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ON GEODESICS OF COMPACT RIEMANNIAN SURFACES

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Abstract

This dissertation is divided into two parts. In part one we deal with the $1/k$ length spectrum of a compact Riemannian manifold. The $1/k$ spectrum was introduced by C. Sormani and has many relations with other geometrical objects. We will show that there exists a class of manifolds with empty $1/k$ length spectrum. In part two we work on the security of a manifold. A compact Riemannian manifold is said to be uniformly secure if there is a number $n \in \mathbb{N}$ such that for any two points the set of geodesics connecting them can be blocked by n point obstacles. A general conjecture is that uniform security implies flatness. We will prove this conjecture for non-simply connected, orientable, two dimensional Riemannian manifolds.

Table of Contents

List of Figures

List of Symbols

$\text{Length}(\gamma)$	The length of a piecewise smooth curve γ, p. 1
$d_g(x,y)$	The geodesic distance between x and y, p. 1
$\text{diam}(M)$	The diameter of M, p. 2
$L(M)$	The length spectrum of M, p. 6
$U_r(A)$	The r-neighborhood of the set A, p. 9
$d_{GH}(X,Y)$	The Gromov-Hausdorff distance between X and Y, p. 9
$L_{1/k}(M)$	The $1/k$ length spectrum of M, p. 11
$\text{Vol}(M)$	The volume of M, p. 15
$\text{sys}(M)$	The systole of M, p. 15
$T_\alpha(t)$	The total rotation of γ_α, p. 22
$T(\tilde{\alpha})$	The first return rotation of γ_α, p. 24
$G(x,y)$	The set of geodesics joining x and y, p. 32
$\Gamma(x,y)$	The set of geodesics connecting x and y, p. 32
$B(x,y)$	The blocking number of the configuration (x,y), p. 32
$B(M)$	The blocking number of M, p. 33
$h_{top}(\phi)$	The topological entropy of the flow ϕ, p. 37
$n_T(x,y)$	The cardinality of the set the geodesic segments joining x and y with length $\leq T$, p. 38

$\mathcal{M}(H)$	The set of all minimal trajectories with respect to H, p. 44
$T_{(a,b)}(x)$	The action that translates the point x by $(a,b) \in \mathbb{Z}^2$, p. 45
$\mathcal{M}_\alpha$	The set of minimal trajectories with rotation number α, p. 47
$\mathcal{M}_\alpha^{per}$	The set of periodic minimal trajectories with rotation number α, p. 47
$\alpha(\gamma)$	The rotation number of γ, p. 47

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Chapter 1

Basic Concepts in Riemannian Geometry

In this beginning chapter we introduce some basic concepts and notions in Riemannian geometry that will be used throughout the dissertation. Most of the notions can be found in a differential geometry book.

Let M be a compact Riemannian manifold. For any point $x \in M$, we write T_xM to indicate the tangent space to M at x and TM denotes the tangent bundle of M. We will denote by g the Riemannian metric on M: for any point $x \in M$, $g|_x$ is a scalar product on T_xM which depends C^∞ on x. Throughout the paper, we will assume that the Riemannian manifold M in question is connected and complete.

Given a continuous piecewise smooth curve $\gamma : [a, b] \to M$, denote by $\text{Length}(\gamma)$ its length, which is given by

$$\text{Length}(\gamma) = \int_a^b \|\gamma'(s)\| ds \tag{1.1}$$

By the chain rule, this length does not depend on the parametrization of γ. However for simplicity we will assume that all curves are parameterized by arc length.

We can make M into a metric space by defining a metric $d_g(\cdot, \cdot)$ on M:

$$d_g(x, y) = \inf\{\text{Length}(\gamma)|\gamma : [a, b] \to M \text{ is continuous}, \gamma(a) = x, \gamma(b) = y\}$$

Our first definition is the diameter of a manifold.

Definition 1.0.1. *The diameter* $\operatorname{diam}(M)$ *of* M *is the supremum of* $d_g(x, y)$ *for* $x, y \in M$.

Given two vector fields V and W, denote by $\nabla_V W$ the covariant derivative of V in the direction of W, then we define a geodesic as follows

Definition 1.0.2. *A smooth curve* γ *in* M *is a geodesic if it satisfies*

$$\nabla_{\gamma'}\gamma' \equiv 0 \tag{1.2}$$

A fundamental property of a geodesic is that given a point $x \in M$ and a vector $v \in T_xM$, there exists a unique geodesic γ_v through x whose tangent vector at x is v. This follows from the fact that the geodesic equation $\nabla_{\gamma'}\gamma' \equiv 0$ is a second order ordinary differential equation, x and v are exactly the initial conditions needed to obtain the uniqueness and existence of the solution γ_v. The exponential map at p $\exp_x : T_xM \to M$ is then defined by $\exp_x(v) = \gamma_v(1)$, and we define the union of these maps over x of the domains of $\exp_x$ to be $\exp : TM \to M$.

Two closed curves γ_0 and γ_1 are said to be *freely homotopic* if there exists $h : \mathbb{S}^1 \to \times[0, 1]$ such that

$$h(\mathbb{S}^1 \times \{0\}) = \gamma_0, \quad h(\mathbb{S}^1 \times \{1\}) = \gamma_1$$

The free homotopy relation divides closed curves into different equivalence classes.

Definition 1.0.3. *A closed geodesic in* M *is a differentiable closed curve such that its lift to the universal cover is a geodesic.*

Here is an elementary result about non-contractible geodesics.

Theorem 1.0.4. *Let* $[\alpha]$ *be a non-trivial free homotopy class in a compact Riemannian manifold* M*, then there exists a closed geodesic* γ_0 *in the* $[\alpha]$*, and the following holds:*

$$\operatorname{Length}(\gamma_0) = \inf\{\operatorname{Length}(\gamma) | \gamma \in [\alpha]\} \tag{1.3}$$

Now let γ be a geodesic in M, a vector field along γ is said to be a *Jacobi field* if it satisfies the Jacobi equation

$$\nabla_{\gamma'}\nabla_{\gamma'}V = R(\gamma', V)\gamma' \tag{1.4}$$

where $R(\cdot,\cdot)\cdot$ is the curvature tensor of M.

Suppose v, w are two vectors that span a plane P in T_xM, we define the sectional curvature $K(P)$ to be

$$K(P) = \frac{<R(v,w)w,v>}{\|v\wedge w\|^2}$$

In the entire paper, the sectional curvature is simply referred to as the curvature. We will say that the curvature is bigger than a constant c if for all plane sections at all points of M the sectional curvature is bigger than this constant c. A manifold is said to be *flat* if its sectional curvatures are identically zero.

The *Ricci curvature* of M is a symmetric bilinear form on each T_xM defined to be the trace of the linear transformation $z \mapsto R(z, x)y$.

Suppose x and y are two points on a geodesic γ. y is *conjugate* to x along a geodesic γ if there exists a non-trivial Jacobi field along γ that vanishes at x and y.

The next concept is the *cut locus* and the *injectivity radius.* The cut locus of x in the tangent space is defined to be the set of all vectors v in T_xM such that $\gamma_v(t) = exp_x(tv)$ is a minimizing geodesic for $t \in [0, 1]$ but is not minimizing for $t \in [0, 1+\epsilon)$ for any $\epsilon > 0$. The cut locus of x in M, denoted by $Cut(x)$, is defined to be image of the cut locus of x in the tangent space under the exponential map at x. In other words, the cut locus of x in M as the points in the manifold where the geodesics beginning at x are no longer minimizing.

Definition 1.0.5. *The injectivity radius of M is defined to be*

$$\text{Inj}(M) = \inf_{x\in M} d_g(x, Cut(x))$$

We can also say that it is the largest radius for which the exponential map at x is a diffeomorphism.

Finally we define an *isometry* between two metric spaces as follows

Definition 1.0.6. *A map $f : M \to N$ between two metric spaces M and N is said to be an isometry if*

$$d_M(x, y) = d_N(f(x), f(y)) \text{ for all } x, y \in M.$$

Chapter 2

Part I: The Spectrum of A Riemannian Manifold

2.1 Introduction

The study of the *spectrum* of a manifold is one of the key areas in Riemannian geometry. This study was motivated by questions in mathematical physics. It deals with eigenvalue problems for the Laplace-Beltrami operator acting on the manifold. The set of eigenvalues with multiplicities of the Laplace-Beltrami operator acting on the manifold is called the *spectrum* of a manifold. Two manifolds are said to be isospectral if they have the same spectrum.

Another important spectrum of a Riemannian manifold is called the *length spectrum*. It is the set of lengths of smooth closed geodesics, counted with multiplicities. The multiplicity of a given length is the number of distinct free homotopy classes of geodesics that contain a closed geodesic of that length. There is also another different notation, which is called the *marked length spectrum*. This consists of the lengths of closed geodesics freely homotopic to a representative of each element in the fundamental group.

These spectra and their relations with the geometry of compact manifolds have been studied extensively. In this chapter we will briefly review some previous results.

2.2 Definitions and Some key results

Let us begin by defining the length spectrum of M.

Definition 2.2.1. *The length spectrum of a compact Riemannian manifold M, denoted by $L(M)$, is the set of lengths of closed geodesics in M, the multiplicity is counted by the number of distinct free homotopy classes that contain a closed geodesic of that length.*

The next spectrum is the marked length spectrum.

Definition 2.2.2. *The marked length spectrum of M is a function such that each free homotopy class $[\alpha]$ is assigned the set of lengths of closed geodesics freely homotopic to $[\alpha]$.*

With this definition, we say that two Riemannian manifolds M and M' (endowed with metrics g and g' respectively) have the same marked length spectrum if there exists an isomorphism between the fundamental groups of M and M' such that the corresponding free homotopy classes contain closed geodesics of the same length. It is clear that two manifolds with the same marked spectrum have the same length spectrum.

The relationship between the length spectrum and the Laplace-Beltrami spectrum arises from the study of the wave equation. A result of Y. Colin de Verdiere states that generically, the Laplace spectrum determines the length spectrum.

Theorem 2.2.3 ([CdV])**.** *If M is a given Riemannian manifold, there exists a generic subset G_M, in the sense of Baire category, of the set of smooth Riemannian metrics on M, such that if $g \in G_M$, the length spectrum of (M, g) can be recovered from the Laplace spectrum. The set G_M contains all metric with sectional curvature less than zero.*

In general the length spectrum does not determine the Laplace spectrum, with the exception of Riemann surfaces. A Riemann surface is a compact orientable surface with constant curvature -1. H. Huber showed that the two spectra are indeed equivalent notions.

Theorem 2.2.4 ([Hu])**.** *If M is a Riemann surface, the Laplace spectrum determines the length spectrum and vice versa.*

There are also many other results that describe the interactions between the Laplace spectrum and the length spectrum. For instance, for flat tori, having the same length spectrum is equivalent to having the same Laplace spectrum. It is also noteworthy that all known examples of isospectral manifolds have the same lengths of closed geodesics.

The marked length spectrum, on the other hand, provide more information about the geometry than the length spectrum. Firstly, if M and N are manifolds that have identical marked length spectra, have dimensions greater than two, and N is of constant negative curvature, then M and N are isometric [BCG]. Later on C. Croke and Otal showed independently that if two compact surfaces of negative curvature have the same marked length spectrum, then they are isometric. However on the other hand, the Zoll surfaces and the standard sphere have the same marked length spectrum, they are not isometric.

Chapter 3

The 1/k Length Spectrum

3.1 Introduction

In the paper [Sor], Sormani studied a totally different length spectrum of a compact Riemannian manifold, it is called the $1/k$ length spectrum. As we shall see later, that unlike the regular length spectrum, the $1/k$ length spectrum persists under the Gromov-Hausdorff convergence. We shall also see that many other geometric results and estimates can be obtained for this special type of length spectrum.

Throughout the whole chapter, we will assume that our Riemannian manifolds are compact and complete, we also assume that all geodesics are parameterized by arc length.

3.2 Gromov-Hausdorff distance

The main feature of the $1/k$ length spectrum is that it persists under the Gromov-Hausdorff convergence. Before we introduce the length spectrum and discuss this feature, let us first provide all backgrounds on the Gromov-Hausdorff distance (and convergence). Loosely speaking, the Gromov-Hausdorff distance measures how close two given compact metric spaces are being isometric. In order to explain what the Gromov-Hausdorff distance is, we need to start with another classical concept: the Hausdorff distance between subsets of a metric space. Let M be a metric space endowed with a metric $d(\cdot,\cdot)$, and suppose that A is a subset of M, we denote by $U_r(A)$ the tubular neighborhood of A with radius r, i.e.,

$$U_r(A) = \{x \in M | \exists a \in A \text{ such that } d(x,a) < r\} \tag{3.1}$$

Definition 3.2.1. *Let M be a metric space, given two subsets A and B of M, we define the Hausdorff distance between A and B to be:*

$$d_H^M(A,B) = \inf\{r | A \subseteq U_r(B) \text{ and } B \subseteq U_r(A)\} \tag{3.2}$$

Let us now define the Gromov-Hausdorff distance between two compact metric spaces, but first of all, let us recall that a map $f : X \to M$ between metric spaces (X, d_X) and (M, d_M) is said to be an isometric embedding if f is an embedding and $d_X(x,y) = d_M(f(x), f(y))$ for all $x, y \in X$.

Definition 3.2.2. *The Gromov-Hausdorff distance between two compact metric spaces X and Y are defined as follows:*

$$d_{GH}(X,Y) = \inf\{d_H^M(f(X), g(Y)) | f : X \to M, g : Y \to M\} \tag{3.3}$$

Where the infimum runs through all metric spaces M, and all isometric embeddings f and g.

It is not immediately clear why the Gromov-Hausdorff distance is preferred over the Hausdorff distance. However for instance suppose one wants to compare two compact surfaces in $\mathbb{R}^3$, endowed with the intrinsic metrics. Even if both surfaces can be embedded in $\mathbb{R}^3$ at all, comparing these two metrics in $\mathbb{R}^3$ may not show the closeness between these two intrinsic metrics.

It is easy to check that d_{GH} is indeed a metric. It also worths noting that the Gromov-Hausdorff distance between spaces are always finite.

With this notation, it is natural to define the Gromov-Hausdorff convergence for metric spaces: a sequence of compact metric spaces $\{X_n\}_{n=1}^{\infty}$ is said to converge to a metric space X in the Gromov-Hausdorff sense if $d_{GH}(X_n, X) \to 0$. X is called the Gromov-Hausdorff limit of the sequence.

Firstly, it is not hard to see that if $d_{GH}(X,Y) = 0$ then the metric spaces X and Y are isometric. Let us also remark that the space of all compact metric spaces, endowed with the metric d_{GH}, is Hausdorff and complete. This resembles the space of compact subsets of $\mathbb{R}^n$.

One advantage that the Gromov-Hausdorff convergence has over the C^k convergence is that the former can compare a larger class of manifolds. As for a sequence of manifolds M_i to converge to M in the C^k sense, the manifolds have to be diffeomorphic to M. However for the Gromov-Hausdorff convergence, the manifolds do not even necessarily have the same dimensions. It also worths mentioning that C^k convergence implies Gromov-Hausdorff convergence.

One of the most celebrated results regarding the Gromov-Hausdorff convergence is *Gromov's compactness theorem*, which is a generalization of the Myer's theorem, the compactness theorem states the following:

Theorem 3.2.3 (Gromov's Compactness Theorem)**.** *The subset of Riemannian manifolds of a given dimension n, with diameter $\leq D$ and the Ricci curvature $\geq -(n-1)$ is precompact.*

Let us now see a couple examples of Gromov-Hausdorff convergence.

Example 1: Let x, y and z be the standard coordinates of $\mathbb{R}^3$ and let c be a real constant. If we let X_c be the surface in $\mathbb{R}^3$ given by the following equation

$$x^2 + y^2 + (z/c)^2 = 1 \tag{3.4}$$

If we let $c = c_i \to \infty$, then X_{c_i} converge in the Gromov-Hausdorff sense to a double disc X, i.e. a singular surface which is two flat discs of radius 1 glued together along their boundaries.

Example 2 If we let X_r be the torus in $\mathbb{R}^3$

$$(1 - \sqrt{x^2 + y^2})^2 + z^2 = r^2 \tag{3.5}$$

Then as we let $r = r_j \to 0$, the sequence of tori X_{r_j} converges in the Gromov-Hausdorff sense to a circle of radius 1.

The convergence in Example 1 is said to be non-collapsing because the dimension of the Gromov-Hausdorff limit is the same as the converging surfaces. Example 2 is collapsing since the dimension of the limit is less than that of the converging surfaces.

3.3 1/k geodesics

We are now ready to introduce the $1/k$ spectrum. Simply put, the $1/k$ spectrum is the set of lengths of $1/k$ geodesics. Given a rectifiable curve γ in a manifold, its length is denoted by $\text{Length}(\gamma)$. According to Sormani, a closed geodesic γ is a $1/k$ *geodesic* if it is distance minimizing on every subsegment of length $\text{Length}(\gamma)/k$, to be precise:

Definition 3.3.1. *A closed geodesic* $\gamma : \mathbb{S}^1 \to M$ *with* $\text{Length}(\gamma) = L$*, is said to be a* $1/k$ *geodesic if the following holds:*

$$d_g(\gamma(t), \gamma(t + 2\pi/k)) = L/k \quad \forall t \in \mathbb{S}^1 \tag{3.6}$$

Definition 3.3.2. *The 1/k length spectrum of* M*, denoted by* $L_{1/k}(M)$*, is the set of lengths of* $1/k$ *geodesics in* M*.*

It is obvious that $L_{1/k}(M) \subset L(M)$, where $L(M)$ is the length spectrum of M. Indeed, the union of the $1/k$ spectra over all $k \in \mathbb{N}$ gives exactly the length spectrum:

Theorem 3.3.3 ([Sor], Theorem 3.1)**.** *Any closed geodesic is a* $1/k$ *geodesic for a sufficiently large number* k*. So*

$$\bigcup_{k=1}^{\infty} L_{1/k}(M) = L(M) \tag{3.7}$$

Since any subsegment of a minimizing geodesic segment is also minimizing, we can see that a $1/k$ geodesic is also a $1/(k+i)$ geodesic, $\forall i \in \mathbb{N}$, the following lemma then follows immediately:

Lemma 3.3.4. $L_{1/k}(M) \subset L_{1/(k+1)}(M), \forall k \geq 2.$

Following Sormani, we also define the minimizing index of a closed geodesic γ, denoted by $\text{minind}(\gamma)$, to be the smallest $k \in \mathbb{N}$ such that γ is a $1/k$ geodesic. With this notation we have the following lemma:

Lemma 3.3.5 ([Sor], Lemma 3.2)**.** *If the diameter satisfies* $diam(M) \leq D$*, then*

$$\text{minind}(\gamma) \geq \text{Length}(\gamma)/D \text{ and } L_{1/k}(M) \subset (0, Dk]$$

Let us also note that if $\gamma : \mathbb{S}^1 \to M$ is a $1/k$ geodesic, the $n-th$ iterate of γ, $\gamma_n : \mathbb{S}^1 \to M$ given by $\gamma_n(t) = \gamma(nt)$, is a $1/(kn)$ geodesic. We will now see an example of $1/k$ spectrum of a manifold.

Example 1: Suppose that M is the standard two dimensional sphere. All of the closed prime geodesics of M are exactly the great circles, each has length of 2π. Since a great circle is minimizing up to the antipodal point, these prime geodesics are $1/2$ geodesics, so $2\pi \in L_{1/2}(M)$. Now as the $k-th$ iterates of these prime geodesics are $1/2k$ geodesics, we have

$$2k\pi \in L_{1/2k}(M) \tag{3.8}$$

Therefore by Lemma 3.3.4,

$$\{2\pi, 4\pi, \ldots, 2k\pi\} \subset L_{1/(2k+1)}(M) \tag{3.9}$$

On the other hand, as $\text{diam}(M) = \pi$, by Lemma 3.3.5, we also have

$$L_{1/j}(M) \subset L(M) \cap (0, j\pi] \tag{3.10}$$

and we conclude that

$$L_{1/2k}(M) = L_{1/2k+1}(M) = \{2\pi, 4\pi, \ldots, 2k\pi\} \tag{3.11}$$

One important feature of the $1/k$ length spectrum which makes it different from the length spectrum is that it persists under the Gromov-Hausdorff convergence. We will now present an example which demonstrates the phenomenon of 'disappearing lengths'.

3.4 An example of disappearing lengths

Let M_j be a Riemannian surface isometric to a standard sphere with four disks of radius $1/j$ removed and attached two 2-handles of intrinsic diameter less than $4/j$. We will see that when $j \to \infty$, these M_j's converge to the standard sphere S in the Gromov-Hausdorff sense.

If we form a finite $100/j$ net of M_j such that the minimizing geodesics between any two points of the net do not enter the handles, we can consider the corresponding net on the surface S. Let us denote this net by N_j, then for each j, N_j isometrically embeds into M_j and S, such that the Hausdorff distances between these metric spaces satisfy:

$$d_H^{M_j}(N_j, M_j) \leq 100/j \tag{3.12}$$

and

$$d_H^S(N_j, S) \leq 100/j \tag{3.13}$$

Therefore by the triangle property of the Gromov-Hausdorff distance, we have

$$d_{GH}(M_j, S) \leq d_{GH}(M_j, N_j) + d_{GH}(N_j, S) \leq 200/j \tag{3.14}$$

and so as $j \to \infty$, M_j converges to S in the Gromov-Hausdorff sense.

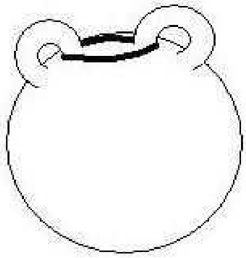
Without loss of generality we can assume that the two handles are say, $\pi/3$ apart. Now on each M_j consider a closed geodesic γ_j that passes through both handles, as in Figure 3.1. Let us set $L_j = \text{Length}(\gamma) \in L(M_j)$, the length spectrum of M_j. As $j \to \infty$, two handles shrink and disappear, the closed geodesics γ_j then converge to a geodesic segment. Therefore the lengths L_j for these geodesics γ_j would converge to $\pi/3$. This length is not in the length spectrum of the Gromov-Hausdorff limit S, as the shortest geodesic in the standard sphere S has length of 2π.

This example shows that the length spectrum does not necessarily persist under Gromov-Hausdorff convergence. However the 1/k length spectrum does, and that is the main advantage of the $1/k$ length spectrum. In fact, Sormani has established the following theorem:

Theorem 3.4.1 ([Sor],Theorem 7.1)**.** *If $M_i \to M$ in the Gromov-Hausdorff sense then $L_{1/k}(M_i)$ converges to a subset of $L_{1/k}(M) \cup \{0\}$ in the Hausdorff sense. That is, for all ϵ, $R > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that*

$$L_{1/k}(M_i)) \cap [0, R] \subset U_\epsilon(L_{1/k}(M) \cup \{0\}) \tag{3.15}$$

Figure 3.1. A geodesic through two handles



Where $U_\epsilon(X)$ is the ϵ-neighborhood of X.

When the manifold M is not simply connected, we can say a lot more about the $1/k$ length spectrum. Indeed, let us see that every non-simply connected manifold has a $1/2$ geodesic.

Suppose that M is not simply connected. Let γ be a shortest homotopically non-trivial closed curve in M. Then γ is a closed geodesic (Theorem 1.0.4). Let us show that γ is a $1/2$ geodesic. Denote the length of γ by l. Reasoning by contradiction, assume that there are two points p, q on γ that are $\frac{l}{2}$ apart along γ and that can be connected by a geodesic γ_1 that is shorter than $\frac{l}{2}$. The points p and q divide γ into two geodesics. Each of them can be closed up by adding γ_1. Hence we represented γ as a product of two loops, each of which is shorter than l. Since γ is homotopically non-trivial, so is at least one of these loops. This contradicts to our assumption that γ is a shortest homotopically non-trivial loop. Since a $1/2$ geodesic is also a $1/k$ geodesic for all $k \in \mathbb{N}$, so we know that every non-simply connected has at least a $1/k$ geodesic. Therefore we have proven the following proposition:

Proposition 3.4.2 (See also Lemma 4.1 of [Sor])**.** *The shortest non-contractible closed geodesic of a manifold is a $1/k$ geodesic, $\forall k \in \mathbb{N}$.*

Given this setting, Sormani proposed the following question: does that exist $k \in \mathbb{N}$, such that every smooth, compact, simply connected manifold has a $1/k$ -geodesic? In the next chapter we will address this question by showing that for each $k \in \mathbb{N}$, there is a manifold with empty $1/k$ length spectrum.

3.5 Relations with other geometric quantities

In the paper [Sor], Sormani has discussed the relationships of the $1/k$ spectrum and various geometric quantities. In this section we will briefly review some of them.

Systoles: Since the $1/k$ length spectrum of a manifold is about the lengths of closed geodesics. It would be natural to pair them up with other metric invariants, the first one is the systole of a manifold.

Definition 3.5.1. *The systole of a compact metric space M, denoted by* $\mathrm{sys}(M)$, *is the length of the shortest non-contractible closed geodesic.*

Since the shortest non-contractible closed geodesic is a $1/2$ geodesic, therefore $\mathrm{sys}(M) \in L_{1/2}(M)$ and in particular, $\min L_{1/2}(M) \leq \mathrm{sys}(M)$. With this estimate and the systolic inequalities, the upper bound for $\min L_{1/2}$ of different manifolds can be obtained. For example, the classical Loewner's inequality for two dimensional tori gives the following:

Theorem 3.5.2. *If M is a 2 dimensional torus, then*

$$(\min L_{1/2}(M))^2 \leq 2\mathrm{Vol}(M)/\sqrt{3} \tag{3.16}$$

Gap Theorems: Sormani also applied the convergence theorem 3.4.1 to prove the existence of gaps in the $1/k$ length spectrum of manifolds with Ricci curvature bounds. For intance, using Colding's sphere stability theorem:

Theorem 3.5.3. *[Co] Given $\epsilon > 0$, there exists $\delta = \delta(n, \epsilon) > 0$ such that if an n-dimensional manifold M satisfies $Ric(M) \geq n-1$ and $Vol(M) > Vol(S^n) - \delta$, then $d_{GH}(M, \mathbb{S}^n) < \epsilon$.*

Sormani obtained the following gap theorem:

Theorem 3.5.4. *([Sor][Theorem 1.1] There exists a function $\Psi : \mathbb{R}^+ \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ such that $\lim_{\delta \to 0} \Psi(\delta, k, n) = 0$ and if M is a compact manifold satisfying $Ric(M) \geq n-1$ and $Vol(M) > Vol(\mathbb{S}^n) - \delta$, then*

$$L_{1/(2k)}(M) \subset [0, \epsilon) \cup (2\pi - \epsilon, 2\pi + \epsilon) \cup \ldots \cup (2k\pi - \epsilon, 2k\pi + \epsilon) \tag{3.17}$$

for $\epsilon = \Psi(\delta, k, n)$

Using similar techniques, many rigidity theorems can be extended to prove the existence of gaps in their respective length spectra.

Chapter 4

Manifolds with empty 1/k spectrum

4.1 Introduction

In this chapter we will answer a question posed by Sormani: does that exist $k \in \mathbb{N}$, such that every smooth, compact, simply connected manifold has a $1/k$ geodesic? We will provide a negative answer to this question, by constructing a metric ρ_k on $\mathbb{S}^2$ for each $k \in \mathbb{N}$, such that $(\mathbb{S}^2, \rho_k)$ has no $1/k$ geodesic. That is, we will prove the following theorems.

Theorem 4.1.1 ([Ho], Theorem 1.1)**.** *There exist a metric ρ_2 on $\mathbb{S}^2$ such that $(\mathbb{S}^2, \rho_2)$ has no $1/2$ geodesic.*

Theorem 4.1.2 ([Ho], Theorem 1.1')**.** *For any fixed $k \in \mathbb{N}$, there exist a metric ρ_k on $\mathbb{S}^2$ such that $(\mathbb{S}^2, \rho_k)$ has no $1/k$ geodesic.*

Before we proceed with the proof of the theorems, we must state an important tool that will be used in the proof, it is the Clairaut integral for surfaces of revolution.

4.2 The Clairaut Integral

The Clairaut integral is a relation which governs the behavior of geodesics on a surface of revolution. Let M be a surface obtained by rotating a regular plane curve C about the z-axis, such a surface is called a surface of revolution and it can be parameterized as follows:

$$x = f(v)\cos u,\ y = f(v)\sin u,\ z = g(v) \tag{4.1}$$

The curve C is called the *generating curve* or the *profile curve* of the surface. The z-axis is called the *axis of revolution*, and the circle described by the points of C are called the *parallels* of the surface.

The geodesic behaviors of a surface of revolution is totally controlled by the Clairaut Integral, this relation is discussed in many differential geometry books. For completeness, we will include the derivation of the relation here.

First of all, let us recall that if $\gamma : I \to M$ a geodesic given by the parametrization $\mathbf{x}(u, v)$, then the following differential equation holds:

$$\begin{aligned} u'' + \Gamma^1_{11}(u')^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}(v')^2 = 0, \\ v'' + \Gamma^2_{11}(u')^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}(v')^2 = 0, \end{aligned} \tag{4.2}$$

Where Γ^k_{ij} are the Christoffel symbols of the surface. Now let us consider the surface of revolution in 4.1, and we will make the computation explicitly. The coefficients of the first fundamental form of M in the parametrization 4.1 are easily computed:

$$E = f^2,\ F = 0,\ G = (f')^2 + (g')^2 \tag{4.3}$$

Now the corresponding partial derivatives of the coefficients are

$$E_u = 0, \qquad E_v = 2ff',$$
$$F_u = F_v = 0, \qquad G_u = 0,$$
$$G_v = 2(f'f'' + g'g'')$$

The Christoffel symbols can then be readily computed:

$$\Gamma^1_{11} = 0, \qquad \Gamma^2_{11} = -\tfrac{ff'}{(f')^2+(g')^2}, \qquad \Gamma^1_{12} = \tfrac{ff'}{f^2},$$
$$\Gamma^2_{12} = 0, \qquad \Gamma^1_{22} = 0, \qquad \Gamma^2_{22} = \tfrac{f'f''+g'g''}{(f')^2+(g')^2}$$

Substitute these values into the differential equation (4.2), the first equation becomes the following:

$$u'' + \frac{2ff'}{f^2}u'v' = 0 \tag{4.4}$$

We can rewrite the equation as

$$(f^2u')' = f^2u'' + 2ff'u'v' = 0 \tag{4.5}$$

and so

$$f^2u' = c \tag{4.6}$$

where c is a constant. On the other hand, if we let θ be the angle of a geodesic on the surface with a parallel than we have

$$\cos\theta = \frac{|< x_u, x_u u' + x_v v' >|}{|x_u|} = |fu'| \tag{4.7}$$

since $f = r$, the radius of a parallel, we obtain the following:

Theorem 4.2.1 (Clairaut Integral)**.** *Let γ be a geodesic on a surface of revolution, if r is the radius of the parallel which the geodesic intersects with, and let θ be the angle of intersection. Then for some constant c the relation*

$$r\cos\theta = c \tag{4.8}$$

holds on the whole geodesic γ.

The Clairaut integral will play a key role in the following construction as it gives a complete description of the geodesics on a surface of revolution.

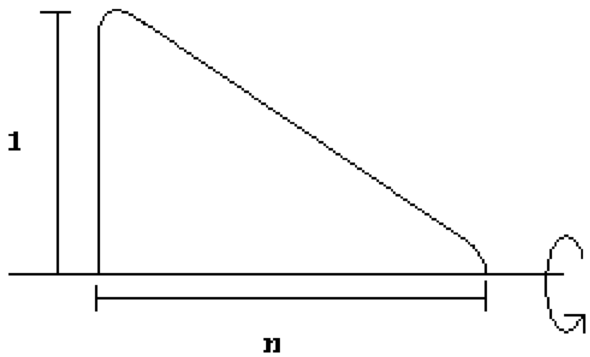
4.3 Construction of the metrics

We will now prove Theorem 4.1.1 and Theorem 4.1.2 by constructing a metric ρ_k on $\mathbb{S}^2$ for each $k \in \mathbb{N}$, such that $(\mathbb{S}^2, \rho_k)$ has no $1/k$ geodesic. Before we proceed, let us note that these metrics on $\mathbb{S}^2$ have non-negative sectional curvature. Furthermore, $\mathrm{diam}(M_k)$ is close to $\sqrt{n^2+1}+1$ and $\mathrm{Vol}(M_k)$ is close to $\frac{\pi n}{3}$, where n is a constant depending on k. In the following, we will construct the manifolds explicitly.

Our goal is to show that, for every integer $k \geqslant 2$, there exists a smooth surface M_k that has no $1/k$ geodesic. In our construction, each M_k will be a surface of revolution. We first start with $k = 2$, and then generalize to all k.

The surface. Consider a curve in ($\mathbb{R}^2$, Euclidean metric) that consists of a straight line joining (0,1) and (n,0) (n to be determined later), and a straight line from (0,1) to (0,0). These are just two sides of a right triangle. If we revolve this curve about the x-axis, we get a cone K with circular base of radius 1 and height n. Now smoothen the two angles on (0,1) and (n,0) by replacing a small neighborhood of each of the angle with a smooth arc, so that when we revolve it about the x-axis we get a smooth surface. The resulting surface is our M_2. For the sake of simplicity, we create M_2 in the way that the longest parallel (the great parallel) has radius 1. Now, M_2 is diffeomorphic to $\mathbb{S}^2$, and looks like a smoothened cone. Actually, since we alter arbitrarily small neighborhoods of the angles, the surface is Gromov-Hausdorff close to K. For instance, such f can be obtained by starting from the midpoint of the hypotenuse. We elongate it by sliding the two ends to sharp angles, followed by a suitable rescaling. Note that M_2 has non-negative sectional curvature. [Figure 4.1]

Figure 4.1. Construction of M_k



The rest of this section is dedicated to proving the following statement:

Proposition 4.3.1. *With n suitably large, M_2 has no $1/2$ geodesic.*

If we can prove Proposition 4.3.1, using the fact that having a $1/k$ geodesic is a scale invariant concept one can get the generalized case by rescaling n. To prove the proposition, we will show that all closed geodesics in M_2 are not 1/2 geodesic. We begin with the following observation:

Lemma 4.3.2. 1/2 *geodesic has no self-intersection.*

Proof: Suppose a closed geodesic γ of length l has self-intersection. Then there exists a segment η with two endpoints coincide, such that η has length $\leq \frac{l}{2}$. To see this, suppose γ has at least one self-intersection. Then this self-intersection separates γ into two geodesics, such that the four endpoints coincide at one point. (Think of the figure 8). It is easy to see that one of them has to have length less or equal to $\frac{l}{2}$. Now, any segment of length $\frac{l}{2}2$ that contains η cannot be distance minimizing. That is because the two endpoints of this segment can be joined by a shorter path, obtained by deleting η from the segment. □

The reason that we consider surfaces of revolution is we can classify all geodesics using Theorem 4.2.1.

With this we have the following lemma:

Lemma 4.3.3. *No closed geodesic can stay on one side of the great parallel (the longest parallel). i.e. it must intersect the great parallel.*

Proof: Firstly, if γ passes either (n,0) or (0,0), then by Theorem 4.2.1 it has to be a meridian, so it cannot stay on one side. Now suppose on the contrary that the non-meridian geodesic γ stays on one side. By compactness of γ, there exist a shortest and longest parallel (with radius r_1 and r_2), such that γ is tangential to both and lies between them. If $r_1 = r_2$, then γ is a parallel. This cannot happen, since any parallel of this kind is generated by the rotation of a point of the profile curve where the tangent is not parallel to the axis of revolution. None of these parallel can be geodesic [Do]. Therefore we must have $r_1 \neq r_2$. This contradicts the *Clairaut integral* since in this case, $c = r_1$ and $c = r_2$. □

So any geodesic is uniquely determined by the following data: the point of intersection with the great parallel and the angle of intersection α. Now by Clairaut integral, the angle α determines the constant $c = c_\alpha$. Denote this geodesic by $\gamma_\alpha(t)$, $\gamma_\alpha(0)$=point of intersection with the great parallel.

Let's investigate all closed geodesics in M_2:

Meridians ($\alpha = \frac{\pi}{2}$) : Meridians cannot be 1/2 geodesic if n is large enough. To see this, fix any meridian, it's length is approximately $2(n+1)$. Now, pick two points p, q that lie on the same parallel and split the meridian into halves. The distance between p and q is approximately half of the length of the parallel and thus is much shorter than the length of half-meridian.

Great parallel ($\alpha = \mathbf{0}$) : The longest parallel (with radius 1) of M_2 cannot be 1/2 geodesic. Fix any two antipodal point p, q on the great parallel. The distance between p and q along the parallel is π. However p and q can be joined by a path across the base. The length of this path equals approximately the diameter of the great parallel. Which means p and q can be joined by a shorter path. Hence the great parallel is not a 1/2 geodesic.

Other closed geodesics ($\alpha \in (0, \frac{\pi}{2})$) require more work. Without loss of generality, we can assume $\gamma'_\alpha(0)$ is pointing into the cone. Let $r_\alpha(t)$ be the radius of parallel intersecting γ_α at $\gamma_\alpha(t)$, and $\theta_\alpha(t)$ be angle of intersection. Observe that when $r_\alpha(t_\alpha) = c_\alpha$, for some $t_\alpha \in [0, l]$, γ_α is tangential to the parallel, and then it will start to return [Sp]. Denote by R_α the parallel where γ_α start to turn back.

Definition 4.3.4. *For each $\alpha \in [0, \frac{\pi}{2})$, define the total rotation $T_\alpha(t),\ t \in [0, l]$ to be the net (oriented) angle of rotation of γ_α about the axis of revolution from $\gamma_\alpha(0)$ to $\gamma_\alpha(t)$.*

Example: When α=0, γ_α is just the great parallel, Therefore $T_\alpha(t) = \pm t$ (depending on the orientation chosen).

Firstly, for any $\alpha \neq \frac{\pi}{2}$, $|T_\alpha(t)|$ is a monotonic increasing function. This is equivalent to saying that any non-meridian geodesic γ rotates only in one direction. To prove this claim, assume on the contrary that γ changes rotational direction at some point. Then at this point, γ should be tangential to a meridian. By the uniqueness of geodesics (in a smooth manifold, a point and a vector uniquely determine a geodesic), γ should coincide with a meridian. This contradicts the assumption that γ is a non-meridian.

Now recall that $\gamma_\alpha(t_\alpha)$ is the point when γ_α turns back, we have the following lemma:

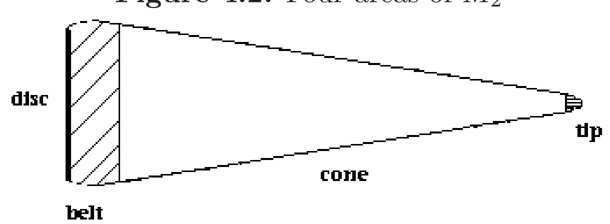
Lemma 4.3.5. *If $|T_\alpha(t_\alpha)| > \pi$, then γ_α has self-intersection.*

Proof: We know from Clairaut integral that γ_α cannot touch the great parallel. So if $|T_\alpha(t_\alpha)| > \pi$, the total rotation of γ_α in the cone area is strictly greater than 2π, which implies there is a self-intersection.□

We are now ready to list all the remaining geodesics in M_2, to simplify our argument, let us divide M_2 into four areas. Recall that in our construction, we smoothen 2 corners of the generating curve. Therefore when we revolve it: There is a curved cap at the tip (the cap), a thin curved belt around the great parallel (the belt), a flat disc at the bottom (the disc) and the long cone (the cone) [Figure 4.2]. Only the cap and the belt have non-zero curvature.

The remaining geodesics can be divided into three types:
a) Geodesics that never leave the belt before returning to the great parallel.
b) Geodesics that enter the cap.
c) Geodesics that enter the cone but miss the cap.

Figure 4.2. Four areas of M_2



There are two parallels which separate the belt and the cone, the cone and the cap. Denote these two parallels by R' and R'' respectively. Now since in constructing M_2, the belt and the cap can be arbitrarily thin. We can choose them to be so thin that for some α' and α'' chosen so that α', $(\frac{\pi}{2} - \alpha'') \ll \frac{\pi}{2}$, we have $R' = R_{\alpha'}$ and $R'' = R_{\alpha''}$. To make the following arguments simpler, we also dilate M_2 proportionally so that $R_{\alpha'}$ has length 1. There is no impact on all previous arguments because they held on all our manifolds regardless of scaling and the region of smoothing. Also we denote the distance between $R_{\alpha'}$ and the great parallel by ϵ, diameter of the cap be ϵ', where $\epsilon, \epsilon' \ll 1$.

The three cases of geodesics are equivalent to:

a) $\alpha \in (0, \alpha')$
b) $\alpha \in [\alpha'', \frac{\pi}{2})$
c) $\alpha \in [\alpha', \alpha'')$

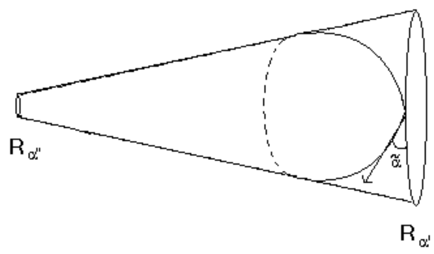
Case a) If γ_α wraps around M_2 twice or more, its winding number about the base's center is greater than 2, so γ_α has self-intersection. Hence by Lemma 4.3.2, γ_α is not 1/2 geodesic. If γ_α only wrap around M_2 once, then it enters each side of the great parallel once. Hence γ_α's length is within $2\pi \pm 10\epsilon$. Therefore γ_α is similar to the great parallel: any two points p, q dividing γ_α into halves can be joined by a path of length $\leq 2 + 10\epsilon$. This is a shorter path. Therefore we conclude that all geodesics in this case are not 1/2 geodesic.

Case b) Now, since γ_α connects the great parallel and some point in the cap, γ_α has at least length of $(2n - \epsilon')$. Then it is just like the meridian case: find two points which are $\frac{2n-\epsilon'}{2}$ apart and lie on the same parallel. When n is large the half-parallel is a shorter path. Hence no geodesic in case b can be 1/2 geodesic.

Case c) If γ_α enters the cone, then it must cross the parallel $R_{\alpha'}$. So there is an angle of intersection $\tilde{\alpha}$ between γ_α and $R_{\alpha'}$. Define $T(\tilde{\alpha})$, the *first return rotation* to be the total rotation of γ_α from $R_{\alpha'}$ and the point when it first hit $R_{\alpha'}$ again [Figure 4.3].

We need the following lemma:

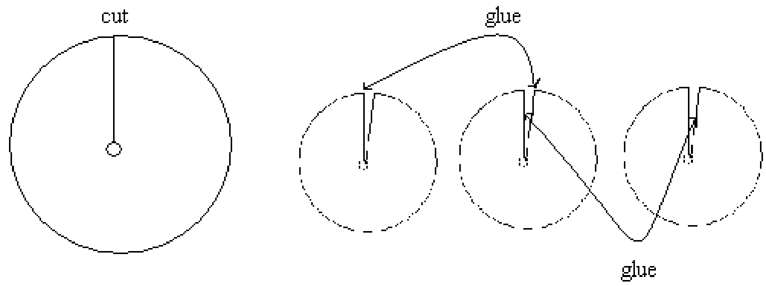
Figure 4.3. $T(\tilde{\alpha}) = 2\pi$



Lemma 4.3.6. *$T(\tilde{\alpha})$ is monotonic increasing in $\tilde{\alpha}$ for all geodesics in case c.*

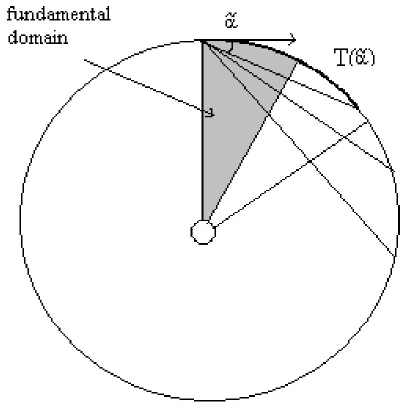
Proof: Consider the universal cover of the cone. Construct it by starting with an annulus, cut through one radius. Then take another copy of the same thing and then glue the left side of the cut from the first copy to the right side of the second copy. Continue infinitely we get the universal cover. It looks like a infinite spiral and is a topological infinite strip. A fundamental domain is a sector [Figure 4.4].

Figure 4.4. The universal cover of the cone



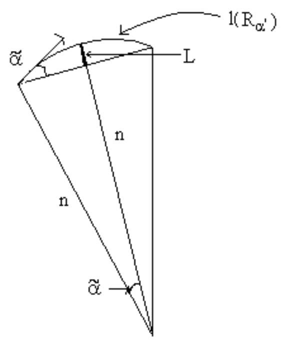
Now this is a development of the cone area, any geodesic segment is a straight line. Also, $\tilde{\alpha}$ is given by the angle of intersection with the outer circle. It is now easy to see that $T(\tilde{\alpha})$ is monotonic increasing in $\tilde{\alpha}$: Since we assume that $R_{\alpha'}$ has length 1, $T(\tilde{\alpha})$ is the length of the arc corresponding to the chord given by γ_α [Figure 4.5]. □

Figure 4.5. $T(\tilde{\alpha})$ is monotonic increasing



Finally, we claim that for any fix $\zeta \geq \epsilon$. When n is large enough, any γ_α not contained in the ζ-neighborhood of the great parallel has self-intersection. To see this, consider the fundamental domain (with arc length $Length(R_{\alpha'}) = 1$). A chord connecting two end points of the arc is a geodesic γ_α with $T(\tilde{\alpha}) = 2\pi$. Denote by L the distance between $R_{\alpha'}$ and γ_α. Elementary calculation shows that $L = n(1 - \sqrt{1 - \sin^2 \frac{1}{2n}}) \to 0$ as $n \to \infty$ [Figure 4.6]. So when n is large enough such that $L = \zeta$, the geodesic that turns back exactly at the boundary of the ζ-neighborhood gives $T(\tilde{\alpha}) = 2\pi$, hence it has self-intersection. Together with Lemma 4.3.6., when γ_α is not contained in the ζ-neighborhood of the great parallel, it has self-intersection. Therefore by Lemma 4.3.2, such geodesic cannot be 1/2 geodesic.

Figure 4.6. $L \to 0$ as $n \to \infty$



Now, the remaining geodesics are those that sit inside the ζ-neighborhood of the great parallel. Take $\zeta \ll 1$, this is a similar case as the geodesics that is contained in the curved belt: any two points p, q dividing γ_α into halves can be joined by a shorter path through the disc.

So if we choose n large enough such that all the previous criteria are met. Then M_2 has no $1/2$ geodesic and we finish the proof of Proposition 4.3.1 and thus Theorem 4.1.1.$\square$

4.4 When $k \geq 3$

Now we move to prove the general case. The construction of M_k is similar to that of M_2, except that we have to use larger n, thinner belt and smaller cap.

Proposition 4.4.1. *For any fixed k, M_k has no $1/k$ geodesic.*

As what we have done before, we will exhibit all possible geodesics. First off, any closed geodesic γ must intersect the great parallel (Lemma 4.3.3). So as before we can use the angle of intersection α to characterize the geodesics. In the following we still assume that γ has length l.

Meridians : Meridians are not $1/k$ geodesic if n is large enough. Again, find two points p, q near the tip that contain a $\frac{1}{k}$ segment and lie on the same parallel. n being large implies $\frac{l}{k}$ is much larger than the length of any parallel. Therefore there is a shorter path joining p, q.

Great parallel : The great parallel has length 2π. Any two points p, q that contains a $1/k$ segment ($\frac{2\pi}{k}$ long) of the great parallel can be joined by a shorter path through the base. This is a chord on the disc plus some small error. For any k, we can make the width of the smoothing to be narrow enough so that the error term is much smaller than $\frac{2\pi}{k}$. Therefore the great parallel is not a $1/k$ geodesic.

Other geodesics : Again, these geodesics can be categorized into 3 types: stays in the belt, goes into the cap, and goes into the cone but not the cap.

1) In the belt: If the geodesic wraps around once, then it is similar to the case of the great parallel: p, q can be joined by a shorter path that is close to a chord of the great parallel. If the geodesic wraps around m times, then for p, q bounding a $\frac{1}{k}$ segment, they are apart by approximately $\frac{2m\pi}{k} > \frac{2\pi}{k}$. Again, p, q can be joined by a shorter path through the disc.

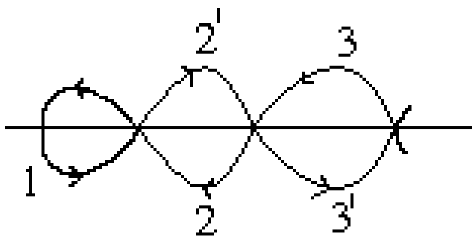
2) Into the cap: Similar to the case of $k = 2$, any geodesic that runs into the cap has length at least $2n - \epsilon'$ for some small ϵ'. We can find p, q near the tip. Such that p, q bound a $\frac{1}{k}$ segment ($\frac{2n-\epsilon'}{k}$ long) of the geodesic, and lie on the same parallel. Then p, q can be joined by a path close to a half-parallel which is a shorter path.

3) Geodesics that run into the cone but miss the cap: Since $k \geq 3$, Lemma 4.3.2 no longer applies here. However, we have the following lemma:

Lemma 4.4.2. *For any γ_α in case 3. If γ_α has $(k+1)$ self-intersections in the cone area. Then γ_α is not a $1/k$ geodesic.*

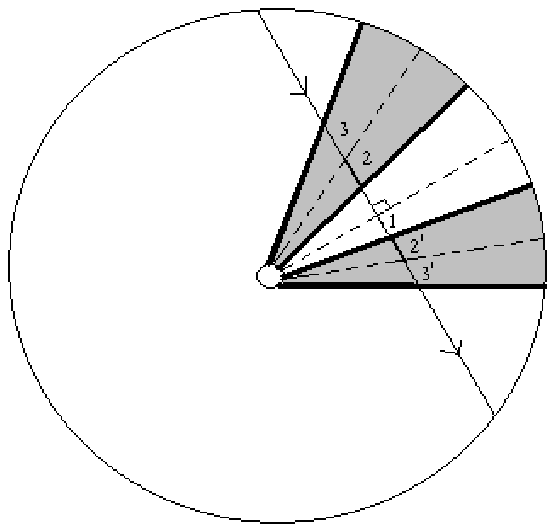
Proof: Suppose γ_α has $(k+1)$ self-intersections in the cone area. Recall that by Clairaut integral, any geodesic of this form is symmetric about the meridian that contains the point where the geodesic starts to turn back. The self-intersections split γ_α into at least $(2k+1)$ segments. Let us label the corresponding segments 1, 2, 2', etc. [Figure 4.7]. Notice that segment 1 forms a loop, segments 2 and 2' form another loop and so on. There are altogether k loops of this kind in the cone area.

Figure 4.7. A geodesic in case 3



Now we consider the universal cover again. Since segment 1 is the only one which is orthogonal to a meridian. This segment has to be strictly shorter than $Length(segment\ i) + Length(segment\ i')$ for $2 \leq i \leq k$ [Figure 4.8]. That means segment 1 is the shortest loop among the k loops in the cone area. Which implies $Length(segment\ 1) < \frac{l}{k}$. Any $\frac{1}{k}$ segment of γ_α containing segment 1 cannot be shortest path. Since we can connect the two endpoints by a shorter path if we jump segment 1 at the point of intersection.$\square$

Figure 4.8. Segment 1 has length $\leq \frac{l}{k}$



Now given any fixed ζ, $\epsilon < \zeta \ll \frac{1}{k}$. Using the same argument as $k = 2$: When n is large enough, the geodesic in figure 4.8 crosses at least $(k+1)$ fundamental domains, therefore $T(\tilde{\alpha}) > 2(k+1)\pi$ for all γ_α not contained inside the ζ-neighborhood of the great parallel. This implies that γ_α has $(k+1)$ self-intersections and by Lemma 4.4.2, γ_α is not $1/k$ geodesic. If γ_α is contained inside the ζ-neighborhood, then $\zeta \ll \frac{1}{k}$ implies γ_α is similar to those in case 1, hence it cannot be $1/k$ geodesic.

So for n large enough, M_k has no $1/k$ geodesic.

We have thus completed the proof of Proposition 4.4.1 and therefore Theorem 4.1.2.

Chapter 5

Part II: Security and Blocking geodesics

5.1 Introduction

In the second part of this dissertation we will investigate the interactions between geodesics and the global geometry of a manifold. We will study that whether or not a given set of geodesics can be blocked by a finite number of point obstacles could give much information about the manifold itself. From now on, we will assume that M is a compact, smooth complete Riemannian manifold.

Given two points on M, we can connect them by geodesic segments. Each geodesic segment can be regarded as a light beam emanating from one point and reaching the other point. We define the *blocking number* between the two points to be the minimal number of points needed to block all geodesic segments connecting them. The blocking number of the manifold M is then defined to be the supremum of the blocking numbers between any two points in M. In some literatures, this number is referred to as the *security threshold* of M. A manifold with finite blocking number is said to be *uniformly secure*.

The notions of blocking and security have been studied by mathematicians such as K. Burns, E. Gutkin, V. Bangert, J.F. Lafont, B. Schmidt, J. Souto, V. Schroeder etc. The origin of this study seems to have originated from polygonal billiards and geometric optics. For example, see [Mon], [Mon1], [Gu], [HeSn]. In this chapter we will provide all backgrounds in this area and present some key results.

5.2 Backgrounds

We will assume that our manifold M is a complete, connected, compact, infinitely differentiable manifold. All geodesics are parameterized by arc length and oriented. By a *geodesic segment* we mean a geodesic $\gamma : [a, b] \to M$ where $Length(\gamma) = b - a$.

Definition 5.2.1. *If $\gamma : [s_0, s_1] \to M$ is a geodesic segment, we call $\gamma(s_0)$ and $\gamma(s_1)$ the* endpoints *of γ and all other points of γ the* interior points *of γ.*

A *configuration* is an ordered pair of points in M, we allow these two points to coincide.

Definition 5.2.2. *We say a geodesic γ* passes through *a point $z \in M$ if z is an interior point of γ. Let $(x, y) \in M \times M$ be a configuration, we say that a geodesic segment* joins *two points x and y in M if the points are the endpoints of γ, and* connects *x and y if further γ does not pass through either x or y.*

Given any configuration (x, y), let us denote by $G(x, y)$ the set of geodesic segments joining x and y, and let $\Gamma(x, y)$ be the set of geodesic segments connecting x and y. Then obviously $\Gamma(x, y) \subset G(x, y)$. These sets can be thought of as light beams that emanate from x and reach y. Firstly, let we recall the following well known result by Serre [Se]:

Theorem 5.2.3. *For any configuration $(x, y) \in M \times M$ where $x \neq y$, the set $G(x, y)$ is infinite.*

On the other hand, $\Gamma(x, y)$ may or may not be infinite. For example, if M is a round sphere and the two points x, y are two non-antipodal points, then $\mathrm{card}(\Gamma(x, y)) = 2$.

Now given a configuration (x, y), we can find a set of points (finite or infinite) such that if $\gamma \in \Gamma(x, y)$, then γ passes through a point from the set. This set of points will be called a *blocking set* of $\Gamma(x, y)$. We confine the blocking set for $\Gamma(x, y)$ to lie in $M \backslash \{x, y\}$. We can now define the *blocking number* of the configuration (x, y).

Definition 5.2.4. *Let* (x, y) *be a configuration in* $M \times M$*. The* blocking number *of* (x, y)*, denoted by* $B(x, y)$*, is a positive integer (could be infinite) that is the minimal number of points from* $M \backslash \{x, y\}$ *that block the set* $\Gamma(x, y)$*.*

One dividing line is whether or not this blocking number is finite.

Definition 5.2.5. *A configuration* (x, y) *is secure if the blocking number* $B(x, y)$ *is finite. Otherwise the configuration is said to be insecure.*

So in terms of geometric optics, a configuration is secure if one of the points can be shaded from the light emanating from the other point by a finite number of point obstacles. Another possible reason that this phenomenon is coined "secure" is that all geodesic paths between two connecting locations can be monitored by a finite number of observation spots.

Now we will define the blocking number of a manifold.

Definition 5.2.6. *The blocking number of a Riemannian manifold* M*, denoted by* $B(M)$*, is the supremum of the blocking numbers between any two points in* M*. i.e.,*

$$B(M) = \sup\{B(x, y) |\ x, y \in M\}$$

$B(M)$ *is also called the* security threshold *of* M*.*

These notions also appear under the context of polygonal billiards. One considers a billiard system in a rational polygon, with the billiard orbits represented by geodesics, then the security question can also be studied. As a matter of fact, this billiard orbits are exactly the geodesics in the associated translation surface. For the study of security in billiards, see [Gu],[Gu1] and [Gu2].

Definition 5.2.7. *A Riemannian manifold* M *is secure if every configuration is secure, otherwise it is insecure.* M *is totally insecure if every configuration is insecure.* M *is said to be* uniformly secure *if* $B(M)$ *is finite.*

Let us now use several manifolds to illustrate the idea of blocking number and security:

Proposition 5.2.8. *If* M *is a Hadamard manifold, then* $B(M) = 1$*.*

Proof: Recall that M is a Hadamard manifold means that M is simply connected and has non-positive sectional curvature everywhere. We see that given any two points in M, the Cartan-Hadamard theorem implies that these two points are connected by a unique geodesic segment. Hence $B(x, y) = 1$ for all configurations $(x, y) \in M \times M$, and so $B(M) = 1$. □

The next example that we consider is flat. In fact, we will see later in the section that all flat manifolds have finite blocking number. It is worth noting that affine transformations do not affect the security of a manifold. Indeed, we have the following lemma:

Lemma 5.2.9. *The blocking number is invariant under affine transformations.*

The second example is the flat 2-torus:

Proposition 5.2.10. *Let M be a flat 2-dimensional torus, then $B(M) = 4$.*

Proof: Since the blocking number is invariant under affine transformations, we can assume that M is the standard flat torus $\mathbb{R}^2/\mathbb{Z}^2$. Now we want to show that for any two points $x, y \in \mathbb{R}^2/\mathbb{Z}^2$ we have $B(x, y) = 4$. Without loss of generality, let us assume that x is a point in M such that its lift $\tilde{x} \in \mathbb{R}^2$ satisifies $\tilde{x} = (0, 0)$. If y is any other point in M with coordinates $(a, b) \in [0, 1] \times [0, 1]$, then y is lifted to points in $\mathbb{R}^2$ with coordinates $(a + m, b + n)$, $m, n \in \mathbb{Z}$. Let us connect each $(a + m, b + n)$ to $\tilde{x}$ by a straight segment $\tilde{\gamma}_{m,n}$. It is easy to see that the projections of $\tilde{\gamma}_{m,n}$ to M coincide with all geodesics connecting x and y. Next, for each (m, n) let us consider the midpoint of $\tilde{\gamma}_{m,n}$ to be its the blocking point. When these blocking points are projected to M they will have coordinates of $(\frac{a}{2}, \frac{b}{2}) + (\frac{1}{2}\mathbb{Z})^2/\mathbb{Z}^2$. It is easy to see there could be at most 4 of such blocking points in M. Therefore we conclude that the set of geodesics connecting x and y can be blocked by 4 points. Since x and y are arbitrary, the blocking number of M is 4. □

Indeed, this proposition can be directly generalized to dimension n.

Proposition 5.2.11. *If M is a flat n-torus, then $B(M) = 2^n$.*

Let us now see a simple example of insecure manifold: the standard sphere. Let M be an $n - sphere$ with the standard round metric, then $B(M) = \mathfrak{c}$, where

$\mathfrak{c}$ stands for the cardinality of the continuum. This is because if x and y are two non-antipodal points, then they are connected by exactly two geodesics: these geodesics form a great circle. Now if x and y are two antipodal points, then they are connected by a family of distinct geodesics, each of which is a half great-circle. Any two of these geodesics do not intersect except at the endpoints. This family of geodesics has cardinality of $\mathfrak{c}$ and so $B(M) = \mathfrak{c}$.

5.3 Security of Locally Symmetric Spaces

In the paper [GuSc], Gutkin and Schroder investigated the security of compact locally symmetric spaces. Recall that a globally symmetric space is a homogeneous space G/H where G is a connected Lie group such that the stabilizer H of a point is an open subgroup of the fixed point set of an involution of G.

The property of having geodesic symmetry at each point is also used to define symmetric spaces. Let M be a connected Riemannian manifold and x be a point of M. A map f defined on a neighborhood of x is said to be a geodesic symmetry, if it fixes the point x and reverses geodesics through that point. It follows that the derivative of the map at x is the negative of the identity map on $T_x(M)$.

M is said to be locally Riemannian symmetric if its geodesic symmetries are isometric, i.e., all symmetries preserve lengths. M is (globally) Riemannian symmetric if in addition its geodesic symmetries are defined on all of M.

Let M be a compact, locally symmetric space, then $M = S/\Gamma$ where S is a simply connected symmetric space, and Γ is a discrete, cocompact group of isometries freely acting on S. The space S decomposes into a product $S = S_0 \times S_- \times S_+$ of simply connected symmetric spaces of *Euclidean type*, *Non-compact type* and *Compact type* respectively. The main work of the Gutkin-Schroeder paper is to study the security of compact, locally symmetric spaces of these types.

Locally symmetric spaces of compact type: A compact locally symmetric space of Euclidean type is of the form $M^n = \mathbb{R}^n/\Gamma$ where Γ is a cocompact subgroup of isometries freely acting on $\mathbb{R}^n$. For these symmetric spaces, Gutkin and Schroeder showed that they are uniformly secure.

Proposition 5.3.1 ([GuSc])**.** *If M is of Euclidean type then it is uniformly secure, and its blocking number is bounded in terms of dim(M).*

Locally symmetric spaces of noncompact type: If $M = S/\Gamma$ is a compact, locally symmetric space of noncompact type. That is, S is a simply connected, noncompact symmetric space, and Γ is the deck group of the covering $\pi : S \to M$, then the following theorem holds:

Theorem 5.3.2 ([GuSc])**.** *If M is a compact locally symmetric space of noncompact type, let $x, y \in M$ and let $F \subset M \backslash \{x, y\}$ be a finite set. Then there exists a geodesic γ connecting x and y such that $\gamma \cap F = \emptyset$.*

In other words, these M are all totally insecure.

Locally symmetric space of compact type: A symmetric space of compact type is a space $S = G/K$, where $G = Iso_0(S)$ is a compact, connected, semisimple Lie group, and $K \subset G$ is the fixed point set of an involution $\sigma : G \to G$. In this case, the following theorem holds:

Theorem 5.3.3. *If M is a compact locally symmetric symmetric space of compact type then it has an open, dense, full measure set of secure configurations, it also has insecure configurations.*

5.4 Security and Entropy

The study of whether or not geodesics can be blocked by finite number of point obstacles also bears information on the geodesic dynamics of a manifold. K. Burns and E. Gutkin have worked in this direction in the paper [BuGu]. In concrete terms, they studied the interactions between security and the topological entropy of the geodesic flow. This direction of study is natural because the study of security concerns about the number of geodesics connecting two points in a manifold, while by the results of such as Gromov and Mañé, this information can relates to the topological entropy and the fundamental group of the manifold. Some results mentioned in this section have also been obtained independently by J.F. Lafont and B. Schmidt, see [LaSc].

Before we proceed, let us review some key concepts involved. We will start with the geodesic flow. Let M be a closed connected Riemannian manifold, let us denote by TM the tangent bundle of M. If we let $\gamma_{x,v}(t)$ to be the geodesic with the initial conditions:

$$\begin{cases} \gamma_{x,v}(0) = x \\ \gamma'_{x,v}(0) = v \end{cases} \tag{5.1}$$

Where $x \in M$ and $v \in T_xM$. This geodesic is uniquely determined for any given pair (x, v). For any given $t \in \mathbb{R}$, we can define a diffeomorphism $\phi_t : TM \to TM$ as follows

$$\phi_t(x, v) = (\gamma_{x,v}(t), \gamma'_{x,v}(t)) \tag{5.2}$$

By the uniqueness of the geodesic with given initial conditions, it can easily seen that $\phi_{t+s} = \phi_t \circ \phi_s$. In other words, ϕ_t is a flow.

Let us also denote by UM the unit tangent bundle of M, i.e., the subset of TM such that if $(x, v) \in UM$ then the norm of v is one. As geodesics travel with constant speed therefore we see that for each $(x, v) \in UM$, $\phi_t(x, v) \in UM$ for all $t \in \mathbb{R}$, or in other words, UM is invariant under ϕ_t. The restriction of ϕ_t on UM is called the geodesic flow of the Riemannian manifold M.

The second concept is the topological entropy of a flow. Let (X, d) be a compact metric space and let $\phi_t : X \to X$ be a continuous flow. For each $T > 0$ let us define a new distance function:

$$d_T(x, y) = max\{d(\phi_t(x), \phi_t(y))|0 \leq t \leq T\} \tag{5.3}$$

Since the metric space X is compact, for any fixed $\epsilon > 0$, there is a finite number of balls with radius ϵ in the metric d_T, let us denote by $N(\epsilon, T)$ the minimal cardinality of these balls needed to cover X. Then we define the following monotone decreasing function

$$h(\phi, \epsilon) = \limsup_{T\to\infty} \frac{1}{T} \log N(\epsilon, T) \tag{5.4}$$

So the *topological entropy* of the flow ϕ, denoted by $h_{top}(\phi)$, is defined as this limit

$$h_{top}(\phi) = \lim_{\epsilon \to 0} h(\phi, \epsilon) \tag{5.5}$$

Our main focus here is the topological entropy of the geodesic flow of a manifold M, this quantity is written as $h_{top}(g)$, where g stands for the given Riemannian metric of the manifold M.

The connecting point between the topological entropy $h_{top}(g)$ and the blocking number $B(M)$ is the so called counting function $n_T(x, y)$, which is defined to be the cardinality of the set the geodesic segments joining x and y with length $\leq T$.

$$n_T(x, y) = \text{card}\{\gamma | \gamma \text{ } joins \text{ } x \text{ } and \text{ } y, Length(\gamma) \leq T\} \tag{5.6}$$

The counting function is a quantity that has been studied extensively. Berger and Bott began this investigation by proving that for each $T > 0$, $n_T(x, y)$ is finite and locally constant on an open subset of $M \times M$ of full Riemannian measure. Let us denote by $d\mu(x)$ the density of this measure. Then the following result by Mañé shows that the topological entropy of the geodesic flow is indeed the mean value over all pairs of points of the exponential factor in the growth of the counting function.

Theorem 5.4.1 ([Mañé]). $h_{top}(g) = \lim_{T \to \infty} \frac{1}{T} \log \int_{M \times M} n_T(x, y) d\mu(x) d\mu(y)$

An immediate interpretation of this estimate is that if a manifold M has positive entropy, then for many configurations $(x, y) \in M \times M$ the counting function $n_T(x, y)$ is exponential.

Mañé then raised the question that whether or not the limit superior of the quantity $\frac{1}{T} \log n_T(x, y)$ equals the topological entropy almost everywhere. K. Burns and G. Paternain provided a negative answer to this question in [BuPa] by constructing an open set of metrics on $\mathbb{S}^2$ for which

$$\limsup_{T \to \infty} \frac{1}{T} \log n_T(x, y) < h_{top}(g) \tag{5.7}$$

for a positive measure set of $(p, q) \in M \times M$. However if the Riemannian manifold has no conjugate point, then $n_T(x, y)$ is always finite and the following equality holds for any $x, y \in M$,

$$h_{top}(g) = \lim_{T\to\infty} \frac{1}{T} \log n_T(x, y) \tag{5.8}$$

See Corollary 1.2 of [Mañé].

For uniformly secure Riemannian manifolds, Burns and Gutkin have proven that the counting function $n_T(x, y)$ is of polynomial growth in T.

Theorem 5.4.2 (Theorem 4.2, [BuGu])**.** *Let M be a compact Riemannian manifold. If M is uniformly secure, then there are positive constants C and d such that for any pair $x, y \in M$ we have*

$$n_T(x, y) \leq CT^d \tag{5.9}$$

Using this estimate and the Mañé formula (Theorem 5.4.1), together with Gromov's theorem which states that a finitely generated group with polynomial growth is virtually nilpotent, one then obtains the following theorem.

Theorem 5.4.3 (Theorem 4.3, [BuGu])**.** *Let M be a compact Riemannian manifold that is uniformly secure. Then the topological entropy of the geodesic flow for M is zero, and the fundamental group of M is virtually nilpotent. If in addition M has no conjugate points, then M is flat.*

For manifolds with no conjugate points, the result can be even stronger. In this case every configuration is insecure. Using Equation 5.8, Burns and Gutkin proved the following theorem

Theorem 5.4.4 (Theorem 4.5,[BuGu])**.** *Let M be a compact Riemannian manifold with no conjugate points whose goeodesic flow has positive topological entropy. Then M is totally insecure.*

Of course with the use of the variational principle this theorem can be extended to manifolds with positive metric entropy. Furthermore if M is a Riemannian manifold of non-positive curvature, then it must be insecure, see [LaSc].

Corollary 5.4.5. *Let M be a compact Riemannian manifold of non-positive curvature. If M is uniformly secure, then M is flat.*

Proof: Assume that M has non-positive curvature and is not flat, then M has no conjugate or focal points. According to a result by Pesin [Pe], this means that the geodesic flow of M has positive entropy, hence the blocking number of M is not finite.□

Let us also mention the following result about security and product manifolds. This proposition is used to construct examples of totally insecure manifolds that have conjugate points.

Proposition 5.4.6 (Proposition 5.2, [BuGu]). *Let M and M' be Riemannian manifolds, then*
1) If one of M and M' is (totally) insecure, then $M \times M'$ is (totally) insecure.
2) $M \times M'$ is (uniformly) secure if and only if both M and M' are (uniformly) secure.

5.5 A Conjecture about Security

In light of Proposition 5.3.1, we know that all compact flat Riemannian manifolds are uniformly secure. While the only known examples of compact Riemannian manifolds that are uniformly secure are the flat manifolds, a natural question to ask is whether or not the converse of the above statement is true.

Conjecture 5.5.1. *A compact Riemannian manifold is uniformly secure if and only if it is flat.*

Our main task in the next chapter is to prove that this conjecture is true for compact, non-simply connected, orientable two dimensional Riemannian manifolds.

5.6 Other Types of Blocking

Under the context of security, B. Schmidt and J.F. Lafont also studied what they called the cross blocking and sphere blocking [LaSc]. The objects of their focus are the compact rank one symmetric spaces (CROSSes). These spaces consists of the round sphere $\mathbb{S}^n$, the projective spaces $K\mathbb{P}^n$ where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and the Cayley projective plane $Ca\mathbb{P}^2$, all endowed with the standard symmetric metrics. These spaces satisfy a blocking condition which is called the cross blocking.

Definition 5.6.1. *A compact Riemannian manifold M is said to have cross blocking if*

$$0 < d_g(x, y) < diam(M) \Longrightarrow B(x, y) \leq 2$$

It was conjectured in [LaSc] that the CROSSes are the only compact manifolds that have cross blocking. Part of the evidences come from the study of Blaschke manifolds.

Definition 5.6.2. *A compact Riemannian manifold M is called a Blaschke manifold if its injectivity radius equals its diameter.*

Lafont and Schmidt showed that a Blaschke manifold has cross blocking, see [LaSc] Proposition 3.6.

The round spheres also satisfies one more property, which is called the sphere blocking

Definition 5.6.3. *A compact manifold M have sphere blocking if $B(x, x) = 1$ for every $x \in M$.*

Schmidt and Souto then proved the following theorem

Theorem 5.6.4 ([ScSo])**.** *A Riemannian manifold that satisfies both the sphere blocking and cross blocking is isometric to the round sphere.*

Chapter 6

Insecurity of Compact, Orientable, Non-Simply Connected Surfaces

6.1 Introduction

In the chapter we will prove Conjecture 5.5.1 for compact, non-simply connected, orientable surfaces. We will show that for a two dimensional torus, uniform security implies flatness, then we will show that this technique can be used to prove that all surfaces of higher genus are insecure. Our main tool to analyze the geodesic behaviors is the Aubry-Mather theory of geodesics on tori discussed by V. Bangert. As a remark, the techniques used in this chapter cannot be directly used to prove the higher dimensional versions of Conjecture 5.5.1, the reason is that many results in the theory are not true for the higher dimensional cases.

We begin the discussion by providing some notations involved, these are some fundamental concepts in Riemannian geometry.

Let M is a geodesic complete, locally compact Riemannian manifold, and let g be a given Riemannian metric on M. Recall that we have defined the metric $d_g(\cdot, \cdot)$ on M by

$$d_g(x, y) = \inf\{\text{Length}(\gamma) | \gamma : [a, b] \to M \text{ is continuous}, \gamma(a) = x, \gamma(b) = y\}$$

In this way (M, d_g) can be considered a complete length space. A continuous curve in M is said to be *distance minimizing* if the distance in terms of d_g between the end points equals to the length of that curve. Next, let us define a *minimal geodesic* to be a continuous curve that is distance minimizing on every subsegment:

Definition 6.1.1. *A* minimal geodesic segment *is a geodesic segment* $\gamma : [a, b] \to M$ *such that for all* $s, s' \in [a, b]$ *we have,*

$$d_g(\gamma(s), \gamma(s')) = |s - s'| \tag{6.1}$$

A geodesic γ such that $\gamma|_{[a,b]}$ is minimal for every interval $[a, b] \subset \mathbb{R}$ is called a *minimal geodesic*. Such type of geodesics has many names, and was first studied by Morse in [Mo].

Using Arzela-Ascoli's Theorem, completeness and compactness of the space (M, d_g), we obtain the following version of the Hopf-Rinow Theorem:

Theorem 6.1.2. *Any two points of* M *can be connected by a minimal geodesic segment.*

It is easy to see that d_g is realized by the length of minimal geodesics, we also call d_g the *geodesic distance* on M.

For our convenience, let us also introduce two more notions about minimal geodesics:

Definition 6.1.3. *An unending geodesic* $\gamma : (-\infty, +\infty) \to M$ *is said to be a* minimal geodesic line *if* $\forall s, s' \in (-\infty, +\infty)$,

$$d_g(\gamma(s), \gamma(s')) = |s - s'| \tag{6.2}$$

A one-sided unending geodesic $\gamma : [a, +\infty) \to M$ is called a *geodesic ray* if it is distance minimizing on every subsegment.

Let us consider now the closed curves in M, i.e. all $\gamma : [a, b] \to M$ satisfying $\gamma(a) = \gamma(b)$. A closed curve that is a geodesic is called a *closed geodesic* or *periodic geodesic*. A periodic geodesic is minimal if it unwraps in the universal cover to a minimal geodesic line. For simplicity, we will identify all higher iterates of a prime periodic geodesic to the geodesic itself.

6.2 Mather Sets for Geodesics on Tori

Our proof of Conjecture 5.5.1 relies on a series of results that relate geodesics on a two dimensional Riemannian torus and the dynamics of monotone twist maps of an annulus. The studies of this interaction began with Hedlund [Hed] and Morse [Mo], then by Mather and later on by Bangert. In this section we will state some key results proven in [Ba].

We will now fix some notations, consider the space $\mathbb{R}^{\mathbb{Z}}$ of bi-infinite sequences of real numbers with the product topology. An element $x \in \mathbb{R}^{\mathbb{Z}}$ will also be denoted by $(x_i)_{x\in\mathbb{Z}}$ and will be called a trajectory. Convergence of a sequence $x^n \in \mathbb{R}^{\mathbb{Z}}$ to $x \in \mathbb{R}^{\mathbb{Z}}$ means that $\lim_{n\to\infty} x_i^n = x_i$ for all $i \in \mathbb{Z}$.

Given a function $H : \mathbb{R}^2 \to \mathbb{R}$, it can be extended to arbitrary finite segments $(x_j, \dots, x_k), j < k$ of trajectories $x \in \mathbb{R}^{\mathbb{Z}}$ by

$$H(x_j, \dots, x_k) = \sum_{i=j}^{k-1} H(x_i, x_{i+1}) \tag{6.3}$$

A segment $(x_j, \dots, x_k)$ is minimal with respect to H if

$$H(x_j, \dots, x_k) \le H(x_j^*, \dots, x_k^*) \tag{6.4}$$

for all $(x_j^*, \dots, x_k^*)$ with $x_j = x_j^*$ and $x_k = x_k^*$. A trajectory $x \in \mathbb{R}^{\mathbb{Z}}$ is said to be minimal with respect to H if every finite segment of x is minimal with respect to H. The set of all minimal trajectories with respect to H is denoted by $\mathcal{M}(H)$.

We also assume that H is continuous. Moreover, in order to relate this model with geodesics on Riemannian tori (and other objects such as the monotone twist maps and the discrete Frenkel-Kontorova model), the functions H that we are considering have to satisify the following four conditions:

(H_1) Periodicity: $\forall(\xi,\eta) \in \mathbb{R}^2, H(\xi+1,\eta+1) = H(\xi,\eta)$.

(H_2) Condition at infinity: $\lim_{|\eta|\to\infty} H(\xi,\xi+\eta) = \infty$ uniformly in ξ.

(H_3) Ordering: If $\xi' < \xi, \eta' < \eta$ then $H(\xi',\eta') + H(\xi,\eta) < H(\xi,\eta') + H(\xi',\eta)$

(H_4) Transversality: If $(x_{-1}, x_0, x_1) \neq (x_{-1}^*, x_0^*, x_1^*)$ are minimal and $x_0 = x_0^*$ then $(x_{-1} - x_{-1}^*)(x_1 - x_1^*) < 0$.

Furthermore, trajectories in $\mathbb{R}^{\mathbb{Z}}$ are partially ordered by:

$$x < x^* \text{ if and only if } x_i < x_i^* \text{ for all } i \in \mathbb{Z}.$$

We also define the α and ω asymptoticity for trajectories:

Definition 6.2.1. *Let $x, x^* \in \mathbb{R}^{\mathbb{Z}}$ be two trajectories, we say that x and x^* are $\alpha-$asymptotic if* $\lim_{i\to-\infty} |x_i - x_i^*| = 0$*, they are $\omega-$asymptotic if* $\lim_{i\to\infty} |x_i - x_i^*| = 0$.

There is an action of the group $\mathbb{Z}^2$ on $\mathbb{R}^{\mathbb{Z}}$ by order-preserving homeomorphisms: if $(a, b) \in \mathbb{Z}^2$, denote by $T_{(a,b)}(x)$ the action that translate the point x by (a, b), i.e.,

$$\text{If } x^* = T_{(a,b)}(x) \text{ then } x_i^* = x_{i-a} + b \tag{6.5}$$

With this notion, we give the definition of a periodic trajectory:

Definition 6.2.2. *$x \in \mathbb{R}^{\mathbb{Z}}$ is periodic with periodic $(q, p) \in \mathbb{Z}\backslash\{0\}\times\mathbb{Z}$ if $T_{(q,p)}x = x$.*

In the framework of geodesics on a torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ endowed with a Riemannian metric g, we will consider the set of geodesics on $\mathbb{R}^2$ with the metric $\tilde{g}$, where $\tilde{g}$ is the lift of the metric g. In order to use this model we need to find a function H such that $\mathcal{M}(H)$ corresponds to the set of minimal geodesics in $\mathbb{R}^2$ with respect to this $\mathbb{Z}^2$ periodic metric $\tilde{g}$. Bangert showed that we can introduce coordinates on $\mathbb{R}^2$ such that the coordinate lines $\{i\} \times \mathbb{R}, i \in \mathbb{Z}$ are minimal geodesics with respect to $\tilde{g}$, we will settle for such coordinates. Then we define

$$H(\xi, \eta) = d_{\tilde{g}}((0, \xi), (1, \eta)) \tag{6.6}$$

where $d_{\tilde{g}}$ is the geodesic distance on $\mathbb{R}^2$ with respect to $\tilde{g}$. Bangert then showed that this H satisfies the properties $(H_1) - (H_4)$.

Lemma 6.2.3 (Lemma 6.4, [Ba])**.** *H satisfies $(H_1) - (H_4)$*

Let us see the relations between these trajectories and the minimal geodesics in $\mathbb{R}^2$. By the $\mathbb{Z}^2$ invariance of the function H, we can see that

$$H(x_j, \ldots, x_k) = \sum_{i=j}^{k-1} d_{\tilde{g}}((i, x_i), (i+1, x_{i+1})) \tag{6.7}$$

Suppose now a minimal trajectory $x = (x_i)_{i \in \mathbb{Z}} \in \mathcal{M}(H) \subset \mathbb{R}^{\mathbb{Z}}$ is given, we can construct a minimal geodesic passing through the points $(i, x_i), \forall i \in \mathbb{Z}$, by simply connecting (i, x_i) and $(i+1, x_{i+1}) \in \mathbb{R}^2$ with minimal geodesic segments for each $i \in \mathbb{Z}$.

Conversely, suppose a minimal geodesic $\gamma(s) = (\xi(s), \eta(s)) \subset \mathbb{R}^2$ is given, then its intersections with the coordinate lines $\{i\} \times \mathbb{R}, i \in \mathbb{Z}$ determine some $x \in \mathcal{M}(H)$ provided $\xi(s)$ is surjective, and this latter condition can always be achieved by a change of coordinates.

Therefore the following lemma is obtained

Lemma 6.2.4. *Let$(x_j, \ldots, x_k)$ be a minimal segment with respect to H and $k-j \geq 2$. Then there exists a unique minimal geodesic segment $c : [0, L] \to \mathbb{R}^2$ which joins (j, x_j) to (k, x_k) passing through the points $(i, x_i), j < i < k$. Conversely, every minimal geodesic segment which is not part of a line $\{i\} \times \mathbb{R}, i \in \mathbb{Z}$ intersects each line $\{i\} \times \mathbb{R}$ at most once and the points of intersection form the graph of a minimal segment with respect to H.*

Hence the study of minimal geodesics on a two dimensional Riemannian torus is equivalent to the study of $\mathcal{M}(H)$. So if a theorem regarding the minimal trajectories $\mathcal{M}(H)$ is established, the corresponding result concerning minimal geodesics on the Riemannian torus can be deduced using this equivalence.

Geometrically, the coordinate lines $\{i\} \times \mathbb{R}, i \in \mathbb{Z}$ project to a closed geodesic on T^2. The above identification means that a minimal geodesic in T^2 can be studied via the *Poincaré return map* for which an orbit is given by the intersection between the minimal geodesic and this closed geodesic, and vice versa. This *Poincaré return map* is in fact an orientation preserving circle homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$.

Lemma 6.2.5 (Lemma 3.15, [Ba])**.** *For every $x \in \mathcal{M}(H)$ there exists a circle map f such that $x_{i+1} = f(x_i)$ for all $i \in \mathbb{Z}$.*

For a circle homeomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$, we can define the *Poincaré rotation number* α. First of all we let G_+ be the set of all orientation preserving circle homeomorphisms, and let

$$\tilde{G}_+ = \{f | f : \mathbb{R} \to \mathbb{R} \text{ continuous, strictly increasing, } f(x+1) = f(x) + 1\}$$

be the group of lifts to $\mathbb{R}$ of these orientation preserving circle homeomorphisms. The Poincaré rotation number $\alpha : G_+ \to \mathbb{S}^1$ has a lift $\tilde{\alpha} : \tilde{G}_+ \to \mathbb{R}$ defined by

$$\tilde{\alpha}(f) = \lim_{|i| \to \infty} \frac{f^i(x)}{i} \tag{6.8}$$

For a real number α, let us denote by $\mathcal{M}_\alpha$ the set of minimal trajectories with rotation number α, and denote by $\mathcal{M}_\alpha^{per}$ the set of periodic minimal trajectories with rotation number α.

For a minimal geodesic $\gamma \in T^2$, there is a equivalent quantity: let $\gamma(s)$ be a minimal geodesic in T^2, then its lift $\tilde{\gamma}(s)$ is a minimal geodesic in $\mathbb{R}^2$. Let us write $\tilde{\gamma}(s) = (\xi(s), \eta(s))$ where ξ and η represent the coordinates of $\tilde{\gamma}$ in $\mathbb{R}^2$. Then the *average slope* or the *rotation number* of $\tilde{\gamma}$, which we denote by α, by setting $\alpha(\tilde{\gamma}) := \lim_{|s| \to \infty} \eta(s)/\xi(s)$. It is easy to see that if α is the rotation number of a minimal geodesic, and $\tilde{\alpha}'$ is the lift of the rotation number of the associated minimal trajectory, then $\alpha = \tilde{\alpha}'$.

Here is a property of the functional α, which is proven in [Ba]:

Theorem 6.2.6 ([Ba],Theorem 6.5)**.** *If the function $\xi(s)$ is surjective, then $\alpha(\tilde{\gamma})$ exists in $(-\infty, \infty)$. If $\xi(s)$ is not surjective then $\xi(s)$ is bounded and we will define $\alpha(\tilde{\gamma}) = \infty$.*

It is clear from the definition that the lifts of two closed curves from the same homotopy class have the same rotation number.

Using the the *Poincaré return map*, Bangert showed that for each rational $\alpha = \frac{p}{q}$, the set of minimal trajectories with rotational number equal to α is non-empty, each minimal trajectory has minimal period (q, p), and the function H is minimal on these trajectories among all trajectories with same endpoints. Hence the following theorem for minimal geodesics is obtained.

Theorem 6.2.7 (see [Ba],Theorem 6.6)**.** *Periodic minimal geodesics are exactly the lifts of closed geodesics on T^2 which have minimal length in their free homotopy class.*

The next important result concerns about neighboring minimal trajectories (or minimal geodesics). Two elements in $\mathcal{M}_\alpha^{per}$ are said to be neighboring if there does not exist another element of $\mathcal{M}_\alpha^{per}$ between them. Now suppose that $x^- < x^+$ are neighboring elements of $\mathcal{M}_\alpha^{per}$, define

$$\mathcal{M}_\alpha^+(x^-,x^+) = \{x \in \mathcal{M}_\alpha | \ x \text{ is } \alpha\text{-asymptotic to } x^- \text{ and } \omega\text{-asymptotic to } x^+\}.$$
$$\mathcal{M}_\alpha^-(x^-,x^+) = \{x \in \mathcal{M}_\alpha | \ x \text{ is } \omega\text{-asymptotic to } x^- \text{ and } \alpha\text{-asymptotic to } x^+\}.$$

It is shown that if α is rational then $\mathcal{M}_\alpha$ is the disjoint union of $\mathcal{M}_\alpha^{per}$, $\mathcal{M}_\alpha^+$ and $\mathcal{M}_\alpha^-$, this in turns implies the following

Theorem 6.2.8 ([Ba], Theorem 6.7)**.** *A minimal geodesic c with $\alpha(c) \in \mathbb{Q}$ is either periodic or is contained in a strip between two periodic minimal geodesics c^-, c^+ with $\alpha(c^-) = \alpha(c^+) = \alpha(c)$. In each direction c is asymptotic to exactly one of c^- and c^+. There are no periodic minimal geodesics in the strip between c^- and c^+.*

Last but not the least, we will also state the existence theorem which will be useful later. Bangert also showed that if $x^- < x^+$ are neighboring elements of $\mathcal{M}_\alpha^{per}$, then both $\mathcal{M}_\alpha^+(x^-,x^+)$ and $\mathcal{M}_\alpha^-(x^-,x^+)$ are non empty, and hence the following theorem

Theorem 6.2.9 ([Ba], Theorem 6.8)**.** *In every strip between two periodic minimal geodesics c^-, c^+ which does not contain other periodic minimal geodesics there exist minimal geodesics c and c^* such that c is α-asymptotic to c^- and ω-asymptotic to c^+ (and vice versa for c^*).*

Before we move on, let us remark that why these results do not generalize to even dimension three. A counterexample is the Hedlund metric on $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. This metric is obtained by taking the standard Euclidean metric, then in small tubes around each of three disjoint lines parallel to the coordinate axes, the metric is decreased by a factor which shrinks to ϵ^2 as one moves toward the line, we obtain the Hedlund metric on T^3.

For example, Theorem 6.2.7 states that on a two dimensional Riemannian torus, a closed geodesics which is shortest in its homotopy class is a minimal geodesic. However the same is not true for T^3 with Hedlund's metric. For more on this, please see [Ba1].

6.3 Geodesics that Characterize Flat Tori

Suppose that $M = T^2$ is a two dimensional Riemannian torus, an effective way to show that M is flat is to demonstrate that there are no conjugate points. In 1948 E.Hopf proved that any two dimensional Riemannian torus without conjugate points is flat.

Theorem 6.3.1 ([Hopf]). *Two dimensional Riemannian tori without conjugate points are flat.*

The higher dimensional version of this theorem was proved by D. Burago and S. Ivanov in 1994.

Theorem 6.3.2 ([BuIv]). *n-dimensional Riemannian tori without conjugate points are flat.*

In order to show that a two dimensional Riemannian torus M is flat, below is a theorem proven by Innami which characterize Riemannian tori without conjugate points.

Theorem 6.3.3 (See Corollary 3.2, [In]). *Let $M = T^2$ be a Riemannian 2-torus, if each non-trivial free homotopy class of closed curves $[\alpha] \in \pi_1(M)$ contains a family of closed geodesics that foliates M, then M has no conjugate points.*

In the next section we will begin our proof of Conjecture 5.5.1 for two dimensional, compact orientable, non-simply connected Riemannian manifolds.

6.4 Insecurity of Non-Simply Connected Compact Surfaces

Throughout this section, we let M be a smooth, compact, complete, orientable 2-dimensional Riemannian manifold. By a *geodesic segment* we mean a geodesic $\gamma : [0, a] \to M$ where a is the length of γ.

Our main result is that Conjecture 5.5.1 is true for orientable, non-simply connected surfaces:

Theorem 6.4.1 ([Ho1]). *Let M be a compact, complete, orientable 2-dimensional Riemannian manifold with $\pi_1(M) \neq \{0\}$, then M is uniformly secure if and only if M is a flat torus.*

Similar results for 2-dimensional tori have also been obtained independently by V. Bangert and E. Gutkin in a preprint posted on ArXiv(ID arXiv:0806.3572), using an analogous but a somewhat different method, please see [BaGu].

The if direction of this theorem has been proven in Proposition 5.3.1, therefore we only need to prove the only if part of this theorem. First of all, let us recall the following classification theorem for compact orientable two dimensional Riemannian manifolds.

Theorem 6.4.2. *The homeomorphism classes of two dimensional manifolds are determined by the genus.*

What we will do is separate the surfaces according to their genus. In the next session we will prove some key lemmas and then we will use the lemmas to investigate surfaces of different genus.

Definition 6.4.3. *Given a Riemannian manifold M, let γ be a periodic minimal geodesic and x be a point not on γ, we define the* distance *between x and γ, denoted by $d(x, \gamma)$, to be the shortest geodesic distance between them as subsets of M, i.e.*

$$d(x, \gamma) = \inf\{d_g(x, y) | y \in \gamma\}$$

Definition 6.4.4. *Let γ be a periodic minimal geodesic and c be a geodesic ray. c is said to be* asymptotic *to γ if c does not intersect γ and for all $\epsilon > 0$, there exists $s_0 > 0$ such that*

$$d(c(s), \gamma) < \epsilon, \text{ whenever } s > s_0$$

The first lemma is a simple fact in differential geometry:

Lemma 6.4.5. *Two minimal geodesics emanating from the same point will not be distance minimizing beyond their first point of intersection.*

For a proof of the lemma, see for instance, [ChEb].

We want to relate the blocking number of M with the properties of periodic minimal geodesics discussed in Session 6.2. We will next present a crucial lemma that shows a condition such that M has infinite blocking number. Let us first remark that if γ is a non-trivial, simple periodic minimal geodesic, then for any sufficiently small $\epsilon > 0$, the set

$$U_\epsilon(\gamma) := \{x \in M | d(x, \gamma) \leq \epsilon\}$$

has γ as its retract. Note that $U_\epsilon(\gamma)$ is diffeomorphic to an annulus, so if we consider the universal cover $\tilde{M}$ of M, together with the lifted metric $\tilde{g}$, $U_\epsilon(\gamma)$ is lifted to a infinite strip. γ will also be lifted to an infinitely long minimal geodesic line $\tilde{\gamma}$ with respect to $\tilde{g}$.

Here we will use a slight abuse of notations, if we say γ is a minimal geodesic of a compact manifold, we mean that its lift $\tilde{\gamma}$ in the universal cover is a minimal geodesic according to definition 6.1.1.

Lemma 6.4.6. *If M has a closed minimal geodesic γ and another minimal geodesic ray c, such that c is asymptotic to γ, then M is insecure.*

Proof: Suppose that M has a closed minimal geodesic γ and another minimal geodesic ray c, such that c is asymptotic to γ. We will show that there is a point x on c and a point y on the closed geodesic γ, such that $B(x, y) = \infty$. To begin with the proof of the claim, let y be any point on γ, and let us consider the lift $\tilde{\gamma}$ of γ to the universal cover $\tilde{M}$, y would then be lifted to a countable discrete set of points in a long strip, let us denote this set by $\{\tilde{y}_i\}$.

At the same time, we see that $\tilde{c}$, the lift of c in $\tilde{M}$, is a minimal geodesic ray asymptotic to $\tilde{\gamma}$. If we let $\tilde{x}$ to be a point on $\tilde{c}$, we can connect $\tilde{x}$ to each $\tilde{y}_i$ by a minimal geodesic $\tilde{\gamma}_i$ in $\tilde{M}$. Each of these $\tilde{\gamma}_i$ projects to M a geodesic γ_i connecting x and y, where x is the projection of $\tilde{x}$ onto M.

Now if we pick the point $\tilde{x} \in \tilde{c}$ that is sufficiently close to $\tilde{\gamma}$, then there must be infinitely many $\tilde{\gamma}_i$ such that each of them approaches $\tilde{\gamma}$ in the same direction as the asymptotic geodesic ray $\tilde{c}$.

We see that none of these $\tilde{\gamma}_i's$ can intersect $\tilde{c}$ at points other than $\tilde{x}$, this is because if we suppose that $\tilde{\gamma}_i$ and $\tilde{c}$ intersect, $\tilde{\gamma}_i$ must intersect $\tilde{c}$ again before it reaches $\tilde{y}_i$. However both $\tilde{\gamma}_i$ and $\tilde{c}$ are minimal, this contradicts Lemma 6.4.5, which states that two minimal geodesics emanating from the same point will not be minimal beyond their first point of intersection.

Since c is asymptotic to γ, the lift $\tilde{c}$ is asymptotic to $\tilde{\gamma}$. In particular if U is any sufficiently small neighborhood of γ, we lift the neighborhood U to $\tilde{U}$, $\tilde{c}$ will enter and stay inside $\tilde{U}$ after some finite length. i.e. There exists an $s_0 > 0$ such that $\tilde{c}(s) \in \tilde{U}$ for all $s > s_0$.

Now let us fix x and y as above and show that $B(x, y) = \infty$. Arguing by contradiction, let us assume that $\{z_i\}$ is a finite set of points that block all geodesics $\{\gamma_i\}$. Throw away several points from $\{z_i\}$ if necessary, we could assume that all z_i are not on γ. Let us lift the set $\{z_i\}$ to $\tilde{M}$, then after re-indexing, denote by $\{\tilde{z}_j\}$ the countable set of points which projects to $\{z_i\}$. The set $\{\tilde{z}_j\}$ should block all geodesics $\{\tilde{\gamma}_i\}$. Since this blocking set is discrete, there exists a neighborhood U of γ in M such that no z_i belongs to U. As mentioned above the lifted geodesic $\tilde{c}$ enters and stay in $\tilde{U}$ after a finite length, and since each $\tilde{\gamma}_i$ cannot intersect $\tilde{c}$ before it hits the point $\tilde{y}_i$, there is a uniform t_0 such that for each i, $\tilde{\gamma}_i(t)$ is in $\tilde{U}$ for all $t > t_0$.

As none of these $\tilde{z}_i$ are in $\tilde{U}$, therefore if the infinite set of geodesics γ_i is to be blocked by the finite set $\{z_i\}$, each γ_i must be blocked by some z_j along the segment $\{\gamma_i|_{[0,t_0]}\}$. As all $\tilde{\gamma}_i(t)$ lies completely in $\tilde{U}$ whenever $t > t_0$. This means that some of $\{\tilde{\gamma}_i|_{[0,t_0]}\}$ have common points (other than $\tilde{x}$) before entering the neighborhood $\tilde{U}$. This contradicts minimality of all $\tilde{\gamma}_i$ and the lemma is proven.□

Let us now assume that M is a two dimensional torus. The metric g is said to be *bumpy* if all periodic geodesics of M are non-degenerate. We will now show that a bumpy torus has infinite blocking number.

Proposition 6.4.7. *If M is a bumpy torus, then it is insecure.*

Proof: Firstly, let γ be a shortest periodic geodesic of M. Then γ is a hyperbolic geodesic due to Morse [Mo]. This implies that γ is isolated. Now we pick another periodic minimizing geodesic γ' in the same homotopy class, such that there is no periodic geodesic from this homotopy class lies in the annulus bounded by γ and γ'. This is possible since γ is isolated: if γ is the only closed geodesic in the homotopy class that we can consider $\gamma' = \gamma$ such that the lift of γ' is next to the lift of γ.

Now we observe that the annulus bounded by γ and γ' does not contain any periodic minimal geodesic. Recall that all higher iterates of a periodic geodesics are identified, so geodesics from all other homotopy class must either intersect γ or itself, but we know that a geodesic that intersect itself cannot be minimizing. Therefore we conclude that for the strip in $\mathbb{R}^2$ between the lift of γ and γ', there cannot exist a curve which is the lift of a periodic minimal geodesic.

Now by Theorem 6.2.9, there exists a minimal geodesic such that c is ω-asymptotic to γ. So in particular, if U is any small neighborhood of γ in T, c would stay inside U after a finite length. By Lemma 6.4.6 we can conclude that M has infinite blocking number and it is insecure.□

Note that Proposition 6.4.7 is an immediate corollary of our main result.

6.5 Proof of The Main Result

We now furnish the proof of Theorem 6.4.1. As stated in Theorem 6.4.2, we could separate the surfaces in term of their genus $\mathfrak{g}$. We will first prove that when $\mathfrak{g} = 1$, if M is uniformly secure then it is a flat torus. We will then prove that when $\mathfrak{g} > 1$, M is insecure.

6.5.1 genus $\mathfrak{g} = 1$

Let M be a topological two dimensional torus, we want to show that if M is uniformly secure, then M must be a flat torus.

Our approaches are as follows: We will argue by first assuming that M is uniformly secure. Then for each free homotopy class $[\alpha]$ of M, we call an annulus *bad* if the annulus is bounded by two periodic minimal geodesics from $[\alpha]$ such that no other periodic minimal geodesics exists in the annulus. We will show that no bad annulus exists in M. Afterwards we prove that having no bad annulus means that M can be foliated by periodic minimal geodesics of the class $[\alpha]$. This in turns will imply that M has no conjugate points and so by Theorem 6.3.1 M must be a flat torus.

Firstly, let us state a lemma that is similar to Lemma 6.4.6.

Lemma 6.5.1. *If M has two periodic minimal geodesics from the same homotopy class, such that the annulus bounded between does not contain other periodic minimal geodesic, i.e. if M has a bad annulus, then M is insecure.*

Proof: Denote one of the periodic minimal geodesics by γ. According to Theorem 6.2.9, there exists a minimal geodesic line c such that c is ω-asymptotic to γ. So in particular, c is asymptotic to γ according to Definition 6.4.4. Therefore we can apply Lemma 6.4.6 to conclude that the blocking number of M is infinite.□

Note that in the above lemma, we can replace the phrase "does not contain other periodic minimal geodesic" by "does not contain other periodic minimal geodesic of the same homotopy class". It is because any periodic geodesic of other homotopy classes must intersect either one of the boundaries of the annulus or is a higher iteration of itself.

Let $[\alpha] \in \pi_1(M)$, there exists a closed geodesic of minimal length in the free homotopy class $[\alpha]$. According to Theorem 6.2.7, this is a periodic minimal geodesic. The following proposition shows that if no periodic minimal geodesics in $[\alpha]$ bound a bad annulus, then M is foliated by periodic minimal geodesics of this homotopy class $[\alpha]$.

Proposition 6.5.2. *Fix $[\alpha] \in \pi_1(M)$, if for any two periodic minimal geodesics γ and γ' of $[\alpha]$ there is a periodic minimal geodesic of that lies in the annulus bounded by γ and γ', then M is foliated by periodic geodesics in $[\alpha]$.*

Proof: Let γ be the shortest periodic geodesic from the free homotopy class $[\alpha]$, this closed geodesic lifts to $(\mathbb{R}^2, \tilde{g})$ to a minimal geodesic line $\tilde{\gamma}$. Let γ' be another periodic minimal geodesic from $[\alpha]$ and $\tilde{\gamma}'$ be the corresponding lift. If there is no such geodesic then we let $\gamma = \gamma'$ and $\tilde{\gamma}'$ be the minimal geodesic in $\mathbb{R}^2$ neighboring $\tilde{\gamma}$ and projects to γ. Next, we assert the following is true.

Claim: For any point x in the strip bounded by $\tilde{\gamma}$ and $\tilde{\gamma}'$, there is a minimal geodesic c_x passing through x such that $\alpha(c_x) = \alpha(\gamma) = \alpha(\gamma')$.

PROOF of the claim: To prove this claim, assume that the minimal geodesics $\tilde{\gamma}$ and $\tilde{\gamma}'$ have minimal period (q, p). Let $x \in (\mathbb{R}^2, \tilde{g})$ be a point that lies in the strip between them. Denote by $x_1 = T_{(q,p)}x$, and $x_{i+1} = T_{(q,p)}x_i$, $\forall i \in \mathbb{N}$. Then we connect each x_i to x by one minimal geodesic c_i.

Note that each c_i cannot touch or cross transversely each of $\tilde{\gamma}$ and $\tilde{\gamma}'$. It is because if c_i crosses say, $\tilde{\gamma}$ transversely, then it has to cross it at least twice, and we know that geodesics that intersect each other twice cannot be minimal beyond the intersections. If c_i touches $\tilde{\gamma}$ then this will contradict the uniqueness of geodesic for a given initial point and tangent vector. This means all c_i stay in the strip.

Now, let v_i be a vector in U_xM, the unit tangent sphere of M at x, such that $v_i = c_i'(0)$. Then $\{v_i\}$ is a set of vectors in a compact sphere, so there is a convergent subsequence with the limit $v = \lim_{i\to\infty} v_i$. Let $c(s)$ be the forward complete geodesic satisfying $c(0) = x$, $c'(0) = v$. We now show that $c(s)$ stays the strip for all $s > 0$. Suppose this is not true, then say, $c(s)$ intersects $\tilde{\gamma}$ transversely at some point. Then there is an $\epsilon > 0$ and $s_0 > 0$ such that $c(s_0)$ is not in the strip and lay outside of the ϵ-neighborhood of $\tilde{\gamma}$.

Since a geodesic is a solution of a second order ODE, the solutions with initial point x continuously depend on the initial vectors v. Also as M is geodesic complete, the geodesic flow $\phi_s(x, v)$ on the unit tangent bundle of T is defined for all $s > 0$. Let $f : UM \to T$ be the composition of the time-s_0 map restricted to U_xM, $\phi_{s_0}(x, \cdot) : U_xM \to UM$ with the projection $\pi : UM \to M$ given by $(x, v) \mapsto x$, i.e. $f = \pi \circ \phi_{s_0}$. Then f continuously depends on v. So for the geodesic c there exists a $\delta > 0$ such that for all geodesics $\bar{c}$ with $\bar{c}(0) = x$ and $\bar{c}'(0) = \bar{v}$ for $\|v - \bar{v}\| \leq \delta$, we have $d_{\tilde{g}}(\bar{c}(s_0) - c(s_0)) \leq \epsilon$.

Recall that $v = \lim_{i\to\infty} v_i$, c_i is the geodesic at x with initial vector v_i. By the arguments above there exists arbitrarily long c_i such that $c_i(s_0)$ lies outside the strip. In particular c_i crosses the boundary of the strip transversely. However this is not possible since c_i connects two points in the strip, if it crosses the boundary then it will intersect the geodesic boundary twice, this contradicts the assumption that c_i is a minimal geodesic.

This means that c must stay inside the strip, it is easy to see that $\alpha(c) = \alpha(\gamma') = \alpha(\gamma)$ and so the claim true.

Now we can apply Theorem 6.2.8, which states that c is either periodic or is contained in a strip between two periodic minimal geodesics c^- and c^+. In the latter case the strip between c^- and c^+ contains no other periodic minimal geodesic.

This will contradict our hypothesis so c can only be periodic. This means each point $x \in M$ lies on a periodic geodesic in $[\alpha]$ and the proposition is proven.$\Box$

We can now finalize the proof of Theorem 6.4.1 when the genus $\mathfrak{g} = 1$.

Proposition 6.5.3. *If M is an uniformly secure 2-dimensional torus , then M is flat.*

Proof: Assume that M is uniformly secure. According to Lemma 6.5.1, if $[\alpha]$ is any free homotopy class, then between any two periodic minimal geodesics of $[\alpha]$ there exists another periodic geodesic. Proposition 6.5.2 then implies that periodic minimal geodesics of $[\alpha]$ foliate M. Since the free homotopy class $[\alpha]$ is arbitrary, we conclude that periodic minimal geodesics in any fixed free homotopy class foliate M. Therefore by Theorem 6.3.3, M does not have conjugate points, and so by Hopf's Theorem 6.3.1, M must be a flat torus.$\Box$

6.5.2 genus $\mathfrak{g} > 1$

We now let M be a closed surface of genus $\mathfrak{g}$, where $\mathfrak{g} \geq 2$. Let g be any Riemannian metric on M.

Let us recall Theorem 5.4.3 that is proven by Burns and Gutkin, it states that a uniformly secure closed Riemannian manifold has zero topological entropy. Let us also recall the following well known fact:

Proposition 6.5.4. *If M is a closed Riemannian surface with genus greater than or equal to 2, then the topological entropy of its geodesic flow is strictly positive.*

Combining Theorem 5.4.3 and Proposition 6.5.4 we obtain the following proposition:

Proposition 6.5.5. *If M is a closed Riemannian surface with genus greater than or equal to 2, then M is insecure.*

In this session, we will provide an alternate proof of Proposition 6.5.5, we will show that the geometrical method used in the torus case can be extended to surfaces of higher genus. Let us begin with a definition.

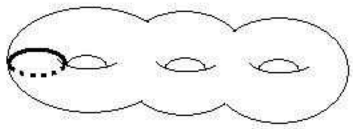
Definition 6.5.6. *Let γ be a closed geodesic on M. A* collar *of γ is the image of a diffeomorphism $f : \mathbb{S}^1 \times [0, a] \to M$ with $f(\mathbb{S}^1 \times \{0\}) = \gamma$, $a \in \mathbb{R}^+$.*

In the followings, a *periodic minimal geodesic* is a closed geodesic with the minimal length in the homotopy class. A *cylinder* is a manifold diffeomorphic to the product of an interval (open or closed) and $\mathbb{S}^1$.

We will show that a surface of higher genus has a closed geodesic and a collar, such that the collar contains an asymptotic geodesic. Once this is established the proof of our desired result is trivial.

Let $[\alpha]$ be the homotopy class of M which represents the cross section of a "handle" of M [Figure 6.1]. We know there is at least one periodic minimal geodesic $\gamma \in [\alpha]$. Let us also note that periodic minimal geodesics of $[\alpha]$ are well ordered:

Figure 6.1. A geodesic from $[\alpha]$



Lemma 6.5.7. *Periodic minimal geodesics of $[\alpha]$ form a totally ordered set.*

Proof: Recall that no two periodic minimal geodesics of $[\alpha]$ intersect, this is because if they do, then their lifts to the universal cover will intersect more than one time, this contradicts the assumption that both geodesics are minimal. Since we also assume that the genus of M is greater than 1, closed geodesics of $[\alpha]$ cannot homotopically pass the portion of M that contains other handles. Therefore we conclude that periodic minimal geodesics of $[\alpha]$ form a totally ordered set.□

Proposition 6.5.8. *If M is a surface of genus $g \geq 2$, then there exists a closed minimal geodesic γ and a collar such that except γ, there is no closed minimal geodesic homotopic to γ that intersects the collar.*

Proof: Just as above, let $[\alpha]$ be a homotopy class of M which represents the cross section of a "handle" of M. Our claim is that there is a periodic minimal geodesic of $[\alpha]$ with a collar such that no geodesic of $[\alpha]$ intersects the collar.

By Lemma 6.5.7, periodic minimal geodesics of $[\alpha]$ form a totally ordered set. We also know that periodic minimal geodesics are the critical points of the length functional, the set of minimal geodesics is closed. Therefore with respect to this ordering on the minimal geodesics of $[\alpha]$, there is a "maximum" γ and "minimum" γ' so that all periodic minimal geodesics of $[\alpha]$ lie in the cylinder bounded by γ and γ'.

If γ and γ' are the same geodesic, then there are two possibilities: either there are more than one or there is only one periodic minimal geodesics in the class $[\alpha]$. If there are more than one periodic minimal geodesic in $[\alpha]$, the geodesic γ has to homotope along the handle and overlap with itself. However since the genus of M is strictly greater than 1, this cannot be done. Therefore we conclude that in the case when γ and γ' are the same geodesic, there could only be one periodic minimal geodesic.

As there is only one periodic minimal geodesic γ in the class $[\alpha]$, we can find a neighborhood of γ such that its retract is γ. Since there is only one minimal geodesic, the neighborhood contains a collar, and the only periodic minimal geodesic contained in the collar is γ itself, thus the Proposition is true when γ and γ' are the same geodesic.

Suppose that γ and γ' are two distinct periodic minimal geodesics. Note again that γ cannot intersect or touch γ'. Hence if we denote by K the cylinder bounded by γ and γ', there is a neighborhood U of K that is diffeomorphic to an open cylinder. Since there is no periodic minimal geodesic from $[\alpha]$ outside K, the set $(U \cup \gamma)\backslash K$ contains a closed geodesic γ and a collar such that except γ, there is no closed geodesic homotopic to γ that intersect the collar, finishing the proof of the Proposition. $\Box$

Proposition 6.5.9. *If M is a surface of genus $g > 1$, then M has a closed minimal geodesic and another minimal geodesic asymptotic to it.*

Proof: According to Proposition 6.5.8, M has a periodic minimal geodesic γ and a collar C such that except γ, there is no closed geodesic homotopic to γ intersects C.

Note again that C is diffeomorphic to a cylinder, let us cut an open neighborhood U of C out such that U is diffeomorphic to an open cylinder, then γ is the shortest closed geodesic in U. After that we smoothly glue a closed Riemannian cylinder to U, we would then obtain a 2 dimensional Riemannian torus.

Since γ is the shortest closed curve in the U, we can glue the cylinder to U such that γ remains to be the shortest closed geodesic in the homotopy class. For instance, we may assume that all non-contractible closed curves in the cylinder have lengths not shorter than γ, which is possible since the length of each boundary of U is not shorter than γ. This way we can guarantee that γ, as a closed curve of the torus, is the shortest in its free homotopy class.

We know that the geodesic behavior of a surface is completely determined by its metric. In particular, the geodesic behavior of C as a subset of M is the same as that of C as a subset of the glued 2 dimensional torus. Now as a subset of the glued torus, C does not contain another periodic minimal geodesic homotopic to γ. If we consider the universal cover $\mathbb{R}^2$ of the glued torus, γ is lifted to a minimal geodesic line $\tilde{\gamma}$ and the collar C is lifted to a strip $\tilde{C}$. Let $\tilde{\gamma}'$ be the lift of an adjacent periodic minimal geodesic from the same homotopy class such that $\tilde{\gamma}$ and $\tilde{\gamma}'$ bound $\tilde{C}$. Then either one of the followings is true:

a) $\tilde{\gamma}'$ is the lift of a distinct periodic minimal geodesic.

b)$\tilde{\gamma}'$ is another lift of γ which is adjacent to $\tilde{\gamma}$.

In case **a** we claim that $\tilde{\gamma}'$ cannot intersect $\tilde{C}$. To see why this is true, let us first recall that periodic minimal geodesics of a torus are exactly the lift of the closed geodesics which have minimal length in their free homotopy class, see Theorem 6.2.7.

If γ_1 is a closed curve homotopic to γ that lies completely in C, then the length of γ_1 is strictly greater than the length of γ, this is because by construction there is no other periodic minimal geodesic in C.

Now we assume that γ_1 is a closed curve that lies partially in C. Note that in the gluing process we have ensured that all closed curve outside C is longer than

γ. Together with the fact that both boundaries of C are not shorter than γ, we see that γ_1 cannot be shorter than γ.

What matters in either case **a** or **b** is that $\tilde{\gamma}'$ and $\tilde{\gamma}$ bound a strip that contains $\tilde{C}$, and the strip does not contain any lift of a periodic minimal geodesic in the same homotopy class. So we can apply Theorem 6.2.9 to show that there is a minimal geodesic c in the strip which is ω-asymptotic to $\tilde{\gamma}$. Now we consider the portion of the minimal geodesic c which stays in $\tilde{C}$, this geodesic projects to a minimal geodesic asymptotic to γ.

Finally, since the geodesic behavior of an area is determined locally by the metric, if we revert the cutting and gluing procedures that change M to a 2 dimensional torus, the geodesic c staying in the collar C remains exactly the same. Therefore on the surface M, c is a minimal geodesic asymptotic to the periodic minimal geodesic γ and the Proposition is proven.□

With the above propositions, we can now prove Proposition 6.5.5.

Proof of Proposition 6.5.5: By Proposition 6.5.9, M has a closed minimal geodesic and another minimal geodesic asymptotic to it. Therefore we can apply Lemma 6.4.6 to conclude that M is insecure. □

Combining the statements of Propositions 6.5.3 and 6.5.5, and using Theorem 6.4.2, we have proven Theorem 6.4.1.

6.6 Spheres with knobs

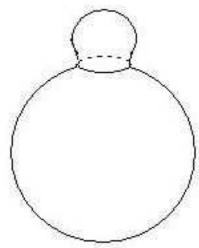
When the surface is diffeomorphic to a sphere, the story is a little bit different. In general the arguments above cannot be applied to Riemannian surfaces diffeomorphic to $\mathbb{S}^2$. For instance, we know from above that a non-flat torus has infinite blocking number because it has a minimal geodesic ray asymptotic to a closed minimal geodesic. On the other hand say, the round sphere, it also has infinite blocking number, but this is completely different to the torus case: the round sphere does not even have a minimal closed geodesic.

Nevertheless, we can say something about the sphere provided that an additional geometrical condition is satisfied. This condition is the possession of knobs [Ri]. We now let M be a Riemannian surface diffeomorphic to $\mathbb{S}^2$.

Definition 6.6.1. *A* knob *is a finite portion K of the surface M such that:*

a) K is diffeomorphic to an Euclidean disc.
b) the geodesics tangent to the boundary ∂K lie interior to K in the immediate neighborhood of the point of contact.
c) boundaries of different knobs have no points in common.

Figure 6.2. $\mathbb{S}^2$ with a knob



Given this setting, we can extend our arguments above to obtain the following theorem:

Theorem 6.6.2. *If M is a Riemannian 2-sphere with at least two knobs K_1 and K_2, then M is insecure.*

Proof: Let $M' = M \setminus (K_1 \cup K_2)$, then M' is a surface diffeomorphic to an annulus. Now by Theorem 5 of [Ri], there exists a periodic minimal geodesic with respect to the annulus M' . Denote by K the set of periodic minimal geodesics of this type, then K is a closed set.

Similar arguments as the non-simply connected case implies that there are two elements γ and γ' in K such that all other elements in K are bounded between γ and γ'. Now neither γ nor γ' could touch $\partial M'$ or this would contradicts condition (b) for the knob. Therefore similar to the cases where genus > 1, we have a collar such that except γ, there is no closed geodesics homotopic to γ that intersect the collar. Hence just as Proposition 6.5.9, there is an minimal geodesic ray asymptotic to γ, finishing the proof of the theorem. $\square$

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