TRANSIENT MARKOV SHIFTS

A Dissertation in

Mathematics

by

Van T. Cyr

© 2010 Van T. Cyr

Submitted in Partial Fulfillment
of the Requirements
for the Degree of

Doctor of Philosophy

August 2010
The dissertation of Van T. Cyr was reviewed and approved* by the following:

Omri Sarig  
Associate Professor of Mathematics  
Dissertation Adviser  
Chair of Committee

John Fricks  
Assistant Professor of Statistics

Anatole Katok  
Raymond N. Shibley Professor of Mathematics

Yakov Pesin  
Distinguished Professor of Mathematics

John Roe  
Professor of Mathematics  
Chair of Mathematics Department

*Signatures are on file in the Graduate School.
Abstract

In this work we investigate questions relating to the existence, commonness and thermodynamic formalism of countable Markov shifts exhibiting the phenomenon of transience. We obtain a complete answer to the question of which countable Markov shifts have transient potentials and give a topological description of their commonness within the set of all potentials of summable variations and finite (Gurevich) pressure. Finally we investigate the thermodynamic formalism of transient Markov shifts. In the locally compact case it is shown that a conformal measure and (generalized) thermodynamic limit always exist.
# Table of Contents

Acknowledgments .................................................. vii

Chapter 1. Introduction ........................................ 1

1.1 Thermodynamic Formalism for Dynamical Systems .......... 1

1.1.1 DLR States .................................................. 3

1.1.2 Thermodynamic Limits .................................... 3

1.1.3 Equilibrium Measures ..................................... 5

1.2 Survey of Thermodynamic Formalism for Countable Markov Shifts . 6

1.2.1 Countable Markov Shifts .................................. 6

1.2.2 Gurevich Pressure ......................................... 7

1.2.3 Modes of Recurrence ...................................... 11

1.2.3.1 Positive Recurrent Potentials ......................... 13

1.2.4 Strong Positive Recurrence ............................... 14

1.2.4.1 Null Recurrent Potentials .......................... 17

1.2.4.2 Transient Potentials ................................. 18

1.3 Summary of Results .......................................... 19

1.3.1 Aim .......................................................... 19

1.3.2 Existence of Transient Potentials ....................... 19

1.3.3 Topology of Transient Potentials ....................... 20

1.3.4 Thermodynamics of Transient Potentials ............... 21
Chapter 2. Existence of Transient Potentials  

2.1 Setting  

2.1.1 Overview of Proof of Theorem 2.1  

2.2 A Finite Uniform Rome implies Recurrence  

2.3 Proof of The Extension Theorem  

2.3.1 Technical Lemmas  

2.3.2 Proof of Theorem 2.2  

2.4 Shifts with Transient Potentials  

2.4.1 Infinite Rays  

2.4.2 Double Bouquets  

2.4.3 Simple Loops with Unbounded Size  

2.5 Proof of Theorem 2.1  

2.6 Appendix A  

2.7 Appendix B  

Chapter 3. Topology of Transient Potentials  

3.1 Strong Positive Recurrence  

3.2 Topology of Transient Potentials  

3.2.1 Proof of Theorem 3.1  

3.3 Proof of theorem 3.2  

Chapter 4. Thermodynamics of Transient Shifts
Acknowledgments

I am deeply grateful to my thesis adviser, Omri Sarig, whose guidance and encouragement made this work possible. I couldn’t have asked for a better adviser nor can I express how much I have enjoyed our conversations, mathematical and otherwise, over the past several years.

I would also like to thank my family, especially my parents, Van and Marcia, and my siblings, Nicole and Chris, for their constant support.
Chapter 1

Introduction

1.1 Thermodynamic Formalism for Dynamical Systems

In a dynamical system\(^1\) \((X, T)\) one of the main problems is to describe the structure of the orbits\(^2\) and the properties of averages along them. Ergodic theory is a branch of dynamical systems which attempts to solve this problem by introducing a \(T\)-invariant\(^3\) probability measure \(\mu\) and, for each \(f \in L^1(\mu)\), viewing the sequence of functions \(\{f, f \circ T, f \circ T^2, \ldots\}\) as a stochastic process (and using probability theory to study its properties). There is \textit{a priori} an issue to address: for a given dynamical system \((X, T)\) there are usually infinitely many \(T\)-invariant probability measures on \(X\) and the ergodic measures are mutually singular (meaning that statistical properties of the stochastic processes arising from one ergodic measure may be very different from those arising from another).

A similar problem is encountered in the theory of thermodynamics in statistical physics.\(^4\) Macroscopic quantities, such as temperature, associated to a system with a large number of particles are computed as weighted averages of microscopic quantities

---

\(^1\)Here \(X\) is a topological space and \(T : X \to X\) is a continuous map.
\(^2\)For \(x \in X\), the \textit{orbit} of \(x\) is the set \(\{x, Tx, T^2x, \ldots\}\).
\(^3\)\(\mu\) is \(T\)-invariant if for every measurable set \(A\), \(\mu(T^{-1}A) = \mu(A)\).
\(^4\)Much of the motivation presented in section 1.1 was discussed in a course attended by the author [21] and is also discussed in [14].
associated with the individual particles (where the average is taken over all of the particles). The question of determining which weights to use is entirely analogous to the issue above, except that here it is possible to empirically study the system to determine the answer.

To be precise, suppose we are studying a many-particle system which is contained in a larger system at equilibrium. The systems can exchange energy but the larger system is so large that the effect of the smaller system on it is negligible. We also assume that the collection of all possible states of the smaller system is finite (but very large so as to approximate an actual physical system). If the total energy of the (smaller) system in state \( x \) is \( U(x) \), then the theory of thermodynamics dictates that the probability that the system will be in state \( x \) is \( P(x) = Ce^{-U(x)/kT} \) (here \( k \) is Boltzmann’s constant, \( T \) is the temperature and \( C \in \mathbb{R} \) is a normalizing constant that depends on \( T \)). This probability distribution is called the canonical ensemble.

Thermodynamic formalism is a branch of ergodic theory that attempts to solve our original problem, selecting the “right” measure for a dynamical system, by analogy with the canonical ensemble (see [13, 14, 22] for example). Of particular interest in this dissertation are dynamical systems modeling a one-dimensional lattice gas (i.e. a semi-infinite array of “sites” each of which can be occupied by one of a number of species of particles called \( 0, 1, \ldots, m - 1 \)). The set of states of the system is \( \{0, \ldots, m - 1\}^\mathbb{N} \) (or some subset of this consisting of “allowable configurations”). Now \( X \) is uncountable so we cannot define the canonical ensemble in its original form (the normalizing constant \( C \) would be zero). Instead, there are three weak versions of the canonical ensemble which are of interest.
1.1.1 DLR States

One approach to defining the “canonical ensemble”, pioneered by Dobrushin [7, 8, 9] and, independently, Lanford and Ruelle [14], is a Gibbs state.

Suppose $U : X \to \mathbb{R}$ is a continuous function, where $U(x_0, x_1, \ldots)$ represents the total energy of particle $x_0$ interacting with $x_1, x_2, \ldots$. Let $T : X \to X$ be the map $T(x_0, x_1, \ldots) := (x_1, x_2, \ldots)$. Instead of the canonical ensemble, we can look for probability measures whose conditional measures satisfy a property similar to the one that appears in its definition:

$$m(x_0, \ldots, x_{n-1} | x_n, x_{n+1}, \ldots)(x) = C(n)e^{-\beta \sum_{k=0}^{n-1} U(T^k x)}$$

where $C(n) \in \mathbb{R}$ is a normalizing constant (since there are only finitely many configurations of the first $n$ sites), $\beta$ is the “inverse temperature,” and

$$m(x_0, \ldots, x_{n-1} | x_n, x_{n+1}, \ldots)(x) := \mathbb{E}_m(1_{\{y \in X : y_i = x_i \text{ for } i = 0, \ldots, n-1\}} | T^{-n} \mathcal{B})(x)$$

is the conditional expectation onto the $n^{th}$ preimage of the Borel $\sigma$-algebra. A (probability) measure whose conditional measures satisfy the above equation for every $n \in \mathbb{N}$ is called a Dobrushin-Lanford-Ruelle (DLR) measure (or Gibbs state).

1.1.2 Thermodynamic Limits

Another approach to defining the canonical ensemble is a thermodynamic limit (see below). The motivation is the following: given a vector $(U_1, \ldots, U_n) \in \mathbb{R}^n$, consider
the function
\[ F(p_1, \ldots, p_n) := \sum_{i=1}^{n} p_i U_i + \frac{1}{\beta} \sum_{i=1}^{n} p_i \log p_i. \]

The vector \( p_i := \frac{e^{-\beta U_i}}{\sum_{j=1}^{n} e^{-\beta U_j}} \) is the unique minimizer of \( F \) over all probability vectors \((p_1, \ldots, p_n) \in \mathbb{R}^n\). If we imagine that we are modeling a many particle system with a finite number of states and the number \( U_i \) gives the energy of the \( i^{th} \) state, then \( F(p_1, \ldots, p_n) \) is interpreted as the *Helmholtz free energy* (at inverse temperature \( \beta \)) of the probability measure \((p_1, \ldots, p_n)\) (here \( \sum_{i=1}^{n} p_i U_i \) is the energy of \((p_1, \ldots, p_n)\) and \( -\sum_{i=1}^{n} p_i \log p_i \) is the entropy). Thus the canonical ensemble (the probability distribution observed in nature) can be characterized as the minimizer of free energy.

One option, then, is to approximate \( X \) by a sequence of finite subspaces \( X_n \), solve the above variational problem on \( X_n \), and take a limit as “\( X_n \) tends to \( X \).” Specifically, fix a “boundary condition” \( w \in X \) and consider the (finite) space
\[ X_n := \{ x \in X : x_{n+i} = w_i \text{ for all } i = 0, 1, \ldots \} \subseteq X. \]

Among probability measures supported on \( X_n \), there is a unique minimizer of the free energy
\[ \int \sum_{i=0}^{n-1} U(T^i x) d\mu + \frac{1}{\beta} \sum_{x_0, \ldots, x_{n-1}} \mu[x_0 \cdots x_{n-1}] \log \mu[x_0 \cdots x_{n-1}] \]
(where $[x_0 \cdots x_{n-1}] := \{y \in X : y_i = x_i \text{ for } i = 0, \ldots, n - 1\}$) given by\textsuperscript{5}

$$\nu_n^{w} := \frac{\sum_{T^n y = w} e^{-\beta(U(y) + U(Ty) + \cdots + U(T^{n-1}y))} \delta_y}{\sum_{T^n y = w} e^{-\beta(U(y) + \cdots + U(T^{n-1}y))}}.$$  

For fixed $w \in X$, a thermodynamic limit (with boundary condition $w$) is any weak-* limit point of the sequence $\{\nu_n^{w}\}_{n \in \mathbb{N}}$.

In a physical system (the one-dimension Ising model of ferromagnetism, for example) a “phase transition” is said to occur when the thermodynamic limit is not unique (see [12]): at equilibrium, the system is indifferent between being in one of several states. We remark that non-uniqueness may be either because the limit depends on the boundary condition $w$ or because $\{\nu_n^{w}\}$ has multiple limit points for fixed $w$.

### 1.1.3 Equilibrium Measures

A final approach to defining the canonical ensemble is an equilibrium measure. Again we consider the average Helmholtz free energy of the first $n$ particles in the array (at inverse temperature $\beta$):

$$\frac{1}{n} \left( \int \sum_{i=0}^{n-1} U(T^i x) d\mu + \frac{1}{\beta} \sum_{x_0 \cdots x_{n-1}} \mu[x_0 \cdots x_{n-1}] \log \mu[x_0 \cdots x_{n-1}] \right).$$

Now, instead of solving the variational problem at this stage, we fix a $T$-invariant probability measure $\mu$ and take a limit as $n \to \infty$. We obtain the limiting equation

\textsuperscript{5}$\delta_y$ denotes Dirac measure concentrated on $y \in X$. 

\[ \int U \, d\mu - \frac{\log(2)}{\beta} h_\mu(T) \] where \( h_\mu(T) \) is the (measure-theoretic) entropy of the transformation \( T \). So minimizing the average free energy per site in \( X \) corresponds to maximizing the quantity \( h_\mu(T) - \frac{\beta}{\log(2)} \int U \, d\mu \). For future reference, we define \( \phi := -\frac{\beta}{\log(2)} U \) and try to maximize \( h_\mu(T) + \int \phi \, d\mu \). A \( T \)-invariant probability measure achieving this maximum is called an *equilibrium measure*.

### 1.2 Survey of Thermodynamic Formalism for Countable Markov Shifts

The dynamical systems under investigation in this work are countable Markov shifts (as the reader will see, these can be thought of as models of a one-dimensional gas with countably many species of molecules allowed at each site). The study of their properties and thermodynamic formalism began with the work of D. Vere-Jones [24] in the 1960’s for infinite matrices and continued since that time (see [11, 15, 16, 17, 18, 19, 20] for example). Here we describe these and collect several known results.

#### 1.2.1 Countable Markov Shifts

Suppose \( \mathcal{G} \) is an aperiodic, irreducible directed graph\(^6\) with countable vertex set \( \mathcal{S}_\mathcal{G} \) (aperiodicity and irreducibility are equivalent to the condition that for any \( a, b \in \mathcal{S}_\mathcal{G} \) there exists \( N(a,b) \in \mathbb{N} \) so that there is a path from \( a \) to \( b \) of length exactly \( n \) for any \( n > N(a,b) \)). For \( a, b \in \mathcal{S}_\mathcal{G} \), a directed edge from \( a \) to \( b \) will be denoted \( a \rightarrow b \).

\(^6\)A directed graph \( \mathcal{G} \) is aperiodic if, for some \( a \in \mathcal{S}_\mathcal{G} \) there are loops based at \( a \) with relatively prime periods. It is irreducible if it is path connected.
define the space of (one-sided) infinite trajectories in \( \mathcal{G} \) to be

\[
X_{\mathcal{G}} := \{(x_0, x_1, \ldots) \in S^\mathbb{N}_{\mathcal{G}} : x_i \rightarrow x_{i+1} \text{ for every } i = 0, 1, 2, \ldots \}.
\]

We topologize \( X_{\mathcal{G}} \) with the metric

\[
d(x, y) := 2^{-\inf\{i \geq 0 : x_i \neq y_i\}}.
\]

This topology is also generated the cylinder sets: all sets of the form

\[
[a_0 \cdots a_{m-1}] := \{x \in X_{\mathcal{G}} : x_i = a_i \text{ for all } i = 0, \ldots, m-1\}
\]

for \((a_0, \ldots, a_{m-1}) \in \bigcup_{n \in \mathbb{N}} S^n_{\mathcal{G}}\) (and in this topology aperiodicity and irreducibility of \( \mathcal{G} \) imply that \( X_{\mathcal{G}} \) is topologically mixing).

1.2.2 Gurevich Pressure

A function \( \phi : X_{\mathcal{G}} \rightarrow \mathbb{R} \) is called a potential. For a potential \( \phi \) we define:

\[
\text{var}_n(\phi) := \sup\{|\phi(x) - \phi(y)| : x_i = y_i \text{ for } i = 0, 1, \ldots, n-1\}.
\]
A potential has **summable variations** if $\sum_{n=2}^{\infty} \text{var}_n(\phi) < \infty$ and is **weakly Hölder continuous** if there exists $\theta \in (0, 1)$ and $A \in \mathbb{R}$ so that $\text{var}_n(\phi) \leq A \cdot \theta^n$ for all $n \geq 2$ (note that we allow $\text{var}_1(\phi) = \infty$). Of course weakly Hölder continuous potentials also have summable variations.

We assume that $\phi : X_G \to \mathbb{R}$ is a potential with summable variations and set $\phi_n(x) := \phi(x) + \phi(Tx) + \cdots + \phi(T^{n-1}x)$. Fix $a \in S_G$ and define the **Gurevich pressure** of $\phi$ to be the quantity

$$P_G(\phi) := \lim_{n \to \infty} \frac{1}{n} \log (Z_n(\phi, a)), \text{ where } Z_n(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_a(x).$$

O. Sarig showed (in [17]) that if $X_G$ is topologically mixing and $\phi$ has summable variations, the limit defining $P_G(\phi)$ exists and is independent of $a \in S_G$.

The Gurevich pressure is of interest because of the following relation to equilibrium measures:

**Theorem 1.1** (Variational Principle [17]). *Suppose $\phi$ has summable variations and $\sup(\phi) < \infty$. Then

$$P_G(\phi) = \sup\{h_\mu(T) + \int \phi d\mu\}$$

where the supremum is over all $T$-invariant Borel probability measures for which $h_\mu(T) + \int \phi \, d\mu$ is not of the form $\infty - \infty$.\n
Ruelle showed [14, Theorem 3.7] that if $G$ is a finite graph (in this case we call $X_G$ a **subshift of finite type**), and $\phi$ is weakly Hölder continuous (with respect to the metric
the set of equilibrium measures for \( \phi \) is exactly the set

\[
\left\{ \sigma : P_G(\phi + \psi) \geq P_G(\phi) + \int \psi d\sigma \text{ for all } \psi \in C(X_G) \right\}.
\]

This can be interpreted as saying that the equilibrium measures for \( \phi \) correspond to tangents to the graph of \( P_G(\cdot) \) at \( \phi \). One way of looking for equilibrium states, then, is to fix some weakly Hölder (bounded) \( \psi \) and check the analyticity properties of the function \( t \mapsto P_G(\phi + t\psi) \). If this function is not differentiable at \( t = 0 \) then there is a non-unique equilibrium measure for \( \phi \). We call this a phase transition of order one (in general if \( P_G(\phi + t\psi) \) is not analytic at \( t = 0 \) we interpret this as a “higher order” phase transition).

Motivated by this result for finite graphs, we make the following definition:

**Definition 1.1.** Suppose \( \phi \in \Phi \) and \( \psi \in \Phi \) is bounded. If \( t \mapsto P_G(\phi + t\psi) \) is not analytic at \( t = 0 \), we say that \( \phi \) exhibits a phase transition.

For Markov shifts associated to finite graphs and Hölder potentials \( \phi \), Ruelle showed [14, Corollary 5.27] that \( t \mapsto P_G(\phi + t\psi) \) is analytic in some \( t \)-neighborhood of zero, so there are no phase transitions. Moreover, Bowen [2] showed that if \( X_G \) is a subshift of finite type and \( \phi \) is Hölder continuous, then a cohomologous potential has a unique DLR state (which is absolutely continuous with respect to the unique equilibrium measure). The construction involves the Ruelle operator defined on functions by

\[
(L_\phi f)(x) := \sum_{Ty=x} e^{\phi(y)} f(y).
\]
He showed that there is a finite measure $\nu$ so that $L_{\phi} : L^1(\nu) \to L^1(\nu)$ and $L^*_{\phi} \nu = e^{P_G(\phi)} \nu$, and a continuous function $h : X_G \to \mathbb{R}$ with $L_{\phi} h = e^{P_G(\phi)} h$ with $\int h d\nu = 1$.

He showed that the measure $dm := h d\nu$ is the unique equilibrium measure for $\phi$, and $\nu$ is the unique DLR state and thermodynamic limit. Sarig showed [17, 18, 19, 20] that for certain potentials on countable Markov shifts, similar results can be obtained (see Theorem 1.2 below).

We remark that the measure $\nu$ constructed by Bowen is of a special class which is of interest. If $\nu$ is non-singular\(^8\) then we can define a new measure which describes how $\nu$ changes under the dynamics of $T$: \[ (\nu \circ T)(E) := \sum_{a \in S_G} \nu(T(E) \cap [a]). \]

A measure (positive and finite on cylinders) for which $\frac{d\nu}{d(\nu \circ T)} = e^{\phi - P_G(\phi)}$ is called $\phi$-conformal. Interest in $\phi$-conformal (or just conformal when there is no confusion) measures arises because a conformal probability measure is a DLR state. Moreover, a measure satisfying $L^*_{\phi} \nu = e^{P_G(\phi)} \nu$ satisfies the definition of a conformal measure. This allows us to convert a measure theoretic problem into a question about eigenmeasures for an operator.

**Remark.** An essential feature in Bowen's construction is that, for a subshift of finite type, the space $X_G$ is compact (so any sequence of probability measures has a weak-* convergent subsequence by the Banach-Alaoglu theorem). In the case of a countable

\(^7\)Here $L^*_{\phi}$ is the dual operator of $L_{\phi}$.

\(^8\)A measure $\nu$ is non-singular for $T$ if $\nu(E) = 0 \Rightarrow \nu(T^{-1} E) = 0$ for every measurable set $E$. 

Markov shift there is no compactness so new arguments are needed and new phenomena can arise.

1.2.3 Modes of Recurrence

For \( a \in S_G \), define the partition function to be

\[
Z_n(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x).
\]

We set \( \varphi_a(x) := 1_{[a]}(x) \inf \{ n > 0 : T^n x \in [a] \} \) and define

\[
Z_n^*(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\varphi_a = n]}(x).
\]

These sequences are related by the following approximate renewal equation:

\[
Z_n(\phi, a) = e^{\pm \sum_{n=2}^{\infty} \var_{\phi}} \left( Z_n^*(\phi, a) + \sum_{i=1}^{n-1} Z_{n-i}^*(\phi, a) Z_i(\phi, a) \right). \tag{1.2.1}
\]

For fixed \( a \in S_G \), we define the generating functions for \( Z_n(\phi, a) \) and \( Z_n^*(\phi, a) \)

\[
t^n_{\phi}(x) := 1 + \sum_{n=1}^{\infty} Z_n(\phi, a) x^n
\]

\[
r^n_{\phi}(x) := \sum_{n=1}^{\infty} Z_n^*(\phi, a) x^n.
\]
The radius of convergence of $t^a_\phi(x)$ is $e^{-P_G(\phi)}$ (by definition of Gurevich pressure). For these functions, equation 1.2.1 gives

$$t^a_\phi(x) - 1 = e^{\pm \sum_{n=2}^\infty \var_n(\phi) t^a_\phi(x) r^a_\phi(x)}, \text{ for any } x \in [0, e^{-P_G(\phi)}].$$

**Definition 1.2.** For $\phi$ with summable variations and $P_G(\phi) < \infty$, we say $\phi$ is

- positive recurrent if $t^a_\phi(e^{-P_G(\phi)}) = \infty$ and $(r^a_\phi)'(e^{-P_G(\phi)}) < \infty$,
- null recurrent if $t^a_\phi(e^{-P_G(\phi)}) = \infty$ and $(r^a_\phi)'(e^{-P_G(\phi)}) = \infty$,
- transient if $t^a_\phi(e^{-P_G(\phi)}) < \infty$.

Additionally, we say $\phi$ is recurrent if it is either positive recurrent or null recurrent.

Interest in recurrence is motivated by the following generalization of Ruelle’s Perron-Frobenius Theorem:

**Theorem 1.2** (Generalized Ruelle-Perron-Frobenius Theorem [18]). $\phi$ is recurrent if and only if there exist $\lambda > 0$, a conservative\(^9\) measure $\nu$, finite and positive on cylinder sets, and a continuous $h : X_G \to \mathbb{R}$ such that $L^\phi \nu = \lambda \nu$ and $L^\phi h = \lambda h$. In this case $\lambda = e^{P_G(\phi)}$ and there is a sequence of natural numbers $a_n \nearrow \infty$ such that for every cylinder $[a_0 \ldots a_{n-1}]$ and $x \in X_G$, we have

$$\frac{1}{a_n} \sum_{k=1}^n \lambda^{-k} (L^k_\phi \nu|[a_0 \ldots a_{n-1}])(x) \xrightarrow{n \to \infty} h(x)\nu([a_0 \ldots a_{n-1}]). \quad (1.2.2)$$

\(^9\)Recall that a (possibly $\sigma$-finite) measure $\nu$ is conservative if for every measurable set $A$ such that $A, T^{-1}A, T^{-2}A, \ldots$ are pairwise disjoint, we have $\nu(A) = 0$. Conservative measures are of interest because they are the $\sigma$-finite measures for which the Poincaré recurrence theorem holds.
Moreover, if \( \phi \) is positive recurrent then \( \int h d\nu < \infty \), \( a_n \propto \text{const} \cdot n \) and for every \([a_0 \cdots a_{m-1}]\) we have \( \lambda^{-n}(L^n_{\phi}[a_0 \cdots a_{m-1}]) \xrightarrow{n \to \infty} h\nu([a_0 \cdots a_{n-1}]) / \int h d\nu \). If \( \phi \) is null recurrent then \( \int h d\nu = \infty \), \( a_n = o(n) \) and for every \([a_0 \cdots a_{m-1}]\) we have \( \lambda^{-n}(L^n_{\phi}[a_0 \cdots a_{m-1}]) \xrightarrow{n \to \infty} 0 \) uniformly on compact sets.

### 1.2.3.1 Positive Recurrent Potentials

Two immediate consequences of the Generalized Ruelle-Perron-Frobenius theorem are the following:

**Corollary 1.1.** Suppose \( \phi \) is as in the Generalized Ruelle-Perron-Frobenius theorem. If \( \phi \) is positive recurrent then the measure \( \nu \) is \( \phi \)-conformal. Moreover, setting \( k := \log \left( \frac{h}{\int h d\nu} \right) \) and \( \tilde{\nu} := e^k d\nu \), we have that \( \tilde{\nu} \) is a DLR state for \( \phi + k - k \circ T \).

**Proof.** The function \( k \) is well-defined since \( \phi \) is positive recurrent (so \( \int h d\nu < \infty \)). Then \( \tilde{\nu} \) is a probability measure and, for any \( f \in L^1(\tilde{\nu}) \),

\[
\int f d\tilde{\nu} = \int f e^k d\nu \\
= \lambda^{-1} \int (L_{\phi}(f e^k)) d\nu \\
= \lambda^{-1} \int \sum_{Ty=x} e^{\phi(y)+k(y)} f(y) d\nu(x) \\
= \lambda^{-1} \int \sum_{Ty=x} e^{\phi(y)+k(y)-k(Ty)} f(y) e^{k(x)} d\nu(x) \\
= \lambda^{-1} \int (L_{\phi+k-k \circ T}f) e^k d\nu \\
= \lambda^{-1} \int (L_{\phi+k-k \circ T}f) d\tilde{\nu}.
\]

So \( \tilde{\nu} \) is \((\phi + k - k \circ T)\)-conformal. A conformal probability measure is a DLR state. \(\square\)
Corollary 1.2. If $\phi$, as in the Generalized Ruelle-Perron-Frobenius theorem, is positive recurrent then $\nu$ is the thermodynamic limit for $\phi$.

Proof. The estimate on $\lambda^{-n}(L_\phi^n [a_0 \cdots a_{m-1}]) (x)$ implies that $\nu$ is the thermodynamic limit. $\square$

Additionally it follows that:

Corollary 1.3 ([5], Proposition 8.1). If $\phi$, as in the Generalized Ruelle-Perron-Frobenius theorem, is positive recurrent and the measure $d\mu = h d\nu$ has finite entropy, then it is (the unique) equilibrium measure for $\phi$.

1.2.4 Strong Positive Recurrence

For positive recurrent potentials, the above results establish the existence of a thermodynamic limit and DLR state (on a cohomologous potential) and, under a finiteness assumption on entropy, the existence of an equilibrium measure. There is a special subset of the positive recurrent potentials where more can be said. First we define this subset.

Fix a vertex $a \in S_G$ and define the countable Markov shift induced on $a$ (or the induced shift) to be:

$$\mathcal{S}_G := \{[a x_1 \cdots x_k] : k \in \mathbb{N} \text{ and } [a x_1 \cdots x_k a] \neq \emptyset \}.$$ 

We make $\mathcal{S}_G$ into the vertex set of a directed graph $\mathcal{G}$ by putting a directed edge between any two elements of $\mathcal{S}_G$. Then $\mathcal{X}_G := X_G$ and $\mathcal{T} : \mathcal{X}_G \to \mathcal{X}_G$ is its left-shift. There is a
natural projection $\pi : X^G \to X_G$ given by

$$\pi([ax_0 \ldots x_k], [ay_0 \ldots y_l], \ldots) := (a, x_0, \ldots, x_k, a, y_0, \ldots, y_l, \ldots) \in X^G.$$ 

Then for any potential $\phi : X_G \to \mathbb{R}$, we can associate the induced potential $\tilde{\phi} : X^G \to X^G$ given by

$$\tilde{\phi} := \left( \sum_{n=0}^{\varphi - 1} \phi \circ T^n \right) \circ \pi.$$ 

If $\phi$ is weakly Hölder continuous, then $\tilde{\phi}$ is too. Interest in the induced system lies in the following definition (and the discussion following it).

**Definition 1.3.** The $a$-discriminant of $\phi$ is the quantity

$$\Delta_a[\phi] := \sup \{ P_G(\phi + p) : p \in \mathbb{R} \text{ and } P_G(\phi + p) < \infty \}.$$ 

The discriminant is of interest because of the following theorem of Sarig:

**Theorem 1.3** (Discriminant Theorem, [19]). If $X_G$ is a topologically mixing countable Markov shift and let $\phi : X_G \to \mathbb{R}$ have summable variations and $P_G(\phi) < \infty$. Let $a \in S_G$ be fixed.

1. The equation $P_G(\phi + p) = 0$ has a unique solution $p(\phi)$ if $\Delta_a[\phi] \geq 0$, and no solution if $\Delta_a[\phi] < 0$. The Gurevich pressure of $\phi$ is given by

$$P_G(\phi) = \begin{cases} 
-p(\phi) & \text{if } \Delta_a[\phi] \geq 0 \\
\limsup \frac{1}{n} \log Z^*(\phi, a) & \text{if } \Delta_a[\phi] < 0.
\end{cases}$$
2. \( \phi \) is positive recurrent if \( \Delta_a[\phi] > 0 \) and transient if \( \Delta_a[\phi] < 0 \). In the case \( \Delta_a[\phi] = 0 \), \( \phi \) is either positive recurrent or null recurrent.

A potential is called strongly positive recurrent if \( \Delta_a[\phi] > 0 \) for some (hence every) \( a \in S_G \).

Thermodynamic properties of strongly positive recurrent potentials were investigated in [5]. There a spectral gap property was defined which says that there exists a rich Banach space \( \mathcal{L} \) where the Ruelle operator \( L_\phi \) acts with spectral gap. It was shown that:

**Theorem 1.4.** If \( X_G \) is topologically mixing and \( \phi : X_G \rightarrow \mathbb{R} \) is weakly H"older continuous with finite Gurevich pressure, then \( \phi \) has the spectral gap property if and only if \( \phi \) is strongly positive recurrent.

Interest in the spectral gap property is motivated by the following theorem:

**Theorem 1.5** ([5], Theorem 1.1). Under the assumption of the above theorem, if \( \phi \) has the spectral gap property then on the Banach space \( \mathcal{L} \), \( L_\phi = \lambda P + N \) where \( \lambda = \exp P_G(\phi) \) and \( PN = NP = 0 \), \( P^2 = P \), \( \dim \text{Im}(P) = 1 \) and the spectral radius of \( N \) is less than \( \lambda \). Moreover (setting \( m_\phi := h\nu \) from the Generalized Ruelle-Perron-Frobenius theorem)

1. There is a \( 0 < \kappa < 1 \) such that for all \( g \in L^\infty(m_\phi) \) and \( f \) bounded H"older continuous, \( \exists C(f,g) > 0 \) s.t. \( \left| \text{Cov}_{m_\phi}(f,g \circ T^n) \right| \leq C(f,g)\kappa^n \). (Cov=covariance)

2. Suppose \( \psi \) is a bounded H"older continuous function such that \( \mathbb{E}_{m_\phi}[\psi] = 0 \). If \( \psi \neq \varphi - \varphi \circ T \) with \( \varphi \) continuous, then \( \exists \sigma > 0 \) s.t. \( \psi_n / \sqrt{n} \) converges in \( (m_\phi - ) \)-distribution to the normal distribution with mean zero and standard deviation \( \sigma \).
3. Suppose $\psi$ is a bounded Hölder continuous function, then $t \mapsto P_G(\phi + t\psi)$ is real analytic on a neighborhood of zero.

Note that this last statement implies that if $\phi$ is strongly positive recurrent, there is no phase transition (in the sense of Definition 1.1). We contrast this with the following (which is a consequence of Theorems 4.1 and 6.1 in [5]):

**Theorem 1.6.** If $\phi$ is a transient potential with summable variations and finite Gurevich pressure, then for any $a \in \mathcal{S}_G$, $\exists t \in \mathbb{R}^+$ s.t. $\phi + t\chi_a$ has a phase transition (in the sense of Definition 1.1).

### 1.2.4.1 Null Recurrent Potentials

For null recurrent potentials we still obtain a $\phi$-conformal measure $\nu$ from the Generalized Ruelle-Perron-Frobenius theorem which can be considered a weak form of a thermodynamic limit. The invariant measure $d\mu = h d\nu$ that it is equivalent to, however, is $\sigma$-finite. This can still be considered a weak version of an equilibrium measure, which we will now explain.

Let $\mathcal{B}$ be the Borel $\sigma$-algebra and, for a measure $\mu$ define its conditional expectation on a $\sigma$-algebra $\mathcal{F}$ to be

$$
\mu([a]|\mathcal{F})(x) := \mathbb{E}_\mu(1_{[a]}|\mathcal{F})(x).
$$

Next define the *information function* associated to $\mu$ to be

$$
I_\mu = - \sum_{a \in \mathcal{S}_G} 1_{[a]} \log \mu([a]|T^{-1}\mathcal{B}).
$$
If $\mu$ is an invariant probability measure then $h_\mu(T) = \int I_\mu \, d\mu$. In this case, the equation $h_\mu(T) + \int \phi \, d\mu \leq P_G(\phi)$ can be reinterpreted as $\int \left( I_\mu + \phi - P_G(\phi) \right) \, d\mu \leq 0$. The advantage of this latter formulation is that it still makes sense even if $\mu$ is $\sigma$-finite (although in this case it is not equivalent to the original equation). The following theorem shows the sense in which $h \nu$ can be considered to be a weak form of an equilibrium measure.

**Theorem 1.7** ([18], Theorem 3). Let $X_G$ be topologically mixing and $\phi$ locally Hölder continuous with finite Gurevich pressure. Assume that $\phi$ is recurrent, let $h$ and $\nu$ be as in the Generalized Ruelle-Perron-Frobenius theorem and set $\phi' := \phi + \log h - \log h \circ T$. Then for every conservative invariant measure $\mu$ which is finite on partition sets, $I_\mu + \phi' - P_G(\phi')$ is one-sided integrable\footnote{If $f$ is one-sided integrable if at least one of $\int |f| \, d\mu < \infty$ or $\int |f| \, 1_{f<0} \, d\mu < \infty$. In this case $\int f \, d\mu$ is well-defined even for $f \notin L^1(\mu)$.} with respect to $\mu$ and

$$-\infty \leq \int \left( I_\mu + \phi' - P_G(\phi') \right) \, d\mu \leq 0. \quad (1.2.3)$$

If $\mu \sim \mu \circ T$, the integral in equation 1.2.3 is equal to zero if and only if $\mu$ is proportional to $h \nu$.

**1.2.4.2 Transient Potentials**

Prior to this work there were no results establishing the existence of thermodynamic quantities for transient potentials on countable Markov shifts. The “if and only if” statement in the Generalized Ruelle-Perron-Frobenius theorem means that a new
approach is needed to treat this case (for example, a transient potential cannot have a conservative conformal measure).

1.3 Summary of Results

1.3.1 Aim

The goal of this thesis is to understand how common the phenomenon of transience is and to develop its thermodynamic formalism. We are specifically interested in three questions:

Question 1: Which countable Markov shifts have transient potentials?

Question 2: For a fixed countable Markov shift $X_G$ that has transient potentials, how common is the phenomenon of transience among potentials with summable variations and finite Gurevich pressure?

Question 3: What thermodynamic properties do transient potentials have?

1.3.2 Existence of Transient Potentials

The main result in this direction is a complete solution to Question 1: we find a (checkable) criterion on a graph $G$ which is necessary and sufficient for the associated countable Markov shift $X_G$ to have a transient potential. This condition is very mild, so we can say that “most” countable Markov shifts have at least one transient potential and hence a (not necessarily transient) potential that admits a phase transition.

Specifically, we make the following definition:
Definition 1.4. Suppose $G$ is a graph and $F \subseteq S_G$. We say that $F$ is a uniform Rome (with parameter $N \in \mathbb{N}$) if the graph $G \setminus F$ (i.e. the graph obtained by deleting all vertices in $F$) does not contain any paths of length greater than $N$. A finite uniform Rome is a uniform Rome with $\#F < \infty$.

Any graph has a uniform Rome (take $F = S_G$) but it is a very strong condition to have a finite uniform Rome (any graph with an infinite ray cannot have a finite uniform Rome, for example). Our result is:

Theorem 1.8. The countable Markov shift $X_G$ has a transient potential with summable variations and finite Gurevich pressure if and only if $G$ does not have a finite uniform Rome.

1.3.3 Topology of Transient Potentials

We next turn our attention to Question 2 and study (among countable Markov shifts that have transient potentials) the relative commonness of transience. More precisely, we put various topologies on the set

$$\Phi := \{ \phi : X \to \mathbb{R} \mid \phi \text{ has summable variations and } P_G(\phi) < \infty \}$$

and ask what topological properties the set $\Phi(T) := \{ \phi \in \Phi \mid \phi \text{ is transient} \}$ has (i.e. open, closed, dense, etc.).

The first main result in this direction is that

$$\Phi(\text{SPR}) := \{ \phi \in \Phi : \phi \text{ is strongly positive recurrent} \}$$
is open and dense in $\Phi$ in the uniform and Lipshitz topologies.\textsuperscript{11} Thus “most” potentials in $\Phi$ have the spectral gap property, DLR states, etc. Moreover this result says that, in the above topologies, $\Phi(T)$ is contained in a closed, nowhere dense set. On the other hand, the result of the previous section says that $\Phi \setminus \Phi(SPR)$ is usually nonempty. A natural question to ask is how large $\Phi(T)$ is as a subset of $\Phi \setminus \Phi(SPR)$. In order to study this set, a new topology is needed.

The next main result is obtained by defining a new localized uniform topology which is strong enough to detect $\Phi \setminus \Phi(SPR)$ as more than a closed, nowhere dense set. In this topology $\Phi(T)$ is open and dense in $\Phi \setminus \Phi(SPR)$ (this topology is stronger than the uniform topology, the set $\Phi(SPR)$ turns out to be open and dense in $\Phi \setminus \Phi(T)$).

These two results provide an answer to Question 2: although strong positive recurrence is the most common phenomenon in $\Phi$, transience is the most important obstruction to it and $\Phi \setminus \Phi(SPR)$ is nonempty precisely when $\Phi(T)$ is nonempty. Said another way, in order to study phase transitions (which cannot occur for strongly positive recurrent potentials), transience is the key phenomenon and its thermodynamic formalism needs to be explored.

\textbf{1.3.4 Thermodynamics of Transient Potentials}

Finally we turn to Question 3 and develop a theory of thermodynamic formalism for transient potentials. It was shown by Buzzi and Sarig [3] that transient potentials cannot have equilibrium measures. They may, however, have conformal measures and DLR states.

\textsuperscript{11}See Chapter 3 for the definitions of these topologies.
The main result in this direction is the construction of a conformal measure for any locally compact countable Markov shift. As previously noted, this was already done for recurrent potentials by Sarig [17, 19] but this is the first construction of conformal measures for transient potentials. A new feature that was not present for recurrent shifts is that this measure may not be unique (examples are given in Chapter 5). This construction differs from constructions of conformal measures for other dynamical systems [6, 17] with the new feature being the construction of a sequence of measures by analogy with the theory of Martin boundaries for transient countable Markov chains. The limiting measure turns out to be conformal.

We also explore the existence of the thermodynamic limit. Again we restrict to the case of a transient locally compact countable Markov shift. Due to the lack of compactness, the ordinary thermodynamic limit may be zero (the measure assigning zero to every cylinder can be the w*-limit of a sequence of probability measures if $X^g$ is not compact). Thus we introduce a normalized sequence of measures and call their w*-limits generalized thermodynamic limits. Our final result is that, in this case, a generalized thermodynamic limit always exists.

### 1.4 Summary and Conclusions

The final picture that emerges is that most countable Markov shifts have transient potentials. Among shifts that do, there is a large set of strongly positive recurrent recurrent potentials (whose thermodynamics are already understood) but the set of transient potentials is open and dense (in an appropriate topology) in the complement of this set; that is, transience is an important obstruction to strong positive recurrence. Finally,
transient potentials on locally finite countable Markov shifts always have at least one (and often many) conformal measure(s), meaning that cohomologous potential has at least one DLR state. In the case of a non-unique conformal measure, we can interpret this as saying that the dynamical system is undergoing a phase transition.
Chapter 2

Existence of Transient Potentials

In this chapter we show that “most” (in a sense to be made precise) countable Markov shifts have transient potentials. The results of this chapter appear in [4].

2.1 Setting

Suppose $\mathcal{G}$ is a directed graph with countable vertex set $S_{\mathcal{G}}$ and $X_{\mathcal{G}}$ is the associated countable Markov shift (see Section 1.2 to recall these definitions). We define

$$Z_n(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x)$$

$$Z^*_n(\phi, a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{\inf\{k > 0 : T^k x \in [a]\} = n, x_0 = a}(x).$$

These sequences are related by the following approximate renewal equation

$$Z_n(\phi, a) = e^{\pm \sum_{n=2}^{\infty} \text{var}_n(\phi)} \left( Z^*_n(\phi, a) + Z^*_n(\phi, a)Z_1(\phi, a) + \cdots + Z^*_n(\phi, a)Z_{n-1}(\phi, a) \right).$$

(2.1.1)

On the level of generating functions, this says

$$t^a_\phi(x) - 1 = e^{\pm \sum_{n=2}^{\infty} \text{var}_n(\phi)} t^a(x) r^a_\phi(x).$$

(2.1.2)
Define

\[ \Phi := \{ \phi : X_G \to \mathbb{R} \mid \phi \text{ has summable variations and } P_G(\phi) < \infty \}, \]

\[ \Phi(T) := \{ \phi \in \Phi \mid \phi \text{ is transient} \}. \]

**Definition 2.1.** A set \( F \subseteq S_G \) is a uniform Rome for \( G \) if there exists \( N \in \mathbb{N} \) so that the graph \( G_F \), whose vertex set is \( S_G \setminus F \) and edge set contains all edges in \( G \) except those to or from vertices in \( F \), contains no paths of length greater than \( N \). A finite uniform Rome is a uniform Rome, \( F \), with \( \#F < \infty \). If the need arises to specify the value of \( N \), we will call \( F \) a uniform Rome with parameter \( N \).

Our main result is the following:

**Theorem 2.1.** The set \( \Phi(T) \) is nonempty if and only if the graph \( G \) does not have a finite uniform Rome.

For several examples of graphs with finite uniform Romes, see Appendix A.

### 2.1.1 Overview of Proof of Theorem 2.1

The proof has three steps.

1. Show that if \( G \) has a finite uniform Rome then any potential in \( \Phi \) must be recurrent. [Section 2.2]

2. Show that for several large classes of graphs, the associated shift has a transient potential. [Sections 2.3 and 2.4]
3. Show that if a graph is not in one of these classes then it has a finite uniform Rome.

[Section 2.5]

We now briefly elaborate on the ideas that go into each of these steps.

**First Step**

For \( \phi \in \Phi(T) \) and \( \psi \in \Phi \) bounded with \( \text{supp}(\psi) \subseteq [a] \) for some \( a \in S_G \) we prove a lemma regarding the pressure function along the family of potentials \( \{ \phi + t\psi \}_{t \in \mathbb{R}} \). The result is a straightforward consequence of this lemma.

**Second Step**

This requires the most effort.

**Definition 2.2.** A potential \( \phi : X_G \to \mathbb{R} \) is called a Markov potential if \( \phi(x) = \phi(x_0, x_1) \) depends only on the first two coordinates.

Note that a Markov potential has summable variations and the approximate renewal equation (1.2.1) becomes an actual equation.

The main tool and starting point for this step is the following:

**Theorem 2.2** (Extension Theorem). If \( \mathcal{H} \subseteq \mathcal{G} \) is a topologically mixing subgraph and \( \phi = \phi(x_0, x_1) \) is a transient Markov potential on \( X_\mathcal{H} \), then \( \phi \) can be extended to a transient Markov potential \( \tilde{\phi} \) on \( X_G \) with \( P_G(\tilde{\phi}) = P_G(\phi) \).

Thus showing that a particular shift \( X_\mathcal{H} \) has a transient potential in fact shows that \( X_G \) has a transient potential for any \( \mathcal{G} \) containing \( \mathcal{H} \) as a subgraph. Thus it suffices to explicitly find a transient potential for several (fairly simple) shifts and then conclude
that any graph containing one of these as a subshift also has a transient potential. Specifically we show:

- If $G$ contains an infinite forward or backward ray\(^1\), then $X_G$ has a transient potential.

- If $G$ has infinitely many disjoint simple loops (based at different base points), then $X_G$ has a transient potential.

- If there is a vertex $a \in S_G$ such that there are simple loops based at $a$ of arbitrarily large size, then $X_G$ has a transient potential.

**Third Step**

We assume that $G$ does not have any of the attributes in the bulleted list above and explicitly find a finite uniform Rome.

### 2.2 A Finite Uniform Rome implies Recurrence

We start by recalling the definition of the $a$-discriminant of a potential $\phi \in \Phi$ (Definition 1.3). For $a \in S_G$, the *countable Markov shift induced on $a$* is the shift obtained

\(^1\)An *infinite forward ray* is an infinite simple path of the form:

\[
v_1 \to v_2 \to v_3 \to \cdots
\]

An *infinite backward ray* is an infinite simple path of the form:

\[
v_1 \leftarrow v_2 \leftarrow v_3 \leftarrow \cdots
\]
from the complete graph on the vertex set

\[ S_G := \{ [ax_1 \cdots x_k] : k \in \mathbb{N} \text{ and } [ax_1 \cdots x_k, a] \neq \emptyset \}. \]

Then \( X_G := X_{\bar{G}} \) and \( T : X_G \to X_G \) is its left-shift. There is a natural projection \( \pi : X_G \to X_G \) given by

\[ \pi([ax_0 \cdots x_k], [ay_0 \cdots y_l], \ldots) := (a, x_0, \ldots, x_k, a, y_0, \ldots, y_l, \ldots) \in X_G. \]

For any potential \( \phi : X_G \to \mathbb{R} \), we associate the induced potential \( \overline{\phi} : X_G \to X_G \) given by

\[ \overline{\phi} := \left( \sum_{n=0}^{\varphi-1} \phi \circ T^n \right) \circ \pi. \]

We define

\[ p^*_a[\phi] := \sup\{ p \in \mathbb{R} : P_G(\overline{\phi} + p) < \infty \}, \]

and set \( \Delta_a[\phi] := \sup\{ P_G(\overline{\phi} + p) : p < p^*_a[\phi] \} \). An important property of the \( a \)-discriminant is the following:

**Lemma 2.1.** \( \Delta_a[\phi + t \cdot 1_{[a]}] = \Delta_a[\phi] + t. \)

*Proof.* \( P_G(\overline{\phi + t \cdot 1_{[a]}} + p) = P_G(\overline{\phi + p}) + t. \) So \( p^*_a[\phi + t \cdot 1_{[a]}] = p^*_a[\phi] \) and \( \Delta_a[\phi + t \cdot 1_{[a]}] = \Delta_a[\phi] + t. \)

This allows us to prove the following useful lemma.
Lemma 2.2. If \( \phi \in \Phi \) is transient and \( \psi \in \Phi \) is non-negative, bounded, and \( \text{supp}(\psi) \subseteq [a] \) for some \( a \in S_G \), then \( \exists t(\phi, \psi) > 0 \) so that \( P_G(\phi + t\psi) = P_G(\phi) \) for every \( 0 \leq t \leq t(\phi, \psi) \).

Proof. Let \( a \in S_G \) be as in the statement of the theorem. Since \( \phi \) is transient, \( \Delta_a[\phi] < 0 \) (by Theorem 1.3). Now \( \Delta_a[\phi + t \cdot 1_{[a]}] = \Delta_a[\phi] + t \) for \( t \in \mathbb{R} \) (Lemma 2.1). So for all \( t < -\frac{\Delta_a[\phi]}{\text{sup}(\psi)} \) we have \( \Delta_a[\phi + t \text{sup}(\psi)1_{[a]}] = \Delta_a[\phi] + t \cdot \text{sup}(\psi) < 0 \) and hence transient. For such a \( t \), by Theorem 1.3, \( P_G(\phi + t \text{sup}(\psi)1_{[a]}) = \limsup_{n} \frac{1}{n} \log Z_n^*(\phi + t \text{sup}(\psi)1_{[a]}) \). Then

\[
P_G(\phi + t \text{sup}(\psi)1_{[a]}) = \limsup_{n} \frac{1}{n} \log Z_n^*(\phi + t \text{sup}(\psi)1_{[a]}, a)
= \limsup_{n} \frac{1}{n} \log Z_n^*(\phi, a)
= P_G(\phi).
\]

Now \( \phi \leq \phi + t\psi \leq \phi + t \text{sup}(\psi)1_{[a]} \) so \( P_G(\phi + t\psi) = P_G(\phi) \) (and \( \phi + t\psi \) is transient). \( \square \)

Theorem 2.3. If \( G \) has a finite uniform Rome then every \( \phi \in \Phi \) is recurrent.

Proof. Suppose \( G \) has a finite uniform Rome \( F \) with parameter \( N \) and \( \phi \in \Phi \). We claim that \( \phi \) is recurrent. For any \( \varepsilon > 0 \),

\[
Z_n(\phi + \varepsilon \cdot 1_F, a) = \sum_{T^n,x=x} e^{(\phi + \varepsilon \cdot 1_F)_n(x)} 1_{[a]}(x)
\geq e^{\lfloor n\varepsilon/N \rfloor} \sum_{T^n,x=x} e^{\phi_n(x)} 1_{[a]}(x)
= e^{\lfloor n\varepsilon/N \rfloor} Z_n(\phi, a)
\]
so that \( P_G(\phi + \varepsilon \cdot 1_F) \geq P_G(\phi) + \varepsilon > P_G(\phi) \). If \( \phi \) were transient, then for all sufficiently small \( \varepsilon \) we would have \( P_G(\phi + \varepsilon \cdot 1_F) = P_G(\phi) \) (repeatedly apply Lemma 2.2 with \( \psi = \varepsilon \cdot 1_{[b]} \) for each \( b \in F \) and take \( \varepsilon \) less than the minimum \( t(\phi, \psi) \)); a contradiction. So \( \phi \) is recurrent.

\[ \Box \]

2.3 Proof of The Extension Theorem

Our goal in this section is to prove theorem 2.2. Starting with a transient Markov potential \( \phi \) on a shift \( X_{\mathcal{H}} \), we want to extend \( \phi \) to a transient potential \( \tilde{\phi} \) on \( X_{\tilde{\mathcal{H}}} \). Our strategy is to “build” \( \tilde{\mathcal{G}} \) from \( \mathcal{H} \) by adding the edges and vertices of \( \mathcal{G} \) to \( \mathcal{H} \) one at a time and, at each step, extend \( \phi \) to this new intermediate shift. Then we check that the common extension (to \( X_{\tilde{\mathcal{H}}} \)) is transient.

2.3.1 Technical Lemmas

To carry out this construction, we will need two tools which are described in lemmas 2.3 and 2.5 and deal with the cases of adding a new edge and a new vertex to \( \mathcal{H} \), respectively.

**Lemma 2.3.** Let \( \phi \) be a transient Markov potential on a shift \( X_{\mathcal{H}} \). Let \( \tilde{\mathcal{H}} \) be the graph \( \mathcal{H} \) with one additional edge added. Then \( \phi \) can be extended to a transient Markov potential \( \tilde{\phi} \) on \( X_{\tilde{\mathcal{H}}} \) with the same pressure as \( \phi \).

Suppose the new edge is \( a \to b \) and let

\[ p : a \xrightarrow{L(p)} b \]
be a path in $\mathcal{H}$ from $a$ to $b$ (possible by topological mixing). We will define a new potential $\tilde{\phi}$ on $X_{\mathcal{H}}$ by setting $\tilde{\phi}(x_0, x_1) = \phi(x_0, x_1)$ whenever $x_0 \to x_1$ is an allowed transition in $\mathcal{H}$ and $\tilde{\phi}(a, b) = -N$ (to be specified later). We claim that for a sufficiently large value of $N$, the potential $\tilde{\phi}$ will be transient and have $P_G(\tilde{\phi}) = P_G(\phi)$.

Note that $Z_n(\tilde{\phi}, a) \geq Z_n(\phi, a)$ since the former sum contains the latter as well as other entries. So $P_G(\tilde{\phi}) \geq P_G(\phi)$. Now observe that we can write

$$Z^*_n(\tilde{\phi}, a) = \sum_{T^n, x = x} e^{\tilde{\phi}_n(x)} 1_{[\phi_a = n]}(x)$$

as

$$= \sum_{T^n, x = x, x \notin X_{\mathcal{H}}} e^{\phi_n(x)} 1_{[\phi_a = n]}(x) + \sum_{T^n, x = x} e^{-N} e^{\phi_n-1(Tx)} 1_{[\phi_a = n]}(x)$$

$$\leq Z^*_n(\phi, a) + e^{-N} e^{-\phi_{k-1}(\bar{x})} Z^*_{n+k-2}(\phi, a)$$

where $\bar{x}$ is a fixed (but arbitrary) element of $[(p)_0, \ldots, (p)_L]$ $\cap X_{\mathcal{H}}$. Therefore,

$$r_{\phi}(x) \leq r_{\tilde{\phi}}(x) \leq r_{\phi}(x) + \frac{C(N)}{\varepsilon^{k-2}} r_{\phi}(x) = \left(1 + \frac{C(N)}{\varepsilon^{k-2}}\right) r_{\phi}(x) \quad (2.3.1)$$

for any $x \in [\varepsilon, R]$, where $\varepsilon > 0$ is arbitrary, $R$ is the radius of convergence of $r_{\phi}(x)$ and $C(N) = e^{-N} e^{-\phi_{k-1}(\bar{x})}$ can be made arbitrarily small by taking $N$ sufficiently large. Therefore $r_{\phi}(x)$ and $r_{\tilde{\phi}}(x)$ have the same radius of convergence: $\lambda^{-1} = e^{-P_G(\phi)}$ (by transience of $\phi$ and equation (2.1.1), this is the radius of convergence of $r_{\phi}(x)$). But, by transience of $\phi$, $t_{\phi}(x) = \frac{1}{1-r_{\phi}(x)}$ and the function converges at $\lambda^{-1}$ so $t_{\phi}(\lambda^{-1}) < 1$, and so for sufficiently large $N$ we have $r_{\tilde{\phi}}(\lambda^{-1}) < 1$ and hence $t_{\tilde{\phi}}(\lambda^{-1}) < \infty$. This
gives us $P_G(\tilde{\phi}) \leq P_G(\phi)$ and our previous estimate shows that $P_G(\tilde{\phi}) = P_G(\phi)$. Since 
$t_{\tilde{\phi}}(e^{-P_G(\tilde{\phi})}) < \infty$, $(X_{\tilde{H}}, \tilde{\phi})$ is transient.

**Lemma 2.4.** Let $\phi$ be a transient Markov potential on a shift $X_{\mathcal{H}}$. Let $\tilde{\mathcal{H}}$ be the graph $\mathcal{H}$ together with a finite collection $v_1, \ldots, v_r$ of new vertices connected by a simple path 

$$a \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_r \rightarrow b$$

for some $a, b \in S_{\mathcal{H}}$. Then $\phi$ can be extended to a transient Markov potential $\tilde{\phi}$ on $X_{\tilde{\mathcal{H}}}$ with $P_G(\tilde{\phi}) = P_G(\phi)$.

The proof is the same as in Lemma 2.3 except we set $\tilde{\phi}(a, v_1) = -N$ and 

$\tilde{\phi}(v_i, v_{i+1}) = \tilde{\phi}(v_r, b) = 0$.

**Remark 1.** We actually get a stronger statement in the previous lemmas. Given any $\varepsilon > 0$ it follows from (2.3.1) that we can take $N$ so large that 

$r_{\tilde{\phi}}^a(e^{-P_G(\tilde{\phi})}) \leq r_{\phi}^a(e^{-P_G(\phi)}) + \varepsilon$

(where $a \in S_{\mathcal{H}}$ is the initial vertex in the new path).

This estimate depends on the vertex $a \in S_{\mathcal{G}}$. We can also obtain a uniform estimate. Fix a base point $o \in S_{\mathcal{G}}$ and define

$$K_m(\tilde{\phi}, o, a, b) := \sum_{n=1}^{\infty} \sum_{T^n x = x} e^{\tilde{\phi}_n(x) - nP_G(\tilde{\phi}) + mN} [\varphi_o = n](x) 1[\# \{0 \leq k < n: T^k x \in [ab] \} = m](x).$$
This quantity is independent of $N$ (it represents the contribution to \( r^o_{\tilde{\phi}} (e^{-P_G(\tilde{\phi})}) \) from all edges except \( a \to b \)). Then we have

\[
\begin{align*}
r^o_{\tilde{\phi}} (e^{-P_G(\tilde{\phi})}) &= \sum_{n=1}^{\infty} Z^n(\tilde{\phi},o)e^{-nP_G(\tilde{\phi})} \\
&= \sum_{m=0}^{\infty} K_m(\tilde{\phi},o,a,b)e^{-mN}
\end{align*}
\]

(where \( a \to b \) is either the new edge added in lemma 2.3 or is the edge \( a \to v_1 \) in lemma 2.4). The sum converges by the preceding remarks, and as \( N \to \infty \) the above sum tends to \( K_0(\tilde{\phi},o,a,b) = r^o_{\phi} (e^{-P_G(\phi)}) \). So for sufficiently large \( N \) we get \( r^o_{\tilde{\phi}} (e^{-P_G(\tilde{\phi})}) \leq r^o_{\phi} (e^{-P_G(\phi)}) + \varepsilon \).

**Lemma 2.5.** Let \( \mathcal{H} \) be a topologically mixing subgraph of \( G \). If \( \phi \) is a transient Markov potential on \( X_{\mathcal{H}} \) and \( v \in S_G \setminus S_{\mathcal{H}} \), then there is a topologically mixing graph \( K \) with

\[
\mathcal{H} \subseteq K \subseteq G
\]

such that \( v \in K \) and \( \phi \) can be extended to a transient Markov potential on \( X_{\mathcal{K}} \) with pressure \( P_G(\phi) \).

Given a vertex \( v \in S_G \setminus S_{\mathcal{H}} \) there are vertices \( a,b \in S_{\mathcal{H}} \) so that there are simple paths (in \( G \)) \( a \xrightarrow{n} v \) and \( v \xrightarrow{m} b \). Together these determine a path, \( p : a \xrightarrow{n+m-1} b \) in \( G \) given by

\[
p : a \to x_1 \to \cdots \to x_{n-2} \to v \to y_1 \to \cdots \to y_{m-2} \to b
\]
and w.l.o.g. we can assume that $x_i, y_j \notin S_H$ for any $i, j$ (otherwise rechoose $a = x_i$ and $b = y_j$ appropriately so that we do have this property).

If $p$ is simple, then we can take $K = H \cup \{p\}$ and apply lemma 2.4 to extend $\phi$ to a transient Markov potential on $X_K$ with pressure $P_G(\phi)$. Otherwise there is a repeated vertex (apart from possibly $a = b$) so there are $i, j$ with $1 \leq i \leq n - 2$ and $1 \leq j \leq m - 2$ such that $x_i = y_j$. Let $i_1$ be the smallest possible index for which there exists $j_1$ with $x_{i_1} = y_j$ (wlog take $j_1$ to be the largest such index). Then the path

$$p_1 : a \rightarrow x_1 \rightarrow \cdots \rightarrow x_{i_1} \rightarrow y_{j_1 + 1} \rightarrow \cdots \rightarrow b$$

has no repeated vertices (by minimality of $i_1$ and maximality of $j_1$) and we can apply lemma 2.4 to extend $\phi$ to a transient Markov potential $\phi[1]$ on $X_{H_1}$ with $P_G(\phi[1]) = P_G(\phi)$, where $H_1 = H \cup \{p\}$. Note that $i_2 := \max \{i \mid x_i \in S_{H_1}\} > \max \{i \mid x_i \in S_H\}$ (where $a = x_0$) and similarly $j_2 := \min \{j \mid y_j \in S_{H_1}\} < \min \{j \mid y_j \in S_H\}$. Take $a_1 = x_{i_2}$, $b_1 = y_{j_2}$ and consider the path

$$\tilde{p} : a_1 \rightarrow x_{i_2 + 1} \rightarrow \cdots \rightarrow x_{n - 2} \rightarrow v \rightarrow y_1 \rightarrow \cdots \rightarrow y_{j_2 - 1} \rightarrow b_1$$

which has $L(\tilde{p}) < L(p)$. Repeat the above process, using $\tilde{p}$ instead of $p$, to produce a graph $H_2$ with $H_1 \subset H_2$ to which we can further extend $\phi[1]$ via lemma 2.4. Continue inductively until we have a sequence of graphs $H_1 \subset H_2 \subset \cdots \subset H_r$ with $v \in H_r$ and a transient Markov potential $\phi[r]$ on $X_{H_r}$ extending $\phi$ with $P_G(\phi[r]) = P_G(\phi)$. Taking $K = H_r$ gives the desired result.
Remark 2. As in the previous remark, given any $\varepsilon > 0$, we arrange for $r_\phi^o(\lambda^{-1}) = r_\phi^o(\lambda^{-1}) + \varepsilon$ (since we only extend $\phi$ through finitely many graphs to get to $\tilde{\phi}$).

2.3.2 Proof of Theorem 2.2

Enumerate the vertices in the graph $G$ as

$$S_G = \{v_1, v_2, v_3, \ldots\}$$

In order to build the potential $\tilde{\phi}$ on $X_G$, we first produce a (possibly infinite) sequence of intermediate graphs

$$H = H_1 \subset H_2 \subset \cdots \subset I$$

$$I = I_1 \subset I_2 \subset \cdots \subset G$$

with the property that $I = \bigcup H_i$, $G = \bigcup I_i$ and

- $S_I = S_G$ (the graph $I$ contains all vertices in the graph $G$);

- $I_{i+1}$ is the graph $I_i$ with one additional edge added [as in lemma 2.3].

Step 1: Producing the sequence of graphs $H_i$.

We produce the graphs $H_i$ by induction. Begin by setting $H_1 := H$ and let $\phi_1 := \phi$. Apply lemma 2.5 to $H_1 \subset G$ with potential $\phi_1$ and vertex $v_1$. Let $H_2$ and $\phi_2$ be the graph and potential (respectively) obtained from the lemma.

Now assume that we have constructed a sequence of graphs

$$H = H_1 \subset H_2 \subset \cdots \subset H_r$$
and transient Markov potentials $\phi_i$ (on $X_{\mathcal{H}_i}$) so that $\phi_i$ is an extension of $\phi_{i-1}$ and $v_{i-1} \in \mathcal{H}_i$ for $2 \leq i \leq r$. Then apply lemma 2.5 to $\mathcal{H}_r \subset \mathcal{G}$ with potential $\phi_r$ and vertex $v_r$. Let $\mathcal{H}_{r+1}$ and $\phi_{r+1}$ be the graph and potential obtained from the lemma.

By induction, we get a sequence of graphs so that

\[ \mathcal{I} := \bigcup_{i=1}^{\infty} \mathcal{H}_i \]

satisfies $\mathcal{S}_\mathcal{I} = \mathcal{S}_\mathcal{G}$.

**Step 2: Adding the vertices of $\mathcal{G}$ to $\mathcal{H}$.**

We will show that $\phi$ can be extended to a transient Markov potential $\varphi$ on $\mathcal{I}$ with $P_{\mathcal{G}}(\varphi) = P_{\mathcal{G}}(\phi)$.

Fix $\epsilon_0 > 0$ so that $r_\varphi(e^{-P_{\mathcal{G}}(\phi)}) + 2\epsilon_0 < 1$ (possible since $r_\varphi(e^{-P_{\mathcal{G}}(\phi)}) < 1$ by transience of $\phi$). Following the remark after lemma 2.5, we can arrange for $r_\varphi^o(\lambda^{-1}) \leq r_\phi^o(\lambda^{-1}) + \epsilon_0$ for all $i$ (here $o \in \mathcal{S}_\mathcal{G}$ is the base point in the remark). Let $\varphi$ be the potential on $\mathcal{I}$ defined by $\varphi(x_0, x_1) = \phi_i(x_0, x_1)$ where $i$ is any index for which $x_0 \to x_1$ is an allowed transition in $\mathcal{H}_i$. We claim that $P_{\mathcal{G}}(\varphi) = P_{\mathcal{G}}(\phi)$ and that $(X_{\mathcal{I}}, \varphi)$ is transient. This will complete step 2.

Suppose $Z^*_n(\varphi, a) < \infty \ \forall n$ but $\varphi$ is recurrent. Set $\lambda = e^{P_{\mathcal{G}}(\phi)}$. Since $P_{\mathcal{G}}(\varphi) \geq P_{\mathcal{G}}(\phi)$ and $\varphi$ is recurrent $r_\varphi^o(e^{-P_{\mathcal{G}}(\phi)}) \geq 1$ (possibly infinite if $P_{\mathcal{G}}(\varphi) > P_{\mathcal{G}}(\phi)$), there is some $R$ such that

\[ \sum_{k=1}^{R} Z^*_k(\varphi, a) \lambda^{-k} > r_\phi(\lambda^{-1}) + 2\epsilon_0. \]
Find $i$ such that $\mathcal{H}_i$ has enough paths so that $Z^*_j(\phi_i, a) \geq Z^*_j(\varphi, a) - \frac{q_0\lambda_j^j}{2^j} \forall 1 \leq j \leq R$ (possible since $Z^*_j(\varphi, a)$ is a sum that can be truncated with small error and the resulting sum is only over finitely many paths – we can find $i$ so that $\mathcal{H}_i$ contains all of these). Then $r_{\phi_i}(\lambda^{-1}) > r_{\varphi}(\lambda^{-1}) + \epsilon_0$ which contradicts the definition of $\phi_i$. Therefore $r_{\varphi}(\lambda^{-1}) < r_{\phi}(\lambda^{-1}) + 2\epsilon_0 < 1$. Since $\varphi$ is Markov, this implies that $\varphi$ is transient on $X_\mathcal{I}$ and since $r_{\varphi}(\lambda^{-1}) < \infty$ we have $P_G(\varphi) = P_G(\phi)$. Thus, it remains only to show that $Z^*_j(\varphi, a) < \infty \forall n$. 

Suppose $\exists n$ such that $Z^*_n(\varphi, a) = \infty$. We can find some finite set $F \subset \{x \mid T^nx = x, \varphi(a)(x) = n\}$ such that 

$$\sum_{x \in F} e^{\varphi_n(x)} > 2e^nPG(\phi).$$

Since $F$ is finite, we can find some $i$ such that $Z^*_n(\phi_i, a) > 2e^nPG(\phi)$ ($F$ contains only finitely many paths). But $\phi_i$ is transient and Markov so $\sum_k Z^*_k(\phi_i, a)e^{-kPG(\phi)} < 1$ (since $r_{\phi_i}(\lambda^{-1}) < 1$) and so $Z^*_n(\phi_i, a) < 2e^nPG(\phi)$. Therefore $Z^*_n(\varphi, a) < \infty \forall n$. 

Therefore, $\varphi$ is a transient Markov potential on $X_\mathcal{I}$ with $P_G(\varphi) = P_G(\phi)$.

**Step 3:** Adding the edges of $\mathcal{G}$ to $\mathcal{I}$.

We want to extend the potential $\varphi$ on $X_\mathcal{I}$ to a potential on $X_\mathcal{G}$. The argument is as above, using lemma 2.3 instead of 2.5. 

\[ \square \]

### 2.4 Shifts with Transient Potentials

In this section we define several classes of graphs and explicitly find transient potentials on their associated shifts. First we address two technical points.
Recall that the shift $X_G$ is assumed to be topologically mixing. We will find a transient potential on $X_G$ by finding a subgraph $\mathcal{H} \subseteq \mathcal{G}$ for which it is easy to define a transient Markov potential on $X_\mathcal{H}$ and use the extension theorem to extend to a transient Markov potential on $X_G$. The graph $\mathcal{H}$ that we construct will often be only path connected (so $X_\mathcal{H}$ is topologically transitive). To force $X_\mathcal{H}$ to be topologically mixing we will assume the existence of an extra edge $a \to a$ for some $a \in S_\mathcal{H}$ even if this edge does not appear in $\mathcal{G}$. The following lemma shows why this procedure does not cause a problem.

**Lemma 2.6.** Let $\mathcal{G}$ be a directed graph and $X_G$ topologically mixing. Let $a \in S_\mathcal{G}$ and define $\tilde{\mathcal{G}}$ to be the graph $\mathcal{G}$ with the additional edge $a \to a$ (if it was not already present). Then:

1. any transient Markov potential $\phi$ on $X_G$ extends to a transient Markov potential $\varphi$ on $X_{\tilde{\mathcal{G}}}$!
2. any transient Markov potential $\varphi$ on $X_{\tilde{\mathcal{G}}}$ restricts to a transient Markov potential $\phi$ on $X_G$.

Thus if we construct a subgraph $\mathcal{H} \subseteq \mathcal{G}$ and $X_\mathcal{H}$ is topologically transitive, we can consider the subgraph $\tilde{\mathcal{H}} \subseteq \tilde{\mathcal{G}}$ obtained by adding the edge $a \to a$ to both $\mathcal{G}$ and $\mathcal{H}$. Now if we construct a transient Markov potential on $X_{\tilde{\mathcal{H}}}$, it can be extended to a transient Markov potential on $X_{\tilde{\mathcal{G}}}$ by theorem 2.2 and restricted to $X_\mathcal{G}$ to get a transient potential on $X_G$. The upshot is that even if we need to assume the existence of an edge in $\mathcal{H}$ that isn’t present in $\mathcal{G}$, any transient potential obtained on this graph gives rise to a transient potential on $X_G$. 
Proof of Lemma 2.6. For the first statement, assume that the edge $a \to a$ was not already present in $G$ (otherwise no “extension” is necessary and we are done). Let $\phi$ be a transient Markov potential on $X_\mathcal{G}$. Define $\varphi$ on $X_{\overline{\mathcal{G}}}$ by $\varphi(x_0, x_1) := \phi(x_0, x_1)$ whenever $x_0 \to x_1$ in $\mathcal{G}$ and $\varphi(a, a) = -M$ (to be specified later). Then defining $r_\phi(x)$ and $r_\varphi(x)$ in terms of $Z_n^*(\phi, a)$ and $Z_n^*(\varphi, a)$ respectively, we get

$$r_\varphi(x) = e^{-M}x + r_\phi(x)$$

(in particular both series have the same radius of convergence). Since $\phi$ and $\varphi$ are Markov, we get

$$t_\varphi(x) = \frac{1}{1 - r_\varphi(x)} = \frac{1}{1 - e^{-M}x - r_\phi(x)}.$$ 

Now $t_\phi(x) = \frac{1}{1 - r_\phi(x)}$ converges at its radius of convergence, so for sufficiently large $M$ we get that $t_\varphi(x)$ converges at its radius of convergence too. So $\varphi$ is transient on $X_{\overline{\mathcal{G}}}$.

The second statement is similar; if $\phi$ is the restriction of $\varphi$ to $X_\mathcal{G}$ then $r_\phi(x)$ and $r_\varphi(x)$ have the same radius of convergence (since we only have deleted one path from the series defining it) and $r_\phi(x) < r_\varphi(x)$ for any $x$ for which both series converge. Thus $t_\phi(x) < t_\varphi(x)$ at their (common) radius of convergence, so $\phi$ is transient on $X_\mathcal{G}$. 

Another technical point is that we often will show the existence of a transient potential on $X_\mathcal{G}$ by finding a potential $\phi \in \Phi$ with $\Delta_a[\phi] < \infty$. It is easy to see that this implies that a transient potential exists, as the next lemma shows.

**Lemma 2.7.** If for some (hence every) $a \in \mathcal{S}_\mathcal{G}$, $\phi \in \Phi$ has $\Delta_a[\phi] < \infty$ then for all sufficiently large $t \in \mathbb{R}$ we have $\phi - t \cdot 1_{\{a\}} \in \Phi(T)$. 


Proof. $\Delta_a[\phi - t \cdot 1_{[a]}] = \Delta_a[\phi] - t$ (by Lemma 3.3). This is negative for all sufficiently large $t$ and by Theorem 1.3 such a potential is transient. □

2.4.1 Infinite Rays

In this section we prove lemma 2.8 which shows that if $G$ contains an infinite ray, then $X_G$ has a transient potential.

An infinite forward ray in a directed graph $G$ is a path

$$v_0 \to v_1 \to v_2 \to v_3 \to \cdots$$

where $v_i \neq v_j$ for any $i \neq j$. Similarly an infinite backward ray in $G$ is a path

$$w_0 \leftarrow w_1 \leftarrow w_2 \leftarrow w_3 \leftarrow \cdots$$

where $w_i \neq w_j$ for any $i \neq j$.

Lemma 2.8. If $G$ contains an infinite forward or backward ray, then $X_G$ has a transient potential.

Suppose $G$ has an infinite forward ray

$$b_0 \to b_1 \to b_2 \to b_3 \to b_4 \to b_5 \to \cdots$$

We make two definitions:
\( R := \{b_i \mid i = 0, 1, 2, \ldots\} \)

\( R_{i,j} := \{b_i, b_{i+1}, \ldots, b_{j-1}, b_j\} \) for any \( i < j \).

The strategy for the proof is as follows. There are two particular graphs whose shifts have transient potentials: the backwards renewal shift and the natural numbers (defined below). Both contain infinite forward rays and turn out to be prototypes for all graphs containing such rays in the sense that any graph \( \mathcal{G} \) (which determines a topologically mixing shift) has a subgraph with enough similarities to one of the prototypes that we can guess a transient potential on it by analogy with the transient potential on the prototype. We show how to find this subgraph and define the transient potential.

**Case 1 (Backwards Renewal Shift):** Suppose there is some \( k_0 \in \mathbb{N}_0^2 \) such that for infinitely many \( l > k_0 \) (say for \( l_1, l_2, \ldots \)) there is a path (w.l.o.g. a simple path)

\[ p_i : b_{l_i} \xrightarrow{L(p_i)} b_{k_0} \]

such that \( (p_i)_j \notin R \) for any \( 0 < j < L(p_i) - 1 \); that is there is a path from \( b_{l_i} \) to \( b_{k_0} \) which is disjoint from the ray except at its endpoints. We claim that \( \mathcal{G} \) contains a subgraph which is analogous to the backwards renewal shift (this directed graph has \( S = \mathbb{N}_0 \), a directed edge from \( n \to (n + 1) \) for every \( n \geq 0 \) and a directed edge \( n \to 0 \) for every \( 2\mathbb{N}_0 := \{0, 1, 2, \ldots\} \).
Without loss of generality we may assume that \( k_0 = 0 \) (otherwise consider the sub-ray starting from \( b_{k_0} \)) and, for notational convenience, we set \( a := b_0 \). Moreover let \( V(p_1, p_2, \ldots, p_k) \) be the set of vertices in the paths \( p_1, \ldots, p_k \).

The paths \( \{p_i\}_{i=0}^{\infty} \) may intersect each other in complicated ways. We will modify the paths so that whenever two paths meet, they coalesce. Set \( \tilde{p}_1 := p_1 \). Now \( (\tilde{p}_1)L(\tilde{p}_1)-1 = a = (p_2)L(\tilde{p}_2)-1 \), so let \( i_0 \) be the minimal index so that \( (p_2)_{i_0} \in V(\tilde{p}_1) \) and define \( j_0 \) so that \( (p_2)_{i_0} = (\tilde{p}_1)_{j_0} \). We define a new path \( \tilde{p}_2 \) from \( b \) to \( a \) by following \( p_2 \) until the first point of intersection with \( \tilde{p}_1 \) and then following \( \tilde{p}_1 \) the rest of the way to \( a \); specifically we set

\[
(\tilde{p}_2)_i = \begin{cases} 
(p_2)_i & \text{for } 0 \leq i \leq i_0 \\
(\tilde{p}_1)_{j_0-i_0+i} & \text{for } i_0 < i \leq L(\tilde{p}_1) + i_0 - i_0 - 1.
\end{cases}
\]

Recursively define \( \tilde{p}_{k+1} \) from \( \tilde{p}_1, \ldots, \tilde{p}_k \) by setting \( \tilde{p}_{k+1} \) to be the path \( p_{k+1} \) until the first point of intersection with \( V(\tilde{p}_1, \ldots, \tilde{p}_k) \) and then following \( \tilde{p}_i \) the rest of the way to \( a \) (where \( i \) is the index of one of the paths met at this point). Note that the graph obtained by taking the union of all the paths \( \tilde{p}_i \) is a subgraph of \( G \) in which every vertex
has outgoing degree one (that is from any vertex there is a unique successor vertex but possibly many predecessor vertices).

Finally let $H$ be the subgraph of $G$ obtained by taking all vertices and edges contained in the ray $a \to b_1 \to b_2 \to \cdots$ and all vertices and edges contained in the paths $\tilde{p}_i$ for $i = 1, 2, \ldots$ (as well as the edge $a \to a$ using lemma 2.6).

The subgraph $H$

By construction, for each $i$ there is a unique first return loop (in $H$) based at $a$ that passes through the first edge of $\tilde{p}_i$ and, moreover, all first return loops can be described in this way (since they must contain at least one edge not contained in $R$ and so must pass through the first edge of some $\tilde{p}_i$). Define a potential

$$
\phi(x_0, x_1) = \begin{cases} 
\log \frac{1}{3k^2} & \text{if } x_0 = b_k, \ x_1 = (\tilde{p}_k)_1 \\
0 & \text{otherwise.}
\end{cases}
$$

We claim that the series $r_\phi(1)$ converges at $x = 1$ and diverges for $x > 1$. First,

$$
r_\phi(1) = \sum_{n=0}^{\infty} Z^*_n(\phi, a) = \sum_{k=1}^{\infty} \frac{1}{3k^2} < 1.
$$
On the other hand, the length of the (unique) first return loop that passes through the edge \( b_{l_k} \rightarrow (p_k)_1 \) has length at least \( k \), so for any \( x > 1 \) we have

\[
    r_{\phi}(x) = \sum_{n=0}^{\infty} Z_n^\phi(\phi, a) x^n \geq \sum_{n=0}^{\infty} \frac{x^n}{3n^2} = \infty.
\]

Thus \( r_{\phi}(x) \) converges at its radius of convergence, and \( r_{\phi}(1) < 1 \) so \( \phi \) is transient by equation (2.1.1).

**Case 2 (The Natural Numbers):** If case 1 doesn’t hold, then for every \( k \) there are at most finitely many \( l > k \) (say \( \lfloor k \rfloor_1 < \lfloor k \rfloor_2 < \cdots < \lfloor k \rfloor_{r_k} \)) such that there is a path \( p_l : b_{l_k} \xrightarrow{L(p_l)} R_{0,k} \)

with \( (p_l)_j \notin R \) for \( 0 < j < L(p_l) - 1 \). Now we claim that \( G \) contains a subgraph which is analogous to the nearest neighbor graph of the natural numbers (we will denote this graph \( G(\mathbb{N}_0) \) – see figure below).

\[
    G(\mathbb{N}_0)
\]

Set \( m_k := \lfloor k \rfloor_{r_k} \) (the maximal index with the above property). Let

\[
p_0 : b_{m_0} \xrightarrow{L(p_0)} a
\]
be a simple path with \((p_0)_i \notin R\) for any \(0 < i < L(p_0) - 1\). Define a sequence \(\{M_i\}\) by 
\[M_0 := m_0 \text{ and } M_{i+1} = m_{M_i}^i.\]
By construction, for every \(k \geq 1\), there is a simple path 
\[p_k : b_{M_k} ^{L(p_k)} \rightarrow R_{0,M_k-1}\]
such that \((p_k)_i \notin R\) for any \(0 < i < L(p_k) - 1\). Moreover, by maximality of \(M_{k-1}\), we have 
\[(p_k)_L(p_k)_{-1} \in R_{1+M_{k-2},M_{k-1}}.\]
Finally note that \(V(p_i) \cap V(p_j) = \emptyset\) if \(|i - j| \neq 1\) and either \(V(p_{i+1}) \cap V(p_i) = \emptyset\) or 
\(V(p_{i+1}) \cap V(p_i) = (p_i)_0 = (p_{i+1})_{L(p_{i+1})-1}\), since otherwise (if \(p_i\) and \(p_j\) meet at some vertex not in \(R\)) we could follow \(p_i\) to this meeting point and then follow \(p_j\) back to \(R\) to obtain a contradiction of the maximality of \(M_j\).

Let \(H\) be the subgraph of \(G\) obtained by taking all vertices and edges in the ray and the paths \(\{p_j\}_{i=0}^\infty\) (and \(a \rightarrow a\) by lemma 2.6).

The subgraph \(H\)

All vertices in \(H\) except for those in the set \(A := \{b_{M_0}^i, b_{M_1}^i, b_{M_2}^i, \ldots\}\) have 
outgoing degree one and each vertex in \(A\) has outgoing degree two (namely, the vertices 
\(b_{M_{k+1}}^i\) and \((p_k)_i\) are the successors of \(b_{M_k}^i\)). Thus a path in \(H\) is completely determined
by a description of which successor is chosen each time a vertex in $A$ is crossed (note the similarity with the natural numbers where a path is just a description of whether to go right or left at any particular vertex).

We now work to obtain a precise description of the first return loops in $H$ based at $a$. Define a map $\alpha : R \to \mathbb{N}_0$ by

$$\alpha(b_i) = \begin{cases} 
0 & \text{if } i = 0 \\
 j & \text{if } b_i \in R_{M_j-1+1,M_j}.
\end{cases}$$

Further, define a function $\beta : X_H \to \mathbb{N}_0^{\mathbb{N}_0}$ by

$$\beta(x)_j = \alpha(x_{k_j})$$

where $k_0, k_1, \ldots$ are the indices for which $x_i \in A$. Note that $\beta$ gives a bijection between paths in $H$ starting at $a$ and paths in $\mathcal{G}(\mathbb{N}_0)$ starting at 0. Moreover, a path in $H$ is a loop (based at $a$) if and only if its image under $\beta$ is a loop in $\mathbb{N}_0$ (based at 0).

Now the potential $\phi(x_0,x_1) \equiv -\log(2)$ has finite discriminant on $X_\mathcal{G}(\mathbb{N}_0)$ and $P_G(\phi) = 0$ (see proof in Appendix B). We define an analogous potential on $X_H$:

$$\tilde{\phi}(x_0,x_1) \equiv -\log(2) \cdot 1_A(x_0)$$

Observe that if $T^n y = y$ and $y_0 \in [a]$ is a loop in $H$ based at $a$, then

$$e^{\tilde{\phi}_n(y)} = e^{\phi_{my}(\beta(y))}$$
where $m_y$ is the length of the loop $\beta(y)$ in $\mathbb{N}$. Moreover the map $\beta$ cannot increase the length of a loop (but can decrease it). Therefore

$$r_\phi^{-1}(1) = r_\phi(1) < \infty,$$

$$r_\phi^{-1}(x) \geq r_\phi(x) = \infty \text{ for } x > 1.$$

Therefore $\tilde{\phi}$ (on $X_H$) has finite discriminant and so, as in the previous case, there is a transient potential on $X_G$.

The case with a backward infinite ray is similar.

2.4.2 Double Bouquets

We now consider graphs without infinite rays. The main result in this section is lemma 2.10 which shows that if $G$ has infinitely many disjoint simple loops, it has a transient potential. This is shown with the use of a special kind of subgraph called a double bouquet defined below.

We say a graph $G$ is a double bouquet if there is some vertex $a \in S_G$ such that $G$ has the following form:

- Primary Loops: There is a set of countably many simple first return loops based at $a$

  $$\{P_1, P_2, \ldots\}.$$

Moreover, there is a constant $C_0$ so that the loops are disjoint apart from their starting point and their last $C_0$ vertices. Specifically:
\[(P_i)_{L(P_i) - k - 1} = (P_j)_{L(P_j) - k - 1}\] for every \(0 \leq k \leq C_0\).

\[(P_i)_k \notin V(P_j)\] for any \(0 < k < L(P_i) - C_0 - 1\).

- Secondary Loops: For each \(i\) there exists \(k_i \notin \{0, L(P_i) - 1\}\) and a simple loop \(S_i\) based at \((P_i)_{k_i}\) whose vertex set is disjoint from \(V(P_j) \cup V(S_j)\) for \(j \neq i\). Moreover \(a \notin V(S_i)\) and in the graph that consists of all edges and vertices in \(P_i\) and \(S_i\), there is exactly one vertex with outgoing degree two.

- There are no other vertices or edges in \(G\) except possibly the edge \(a \rightarrow a\).

\[\text{A Double Bouquet \text{(with } C_0 = 0\text{)}}\]

**Lemma 2.9.** If \(G\) is a double bouquet and \(X_G\) is topologically mixing, then \(X_G\) has a transient Markov potential.
Define $b_k := (P_k)_1$ so that the edge $a \to b_k$ is not an edge in $P_i$ for any $i \neq k$ (and is not an edge in any $S_i$). Next let $c_k, d_k \in V(S_k)$ be vertices such that $c_k \to d_k$ is an edge in $S_k$ but not in $P_k$. Define a potential $\phi$ by

$$\phi(x) = \phi(x_0, x_1) := \begin{cases} 
\log\left(\frac{k}{k+1}\right) & \text{if } x_0 = c_k, x_1 = d_k \\
\log\left(\frac{1}{k^3}\right) & \text{if } x_0 = a, x_1 = b_k \\
0 & \text{otherwise.}
\end{cases}$$

Note that if $\ell$ is a first return loop (of length greater than one) in $X_\mathcal{G}$ based at $a$, then $(\ell)_1 = b_i$ for some $i$. Moreover, by definition of the loop $S_i$, there is some number $n$ so that $\ell$ is the loop running around $P_i$ once and $S_n$ $n$ times (it cannot contain any other $P_j$ or $S_j$ as a subloop since it is a first return loop). We will show that that $r^a(x)$ converges for $x = 1$ and diverges for $x > 1$ (and hence has finite discriminant and finite pressure).

Recall that $r_\phi(1) = \sum_{n=1}^\infty Z_n^*(\phi, a)$. The contribution to the sum from the $k^{th}$ component of $\mathcal{G}$ (namely $P_k \cup S_k$) is

$$\sum_{m=0}^\infty e^{\phi(a, b_k)} e^{m\phi(c_k, d_k)}.$$ 

So the sum in question is

$$r_\phi(1) = 1 + \sum_{k=1}^\infty \sum_{m=0}^\infty \frac{1}{k^3} \left(\frac{k}{k+1}\right)^m = 1 + \zeta(2) + \zeta(3).$$
Finally, we show that $r^a(x)$ diverges for $x > 1$. Note that $L(P_k), L(S_k) \geq 1$.

\[
r_\phi(x) = x + \sum_{k=1}^{\infty} \frac{1}{k^3} x L(P_k) + \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{k^3} \left( \frac{k}{k+1} \cdot x \right)^m L(S_k) x L(P_k)
\]

\[
\geq \sum_{k=1}^{\infty} \frac{1}{k^3} x L(P_k) \sum_{m=1}^{\infty} \left( \frac{k}{k+1} \cdot x \right)^m, \text{ when } x > 1
\]

\[
\geq \frac{1}{k^3} \sum_{m=1}^{\infty} \left( \frac{k_0}{k_0+1} \cdot x \right)^m = \infty
\]

where $k_0$ is some integer with $\left( \frac{k_0}{k_0+1} \cdot x \right) > 1$.

So the series $r_\phi(x)$ converges at its radius of convergence and so $\phi$ has finite discriminant and therefore a transient potential by lemma 2.7.

**Lemma 2.10.** If $G$ has infinitely many pairwise disjoint simple loops (based at different points), then $X_G$ has a transient Markov potential.

We will assume that $G$ does not contain an infinite ray (otherwise we are done by lemma 2.8) and show that $G$ contains a double bouquet (and so a transient potential by lemma 2.9). The proof involves a number of inductive constructions, so we first give a brief overview of the steps.

**Overview of Steps.** We attempt to build a double bouquet. The idea is to view the infinite collection of disjoint loops (call them $\ell_1, \ell_2, \ldots$) as the “secondary” loops and to build the primary loops.
Recall the properties of the primary loops \( \{P_1, P_2, \ldots \} \):

1. The loops are simple and are all based at a common vertex \( a \);

2. The loop \( P_i \) intersects \( S_i \) so that any first return loop in \( P_i \cup S_i \) based at \( a \) is obtained by going around \( P_i \) once and \( S_i \) some number of times;

3. For any \( i, j \) the final \( C_0 \) vertices of \( P_i \) and \( P_j \) coincide;

4. For any \( i \neq j \), \( P_i \) and \( P_j \) are disjoint apart from their starting point and their final \( C_0 \) vertices.

**Step 1:** We fix \( a \in S \) and construct a path \( \tilde{p}_i \) from \( a \) to \( \ell_i \) for each \( i \). We ensure that \( V(\tilde{p}_i) \cap V(\tilde{p}_j) = \{a\} \) for \( i \neq j \) and that \( \tilde{p}_i \) does not intersect \( \ell_j \) for any \( j \neq i \).

**Step 2:** We construct a path \( \tilde{\rho}_i \) from \( \ell_i \) to \( a \) for each \( i \). We ensure that for some constant \( C_0 \), \( V(\tilde{p}_i) \cap V(\tilde{\rho}_j) \) is exactly the final \( C_0 \) vertices of \( \tilde{p}_i \) for each \( i \neq j \).

We would now like to let \( P_i \) be the loop obtained by following \( \tilde{p}_i \) from \( a \) to \( \ell_i \), then following \( \ell_i \) from the end of \( \tilde{p}_i \) to the beginning of \( \tilde{\rho}_i \), and finally following \( \tilde{\rho}_i \) back to \( a \). Unfortunately these loops may not be of the proper form for two reasons: first the loops might not be disjoint outside of their starting point and their final \( C_0 \) vertices and, second, the loops might not be simple.

**Step 3:** We pass to a subsequence of these loops so that they are disjoint apart from their initial vertex and their final \( C_0 \) vertices.
**Step 4**: We modify the graph obtained from $\tilde{p}_i$, $\tilde{\rho}_i$ and $\ell_i$ so that the primary loop is simple (in some cases, this will require us to let the secondary loop be a different loop in this graph than $\ell_i$).

The result of these four steps will give us a double bouquet.

**Step 1**: The first step is to build a path from $a$ to each of the “secondary” loops and give these paths a simple graph structure.

Fix some $a \in S_g$ and suppose there are infinitely many disjoint loops in $X$; enumerate these as $\ell_1, \ell_2, \ldots$ (wlog assume that $a \notin \ell_i$ for any $i$). For each $i$, let

$$p_i : a \xrightarrow{L(p_i)} V(\ell_i)$$

be a simple path from $a$ to some vertex in $\ell_i$. We will assume that $V(p_i) \cap V(\ell_j) = \emptyset$ for any $i \neq j$. This causes no loss of generality, we will see this by induction. Since the loops $\ell_i$ are all disjoint, the path $p_1$ intersects at most finitely many of them; pass to the (infinite) subsequence of indices $n_1, n_2, \ldots$ so that $n_1 = 1$ and $V(p_1) \cap V(\ell_{n_1}) = \emptyset$ for $i > 1$. Now there are three possibilities. First, it could be that infinitely many paths $p_{n_j}$ do not intersect the loop $\ell_1$; then pass to the (sub)sub-sequence consisting of only $n_1$ and indices corresponding to these paths. The result is a subsequence of indices (which includes the index 1) so that $V(p_1) \cap V(\ell_{n_j}) = V(p_{n_j}) \cap V(\ell_1) = \emptyset$ for $j > 1$. Otherwise, all but finitely many paths $p_{n_j}$ intersect $\ell_1$; for each such path, let $v_{n_j}$ be the last vertex in $V(p_1) \cup V(\ell_1)$ which is included in the path $p_{n_j}$. The second possibility is that this occurs and $v_{n_j} \notin V(\ell_1)$ for infinitely many $j$; then pass to the subsequence of only those
indices for which \( v_{n_j} \notin V(\ell_1) \) (as well as index 1). Now modify the paths \( p_{n_j} \) to follow \( p_1 \) until vertex \( v_{n_j} \) and then follow \( p_{n_j} \) the rest of the way to \( \ell_{n_j} \). The result is a (modified) collection of paths which do not intersect \( \ell_1 \). The third possibility is that all but finitely many \( v_{n_j} \) correspond to vertices in \( V(\ell_1) \). In this case, set \( u_1 \in V(\ell_1) \) to be a vertex so that \( v_{n_j} = u_1 \) for infinitely many \( j \); pass to the subsequence of indices for which this is true (as well as index 1) and modify the paths to follow \( p_1 \) and \( \ell_1 \) until vertex \( u_1 \) and then continue as the path originally did.

At this stage, we have either produced a subsequence of indices (and possibly modified paths) so that \( V(p_1) \cap V(\ell_{n_j}) = V(p_{n_j}) \cap V(\ell_1) = \emptyset \) for \( j > 1 \) or we have produced a subsequence so that all paths follow \( p_1 \) to \( u_1 \in V(\ell_1) \) and then continue to their respective loops \( \ell_{n_j} \). Now we repeat this procedure for \( \ell_{n_2} \) and \( p_{n_2} \) and continue inductively. At each step, one of the three possibilities above occurs. If either the first or second possibilities occurs infinitely many times then pass to the subsequence consisting of only these indices and the above procedure produces a collection of paths with the desired property (i.e. \( V(p_i) \cap V(\ell_j) = \emptyset \) for any \( i \neq j \)). Otherwise the third possibility occurs all but finitely many times (for simplicity, say it occurs always). In this case, the graph \( G \) contains and infinite forward ray obtained by following \( p_1 \) to vertex \( u_1 \), then following \( p_{n_2} \) to vertex \( u_2 \) and so on. Since we assumed that \( G \) does not contain a forward infinite ray, this cannot be the case and our assumption is justified.

The paths \( \{p_i\} \) may intersect each other in complicated ways. We will define a new collection of paths \( \{\tilde{p}_i\} \) so that \( \tilde{p}_k \) connects \( a \) to \( V(\ell_k) \) and the union of \( \{\tilde{p}_i\} \) has a “branching” structure (the paths will be initially identical for a fixed number of vertices and then they will “branch off” from each other and be disjoint thereafter).
Begin by setting $\tilde{p}_1 := p_1$. Now we proceed inductively. Suppose we have constructed simple paths $\tilde{p}_1, \ldots, \tilde{p}_k$ with $V(\tilde{p}_1, \ldots, \tilde{p}_k) \subseteq V(p_1, \ldots, p_k)$ so that the graph $V(\tilde{p}_1) \cup \cdots \cup V(\tilde{p}_k)$ has the structure of a tree, we construct $\tilde{p}_{k+1}$. Since $(p_{k+1})_{L(p_{k+1} - 1)} \notin V(\tilde{p}_1, \ldots, \tilde{p}_k)$ let $i_{k+1}$ be the maximal index such that $(p_{k+1})_{i_{k+1}} \in V(\tilde{p}_1, \ldots, \tilde{p}_k)$, say $(p_{k+1})_{i_{k+1}} = (\tilde{p}_j)_n$ for some $j, n$. Define $\tilde{p}_{k+1}$ to be the (simple) path that follows $\tilde{p}_j$ until vertex $n$ and then follows $p_{k+1}$ the rest of the way to $\ell_{k+1}$.

The paths $p_i$

Note that for any $i \neq j$ the paths $\tilde{p}_i$ and $\tilde{p}_j$ are initially the same path and then split. Specifically, there is some index $s_{i,j}$ such that

$$(\tilde{p}_i)_k = (\tilde{p}_j)_k \text{ for all } 0 \leq k \leq s_{i,j}$$

$$(\tilde{p}_i)_k \notin V(\tilde{p}_j) \text{ for any } k > s_{i,j}$$

$$(\tilde{p}_j)_k \notin V(\tilde{p}_i) \text{ for any } k > s_{i,j}.$$

Our goal is now to find a subsequence of paths for which $s_{i,j}$ is independent of $i$ and $j$ (that is, there is a single vertex up to which all paths are the same and after which all paths are disjoint). To start, note that $s_{1,j} = L(\tilde{p}_1)$ for all $j$, so let $s_1$ be the largest index such that $s_{1,j} = s_1$ for infinitely many $j \in \mathbb{N}$. Let $I_1$ be the set of indices
\( I_1 := \{ j \mid s_{1,j} = s_1 \} \). Suppose we have constructed index sets

\[
I_1 \supseteq I_2 \supseteq \cdots \supseteq I_r
\]

such that \( I_k \) is infinite for all \( k = 1, \ldots, r \) and \( s_{k,j} = \text{const} \) for all \( j \in I_k \); set \( s_k \) to be this constant. Since \( I_r \) is infinite there is some \( s_{r+1} \) so that \( s_{r+1,j} = s_{r+1} \) for infinitely many \( j \in I_r \) (note that \( s_{r+1} \geq s_r \) since any two paths in \( I_r \) have the same first \( s_r \) vertices). Let \( I_{r+1} \subseteq I_r \) be the set of indices for which this is true. By induction we obtain an infinite descending sequence of index sets

\[
I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots
\]

Let \( k_1 < k_2 < k_3 < \cdots \) be a sequence of indices such that \( k_i \in I_i \) for all \( i \). Then for fixed \( i \) and any \( j > i \) we have \( s_{k_i,k_j} = s_{k_i} \). By construction, if the sequence \( s_{k_1}, s_{k_2}, \ldots \) is unbounded then \( X \) contains an infinite forward ray; contradicting our assumption that it does not (we reduced to this case since otherwise we are done by lemma 2.8).

If the sequence \( \{s_{k_i}\} \) is bounded, say \( N \) is such that \( s_{k_i} = s_{k_j} \) for all \( i, j > N \), then we consider the subsequence of paths with index set

\[
\mathcal{I} := \{ k_{N+i} \mid i = 1, 2, \ldots \}
\]

Then there is some constant \( C_1 \) such that for any \( i, j \in \mathcal{I} \),
\[ (\tilde{p}_i)_k = (\tilde{p}_j)_k \text{ for all } 0 \leq k \leq C_1 \]
\[ (\tilde{p}_i)_k \notin V(\tilde{p}_j) \text{ for any } k > C_1 \]
\[ (\tilde{p}_j)_k \notin V(\tilde{p}_i) \text{ for any } k > C_1. \]

Without loss of generality we can assume that \( C_1 = 0 \) (by redefining \( a \) to be the vertex \((\tilde{p}_i)_C_1\)). This is the “branching” structure we had desired – the paths \( \tilde{p}_i \) connect \( a \) to loops \( \ell_i \) and are disjoint except at their initial vertex.

**Step 2:** We now build the rest of the “primary” loop by finding a path back from each “secondary” loop to \( a \). In this step we will build these paths and give them a similar “branching” structure to those in the previous step.

For each \( i \in I \) (the index set from the previous step) let \( \rho_i \) be a simple path from some vertex in \( \ell_i \) to \( a \):

\[ \rho_i : V(\ell_i) \xrightarrow{L(\rho_i)} a \]

such that \( (\rho_i)_j \notin V(\ell_i) \) for any \( j > 0 \) (i.e. only the first vertex of \( \rho_i \) is in \( V(\ell_i) \)). As before, we may assume that \( V(\rho_i) \cap V(\ell_j) = \emptyset \) for every \( i \neq j \) (otherwise \( \rho_i \) intersects at most finitely many \( \ell_j \) and we can inductively produce a subsequence where the condition holds). It may, again, be the case that the paths \( \rho_{ij} \) intersect each other in complicated ways. As before we will define a new collection of paths \( \tilde{\rho}_j \) so that \( \tilde{\rho}_j \) connects \( \ell_j \) to \( a \) and this time the paths \( \{\tilde{\rho}_i\} \) have a type of “reverse branching” structure (the paths are initially disjoint and all coalesce at a common vertex).
For convenience write \( I = \{i_1, i_2, \ldots \} \) with \( i_1 < i_2 < \cdots \). Set \( \tilde{\rho}_{i_1} := \rho_{i_1} \). As before, we proceed by induction. Suppose we have constructed simple paths \( \tilde{\rho}_{i_1}, \ldots, \tilde{\rho}_{i_k} \), we will show how to construct \( \tilde{\rho}_{i_{k+1}} \). Let \( j_{k+1} \) be the least index so that \( (\rho_{i_{k+1}})_{j_{k+1}} \in V(\tilde{\rho}_{i_1}, \ldots, \tilde{\rho}_{i_k}) \), say \( (\rho_{i_{k+1}})_{j_{k+1}} = (\tilde{\rho}_l)_{m} \) for some \( l, m \). Define \( \tilde{\rho}_{i_{k+1}} \) to be the simple path that follows \( \rho_{i_{k+1}} \) for the first \( j_{k+1} \) vertices and then follows \( \tilde{\rho}_l \) thereafter.

Then, as before, for any \( i, j \in I \) with \( i \neq j \), there is some index \( \tilde{s}_{i,j} \) so that

\[
(\tilde{\rho}_{i})_{L(\tilde{\rho}_{i})} - k - 1 = (\tilde{\rho}_{j})_{L(\tilde{\rho}_{j})} - k - 1 \quad \text{for all} \quad 0 \leq k \leq \tilde{s}_{i,j};
\]

\[
(\tilde{\rho}_{i})_{k} \notin V(\tilde{\rho}_{j}) \quad \text{for any} \quad k < L(\tilde{\rho}_{i}) - \tilde{s}_{i,j} - 1;
\]

\[
(\tilde{\rho}_{j})_{k} \notin V(\tilde{\rho}_{i}) \quad \text{for any} \quad k < L(\tilde{\rho}_{j}) - \tilde{s}_{i,j} - 1.
\]

This means precisely that the paths are initially disjoint and at some point coalesce.

Similar to before, either there is a backward infinite ray contained in \( G \) (contradicting our assumption) or we can pass to a subsequence with index set \( J \subseteq I \) so that the number \( \tilde{s}_{i,j} \) is independent of \( i, j \in J \). This gives exactly the “reverse branching” structure that we desired. Define \( T \) to be the common set of vertices shared by the paths \( \{\tilde{\rho}_{i} \mid i \in J\} \). Note that \( T \) is the final \( |T| \) vertices in each path \( \tilde{\rho}_{i} \) \( i \in J \). Finally, we
will be working only with the subsequence determined by $J$ for the remainder of the proof, so for notational convenience we will assume that $J = \mathbb{N}$ (by renaming our paths if necessary).

**Step 3:** As previously stated, we would like for the “primary” loops in our construction to be the loops determined by the paths $\tilde{p}_i$ and $\tilde{\rho}_i$ (and the vertices in $\ell_i$ needed to connect these two paths). There are two issues left to consider. First these “primary” loops may not be simple if $\tilde{p}_i$ and $\tilde{\rho}_i$ intersect in places other than their endpoints. We will resolve this in step four. The other issue is that a path $\tilde{p}_i$ may intersect a number of different $\tilde{\rho}_j$ (or vice versa) so that the “primary” loops are not disjoint outside of their final $|T| (=: C_0)$ vertices. We resolve this issue here.

We will pass to a subsequence of triples $\{((\ell_i, \tilde{p}_i, \tilde{\rho}_i))\}_{i \in \mathbb{N}}$ so that $V(\tilde{p}_i) \cap V(\tilde{\rho}_j) = \{a\}$ for every $i \neq j$. We accomplish this inductively: take $c_1 := 1$ and observe that $\tilde{p}_{c_1}$ intersects at most finitely many $\tilde{\rho}_k$ outside of the set $\{a\}$ and similarly $\tilde{\rho}_{c_1}$ intersects at most finitely many $\tilde{p}_k$ outside of the set $T$ (defined in the previous step). Let

$$K_1 := \{ k \in \mathbb{N} \mid V(\tilde{p}_{c_1}) \cap V(\tilde{\rho}_k) \subseteq T \text{ and } V(\tilde{\rho}_{c_1}) \cap V(\tilde{p}_k) = \{a\} \}.$$ 

We construct sets $K_i$ by induction. Assume that we have constructed a sequence of infinite index sets

$$K_1 \supseteq \cdots \supseteq K_r.$$
and integers \( c_1 < \cdots < c_r \) so that \( c_i \in K_{i-1} \) (we consider \( K_0 := \mathbb{N} \)) so that whenever \( i \in K_n \), we have \( V(\tilde{p}_n) \cap V(\tilde{\rho}_i) \subseteq T \) and \( V(\tilde{p}_n) \cap V(\tilde{\rho}_i) = \{a\} \). Let \( c_{r+1} \in K_r \) be some element with \( c_{r+1} > c_r \). Define

\[
K_{r+1} := \{ k \in K_r \mid V(\tilde{p}_{c_{r+1}}) \cap V(\tilde{\rho}_k) \subseteq T \text{ and } V(\tilde{\rho}_{c_{r+1}}) \cap V(\tilde{p}_k) = \{a\} \}.
\]

This set is infinite since \( \tilde{p}_{r+1} \) intersects at most finitely many \( \tilde{\rho}_k \) outside of \( T \) and \( \rho_{r+1} \) intersects at most finitely many \( \rho_k \) outside of \( \{a\} \). Finally let \( \mathcal{K} := \{c_1, c_2, \ldots \} \). By construction, if \( i, j \in \mathcal{K} \) and \( i < j \) then \( V(\tilde{p}_i) \cap V(\tilde{\rho}_j) \subseteq T \) and \( V(\tilde{\rho}_i) \cap V(\tilde{p}_j) = \{a\} \).

But there are only finitely many \( j \in \mathcal{K} \) for which \( V(\tilde{p}_j) \cap (T \setminus \{a\}) \neq \emptyset \) (since the paths \( \tilde{p}_i \) are disjoint apart from their common starting point, so each vertex in \( T \setminus \{a\} \) is contained in at most one \( \tilde{p}_i \)). Let \( \mathcal{K}' \subseteq \mathcal{K} \) be the (infinite) set of vertices for which \( V(\tilde{p}_i) \cap V(\tilde{\rho}_j) = \{a\} \) for every \( i, j \in \mathcal{K}' \) with \( i \neq j \).

It is now the case that if \( i, j \in \mathcal{K}' \) and \( i \neq j \), then \( V(\tilde{p}_i) \cap V(\tilde{\rho}_j) = \{a\} \), \( V(\tilde{p}_i) \cap V(\tilde{\rho}_j) = T \) and \( V(\ell_i) \cap V(\tilde{\rho}_j) = \emptyset \). That is, if \( \tilde{P}_i \) is the loop formed by following \( \tilde{p} \) from \( a \) to \( \ell_i \), then following \( \ell_i \) to from the last vertex in \( \tilde{p}_i \) to the first vertex in \( \tilde{p}_i \) and finally following \( \tilde{\rho}_i \) back to \( a \), then the loops \( \tilde{P}_i \) are all based at a common vertex \( a \) and are all disjoint apart from \( \{a\} \) and their final \( C_0 = |T| \) vertices.

**Step 4:** We finally must ensure that the paths \( \tilde{P}_i \) we have constructed are simple.

Unfortunately this may not be the case – this situation is pictured below. We modify the triples \( (\tilde{p}_i, \tilde{\rho}_i, \ell_i) \) so that the “primary” loops are simple.
Given a triple \((\ell_i, \tilde{p}_i, \tilde{\rho}_i)\), if \(V(\tilde{p}_i) \cap V(\tilde{\rho}_i) \subseteq \{a\} \cup \{\tilde{\rho}_i, 0\}\) then we do nothing (take \(P_i = \tilde{P}_i\) and \(S_i = \ell_i\)). Otherwise there is some \(0 < k < L(\tilde{\rho}_i) - 1\) such that \((\tilde{\rho}_i)_k \in V(\tilde{p}_i)\). Let \(i_0 < L(\tilde{\rho}_i) - 1\) be the largest such index and let \(j_0\) be the index of \(\tilde{\rho}_i\) such that \((\tilde{\rho}_i)^{i_0}_1 = (\tilde{\rho}_i)^{j_0}_1\). In this case, take the “primary” loop \(P_i\) to be the path \(\tilde{p}_i\) for its first \(j_0\) vertices and then follow \(\tilde{\rho}_i\) back to \(a\). Note that \(P_i\) is simple by maximality of \(i_0\). Next let \(i_1\) be the least index of \(\tilde{\rho}_i\) so that \((\tilde{\rho}_i)^{i_1}_1 \in \{(\tilde{\rho}_i)^{i_0}_1, \ldots, (\tilde{\rho}_i)^{j_0}_1\}\) (it could be the case that \(i_0 = i_1\)). Finally let \(i_2 < i_1\) be the index of \(\tilde{\rho}_i\) so that \((\tilde{\rho}_i)^{i_2}_1 \in V(\tilde{p}_i)\) and the path from \((\tilde{p}_i)^{i_2}_1\) to \((\tilde{\rho}_i)^{i_1}_1\) is disjoint from \(V(\tilde{p}_i)\) or \(i_2 = 0\) if no such index exists (i.e. \(i_2\) is the index of the meet of \(\tilde{\rho}_i\) and \(\tilde{p}_i\) just before \(i_1\)). Set \(S_i\) to be the loop that follows \(\tilde{p}_i\) (and possibly part of \(\ell_i\)) from \((\tilde{\rho}_i)^{i_2}_1\) to \((\tilde{\rho}_i)^{i_1}_1\) and then follows \(\tilde{\rho}_i\) back to \((\tilde{\rho}_i)^{i_1}_1\). This loop is simple (by construction), contains \(a\) and the graph \(P_i \cup S_i\) has exactly one vertex with outgoing degree two. In this case we modify the triple \((\tilde{p}_i, \tilde{\rho}_i, \ell_i)\) to be the graph \(P_i \cup S_i\).

The original triple \((\ell_i, \tilde{p}_i, \tilde{\rho}_i)\)          The modified graph

Finally let \(H\) be the graph obtained by taking all edges and vertices in the loops \(P_i\) and \(S_i\) (and the edge \(a \rightarrow a\) using lemma 2.6), which is a double bouquet. By lemma 2.9 there is a transient potential on \(X_H\) and hence also on \(X_{\tilde{G}}\).
2.4.3 Simple Loops with Unbounded Size

The main result of this section is lemma 2.12 which shows that if $G$ has simple loops of arbitrarily large size based at a common vertex, then $X_G$ has a transient potential.

For the construction that we will perform, we will need some notation.

For some fixed $a \in S_G$ and a collection of simple loops $\ell, \ell_1, \ell_2, \ldots, \ell_n$ based at $a$, we say that $\ell$ is *semi-disjoint* from the collection $\{\ell_1, \ldots, \ell_n\}$ if there exists $i = i(\ell)$ and non-negative integers $A(\ell, \ell_i), B(\ell, \ell_i)$ such that

- $A(\ell, \ell_i) < L(\ell) - B(\ell, \ell_i) - 1$;

- $(\ell)_j = (\ell_i)_j$ for all $0 \leq j \leq A(\ell, \ell_i)$;

- $(\ell)_{L(\ell) - j} = (\ell_i)_{L(\ell_i) - j} - 1$ for all $0 \leq j \leq B(\ell, \ell_i)$;

- $(\ell)_j \notin \left(V(\ell_1) \cup \cdots \cup V(\ell_n)\right)$ for any $A(\ell, \ell_i) < j < L(\ell) - B(\ell, \ell_i) - 1$.

For a simple loop $\ell$, if $l$ is another simple loop then we define

\[
A^*_\ell(l) = \max\{i \mid (\ell)_j = (l)_j \text{ for all } j \leq i\}
\]

\[
B^*_\ell(l) = \max\{i \mid (\ell)_{L(\ell) - j} = (l)_{L(l) - j} - 1 \text{ for all } j \leq i\}.
\]
The statement that $\ell$ is \textit{not} semi-disjoint from $l$ means exactly that there is some $i$ with $A^\times_\ell(l) < i < L(\ell) - B^\times_\ell(l) - 1$ such that $(\ell)_i = (l)_i$.

For fixed $a \in S_G$, a simple loop $\ell$ based at $a$ determines a linear ordering on its vertices:

$$a = (\ell)_0 < (\ell)_1 < \cdots < (\ell)_{L(\ell)-2}$$

where we take $a$ to be the least element in this ordering rather than the greatest. We denote the ordering determined by $\ell$ with $\leq_\ell$.

Note that for a sequence of simple loops $p_1, p_2, \ldots$ based at a common point $a \in S_G$, exactly one of the following three conditions holds:

\begin{itemize}
  \item \textbf{C1:} There are infinitely many $k$ such that $p_k$ is semi-disjoint from $\{p_1\}$.
  \item \textbf{C2:} C1 does not hold, but for infinitely many $k$, the set $V(p_1) \cap V(p_k)$ is ordered in the same way by $\leq_{p_1}$ and $\leq_{p_k}$.
  \item \textbf{C3:} C1 and C2 do not hold and there exist distinct vertices $v_1, v_2$ in $p_1$ such that $v_1 \leq_{p_1} v_2$ but for infinitely many $k$, $v_2 \leq_{p_k} v_1$.
\end{itemize}

The following is a technical lemma which will allow us to prove lemma 2.12.

\textbf{Lemma 2.11.} Suppose $G$ does not have an infinite ray. If there is a vertex $a \in S_G$ and a sequence $p_1, p_2, \ldots$ of simple loops based at $a$ with $L(p_{i+1}) > L(p_i)$ $\forall i$, then there exist integers $K_1, K_2$ and a subsequence $p_{a_0}, p_{a_1}, p_{a_2}, \ldots$ such that $p_{a_k}$ is semi-disjoint from $\{p_{a_0}\}$, $A(p_{a_k}, p_{a_0}) = K_1$ and $B(p_{a_k}, p_{a_0}) = K_2$ for all $k \geq 1$.

There are three cases to consider.
Case 1: If the list $p_1, p_2, \ldots$ has property C1 then find integers $K_1, K_2$ such that for infinitely many $k$, $p_k$ is semi-disjoint from $p_1$ and $A(p_k, p_1) = K_1$, $B(p_k, p_1) = K_2$ (recall that $A(p_k, p_1), B(p_k, p_1)$ can take on only finitely many different values since they correspond to vertices in $p_1$). Pass to the subsequence including only $p_1$ and loops of this type.

Case 2: If the list $p_1, p_2, \ldots$ has property C2, we claim that a subsequence has property C1. Begin by finding integers $K'_1, K'_2$ such that $A^*(p_k)_{p_1} = K'_1$ and $B^*(p_k)_{p_1} = K'_2$ for infinitely many $k$ (again $A^*(p_k)_{p_1}, B^*(p_k)_{p_1}$ can take on only finitely many different values). Define

$$I := \{1\} \cup \{k \mid A^*(p_k)_{p_1} = K'_1 \text{ and } B^*(p_k)_{p_1} = K'_2\}$$

to be the index set of the subsequence of loops of this type (and also $p_1$).

For each $k \in I$ define

$$\{i_{1, \ldots, i_{r(k)}}^k\} := \{i \mid (p_k)_i \in V(p_1)\}$$

$$\{j_{1, \ldots, j_{r(k)}}^k\} := \{j \mid (p_k)_j \in V(p_k)\},$$

ordered so that $(p_k)_i^n = (p_1)_j^n$ for all $n$. Then for each $k$, there is some $m$ such that the (simple) loop:

$$a \xrightarrow{[p_1]} (p_1)_i^k \xrightarrow{[p_k]} (p_1)_i^m \xrightarrow{[p_k]} (p_1)_i^{m+1} \xrightarrow{[p_1]} a$$
has length greater than or equal to \( \frac{L(p_k)}{L(p_1)} \) (this is the loop \( p_1 \) with one “excursion” through part of \( p_k \)); where \( v_1 \xrightarrow{[p]} v_2 \) denotes the segment of path \( p \) from \( v_1 \) to \( v_2 \).

This is possible to do because the segment \( (p_k)^m \xrightarrow{[p_k]} (p_k)^m_{m+1} \) must have length at least \( \frac{L(p_k)}{L(p_1)} \) for some \( m \) (since the entire path \( p_k \) is made up of segments of this form).

Denote this loop by \( \tilde{p}_k \). Note that \( \tilde{p}_k \) is semi-disjoint from \( p_1 \) and \( A(\tilde{p}_k, p_1) \geq A^* (p_1) \) (similarly for the \( B \)'s). Pass to a subsequence of \( p_1, \tilde{p}_2, \tilde{p}_3, \ldots \) (that includes \( p_1 \)) for which the lengths of the paths are strictly increasing. This subsequence has property C1.

\textit{Case 3:} If the list \( p_1, p_2, \ldots \) has property C3 we first ask whether one of the subsequences \( p_k, p_{k+1}, p_{k+2}, \ldots \) (just the indices greater than or equal to \( k \)) has either property C1 or C2. If so, then we are reduced to one of the previous cases on this subsequence. Otherwise, for every \( k \), there are distinct vertices \( b_k, c_k \in V(p_k) \) such that

\[ b_k \leq p_k c_k \quad \text{but} \quad c_k \leq p_i b_k \quad \text{for infinitely many} \quad i. \]

We inductively define a nested collection of subsequences of \( \{p_k\}_{k=1}^{\infty} \). Let

\[ I_1 := \{ j \in \mathbb{N} \mid c_1 \leq p_j b_1 \} \]

and let \( s_1 \) be the least element of \( I_1 \). Now suppose we have constructed \textit{infinite} sets

\[ I_1 \supseteq \cdots \supseteq I_r \]
such that $I_i := \{ j \in I_{i-1} \mid j > s_{i-1} \text{ and } c_{s_{i-1}} \leq b_{s_{i-1}} \}$ where $s_{i-1}$ is the least element of $I_{i-1}$ (here we take $I_0 := \mathbb{N}$). If there are infinitely many $k \in I_r$ such that $c_{s_r} \leq b_{s_r}$ then let $I_{r+1} := \{ j \in I_r \mid c_{s_r} \leq b_{s_r} \}$. Otherwise the construction terminates.

If the construction terminates after $N$ steps then the subsequence $\{p_j \mid j \in I_N\}$ has either property C1 or C2, whence a sub-subsequence is of type C1 (and the lemma follows). Otherwise the process continues indefinitely and we obtain a sequence of index sets

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$$

and integers $s_1, s_2, s_3, \ldots$ (recall $s_i$ is the least element in $I_i$) so that

$$c_{s_k} \leq b_{s_k} \text{ for all } j \in I_{k+1}.$$ 

Note that $s_{k+1} > s_k$ and $b_{s_k} \leq c_{s_k}$ by definition (also note $b_{s_k} \neq c_{s_k}$ by property C3). The remainder of the proof is to show that, under these conditions, there is an infinite forward or backward ray contained in $G$, in contradiction to the assumptions of the lemma.

**Claim.** At least one of the sets $\{b_{s_1}, b_{s_2}, \ldots\}$, $\{c_{s_1}, c_{s_2}, \ldots\}$ is infinite.

**Proof.** We begin by noting that if $\{b_{s_i}, c_{s_i}\} = \{b_{s_j}, c_{s_j}\}$ and $i < j$ then either $(b_{s_i}, c_{s_i}) = (b_{s_j}, c_{s_j})$ or $(b_{s_i}, c_{s_i}) = (c_{s_j}, b_{s_j})$ (as ordered lists). The first case is impossible since $b_{s_j} \leq p_{s_j} c_{s_j}$ but $c_{s_i} \leq p_{s_j} b_{s_i}$ which implies that $b_{s_i} = c_{s_i}$; a contradiction. The
second case is also impossible since \( c_{s_i} \leq p_{s_j+1} b_{s_i} \) and \( c_{s_j} \leq p_{s_j+1} b_{s_j} \) gives a similar contradiction.

On the other hand, if both \( \{b_{s_1}, b_{s_2}, \ldots\} \) and \( \{c_{s_1}, c_{s_2}, \ldots\} \) are finite (say all distinct elements are exhausted in both sets by index \( r \)) then the set \( \{b_{s_{r^2+1}}, c_{s_{r^2+1}}\} \) must be equal to \( \{b_i, c_i\} \) for some \( 1 \leq i \leq r^2 \). This establishes the claim.

Without loss of generality, suppose the set \( \{b_{s_1}, b_{s_2}, \ldots\} \) is infinite and, by passing to a subsequence, assume that \( b_{s_i} \neq b_{s_j} \) for \( i \neq j \). Recall from the construction of the sets \( I_j \) that

\[
 b_{s_1}, \ldots, b_{s_n} \in V(p_{s_n}) \text{ for every } n.
\]

Therefore the set \( \{b_{s_1}, b_{s_2}, \ldots\} \) has the property that if one of its elements \( b_{s_i} \) occurs in some path \( p_{s_j} \), it also occurs in \( p_{s_k} \) for every \( k \geq j \). Any subsequence of the paths \( \{p_{s_i}\}_{i=1}^{\infty} \) also has this property. Our goal is to produce a subsequence of \( \{p_{s_i}\}_{i=1}^{\infty} \), say \( p_{a_1}, p_{a_2}, \ldots \) with the property that for any \( k \) and any \( v \in V(p_{a_k}) \) either \( v \in V(p_{a_n}) \) for all \( n > k \) or \( v \notin V(p_{a_k}) \) for any \( n > k \). We will then produce a set

\[
 A = \bigcup_{k=1}^{\infty} \{v \in V(p_{a_k}) \mid v \in V(p_{a_n}) \text{ for all } n \geq k\}
\]

which contains all vertices of the first type (this set will be infinite since \( b_{s_i} \in A \) for all \( i \)) and use it to construct an infinite ray in \( G \).

We begin by producing the subsequence – this is done one path at a time. Enumerate the vertices of \( p_{s_1} \) by \( v_i = (p_{s_1})_i \). Define a set of indices for \( v_1 \) as follows. If \( v_1 \in V(p_{s_k}) \) for only finitely many \( k > 1 \) then let \( J_{v_1} \) be the set of \( s_k \) for which
v_1 \not \in V(p_{s_k}). Otherwise v_1 \in V(p_{s_k}) for infinitely many k > 1; in this case let J_{v_1} be
the set of s_k for which v \in V(p_{s_k}). Now suppose we have constructed infinite sets (i.e.
subsequences)

\[ J_{v_1} \supseteq \cdots \supseteq J_{v_r} \]

so that for every 1 \leq i \leq r, either v_i \in V(p_j) for all j \in J_{v_i} or v_i \not \in V(p_j) for any
j \in J_{v_i}. If r < L(p_{s_1}) - 1 then

\[ J_{v_{r+1}} := \begin{cases} 
\{ j \in J_{v_r} \mid v_{r+1} \not \in V(p_j) \} & \text{if } v_{r+1} \in V(p_{s_k}) \text{ for only finitely many } k \in J_{v_r} \\
\{ j \in J_{v_r} \mid v_{r+1} \in V(p_j) \} & \text{otherwise.}
\end{cases} \]

Define J := \{s_1\} \cup J_{v_{L(p_{s_1})-1}} (notice that s_1 is the least element of J). Then for every
v \in V(p_{s_1}), either v \in V(p_i) for all i \in J \setminus \{s_1\} or v \not \in V(p_i) for any i \in J \setminus \{s_1\} (that
is, the vertices in p_{s_1} occur either in all subsequent paths in this subsequence or in none
of them). Set a_1 := s_1.

Next, let a_2 := \min\{i \in J \mid i > a_1\}, enumerate the vertices in p_{a_2}, and perform
the above procedure for p_{a_2} : namely produce an infinite subset J' \subseteq J so that a_1, a_2 \in J'
and for each v \in V(p_{a_2}), either v \in V(p_i) for all i \in J' \setminus \{a_1, a_2\} or v \not \in V(p_i) for any
i \in J' \setminus \{a_1, a_2\}. Continue this process inductively to obtain a subsequence with index
set J = \{a_1, a_2, \ldots\} so that for any k, if v \in V(p_{a_k}) then either v \in V(p_{a_m}) for all
m \geq k or v \not \in V(p_{a_m}) for all m \geq k. We draw special attention to the first type of
vertices by setting

\[ \{ d_{k_1}^1, \ldots, d_{k_n}^k \} := \{ v \in V(p_{a_k}) \mid v \in V(p_{a_m}) \text{ for all } m \geq k \}, \]
where $n_k$ is the number of elements in the set on the right. Then the set

$$A := \{d_j^i \mid i \in \mathbb{N}, j \leq n_i \} \supset \{b_{s_1}, b_{s_2}, \ldots \}$$

is infinite. We will use $A$ to produce an infinite ray. In order to do this, we introduce an ordering on the elements of $A$; this is done inductively as follows.

Since $d_{1}^{1}, \ldots, d_{n_1}^{1} \in V(p_{a_k})$ for all $k \geq 1$ and this (finite) set has only finitely many possible orderings, there is a permutation $\sigma$ such that

$$\begin{align*}
  d_{1}^{1} \leq_{\sigma(1)} p_{a_k} & \leq_{\sigma(2)} d_{1}^{1} \leq_{\sigma(n_1)} \cdots \leq_{\sigma(n_1)} d_{1}^{1}
\end{align*}$$

for infinitely many $k \in \mathbb{N}$. Let $\mathcal{K}_1$ be the set of indices for which this is true, then all of the orderings $\{\leq_{p_j} \mid j \in \mathcal{K}_1\}$ agree on the set $\{d_{1}^{1}, \ldots, d_{n_1}^{1}\}$.

Now inductively construct infinite sets $\mathcal{K}_1 \supseteq \mathcal{K}_2 \supseteq \cdots$ so that for every $i$ the orderings $\{\leq_{p_j} \mid j \in \mathcal{K}_i\}$ agree on the set

$$\{d_{1}^{1}, \ldots, d_{n_1}^{1}, \ldots, d_{1}^{i}, \ldots, d_{n_i}^{i}\}$$

$(\mathcal{K}_{i-1}$ has infinitely many elements and $\{d_{1}^{1}, \ldots, d_{n_1}^{1}, \ldots, d_{1}^{i}, \ldots, d_{n_i}^{i}\}$ has only finitely many orderings). Finally let $\leq_{\infty}$ be an ordering of $A$ which, for any $i \in \mathbb{N}$ and $j \in \mathcal{K}_i$ is equivalent to $\leq_{p_j}$ on $\{d_{1}^{1}, \ldots, d_{n_1}^{1}, \ldots, d_{1}^{i}, \ldots, d_{n_i}^{i}\}$.

Suppose there is a sequence of distinct vertices $d_{1}, d_{2}, \cdots \in A$ such that

$$d_{1} \leq_{\infty} d_{2} \leq_{\infty} \cdots.$$
For each \( i \), find \( j(i) \) so that \( \{d_1, \ldots, d_i\} \subseteq \{d_1^{j(1)}, \ldots, d_i^{j(i)}\} \). Let \( e_i \in K_{j(i)} \) (chosen so that \( e_1 < e_2 < \cdots \)). Then \( d_1, \ldots, d_i \in V(p_{e_i}) \) and \( \leq_{p_{e_i}} \) agrees with \( \leq_{\infty} \) on \( \{d_1, \ldots, d_i\} \). Then form the path

\[
a \xrightarrow{[p_{e_1}]} d_1 \xrightarrow{[p_{e_2}]} d_2 \xrightarrow{[p_{e_3}]} d_3 \rightarrow \cdots \tag{2.4.1}
\]

where if \( v_1, v_2 \in V(p_{e_i}) \) then \( v_1 \xrightarrow{[p_{e_i}]} v_2 \) denotes segment of \( p_{e_i} \) from \( v_1 \) to \( v_2 \).

We claim that any vertex in this path repeats at most finitely many times. Indeed suppose \( v \) is a vertex that is encountered on the section \( d_i \xrightarrow{[p_{e_i+1}]} d_{i+1} \). If \( v \notin A \), then \( v \notin V(p_{e_j}) \) for any \( j > i + 1 \) (so \( v \) repeats at most finitely many times). Otherwise \( v \in A \) and there are two possibilities: either \( d_j \leq_{\infty} v \) for all \( j \) or there is some \( j \) so that \( v \leq_{\infty} d_j \). In the first case (\( d_j \leq_{\infty} v \) for all \( j \)), since \( v \in A \) there is some \( N \) so that \( v \in V(p_{e_n}) \) for all \( n > N \) and some \( M \geq N \) so that \( d_m \xrightarrow{[p_{e_m+1}]} v \) for all \( m > M \) (we take \( M \) so that \( d_{M+1} \leq_{\infty} v \) implies that \( d_{M+1} \leq_{p_{e_{M+1}}} v \)). In this case \( v \) cannot occur in the segment \( d_m \xrightarrow{[p_{e_{m+1}}]} d_{m+1} \) for any \( m > M \), so in the entire path (2.4.1) it occurs at most finitely many times. In the second case (there is some \( j \) so that \( v \leq_{\infty} d_j \)), we have \( v \leq_{p_{e_i}} d_j \leq_{p_{e_i}} d_i \) for all sufficiently large \( i \) (i.e. large enough that \( \leq_{\infty} \) agrees with \( \leq_{e_i} \) on the set \( \{v, d_j, d_i\} \)). Again \( v \) cannot occur in the segment \( d_i \xrightarrow{[p_{e_i}]} d_{i+1} \) for such \( i \), so it occurs at most finitely many times in (2.4.1).

Since every vertex repeats at most finitely many times in this path, it has a subpath that is an infinite forward ray which contradicts the assumptions of the lemma.
Otherwise there is a sequence $d_1, d_2, \cdots \in A_\infty$ such that

$$d_1 \geq \infty \geq d_2 \geq \infty \cdots$$

We argue as above to produce a backwards infinite ray and obtain a similar contradiction.

**Lemma 2.12.** If there is a vertex $a \in S_G$ such that there are simple loops based at $a$ of arbitrarily large size, then $X_G$ has a transient Markov potential.

By lemma 2.8 it is enough to consider the case when $G$ does not have an infinite ray. Our goal is to produce an infinite sequence of loops $\ell_0, \ell_1, \ldots$ such that for every $n$, $\ell_n$ is semi-disjoint from $\{\ell_1, \ldots, \ell_{n-1}\}$, $i(\ell_n) = n - 1$, and the sequences $\{A(\ell_n, \ell_{n-1})\}_n$ and $\{B(\ell_n, \ell_{n-1})\}_n$ are (not necessarily strictly) increasing (see definition of semi-disjoint to recall these numbers). This situation is pictured below.

The subgraph $\mathcal{H}$ (to be constructed)

Enumerate the simple loops based at $a$ by $p_1, p_2, \ldots$ By passing to a subsequence, we may assume that $L(p_{n+1}) > L(p_n)$ for every $n$. We proceed to construct the sequence $\{\ell_k\}_k$ by induction.
Apply lemma 2.11 to the sequence \( p_1, p_2, \ldots \) to obtain integers \( K_1^1, K_1^2 \) and an index set \( A_1 \subseteq \mathbb{N} \) (with minimal element \( a_1 \)) so that \( p_k \) is semi-disjoint from \( \{ p_{a_1} \} \), \( A(p_k, p_{a_1}) = K_1^1 \) and \( B(p_k, p_{a_1}) = K_1^2 \), for every \( k \in A_1 \setminus \{ a_1 \} \). Set \( \ell_1 := p_{a_1} \).

Next apply lemma 2.11 to the sequence \( \{ p_i \mid i \in A_1 \setminus \{ a_1 \} \} \) to obtain integers \( K_2^1, K_2^2 \) and an index set \( A_2 \subseteq A_1 \) (with minimal element \( a_2 \)) so that \( p_k \) is semi-disjoint from \( \{ p_{a_2} \} \), \( A(p_k, p_{a_2}) = K_2^1 \) and \( B(p_k, p_{a_2}) = K_2^2 \), for every \( k \in A_2 \setminus \{ a_2 \} \). Set \( \ell_2 := p_{a_2} \). Note that \( K_2^1 \geq K_1^1 \) since for any \( k \in A_2 \subseteq A_1 \), \( p_k \) follows \( \ell_1 \) for the first \( K_1^1 \) vertices whence also \( \ell_2 \) (since it follows \( \ell_1 \) for the first \( K_1^1 \) vertices). Similarly \( K_2^2 \geq K_1^1 \).

Thus, for any \( k \in A_2 \setminus \{ a_2 \}, p_k \) is semidisjoint from \( \{ \ell_1, \ell_2 \} \) and \( i(p_k) = 2 \). Continue this process inductively to construct loops \( \ell_1, \ldots, \ell_r \) and infinite sets of indices

\[
A_1 \supseteq \cdots \supseteq A_r
\]

so that

- \( \ell_i \) is semi-disjoint from \( \{ \ell_1, \ldots, \ell_{i-1} \} \) for every \( 2 \leq i \leq r \)
- \( i(\ell_i) = i-1 \) for every \( 2 \leq i \leq r \) and the sequences \( \{ A(\ell_n, \ell_{n-1}) \} \) and \( \{ B(\ell_n, \ell_{n-1}) \} \) are nondecreasing.

Thus we obtain our desired sequence by induction. So, the graph \( \mathcal{H} \) obtained by taking all of the vertices and edges in \( \{ \ell_i \}_{i=0}^{\infty} \) (and \( a \rightarrow a \) by lemma 2.6) is of the desired type.
We now show how to define a potential with finite discriminant on $X_{\mathcal{H}}$. Define

$$\phi(x_0, x_1) := \begin{cases} 
\log \left( \frac{1/n^2}{\sum_{k=1}^{n-1} 1/k^2} \right) & \text{if } x_0 = (\ell_n)K_n^n \text{ and } x_1 = (\ell_n)K_n^n+1 \\
0 & \text{otherwise.}
\end{cases}$$

By construction, the only first return loops on $\mathcal{H}$ are the loops $\ell_i$ (and $a \to a$) and the contribution of each loop $\ell_n$ to the sum defining $r_\phi(x)$ is $\propto \frac{1}{n^2} L(\ell_n)$ (it is no smaller than $\frac{1}{\zeta(2)n^2} L(\ell_n)$ and no larger than $\frac{1}{n^2} L(\ell_n)$). Therefore $r_\phi(x)$ converges at $x = 1$ which is its radius of convergence. Therefore $\phi$ has finite discriminant and so there exists a transient potential on $X_{\mathcal{H}}$ and so also on $X_G$.

### 2.5 Proof of Theorem 2.1

By theorem 2.3 a graph with a finite uniform Rome cannot have a transient potential. We now prove the converse.

Suppose $X_G$ has no transient potentials then:

- $G$ cannot have infinitely many disjoint simple loops (Lemma 2.10);
- For any $a \in S_G$ the length of the simple loops based at $a$ is bounded. (Lemma 2.12)

We will show that any graph that has these two properties has a finite uniform Rome.

Enumerate all simple loops in $G$ as $\ell_1, \ell_2, \ell_3, \ldots$ (ignoring the base point). Since there are not infinitely many disjoint simple loops in $G$, there must be some collection $\ell_{k_1}, \ldots, \ell_{k_r}$ such that for any $i$ there exists $j$ with $V(\ell_i) \cap V(\ell_{k_j}) \neq \emptyset$. Define

$$F := V(\ell_{k_1}) \cup \cdots \cup V(\ell_{k_r}).$$
Observe that $F$ is a finite set and every simple loop in $G$ has at least one vertex in $F$. For each $v \in F$ the lengths of the simple loops based at $v$ are bounded and $F$ is finite, so there is some $N \in \mathbb{N}$ such that every simple loop in $G$ has length less than $N$.

We claim that $F$ is a uniform Rome for $G$. Consider an arbitrary path $p$ in $X \setminus F$

$$p : b_0 \rightarrow b_1 \rightarrow b_2 \rightarrow b_3 \rightarrow \cdots \rightarrow b_n$$

We aim to bound the size of $n$ (so that all paths in $X \setminus F$ have uniformly bounded length). Our first observation is that $p$ is simple since if any vertex repeats then $p$ contains a simple loop and, therefore, at least one vertex in $F$. Let $i_1$ be the least index so that there is a path

$$p_1 : b_n \xrightarrow{L(p_1)} b_{i_1}$$

with $(p_1)_j \notin \{b_0, \ldots, b_n\}$ for any $0 < j < L(p_1) - 1$ (note that $i_1 < n$ by topological mixing). Let $i_2$ be the least index so that there is a path from one of the vertices $b_{i_1}, \ldots, b_{n-1}$ to $b_{i_2}$:

$$p_2 : \{b_{i_1}, \ldots, b_{n-1}\} \xrightarrow{L(p_2)} b_{i_2}$$

with $(p_2)_j \notin \{b_0, \ldots, b_n\}$ for any $0 < j < L(p_2) - 1$. Note that $i_2 < i_1$ since there is a path from $b_n$ to $b_0$ and it must pass through the set $\{b_{i_1}, \ldots, b_{n-1}\}$ before it can get to any of the vertices $\{b_0, \ldots, b_{i_1-1}\}$ (by choice of $i_1$). Now suppose we have constructed indices $i_1 > \cdots > i_k$ and paths $p_1, \ldots, p_k$ so that $i_k$ is the least index such that there is a path

$$q : \{b_{i_{k-1}}, \ldots, b_{i_{k-2}}\} \xrightarrow{L(q)} b_{i_k}$$
with \((q)_j \notin \{b_0, \ldots, b_n\}\) for any \(0 < j < L(q) - 1\), and where \(p_k\) is such a path. If \(i_k > 0\) then let \(i_{k+1}\) be the least index such that there is a path

\[
q: \{b_{i_k}, \ldots, b_{i_{k-1}-1}\} \xrightarrow{L(q)} b_{i_{k+1}}
\]

with \((q)_j \notin \{b_0, \ldots, b_n\}\) for any \(0 < j < L(q) - 1\). Note that \(i_{k+1} < i_k\) since there is a path from \(b_n\) to \(b_0\) and it must pass through the set \(\{b_{i_k}, \ldots, b_{i_{k-1}-1}\}\) before it can get to any of the vertices \(\{b_0, \ldots, b_{i_{k-1}}\}\). Continue this process as long as \(i_k > 0\).

This process cannot continue indefinitely, so let \(i_1 > \cdots > i_r = 0\) and \(p_1, \ldots, p_r\) be the indices and paths so constructed. Observe that if \(|i - j| > 1\) then \(V(p_i) \cap V(p_j) = \emptyset\) since if \(i > j + 1\) then \(V(p_i) \cap V(p_j)\) cannot contain any vertices in \(\{b_0, \ldots, b_n\}\) (by definition of the indices \(i_1, \ldots, i_r\)) and cannot contain any vertices not in the path \(p\) (otherwise we could follow the path \(p_i\) until it meets \(p_j\) and then follow \(p_j\) to obtain a path that violates minimality of the indices \(i_1, \ldots, i_r\)). Therefore the simple loops

\[
\ell_k: b_{i_k} \rightarrow b_{i_{k+1}} \rightarrow \cdots \rightarrow b_{i_{k-1}} \xrightarrow[p_k]{p_k} b_{i_k}
\]

have \(V(\ell_{2n}) \cap V(\ell_{2m}) = \emptyset\) for all \(n \neq m\) (similarly for the odd-indexed loops). Moreover, every vertex in \(\{b_0, \ldots, b_n\}\) is contained in one of these loops. But the total number of vertices in these (two sets of pairwise disjoint, simple) loops is bounded, so we get

\[
(n + 1) \leq 2N \cdot (\text{maximal number of pairwise disjoint loops}) \leq 2N \cdot |F|.
\]

Therefore \(X\) has a finite uniform Rome. □
2.6 Appendix A

Here we give examples of graphs with finite uniform Romes (see Definition 2.1). First note that any graph has a uniform Rome by taking $F = \mathcal{S}$ and $N = 1$. On the other hand, it is a strong restriction on $\mathcal{G}$ to require a finite uniform Rome.

**Example 1:** Any finite graph has a finite uniform Rome by taking $F = \mathcal{S}$ and $N = 1$ (so subshifts of finite type cannot have transient potentials).

**Example 2:** A topologically mixing infinite bouquet of first return loops of length less than or equal to a fixed integer. Below is pictured an infinite bouquet of first return loops of length three (and an edge $a \to a$). We can take $F = \{a\}$ and $N = 2$.

![Diagram](https://via.placeholder.com/150)

**Example 3:** Finitely many bouquets can be joined together (see below). Here we take $F = \{a, b, c\}$ and $N = 3$. 
Example 4: A finite graph can be joined to an infinite bouquet. Here take $F = \{a, b, c, d, e, f, g, h, i\}$ and $N = 2$.

2.7 Appendix B

The following result follows from the discussion of the symmetric random walk on $\mathbb{Z}$ in [10] (p. 73-78). Let $G(N_0)$ be the graph obtained from $\mathbb{N} \cup \{0\}$ by putting a directed edge from $0 \to 0$ as well as directed edges of the form $n \to (n + 1)$ and $(n + 1) \to n$ for all $n \geq 0$.

Lemma 2.13. The potential $\phi(x_0, x_1) = -\log(2)$ on $X_{G(N_0)}$ has finite discriminant.
For the symmetric random walk on \( Z \), the probability that a trajectory starting at zero returns at time \( 2n \) is

\[
p_{2n} = \left( \frac{1}{2} \right)^{2n} \binom{2n}{n} \times \frac{1}{\sqrt{\pi n}}
\]

by Stirling’s formula. Let \( f_{2n} \) be the probability that a trajectory starting at zero will return for the first time at time \( 2n \). Then \( f_{2n} = p_{2n-2} - p_{2n} = \frac{1}{2n-1} \cdot p_{2n} \) \([10, \text{Lemma 1}]\) (p. 76). Therefore, the probability that a trajectory starting at zero will stay in \( Z^+ \) and return for the first time at time \( 2n \) is \( \propto \frac{1}{(4n-2)\sqrt{\pi n}} \). On the other hand, this probability is exactly \( (\frac{1}{2})^{2n} \cdot \#(\text{paths in } Z^+ \text{ that first return at time } 2n) \).

Now consider the potential \( \phi \) on \( G(\mathbb{N}_0) \). We have

\[
Z^*_{2n}(\phi, 0) = \left( \frac{1}{2} \right)^{2n} \cdot \#(\text{first return loops of length } 2n \text{ based at } 0 \text{ in } G(\mathbb{N}_0)) \times \frac{C}{n^{3/2}}
\]

for some constant \( C \). Moreover, \( Z^*_1(\phi, 0) = \frac{1}{2} \) and \( Z^*_{2n+1}(\phi, 0) = 0 \) for all \( n > 0 \) so \( r_\phi(1) < \infty \) and \( r_\phi(x) = \infty \) for all \( x > 1 \); that is, \( \phi \) has finite discriminant and \( P_G(\phi) = 0 \).
Chapter 3

Topology of Transient Potentials

The material in this section summarizes a number of results from [5].

In the previous chapter we saw that a countable Markov shifts $X_G$ has a nonempty set of transient potentials with summable variations and finite pressure if and only if the graph $G$ does not have a finite uniform Rome. The goal of this chapter is to describe the relative commonness of transience.

Fix a graph $G$ that does not have a finite uniform Rome and consider the set

$$\Phi := \{ \phi : X_G \to \mathbb{R} : \phi \text{ has summable variations} \}.$$

In this case

$$\Phi(T) := \{ \phi \in \Phi : \phi \text{ is transient} \} \neq \emptyset.$$

Our interest is in describing how large the set $\Phi(T)$ is as a subset of $\Phi$. In order to do this, we need to recall the definition of a set (the strongly positive recurrent potentials in $\Phi$) which will turn out to be open and dense in $\Phi$ in several natural topologies.

3.1 Strong Positive Recurrence

The content of this section is classical and is included for completeness (see Sarig [19]).
If $X_G$ is a topologically mixing countable Markov shift and $a \in S_G$. Set

$$S_G := \{ [ax_1 \ldots x_k] : k \in \mathbb{N}, x_i \neq a, [ax_1 \ldots x_k a] \neq \emptyset \}.$$ 

The full shift on $S_G$ (i.e. the countable Markov shift arising from the graph with a directed edge between any two elements of $S_G$) is called the induced shift (induced on state $a$) and is denoted $X_G$. If $\phi : X_G \rightarrow \mathbb{R}$ is a potential with summable variations, then we associate the induced potential $\phi : X_G \rightarrow \mathbb{R}$ by defining

$$\phi([ax_{11} \cdots x_{1k_1}], [ax_{21} \cdots x_{2k_2}], \ldots) := \phi_{k_1}(ax_{11} \cdots x_{1k_1} ax_{21} \cdots x_{2k_2} \cdots).$$

An induced potential give information about the recurrence properties of a potential $\phi$ via its discriminant:

**Definition 3.1.** Let $p^* \phi := \sup \{ p : P_G(\phi + p) < \infty \}$. The $a$-discriminant of $\phi$ is defined to be

$$\Delta_a[\phi] := \sup \{ P_G(\phi + p) : p < p^* \phi \}.$$ 

The connection between $\Delta_a[\phi]$ and the recurrence properties of $\phi$ comes from the following.

**Lemma 3.1** (Sarig; Theorem 2 in [19]). Let $X_G$ be a topologically mixing countable Markov shift and let $\phi : X_G \rightarrow \mathbb{R}$ be some function with summable variations and finite Gurevich pressure. Let $a \in S_G$ be some arbitrary fixed state. The equation $P_G(\phi + p) = 0$ has a unique solution $p(\phi)$ if $\Delta_a[\phi] \geq 0$ and no solution if $\Delta_a[\phi] < 0$. The Gurevich
The pressure of $\phi$ is given by

$$P_G(\phi) = \begin{cases} 
-p(\phi) & \Delta_a[\phi] \geq 0 \\
\limsup_{n \to \infty} \frac{1}{n} \log Z_n^*(\phi, a) & \Delta_a[\phi] < 0.
\end{cases}$$

Moreover, $\Delta_a[\phi] \geq 0$ if and only if $\phi$ is recurrent.

The connection between $\Delta_a[\phi]$ and the exponential growth rate of $Z_n^*(\phi, a)$ (in the transient case) is further clarified by the following.

**Lemma 3.2** (Sarig; Proposition 3 in [19]). Let $X_G$ be a topologically mixing countable Markov shift, let $\phi$ be a function with summable variations and finite Gurevich pressure. Let $\overline{X}_G$ and $\overline{\phi}$ denote the induced pair with respect to $a \in S_G$. Then $P_G(\overline{\phi} + p)$ is convex, strictly increasing and continuous on $(-\infty, \limsup_{n \to \infty} \frac{1}{n} \log Z_n^*(\overline{\phi}, [a]))$. Moreover

$$\Delta_a[\phi] = P_G(\overline{\phi} + \limsup_{n \to \infty} \frac{1}{n} \log Z_n^*(\overline{\phi}, [a])).$$

**Definition 3.2.** A potential $\phi \in \Phi$ is called strongly positive recurrent if for some (hence every) $a \in S_G$ we have $\Delta_a[\phi] > 0$.

For later use, we state an elementary property of the discriminant.

**Lemma 3.3.** If $a \in S_G$ and $\phi$ has summable variations, then $\Delta_a[\phi + t \cdot 1_{[a]}] = \Delta_a[\phi] + t$.

**Proof.** For any $a \in S_G$, note that

$$P_G(\overline{\phi} + t \cdot 1_{[a]} + p) = P_G(\overline{\phi} + p + t) = P_G(\overline{\phi} + p) + t$$

which gives that $p^*_a[\phi + t \cdot 1_{[a]}] = p^*_a[\phi]$ (recall these definitions from Section 3.1). Therefore $\Delta_a[\phi + t \cdot 1_{[a]}] = \Delta_a[\phi] + t$. \qed
3.2 Topology of Transient Potentials

We quickly recall two natural topologies on the set $\Phi$. The first is the uniform topology (also called the $\| \cdot \|_\infty$-topology). We set $\| \phi \|_\infty := \sup_{x \in X_G} |\phi(x)|$ and define the uniform topology to be the topology on $\Phi$ generated by sets of the form

$$U(\phi, \varepsilon) := \{ \psi \in \Phi : \| \phi - \psi \|_\infty < \varepsilon \}$$

(note that we do not require that $\| \phi \|_\infty < \infty$). The other is the Lipshitz topology (also called the $\| \cdot \|_{\text{Lip}}$-topology). We set

$$\| \phi \|_{\text{Lip}} := \| \phi \|_\infty + \sup_{x \neq y \in X_G} \frac{d(T_x, T_y)}{d(x, y)}$$

(where $d(\cdot, \cdot)$ is the standard metric on $X_G$). The Lipshitz topology is the topology on $\Phi$ generated by sets of the form

$$V(\phi, \varepsilon) := \{ \psi \in \Phi : \| \phi - \psi \|_{\text{Lip}} < \varepsilon \}.$$  

Note that any $\| \cdot \|_\infty$-open set is $\| \cdot \|_{\text{Lip}}$-open and any $\| \cdot \|_{\text{Lip}}$-dense set is $\| \cdot \|_\infty$-dense.

The first main result is the following.

**Theorem 3.1.** The set

$$\Phi(\text{SPR}) := \{ \phi \in \Phi : \phi \text{ is strongly positive recurrent} \}$$
is $\| \cdot \|_\infty$-open and $\| \cdot \|_{\text{Lip}}$-dense in $\Phi$.

This should be interpreted as saying that “most” potentials with summable variations are recurrent and that transient potentials can be uniformly approximated by recurrent potentials. Together with the results of the previous chapter, the picture is that “most” (sometimes even all) potentials on a given countable Markov shift are recurrent, but “most” shifts admit at least one transient potential. In order to say more precisely how large the set $\Phi(T)$ is, we need to define a stronger topology capable of detecting it.

**Definition 3.3.** For a given finite set $B \subseteq S^G$, we define the uniform topology localized at $B$ (denoted the $\mathcal{LU}(B)$-topology) on $\Phi$ to be the topology generated by sets of the form

$$U(\phi; \varepsilon) := \{ \psi \in \Phi : \| \phi - \psi \|_\infty < \varepsilon, \phi(x) = \psi(x) \text{ whenever } x_0 \notin B \}$$

(i.e. the topology of uniform perturbations supported inside the set $\bigcup_{a \in B} [a]$).

The next main result is the following.

**Theorem 3.2.** For any finite $B \subseteq S^G$, the sets $\Phi(T)$ and $\Phi(\text{SPR})$ are both $\mathcal{LU}(B)$-open. Moreover $\Phi(T)$ is $\mathcal{LU}(B)$-dense in $\Phi \setminus \Phi(\text{SPR})$ and $\Phi(\text{SPR})$ is $\mathcal{LU}(B)$-dense in $\Phi \setminus \Phi(T)$.

This result may be interpreted as saying that transience is the most important obstruction to strong positive recurrence (of course $\Phi \setminus \Phi(\text{SPR})$ also contains recurrent potentials with a-discriminant zero). Thus, although most potentials in $\Phi$ are strongly positive recurrent, the phenomenon of transience can still be considered common.
3.2.1 Proof of Theorem 3.1

We begin by showing that recurrent potentials are \( \| \cdot \|_{Lip} \)-dense in \( \Phi \) (and will later show that \( \Phi(SPR) \) is \( \| \cdot \|_{Lip} \)-dense in the set of recurrent potentials). This relies on the following two lemmas.

Lemma 3.4. If \( \phi \in \Phi \) is a transient potential, \( a \in S \), and \( \psi \) is a non-positive bounded weakly Hölder function s.t. \( \text{supp}(\psi) \subset [a] \), then \( \phi + \psi \in \Phi \), \( \phi + \psi \) is transient, and \( P_G(\phi + \psi) = P_G(\phi) \).

Proof. Since \( \phi + \psi \leq \phi \) we have \( P_G(\phi + \psi) \leq P_G(\phi) \). To see the other inequality, we note that since \( \phi \) is transient,

\[
P_G(\phi) = \limsup_{n \to \infty} \frac{1}{n} \log Z^*_n(\phi, a) \quad \text{(Theorem 3.1)}
\]

\[
= \limsup_{n \to \infty} \frac{1}{n} \log Z^*_n(\phi + \psi, a) \quad \text{(supp}(\psi) \subset [a] \text{ and sup } |\psi| < \infty) \]

\[
\leq P_G(\phi + \psi). \quad \text{(Theorem 3.1)}
\]

This shows that \( P_G(\phi) = P_G(\phi + \psi) \).

Using the transience of \( \phi \) and the non-positivity of \( \psi \), we see that

\[
\sum_{n=0}^{\infty} e^{-nP_G(\phi + \psi)} Z_n(\phi + \psi, a) = \sum_{n=0}^{\infty} e^{-nP_G(\phi)} Z_n(\phi + \psi, a) \leq \sum_{n=0}^{\infty} e^{-nP_G(\phi)} Z_n(\phi, a) < \infty,
\]

so \( \phi + \psi \) is transient.

\( \square \)

Lemma 3.5. Suppose \( \phi \in \Phi \) is transient, then for any \( \epsilon > 0 \) there exists a recurrent \( \varphi \in \Phi \) so that \( \| \varphi - \phi \|_{\infty} \leq \epsilon \) and \( \text{var}_1[\varphi - \phi] = 0 \).
Proof. Recall that $\mathcal{S}_G$ denotes the set of states. We write $a \xrightarrow{k} b$ for $a, b \in \mathcal{S}$ if there is a path in $G$ with $k + 1$ vertices which starts with $a$ and ends with $b$.

Fix $\epsilon > 0$ and $b \in \mathcal{S}$. We construct finite sets of states $\{c^k_1, \ldots, c^k_{r_k}\}$ ($k \geq 0$) by induction as follows. When $k = 0$, let $r_0 := 1$, and $c^0_1 := b$. Now suppose we have carried the construction for each $\ell < k$. Let $b^k_1, b^k_2, b^k_3, \ldots$ be the list of all different states $c$ for which $b^\ell \rightarrow c$ for $\ell \leq k$. If this collection is finite, let $r_k$ be its size, and set $\{c^k_1, \ldots, c^k_{r_k}\} := \{b^k_1, \ldots, b^k_{r_k}\}$. If it is infinite, observe that

$$Z_n^\ast \left( \phi + \epsilon \sum_{i=1}^\infty 1_{[b^k_i]}, b \right) \geq e^{n \epsilon} Z_n^\ast (\phi, b) \quad (1 \leq n \leq k).$$

since for any $x$ with $T^n x = x$ and $x_0 = b$ we have added an extra factor of $\epsilon$ to the potential at states $x_0, x_1, \ldots, x_{n-1}$. Therefore we can find $s_k \in \mathbb{N}$ such that

$$Z_n^\ast \left( \phi + \epsilon \sum_{i=1}^{s_k} 1_{[b^k_i]}, b \right) \geq e^{n \epsilon} Z_n^\ast (\phi, b) \quad (1 \leq n \leq k). \quad (3.2.1)$$

We let $\{c^k_1, \ldots, c^k_{r_k}\}$ be the set $\{c^{k-1}_1, \ldots, c^{k-1}_{r_{k-1}}, b^k_1, \ldots, b^k_{s_k}\}$ where, in this case, $r_k$ is the number of different states $c^k_i$ so defined.

Set $\phi[0] := \phi$, and define for $k \geq 1$

$$\phi[k] = \phi + \epsilon \sum_{i=1}^{r_k} 1_{[c^k_i]}.$$
We interpolate these potentials. Observe that for all $k \geq 1$,

$$\phi[k] = \phi[k-1] + \epsilon \sum_{i=1}^{m_k} 1_{[d^k_i]}$$

where $\{d^k_1, \ldots, d^k_{m_k}\} = \{c^k_1, \ldots, c^k_{r_k}\} \setminus \{c^{k-1}_1, \ldots, c^{k-1}_{r_{k-1}}\}$,

with $m_k$ defined by the above identity. Define for $k \geq 1$ and $0 \leq i \leq m_k$

$$\phi[k, i] := \phi[k-1] + \epsilon \sum_{j=1}^{i} 1_{[d^k_j]}.$$  

Then $\phi[k, 0] = \phi[k-1]$, and $\phi[k, m_k] = \phi[k]$.

We claim that there must be some $k, i$ such that $\phi[k, i]$ is recurrent. Assume by way of contradiction that this is not the case: $\phi[k, i]$ is transient for all $k, i$.

In this case, the sequence

$$\phi[k] = \phi[k, m_k] \geq \phi[k, m_k - 1] \geq \cdots \geq \phi[k, 1] \geq \phi[k - 1, m_{k-1}] \geq \cdots$$

is a decreasing sequence of transient potentials where each term is equal to its predecessor minus $\epsilon$ times the indicator of some partition set. By lemma 3.4, all terms in the sequence have the same Gurevich pressure. Since the sequence terminates after finitely many steps at $\phi[0] = \phi$,

$$P_G(\phi[k]) = P_G(\phi) \text{ for all } k.$$  \hfill (3.2.2)
Consider now the power series

\[ t_k(x) := 1 + \sum_{i=1}^{\infty} Z_i(\phi[k], b)x^i \]

\[ r_k(x) := \sum_{i=1}^{\infty} Z_i^*(\phi[k], b)x^i \]

Both have radius of convergence \( \exp[-P_G(\phi)] \): the first by the definition of the Gurevich pressure and (3.2.2), and the second because of the assumption that \( \phi[k] \) is transient (Part 1 of Theorem 3.1). They are related by the following inequality for all \( 0 < x < \exp[-P_G(\phi)] \) (2.1.1)):

\[
\frac{1}{B^2}[t_k(x) - 1] \leq t_k(x)r_k(x) \leq B^2[t_k(x) - 1], \quad \text{where} \quad B := \exp \sum_{n=2}^{\infty} \text{var} \ n \phi. \quad (3.2.3)
\]

By (3.2.3), \( r_k(x) \leq B^2 \) for all \( 0 < x < \exp[-P_G(\phi)] \) and \( k \geq 1 \).

But this cannot be the case, because for \( \exp[-P_G(\phi) - \frac{\epsilon}{2}] < x < \exp[-P_G(\phi)] \)

\[
r_k(x) \geq \sum_{n=1}^{k} Z_n^*(\phi[k], b)x^n \geq \sum_{n=1}^{k} e^{n\cdot \frac{\epsilon}{2}} Z_n^*(\phi, b)x^n \quad \text{(by (3.2.1))}
\]

\[
\lim_{k \to \infty} \sum_{n=1}^{\infty} Z_n^*(\phi, b)(e^{\epsilon/2}x)^n = \infty.
\]

This contradiction shows that there must be some \( k_0, i_0 \) for which \( \varphi := \phi[k_0, i_0] \) is recurrent. By construction \( \varphi \in \Phi, \text{var} [\varphi - \phi] = 0, \text{and} \|\varphi - \phi\|_\infty = \epsilon. \quad \square \)

**Proof of Theorem 3.1**

The proof has two parts:
(a) If $\phi \in \Phi$, then for every $\epsilon > 0$ there is a strongly positive recurrent potential $\varphi \in \Phi$ s.t. $\|\varphi - \phi\|_\infty < \epsilon$ and $\text{var}_1[\varphi - \phi] = 0$.

(b) The set of strongly positive recurrent potentials is open w.r.t the sup norm on $\Phi$.

**Part 1.** Approximating general potentials by strongly positive recurrent potentials.

Fix $\phi \in \Phi$ and $\epsilon > 0$. By Lemma 3.5, there exists a recurrent $\psi \in \Phi$ such that $\|\phi - \psi\|_\infty < \epsilon/2$ and $\text{var}_1[\phi - \psi] = 0$.

We now appeal to the discriminant theorem (3.1): Fix some $a \in S$, then the recurrence of $\psi$ implies that $\Delta_a[\phi] \geq 0$. If $\varphi := \psi + \frac{\epsilon}{2} \cdot 1[a]$, then

$$\Delta_a[\varphi] = \Delta_a[\psi] + \frac{\epsilon}{2} \quad \text{(Lemma 3.3)},$$

so $\varphi$ is strongly positive recurrent. It is obvious that $\|\phi - \varphi\|_\infty < \epsilon$ and $\text{var}_1[\varphi - \psi] = \text{var}_1[\psi - \phi] = 0$.

**Part 2.** The following is a simple corollary of lemmas 3.1 and 3.2.

**Corollary 3.1.** If $X_G$ is a topologically mixing countable Markov shift $a \in S_G$ and $\phi : X_G \to \mathbb{R}$ has summable variations and finite Gurevich pressure, then $\phi$ is strongly positive recurrent if and only if $e^{-P_G(\phi)} < R$ where $R$ is the radius of convergence of the power series $r^a_\phi(x)$.

**Proof.** The function $P_G(\phi + p)$ is continuous and strictly increasing. The (unique) solution to $P_G(\phi + p) = 0$ is $-P_G(\phi)$ and $P_G(\limsup \frac{1}{n} \log Z^*(\phi, a)) > 0$.

Now let $\epsilon > 0$ and suppose $\phi$ is strongly positive recurrent and $\psi : X_G \to \mathbb{R}$ is a potential with $\|\psi - \phi\|_\infty < \epsilon$. Then for any $x \in X_G$, $\phi(x) - \epsilon \leq \psi(x) \leq \phi(x) + \epsilon$. In
particular $P_G(\psi) = P_G(\phi) \pm \varepsilon$. Moreover, if $R_\phi$ and $R_\psi$ are the radii of convergence of $r^a_\phi(x)$ and $r^a_\psi(x)$, respectively, then $R_\psi = e^{\pm \varepsilon} R_\phi$. In particular, for all sufficiently small $\varepsilon$, $\psi$ is also strongly positive recurrent by corollary 3.1.

Thus, the set of strongly positive recurrent potentials is open in the $\| \cdot \|_\infty$-topology.

### 3.3 Proof of theorem 3.2

By lemma 3.3, recall that $\Delta_a [\phi + t \cdot 1_{[a]}] = \Delta_a [\phi] + t$.

Suppose $B = \{a_1, \ldots, a_r\} \subseteq \mathcal{S}_G$ and $\phi \in \Phi$ is transient. Then $\Delta_{a_1} [\phi] < 0$. Find $\epsilon_1 > 0$ s.t. $\phi^{(1)} := \phi + \epsilon_1 \cdot 1_{[a_1]}$ satisfies $\Delta_{a_1} [\phi^{(1)}] < 0$. Then $\phi^{(1)}$ is transient.

The transience of $\phi^{(1)}$ means that $\Delta_{a_2} [\phi^{(1)}] < 0$, so we can find $\epsilon_2 > 0$ s.t. $\phi^{(2)} := \phi^{(1)} + \epsilon_2 \cdot 1_{[a_2]}$ satisfies $\Delta_{a_2} [\phi^{(2)}] < 0$. So $\phi^{(2)}$ is also transient. Continuing in this manner, we obtain $\epsilon_1, \ldots, \epsilon_r > 0$ s.t.

$$\psi := \phi^{(r)} = \phi + \sum_{i=1}^r \epsilon_i \cdot 1_{[a_i]}$$

is transient.

Take $\delta := \min \{\epsilon_1, \ldots, \epsilon_r\}$. Then every $\varphi \in \Phi$ such that $\| \varphi - \psi \|_\infty < \delta$ and $\varphi|_{X \setminus B} = \varphi|_{X \setminus B}$ is transient. To see this, observe that $\varphi$ can be obtained from $\psi$ by subtracting the $r$ non-negative functions $(\psi - \varphi)1_{[a_i]}$. By lemma 3.4 each subtraction preserves transience, so the end result $\varphi$ is transient.

This proves that the set of transient potentials is $\mathcal{LU}(B)$-open. In fact it is dense in the complement of the strongly positive recurrent potentials. To see this, it is enough to show that every $\phi \in \Phi$ s.t. $\Delta_{a_1} [\phi] = 0$ can be approximated in $\mathcal{LU}(B)$ by a transient potential. Take $\phi + t \cdot 1_{[a_1]}$ with $t \to 0^-$. □
Chapter 4

Thermodynamics of Transient Shifts

In this chapter, we turn our attention to the case of countable Markov shifts arising from locally finite graphs\(^1\) \(\mathcal{G}\) with vertex set \(S_{\mathcal{G}}\). Our goal is to construct a conformal measure for any \(\phi \in \Phi(T)\). This will be done by constructing a sequence of measures and taking a weak-* limit. The construction of the sequence is inspired by the idea of a Martin boundary for transient Markov chains. We briefly review the idea of this boundary.

4.1 Martin Boundary of a Transient Markov Chain

The exposition from this section follows Woess [25].

Suppose \(\mathcal{G}\) is locally finite, irreducible and aperiodic and suppose \(A\) is an infinite, non-negative, stochastic matrix whose rows and columns are labeled by elements of \(S_{\mathcal{G}}\) such that \(a_{i,j} \neq 0\) if and only if \(i \rightarrow j\). We associate a random walk on \(\mathcal{G}\) to \(A\) by setting the probability that a random walker at vertex \(i\) will move to vertex \(j\) to be \(a_{i,j}\). We

\(^{1}\)Our convention is that a countable directed graph is locally finite if each vertex has finite outgoing degree (but may have infinite incoming degree). This is exactly the condition required for \(X_{\mathcal{G}}\) to be locally compact.
define

\[
p(i, j) := a_{i,j},
\]

\[
p^n(i, j) := \sum_{x_1, \ldots, x_{n-2} \in \mathcal{S}_G} a_{x_1} a_{x_2} \cdots a_{x_{n-3}} a_{x_{n-2}} a_{x_{n-2}, j}
\]

(and define \(p^0(a, b)\) to be the Kronecker delta function: \(p^0(a, b) := \delta_{a,b}\)). We say the Markov chain defined by \(A\) is transient if \(\sum_{n=1}^{\infty} p^n(a, a) < \infty\) for some (hence every) \(a \in \mathcal{S}_G\). The Martin compactification (defined below) is a compactification of the vertex set \(\mathcal{S}_G\) whose boundary points can be put in one-to-one correspondence with positive functions \(f: \mathcal{S}_G \to \mathbb{R}\) for which (for every \(a \in \mathcal{S}_G\))

\[
f(a) = \sum_{b: a \to b} p(a, b) f(b)
\]

(such a function is called harmonic). As we will see in the next section, harmonic functions (as interpreted in the context of countable Markov shifts) are in one-to-one correspondence with conformal measures. This justifies our current interest in their classification.

If \(A\) is transient, we define the Green’s function \(G: \mathcal{S}_G \times \mathcal{S}_G \to \mathbb{R}\) by

\[
G(a, b) := \sum_{n=0}^{\infty} p^n(a, b).
\]

This definition of transience coincides with that previously defined if, for \(\phi(x_0, x_1) := \log(a_{x_0, x_1})\) whenever \(a_{x_0, x_1} \neq 0\), we have \(P_\phi(\phi) = 0\).
By transience, $G(a, b) < \infty$ and by irreducibility of $\mathcal{G}$, $G(a, b) > 0$. Now fix a base point $o \in \mathcal{S}_\mathcal{G}$ and define the Martin kernel $K : \mathcal{S}_\mathcal{G} \times \mathcal{S}_\mathcal{G} \rightarrow \mathbb{R}$ by

$$K(a, b) := \frac{G(a, b)}{G(o, b)}.$$ 

Finally define the operator $\tilde{P}_\mathcal{G} : C(\mathcal{S}_\mathcal{G}) \rightarrow \mathcal{S}_\mathcal{G}$ (with $\mathcal{S}_\mathcal{G}$ a discrete space) by

$$(\tilde{P}_\mathcal{G} f)(a) := \sum_{b : a \rightarrow b} p(a, b) f(b).$$

Following the notation of Woess, we define the space of (normalized) positive $\tilde{P}_\mathcal{G}$-subharmonic functions:

$$\mathcal{B}^+ := \{ f \in C(\mathcal{S}_\mathcal{G}) : f \geq 0, f(o) = 1, \tilde{P}_\mathcal{G} f \leq f \}.$$ 

The following result is proven in [25]:

**Theorem 4.1.** Every sequence $\{f_n\}_{n \in \mathbb{N}}$ of elements of $\mathcal{B}^+$ has a pointwise convergent subsequence. Moreover $K(\cdot, b) \in \mathcal{B}^+$ for every $b \in \mathcal{S}_\mathcal{G}$.

We define the Martin compactification of $A$ to be the closure of the family of functions $\{K(\cdot, b)\}_{b \in \mathcal{S}_\mathcal{G}}$ in the (compact) space $\mathcal{B}^+$ (in the topology of pointwise convergence). Under the identification $b \mapsto K(\cdot, b)$ this can be viewed as a compactification of $\mathcal{S}_\mathcal{G}$. Interest in the Martin compactification is the following:

**Theorem 4.2.** The Martin compactification of $A$ contains all $\tilde{P}_\mathcal{G}$-harmonic functions.
Thus harmonic functions can always (in the setting of a locally finite graph $G$) be realized as pointwise limits of a sequence of Martin kernels.

## 4.2 Existence of Conformal Measures for Transient Shifts

### 4.2.1 Motivation

We briefly explain the connection between harmonic functions (of the previous section) and conformal measures.

Suppose $\phi : X \to \mathbb{R}$ is a transient potential such that $\phi(x) = \phi(x_0, x_1)$ is a function only of the first two coordinates and $P_G(\phi) = 0$. Suppose $\nu$ is a conformal measure for $\phi$ (so $L^\phi \nu = \nu$). Then for any $a \in S_G$,

$$
\nu([a]) = \int 1_{[a]} \, d\nu
= \int L_\phi 1_{[a]} \, d\nu
= \int \sum_{T_y = x} e^{\phi(y)} 1_{[a]}(y) \, d\nu(x)
= \sum_{b : a \rightarrow b} \int e^{\phi(ax)} \, d\nu(x)
= \sum_{b : a \rightarrow b} e^{\phi(a, b)} \nu([b]).
$$

This suggests defining the operator $P_G : C(S_G) \to C(S_G)$ by

$$(P_G f)(a) := \sum_{b : a \rightarrow b} e^{\phi(a, b)} f(b)$$
(compare this with the operator $\tilde{P}_G$ in the previous section). We will call $f$ harmonic if $P_G f = f$. The calculation above shows that for any conformal measure $\nu$, the function $f_\nu : S_G \to S_G$ defined by $f_\nu(a) := \nu([a])$ is $P_G$-harmonic.

Conversely, suppose $f$ is a $P_G$-harmonic function. Then define $\tilde{f} : \bigcup_{n=1}^\infty S_n^G$ by

$$\tilde{f}(a_0, \ldots, a_{m-1}) := e^{\phi(a_0, a_1) + \phi(a_1, a_2) + \cdots + \phi(a_{m-2}, a_{m-1})} f(a_{m-1}).$$

Then $\sum_{a_m, a_{m-1} \to a_m} f(a_0, \ldots, a_{m-1}, a_m) = f(a_0, \ldots, a_{m-1})$. We can, therefore, define a measure $\nu_f$ by setting $\nu_f([a_0 \cdots a_{m-1}]) := f(a_0, \ldots, a_{m-1})$ and extending this to a Borel measure via the Carathéodory extension theorem. By definition of $\tilde{f}$ this measure satisfies $\sum_{b : a \to b} e^{\phi(a, b)} \nu([b]) = \nu([a])$ and hence

$$\int 1_{[a_0 \cdots a_{m-1}]} d\nu = \int L_\phi 1_{[a_0 \cdots a_{m-1}]} d\nu$$

for any cylinder $[a_0 \cdots a_{m-1}]$. This implies that $\nu_f$ is $\phi$-conformal.

### 4.2.2 Construction of Conformal Measures

Here we use our intuition from Markov potentials (as in the previous section) and construct “Martin kernels” for general transient $\phi$. We show that their pointwise limits give rise to functions that can be converted into conformal functions, as above. We encourage the reader to keep the specific example from section 5.1 (in the next chapter) in mind while working through the definitions presented here.

Suppose $G$ is locally finite, irreducible and aperiodic, and $\phi : X_G \to \mathbb{R}$ is a transient potential with summable variations. Assume that $P_G(\phi) = 0$ (otherwise pass
to $\phi - P_G(\phi)$. Fix $o \in S_G$. We claim that $G$ contains an infinite forward ray. To see this let $a \xrightarrow{n} b$ denote the existence of a path from $a$ to $b$ of length exactly $n$ and set

$$A_n := \{v \in S_G : o \xrightarrow{n} v \text{ but } o \not\xrightarrow{s} v \text{ for any } s < n\}.$$ 

By local finiteness, $|A_n| < \infty$ for all $n$ and since $S_G$ is countable, $A_n \neq \emptyset$ for arbitrarily large $n \in \mathbb{N}$ (and hence $A_n \neq \emptyset$ for every $n$). For each $v \in \bigcup_n A_n =: V$ let $n_v \in \mathbb{N}$ be the natural number such that $v \in A_{n_v}$. Let $p_v : o \xrightarrow{n_v} v$ be a path in $G$ (by minimality of $n_v$, $p_v$ is simple and $(p_v)_i \in A_i$ for $i = 1, \ldots, n_v$). Now since $A_1$ is finite, there is some $v_1 \in A_1$ so that $(p_v)_1 = v_1$ for infinitely many $v \in V$. Set $V_1 := \{v \in V : (p_v)_1 = v_1\}$. But since $V_1$ is countable, $A_2$ is finite and $(p_v)_2 \in A_2$ for every $v \in V_1$ there is some $v_2 \in A_2$ so that $(p_v)_2 = v_2$ for infinitely many $v \in V_1$. Inductively, assume we have constructed vertices $v_1, \ldots, v_n \in V$ and infinite sets $V_n \subseteq \cdots \subseteq V_1$ so that $v_i \in A_i$ and $(p_v)_i = v_i$ for every $v \in V_i$. Since $V_n$ is infinite and $A_{n+1}$ is finite there is some $v_{n+1} \in A_{n+1}$ so that $(p_v)_{n+1} = v_{n+1}$ for infinitely many $v \in V_n$. Set $V_{n+1} := \{v \in V_n : (p_v)_{n+1} = v_{n+1}\}$. Recursively this allows us to construct a sequence of vertices

$$o \rightarrow v_1 \rightarrow v_2 \rightarrow v_3 \rightarrow \cdots$$

which is the desired forward infinite ray.
Now, fix some (forward) infinite ray \( b_1 \to b_2 \to b_3 \to \cdots \) set \( x := (b_1, b_2, \ldots) \in X_G \). Set \( W := \bigcup_{n=1}^{\infty} S^n_G \) and define a collection of functions: \( f_k : W \to \mathbb{R} \) by:

\[
f_k(a_0, \ldots, a_{m-1}) := \frac{\sum_{r=0}^{\infty} (L^r_{\phi}1_{[a_0 \cdots a_{m-1}]})(T^k x)}{\sum_{r=0}^{\infty} (L^r_{\phi}1_{[a]})(T^k x)}
\]

Note that by transience of \( \phi \), the numerator and denominator of \( f_k(\cdot) \) are both finite\(^3\) and by topological mixing of \( X_G \), \( f_k(a_0, \ldots, a_{m-1}) \neq 0 \) if and only if \( (a_0, \ldots, a_{m-1}) \) is an allowed word in the graph \( G \).

**Lemma 4.1.** For any \( k, p, q \in \mathbb{N} \) and any word \( (a_0, \ldots, a_{p-1}) \in W \),

\[
f_k(a_0, \ldots, a_{p-1}) = \sum_{a_p, \ldots, a_{p+q} \in S_G} f_k(a_0, \ldots, a_{p-1}, a_p, \ldots, a_{p+q}).
\]

**Proof.** We calculate

\[
\sum_{a_p, \ldots, a_{p+q} \in S_G} f_k(a_0, \ldots, a_{p+q}) = \sum_{a_p, \ldots, a_{p+q} \in S_G} \frac{\sum_{r=0}^{\infty} (L^r_{\phi}1_{[a_0 \cdots a_{p+q}]})(T^k x)}{\sum_{r=0}^{\infty} (L^r_{\phi}1_{[a]})(T^k x)}
\]

\[
= \frac{\sum_{r=0}^{\infty} \sum_{a_p, \ldots, a_{p+q} \in S_G} (L^r_{\phi}1_{[a_0 \cdots a_{p+q}]})(T^k x)}{\sum_{r=0}^{\infty} (L^r_{\phi}1_{[a]})(T^k x)}
\]

\[
= \frac{\sum_{r=0}^{\infty} (L^r_{\phi}1_{[a_0 \cdots a_{p-1}]})(T^k x)}{\sum_{r=0}^{\infty} (L^r_{\phi}1_{[a]})(T^k x)}
\]

\[
= f_k(a_0, \ldots, a_{p-1}).
\]

\(^3\)For every \( a \in S_G \) there exist constants \( C_a, k \in \mathbb{N} \) so that \( C_a^{-1}Z_{n-k}(\phi, o) \leq (L^n_{\phi}1_{[a]})(x_a) \leq C_aZ_{n+k}(\phi, o) \) where \( x_a \in [a] \).
Note in this case that there are only finitely many words \((a_p, \ldots, a_{p+q})\) for which 
\(f_k(a_0, \ldots, a_{p-1}, a_p, \ldots, a_{p+q})\) is nonzero. \(\Box\)

Next we show that for any word \((a_0, \ldots, a_{m-1}) \in \mathcal{W}\) (for which \([a_0 \cdots a_{m-1}] \neq \emptyset\)),
\[
0 < \liminf_{k \to \infty} f_k(a_0, \ldots, a_{m-1}) \leq \limsup_{k \to \infty} f_k(a_0, \ldots, a_{m-1}) < \infty.
\]

This is accomplished with three lemmas. First we fix \(z_c \in [c]\) for each \(c \in S_g\). We start by showing that \(f_k(\cdot)\) is "almost-subharmonic" (in a sense that we now explain) on words of length one.

**Definition 4.1.** For the discrete space \(S_g\), define the operator \(P : C(S_g) \to C(S_g)\) by
\[
(Pf)(a) := \sum_{c : a \to c} e^{\phi(a z_c)} f(c).
\]

**Lemma 4.2.** For any \(a \in S_g\),
\[
(Pf_k)(a) \leq e^{\text{var}_2(\phi)} \cdot f_k(a).
\]
Proof. Again we calculate

\[
(Pf_k)(a) = \sum_{c:a\rightarrow c} e^{\phi(a_Zc)} \frac{\sum_{r=0}^{\infty} (L^r 1_{[c]})(T^k x)}{\sum_{r=0}^{\infty} (L^r 1_{[a]})(T^k x)}
\]

\[
= \sum_{r=0}^{\infty} \frac{\sum_{c:a\rightarrow c} e^{\phi(a_Zc)} (L^r 1_{[c]})(T^k x)}{\sum_{r=0}^{\infty} (L^r 1_{[a]})(T^k x)}
\]

\[
\leq \sum_{r=0}^{\infty} e^{\operatorname{var}_2(\phi)} \frac{(L^r 1_{[a]})(T^k x)}{\sum_{r=0}^{\infty} (L^r 1_{[a]})(T^k x)}
\]

\[
\leq e^{\operatorname{var}_2(\phi)} \cdot f_k(a).
\]

Lemma 4.3. For any \(a \in S_G\),

\[0 < \liminf_{k \rightarrow \infty} f_k(a) \leq \limsup_{k \rightarrow \infty} f_k(a) < \infty\]

Proof. First note that \(f_k(o) = 1\) for every \(k\). Fix \(a \in S_G\) and suppose

\[p : o \xrightarrow{r} a\]

\[q : a \xrightarrow{s} o\]
are (fixed) paths in $G$. Then,

$$e^{r \cdot \text{var}_2(\phi)} f_k(a) \geq (P^r f_k)(a) \geq e^{\phi_r(p_0 \cdots p_r - 2^z p_{r-1}^z)} f_k(a),$$

$$e^{s \cdot \text{var}_2(\phi)} f_k(a) \geq (P^s f_k)(a) \geq e^{\phi_s(q_0 \cdots q_s - 2^z q_{s-1}^z)} f_k(o).$$

(the first rightmost inequality comes from the fact that $(P^r f_k)(o)$ is a sum over all paths of length $r$ starting at $o$; one such path is the path $p$. Similarly for $a$ in the second inequality). So we have, for any $k$,

$$e^{\phi_s(q_0 \cdots q_s - 2^z q_{s-1}^z) - \text{var}_s(\phi) - s \cdot \text{var}_2(\phi)} \leq f_k(a) \leq e^{r \cdot \text{var}_2(\phi) + \varphi_r(p_0 \cdots p_r - 2^z p_{r-1}^z)}.$$

\[\square\]

**Lemma 4.4.** For any $(a_0, \ldots, a_{m-1}) \in \mathcal{W}$ such that $[a_0 \cdots a_{m-1}] \neq \emptyset$

$$0 < \liminf_{k \to \infty} f_k(a_0, \ldots, a_{m-1}) \leq \limsup_{k \to \infty} f_k(a_0, \ldots, a_{m-1}) < \infty.$$  

**Proof.** Fix $(a_0, \ldots, a_{m-1}) \in \mathcal{W}$ and find $N$ so that for each $i = 0, \ldots, m - 1$ we have $T^k x \notin [a_i]$ for any $k > N$ (this is possible since $x = (b_1, b_2, \ldots)$ never repeats any vertices). If $k > N$, then we have $(L^{|a_0 \cdots a_{m-1}|}_\phi(T^k x) = 0$ for any $n < m$. For the remainder of the proof we assume that $k > N$. 


We calculate, for $n > m$,

$$
(L^n_{\phi_1[a_0\ldots a_{m-1}]})_{T^k x} = \sum_{T^ny = T^kx} e^{\phi_n(y)} \sum_{a_0\ldots a_{m-1}} (y)
$$

$$
= e^{B(m)\pm C(m)} (L^{n-m}_{\phi_1[a_{m-1}]})_{T^k x}.
$$

where $B(m)$ and $C(m)$ are the constants

$$
B(m) := \phi(a_0 \ldots a_{m-2} a_{m-1}) + \phi(a_1 \ldots a_{m-2} a_{m-1}) + \cdots + \phi(a_{m-1})
$$

$$
C(m) := \text{var}_2(\phi) + \text{var}_3(\phi) + \cdots + \text{var}_m(\phi).
$$

We obtain

$$
f_k(a_0, \ldots, a_{m-1}) = \frac{\sum_r \sum_{R^1_{\phi_1[a_0\ldots a_{m-1}]}} (T^k x)}{\sum_r \sum_{R^1_{\phi_0[a]}} (T^k x)}
$$

$$
= \frac{\sum_{r=m} \sum_{R^1_{\phi_1[a_0\ldots a_{m-1}]}} (T^k x)}{\sum_r \sum_{R^1_{\phi_0[a]}} (T^k x)}
$$

$$
= \frac{\sum_{r=0} \sum_{R^1_{\phi_1[a_{m-1}]}} (T^k x)}{\sum_r \sum_{R^1_{\phi_0[a]}} (T^k x)} \cdot e^{B(m)\pm C(m)}
$$

$$
= e^{B(m)\pm C(m)} \cdot f_k(a_{m-1}).
$$

By lemma 4.3 applied to $f_k(a_{m-1})$ we obtain the desired result.

By the diagonal argument, there exists a subsequence $k_i \not\to \infty$ so that for every $(a_0, \ldots, a_{m-1}) \in W$, the sequence $\{f_{k_i}(a_0, \ldots, a_{m-1})\}_{i \in \mathbb{N}}$ converges to a positive and
finite limit. We define the function \( f_\infty : \mathcal{W} \to \mathbb{R} \) by

\[
f_\infty(a_0, \ldots, a_{m-1}) := \lim_{i \to \infty} f_{k_i}(a_0, \ldots, a_{m-1}).\]

Since \( \mathcal{G} \) is locally finite, we immediately get from lemma 4.1 that for any \( p, q \in \mathbb{N} \),

\[
\sum_{a_p \ldots a_{p+q}} f_\infty(a_0, \ldots, a_{p+q}) = f_\infty(a_0, \ldots, a_{p-1}).
\]

**Lemma 4.5.** For given \( n, k \ (n \geq 1) \),

\[
\sum_{c : d \to c} e^{\phi(a_0 \ldots a_{m-1} z_c)} (L^n_\phi [a_1 \ldots a_{m-1}]) (T^k x)
= e^{\pm \text{var} + 1} (L^{n+1}_\phi [a_0 \ldots a_{m-1}]) (T^k x).
\]
Proof. This is immediate from

\[
(L^n_{\phi} 1_{[a_0 \cdots a_{m-1}]}) (T^k x) = \sum_{T^n+1 y = T^k x} e^{\phi_n+1(y)} 1_{[a_0 \cdots a_{m-1}]}(y)
\]

\[
= \sum_{c a_{m-1} \rightarrow c} \sum_{T^n+1 y = T^k x} e^{\phi_n+1(y)} e^{\phi(Ty)} 1_{[a_0 \cdots a_{m-1}]}(y)
\]

\[
= e^{\pm \text{var}_m+1(\phi)} \sum_{c a_{m-1} \rightarrow c} \times \sum_{T^n+1 y = T^k x} e^{\phi(a_0 \cdots a_{m-1} z_c)} e^{\phi_n(Ty)} 1_{[a_1 \cdots a_{m-1}]}(Ty)
\]

\[
= e^{\pm \text{var}_m+1(\phi)} \sum_{c a_{m-1} \rightarrow c} e^{\phi(a_0 \cdots a_{m-1} z_c)} (L^n_{\phi} 1_{[a_1 \cdots a_{m-1}]})(T^k x).
\]

From this calculation we have

\[
\sum_{c a_{m-1} \rightarrow c} e^{\phi(a_0 \cdots a_{m-1} z_c)} f_k(a_1, \ldots, a_{m-1}, c)
\]

\[
= e^{\pm \text{var}_m+1(\phi)} f_k(a_0, \ldots, a_{m-1}) - \frac{1_{[a_0 \cdots a_{m-1}]}(T^k x)}{\sum_{r=0}^{\infty} (L^n_{\phi} 1_{[a]})(T^k x)}.
\]

(4.2.1)

**Definition 4.2.** Let \( \nu \) be the (possibly \( \sigma \)-finite) measure obtained via the Carathéodory extension theorem by setting \( \nu([a_0 \cdots a_{m-1}]) := f_{\infty}(a_0, \ldots, a_{m-1}) \) for every nonempty cylinder \([a_0 \cdots a_{m-1}]\).

**Theorem 4.3.** The measure \( \nu \) defined above is \( \phi \)-conformal.
Proof. It is enough to show that for any nonempty cylinder \([a_0 \cdots a_{m-1}]\),

\[
\int 1_{[a_0\cdots a_{m-1}]} d\nu = \int L_\phi 1_{[a_0\cdots a_{m-1}]} d\nu. \tag{4.2.2}
\]

Fix a word \((a_0, \ldots, a_{m-1}) \in \mathcal{W}\) and \(\varepsilon > 0\). Find \(r\) so that \(\text{var}_{m+r}(\phi) < \frac{\varepsilon}{2}\). Find \(M\) so that for any \(i > M\) we have, for \(S := \#\{c : a_0 \overset{s}{\rightarrow} c \text{ for some } s \leq m + r + 1\},\)

\[
\left| f_{k_i}(a_0, y_1, \ldots, y_t) - f_{\infty}(a_0, y_1, \ldots, y_t) \right| < \frac{\varepsilon}{2 \cdot S \cdot e^{\sup(\phi)}}
\]

for any nonempty cylinder \([a_0 y_1 \cdots y_t]\) with \(t \leq m + r\) (possible since \(\mathcal{G}\) is locally finite) and such that \(T^k x \notin [a_0 \cdots a_{m-1}]\) (possible since \(x\) does not repeat any vertices). We will show that

\[
\int 1_{[a_0\cdots a_{m-1}]} d\nu = e^{\pm \varepsilon} \int L_\phi 1_{[a_0\cdots a_{m-1}]} d\nu \pm \varepsilon.
\]
To see this, we calculate

\[ \int L_1^1[a_0 \ldots a_{m-1}] d\nu = \sum_{a_m, \ldots, a_{m+r-1}} \int L_1^1[a_0 \ldots a_{m+r-1}] d\nu \]

\[ = \sum_{a_m, \ldots, a_{m+r-1}} \int \sum_{T^1 = w} e^{\varphi_1[a_0 \ldots a_{m+r-1}]}(w) d\nu(w) \]

\[ = \sum_{a_m, \ldots, a_{m+r-1}} \sum_{c:a_m+r-1 \rightarrow c} e^{\phi(a_0 w)} d\nu(w) \]

\[ = \sum_{a_m, \ldots, a_{m+r-1}} \sum_{c:a_m+r-1 \rightarrow c} e^{\phi(a_0 \ldots a_{m+r-1} z_c)} \pm \var{m+r+1(\phi)} \times \nu([a_1 \ldots a_{m+r-1}]) \]

\[ = e^{\pm \var{m+r+1(\phi)}} \left[ \sum_{a_m, \ldots, a_{m+r-1}} \nu([a_0 \ldots a_{m+r-1}]) \pm \varepsilon \right] . \]

Now if \( i > M \) is so large that the term \( \frac{1_{[a_0 \ldots a_{m-1}]}(T^k x)}{\sum_{r=0}^{\infty} (L_1^1[x])^r(T^k x)} \) vanishes (recall that the ray \( x \) does not repeat any vertices), then from (4.2.1) the right-hand side of the above string of equalities becomes

\[ = e^{\pm 2\var{m+r+1(\phi)}} \left[ \sum_{a_m, \ldots, a_{m+r-1}} f_{k_1}([a_0, \ldots, a_{m+r-1}]) \pm \varepsilon \right] \]

\[ = e^{\pm 2\var{m+r+1(\phi)}} \left[ \sum_{a_m, \ldots, a_{m+r-1}} \nu([a_0 \ldots a_{m+r-1}]) \pm \varepsilon \right] \]

\[ = e^{\pm 2\var{m+r+1(\phi)}} (\nu([a_0 \ldots a_{m-1}]) \pm \varepsilon) \]

\[ = e^{\pm \varepsilon \left[ \int 1_{[a_0 \ldots a_{m-1}]} d\nu \pm \varepsilon \right] .} \]

Now letting \( \varepsilon \to 0 \) gives us equation (4.2.2).
4.3 Existence of (Generalized) Thermodynamic Limit for Transient Shifts

In this section we investigate the existence of thermodynamic limits for transient shifts. As before we are assuming that $G$ is an irreducible, aperiodic, locally finite directed graph, $X_G$ its countable Markov shift and $\phi : X_G \to \mathbb{R}$ a transient potential with summable variations.

The standard thermodynamic limit, with boundary condition $x \in X_G$, is any weak* limit of the sequence of probability measures

$$\nu^n_x := \frac{\sum T^n y = x e^{\phi_n(y)} \delta_y}{\sum T^n z = x e^{\phi_n(z)}}.$$

In the transient case, this limit may be zero (recall that $X_G$ is not compact). Thus, we introduce a normalized sequence (here we choose the base point $o := x_0$):

$$\mu^n_x := \frac{\sum T^n y = x e^{\phi_n(y)} \delta_y}{\sum T^n y = x e^{\phi_n(y)} 1_{[o]}(y)}.$$

The measures $\{\mu^n_x\}$ are not probability measures, but rather finite measures normalized so that they give the cylinder $[o]$ measure one. A weak* limit of this sequence will be called a generalized thermodynamic limit. If $\phi$ is positive recurrent then the generalized thermodynamic limit exists and is just a scalar multiple of the ordinary thermodynamic
limit (this follows from Theorem 1.2). We also introduce the measures

\[ \mu_{n,a}^x := \frac{\sum_{T^n y=x} e^{\phi_n(y)} 1_{[a]}(y)}{\sum_{T^n y=x} e^{\phi_n(y)} 1_{[o]}(y)}. \]

Thus \( \mu_n^x = \sum_{a \in S_G} \mu_{n,a}^x \) (the advantage here is that each \( \mu_{n,a}^x \) gives full measure to the compact set \([a]\)).

**Lemma 4.6.** If \( X_G \) is locally compact, \( \phi : X_G \to \mathbb{R} \) has summable variations and finite Gurevich pressure and \( x \in X_G \), then there is a subsequence \( n_k \nearrow \infty \) such that, for every \( a \in S_G \), \( \{ \mu_{n_k,a}^x \}_{k \in \mathbb{N}} \) is tight and

\[ 0 < \liminf_{k \to \infty} \mu_{n_k,a}^x (X_G) \leq \limsup_{k \to \infty} \mu_{n_k,a}^x (X_G) < \infty. \]

**Proof.** There are two things to check:

1. \( 0 < \liminf_{k \to \infty} \mu_{n_k,a}^x (X_G) \leq \limsup_{k \to \infty} \mu_{n_k,a}^x (X_G) < \infty; \)

2. For every \( \varepsilon > 0 \) there is a compact \( F_{\varepsilon} \subseteq X_G \) such that \( \mu_{n,a}^x (F_{\varepsilon}^c) < \varepsilon \) for every \( n \).

In our case, the second condition is trivial since the measures \( \mu_{n,a}^x \) are all supported inside the compact set \([a]\). The main work here is to show that the first condition holds.

Fix \( a \in S_G \) and find \( m = m(a) \in \mathbb{N} \) so that there are admissible words (i.e. the cylinders in \( X_G \) that they determine are nonempty) \( (a, p_1, \ldots, p_{m-2}, a), (a, q_1, \ldots, q_{m-2}, o) \in \mathcal{W} \) and such that \( Z_m^* (\phi, o) \neq 0 \) (since \( G \) is locally finite \( Z_m^* > 0 \) for arbitrarily large \( m \)).
Then

\[ e^{\phi_m (aq_1 \cdots q_m - 2z_o)} Z_{n-m} (\phi, o) \leq (L_{\phi [a]}^n)(x) \leq e^{-\phi_m (op_1 \cdots p_m - 2z_a)} Z_{n+m} (\phi, o) \]

for every \( n > m \) (and, of course, \( (L_{\phi [a]}^n)(x) = Z_n (\phi, o) \) by choice of \( o \)). Therefore the ratio \( \mu^x_{n,a} (X_G) = \frac{(L_{\phi [a]}^n)(x)}{(L_{\phi [a]}^n)(x)} \) is bounded from above (in \( n \)) as long as \( \frac{Z_{n+m} (\phi, o)}{Z_n (\phi, o)} \) is bounded from above. Moreover, if the ratio

\[ \frac{Z_{n_s+k} (\phi, o)}{Z_{n_s} (\phi, o)} \]

is (not necessarily uniformly) bounded above for each (fixed) \( k \) such that \( Z^*_k > 0 \) along some subsequence \( \{n_s\} \), then \( \limsup_{s} \mu^x_{n_s,a} (X_G) < \infty \) for every \( a \in S_G \). In fact, it is enough to show this only for \( k \) such that \( 0 < Z^*_k < 1 \) since \( \phi \) has summable variations.

We aim to show that such a subsequence exists.

For ease of notation, set \( Z_n := Z_n (\phi, o) \) and \( Z^*_n := Z^*_n (\phi, o) \). Recall that

\[ \frac{Z_{n+k} (\phi, o)}{Z_n (\phi, o)} = \frac{Z^*_n + Z^*_n Z_1 + \cdots + Z^*_n Z_k + \cdots + Z^*_n Z_{n+k-1}}{Z_n} \geq Z^*_k \]

for every \( n \). We claim that there is a subsequence \( \{n_i\} \) so that for every \( k \) such that \( Z^*_k > 0 \), the ratio \( \frac{Z_{n_i+k}}{Z_{n_i}} \) is bounded (by a constant depending on \( k \)). To see this we need the following:
Claim. For fixed $k$ (such that $0 < Z_k^* < 1$) and given $\varepsilon > 0$ there exists $R$ so that

$$\limsup_{n \to \infty} \frac{\#\{0 \leq r \leq n | \frac{Z_r + k}{Z_r} \geq R\}}{n} < \varepsilon.$$ 

Proof. We proceed by contradiction. Suppose that there is some $\lambda > 0$ such that

$$\limsup_{n \to \infty} \frac{\#\{0 \leq r \leq n | \frac{Z_r + k}{Z_r} \geq (Z_k^*)^{-N}\}}{n} > \lambda$$

for every $N$ (we will later choose $N$ to be very large). Set $A_n := \{ r \in [0, n] \cap \mathbb{N} | \frac{Z_r + k}{Z_r} \geq (Z_k^*)^{-N}\}$, then for any $r \in A_n$, $\frac{Z_r + k}{Z_r} \geq (Z_k^*)^{-N}$ and for any $r \notin A_n$, $\frac{Z_r + k}{Z_r} \geq Z_k^*$. This allows us to estimate,

$$\begin{align*}
\prod_{i=k(n-1)+1}^{kn} Z_i &= \left( \prod_{j=1}^{k} Z_j \right) \prod_{s=1}^{n-1} \frac{\prod_{t=k(s+1)+1}^{(s+1)k} Z_t}{\prod_{t=k(s)+1}^{sk} Z_t} \\
&= \left( \prod_{j=1}^{k} Z_j \right) \prod_{u=1}^{k} \prod_{s=0}^{n-2} \frac{Z_{(ks+u)+k}}{Z_{(ks+u)}} \\
&\geq \left( \prod_{j=1}^{k} Z_j \right) \left( (Z_k^*)^{-1} \right)^N \left| A_{k(n-1)} \right| - (k(n-1) - \left| A_{k(n-1)} \right|)
\end{align*}$$

Then

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \prod_{j=k(n-1)+1}^{kn} Z_j \right)$$
\[
\geq \log \left( \prod_{j=1}^{k} Z_j \right) + \limsup_{n \to \infty} \frac{(N + 1) A_{k(n-1)} - k(n-1)}{n} \log((Z^*_k)^{-1})
\]

\[
\geq \log \left( \prod_{j=1}^{k} Z_j \right) - k \log((Z^*_k)^{-1}) + \frac{\lambda(N + 1)}{k} \log((Z^*_k)^{-1}).
\]

If \( N \) is sufficiently large, this is greater than \( k P_G(\phi) \) (recall that \( 0 < Z^*_k < 1 \)). But

\[
\lim_{n \to \infty} \frac{1}{n} \log(Z_n(\phi,0)) = P_G(\phi)
\]

and hence

\[
\lim_{n \to \infty} \frac{1}{n} \log \left( \prod_{j=k(n-1)}^{kn} Z_j \right) = k P_G(\phi);
\]

a contradiction.

Fix \( \varepsilon > 0 \) and let \( R(k) \) be the constant produced by the previous claim so that

\[
\limsup_{n \to \infty} \frac{\# \{ 0 \leq r \leq n | \frac{Z_r + k}{Z_r} \geq R(k) \}}{n} < \frac{\varepsilon}{2^k}
\]

and let \( \tilde{R}(k) \geq R(k) \) be a constant so that

\[
\sup_{n} \frac{\# \{ 0 \leq r \leq n | \frac{Z_r + k}{Z_r} \geq \tilde{R}(k) \}}{n} < \frac{\varepsilon}{2^k}.
\]
So

$$\sup_n \frac{\# \{0 \leq r \leq n | \frac{Z_{r+i}}{Z_r} \geq \bar{R}(i) \text{ for some } i \}}{n} \leq \sum_{i=1}^{\infty} \frac{\# \{0 \leq s \leq n | \frac{Z_{s+i}}{Z_s} \geq \bar{R}(i) \}}{n} \leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^{i+1}} = \varepsilon.$$ 

Therefore

$$\inf_n \frac{\# \{0 \leq r \leq n | \frac{Z_{r+i}}{Z_r} \leq \bar{R}(i) \text{ for every } i \}}{n} \geq 1 - \varepsilon$$ 

and in particular \( \{ n \in \mathbb{N} | \frac{Z_{n+i}}{Z_n} \leq \bar{R}(i) \text{ for every } i \} \) has infinitely many elements (in fact it has lower density greater than \( 1 - \varepsilon \) in \( \mathbb{N} \)). Call the subsequence defined by this set \( \{ n_k \} \), then for every \( a, \mu_{n_k, a}^x (X_G) \) is uniformly bounded (by constants depending on \( a \in \mathcal{S}_G \)) away from infinity.

A similar argument produces a subsequence of lower density greater than \( 1 - \varepsilon \) along which the ratios \( \frac{Z_{n_k-s-k}(\phi,o)}{Z_{n_k-s}(\phi,o)} \) are bounded below. Thus there is a common subsequence of lower density greater than \( 1 - 2\varepsilon \) along which \( 0 < \liminf_s \mu_{n_k, a}^x (X_G) \leq \liminf_s \mu_{n_k, a}^x (X_G) \leq \infty \) for every \( a \in \mathcal{S}_G \).

Since the sequence \( \{ \mu_{n,a}^x \} \) is tight for every \( a \) we can choose a subsequence (by the diagonal argument) so that \( \{ \mu_{n,a}^x \} \) has a weak* limit for every \( a \in \mathcal{S}_G \). For such a subsequence, define \( \mu_a^x = \lim_{n_k\to\infty} \mu_{n_k,a}^x \) and set

$$\mu = \sum_{a \in \mathcal{S}_G} \mu_a^x.$$ 

This measure is positive and finite on cylinder sets. This gives us the following.
Theorem 4.4. If \( X_G \) is locally compact and \( \phi \) has summable variations and finite Gurevich pressure, there is at least one generalized thermodynamic limit for \( \phi \).

Remark. We note that the proof that
\[
0 < \liminf_s \mu^{x}_{n_s,a} (X_G) \leq \limsup_s \mu^{x}_{n_s,a} (X_G) < \infty
\]
holds even if we replace the assumption that \( G \) is locally finite with the assumption that
\[
Z^*(\phi, o) > 0 \text{ for arbitrarily large } k. \quad \text{In this case, however, an argument would be needed to show tightness.}
\]
Chapter 5

Examples

In this chapter we present several examples to show the reader, concretely, how the conformal measures for a transient Markov shift can be found. First we will recall several of the objects that we have been using throughout this work.

The partition function of $\phi : X_G \to \mathbb{R}$ at the vertex $a \in S_G$ is

$$Z_n(\phi,a) = \sum_{T^n x = x} e^{\phi_n(x)} 1_{[a]}(x).$$

The generating function associated to the sequence $\{Z_n(\phi,a)\}$ is

$$t^a(\phi;x) := 1 + \sum_{n=1}^{\infty} Z_n(\phi,a)x^n.$$

The Gurevich pressure of $\phi$ is $P_{G,\phi} := \lim\frac{1}{n} \log(Z_n(\phi,a))$ (which is independent of $a \in S_G$ as long as $\phi$ has summable variations and $X_G$ is topologically mixing). The radius of convergence of $t^a(\phi;x)$ is $e^{-P_{G,\phi}}$.

It is often difficult to explicitly find the value of $Z_n(\phi,a)$, but in many examples it is possible to find

$$Z^*_n(\phi,a) := \sum_{T^n x = x} e^{\phi_n(x)} 1_{[\varphi_a = n]}(x),$$

where $\varphi_a(x) = 1_{[a]}(x) \inf\{m > 0 : T^m x \in [a]\}$. 
If \( \phi \) is a Markov potential (i.e. \( \phi(x) = \phi(x_0, x_1) \) depends only on the first two coordinates of \( x \in X_G \)) then we have a renewal equation relating \( \{Z_n(\phi, a)\} \) to \( \{Z^*_n(\phi, a)\} \):

\[
Z_n(\phi, a) = Z^*_n(\phi, a) + Z^*_{n-1}(\phi, a)Z_1(\phi, a) + \cdots + Z^*_1(\phi, a)Z_{n-1}(\phi, a).
\]

Thus \( t^a_\phi(x) = \frac{1}{1 - r^a_\phi(x)} \) for any \( x \in [0, e^{-P_G(\phi)}) \).

For a Markov potential, conformal measures are in one-to-one correspondence with \( P_G \)-harmonic functions; i.e. functions \( h : S_G \to \mathbb{R} \) satisfying

\[
h(a) = \sum_{c:a\to c} e^{\phi(ac)} h(c) =: (P_G h)(a)
\]

(the correspondence is outlined in section 5.1.3). In practice, finding \( P_G \)-harmonic functions is the easiest way to (explicitly) find conformal measures.

We also will calculate the Martin kernels which were used in section 4.2 to define conformal measures (in fact in section 4.2 a family of functions on all finite words was defined – the Martin kernels we compute are the restriction of these functions to words of length one). Suppose \( \phi \in \Phi \) and \( P_G(\phi) = 0 \), then the Green’s function associated to the shift \( X_G \) is a function \( G : S_G \times S_G \to \mathbb{R} \) given by

\[
G(a, b) := \sum_{n=0}^{\infty} (L^n \phi(a))(x_b)
\]

\(^1\)The operator \( P_G \) acts on functions by \( (P_G f)(a) = \sum_{c:a\to c} e^{\phi(ac)} f(c) \) (here \( a \to c \) means there is a directed edge from \( a \) to \( c \) in \( G \)). Later we will consider a subgraph \( H \subseteq G \) in which case we will also have the operator \( P_H \) which acts on functions by \( (P_H f)(a) = \sum_{c:a\to c} e^{\phi(ac)} f(c) \) (now \( a \to c \) means there is a directed edge from \( a \) to \( c \) in \( H \)).
where $x_b \in [b]$ (since $\phi$ is Markov, the value of $G(a, b)$ is independent of choice of $x_b$).

We fix a base point $o \in S_G$ and normalize the Green’s functions to be the Martin kernels:

$$K(a, b) := \frac{G(a, b)}{G(o, b)}$$

(the denominator is nonzero by irreducibility of $G$). On a locally finite graph $G$ if $b_n$ is a sequence of vertices which diverges to infinity and along which $K(\cdot, b_n)$ converges point-wise, its pointwise limit is a harmonic function (and hence corresponds to a conformal measure).

### 5.1 A Shift with Non-unique Conformal Measure

Let $\mathcal{H}$ be the graph:

and let $\phi(x) := \phi(x_0, x_1)$ be the potential

$$\phi(x_0, x_1) := \begin{cases} 
\log \left( \frac{1}{2\zeta(2)+2} \cdot \frac{1}{(n+1)^3} \right) & \text{if } x_0 = n, x_1 = 0, n > 0; \\
\log \left( \frac{1}{\zeta(2)+1} \right) & \text{if } x_0 = x_1 = 0; \\
0 & \text{otherwise.}
\end{cases}$$
\[ Z^*_n(\phi, 0) = \frac{1}{\zeta(2)+1} \cdot \frac{1}{n^3} \] for every \( n \geq 1 \), so \( r_\phi(x) = \frac{1}{\zeta(2)+1} \sum_{n=1}^{\infty} \frac{x^n}{n^3} \). The radius of convergence of \( r_\phi(x) \) is one and \( r_\phi(1) = \frac{\zeta(3)}{\zeta(2)+1} < 1 \) so \( t_\phi(x) = \frac{1}{1-r_\phi(x)} \) converges to

\[ t_\phi(1) = \sum_{n=0}^{\infty} Z_n(\phi, 0) = \frac{\zeta(2) + 1}{\zeta(2) - \zeta(3) + 1} \quad (5.1.1) \]

at its radius of convergence. Thus, \( \phi \) is transient and \( P_G(\phi) = 0 \).

\section*{5.1.1 Harmonic Functions}

Suppose \( h : \mathcal{S}_\mathcal{H} \to \mathbb{R} \) is positive and \( P_\mathcal{H} \)-harmonic and normalized so that \( h(0) = 1 \). Harmonicity (at zero) implies

\[ h(1) = h(0) - h(-1) - \frac{h(0)}{\zeta(2)+1} = 1 - h(-1) - \frac{1}{\zeta(2)+1}. \]

If \( n > 0 \) then \( h(n) = \frac{1}{2\zeta(2)+2} \cdot \frac{1}{(n+1)^3} \cdot h(0) + h(n+1) \) so, inductively,

\[ h(n) = 1 - h(-1) - \frac{1}{2\zeta(2)+2} - \frac{1}{2\zeta(2)+2} \sum_{k=1}^{n} \frac{1}{k^3}. \quad (5.1.2) \]

Similarly, if \( n < 0 \), then \( h(n) = \frac{1}{2\zeta(2)+2} \cdot \frac{1}{(n+1)^3} \cdot h(0) + h(n-1) \) and we get

\[ h(n) = h(-1) + \frac{1}{2\zeta(2)+2} - \frac{1}{2\zeta(2)+2} \sum_{k=1}^{n} \frac{1}{k^3}. \quad (5.1.3) \]

To ensure positivity of \( h \) for positive values of \( n \) (5.1.2) forces \( h(-1) \leq \frac{2\zeta(2) - \zeta(3) + 1}{2\zeta(2)+2} \).

Similarly, for negative values of \( n \) (5.1.3) forces \( h(-1) \geq \frac{\zeta(3) - 1}{2\zeta(2)+2} \).
To summarize, there is a one parameter family of positive $P_H$-harmonic functions on $S_H$ given by

$$h(n) = \begin{cases} 
1 - h(-1) - \frac{1}{2\zeta(2)+2} - \frac{1}{2\zeta(2)+2} \sum_{k=1}^{n} \frac{1}{k^3} & \text{if } n > 0; \\
h(-1) + \frac{1}{2\zeta(2)+2} - \frac{1}{2\zeta(2)+2} \sum_{k=1}^{n} \frac{1}{k^3} & \text{if } n < 0; \\
1 & \text{if } n = 0
\end{cases}$$

where $\frac{\zeta(3)-1}{2\zeta(2)+2} \leq h(-1) \leq \frac{2\zeta(2)-\zeta(3)+1}{2\zeta(2)+2}$. This is a convex set with the extremal points corresponding to $h(-1) = \frac{\zeta(3)-1}{2\zeta(2)+2}$ and $h(-1) = \frac{2\zeta(2)-\zeta(3)+1}{2\zeta(2)+2}$ respectively.

### 5.1.2 Martin Boundary

We will compute the Martin kernels and find their pointwise limits. First we analyze the case when $b > 0$.

If $a = 0$ then $(L_{\phi}^{n}1_{|0|})(x_{b}) = Z_{n-b}(\phi, 0)$. Thus

$$G(0, b) = \sum_{n=b}^{\infty} Z_{n-b}(\phi, 0) = \frac{\zeta(2) + 1}{\zeta(2) - \zeta(3) + 1}$$

by (5.1.1).

If $a \neq 0$ then

$$(L_{\phi}^{n}1_{|a|})(x_{b}) = 1_{0 < a \leq b, n = b-a} + \sum_{r=1}^{n-b} \frac{1}{2\zeta(2) + 2} \cdot \frac{1}{(|a| + r)^3} \cdot Z_{n-b-r}(\phi, 0)$$
using the fact that, except for a possible path \( a \to (a + 1) \to \cdots \to (b - 1) \to b \), every path from \( a \) to \( b \) decomposes into a path \( a \to (a + 1) \to \cdots \to (a + r - 1) \to 0 \), then a loop of length \( n - r - b \) at zero and finally the path \( 0 \to 1 \to \cdots \to b \). This calculation gives

\[
G(a, b) = \sum_{n=0}^{\infty} (L^n_{\phi, [a]}) \phi \bigg|_{b}
\]

\[
= 1_{[0 < a \leq b]} + \sum_{n=b+1}^{\infty} \sum_{r=1}^{n-b} \frac{1}{2\zeta(2) + 2} \cdot \frac{1}{(|a| + r)^3} \cdot Z_{n-b-r}(\phi, 0)
\]

\[
= 1_{[0 < a \leq b]} + \sum_{r=1}^{\infty} \sum_{n=b+r}^{\infty} \frac{1}{2\zeta(2) + 2} \cdot \frac{1}{(|a| + r)^3} \cdot Z_{n-b-r}(\phi, 0)
\]

\[
= 1_{[0 < a \leq b]} + \frac{1}{2\zeta(2) - 2\zeta(3) + 2} \sum_{r=|a|+1}^{\infty} \frac{1}{r^3}
\]

(here we have used the fact that \( \sum_{n=b+r}^{\infty} Z_{n-b-r}(\phi, 0) = \frac{\zeta(2) + 1}{\zeta(2) - \zeta(3) + 1} \) by (5.1.1)).

Finally we get \( K(a, b) = \frac{G(a, b)}{G(0, b)} \), so

\[
K(a, b) = \begin{cases} 
\frac{(2\zeta(2) - 2\zeta(3) + 2)1_{[0 < a \leq b]} + \zeta(3) - \frac{1}{2} \sum_{s=1}^{[a]} \frac{1}{s^3}}{\zeta(2) + 1} & \text{if } a \neq 0; \\
1 & \text{if } a = 0.
\end{cases}
\]

As \( b \to +\infty \) the functions \( K(\cdot, b) \) converge pointwise to the harmonic function determined by \( h(-1) = \frac{\zeta(3) - 1}{2\zeta(2) + 2} \) in the previous section. A similar analysis when \( b < 0 \) shows that as \( b \to -\infty \) the pointwise limit of the Martin kernels is the harmonic function determined by
h(-1) = \frac{2\zeta(2) - \zeta(3) + 1}{2\zeta(2) + 2}. Thus the extremal harmonic functions are obtained as pointwise limits of Martin kernels “approaching infinity in different directions.”

5.1.3 Conformal Measures

Each positive \( P_H \)-harmonic function above defines a \( \phi \)-conformal measure as follows. Suppose \( h: S_H \to \mathbb{R} \) is \( P_H \)-harmonic. We define a function \( f_h: \bigcup_{n=1}^{\infty} S^n_H \to \mathbb{R} \) by setting \( f_h(a) := h(a) \) for every \( a \in S_H \) and, inductively,

\[
f_h(a_0, \ldots, a_m) := e^{\phi(a_0a_1)} f_h(a_1, \ldots, a_m)
\]

for any \( m \geq 1 \) (here we are using the fact that \( \phi(x) = \phi(x_0, x_1) \)). This gives

\[
f_h(a_0, \ldots, a_m) = e^{\phi_{m-1}(a_0a_1 \cdots a_m)} h(a_m).
\]

One checks that for any \( t \geq 0 \),

\[
\sum_{a_{m+t}} f_h(a_0, \ldots, a_{m+t}) = f_h(a_0, \ldots, a_{m-1})
\]

so that \( f_h \) defines a \( \sigma \)-additive set of functions on the semi-algebra of cylinders of \( X_H \).

Moreover for any \( m \geq 0 \),

\[
f_h(a_0, \ldots, a_{m-1}) = \sum_{c:a_{m-1} \rightarrow c} e^{\phi(a_0a_1)} f_h(a_1, \ldots, a_{m-1}, c).
\]
It follows that the (unique) measure $\nu_h$ obtained on $X_H$ from $f_h$ by the Carathéodory extension theorem is $\phi$-conformal.

Conversely, if $\nu$ is any $\phi$-conformal measure on $X_H$ then

$$
\nu([a_0 \cdots a_{m-1}]) = \int 1_{[a_0 \cdots a_{m-1}]} \, d\nu
= \int L^{\phi} 1_{[a_0 \cdots a_{m-1}]} \, d\nu
= \int \sum_{T^i y = x} e^{\phi(y)} 1_{[a_0 \cdots a_{m-1}]}(y) \, d\nu(x)
= \sum_{c: a_{m-1} \rightarrow c} \int 1_{[a_0 \cdots a_{m-1}]} \, e^{\phi} \, d\nu
= \sum_{c: a_{m-1} \rightarrow c} e^{\phi(a_0 a_1)} \nu([a_1 \cdots a_{m-1}])
$$

(where we interpret $a_1 := c$ if $m = 1$). Thus the function $f_{\nu}(a_0, \ldots, a_{m-1}) := \nu([a_0 \cdots a_{m-1}])$ is one of the functions above. Therefore we have a complete classification of the $\phi$-conformal measures on $X_H$.

Remark. A $\phi$-conformal probability measure is an example of a Dobrushin-Lanford-Ruelle (DLR) state for $\phi$ on $X_G$. We note that if $\nu$ is $\phi$-conformal and $h$ is a density with respect to which then $\nu$ is a probability measure, then the measure $dm = h \, d\nu$ is a DLR state for the potential $\phi + \log(h) - \log(h \circ T)$.

In the example above, if $h : S_G \rightarrow \mathbb{R}$ is an even function (i.e. $h(n) = h(-n)$) with respect to which one of the extremal conformal measures is a probability measure then, by symmetry, all of the above measures are. That is, $\phi$ can be changed by a coboundary so that it has a non-unique DLR state.
5.2 A Non-locally Finite Shift with No Conformal Measure

Suppose $G$ is the graph $H$ with the additional edges $0 \rightarrow n$ for every $n \neq 0$ (note that $G$ is locally finite at every vertex except zero).

Let $\varphi(x) := \varphi(x_0, x_1)$ be the potential

$$
\varphi(x_0, x_1) := \begin{cases} 
\log \left( \frac{1}{2\zeta(2)+2} \cdot \frac{1}{(n+1)^3} \right) & \text{if } x_0 = n, x_1 = 0, n > 0; \\
\log \left( \frac{1}{\zeta(2)+1} \right) & \text{if } x_0 = x_1 = 0; \\
0 & \text{otherwise.}
\end{cases}
$$

We remark that $\varphi|_{X_H} = \phi$. Note that $Z^*(\varphi, 0) = \frac{1}{\zeta(2)+1}$ and

$$Z^*_n(\phi, 0) = \frac{1}{\zeta(2)+1} \sum_{k=0}^{\infty} \frac{1}{(n+k)^3}, \text{ for } n \geq 2.$$
Then

\[ r^0(\varphi)(x) = \sum_{n=1}^{\infty} Z^n(\varphi,0)x^n \]

\[ = \frac{1}{\zeta(2)+1} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{x^n}{(n+k)^3} \]

\[ = \frac{1}{\zeta(2)+1} \sum_{n=1}^{\infty} \sum_{r=n}^{\infty} \frac{x^n}{r^3} \]

\[ = \frac{1}{\zeta(2)+1} \sum_{r=1}^{\infty} \sum_{n=1}^{r} \frac{x^n}{r^3}. \]

Then \( r^0(1) = \frac{\zeta(2)}{\zeta(2)+1} < 1 \) and \( r^0(\varphi)(x) \) diverges for \( x > 1 \). Thus \( \varphi \) is transient and \( P_G(\varphi) = P_G(\phi) = 0 \). We claim that there are no positive \( P_G \)-harmonic functions.

Suppose for contradiction that \( h : S \rightarrow \mathbb{R} \) is a positive \( P_G \)-harmonic function.

Then

\[ 1 = h(0) = \sum_{n=-\infty}^{-1} h(n) + \frac{1}{\zeta(2)+1} h(0) + \sum_{n=1}^{\infty} h(n). \quad (5.2.1) \]

If \( n > 0 \) then, by harmonicity (at \( n \)), \( h(n+1) = h(n) - \frac{1}{2\zeta(2)+2} \cdot \frac{1}{(n+1)^3} \). Inductively we get

\[ h(n) = h(1) - \frac{1}{2\zeta(2)+2} \sum_{k=2}^{n} \frac{1}{k^3}. \]

Similarly if \( n < 0 \) then we get

\[ h(n) = h(-1) - \frac{1}{2\zeta(2)+2} \sum_{k=2}^{n} \frac{1}{k^3}. \]
If the two infinite sums appearing in (5.2.1) converge, we must have \( h(n) \xrightarrow{n \to \pm \infty} 0; \) that is \( h(-1) = h(1) = \frac{\zeta(3)-1}{2\zeta(2)+2}. \) Therefore,

\[
h(n) = \begin{cases} 
\frac{1}{2\zeta(2)+2} \sum_{k=|n|+1}^{\infty} \frac{1}{k^3} & \text{if } n \neq 0; \\
1 & \text{if } n = 0.
\end{cases}
\]

Then

\[
\sum_{n=-\infty}^{-1} h(n) = \frac{1}{2\zeta(2)+2} \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \frac{1}{k^3} = \frac{1}{2\zeta(2)+2} \sum_{k=2}^{\infty} \sum_{n=1}^{k-1} \frac{1}{k^3} = \frac{\zeta(2) - \zeta(3)}{2\zeta(2)+2}
\]

Similarly, \( \sum_{n=1}^{\infty} h(n) = \frac{\zeta(2)-\zeta(3)}{2\zeta(2)+2}. \) Thus the RHS of (5.2.1) is \( \frac{\zeta(2)-\zeta(3)+1}{\zeta(2)+1} \neq 1; \) there can be no positive \( P_G^- \)-harmonic function.

### 5.3 A Non-locally Finite Shift with a Conformal Measure

In this section we show a shift that is not locally finite and has a conformal measure. Interest in this example is the following (here \( G \) and \( \varphi \) are as in the previous example):

- We construct a Markov potential \( \psi : X_G \to \mathbb{R} \) with \( P_G(\psi) = P_G'(\varphi) = P_G(\phi) = 0; \)
- \( \psi|_{X_H} = \varphi|_{X_H} = \phi; \)
• \( \psi \) has a conformal measure.

That is, \( \varphi \) and \( \psi \) are nearly identical (they have the same restriction to \( X_\mathcal{H} \) which can be viewed as “big” in the sense that \( P_G(\psi) = P_G(\varphi) = P_G(\phi) \)). Nevertheless these two potentials have very different properties with regard to conformal measures.

Let \( \psi(x) := \psi(x_0, x_1) \) be the potential on \( X_g \) (as in the previous example) given by:

\[
\psi(x_0, x_1) := \begin{cases} 
\log \left( \frac{1}{2^\zeta(2)+2} \cdot \frac{1}{(n+1)^3} \right) & \text{if } x_0 = n, x_1 = 0, n > 0; \\
\log \left( \frac{1}{\zeta(2)+1} \right) & \text{if } x_0 = x_1 = 0; \\
\log \left( \frac{1}{2\pi} \right) & \text{if } x_0 = 0, \abs{x_1} = n, n > 0; \\
0 & \text{otherwise.}
\end{cases}
\]

Note that \( \psi \big|_{X_\mathcal{H}} = \phi \) (so \( P_G(\psi) \geq 0 \)) and \( \psi \leq \varphi \) (so \( P_G(\psi) \leq 0 \)); so \( P_G(\psi) = 0 \).

Suppose \( h: S_g \to \mathbb{R} \) is positive and \( P_G \)-harmonic. Then

\[
1 = h(0) = \sum_{n=-\infty}^{-1} \frac{h(n)}{2^n} + \frac{1}{\zeta(2)+1} + \sum_{n=1}^{\infty} \frac{h(n)}{2^n}. \tag{5.3.1}
\]

As above, we obtain

\[
h(n) = h(1) - \frac{1}{2\zeta(2)+2} \sum_{k=2}^{n} \frac{1}{k^3} \quad \text{if } n > 0; \\
h(n) = h(-1) - \frac{1}{2\zeta(2)+2} \sum_{k=2}^{|n|} \frac{1}{k^3} \quad \text{if } n < 0.
\]
Therefore (5.3.1) becomes

\[
1 = \frac{h(-1)}{2} + \sum_{n=2}^{\infty} \frac{h(-1) - \frac{1}{2\zeta(2)+2} \sum_{k=2}^{n} \frac{1}{k^3}}{2^n} + \frac{1}{\zeta(2) + 1} + \\
+ \frac{h(1)}{2} + \sum_{n=2}^{\infty} \frac{h(1) - \frac{1}{2\zeta(2)+2} \sum_{k=2}^{n} \frac{1}{k^3}}{2^n}
\]

\[
= h(-1) + h(1) - \frac{1}{\zeta(2) + 1} \sum_{n=2}^{\infty} \sum_{k=2}^{n} \frac{1}{k^3 2^n} + \frac{1}{\zeta(2) + 1}
\]

\[
= h(-1) + h(1) - \frac{1}{\zeta(2) + 1} \sum_{k=2}^{\infty} \frac{1}{k^3} \cdot \frac{1}{2^{k-1}} + \frac{1}{\zeta(2) + 1}
\]

So any pair \((h(-1), h(1)) \in \mathbb{R}^2\) with \(0 < h(-1), 0 < h(1)\) and

\[
h(-1) + h(1) = \frac{1}{2\zeta(2)+2} \left( \zeta(2) + \sum_{k=2}^{\infty} \frac{1}{k^3} \cdot \frac{1}{2^{k-1}} \right)
\]

produces a positive \(P_G\)-harmonic function.

**Remark.** The functions \(\phi\) and \(\psi\) of the last two examples are both extensions of \(\phi\) from the shift \(X_H\) to the shift \(X_G\) and \(P_G(\phi) = P_G(\psi) = P_G(\phi)\). The existence of a conformal measure is, therefore, not purely a property of the shift \(X_G\) but depends on the (transient) potential.
References


Vita
Van Cyr

Born

April 3, 1983 in Rochester, New York, USA

Education

The Pennsylvania State University
Ph.D., Mathematics, 2010; Advisor: Omri Sarig

University at Buffalo, The State University of New York
B.S., Mathematics, summa cum laude, May 2005

Academic Employment

Boas Assistant Professor (Northwestern University) 2010-2013

Papers
