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TOPICS ON STABILITY OF COMPLEX FLUID MODELS

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Abstract

In this thesis, we study analytical problems related to two models in the hydrodynamics of complex fluids. The first is the general Ericksen-Leslie system, which models nematic liquid crystal flow, while the latter is a diffuse-interface model for the mixture of two incompressible fluids. Both models are based on a special coupling between the induced elastic stress and transport property of microstructures. Both models can be derived in the energetic variational framework which demonstrates the consistent exchange of the kinetic energy of the fluid and internal energy due to elastic effects.

For the general Ericksen-Leslie system, in both the two and three dimensional cases, we develop the existence theory for global classical solutions with various assumptions on physical relations between viscosity coefficients. Meanwhile, we study the asymptotic behavior of global bounded solutions as time goes to infinity and show that the asymptotic limit is unique. More importantly, we reveal the various roles of physical relations on corresponding analytical results. For the diffuse-interface model, within the study of axisymmetric solutions, we construct perturbations to near infinite-energy solutions.

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Chapter 1

Introduction

This thesis is devoted to the study of analytical problems in hydrodynamics of complex fluids. Our main study is of the general Ericksen-Leslie system (2.1.10)–(2.1.14), which models nematic liquid crystal flow and a diffuse-interface model (5.1.1)-(5.1.3) for the mixture of two incompressible fluids. From the energetic variational point of view, both are models whose equations of conservation of momentum can be derived via a calculus of variations. To have a better understanding of their importance and relations, we start from the general system of equations in fluid mechanics.

1.1 Constitutive law

In the context of fluid mechanics, compressible fluids are described by the following hydrodynamic system [1]:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ \rho(u_t + u \cdot \nabla u) = \nabla \cdot \sigma. \end{cases} \tag{1.1.1}$$

Here ρ is the fluid density, u is the fluid velocity, and σ is the Cauchy stress tensor, all of which are functions of time variable t and space variable x in Eulerian coordinates. Throughout this thesis, the notation ∇ is used to represent the gradient with respect to spatial variables. In the system (1.1.1), the first equation stands for conservation of mass, which is also referred to as the continuity equation; the second equation represents conservation of linear momentum. Analogously, the hydrodynamic system describing incompressible fluids is (assuming the density before deformation is 1):

$$\begin{cases} \nabla \cdot u = 0, \\ u_t + u \cdot \nabla u = \nabla \cdot \sigma. \end{cases} \tag{1.1.2}$$

The stress tensor can be decomposed as $\sigma = -pI + \tau$, where $-pI$ is the normal part. For incompressible fluids, p is the Lagrangian multiplier due to the incompressibility condition, and τ is the tangential part. For simple fluids (inviscid case), the tangential part of stress tensor is zero, hence $\sigma = -pI$. While for complex fluids (inviscid case), the corresponding tangential part is nonzero, hence the system (1.1.2) can be written as

$$\begin{cases} \nabla \cdot u = 0, \\ u_t + u \cdot \nabla u + \nabla p = \nabla \cdot \tau. \end{cases} \tag{1.1.3}$$

From here on, we discuss the incompressible case only.

If the tangential part of the stress tensor is zero in (1.1.3) (inviscid simple fluid), we get the following classical Eulerian system of equations:

$$\begin{cases} \nabla \cdot u = 0, \\ u_t + u \cdot \nabla u + \nabla p = 0. \end{cases} \tag{1.1.4}$$

For viscous Newtonian fluids, τ depends on ∇u linearly:

$$\tau = 2\mu D(u), \tag{1.1.5}$$

where

$$D(u) = \frac{\nabla u + \nabla u^T}{2}$$

is called Cauchy strain tensor and μ is the fluid viscosity coefficient ($\mu > 0$). The relation (1.1.5) that connects stress and strain is usually called the constitutive law. In this case, (1.1.3) becomes

$$\begin{cases} \nabla \cdot u = 0, \\ u_t + u \cdot \nabla u + \nabla p = \mu \Delta u, \end{cases} \tag{1.1.6}$$

which is the well-known incompressible Navier-Stokes equations.

The complex fluids studied in the present thesis are non-Newtonian fluids that do not satisfy the Newtonian Law (1.1.5). For such fluids there is no general constitutive law to describe the dependence of the stress tensor on the strain or the deformation. Both nematic liquid crystal flow and the mixture of two incompressible fluids studied in this thesis exhibit certain elastic property and the induced elastic stress is contained in the equation of conservation of linear momentum.

1.2 Energetic variational formulation

From the energetic point of view, there is special coupling between the induced elastic stress and the transport property of nontrivial microstructures in complex fluids. The energetic variational principles used in this thesis combine the maximum dissipation principle (for long time dynamics) and the least action principle, or equivalently, the principle of virtual work (for intrinsic and short time dynamics) into a force balance law that expands the law of conservation of momentum to include dissipation. This procedure is a modern reworking of Rayleigh's dissipation principle in [2] motivated by Onsager's treatment of dissipation (c.f. [3, 4]). This procedure optimizes both the action functional of classical mechanics and the dissipation functional. The least action principle gives us the Hamiltonian (reversible) part of the system related to conservative forces, while the maximal dissipation principle provides the dissipative (irreversible) part of the system related to dissipative forces.

The energy variational treatment of complex fluids starts with the energy dissipative law for the whole coupled system:

$$\frac{dE^{total}}{dt} = -\mathcal{D},$$

where E^{total} is the total energy of the system and $\mathcal{D}$ is the dissipation function of Onsager, which usually consists of a linear combination of the squares of various rate functions, such as velocity and rate of strain, etc. (cf. [3–5]). In a classical Hamiltonian conservative system, the energy $E^{total} = E^{kinetic} + E^{internal}$ is the sum of kinetic and internal energies.

In the context of hydrodynamics, the basic variable is the flow map (particle trajectory), $x(X, t)$. (X is the original labeling (the Lagrangian coordinate) of the particle, which is also referred to as the material coordinate. x is the current (Eulerian) coordinate, which is also called the reference coordinate.) For a given velocity field $v(x, t)$, the flow map is defined by the ODE:

$$x_t = v(x(X, t), t), \quad x(X, 0) = X.$$

The deformation tensor (strain) of the flow map is given by

$$\mathcal{F}(x(X, t)) = \frac{\partial x(X, t)}{\partial X},$$

which satisfies the following transport equation:

$$\mathcal{F}_t + v \cdot \nabla \mathcal{F} = \nabla v \mathcal{F}.$$

Here, $\mathcal{F}$ carries all information of microstructures and configurations. It is noted that all evolutions are based on the above relations of the flow map between the reference domain, Ω_0, at time 0 and the current domain, Ω_t, at time t.

We take a simple example in dealing with simple fluids to illustrate the main idea of the energetic variational approach. A simple fluid is described by the incompressible Navier-Stokes equations given by (1.1.6) with suitable boundary and initial conditions. One can directly arrive at the following dissipative energy law:

$$\frac{d}{dt}\int \frac{1}{2}|u|^2 dx = -\int \mu|\nabla u|^2 dx. \tag{1.2.7}$$

Conversely, based on the dissipative energy law (1.2.7), we can derive the system (1.1.6) through energetic variational approaches. It follows from (1.2.7) that the total energy E^{total} and the dissipation $\mathcal{D}$ are given by

$$E^{total} = \int \frac{1}{2}|u|^2 dx, \quad \mathcal{D} = \int \mu|\nabla u|^2 dx, \tag{1.2.8}$$

respectively.

We define the action functional $\mathcal{A}$ as

$$\mathcal{A} = \int_0^T \int_{\Omega_t} \frac{1}{2}|u|^2 dx dt. \tag{1.2.9}$$

After pulling back the current domain Ω_t to the reference domain Ω_0 through the flow map and using the incompressibility condition, the action functional becomes

$$\mathcal{A}(x) = \mathcal{A} = \int_0^T \int_{\Omega_0} \frac{1}{2}|x_t|^2 dX dt. \tag{1.2.10}$$

We choose a family of volume preserving diffeomorphism, x^ε, such that the infinitesimal generator is $\frac{dx^\varepsilon}{d\varepsilon}|_{\varepsilon=0} = y$. It is noted that the volume preserving assumption implies $\nabla \cdot y$=0. Then the variation with respect to x (least action principle), yields the Hamiltonian part of the system. The least action principle tells

$$\begin{aligned} & \frac{\delta \mathcal{A}(x)}{\delta x} = 0 \\ \Rightarrow \quad & \frac{d\mathcal{A}(x^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = 0 \\ \Rightarrow \quad & 0 = \int_0^T \int_{\Omega_0} (x_t, y_t) dX dt = -\int_0^T \int_{\Omega_t} (u_t + u\cdot\nabla u, y) dx dt, \end{aligned}$$

where we used integration by parts and pushed forward from Ω_0 to Ω_t in the last step. Since y is divergence-free, we need to add a pressure term as a Lagrange multiplier, thus obtaining the Euler equation as the momentum equation

$$\begin{aligned} u_t + u\cdot\nabla u + \nabla p &= 0, \\ \nabla\cdot u &= 0. \end{aligned} \tag{1.2.11}$$

On the other hand, consider the dissipation functional

$$\mathcal{D} = \int_0^T \int_{\Omega_t} \frac{\mu}{2} |\nabla u|^2 dx dt. \tag{1.2.12}$$

Choose a family of rate functions $u^\varepsilon = u + \varepsilon v$, with $\nabla \cdot v = 0$. Then, the variation with respect to the rate function (Onsager's maximum dissipation law), gives the irreversible part of the system.

$$\begin{aligned} & \frac{\delta \mathcal{D}}{\delta u} = 0 \\ \Rightarrow \quad & 0 = \int_0^T \int_{\Omega_t} (\nabla u, \nabla v) dx dt = -\int_0^T \int_{\Omega_t} (\Delta u, v) dx dt. \end{aligned}$$

As a result, we obtain Stokes equation:

$$\begin{aligned} \mu \Delta u &= \nabla p, \\ \nabla \cdot u &= 0. \end{aligned} \tag{1.2.13}$$

Remark 1.2.1. *One can infer from the derivations of* (1.2.11) *and* (1.2.13) *that the term* $\mu \Delta u$ *serves as dissipative force.*

1.3 Summary of mathematical work

1.3.1 Liquid crystal model

The first model we consider is the general Ericksen-Leslie system (2.1.10)–(2.1.12). Physically, the general Ericksen-Leslie system is a coupled system used to model nematic liquid crystal flows. It is a macroscopic continuum description of the time evolutions of these materials influenced by both the flow field, $v(x,t)$, and the microscopic orientational configuration, $d(x,t)$, which can be derived from the coarse graining of the directions of rod-like liquid crystal molecules. The problem contains three variables: the velocity vector, $v = (v_1, v_2, v_3)^T$, the director vector, $d = (d_1, d_2, d_3)^T$, and the hydrostatic pressure function P. There are generally two types of relations between these viscosity coefficients in stress terms: Leslie's relations and Parodi's relation. The former are necessary conditions for the liquid crystal system (c.f. [6, 7]). While the latter, namely Parodi's relation, is derived directly from the Onsager reciprocal relation (cf. [8]) expressing the equality of certain relations between flows and forces in thermodynamic systems out of equilibrium (cf. [3]). However, Onsager's relation has not been widely accepted.

Mathematically speaking, due to its complex mathematical structures, only some simplified models or special cases have been studied (c.f. [9, 10] and references therein).

The main contribution of our work, discussed in Chapters 2-4, is to study the well-posedness of the general system with additional assumptions on coefficients, as well as the asymptotic behavior of global solutions. More importantly, we reveal the various roles played by physical relations on mathematical results. For instance, Parodi's relation serves as a stability condition in the liquid crystal system.

Our proof of the existence of global classical solutions relies on a modified Galerkin method introduced in [9]. After generating a sequence of approximate solutions (v_m, d_m), $m = 1, 2, \cdots$ and denoting higher order energy terms $A_m(t) = \|\nabla v_m\|^2 + \|\Delta d_m - f(d_m)\|^2$, one can get a uniform high-order energy estimate for the approximate system, under the large viscosity assumption of one viscosity coefficient μ_4.

Based on the uniform higher order energy control, we have shown uniform estimates for $A_m(t)$. The uniform bound on $A_m(t)$ enables us to pass to the limit. Furthermore, a weak solution together with high-order derivative estimates implies the existence of a strong solution. Finally, a bootstrap argument leads to the existence of classical solutions. The main result in shown in Theorem 2.3.2.

After establishing the existence of globally bounded solutions, the problem of whether the solutions will converge to single equilibria as time tends to infinity becomes a problem of interest. It is well known that the structure of the set of equilibria can be nontrivial and may form a continuum for certain physically reasonable nonlinearities in higher dimensional cases. In particular, under current periodic boundary conditions of the liquid crystal system (2.1.10)-(2.1.12) in n dimensional space, one may expect that the dimension of the set of equilibria is at least n. This is because a shift in each variable should give another steady state (cf. also [11]), e.g., in our case, if $d^*(\cdot)$ is a steady state solution, so is $d^*(\cdot + \tau e_i)$, $1 \leq i \leq n$, $\tau \in \mathbb{Z}^+$. Moreover, we note that for our system, every constant vector d with unit-length serves as an absolute minimizer of the elastic energy functional. As a result, it is highly nontrivial to decide whether a given trajectory will converge to a single equilibrium. To this end, we apply the so-called Łojasiewicz–Simon approach to obtain our goal. Simon's idea relies on a nontrivial generalization of the Łojasiewicz inequality (cf. [12,13]) for analytic functions defined in the finite dimensional space $\mathbb{R}^m$ to infinite dimensional spaces. We refer to [11,14–22] and the references therein for applications to various evolution equations. In order to apply the Łojasiewicz–Simon approach to our problem (2.1.10)–(2.1.14), we need to introduce a suitable Łojasiewicz–Simon type inequality for vector functions with periodic boundary conditions. We prove that, although different kinematic transports for the liquid crystal molecules will yield different dynamics of the hydrodynamical system, the global solutions to the system have uniform long-time behavior under different kinematic transports, i.e., convergence to equilibrium with a uniform convergence rate. The main result is shown in Theorem 3.3.1.

As mentioned previously, Parodi's relation is a direct result of the Onsager reciprocal relations, which are, nevertheless, independent of the second law of thermodynamics. The thermodynamic basis of Onsager reciprocal relations have been criticized(c.f. [23]), but it is admitted meanwhile that, there are certain physical hints indicating that for particular materials, Onsager relations and their counterparts may serve as stability conditions (c.f. [23, 24]). In this work, we provide a mathematical verification of this physical hint for nematic liquid crystal material.

For the dissipative system, the internal energy functional is

$$E(d) = \frac{1}{2}\|\nabla d\|^2 + \int_Q \frac{(|d|^2-1)^2}{4}dx.$$

Suppose d^* is a local energy minimizer of $E(d)$. If the initial velocity, v_0, is close to zero and the initial orientational director, d_0, is close to d^*, we prove that under Parodi's relation, the local energy minimizer, d^*, is Lyaponov stable.

1.3.2 Diffuse-interface model

The study of analytic results on the nematic liquid crystal model stems from the work in [9], where the following highly simplified system was investigated:

$$\begin{aligned} v_t + v\cdot\nabla v + \nabla p &= \mu\Delta v - \lambda\nabla\cdot(\nabla d\otimes\nabla d), && (1.3.14)\\ \nabla\cdot v &= 0, && (1.3.15)\\ d_t + v\cdot\nabla d &= \gamma(\Delta d - f). && (1.3.16) \end{aligned}$$

If we replace the vector function, d, with a scalar function ϕ, then the system, (1.3.14)-(1.3.14), becomes a coupled Navier-Stokes/Allen-Cahn system of equations in $\mathbb{R}^3$, which can be viewed as a phase-field model describing the motion of a mixture of two incompressible viscous fluids (see [25]). In the past, there have been many studies of the dynamic stability problem near zero or equilibrium. In contrast, the problems we are interested in are for point-wise solutions with infinite energy. To this end, in [26] we are concerned with the axisymmetric solutions only.

The consideration of axisymmetry makes it possible to reduce the three dimensional ($3D$) problem to a $2D$ problem. Meanwhile, it should be noted that our system contains the Navier-Stokes equations as a subsystem. By the well-known Caffarelli-Kohn-Nirenberg theory in [27], the singularity set of any suitable weak solution of the $3D$ Navier-Stokes equations has one-dimensional Hausdorff measure zero. Thus, in the case of $3D$ axisymmetric equations, if there is any singularity, it must be along the symmetric axis. This motivates focusing our work on a special study of the system near the z axis.

Differing from the method of asymptotic expansion in [28], we use a more natural and straightforward method: separation of variables in the framework of axisymmetry-namely, in the radial component r and the height z-to derive a $1D$ system of equations. This $1D$ system of equations approximates the $3D$ system along the z axis. We also establish the regularity of global solutions $(u_1^*(z,t), \omega_1^*(z,t), \psi_1^*(z,t), \phi_0^*(z,t))$ for the $1D$ system. Based on the $1D$ solutions, we obtain exact solutions, $(ru_1^*, r\omega_1^*, r\psi_1^*, \phi_0^*)$, to our $3D$ coupled system, though they have infinite energy.

We make an improvement to the infinite energy solutions by adding a cut-off function $\chi(r)$ and perturbation terms $(ru_1, r\omega_1, r\psi_1, \phi_1)$, which ensure that the solutions constructed have finite energy:

$$\tilde{u}(r,z,t) = r(u_1^*(z,t)\chi(r) + u_1(r,z,t)), \tag{1.3.17}$$

$$\tilde{\omega}(r,z,t) = r(\omega_1^*(z,t)\chi(r) + \omega_1(r,z,t)), \tag{1.3.18}$$

$$\tilde{\psi}(r,z,t) = r(\psi_1^*(z,t)\chi(r) + \psi_1(r,z,t)), \tag{1.3.19}$$

$$\tilde{\phi}(r,z,t) = \phi_0^*(z,t)\chi(r) + \phi_1(r,z,t). \tag{1.3.20}$$

Since we are interested in smooth solutions only, one major concern here is the regularity of the perturbation terms. To solve this, we introduce certain higher order energy terms and prove their boundedness. The main mathematical difficulty lies in how to control a series of weighted norms. We discuss it into two subcases, namely the large viscosity case and small initial data case. And the main results are shown in Theorem 5.1.1 and Theorem 5.1.2.

1.4 Some useful lemmas and inequalities

In this section we list some inequalities which are frequently used in the proofs throughout this thesis.

Theorem 1.4.1. (Sobolev Imbedding Theorem) *Assume that Ω is a bounded domain of class C^m. Then we have*
(1) If $mp < n$, then $W^{m,p}(\Omega)$ is continuously imbedded in $L^{q^*}(\Omega)$ with $\frac{1}{q^*} = \frac{1}{p} - \frac{m}{n}$:

$$W^{m,p}(\Omega) \hookrightarrow L^{q^*}(\Omega). \tag{1.4.21}$$

Moreover, the imbedding operator is compact for any q, $1 \le q < q^*$.
(2) If $mp = n$, then $W^{m,p}(\Omega)$ is continuously imbedded in $L^q(\Omega)$, $\forall q$, $1 \le q < \infty$:

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega). \tag{1.4.22}$$

In addition, the imbedding operator is compact for any q, $1 \le q < q^*$. If $p = 1, m = n$, then the above still holds for $q = \infty$.

(3) If $k < m - \frac{n}{p} < k+1$, $k \in N$, *then writing* $m - \frac{n}{p} = k + \alpha$, $k \in N$, $0 < \alpha < 1$, $W^{m,p}(\Omega)$ *is continuously imbedded in* $C^{k,\alpha}(\bar{\Omega})$*:*

$$W^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\bar{\Omega}), \tag{1.4.23}$$

where $C^{k,\alpha}(\bar{\Omega})$ *is the space of functions in* $C^k(\bar{\Omega})$ *whose derivative s of order* k *are Hölder continuous with exponent* α*. In addition, if* $n = m - k - 1$, *and* $\alpha = 1, p = 1$, *then* (1.4.23) *holds for* $\alpha = 1$, *and the imbedding operator is compact from* $W^{m,p}(\Omega)$ *to* $C^{k,\beta}(\bar{\Omega})$, $\forall 0 \leq \beta < \alpha$.

Theorem 1.4.2. (Gargliardo-Nirenberg inequality) *Let* j *and* m *be integers satisfying* $0 \leq j < m$, *and let* $1 \leq q, r \leq \infty$ *and* $p \in \mathbb{R}$, $\frac{j}{m} \leq a \leq 1$ *such that*

$$\frac{1}{p} - \frac{j}{n} = a(\frac{1}{r} - \frac{m}{n}) + (1-a)\frac{1}{q}. \tag{1.4.24}$$

Then
(1) For any $u \in W^{m,r}(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$, *there is a positive constant* C *depending only on* n, m, j, q, r, a *such that the following inequality holds:*

$$|D^j u|_p \leq C|D^m u|_r^a |u|_q^{1-a} \tag{1.4.25}$$

with the following exception: if $1 < r < \infty$ *and* $m - j - \frac{n}{r}$ *is a nonnegative integer, then* (1.4.25) *holds only for a satisfying* $\frac{j}{m} \leq a < 1$.
(2) For any $u \in W^{m,r}(\Omega) \cap L^q(\Omega)$ *where* Ω *is a bounded domain with smooth boundary, there are two positive constants* C_1, C_2 *such that the following inequality holds:*

$$|D^j u|_{p,\Omega} \leq C|D^m u|_{r,\Omega}^a |u|_{q,\Omega}^{1-a} + C_2|u|_{q,\Omega}. \tag{1.4.26}$$

with the same exception as in (1).
In particular, for any $u \in W_0^{m,r}(\Omega) \cap L^q(\Omega)$, *the constant* C_2 *in* (1.4.26) *can be taken as zero.*

Theorem 1.4.3. (Poincaré Inequality I)*Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $u \in H_0^1(\Omega)$*. Then there is a positive constant* C *depending only on* Ω *and* n *such that*

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{1.4.27}$$

Theorem 1.4.4. (Poincaré Inequality II)*Let* Ω *be a bounded domain of* C^1 *in* $\mathbb{R}^n$*. There is a positive constant* C *depending only on* Ω, n *such that for any* $u \in H^1(\Omega)$,

$$\|u\|_{L^2(\Omega)} \leq C\left(\|\nabla u\|_{L^2(\Omega)} + \left|\int_\Omega u dx\right|\right). \tag{1.4.28}$$

Theorem 1.4.5. (Gronwall Inequality) *Let $\eta(t)$ be a nonnegative, absolutely continuous function on $[0,T]$, which satisfies for a.e. t the differential inequality*

$$\eta'(t) \leq \phi(t)\eta(t) + \psi(t),$$

where $\phi(t)$ and $\psi(t)$ are nonnegative, summable functions on $[0,T]$. Then

$$\eta(t) \leq e^{\int_0^t \phi(s)ds} \left[\eta(0) + \int_0^t \psi(s)ds\right] \tag{1.4.29}$$

for all $0 \leq t \leq T$. In particular, if

$$\eta' \leq \phi\eta \quad on\ [0,T]\ and\ \eta(0) = 0,$$

then

$$\eta \equiv 0 \quad on\ [0,T]. \tag{1.4.30}$$

The remaining part of the thesis is organized as follows:

Chapter 2 is devoted to the derivation of the liquid crystal model and the proof of existence of global solutions in the general Ericksen-Leslie system. After the problem setting in the first section, we make a formal physical derivation of the system based on the basic energy law. Meanwhile, through various energetic variational approaches, we distinguish the Hamiltonian and dissipative parts among induced elastic stress. In the third section, we first show the existence of local solutions of an approximate system using a modified Galerkin method, then we provide the proof of a higher order energy inequality under the assumption of one large viscosity coefficients, say, μ_4, which leads to the existence of global classical solutions of the liquid crystal system.

In Chapter 3, we study the long time behavior of the global solution of the liquid crystal system by virtue of a Łojaciewicz-Simon type inequality. In the first section we give a brief discussion of the application of the Łojaciewicz inequality in finite-dimensional-Euclidean space. In the second section, we prove that the global solution will converge to single steady states as time goes to infinity, which implies the unique asymptotic limit of the solution. It is an improvement of the result in [10], where only sequence convergence is obtained. In the last section, by using suitable energy estimates and constructing proper differential inequalities, we provide the estimates on convergence rate in both higher and lower order norms.

In Chapter 4, we investigate the relation between Parodi's relation and the stability of the liquid crystal system, which shows Parodi's relation serves as a stability condition of the system. In the first section, we provide a suitable higher-order energy inequality, which implies the local existence of strong solutions and the global existence provided that the initial data is near equilibrium. In the second section, we prove that if Parodi's relation is satisfied, if the initial velocity is close to zero, and if the initial molecule director is sufficiently close to a *local* minimizer of the elastic energy, then the solution will stay close to the minimizer for all time (Lyaponov stability).

In Chapter 5, we focus on the discussion of axisymmetric solutions to a diffuse-interface model coupling the Navier-Stokes equations and Allen/Cahn equations in $\mathbb{R}^3$, where we construct smooth solutions, which can be considered as perturbations near infinite-energy solutions. In section 1 we give the problem settings in the framework of axisymmetry. Then a $1D$ system of equations is derived in section 2, which approximats the $3D$ system along its symmetry axis. Some useful lemmas are given in section 3. Then based on the solutions to the $1D$ system, by adding perturbation terms, we construct finite energy solutions to the $3D$ system, and we study the global regularity of the constructed solutions in both large viscosity and small initial data cases in section 4.

Chapter 2

Existence of Global Solutions to the General Ericksen-Leslie System

Liquid crystal is often viewed as the fourth state of the matter besides the gas, liquid and solid, or as an intermediate state between liquid and solid. It possesses no or partial positional order, while at the same time, displays an orientational order. The nematic phase is the simplest of liquid crystal phases and is close to the liquid phase. The molecules float around as in a liquid phase, but have the tendency of aligning along a preferred direction due to their orientation. The hydrodynamic theory of liquid crystals due to Ericken and Leslie was developed around 1960's [6, 29, 30]. However, the first rigorous mathematical analysis of the Ericksen-Leslie system was made recently [10] (see [9, 31] for a simplified system which carried important mathematical difficulties of the original Ericksen-Leslie system, except the kinematic transport of the director field). In the following context, after the introduction of the problem setting and related results in section 2.1, the formal physical derivation of the model is made via energetic variational approaches in section 2.2. And the proof of wellposedness of the liquid crystal system under large viscosity case is provided in section 2.3.

2.1 Problem settings and related results

A well established model for nematic liquid crystal flow is the Ericksen–Leslie system consisting of the following equations (cf. [6, 7, 30, 32, 33]):

$$\rho_t + v \cdot \nabla\rho \quad = \quad 0, \tag{2.1.1}$$

$$\rho\dot{v} = \rho F + \nabla\cdot\hat{\sigma}, \quad (2.1.2)$$
$$\rho_1\dot{\omega} = \rho_1 G + \hat{g} + \nabla\cdot\pi. \quad (2.1.3)$$

Equations (2.1.1)-(2.1.3) represent the conservation of mass, linear momentum, and angular momentum respectively, with the anisotropic feature of liquid crystal materials exhibited in (2.1.3) and its nonlinear coupling in (2.1.2) (cf. [6,10]). ρ is the fluid density, ρ_1 is a (positive) inertial constant. $v = (v_1, v_2, v_3)^T$ is flow velocity and $d = (d_1, d_2, d_3)^T$ represents the director of molecules. $\hat{g}$ is the intrinsic force associated with d and π is the director stress. F and G are the external body force and external director body force. The superposed dot denotes the material derivative. The notations

$$A = \frac{1}{2}(\nabla v + \nabla^T v),\ \ \Omega = \frac{1}{2}(\nabla v - \nabla^T v),\ \ \omega = \dot{d} = d_t + (v\cdot\nabla)d,\ \ N = \omega - \Omega\, d$$

denote the rate of strain tensor, skew-symmetric tensor, the material derivative of d, the changing rate of the director relative to fluid, respectively. Here, we consider the flow of an incompressible liquid, namely, $\nabla\cdot v = 0$. We have the following constitutive relations in the system (2.1.1)–(2.1.3) for $\hat{\sigma}$, π and $\hat{g}$:

$$\hat{\sigma}_{ij} = -P\delta_{ij} - \rho\frac{\partial W}{\partial d_{k,i}}d_{k,j} + \sigma_{ij}, \quad (2.1.4)$$
$$\pi_{ij} = \beta_i d_j + \rho\frac{\partial W}{\partial d_{j,i}}, \quad (2.1.5)$$
$$\hat{g}_i = \gamma d_i - \beta_j d_{i,j} - \rho\frac{\partial W}{\partial d_i} + g_i. \quad (2.1.6)$$

P is a scalar function representing the pressure. The vector $\beta = (\beta_1, \beta_2, \beta_3)^T$ and the scalar function γ (called director tension) are the Lagrangian multipliers for the constraint on the length of director $|d| = 1$, with the Oseen-Frank energy functional W for the equilibrium configuration of a unit director field:

$$\begin{aligned} W = {} & \frac{k_1}{2}(\nabla\cdot d)^2 + \frac{k_2}{2}|d\times(\nabla\times d)|^2 + \frac{k_3}{2}|d\cdot(\nabla\times d)|^2 \\ & +(k_2+k_4)[\mathrm{tr}(\nabla d)^2 - (\nabla\cdot d)^2]. \end{aligned} \quad (2.1.7)$$

The kinematic transport of the director d (denoted by g) is given by:

$$g_i = \lambda_1 N_i + \lambda_2 d_j A_{ji}, \quad (2.1.8)$$

while the stress tensor σ has the following form:

$$\sigma_{ji} = \mu_1 d_k d_p A_{kp} d_i d_j + \mu_2 d_j N_i + \mu_3 d_i N_j + \mu_4 A_{ij} + \mu_5 d_j d_k A_{ki} + \mu_6 d_i d_k A_{kj}. \quad (2.1.9)$$

The (independent) coefficients $\mu_1, ..., \mu_6$ that may depend on material and temperature are called Leslie coefficients, which are related to certain local correlations in the fluid (cf. [33]).

In order to reduce the higher-order nonlinearities in the model (in the Lagrangian multipliers β, γ for the nonlinear constraint $|d| = 1$), one frequently used method is to introduce a proper penalty approximation, namely, we add the term $\mathcal{F}(d) = \frac{1}{4\varepsilon^2}(|d|^2-1)^2$ in W, which holds the information on the extensibility of the molecules. After the discussions for each $\varepsilon > 0$, we then take the limit as $\varepsilon \to 0$. This method is motivated by the work on the gradient flow of harmonic maps into the sphere (cf. [34, 35]) and it has been successfully used for other problems (cf. [9, 10, 31, 34]). The reformulated system with penalty approximation also has natural physical interpretations. It is similar to that proposed by Leslie in [7] for the flow of an anisotropic liquid with varying director length. Mathematically, it is also quite similar to the system in [32] for nematic liquid crystals with variable degree of orientation, despite some definite physical differences. We refer to [9] for more discussions.

For the sake of simplicity, we set

$$W = \frac{1}{2}|\nabla d|^2 + \frac{1}{4\varepsilon^2}(|d|^2 - 1)^2.$$

The current choice of W corresponds to the elastically isotropic situation, i.e., $k_1 = k_2 = k_3 = 1$, $k_4 = 0$. The case with more general Oseen-Frank energy (2.1.7) can be treated in the same way, but the argument is more involved. Under the choice of penalized energy W, we can remove the Lagrangian multipliers and set $\gamma = \beta_j = 0$. Since the inertial constant ρ_1 is usually very small, we take $\rho_1 = 0$. Moreover, we assume that the density is constant and external forces vanish, namely, $\rho = 1,\ F = 0,\ G = 0$ (cf. [10]). Note that $F = 0$ is equivalent to the assumption that the exterior forces are conservative (thus can be absorbed into pressure).

Now the system (2.1.1)–(2.1.3) is reformulated to

$$\begin{aligned}
v_t + v \cdot \nabla v + \nabla P &= -\nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot \sigma, && (2.1.10)\\
\nabla \cdot v &= 0, && (2.1.11)\\
d_t + (v \cdot \nabla) d - \Omega d + \frac{\lambda_2}{\lambda_1} A d &= -\frac{1}{\lambda_1}\left(\Delta d - f(d)\right), && (2.1.12)
\end{aligned}$$

where $f(d) = \mathcal{F}'(d) = \frac{1}{\varepsilon^2}(|d|^2 - 1)d$ and σ is given by (2.1.9). We denote by $\nabla d \odot \nabla d$ the $n \times n$-matrix $(n = 2, 3)$ whose (i, j)-entry is $\nabla_i d \cdot \nabla_j d$, $1 \le i, j \le n$. In the following text, we just set $\varepsilon = 1$ and our results indeed hold for all $\varepsilon > 0$. We consider the system

(2.1.10)–(2.1.12) subject to the periodic boundary conditions (i.e., in torus $\mathbb{T}^n$, $n = 2, 3$):

$$v(x + e_i, t) = v(x, t), \quad d(x + e_i, t) = d(x, t), \quad \text{for } (x, t) \in \partial Q \times \mathbb{R}^+, \tag{2.1.13}$$

and to the initial conditions

$$v|_{t=0} = v_0(x), \quad \text{with } \nabla \cdot v_0 = 0, \quad d|_{t=0} = d_0(x), \quad \text{for } x \in Q, \tag{2.1.14}$$

where Q is a unit square in $\mathbb{R}^n$ $(n = 2, 3)$.

Due to Leslie coefficients' temperature dependence, there are differences in behavior between the various coefficients (cf. [33]): μ_4, which does not involve the alignment properties, is a rather smooth function of temperature; but all the other $\mu's$ describe couplings between orientation and flow, and are thus affected by a decrease in the nematic order. The special role played by μ_4 on the well-posedness of the liquid crystal system will be shown in Theorems 2.3.2 and 3.3.1. In order to reduce the complexity of mathematical analysis, we ignore the thermal effect in the subsequent sections so that $\mu's$ are assumed to be constants. Next, the following relations are introduced in the literature (cf. [7])

$$\lambda_1 = \mu_2 - \mu_3, \quad \lambda_2 = \mu_5 - \mu_6, \tag{2.1.15}$$

$$\mu_2 + \mu_3 = \mu_6 - \mu_5. \tag{2.1.16}$$

(2.1.15) is achieved from the hydrodynamic point of view in order to guarantee the entropy condition, that is, the second law of thermodynamics. (2.1.16) is called *Parodi's relation* (cf. [8]), which is derived from the Onsager reciprocal relations expressing the equality of certain relations between flows and forces in thermodynamic systems out of equilibrium (cf. [3]). Under the assumption of Parodi's relation, we see that the dynamics of an incompressible nematic liquid crystal flow involves five independent viscous coefficients.

Since the mathematical structure of E-L system is quite complicated, in the past there were only some works on its simplified versions (cf. [9,31,36–38]). As far as the general E-L system is concerned, the only known result in analysis is [10]. In particular, well-posedness of the general E-L system (2.1.10)-(2.1.12) subject to Dirichlet boundary conditions was proved under the special assumption $\lambda_2 = 0$, which brings another extra constraint on the Leslie coefficients. Although the physical meaning of this assumption is unclear, it brings great convenience in mathematical analysis such that the maximum principle for $|d|$ holds (cf. [10, Theorem 3.1]). For the system (2.1.10)-(2.1.12), the maximum principle for $|d|$ fails when $\lambda_2 \neq 0$. This leads to extra difficulties in the study of well-posedness that we are not able to handle those highly nonlinear stress terms as in [10]. Even in the 2D case, one fails to obtain global existence of solutions without any further restriction on the viscous coefficients. This is rather different from all the cases studied in the literature.

On the other hand, we have to confine ourselves to the periodic boundary conditions, because one cannot get rid of certain boundary terms when performing integration by parts in the derivation of higher-order energy inequalities.

2.2 Derivation of model via energetic variational approaches

In the general case with $\lambda_2 \neq 0$, after imposing some additional constraints, we can still formally establish the basic energy law which governs the dynamics of the general system (2.1.10)–(2.1.12). Conversely, given the basic energy law, one can also recover the general Ericksen–Leslie system by energetic variational approaches. Furthermore, through different types of energetic methods, say, Onsager's maximal dissipation principle and Least Action Principle, we are able to distinguish the dissipative part and conservative part among all stress terms.

2.2.1 Derivation of basic energy law

It has been pointed out [10] that the Ericksen–Leslie system (2.1.10)–(2.1.14) obeys certain dissipative energy law under proper assumptions on the physical coefficients, which plays an important role in the study of hydrodynamical motions of liquid crystal flows (cf. [9, 10]). Generally speaking, the physical singularities tracked by people are those energetically admissible ones (cf. [36]). Denote the total energy of the system (2.1.10)–(2.1.14) by

$$\mathcal{E}(t) = \frac{1}{2}\|v\|^2 + \frac{1}{2}\|\nabla d\|^2 + \int_Q \mathcal{F}(d)dx. \tag{2.2.1}$$

By a direct (formal) calculation with smooth solutions (v, d) to the system (2.1.10)–(2.1.14), we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= -\int_Q \left[\mu_1|d^T Ad|^2 + \frac{\mu_4}{2}|\nabla v|^2 + (\mu_5+\mu_6)|Ad|^2\right] dx \\ &\quad +\lambda_1\|N\|^2 + (\lambda_2-\mu_2-\mu_3)(N, Ad). \end{aligned} \tag{2.2.2}$$

We note that the assumption (2.1.15) is sufficient to guarantee the existence of the Lyapunov-type functional. However, the Parodi's relation (2.1.16) is not necessary in the derivation of (2.2.2). If (2.1.16) is supposed, we immediately arrive at the energy inequality obtained in [10, Theorem 2.1]. Here and after, we always assume that

$$\lambda_1 < 0, \tag{2.2.3}$$

$$\mu_5+\mu_6 \geq 0, \tag{2.2.4}$$

$$\mu_1 \geq 0, \quad \mu_4 > 0. \tag{2.2.5}$$

These assumptions are supposed in [7,39] to provide necessary conditions for the dissipation of the director field. If $\lambda_2 = 0$, it follows from (2.2.2)–(2.2.5) that $\mathcal{E}(t)$ is decreasing in time, which is exactly the case studied in [10].

Lemma 2.2.1. *Suppose that* (2.1.15), (2.1.16), (2.2.3), (2.2.4) *and* (2.2.5) *are satisfied. In addition, if we assume*

$$\frac{(\lambda_2)^2}{-\lambda_1} \le \mu_5 + \mu_6, \tag{2.2.6}$$

then the total energy $\mathcal{E}(t)$ *is decreasing in time such that*

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(t) &= -\int_Q \Big[\mu_1|d^T Ad|^2 + \frac{\mu_4}{2}|\nabla v|^2\Big]\, dx + \frac{1}{\lambda_1}\|\Delta d - f(d)\|^2 \\ &\quad -\Big(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1}\Big)\|Ad\|^2 \le 0. \end{aligned} \tag{2.2.7}$$

Proof. By (2.1.16), i.e., $\lambda_2 = -(\mu_2 + \mu_3)$, we infer from the transport equation of d (cf. (2.1.12)) that

$$\begin{aligned} & \lambda_1\|N\|^2 + (\lambda_2 - \mu_2 - \mu_3)(N, Ad) = (N, \lambda_1 N + \lambda_2 Ad) + \lambda_2(N, Ad) \\ = \;& (N, -\Delta d + f) + \lambda_2(N, Ad) = \frac{1}{\lambda_1}\|\Delta d - f(d)\|^2 + \frac{\lambda_2}{\lambda_1}(Ad, \Delta d - f + \lambda_1 N) \\ = \;& \frac{1}{\lambda_1}\|\Delta d - f(d)\|^2 - \frac{(\lambda_2)^2}{\lambda_1}\|Ad\|^2. \end{aligned} \tag{2.2.8}$$

Inserting the above result into (2.2.2), we arrive at our conclusion. □

On the contrary, if the Parodi's relation (2.1.16) does not hold, alternative assumptions will be required to ensure the dissipation of the total energy.

Lemma 2.2.2. *Suppose that* (2.1.15), (2.2.3), (2.2.4) *and* (2.2.5) *are satisfied. If we also assume that*

$$|\lambda_2 - \mu_2 - \mu_3| \le 2\sqrt{-\lambda_1}\sqrt{\mu_5 + \mu_6}, \tag{2.2.9}$$

then the following energy inequality holds:

$$\frac{d}{dt}\mathcal{E}(t) \le -\int_Q \Big[\mu_1|d^T Ad|^2 + \frac{\mu_4}{2}|\nabla v|^2\Big]\, dx \le 0. \tag{2.2.10}$$

Moreover, if

$$|\lambda_2 - \mu_2 - \mu_3| < 2\sqrt{-\lambda_1}\sqrt{\mu_5 + \mu_6}, \tag{2.2.11}$$

then the dissipation in (2.2.10) *will be stronger in the sense that there exists a small* *constant* $\eta > 0$,

$$\frac{d}{dt}\mathcal{E}(t) \le -\int_Q \left[\mu_1|d^T A d|^2 + \frac{\mu_4}{2}|\nabla v|^2\right] dx - \eta(\|Ad\|^2 + \|N\|^2) \le 0. \tag{2.2.12}$$

Proof. The conclusion easily follows from (2.2.2) and the Cauchy–Schwarz inequality. □

2.2.2 Energetic variational approaches

From the energetic point of view, the system is the coupling between the transport of the director d in the macroscopic velocity field v and the averaged microscopic effect in the form of induced macroscopic elastic stress. This indicates some interesting hydrodynamic and rheological properties of the liquid crystal flows. Based on the basic energy law in Section 2, and due to the special feature of nematic liquid crystal flow such that the molecular orientations are transported and deformed by the flow under parallel transport, we develop a formal physical derivation of the induced elastic stress through energetic variational approaches. This provides us with a better understanding of the competition between hydrodynamic kinetic energy and internal elastic energy due to the presence of the orientational field d.

In the context of hydrodynamics, the basic variable is the flow map (particle trajectory) $x(X,t)$. X is the original labeling (the Lagrangian coordinate) of the particle, which is also referred to as the material coordinate. x is the current (Eulerian) coordinate, and is also called the reference coordinate. For a given velocity field $v(x,t)$, the flow map is defined by the ODE:

$$x_t = v(x(X,t),t), \quad x(X,0) = X.$$

We define the director field

$$d(x(X,t),t) = \mathbb{E}d_0(X)$$

with $d_0(X)$ being the initial condition. For general ellipsoid shaped liquid crystal molecules, the deformation $\mathbb{E}$ carries all the information of micro structures and configurations and it satisfies (cf. [36, 38, 40])

$$\dot{\mathbb{E}} = \Big(\alpha\nabla v + (1-\alpha)(-\nabla^T v)\Big)\mathbb{E}, \tag{2.2.1}$$

which can also be reformulated into a combination of a symmetric part and a skew part:

$$\dot{\mathbb{E}} = \Omega\mathbb{E} + (2\alpha - 1)A\mathbb{E}, \tag{2.2.2}$$

where $2\alpha - 1 = \frac{r^2-1}{r^2+1} \in [-1,1]$ and $r \in \mathbb{R}$ is the aspect ratio of the ellipsoids (cf. [38, 40]). In our present case, $\alpha = \frac{1}{2}(1 - \frac{\lambda_2}{\lambda_1})$ and we deduce from either (2.2.1) or (2.2.2) that the total (pure) transport equation of d is

$$d_t + v \cdot \nabla d - \alpha \nabla v\, d + (1-\alpha)(\nabla^T v) d = d_t + v \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d = 0. \qquad (2.2.3)$$

The energetic variational treatment of complex fluids starts with the energy dissipative law for the whole coupled system [41]:

$$\frac{dE^{tot}}{dt} = -\mathcal{D},$$

where E^{tot} is the total energy of the system such that $E^{tot} = E^{kinetic} + E^{int}$ and $\mathcal{D}$ is the dissipation function of Onsager, which usually consists of a linear combination of the squares of various rate functions such as velocity and rate of strain etc. (cf. [3–5]). Our dissipation functional (like in [42]) departs from Onsager's-loosely defined between Eqs. 5.6 and 5.7 on p. 2227 in [4]-because we use variations with respect to two functions (cf. [43, 44]). The dissipative part uses a variation with respect to the rate function (velocity v) while the conservative (Hamiltonian) part with respect to the domain (position x). In what follows, we recover the system (2.1.10)–(2.1.12) from the basic energy law in the case that (2.1.15) and the Parodi's relation (2.1.16) are satisfied.

The kinetic energy and internal elastic energy of the system (2.1.10)–(2.1.12) are given by

$$E^{kinetic} = \frac{1}{2}\|v\|^2, \quad E^{int} = E(d) = \frac{1}{2}\|\nabla d\|^2 + \int_Q \mathcal{F}(d) dx.$$

The Legendre transformation gives the action of the trajectories of the particles in terms of the flow map $x(X, t)$:

$$\mathbb{A} = \int_0^T (E^{kinetic} - E^{int}) dt.$$

The least action principle optimizes the action $\mathbb{A}$ with respect to all trajectories $x(X, t)$ by setting its variation with respect to domain to zero, namely $\delta_x \mathbb{A} = 0$, with incompressibility of flow and the pure transport equation of d (2.2.3). Then we obtain the weak variational from of the conservative force balance equation of classical Hamiltonian mechanics and recover the conservative (Hamiltonian) part of the full system (2.1.10)–(2.1.12) (see appendix for the detailed calculations). We just formally write down the strong form:

$$\begin{aligned} & v_t + v \cdot \nabla v + \nabla P = -\nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot \tilde{\sigma}, \\ & \text{where} \quad \tilde{\sigma} = \mu_2 N \otimes d + \mu_3 d \otimes N + \eta_5 A d \otimes d + \eta_6 d \otimes A d, \end{aligned}$$

with constants

$$\mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2), \quad \mu_3 = -\frac{1}{2}(\lambda_1 + \lambda_2), \quad \eta_5 = \frac{\lambda_2}{2} - \frac{(\lambda_2)^2}{2\lambda_1}, \quad \eta_6 = -\frac{\lambda_2}{2} + \frac{(\lambda_2)^2}{2\lambda_1}.$$

Here, we use $\otimes$ for the usual Kronecker multiplication, namely, $(a \otimes b)_{i,j} = a_i b_j$ for $a, b \in \mathbb{R}^n$, $n = 2, 3$ and $1 \le i, j \le n$ (cf. e.g., [9,38]).

Taking the elastic dissipation into account in the transport equation (2.2.3), we get

$$d_t + v \cdot \nabla d - \Omega\, d + \frac{\lambda_2}{\lambda_1} A\, d = \frac{1}{\lambda_1}\frac{\delta E}{\delta d} = -\frac{1}{\lambda_1}(\Delta d - f(d)). \qquad (2.2.4)$$

The dissipation functional to the system (2.1.10)–(2.1.14) is in terms of the variables A and N (cf. (2.2.2))

$$\mathcal{D} = \mu_1 \|d^T A d\|^2 + \frac{\mu_4}{2}\|\nabla v\|^2 + (\mu_5 + \mu_6)\|Ad\|^2 + \lambda_1 \|N\|^2 + (\lambda_2 - \mu_2 - \mu_3)(N, Ad).$$

Moreover, under the Parodi's relation, it can be transformed into the following form (cf. (2.2.8)):

$$\mathcal{D} = \mu_1 \|d^T A d\|^2 + \frac{\mu_4}{2}\|\nabla v\|^2 - \frac{1}{\lambda_1}\|\Delta d - f(d)\|^2 + \Big(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1}\Big)\|Ad\|^2. \qquad (2.2.5)$$

According to the maximum dissipation principle, we treat the dissipation functional by performing a variation with respect to the velocity v in the Eulerian coordinates. Letting $\delta_v(\frac{1}{2}\mathcal{D}) = 0$ with incompressibility of flow, we obtain a weak variational form of the dissipative force balance law (see appendix for the detailed calculations). Then we formally state its strong form:

$$\begin{aligned} &\nabla \cdot (\nabla d \odot \nabla d) - \nabla \cdot \sigma + \nabla P = 0, \quad \text{where} &(2.2.6)\\ &\sigma = \mu_1 (d^T A d) d \otimes d + \mu_2 N \otimes d + \mu_3 d \otimes N + \mu_4 A + \mu_5 Ad \otimes d + \mu_6 d \otimes Ad, \end{aligned}$$

with constants

$$\mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2), \quad \mu_3 = -\frac{1}{2}(\lambda_1 + \lambda_2).$$

We have thus derived the induced stress term in the momentum equation (2.1.10) and recovered the dissipative part of (2.1.10).

The most surprising fact from the above derivation is that the induced stress terms

$$-\nabla \cdot (\nabla \phi \odot \nabla \phi) + \mu_2 \nabla \cdot (N \otimes d) + \mu_3 \nabla \cdot (d \otimes N) + \eta_5 \nabla \cdot (Ad \otimes d) + \eta_6 \nabla \cdot (d \otimes Ad)$$

can be derived either by the least action principle or the maximum dissipation principle. Therefore, we are not able to specify them as either conservative forces or dissipative forces. However, the remaining part

$$\mu_1\nabla\cdot[(d^TAd)d\otimes d]+\mu_4\nabla\cdot A+(\mu_5-\eta_5)\nabla\cdot(Ad\otimes d)+(\mu_6-\eta_6)\nabla\cdot(d\otimes Ad) \quad (2.2.7)$$

can only be derived by the maximum dissipation principle. This indicates that it can be identified as dissipative force.

2.3 Existence of global classical solutions in large viscosity case

This section is devoted to the proof of existence of global classical solutions of the system (2.1.10)–(2.1.12) under the assumption of large μ_4. The main theorem in this section also indicates that Parodi's relation is not necessary in the wellposedness of the system under large viscosity assumption.

2.3.1 Preliminary

First, we recall the well established functional setting for periodic problems (cf. [9, 45]):

$$\begin{aligned}
H^m_p(Q) &= \{u\in H^m(\mathbb{R}^n,\mathbb{R}) \mid u(x+e_i)=u(x)\},\\
\dot{H}^m_p(Q) &= H^m_p(Q)\cap\left\{u: \int_Q u(x)dx=0\right\},\\
H &= \{v\in \mathbf{L}^2_p(Q),\ \nabla\cdot v=0\}, \quad \text{where } \mathbf{L}^2_p(Q)=\mathbf{H}^0_p(Q),\\
V &= \{v\in \dot{\mathbf{H}}^1_p(Q),\ \nabla\cdot v=0\},\\
V' &= \text{the dual space of } V.
\end{aligned}$$

For any Banach space X, we denote by $\mathbf{X}$ the space $(X)^r$, $r\in\mathbb{N}$, endowed with the product norms. For the sake of simplicity, we denote the inner product on $L^2_p(Q)$ (or $\mathbf{L}^2_p(Q)$) as well as H by $(\cdot,\cdot)$ and the associated norm by $\|\cdot\|$. The space $H^m(Q)$ will be shorthanded by H^m and the H^m-inner product ($m\in\mathbb{N}$) can be given by $\langle v,u\rangle_{H^m}=\sum_{|\kappa|=0}^m(D^\kappa v, D^\kappa u)$, where $\kappa=(\kappa_1,...,\kappa_n)$ is a multi-index of length $|\kappa|=\sum_{i=1}^n\kappa_i$ and $D^\kappa=\partial^{\kappa_1}_{x_1},,,\partial^{\kappa_n}_{x_n}$. We denote by C the genetic constant possibly depending on $\lambda_i's,\mu_i's,\nu,Q,f$ and the initial data. Special dependence will be pointed out explicitly in the text if necessary. Throughout our work, the Einstein summation convention will be used.

As mentioned earlier, we are now using the Ginzburg–Landau approximation to reduce the order of nonlinearities caused by the constraint $|d|=1$. We note that either for the highly simplified liquid crystal model (cf. [9]), or for the general Ericksen–Leslie system (2.1.10)–(2.1.14) with additional assumption $\lambda_2=0$, the maximum principle holds for d-equation (cf. [9, 10]). In these case, one can deduce from the basic energy law that

$$v \in L^2(0,T;V) \cap L^\infty(0,T;H), \quad d \in L^2(0,T;\mathbf{H}^2) \cap L^\infty(0,T;\mathbf{H}^1), \tag{2.3.1}$$

which is sufficient for the following formulation of weak solutions:

Definition 2.3.1. *(v, d) is called a weak solution of* (2.1.10)–(2.1.12) *in $Q_T = Q \times (0,T)$ if it satisfies* (2.3.1) *and for any smooth function $\psi(t)$ with $\psi(T) = 0$ and $\phi(x) \in H^1_p$, the following weak formulation together with the initial and boundary conditions* (2.1.13) *and* (2.1.14) *hold:*

$$\begin{aligned} & -\int_0^T (v, \psi_t \phi) dt + \int_0^T (v \cdot \nabla v, \psi\phi) dt \\ = \quad & -(v_0, \phi)\psi(0) + \int_0^T (\nabla d \odot \nabla d, \psi \nabla \phi) dt - \int_0^T (\sigma, \psi \nabla \phi) dt, \end{aligned}$$

where σ is defined in (2.1.4), *and*

$$\begin{aligned} & -\int_0^T (d, \psi_t \phi) dt + \int_0^T (v \cdot \nabla d, \psi\phi) dt - \int_0^T (\omega d, \psi\phi) dt \\ & + \frac{\lambda_2}{\lambda_1} \int_0^T (Ad, \psi\phi) dt = -(d_0, \phi)\psi(0) - \frac{1}{\lambda_1} \int_0^T (\Delta d - f(d), \psi\phi) dt. \end{aligned}$$

Thanks to the maximum principle, one can also derive the existence of weak solutions by applying a semi-Galerkin procedure (cf. [9,10]). However, as it has been pointed in [10], although the artificial assumption $\lambda_2 \neq 0$ brings convenience in analysis, its physical meaning is unclear. For the more general case considered in the present paper, we no longer assume $\lambda_2 = 0$, thus the kinetic transport includes the stretching effect that leads to the loss of maximum principle for the molecule director d. As a result, the extra stress term $\nabla \cdot \sigma$ can not be well-defined in the weak formulation Definition 2.3.1. This suggests the requirement for higher-order regularity of the solution, i.e., $d \in L^\infty(0,T;\mathbf{L}^\infty)$.

2.3.2 Galerkin approximation

We shall apply the semi-Galerkin method (cf. [9, 10]) to prove the existence of solution to system (2.1.10)–(2.1.14) .

In the periodic setting, one can define mapping S associated with the Stokes problem:

$$Su = -\Delta u, \quad \forall\, u \in D(S) = \{u \in H, Su \in H\} = \dot{\mathbf{H}}^2_p \cap H.$$

The operator S can be seen as an unbounded positive linear self-adjoint operator on H. If $D(S)$ is endowed with the norm induced by $\dot{\mathbf{H}}^0_p(Q)$, then S becomes an isomorphism from $D(S)$ onto H. We take $\{\phi_i\}_{i=1}^{\infty}$ with $\|\phi_i\| = 1$ be the eigenvectors of the Stokes operator in the periodic case with zero mean,

$$-\Delta\phi_i + \nabla P_i = \kappa_i\phi_i, \quad \nabla\cdot\phi_i = 0 \text{ in Q}, \quad \int_Q \phi_i(x)\,dx = 0,$$

where $P_i \in L^2(Q)$ and $0 < \kappa_1 \le \kappa_2 \le ....$ The eigenfunctions ϕ_i are smooth and $\{\phi_i\}_{i=1}^{\infty}$ forms an orthogonal basis of H (cf. [45]). Let $P_m : H \to H_m \doteq span\{\phi_1, \cdots, \phi_m\}$, $m \in \mathbb{N}$.

We consider the following (variational) approximate problem: $\forall\ u_m \in H_m$,

$$(\partial_t v_m, u_m) + (v_m\cdot\nabla v_m, u_m) = (\nabla d_m \odot \nabla d_m, \nabla u_m) - (\sigma_m, \nabla u_m), \tag{2.3.2}$$

$$N_m + \frac{\lambda_2}{\lambda_1}A_m\cdot d_m = -\frac{1}{\lambda_1}\Delta d_m - f(d_m), \tag{2.3.3}$$

$$v_m(x,0) = P_m v_0(x), \quad d_m(x,0) = d_0(x), \tag{2.3.4}$$

$$v_m(x+e_i,t) = v_m(x,t), \quad d_m(x+e_i,t) = d_m(x,t), \tag{2.3.5}$$

where

$$\begin{aligned}\Omega_m &= \frac{\nabla v_m - \nabla^T v_m}{2}, \quad A_m = \frac{\nabla v_m + \nabla^T v_m}{2}, \quad N_m = \partial_t d_m + v_m\cdot\nabla d_m + \Omega_m\cdot d_m\\ \sigma_m &= \mu_1 d_m^T A_m d_m d_m\otimes d_m + \mu_2 N_m\otimes d_m + \mu_3 d_m\otimes N_m + \mu_4 A_m + \mu_5 A_m d_m\otimes d_m\\ &\quad +\mu_6 d_m\otimes A_m d_m.\end{aligned}$$

Here and after, we assume that the initial data satisfy

$$v_0 \in V, \quad d_0 \in \mathbf{H}^2(Q). \tag{2.3.6}$$

For $v_m \in H_m$, we have the expansion $v_m(x,t) = \sum_{i=1}^m g_m^i(t)\phi_i(x)$. Then, (2.3.2) can be reduced to the following ODE system

$$\frac{d}{dt}g_m^i(t) = \frac{1}{2}\mu_4\kappa_i g_m^i(t) + \mathfrak{A}_k(t)g_m^k(t) + \mathfrak{B}_{jk}^i g_m^k(t)g_m^j(t) + \mathfrak{D}_m^i(t), \quad i = 1,...,m, \tag{2.3.7}$$

subject to the initial conditions $g_m^i(0) = (v_0, \phi_i)$, where for $j,k = 1,...,m$,

$$\begin{aligned}\mathfrak{A}_k(t) &= -\mu_1\Big(d_m^T\cdot\frac{\nabla\phi_k + \nabla^T\phi_k}{2}\cdot d_m,\ d_m^T\cdot\nabla\phi_i\cdot d_m\Big)\\ &\quad +\Big(\frac{\lambda_2\mu_2}{\lambda_1} - \mu_5\Big)\Big(\frac{\nabla\phi_k + \nabla^T\phi_k}{2}, d_m\cdot\nabla\phi_i\Big)\\ &\quad +\Big(\frac{\lambda_2\mu_2}{\lambda_1} - \mu_6\Big)\Big(\frac{\nabla\phi_k + \nabla^T\phi_k}{2}, d_m\cdot\nabla^T\phi_i\Big),\\ \mathfrak{B}_{jk}^i &= -\big(\phi_j\cdot\nabla\phi_k, \phi_i\big),\end{aligned}$$

$$
\begin{aligned}
\mathfrak{D}^i_m(t) &= \Big(\nabla d_m, \nabla\phi_i\cdot\nabla d_m\Big)dx + \frac{\mu_2}{\lambda_1}\int_Q \big(\Delta d_m - f(d_m), d_m\cdot\nabla\phi_i\big)\\
&\quad + \frac{\mu_3}{\lambda_1}\Big(\Delta d_m - f(d_m), d_m\cdot\nabla^T\phi_i\Big).
\end{aligned}
$$

We have the following local existence result for the approximate problem.

Theorem 2.3.1. *For any $m > 0$, $v_0 \in V$ and $d_0 \in \mathbf{H}^2$, there is a $T_0 > 0$ depending on v_0, d_0, Q and m such that the approximate problem (2.3.2)–(2.3.5) admits a unique strong solution (v_m, d_m) such that $v_m \in L^\infty(0, T_0; V) \cap L^2(0, T_0; \mathbf{H}^2)$, $d_m \in L^\infty(0, T_0; \mathbf{H}^2) \cap L^2(0, T_0; \mathbf{H}^3)$, and (2.3.2)–(2.3.5) are satisfied a.e. in $Q_{T_0} := Q \times [0, T_0]$. Besides, (v_m, d_m) is smooth in the interior of Q_{T_0}.*

Proof. The local existence of weak solutions to (2.3.2)–(2.3.5) follows from the semi-Galerkin procedure with a fixed point argument (cf. [9, 10, 38]). We just point the difference in the proof.

Step 1. For $0 < T \le 1$, given $u = \sum_{i=1}^m g^i_m(t)\phi_i(x)$ with $g^i_m(0) = (v_0, \phi_i)$ and $\sum_{i=1}^m |g^i_m(t)|^2 \le M = 2 + 2\sum_{i=1}^m |(v_0, \phi_i)|^2$ on $[0, T]$, we consider the parabolic equation (2.3.3) for d_m with $v_m = u$ and $d_m(0) = d_0$. The existence of d_m easily follows from the standard parabolic equation. Moreover, from the observation that u is smooth in space, we have

$$
\begin{aligned}
&\frac{d}{dt}\int_Q \left(\frac{1}{2}|\nabla d_m|^2 + \mathcal{F}(d_m)\right)dx - \frac{1}{2\lambda_1}\|-\Delta d_m + f(d_m)\|^2\\
\le\ & C\|u\|^2_{\mathbf{L}^\infty}\|\nabla d_m\|^2 + C\|\nabla u\|^2_{\mathbf{L}^\infty}\|d_m\|^2 \le C\int_Q\left(\frac{1}{2}|\nabla d_m|^2 + \mathcal{F}(d_m)\right)dx + C,
\end{aligned}
$$

where C depends on M, Q, m and coefficients of the system. By the Gronwall inequality,

$$
\|d_m(t)\|^2_{\mathbf{H}^1} \le C(\|d_0\|_{\mathbf{H}^1})e^C, \quad \forall\, t\in[0,T]. \tag{2.3.8}
$$

Besides, apply Δ to (2.3.3) and test it by Δd_m, we infer from the Sobolev embedding theorem that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\Delta d_m\|^2 - \frac{1}{\lambda_1}\|\nabla\Delta d_m\|^2\\
\le\ & C\|\nabla\Delta d_m\|(\|\nabla f(d_m)\| + \|\nabla u\|_{\mathbf{L}^\infty}\|\nabla d_m\| + \|u\|_{\mathbf{L}^\infty}\|\Delta d_m\| + \|\Delta u\|_{\mathbf{L}^\infty}\|d_m\|)\\
\le\ & -\frac{1}{2\lambda_1}\|\nabla\Delta d_m\|^2 + C\|\Delta d_m\|^2 + C,
\end{aligned} \tag{2.3.9}
$$

where C depends on M, Q, m and coefficients of the system. The Gronwall inequality together with (2.3.8) yields that holds

$$
\|d_m(t)\|^2_{\mathbf{H}^2} \le C(\|d_0\|_{\mathbf{H}^2})e^C, \quad \forall\, t\in[0,T],
$$

which implies the $\mathbf{L}^\infty$ estimate of d_m:

$$\|d_m(t)\|_{\mathbf{L}^\infty} \le Ce^C, \quad \forall\, t \in [0,T].$$

Step 2. We substitute d_m into (2.3.2) and solve the ODE (2.3.7). By the above estimates on d_m, we see that the $\mathfrak{A}_k, \mathfrak{B}^i_{jk}, \mathfrak{D}^i_m$ are all bounded by a constant that depends on M, Q, m and coefficients of the system. Then we can see that (2.3.7) admits a unique local solution $\tilde g^i_m(t)$ with $\tilde g^i_m(0) = (v_0, \phi_i)$.

Denote $v = \sum_{i=1}^m \tilde g^i_m(t)\phi_i(x)$ with $\tilde g^i_m(0) = (v_0, \phi_i)$. Then we can argue exactly as in [9, 38] that for sufficiently small T_0, the mapping $\mathfrak{L} : u \mapsto v$ admits a fixed point in the space $V(T_0) = \{v(x,t) = \sum_{i=1}^m g^i_m(t)\phi(x) : \sum_{i=1}^m (g^i_m(t))^2 \le M \text{ for } 0 \le t \le T_0, g^i_m(0) = (v_0, \phi_i)\}$, which completes the proof of existence. The regularity of solutions in the interior of Q_{T_0} follows from the regularity theory for parabolic equations and a bootstrap argument (cf. [46]). Then the uniqueness of smooth/regular solutions can be proved in a standard way by Gronwall's inequality. □

2.3.3 Uniform *a prior* estimates

In order to prove the (global) existence of solutions to our original problem (2.1.10)–(2.1.14), we have to derive some uniform (in time) estimates that are independent of approximation parameter m and time T_0. These uniform estimates enable us to (i) pass to the limit as $m \to \infty$ to obtain a solution to system (2.1.10)–(2.1.14) in proper Sobolev spaces; (ii) extend the local solution to a global one on $(0, +\infty)$. Besides, the higher-order estimates allow us to prove the uniqueness of the solution. We note that the advantage of above mentioned semi-Galerkin scheme is that the approximate solutions satisfy the same basic energy law and higher-order differential inequalities for smooth solutions to system (2.1.10)–(2.1.14). For the sake of simplicity, the following calculations are carried out formally for smooth solutions. However, they can be justified by using the approximate solutions to (2.3.2)–(2.3.5) and then pass to limit.

The basic energy law plays an important role in the derivation of uniform estimates. According to the discussions in Section **2.2**, we shall consider two cases, in which the basic energy law holds:

- **Case I.** with Parodi's relation: $\lambda_2 \neq 0$, (2.1.15), (2.1.16), (2.2.3)–(2.2.6);
- **Case II.** without the Parodi's relation: $\lambda_2 \neq 0$, (2.1.15), (2.2.3)–(2.2.5) and (2.2.11).

We first consider **Case I**. It follows from Lemma 2.2.1 that

$$\frac{d}{dt}\mathcal{E}(t) \le -\int_\Omega \mu_1 (A_{kp} d_k d_p)^2 dx - \frac{\mu_4}{2}\|\nabla v\|^2 + \frac{1}{\lambda_1}\|\Delta d - f(d)\|^2, \quad \forall\, t \ge 0.$$

This implies the uniform estimates

$$\|v\| \le C, \quad \|d\|_{\mathbf{H}^1} \le C, \quad t \ge 0, \tag{2.3.10}$$

$$\int_0^\infty \left[\int_Q \mu_1 (A_{kp} d_k d_p)^2 dx + \frac{\mu_4}{2} \|\nabla v\|^2 - \frac{1}{\lambda_1} \|\Delta d - f(d)\|^2 \right] dt \le C, \tag{2.3.11}$$

where the constant $C > 0$ depends only on $\|v_0\|$ and $\|d_0\|_{\mathbf{H}^1}$.

Next, we try to derive a new type high-order energy inequality, which turns out to be useful in the study of some (simplified) liquid crystal models on the global existence of regular solutions as well as the long-time behavior (cf. [9, 10, 38, 47, 48]). Define

$$\mathcal{A}(t) = \|\nabla v(t)\|^2 + \|\Delta d(t) - f(d(t))\|^2. \tag{2.3.12}$$

A direct calculation yields that

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \mathcal{A}(t) &= -(\Delta v, v_t) + (\Delta d - f, \Delta d_t - f'(d) d_t) \\
&= (\Delta v, v \cdot \nabla v) + (\Delta v, \nabla d \Delta d) + (\nabla \cdot \sigma, -\Delta v) + \frac{1}{\lambda_1} \|\nabla(\Delta d - f)\|^2 \\
&\quad - \big(\Delta d - f, \Delta(v \cdot \nabla d)\big) + \big(\Delta d - f, \Delta(\Omega d)\big) - \frac{\lambda_2}{\lambda_1} \big(\Delta d - f, \Delta(Ad)\big) \\
&\quad + \left(\Delta d - f, f'(d) \Big(\frac{1}{\lambda_1} (\Delta d - f) + v \cdot \nabla d + \Omega d + \frac{\lambda_2}{\lambda_1} Ad \Big) \right).
\end{aligned} \tag{2.3.13}$$

We expand the right-hand side of (2.3.13) term by term.

$$\begin{aligned}
(\nabla \cdot \sigma, -\Delta v) &= -\int_Q \nabla_j \sigma_{ij} \nabla_l \nabla_l v_i dx = -\int_Q \nabla_l \sigma_{ij} \nabla_l \nabla_j v_i dx \\
&= -\mu_1 \int_Q \nabla_l (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_j v_i dx - \mu_4 \int_Q \nabla_l (A_{ij}) \nabla_l \nabla_j v_i dx \\
&\quad -\mu_2 \int_Q \nabla_l (d_j N_i) \nabla_l \nabla_j v_i dx - \mu_3 \int_Q \nabla_l (d_i N_j) \nabla_l \nabla_j v_i dx \\
&\quad -\mu_5 \int_Q \nabla_l (d_j d_k A_{ki}) \nabla_l \nabla_j v_i dx - \mu_6 \int_Q \nabla_l (d_i d_k A_{kj}) \nabla_l \nabla_j v_i dx,
\end{aligned}$$

then we have

$$\begin{aligned}
& -\mu_1 \int_Q \nabla_l (d_k d_p A_{kp} d_i d_j) \nabla_l \nabla_j v_i dx \\
= & -\frac{\mu_1}{2} \int_Q d_k d_p \nabla_l A_{kp} d_i d_j \nabla_l (\nabla_j v_i + \nabla_i v_j) dx - 2\mu_1 \int_Q A_{kp} \nabla_l d_k d_p d_i d_j \nabla_l \nabla_j v_i dx \\
& -2\mu_1 \int_Q A_{kp} d_k d_p d_i \nabla_l d_j \nabla_l A_{ij} dx \\
= & -\mu_1 \int_Q (d_k d_p \nabla_l A_{kp})^2 dx - 2\mu_1 \int_Q A_{kp} \nabla_l d_k d_p d_i d_j \nabla_l \nabla_j v_i dx
\end{aligned}$$

$$-2\mu_1 \int_Q A_{kp} d_k d_p d_i \nabla_l d_j \nabla_l A_{ij} dx. \tag{2.3.14}$$

By the incompressibility condition, we see that

$$-\mu_4 \int_Q \nabla_l (A_{ij}) \nabla_l \nabla_j v_i dx = -\mu_4 \int_Q \nabla_j (A_{ij}) \nabla_l \nabla_l v_i dx = -\frac{\mu_4}{2} \|\Delta v\|^2, \tag{2.3.15}$$

$$\begin{aligned} & -\mu_2 \int_Q \nabla_l (d_j N_i) \nabla_l \nabla_j v_i dx - \mu_3 \int_Q \nabla_l (d_i N_j) \nabla_l \nabla_j v_i dx \\ = \ & \mu_2 \int_Q d_j N_i \Delta(A_{ij} + \Omega_{ij}) dx + \mu_3 \int_Q d_i N_j \Delta(A_{ij} + \Omega_{ij}) dx \\ = \ & (\mu_2 + \mu_3) \int_Q d_j N_i \Delta(A_{ij}) dx - (\mu_2 - \mu_3) \int_Q d_i N_j, \Delta(\Omega_{ij}) dx. \end{aligned} \tag{2.3.16}$$

$$\begin{aligned} & -\mu_5 \int_Q \nabla_l (d_j d_k A_{ki}) \nabla_l \nabla_j v_i dx - \mu_6 \int_Q \nabla_l (d_i d_k A_{kj}) \nabla_l \nabla_j v_i dx, \\ = \ & \mu_5 \int_Q d_j d_k A_{ki} \Delta(A_{ij} + \Omega_{ij}) dx + \mu_6 \int_Q d_j d_k A_{ki} \Delta(A_{ij} - \Omega_{ij}) dx, \\ = \ & (\mu_5 + \mu_6) \int_Q d_j d_k A_{ki} \Delta A_{ij} dx + (\mu_5 - \mu_6) \int_Q d_j d_k A_{ki} \Delta \Omega_{ij} dx, \\ = \ & -(\mu_5 + \mu_6) \int_Q \nabla_l A_{ki} d_k \nabla_l A_{ij} d_j dx - (\mu_5 + \mu_6) \int_Q A_{ki} d_k \nabla_l d_j \nabla_l A_{ij} dx \\ & -(\mu_5 + \mu_6) \int_Q A_{ki} \nabla_l d_k d_j \nabla_l A_{ij} dx + (\mu_5 - \mu_6) \big(Ad, \Delta \Omega d\big) \\ = \ & -(\mu_5 + \mu_6) \|\nabla A \cdot d\|^2 - (\mu_5 + \mu_6) \int_Q A_{ki} d_k \nabla_l d_j \nabla_l A_{ij} dx \\ & -(\mu_5 + \mu_6) \int_Q A_{ki} \nabla_l d_k d_j \nabla_l A_{ij} dx + (\mu_5 - \mu_6) \big(Ad, \Delta \Omega d\big). \end{aligned} \tag{2.3.17}$$

We have

$$\begin{aligned} & (\Delta d - f, \Delta(\Omega\, d)) \\ = \ & (\Delta d - f, \Delta \Omega\, d) + 2(\Delta d - f, \nabla \Omega \nabla d) + (\Delta d - f, \Omega \Delta d) \\ = \ & -\lambda_1 \int_Q d_j N_i \Delta \Omega_{ij}\, dx - \lambda_2 \big(Ad, \Delta \Omega d\big) + 2(\Delta d - f, \nabla \Omega \nabla d) + (\Delta d - f, \Omega \Delta d), \\ = \ & -\lambda_1 \int_Q d_j N_i \Delta \Omega_{ij}\, dx - \lambda_2 \big(Ad, \Delta \Omega d\big) - (\nabla(\Delta d - f), \Omega \nabla d) + (\Delta d - f, \nabla \Omega \nabla d), \end{aligned} \tag{2.3.18}$$

and

$$\begin{aligned} & -\frac{\lambda_2}{\lambda_1} \big(\Delta d - f, \Delta(Ad)\big) = \lambda_2 \big(N, \Delta(Ad)\big) + \frac{(\lambda_2)^2}{\lambda_1} \big(Ad, \Delta(Ad)\big) \\ = \ & \lambda_2 \big(N, \Delta A\, d\big) + 2\lambda_2 (N, \nabla A \nabla d) + \lambda_2 (N, A \Delta d) \end{aligned}$$

$$-\frac{{\lambda_2}^2}{\lambda_1}\int_Q |\nabla A\cdot d|^2 dx - \frac{{\lambda_2}^2}{\lambda_1}\int_Q |A\cdot\nabla d|^2 dx. \tag{2.3.19}$$

By condition (2.1.15), the first term on the right-hand side of (2.3.18) cancels with the second term of the right-hand side of (2.3.16) and the second term on the right-hand side of (2.3.18) cancels with the fourth term of the right-hand side of (2.3.17). By (2.1.16), the first term of the right-hand side of (2.3.19) cancels with the first term of the right-hand side of (2.3.16).

For the fifth term on the right-hand side of (2.3.13), we have

$$-(\Delta d - f, \Delta(v\cdot\nabla d)) = -(\Delta d - f, \Delta v\cdot\nabla d) - 2(\Delta d - f, \nabla v\cdot\nabla^2 d) - (\Delta d - f, v\cdot\nabla\Delta d),$$

and due to the incompressibility of v,

$$\begin{aligned}
-(\Delta d - f, \Delta v\cdot\nabla d) &= -(\Delta v, \nabla d\Delta d) + (\Delta v, \nabla\mathcal{F}(d)) \\
&= -(\Delta v, \nabla d\Delta d), \\
-2(\Delta v - f, \nabla v\cdot\nabla^2 d) &= 2(\nabla(\Delta v - f), \nabla v\cdot\nabla d), \\
-(\Delta d - f, v\cdot\nabla\Delta d) &= -(\Delta d - f, v\cdot\nabla(\Delta d - f)) - (\Delta d - f, v\cdot\nabla f) \\
&= -(\Delta d - f, v\cdot\nabla f).
\end{aligned} \tag{2.3.20}$$

Hence,

$$\begin{aligned}
& -(\Delta d - f, \Delta(v\cdot\nabla d)) + (\Delta v, \nabla d\Delta d) \\
& +\left(\Delta d - f, f'(d)\left(\frac{1}{\lambda_1}(\Delta d - f) + v\cdot\nabla d - \Omega\, d + \frac{\lambda_2}{\lambda_1}A\, d\right)\right) \\
= \; & \frac{1}{\lambda_1}\int_Q f'(d)|\Delta d - f|^2 dx - \left(\Delta d - f, f'(d)\left(\;\Omega\, d - \frac{\lambda_2}{\lambda_1}A\, d\right)\right) \\
& +2(\nabla(\Delta v - f), \nabla v\cdot\nabla d).
\end{aligned} \tag{2.3.21}$$

Summing up, we infer from (2.3.14)–(2.3.21) that

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\mathcal{A}(t) + \mu_1\|d^T\cdot\nabla A\cdot d\|^2 + \frac{\mu_4}{2}\|\Delta v\|^2 + (\mu_5+\mu_6)\|\nabla A\cdot d\|^2 - \frac{1}{\lambda_1}\|\nabla(\Delta d - f)\|^2 \\
= \; & -2\mu_1\int_Q A_{kp}\nabla_l d_p d_k d_i d_j \nabla_l A_{ij} dx - 2\mu_1\int_Q A_{kp} d_p d_k d_i \nabla_l d_j \nabla_l A_{ij} dx \\
& -(\mu_5+\mu_6)\int_Q A_{ik} d_k \nabla_l d_j \nabla_l A_{ij} dx - (\mu_5+\mu_6)\int_Q A_{ik}\nabla_l d_k d_j \nabla_l A_{ij} dx \\
& -(\nabla(\Delta d - f), \Omega\nabla d) + (\Delta d - f, \nabla\Omega\nabla d) + 2\lambda_2(N, \nabla A\nabla d) \\
& +\lambda_2(N, A\Delta d) - \frac{(\lambda_2)^2}{\lambda_1}\int_Q |\nabla A\cdot d|^2 dx - \frac{(\lambda_2)^2}{\lambda_1}\int_Q |A\cdot\nabla d|^2 dx + (\Delta v, v\cdot\nabla v) \\
& +\frac{1}{\lambda_1}\int_Q f'(d)|\Delta d - f|^2 dx - \left(\Delta d - f, f'(d)\left(\Omega d - \frac{\lambda_2}{\lambda_1}A\, d\right)\right) \\
& +2(\nabla(\Delta v - f), \nabla v\cdot\nabla d) \\
\triangleq \; & I_1 + \ldots + I_{14}.
\end{aligned} \tag{2.3.22}$$

Lemma 2.3.1. *Assume $n = 2$ or $n = 3$. Under assumption of **Case I** we have the following inequality:*

$$\frac{d}{dt}\mathcal{A}(t) \leq -\Big(\frac{\mu_4}{2} - C_1 \mu_4{}^{\frac{1}{2}} \tilde{\mathcal{A}}(t)\Big) \|\Delta v\|^2 - \left(\frac{1}{-2\lambda_1} - C_2 \mu_4^{-\frac{1}{4}} \tilde{\mathcal{A}}(t)\right) \|\nabla(\Delta d - f)\|^2 + C_3 \mathcal{A}(t), \tag{2.3.23}$$

where $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) + 1$, C_i $(i = 1, 2, 3)$ are constants depending on $Q, f, \|v_0\|, \|d_0\|_{\mathbf{H}^1}, \lambda' s$, and $\mu' s$ but except μ_4.

Proof. Without loss of generality, we assume $\mu_4 \geq 1$. Our argument is valid for any $\mu_4 \geq \underline{\mu} > 0$. Below we only give the proof for $n = 3$, the proof for $n = 2$ can be done in a similar way with minor modifications. We now estimate the right-hand side of (2.3.22) term by term.

$$I_1 = -2\mu_1 \int_Q A_{kp} \nabla_l d_p d_k d_i d_j \nabla_l A_{ij} dx \leq \frac{\mu_1}{4} \|d^T \cdot \nabla A \cdot d\|^2 + C \|d\|^2_{\mathbf{L}^\infty} \|\nabla v\|^2_{\mathbf{L}^3} \|\nabla d\|^2_{\mathbf{L}^6}.$$

By the lower-order estimate (2.3.10), we can apply the Agmon's inequality that

$$\|d\|_{\mathbf{L}^\infty} \leq C(1 + \|\Delta d\|^{\frac{1}{2}}). \tag{2.3.24}$$

Besides, from (2.3.10), the Gagliardo–Nirenberg inequality and (2.3.24), we obtain

$$\begin{aligned}
\|\nabla v\|_{\mathbf{L}^3} &\leq \|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{1}{2}}, \quad \|\nabla v\|_{\mathbf{L}^4} \leq \|\nabla v\|^{\frac{1}{4}} \|\Delta v\|^{\frac{3}{4}}, \quad \|\nabla d\|_{\mathbf{L}^6} \leq C(\|\Delta d\| + 1), \\
\|\Delta d\| &\leq \|\Delta d - f(d)\| + \|f(d)\| \leq \|\Delta d - f(d)\| + C, \\
\|\nabla \Delta d\| &\leq \|\nabla(\Delta d - f(d)\| + \|\nabla f(d)\| \leq \|\nabla(\Delta d - f(d)\| + \|f'(d)\|_{\mathbf{L}^\infty} \|\nabla d\| \\
&\leq \|\nabla(\Delta d - f(d)\| + C(1 + \|d\|^2_{\mathbf{L}^\infty}) \leq \|\nabla(\Delta d - f(d)\| + C(1 + \|\Delta d\|) \\
&\leq \|\nabla(\Delta d - f(d)\| + C(1 + \|\nabla \Delta d\|^{\frac{1}{2}} \|\nabla d\|^{\frac{1}{2}} + \|\nabla d\|) \\
&\leq \|\nabla(\Delta d - f(d)\| + \frac{1}{2} \|\nabla \Delta d\| + C.
\end{aligned} \tag{2.3.25}$$

As a consequence,

$$\begin{aligned}
& \|d\|^2_{\mathbf{L}^\infty} \|\nabla v\|^2_{\mathbf{L}^3} \|\nabla d\|^2_{\mathbf{L}^6} \leq C \|\nabla v\| \|\Delta v\| (\|\Delta d - f\|^3 + 1) \\
\leq\ & \Big(\mu_4^{\frac{1}{2}} + \mu_4^{\frac{1}{2}} \|\Delta d - f\|^2\Big) \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 (1 + \|\Delta d - f\|^4) \\
\leq\ & \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 \Big(\|\nabla \Delta d\|^{\frac{1}{2}} \|\nabla d\|^{\frac{1}{2}} + \|\nabla d\| + C\Big)^4 \\
\leq\ & \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 + C \mu_4^{-\frac{1}{2}} \|\nabla v\|^2 (\|\nabla(\Delta d - f)\|^2 + 1) \\
\leq\ & \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}.
\end{aligned} \tag{2.3.26}$$

This implies that

$$I_1 \leq \frac{\mu_1}{4} \|d^T \cdot \nabla A \cdot d\|^2 + \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C \mu_4^{-\frac{1}{2}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}. \tag{2.3.27}$$

For I_2, using integration by parts, we obtain

$$
\begin{aligned}
I_2 &= -2\mu_1 \int_Q A_{kp} d_p d_k d_i \nabla_l d_j \nabla_l A_{ij} dx \\
&= 2\mu_1 \int_Q \nabla_l A_{kp} d_p d_k d_i \nabla_l d_j A_{ij} dx + 4\mu_1 \int_Q A_{kp} d_p \nabla_l d_k A_{ij} d_i \nabla_l d_j dx \\
&\quad +2\mu_1 \int_Q A_{kp} d_k \nabla_l d_i A_{ij} d_p \nabla_l d_j dx + 2\mu_1 \int_Q A_{kp} d_p d_k d_i \Delta d_j A_{ij} dx \\
&\le \frac{\mu_1}{4} \|d^T \cdot \nabla A \cdot d\|^2 + C\|d\|^2_{\mathbf{L}^\infty} \|\nabla v\|^2_{\mathbf{L}^3} \|\nabla d\|^2_{\mathbf{L}^6} + C\|\nabla v\|^2_{\mathbf{L}^4} \|\Delta d\| \|d\|^3_{\mathbf{L}^\infty},
\end{aligned}
\tag{2.3.28}
$$

where

$$
\begin{aligned}
& C\|\nabla v\|^2_{\mathbf{L}^4} \|\Delta d\| \|d\|^3_{\mathbf{L}^\infty} \le C\|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{3}{2}} (\|\Delta d - f\|^{\frac{5}{2}} + 1) \\
\le\ & {\mu_4}^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C{\mu_4}^{-\frac{3}{2}} \|\nabla v\|^2 (1 + \|\Delta d - f\|^4),
\end{aligned}
\tag{2.3.29}
$$

and now the right-hand side in (2.3.29) can be estimated exactly as (2.3.26). Therefore,

$$
I_2 \le \frac{\mu_1}{4} \|d^T \cdot \nabla A \cdot d\|^2 + \mu_4^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C\mu_4^{-\frac{1}{2}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}. \tag{2.3.30}
$$

Using integration by parts, we deduce from (2.3.25) that (cf. the argument in (2.3.26))

$$
\begin{aligned}
I_3 + I_4 &= -(\mu_5 + \mu_6) \int_Q A_{ik} d_k \nabla_l d_j \nabla_l A_{ij} dx - (\mu_5 + \mu_6) \int_Q A_{ik} \nabla_l d_k d_j \nabla_l A_{ij} dx \\
&= (\mu_5 + \mu_6) \int_Q A_{ik} \nabla_l d_k \nabla_l d_j A_{ij} dx + (\mu_5 + \mu_6) \int_Q A_{ik} d_k \Delta d_j A_{ij} dx \\
&\le C\|\nabla v\|^2_{\mathbf{L}^4} \|\nabla d\|^2_{\mathbf{L}^4} + C\|\nabla v\|^2_{\mathbf{L}^4} \|\Delta d\| \|d\|_{\mathbf{L}^\infty} \\
&\le C\|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{3}{2}} \big(\|\Delta d - f\|^{\frac{3}{2}} + 1\big) \\
&\le {\mu_4}^{\frac{1}{2}} \tilde{\mathcal{A}} \|\Delta v\|^2 + C{\mu_4}^{-\frac{3}{2}} \|\nabla v\|^2.
\end{aligned}
\tag{2.3.31}
$$

Next,

$$
\begin{aligned}
I_5 &= -(\nabla(\Delta d - f), \Omega \nabla d) \le C\|\nabla(\Delta d - f)\| \|\nabla v\|_{\mathbf{L}^3} \|\nabla d\|_{\mathbf{L}^6} \\
&\le C\|\nabla(\Delta d - f)\| \|\nabla v\|^{\frac{1}{2}} \|\Delta v\|^{\frac{1}{2}} (\|\Delta d - f\| + 1) \\
&\le \mu_4^{\frac{1}{4}} \|\nabla v\| \|\Delta v\| + C\mu_4^{-\frac{1}{4}} \tilde{\mathcal{A}} \|\nabla(\Delta d - f)\|^2 \\
&\le \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C\mu_4^{-\frac{1}{4}} \tilde{\mathcal{A}} \|\nabla(\Delta d - f)\|^2 + C\|\nabla v\|^2.
\end{aligned}
\tag{2.3.32}
$$

$$
\begin{aligned}
I_6 &= (\Delta d - f, \nabla \Omega \nabla d) \le \|\nabla \Omega\| \|\Delta d - f\| \|\nabla d\|_{\mathbf{L}^\infty} \\
&\le C\|\Delta v\| \|\Delta d - f\| (\|\nabla(\Delta d - f)\|^{\frac{3}{4}} + 1) \\
&\le \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C\mu_4^{-\frac{2}{3}} \|\Delta d - f\|^2 \|\nabla(\Delta d - f)\|^2 + C\|\Delta d - f\|^2 \\
&\le \mu_4^{\frac{1}{2}} \|\Delta v\|^2 + C\mu_4^{-\frac{2}{3}} \mathcal{A} \|\nabla(\Delta d - f)\|^2 + C\mathcal{A}.
\end{aligned}
\tag{2.3.33}
$$

Using integration by parts and (2.1.12), we get

$$\begin{aligned}I_7+I_8+I_9 &= 2\lambda_2(N,\nabla A\nabla d)+\lambda_2(N,A\Delta d)-\frac{(\lambda_2)^2}{\lambda_1}\int_Q |A\nabla d|^2dx\\ &= \lambda_2(N,\nabla A\nabla d)-\lambda_2(\nabla N,A\nabla d)-\frac{(\lambda_2)^2}{\lambda_1}\int_Q |A\nabla d|^2dx\\ &= -\frac{\lambda_2}{\lambda_1}(\Delta d-f,\nabla A\nabla d)-\frac{(\lambda_2)^2}{\lambda_1}(Ad,\nabla A\nabla d)\\ &\quad+\frac{\lambda_2}{\lambda_1}\big(\nabla(\Delta d-f),A\nabla d\big)+\frac{(\lambda_2)^2}{\lambda_1}(\nabla Ad,A\nabla d). \qquad (2.3.34)\end{aligned}$$

Then we estimate the four terms on the right-hand side of (2.3.34). Similar to (2.3.32) and (2.3.33), we have

$$\begin{aligned}&-\frac{\lambda_2}{\lambda_1}(\Delta d-f,\nabla A\nabla d)+\frac{\lambda_2}{\lambda_1}\big(\nabla(\Delta d-f),A\nabla d\big)\\ \le\ & C\|\Delta v\|\|\Delta d-f\|\|\nabla d\|_{\mathbf{L}^\infty}+C\|\nabla(\Delta d-f)\|\|\nabla v\|_{\mathbf{L}^3}\|\nabla d\|_{\mathbf{L}^6}\\ \le\ & \mu_4^{\frac12}\|\Delta v\|^2+C\mu_4^{-\frac14}\mathcal{A}\|\nabla(\Delta d-f)\|^2+C\mathcal{A},\end{aligned}$$

and a similar argument as in (2.3.31) yields

$$-\frac{(\lambda_2)^2}{\lambda_1}(Ad,\nabla A\nabla d)+\frac{(\lambda_2)^2}{\lambda_1}(\nabla A\,d,A\nabla d)\le \mu_4{}^{\frac12}\tilde{\mathcal{A}}\|\Delta v\|^2+C\mu_4{}^{-\frac32}\|\nabla v\|^2,$$

hence

$$I_7+I_8+I_9\le \mu_4^{\frac12}\tilde{\mathcal{A}}\|\Delta v\|^2+C\mu_4^{-\frac14}\mathcal{A}\|\nabla(\Delta d-f)\|^2+C\mathcal{A}. \qquad (2.3.35)$$

(2.2.4) and (2.2.6) indicate that

$$I_{10}=-\frac{(\lambda_2)^2}{\lambda_1}\int_Q|\nabla A\,d|^2dx\le(\mu_5+\mu_6)\int_Q|\nabla A\,d|^2dx. \qquad (2.3.36)$$

Furthermore,

$$\begin{aligned}I_{11} &= (\Delta v,v\cdot\nabla v)\le\|v\|_{\mathbf{L}^4}\|\nabla v\|_{\mathbf{L}^4}\|\Delta v\|\le C\|v\|^{\frac14}\|\nabla v\|^{\frac34}\|\nabla v\|^{\frac14}\|\Delta v\|^{\frac34}\|\Delta v\|\\ &\le \mu_4{}^{\frac12}\|\Delta v\|^2+\mu_4{}^{\frac12}\|\nabla v\|^2\|\Delta v\|^2+C\mu_4{}^{-\frac72}\|\nabla v\|^2\\ &\le \mu_4{}^{\frac12}\tilde{\mathcal{A}}\|\Delta v\|^2+C\mathcal{A}. \qquad (2.3.37)\\ I_{12} &= \frac{1}{\lambda_1}\int_Q f'(d)|\Delta d-f|^2dx\le C(\|d\|^2_{\mathbf{L}^6}+1)\|\Delta d-f\|^2_{\mathbf{L}^3}\\ &\le C\Big(\|\Delta d-f\|\|\nabla(\Delta d-f)\|+\|\Delta d-f\|^2\Big)\\ &\le -\frac{1}{2\lambda_1}\|\nabla(\Delta d-f)\|^2+C\|\Delta d-f\|^2. \qquad (2.3.38)\\ I_{13} &= -\left(\Delta d-f,f'(d)\left(\Omega\,d-\frac{\lambda_2}{\lambda_1}Ad\right)\right)\le C\|f'(d)d\|\|\Delta d-f\|_{\mathbf{L}^3}\|\nabla v\|_{\mathbf{L}^6}\\ &\le C\Big(\|\nabla(\Delta d-f)\|+\|\Delta d-f\|\Big)\|\nabla v\|^{\frac12}\|\Delta v\|^{\frac12}\end{aligned}$$

$$\leq \quad \mu_4^{\frac{1}{2}}\|\Delta v\|^2+\mu_4{}^{-\frac{1}{4}}\|\nabla(\Delta d-f)\|^2+C\mathcal{A}. \tag{2.3.39}$$

For I_{14}, the estimate is exactly the same as (2.3.32) such that

$$I_{14}\leq \mu_4{}^{\frac{1}{2}}\|\Delta v\|^2+C\mu_4^{-\frac{1}{4}}\tilde{\mathcal{A}}\|\nabla(\Delta d-f)\|^2+C\|\nabla v\|^2. \tag{2.3.40}$$

Collecting all the above estimates together, we obtain (2.3.23). □

Lemma 2.3.2. *Under the assumption **Case I**, for any initial data $(v_0,d_0)\in V\times\mathbf{H}^2(Q)$, if the viscosity μ_4 is properly large, i.e., $\mu_4\geq\mu_4^0(\mu_i,\lambda_1,\lambda_2,v_0,d_0), i=1,2,3,5,6,$ we have*

$$\mathcal{A}(t)\leq C,\quad \forall t\ \geq 0. \tag{2.3.41}$$

The uniform bound C is a constant depending only on $f,Q,\|v_0\|_V,\|d_0\|_{\mathbf{H}^2},\mu's,\lambda's.$

Proof. It follows from (2.3.23) that

$$\frac{d}{dt}\tilde{\mathcal{A}}(t)+\left(\frac{\mu_4}{2}-C_1\mu_4{}^{\frac{1}{2}}\tilde{\mathcal{A}}(t)\right)\|\Delta v\|^2+\left(\frac{1}{-2\lambda_1}-C_2\mu_4^{-\frac{1}{4}}\tilde{\mathcal{A}}(t)\right)\|\nabla(\Delta d-f)\|^2\leq C_3\tilde{\mathcal{A}}(t). \tag{2.3.42}$$

On the other hand, by (2.3.11), there exists a positive constant M depending only on $\mu_i's,\lambda_i's$, $\|v_0\|$, $\|d_0\|_{\mathbf{H}^1}$, such that

$$\int_t^{t+1}\tilde{\mathcal{A}}(\tau)d\tau\leq\int_t^{t+1}\mathcal{A}(\tau)d\tau+1\leq M,\quad\forall t\geq 0. \tag{2.3.43}$$

Now we choose μ_4 large enough satisfying

$$\mu_4{}^{\frac{1}{2}}\geq 2C_1(\tilde{\mathcal{A}}(0)+4M+C_3M)+4\lambda_1^2C_2^2(\tilde{\mathcal{A}}(0)+4M+C_3M)^2+1. \tag{2.3.44}$$

As a result, there must be some $T_0>0$ such that

$$\frac{\mu_4}{2}-C_1\mu_4{}^{\frac{1}{2}}\tilde{\mathcal{A}}(t)\geq 0,\quad -\frac{1}{2\lambda_1}-C_2\mu_4{}^{-\frac{1}{4}}\tilde{\mathcal{A}}(t)\geq 0,\quad\text{for all } t\in[0,T_0].$$

Moreover, on $[0,T_0]$,

$$\frac{d}{dt}\tilde{\mathcal{A}}(t)\leq C_3\tilde{\mathcal{A}}(t). \tag{2.3.45}$$

Denote $T_*=\sup T_0$. First we show that $T_*\geq 1$ by a contradiction argument.
If $T_*<1$, then

$$\tilde{\mathcal{A}}(T_*)\leq\tilde{\mathcal{A}}(0)+C_3\int_0^1\tilde{\mathcal{A}}(t)dt\leq\tilde{\mathcal{A}}(0)+C_3M.$$

On the other hand, from the definition of T_*, we have for $t=T_*$

$$\mu_4^{\frac{1}{2}}<\max\{2C_1\tilde{\mathcal{A}}(T_*),\ 4\lambda_1^2C_2^2\tilde{\mathcal{A}}^2(T_*)\}, \tag{2.3.46}$$

which contradicts (2.3.44).

Next, if $T_* < +\infty$, (2.3.43) implies that there is a $t_1 \in [T_* - \frac{1}{2}, T_*]$ such that

$$\tilde{\mathcal{A}}(t_1) \leq 4M.$$

As a result,

$$\tilde{\mathcal{A}}(T_*) \leq 4M + C_3 \int_{t_1}^{T_*} \tilde{\mathcal{A}}(\tau) d\tau \leq 4M + C_3 M. \tag{2.3.47}$$

Then from the definition of T_*, we again have (2.3.46), which together with (2.3.47) yields a contradiction with (2.3.44). Hence, we have the uniform estimate

$$\tilde{\mathcal{A}}(t) \leq \min \left\{ \frac{\mu_4{}^{\frac{1}{2}}}{2C_1}, \frac{\mu_4{}^{\frac{1}{4}}}{-2\lambda_1 C_2} \right\}, \quad \forall\, t \geq 0.$$

The proof is complete.

$\square$

Next, we briefly discuss **Case II**.

Corollary 2.3.1. *For $n = 2, 3$, under the assumption **Case II**, we still have inequality* (2.3.23).

Proof. If Parodi's relation (2.1.16) doesn't hold, i.e., $\lambda_2 + (\mu_2 + \mu_3) \neq 0$, then in the proof of Theorem 2.3.1, the first term of the right-hand side of (2.3.19) does not cancel with the first term of the right-hand side of (2.3.16). Consequently, there is one extra term:

$$(\lambda_2 + \mu_2 + \mu_3)\Big(N_i d_j,\, \Delta(A_{ij})\Big).$$

Besides, since we no longer have (2.2.6) in **Case II**, we have to redo the estimate for (2.3.36). Using the d equation (2.1.12) and integration by parts, we get

$$\begin{aligned}
& (\lambda_2 + \mu_2 + \mu_3)\Big(N_i d_j,\, \Delta(A_{ij})\Big) \\
= \ & \frac{\lambda_2 + \mu_2 + \mu_3}{\lambda_1}\Big(\nabla(\Delta d - f)d,\, \nabla A\Big) + \frac{\lambda_2 + \mu_2 + \mu_3}{\lambda_1}\Big((\Delta d - f)\nabla d,\, \nabla A\Big) \\
& + \frac{\lambda_2(\lambda_2 + \mu_2 + \mu_3)}{\lambda_1}\Big(\nabla A d,\, \nabla A d\Big) + \frac{2\lambda_2(\lambda_2 + \mu_2 + \mu_3)}{\lambda_1}\Big(A\nabla d,\, \nabla A d\Big).
\end{aligned} \tag{2.3.48}$$

We estimate the the right-hand side of (2.3.48). First, we notice that the second term can be estimated as (2.3.33), while the fourth term of (2.3.48) is similar to (2.3.31). For the first term, we have

$$\begin{aligned}
& \frac{\lambda_2 + \mu_2 + \mu_3}{\lambda_1}\Big(\nabla(\Delta d - f)d,\, \nabla A\Big) \\
\leq \ & C\|d\|_{\mathbf{L}^\infty}\|\nabla(\Delta d - f)\|\|\Delta v\| \leq C(\|\Delta d - f\|^{\frac{1}{2}} + 1)\|\nabla(\Delta d - f)\|\|\Delta v\| \\
\leq \ & \mu_4{}^{\frac{1}{4}}(1 + \|\Delta d - f\|)\|\Delta v\|^2 + \frac{C}{\mu_4{}^{\frac{1}{4}}}\|\nabla(\Delta d - f)\|^2 \\
\leq \ & \mu_4{}^{\frac{1}{2}}\tilde{\mathcal{A}}\|\Delta v\|^2 + C\mu_4^{-\frac{1}{4}}\|\nabla(\Delta d - f)\|^2.
\end{aligned} \tag{2.3.49}$$

Finally, concerning the third term in (2.3.48) and the two terms in (2.3.36), we infer from (2.1.15) and (2.2.11) that

$$\begin{aligned}
&\frac{\lambda_2(\lambda_2+\mu_2+\mu_3)}{\lambda_1}-\frac{(\lambda_2)^2}{\lambda_1}-(\mu_5+\mu_6)=-\frac{1}{\lambda_1}[\lambda_1(\mu_5+\mu_6)-\lambda_2(\mu_2+\mu_3)]\\
<\quad &-\frac{1}{\lambda_1}\left[-\frac{1}{2}(\lambda_2-\mu_2-\mu_3)^2-\lambda_2(\mu_2+\mu_3)\right]=\frac{1}{2\lambda_1}[(\lambda_2)^2+(\mu_2+\mu_3)^2]\leq 0,
\end{aligned}$$

which yields

$$\left[\frac{\lambda_2(\lambda_2+\mu_2+\mu_3)}{\lambda_1}-\frac{(\lambda_2)^2}{\lambda_1}-(\mu_5+\mu_6)\right]\int_Q|\nabla Ad|^2dx\leq 0.$$

Combining the other estimates in the proof of Theorem 2.3.1, we obtain the inequality (2.3.23) under assumption **Case II**. □

Corollary 2.3.2. *Under the assumption **Case II**, for any initial data $(v_0,d_0)\in V\times\mathbf{H}^2(Q)$, if the viscosity μ_4 is properly large, i.e., $\mu_4\geq\mu_4^0(\mu_i,\lambda_1,\lambda_2,v_0,d_0)$, $i=1,2,3,5,6$, we have $\mathcal{A}(t)\leq C$ for $t\geq 0$ with C being a constant depending only on f, Q, $\|v_0\|_V$, $\|d_0\|_{\mathbf{H}^2}$, μ's, λ's.*

2.3.4 Existence and uniqueness of classical solutions

Under both **Case I** and **Case II**, the uniform estimates we have obtained are independent of the approximation parameter m and time t. This indicates that for both cases, (v_m,d_m) is a global solution to the approximate problem (2.3.2)–(2.3.4):

$$v_m\in L^\infty(0,+\infty;V)\cap L^2_{loc}(0,+\infty;\mathbf{H}^2),\quad d_m\in L^\infty(0,+\infty;\mathbf{H}^2)\cap L^2_{loc}(0,+\infty;\mathbf{H}^3),\tag{2.3.50}$$

which further implies that

$$\partial_t v_m\in L^2_{loc}(0,+\infty;\mathbf{L}^2),\quad \partial_t d_m\in L^2_{loc}(0,+\infty;\mathbf{L}^2).\tag{2.3.51}$$

The uniform estimates enable us to pass to the limit for (v_m,d_m) as $m\to\infty$. By a similar argument to [9,38], we can show that there exist limit functions (v,d) satisfying

$$v\in L^\infty(0,\infty;V)\cap L^2_{loc}(0,+\infty;\mathbf{H}^2),\quad d\in L^\infty(0,+\infty;\mathbf{H}^2)\cap L^2_{loc}(0,+\infty;\mathbf{H}^3),\tag{2.3.52}$$

such that (v,d) is a strong solution and system (2.1.10)–(2.1.12) are satisfied a.e. in Q_T for arbitrary $T>0$. A bootstrap argument based on Serrin's result [49] and Sobolev embedding theorems leads to the existence of classical solutions.

To prove the uniqueness of regular solutions to problem (2.1.10)–(2.1.14), we need the following continuous dependence on initial data. The proof is lengthy but quite standard, hence we omit the proof here. The corresponding proof in a simplified case can be found in [48].

Lemma 2.3.3. *Suppose that assumptions in Theorem 2.3.2 are satisfied. (v_i,d_i) $(i=1,2)$ are global solutions to problem* (2.1.10)–(2.1.14) *corresponding to initial data $(v_{0i},d_{0i})\in V\times\mathbf{H}^2_p(Q)$ $(i=1,2)$. Moreover, we assume that for any $T>0$, the following estimate holds*

$$\|v_i(t)\|_{H^1}+\|d_i(t)\|_{H^2}\leq M,\quad\forall\ t\in[0,T].\tag{2.3.53}$$

Then for any $t \in [0,T]$*, we have*

$$\begin{aligned} & \|(v_1 - v_2)(t)\|^2 + \|(d_1 - d_2)(t)\|^2_{\mathbf{H}^1} \\ & + \int_0^t \left(\frac{\mu_4}{2} \|\nabla(v_1 - v_2)(\tau)\|^2 + \|\Delta(d_1 - d_2)(\tau)\|^2 \right) d\tau \\ \leq \; & 2e^{Ct}(\|v_{01} - v_{02}\|^2 + \|d_{01} - d_{02}\|^2_{\mathbf{H}^1}), \end{aligned}$$

where C *is a constant depending on* $\|v_0\|_V, \|d_0\|_{\mathbf{H}^2}, \mu' s, \lambda' s$ *but not on* t*.*

Corollary 2.3.3. *The global solution* (v,d) *obtained in Lemma 2.3.2 or Corollary 2.3.2 is unique.*

Proof. Since the global classical solution (v,d) to the problem (2.1.10)–(2.1.14) obtained in both Lemma 2.3.2 and Corollary 2.3.2 is uniformly bounded in $V \times H^2$, it follows immediately from Lemma 2.3.3 that the solution is unique. □

Summing up, we have proved the following main theorem in this chapter.

Theorem 2.3.2. *Let* $n = 2,3$*. We assume that either the conditions in* ***Case I*** *or in* ***Case II*** *are satisfied. For any* $(v_0, d_0) \in V \times \mathbf{H}^2(Q)$*, under the large viscosity assumption* $\mu_4 \geq \mu_4^0(\mu_i, \lambda_1, \lambda_2, v_0, d_0)$*,* $i = 1,2,3,5,6$*, the problem* (2.1.10)–(2.1.14) *admits a unique global solution in the sense that*

$$v \in L^\infty(0,\infty;V) \cap L^2_{loc}(0,+\infty;\mathbf{H}^2), \quad d \in L^\infty(0,+\infty;\mathbf{H}^2) \cap L^2_{loc}(0,+\infty;\mathbf{H}^3). \tag{2.3.54}$$

.

Remark 2.3.1. *Due to the complexity of the system and the appearance of highly nonlinear stress terms, we have to impose the large viscosity assumption even in* $2D$ *case, which is quite different from all existing related results.*

Remark 2.3.2. *If in addition, the assumption (i)* $\mu_1 = 0, \lambda_2 \neq 0$*, or (ii)* $\mu_1 \geq 0, \lambda_2 = 0$ *is supposed, the same result holds true in* $2D$ *without assuming the viscosity* μ_4 *to be large. For (i), we notice that the nonlinearity of the highest-order vanishes. In particular, this applies for the rod-like system in [38], which is a simplified version of the general Ericksen–Leslie model. On the other hand, for (ii), one can apply the maximum principle for* d *to obtain its* $\mathbf{L}^\infty$*-bound, which makes the proof much easier (cf. [10]).*

Chapter 3

Long Time Behavior for Global Solutions to the General Ericksen-Leslie System

Generally speaking, the study of long behavior of solutions to nonlinear dissipative evolution equations can be divided into two categories: In the first category, it is concerned with the long behavior of a global solution (or a single orbit) for any given initial datum, i.e., whether the global solution will converge to an equilibrium as time goes to infinity. In the second category, it is concerned with the asymptotic behavior of a family of global solutions (or orbits) for initial data starting from any bounded set in certain Sobolev space, i.e., whether the family of global solutions will converge to a compact invariant set, namely, a global attractor. In this chapter for the global solution to the liquid crystal system, we are concerned with the first category. With the help of a suitable Łojaciewicz–Simon type inequality, we prove that although different kinematic transports for the liquid crystal molecules will yield different dynamics of the hydrodynamical system, we show that global solutions to our system have uniform long-time behavior under different kinematic transports, i.e., convergence to equilibrium with a uniform convergence rate. Section 3.1 is a brief discussion of the application of Łojasiewicz inequality in finite dimensional Euclidean space. Section 3.2 is devoted to the proof of convergence of global solutions to single equilibrium states as times goes to infinity, and Section 3.3 provides with the estimate on the convergence rate.

3.1 Application of finite dimensional Łojasiewicz inequality

The key ingredient in this chapter is the application of the Łojasiewicz–Simon approach. Since it is a generalization of the Łojasiewicz inequality in finite dimensional space $\mathbb{R}^m$ for analytic functions, to understand it better, let us briefly recall the applications in the finite dimensional case first.

In the 1960's, Łojaciewicz proved the following fundamental inequality for gradient systems of analytic functions in finite dimensional Euclidean spaces [12, 13].

Theorem 3.1.1 (Łojasiewicz inequality). *Suppose that $F : \mathbb{R}^m \to \mathbb{R}$ is an analytic function near its critical point a (i.e., $\nabla F(a) = 0$). Then there is a positive constant σ and $\theta \in (0, \frac{1}{2})$ depending on a, such that when $\|x - a\|_{\mathbb{R}^m} \le \sigma$,*

$$|F(x) - F(a)|^{1-\theta} \le \|\nabla F(x)\|_{\mathbb{R}^m}. \tag{3.1.1}$$

The Łojasiewicz inequality is a powerful tool to study the asymptotic behavior of solutions to gradient systems. To describe the idea, let us recall a simple example discussed in [19] (for other applications on ODEs, cf. e.g. [50]).

Consider the ODE system

$$\begin{cases} x_t &= & -\nabla f(x), \qquad x \in \mathbb{R}^N, \\ x(0) &= & x_0. \end{cases} \tag{3.1.2}$$

We assume that f is analytic in x, $f \ge 0$. We also assume that the ODE system (3.1.2) admits a bounded smooth solution $x(t)$, defined for all $t \ge 0$. For brevity we denote $F(t) = f(x(t))$, $t \ge 0$. Multiplying both sides of (3.1.2) with $x_t(t)$, we know

$$\frac{dF(t)}{dt} = -\|\nabla f(x(t))\|^2_{\mathbb{R}^N} = -\|x_t(t)\|^2_{\mathbb{R}^N} \le 0, \quad \forall\, t \ge 0. \tag{3.1.3}$$

Therefore, the nonnegative function $F(t)$ is decreasing on $[0, +\infty)$ and serves as a Lyapunov function for (3.1.2). Then integrating (3.1.3) from 0 to t, we have

$$F(t) + \int_0^t \|x_t(\tau)\|^2_{\mathbb{R}^N} d\tau = F(0). \tag{3.1.4}$$

For the gradient system (3.1.2), we infer that the ω-limit set of $x(t)$, is nonempty and consists of equilibria (cf. [51]). Namely, there exists an increasing unbounded sequence $\{t_n\}_{n\in N}$ and an equilbrium $x_\infty \in \mathbb{R}^N$, such that

$$\lim_{t_n \to \infty} \|x(t_n) - x_\infty\|_{\mathbb{R}^N} = 0. \tag{3.1.5}$$

Consequently, $F(t_n) \ge f(x_\infty)$ and

$$\lim_{t_n \to \infty} F(t_n) = F_\infty = f(x_\infty) \ge 0. \tag{3.1.6}$$

Our goal is to prove

$$\lim_{t_n \to \infty} \|x(t) - x_\infty\|_{\mathbb{R}^N} = 0. \tag{3.1.7}$$

We can discuss in two subcases.

Case 1. If there exists some $t_0 \ge 0$, such that $F(t_0) = f(x_\infty)$, then we deduce from (3.1.3) that $\forall\, t \ge t_0$, $x(t) \equiv x(t_0)$.

Case 2. If $\forall t > 0$, $F(t) > f(x_\infty)$, due to (3.1.3), (3.1.6), we have

$$\lim_{t\to\infty} F(t) = f(x_\infty). \tag{3.1.8}$$

Let $\varepsilon = (\frac{\sigma\theta}{4})^{\frac{1}{\theta}}$, it follows from (3.1.5) and (3.1.8) that there exists an integer K such that for all $n > K$.

$$\|x(t_n) - x_\infty\|_{\mathbb{R}^N} < \frac{\sigma}{4}, \quad 0 < F(t_n) - f(x_\infty) < \varepsilon. \tag{3.1.9}$$

Define

$$\bar{t}_n = \sup\{t > t_n \big| \, \|x(s) - x_\infty\|_{\mathbb{R}^N} < \sigma, \ \forall\, s \in [t_n, t].\}$$

In what follows, we recall the simple argument introduced in [14], which provides a convenient way to apply the Łojasiewicz inequality.

Proposition 3.1.1. *There exists $n_0 \geq K$, such that $\bar{t}_{n_0} = +\infty$.*

Proof. The proof follows from the contradiction argument in [14]. Suppose $\forall\, n \geq K$, $t_n < \bar{t}_n < +\infty$. We can apply Theorem 3.1.1 on interval $[t_n, \bar{t}_n]$. As a consequence, the length of the trajectory $x(t)$ between $[t_n, \bar{t}_n]$ is

$$\begin{aligned}
&\int_{t_n}^{\bar{t}_n} \|x_t(\tau)\|_{\mathbb{R}^N} d\tau = \int_{t_n}^{\bar{t}_n} \frac{1}{\|\nabla f(x(\tau))\|_{\mathbb{R}^N}} (-\frac{d}{d\tau} F(\tau)) d\tau \\
\leq\ & \int_{t_n}^{\bar{t}_n} \frac{1}{|F(\tau) - f(x_\infty)|^{1-\theta}} (-\frac{d}{d\tau} F(\tau)) d\tau \\
=\ & \frac{1}{\theta}\Big[(F(t_n) - f(x_\infty))^\theta - (F(\bar{t}_n) - f(x_\infty))^\theta\Big] \\
<\ & \frac{1}{\theta}(F(t_n) - f(x_\infty))^\theta < \frac{1}{\theta}\varepsilon^\theta < \frac{\sigma}{4}.
\end{aligned} \tag{3.1.10}$$

Therefore,

$$\begin{aligned}
\|x(\bar{t}_n) - x_\infty\|_{\mathbb{R}^N} &\leq \|x(\bar{t}_n) - x(t_n)\|_{\mathbb{R}^N} + \|x(t_n) - x_\infty\|_{\mathbb{R}^N} \\
&< \int_{t_n}^{\bar{t}_n} \|x_t(\tau)\|_{\mathbb{R}^N} d\tau + \frac{\sigma}{4} < \frac{\sigma}{2},
\end{aligned} \tag{3.1.11}$$

which is a contradiction to the definition of $\bar{t}_n$. $\square$

Since $\|x(t) - x_\infty\|_{\mathbb{R}^N} < \sigma$, $\forall\, t \geq t_{n_0}$, we infer that

$$\int_0^{+\infty} \|x_t(\tau)\|_{\mathbb{R}^N} d\tau < +\infty.$$

Consequently,

$$\|x(t_1) - x(t_2)\|_{\mathbb{R}^N} \leq \int_{t_1}^{t_2} \|x_t(\tau)\|_{\mathbb{R}^N} d\tau \to 0, \quad \text{as } t_1,\ t_2 \to 0.$$

Hence, $x(t)$ is uniformly convergent in $\mathbb{R}^N$. Combined with (3.1.5), one arrives at our goal (3.1.7).

Finally, let us study the convergence rate of $x(t)$ to x_∞. We only have to consider Case 2, since Case 1 is trivial. We know from Theorem 3.1.1 that $\forall\, t \ge t_{n_0}$,

$$-\frac{d}{dt}[F(t)-f(x_\infty)]^\theta \quad = \quad \theta\|x_t\|^2_{\mathbb{R}^N}[F(t)-f(x_\infty)]^{\theta-1} \ge \theta\|F(t)-f(x_\infty)\|^{1-\theta}_{\mathbb{R}^N},$$

which indicates (cf. [15, 22])

$$F(t)-f(x_\infty) \le C(1+t)^{-\frac{1}{1-2\theta}}, \qquad \forall\, t \ge t_{n_0}. \tag{3.1.12}$$

As a result,

$$\int_t^\infty \|x_t(\tau)\|_{\mathbb{R}^N}d\tau \le -\frac{1}{\theta}\int_t^\infty \frac{d}{d\tau}[F(t)-f(x_\infty)]^\theta d\tau \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \ge t_{n_0}.$$

After choosing proper constant C, we can get

$$\|x(t)-x_\infty\|_{\mathbb{R}^N} \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \qquad \forall\, t \ge 0. \tag{3.1.13}$$

3.2 Convergence to Equilibrium

In this section we prove that the global bounded solution obtained in (2.3.2) will converge uniformly to a single equilibrium state solution as time goes to infinity, which implies the uniqueness of the asymptotic limit of the system (2.1.10)–(2.1.14).

For brevity, we only discuss the situation in $2D$ case here, because the analysis in $3D$ can be treated similarly.

For the system (2.1.10)-(2.1.14), we have the Lyapunov functional to the system

$$\mathcal{E}(t) = \frac{1}{2}\|v(t)\|^2 + \frac{1}{2}\|\nabla d(t)\|^2 + \int_\Omega F(d(t))dx \tag{3.2.1}$$

satisfying the basic energy law (2.2.7), which immediately yields $\mathcal{A}(t) \in L^1(0,+\infty)$. Combined with the uniform bounds of $\mathcal{A}(t)$ itself and the time derivative of $\mathcal{A}(t)$, provided by Lemma 2.3.2 and Corollary 2.3.2 respectively, we can easily prove

Lemma 3.2.1. *For any $t \ge 0$, in both $2D$ and $3D$ cases, under the large viscosity assumption* (2.3.44)*, for the unique global solution $(v(t), d(t))$, it holds*

$$\lim_{t\to+\infty}(\|v(t)\|_{H^1} + \|-\Delta d(t) + f(d(t))\|) = 0. \tag{3.2.2}$$

It follows immediately from Lemma 3.2.1 that the velocity v is uniformly convergent to 0 as time goes to infinity. To prove the convergence of the director d, first let $\mathcal{S}$ be the set

$$\mathcal{S} = \{u \mid -\Delta u + f(u) = 0, \text{ in } Q, \ u(x+e_i) = u(x) \text{ on } \partial Q\}.$$

The ω-limit set of $(v_0, d_0) \in V \times H^2_p(Q) \subset L^2_p(Q) \times H^1_p(Q)$ is defined as follows:

$$\omega(v_0, d_0) \quad = \quad \{(v_\infty(x), d_\infty(x)) \mid \text{ there exists } \{t_n\} \nearrow \infty \text{ such that}$$

$$(v(t_n), d(t_n)) \to (v_\infty, d_\infty) \text{ in } L^2(Q) \times H^1(Q), \text{ as } t_n \to +\infty\}.$$

We infer from the uniform bound of $(v(t), d(t))$ and Lemma 3.2.1 that

Proposition 3.2.1. *$\omega(v_0, d_0)$ is a nonempty bounded subset in $H^1_p(Q) \times H^2_p(Q)$, which is compact in $L^2_p(Q) \times H^1_p(Q)$. Besides, all asymptotic limiting points (v_∞, d_∞) of the problem* (2.1.10) *-* (2.1.14) *satisfy that $v_\infty = 0$ and $d_\infty \in \mathcal{S}$.*

Proposition 3.2.1 implies that there is an increasing unbounded sequence $\{t_n\}_{n\in\mathbb{N}}$ and a function $d_\infty \in \mathcal{S}$ such that

$$\lim_{t_n \to +\infty} \|d(t_n) - d_\infty\|_{H^1} = 0. \tag{3.2.3}$$

Moreover, d_∞ satisfies the equation

$$-\Delta d_\infty + f(d_\infty) = 0, \quad x \in \Omega, \quad d_\infty(x + e_i) = d_\infty(x) \text{ on } \partial Q. \tag{3.2.4}$$

Let us look at the following elliptic periodic boundary value problem

$$\begin{cases} -\Delta d + f(d) = 0, & x \in \Omega, \\ d(x + e_i) = d(x), & x \text{ on } \partial\Omega. \end{cases} \tag{3.2.5}$$

Define

$$E(d) := \frac{1}{2}\|\nabla d\|^2 + \int_Q F(d)dx. \tag{3.2.6}$$

It is not difficult to see that the solution to (3.2.5) is a critical point of $E(d)$ and conversely the critical point of $E(d)$ is a solution to (3.2.5).

In order to apply the Łojasiewicz–Simon approach to prove the convergence to equilibrium, we shall introduce a suitable Łojasiewicz–Simon type inequality that is related to our problem. In particular, we have (cf. [21])

Lemma 3.2.2. [Łojasiewicz–Simon inequality] *Let ψ be a critical point of $E(d)$. Then there exist constants $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ depending on ψ such that for any $d \in H^1_p(Q)$ satisfying $\|d - \psi\|_{H^1_p(Q)} < \beta$, it holds*

$$\| - \Delta d + f(d)\|_{(H^1_p(Q))'} \geq |E(d) - E(\psi)|^{1-\theta}, \tag{3.2.7}$$

where $(H^1_p(Q))'$ is the dual space of $H^1_p(Q)$.

Remark 3.2.1. *Lemma 3.2.2 can be viewed as an extended version of Simon's result [52] for scalar functions under the L^2-norm.*

We prove the convergence result following a simple argument first introduced in [14] in which the key observation is that after a certain time t_0, $d(t)$ will fall into a certain small neighborhood of d_∞ and stay there forever.

From the basic energy law (2.2.7), we can see that $\mathcal{E}(t)$ is decreasing on $[0, \infty)$, and it has a finite limit as time goes to infinity because it is nonegative. Therefore, it follows from (3.2.2) and (3.2.3) that

$$\lim_{t_n \to +\infty} \mathcal{E}(t_n) = E(d_\infty). \tag{3.2.8}$$

On the other hand, we can infer from (2.2.7) that $\mathcal{E}(t) \geq E(d_\infty)$, for all $t > 0$, and the equal sign holds if and only if, for all $t > 0$, $v = 0$ and d solves problem (3.2.4).

We now consider all possibilities.

Case 1. If there is a $t_0 > 0$ such that $\mathcal{E}(t_0) = E(d_\infty)$, then for all $t > t_0$, we deduce from (2.2.7) that

$$\|v\| = \|\nabla v\| \equiv 0, \quad \| -\Delta d + f(d)\| \equiv 0. \tag{3.2.9}$$

It follows from (2.1.12), (3.2.9) and the Sobolev embedding theorem that for $t \geq t_0$,

$$\begin{aligned} 0 \leq \|d_t\| &\leq -\frac{1}{\lambda_1}\| -\Delta d + f(d)\| + |\frac{\lambda_2}{\lambda_1}|\|Ad\| + \|v \cdot \nabla d\| + \|\Omega d\| \\ &\leq \|v\|_{L^4}\|\nabla d\|_{L^4} + \|\Omega\|\|d\|_{L^\infty} \\ &\leq C\|\nabla v\|\|\nabla d\|_{L^4} + C\|\nabla v\| = 0. \end{aligned} \tag{3.2.10}$$

Namely, d is independent of time for all $t \geq t_0$. Due to (3.2.3), we conclude that $d(t) \equiv d_\infty$ for $t \geq t_0$.

Case 2. For all $t > 0$, we suppose that $\mathcal{E}(t) > E(d_\infty)$. First we assume that the following claim holds true.

Proposition 3.2.2. *There is a $t_0 > 0$ such that for all $t \geq t_0$, $\|d(t) - d_\infty\|_{H^1} < \beta$. Namely, for all $t \geq t_0$, $d(t)$ satisfies the condition in Lemma 3.2.2.*

In this case, it follows from Lemma 3.2.2 that

$$|E(d) - E(d_\infty)|^{1-\theta} \leq \| -\Delta d + f(d)\|_{(H^1_p(Q))'} \leq \| -\Delta d + f(d)\|, \quad \forall\, t \geq t_0. \tag{3.2.11}$$

The fact $\theta \in (0, \frac{1}{2})$ implies that $0 < 1-\theta < 1$, $2(1-\theta) > 1$. Consequently,

$$\|v\|^{2(1-\theta)} = \|v\|^{2(1-\theta)-1}\|v\| \leq C\|v\|.$$

Then we infer from the basic inequality

$$(a+b)^{1-\theta} \leq a^{1-\theta} + b^{1-\theta}, \qquad \forall\, a, b \geq 0$$

that

$$\begin{aligned} (\mathcal{E}(t) - E(d_\infty))^{1-\theta} &\leq \left(\frac{1}{2}\|v\|^2 + |E(d) - E(d_\infty)|\right)^{1-\theta} \\ &\leq \left(\frac{1}{2}\|v\|^2 + \| -\Delta d + f(d)\|^{\frac{1}{1-\theta}}\right)^{1-\theta} \\ &\leq \left(\frac{1}{2}\right)^{1-\theta}\|v\|^{2(1-\theta)} + \| -\Delta d + f(d)\| \\ &\leq C\|v\| + \| -\Delta d + f(d)\|. \end{aligned} \tag{3.2.12}$$

Therefore, a direct calculation yields that

$$\begin{aligned} -\frac{d}{dt}(\mathcal{E}(t)-E(d_\infty))^\theta &= -\theta(\mathcal{E}(t)-E(d_\infty))^{\theta-1}\frac{d}{dt}\mathcal{E}(t) \\ &\geq \frac{C\theta(\|\nabla v\|+\|-\Delta d+f(d)\|)^2}{C\|v\|+\|-\Delta d+f(d)\|} \\ &\geq C_1(\|\nabla v\|+\|-\Delta d+f(d)\|), \quad \forall\, t\geq t_0, \end{aligned} \tag{3.2.13}$$

where C_1 is a constant depending on $\mu_4, \lambda_1, v_0, d_0, Q, \theta$.
Integrating from t_0 to t, we get

$$\begin{aligned} & (\mathcal{E}(t)-E(d_\infty))^\theta + C_1\int_{t_0}^t (\|\nabla v(\tau)\|+\|-\Delta d(\tau)+f(d(\tau))\|)d\tau \\ \leq\ & (\mathcal{E}(t_0)-E(d_\infty))^\theta < \infty, \quad \forall\, t\geq t_0. \end{aligned} \tag{3.2.14}$$

Since $\mathcal{E}(t)-E(d_\infty)\geq 0$, we conclude that

$$\int_{t_0}^\infty (\|\nabla v(\tau)\|+\|-\Delta d(\tau)+f(d(\tau))\|)d\tau < \infty. \tag{3.2.15}$$

On the other hand, it follows from equation (2.1.12) and Sobolev embedding theorems that

$$\begin{aligned} \|d_t\| &\leq \|v\cdot\nabla d\|+\|\Omega d\|+|\frac{\lambda_2}{\lambda_1}|\|Ad\|-\frac{1}{\lambda_1}\|-\Delta d+f(d)\| \\ &\leq \|v\|_{L^4}\|\nabla d\|_{L^4}+\|d\|_{L^\infty}\|\Omega\|+|\frac{\lambda_2}{\lambda_1}|\|\nabla v\|\|d\|_{L^\infty}-\frac{1}{\lambda_1}\|-\Delta d+f(d)\| \\ &\leq C\|\nabla v\|+C\|-\Delta d+f(d)\|. \end{aligned} \tag{3.2.16}$$

Hence,

$$\int_{t_0}^\infty \|d_t(\tau)\|d\tau < +\infty, \tag{3.2.17}$$

which easily implies that as $t\to+\infty$, $d(t)$ converges in $L^2(Q)$. This and (3.2.3) indicate that

$$\lim_{t\to+\infty}\|d(t)-d_\infty\| = 0. \tag{3.2.18}$$

Since $d(t)$ is uniformly bounded in $H^2(Q)$, by standard interpolation inequality we have

$$\lim_{t\to+\infty}\|d(t)-d_\infty\|_{H^1} = 0. \tag{3.2.19}$$

On the other hand, the uniform bound of d in $H^2(Q)$ implies the weak convergence

$$d(t)\rightharpoonup d_\infty, \quad \text{in } H^2(Q). \tag{3.2.20}$$

However, the decay property of the quantity $A(t)$ (cf. Lemma 3.2.1) could tell us more. Namely, we could get strong convergence of d in H^2. To see this, we keep in mind that

$$\|\Delta d-\Delta d_\infty\| \leq \|\Delta d-\Delta d_\infty-f(d)+f(d_\infty)\|+\|f(d)-f(d_\infty)\|$$

$$
\begin{aligned}
&\le \|\Delta d - f(d)\| + \|f'(\xi)\|_{L^4}\|d - d_\infty\|_{L^4} \\
&\le \|\Delta d - f(d)\| + C\|d - d_\infty\|_{H^1}. \qquad (3.2.21)
\end{aligned}
$$

The above estimate together with (3.2.2) and (3.2.19) yields

$$\lim_{t\to+\infty} \|d(t) - d_\infty\|_{H^2} = 0. \qquad (3.2.22)$$

To finish the proof, it remains to show that Proposition 3.2.2 always holds true for the global solution $d(t)$ to the system (2.1.10)-(2.1.14). Define

$$\bar{t}_n = \sup\{\ t > t_n |\ \|d(s) - d_\infty\|_{H^1} < \beta,\ \forall\ s \in [t_n, t]\}. \qquad (3.2.23)$$

It follows from (3.2.3) that for any $\varepsilon \in (0, \beta)$, there exists an integer N such that when $n \ge N$,

$$
\begin{aligned}
&\|d(t_n) - d_\infty\|_{H^1} < \varepsilon, \qquad (3.2.24)\\
&\frac{1}{C_1}(\mathcal{E}(t_n) - E(d_\infty))^\theta < \varepsilon. \qquad (3.2.25)
\end{aligned}
$$

On the other hand, we know that the orbit of the classical solution d is continuous in H^1. It follows from the uniform bound of $\|d(t)\|_{H^2}$ that $d \in L^2(t, t+1; H^2)$ for any $t \ge 0$. The basic energy law and (3.2.16) imply $d_t \in L^2(t, t+1; L^2)$. Thus, for any $t \ge 0$, it holds $d \in C([t, t+1]; H^1)$. The continuity of the orbit of d in H^1 and (3.2.24) yield that

$$\bar{t}_n > t_n, \quad \text{for all} \ \ n \ge N. \qquad (3.2.26)$$

Then there are two possibilities:

(i). If there exists $n_0 \ge N$ such that $\bar{t}_{n_0} = +\infty$, then from the previous discussions in Case 1 and Case 2, the theorem is proved.

(ii) Otherwise, for all $n \ge N$, we have $t_n < \bar{t}_n < +\infty$, and for all $t \in [t_n, \bar{t}_n]$, $E(d_\infty) < \mathcal{E}(t)$. Then from (3.2.14) with t_0 being replaced by t_n, and t being replaced by $\bar{t}_n$, we obtain from (3.2.25) that

$$\int_{t_n}^{\bar{t}_n} (\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|) d\tau < \varepsilon. \qquad (3.2.27)$$

Thus, it follows that (cf. (3.2.16))

$$
\begin{aligned}
\|d(\bar{t}_n) - d_\infty\| &\le \|d(t_n) - d_\infty\| + C\int_{t_n}^{\bar{t}_n} \|d_t(\tau)\| d\tau \\
&\le \|d(t_n) - d_\infty\| + C\int_{t_n}^{\bar{t}_n} (\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|) d\tau \\
&< C\varepsilon, \qquad (3.2.28)
\end{aligned}
$$

which implies that $\lim_{n\to+\infty} \|d(\bar{t}_n) - d_\infty\| = 0$. Since $d(t)$ is relatively compact in H^1, there exists a subsequence of $\{d(\bar{t}_n)\}$, still denoted by $\{d(\bar{t}_n)\}$ converging to d_∞ in H^1, i.e., when n is sufficiently large,

$$\|d(\bar{t}_n) - d_\infty\|_{H^1} \ < \beta,$$

which contradicts the definition of $\bar{t}_n$ that $\|d(\bar{t}_n) - d_\infty\|_{H^1} = \beta$.

Summing up, we have considered all the possible cases and prove the conclusion that

$$\lim_{t\to+\infty}(\|v(t)\|_{H^1} + \|d(t) - d_\infty\|_{H^2}) = 0, \tag{3.2.29}$$

3.3 Convergence rate

Once we prove the convergence to an equilibrium, a natural question is the convergence rate. It is well known that an estimate in certain lower–order norm can usually be obtained directly from the Łojasiewicz–Simon approach (see, e.g., [15]). One can then in a straightforward way, obtain estimates in higher–order norms by using interpolation inequalities (cf. [15]), and consequently, the decay exponent deteriorates. We shall show that by using suitable energy estimates and constructing proper differential inequalities, it is possible to obtain the same estimates on the convergence rate in both higher and lower order norms. This procedure can be achieved by three steps.

Step 1. As has been shown in the literature (cf. [15]), an estimate on the convergence rate in certain lower–order norm could be obtained directly from the Łojasiewicz–Simon approach. From Lemma 3.2.2 and (3.2.13), we have

$$\frac{d}{dt}(\mathcal{E}(t) - E(d_\infty)) + C_1(\mathcal{E}(t) - E(d_\infty))^{2(1-\theta)} \le 0, \quad \forall\ t \ge t_0, \tag{3.3.1}$$

which implies

$$\mathcal{E}(t) - E(d_\infty) \le C(1+t)^{-\frac{1}{1-2\theta}}, \quad \forall\ t \ge t_0. \tag{3.3.2}$$

Integrating (3.2.13) on (t,∞), where $t \ge t_0$, it follows from (3.2.16) that

$$\int_t^\infty \|d_t(\tau)\|d\tau \le \int_t^\infty C(\|\nabla v(\tau)\| + \| - \Delta d(\tau) + f(d(\tau))\|)d\tau \le C(1+t)^{-\frac{\theta}{1-2\theta}}. \tag{3.3.3}$$

By adjusting the constant C properly, we obtain

$$\|d(t) - d_\infty\| \le C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad t \ge 0. \tag{3.3.4}$$

Step 2. The steady state corresponding to problem (2.1.10)-(2.1.14) satisfies the following system:

$$\frac{\partial v_\infty}{\partial t} + v_\infty\cdot\nabla v_\infty + \nabla P_\infty = -\nabla\cdot(\nabla d_\infty \odot \nabla d_\infty) + \nabla\cdot\sigma_\infty, \tag{3.3.5}$$

$$\nabla\cdot v_\infty = 0, \tag{3.3.6}$$

$$-\lambda_1\Big(d_{\infty t} + v_\infty\cdot\nabla d_\infty - \Omega_\infty d_\infty + \frac{\lambda_2}{\lambda_1}A_\infty d_\infty\Big) = \Delta d_\infty - f(d_\infty), \tag{3.3.7}$$

$$v_\infty(x+e_i) = v_\infty(x), \qquad d_\infty(x+e_i) = d_\infty(x). \tag{3.3.8}$$

where

$$\begin{aligned}
A_\infty &= \frac{\nabla v_\infty + \nabla^T v_\infty}{2}, \quad \Omega_\infty = \frac{\nabla v_\infty - \nabla^T v_\infty}{2}, \\
N_\infty &= \frac{\partial d_\infty}{\partial t} + v_\infty \cdot \nabla d_\infty - \Omega_\infty d_\infty. && (3.3.9) \\
\sigma_\infty &= \mu_1 d_\infty^T A_\infty d_\infty d_\infty \odot d_\infty + \mu_2 N_\infty \odot d_\infty + \mu_3 d_\infty \odot N_\infty + \mu_4 A_\infty \\
&\quad + \mu_5 A_\infty d_\infty \odot d_\infty + \mu_6 d_\infty \odot A_\infty d_\infty. && (3.3.10)
\end{aligned}$$

Lemma 3.2.1 implies that all limiting points of system (2.1.10)-(2.1.14) satisfy $v_\infty = 0$ and $d_\infty \in \mathcal{S}$. As a result, system (3.3.5)–(3.3.8) can be reduced to

$$\begin{aligned}
\nabla P_\infty + \nabla \left(\frac{|\nabla d_\infty|^2}{2} \right) &= -\nabla d_\infty \cdot \Delta d_\infty, && (3.3.11) \\
\nabla \cdot v_\infty &= 0, && (3.3.12) \\
-\Delta d_\infty + f(d_\infty) &= 0, && (3.3.13) \\
v_\infty(x + e_i) &= v(x), \quad d_\infty(x + e_i) = d(x), \quad \text{for } x \in \partial Q && (3.3.14)
\end{aligned}$$

In (3.3.11), we use the fact that

$$\nabla \cdot (\nabla d_\infty \odot \nabla d_\infty) = \nabla \left(\frac{|\nabla d_\infty|^2}{2} \right) + \nabla d_\infty \cdot \Delta d_\infty.$$

Subtracting the stationary problem (3.3.11)-(3.3.14) from the evolution problem (2.1.10)-(2.1.14), we obtain that

$$\begin{aligned}
& v_t + v \cdot \nabla v - \nu \Delta v + \nabla (P - P_\infty) + \nabla \left(\frac{|\nabla d|^2}{2} - \frac{|\nabla d_\infty|^2}{2} \right) \\
&= -\nabla d \cdot \Delta d + \nabla d_\infty \cdot \Delta d_\infty + \nabla \cdot \sigma, && (3.3.15) \\
& \nabla \cdot v = 0, && (3.3.16) \\
& -\lambda_1 (d_t + v \cdot \nabla d - \Omega d + \frac{\lambda_2}{\lambda_1} A d) = \Delta (d - d_\infty) - f(d) + f(d_\infty), && (3.3.17) \\
& v_\infty(x + e_i) = v_\infty(x), \qquad d_\infty(x + e_i) = d_\infty(x). && (3.3.18)
\end{aligned}$$

Multiplying (3.3.15) by v and (3.3.17) by $\frac{1}{\lambda_1}(\Delta d - f) = \frac{1}{\lambda_1}\Delta(d - d_\infty) - \frac{1}{\lambda_1}(f(d) - f(d_\infty))$, respectively, integrating over Q, and adding the results together, we have

$$\begin{aligned}
& \frac{d}{dt} \left(\frac{1}{2}\|v\|^2 + \frac{1}{2}\|\nabla d - \nabla d_\infty\|^2 + \int_Q [F(d) - F(d_\infty) - f(d_\infty)(d - d_\infty)] dx \right) \\
& + \frac{\mu_4}{2}\|\nabla v\|^2 - \frac{1}{\lambda_1}\|\Delta d - f(d)\|^2 + \mu_1 \|d^T \nabla A d\|^2 + \left(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1} \right) \|Ad\|^2 \\
= \; & (v, \nabla d_\infty \cdot \Delta d_\infty) \\
= \; & (v, \nabla d_\infty \cdot (\Delta d_\infty - f(d_\infty))) + (v \cdot \nabla d_\infty, f(d_\infty)) \\
= \; & 0. && (3.3.19)
\end{aligned}$$

Multiplying (3.3.17) by $-\frac{1}{\lambda_1}(d-d_\infty)$ and integrating in Q, we have

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\|d-d_\infty\|^2 - \frac{1}{\lambda_1}\|\nabla(d-d_\infty)\|^2 \\
= \quad & -(v\cdot\nabla d, d-d_\infty) + (\Omega d, d-d_\infty) - \frac{\lambda_2}{\lambda_1}(Ad, d-d_\infty) - \frac{1}{\lambda_1}(f(d)-f(d_\infty), d-d_\infty) \\
:= \quad & I_1. \qquad (3.3.20)
\end{aligned}$$

Using Sobolev embedding theorem, the right hand side can be estimated as follows

$$\begin{aligned}
|I_1| \quad \leq \quad & \|v\|_{L^4}\|\nabla d\|_{L^4}\|d-d_\infty\| + \|\Omega\|\|d\|_{L^\infty}\|d-d_\infty\| + |\frac{\lambda_2}{\lambda_1}|\|A\|\|d\|_{L^\infty}\|d-d_\infty\| \\
& +\|f'(\zeta)\|_{L^\infty}\|d-d_\infty\|^2 \\
\leq \quad & C\|\nabla v\|\|d-d_\infty\| + C\|\nabla v\|\|d-d_\infty\| + C\|d-d_\infty\|^2 \\
\leq \quad & \varepsilon_1\|\nabla v\|^2 + C\|d-d_\infty\|^2. \qquad (3.3.21)
\end{aligned}$$

where $\zeta = ad + (1-a)d_\infty$ with $a \in [0,1]$.
Multiplying (3.3.20) by $\alpha > 0$ and adding the resultant to (3.3.19), using (3.3.21), we get

$$\begin{aligned}
& \frac{d}{dt}\bigg(\frac{1}{2}\|v\|^2 + \frac{1}{2}\|\nabla d - \nabla d_\infty\|^2 + \frac{\alpha}{2}\|d-d_\infty\|^2 + \int_\Omega (F(d)-F(d_\infty))dx \\
& \quad - \int_\Omega f(d_\infty)(d-d_\infty)dx\bigg) + (\frac{\mu_4}{2} - \alpha\varepsilon_1)\|\nabla v\|^2 - \frac{1}{\lambda_1}\|\Delta d - f(d)\|^2 \\
& -\frac{\alpha}{\lambda_1}\|\nabla(d-d_\infty)\|^2 + \mu_1\|d^T Ad\|^2 + \Big(\mu_5 + \mu_6 + \frac{(\lambda_2)^2}{\lambda_1}\Big)\|Ad\|^2 \\
\leq \quad & C\alpha\|d-d_\infty\|^2. \qquad (3.3.22)
\end{aligned}$$

On the other hand, by the Taylor's expansion, we have

$$F(d) = F(d_\infty) + f(d_\infty)(d-d_\infty) + f'(\xi)(d-d_\infty)^2, \qquad (3.3.23)$$

where $\xi = bd + (1-b)d_\infty$ with $b \in [0,1]$.
Then we deduce that

$$\begin{aligned}
& \left|\int_\Omega [F(d) - F(d_\infty)dx - f(d_\infty)(d-d_\infty)]dx\right| = \left|\int_\Omega f'(\xi)(d-d_\infty)^2 dx\right| \\
\leq \quad & \|f'(\xi)\|_{L^\infty}\|d-d_\infty\|^2 \leq C_2\|d-d_\infty\|^2. \qquad (3.3.24)
\end{aligned}$$

Let us define now, for $t \geq 0$,

$$\begin{aligned}
y(t) \quad = \quad & \frac{1}{2}\|v(t)\|^2 + \frac{1}{2}\|\nabla d(t) - \nabla d_\infty\|^2 + \frac{\alpha}{2}\|d(t)-d_\infty\|^2 + \int_Q (F(d(t))dx - F(d_\infty))dx \\
& - \int_\Omega f(d_\infty)(d(t)-d_\infty)dx. \qquad (3.3.25)
\end{aligned}$$

In (3.3.22) and (3.3.25), we choose

$$\alpha \geq 1 + 2C_2 > 0, \quad \varepsilon_1 = \frac{\mu_4}{4\alpha}. \tag{3.3.26}$$

As a result,

$$y(t) + C_2\|d - d_\infty\|^2 \geq \frac{1}{2}(\|v\|^2 + \|d - d_\infty\|^2_{H^1}). \tag{3.3.27}$$

Furthermore, we infer from (3.3.27) that for certain constants $C_3, C_4 > 0$,

$$\frac{d}{dt}y(t) + C_3 y(t) \leq C_4\|d - d_\infty\|^2 \leq C(1+t)^{-\frac{2\theta}{1-2\theta}}. \tag{3.3.28}$$

By Gronwall's inequality, we have (cf. [17,20])

$$y(t) \leq C(1+t)^{-\frac{2\theta}{1-2\theta}}, \quad \forall\, t \geq 0, \tag{3.3.29}$$

which together with (3.3.27) and (3.3.4) implies that

$$\|v(t)\| + \|d(t) - d_\infty\|_{H^1} \leq C(1+t)^{-\frac{\theta}{1-2\theta}}, \quad \forall\, t \geq 0. \tag{3.3.30}$$

To sum up, we have proved the main theorem of this chapter:

Theorem 3.3.1. *Under the assumption of Theorem 2.3.2, the global solution (v, d) satisfies*

$$\|v(t)\|_V + \|d(t) - d_\infty\|_{\mathbf{H}^2} \leq C(1+t)^{-\frac{\theta}{(1-2\theta)}}, \quad \forall\, t \geq 0, \tag{3.3.31}$$

where d_∞ is a solution to (3.2.5)*, C is a constant depending on $v_0, d_0, f, Q, \mu_i's, \lambda_i's, d_\infty$ and $\theta \in (0, \frac{1}{2})$ depends on d_∞.*

Chapter 4

Parodi's Relation and Stability in the General Ericksen-Leslie System

To derive the liquid crystal system the influence of spatial symmetry on the phenomenological equations was considered. Later on the influence of the property of time reversal invariance of the equations of motion of the individual particles was also taken into account. This property of time reversal invariance expresses the fact that the mechanical equations of motion (classical as well as quantum mechanical) of the particles are symmetric with respect to time. It implies that the particles retrace their former paths if all velocities are reversed.

It is due to Onsager in [3] and [4] that a macroscopic theorem was concluded from this microscopic property, where the connection between the theory of irreversible processes and the spontaneous fluctuations of thermodynamic variables of equilibrium systems was discussed. The fluctuation theory was brought in to prove a theorem for irreversible processes, the reciprocal relations: the symmetry of the matrix of coefficients of the set of linear equations relating thermodynamic "forces" and "fluxes". The connection was made by postulating that the decay of a system from a given non-equilibrium state produced by a spontaneous fluctuation obeys, on the average, the (empirical) law for the decay from the same state back to equilibrium. Consequently, when Onsager's reciprocal relations were applied to the liquid crystal model, one can get Parodi's relation (2.1.16) in a direct way (c.f. [8]). As a matter of fact, if (2.1.16) is accepted, then the Ericksen-Leslie system is also referred to as the Ericksen-Leslie-Parodi system.

In this chapter we demonstrate that Parodi's relation is a stability condition for the nematic liquid crystal flow. In section 4.1 we provide the higher-order energy law under Parodi's relation, which implies the local existence of strong solutions and the global existence provided that the initial data is near equilibrium. In section 4.2, we obtain some information on the stability of local energy minimizers. From the mathematical point of view, the Parodi's relation actually serves as a stability condition.

4.1 Higher-order energy inequality under Parodi's relation

The results obtained in Chapter 2 indicate that in both **Case I** (with Parodi's relation) and **Case II** (without Parodi's relation), the viscosity μ_4 plays a crucial role. Recall Navier–Stokes equation in $3D$ (with periodic boundary condition and $v_0 \in H$), we can easily derive that

$$\frac{d}{dt}\|\nabla v\|^2 + \left(\frac{1}{2}\mu_4 - \frac{C}{\mu_4}\|\nabla v\|^2\right)\|\Delta v\|^2 \le 0,$$

which implies that the large viscosity assumption is equivalent to small initial data in $\mathbf{H}^1$-norm. However, this is not the case for our complicated coupling system (2.1.10)–(2.1.14) (see e.g. (2.3.44)). Actually, we do not have the large viscosity/small initial data alternative relation even for the simplified liquid crystal systems [9, 38].

In this section, we shall see that the Parodi's relation (2.1.16) does play an important role in the well-posedness and stability of the system (2.1.10)–(2.1.14), if we do not suppose additional requirement on the viscosity μ_4. In particular, under assumptions in **Case I**, we are able to prove a suitable higher-order energy inequality which implies the local existence of strong solutions and the global existence provided that the initial data v_0 is near zero and d_0 is close to a local energy minimizer d^* of the elastic energy $E(d)$.

First, we derive the higher-order energy inequality under assumptions in **Case I**:

Lemma 4.1.1. *Let $n = 2, 3$. Suppose that the conditions in **Case I** are satisfied. Then the following higher-order energy inequality holds:*

$$\frac{d}{dt}\mathcal{A}(t) \le -\frac{\mu_1}{2}\|d^T \cdot \nabla A \cdot d\|^2 - \frac{\mu_4}{8}\|\Delta v\|^2 + \frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C_*(\mathcal{A}^6(t) + \mathcal{A}(t)), \qquad (4.1.1)$$

where C_* is a constant that only depends on $\mu's$, $\lambda's$, $\|v_0\|$ and $\|d_0\|_{\mathbf{H}^1}$.

Proof. We only give the proof in $3D$ case and the proof for $2D$ is similar. The basic energy law (2.2.7) implies that

$$\|v(t)\| + \|d(t)\|_{\mathbf{H}^1} \le C, \quad \forall\, t \ge 0. \qquad (4.1.2)$$

We re-estimate the right-hand side of differential equality (2.3.22) which was derived under conditions in **Case I**. We note that estimates (2.3.24)–(2.3.25) are still valid.

$$\begin{aligned}
I_1 &= -2\mu_1 \int_Q (A_{kp}\nabla_l d_p d_k, d_i d_j \nabla_l A_{ij})dx \\
&\le \frac{\mu_1}{4}\|d^T \cdot \nabla A \cdot d\|^2 + C\|d\|^2_{\mathbf{L}^\infty}\|\nabla v\|^2_{\mathbf{L}^3}\|\nabla d\|^2_{\mathbf{L}^6}. \\
&\le \frac{\mu_1}{4}\|d^T \cdot \nabla A \cdot d\|^2 + C(\|\Delta d - f\|^3 + 1)\|\nabla v\|\|\Delta v\| \\
&\le \frac{\mu_1}{4}\|d^T \cdot \nabla A \cdot d\|^2 + \frac{\mu_4}{32}\|\Delta v\|^2 + C(\|\nabla v\|^2\|\Delta d - f\|^6 + \|\nabla v\|^2) \\
&\le \frac{\mu_1}{4}\|d^T \cdot \nabla A \cdot d\|^2 + \frac{\mu_4}{32}\|\Delta v\|^2 + C\mathcal{A}^4 + C\mathcal{A}. \qquad (4.1.3)
\end{aligned}$$

Corresponding to (2.3.28),

$$\begin{aligned}
I_2 &= -2\mu_1 \int_Q (A_{kp} d_p d_k d_i \nabla_l d_j, \nabla_l A_{ij}) dx \\
&\leq \frac{\mu_1}{4}\|d^T \cdot \nabla A \cdot d\|^2 + C\|d\|^2_{\mathbf{L}^\infty}\|\nabla v\|^2_{\mathbf{L}^3}\|\nabla d\|^2_{L^6} + C\|d\|^3_{\mathbf{L}^\infty}\|\nabla v\|^2_{\mathbf{L}^4}\|\Delta d\| \\
&\leq \frac{\mu_1}{4}\|d^T \cdot \nabla A \cdot d\|^2 + \frac{\mu_4}{32}\|\Delta v\|^2 + C\mathcal{A}^4 + C\mathcal{A} + C(\|\Delta d - f\|^{\frac{5}{2}} + 1)\|\nabla v\|^{\frac{1}{2}}\|\Delta v\|^{\frac{3}{2}} \\
&\leq \frac{\mu_1}{4}\|d^T \cdot \nabla A \cdot d\|^2 + \frac{\mu_4}{16}\|\Delta v\|^2 + C\mathcal{A}^6 + C\mathcal{A}. \qquad (4.1.4)
\end{aligned}$$

Concerning (2.3.31), we have

$$\begin{aligned}
I_3 + I_4 &\leq C\|\nabla v\|^2_{\mathbf{L}^4}\|\nabla d\|^2_{\mathbf{L}^4} + C\|d\|_{\mathbf{L}^\infty}\|\nabla v\|^2_{\mathbf{L}^4}\|\Delta d\| \leq C\|\nabla v\|^{\frac{1}{2}}\|\Delta v\|^{\frac{3}{2}}(\|\Delta d - f\|^{\frac{3}{2}} + 1) \\
&\leq \frac{\mu_4}{32}\|\Delta v\|^2 + C\mathcal{A}^4 + C\mathcal{A}. \qquad (4.1.5)
\end{aligned}$$

Next, by (2.3.32) and (2.3.33) respectively, we obtain

$$\begin{aligned}
I_5 &= -\int_Q (\nabla(\Delta d - f), \Omega \nabla d) dx \leq C\|\nabla(\Delta d - f)\|\|\nabla v\|^{\frac{1}{2}}\|\Delta v\|^{\frac{1}{2}}(\|\Delta d - f\| + 1) \\
&\leq \frac{\mu_4}{32}\|\Delta v\|^2 - \frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\|\nabla v\|^2(\|\Delta d - f\|^4 + 1) \\
&\leq \frac{\mu_4}{32}\|\Delta v\|^2 - \frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^3 + C\mathcal{A}, \qquad (4.1.6) \\
I_6 &= \int_Q (\Delta d - f, \nabla \Omega \nabla d) dx \leq C\|\Delta v\|\|\Delta d - f\|((\|(\nabla(\Delta d - f)\|^{\frac{3}{4}} + 1) \\
&\leq \frac{\mu_4}{32}\|\Delta v\|^2 - \frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^4 + C\mathcal{A}. \qquad (4.1.7)
\end{aligned}$$

Corresponding to (2.3.34), a similar argument to (4.1.5)–(4.1.7) yields

$$\begin{aligned}
-\frac{\lambda_2}{\lambda_1}\int_Q (\Delta d - f, \nabla A \cdot \nabla d) dx &\leq \frac{\mu_4}{32}\|\Delta v\|^2 - \frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^4 + C\mathcal{A}, \\
\frac{\lambda_2}{\lambda_1}\int_Q \big(\nabla(\Delta d - f), A \cdot \nabla d\big) dx &\leq \frac{\mu_4}{32}\|\Delta v\|^2 - \frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^3 + C\mathcal{A},
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{(\lambda_2)^2}{\lambda_1}\int_Q (Ad, \nabla A \cdot \nabla d) dx + \frac{(\lambda_2)^2}{\lambda_1}\int_Q (\nabla A \cdot d, A \cdot \nabla d) dx \\
=\ & \frac{(\lambda_2)^2}{\lambda_1}\int_Q (A \cdot \nabla d, A \cdot \nabla d) dx + \frac{(\lambda_2)^2}{\lambda_1}\int_Q (Ad, A\Delta d) dx \\
\leq\ & \frac{\mu_4}{32}\|\Delta v\|^2 + C\mathcal{A}^4 + C\mathcal{A},
\end{aligned}$$

which implies that

$$I_7 + I_8 + I_9 \leq \frac{3\mu_4}{32}\|\Delta v\|^2 - \frac{1}{4\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^4 + C\mathcal{A}.$$

The remaining terms can be estimated in a straightforward way.

$$
\begin{aligned}
I_{10} &= -\frac{(\lambda_2)^2}{\lambda_1}\int_Q |\nabla A\, d|^2 dx \le (\mu_5+\mu_6)\int_Q |\nabla A\, d|^2 dx,\\
I_{11} &\le |(\Delta v, v\cdot\nabla v)| \le C\|\Delta v\|^{\frac{7}{4}}\|\nabla v\| \le \frac{\mu_4}{32}\|\Delta v\|^2 + C\|\nabla v\|^8,\\
I_{12} &\le \frac{1}{\lambda_1}\int_Q f'(d)|\Delta d - f|^2 dx \le C(\|d\|^2_{\mathbf{L}^6}+1)\|\Delta d - f\|^2_{\mathbf{L}^3}\\
&\le C\Big(\|\Delta d - f\|\|\nabla(\Delta d - f)\| + \|\Delta d - f\|^2\Big)\\
&\le -\frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\mathcal{A},\\
I_{13} &= -\int_Q f'(d)\Big(\Delta d - f, \Omega\, d - \frac{\lambda_2}{\lambda_1}Ad\Big)dx \le C\|f'(d)d\|\|\Delta d - f\|_{\mathbf{L}^6}\|\nabla v\|_{L^3}\\
&\le C\|\nabla v\|^{\frac{1}{2}}\|\Delta v\|^{\frac{1}{2}}\Big(\|\nabla(\Delta d - f)\| + \|\Delta d - f\|\Big)\\
&\le \frac{\mu_4}{32}\|\Delta v\|^2 - \frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^3 + C\mathcal{A},
\end{aligned}
$$

Finally, the estimate of I_{14} is similar to (4.1.6):

$$
I_{14} \le -\frac{1}{8\lambda_1}\|\nabla(\Delta d - f)\|^2 + C\mathcal{A}^3 + C\mathcal{A}.
$$

Collecting all the estimates above, we can conclude (4.1.1). □

As a direct consequence, we have the local existence result:

Theorem 4.1.1. *Let $n = 2, 3$. We suppose that the conditions in **Case I** are satisfied. For any $(v_0, d_0) \in V \times \mathbf{H}^2(Q)$, there exists $T > 0$ such that problem (2.1.10)–(2.1.14) admits a unique local solution satisfying $v \in L^\infty(0, T; V) \cap L^2(0, T; \mathbf{H}^2)$, $d \in L^\infty(0, T; \mathbf{H}^2) \cap L^2(0, T; \mathbf{H}^3)$.*

Remark 4.1.1. *Unfortunately, we are not able to prove an corresponding local well-posedness result under the conditions in **Case II**. This is because when the Parodi's relation does not hold, the higher-order energy inequality (4.1.1) is not available. One direct difficulty is that we lose control of some higher-order nonlinearities which will vanish due to cancelation under Parodi's relation (see, e.g., (2.3.49)).*

4.2 Close to local minimizers: well-posedness and stability

In this section, we obtain some information on the stability of local energy minimizers of $E(d)$. From the mathematical point of view, Parodi's relation actually serves as a stability condition for the Ericksen–Leslie system.

First we can deduce the following property based on the higher-order energy inequality.

Proposition 4.2.1. *Suppose the assumptions in **Case I** are satisfied. For any $(v_0, d_0) \in V \times \mathbf{H}^2_p(Q)$, if*

$$
\|\nabla v\|^2(0) + \|\Delta d - f(d)\|^2(0) \le R, \tag{4.2.1}
$$

where $R > 0$ being a constant, there exists a positive constant ε_0 depending on $\mu's$, $\lambda's$, $\|v_0\|$, $\|d_0\|_{\mathbf{H}^1}$, f, Q and R, such that the following property holds: for the (unique) local strong solution (v, d) of system (2.1.10)–(2.1.14) which exists on $[0, T^*]$, if $\mathcal{E}(t) \geq \mathcal{E}(0) - \varepsilon_0$, for any $t \in [0, T^*]$, then the local strong solution (v, d) can be extended beyond T^*.

Proof. Inspired by Lemma 4.1.1, we consider the following initial value problem of an ODE:

$$\frac{d}{dt}Y(t) = C_*(Y(t)^6 + Y(t)), \quad Y(0) = R \geq \mathcal{A}(0). \tag{4.2.2}$$

We denote by $I = [0, T_{max})$ the maximal existence interval of $Y(t)$ such that $\lim_{t \to T_{max}^-} Y(t) = +\infty$. On the other hand, it is easy to see from the comparison principle that for any $t \in I$, $0 \leq \mathcal{A}(t) \leq Y(t)$. Consequently, $\mathcal{A}(t)$ exists on I. Moreover, T_{max} is determined by $Y(0) = R$ and C_* such that $T_{max} = T_{max}(Y(0), C_*)$ is increasing when $Y(0) \geq 0$ is decreasing. We can take $t_0 = \frac{3}{4}T_{max}(R, C_*) > 0$. Then it follows that

$$0 \leq \mathcal{A}(t) \leq Y(t) \leq K, \quad \forall t \in [0, t_0], \tag{4.2.3}$$

where K is a constant that only depends on R, C_*, t_0. This fact together our previous Galerkin approximate scheme will imply the local existence of a unique strong solution of system (2.1.10)–(2.1.14) at least on $[0, t_0]$. (This also provides a proof of Theorem 4.1.1.)

The above argument suggests that $T^* \geq t_0$. Now if $\mathcal{E}(t) \geq \mathcal{E}(0) - \varepsilon_0$ for all $t \in [0, T^*]$, we infer from Lemma 2.2.1 that

$$\int_0^{T^*} \int_Q \left(\frac{\mu_4}{2} |\nabla v(t)|^2 - \frac{1}{\lambda_1} |\Delta d(t) - f(d(t))|^2 \right) dxdt \leq \varepsilon_0, \quad \forall\, t \geq 0.$$

Hence, there exists a $t_* \in [T^* - \frac{t_0}{3}, T^*]$ such that

$$\|\nabla v(t_*)\|^2 + \|\Delta d(t_*) - f(d(t_*))\|^2 \leq \max\left\{ \frac{2}{\mu_4}, -\lambda_1 \right\} \frac{3\varepsilon_0}{t_0}.$$

Choosing $\varepsilon_0 > 0$ such that

$$\max\left\{ \frac{2}{\mu_4}, -\lambda_1 \right\} \frac{3\varepsilon_0}{t_0} = R, \tag{4.2.4}$$

we have $\mathcal{A}(t_*) \leq R$. Taking t_* as the initial time in ODE (4.2.2), we infer from the above argument that $Y(t)$ (and thus $\mathcal{A}(t)$) is uniformly bounded at least on $[0, t_* + t_0] \supset [0, T^* + \frac{2}{3}t_0]$. Thus, we can extend the local unique strong solution (v, d) from $[0, T^*]$ to $[0, T^* + \frac{2}{3}t_0]$. □

Remark 4.2.1. *Proposition 4.2.1 implies that, for the (local) strong solution (v, d) of (2.1.10)–(2.1.14), if $\mathcal{E}(t)$ does not drop too fast on its existence interval $[0, T]$, then it can be extended beyond T. We note that stronger results have been obtained in [9, 10] for various simplified Ericksen–Leslie systems. In those cases, the existence of weak solutions can be proved and the associated total energy $\mathcal{E}(t)$ is well-defined on $[0, +\infty)$. Then one can show the alternative relation: either there exists a $T < +\infty$ such that $\mathcal{E}(T) < \mathcal{E}(0) - \varepsilon_0$ or the system admits a (unique) global strong solution.*

Concerning the elastic energy functional

$$E(d) = \frac{1}{2}\|\nabla d\|^2 + \int_Q \mathcal{F}(d)dx \geq 0, \tag{4.2.5}$$

it is easy to see that the above hypothesis on the changing rate of $\mathcal{E}(t)$ can be fulfilled if the initial velocity v_0 is close to zero and the initial molecule director d_0 is sufficiently close to an *absolute* minimizer of $E(d)$. We refer to [9,10,47] for special simplified cases of the Ericksen–Leslie system. Of course, the same result holds for our current general case if we make the same assumption.

The condition being closed to absolute minimizer can be greatly improved in the following sense. Under the Parodi's relation, we can show that if the initial velocity v_0 is close to zero and the initial molecule director d_0 is sufficiently close to a *local* minimizer of $E(d)$, then the total energy $\mathcal{E}$ will never drop too much. Actually, we shall see that the global solution will stay close to the minimizer for all time (Lyaponov stability) and $\mathcal{E}(t)$ will converge to the same energy level of the local minimizer. This result applies to all the simplified systems considered in [9, 10, 38, 47, 48].

Definition 4.2.1. *$d^* \in \mathbf{H}^1_p(Q)$ is called a local minimizer of $E(d)$, if there exists $\sigma > 0$, such that for any $d \in \mathbf{H}^1_p(Q)$ satisfying $\|d - d^*\|_{\mathbf{H}^1} \leq \sigma$, it holds $E(d) \geq E(d^*)$.*

Remark 4.2.2. *Since any minimizer of $E(d)$ is also a critical point of $E(d)$, it satisfies the Euler–Lagrange equation*

$$-\Delta d + f(d) = 0, \quad x \in \Omega, \quad d(x) = d(x + e_i), \quad x \in \partial Q. \tag{4.2.6}$$

From the elliptic regularity theorem and bootstrap argument, we easily see that if the solution $d \in \mathbf{H}^1_p(Q)$, then d is smooth.

Now we state the main result of this Chapter:

Theorem 4.2.1. *We suppose $n = 2, 3$ and the conditions in **Case I** are satisfied. Let $d^* \in \mathbf{H}^2_p(Q)$ be a local minimizer of $E(d)$. There exist positive constants σ_1, σ_2, which may depend on $\lambda_i's$, $\mu_i's$, Q, σ and d^*, such that for any initial data $(v_0, d_0) \in V \times \mathbf{H}^2_p(Q)$ satisfying $\|v_0\|_{\mathbf{H}^1} \leq 1$, $\|d_0 - d^*\|_{\mathbf{H}^2} \leq 1$ and $\|v_0\| \leq \sigma_1$, $\|d_0 - d^*\|_{\mathbf{H}^1} \leq \sigma_2$, there holds*

(i) problem (2.1.10)–(2.1.14) admits a unique global strong solution (v, d),

(ii) (v, d) enjoys the same long-time behavior as in Theorem 3.3.1. Moreover,

$$\lim_{t \to +\infty} \mathcal{E}(t) = E(d_\infty) = E(d^*). \tag{4.2.7}$$

Proof. Without loss of generality, we assume that the constant σ in Definition 4.2.1 satisfies $\sigma \leq 1$. Throughout the proof, C_i, $i = 1, 2, \cdots$ denote generic constants depending only on $\mu_i's$, $\lambda_i's$, σ and d^*. By our assumptions, we easily see that

$$\|v(t)\| + \|d(t)\|_{\mathbf{H}^1} \leq C_1, \quad \forall\, t \geq 0, \quad \text{and} \quad \mathcal{A}(0) = \|\nabla v_0\|^2 + \|\Delta d_0 - f(d_0)\|^2 \leq C_2. \tag{4.2.8}$$

Now we recall Proposition 4.2.1 and its proof. For our current case, we take $R = C_2$ and the constant C_* in (4.2.2) can be fixed by C_1 and $\mu's, \lambda's$. Then we set $t_0 = \frac{3}{4}T_{max}(C_2, C_*)$ and take

$T^* = t_0$. We see from (4.2.3) that $\mathcal{A}(t)$ is uniformly bounded on $[0, t_0]$, which implies

$$\|v(t)\|_V + \|d(t)\|_{\mathbf{H}^2} \le C_3, \quad \forall\, t \in [0, t_0]. \tag{4.2.9}$$

Finally, the critical constant ε_0 can be determined by (4.2.4).

Next, we extend the local strong solution not only beyond $T^* = t_0$ but to infinity by using the Łojasiewicz–Simon approach. Since the minimizer d^* is a critical point of $E(d)$, in the Łojasiewicz–Simon inequality (cf. Lemma 3.2.2), we take $\psi = d^*$, then the constants $\beta > 0, \theta \in (0, \frac{1}{2})$ are determined by d^* and (3.2.7) holds.

The proof consists of several steps.

Step 1. In order to apply Proposition 4.2.1 with $T^* = t_0$, it suffices to show that

$$\mathcal{E}(t) - \mathcal{E}(0) \ge -\varepsilon_0, \quad \forall t \in [0, t_0]. \tag{4.2.10}$$

Using (4.2.8) and Sobolev embedding theorem, we have $|E(d_0) - E(d^*)| \le C_4 \|d_0 - d^*\|_{\mathbf{H}^1}$, which implies that

$$\begin{aligned}
\mathcal{E}(t) - \mathcal{E}(0) &= \frac{1}{2}\|v(t)\|^2 - \frac{1}{2}\|v_0\|^2 + E(d(t)) - E(d_0) \\
&\ge -\frac{1}{2}\|v_0\|^2 + E(d(t)) - E(d^*) + E(d^*) - E(d_0) \\
&\ge -\frac{1}{2}\|v_0\|^2 - C_4\|d_0 - d^*\|_{H^1} + E(d(t)) - E(d^*).
\end{aligned} \tag{4.2.11}$$

If we take

$$\sigma_1 \le \min\left\{\varepsilon_0^{\frac{1}{2}}, 1\right\}, \quad \sigma_2 \le \min\left\{\frac{\varepsilon_0}{2C_4}, 1\right\}, \tag{4.2.12}$$

then by (4.2.11), (4.2.10) will be satisfied if we can prove

$$E(d(t)) - E(d^*) \ge 0, \quad \forall\, t \in [0, t_0]. \tag{4.2.13}$$

By the definition of d^*, it reduces to prove that

$$\|d(t) - d^*\|_{\mathbf{H}^1} \le \sigma, \quad \forall\, t \in [0, t_0]. \tag{4.2.14}$$

Actually, we will prove a slightly stronger conclusion such that

$$\|d(t) - d^*\|_{\mathbf{H}^1} < \omega := \frac{1}{2}\min\{\sigma, \beta\}, \quad \forall\, t \in [0, t_0]. \tag{4.2.15}$$

Let $\sigma_2 \le \frac{1}{4}\omega$. If (4.2.15) is not true, then by the continuity of d that $d \in C([0, t_0]; \mathbf{H}^1)$, there exists a minimal time $T_0 \in (0, t_0]$, such that $\|d(T_0) - d^*\|_{\mathbf{H}^1} = \omega$. We observe that $\mathcal{E}(t) = \frac{1}{2}\|v(t)\|^2 + E(d(t)) \ge E(d^*)$ for any $0 \le t \le T_0$. First, we consider the trivial case that if for some $T \le T_0$, $\mathcal{E}(T) = E(d^*)$, then the definition of local minimizer indicates that for $t \ge T$, $\mathcal{E}$ cannot drop and will remain $E(d^*)$. Thus, we infer from the basic energy law (2.2.7) that the evolution will be stationary and the conclusion easily follows. So in the following, we just assume

$\mathcal{E}(t) > E(d^*)$ for any $0 \le t \le T_0$. By Lemma 3.2.2 with $\psi = d^*$, we can compute

$$\begin{aligned} -\frac{d}{dt}[\mathcal{E}(t) - E(d^*)]^\theta &= -\theta[\mathcal{E}(t) - E(d^*)]^{\theta-1}\frac{d}{dt}\mathcal{E}(t) \ge \frac{\theta\left(\frac{\mu_4}{2}\|\nabla v\|^2 - \frac{1}{\lambda_1}\|\Delta d - f\|^2\right)}{\|\Delta d - f\|} \\ &\ge C_5(\|\nabla v\| + \|\Delta d - f\|). \end{aligned}$$

On the other hand, it follows from (2.1.12) and (4.2.9) that

$$\begin{aligned} \|d_t\| &\le \|v \cdot \nabla d\| + \|\Omega d\| + \left|\frac{\lambda_2}{\lambda_1}\right| \|Ad\| - \frac{1}{\lambda_1}\|\Delta d - f\| \\ &\le C_6(\|v\|_{\mathbf{L}^6}\|\nabla d\|_{\mathbf{L}^3} + \|\nabla v\|\|d\|_{\mathbf{L}^\infty} + \|\Delta d - f\|) \\ &\le C_7(\|\nabla v\| + \|\Delta d - f\|), \quad \forall t \in [0, t_0]. \end{aligned} \tag{4.2.16}$$

As a consequence,

$$\begin{aligned} \|d(T_0) - d_0\|_{\mathbf{H}^1} &\le C_8\|d(T_0) - d_0\|^{\frac{1}{2}}\|d(T_0) - d_0\|_{\mathbf{H}^2}^{\frac{1}{2}} \\ &\le C_9\left(\int_0^{T_0}\|d_t\|dt\right)^{\frac{1}{2}} \le C_{10}[\mathcal{E}(0) - E(d^*)]^{\frac{\theta}{2}} \\ &\le C_{10}\left(\frac{1}{2}\|v_0\|^2 + C_4\|d_0 - d^*\|_{\mathbf{H}^1}\right)^{\frac{\theta}{2}} \\ &\le C_{11}(\|v_0\|^\theta + \|d_0 - d^*\|_{\mathbf{H}^1}^{\frac{\theta}{2}}). \end{aligned} \tag{4.2.17}$$

Finally, choosing (also taking previous assumptions into account)

$$\sigma_1 = \min\left\{\varepsilon_0^{\frac{1}{2}}, \left(\frac{\omega}{4C_{11}}\right)^{\frac{1}{\theta}}, 1\right\}, \quad \sigma_2 = \min\left\{\frac{\varepsilon_0}{2C_4}, \left(\frac{\omega}{4C_{11}}\right)^{\frac{2}{\theta}}, \frac{\omega}{4}, 1\right\}, \tag{4.2.18}$$

we can see that

$$\|d(T_0) - d^*\|_{\mathbf{H}^1} \le \|d(T_0) - d_0\|_{\mathbf{H}^1} + \|d_0 - d^*\|_{\mathbf{H}^1} \le \frac{\omega}{4} + \frac{\omega}{4} + \frac{\omega}{4} < \omega,$$

which leads to a contradiction with the definition of T_0. Thus, (4.2.15) is true and so is (4.2.13), which implies that (4.2.10) are satisfied.

As in the proof of Proposition 4.2.1, there exists $t_* \in [\frac{2t_0}{3}, t_0]$, such that $\mathcal{A}(t^*) \le R$. Then we can conclude that $\mathcal{A}(t)$ is uniformly bounded on $[0, t^* + t_0] \supset \left[0, \frac{5t_0}{3}\right]$ (with *the same bound* as on $[0, t_0]$). This implies the important fact that the bound only depends on R, C_*, t_0 but not on the length of existence interval.

Step 2. We take $T^* = \frac{5}{3}t_0$. By same argument as in Step 1, we can show that $\mathcal{E}(t) - \mathcal{E}(0) \ge -\varepsilon_0$, for $t \in [0, T^*]$. Again, we can show that $\mathcal{A}(t)$ is uniformly bounded on $\left[0, T^* + \frac{2}{3}t_0\right]$ (with *the same bound* as on $[0, t_0]$).

By iteration, we can see that the local strong solution can be extended by a fixed length $\frac{2}{3}t_0$ at each step and $\mathcal{A}(t)$ is uniformly bounded by a constant only depending on R, C_*, t_0.

As a consequence, we can show that (v,d) is indeed a global strong solution. Moreover,

$$\|v(t)\|_{\mathbf{H}^1} + \|d(t)\|_{\mathbf{H}^2} \le K, \quad \forall\, t \ge 0, \tag{4.2.19}$$

where K depends on C_1, R, C_*, t_0. The conclusion (i) is proved.

Based on uniform estimate (4.2.19), a similar argument to Theorem 3.3.1 yields that there exists a d_∞ satisfying (3.2.5), such that

$$\lim_{t\to+\infty}(\|v(t)\|_V + \|d(t) - d_\infty\|_{\mathbf{H}^2}) = 0$$

with the convergence rate (3.3.31) (in (3.3.31), the Łojasiewicz exponent θ is determined by d_∞, which is different from the one we have used in Step 1).

By repeating the argument in Step 1, we are able to show that $\|d(t) - d^*\|_{\mathbf{H}^1} \le \omega$ for all $t \ge 0$. Then for t sufficiently large, we have

$$\|d_\infty - d^*\|_{\mathbf{H}^1} \le \|d_\infty - d(t)\|_{\mathbf{H}^1} + \|d(t) - d^*\|_{\mathbf{H}^1} \le \frac{3}{2}\omega < \min\{\beta, \sigma\}. \tag{4.2.20}$$

Applying Lemma 3.2.2 with $d = d_\infty$ and $\psi = d^*$, we obtain

$$|E(d_\infty) - E(d^*)|^{1-\theta} \le \| - \Delta d^* + f(d^*)\| = 0,$$

which together with (4.2.20) implies that d_∞ is also a local minimizer of $E(d)$. $\square$

Remark 4.2.3. *We note that in the assumptions $\|v_0\|_{\mathbf{H}^1} \le 1$, $\|d_0 - d^*\|_{\mathbf{H}^2} \le 1$, the bound 1 is not crucial and it can be replaced by any fixed positive constant M. Of course, the constants in the proof of Theorem 4.2.1 will depend also on M.*

Remark 4.2.4. *Since ω in the proof of of Theorem 4.2.1 can be arbitrary small positive constant satisfying $\omega \le \frac{1}{2}\min\{\sigma, \beta\}$, by our choice of σ_1, σ_2, we actually have shown that the local minimizer d^* is Lyapunov stable. Moreover, if d^* is an isolated local minimizer, then $d_\infty = d^*$ and d^* is asymptotic stable.*

Chapter 5

Axisymmetric Solutions to a diffuse-interface Model

The hydrodynamic of mixture of different materials plays an important role in many scientific and engineering applications. Among them, the interfacial dynamics is one of the fundamental issues in hydrodynamics and rheology [53–55]. Conventionally, the model for the mixture consists of separate hydrodynamic system of each component, together with the free interface that separates them. In another point of view, the mixture can be treated as a special type of non-newtonian fluids, whose rheology property reflects the competition between the kinetic energy and the "elastic" mixing energy.

In classical approaches, the interface is usually considered to be a free surface that evolves in time with the fluid. The hydrodynamic system describing the mixture of two Newtonian fluids with a free interface will be the usual Navier-Stokes equations in each of the fluid domains (possibly with different density and viscosity) together with the kinematic and force balance (traction free) boundary conditions on the interface. From the statistical (phase field approach) point of view, the interface represents a continuous, but steep change of the properties (density, viscosity, etc.) of two fluids. Within this "thin" transition region, the fluid is mixed and has to store certain amount of "mixing energy". In recent years, much work has been done in various fluid environments using the phase field approach.

The diffuse-interface model studied in this chapter can be viewed as a physically motivated level-set method. Instead of choosing an artificial smoothing function for the interface, the diffuse-interface model describes the interface by a mixing energy, whose idea can be traced back to [56]. The structure of the interface is determined by molecular forces: the tendencies for mixing and demixing are balanced through the non local mixing energy. When the capillary width approaches zero, the diffuse-interface model becomes identical to a sharp-interface level-set formulation.

This chapter is organized as follows: In section 1 we provide the basic problem settings and the statement of main results. In section 2 a system of $1D$ equations is derived by separation of variables. In section 3, some useful lemmas and estimates are prepared, in order to prove the regularity of perturbation terms later. In section 4, the proof of global regularity of the solutions to the $3D$ system in both large viscosity and small initial data cases is discussed.

5.1 Problem settings and main results

In this chapter, we shall study the following coupled Navier-Stokes/Allen-Cahn equations in $\mathbb{R}^3 \times (0, +\infty)$:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} - \lambda \nabla \cdot (\nabla \phi \otimes \nabla \phi), \tag{5.1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{5.1.2}$$

$$\phi_t + (\mathbf{u} \cdot \nabla)\phi = \gamma(\Delta \phi - f(\phi)). \tag{5.1.3}$$

We assume $\mathbf{u}$ and $\nabla\phi$ decay sufficiently fast in the infinity. Here $\mathbf{u}$ is a vector function, ϕ and p are scalar functions, and $f(\phi) = \frac{1}{\eta^2}(\phi^3 - \phi)$, ν, λ, γ, η are positive constants. In addition, $\nabla\phi \otimes \nabla\phi$ is a tensor product–e.g., $(\nabla\phi \otimes \nabla\phi)_{ij} = (\nabla\phi)_i(\nabla\phi)_j$, $1 \le i,\ j \le 3$.

Multiplying (5.1.1) by $\mathbf{u}$, (5.1.3) by $\lambda(-\Delta\phi + f(\phi))$, then adding them up, and using integration by parts combined with (5.1.2), we get the basic energy law

$$\frac{1}{2}\frac{d}{dt}\left(\|\mathbf{u}\|^2 + \lambda\|\nabla\phi\|^2 + \frac{\lambda}{2\eta^2}\|(\phi^2-1)\|^2\right) = -(\nu\|\nabla\mathbf{u}\|^2 + \lambda\gamma\|\Delta\phi - f(\phi)\|^2), \tag{5.1.4}$$

where $\|\cdot\|$ denotes the L^2 norm in $3D$ space $(\int_{\mathbb{R}^3} |\cdot|^2 dx)^{\frac{1}{2}}$.

The system (5.1.1)-(5.1.3) can be viewed as a phase field model describing the motion of a mixture of two incompressible viscous fluids (see [25]). The fluids are separated by a thin interface of width η. The velocity vector of the mixture is represented by $\mathbf{u}$, the pressure by p, the fluid kinetic viscosity by ν, and the phase of the fluid components by ϕ. The phase ϕ takes the value 1 in one bulk phase and -1 in the other. In the interfacial region, it undergoes rapid but smooth variation. It is assumed that the interface possesses a free energy $E_\eta = \int_\Omega \frac{1}{4\eta^2}(\phi^2-1)^2 + \frac{1}{2}|\nabla\phi|^2 dx$ caused by the mixing of fluids. Motion of the interface is caused by energy dissipation, which is given by $\phi_t = -\delta E_\eta/\delta\phi$. The term $\nabla\phi\otimes\nabla\phi$ in the momentum equation is the induced elastic stress due to the mixing of fluids. Finally, λ corresponds to the surface tension and γ the elastic relaxation time. We want to point out that this model is based on an energetic variational formulation and interested reader can refer to [26] for more details.

In this chapter, we study only axisymmetric solutions to (5.1.1)-(5.1.3). There have been some interesting developments in the study of axisymmetric solutions to the $3D$ Navier-Stokes equations, see for example [57], [58], [59], and [60]. The $2D$ Boussinesq equations are closely related to the $3D$ Navier-Stokes equations with swirl (away from the symmetry axis). Recently in [57], [61], the authors have independently proved the existence of solutions to the $2D$ global viscous Boussinesq equations with viscosity entering only in the fluid equation. And most interestingly, in [28], the authors constructed a smooth solution to the Navier-Stokes equations, with initial conditions $\mathbf{u}_0 = \mathbf{u}(r, z, 0)$ satisfying

$$\|\mathbf{u}_0\|_{L^2(\Omega)} \approx \frac{A}{\sqrt{M}}, \qquad \|\nabla\mathbf{u}_0\|_{L^2(\Omega)} \approx A\sqrt{M},$$

where A and M are constants to be determined. Since $\|\mathbf{u}_0\|_{L^2(\Omega)}\|\nabla\mathbf{u}_0\|_{L^2(\Omega)} \approx A^2$, by choosing

A large enough, $\|\mathbf{u}_0\|_{L^2(\Omega)}\|\nabla\mathbf{u}_0\|_{L^2(\Omega)}$ can be made arbitrarily large. Thus, it violates the smallness condition that guarantees the existence of global classical solutions to $3D$ Navier-Stokes equations.

Motivated by these results, it seems natural to study the properties of $3D$ axisymmetric solutions to our system (5.1.1)-(5.1.3). For this system, we construct a family of global classical solutions with finite energy, which can also be regarded as perturbations of near infinite-energy solutions.

In contrast to the asymptotic expansion method in [28], we use the much more straightforward method of separation of variables to derive a system of $1D$ equations. Then, based on the solutions to these equations, using cutoff functions, we construct a family of finite energy solutions to the $3D$ system (5.1.1)-(5.1.3). After that, through a detailed study of weighted norm inequalities, we prove the global regularity of the solutions we construct in the case of large viscosity and small initial data.

5.1.1 Basic settings and 1D special configurations

Let

$$\mathbf{e_r} = (\frac{x}{r}, \frac{y}{r}, 0),\ \ \mathbf{e}_\theta = (-\frac{y}{r}, \frac{x}{r}, 0),\ \ \mathbf{e_z} = (0,0,1) \tag{5.1.5}$$

be three unit vectors, where $r = \sqrt{x^2+y^2}$. We can decompose the velocity field as

$$\mathbf{u} = v^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + v^z(r,z,t)e_z.$$

The vorticity field is expressed similarly as

$$\omega = -(u^\theta)_z(r,z,t)e_r + \omega^\theta(r,z,t)e_\theta + \frac{1}{r}(ru^\theta)_r(r,z,t)e_z,$$

where $\omega^\theta = (v^r)_z - (v^z)_r$. To simplify our notation, we will use u and ω to denote u^θ and ω^θ in the rest of our paper.

Throughout this chapter, ∇^2, Δ, and ∇ will stand for the Laplace, modified Laplace, and gradient operators, respectively in cylindrical coordinates,

$$\begin{aligned}
\nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{\partial_r}{r} + \frac{\partial^2}{\partial z^2}, && (5.1.6)\\
\Delta &= \frac{\partial^2}{\partial r^2} + \frac{3\partial_r}{r} + \frac{\partial^2}{\partial z^2} \equiv \Delta_r + \frac{\partial^2}{\partial z^2}, && (5.1.7)\\
\nabla &= \partial_r \mathbf{e_r} + \partial_z \mathbf{e_z}. && (5.1.8)
\end{aligned}$$

Rewriting (5.1.1)–(5.1.3) into cylindrical coordinates, we obtain the equivalent system

$$\begin{aligned}
u_t + v^r u_r + v^z u_z &= \nu(\nabla^2 - \frac{1}{r^2})u - \frac{1}{r}v^r u, && (5.1.9)\\
\omega_t + v^r\omega_r + v^z\omega_z &= \nu(\nabla^2 - \frac{1}{r^2})\omega + \frac{1}{r}(u^2)_z - \frac{1}{r}v^r\omega \\
&\quad +\lambda(\phi_z\nabla^2\phi_r - \phi_r\nabla^2\phi_z - \frac{1}{r^2}\phi_r\phi_z), && (5.1.10)
\end{aligned}$$

$$-(\nabla^2 - \frac{1}{r^2})\psi = \omega, \tag{5.1.11}$$

$$(v^r)_r + \frac{v^r}{r} + (v^z)_z = 0, \tag{5.1.12}$$

$$\phi_t + v^r\phi_r + v^z\phi_z = \gamma(\nabla^2\phi - \frac{1}{\eta^2}\phi^3 + \frac{1}{\eta^2}\phi). \tag{5.1.13}$$

Here u and ω stand for θ components of velocity $\mathbf{u}$ and vorticity ω respectively, and v^r and v^z are the other two components of $\mathbf{u}$. ψ is the angular stream function, which is related to v^r and v^z as follows:

$$v^r = -\frac{\partial\psi}{\partial z}, \quad v^z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi). \tag{5.1.14}$$

One can alternatively derive the following $1D$ equations :

$$(u_1^*)_t + 2\psi_1^*(u_1^*)_z = \nu(u_1^*)_{zz} + 2(\psi_1^*)_z u_1^*, \tag{5.1.15}$$

$$(\omega_1^*)_t + 2\psi_1^*(\omega_1^*)_z = \nu(\omega_1^*)_{zz} + (u_1^{*2})_z, \tag{5.1.16}$$

$$-(\psi_1^*)_{zz} = \omega_1^*, \tag{5.1.17}$$

$$(\phi_0^*)_t + 2\psi_1^*(\phi_0^*)_z = \gamma(\phi_0^*)_{zz} - \frac{\gamma}{\eta^2}(\phi_0^*)^3 + \frac{\gamma}{\eta^2}\phi_0^*. \tag{5.1.18}$$

Here u_1^*, ω_1^*, ψ_1^* and ϕ_0^* are functions of only z and t.

We will consider solutions with periodic boundary conditions in the z direction with period 1, hence in the rest of this chapter we set

$$\begin{aligned}
\Omega &= [0,\infty)\times[0,1],\ 0\le r\le\infty,\ 0\le z\le 1. \\
\|\cdot\| &= \|\cdot\|_{L^2(\Omega)} = \left(\int_0^1\int_0^\infty |\cdot|^2 r dr dz\right)^{\frac{1}{2}}. \\
\|\cdot\|_{L^4} &= \|\cdot\|_{L^4(\Omega)} = \left(\int_0^1\int_0^\infty |\cdot|^4 r dr dz\right)^{\frac{1}{4}}. \\
L^\infty(0,\infty;\mathbf{X}) &= \left\{x(t)\in\mathbf{X} \text{ for a.e. t} \middle|\ sup_{t\in(0,\infty)}\|x\|_{\mathbf{X}} < \infty\right\}.
\end{aligned}$$

5.1.2 Construction of solutions to the $3D$ system and main results

By the $1D$ equations to (5.1.15)-(5.1.18), we can construct a family of exact solutions to the $3D$ system. If (u_1^*, ω_1^*, ψ_1^*, ϕ_0^*) is a solution to the $1D$ equations, then ($ru_1^*(z,t)$, $r\omega_1^*(z,t)$, $r\psi_1^*(z,t)$, $\phi_0^*(z,t)$) is an exact solution to the $3D$ system. Therefore, it is reasonable to think that the $1D$ equations retain some essential nonlinear features of the $3D$ system. However, ($ru_1^*(z,t)$, $r\omega_1^*(z,t)$, $r\psi_1^*(z,t)$, $\phi_0^*(z,t)$) is an exact solution with infinite energy. Thus, we want to look for global classical solutions to (5.1.9)-(5.1.13) with finite energy. To this end, we study solutions of the following form :

$$\tilde{u}(r,z,t) = r(u_1^*(z,t)\chi(r) + u_1(r,z,t)), \tag{5.1.19}$$

$$\tilde{\omega}(r,z,t) = r(\omega_1^*(z,t)\chi(r) + \omega_1(r,z,t)), \tag{5.1.20}$$

$$\tilde{\psi}(r,z,t) = r(\psi_1^*(z,t)\chi(r) + \psi_1(r,z,t)), \tag{5.1.21}$$

$$\tilde{\phi}(r,z,t) = \phi_0^*(z,t)\chi(r) + \phi_1(r,z,t), \tag{5.1.22}$$

where $\tilde{u}(r,z,t)$, $\tilde{\omega}(r,z,t)$, $\tilde{\psi}(r,z,t)$ are the θ components of velocity, vorticity and stream function, respectively, and $\chi(r)$ is a cut-off function, which ensures the solution has finite energy. Here, u_1, ω_1, ψ_1 and ϕ_1 are considered as perturbation terms.

Using a priori estimates of solutions to the $1D$ equations and delicate energy estimates, we prove that if the viscosity ν is large enough, then there exists a family of global classical functions $u_1(r,z,t)$, $\omega_1(r,z,t)$, $\psi_1(r,z,t)$ and $\phi_1(r,z,t)$ such that $\tilde{u}$, $\tilde{\omega}$, $\tilde{\psi}$ and $\tilde{\phi}$ are global classical solutions to the $3D$ system.

Since our system contains the $3D$ axisymmetric Navier-Stokes equation as a sub-system, one can not expect better results. In fact, we get theorems in both the large viscosity and small initial data cases. Our main theorems are stated as follows.

Theorem 5.1.1. *For the $3D$ system* (5.1.1)-(5.1.3)*, assume $u_1^*(z,0)$, $\psi_1^*(z,0)$, $\omega_1^*(z,0)$, and $\phi_0^*(z,0)$ are smooth functions which are periodic in z with period* 1*. Then there exists a global classical solution in the form of* (5.1.19)-(5.1.22)*, if initial conditions $\tilde{\mathbf{u}}_\mathbf{0} \triangleq \tilde{\mathbf{u}}(r,z,0) \in H^1(\Omega)$, $\tilde{\phi}_0 \triangleq \tilde{\phi}(r,z,0) \in H^2(\Omega)$ and $\nu \geq \nu_0(\gamma,\lambda,\tilde{\mathbf{u}}_\mathbf{0},\tilde{\phi}_0)$.*

In addition, without the assumption of large viscosity ν, if we assume $u_1^*(z,0)$, $\psi_1^*(z,0)$, $\omega_1^*(z,0)$, $\phi_0^*(z,0)$ are odd, periodic functions in the z direction with period 1, after some delicate analysis, we can also get a global smooth solution, provided the initial data is small enough.

Theorem 5.1.2. *Suppose the initial conditions for u_1, ω_1, ψ_1, and ϕ_1 are smooth functions with compact support and odd in z. Moreover, assume that $\eta > 1$, and $\|\tilde{\mathbf{u}}(0)\|^2 + \lambda\|\nabla\tilde{\phi}(0)\|^2 + \frac{\lambda}{2\eta^2}\|\tilde{\phi}(0)^2 - 1\|^2 \leq \frac{C}{\sqrt{M}}$. For any given $\nu > 0$, there exists $C(\nu) > 0$, such that if $M \geq C(\nu)$ and $H(0) \leq 1$ where $H^2(t) = \|r\nabla u_1\|^2 + \|r\triangle\psi_1\|^2 + \|\nabla^2\phi_1\|^2$. Then, solutions to the 3D system* (5.1.1)-(5.1.3) *in the form of* (5.1.19)-(5.1.22) *are globally smooth.*

5.2 Derivation of the 1D system of equations

In this section, we use the method of separation of variables to derive the 1D equations. Moreover, the regularity of solutions to the 1D equations is investigated. In the end, we present a key observation of the connection between solutions to the 1D equations and those to the $3D$ axisymmetric system.

Assume

$$\begin{aligned}
u(r,z,t) &= \bar{u}(r)u_1^*(z,t), \\
v^r(r,z,t) &= \bar{v}^r(r)a(z,t), \\
v^z(r,z,t) &= \bar{v}^z(r)b(z,t), \\
\psi(r,z,t) &= \bar{\psi}^r(r)\psi_1^*(z,t), \\
\omega(r,z,t) &= \bar{\omega}(r)\omega_1^*(z,t), \\
\phi(r,z,t) &= \phi_0^*(z,t).
\end{aligned}$$

Then (5.1.12) gives

$$\left[(\bar{v}^r)_r + \frac{\bar{v}^r}{r}\right] a(z,t) + \bar{v}^z b'(z,t) = 0,$$

implying

$$
\begin{aligned}
a(z,t) &= b'(z,t), && (5.2.1)\\
(\bar{v}^r)_r + \frac{\bar{v}^r}{r} + \bar{v}^z &= 0. && (5.2.2)
\end{aligned}
$$

Since $v^r = -\psi_z$, $v_z = \frac{1}{r}\frac{\partial}{\partial r}(r\psi)$ and $\omega = (v^r)_z - (v^z)_r$, by (5.2.1) we get

$$
\begin{aligned}
\bar{\psi}^r(r) &= \bar{v}^r(r), && (5.2.3)\\
a(z,t) &= -\psi^*_{1z}(z,t), && (5.2.4)\\
b(z,t) &= -\psi^*_1(z,t), && (5.2.5)\\
w &= -\bar{v}^r\psi^*_{1zz} + (\bar{v}^z)_r\psi^*_1, && (5.2.6)
\end{aligned}
$$

then plugging (5.2.4) and (5.2.5) into (5.1.9) and (5.1.10), one arrives at

$$
u^*_{1t} - \nu u^*_{1zz} = \bar{v}^z\psi^*_1 u^*_{1z} + \nu\left[\frac{(\bar{u})_{rr}}{\bar{u}} + \frac{(\bar{u})_r}{r\bar{u}} - \frac{1}{r^2}\right]u^*_1 + \left(\frac{1}{r} + \frac{(\bar{u})_r}{\bar{u}}\right)\bar{v}^r\psi^*_{1z}u^*_1, \tag{5.2.7}
$$

$$
\begin{aligned}
\omega^*_{1t} - \nu\omega^*_{1zz} &= \left(\frac{\bar{v}^r(\bar{\omega})_r}{\bar{\omega}} - \frac{\bar{v}^r}{r}\right)\psi^*_{1z}\omega^*_1 + \bar{v}^z\psi^*_1\omega^*_{1z} + \frac{2}{r}\frac{(\bar{u})^2}{\bar{\omega}}u^*_1u^*_{1z}\\
&\quad +\nu\left[\frac{(\bar{\omega})_{rr}}{\bar{\omega}} + \frac{(\bar{\omega})_r}{r\bar{\omega}} - \frac{1}{r^2}\right]\omega^*_1. && (5.2.8)
\end{aligned}
$$

Comparing the r and z components in (5.2.8), we know immediately that $\bar{v}^z$ is a constant. From (5.2.6), we have

$$
\bar{\omega} = \bar{v}^r, \quad \omega^*_1 = -\psi^*_{1zz},
$$

Comparing the r and z components in (5.2.8) again, it follows that

$$
\frac{\bar{v}^r(\bar{\omega})_r}{\bar{\omega}} - \frac{\bar{v}^r}{r} = (\bar{\omega})_r - \frac{\bar{\omega}^r}{r}, \quad \frac{2}{r}\frac{(\bar{u})^2}{\bar{\omega}}, \quad \frac{(\bar{\omega})_{rr}}{\bar{\omega}} + \frac{(\bar{\omega})_r}{r\bar{\omega}} - \frac{1}{r^2}
$$

are all constants. As a result,

$$
\bar{\omega} = \bar{v}^r = \bar{u} = r,
$$

together with (5.2.2) we obtain

$$
\bar{v}^z = -2.
$$

Consequently,

$$
\begin{aligned}
u(r,z,t) &= ru^*_1(z,t), && (5.2.9)\\
v^r(r,z,t) &= -r\psi^*_{1z}(z,t), && (5.2.10)\\
v^z(r,z,t) &= 2\psi^*_1(z,t), && (5.2.11)\\
\psi(r,z,t) &= r\psi^*_1(z,t), && (5.2.12)\\
\omega(r,z,t) &= r\omega^*_1(z,t). && (5.2.13)
\end{aligned}
$$

Plugging these into equations (5.1.9)-(5.1.13), one derives (5.1.15)-(5.1.18) of the $1D$ system.

Let $v_1^* = -(\psi_1^*)_z$. Integrating the ω_1^* equation with respect to z and using $-(\psi_1^*)_{zz} = \omega_1^*$, an equation for v_1^* is derived,

$$(v_1^*)_t + 2\psi_1^*(v_1^*)_z = \nu(v_1^*)_{zz} + u_1^{*2} - v_1^{*2} - c(t), \tag{5.2.14}$$

where $c(t)$ is an integration constant, which enforces that the mean value of v_1^* be zero. For instance, if ψ_1^* is periodic with period 1 in z, then $c(t) = 3\int_0^1 v_1^{*2}dz - \int_0^1 u_1^{*2}dz$. We point out that the equation for ω_1^* is equivalent to that for v_1^*. Using (5.1.15), (5.2.14) and the result in [28], we can get some regularity results for the $1D$ equations in the following lemmas.

Lemma 5.2.1. *Assume $u_1^*(z,0)$, $\psi_1^*(z,0)$, $\omega_1^*(z,0)$ are smooth and periodic functions with period 1, then $\psi_1^*(z,t)$, $\psi_{1z}^*(z,t)$, $u_1^*(z,t)$, $u_{1z}^*(z,t)$, and $\omega_1^*(z,t)$ are uniformly bounded.*

Lemma 5.2.2. *Assume $\phi^*(z,0)$ is a smooth and periodic function with period 1, then $\phi_0^*(z,t)$ and its derivatives are uniformly bounded.*

Proof. Multiplying (5.1.18) by ϕ_0^*, then integrating with respect to z over $[0,1]$,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_0^*\|_{L^2(0,1)}^2 + \frac{\gamma}{\eta^2}\int_0^1(\phi_0^*)^4dz + \gamma\|\phi_{0z}^*\|_{L^2(0,1)}^2 \\
= \;& -2\int_0^1 \psi_1^*\phi_{0z}^*\phi_0^*dz + \frac{\gamma}{\eta^2}\|\phi_0^*\|_{L^2(0,1)}^2 \le \frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + (\frac{C}{\gamma} + \frac{\gamma}{\eta^2})\|\phi_0^*\|_{L^2(0,1)}^2 \\
\le \;& \frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + (\frac{C}{\gamma} + \frac{\gamma}{\eta^2})\|\phi_0^*\|_{L^4(0,1)}^2 \\
\le \;& \frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + (\frac{C}{\gamma} + \frac{\gamma}{2\eta^2})\varepsilon\int_0^1(\phi_0^*)^4dz + C(\varepsilon).
\end{aligned} \tag{5.2.15}$$

Multiplying (5.1.18) by ϕ_{0t}^*, and integrating with respect to z over $[0,1]$, it follows that

$$\begin{aligned}
&\|\phi_{0t}^*\|_{L^2(0,1)}^2 + \frac{d}{dt}\left[\frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + \frac{1}{4}\int_0^1(\phi_0^*)^4dz - \frac{\gamma}{2\eta^2}\|\phi_0^*\|_{L^2(0,1)}^2\right] \\
= \;& -2\int_0^1 \psi_1^*\phi_{0z}^*\phi_{0t}^*dz \le \|\phi_{0t}^*\|_{L^2(0,1)}^2 + C\|\phi_{0z}^*\|_{L^2(0,1)}^2 .
\end{aligned} \tag{5.2.16}$$

Multiplying (5.2.15) by a constant $\tilde{C}$, then summing up with (5.2.16), one arrives at

$$\begin{aligned}
&\frac{d}{dt}\left[\frac{\gamma}{2}\|\phi_{0z}^*\|_{L^2(0,1)}^2 + \frac{1}{4}\int_0^1(\phi_0^*)^4dz + (\frac{\tilde{C}}{2} - \frac{\gamma}{2\eta^2})\|\phi_0^*\|_{L^2(0,1)}^2\right] \\
&+(\frac{\tilde{C}\gamma}{2} - C)\|\phi_{0z}^*\|_{L^2(0,1)}^2 + \tilde{C}\left[\frac{\gamma}{\eta^2} - (\frac{C}{\gamma} + \frac{\gamma}{2\eta^2})\varepsilon\right]\int_0^1(\phi_0^*)^4dz \le \tilde{C}C(\varepsilon).
\end{aligned} \tag{5.2.17}$$

Choosing $\tilde{C}$ large enough and ε small enough that $\frac{\tilde{C}\gamma}{2} - C > 0$, $\frac{\gamma}{\eta^2} - (\frac{C}{\gamma} + \frac{\gamma}{2\eta^2})\varepsilon > 0$, we get

$$\frac{d}{dt}\left[\|\phi_0^*\|_{H^1(0,1)}^2 + \int_0^1(\phi_0^*)^4dz\right] + \|\phi_0^*\|_{H^1(0,1)}^2 + \int_0^1(\phi_0^*)^4dz \le C.$$

Applying Gronwall's lemma, we have that

$$\phi_0^*(z,t) \in L^\infty\left(0,\infty; H^1(0,1)\right).$$

$\square$

Theorem 5.2.1. *Let u_1^*, ω_1^*, ψ_1^*, ϕ_0^* be the solution to the $1D$ equations* (5.1.15)-(5.1.18). *Define*

$$\begin{aligned} u(r,z,t) &= ru_1^*(z,t), \\ \omega(r,z,t) &= r\omega_1^*(z,t), \\ \psi(r,z,t) &= r\psi_1^*(z,t), \\ \phi(r,z,t) &= \phi_0^*(z,t). \end{aligned} \tag{5.2.18}$$

Then $(u(r,z,t),\ \omega(r,z,t),\ \psi(r,z,t),\ \phi(r,z,t))$ is an exact solution to the $3D$ system (5.1.1)-(5.1.3).

Remark 5.2.1. *The exact solution $(u(r,z,t),\ \omega(r,z,t),\ \psi(r,z,t),\ \phi(r,z,t))$ in Theorem 5.2.1 has infinite energy in $\mathbb{R}^3$.*

5.3 Some useful lemmas

In this section, using the solution to the $1D$ equations (5.1.15)-(5.1.18), we construct a global classical solution to the $3D$ axisymmetric system with finite energy. To do this, some preliminary work is necessary.

Denoting (u_1^*(z,t), ω_1^*(z,t), ψ_1^*(z,t), $\phi_0^*(z,t)$) as the solution to the $1D$ equations, ($\tilde{u}$(r,z,t), $\tilde{\omega}$(r,z,t), $\tilde{\psi}$(r,z,t), $\tilde{\phi}(r,z,t)$) as the solution to the 3D system. Further, we define

$$\tilde{u}_1 = \frac{\tilde{u}}{r}, \ \ \tilde{\omega}_1 = \frac{\tilde{\omega}}{r}, \ \ \tilde{\psi}_1 = \frac{\tilde{\psi}}{r}.$$

Let $\chi(r) = \chi_0(\frac{r}{R_0})$ be a smooth cut-off function, where $\chi_0(r) = 1$, if $0 \le r \le \frac{1}{2}$, and $\chi_0(r) = 0$, if $r \ge 1$. Our idea is to construct a global classical function $(ru_1,\ r\omega_1,\ r\psi_1,\ \phi_1)$, which is periodic in z direction with periods 1, such that

$$\tilde{u}(r,z,t) = r(u_1^*(z,t)\chi(r) + u_1(r,z,t)) = ru_1^*\chi + u, \tag{5.3.1}$$

$$\tilde{\omega}(r,z,t) = r(\omega_1^*(z,t)\chi(r) + \omega_1(r,z,t)) = r\omega_1^*\chi + \omega, \tag{5.3.2}$$

$$\tilde{\psi}(r,z,t) = r(\psi_1^*(z,t)\chi(r) + \psi_1(r,z,t)) = r\psi_1^*\chi + \psi, \tag{5.3.3}$$

$$\tilde{\phi}(r,z,t) = \phi_0^*(z,t)\chi(r) + \phi_1(r,z,t), \tag{5.3.4}$$

is a global classical solution to the $3D$ axisymmetric system. From (5.3.1)-(5.3.4), we also know

$$\tilde{v}^r = -\tilde{\psi}_z = -r\psi_{1z}^*\chi - r\psi_{1z} = -r\psi_{1z}^*\chi + v^r, \tag{5.3.5}$$

$$\tilde{v}^z = \frac{(r\tilde{\psi})_r}{r} = r\psi_1^*\chi_r + 2\psi_1^*\chi + 2\psi_1 + r\psi_{1r} = r\psi_1^*\chi_r + 2\psi_1^*\chi + v^z. \tag{5.3.6}$$

Here v^r, v^z are considered as perturbation terms of radial and z-axis velocity components respectively.

From (5.1.15)-(5.1.18) of the $1D$ equations about $(u_1^*,\ \omega_1^*,\ \psi_1^*,\ \phi_0^*)$, and equations (5.1.9)-(5.1.13) of the $3D$ system on $(\tilde{u},\ \tilde{\omega},\ \tilde{\psi},\ \tilde{\phi})$, one can derive the equations for $(u_1,\ \omega_1,\ \psi_1,\ \phi_1)$ as follows :

$$\begin{aligned}
u_{1t}+\tilde{v}^r u_{1r}+\tilde{v}^z u_{1z} &= \nu\Delta u_1+2\left(\tilde{\psi}_{1z}\tilde{u}_1-\chi\psi_{1z}^*u_1^*\right)-\tilde{v}^r u_1^*\chi_r \\
&\quad -\chi\left(\left[r\chi_r+2(\chi-1)\right]\psi_1^*+v^z\right)u_{1z}^*+\nu u_1^*\Delta_r\chi, \qquad (5.3.7)\\
\omega_{1t}+\tilde{v}^r\omega_{1r}+\tilde{v}^z\omega_{1z} &= \nu\Delta\omega_1+\left[(u_1^*\chi+u_1)^2\right]_z-(u_1^{*2})_z\chi-\tilde{v}^r\omega_1^*\chi_r \\
&\quad -\chi([r\chi_r+2(\chi-1)]\psi_1^*+v^z)\omega_{1z}^*+\nu\omega_1^*\Delta_r\chi \\
&\quad +\frac{\lambda}{r}\left(\phi_{0z}^*\chi+\phi_{1z}\right)\left[(\nabla^2(\phi_0^*\chi))_r+(\nabla^2\phi_1)_r\right] \\
&\quad -\frac{\lambda}{r}(\phi_0^*\chi_r+\phi_{1r})\left[(\nabla^2(\phi_0^*\chi))_z+(\nabla^2\phi_1)_z\right], \qquad (5.3.8)\\
\phi_{1t}+\tilde{v}^r\phi_{1r}+\tilde{v}^z\phi_{1z} &= \gamma\nabla^2\phi_1-\frac{\gamma}{\eta^2}\left(\phi_1^3+3\phi_0^*\phi_1^2\chi+3\phi_0^{*2}\phi_1\chi^2-\phi_1\right) \\
&\quad +\gamma\phi_0^*\left(\chi_{rr}+\frac{\chi_r}{r}\right)-\frac{\gamma}{\eta^2}\phi_0^{*3}\left(\chi^3-\chi\right)-\phi_0^*\tilde{v}^r\chi_r \\
&\quad +2\psi_1^*\phi_{0z}^*\chi-\phi_{0z}^*\tilde{v}^z\chi. \qquad (5.3.9)
\end{aligned}$$

From the basic energy law (5.1.4), we know actually

$$\mathbf{u}\in L^\infty(0,\infty;L^2(\Omega)),\quad \phi_1\in L^\infty(0,\infty;H^1(\Omega)). \qquad (5.3.10)$$

Therefore, from the result in [49] and standard bootstrap arguments, all we need is to prove

$$\mathbf{u}\in L^\infty(0,\infty;H^1(\Omega)),\quad \phi_1\in L^\infty(0,\infty;H^2(\Omega)). \qquad (5.3.11)$$

We assume $\mathbf{u}$ and $\nabla\tilde{\phi}$ decay sufficiently fast in the infinity $r=\infty$, and has periodic boundary conditions in z direction, with period 1, then so do the perturbation terms u_1, ω_1, ψ_1 and ϕ_1. Due to the periodicity of our boundary conditions in z. Using the boundary conditions, we can get the following useful lemmas.

Lemma 5.3.1.

$$\|u_1\|\le\|ru_{1r}\|.$$

Proof.

$$\begin{aligned}
\int_0^1\int_0^\infty(u_1)^2 r\,dr\,dz &= \int_0^1\int_0^\infty(u_1)^2 d\left(\frac{r^2}{2}\right)dz=-\int_0^1\int_0^\infty\frac{r^2}{2}2u_1u_{1r}\,dr\,dz \\
&= -\int_0^1\int_0^\infty u_1(u_{1r}r)r\,dr\,dz\le\|u_1\|\|ru_{1r}\|,
\end{aligned}$$

hence $\|u_1\|\le\|ru_{1r}\|$. □

Lemma 5.3.2.

$$\|r\nabla u_1\|\le\|r\Delta u_1\|.$$

Proof.

$$
\begin{aligned}
-\int_0^1\int_0^\infty u_1\triangle u_1 r^2 rdrdz &= -\int_0^1\int_0^\infty u_1u_{1rr}r^2rdrdz - 3\int_0^1\int_0^\infty u_1u_{1r}r^2drdz \\
&\quad -\int_0^1\int_0^\infty u_1u_{1zz}r^2rdrdz \\
&= \int_0^1\int_0^\infty (u_{1r}r)^2rdrdz + \int_0^1\int_0^\infty (u_{1z}r)^2rdrdz.
\end{aligned}
$$

On the other hand,

$$
-\int_0^1\int_0^\infty u_1\Delta u_1 r^2 rdrdz \le \|ru_1\|\|r\Delta u_1\|,
$$

hence

$$
\|r\nabla u_1\|^2 \le \|ru_1\|\|r\Delta u_1\| \le \|r\nabla u_1\|\|r\Delta u_1\|.
$$

□

Lemma 5.3.3.

$$
\|ru_{1zz}\| + \|ru_{1zr}\| + \|ru_{1rr}\| + 3\|u_{1r}\| \le \|r\Delta u_1\|.
$$

Proof.

$$
\begin{aligned}
\int_0^1\int_0^\infty \Delta u_1 u_{1zz}r^2rdrdz &= \|ru_{1zz}\|^2 + 3\int_0^1\int_0^\infty u_{1r}u_{1zz}r^2drdz \\
&\quad + \int_0^1\int_0^\infty u_{1rr}u_{1zz}r^2rdrdz \\
&= \|ru_{1zz}\|^2 - 3\int_0^1\int_0^\infty u_{1zr}u_{1z}r^2drdz + \|ru_{1zr}\|^2 \\
&\quad +3\int_0^1\int_0^\infty u_{1zr}u_{1z}r^2drdz \\
&= \|ru_{1zz}\|^2 + \|ru_{1zr}\|^2.
\end{aligned}
$$

On the other hand,

$$
\int_0^1\int_0^\infty \Delta u_1 u_{1zz}r^2rdrdz \le \|r\Delta u_1\|\|ru_{1zz}\| \le \frac{1}{2}\|ru_{1zz}\|^2 + \frac{1}{2}\|r\Delta u_1\|^2,
$$

hence

$$
\|ru_{1zz}\|^2 + \|ru_{1zr}\|^2 \le \frac{1}{2}\|ru_{1zz}\|^2 + \frac{1}{2}\|r\Delta u_1\|^2,
$$

which implies

$$
\|ru_{1zz}\|^2 + \|ru_{1zr}\|^2 \le \|r\Delta u_1\|^2.
$$

Similarly,

$$
\int_0^1\int_0^\infty \Delta u_1 \Delta_r u_1 r^2 rdrdz = \|r\Delta_r u_1\|^2 + \int_0^1\int_0^\infty u_{1zz}u_{1rr}r^2rdrdz
$$

$$
\begin{aligned}
& +3\int_0^1\int_0^\infty u_{1r}u_{1zz}r^2drdz \\
= \quad & \|r\Delta_r u_1\|^2+\|ru_{1zr}\|^2,
\end{aligned}
$$

therefore

$$
\|\Delta_r u_1 r\|^2+\|u_{1zr}r\|^2\le\|\Delta u_1 r\|\|\Delta_r u_1 r\|\le\frac{1}{2}\|\Delta u_1 r\|^2+\frac{1}{2}\|\Delta_r u_1 r\|^2,
$$

which tells us

$$
\|r\Delta_r u_1\|^2+\|ru_{1zr}\|^2\le\|r\Delta u_1\|^2.
$$

Since

$$
\begin{aligned}
\|r\Delta_r u_1\|^2 &= \int_0^1\int_0^\infty(\Delta_r u_1)^2r^2rdrdz \\
&= \|ru_{1rr}\|^2+9\int_0^1\int_0^\infty(u_{1r})^2rdrdz+6\int_0^1\int_0^\infty u_{1rr}u_{1r}r^2drdz \\
&= \|ru_{1rr}\|^2+3\|ru_{1r}\|^2+3\int_0^1\int_0^\infty[(ru_{1r})^2]_rdrdz \\
&= \|ru_{1rr}\|^2+3\|ru_{1r}\|^2,
\end{aligned}
$$

we finish the proof. □

Analogously, we can get

Lemma 5.3.4.

$$
\|\psi_{1z}\|\le\|r\psi_{1zr}\|,\ \ \|\psi_{1r}\|\le\|r\psi_{1rr}\|,\ \ \|\Delta\psi_1\|\le\|r\nabla(\Delta\psi_1)\|.
$$

Lemma 5.3.5.

$$
\|r\psi_{1zz}\|+\|r\psi_{1zr}\|+\|r\Delta_r\psi_1\|\le 2\|r\Delta\psi_1\|.
$$

Lemma 5.3.6.

$$
\|r\psi_{1rrz}\|+\|\psi_{1rz}\|+\|r\psi_{1zzr}\|+\|r\psi_{1rrr}\|+\|r\psi_{1zzz}\|\le 3\|r\nabla(\Delta\psi_1)\|.
$$

Lemma 5.3.7.

$$
\|\phi_{1zz}\|+\|\phi_{1zr}\|+\|\phi_{1rr}\|\le\|\nabla^2\phi_1\|.
$$

Lemma 5.3.8.

$$
\|\phi_{1rrz}\|+\|\phi_{1zzr}\|+\|\phi_{1rrr}\|+\|\phi_{1zzz}\|\le 3\|\nabla(\nabla^2\phi_1)\|.
$$

In all, we conclude from the lemmas above, that to prove (5.3.11), it is sufficient to prove

$$
r\nabla u_1,\ r\Delta\psi_1,\nabla^2\phi_1\in L^\infty\left(0,\infty,L^2(\Omega)\right). \tag{5.3.12}
$$

5.4 Regularity of perturbation terms

In this section we are discussing the proofs of Theorem 5.1.1 and Theorem 5.1.2.

5.4.1 Large viscosity case

Proof. We begin to do estimates term by term, where Hölder inequality and Sobolev interpolation inequalities are used at times.

Multiplying (5.3.7) with $-r^2\Delta u_1$, then integrating over Ω, we have

$$\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int \left[(u_{1z})^2 + (u_{1r})^2\right] r^2 rdrdz \\
= \ & -\nu\int(\Delta u_1)^2 r^2 rdrdz + \int \tilde{v}^r u_{1r}\Delta u_1 r^2 rdrdz + \int \tilde{v}^z u_{1z}\Delta u_1 r^2 rdrdz \\
& -2\int \tilde{v}^r u_1 \Delta u_1 r^2 rdrdz - 2\int \chi u_1^* \psi_{1z}\Delta u_1 r^2 rdrdz - 2\int (\chi^2-\chi)\psi_{1z}^* u_1^* \Delta u_1 r^2 rdrdz \\
& +\int \tilde{v}^r u_1^* \chi_r \Delta u_1 r^2 rdrdz + \int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right) u_{1z}^* \Delta u_1 r^2 rdrdz \\
& -\nu\int u_1^* \Delta_r \chi \Delta u_1 r^2 rdrdz \\
\equiv \ & -\nu\int(\Delta u_1)^2 r^2 rdrdz + I_a + I_b + I_c + I_d + I_e + I_f + I_g + I_h.
\end{aligned} \tag{5.4.1}$$

Estimates for u_1 equation
From (5.3.5), Lemma 5.2.1 and 5.3.2, we know

$$\begin{aligned}
|I_a| &= \left|\int \left(-r\psi_{1z}^*\chi - r\psi_{1z}\right) u_{1r}\Delta u_1 r^2 rdrdz\right| \\
&\le C\left\|\psi_{1z}^*\right\|_{L^\infty}\|ru_{1r}\|\|r\Delta u_1\| + \left|\int \psi_{1z}u_{1r}\Delta u_1 r^3 rdrdz\right| \\
&\le C\|r\Delta u_1\|^2 + \left|\int \psi_{1z}u_{1r}\Delta u_1 r^3 rdrdz\right|,
\end{aligned}$$

where the second term

$$\begin{aligned}
& \left|\int \psi_{1z}u_{1r}\Delta u_1 r^3 rdrdz\right| \\
\le \ & \|r\psi_{1z}\|_{L^4}\|ru_{1r}\|_{L^4}\|r\Delta u_1\| \ \le \ \|r\psi_{1z}\|^{\frac14}\|\nabla(r\psi_{1z})\|^{\frac34}\|ru_{1r}\|^{\frac14}\|\nabla(ru_{1r})\|^{\frac34}\|r\Delta u_1\| \\
\le \ & \|r\psi_{1z}\|^{\frac14}\left(\|r\psi_{1zz}\| + \|r\psi_{1zr}+\psi_{1z}\|\right)^{\frac34}\|ru_{1r}\|^{\frac14}\left(\|ru_{1rz}\| + \|ru_{1rr}+u_{1r}\|\right)^{\frac34}\|r\Delta u_1\| \\
\le \ & C\left(\|r\psi_{1zz}\| + \|r\psi_{1zr}\|\right)^{\frac34}\|ru_{1r}\|^{\frac14}\left(\|ru_{1rz}\| + \|ru_{1rr}\|\right)^{\frac34}\|r\Delta u_1\| \\
\le \ & C\|r\Delta\psi_1\|^{\frac34}\|ru_{1r}\|^{\frac14}\|r\Delta u_1\|^{\frac74} \ \le \ C\|r\Delta\psi_1\|^{\frac67}\|r\Delta u_1\|^2 + C\|ru_{1r}\|^2 \\
\le \ & C\left(\|r\Delta\psi_1\|^2+1\right)\|r\Delta u_1\|^2 + C\|ru_{1r}\|^2 \\
\le \ & C\left(\|r\Delta\psi_1\|^2+1\right)\|r\Delta u_1\|^2,
\end{aligned}$$

here we used (5.3.10) and Lemma 5.3.2, Lemma 5.3.3, Lemma 5.3.5 and Young's Inequality. As a result,

$$|I_a| \ \le \ C\left(\|r\Delta\psi_1\|^2+1\right)\|r\Delta u_1\|^2. \tag{5.4.2}$$

Similar to I_a, we get

$$|I_b| = \left|\int \tilde{v}^z u_{1z}\Delta u_1 r^2 rdrdz\right| \le C\left(\|r\Delta\psi_1\|^2+1\right)\|r\Delta u_1\|^2. \tag{5.4.3}$$

Due to (5.3.5), (5.3.10), we have

$$|I_c| = 2\left|\int \tilde{v}^r u_1\Delta u_1 r^2 rdrdz\right| \quad \le \quad C\left(\|r\Delta u_1\|^2+1\right) + 2\|r\psi_{1z}\|_{L^4}\|ru_1\|_{L^4}\|r\Delta u_1\|,$$

Similar to I_a,

$$|I_c| \le C\left(\|r\Delta\psi_1\|^2+1\right)\|r\Delta u_1\|^2 + C. \tag{5.4.4}$$

For estimates from I_d to I_h, with the help of (5.3.5), (5.1.4), (5.3.10) and Lemma 5.2.1, we obtain

$$|I_d| \;=\; \left|-2\int \chi u_1^*\psi_{1z}\Delta u_1 r^2 rdrdz\right| \le 2\|u_1^*\|_{L^\infty}\|r\psi_{1z}\|\|r\Delta u_1\| \le C\|r\Delta u_1\|^2 + C. \tag{5.4.5}$$

$$\begin{aligned}
|I_e| &= \left|-2\int (\chi^2-\chi)\psi_{1z}^* u_1^*\Delta u_1 r^2 rdrdz\right| \le C\|\psi_{1z}^*\|_{L^\infty}\|u_1^*\|_{L^\infty}\|r\Delta u_1\| \\
&\le C\|r\Delta u_1\|^2 + C. && (5.4.6)\\
|I_f| &= \left|\int \tilde{v}^r u_1^*\chi_r\Delta u_1 r^2 rdrdz\right| \le C\|u_1^*\|_{L^\infty}\|\tilde{v}^r\|\|r\Delta u_1\| \le C\|r\Delta u_1\|^2 + C. && (5.4.7)\\
|I_g| &= \left|\int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right)u_{1z}^*\Delta u_1 r^2 rdrdz\right| \\
&\le C\|\psi_1^*\|_{L^\infty}\|u_{1z}^*\|_{L^\infty}\|r\Delta u_1\| + C\|u_{1z}^*\|_{L^\infty}\|v^z\|\|r\Delta u_1\| \\
&\le C\|r\Delta u_1\|^2 + C. && (5.4.8)\\
|I_h| &= \left|-\nu\int u_1^*\Delta_r\chi\Delta u_1 r^2 rdrdz\right| \le \frac{\nu}{2}\|r\Delta u_1\|^2 + C. && (5.4.9)
\end{aligned}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|r\nabla u_1\|^2 \le -\left[\frac{\nu}{2} - C\left(\|r\Delta\psi_1\|^2+1\right)\right]\|r\Delta u_1\|^2 + C. \tag{5.4.10}$$

Multiplying (5.3.8) with $-r^2\Delta\psi_1$, and integrating over Ω, since $-\omega_1 = \Delta\psi_1 + (\Delta_r\phi)\psi_1^*$, by (5.1.12), we know the fact that

$$\begin{aligned}
& -\int \tilde{v}^r(\Delta\psi_1)_r\Delta\psi_1 r^2 rdrdz - \int \tilde{v}^z(\Delta\psi_1)_z\Delta\psi_1 r^2 rdrdz \\
=\; & -\frac{1}{2}\int \tilde{v}^r\left[(\Delta\psi_1)^2\right]_r r^3 drdz - \frac{1}{2}\int \tilde{v}^z\left[(\Delta\psi_1)^2\right]_z r^3 drdz \\
=\; & \frac{1}{2}\int (\Delta\psi_1)^2\left[\left(r^3\tilde{v}^r\right)_r + (r^3\tilde{v}^z)_z\right] drdz \\
=\; & \frac{1}{2}\int (\Delta\psi_1)^2\left[(r\tilde{v}^r)_z + (r\tilde{v}^z)_z + 2\tilde{v}^r\right] r^2 drdz \\
=\; & \int (\Delta\psi_1)^2\tilde{v}^r r^2 drdz,
\end{aligned}$$

consequently, one arrives at

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\int (\Delta\psi_1)^2 r^2 r drdz \\
= \quad & -\nu\int \left[((\Delta\psi_1)_r)^2 + ((\Delta\psi_1)_z)^2\right] r^3 drdz - \int \Delta_r\chi\psi_{1t}^*\Delta\psi_1 r^3 drdz + \int (\Delta\psi_1)^2\tilde{v}^r r^2 drdz \\
& -\int \tilde{v}^r(\Delta_r\chi)_r\psi_1^*\Delta\psi_1 r^3 drdz - \int \tilde{v}^z\Delta_r\chi\psi_{1z}^*\Delta\psi_1 r^3 drdz + \nu\int \Delta\left(\Delta_r\chi\psi_1^*\right)\Delta\psi_1 r^3 drdz \\
& -2\int \left[u_1^* u_{1z}^*(\chi^2-\chi) + u_{1z}^*\chi u_1 + u_1^*\chi u_{1z}\right]\Delta\psi_1 r^3 drdz - 2\int u_1 u_{1z}\Delta\psi_1 r^3 drdz \\
& +\int \tilde{v}^r\omega_1^*\chi_r\Delta\psi_1 r^3 drdz + \int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right)\omega_{1z}^*\Delta\psi_1 r^3 drdz \\
& -\nu\int \omega_1^*\Delta_r\chi\Delta\psi_1 r^3 drdz - \lambda\int (\phi_{0z}^*\chi + \phi_{1z})\left[\left(\nabla^2(\phi_0^*\chi)\right)_r + (\nabla^2\phi_1)_r\right]\Delta\psi_1 r^2 drdz \\
& +\lambda\int (\phi_0^*\chi_r + \phi_{1r})\left[\left(\nabla^2(\phi_0^*\chi)\right)_z + (\nabla^2\phi_1)_z\right]\Delta\psi_1 r^2 drdz \\
\equiv \quad & -\nu\int \left[\nabla(\Delta\psi_1)\right]^2 r^3 drdz + J_a + J_b + J_c + J_d + J_e + J_f + J_g \\
& +J_h + J_i + J_j + J_k + J_l. \qquad (5.4.11)
\end{aligned}
$$

Estimates for ω_1 equation

Equation (5.1.17) infers $(\psi_{1t}^*)_{zz} = -\omega_{1t}^*$. By (5.1.16) and Lemma 5.2.1, we conclude $\|\psi_{1t}^*\|_{L^\infty} \le C$. As a result,

$$
|J_a| = \left|\int \Delta_r\chi\psi_{1t}^*\Delta\psi_1 r^3 drdz\right| \le C\|r\Delta\psi_1\| \le C\left(\|r\Delta\psi_1\|^2 + 1\right). \qquad (5.4.12)
$$

By (5.3.5), Lemma 5.2.1, 5.3.4, 5.3.5, 5.3.6, and Young's Inequality, it follows that

$$
\begin{aligned}
|J_b| \quad &= \quad \left|\int (\Delta\psi_1)^2\tilde{v}^r r^2 drdz\right| \\
&= \quad \left|\int -\psi_{1z}^*\chi(\Delta\psi_1)^2 r^3 drdz - \int \psi_{1z}(\Delta\psi_1)^2 r^3 drdz\right| \\
&\le \quad C\|r\Delta\psi_1\|^2 + \|\psi_{1z}\|_{L^4}\|r\Delta\psi_1\|_{L^4}\|r\Delta\psi_1\| \\
&\le \quad C\|r\Delta\psi_1\|^2 + C\|\psi_{1z}\|^{\frac{1}{4}}\|\nabla\psi_{1z}\|^{\frac{3}{4}}\|r\Delta\psi_1\|^{\frac{1}{4}}\|\nabla(r\Delta\psi_1)\|^{\frac{3}{4}}\|r\Delta\psi_1\| \\
&\le \quad C\|r\Delta\psi_1\|^2 + C\|r\psi_{1rz}\|^{\frac{1}{4}}\left(\|r\psi_{1zzr}\| + \|r\psi_{1rzz}\|\right)^{\frac{3}{4}}\|r\Delta\psi_1\|^{\frac{5}{4}}\|r\nabla(\Delta\psi_1)\|^{\frac{3}{4}} \\
&\le \quad C\|r\Delta\psi_1\|^2 + C\|r\Delta\psi_1\|^{\frac{3}{2}}\|r\nabla(\Delta\psi_1)\|^{\frac{3}{2}} \\
&\le \quad C\|r\Delta\psi_1\|^2 + C\|r\Delta\psi_1\|\|r\nabla(\Delta\psi_1)\|^2 \\
&\le \quad C\|r\Delta\psi_1\|^2 + C\left(\|r\Delta\psi_1\|^2 + 1\right)\|r\nabla(\Delta\psi_1)\|^2. \qquad (5.4.13)
\end{aligned}
$$

Using basic energy law (5.1.4) and Lemma 5.2.1, 5.3.2, one can get estimates of J_c to J_f as

$$
\begin{aligned}
|J_c| \quad &= \quad \left|-\int \tilde{v}^r(\Delta_r\chi)_r\psi_1^*\Delta\psi_1 r^3 drdz\right| \le C\|\tilde{v}^r\|\|r\Delta\psi_1\| \le C\left(\|r\Delta\psi_1\|^2 + 1\right). \qquad (5.4.14) \\
|J_d| \quad &= \quad \left|-\int \tilde{v}^z\Delta_r\chi\psi_{1z}^*\Delta\psi_1 r^3 drdz\right| \le C\|\tilde{v}^z\|\|r\Delta\psi_1\| \le C\left(\|r\Delta\psi_1\|^2 + 1\right). \qquad (5.4.15)
\end{aligned}
$$

$$
\begin{aligned}
|J_e| &= \left|\nu \int \Delta(\Delta_r \chi \psi_1^*)\Delta\psi_1 r^3 drdz\right| \\
&\le \nu \left| \int \left[(\Delta_r\chi\psi_1^*)_{rr} + (\Delta_r\chi\psi_1^*)_{zz}\right](\Delta\psi_1) r^3 drdz + 3\int (\Delta_r\chi\psi_1^*)_r(\Delta\psi_1) r^2 drdz \right| \\
&= \left| -\nu \int \Delta_r\chi\psi_{1z}^*(\Delta\psi_1)_z r^3 drdz - \nu \int (\Delta_r\chi)_r\psi_1^*(\Delta\psi_1)_r r^3 drdz \right| \\
&\le \nu(\|(r^2\Delta_r\chi)\psi_{1z}^*\|_{L^\infty} + \|r(\Delta_r\chi)_r\psi_1^*\|_{L^\infty})\|r\nabla\Delta\psi_1\| \\
&\le \frac{\nu}{2}\|r\nabla\Delta\psi_1\|^2 + C. \qquad (5.4.16) \\
|J_f| &= \left| -2\int \left[u_1^* u_{1z}^*(\chi^2-\chi) + u_{1z}^*\chi u_1 + u_1^*\chi u_{1z}\right]\Delta\psi_1 r^3 drdz \right| \\
&\le C\|r\Delta\psi_1\| + C\|ru_1\|\|r\Delta\psi_1\| + C\|ru_{1z}\|\|r\Delta\psi_1\| \\
&\le C\|ru_{1z}\|^2 + C\|r\Delta\psi_1\|^2 + C \\
&\le C\|r\Delta u_1\|^2 + C\|r\Delta\psi_1\|^2 + C. \qquad (5.4.17)
\end{aligned}
$$

By Lemma 5.3.1, 5.3.2, 5.3.5, we have

$$
\begin{aligned}
|J_g| &= \left| -2\int u_1 u_{1z}\Delta\psi_1 r^3 drdz \right| \\
&\le C\|u_1\|_{L^4}\|ru_{1z}\|_{L^4}\|r\Delta\psi_1\| \le C\|u_1\|^{\frac14}\|\nabla u_1\|^{\frac34}\|ru_{1z}\|^{\frac14}\|\nabla(ru_{1z})\|^{\frac34}\|r\Delta\psi_1\| \\
&\le C\|ru_{1r}\|^{\frac14}\left(\|ru_{1rr}\| + \|ru_{1zz}\|\right)^{\frac34}\|ru_{1z}\|^{\frac14}\left(\|ru_{1rz}\| + \|ru_{1zz}\|\right)^{\frac34}\|r\Delta\psi_1\| \\
&\le C\|ru_{1r}\|^{\frac14}\|ru_{1z}\|^{\frac14}\|r\Delta u_1\|^{\frac32}\|r\Delta\psi_1\| \\
&\le C\|ru_{1r}\|^{\frac12}\|ru_{1z}\|^{\frac12}\|r\Delta u_1\| + C\|r\Delta u_1\|^2\|r\Delta\psi_1\|^2 \\
&\le C\left(\|ru_{1r}\|^2 + \|ru_{1z}\|^2\right) + C\|r\Delta u_1\|^2\left(\|r\Delta\psi_1\|^2 + 1\right) \\
&\le C\|r\Delta u_1\|^2\left(\|r\Delta\psi_1\|^2 + 1\right). \qquad (5.4.18)
\end{aligned}
$$

Lemma 5.2.1 and basic energy law (5.1.4) tell us

$$
\begin{aligned}
|J_h| &= \left| \int \tilde{v}^r\omega_1^*\chi_r\Delta\psi_1 r^3 drdz \right| \le C\|\tilde{v}^r\|\|r\Delta\psi_1\| \le C\|r\Delta\psi_1\|^2 + C. \qquad (5.4.19) \\
|J_i| &= \left| \int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right)\omega_{1z}^*\Delta\psi_1 r^3 drdz \right| \\
&\le \left| \int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v_z^z\right)\omega_1^*\Delta\psi_1 r^3 drdz \right| \\
&\quad + \left| \int \chi\left([r\chi_r + 2(\chi-1)]\psi_1^* + v^z\right)\omega_1^*(\Delta\psi_1)_z r^3 drdz \right| \\
&\le C\|r\Delta\psi_1\| + C\|r\nabla(\Delta\psi_1)\| + \left| \int \chi v_z^z\omega_1^*\Delta\psi_1 r^3 drdz \right|, \qquad (5.4.20)
\end{aligned}
$$

where the estimate of the third term can be derived from (5.3.6) and Lemma 5.3.5.

$$
\begin{aligned}
\left| \int \chi v_z^z\omega_1^*\Delta\psi_1 r^3 drdz \right| &= \left| \int \chi\left(2\psi_{1z} + r\psi_{1rz}\right)\omega_1^*\Delta\psi_1 r^3 drdz \right| \\
&\le C\|r\Delta\psi_1\| + C\|r\psi_{1rz}\|\|r\Delta\psi_1\| \le C\|r\Delta\psi_1\| + C\|r\Delta\psi_1\|^2,
\end{aligned}
$$

consequently,

$$|J_i| \leq C\|r\Delta\psi_1\|^2 + C\|r\nabla(\Delta\psi_1)\|^2 + C. \tag{5.4.21}$$

Lemma 5.2.1 infers that

$$|J_j| \;=\; \left|\nu\int \omega_1^*\Delta_r\chi\Delta\psi_1 r^3 drdz\right| \leq C\nu\|r\Delta\psi_1\| \leq \frac{\nu}{2}\|r\Delta\psi_1\|^2 + C. \tag{5.4.22}$$

For the estimate of J_k, using basic energy law (5.1.4) and Lemma 5.2.2, one derives

$$\begin{aligned}
|J_k| &= \left|-\lambda\int(\phi_{0z}^*\chi+\phi_{1z})\left[\left(\nabla^2(\phi_0^*\chi)\right)_r + (\nabla^2\phi_1)_r\right]\Delta\psi_1 r^2 drdz\right| \\
&\leq \lambda\int|\phi_{0z}^*\chi\left(\nabla^2(\phi_0^*\chi)\right)_r\Delta\psi_1|r^2drdz + \lambda\int|\phi_{1z}\left(\nabla^2(\phi_0^*\chi)\right)_r\Delta\psi_1|r^2drdz \\
&\quad +\lambda\int|\phi_{0z}^*\chi(\nabla^2\phi_1)_r\Delta\psi_1|r^2drdz + \lambda\int|\phi_{1z}\left(\nabla^2\phi_1\right)_r\Delta\psi_1|r^2drdz \\
&\leq C\left(1+\|\phi_{1z}\|+\|(\nabla^2\phi_1)_r\|\right)\|r\Delta\psi_1\| + C\|\phi_{1z}\|_{L^4}\|(\nabla^2\phi_1)_r\|\|r\Delta\psi_1\|_{L^4} \\
&\leq C\left(\|r\Delta\psi_1\|^2+1\right) + C\|\phi_{1z}\|_{L^4}\|(\nabla^2\phi_1)_r\|\|r\Delta\psi_1\|_{L^4},
\end{aligned}$$

where the estimate of the second term is obtained by using (5.1.4), Lemma 5.3.6, 5.3.7, and Young's Inequality,

$$\begin{aligned}
&\|\phi_{1z}\|_{L^4}\|(\nabla^2\phi_1)_r\|\|r\Delta\psi_1\|_{L^4} \\
\leq\;& C\|\nabla^2\phi_1\|^{\frac{3}{4}}\|\nabla(\nabla^2\phi_1)\|\|r\Delta\psi_1\|^{\frac{1}{4}}\|\nabla(r\Delta\psi_1)\|^{\frac{3}{4}} \\
\leq\;& C\left(\nu^{\frac{3}{16}}\|\nabla^2\phi_1\|^{\frac{3}{4}}\|r\nabla(\Delta\psi_1)\|^{\frac{3}{4}}\right)\left(\nu^{\frac{1}{16}}\|r\Delta\psi_1\|^{\frac{1}{4}}\right)\left(\frac{1}{\nu^{\frac{1}{4}}}\|\nabla(\nabla^2\phi_1)\|\right) \\
\leq\;& C\nu^{\frac{1}{2}}\|r\Delta\psi_1\|^2 + C\nu^{\frac{1}{2}}\|\nabla^2\phi_1\|^2\|r\nabla(\Delta\psi_1)\|^2 + \frac{C}{\nu^{\frac{1}{2}}}\|\nabla(\nabla^2\phi_1)\|^2,
\end{aligned}$$

therefore,

$$|J_k| \leq C\nu^{\frac{1}{2}}\left(1+\|\nabla^2\phi_1\|^2\right)\|r\nabla(\Delta\psi_1)\|^2 + \frac{C}{\nu^{\frac{1}{2}}}\|\nabla(\nabla^2\phi_1)\|^2 + C\left(\|r\Delta\psi_1\|^2+1\right). \tag{5.4.23}$$

Similarly, we have

$$\begin{aligned}
|J_l| &= \left|\lambda\int(\phi_0^*\chi_r+\phi_{1r})\left[\left(\nabla^2(\phi_0^*\chi)\right)_z + (\nabla^2\phi_1)_z\right]\Delta\psi_1 r^2 drdz\right| \\
&\leq C\nu^{\frac{1}{2}}\|\nabla^2\phi_1\|^2\|r\nabla(\Delta\psi_1)\|^2 + C\nu^{-\frac{1}{2}}\|\nabla^2\phi_1\|^2\|\nabla(\nabla^2\phi_1)\|^2 \\
&\quad + C\left(\|r\Delta\psi_1\|^2 + \|\nabla(\nabla^2\phi_1)\|^2 + 1\right).
\end{aligned} \tag{5.4.24}$$

In sum,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|r\Delta\psi_1\|^2 \;\leq\; & -\left[\frac{\nu}{2} - \nu^{\frac{1}{2}}\|\nabla^2\phi_1\|^2 - C\left(\|r\Delta\psi_1\|^2+1\right)\right]\|r\nabla(\Delta\psi_1)\|^2 \\
& + C\nu^{-\frac{1}{2}}\|\nabla^2\phi_1\|^2\|\nabla(\nabla^2\phi_1)\|^2 + C\left(\|r\Delta\psi_1\|^2+1\right)\|r\Delta u_1\|^2
\end{aligned}$$

$$+C\left(\|r\Delta\psi_1\|^2+\|\nabla(\nabla^2\phi_1)\|^2+\|r\Delta u_1\|^2+1\right). \qquad (5.4.25)$$

Multiplying (5.3.9) with $\nabla^2(\nabla^2\phi_1)$, then integrating over Ω, one arrives at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla^2\phi_1|^2 rdrdz\\
= \ & -\gamma\int\left[\nabla(\nabla^2\phi_1)\right]^2 rdrdz-\int\tilde{v}^r\phi_{1r}\nabla^2(\nabla^2\phi_1)rdrdz-\int\tilde{v}^z\phi_{1z}\nabla^2(\nabla^2\phi_1)rdrdz\\
&-\frac{\gamma}{\eta^2}\int\left(\phi_1^3+3\phi_0^*\phi_1^2\chi+3\phi_0^{*2}\phi_1\chi^2-\phi_1\right)\nabla^2(\nabla^2\phi_1)rdrdz\\
&+\gamma\int\phi_0^*\left(\chi_{rr}+\frac{\chi_r}{r}\right)\nabla^2(\nabla^2\phi_1)rdrdz-\int\frac{\gamma}{\eta^2}\phi_0^{*3}\left(\chi^3-\chi\right)\nabla^2(\nabla^2\phi_1)rdrdz\\
&-\int\phi_0^*\tilde{v}^r\chi_r\nabla^2(\nabla^2\phi_1)rdrdz+2\int\psi_1^*\phi_{0z}^*\chi\nabla^2(\nabla^2\phi_1)rdrdz\\
&-\int\phi_{0z}^*\tilde{v}^z\chi\nabla^2(\nabla^2\phi_1)rdrdz.\\
\equiv \ & -\gamma\int\left[\nabla(\nabla^2\phi_1)\right]^2 rdrdz+K_a+K_b+K_c+K_d+K_e+K_f+K_g+K_h. \qquad (5.4.26)
\end{aligned}$$

Estimates for ϕ_1 equation

$$\begin{aligned}
K_a &= \int\left[(\tilde{v}^r)_r\phi_{1r}(\nabla^2\phi_1)_r+\tilde{v}^r\phi_{1rr}(\nabla^2\phi_1)_r+(\tilde{v}^r)_z\phi_{1r}(\nabla^2\phi_1)_z+\tilde{v}^r\phi_{1rz}(\nabla^2\phi_1)_z\right]rdrdz\\
&\equiv a_1+b_1+c_1+d_1,
\end{aligned}$$

where estimates of a_1 to d_1 can be derived through (5.3.5), basic energy law (5.1.4), Lemma 5.2.1, 5.3.4, 5.3.5, 5.3.7, 5.3.8, and Young's Inequality.

$$\begin{aligned}
|a_1| &= \left|\int(\tilde{v}^r)_r\phi_{1r}(\nabla^2\phi_1)_r rdrdz\right|\\
&\le C\|\phi_{1r}\|\|\nabla(\nabla^2\phi_1)\|+C\|\psi_{1z}+r\psi_{1rz}\|_{L^4}\|\phi_{1r}\|_{L^4}\|\nabla(\nabla^2\phi_1)\|\\
&\le C\|\nabla(\nabla^2\phi_1)\|+C\|\psi_{1z}+r\psi_{1rz}\|^{\frac14}\|\nabla(\psi_{1z}+r\psi_{1rz})\|^{\frac34}\|\phi_{1r}\|^{\frac14}\|\nabla(\phi_{1r})\|^{\frac34}\|\nabla(\nabla^2\phi_1)\|\\
&\le C\|\nabla(\nabla^2\phi_1)\|+C\|r\Delta\psi_1\|^{\frac14}\|r\nabla(\Delta\psi_1)\|^{\frac34}\|\nabla^2\phi_1\|^{\frac34}\|\nabla(\nabla^2\phi_1)\|\\
&\le C\|\nabla(\nabla^2\phi_1)\|+C\left(\nu^{\frac{1}{16}}\|r\Delta\psi_1\|^{\frac14}\right)\left(\nu^{\frac{3}{16}}\|r\nabla(\Delta\psi_1)\|^{\frac34}\|\nabla^2\phi_1\|^{\frac34}\right)\left(\frac{1}{\nu^{\frac14}}\|\nabla(\nabla^2\phi_1)\|\right)\\
&\le \frac{C}{\sqrt{\nu}}\|\nabla(\nabla^2\phi_1)\|^2+\sqrt{\nu}\|r\nabla(\Delta\psi_1)\|^2\|\nabla^2\phi_1\|^2+C\left(\|r\Delta\psi_1\|^2+1\right).
\end{aligned}$$

And

$$\begin{aligned}
|b_1| &= \left|\int(\tilde{v}^r)\phi_{1rr}(\nabla^2\phi_1)_r rdrdz\right|\\
&\le C\|\phi_{1rr}\|\|\nabla(\nabla^2\phi_1)\|+C\|r\psi_{1z}\|_4\|\phi_{1rr}\|_4\|(\nabla^2\phi_1)_r\|\\
&\le C\|\nabla^2\phi_1\|\|\nabla(\nabla^2\phi_1)\|+C\|r\psi_{1z}\|^{\frac14}\|\nabla(r\psi_{1z})\|^{\frac34}\|\phi_{1rr}\|^{\frac14}\|\nabla(\phi_{1rr})\|^{\frac34}\|\nabla(\nabla^2\phi_1)\|\\
&\le C\|\nabla^2\phi_1\|\|\nabla(\nabla^2\phi_1)\|+C\|\Delta\psi_1\|^{\frac34}\|\nabla^2\phi_1\|^{\frac14}\|\nabla(\nabla^2\phi_1)\|^{\frac74}\\
&\le \frac{C}{\sqrt{\nu}}(\|r\Delta\psi_1\|^2+\|\nabla^2\phi_1\|^2+1)\|\nabla(\nabla^2\phi_1)\|^2+C.
\end{aligned}$$

Similar to a_1,

$$
\begin{aligned}
|c_1| &= \left| \int (\tilde{v}^r)_z \phi_{1r} (\nabla^2 \phi_1)_z r dr dz \right| \\
&\le \frac{C}{\sqrt{\nu}} \|\nabla(\nabla^2 \phi_1)\|^2 + \sqrt{\nu} \|r\nabla(\Delta\psi_1)\|^2 \|\nabla^2\phi_1\|^2 + C\left(\|r\Delta\psi_1\|^2 + 1\right).
\end{aligned}
$$

Similar to b_1,

$$
\begin{aligned}
|d_1| &= \left| \int (\tilde{v}^r) \phi_{1rz} (\nabla^2 \phi_1)_z r dr dz \right| \le \frac{C}{\sqrt{\nu}} (\|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2 + 1) \|\nabla(\nabla^2\phi_1)\|^2 \\
&\le \frac{C}{\sqrt{\nu}} (\|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2 + 1) \|\nabla(\nabla^2\phi_1)\|^2 + C.
\end{aligned}
$$

To sum up, we conclude

$$
\begin{aligned}
|K_a| \le\ & \frac{C}{\sqrt{\nu}} (\|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2 + 1) \|\nabla(\nabla^2\phi_1)\|^2 \\
& + \sqrt{\nu} \|r\nabla(\Delta\psi_1)\|^2 \|\nabla^2\phi_1\|^2 + C\left(\|r\Delta\psi_1\|^2 + 1\right).
\end{aligned} \tag{5.4.27}
$$

For K_b, we can get the same estimate like K_a. After expanding K_c, we get

$$
\begin{aligned}
K_c &= -\frac{\gamma}{\eta^2} \int \left(\phi_1^3 + 3\phi_0^* \phi_1^2 \chi + 3\phi_0^{*2} \phi_1 \chi^2 - \phi_1\right) \nabla^2(\nabla^2\phi_1) r dr dz \\
&= \frac{\gamma}{\eta^2} \int \nabla \left(\phi_1^3 + 3\phi_0^* \phi_1^2 \chi + 3\phi_0^{*2} \phi_1 \chi^2 - \phi_1\right) \nabla(\nabla^2\phi_1) r dr dz \\
&= \frac{3\gamma}{\eta^2} \int (3\phi_1^2 - 1) \nabla\phi_1 \nabla(\nabla^2\phi_1) r dr dz + \frac{3\gamma}{\eta^2} \int \phi_1^2 \nabla(\phi_0^* \chi) \nabla(\nabla^2\phi_1) r dr dz \\
&\quad + \frac{3\gamma}{\eta^2} \int \phi_1 \nabla(\phi_0^{*2}\chi^2) \nabla(\nabla^2\phi_1) r dr dz + \frac{6\gamma}{\eta^2} \int \phi_0^* \chi \phi_1 \nabla\phi_1 \nabla(\nabla^2\phi_1) r dr dz \\
&\quad + \frac{3\gamma}{\eta^2} \int \phi_0^{*2} \chi^2 \nabla\phi_1 \nabla(\nabla^2\phi_1)\, r dr dz,
\end{aligned}
$$

using basic energy law (5.1.4) and Lemma 5.2.2, it is easy to obtain

$$
|K_c| \le \frac{1}{\sqrt{\nu}} \|\nabla(\nabla^2\phi_1)\|^2 + C. \tag{5.4.28}
$$

Similarly,

$$
|K_d| \le \frac{1}{\sqrt{\nu}} \|\nabla(\nabla^2\phi_1)\|^2 + C. \tag{5.4.29}
$$

$$
|K_e| \le \frac{1}{\sqrt{\nu}} \|\nabla(\nabla^2\phi_1)\|^2 + C. \tag{5.4.30}
$$

$$
|K_g| \le \frac{1}{\sqrt{\nu}} \|\nabla(\nabla^2\phi_1)\|^2 + C. \tag{5.4.31}
$$

For K_f, we use (5.3.5) and Lemma 5.3.4, 5.3.5,

$$
\begin{aligned}
|K_f| &= \left| -\int (\phi_0^* \tilde{v}^r \chi_r) \nabla^2 (\nabla^2 \phi_1) r dr dz \right| \\
&\leq \left| \int \tilde{v}^r \nabla(\chi_r \phi_0^*) \cdot \nabla(\nabla^2 \phi_1) r dr dz \right| + \left| \int \phi_0^* \chi_r \nabla(\tilde{v}^r) \cdot \nabla(\nabla^2 \phi_1) r dr dz \right| \\
&\leq C\|\tilde{v}^r\| \|\nabla(\nabla^2 \phi_1)\| + C\|\nabla(\tilde{v}^r)\| \|\nabla(\nabla^2 \phi_1)\| \\
&\leq C\|\nabla(\nabla^2 \phi_1)\| + C\|\nabla(r\psi_{1z})\| \|\nabla(\nabla^2 \phi_1)\| \\
&\leq C\|\nabla(\nabla^2 \phi_1)\| + C\|r\Delta\psi_1\| \|\nabla(\nabla^2 \phi_1)\| \\
&\leq \frac{1}{\sqrt{\nu}} \|\nabla(\nabla^2 \phi_1)\|^2 + \frac{1}{\sqrt{\nu}} \|r\Delta\psi_1\|^2 + C. \qquad (5.4.32)
\end{aligned}
$$

Similarly for K_h,

$$
|K_h| \leq \frac{1}{\sqrt{\nu}} \|\nabla(\nabla^2 \phi_1)\|^2 + \frac{1}{\sqrt{\nu}} \|r\Delta\psi_1\|^2 + C. \qquad (5.4.33)
$$

Thus,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} \left\|\nabla^2 \phi_1\right\|^2 \leq & -\left[\gamma - \frac{C}{\sqrt{\nu}}(\|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2 + 1)\right] \|\nabla(\nabla^2\phi_1)\|^2 \\
& + C\nu^{\frac{1}{2}} \left(\|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2\right) \|r\nabla(\Delta\psi_1)\|^2 + C\|r\Delta\psi_1\|^2 + C. \qquad (5.4.34)
\end{aligned}
$$

Adding up estimates (5.4.10), (5.4.25) and (5.4.34), and denoting

$$
\begin{aligned}
H^2(t) &= \|r\nabla u_1\|^2 + \|r\Delta\psi_1\|^2 + \|\nabla^2\phi_1\|^2, \qquad (5.4.35) \\
E^2(t) &= \|r\Delta u_1\|^2 + \|r\nabla(\Delta\psi_1)\|^2 + \|\nabla(\nabla^2\phi_1)\|^2. \qquad (5.4.36)
\end{aligned}
$$

Then we get

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt} H^2(t) \leq & -\left[\nu - CH^2(t) - C\right] \|r\Delta u_1\|^2 \\
& - \left[\nu - \nu^{\frac{1}{2}}(H^2(t) + C)\right] \|r\nabla(\Delta\psi_1)\|^2 \\
& - \left[\gamma - \frac{C}{\sqrt{\nu}} H^2(t) - \frac{C}{\sqrt{\nu}}\right] \|\nabla(\nabla^2\phi_1)\|^2 + CH^2(t) + C. \qquad (5.4.37)
\end{aligned}
$$

Following the steps in [9], we can prove when ν is large enough, H(t) is uniformly bounded for all $t > 0$. It follows that

$$
\mathbf{u} \in L^\infty(0, \infty; H^1(\Omega)), \quad \phi \in L^\infty(0, \infty; H^2(\Omega)),
$$

which is actually a classical solution. □

5.4.2 Discussion in the small initial data case

In this section we briefly discuss the proof of Theorem 5.1.2.

We choose the initial data for the $1D$ system as:

$$\psi_1^*(z,0)=\frac{1}{M^2}\overline{\psi_1}(zM),\quad u_1^*(z,0)=\frac{1}{M}\overline{U_1}(zM),$$
$$\omega_1^*(z,0)=\overline{W_1}(zM),\quad \phi_0^*(z,0)=\frac{1}{M^3}\overline{\phi}(zM), \tag{5.4.1}$$

where M is a positive constant to be determined, $\overline{\psi_1}$, $\overline{U_1}$, $\overline{W_1}$, $\overline{\phi}$ are smooth, periodic functions in y with period 1. Moreover, we assume $\overline{\psi_1}$, $\overline{U_1}$, $\overline{\phi}$ are odd functions in y. By (5.1.17), $\overline{W_1}=-\overline{\psi_1}_{zz}$, hence it is also a smooth, periodic, and odd function in y. In particular, $u_1^*(z,t)$, $\psi_1^*(z,t)$, $\omega_1^*(z,t)$ and $\phi_0^*(z,t)$ are periodic functions in z with period $\frac{1}{M}$ and odd in z within each period. Therefore, a priori estimates for the solutions to the $1D$ equations are modified from Lemma 5.2.1 as follows

$$\|\psi_1^*\|_{L^\infty}\le\frac{C_0}{M^2}, \tag{5.4.2}$$

$$\|\psi_{1z}^*\|_{L^\infty}\le\frac{C_0}{M},\quad \|u_1^*\|_{L^\infty}\le\frac{C_0}{M}, \tag{5.4.3}$$

$$\|\omega_1^*\|_{L^\infty}\le C_0,\quad \|u_{1z}^*\|_{L^\infty}\le C_0. \tag{5.4.4}$$

Let $R_0=M^{\frac{1}{4}}$, from the above inequalities (5.4.3), (5.4.4), we know

$$\|ru_1^*\|\le\frac{C}{\sqrt{M}},\quad \|\nabla(ru_1^*)\|\le C\sqrt{M},\quad \|r\psi_{1z}^*\|\le\frac{C}{\sqrt{M}}. \tag{5.4.5}$$

As long as $\eta>\eta_0>1$, the right hand side of (5.2.15) can be refined as

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\phi_0^*\|^2+\frac{\gamma}{\eta^2}\int_0^1(\phi_0^*)^4dz+\gamma\|\phi_{0z}^*\|^2\\
=\;&-2\int_0^1\psi_1^*\phi_{0z}^*\phi_0^*dz+\frac{\gamma}{\eta^2}\|\phi_0^*\|^2\\
\le\;&\left(1-\frac{1}{\eta_0^2}\right)\gamma\|\phi_{0z}^*\|^2+\left(\frac{C(\eta_0)}{\gamma M^2}+\frac{\gamma}{\eta^2}\right)\|\phi_0^*\|^2\\
\le\;&\left(\gamma-\frac{\gamma}{\eta_0^2}+\frac{C}{\gamma M^2}+\frac{\gamma}{\eta^2}\right)\|\phi_{0z}^*\|^2.
\end{aligned} \tag{5.4.6}$$

Also, the right hand side of (5.2.16) is refined as

$$\begin{aligned}
&\|\phi_{0t}^*\|^2+\frac{d}{dt}\left[\frac{\gamma}{2}\|\phi_{0z}^*\|^2+\frac{1}{4}\int_0^1(\phi_0^*)^4dz-\frac{\gamma}{2\eta^2}\|\phi_0^*\|^2\right]\\
=\;&-2\int_0^1\psi_1^*\phi_{0z}^*\phi_{0t}^*dz\\
\le\;&\|\phi_{0t}^*\|^2+\frac{C}{M^4}\|\phi_{0z}^*\|^2.
\end{aligned} \tag{5.4.7}$$

Multiplying (5.4.6) by $\frac{\gamma}{\eta^2}$, then adding the resultant with (5.4.7), it infers that

$$\frac{d}{dt}\left[\frac{\gamma}{2}\|\phi_{0z}^*\|^2+\frac{1}{4}\int_0^1(\phi_0^*)^4dz\right]$$

$$\leq -\left[\frac{\gamma^2}{\eta^2\eta_0^2} - \frac{\gamma^2}{\eta^4} - \frac{C}{\eta^2 M^2} - \frac{C}{M^4}\right] \|\phi_{0z}^*\|^2 . \tag{5.4.8}$$

Since $\eta > \eta_0 > 1$, if M is chosen large enough, it follows that

$$\frac{d}{dt}\left[\|\phi_{0z}^*\|^2 + \int_0^1 (\phi_0^*)^4 dz\right] + \|\phi_{0z}^*\|^2 \leq 0. \tag{5.4.9}$$

Hence we have the uniform bound

$$\|\phi_0^*\|_{H^1[0,1]}(t) \leq \|\phi_0^*\|_{H^1[0,1]}(0) \leq \frac{C}{M^2}. \tag{5.4.10}$$

Similarly, one can derive the uniform bound of ϕ_{0z}^* in H^1 norm,

$$\|\phi_{0zz}^*\|_{[0,1]}^2(t) \leq \|\phi_{0zz}^*\|_{[0,1]}^2(0) \leq \frac{C}{M}. \tag{5.4.11}$$

From (5.4.10), (5.4.11) and Morrey's inequality, the uniform L^∞ bounds for ϕ_0^* are

$$\|\phi_0^*\|_{L^\infty[0,1]} \leq \|\phi_0^*\|_{H^1[0,1]}(t) \leq \frac{C}{M^2}, \qquad \|\phi_{0z}^*\|_{L^\infty[0,1]} \leq \|\phi_0^*\|_{H^2[0,1]}(t) \leq \frac{C}{M}. \tag{5.4.12}$$

On the other hand, we assume the initial conditions of the $3D$ velocity vector $\tilde{\mathbf{u}}$, and the phase function ϕ as

$$\|\tilde{\mathbf{u}}(0)\|^2 + \lambda\|\nabla\phi(0)\|^2 + \frac{\lambda}{2\eta^2}\|\phi(0)^2 - 1\|^2 \leq \frac{C}{\sqrt{M}}. \tag{5.4.13}$$

From the basic energy law (5.1.4),

$$\|\tilde{\mathbf{u}}(t)\| \leq \frac{C}{\sqrt{M}}, \quad \|\nabla\phi(t)\| \leq \frac{C}{\sqrt{M}}. \tag{5.4.14}$$

By (5.4.5), (5.4.10) and (5.4.14), we get a priori bounds for the perturbed velocity and the phase function in L^2 norm :

$$\|ru_1\| \leq \frac{C}{\sqrt{M}}, \quad \|v^r\| \leq \frac{C}{\sqrt{M}}, \quad \|v^z\| \leq \frac{C}{\sqrt{M}}, \quad \|\nabla\phi_1\| \leq \frac{C}{\sqrt{M}}. \tag{5.4.15}$$

Now we give a sketch of proof for Theorem 5.1.2.

Proof. Under these conditions (5.4.2)-(5.4.4), (5.4.10)-(5.4.15), we shall refine all estimates in the proof of Theorem 5.1.1 (c.f. [62] for details). By adding all the estimates about u_1, ψ_1 and ϕ_1 equations, we find except for dissipation terms, all other terms are bounded by

$$\frac{C}{M^\beta}E^2 + \frac{1}{M^\beta}g(H),$$

where $g(H)$ is a polynomial of H with positive exponents and coefficients, and β denotes a positive constant. Choose M large enough, then

$$\frac{d}{dt}H^2 \leq -\frac{\mu}{2}E^2 + \frac{1}{M^\beta}g(H) \leq -\frac{\mu}{2}H^2 + \frac{1}{M^\beta}g(H) \tag{5.4.16}$$

since $H \leq E$, and here $\mu = min(\nu, \gamma)$.

We can choose M large enough such that

$$-\frac{\mu}{2} + \frac{1}{M^{\beta}} g(1) \leq 0. \tag{5.4.17}$$

Therefore, if the initial conditions are small so that $H(0) \leq 1$, we will get the uniform bound of $H(t)$, such that

$$H(t) \leq 1, \quad for\ all \;\; t > 0. \tag{5.4.18}$$

which indicates (5.3.12) holds, hence the proof is complete. □

Appendix A

Detailed calculations for energetic variational approaches

Here we provide some detailed computations in the previous sections.

A.1 Least action principle

The action functional takes the form

$$\mathbb{A}(x) = \int_0^T \int_{\Omega_0} \left[\frac{1}{2}|x_t(X,t)|^2 - \left(\frac{1}{2}|\mathbb{F}^{-T}\nabla_X \mathbb{E} d_0(X)|^2 + \mathcal{F}(\mathbb{E}d_0(X)) \right) \right] J\, dX dt, \qquad \text{(A.1.1)}$$

where $\Omega_0 = Q$ is the original domain occupied by the material, $\mathbb{E}$ is the deformation tensor satisfying (2.2.1) and the Jacobian $J = \det\mathbb{F} = 1$. The above expression includes all the kinematic transport property of the molecular director d. With different kinematic transport relations, we will obtain different action functionals, even though the energies may have the same expression in the Eulerian coordinate.

We take any one-parameter family of volume preserving flow map $x^\epsilon(X,t)$ with $x^0 = x$, $\frac{dx^\epsilon}{d\epsilon}\big|_{\epsilon=0} = y$ and the volume-preserving constraint $\nabla_x \cdot y = 0$ (or $J^\epsilon = \det\mathbb{F}^\epsilon = 1$). Apply the least action principle $\delta\mathbb{A} = 0$, we have $\frac{d\mathbb{A}(x^\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = 0$ such that

$$\begin{aligned} 0 \;=\; & \int_0^T \int_{\Omega_0} x_t \cdot y_t dX dt - \int_0^T \int_{\Omega_0} \left(\mathbb{F}^{-T}\nabla_X \mathbb{E} d_0\right) : \left[\frac{d}{d\epsilon}\Big|_{\epsilon=0} (\nabla_{x^\epsilon} d(x^\epsilon, t)) \right] dX dt \\ & - \int_0^T \int_{\Omega_0} f(\mathbb{E}d_0) \cdot \left(\frac{d\mathbb{E}^\epsilon}{d\epsilon}\Big|_{\epsilon=0} d_0 \right) dX dt := I_1 + I_2 + I_3, \end{aligned} \qquad \text{(A.1.2)}$$

where $\mathbb{E}^\epsilon = \mathbb{E}(x^\epsilon(X,t),t)$.

Pushing forward to the Eulerian coordinate, we have

$$I_1 = -\int_0^T \int_{\Omega_0} x_{tt} \cdot y dX dt = -\int_0^T \int_{\Omega_t} \dot{v} \cdot y dx dt = -\int_0^T \int_{\Omega_t} (v_t + v \cdot \nabla v) \cdot y dx dt, \qquad \text{(A.1.3)}$$

where Ω_t is the domain occupied by the material at time t. From the definition of $\mathbb{E}^\epsilon$, we have

$$\frac{d\mathbb{E}^\epsilon}{d\epsilon}\bigg|_{\epsilon=0} d_0 = \left(\frac{1}{2}(\nabla y - \nabla^T y) - \frac{\lambda_2}{2\lambda_1}(\nabla y + \nabla^T y)\right)\mathbb{E}d_0, \tag{A.1.4}$$

which implies that

$$\begin{aligned} I_2 &= -\int_0^T\int_{\Omega_0}\left(\mathbb{F}^{-T}\nabla_X\mathbb{E}d_0\right) : \left[\frac{d(\mathbb{F}^\epsilon)^{-T}}{d\epsilon}\bigg|_{\epsilon=0}\nabla_X\mathbb{E}d_0 + \mathbb{F}^{-T}\nabla_X\left(\frac{d\mathbb{E}^\epsilon}{d\epsilon}\bigg|_{\epsilon=0} d_0\right)\right] dXdt \\ &= -\int_0^T\int_{\Omega_t}\nabla d : \left\{-\nabla^T y\nabla d + \nabla\left[\left(\frac{1}{2}(\nabla y - \nabla^T y) - \frac{\lambda_2}{2\lambda_1}(\nabla y + \nabla^T y)\right) d\right]\right\} dxdt \\ &= -\int_0^T\int_{\Omega_t}[\nabla\cdot(\nabla d\odot\nabla d)]\cdot y dxdt + \frac{1}{2}\left(1-\frac{\lambda_2}{\lambda_1}\right)\int_0^T\int_{\Omega_t}[\nabla\cdot(\Delta d\otimes d)]\cdot y dxdt \\ &\quad -\frac{1}{2}\left(1+\frac{\lambda_2}{\lambda_1}\right)\int_0^T\int_{\Omega_t}[\nabla\cdot(d\otimes\Delta d)]\cdot y dxdt, \end{aligned} \tag{A.1.5}$$

$$\begin{aligned} I_3 &= -\int_0^T\int_{\Omega_0} f(d)\cdot\left[\left(\frac{1}{2}(\nabla y - \nabla^T y) - \frac{\lambda_2}{2\lambda_1}(\nabla y + \nabla^T y)\right) d\right] dXdt \\ &= \int_0^T\int_{\Omega_t}\left[-\frac{1}{2}\left(1-\frac{\lambda_2}{\lambda_1}\right)\nabla\cdot(f(d)\otimes d) + \frac{1}{2}\left(1+\frac{\lambda_2}{\lambda_1}\right)\nabla\cdot(d\otimes f(d))\right]\cdot y\ dxdt. \end{aligned} \tag{A.1.6}$$

Inserting (A.1.3), (A.1.5) and (A.1.6) into (A.1.2), we arrive at

$$\int_0^T\int_{\Omega_t}(v_t + v\cdot\nabla v + \nabla\cdot(\nabla d\odot\nabla d) - \nabla\cdot\tilde{\sigma})\cdot y dxdt = 0, \tag{A.1.7}$$

where

$$\tilde{\sigma} = -\frac{1}{2}\left(1-\frac{\lambda_2}{\lambda_1}\right)(\Delta d - f(d))\otimes d + \frac{1}{2}\left(1+\frac{\lambda_2}{\lambda_1}\right) d\otimes(\Delta d - f(d)). \tag{A.1.8}$$

Since y is an arbitrary divergence free vector field, integration by parts we formally derive the momentum equation (Hamiltonian/conservative part)

$$v_t + v\cdot\nabla v + \nabla P = -\nabla\cdot(\nabla d\odot\nabla d) + \nabla\cdot\tilde{\sigma}, \tag{A.1.9}$$

where the pressure P serves as a Lagrangian multiplier for the incompressibility of the fluid.

A.2 Maximum dissipation principle

Using the transport equation of d (2.2.4), we can manipulate the dissipation (2.2.5) in terms of a rate in time

$$\mathcal{D} = \int_Q \mu_1|A_{kp}d_kd_p|^2dx + \frac{1}{2}\int_Q\mu_4|\nabla v|^2dx - \lambda_1\int_Q\left|d_t + v\nabla d - \Omega\, d + \frac{\lambda_2}{\lambda_1}A\, d\right|^2dx$$

$$+\Big(\mu_5+\mu_6+\frac{(\lambda_2)^2}{\lambda_1}\Big)\int_Q |Ad|^2 dx. \tag{A.2.1}$$

Let $v^\epsilon = v+\epsilon u$, where u is an arbitrary regular function with $\nabla\cdot u = 0$. Set $\frac{\delta(\frac{1}{2}\mathcal{D})}{\delta v} = 0$. Then we have

$$\begin{aligned}
0 &= \frac{1}{2}\frac{d\mathcal{D}(v^\epsilon)}{d\epsilon}\Big|_{\epsilon=0}\\
&= \frac{\mu_4}{2}\int_Q \nabla v:\nabla u dx + \mu_1\int_Q d_k A_{kp} d_p d_i \frac{\nabla_i u_j+\nabla_j u_i}{2} d_j dx\\
&\quad -\lambda_1\int_Q \Big(d_t + v\nabla d - \Omega\, d + \frac{\lambda_2}{\lambda_1} A\, d\Big)\cdot\Big(u\cdot\nabla d - \frac{\nabla u - \nabla^T u}{2} d + \frac{\lambda_2}{\lambda_1}\frac{\nabla u+\nabla^T u}{2} d\Big)dx\\
&\quad +\Big(\mu_5+\mu_6+\frac{(\lambda_2)^2}{\lambda_1}\Big)\int_Q A_{ij} d_j \frac{\nabla_i u_k + \nabla_k u_i}{2} d_k dx,\\
&:= I_1+I_2+I_3+I_4.
\end{aligned} \tag{A.2.2}$$

Integration by parts, we get

$$I_1 = -\frac{\mu_4}{2}(\Delta v, u), \tag{A.2.3}$$

$$I_2 = -\mu_1\left(\nabla\cdot\left[d^T A d(d\otimes d)\right], u\right), \tag{A.2.4}$$

$$\begin{aligned}
I_4 &= -\frac{1}{2}\left(\mu_5+\mu_6+\frac{(\lambda_2)^2}{\lambda_1}\right)\int_Q \Big(u_k\nabla_i(d_k A_{ij} d_j) + u_i\nabla_k(A_{ij} d_j d_k)\Big)dx,\\
&= -\frac{1}{2}\left(\mu_5+\mu_6+\frac{(\lambda_2)^2}{\lambda_1}\right)\left[\big(\nabla\cdot(d\otimes Ad)\,,\, u\big) + \big(\nabla\cdot(Ad\otimes d)\,,\, u\big)\right].
\end{aligned} \tag{A.2.5}$$

Using the transport equation (2.2.4) of d and the incompressibility of u, we infer that

$$\begin{aligned}
I_3 &= \left(\Delta d - f(d), u\cdot\nabla d - \frac{1}{2}\Big(1-\frac{\lambda_2}{\lambda_1}\Big)\nabla u d + \frac{1}{2}\Big(1+\frac{\lambda_2}{\lambda_1}\Big)\nabla^T u d\right)\\
&= \Big(u, -\nabla F(d) + \nabla\cdot(\nabla d\odot\nabla d) - \frac{1}{2}\nabla(|\nabla d|^2)\Big)\\
&\quad +\Big(1-\frac{\lambda_2}{\lambda_1}\Big)\Big(u,\nabla\cdot[(\Delta d - f(d))\otimes d]\Big) - \Big(1+\frac{\lambda_2}{\lambda_1}\Big)\Big(u,\nabla\cdot[d\otimes(\Delta d - f(d))]\Big)\\
&= \big(u,\nabla\cdot(\nabla d\odot\nabla d)\big) - \mu_2\big(u,\nabla\cdot(N\otimes d)\big) - \mu_3\big(u,\nabla\cdot(d\otimes N)\big)\\
&\quad -\eta_5\big(u,\nabla\cdot(A\, d\otimes d)\big) - \eta_6\big(u,\nabla\cdot(d\otimes A\, d)\big),
\end{aligned} \tag{A.2.6}$$

with the coefficients

$$\mu_2 = \frac{1}{2}(\lambda_1-\lambda_2),\quad \mu_3 = -\frac{1}{2}(\lambda_1+\lambda_2),\quad \eta_5 = \frac{\lambda_2}{2} - \frac{(\lambda_2)^2}{2\lambda_1},\quad \eta_6 = -\frac{\lambda_2}{2} - \frac{(\lambda_2)^2}{2\lambda_1}. \tag{A.2.7}$$

It follows from (A.2.3)–(A.2.7) that

$$0 = \frac{1}{2}\frac{d\mathcal{D}}{d\epsilon}\Big|_{\epsilon=0} = \big(u,\nabla\cdot(\nabla d\odot\nabla d)\big) - (u,\nabla\cdot\sigma). \tag{A.2.8}$$

The stress tensor σ is given by

$$\sigma = \mu_1(d^T Ad)d \otimes d + \mu_2 N \otimes d + \mu_3 d \otimes N + \mu_4 A + \tilde{\mu}_5 Ad \otimes d + \tilde{\mu}_6 d \otimes Ad,$$

with constants

$$\mu_2 = \frac{1}{2}(\lambda_1 - \lambda_2),\ \mu_3 = -\frac{1}{2}(\lambda_1 + \lambda_2),\quad \tilde{\mu}_5 = \frac{1}{2}(\lambda_2 + \mu_5 + \mu_6),\quad \tilde{\mu}_6 = \frac{1}{2}(-\lambda_2 + \mu_5 + \mu_6).$$

Since u is an arbitrary function with $\nabla \cdot u = 0$, we arrive at the dissipative force balance equation

$$\nabla P = -\nabla \cdot (\nabla d \odot \nabla d) + \nabla \cdot \sigma, \tag{A.2.9}$$

where the pressure P serves as a Lagrangian multiplier for the incompressibility of the fluid.

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