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}$, where $m(5b)$ represents the multiplicity of the representations of Case $5b$ in $L^2(\Gamma\backslash G)$.

Case $5d$

The irreducible subrepresentation $(\rho, V) = L(\nu\xi_0, \xi_0 \rtimes \nu^{-\frac{1}{2}}\sigma)$ with $\sigma^2 = id$ of $Ind(\nu^{-\frac{1}{2}}\xi_0 1_{GL_2} \rtimes \nu^{\frac{1}{2}}\xi_0\sigma)$, which is a subrepresentation of $Ind(\nu^{-1}\xi_0 \times \xi_0 \rtimes \nu^{\frac{1}{2}}\xi_0\sigma)$.

Let $f_\alpha(x)$ be the function in V supported on the coset $P\alpha I$ with $f_\alpha(\alpha I) = 1$. Let $f_1 := f_{id}, f_2 := f_{s_2}$, $f_3 := f_{s_2s_1}, f_4 := f_{s_2s_1s_2}$. From [12], the Iwahori invariant vector space V^I has $i_1 = f_4 + \frac{1}{2}(1-q)f_3 + \frac{1}{2}(1-q)f_2 + \frac{1}{2}(1+q^2)f_1$

and $i_2 = -\frac{1}{2}(1-q)f_4 + qf_3 + qf_2 + \frac{1}{2}(1-q)qf_1$ as a basis. Since $s_1 \in E_1$, the E_1-invariant space V^{E_1} has a basis consisting of i_1, i_2. Since $s_2 \in E_2$, the E_2-invariant space V^{E_2} has a basis $i_1 + i_2$. The $K-$invariant space V^K has a basis $i_1 + i_2$.

With the basis listed above, we obtain the actions of L_I, L_{E_1}, L_{E_2}: $L_I(i_1) = q\sigma(\pi)i_2$, $L_I(i_2) = -q^2\sigma(\pi)i_1$. $L_{E_1}(i_1) = -q^2\sigma(\pi)i_1$, $L_{E_1}(i_2) = (-q+1)q^2\sigma(\pi)i_1 + q^2\sigma(\pi)i_2$. $L_{E_2}(i_1+i_2) = -q^3\sigma(\pi)(i_1+i_2)$. Hence, the eigenvalues of L_I on V^I are $\pm\sqrt{-q^3\sigma^2(\pi)}$. The eigenvalues of L_{E_1} on V^{E_1} are $\pm q^2\sigma^2(\pi)$. The eigenvalue of L_{E_2} on V^{E_2} is $-q^3\sigma(\pi)$.

Replace χ_1 by $\nu^{-1}\xi_0$, χ_2 by ξ_0, σ by $\nu^{\frac{1}{2}}\xi_0\sigma$ and apply results in Case 1; the inverse of the roots of $det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)$ are $\pm q^2\sigma(\pi), \pm q\sigma(\pi)$.

With a similar discussion to the previous cases, the zeta identity for Case $5d$ is

$\frac{(1-q^2u^2)^{m(5d)}}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}$, where $m(5d)$ represents the multiplicity of the representations of Case $5d$ in $L^2(\Gamma\backslash G)$.

Case $5c$

The irreducible subrepresentation $(\rho, V) = L(\nu^{\frac{1}{2}}\xi_0 St_{GL_2}, \nu^{-\frac{1}{2}}\xi_0\sigma)$ with $\sigma^2 = id$ of $Ind(\nu^{\frac{1}{2}}\xi_0 1_{GL_2} \rtimes \nu^{-\frac{1}{2}}\sigma)$, which is a subrepresentation of $Ind(\xi_0 \times \nu\xi_0 \rtimes \nu^{-\frac{1}{2}}\sigma)$.

Let $f_\alpha(x)$ be the function in V supported on the coset $P\alpha I$ with $f_\alpha(\alpha I) = 1$. Let $f_1 := f_{id}, f_2 := f_{s_2},\ f_3 := f_{s_2s_1}, f_4 := f_{s_2s_1s_2}$. A similar argument as Case 5b in [12] shows that the expressions of a basis of V^I, V^{E_1}, V^{E_2} are the same as Case 5b. With similar computation, we get the eigenvalues of L_I on V^I are $\pm\sqrt{-q^2\sigma^2(\pi)}$. The eigenvalue of L_{E_1} on V^{E_1} is $q\sigma(\pi)$. The eigenvalue of L_{E_2} on V^{E_2} is $-q^2\sigma^2(\pi)$.

With a similar discussion to the previous cases, the zeta identity for Case 5c is

$\frac{(1-q^2u^2)^{-\frac{1}{2}m(5c)}}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}$, where $m(5c)$ represents the multiplicity of the representations of Case 5c in $L^2(\Gamma\backslash G)$.

Case 5a

The irreducible subrepresentation $(\rho, V) = \delta([\xi_0, \nu\xi_0], \nu^{-\frac{1}{2}}\sigma)$ with $\sigma^2 = id$ of $Ind(\nu^{\frac{1}{2}}\xi_0 St_{GL_2} \rtimes \nu^{-\frac{1}{2}}\sigma)$, which is a subrepresentation of $Ind(\nu\xi_0 \times \xi_0 \rtimes \nu^{-\frac{1}{2}}\sigma)$.

In Case 2a, we know the $I-$invariant space of $Ind(\nu^{\frac{1}{2}}\xi_0 St_{GL_2} \rtimes \nu^{-\frac{1}{2}}\sigma)$ has a basis of $f_1 := f_{s_1} - qf_{id}, f_2 := f_{s_1s_2} - qf_{s_2},\ f_3 := f_{s_1s_2s_1} - qf_{s_2s_1}, f_4 := f_{s_1s_2s_1s_2} - qf_{s_2s_1s_2}$, where $f_\alpha(x)$ is the function in V supported on the coset $B\alpha I$ with $f_\alpha(\alpha I) = 1$. The E_2-invariant space V^{E_2} has dimension 1 [11], say i_1. Since $s_2 \in E_2$, i_1 has the expression $i_1 = a(f_4+f_3)+(f_2+f_1)$, where a is to be determined. The $I-$invariant space V^I has dimension 2, say i_1 and

i_2. We may assume $i_2 = \frac{1}{\sigma(\pi)} L_I(i_1) = af_4 - aqf_3 + (a(q-1)q-1)f_2 + q^2 f_1$. And $\frac{1}{\sigma(\pi)} L_I(i_2) = -aqf_4 - aqf_3 - (aq^3 - aq^2 + q^2)f_2 - (aq^3 - aq^2 + q^2)f_1$, which should be equal to $-qi_1$. This shows that $a = -\frac{1}{q}$. Hence we get $i_1 = -\frac{1}{q}(f_4 + f_3) + (f_2 + f_1)$, $i_2 = -f_4 + qf_3 - q^2 f_2 + q^3 f_1$. There is no E_2- or $K-$invariant vector [11].

A straightforward computation shows the eigenvalues of L_I on V^I are $\pm\sqrt{-q\sigma^2(\pi)}$, the eigenvalue of L_{E_2} on V^{E_2} is $-q\sigma^2(\pi)$.

After the cancelation of these eigenvalues, the zeta identity for Case $5a$ is

$$\frac{1}{det(I - A_1 u + qA_2 u^2 - q^3 A_1 u^3 + q^6 I u^4)} = \frac{det(I - L_I u)}{det(I - L_{E_1} u) det(I - L_{E_2} u^2)}.$$

Case $6b$

The irreducible subrepresentation $(\rho, V) = \tau(T, \nu^{-\frac{1}{2}}\sigma)$ with $\sigma^2 = id$ of $Ind(\nu^{\frac{1}{2}} 1_{GL_2} \rtimes \nu^{-\frac{1}{2}}\sigma)$, which is a subrepresentation of $Ind(1_{F^*} \times \nu \rtimes \nu^{-\frac{1}{2}}\sigma)$.

Let $f_\alpha(x)$ be the function in V supported on the coset $P\alpha I$ with $f_\alpha(\alpha I) = 1$. Let $f_1 := f_{id}, f_2 := f_{s_2}$, $f_3 := f_{s_2 s_1}, f_4 := f_{s_2 s_1 s_2}$. The $I-$invariant space V^I has a basis of $i_1 = f_4 - qf_3 - qf_2 + q^2 f_1$ [12]. Since $s_1 \in E_1$, the E_1-invariant space V^{E_1} also has a basis of i_1. There are neither E_2- nor $K-$invariant vectors [11].

A straightforward computation shows that the eigenvalue of L_I on V^I is $-q\sigma(\pi)$, the eigenvalue of L_{E_1} on V^{E_1} is $q\sigma(\pi)$.

After the cancelation of these eigenvalues, the zeta identity for Case $6b$ is

$$\frac{1}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}.$$

Case $6c$

The irreducible subrepresentation $(\rho, V) = L(\nu^{\frac{1}{2}}St_{GL_2}, \nu^{-\frac{1}{2}}\sigma)$ with $\sigma^2 = id$ of $Ind(1_{F^*} \rtimes \sigma 1_{GSP_2})$, which is a subrepresentation of $Ind(1_{F^*} \times \nu^{-1} \rtimes \nu^{\frac{1}{2}}\sigma)$.

Let $f_\alpha(x)$ be the function in V supported on the coset $Q\alpha I$ with $f_\alpha(\alpha I) = 1$. Let $f_1 := f_{id}, f_2 := f_{s_1}, f_3 := f_{s_1s_2}, f_4 := f_{s_1s_2s_1}$. The $I-$invariant space V^I has a basis of $i_1 = f_4 - qf_3 - qf_2 + q^2f_1$ [12]. Since $s_2 \in E_2$, the E_2-invariant space V^{E_2} also has a basis of i_1. There are neither E_1- nor $K-$invariant vectors [11].

A straightforward computation shows that the eigenvalue of L_I on V^I is $q\sigma(\pi)$, the eigenvalue of L_{E_2} on V^{E_2} is $q^2\sigma^2(\pi)$.

With a similar discussion to the previous cases, the zeta identity for Case $6c$ is

$\frac{(1-q^2u^2)^{-\frac{1}{2}m(6c)}}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}$, where $m(6c)$ represents the multiplicity of the representations of Case $6c$ in $L^2(\Gamma\backslash G)$.

Case $6d$

The irreducible subrepresentation $(\rho, V) = L(\nu, 1_{F^*} \rtimes \nu^{-\frac{1}{2}}\sigma)$ with $\sigma^2 = id$

of $Ind(\nu^{-\frac{1}{2}}1_{GL_2} \rtimes \nu^{\frac{1}{2}}\sigma)$, which is a subrepresentation of $Ind(\nu^{-1} \times 1_{F^*} \rtimes \nu^{\frac{1}{2}}\sigma)$.

Let $f_\alpha(x)$ be the function in V supported on the coset $P\alpha I$ with $f_\alpha(\alpha I) = 1$. Let $f_1 := f_{id}, f_2 := f_{s_2}$, $f_3 := f_{s_2 s_1}, f_4 := f_{s_2 s_1 s_2}$. The $I-$invariant space V^I has a basis of $i_1 = f_4 + f_3$, $i_2 = f_2 + f_1$, $i_3 = f_3 + qf_1$ [12]. Since $s_1 \in E_1$ and the dimension of V^{E_1} is 2 [11], observe that V^{E_1} has a basis of $e_1 = i_1 + i_2$, $e_2 = i_1 + qi_2 + (q-1)i_3 = f_4 + qf_3 + qf_2 + q^2 f_1$. Since $s_2 \in E_2$ and the dimension of V^{E_2} is 2 [11], observe that V^{E_2} has a basis of i_1, i_2. The $K-$invariant vector space V^K has a basis of e_1.

With the basis listed above, we obtain the actions of L_I, L_{E_1}, L_{E_2}:

$L_I(i_1) = q\sigma(\pi)i_1 + (q-1)q\sigma(\pi)i_2 + (q-1)q\sigma(\pi)i_3$, $L_I(i_2) = q\sigma(\pi)i_2$, $L_I(i_3) = q\sigma(\pi)i_1 + (q^2\sigma(\pi) + (q-1)q\sigma(\pi))i_2 - q\sigma(\pi)i_3$.

$L_{E_1}(e_1) = q^2\sigma(\pi)e_2$, $L_{E_1}(e_2) = -q^2\sigma(\pi)e_1 + 2q^2\sigma(\pi)e_2$.

$L_{E_2}(i_1) = q^3\sigma^2(\pi)i_1 + ((q-1)q^3\sigma^2(\pi) + (q^2-1)q^2\sigma^2(\pi))i_2$, $L_{E_2}(i_2) = q^2\sigma^2(\pi)i_2$.

Hence the eigenvalues of L_I on V^I are $\pm\sqrt{q^3\sigma^2(\pi)}, q\sigma(\pi)$. The eigenvalues of L_{E_1} on V^{E_1} are $q^2\sigma(\pi), q^2\sigma(\pi)$. The eigenvalues of L_{E_2} on V^{E_2} are $q^3\sigma^2(\pi), q^2\sigma^2(\pi)$.

Replace χ_1 by ν^{-1}, χ_2 by 1, σ by $\nu^{\frac{1}{2}}\sigma$ and apply results in Case 1; the inverse of the roots of $det(I - A_1u + qA_2u^2 - q^3A_1u^3 + q^6Iu^4)$ are $q^2\sigma(\pi), q^2\sigma(\pi), q\sigma(\pi), q\sigma(\pi)$.

With a similar discussion to the previous cases, the zeta identity for Case $6d$ is

$\frac{(1-q^2u^2)^{\frac{1}{2}m(6d)}}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}$, where $m(6d)$ represents the multiplicity of the representations of Case $6d$ in $L^2(\Gamma\backslash G)$.

Case $6a$

The irreducible subrepresentation $(\rho, V) = \tau(S, \nu^{-\frac{1}{2}}\sigma)$ with $\sigma^2 = id$ of $Ind(\nu^{\frac{1}{2}}St_{GL_2} \rtimes \nu^{-\frac{1}{2}}\sigma)$, which is a subrepresentation of $Ind(\nu \times 1_{F^*} \rtimes \nu^{-\frac{1}{2}}\sigma)$.

In Case $2a$, we know that the $I-$invariant space of $Ind(\nu^{\frac{1}{2}}St_{GL_2} \rtimes \nu^{-\frac{1}{2}}\sigma)$ has a basis of $f_1 := f_{s_1} - qf_{id}$, $f_2 := f_{s_1s_2} - qf_{s_2}$, $f_3 := f_{s_1s_2s_1} - qf_{s_2s_1}$, $f_4 := f_{s_1s_2s_1s_2} - qf_{s_2s_1s_2}$, where $f_\alpha(x)$ is the function in V supported on the coset $B\alpha I$ with $f_\alpha(\alpha I) = 1$. The $I-$invariant space V^I has dimension 3, the E_1-invariant space V^{E_1} has dimension 1, the E_2-invariant space V^{E_2} has dimension 1 and there is no $K-$invariant vector [11]. Since $s_1 \in E_1$, observe that V^{E_1} has a basis $i_1 = f_2 + f_3$. Since $s_2 \in E_2$, observe that V^{E_2} has a basis of the form $i_2 = (f_4+f_3)+a(f_2+f_1)$, where a is to be determined. Let $i_3 = \frac{1}{\sigma(\pi)}L_I(i_2) = f_4+qf_3+(-q^2+q+a)f_2+aq^2f_1$ so that $\{i_1, i_2, i_3\}$ is a basis of V^I. Then $L_I(i_3) = q\sigma(\pi)(f_4+f_3+(-q^2+q+aq)f_2+(-q^2+q+aq)f_1)$, which should be equal to $q\sigma(\pi)i_2$. Hence $a = q$. For the purpose of computation, we shall change the basis i_1, i_2, i_3 to $e_1 = f_4+qf_2$, $e_2 = f_2+f_3$, $e_3 = f_3+qf_1$. Hence V^{E_1} has a basis of e_2, V^{E_2} has a basis of $e_1 + e_3$.

With the basis listed above, we obtain the actions of L_I, L_{E_1}, L_{E_2}: $L_I(e_1) = q\sigma(\pi)e_3$, $L_I(e_2) = \sigma(\pi)e_1 - q^2\sigma(\pi)e_2 + q^2\sigma(\pi)e_3$, $L_I(e_3) = \sigma(\pi)e_1 - (q^2 - q)\sigma(\pi)e_2 + (q^2-q)\sigma(\pi)e_3$. $L_{E_1}(e_2) = q\sigma(\pi)e_2$. $L_{E_2}(e_1+e_3) = q\sigma^2(\pi)(e_1+e_3)$. Hence, the eigenvalues of L_I on V^I are $\pm\sqrt{q\sigma^2(\pi)}, -q\sigma(\pi)$. The eigenvalue of L_{E_1} on V^{E_1} is $q\sigma(\pi)$. The eigenvalue of L_{E_2} on V^{E_2} is $q\sigma^2(\pi)$.

After the cancelation of these eigenvalues, the zeta identity for Case $6a$ is

$$\frac{1}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}.$$

4.2 Main results

Theorem 4.2. *The table below summarizes the eigenvalues of each operator from each type of the representations in $L^2(\Gamma\backslash G)$:*

det	$I-L_Iu$	$I-L_{E_1}u$	$I-L_{E_2}u$	$I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4$
1	$\pm\sqrt{q^2\chi_1\sigma^2(\pi)}$ $\pm\sqrt{q^2\chi_2\sigma^2(\pi)}$ $\pm\sqrt{q^2\chi_1\chi_2^2\sigma^2(\pi)}$ $\pm\sqrt{q^2\chi_1^2\chi_2\sigma^2(\pi)}$	$q^{\frac{3}{2}}\chi_1\sigma(\pi)$ $q^{\frac{3}{2}}\chi_2\sigma(\pi)$ $q^{\frac{3}{2}}\chi_1\chi_2\sigma(\pi)$ $q^{\frac{3}{2}}\sigma(\pi)$	$q^2\chi_1\sigma^2(\pi)$ $q^2\chi_2\sigma^2(\pi)$ $q^2\chi_1\chi_2^2\sigma^2(\pi)$ $q^2\chi_1^2\chi_2\sigma^2(\pi)$	$q^{\frac{3}{2}}\chi_1\sigma(\pi)$ $q^{\frac{3}{2}}\chi_2\sigma(\pi)$ $q^{\frac{3}{2}}\chi_1\chi_2\sigma(\pi)$ $q^{\frac{3}{2}}\sigma(\pi)$
$2a$	$\pm\sqrt{q^{\frac{3}{2}}\chi\sigma^2(\pi)}$ $\pm\sqrt{q^{\frac{3}{2}}\chi^3\sigma^2(\pi)}$	$q\chi\sigma(\pi)$	$q^{\frac{3}{2}}\chi\sigma^2(\pi)$ $q^{\frac{3}{2}}\chi^3\sigma^2(\pi)$	*none*
$2b$	$\pm\sqrt{q^{\frac{5}{2}}\chi\sigma^2(\pi)}$ $\pm\sqrt{q^{\frac{5}{2}}\chi^3\sigma^2(\pi)}$	$q^{\frac{3}{2}}\chi^2\sigma(\pi)$ $q^{\frac{3}{2}}\sigma(\pi)$ $q^2\chi\sigma(\pi)$	$q^{\frac{5}{2}}\chi\sigma^2(\pi)$ $q^{\frac{5}{2}}\chi^3\sigma^2(\pi)$	$q^{\frac{3}{2}}\chi^2\sigma(\pi)$ $q^{\frac{3}{2}}\sigma(\pi)$ $q^2\chi\sigma(\pi), q\chi\sigma(\pi)$
$3a$	$\pm\sqrt{q\chi\sigma^2(\pi)}$ $-q\chi\sigma(\pi), -q\sigma(\pi)$	$q\chi\sigma(\pi)$ $q\sigma(\pi)$	$q\chi\sigma^2(\pi)$	*none*
$3b$	$\pm\sqrt{q^3\chi\sigma^2(\pi)}$ $q\chi\sigma(\pi), q\sigma(\pi)$	$q^2\chi\sigma(\pi)$ $q^2\sigma(\pi)$	$q^3\chi\sigma^2(\pi)$ $q^2\chi^2\sigma^2(\pi), q^2\sigma^2(\pi)$	$q^2\chi\sigma(\pi), q\chi\sigma(\pi)$ $q^2\sigma(\pi), q\sigma(\pi)$
$4a$	$-\sigma(\pi)$	*none*	*none*	*none*
$4d$	$q^2\sigma(\pi)$	$q^3\sigma(\pi)$	$q^4\sigma^2(\pi)$	$q^3\sigma(\pi), q^2\sigma(\pi), q\sigma(\pi), \sigma(\pi)$
$5a$	$\pm\sqrt{-q\sigma^2(\pi)}$	*none*	$-q\sigma^2(\pi)$	*none*
$5b$	$\pm\sqrt{-q^2\sigma^2(\pi)}$	$-q\sigma(\pi)$	$-q^2\sigma^2(\pi)$	*none*
$5c$	$\pm\sqrt{-q^2\sigma^2(\pi)}$	$q\sigma(\pi)$	$-q^2\sigma^2(\pi)$	*none*
$5d$	$\pm\sqrt{-q^3\sigma^2(\pi)}$	$\pm q^2\sigma(\pi)$	$-q^3\sigma^2(\pi)$	$\pm q^2\sigma(\pi), \pm q\sigma(\pi)$
$6a$	$\pm\sqrt{q\sigma^2(\pi)}, -q\sigma(\pi)$	$q\sigma(\pi)$	$q\sigma^2(\pi)$	*none*
$6b$	$-q\sigma(\pi)$	$q\sigma(\pi)$	*none*	*none*
$6c$	$q\sigma(\pi)$	*none*	$q^2\sigma^2(\pi)$	*none*
$6d$	$\pm\sqrt{q^3\sigma^2(\pi)}, q\sigma(\pi)$	$q^2\sigma(\pi), q^2\sigma(\pi)$	$q^3\sigma^2(\pi), q^2\sigma^2(\pi)$	$q^2\sigma(\pi), q^2\sigma(\pi), q\sigma(\pi), q\sigma(\pi)$

Summarizing the previous results, the operators satisfy

$$\frac{(1-u^2)^{\chi(X_\Gamma)}(1-q^2u^2)^n}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)},$$

where $n = \frac{1}{2}(-m(2a)+m(2b)+2-m(5b)-m(5c)+2m(5d)-m(6c)+m(6d))$.

Theorem 4.3. [3]. *Each Steinberg representation of Case* $4a$ *occurs in* $L^2(\Gamma\backslash G)$ *with multiplicity* $\chi(X_\Gamma)-1$, *where* $\chi(X_\Gamma)$ *is the Euler characteristic of* X_Γ.

Proof. Let m be the multiplicity of each Steinberg representation of Case $4a$ occurs in $L^2(\Gamma\backslash G)$. It is shown that dim $H^2(\Gamma, \mathrm{R}) = m$ in [3] and dim $H^1(\Gamma, \mathrm{R}) = 0$ by Kazdan. By definition dim $H^0(\Gamma, \mathrm{R}) = 1$. Hence, $m = \chi(\Gamma) - 1$, where $\chi(\Gamma) =$ dim $H^2(\Gamma, \mathrm{R})-$ dim $H^1(\Gamma, \mathrm{R})+$ dim $H^0(\Gamma, \mathrm{R})$. Moreover, it is shown by Serre that $\chi(X_\Gamma) = \chi(\Gamma)$ when X_Γ is a finite CW-complex and $\pi_1(X_\Gamma) = \Gamma$, $\pi_n(X_\Gamma) = 0$ $\forall n > 1$. Therefore $m = \chi(X_\Gamma) - 1$.

Denote by N_0, N_1 and N_2 the number of vertices, edges and chambers in X_Γ, respectively. In the following lemma, we find the relations of these numbers with the dimensions of the L^2-spaces.

Lemma 4.1. 1. *Let m be the number of primitive vertices in X_Γ. Then $dim L^2(\Gamma \setminus G/K) = 2m$. The number of nonspecial vertices is $(q^2 + 1)m$. Hence $N_0 = (q^2 + 3)m$.*

2. *$dim L^2(\Gamma \setminus G/E_1) = 2N_1'$ and $dim L^2(\Gamma \setminus G/E_2) = N_1''$, where N_1' is the number of type-one edges, N_1'' is the number of type-two edges and $N_1' + N_1'' = N_1$.*

3. *$dim L^2(\Gamma \setminus G/I) = 2N_2$.*

Proof. 1. The primitive vertices and nonprimitive vertices are in one-to-one correspondence via $K \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} K$. Each special vertex is adjacent to $q^3 + q^2 + q + 1$ nonspecial vertices. Each nonspecial vertex is adjacent to $q+1$ primitive or nonprimitive special vertices [Set]. Hence the number of nonspecial vertices is $(q^2+1)m$.

2 and 3. This is because τE_1 and E_1 represent the same undirected edge, and τI and I represent the same undirected chamber.

To finish the proof of the main theorem, observe the degree on both sides of the identity

$$\frac{(1-u^2)^{\chi(X_\Gamma)}(1-q^2u^2)^n}{det(I-A_1u+qA_2u^2-q^3A_1u^3+q^6Iu^4)} = \frac{det(I-L_Iu)}{det(I-L_{E_1}u)det(I-L_{E_2}u^2)}.$$

We get $2\chi(X_\Gamma)+2n-8m = 2N_2-(2N_1'+2N_1'')$, which implies $n=-(q^2-1)m$ by Lemma 4.1.

The complex X_Γ is called Ramanujan if the nontrivial irreducible unitary representations occurring in $L^2(\Gamma\setminus G/I)$ are tempered. By observing the eigenvalues of each operator on each representation space, we get equivalent conditions for X_Γ to be Ramanujan.

Corollary 4.1. *(F-Li-Wang) The following statements are equivalent:*

(1) X_Γ *is Ramanujan.*

(2) *The nontrivial zeros of* $det(I-L_{E_2}u)$ *do not have absolute values* q^{-3} *or* $q^{-\frac{5}{2}}$. *When the nontrivial zeros of* $det(I-L_{E_2}u)$ *have absolute value* q^{-2}, *the corresponding eigenspace must contain* K*-invariant vectors.*

(3) *The nontrivial zeros of* $det(I-L_Iu)$ *do not have absolute values* $q^{-\frac{3}{2}}$

or $q^{-\frac{5}{4}}$. *When the nontrivial zeros of* $\det(I - L_I u)$ *are equal to* $\pm q^{-1} i$, *the corresponding eigenspace must contain* K*-invariant vectors. When the nontrivial zeros of* $\det(I - L_I u)$ *are equal to* $\pm q^{-1}$, *the corresponding eigenspace must contain* E_1*-invariant vectors.*

Chapter 5

Nonspecial vertices

Since A_1 and A_2 act on special vertices, it is natural to ask whether there is a similar identity involving operators on nonspecial vertices. So far we have not succeeded in answering this question because $P'_{02}-$invariant spaces have lower dimensions. We record our work on nonspecial vertices in this chapter.

5.1 Representations containing P'_{02}-invariant vectors

Listed below are the cases of the irreducible representation spaces which contain $P'_{02}-$invariant vectors:

Since $\begin{pmatrix} 0 & 0 & 0 & -\pi^{-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pi & 0 & 0 & 0 \end{pmatrix}$, s_2 and $\tau \in P'_{02}$, a vector f in the P'_{02}-invariant subspace of $Ind(\chi_1 \times \chi_2 \rtimes \sigma)$ satisfies the following: $f(P'_{02}) = f(s_2P'_{02})$, $f(s_1P'_{02}) = f(s_1s_2P'_{02}) = q^{-\frac{1}{2}}\chi_1\sigma(\pi)f(P'_{02})$, $f(s_2s_1P'_{02}) = f(s_2s_1s_2p'_{02}) =$

$q^{-\frac{3}{2}}\chi_1\chi_2\sigma(\pi)f(P'_{02})$, $f(s_1s_2s_1P'_{02}) = f(s_1s_2s_1s_2P'_{02}) = q^{-2}\chi_1(\pi)f(P'_{02})$. Hence $f = f_1+f_3+q^{-\frac{1}{2}}\chi_1\sigma(\pi)f_2+q^{-\frac{1}{2}}\chi_1\sigma(\pi)f_4+q^{-\frac{3}{2}}\chi_1\chi_2\sigma(\pi)f_5+q^{-\frac{3}{2}}\chi_1\chi_2\sigma(\pi)f_7+q^{-2}\chi_1(\pi)f_6 + q^{-2}\chi_1(\pi)f_8$ is a basis of the P'_{02}-invariant subspace of Case 1.

Case $2b$: The subrepresentation of $Ind(\nu^{-\frac{1}{2}}\chi \times \nu^{\frac{1}{2}}\chi \rtimes \sigma)$. A vector f in the I-invariant space satisfies $f(s_1g) = f(g)$. So do vectors in the P'_{02}-invariant subspace. Hence we obtain the following equalities: $q^{-\frac{1}{2}}\chi_1\sigma(\pi) = 1$, $q^{-2}\chi_1(\pi) = q^{-\frac{3}{2}}\chi_1\chi_2\sigma(\pi)$. Simplify and we get $\chi\sigma(\pi) = 1$.

Case $2a$: The subrepresentation of $Ind(\nu^{\frac{1}{2}}\chi \times \nu^{-\frac{1}{2}}\chi \rtimes \sigma)$. A vector f in the I-invariant space satisfies $f(1) = -qf(s_1)$, $f(s_2) = -qf(s_1s_2)$, $f(s_2s_1) = -qf(s_1s_2s_1)$, $f(s_2s_1s_2) = -qf(s_1s_2s_1s_2)$. So do vectors in the P'_{02}-invariant subspace. Hence we obtain the following equalities $1 = (-q)(q^{-\frac{1}{2}}\chi_1\sigma(\pi))$, $q^{-\frac{3}{2}}\chi_1\chi_2\sigma(\pi) = (-q)(q^{-2}\chi_1(\pi))$. Simplify and we get $\chi\sigma(\pi) = -1$.

Case $3b$: The subrepresentation of $Ind(\chi \times \nu^{-1} \rtimes \nu^{\frac{1}{2}}\sigma)$. A vector f in the I-invariant space satisfies $f(s_2g) = f(g)$. So do vectors in the P'_{02}-invariant subspace. Hence $f = f_1 + f_3 + q^{-1}\chi\sigma(\pi)f_2 + q^{-1}\chi\sigma(\pi)f_4 + q^{-1}\chi\sigma(\pi)f_5 + q^{-1}\chi\sigma(\pi)f_7+q^{-2}\chi(\pi)f_6+q^{-2}\chi(\pi)f_8$ is a basis of the P'_{02}-invariant subspace.

Case $4d$: The subrepresentation of $Ind(\nu^{-2} \times \nu^{-1} \rtimes \nu^{\frac{3}{2}}\sigma)$. A vector f in the I-invariant space satisfies $f(g) = f(h), \forall g, h \in G$. So do vectors in the P'_{02}-invariant subspace. Simplify and we get $\sigma(\pi) = 1$.

Case $5b$: The subrepresentation of $Ind(\xi_0 \times \nu\xi_0 \rtimes \nu^{-\frac{1}{2}}\xi_0\sigma)$. A vector

f in the E_2-invariant space satisfies $f(1) = f(s_1) = f(s_2) = f(s_1s_2) = -q^2f(s_2s_1) = -q^2f(s_1s_2s_1) = -q^2f(s_2s_1s_2) = -q^2f(s_1s_2s_1s_2)$. Simplify the equalities and we get $\sigma(\pi) = 1$.

Case 5c: The subrepresentation of $Ind(\xi_0 \times \nu\xi_0 \rtimes \nu^{-\frac{1}{2}}\sigma)$. A vector f in the E_2-invariant space satisfies $f(1) = f(s_1) = f(s_2) = f(s_1s_2) = -q^2f(s_2s_1) = -q^2f(s_1s_2s_1) = -q^2f(s_2s_1s_2) = -q^2f(s_1s_2s_1s_2)$. Hence we simplify the equalities and get $\sigma(\pi) = -1$.

Case 6c: The subrepresentation of $Ind(1_{F^*} \times \nu^{-1} \rtimes \nu^{\frac{1}{2}}\sigma)$. A vector f in the E_2-invariant space satisfies $f(1) = f(s_2) = -qf(s_1) = -qf(s_1s_2) = -qf(s_2s_1) = -qf(s_2s_1s_2) = q^2f(s_1s_2s_1) = q^2f(s_1s_2s_1s_2)$. Hence we simplify the equalities and get $\sigma(\pi) = -1$.

Case 6d: The subrepresentation of $Ind(\nu^{-1} \times 1_{F^*} \rtimes \nu^{\frac{1}{2}}\sigma)$. A vector f in the E_2-invariant space satisfies $f(1) = f(s_1) = f(s_2) = f(s_1s_2)$, $f(s_2s_1) = f(s_1s_2s_1) = f(s_2s_1s_2) = f(s_1s_2s_1s_2)$. Hence we simplify the equalities and get $\sigma(\pi) = 1$.

5.2 Operators on nonspecial vertices

Definition 5.1. *The operators B_1 and B_2 are defined in a similar way as A_1 and A_2. Let* $\overline{B_1} = P'_{02}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix}P'_{02} = \bigcup_{a,b,c\in F_q}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \pi b & a & \pi & 0 \\ \pi c & \pi b & 0 & \pi \end{pmatrix}P'_{02}$

$$\bigcup_{c,\alpha\in F_q}\begin{pmatrix}1&0&0&0\\-\pi\alpha&\pi&0&0\\0&0&1&0\\\pi c&0&\pi\alpha&\pi\end{pmatrix}P'_{02}\bigcup_{a\in F_q}\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&\pi&a&0\\\pi&0&0&0\end{pmatrix}P'_{02}\bigcup\begin{pmatrix}0&0&0&1\\0&0&\pi&0\\0&1&0&0\\\pi&0&0&0\end{pmatrix}P'_{02}$$

$$\bigcup_{b\in F_q^*}\begin{pmatrix}0&1&0&0\\\pi&-\alpha&0&0\\0&b&0&1\\\pi b&0&\pi&\alpha\end{pmatrix}P'_{02}\bigcup_{\alpha\in F_q^*}\begin{pmatrix}0&1&0&0\\\pi&-\alpha&0&0\\0&0&0&1\\0&0&\pi&\alpha\end{pmatrix}P'_{02}.$$

Let $\overline{B_1}=B_1\cup(q^2-1)I$. B_1 *could also be regarded as an operator acting on* $C(\Gamma\backslash G/P'_{02})$.

Let $\overline{B_2}=P'_{02}\begin{pmatrix}\pi^{-1}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&\pi\end{pmatrix}P'_{02}=\bigcup_{b,c,c',\alpha\in F_q}\begin{pmatrix}\pi^{-1}&0&0&0\\-\alpha&1&0&0\\b&0&1&0\\\pi c+c'&\pi b&\pi\alpha&\pi\end{pmatrix}P'_{02}$

$$\bigcup_{b,\alpha\in F_q,c\in F_q^*}\begin{pmatrix}1&0&0&0\\-\alpha&1&0&0\\b&0&1&0\\c&b&\alpha&1\end{pmatrix}P'_{02}\bigcup_{a\in F_q,b\in F_q^*}\begin{pmatrix}0&1&0&0\\1&0&0&0\\a&b&0&1\\b&0&1&0\end{pmatrix}P'_{02}$$

$$\bigcup_{\alpha\in F_q^*}\begin{pmatrix}0&1&0&0\\0&-\alpha&0&-1\\1&0&0&0\\\alpha&0&1&0\end{pmatrix}P'_{02}\bigcup\begin{pmatrix}0&0&0&-1\\0&0&-1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}P'_{02}\bigcup_{a,a',b\in F_q}\begin{pmatrix}1&0&0&0\\0&\pi^{-1}&0&0\\\pi b&\pi a+a'&\pi&0\\0&b&0&1\end{pmatrix}P'_{02}$$

$$\bigcup_{b\in F_q,a\in F_q^*}\begin{pmatrix}1&0&0&0\\0&1&0&0\\b&\pi^{-1}a&1&0\\0&b&0&1\end{pmatrix}P'_{02}\bigcup_{\alpha\in F_q}\begin{pmatrix}1&0&0&0\\-\pi\alpha&\pi&0&0\\0&0&\pi^{-1}&0\\0&0&\alpha&1\end{pmatrix}P'_{02}$$

$$\bigcup_{a,b,c\in F_q,\alpha\in F_q^*}\begin{pmatrix}0&\pi^{-1}&0&0\\1&-\pi^{-1}\alpha&0&0\\a&b-\pi^{-1}a\alpha&0&1\\\pi b&c&\pi&\alpha\end{pmatrix}P'_{02}\bigcup_{c,\alpha\in F_q,b\in F_q^*}\begin{pmatrix}0&\pi^{-1}&0&0\\\pi&-\alpha&0&0\\0&\pi^{-1}b&0&\pi^{-1}\\\pi b&c&\pi&\alpha\end{pmatrix}P'_{02}.$$

Let $\overline{B_2}=B_2\cup(q-1)B_1\cup(q^3-2q^2-q)I$. B_2 *could also be regarded as an operator acting on* $C(\Gamma\backslash G/P'_{02})$.

Definition 5.2. *Let* $L_2=E_2\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\pi&0\\0&0&0&\pi\end{pmatrix}E_2$

$$= \bigcup_{a,b,c\in F_q} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \pi b & a & \pi & 0 \\ \pi c & \pi b & 0 & \pi \end{pmatrix} E_2 \bigcup_{c,\alpha\in F_q} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\pi\alpha & \pi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \pi c & 0 & \pi\alpha & \pi \end{pmatrix} E_2,$$

which could also be regarded as an operator acting on $C(\Gamma\backslash G/E_2)$.

Definition 5.3. *Let* $L_3 = I \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \pi & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix} I = \bigcup_{a,b,c\in F_q} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \pi b & \pi a & \pi & 0 \\ \pi c & \pi b & 0 & \pi \end{pmatrix} I,$ *which could also be regarded as an operator acting on* $C(\Gamma\backslash G/I)$.

5.3 Eigenvalues

Case 1: With the basis listed before, a straightforward computation shows that L_3 on V^I has eigenvalues $q^{\frac{3}{2}}\chi_1\sigma(\pi)$, $q^{\frac{3}{2}}\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\chi_1\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\sigma(\pi)$, $q^{\frac{3}{2}}\chi_1\sigma(\pi)$, $q^{\frac{3}{2}}\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\chi_1\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\sigma(\pi)$. L_2 on V^{E_2} has eigenvalues $q^{\frac{3}{2}}\chi_1\sigma(\pi)$, $q^{\frac{3}{2}}\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\chi_1\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\sigma(\pi)$. The inverse of the roots of $det(I - B_1u + qB_2u^2 - q^3B_1u^3 + q^6Iu^4)$ on $V^{P'_{02}}$ are $q^{\frac{3}{2}}\chi_1\sigma(\pi)$, $q^{\frac{3}{2}}\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\chi_1\chi_2\sigma(\pi)$, $q^{\frac{3}{2}}\sigma(\pi)$.

Case 2b: L_3 on V^I has eigenvalues $q^{\frac{3}{2}}\sigma(\pi), q^2\chi\sigma(\pi), q^2\chi\sigma(\pi), q^{\frac{3}{2}}\chi^2\sigma(\pi)$. L_2 on V^{E_2} has eigenvalues $q^{\frac{5}{2}}\sigma(\pi), q^{\frac{5}{2}}\chi^2\sigma(\pi)$. Here $\chi\sigma(\pi) = \pm 1$.

The inverse of the roots of $det(I - B_1u + qB_2u^2 - q^3B_1u^3 + q^6Iu^4)$ on $V^{P'_{02}}$ are $q^{\frac{5}{2}}\sigma(\pi), q^{\frac{5}{2}}\chi^2\sigma(\pi)$, $q^{\frac{3}{2}}\sigma(\pi), q^{\frac{3}{2}}\chi^2\sigma(\pi)$, where $\chi\sigma(\pi) = 1$.

Observing these eigenvalues, there is no way we could achieve an identity with only trivial eigenvalues left since the eigenvalues from the $P'_{02}-$invariant space is only half as many as other operators. And the situation is similar for other representation spaces.

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