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LOOP GRAVITY: AN APPLICATION AND AN EXTENSION

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Abstract

In this thesis we address two issues in the area of loop quantum gravity. The first concerns the semiclassical limit in loop quantum cosmology via the use of so-called effective equations. In loop quantum cosmology the quantum dynamics is well understood. We can approximate the full quantum dynamics in the infinite dimensional Hilbert space by projecting it on a finite dimensional submanifold thereof, spanned by suitably chosen semiclassical states. This submanifold is isomorphic with the classical phase space and the projected dynamical flow provides effective equations incorporating the leading quantum corrections to the classical equations of motion. Numerical work has been done in the full theory using quantum states which are semiclassical at late times. These states follow the classical trajectory until the density is on the order of 1% of the Planck density then deviate strongly from the classical trajectory. The effective equations we obtain reproduce this behavior to surprising accuracy.

The second issue concerns generalizations of the classical action which is the starting point for loop quantum gravity. In loop quantum gravity one begins with the Einstein-Hilbert action, modified by the addition of the so-called Holst term. Classically, this term does not affect the equations of motion, but it leads to a well-known quantization ambiguity in the quantum theory parametrized by the Barbero–Immirzi parameter, which rescales the eigenvalues of the area and volume operators. We consider the theory obtained by promoting the Barbero–Immirzi parameter to a field. The resulting theory, called Modified Holst Gravity,is equivalent to General Relativity coupled to a pseudo-scalar field. However, this theory turns out to have an unconventional kinetic term for the Barbero-Immirzi field and a rather unnatural coupling with fermions. We then propose a further generalization of the Holst action, which we call Modified Nieh–Yan Gravity, which yields a theory of gravity and matter with a more natural coupling to the Barbero–Immirzi field. We conclude by commenting on possible implications for cosmology,

induced by the existence of this new pseudo-scalar field.

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Acknowledgments

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To my parents.

Chapter 1

Introduction

Albert Einstein proposed his general theory of relativity in 1915 [38]. The theory makes dramatic new predictions:the big bang and the subsequent evolution of the universe in cosmology, geometry of black holes, and existence of gravitational waves [82]. There is much evidence for the first two. The Hulse-Taylor pulsar provides indirect evidence for the third and with the future launching of Advanced LIGO and later LISA, direct evidence for the third is also soon expected. However, the general theory of relativity also showcases its limitations since the theory breaks down at singularities. Singularities have been shown to be a generic feature of solutions to the theory [46]. But the singularities occur in regimes where quantum effects are expected to dominate, e.g. at the high densities near the big bang and the strong curvatures inside a black hole. Thus, a quantum theory of gravity is expected to be necessary to describe the true physics in such regions. This problem is still open and candidate theories have been put forward. Arguably the most notable candidates are loop quantum gravity (LQG) [14, 73, 80] and string theory [69, 70, 86]. This thesis is focused on ideas and developments in the former.

Loop quantum gravity is a background independent, non-perturbative candidate for a quantum theory of gravity. Although the program is still incomplete, the theory has had many successes some of which we list here. One is the so-called 'LOST theorem' on uniqueness of the holonomy-flux algebra used for quantization [52]. Furthermore, much work has been done in analysis of symmetry reduced models such as FRW [16, 17, 18], Bianchi I and II [23, 22], the Gowdy models [54] and black holes [43]. Lastly there are the well known results on computations of the black hole entropy [10, 9].

In spite of these successes there, remain many open issues. In this thesis we address two of these and briefly comment on a third. First, making contact with the semiclassical limit of the theory, at least in the context of loop quantum cosmology, via effective equations. The second is to attempt to make contact with observational cosmology via an extension of the Holst gravity framework by promoting the Barbero-Immirzi parameter to a scalar field. Finally, at the end of this introductory chapter we briefly discuss black hole evaporation and the information loss problem for 2 dimensional black holes. However this analysis is not a part of the thesis, because it involves concepts and techniques that are very different from those used in the analysis of the other two issues.

1.1 Loop Quantum Cosmology corrections to the Friedmann equations

An open issue in loop quantum gravity is that of the semiclassical limit. That is, is there a precise sense in which the quantum theory has solutions that closely approximate solutions to the classical theory in an appropriate limit? A priori, this need not be the case. For instance, currently it is not known if the full theory admits states that can be said to closely approximate the Schwarzchild spacetime, for instance. Preliminary, work has been done on so-called weave states [19] but the problem is still largely open.

However, in the context of loop quantum cosmology concrete results can be obtained because the symmetry reduction in the passage to cosmological models brings out enormous technical simplifications. There, one can use the geometric formulation of quantum mechanics to show that there are semiclassical states in the quantum theory that closely approximate solutions in the classical theory. More precisely, one is able to use the geometric quantum mechanics framework to embed the classical phase space into the Hilbert space of the quantum theory. Furthermore, one is able to obtain a consistent set of effective equations incorporating the leading quantum corrections to the classical equations. (This has been previously shown for a dust-filled, spatially flat, Friedmann-Robertson Walker (FRW) universe [85] and for a spatially flat, FRW universe with a massless scalar field (unpublished) [77] in the so-called μ_0 scheme [17].)

By comparing with the numerical work [17] done previously in the full loop quantum cosmology theory, one can show that these leading quantum corrections to the classical equations of motion are *sufficient* to capture the physics of the full quantum dynamics. That is, one can start with a semiclassical state at late times in the the full theory and evolve it backwards using the full quantum equations. It follows the trajectory given by the classical solution until the scalar field density is on the order of 1% Planck density ρ_p where it then follows the trajectory given by the effective equations *instead of* proceeding to the classical singularity.

Obtaining these effective equations from the geometric quantum mechanics framework has several advantages. First, they provide a direct route to the effective equations directly from the Hamiltonian constraint. Secondly, one can obtain the effective equations without having to deparameterize the theory by choosing an internal clock. However, there are also some limitations. First, as we will see in chapter 2, the route to the these effective equations from geometric quantum mechanics involves assumptions on the form of the state. The 0th order choice one can make is to choose states from the family of Gaussian coherent states. Then one has to make assumptions on the various physical parameters in the state. For the late-time, large-volume, semiclassical analysis in chapter 2, this is be sufficient but these assumptions do not hold in highly quantum regimes in general, or in more complicated models. Secondly, because of the choice of state made here, one is neglecting the higher order moments of the state. One can loosen the assumption on the form of state by specifying a state that is not Gaussian but that includes these higher moments. However, then the analysis in chapter 2 becomes much more difficult very quickly.

An alternative method to obtaining effective equations is presented in [30]. There, they make no assumption on the form of the state but instead characterize it in terms of its infinite moments. The moments then gain their own equations of motion which must be solved simultaneously by, e.g. truncating them to finite order. However, physical meaning of the truncations is not transparent and the results are again tied to the choice of the initial state. The two methods are complementary. The status can be loosely explained through an analogy to atomic physics: the second method is similar to the development of a systematic perturbation series, while the first is similar to the "variational principle" which, at the outset requires an input that has to be guessed from one's experience, but which can lead to surprisingly accurate predictions.

1.2 Barbero-Immirzi parameter as a scalar field

The second part of this thesis concerns with the generalization of the Barbero-Immirzi parameter in the Holst action to a dynamical scalar field. The Holst action is the classical point of departure for loop quantum gravity. It consists of the standard Einstein-Hilbert contribution with a new Holst term who's strength is controlled by the value of the Barbero-Immirzi parameter. Classically, the Holst term does not modify the equations of motion as its contribution vanishes due to the Bianchi identities; the Holst action just reproduces Einstein gravity. However, it induces a well-known quantization ambiguity in the quantum theory through an undetermined multiplicative factor in the spectra of geometrical operators.

"Scalarization" of the Barbero-Immirzi parameter has two primary motivations. The first is the possible variation of the universal physical constants that describe our universe. The most most famous example is arguably the Jordan-Brans-Dicke theory which can be thought of as an alternative theory of gravity where Newton's constant,G, is replaced by a dynamical scalar field [50, 31]. Similarly, one can replace the Barbero-Immirzi parameter. Then, in the variational principle, one must vary the action also with respect to this new spacetime field. The new field then acquires its own equations of motion, and it interacts with the gravitational field thus modifying the usual dynamics generated by the Einstein equations. One can then study solutions to this theory and possibly compare them with observations to place bounds on the coupling between this new field and other matter.

It turns out that this new theory of gravity, which we call Modified Holst Gravity, has a rather complicated and unnatural coupling between gravity and the Barbero-Immirzi field. At first one may think that, because of this unnatural coupling between gravity and the kinetic term for the Barbero-Immirzi field, the so-called k-inflation scenario of [4] may be realized. We discuss why this fails to be true. The unnaturalness of the coupling becomes even more apparent when one includes fermions. The reason is because a spacetime dependent Barbero-Immirzi field yields a theory of gravity with non-vanishing torsion. The coupling to fermions also generates non-zero torsion. Thus, in this modified theory of gravity the second Cartan structure equation must be solved and one must pullback the action to the space of solutions to the second Cartan structure equation to obtain the effective theory.

The main problem with the Modified Holst Gravity theory is that it generates non-zero torsion with or without fermions. However, there is an important term quadratic in torsion that is neglected in Modified Holst Gravity. This term was originally neglected because the Holst term by itself is not topological but vanishes because of the Bianchi identity, which is modified in the presence of torsion. Thus in a theory of gravity with torsion, the Holst term needs to be generalized. We propose that it be generalized to the Nieh–Yan density which has the properties ascribable to a topological term. This analysis is completed in chapter 4 where we include fermions and argue that Modified Nieh–Yan Gravity is the more natural choice for a theory of gravity containing a Barbero-Immirzi field.

We conclude by analyzing the equations of motion for this new theory, which impose on us that this new scalar field must be a pseudoscalar. Furthermore analysis of the effective theory shows that there is a chiral anomaly coupling this new pseudoscalar field to photons and to a Chern-Simons gravity term. This has possible observable implications for this new pseudoscalar field because bounds on such couplings have been discussed in the literature [71, 1, 76]. In particular the coupling of pseudoscalars to photons is severely constrained by the cosmic microwave background. However, using the fact that the coupling of the Barbero-Immirzi field with photons involves the Planck scale, we show that it lies within the bounds presented in [71]. Lastly, we discuss the interaction of the Barbero-Immirzi field with the Chern-Simons gravity term. Such a coupling is of physical interest because such a coupling can perturb [1] and generate [76] gravitational waves.

1.3 The issue of information loss and CGHS black holes

Although, not included in this thesis. The author has contributed to work done on resolving the information loss puzzle in 2 dimensional black holes discovered by Callan-Giddings-Harvey-Strominger (CGHS)[33].

Ever since Hawking showed that black holes are not black but that they emit so-called thermal Hawking radiation and evaporate [45] the information loss puzzle has been with us. Indeed, one may consider a scalar field and prescribe a pure state at past null infinity. This state then evolves and creates a geometry describing the collapse to a black hole. However, the state at future null infinity would appear to be a mixed state since the Hawking radiation is completely thermal. Thus a pure state evolves to a mixed state and information appears to be lost. Many attempts have been made to try to resolve this problem in the full 3+1 dimensional theory but it remains an open issue.

However, the 1+1 dimensional CGHS black holes [33] are mathematically much simpler than the 3+1 dimensional case but retain most of the conceptual issues. Much work has been done on these analytically and numerically [53, 68] over the years but the issue of information loss was never satisfactorily resolved, because even in these approaches the spacetime still contains a spacelike singularity. Indeed, one may prescribe a pure state at past null infinity and attempt to evolve it to future null infinity via either analytical or numerical methods. However, in any spacetime with a singularity, to obtain a state at future null infinity one has to trace over the part of the state that "fell into the singularity". Thus, in any approach where the classical singularity remains, it would appear that information would be lost.

Nonetheless, concrete results can be proven using non-perturbative ideas [21] which we now summarize. In the CGHS model one can show that there is only one true degree of freedom in the scalar field f that collapses to form the black hole. All of the other fields in the theory are determined via the equations of motion. In two dimensions the wave equation for f naturally decomposes into left-moving and right-moving modes. Thus one has the product Fock space associated with the left-movers and the right-movers. One then promotes the equations of motion to operator equations on this Fock space. The main result of [21] is to show that the full *quantum space time* thus obtained is well-defined even beyond the classically singular region. Thus spacetime doesn't end at the singularity but extends beyond it. Thus in a precise sense, pure states at past null infinity evolve to pure states at future null infinity. That is, *information is not lost*.

While this issue is important for fundamental physics, it requires conceptual structures and technical tools that are completely different from those used in chapters 2-4. To maintain an overall coherence, and balance we decided not to include this contribution in the detailed discussion.

Chapter 2

Loop Quantum Cosmology corrections to the Friedmann equations

2.1 Introduction

Loop Quantum Gravity (see [14, 73, 80] for reviews) is a nonperturbative approach to the problem of quantizing gravity[1]. An open problem is that of the semiclassical limit, i.e. are there solutions to LQG which closely approximate solutions to the classical Einstein's equations? Although this remains an open problem, concrete results can be achieved in the context of Loop Quantum Cosmology (for recent review see [28, 8]) in the spatially flat case. Loop Quantum Cosmology is a symmetry reduction of LQG, i.e. a quantization of a symmetry reduced sector of general relativity. There, it is the case that one can indeed find semiclassical solutions in the quantum theory that closely approximate solutions to the classical Einstein equations, namely, the Friedmann equations at late times. Our goal is to to not just find semiclassical solutions but we would like to be able to go further and find effective equations incorporating the leading quantum corrections.

It is a common misconception that canonically quantizing general relativity would just reproduce Einstein's equations without any modifications. The classi-

[1]This chapter is based on the work by the author in [78]

cal equations are in fact modified by quantum corrections. Using the geometric quantum mechanics framework [75, 20] this has been previously shown for a dust-filled, spatially flat Friedmann universe [85] in the so-called μ_0 framework (see reference [18]). In this chapter we work in the so-called $\bar{\mu}$ framework and show that this is also the case for a spatially flat Friedmann universe with a free scalar field.

We can find effective equations arising from the geometric quantum mechanics framework [75, 20] ('effective equations' for short). These semiclassical states follow the classical trajectory until the scalar field density is on the order of 1% of the Planck density where deviations from the classical trajectory start to occur. Then there are major deviations from the classical theory. However, states which are semiclassical at late times continue to remain sharply peaked *but* on the trajectory given by the 'effective equations'. Thus, we can approximate the full quantum dynamics in the (infinite dimensional) Hilbert space by a system of 'effective equations', incorporating the leading quantum corrections, on a finite dimensional submanifold thereof isomorphic to the classical phase space. Furthermore, we know from comparison with the numerical work that the leading corrections are sufficient to capture essential features of the full quantum dynamics [16, 17, 18].

We begin by briefly summarizing the classical theory in 2.2. We then provide an overview of the framework for effective theories that we use to obtain the 'effective equations' in 2.3. Then in the quantum theory, 2.4 we choose a family of candidate semiclassical states, verify that they are indeed sharply peaked, and compute the effective Hamiltonian constraint. We obtain the 'effective equations' in 2.5 and show that they are self-consistent to within our order of approximation. Finally, we discuss how much this approximation can be trusted near the bounce point in the Appendix and conclude.

Additionally, in this chapter we use the choice of convention: $l_p^2 = G\hbar$, $c_{light} = 1$, $\kappa = 8\pi G$, and FRW parameter $k = 0$ since we work in the spatially flat case.

2.2 Classical Theory

In the classical theory, the phase space Γ consists of pairs (A_a^i, E_i^a) where A_a^i is an $SU(2)$ connection and the E_i^a its canonically conjugate field. Due to isotropy and homogeneity we have a fiducial background triad ${}^oe^a_i$ and connection ${}^o\omega^i_a$. Additionally, we may fix a fiducial cell with volume ${}^oV$ and restrict our calculations to that cell. There, we can write A^i_a and E^a_i as [11]

$$A^i_a = c\, {}^oV^{-\frac{1}{3}}\, {}^o\omega^i_a \tag{2.1}$$

$$E^a_i = p\sqrt{{}^oq}\, {}^oV^{-\frac{2}{3}}\, {}^oe^a_i \tag{2.2}$$

where ${}^oq$ is the determinant of the fiducial metric. Due to homogeneity and isotropy all the nontrivial information in A^i_a and E^a_i is contained in the variables c and p. The Hamiltonian constraint in these variables for a Friedmann universe with a free scalar field ϕ, in the regime $p > 0$ is given by,

$$C = -\frac{3}{\kappa\gamma^2}c^2p^{\frac{1}{2}} + \frac{1}{2}\frac{p_\phi^2}{p^{\frac{3}{2}}} = 0 \tag{2.3}$$

where p_ϕ is the conjugate momentum of the massless scalar field ϕ.

For convenience we switch to variables given by $\beta = c/\sqrt{p}$ and $V = p^{\frac{3}{2}}$ in which the Hamiltonian is given by

$$C = -\frac{3}{\kappa\gamma^2}\beta^2V + \frac{1}{2}\frac{p_\phi^2}{V} = 0 \tag{2.4}$$

These new canonically conjugate variables are related to the old geometrodynamics variables via

$$\beta = \gamma\frac{\dot{a}}{a}, \tag{2.5}$$

$$V = a^3, \tag{2.6}$$

With the inclusion of a scalar field, the symplectic structure now has an extra part,

$$\Omega = \frac{2}{\kappa\gamma}d\beta \wedge dV + d\phi \wedge dp_\phi, \tag{2.7}$$

where the scalar field is ϕ and it's conjugate momentum is p_ϕ. The Poisson bracket on the phase space is given by,

$$\{f,g\} = \frac{\kappa\gamma}{2}\left(\frac{\partial f}{\partial\beta}\frac{\partial g}{\partial V} - \frac{\partial g}{\partial\beta}\frac{\partial f}{\partial V}\right) + \frac{\partial f}{\partial\phi}\frac{\partial g}{p_\phi} - \frac{\partial g}{\partial\phi}\frac{\partial f}{\partial p_\phi}. \tag{2.8}$$

where the phase space Γ consists of all possible points $\{\beta, V, \phi, p_\phi\}$. The allowed pairs of points in Γ are those satisfying the Hamiltonian constraint

$$C(\beta, V, \phi, p_\phi) = 0. \tag{2.9}$$

Recall, that for a function that depends on the canonical variables, its time dependence is given by its Poisson bracket with the Hamiltonian. Thus the classical equations of motion are given by:

$$\dot{\beta} = \{\beta, C\} = -\frac{3}{2}\frac{\beta^2}{\gamma} - \frac{\kappa\gamma}{4}\frac{p_\phi^2}{V^2}, \tag{2.10}$$

$$\dot{V} = \{p, C\} = 3\frac{\beta}{\gamma}V, \tag{2.11}$$

$$\dot{\phi} = \{\phi, C\} = \frac{p_\phi}{V}, \tag{2.12}$$

$$\dot{p_\phi} = \{p_\phi, C\} = 0. \tag{2.13}$$

We can verify that (2.10) and (2.11) are equivalent to the Friedmann equations for a free scalar field when written in terms of ordinary ADM variables. Looking first at (2.11) we see that it gives that,

$$\beta = \gamma\frac{\dot{a}}{a}. \tag{2.14}$$

Putting in this expression in (2.9) we obtain,

$$-\frac{3}{\kappa}\dot{a}^2 a + \frac{1}{2}\frac{p_\phi^2}{a^3} = 0, \tag{2.15}$$

which can be rewritten as the Friedmann's equations,

$$3\frac{\dot{a}^2}{a^2} = \kappa\left(\frac{1}{2}\frac{p_\phi^2}{a^3}\right)\frac{1}{a^3}. \tag{2.16}$$

or in terms of the Hubble parameter and the scalar field density,

$$H^2 = \frac{\kappa}{3}\rho \tag{2.17}$$

Now putting in the equation for β into (2.9) we can obtain the Raychaudhuri equation,

$$3\frac{\ddot{a}}{a} = -2\kappa\rho \tag{2.18}$$

Now (2.12) and (2.13) give us, respectively,

$$\dot{\phi} = \frac{p_\phi}{p^{\frac{3}{2}}}, \tag{2.19}$$

and

$$p_\phi = \text{const}, \tag{2.20}$$

since C does not depend on ϕ for a free scalar field. These are just the equations of motion for a free scalar field. It is to the Hamiltonian constraint and these 4 equations, i.e. (2.10), (2.11), (2.12), (2.13), that we wish to find the corrections due to quantum gravity effects.

2.3 Framework for Effective Theories

Fortunately, there is the so-called geometric quantum mechanics framework which gives us a framework in which we can obtain these effective equations. In this section we briefly review the framework for effective theories. A more through review may be found in [85] and [75]. This framework is especially suitable because it provides a direct route to the effective equations from the Hamiltonian constraint without having to deparameterize the theory. That is, in background independent theories there is no canonical notion of time so one usually deparameterizes the theory by choosing one of the fields as an internal clock and then proceeds by considering the dynamics of the other fields with respect to the field serving as a clock. In particular, this model is deparamaterized and analyzed in [16, 17, 18] using the scalar field ϕ as an internal clock.

We take a brief detour and present the main idea behind the procedure for the simpler example of a particle moving in a potential along the real line. The proof that quantum mechanics has the correct semiclassical limit is usually based on an appeal to Ehrenfest's theorem, which is usually expressed in the form, for a particle in a potential,

$$m\frac{\partial^2\langle x\rangle}{\partial t^2} = -\left\langle \frac{\partial V(x)}{\partial x}\right\rangle. \tag{2.21}$$

However, this holds for all states, not just semiclassical ones. To recover classical equations for the expectation values $\langle x\rangle$ we would like to be able to pull the derivative outside of the expectation value,

$$m\frac{\partial^2\langle x\rangle}{\partial t^2} \simeq -\frac{\partial V(\langle x\rangle)}{\partial\langle x\rangle}. \tag{2.22}$$

Where in pulling the derivative outside, we have replaced $\frac{\partial}{\partial x}$ with $\frac{\partial}{\partial\langle x\rangle}$, since after taking the expectation value the right hand side is now not a function of x but of $\langle x\rangle$. Having done this we have obtained an approximate equation for the expectation values. However, this equation does not hold exactly but there are quantum corrections to the right hand side. We would like to be able to express the corrections to the right hand side of the (2.22) in terms of a corrected potential (or in our cosmological case, correction terms due to quantum geometry effects),

$$m\frac{\partial^2\langle x\rangle}{\partial t^2} = -\frac{\partial}{\partial\langle x\rangle}\left[V(\langle x\rangle) + \delta V(\langle x\rangle)\right] + \ldots. \tag{2.23}$$

That is, we are looking for an equation of motion satisfied by the expectation values which includes quantum corrections to the classical equations of motion. A priori, it is not certain if we can do this. However, the geometrical formulation of quantum mechanics provides a framework in which we can answer this question. Therefore, we quickly review this framework. A more thorough review can be found in [75].

In the quantum theory quantum states are represented by elements of a Hilbert space $\mathcal{H}$. More precisely, the quantum phase space consists of rays in $\mathcal{H}$. The space $\mathcal{H}$ can be made into a symplectic space via its inner product, which can be decomposed into its real and imaginary parts,

$$\langle\phi|\psi\rangle = \frac{1}{2\hbar}G(\phi,\psi) + \frac{i}{2\hbar}\Omega(\phi,\psi), \tag{2.24}$$

where G gives a Riemannian metric and Ω a symplectic form on $\mathcal{H}$. Given a function f on the phase space, one can construct the Hamiltonian vector field associated with f by using the symplectic form,

$$X^a_{(f)} = \Omega^{ab}\nabla_b f \tag{2.25}$$

It turns out that the flow generated on the phase space by the Hamiltonian vector field $X^a_{(f)}$ corresponds to Schrodinger evolution in the quantum theory generated by $\hat{f}$ [75].

Taking expectation values of the operators corresponding to the canonical variables provides a natural projection, π, from states in the Hilbert space to points in the classical phase space. Thus, the Hilbert space, $\mathcal{H}$ is naturally viewed as a fiber bundle over the classical phase space Γ (see Fig 2.1),

$$\pi : \mathcal{H} \to \Gamma. \tag{2.26}$$

Because of the fiber bundle structure, the classical phase space can be viewed as a cross section of the Hilbert space (see Fig 2.2).

We can use this symplectic form to define a notion of horizontal vectors in $\mathcal{H}$. Namely, vertical vectors are vectors whose components are in the directions in which the expectation values don't change and horizontal vectors are vectors orthogonal to the vertical vectors. It turns out that one has to use Ω rather than G to define orthogonality. That is, two vectors, ϕ and ψ, are orthogonal if

$$\Omega(\phi,\psi) = 0. \tag{2.27}$$

To summarize, Ω gives us a notion of horizontal vectors which we use to construct horizontal sections of the Hilbert space.

Because of the fiber bundle structure, *any* horizontal section can be identified with the classical phase space. At the kinematical level there is no natural section

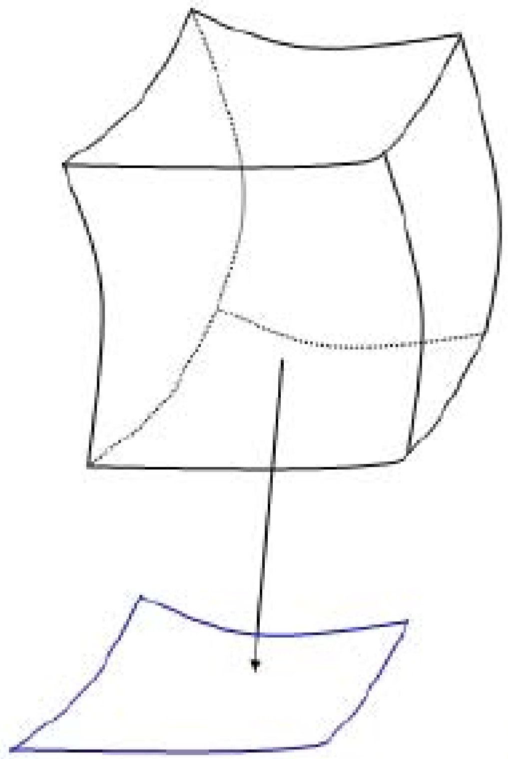

Figure 2.1. A schematic showing the fiber bundle structure of $\mathcal{H}$ (in black) with the projection to Γ (in blue)

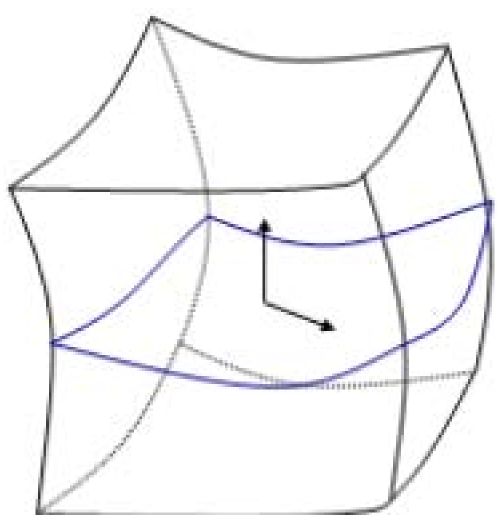

Figure 2.2. A schematic showing the embedding of Γ (in blue) in $\mathcal{H}$ (in black)

for us to choose. However, later when we consider dynamics in section 2.5 we can look for a natural section that is approximately preserved by the flow of the Hamiltonian constraint in a precise sense.

A priori, we have no way of knowing whether such a section exists. However, *if* we can find such a section then the quantum dynamics on such a section can be expressed in terms of an effective Hamiltonian which is simply the expectation value of the quantum Hamiltonian operator [75]. The expectation value yields the classical term as the leading term and gets corrections due to quantum effects in the subleading terms. This is the key idea behind our calculation.

This can be done exactly for, e.g., the harmonic oscillator and approximately for several other physically interesting systems, including a dust-filled Friedmann universe [85]. The main result of this chapter is proving that this can also be done approximately for a Friedmann universe with a free scalar field and thus obtaining effective equations for this model. In the remainder of this work we will call these 'effective equations' because they are not effective equations in the traditional sense. These are on a different footing from traditional effective equations since there is no precise definition of approximately horizontal sections. Nonetheless, we show that we can obtain these 'effective equations' in a well-controlled approximation

2.4 Quantum Theory

Recall that in the full theory the elementary variables are the holonomies of the connection and the electric fluxes. Recall that in loop quantum cosmology there exists no operator $\hat{c}$ [11] corresponding to the connection, and thus $\hat{\beta}$ also by extension. However, the holonomy operator, $\exp(\frac{i}{2}\bar{\mu}c) = \exp(\frac{i}{2}\sqrt{\Delta}\beta)$ does exist. Therefore in the quantum theory we work with the algebra generated by $\exp(\frac{i}{2}\sqrt{\Delta}\beta)$ and $\hat{V}$

2.4.1 Coherent State

In the previous section, we saw that the taking of expectation values provides a natural projection from the Hilbert space to the classical phase space. These expectation values can be taken in any state. However, for our investigation of the semiclassical limit it is natural to choose a semiclassical state. Simplest candidates are the Gaussian coherent states. They are natural because we can choose them to be sharply peaked at classical values of the canonical variables, i.e. with small spread in both canonically conjugate variables. Let v and v' denote the parameters proportional to the eigenvalues of the volume operator, related to the volume eigenvalues V and V' via $V = (\frac{8\pi\gamma}{6})^{\frac{3}{2}}\frac{l_p^3}{K}v$ and $V' = (\frac{8\pi\gamma}{6})^{\frac{3}{2}}\frac{l_p^3}{K}v'$ where $K = \frac{2\sqrt{2}}{3\sqrt{3\sqrt{3}}}$. Then a Gaussian coherent state $(\psi|$, with Gaussian spreads ϵ and ϵ_ϕ in the gravitational sector and scalar field sector of the state, peaked at some classical values $\beta', V', \phi', p'_\phi$ is given by,

$$\begin{aligned}(\psi_{\beta',V';\phi',p'_\phi}| &= \int dp_\phi \sum_v e^{-\frac{1}{2}\epsilon^2(v-v')^2} e^{\frac{i}{2}\sqrt{\Delta}\beta'(v-v')} \\ &\times e^{-\frac{1}{2}\epsilon_\phi^2(p_\phi-p'_\phi)^2} e^{-i\phi'(p_\phi-p'_\phi)}(v;p_\phi| \\ &=: \int dp_\phi \sum_v \overline{\psi_n}(p_\phi)(v;p_\phi|. \qquad (2.28)\end{aligned}$$

As usual, ψ is defined on a lattice and the summation index v runs over the integers. It is important to note that the spread ϵ is not constant but is a function of the phase space point. We will provide the reason and more details on the functional dependence in the next section. Recall that solutions to the constraints do not lie in the kinematical Hilbert space, but rather in its algebraic dual [15]. Physical states, such as the semiclassical state given in (2.28) should thus lie in the dual space and thus we write it as a so-called 'bra' state $(\psi|$. At first it would seem that working in the dual space would be unmanageable but it does not pose computational difficulties. However, because one fortunately has the "shadow state framework" to carry out calculations [12, 13].

We will use $(\psi|$ as our semiclassical state and calculate all of our expectation values in this state. The interpretation of the basis ket is that the universe in the state $|v; p_\phi\rangle$ has physical volume v in Planckian units and scalar field momentum p_ϕ. Our choice for $(\psi|$ as a Gaussian coherent state is not the most general but, rather, is the 'simplest' choice one can make to obtain these 'effective equations' in a late-time, large-volume approximation. For more general considerations it is possible to characterize the state in terms of all its 'moments' as in [30]. The higher moments acquire their own equations of motion which must be solved simultaneously with the equations of motion for the canonical variables.

2.4.2 Operators in the Quantum Theory

In order to take expectation values, we need to define the operators in the quantum theory corresponding to the canonical variables. We now construct an approximate operator for β in terms of exponentiated β variables in order to verify that the state Ψ is indeed sharply peaked at classical value of β'. Our point of departure is the classical expression

$$\beta \simeq \frac{1}{i\sqrt{\Delta}}\left(e^{\frac{i}{2}\sqrt{\Delta}\beta} - e^{-\frac{i}{2}\sqrt{\Delta}\beta}\right) \tag{2.29}$$

which is exact in the limit $\sqrt{\Delta}\beta \to 0$. However, our experience in the full theory tells us that in the quantum theory we should not be taking this limit to 0 but to the area gap $\Delta = 4\sqrt{3}\pi\gamma l_p^2$. So in our quantum theory we take

$$\hat{\beta}_\Delta = \frac{1}{i\sqrt{\Delta}}\left(\widehat{e^{\frac{i}{2}\sqrt{\Delta}\beta}} - \widehat{e^{-\frac{i}{2}\sqrt{\Delta}\beta}}\right). \tag{2.30}$$

Thus, our operator $\hat{\beta}_\Delta$ agrees approximately with the classical β in the regime $\sqrt{\Delta}\beta \ll 1$. The choice for $\hat{\beta}$ is not unique but this is the simplest choice which is self adjoint (others have been considered in the literature, e.g. [62]). For the rest of this chapter we use this as our approximate β operator and drop the subscript Δ. Its action on our basis kets is given by,

$$\hat{\beta}|v;p_\phi\rangle \;=\; \frac{1}{i\sqrt{\Delta}}\left(|v+1;p_\phi\rangle - |v-1;p_\phi\rangle\right). \tag{2.31}$$

The action of the other operators is straightforward, $\hat{V}$ and $\hat{p}_\phi$ act by multiplication,

$$\hat{V}|v;p_\phi\rangle \;=\; \left(\frac{8\pi\gamma}{6}\right)^{\frac{3}{2}}\frac{l_p^3}{K}v|v;p_\phi\rangle, \tag{2.32}$$

$$\hat{p}_\phi|v;p_\phi\rangle \;=\; p_\phi|v;p_\phi\rangle, \tag{2.33}$$

and since we work in the p_ϕ representation ϕ acts by differentiation

$$\hat{\phi} = -\frac{\hbar}{i}\frac{\partial}{\partial p_\phi}. \tag{2.34}$$

2.4.3 Restrictions on Parameters in Coherent State

Before we move to computation we impose some physically motivated restrictions on the parameters appearing in the coherent state (2.28). The first pair of restrictions,

$$v' \gg 1, \tag{2.35}$$

$$\sqrt{\Delta}\beta' \ll 1, \tag{2.36}$$

corresponds to late times $V' \gg l_p^3$ and $\dot{a} \ll 1$ respectively. Namely, that the scale factor be much larger than the Planck length and that the rate of change of the scale factor be much smaller than the speed of light. We will see later that (2.36) also holds well even at early times.

The next pair of restrictions demands that the spreads in $\hat{V}$ and $\hat{\beta}$ be small, $\frac{\Delta V}{V} \ll 1$ and $\frac{\Delta \beta}{\beta} \ll 1$, or equivalently:

$$v'\epsilon \gg 1, \tag{2.37}$$

$$\epsilon \ll \sqrt{\Delta}\beta'. \tag{2.38}$$

The last pair of restrictions on parameters demands that the spreads in ϕ and p_ϕ are small, $\frac{\Delta \phi}{\phi} \ll 1$ and $\frac{\Delta p_\phi}{p_\phi} \ll 1$, or equivalently:

$$\phi \gg \epsilon_\phi, \tag{2.39}$$

$$p_\phi \epsilon_\phi \gg 1. \tag{2.40}$$

We use these physically motivated restrictions in our calculations in the remainder of this paper. We now return to showing that ψ is sharply peaked.

2.4.4 Verifying that ψ is Sharply Peaked

Now that we have a candidate semiclassical state, we can calculate the expectation values of the canonical variables and verify that the state ψ is sharply peaked at classical values of the canonical variables.

Calculating $\langle\hat{\beta}\rangle$ we obtain,

$$\langle\hat{\beta}\rangle = \frac{2}{\sqrt{\Delta}} e^{-\frac{1}{4}\epsilon^2} \sin(\frac{1}{2}\sqrt{\Delta}\beta'). \tag{2.41}$$

Thus we must take $\epsilon \ll 1$ for $\langle\hat{\beta}\rangle$ to approximately agree with the classical β. Similarly,

$$\langle\hat{\beta}^2\rangle = \frac{2}{\Delta}\left[1 - e^{-\epsilon^2}\cos(\sqrt{\Delta}\beta')\right]. \tag{2.42}$$

Thus,

$$\Delta\beta^2 \simeq \frac{2\epsilon^2}{\Delta}, \tag{2.43}$$

where to get the last line we have used $\cos(\sqrt{\Delta}\beta') \simeq 1$ and $\epsilon \ll 1$. Now we obtain the expectation value and spread of V,

$$\langle\hat{V}\rangle \simeq V', \tag{2.44}$$

where to obtain the last line we have done Poisson resummation on the sum over n. Similarly,

$$\langle\hat{V}^2\rangle = V'^2 + \frac{1}{2\epsilon^2}\frac{l_p^6}{K^2}\left(\frac{8\pi\gamma}{6}\right)^3, \tag{2.45}$$

$$\Delta V^2 = \frac{1}{2\epsilon^2}\frac{l_p^6}{K^2}\left(\frac{8\pi\gamma}{6}\right)^3. \tag{2.46}$$

Thus, up to the approximation used to arrive at (2.43) the product of uncertainties is the minimum possible,

$$\Delta\beta\Delta V \simeq \frac{\kappa\gamma}{2}\frac{\hbar}{2}. \tag{2.47}$$

We make a brief detour to show that we can satisfy both of the conditions (2.37) and (2.38) on the phase space if the width ϵ of the coherent state is properly chosen as a function of the phase space point. Notice that (2.37) requires that ϵ satisfy $v'\epsilon \gg 1$ and (2.38) requires that $\epsilon \ll \sqrt{\Delta}\beta$, but $\sqrt{\Delta}\beta \ll 1$ so it appears that there is a tension between these 2 conditions. Nonetheless, if p'_ϕ is large, which is reasonable for a universe that would increase to macroscopic size then we can choose $\epsilon(v')$ such that both of the conditions $\frac{\Delta V}{V}$ and $\frac{\Delta\beta}{\beta}$ are satisfied. Looking at the relative uncertainties, we get

$$\frac{\Delta V}{V} = \frac{1}{\sqrt{2}\epsilon V'}\frac{l_p^3}{K}\left(\frac{8\pi\gamma}{6}\right)^3 \ll 1 \tag{2.48}$$

$$\frac{\Delta\beta}{\beta} = \frac{\sqrt{2}\epsilon}{\Delta\beta'} \ll 1 \tag{2.49}$$

The first condition shows that we need to choose ϵ as a function of the phase phase variable V' s.t. $\epsilon = \frac{\lambda}{v'}$ (for $\lambda \ll (\frac{8\pi\gamma}{6})^3 K^{-1}$) in order to satisfy the inequality. As for the 2nd condition we use the the classical form for $\beta^2 = \frac{\kappa\gamma^2}{3}\rho$, where $\rho = \frac{p_\phi^2}{2V^2}$ for a scalar field, to show that the inequality is equivalent to taking $\sqrt{3\pi}\hbar \ll p_\phi$. For a universe that grows to macroscopic size such that p_ϕ satisfies this then we can choose λ in such a way to satisfy $\lambda \ll (\frac{8\pi\gamma}{6})^3 K^{-1}$, e.g. $\lambda = 70, p_\phi = 5000$. Therefore we can satisfy these two conditions simultaneously.

Continuing, taking the expectation values of $\hat{\phi}$ and $\hat{p}_\phi$ yields,

$$\langle\phi\rangle = \phi', \tag{2.50}$$

and

$$\langle\phi^2\rangle = \phi'^2 + \frac{1}{2}\epsilon_\phi^2, \tag{2.51}$$

so

$$\Delta\phi = \frac{\epsilon_\phi}{\sqrt{2}}. \tag{2.52}$$

Thus, to have $\frac{\Delta\phi}{\phi} \ll 1$ we must have $\epsilon_\phi \ll \phi$. Similarly for p_ϕ and p_ϕ^2,

$$\langle p_\phi\rangle = p'_\phi, \tag{2.53}$$

$$\langle p_\phi^2\rangle = p_\phi'^2 + \frac{1}{2\epsilon_\phi^2}, \tag{2.54}$$

therefore we have for Δp_ϕ^2,

$$\Delta p_\phi = \frac{1}{\sqrt{2}\epsilon_\phi}. \tag{2.55}$$

Thus, to have $\frac{\Delta p_\phi}{p_\phi} \ll 1$ we must take $p_\phi \epsilon_\phi \gg 1$.

Notice that this coherent state saturates the Heisenberg uncertainty bound as it should),

$$\Delta\phi \Delta p_\phi = \frac{1}{2}. \tag{2.56}$$

Therefore we have confirmed that the expectation values of the operators corresponding to the canonical variables agree with their classical values. Additionally, we have also shown that this coherent state saturates the Heisenberg bound, as it should. Therefore it is a natural kinematical semiclassical state to work with in order to obtain these 'effective equations' from the geometric quantum mechanics framework.

2.4.5 Expectation Value of the Hamiltonian Constraint Operator

Having verified that ψ is indeed sharply peaked around classical values of the canonical variables, we proceed to calculate the expectation value of the Hamiltonian constraint operator. Recall that, if we can find an approximately horizontal section of the Hilbert space then the effective quantum dynamics on that section is generated by the effective Hamiltonian which is just the expectation value of the Hamiltonian constraint operator. We calculate this expectation value now, leaving the proof that the section is approximately horizontal to a later section.

Now notice that the self-adjoint constraint is given by

$$C = \frac{1}{16\pi G} C_{grav} + \frac{1}{2} C_\phi \tag{2.57}$$

where

$$\begin{aligned}\hat{C}_{grav} &= \sin(\bar{\mu}c) \left\{ \frac{24i \,\mathrm{sgn}(\mu)}{8\pi\gamma^3\bar{\mu}^3 l_p^2} \left[\sin\left(\frac{\bar{\mu}c}{2}\right) \hat{V} \cos\left(\frac{\bar{\mu}c}{2}\right) \right.\right. \\ &\quad \left.\left. - \cos\left(\frac{\bar{\mu}c}{2}\right) \hat{V} \sin\left(\frac{\bar{\mu}c}{2}\right) \right] \right\} \sin(\bar{\mu}c) \end{aligned} \tag{2.58}$$

$$\hat{C}_\phi = \widehat{p^{-\frac{3}{2}}} \otimes p_\phi^2 \tag{2.59}$$

The action of the these on our basis kets is given by,

$$\hat{C}_{grav}|v;p_\phi\rangle = f_+(v)|v+4;p_\phi\rangle + f_0(v)|v;p_\phi\rangle + f_-(v)|v-4;p_\phi\rangle \quad (2.60)$$

$$\hat{C}_\phi|v;p_\phi\rangle = \frac{1}{2}\left(\frac{6}{8\pi\gamma l_p^2}\right)^{\frac{3}{2}} B(v)p_\phi^2|v;p_\phi\rangle \quad (2.61)$$

$$(2.62)$$

where

$$f_+(v) = \frac{27}{16}\sqrt{\frac{8\pi}{6}}\frac{Kl_p}{\gamma^{\frac{3}{2}}}|v+2|||v+1|-|v+3|| \quad (2.63)$$

$$f_-(v) = f_+(v-4) \quad (2.64)$$

$$f_0(v) = -f_+(v) - f_-(v) \quad (2.65)$$

and

$$B(v) = \left(\frac{3}{2}\right)^3 K|v|||v+1|^{\frac{1}{3}} - |v-1|^{\frac{1}{3}}|^3 \quad (2.66)$$

and

$$\widehat{p^{-\frac{3}{2}}}|v;p_\phi\rangle = \left(\frac{6}{8\pi\gamma l_p^2}\right)^{\frac{3}{2}} B(v)|v;p_\phi\rangle \quad (2.67)$$

Using these one can go ahead and compute the expectation value of the Hamiltonian constraint $\langle\hat{C}\rangle$. Note that such a summation includes a summation over negative v even though we are considering late times $v \gg 1$. However the contribution introduced from these terms is exponentially suppressed. For a proof of this see [85] and the appendix in [11]. We can evaluate the sum by Poisson resummation and find an asymptotic expansion for the first term in the series obtaining the effective constraint to leading and subleading order.

$$\langle\hat{C}\rangle = -\frac{3}{16\pi G\gamma^2\bar{\mu}'^2}p^{\frac{1}{2}}\left[1 + e^{-4\epsilon^2}\left(2\sin^2(\sqrt{\Delta}\beta') - 1\right)\right] + \frac{1}{2}\left(p_\phi'^2 + \frac{1}{2\epsilon_\phi^2}\right)\left(\frac{6}{8\pi\gamma l_p^2}\right)^{\frac{3}{2}} K\left[\frac{1}{v'} + O(v'^{-3}, v'^{-3}\epsilon^{-2})\right] \quad (2.68)$$

2.4.6 Equations of Motion

Recall that to evaluate $\dot{O}$ for some operator O we simply take the commutator between O and the Hamiltonian and divide by $i\hbar$. We begin with $\langle\dot{\beta}\rangle$,

$$\langle\dot{\beta}\rangle = \frac{\langle\psi|\frac{1}{i\hbar}\left(\frac{1}{16\pi G}[\beta, C_{grav}] + \frac{1}{2}[\beta, C_\phi]\right)|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{2.69}$$

which can be computed using a similar expansion to that used to find the expectation value of the Hamiltonian constraint to yield,

$$\begin{aligned}\langle\dot{\beta}\rangle &\simeq -\frac{1}{16\pi G}\frac{27}{16\hbar}\left(\frac{8\pi\gamma}{6}\right)^{\frac{1}{2}}\frac{Kl_p}{\gamma^2\sqrt{\Delta}}\left[4e^{-\frac{25}{4}\epsilon^2}\cos\left(\frac{5}{2}\sqrt{\Delta}\beta'\right)\right.\\ &+ \left. 4e^{-\frac{9}{4}\epsilon^2}\cos\left(\frac{3}{2}\sqrt{\Delta}\beta'\right) - 8e^{-\frac{1}{4}\epsilon^2}\cos\left(\frac{1}{2}\sqrt{\Delta}\beta'\right)\right]\\ -\left(p_\phi'^2 + \frac{1}{2\epsilon_\phi^2}\right)\frac{1}{2V'^2}\end{aligned} \tag{2.70}$$

Now for $\langle\dot{V}\rangle$

$$\langle\dot{V}\rangle = \frac{\langle\psi|\frac{1}{i\hbar}\left(\frac{1}{16\pi G}[V, C_{grav}] + \frac{1}{2}[V, C_\phi]\right)|\psi\rangle}{\langle\psi|\psi\rangle} \tag{2.71}$$

Notice, V commutes with C_ϕ so there is no contribution dependent on the scalar field and we just retain the contribution from the commutator with C_{grav} which yields

$$\langle\dot{V}\rangle \simeq \frac{3V'}{\gamma}e^{-4\epsilon^2}\frac{\sin(2\sqrt{\Delta}\beta')}{2\sqrt{\Delta}} \tag{2.72}$$

Now focus on $\langle\dot{\phi}\rangle$ and $\langle\dot{p}_\phi\rangle$,

$$\langle\dot{\phi}\rangle = \frac{\langle\psi|\frac{1}{2i\hbar}[\phi, C_\phi]|\psi\rangle}{\langle\psi|\psi\rangle}, \tag{2.73}$$

since ϕ commutes with C_{grav}

$$\langle\dot{\phi}\rangle \simeq \frac{p_\phi'}{V'} + O\left(\frac{1}{V'^3}\right) \tag{2.74}$$

Similarly,

$$\langle \dot{p_\phi} \rangle = \frac{\langle\psi| \frac{1}{i\hbar}\left(\frac{1}{16\pi G}[p_\phi, C_{grav}] + \frac{1}{2}[p_\phi, C_\phi]\right)|\psi\rangle}{\langle\psi|\psi\rangle} = 0 \tag{2.75}$$

since p_ϕ commutes with both C_{grav} and C_ϕ.

From these expressions we obtain the 'effective equations'.

2.5 Effective Equations

Recall that in the geometric quantization picture we take as our basic observables, the expectation values.

$$\bar{\beta} = \langle\beta\rangle = \frac{2}{\sqrt{\Delta}} e^{-\frac{1}{4}\epsilon^2} \sin(\frac{1}{2}\sqrt{\Delta}\beta') \tag{2.76}$$

$$\bar{V} = \langle V\rangle = V' \tag{2.77}$$

$$\bar{\phi} = \langle\phi\rangle = \phi' \tag{2.78}$$

$$\bar{p}_\phi = \langle p_\phi\rangle = p'_\phi \tag{2.79}$$

Recall, that the coordinates on the classical phase space are the expectation values, thus these barred variables are the coordinates on the classical phase space. So we search for an effective description in terms of these variables.

We thus invert these (2.76) - (2.79) for the primed variables and express the evolution equations in terms of the barred variables. We look at the first few terms of these asymptotic expansions to get the leading and next to leading behavior and apply the approximations listed above in 2.4.3. Doing this we obtain the 'effective equations of motion' (the main result of this chapter),

$$\bar{C} = -\frac{3}{\kappa\gamma^2}\bar{V}\bar{\beta}^2\left(1 - \frac{1}{4}\Delta\bar{\beta}^2\right) - \frac{6\epsilon^2}{\kappa\gamma^2}\frac{\bar{V}}{\Delta} + \frac{\bar{p}_\phi^2}{2\bar{V}}\left[1 + O(\bar{V}^{-2}, \bar{V}^{-2}\epsilon^{-2})\right] \tag{2.80}$$

$$\dot{\bar{\beta}} = \frac{3}{4\gamma}\sqrt{1 - \frac{1}{4}\Delta\bar{\beta}^2}\left[-2\bar{\beta}^2 + \Delta\bar{\beta}^4\right]$$

$$-\frac{\kappa}{4}\sqrt{1-\frac{1}{4}\Delta\bar{\beta}^2}\frac{\bar{p}_\phi^2}{V'^2}[1+O(\bar{V}^{-2},\bar{V}^{-2}\epsilon^{-2})] \tag{2.81}$$

$$\dot{\bar{V}} = 3\frac{\bar{\beta}}{\gamma}\bar{V}\sqrt{1-\frac{\Delta\bar{\beta}^2}{4}}\left(1-\frac{1}{2}\Delta\bar{\beta}^2\right) \tag{2.82}$$

$$\dot{\bar{\phi}} = \frac{\bar{p}_\phi}{\bar{V}} + O(\bar{V}^{-3}) \tag{2.83}$$

$$\dot{\bar{p}}_\phi = 0 \tag{2.84}$$

As discussed in Appendix A, Eqs ((2.80) and (2.82) imply that the Hubble parameter $H := \frac{\dot{a}}{a} \equiv \frac{\dot{V}}{V}$ satisfies an effective Friedmann equation,

$$H^2 = \frac{\kappa\rho}{3}\left(1-\frac{\rho}{\rho_{crit}}\right) + O(\epsilon^2) \tag{2.85}$$

where

$$\rho_{crit} = \frac{3}{\kappa\gamma^2\Delta} \tag{2.86}$$

which incorporates the leading quantum corrections. Somewhat surprisingly, as numerical simulations show, these corrections are already sufficient to correctly reproduce the main features of full quantum dynamics. These equations have been used in the literature to make certain phenomenological predictions (see,e.g., [61]).

One may ask, what is the error involved in making the approximations that led to equations (2.80)-(2.84). In these equations we are keeping corrections of $O(\Delta\beta^2)$ relative to the classical expressions but ignoring terms of $O(\Delta^2\beta^4)$. How much of an error is one making by ignoring these terms? Let us compute, $\Delta\beta^2$,

$$\Delta\beta^2 = \Delta\gamma^2\frac{\dot{a}^2}{a^2} = \frac{\rho}{\rho_c}\left(1-\frac{\rho}{\rho_c}\right) \tag{2.87}$$

Notice, this function starts off at 0 at late times since $\rho/\rho_c \sim 0$, it slowly increases reaches a maximum value of 1/4 at $\rho = \rho_c/2$ and then again goes to 0 at the point $\rho/\rho_c = 1$. Therefore, at the worst we are making an error of about 6% in ignoring terms of the order $\Delta^2\beta^4$ and this occurs at $\rho = \rho_c/2$ (see Fig 2.3). At other times, i.e. at late times and at times near the bounce point, this approximation is very good.

Additionally, one can pullback the symplectic structure Ω to our candidate horizontal section and analyze the dynamics generated by this effective Hamiltonian.

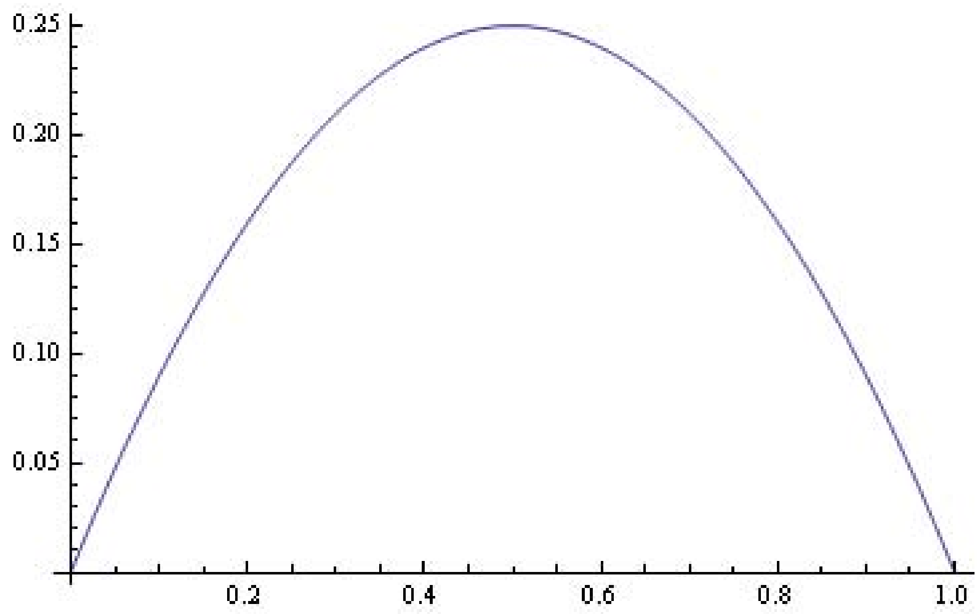

Figure 2.3. A plot of $\Delta\beta^2$ (2.87) in units of ρ_p

Does it give the equations (2.81)-(2.84)? One can pullback Ω and verify that it is given by

$$\Omega = \frac{2}{\kappa\gamma}\frac{1}{\sqrt{1-\frac{1}{4}\Delta\bar{\beta}^2}}d\bar{\beta}\wedge d\bar{V} + d\bar{\phi}\wedge d\bar{p}_\phi \tag{2.88}$$

and that its associated Poisson bracket is

$$\begin{aligned}\{f,g\} &= \frac{\kappa\gamma}{2}\sqrt{1-\frac{1}{4}\Delta\bar{\beta}^2}\left(\frac{\partial f}{\partial\bar{\beta}}\frac{\partial g}{\partial\bar{V}} - \frac{\partial g}{\partial\bar{\beta}}\frac{\partial f}{\partial\bar{V}}\right)\\ &+ \frac{\partial f}{\partial\bar{\phi}}\frac{\partial g}{\partial\bar{p}_\phi} - \frac{\partial g}{\partial\bar{p}_\phi}\frac{\partial f}{\partial\bar{\phi}}\end{aligned} \tag{2.89}$$

We can verify that our 'effective equations' are consistent by checking that the Poisson brackets in terms of the barred variables hold, to within our order of approximation. That is, $(\bar{\beta},\bar{V},\bar{\phi},\bar{p}_\phi)$ are the coordinates on the horizontal section, Γ, and we wish to know whether they are preserved, i.e. is the vector $(\dot{\bar{\beta}},\dot{\bar{V}},\dot{\bar{\phi}},\dot{\bar{p}}_\phi)$ tangent to Γ or off it? If it is not tangent to Γ then is it approximately tangent? That is, are its components off Γ small relative to the tangential components? This is indeed the case, since to our order of approximation the equations of motion in terms of the barred variable hold. Indeed, it can be verified that the following equations hold in terms of the pullback of the symplectic structure to the barred variables.

$$
\begin{aligned}
\dot{\bar{\beta}} &= \{\bar{\beta}, \bar{C}\} \\
\dot{\bar{V}} &= \{\bar{V}, \bar{C}\} \\
\dot{\bar{\phi}} &= \{\bar{\phi}, \bar{C}\} \\
\dot{\bar{p}}_\phi &= \{\bar{p}_\phi, \bar{C}\}
\end{aligned}
\tag{2.90}
$$

This is a highly nontrivial consistency check on the formalism as well as approximations used at intermediate steps.

2.6 Conclusion

Thus, we have constructed kinematical quantum states that closely approximate solutions to the classical Einstein equations. Therefore, at least for the case of a Friedmann universe with a free scalar field in the context of loop quantum cosmology, it is the case that there exist suitable quantum states that closely approximate solutions to the classical Einstein equations and *remain* sharply peaked along a quantum corrected, semiclassical trajectory.

Furthermore we note, that we have shown that the classical Einstein's equations are not just reproduced identically in the quantum theory but they do indeed pick up corrections due to quantum effects. In this work, we have used the geometric quantum mechanics framework and approximated the full quantum dynamics in the (infinite dimensional) Hilbert space by a system of 'effective equations', incorporating the leading quantum corrections, on a finite dimensional submanifold isomorphic to the classical phase space.

We have also discussed how accurate this approximation is near the bounce point even though it is a priori not expected to be a good approximation there by comparison with the numerical work in [18].

Chapter 3

Modified Holst Gravity

3.1 Introduction

The failure of general relativity (GR) to explain the nature of spacetime and cosmological singularities begs for a completion of the theory[1]. One path toward this completion is the unification of GR and quantum mechanics, through the postulate that spacetime itself is discrete – loop quantum gravity (LQG) [14, 80, 73]. This formalism is most naturally developed within the first-order approach [72] in terms of a generic connection and its conjugate electric field. When cast in these new variables, the quantization of the Einstein-Hilbert action resembles that of quantum electrodynamics, and thus, tools from field and gauge theories can be employed.

Currently, two versions exist of the connection variables: a selfdual $SL(2,\mathbb{C})$ Yang-Mills-like connection and a real $SU(2)$ connection. The first kind is the so-called Ashtekar connection, which was the first employed to develop LQG and which must satisfy some reality conditions [5]. The second type is the so-called Barbero connection and it was constructed to avoid these reality conditions [24]. Both the Ashtekar or Barbero formalisms can be obtained directly from the so-called Holst action, which consists of the Einstein-Hilbert piece plus a new term that depends on the dual of the curvature tensor [47]. The Barbero-Immirzi (BI) parameter γ arises in the Holst action as a multiplicative constant that controls the

[1]This chapter is based on the work of the author in [79]

strength of the dual curvature correction. In the quantum theory, it determines the minimum eigenvalue of the discrete area and discrete volume operators [48].

The Holst action reduces to the Einstein-Hilbert action upon imposition of the field equations obtained through the action-principle. Variation with respect to the connection reduces to a torsion-free condition, and when this is used at the level of the action, the dual curvature piece vanishes due to the Bianchi identities. Therefore, the Holst action leads to the same dynamical field equations as the Einstein-Hilbert action, with modifications only in the quantum regime. In the presence of matter, such as fermions, the dual curvature piece does not vanish identically since the variation of the action with respect to the connection leads to a non-vanishing torsion tensor [67, 39].

In this chapter, we consider a naive generalization of GR, modified Holst gravity, where we *scalarize* the BI parameter in the Holst action, i.e. we promote the BI parameter to a field under the integral of the dual curvature term. Allowing the BI field to be dynamical implies that derivatives of this field can no longer be set to zero when one varies the action and integrates it by parts. These derivatives generically lead to a torsion-full condition that produces non-trivial modifications to the field equations.

Scalarization is motivated in two different ways. One such way is the study of the possible variation of what we believe to be "universal physical constants." The study of models that allow non-constant couplings have a long history, one of the most famous of which, perhaps, is the so-called Jordan-Brans-Dicke theory [50, 31]. In this model, the universal gravitation constant G is effectively replaced by a time varying coupling field, such that $\dot{G} \neq 0$ (see eg. [84] and references therein for a review). Along these same lines, one could consider the possibility of a non-constant Holst coupling, where the variation could arise for example due to renormalization of the quantum theory. Such a possibility could and does lead to interesting corrections to the dynamics of the field equations that deserve further consideration.

Another motivation for scalarization is rooted in a recent proposal of a parity-violating correction to GR in 4 dimensions: Chern-Simons modified gravity [49]. In this model, a Pontryagin density is added to the Einstein-Hilbert action, multiplied by a θ-field that controls the strength of the correction. The Pontryagin density, however, can also be written as the divergence of some current (the so-called Pontryagin current). Thus, upon integrating the action by parts, the Chern-Simons correction can be thought of as the projection of the Pontryagin current along the *embedding* vector $\partial\theta$. Modified Holst gravity can also be interpreted as the embedding of a certain current along the direction encoded in the exterior derivative of the BI field. In this topological view, $\partial\gamma$ acts as an embedding coordinate that projects a certain current, given by the functional integral of the dual curvature tensor.

We shall here view modified Holst gravity as a viable, falsifiable model that allows us to study the dynamical consequences of promoting the BI parameter to a dynamical quantity. This model is a formal enlargement or deformation of GR in the phase space of all gravitational theories, in which the BI scalar acts as a dynamical deformation parameter, where γ = const. corresponds to GR. Arbitrarily close to this fixed point, one encounters deviations from GR, which originate from a modified torsional constraint that arises upon variation of the modified Holst action with respect to the spin connection. We shall explicitly solve this constraint to find that the torsion and contorsion tensors become proportional to first derivatives of the BI field. In the absence of matter, we find no parity violation induced by such a torsion tensor, as opposed to other modifications of GR that do include matter [67, 39, 2].

The variation of the action with respect to the tetrad yields the field equations, which differ from those of GR due to the non-vanishing contorsion tensor. The modification to the field equations are found to be quadratic in the first-derivatives of the BI field, and in fact equivalent to a scalar field stress-energy tensor with no potential and non-trivial kinetic energy. This stress energy is shown to be covariantly conserved, provided the BI field satisfies the equation of motion derived from the variation of the action with respect to this field. Since the BI field now possesses equations of motion, it is dynamically determined and not fixed *a priori*. Moreover, the motion of point particles is still determined by the divergence of their stress-energy tensor and unaffected by the Holst modification, allowing the modified theory to satisfy the strong equivalence principle.

Solutions of the modified theory are also studied, both for slowly-varying and arbitrarily-fast, time-varying BI fields. Since the modification to the field equations depends on derivatives of the BI scalar, every GR solution remains a solution of the modified model for constant γ. For slowly-varying BI fields, we find that GR solutions remain solutions to the modified theory up to second order in the variation of the BI parameter, due to the structure of the stress-energy tensor. In fact, gravitational waves in a Minkowski or Friedmann-Roberston-Walker (FRW) background remain unaffected by the Holst modification, and the BI field is seen to satisfy a wave equation. For arbitrarily-fast, time-varying BI fields, cosmological solutions are considered and the scalar field stress energy tensor induced by the Holst modification is found to reduce to that of a pressureless perfect fluid in a comoving reference frame. For a flat FRW background and in the absence of other fields, the scale factor is shown to evolve in the same way as in the presence of a stiff perfect fluid.

Next, an effective action is constructed by reinserting the solution to the torsional constraint into the modified Holst action, which is found to lead to the same dynamics as the full action. This effective action corresponds again to that of a scalar field with no potential but non-trivial kinetic energy. Such non-trivial kinetic terms in the action prompt the comparison of modified Holst gravity to k-inflationary models, in which the inflaton is driven not by a potential but by non-standard kinetic terms. Modified Holst gravity only contains non-trivial quadratic first-derivatives of the scalar field, which in itself is insufficient to lead to inflation in the k-inflationary scenario [4]. However, inflationary solutions are found to be allowed provided quadratic curvature corrections are added to the modified Holst action, which are prone to arise upon a UV completion of the theory.

Finally, we reexamine the effective theory obtained which was shown to be equivalent to GR coupled nonlinear to a pseudoscalar field β. This theory can be shown as equivalent to GR coupled to a scalar field $\phi \sim \sinh\beta$ with the canonical coupling in which case it is more obvious that there would be no k-inflation without including higher order terms in the Riemann curvature in the action. Furthermore, this coupling is a bit unnatural due to the solution to the torsion condition and can be traced back to the naive generalization of the Holst action neglecting a contribution from a term quadratic in the torsion that should naturally be included in the action. This quadratic term in the torsion is part of the so-called Nieh-Yan invariant and should not be neglected in the action since Modified Holst Gravity yields a theory with non-vanishing torsion depending on derivatives of the Barbero-Immirzi field. Thus, we propose a more natural generalization of the Holst action in the next chapter. In this chapter we study Modified Holst Gravity as a theory in its own right and return to the more natural generalization, Modified Nieh–Yan Gravity, in Chapter 4.

The remainder of this chapter deals with details that establish the results summarized above. We shall here adopt the following conventions. Lowercase Latin letters $a, b, \ldots = 0, 1, 2, 3$ stand for internal Lorentz indices, while lower Greek letters $\mu, \nu, \ldots = 0, 1, 2, 3$ stand for spacetime indices. Spacetime indices are usually suppressed in favor of wedge products and internal indices. We also choose the Lorentzian metric signature $(-, +, +, +)$ and the Levi-Cività symbol convention $\tilde{\eta}_{0123} = +1$, which implies $\tilde{\eta}^{0123} = -1$. Note that in the next chapter with fermions we use the opposite metric signature strictly for mathematical convenience. In this chapter we keep the $(-, +, +, +)$ signature convention to make the comparison with the usual Holst framework more transparent. Square brackets around indices stand for antisymmetrization, such as $A_{[ab]} = (A_{ab} - A_{ba})/2$. Other conventions and notational issues are established in the next section and in the appendices.

3.2 Modified Holst Gravity

In this section we introduce modified Holst gravity and establish some notation. Let us consider the following action in first-order form:

$$
\begin{aligned}
S &= \frac{1}{4\kappa} \int \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} \\
&+ \frac{1}{2\kappa} \int \beta \, e^a \wedge e^b \wedge R_{cd} + S_{\text{mat}},
\end{aligned} \tag{3.1}
$$

where $\kappa = 8\pi G$, S_{mat} is the action for possible additional matter degrees of freedom, ϵ_{abcd} is the Levi-Civitàa tensor, e is the determinant of the tetrad e^a and e_a is its inverse. In Eq. (3.1), the quantity R^{ab} is the curvature tensor of the Lorentz spin connection ω^{ab}, while $\beta = 1/\gamma$ is a coupling field, with γ the BI field. We use β instead of γ^{-1}, both, for mathematical simplicity and to avoid confusing with the

Dirac γ matrices in the next chapter.

Note that the first term in Eq. (3.1) is the standard Einstein-Hilbert piece, while the second term reduces to the standard Holst piece in the limit $\beta=$ const. (or $\gamma=$ const.). Also note that in the modified theory there are three independent degrees of freedom, namely the tetrad, the spin connection and the coupling field.

Varying the action with respect to the degrees of freedom one obtains the field equations of the modified theory. Assuming that the additional matter action does not depend on the connection and varying the full action with respect to this quantity, one obtains

$$\epsilon_{abcd}T^a \wedge e^b = -e_c \wedge e_d \wedge D\beta - 2\,\beta\,T_{[c} \wedge e_{d]}, \tag{3.2}$$

where D stands for covariant differentiation with respect to the spin connection and the torsion tensor is defined as $T^a = De^a$. One can arrive at Eq. (3.2) by noting that $\delta_\omega R^{ab} = D\delta\omega^{ab}$, where δ_ω is shorthand for the variation with respect to the spin connection, and integrating by parts. We shall here ignore boundary contributions that arise when integrating by parts, since they shall not contribute to the scenarios we shall investigate in later sections (gravitational waves and cosmological solutions). For the case of black hole solutions, such boundary terms could modify black hole thermodynamics in the quantum theory, but this goes beyond the scope of this work.

The remaining field equations can be obtained by varying the modified Holst action with respect to the tetrad and the coupling field. Varying first with respect to the tetrad we find

$$\epsilon_{abcd}e^b \wedge R^{cd} = -2\beta e^b \wedge F_{ab}, \tag{3.3}$$

where again we have assumed the additional matter degrees of freedom do not depend on the tetrad. Varying now the action with respect to the coupling field we find

$$\frac{\delta S_{\rm mat}}{\delta\beta} = -\frac{1}{2\kappa}e^a \wedge DT_a, \tag{3.4}$$

where we have assumed the matter degrees of freedom could contain a contribution that depends on β and thus the BI field.

The field equations of modified Holst gravity are then Eqs. (3.2)-(3.4). Note that for any non-constant value of β, the Holst modification leads to a torsion theory of gravity. More interestingly, even if $\beta = 0$, modified Holst gravity also leads to torsion provided the derivatives of the BI parameter are non-vanishing. In fact, Eq. (3.3) resembles the Einstein equations in the presence of matter, where the matter stress-energy is given by the covariant derivative of the torsion tensor. Such a resemblance is somewhat deceptive because the curvature tensor is *not* the Riemann tensor, but a generalization thereof, which also contains torsion-dependent pieces. Thus, the full modified field equations can only be obtained once Eq. (3.2) is solved for the torsion and contorsion tensors.

3.3 Torsion and Contorsion in Modified Holst Gravity

In this section we solve for the torsion and contorsion tensors inherent to modified Holst gravity. Equation (3.2) is difficult to solve in its standard form, so instead of addressing it directly we shall follow the method introduced by [67].

Let us then simplify Eq. (3.1) in the following manner

$$
\begin{aligned}
S &= \frac{1}{4\kappa}\int \epsilon_{abcd} e^a \wedge e^b \wedge e^m \wedge e^n \frac{1}{2} R^{cd}{}_{mn} \\
&+ \frac{1}{2\kappa}\int \beta\, e^a \wedge e^b \wedge e^c \wedge e^d \frac{1}{2} R_{abcd} + S_{\rm mat}, \qquad (3.5)\\
&= \frac{1}{8\kappa}\int \epsilon_{abcd}\,(-\tilde{\sigma})\,\epsilon^{abmn} R^{cd}{}_{mn} \\
&+ \frac{1}{4\kappa}\int \beta\,(-\tilde{\sigma})\,\epsilon^{abcd} R_{abcd} + S_{\rm mat}, \qquad (3.6)\\
&= \frac{1}{8\kappa}\int (-4)\,(-\tilde{\sigma})\,\delta^{[mn]}_{cd} R^{cd}{}_{mn} \\
&+ \frac{1}{4\kappa}\int \beta\,(-\tilde{\sigma})\,\epsilon^{abcd} R_{abcd} + S_{\rm mat}, \qquad (3.7)\\
&= \frac{1}{2\kappa}\int \tilde{\sigma}\left[\delta^{[mn]}_{cd} R^{cd}{}_{mn} - \frac{\beta}{2}\epsilon^{abcd} R_{abcd}\right] + S_{\rm mat},
\end{aligned}
$$

where $\tilde{\sigma} = d^4x\sqrt{-g} = d^4x\; e$. In Eq. (3.5) we reinstated all indices of the curvature tensor following the conventions in the Appendix. In Eq. (3.6), we have used the following identity:

$$e^a \wedge e^b \wedge e^c \wedge e^d = -\tilde{\sigma}\epsilon^{abcd}, \tag{3.8}$$

which derives from the relation $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = 1/4!\ \tilde{\eta}_{abcd}\ e^a e^b e^c e^d$, where $\tilde{\eta}_{abcd}$ is the Levi-Civitā symbol. Equation (3.7) makes use of the $\delta - \epsilon$ relation, which in four-dimensions reduces to

$$\epsilon^{abcd}\epsilon_{abmn} = -4\delta^{[cd]}_{mn} := -4\delta^{[c}_{m}\delta^{d]}_{n} = -2\left(\delta^c_m\delta^d_n - \delta^d_m\delta^c_n\right). \tag{3.9}$$

The modified Holst action can thus be recast as follows:

$$S = \frac{1}{2\kappa}\int d^4x\ e\ p^{ab}{}_{cd}\ e^\mu_a e^\nu_b R^{cd}{}_{\mu\nu}, \tag{3.10}$$

where the operator $p^{ab}{}_{cd}$ is given by

$$p_{ab}{}^{cd} = \delta^{[c}_{a}\delta^{d]}_{c} - \frac{\beta}{2}\epsilon_{ab}{}^{cd}. \tag{3.11}$$

In terms of this operator, Eq. (3.2) becomes

$$p^{ab}{}_{cd}D\left(e^\mu_a e^\nu_b\right) = \frac{1}{2}e^\mu_a e^\nu_b \epsilon^{ab}{}_{cd}D\beta, \tag{3.12}$$

and after isolating the torsion tensor we obtain

$$\begin{aligned} 2T_{[a} \wedge e_{b]} &= \frac{\partial_q\beta}{2\beta^2+2}\left[\epsilon_{mnab}e^m \wedge e^n \wedge e^q\right. \\ &- \left. 2\beta e_a \wedge e_b \wedge e^q\right], \end{aligned} \tag{3.13}$$

where we have employed the inverted projection tensor

$$(p^{-1})_{cd}{}^{ab} = \frac{1}{\beta^2+1}\left(\delta^{[a}_{c}\delta^{b]}_{d} + \frac{\beta}{2}\epsilon_{cd}{}^{ab}\right). \tag{3.14}$$

The torsion tensor can now be straightforwardly computed by solving the torsion condition [Eq. (3.13)] to find

$$T^a = \frac{1}{2}\frac{1}{\beta^2+1}\left[\epsilon^a{}_{bcd}\partial^d\beta + \beta\ \delta^a_{[b}\partial_{c]}\beta\right] e^b \wedge e^c \tag{3.15}$$

This expression can be shown to solve Eq. (3.2), thus satisfying the field equation associated with the variation of the action with respect to the spin connection.

Before we can address the modified field equations for the tetrad fields, we must first calculate the contorsion tensor. This tensor plays a critical role in the construction of the spin curvature, correcting the Riemann curvature through torsion-full terms. Let us then split this connection into a torsionless tetrad-compatible piece ${}^{o}\omega^{ab}$ and an antisymmetric piece K^{ab}, called the contorsion:

$$\omega^{ab} = {}^{o}\omega^{ab} + K^{ab}, \tag{3.16}$$

In the Appendix, we derive the relation between the contorsion and torsion tensor, so in this section it suffices to mention that they satisfy

$$K_{abc} = -\frac{1}{2}\left(T_{abc} + T_{bca} + T_{cba}\right), \tag{3.17}$$

where here we have converted the suppressed spacetime index into an internal one with the tetrad.

The contorsion tensor is then simply

$$K_{ab} = -\frac{1}{2}\frac{1}{\beta^2+1}\left(\epsilon_{abcd}e^c\partial^d\beta - 2\beta e_{[a}\partial_{b]}\beta\right). \tag{3.18}$$

One can verify that this tensor indeed satisfies the required condition $T_{abc} = -2K_{a[bc]}$.

3.4 Field Equations in Modified Holst Gravity

The field equations in modified Holst gravity are given by Eqs. (3.2)-(3.4), the first of which (the torsion condition) was already solved for in the previous section. We are then left with two sets of coupled partial differential equations, one equation for the reciprocal of the BI field β [Eq. (3.4)] and ten equations for the tetrad fields [Eq. (3.3)].

Let us begin with the equation of motion for β. We can compute the right-hand side of Eq. (3.4), by first calculating the covariant derivative of the torsion tensor.

Upon contraction with a tetrad, this quantity is given by

$$e_a \wedge DT^a = -3\tilde{\sigma}\frac{\beta}{\left(\beta^2+1\right)^2}\left(\partial\beta\right)^2 + 3\tilde{\sigma}\frac{1}{\beta^2+1}\Box\beta, \tag{3.19}$$

where $\Box = D_a D^a = g^{\mu\nu}D_\mu D_\nu$ is the covariant D'Alembertian operator, $\tilde{\sigma} = \sqrt{-g} = e$ is the volume element and $\left(\partial\beta\right)^2 := (\partial_a\beta)(\partial^a\beta) = g^{\mu\nu}(\partial_\mu\beta)(\partial_\nu\beta)$.

The equation of motion for β then becomes

$$\frac{\delta S_{\text{mat}}}{\delta\beta} = \frac{3\tilde{\sigma}}{2\kappa}\frac{\beta}{\left(\beta^2+1\right)^2}\left(\partial\beta\right)^2 - \frac{3\tilde{\sigma}}{2\kappa}\frac{1}{\beta^2+1}\Box\beta. \tag{3.20}$$

Even in the absence of other matter degrees of freedom, the field equations themselves guarantee that the BI coupling be dynamical. In a later section we shall study solutions to the equations of motion in different backgrounds that are approximate solutions to the modified field equations in the new theory.

In order to obtain the modified field equations let us contract $e^S\wedge$ into Eq. (3.3). Doing so we obtain,

$$\bar{R}^s{}_a - \frac{1}{2}\delta^S_a\bar{R} = -\frac{\beta}{2}\epsilon^{sbpq}\bar{F}_{abpq} - \frac{1}{4}\bar{R}e^s \wedge e_a, \tag{3.21}$$

where note that the last term will later vanish because it is totally antisymmetric and we shall drop it henceforth. The overhead bar in Eq. (3.21) is a reminder that no indices have been suppressed, and thus, here $\bar{R}_{ab} := \delta_c{}^d\bar{R}^c{}_{ads}$ and $\bar{R} = \delta^{[pq]}_{cd}\bar{R}^{cd}{}_{pq} = \delta^{ab}\bar{R}_{ab}$.

With this notation, the modified field equations resemble that of GR, except that here the $\bar{R}$ tensor is not the Ricci curvature but it also contains corrections due to torsion. Let us then decompose the curvature tensor into the Riemann curvature plus additional terms that depend on the contorsion tensor:

$$R^{ab}{}_{cd} = {}^oR^{ab}{}_{cd} + H^{ab}{}_{cd}, \tag{3.22}$$

where $H^{ab}{}_{cd}$ stands for the Holst correction tensor

$$H^{ab}{}_{cd} := 2{}^oD_{[c}K^{ab}{}_{d]} + 2K^a{}_{m[c}K^{mb}{}_{d]} \tag{3.23}$$

with ${}^oD$ the covariant derivative associated with the symmetric connection. We then find that Eq. (3.21) reduces to

$$G^s{}_a = -\left(H^s{}_a - \frac{1}{2}\delta^s_a H\right) - \frac{\beta}{2}\epsilon^{scpq} H_{abpq}, \tag{3.24}$$

where again $H_{ab} := \delta^{cd} H_{cadb}$ and $H = \delta^{[pq]}_{cd} H^{cd}{}_{pq}$. The right-hand side of Eq. (3.24) acts as a stress energy tensor for the reciprocal of the BI scalar.

The remainder of the calculation reduces to the explicit calculation of the Holst correction tensor for the contorsion found in the previous section. In a sense, Eq. (3.24) is similar to the decomposition of the correction into irreducible pieces: a trace, a symmetric piece and an antisymmetric piece. The calculation of these pieces is simplified if we first calculate the covariant derivative of the contorsion and the contorsion squared, namely

$$\begin{aligned}
{}^oD_m C^{cd}{}_n &= -\frac{1}{2}\frac{1}{\beta^2+1}\left\{\left[-\frac{2\beta}{\beta^2+1}\partial_m\beta\partial^q\beta\right.\right. \\
&+ \left. {}^oD_m\partial^q\beta\right]\epsilon^{cd}{}_{nq} + \left[2\frac{\beta^2-1}{\beta^2+1}\partial_m\beta\right. \\
&- \left.\left. 2\beta{}^oD_m\right]\delta^{[c}_n\partial^{L]}\beta\right\} \\
K^c{}_{m[q}K^{md}{}_{t]} &= \frac{1}{4}\frac{1-\beta^2}{\left(\beta^2+1\right)^2}\left[\left(\partial\beta\right)^2\delta^c_{[q}\delta^d_{t]} + 2\left(\partial^{[c}\beta\right)\right. \\
&\times \delta^{d]}_{[q}\left(\partial_{t]}\beta\right) \\
&+ \left.\frac{2\beta}{1-\beta^2}\left(\partial^{[c}\beta\right)\epsilon^{d]}{}_{tqs}\left(\partial^s\beta\right)\right].
\end{aligned} \tag{3.25}$$

With these expressions at hand, the Holst correction tensor is given by

$$\begin{aligned}
H^a{}_b &= {}_1H^a{}_b + {}_2H^a{}_b \\
{}_1H^a{}_b &:= 2{}^oD_{[c}C^{ca}{}_{b]} \\
&= \frac{1}{\beta^2+1}\left\{\frac{\beta^2-1}{\beta^2+1}\left[\left(\partial_a\beta\right)\left(\partial^b\beta\right) + \frac{1}{2}\delta^a_b\left(\partial\beta\right)^2\right]\right. \\
&- \left.\beta\left[D_b\partial^a\beta + \frac{1}{2}\Box\beta\delta^a_b\right]\right\} \\
{}_2H^a{}_b &:= 2K^c{}_{m[c}K^{ma}{}_{b]}
\end{aligned}$$

$$= \quad \frac{1}{2}\frac{1-\beta^2}{(1+\beta^2)^2}\left[(\partial\beta)^2\,\delta^a_b - (\partial^a\beta)\,(\partial_b\beta)\right], \tag{3.26}$$

and its trace is then simply

$$H = \frac{3}{2}\frac{\beta^2-1}{(\beta^2+1)^2}\,(\partial\beta)^2 - \frac{3\,\beta\,\Box\beta}{\beta^2+1}. \tag{3.27}$$

Finally, the antisymmetric part of this tensor is given by

$$\begin{aligned}\epsilon_{sb}{}^{pq} H^{ab}{}_{pq} \quad &= \quad -\frac{6\beta}{(\beta^2+1)^2}\left[(\partial_s\beta)\,(\partial^a\beta) - \frac{1}{2}\,(\partial\beta)^2\,\delta^a_b\right] \\ &- \quad \frac{2}{\beta^2+1}\,(\delta^a_s\Box\beta - D^a\partial_s\beta)\end{aligned} \tag{3.28}$$

We have now all the machinery in place to compute the modification to the field equations [ie. the right-hand side of Eq. (3.24)]. Combining all irreducible pieces of the Holst correction tensor, we find

$$G^s{}_a = \frac{3}{2}\frac{1}{\beta^2+1}\left[(\partial^s\beta)\,(\partial_a\beta) - \frac{1}{2}\delta^s_a\,(\partial\beta)^2\right], \tag{3.29}$$

or in terms of spatial indices

$$G_{\mu\nu} = \frac{3}{2}\frac{1}{\beta^2+1}\left[(\partial_\mu\beta)\,(\partial_\nu\beta) - \frac{1}{2}g_{\mu\nu}\,(\partial\beta)^2\right]. \tag{3.30}$$

Remarkably, the second derivatives of β have identically cancelled upon substitution of the solution to the torsional constraint. Perhaps even more remarkably, we find that modified Holst gravity is exactly equivalent to GR in the presence of an BI field with stress-energy tensor

$$T_{\mu\nu} = \frac{3}{2\kappa}\frac{1}{\beta^2+1}\left[(\partial_\mu\beta)\,(\partial_\nu\beta) - \frac{1}{2}g_{\mu\nu}\,(\partial\beta)^2\right]. \tag{3.31}$$

Such a stress-energy tensor is similar to that of a scalar field, except for a scalar-field dependent prefactor and the fact that β obeys a more complicated and non-linear evolution equation [Eq. (3.20)] than the scalar field one.

Before moving on to discuss solutions to this theory we wish to briefly comment on the equations (3.30)-(3.31). Not only does β obey a more complicated equation of motion but it is also has a complicated stress-energy tensor (3.31). Thus one has to do the field redefinition $\phi = \sqrt{3}\sinh\beta$ to obtain a scalar field with the canonical stress energy tensor. This is ok in principle but becomes unnatural when one includes the minimal coupling to fermions and when one wants to make contact with observational cosmology. This is shown in the next chapter but it is instructive to also comment on it here.

3.5 Solutions in Modified Holst Gravity

Now that the field equations have been obtained, one can study whether well-known solutions in GR are still solutions in modified Holst gravity. Formally, in the limit $\beta =$ const., all GR solutions remain solutions of the modified theory. If one adopts the view that derivatives of the BI field are small, then to first order in these derivatives, all standard solutions in GR also remain solutions of modified Holst gravity. This is because the stress energy tensor found in the previous section depends quadratically on derivatives of the BI field, and thus, can be neglected to first order.

Perturbations of standard solutions that solve the linearized Einstein equations, however, need not in general be also solutions to the linearized modified Holst field equations. For example, perturbations of the Schwarzschild spacetime will now acquire a source that depends on the BI field. This source could in turn modify the gravitational wave emission of such perturbed spacetime, and thus, the amount of energy-momentum carried by such waves.

Exact solutions to the modified field equations are difficult to find, due to the non-trivial coupling of the BI scalar to all metric components. We can however study some of the perturbative features of this theory to first order. In the next subsections we shall do so for spacetimes with propagating gravitational waves and FRW metrics.

3.5.1 Gravitational Waves and Other Approximate Solutions

Let us begin by assuming a flat background metric with a gravitational wave perturbation

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{3.32}$$

In the limit $\beta =$ const., all solution of the Einstein equations are also solutions of the modified Holst equations, and thus, Minkowski is also a solution. The Minkowski metric then is the background solution we shall employ.

Let us now concentrate on the first-order evolution of β in a Minkowski background. Since we are assuming the BI field varies slowly, terms quadratic in $\partial\beta$ can be neglected, and the equation of motion for β becomes

$$\left(-\partial_t^2 + \partial_k \partial^k\right)\beta = 0, \tag{3.33}$$

whose solution is

$$\beta = \beta_C \cos\left(\omega t - k_i x^i\right) + \beta_S \sin\left(\omega t - k_i x^i\right), \tag{3.34}$$

where $\beta_{C,S}$ are constants of integration, while $\omega^2 = k_i k^i$ is the dispersion relation, with ω the angular velocity and k_i the wave-number vector in the direction of propagation.

Now that the evolution of the BI field has been determined to first order, one can study first-order gravitational wave perturbations about Minkowski spacetime. In doing so, the modified field equations become

$$G_{\mu\nu}[h_{\sigma\delta}] = \mathcal{O}(\omega/\Omega)^2, \tag{3.35}$$

where Ω is the gravitational wave frequency. Note that the left-hand side stands for differential operators acting on the metric perturbation (*i.e.* in the Lorentz gauge, this operator would be the flat-space Laplacian), while the right-hand side stands for terms of second-order in the variation of the BI field. Thus, in modified Holst gravity, gravitational waves obey the same wave equation as in GR, to leading order in the variation of the BI field.

The equation of motion for β can also be solved exactly in a flat background, namely

$$\beta = \beta_0 \ln \left(1 + k_\mu x^\mu\right) \tag{3.36}$$

where β_0 and k^μ are constants of integration. One can check that for $c_\mu x^\mu \ll 1$ one recovers the linearized version of the wave solution presented above. Of course, if all orders in the derivatives of the BI field are retained, gravitational wave perturbations will be modified, but then one must treat the coupled system simultaneously. Such a study is beyond the scope of this work.

The result presented above is of course not dependent on the background chosen. For example, let us consider a Friedmann-Robertson-Walker (FRW) background in comoving coordinates

$$ds^2 = a(\eta)\left(-d\eta^2 + d\chi_i d\chi^i\right), \tag{3.37}$$

where $a(\eta)$ is the conformal factor, η is conformal time and χ^i are comoving coordinates. As before, to zeroth order in the derivatives of the BI field, the FRW metric remains an exact solution of the modified theory. Neglecting quadratic first derivatives of the BI field, its evolution is still governed by a wave equation, but this time about an FRW background:

$$-\partial_\eta^2 \beta - 2\mathcal{H}\partial_\eta \beta + \partial_i \partial^i \beta = 0, \tag{3.38}$$

where $\mathcal{H} := \partial_\eta a/a$ is the conformal Hubble parameter. The solution to this equation is still obviously a wave, with comoving angular velocity and wavelength, *i.e.* Eq. (3.36) with $k \to \tilde{k} = a(\eta)k$ and $\omega \to \tilde{\omega} = a(\eta)\omega$.

The argument presented above can be generalized to other exact solutions. For example, the Schwarzschild and the Kerr metrics remain exact solutions to the modified Holst field equations to zeroth order in the derivatives of the BI field. In turn, β is constrained to obey a wave equation in these background, neglecting its quadratic derivatives. This field would then source corrections to the background that would appear as modifications to the perturbation equations, but we shall not study these perturbations here.

3.5.2 Cosmological Solutions

Let us now consider the evolution of the universe in modified Holst gravity. Let us then consider the FRW line-element in cosmological non-comoving coordinates

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right)\right], \tag{3.39}$$

where $a(t)$ is the scale factor, t is cosmological time and k is the curvature parameter.

First, let us study the evolution of β in this background. In GR, the evolution of any cosmological stress-energy is given by the divergence of $T_{\mu\nu}$, namely $\nabla_\mu T^\mu{}_\nu$. The zeroth component of this equation is usually used to determine the scale-factor dependance of the energy density, once an equation of state is posed. In modified Holst gravity, we find that energy conservation is automatically guaranteed, provided β satisfies its own equation of motion [Eq. (3.20)], which in the absence of exterior source it reduces to

$$\Box\beta = \frac{\beta}{\beta^2+1}\left(\partial\beta\right)^2. \tag{3.40}$$

In order to make progress, we shall assume that the BI field depends only on time, such that the equation of motion of its reciprocal reduces to

$$\ddot{\beta} + 3H\dot{\beta} = \frac{\beta}{\beta^2+1}\dot{\beta}^2, \tag{3.41}$$

where overhead dots stand for partial derivatives with respect to cosmological time and $H := \dot{a}/a$. This equation can be solved exactly to find

$$\frac{\dot{\beta}}{\left(1+\beta^2\right)^{1/2}} = \frac{L_0^2}{a^3}, \tag{3.42}$$

where L_0 is a constant of integration needed for dimensional consistency. Equation (3.42) can be inverted to render

$$\beta(t) = \sinh\mathcal{A}, \tag{3.43}$$

where we have defined

$$\mathcal{A}(t) := \int \frac{L_0^2}{a^3(t)} dt, \tag{3.44}$$

which contains a hidden constant of integration. We see then that the BI field depends on the integrated history of the inverse volume element of spacetime. Naturally, as spacetime contracts (near the spacelike singularity where $a(t) \to 0$), $\beta \to \infty$ and the BI scalar tends to zero.

Let us now return to the modified field equations. Due to the symmetries of the background, there are only two independent modified field equations, namely

$$-3\frac{\ddot{a}}{a} = \frac{3}{2}\frac{\dot{\beta}^2}{\beta^2+1}, \tag{3.45}$$

$$\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2} = 0. \tag{3.46}$$

We can simplify Eq. (3.45) with both Eqs. (3.46) and (3.42) to find

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{L_0^4}{4a^6} - \frac{k}{a^2}. \tag{3.47}$$

Equations (3.46) and (3.47) are the only two independent modified field equations and they reduce to the Raychaudhuri and Friedmann equations respectively in the limit $\beta =$ const. The flat ($k = 0$) solution to the Holst-modified Friedmann-Raychaudhuri equations is simply $a \propto L_0^{2/3}(t - t_0)^{1/3}$, where t_0 is an integration constant, associated with the classical singularity.

Interestingly, one can now reinsert this solution into Eq. (3.44) to study the temporal behavior of the BI scalar. Doing so, one finds that $\beta \propto [(t-t_0)/t_1]^{2/3} - [(t-t_0)/t_1]^{-2/3}$, where t_1 is the hidden constant of integration of Eq. (3.44), which is fixed via initial conditions on β. Such a solution implies that as $t \to t_0$ or $t \to \infty$, $\beta \to \infty$, which forces the BI scalar γ to asymptotically approach zero.

Such results, however, are at this point premature since modified Holst gravity is a *classical* theory and one must analyze its quantization more carefully to determine what the β field represents in terms of the spectrum of quantum geometric operators. If we make the naive assumption that this field plays the same role in the quantized modified theory as in LQG, then in the infinite future limit

$t \to \infty$, the spectrum of quantum geometric operators would become continuous. Surprisingly, in the infinite-past limit $t \to t_0$, the spectrum of geometric operators also approaches continuity, which could indicate that the BI scalar becomes asymptotically free.

The Holst modification with time-dependent BI field is then equivalent to GR in the presence of a perfect fluid. The stress-energy tensor of such fluids is given by $T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p g_{\mu\nu}$, where p is the pressure, ρ is the energy density and u_μ is the 4-velocity of the fluid. In this case, the Holst modification is equivalent to a pressureless perfect fluid in a comoving reference frame $u_\mu = [-1, 0, 0, 0]$, with energy density

$$T_{00} = \rho = \frac{3L_0^2}{4\kappa a^6}. \tag{3.48}$$

Such a stress energy energy in fact also leads to the same scale-factor evolution as a pressureful perfect fluid in a comoving reference frame with equation of state $p = w\rho$ and $w = +1$. Such an equation of state is called *stiff* in the literature.

3.6 Effective Action and Inflation

The structure of the torsion and contorsion tensors remind us of the Klein-Gordon scalar field. For this reason, it is interesting to study the correction to the effective action obtained by reinserting these tensors into Eq. (3.10). In doing so, one obtains

$$S_{\text{eff}} = \frac{1}{2\kappa} \int \tilde{\sigma} \left[R - \frac{3}{2} \frac{1}{\beta^2 + 1} (\partial\beta)^2 \right], \tag{3.49}$$

where again we see that the second derivatives have identically vanished. In general, the insertion of the solution to the torsion condition into the action and its variation to obtain field equations need not commute. In this case, however, they do as one can trivially check by varying Eq. (3.49) with respect to the metric. Similarly, from this effective action one can recompute the stress-energy tensor of β to obtain Eq. (3.31).

Non-trivial kinetic terms in the action, similar to those in Eq. (3.31), are the pillars of the k-inflationary model. In this model, inflation and the inflaton field are driven by such terms, instead of a potential. More precisely, Ref. [4] considers the following action:

$$S_{\rm k} = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left[R - \kappa K(\psi) \left(\partial\psi\right)^2 - \frac{\kappa}{2} L(\psi) \left(\partial\psi\right)^4 \right] , \tag{3.50}$$

where ψ is the inflaton, while $K(\psi)$ and $L(\psi)$ are non-trivial arbitrary functions of the scalar field ψ. Ref. [4] shows that this modified action is equivalent to GR with a perfect fluid, whose stress energy tensor $T_{\mu\nu} = (\rho + p)\, u_\mu u_\nu - p g_{\mu\nu}$ and its energy density and pressure are given by

$$\begin{aligned} \rho &= \frac{1}{2} K(\psi) \left(\partial\psi\right)^2 + \frac{3}{4} L(\psi) \left(\partial\psi\right)^4 , \\ p &= \frac{1}{2} K(\psi) \left(\partial\psi\right)^2 + \frac{1}{2} L(\psi) \left(\partial\psi\right)^4 , \end{aligned} \tag{3.51}$$

with four-velocity

$$u_\mu = \frac{1}{\sqrt{\left(\partial\psi\right)^2}} \partial_\mu \psi, \tag{3.52}$$

and with $\dot{\psi} > 0$. Inflation then arises provided $w = p/\rho = -1$, which corresponds to

$$\frac{K}{L} = - \left(\partial\psi\right)^2 . \tag{3.53}$$

One then discovers that if non-trivial quadratic and quartic kinetic terms are present in the action, inflation can arise naturally without the presence of a potential.

The k-inflationary scenario can be compared now to modified Holst gravity. Doing so, one finds that the modified Holst contribution to the effective action is equivalent to the one considered in [4], where, modulo a conventional overall minus sign

$$K = \frac{3}{2\kappa} \frac{1}{\beta^2 + 1}, \qquad L = 0. \tag{3.54}$$

Note that the functional K is always positive, provided the BI field is real. If β is complex (which is allowed provided $\beta \neq i$), then the K functional could in fact change signs.

One is thus tempted to arrive at the perhaps surprising identification of the BI field as the inflaton of early cosmology. However, modified Holst gravity as analyzed here (without external potential contributions) is not sufficient to lead to an inflationary solution. One has already seen this in the previous section, where we found that $a(t) \propto t^{1/3} \neq e^{Ht}$. In other words, since $L = 0$ the energy density of the analog perfect fluid would be equal to its pressure, thus leading to a so-called "hard" or "”stiff“" equation of state and $a(t) \propto t^{1/3}$.

Two paths can lead to inflation in modified Holst gravity. The first path is to include a potential or kinetic contribution for the BI field to the action or matter Lagrangian density (the $S_{\rm mat}$ considered earlier). The obvious choice would be to simply add a quartic term of the form $N(\beta)\,(\partial\beta)^4$. Another less trivial possibility would be to include a term of the form

$$S_{\rm mat} = \frac{1}{2}\int \epsilon_{IJKL} e^I \wedge e^J \wedge F^{KL} \left[(\partial\beta)^2 + V(\beta)\right], \tag{3.55}$$

such that the full Lagrangian density became $\mathcal{L} = \mathcal{L}_{EH}\left[1 + (\partial\beta)^2 + V(\beta)\right]$. Since the Einstein-Hilbert Lagrangian density contains non-trivial kinetic terms, such an additional kinetic piece would lead to quartic first derivatives and thus non-vanishing $L(\psi)$.

Another much more natural route to produce quartic terms that does not involve adding arbitrary potential or kinetic contributions to the action is the inclusion of higher-order curvature corrections to the action. The modified Holst action corrects GR at an infrared level, without producing ultraviolet corrections. However, the effective quantum gravitational model represented by modified Holst gravity might require UV completion, just as string theory does. In string theory, such completion arises naturally in the form of effective Gauss-Bonnet and Chern-Simons terms. Such terms are topological in 4-dimensions and are thus usually integrated by parts and the boundary contribution set to zero. However, in modified Holst gravity, such terms will generically be non-vanishing. For example, a Ricci scalar squared correction to the action would lead to a new term in the effective action of the form

$$S_{\rm eff}^{R^2} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\frac{9}{4}\frac{1}{\left(\beta^2+1\right)^2}\left(\partial\beta\right)^4 \tag{3.56}$$

With this correction, the $L(\psi)$ function is not vanishing and in fact reduces to

$$L = -\frac{9}{2\kappa}\frac{1}{\left(\beta^2+1\right)^2}. \tag{3.57}$$

The ratio of functionals then becomes

$$\frac{K}{L} = -\frac{1}{3}\left(1+\beta^2\right), \tag{3.58}$$

which could generically lead to inflation. We thus conclude that although plain modified Holst gravity does not lead directly to inflation, it does allow for k-type inflation provided UV motivated, dimension-four corrections to the modified Holst action, such as a Gauss-Bonnet one, are also included in the action.

3.7 Conclusions

We have studied an LQG-inspired generalization of GR, where the Holst action is modified by promoting the BI parameter to a dynamical scalar field. Three sets of field equations were obtained from the variation of the action with respect to the degrees of freedom of the model. The first one is a non-linear, wave-like equation of motion for the reciprocal of the BI field, obtained from the variation of the action with respect to this field. The second one is a torsional constraint, obtained from the variation of the action with respect to the spin connection, which forces the spin connection to deviate from the Christoffel one. The third set corresponds to the modified field equations (a modification to the Einstein equations), obtained by varying the action with respect to the metric.

The torsional constraint was found to generically lead to Riemann-Cartan theory, with a torsion-full connection that we calculated explicitly in terms of derivatives of the BI field. From this torsion tensor, we computed the contorsion tensor, which allowed us to calculate the correction to the curvature tensor. Once this correction was obtained, we found explicit expressions for both the equation of motion for the reciprocal of the BI field as well as the modified field equations. The structure of the latter was in fact found equivalent to GR in the presence of a scalar field stress-energy tensor. This tensor was then seen to be covariantly conserved in the modified theory via the equation of motion of the BI field, thus satisfying the strong equivalence principle.

Typically the value of the BI parameter $\gamma = \beta^{-1}$ is determined by black hole thermodynamics and takes the value $\gamma \simeq 0.24$. However, in modified Holst gravity the BI parameter is determined dynamically via its own equation of motion. In this sense, β can take on an infinite number of values in our universe (the space of solutions of β is infinite-dimensional) and its precise value depends on the solution to a coupled system of partial differential equations for β and the metric. For instance, int he cosmological context discussed in Sec. V B neglecting backreaction and for a BI scalar that is isotropic and homogeneous, the solution we found for γ approaches 0 and not 0.24 as in the black hole case. Therefore, in this context, one would need to introduce a suitable effective potential for β to drive it to the black hole value. We have here only discussed the possibility of such a relaxation mechanism for the BI scalar in modified Holst gravity, but much more work remains to be done to understand full the nonlinear behavior of γ and to explain the inclusion of an effective potential.

Solutions were next studied in the modified theory. Since the correction to the field equations is in the form of quadratic first-order derivatives of β, all solutions of GR are also solutions to the modified theory if these derivatives are treated as small in some well-defined sense. Gravitational wave perturbations about a Minkowski and FRW background were also studied and found to still be solutions of the modified theory without any additional modifications. The reciprocal of the BI field in such backgrounds was seen to perturbatively satisfy the wave equation.

Cosmological solutions were also investigated in the modified theory for an FRW background. The equations of motion for the reciprocal of the BI field were solved exactly to find hyperbolic sinusoidal solutions. The modified Friedmann equations were then derived and solved to find a scale factor evolution corresponding to that of a stiff equation of state. In fact, in an FRW background and for a time-like BI field, the modified theory was found equivalent to GR in the presence of a pressureless perfect fluid in a comoving reference frame.

Next, an effective action was derived by inserting the solution to the torsional constrained into the modified Holst action. The effective action was found to be equivalent to the standard kinetic part of a scalar field action, with a non-trivial prefactor. Such an action was then compared to the ones studied in the k-inflationary model, where the inflaton is driven by such non-trivial kinetic terms. As considered here, modified Holst gravity is insufficient to drive inflation, since the BI field is found to be too stiff. However, upon UV completion, quartic kinetic terms should naturally arise due to torsion contributions that are quadratic in the modified theory. The combination of such non-trivial quadratic and quartic kinetic terms could generically allow for inflationary fixed points in the phase space of solutions.

Whether such inflationary solutions are truly realized remains to be studied further, but such a task is difficult on many fronts. First lack of a UV-complted modified Holst gravity theory forces one to draw physical inspiration from UV completions in string theory, such as Gauss-Bonnet or Chern-Simons like terms. The inclusion of a new Gauss-Bonnet term would require the addition of three new terms to the modified Holst action, including quadratic curvature tensor pieces, which would render the new equations of motion greatly nonlinear. The solution to this new system would thus necessarily have to be fully numerical and also raises the questions about the proper choice of intitial conditions.

Even if such a UV completion leads to a tractable system and a solution were found, its mere existence is not sufficient to render the model viable as an inflationary scenario. One would necessarily also have to study the duration of the inflationary period (the number of e-folds), the spectrum of perturbations, and other tests that the standard inflationary model passes. This work lays the foundations for a new set of ideas that could potentially tie together phenomoligical k-inflationary scenarios to quantum gravitational foundations. The tools developed here will hopefully allow researchers to consider this model more carefully and finally contrast it with experimental data.

Finally, we saw that the effective theory obtained in this chapter was shown to be equivalent to GR coupled nonlinearly to a pseudoscalar field β. This theory is equivalent to GR coupled to a an unnatural scalar field $\phi \sim \sinh \beta$. This coupling originates from the naive generalization of the Holst action neglecting a contribution from a term quadratic in the torsion making up part of the so-called Nieh-Yan invariant. In the next chapter we study the theory obtained without neglecting this term in the action since this term contributes due to the non-vanishing torsion which contains derivatives of the Barbero-Immirzi field. This proposed generalization of the Holst action is more general and we now proceed to motivate and study it.

Chapter 4

Modified Nieh–Yan Gravity

4.1 Introduction

Loop Quantum Gravity [14, 73, 80] is a non-perturbative and mathematically rigorous formulation of a quantum theory of gravity [1]. It is the result of the Dirac quantization procedure [37] applied to the Ashtekar–Barbero canonical constraints [6, 7, 26, 25] of General Relativity (GR), which are classically equivalent to the usual constraints of canonical tetrad gravity in the temporal gauge. The equivalence of the two formulations relies on the fact that they essentially describe the same classical system in two different sets of fundamental variables, related by a one-parameter canonical transformation. According to Rovelli and Thiemann [74], this canonical transformation cannot be unitarily implemented in the quantum theory, so that the quantization procedure necessarily generates a one-parameter family of unitarily inequivalent representations of the quantum commutation relations. As a result, the spectra of the quantum geometrical observables are not uniquely determined, being affected by the presence of a constant, known as Barbero–Immirzi (BI) parameter. Specifically, the area spectrum is

$$A_\gamma = 8\pi\gamma\ell_P^2 \sum_k \sqrt{j_k\left(j_k+1\right)}\,, \tag{4.1}$$

[1] This chapter is based on the work of the author in [60]

where γ is the BI parameter, while ℓ_P is the Planck length. The numerical value of the BI parameter can be determined by studying the modes of isolated horizons of non-rotating black holes [9, 10], but its physical origin is still an argument of discussions.

Recently, the idea that the BI parameter has a topological origin, has shed light on the nature of this ambiguity, reinforcing the analogy with the so-called θ -angle of Yang–Mills gauge theories as initially suggested by Gambini, Obregon and Pullin [42], and lately supported by other works [57, 36, 58].

It is worth remarking that in pure gravity this interpretation is not completely convincing, because the Holst action, which is the Lagrangian counterpart of the Ashtekar–Barbero canonical formulation of gravity, does not contain any topological term. In fact, the so called Holst modification [47],[2]

$$S_{\text{Hol}}\left[e,\omega\right] = \frac{1}{2\kappa\gamma}\int e_a \wedge e_b \wedge R^{ab}\,, \tag{4.2}$$

where e^a is the gravitational field 1-form and R^{ab} is the Riemann curvature 2-form defined as $R^{ab} = d\omega^{ab} + \omega^a_{\ c} \wedge \omega^{cb}$ (ω^{ab} being the Ricci spin connection 1-form), does not have the properties ascribable to a topological density, rather it is an on-(half)shell identically vanishing term. Nevertheless, the Holst framework can be further generalized and a true topological term can be naturally introduced in the action [36, 58, 32], providing new interesting insights into the physical origin of the BI parameter [58].

In order to clarify this point, let us summarize the main motivations to consider a generalization of the Holst framework. Generally, the coupling of fermion fields with first order Palatini gravity has a non-trivial effect on the geometry of spacetime: it generates a non-vanishing torsion tensor proportional to the spinor axial current. This fact has an interesting implication if we wish to describe the gravitational sector of the theory by using the Holst action, instead of the usual Hilbert–Palatini one. It turns out that in fact, in this more general case, the Holst modification no longer vanishes on-shell, consequently, the effective action differs from that of the Einstein–Cartan theory [39, 67, 55, 56, 29]. On one hand, the

[2]The signature throughout this chapter is now set to $+,-,-,-$, we set $\hbar = c = 1$ and $\kappa = 8\pi G$.

new effective action has the remarkable property to depend explicitly on the BI parameter, which acquires a classical meaning [67]. On the other hand, the geometrical properties of the torsion tensor associated to the Holst gravity coupled with spinors leads to a modification of the effective theory. The resulting modification diverges as soon as we set $\gamma = \pm\, i$, which correspond to the self and anti-self dual formulation of gravity [6, 7], respectively. This induces one to look for a different formulation [55].

Interestingly enough, the Holst term has a fermionic counter-term. In other words, it is possible to modify the Dirac as well as the gravitational action in such a way that the effective action exactly corresponds to the usual Einstein–Cartan one [55, 56]. Basically, the two modifications sum up reconstructing the so-called Nieh–Yan topological density [63]. This intriguing result [55] was later confirmed by Kaul in the framework of supergravities [51], demonstrating that to preserve supersymmetry, a possible Holst modification of the gravitational sector has to be counterbalanced by a specific modification of the fermionic sector, which exactly corresponds to the one previously argued for the ordinary theory.

These results strongly suggested that the Nieh–Yan density could play an important role in gravity. In particular, its role seems to reflect that of the Pontryagin densities in Yang–Mills gauge theories [83, 64, 5]. Moreover, as advertised before, in this extended framework, the BI parameter can in fact be interpreted as a topological ambiguity analogous to the θ-angle of Yang–Mills gauge theories [57, 36, 58], but at the price of enlarging the local gauge group. Specifically, such an ambiguity can be correlated to a specific large gauge sector of tetrad gravity in the temporal gauge, but requires a suitably extension of the the local gauge group, which corresponds to a compactification of the tangent bundle [58]. This can clarify the relation of the Nieh–Yan density with the Pontryagin classes, suggesting also why the appearance of the BI parameter is an unavoidable feature of the AB formulation of gravity.

So, we claim that a natural generalization of the Holst framework can be easily obtained by adding to the usual Hilbert–Palatini action, the Nieh–Yan density, i.e. [36, 32, 59, 58]

$$S\,[e,\omega] = -\frac{1}{2\kappa}\int e_a \wedge e_b \wedge \star R^{ab} - \frac{\beta}{2k}\int \left(e_a \wedge e_b \wedge R^{ab} - T^a \wedge T_a\right)\,. \tag{4.3}$$

where for later convenience we defined $\beta = -\frac{1}{\gamma}$. Above, T^a is the torsion 2-form defined as $T^a = de^a + \omega^a{}_b \wedge e^b$, while the symbol "$\star$" denotes the Hodge dual operator. It is worth remarking that the above action is classically equivalent to the usual Hilbert–Palatini action, the Nieh–Yan being reducible to a total divergence. Furthermore, this result can be extended to torsional spacetimes, in particular, action (4.3) is dynamically equivalent to the unmodified Einstein–Cartan action, as can be easily demonstrated by considering in (4.3) spinor matter fields minimally coupled to gravity [36].

Recently, it has been proposed to promote the BI parameter to be a field. This proposal was initially considered in the Holst framework [79] (see also [81]).[3] Here, in accordance with [59, 32], we claim that the new generalized action (4.3) is a more natural starting point to study the dynamics of the BI field, i.e.

$$S[e,\omega,\beta] = -\frac{1}{2\kappa}\int e_a \wedge e_b \wedge \star R^{ab} - \frac{1}{2k}\int \beta(x)\left(e_a \wedge e_b \wedge R^{ab} - T^a \wedge T_a\right) . \quad (4.4)$$

In fact, the presence of the field $\beta = \beta(x)$ generates torsion on spacetime, so that we consider the above action, with the Nieh–Yan invariant a more natural choice to incorporate the dynamics of the BI field. More precisely, we recall that starting from the Holst action and promoting the BI parameter to be a field, it is $\varphi = \sinh\beta$ to play, in the effective action, the role of a scalar field rather than β itself. Moreover, considering the presence of fermions, the Holst framework generates an unnatural coupling between the BI field and the fermionic matter fields [81], motivating the choice of a different starting point to describe the system. We note that in this framework the BI parameter can be associated with the expectation value of the field β [59], as a consequence its value can be determined by the dynamics, e.g. through a sort of Peccei–Quinn mechanism [59].

The organization of this paper is the following: In section 4.2 we study the effective dynamics of the BI field, considering as a starting point the action (4.4). In section 4.3 we discuss possible physical effects produced by the interaction of the

[3]Interestingly enough, the same model was considered a long time ago by Castellani, D'Auria and Frè [34], who in a completely different framework, mainly suggested by String Theory, proposed the idea of considering a field interacting with gravity through the Holst modification. It is worth remarking that this proposal was precedent to the papers by Barbero and Immirzi: in this sense it has not any relation with the BI field considered in the recent works [79] and [81].

BI field with matter, in particular we consider the perturbations it could produce on the cosmic microwave background (CMB), and digress on the known experimental limit on the magnitude of this effect. In the Appendix B.3, we further generalize the theory by relaxing the scale of the interaction between the BI field and gravity, introducing a dimensional parameter M. Surprisingly enough, this generalization does not change the scale of the effective interaction between the BI field and matter, which results to be determined by the theory itself. Appendix B.4 is devoted to a brief description of the effective dynamics of the BI field coupled to the original Holst modification, so that the outcomes of the different models can be easily compared. A discussion of the proposed model concludes the paper.

4.2 Effective Dynamics of Nieh–Yan Modified Gravity with Fermions

In order to study the dynamics of the BI field β, we consider a further generalization of the action (4.4). Specifically, in our argumentation fermion fields will play an essential role. So, let us generalize the theory by coupling spinor field to gravity through the usual minimal coupling, i.e.

$$S\left[e,\omega,\beta,\psi,\overline{\psi}\right] = -\frac{1}{2\kappa}\int e_a\wedge e_b\wedge\star R^{ab} - \frac{1}{2\kappa}\int\beta(x)\left(e_a\wedge e_b\wedge R^{ab} - T^a\wedge T_a\right) + \frac{i}{2}\int\star e_a\wedge\left(\overline{\psi}\gamma^a D\psi - \overline{D\psi}\gamma^a\psi + \frac{i}{2}\,me^a\overline{\psi}\psi\right). \tag{4.5}$$

The covariant derivatives are defined as

$$D\psi = d\psi - \frac{i}{4}\omega^{ab}\Sigma_{ab}\psi\,, \tag{4.6a}$$

$$\overline{D\psi} = d\overline{\psi} + \frac{i}{4}\overline{\psi}\Sigma_{ab}\omega^{ab}\,, \tag{4.6b}$$

where $\Sigma^{ab} = \frac{i}{2}\left[\gamma^a,\gamma^b\right]$ are the generators of the Lorentz group.

Now by varying the action with respect to the dynamical fields, we can extract the equations of motion. First, let us calculate the equation resulting from the variation with respect to the connection 1-form ω^{ab}, i.e.

$$\frac{1}{4\kappa}\epsilon_{abcd}D(e^a \wedge e^b) + \frac{1}{8} \star e_a \bar{\psi}\{\gamma^a, \Sigma_{cd}\}\psi + \frac{1}{2\kappa} e_c \wedge e_d \wedge d\beta = 0\,, \tag{4.7}$$

which, by using the formula $\left\{\gamma^a, \Sigma^{bc}\right\} = 2\epsilon^{abc}{}_d \gamma^5 \gamma^d$, can be rewritten as

$$\frac{1}{4\kappa}\,\epsilon_{abcd}(T^a \wedge e^b - e^a \wedge T^b) - \frac{1}{4}\,\epsilon_{abcd} \star e^a J^b_{(A)} + \frac{1}{2\kappa} e_c \wedge e_d \wedge d\beta = 0\,, \tag{4.8}$$

where $J^b_{(A)} = \bar{\psi}\gamma^b\gamma^5\psi$ is the spinor axial current. This equation can be further reduced with some algebra; we have,

$$T^a \wedge e^b - e^a \wedge T^b - \star e^{[a}\left(\kappa J^{b]}_{(A)} - 2\eta^{b]c}\partial_c \beta\right) = 0\,. \tag{4.9}$$

The variation of the action in Eq. (4.5) with respect to β, $\overline{\psi}$, ψ and e^a respectively yields,

$$e_a \wedge e_b \wedge R^{ab} - T^a \wedge T_a = 0\,, \tag{4.10a}$$

$$\star e_a \wedge \left(i\gamma^a D\psi - \frac{m}{4} e^a \psi\right) = 0\,, \tag{4.10b}$$

$$\star e_a \wedge \left(i\overline{D\psi}\gamma^a + \frac{m}{4} e^a \overline{\psi}\right) = 0\,, \tag{4.10c}$$

$$\frac{1}{\kappa} e_b \wedge \star R^{ab} - \frac{1}{\kappa} d\beta \wedge T^a - \frac{i}{4} \star (e^a \wedge e_b) \wedge \left(\overline{\psi}\gamma^b D\psi - \overline{D\psi}\gamma^b\psi\right) - \star e^a m\overline{\psi}\psi = 0\,, \tag{4.10d}$$

The set of the equations of motion is complicated, but interesting consequences can be extracted from the effective dynamics. In order to extract the effective dynamics, we must re-express the connection 1-form ω^{ab} as function of the other dynamical fields. In this respect, we recall that the connection ω^{ab} has to satisfy the second Cartan structure equation

$$de^a + \omega^a{}_b \wedge e^b = T^a, \tag{4.11}$$

which, being a linear equation, admits a natural decomposition of the connection. Specifically, recalling that the contorsion 1-form K^{ab} captures the part of the gravitational connection that depends on torsion, a natural decomposition of the connection 1-form ω^{ab} is

$$\omega^{ab} = {}^{o}\omega^{ab}(e) + K^{ab} \,. \tag{4.12}$$

Where ${}^{o}\omega^{ab}(e)$ is the usual Ricci spin connection and satisfies the homogeneous structure equation, namely

$$de^{a} + {}^{o}\omega^{a}{}_{b} \wedge e^{b} = 0 \,, \tag{4.13}$$

whereas the contorsion 1-form is related to the torsion 2-form as follows

$$K^{a}{}_{b} \wedge e^{b} = T^{a} \,, \tag{4.14}$$

so that the full connection 1-form in (4.12) satisfies the inhomogeneous structure equation.

The first step to calculate the expression of the connection 1-form is to extract the expression of the torsion 2-form from Eq. (4.9): we easily obtain

$$T^{a} = -\frac{1}{2} \star \left[e^{a} \wedge e_{b} \left(\kappa J^{b}_{(A)} - 2\eta^{bf} \partial_{f} \beta \right) \right] = -\frac{1}{4} \, \epsilon^{a}{}_{bcd} \left(\kappa J^{b}_{(A)} - 2\eta^{bf} \partial_{f} \beta \right) e^{c} \wedge e^{d} \,. \tag{4.15}$$

It is important to note that, in order to preserve the standard transformation properties of the torsion tensor under the Lorentz group, the field β has to be a pseudo-scalar [32, 59]. In other words, the geometrical content of the theory suggests the pseudo-scalar nature of the BI field β, which is a consequence of the peculiar interaction with the Nieh–Yan density and was not assumed *a priori*. The explicit expression for torsion in (4.15) corresponds to the following expression for the contortion 1-form,

$$K^{ab} = \frac{1}{4} \, \epsilon^{ab}{}_{cd} e^{c} \left(\kappa J^{d}_{(A)} - 2\eta^{df} \partial_{f} \beta \right) \,. \tag{4.16}$$

Hence, the connection 1-form satisfying the second Cartan structure equation is

$$\omega^{ab} = {}^{o}\omega^{ab}(e) + \frac{1}{4} \, \epsilon^{ab}{}_{cd} e^{c} \left(\kappa J^{d}_{(A)} - 2\eta^{df} \partial_{f} \beta \right) \,. \tag{4.17}$$

Now, we can substitute the solution (4.17) into the other equations of motion to study the effective dynamics. It is particularly interesting to note that by substituting the solution (4.17) into the equation (4.10a) we obtain

$$d\star d\beta = \frac{\kappa}{2}\, d\star J_{(A)} = m\kappa\,\overline{\psi}\gamma^5\psi dV\,, \tag{4.18}$$

where we have used $\star d\star J_{(A)} = -2m\overline{\psi}\gamma^5\psi$, $J_{(A)} = J^a_{(A)}e_a$ being the axial current 1-form and dV the natural volume element. This equation establishes a dynamical relation between the BI field and the spinor axial current, implying that the BI field has to be a pseudo-scalar as previously suggested by geometrical arguments.

It is worth noting, that the same effective equations can be obtained by pulling back the action (4.5) on the solution of the structure equation (4.17), obtaining the effective action, namely

$$\begin{aligned} S_{eff} = &-\frac{1}{2\kappa}\int e_a\wedge e_b\wedge\star^oR^{ab} + \frac{i}{2}\int\star e_a\wedge\left(\overline{\psi}\gamma^a\,{}^oD\psi - \overline{{}^oD\psi}\gamma^a\psi + \frac{i}{2}e^a m\overline{\psi}\psi\right)\\ &+\frac{3}{16}\kappa\int\star J_{(A)}\wedge J_{(A)} + \frac{3}{4\kappa}\int\star d\beta\wedge d\beta - \frac{3}{4}\int\star J_{(A)}\wedge d\beta\,, \end{aligned} \tag{4.19}$$

and varying it with respect to the other dynamical fields. It is worth remarking that the above action contains only torsionfree objects, in particular the symbol "o" denotes that the 2-form ${}^oR^{ab}$ is the curvature associated with the Ricci spin connection ${}^o\omega^{ab}$ and, analogously, the covariant derivatives action on spinors are defined as in (4.6), where the full connection is replaced by ${}^o\omega^{ab}$. The non-vanishing torsion tensor contributes to the kinetic term of the field β and generates the four fermions Fermi-like term as well as the interaction between the field β and the spinor axial current. The pseudo-scalar nature of the BI field prevents the theory from any possible parity violation, in contrast with what was argued elsewhere in the literature [81].

Finally, in order to reabsorb the constant factor in front of the kinetic term for the BI field, we can define the new dimensional field ϕ by rescaling the original dimensionless field β in a suitable way, i.e.

$$\phi := \sqrt{\frac{3}{2\kappa}}\beta\,, \tag{4.20}$$

so that the last two terms of the effective action can be rewritten as follows

$$\frac{3}{4\kappa}\int \star d\tilde{\beta}\wedge d\tilde{\beta} - \frac{3}{4}\int \star J_{(A)}\wedge d\tilde{\beta} = \frac{1}{2}\int \star d\phi \wedge d\phi - \frac{\sqrt{6\kappa}}{4}\int \star J_{(A)}\wedge d\phi\,. \quad (4.21)$$

The last term above contains interesting dynamical information about the interaction of the rescaled BI field and matter: the study of this interaction will be the focus of the next section.

4.3 Pseudo Scalar Perturbations

The interaction between the rescaled BI field ϕ and spinor matter through the last term in (4.21) produces interesting effects as soon as we consider the quantum effective dynamics of this system. In this respect, let us specify the framework we are going to consider. Let us initially assume that the spinor fields describe charged leptons, which minimally interact with the gravitational as well as electromagnetic field. According to this assumption, we generalize the effective action to contain the interaction with the electromagnetic field, i.e.

$$\begin{aligned} S_{eff} = &-\frac{1}{2\kappa}\int e_a\wedge e_b\wedge \star {}^{o}R^{ab} - \frac{1}{4}\int \star F\wedge F + \frac{1}{2}\int \star d\phi\wedge d\phi \\ &+\frac{i}{2}\int \star e_a \wedge \sum_l \left(\overline{\psi}_l\gamma^a\mathcal{D}\psi_l - \overline{\mathcal{D}\psi}_l\gamma^a\psi_l + \frac{i}{2}e^a m_l\overline{\psi}_l\psi_l\right) \\ &+\frac{3}{16}\kappa\int \star J_{(A)}\wedge J_{(A)} - \frac{\sqrt{6\kappa}}{4}\int \star J_{(A)}\wedge d\phi\,, \end{aligned} \quad (4.22)$$

where the sum is extended over the charged leptons and $J_{(A)} = \sum_l j_l^5$, where $j_l^5 = \overline{\psi}_l\gamma^a\gamma^5\psi_l e_a$ is the axial current 1-form associated with the lepton l. In the action above we have defined the new covariant derivative $\mathcal{D} = {}^{o}\mathcal{D} + igA$, where A is the connection 1-form associated with the electromagnetic field and $F = dA$ its curvature 2-form; g denotes the charge of the particle.

Let us now make a detour from the classical theory. Specifically, as soon as we consider the quantum theory, at an effective level, we have to take into account that the chiral anomaly contains a topological contribution besides the term proportional to the mass of the particle. To be more precise, let us suppose that we quantize the theory by using the path-integral method. The fermionic measure is not invariant under a chiral rotation [40, 41], generating a contribution to the chiral anomaly which can be easily evaluated. Specifically, it turns out to depend on the Pontryagin densities associated with the gauge fields contained in the theory.

So far we have only considered charged leptons interacting with the gravitational and electromagnetic field, so that we have [27]:

$$d\star j^5_l = 2m_l\overline{\psi}_l\gamma^5\psi_l dV + \frac{\alpha}{2\pi}F\wedge F + \frac{1}{8\pi^2}R^{ab}\wedge R_{ab}\,, \tag{4.23}$$

where $\alpha = g^2/4\pi$ is the fine structure constant. So, the existence of the chiral anomaly contributes to the effective action producing an interaction between the BI and gauge fields. In other words, through the chiral anomaly, in the effective semi-classical action, a non-trivial interaction between ϕ and the electromagnetic as well as the gravitational field appears, i.e.

$$\begin{aligned} S_{eff} = &-\frac{1}{2\kappa}\int e_a\wedge e_b\wedge {}^oR^{ab} - \frac{i}{4}\int \star F\wedge F + \frac{1}{2}\int \star d\phi\wedge d\phi \\ &+\frac{i}{2}\int \star e_a \wedge \sum_l \left[\overline{\psi}_l\gamma^a\mathcal{D}\psi_l - \overline{\mathcal{D}\psi}_l\gamma^a\psi_l + \frac{i}{2}e^a m_l\overline{\psi}_l\left(1+\frac{\sqrt{6\kappa}}{2}\phi\gamma^5\right)\psi_l\right] \\ &+\frac{3}{16}\kappa\int\star J_{(A)}\wedge J_{(A)} - \frac{1}{2f_\phi}\int\phi F\wedge F - \frac{1}{2r_\phi}\int \phi R^{ab}\wedge R_{ab}\,, \end{aligned} \tag{4.24}$$

where $f_\phi^{-1} = \frac{3\alpha}{4\pi}\sqrt{6\kappa}$ and $r_\phi^{-1} = \frac{3}{16\pi^2}\sqrt{6\kappa}$ determines the scale of the interactions. It is worth noting that the pseudo-scalar nature of the BI field protects the theory from parity violation. Additionally, it is interesting to note that in the effective action the first and last terms in (4.24) reproduce Chern-Simons Modified gravity [49, 2]. Thus the effective action reproduces Chern-Simons Modified gravity coupled to fermions and Maxwell theory

As soon as we consider also QCD in this picture, another massless pseudo-scalar field, namely the axion, $a(x)$, is present in the dynamics. The presence of the axion allows to solve in a natural way the so-called *strong CP problem*, while it acquires an anomaly-induced mass term: $m_a \approx m_\pi F_\pi/g_a$, where $m_\pi \approx 135$MeV is the mass of the pion, $F_\pi \approx 93$MeV denotes the pion decay constant and g_a denotes the scale of the interaction. We expect that the axion and the BI field combine and naturally interact via the chiral anomaly with the gauge bosons in a linear combination, schematically [71]:

$$\mathcal{L}_{\text{int}} = \left(\frac{\phi}{2g_\phi} + \frac{a}{2g_a}\right) \text{tr} G \wedge G + \left(\frac{\phi}{2f_\phi} + \frac{a}{2f_a}\right) F \wedge F + \left(\frac{\phi}{2r_\phi} + \frac{a}{2r_a}\right) R^{ab} \wedge R_{ab}\,, \tag{4.25}$$

where G is the curvature two form of the $SU(3)$ valued connection 1-form associated to the strong interaction. The mechanism that induces a mass is peculiar and only one linear combination of the two pseudo-scalar fields acquires a mass [3]. At an effective level, this fact implies that besides the usual QCD term for the massive physical axion, one has a massless additional pseudo-scalar field, Φ, which interacts with the electromagnetic as well as the gravitational field as follows:

$$\mathcal{L}_{\text{int}} = \mathcal{L}_{\text{axion}} + \frac{\Phi}{2f_\Phi} F \wedge F + \frac{\Phi}{2r_\Phi} R^{ab} \wedge R_{ab}\,, \tag{4.26}$$

where f_Φ^{-1} and r_Φ^{-1} denote the scale of the respective interactions.

The presence of the coupling with photons induces a Faraday rotation of the polarization, ε, of an electromagnetic wave, according to the following expression [44]

$$\Delta\varepsilon = \frac{\Delta\Phi}{f_\Phi}\,, \tag{4.27}$$

where $\Delta\Phi$ is the spacetime variation of the massless pseudo-scalar field. Additionally, the coupling with ${}^oR_{ab} \wedge {}^oR^{ab}$ has not only been shown to perturb gravitational wave propagation and by inducing an amplitude birefringence in the propagation of gravitational waves [1], but also to perturb the generation of gravitational waves in, e.g., binary inspirals [76]. We will not say anything more in this work about this term but focus on the $F \wedge F$ term for the remainder of the discussion.

So the existence of the BI field, motivated by the necessity of reabsorbing a divergence in the chiral anomaly in torsional spacetime as recently argued in [59], combined with the pseudo-scalar field associated to the additional Peccei–Quinn symmetry [66, 65], leads to the existence of a massless pseudo-scalar field Φ. Interestingly enough, from a cosmological point of view, the existence of this massless state super-weakly interacting with photons has interesting effects on the polarization of CMB. In particular, by studying the polarization anisotropies of the observed spectra of CMB, it is possible to put a stringent lower bound on the scale parameter f_Φ, as recently demonstrated by Pospelov, Ritz, and Skordis [71], who have also proved that this method can efficiently probe new models containing such new pseudo-scalar fields. They have found that $f_\Phi > 2.4 \times 10^{14}$GeV, fixing a constraint on massless pseudo-scalars more stringent of at least two order of magnitude with respect to previously existing limits.

4.4 Discussion

In this paper, according to previous works [79, 81, 32, 59], we proposed to promote the BI parameter to a field. Initially, this idea was motivated by the hope of correlating the value of the BI parameter to a possible dynamical mechanism. But, simultaneously, the presence of this new field in the action allows to solve another problem correlated with the chiral anomaly on a torsional spacetime [59]. In particular, according to a result of Chandía and Zanelli [35], the chiral anomaly diverges on spacetimes with torsion, the divergence being correlated to the Nieh–Yan contribution to the anomaly itself. This divergence can be reabsorbed in the definition of the BI field, so its presence naturally solves this problem. It is worth remarking that an analogous redefinition could work even if we did not consider the possibility that β is actually a field. In other words, we could imagine to reabsorb the divergence in the physical BI parameter, considering the β appearing in the gravitational action as a “bare” vacuum parameter. But this redefinition would involve a shift of the parameter and to work properly requires a sort of “fine tuning”. So the solution with the BI parameter seems to be preferable and based on a shift symmetry of the theory immediately recognizable looking at action (4.5) or (4.19).

The dynamical equations show that this new field is a pseudo-scalar, suggesting an interesting perspective. In fact, another pseudo-scalar field not yet observed is supposed to exist, as it allows to solve in a very natural way the strong CP problem, this is the axion. Considering this fact, it is possible that the physical pseudo-scalar states associated to the two particles are a linear superposition of the BI and the axion field. This possibility is interesting from a cosmological point of view, because if the two fields interact with the gluons as well as photons through a linear superposition, only one of the two physical pseudo-scalar degrees of freedom can acquire an anomaly-induced mass, the other remaining massless. As a consequence, the massless field, interacting with photons generates a rotation of the polarization angle of electromagnetic waves and this effect can be probed by studying the polarization anisotropies of CMB.

In general, any massless pseudo-scalar field super-weakly coupled to photons generates such an effect, which, compared to the available data on the B-mode, allows to fix a limit on the strength of the pseudo-scalar coupling to photons. Many new high-energy physical theories contains or predict two or more light pseudo-scalar fields, which can generate such an effect. Our model with the BI field represents a possible new theoretical framework in which a new pseudo-scalar particle is present and can, in fact, possibly encompass the physics generating a rotation of the polarization angle of CMB in a fairly standard way. There is also the possibility of detecting it by its coupling to gravity and producing an amplitude birefringence in the propagation of gravitational waves.

Appendix A

Validity of Effective Equations Near the Bounce

It is instructive to write the 'effective equations' in standard form. Therefore, in this appendix we address the question: what is the form of the corrected Friedmann and Raychaudhuri equations?

We can obtain the corrected Friedmann equation by, working to next-to leading order, solving (2.80) for $\bar{\beta}$, substituting into (2.82), and then computing the Hubble parameter $H = \frac{\dot{a}}{a} = \frac{\dot{V}}{3V}$,

$$H^2 = \frac{\kappa\rho}{3}\left(1 - \frac{\rho}{\rho_{crit}}\right) + O(\epsilon^2) \tag{A.1}$$

where

$$\rho_{crit} = \frac{3}{\kappa\gamma^2\Delta} \tag{A.2}$$

Notice, in addition to recovering the classical $\frac{\kappa}{3}\rho$ term we get a correction term that becomes important when the density ρ becomes very large. We see that the minus sign allows H, and hence $\dot{a}$ to be zero. Thus allowing the possibility of a bounce. Thus, even though these equations hold at late times since we are working in the semiclassical regime, we already see that gravity is becoming repulsive and allowing the scale factor to bounce when we evolve backwards in time and approach the classical big bang singularity.

On (A.1) we must make two comments. First, a priori one would not expect

(A.1) to describe the correct dynamics near the bounce point because the bounce point $\rho/\rho_c = 1$ lies outside of the regime of our approximation. That is, the natural domain of applicability of these 'effective equations' is a late-time, large-volume approximation and the point $\rho/\rho_c = 1$ lies well outside of that regime since the condition (2.38) fails badly at that point. However, numerical work (see reference [17]) has shown that the dynamics is indeed described well by (A.1) even at the bounce and hence this approximation and the results obtained in this work continue to be good even beyond their expected regime. Secondly, near the bounce point the term in parentheses in (A.1)is approaching 0, but since the approximation (2.38) is breaking down there it is not clear that the $O(\epsilon^2)$ term is negligible there. Thus (A.1) appears to break down there, but the numerical work has shown that it holds very well with negligible $O(\epsilon^2)$ corrections. As is not uncommon in physics, the effective theory is valid well beyond the domain for which it was constructed.

A more detailed analysis using the methods of [30] can analyze the $O(\epsilon^2)$ terms, independently of the assumptions on the form of the state as we have done, more accurately near the bounce but that is beyond the scope of this work.

Returning to the effective equations from geometric quantum mechancis. So now we have the effective Friedmann equation in this model. It would be instructive, for completeness, to also obtain the conservation equation and the corrected Raychaudhuri equation for this effective theory. Recalling that $\rho = p$ for a massless scalar field, classically these equations are

$$\dot{\rho} = -6\frac{\dot{a}}{a}\rho, \tag{A.3}$$

$$3\frac{\ddot{a}}{a} = -2\kappa\rho \tag{A.4}$$

Let us compute the corrected versions of these equations. Using the modified Poisson bracket, we can compute $\dot{\rho} = \{\rho, \bar{C}\}$. Doing this we obtain precisely (A.3), so the continuity equation is not modified. Similarly, we can take (2.82) and compute $\ddot{V} = \{\dot{V}, \bar{C}\}$, expressing it in terms of the scale factor a, and solving for $3\frac{\ddot{a}}{a}$. One obtains

$$3\frac{\ddot{a}}{a} = -2\kappa\rho\left(1 - \frac{5}{2}\frac{\rho}{\rho_c}\right) + O(\epsilon^2) \tag{A.5}$$

Notice that classically this expression is always negative. However, now there is a correction of $O(\frac{\rho}{\rho_c})$. Notice, that the corrected Friedmann equation (A.1) tells us there is a bounce at $\rho/\rho_c = 1$. At that density the corrected Raychaudhuri equation (A.5) tells us that $\ddot{a}$ is positive as it should be if there is to be a bounce.

Appendix B

Conventions and Notations for Chapters 4 and 5

B.1 First Order Formalism

In this appendix we establish the notation for the first order formalism used in chapters 3 and 4. Let us first note that all spacetime indices are suppressed, and if reinstated, they are to be added after the internal ones. We use lowercase Greek letters, e.g. $\mu\nu\lambda$, to represent spacetime indices and lowercase Latin letters, abc to represent internal indices. It then follows that the tetrad e^i and the spin connection ω^{ab} are 1-forms on the base manifold, while the curvature tensor associated with it, F^{ab}, is a 2-form on the base manifold.

Spacetime indices are reinstated through wedge product operators, where the latter are defined by the operation

$$(A \wedge B)_{\mu\nu} := \frac{(p+q)!}{p!q!} A_{[\mu_1 \ldots \mu_p} B_{\nu_1 \ldots \nu_q]} \tag{B.1}$$

with A and B p- and q-forms respectively. Note that the wedge product satisfies the following chain rule

$$D\left(A \wedge B\right) = \left(DA\right) \wedge B + (-1)^q A \wedge \left(DB\right), \tag{B.2}$$

and the following commutativity relation

$$A \wedge B = (-1)^{pq} B \wedge A. \tag{B.3}$$

Thus, for example,

$$T^a = T^a{}_{\mu\nu} = T^a{}_{bc} e^b_\mu e^c_\nu = \frac{1}{2} T^a{}_{bc} e^b \wedge e^c. \tag{B.4}$$

Since the wedge product acts on spacetime indices only, it acts on the base manifold and not on the internal fiber structure.

With this in mind, the covariant derivative only acts on internal indices as follows:

$$DA^{ab} := dA^{ab} + \omega^{ac} \wedge A_c{}^b + \omega^{bc} \wedge A^a{}_c, \tag{B.5}$$

$$DA_{ab} := dA_{ab} - \omega_a{}^c \wedge A_{cb} - \omega_b{}^c \wedge A_{bc}, \tag{B.6}$$

where the exterior derivative operator d acts on spacetime indices only, namely

$$dA^{ab} := 2\partial_{[\mu} A^{ab}{}_{\nu]}. \tag{B.7}$$

From the anticommutator of covariant derivatives, one can define the curvature tensor associated with the spin connection

$$R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb}. \tag{B.8}$$

With this definition at hand, one can easily show by direct computation that the variation of the curvature tensor with respect to the connection is given by

$$\delta_\omega R^{ab} = D\delta\omega^{ab}. \tag{B.9}$$

We choose here to work with a spin connection that is internally compatible. In other words, we demand $D\eta^{ab} = 0$, which then forces the spin connection to be fully antisymmetric on its internal indices $\omega^{(ab)} = 0$. From this connection and the tetrad, one can also construct the torsion tensor defined as

$$T^a := De^a = de^a + \omega^a{}_b \wedge e^b, \tag{B.10}$$

which is equivalent to $T^a{}_{\mu\nu} = 2D_{[\mu}e^a_{\nu]}$, or when spacetime indices are reinstated

$$T^\sigma{}_{\mu\nu} = 2\Gamma^\sigma_{[\mu\sigma]}. \tag{B.11}$$

Note that internal metric compatibility is not equivalent to a torsion-free condition.

The contorsion tensor can be obtained from the definition of the torsion tensor. We thus split the spin connection into a tetrad compatible piece ${}^o\omega^a{}_b$ solving the homogeneous Cartan structure equation

$$de^a + {}^o\omega^a{}_b \wedge e^b = 0 \tag{B.12}$$

and a part $K^a{}_b$, called the contorsion, satisfying the nonhomogeneous part of the Cartain structure equation $De^a = T^a$:

$$T^a = K^a{}_b \wedge e^b, \tag{B.13}$$

or simply $T^a{}_{bc} = -2K^a{}_{[bc]}$. These equations can be inverted to find

$$K_{abc} = -\frac{1}{2}\left(T_{abc} + T_{bca} + T_{cba}\right). \tag{B.14}$$

Note that the contorsion is fully antisymmetric on its first two indices, while the torsion tensor is fully antisymmetric on its last two indices. Also note that Eq. (B.11) can be obtained by converting Eq. (B.10) to spacetime indices and using the transformation law from spin to spacetime connection established by ${}^oDe^a = 0$ (this relation is sometimes referred to as "the tetrad postulate").

With the contorsion tensor, we can now express the curvature tensor in terms of the Riemann tensor ${}^oR^{ab}$ and terms proportional to the contorsion

$$R^{ab} = {}^oR^{ab} + {}^oDK^{ab} + K^a{}_c \wedge K^{cb}, \tag{B.15}$$

where ${}^oD$ is the connection compatible with the torsion-free connection. One can also check that the Bianchi identities in first order form become

$$DT^a = {}^oR^a{}_b \wedge e^b, \qquad D^oR^{ab} = 0. \tag{B.16}$$

Finally, it is sometimes useful to control the expression of the volume form in the first-order formalism. This quantity is given by

$$\tilde{\sigma} := \sqrt{-g}\; d^4x = \frac{1}{4!}\epsilon_{abcd}e^a e^b e^c e^d \tag{B.17}$$

and it allows one to rewrite the contraction of the Levi-Civitá tensor with tetrad vectors in terms of e.

B.2 Other useful formulae

In this appendix we present a compendium of other useful formulae, where the first expression corresponds to suppressed spacetime indices, followed by a second expression with spacetime indices reinstated, but transformed to internal ones with the tetrad.

We begin with the torsion tensor

$$\begin{aligned} T^a &= \frac{1}{2}\frac{1}{\beta^2+1}\left[\epsilon^a{}_{bcd}\partial^d\beta + \beta\delta^a_{[b}\partial_{c]}\beta\right] e^b \wedge e^c, \\ T_{abc} &= \frac{1}{\beta^2+1}\left[\epsilon_{abc}{}^d\partial_d\beta + \beta\delta_{a[b}\partial_{c]}\beta\right]. \end{aligned} \tag{B.18}$$

and the contorsion tensor

$$\begin{aligned} C_{ab} &= -\frac{1}{2}\frac{1}{\beta^2+1}\left(\epsilon_{abcd}e^c\partial^d\beta - 2\beta e_{[a}\partial_{b]}\beta\right), \\ C_{abc} &= -\frac{1}{2}\frac{1}{\beta^2+1}\left[\epsilon_{abcd}\partial^d\beta - 2\beta\delta_{c[a}\partial_{b]}\beta\right]. \end{aligned} \tag{B.19}$$

B.3 On the Role of Scale Parameter

Here we consider a further generalization of the action (4.5) for Nieh–Yan gravity by introducing a dimensionful parameter M,

$$S\left[e,\omega,\beta,\psi,\overline{\psi}\right] = -\frac{1}{2\kappa}\int e_a\wedge e_b\wedge\star R^{ab} - M\int\tilde{\beta}(x)\left(e_a\wedge e_b\wedge R^{ab} - T^a\wedge T_a\right)$$
$$+\frac{1}{2}\int\star d\tilde{\beta}(x)\wedge d\tilde{\beta}(x) + \frac{i}{2}\int\star e_a\wedge\left(\overline{\psi}\gamma^a D\psi - \overline{D\psi}\gamma^a\psi + \frac{i}{2}me^a\overline{\psi}\psi\right). \quad \text{(B.20)}$$

The introduction of the parameter M is extremely natural and generalizes the action in Eq. (4.5) by relaxing the scale of the interaction between matter and the BI field β. In fact, we saw that the BI field couples to the Nieh–Yan density via a term of the form

$$\frac{1}{2\kappa}\int\beta(x)(e_a\wedge e_b\wedge R^{ab} - T^a\wedge T_a)\,, \quad \text{(B.21)}$$

and the coupling scale is uniquely determined by the factor $\frac{1}{2\kappa}$ ($\kappa^{-1} = M_p^2/8\pi$ in natural units, i.e. $c = \hbar = 1$, where M_p is the Planck mass). The presence of the parameter M allows us to relax the scale at which the coupling between gravity and $\tilde{\beta}$ occurs.[1]

One can then proceed with this action along the lines in the main body of the paper and obtain the effective action,

$$S_{eff} = -\frac{1}{2\kappa}\int e_a\wedge e_b\wedge {}^oR^{ab} + \frac{i}{2}\int\star e_a\wedge\left(\overline{\psi}\gamma^a\,{}^oD\psi - \overline{{}^oD\psi}\gamma^a\psi + \frac{i}{2}e^a m\overline{\psi}\psi\right) \quad \text{(B.22)}$$

$$+\frac{3}{16}\kappa\int\star J_{(A)}\wedge J_{(A)} + 3\kappa M^2\int\star d\tilde{\beta}\wedge d\tilde{\beta} - \frac{3}{2}\kappa M\int\star J_{(A)}\wedge d\tilde{\beta}\,. \quad \text{(B.23)}$$

Then the last two terms in the effective action suggest a field redefinition of the form

$$\phi = \sqrt{6\kappa}M\tilde{\beta}\,, \quad \text{(B.24)}$$

[1]Note that now the BI field $\tilde{\beta}$ carries dimensions of energy.

upon which the effective action takes the form

$$S_{eff} = -\frac{1}{2\kappa}\int e_a \wedge e_b \wedge \star {}^o\!R^{ab} + \frac{i}{2}\int \star e_a \wedge \left(\overline{\psi}\gamma^a\, {}^o\!D\psi - \overline{{}^o\!D\psi}\gamma^a\psi + \frac{i}{2}e^a m\overline{\psi}\psi\right) \tag{B.25}$$

$$+\frac{3}{16}\kappa\int \star J_{(A)} \wedge J_{(A)} + \frac{1}{2}\int \star d\phi \wedge d\phi - \frac{\sqrt{6}}{4}\kappa\int d\star J_{(A)} \wedge \phi\,. \tag{B.26}$$

Surprisingly, one obtains a coupling between ϕ and fermionic matter which is *independent of the parameter M*. Thus, even though we have tried to allow an arbitrary energy scale for the coupling of the BI field, it turns out that the scale of the interaction is fixed by the theory to the Planck energy.

B.4 On BI field in Holst Gravity

B.4.1 Effective Dynamics

In [79] and chapter 3 a different model for the BI field was considered. The same model was already studied much earlier in [34], before the Holst, Barbero, and Immirzi proposals became popular in the Quantum Gravity community. The model in [34] was most likely motivated by String Theory considerations and led to a theory of General Relativity coupled to a pseudo-scalar field β, which remarkably was not the consequence of the "scalarization" of the BI parameter.

Here we compare the dynamical features of the BI field as resulting from Holst gravity with that presented in this paper and explain why the present model is more appealing and, to a great extent, more natural. Note the changes in convention: our field β corresponds to the field $-\bar{\gamma}$ of [79], lowercase internal indexes abc correspond to uppercase indexes IJK of [79], our contorsion tensor is denoted K^{ab} rather than C^{IJ}, and finally our metric signature is $+,-,-,-$ whereas $-+++$ was used in [79]. Moreover, we stress that in [79] the Riemann 2-form was denoted by F^{ab}, using the symbol R^{ab} for the torsion-free curvature; here we used the usual symbol R^{ab} for the full curvature and ${}^o\!R^{ab}$ for the torsion-free Riemann 2-form.

Neglecting matter, the action considered in chapter 4 and [79] is

$$S = -\frac{1}{4\kappa}\int \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} - \frac{1}{2\kappa}\int \beta e_a \wedge e_b \wedge R^{ab} \tag{B.27}$$

At once one notices that the second term in (B.27) is the original Holst modification, but the $T^a \wedge T_a$ contribution, which makes up the other part of the so-called Nieh–Yan invariant, is missing. This is significant because (B.27) yields a theory with torsion as shown in [79] and the $T^a \wedge T_a$ term affects the dynamical outcomes when β is non-constant.

One can then vary the action with respect to ω^{ab} and solve the Cartan structure equation for the torsion 2-form thus obtaining equation (15) in [79],

$$T^a = \frac{1}{2}\frac{1}{1+\beta^2}\left(\epsilon^a{}_{bcd}\partial^d\beta + \beta\delta^a_{[b}\partial_{c]}\beta\right) e^b \wedge e^c \tag{B.28}$$

and the corresponding contorsion 1-form

$$K^{ab} = -\frac{1}{2}\frac{1}{1+\beta^2}\left(\epsilon^{ab}{}_{cd} e^c \partial^d\beta + 2\beta e^{[a}\partial^{b]}\beta\right) \tag{B.29}$$

There are two main differences between these formulas obtained in chapter 3 and [79] and the corresponding expressions obtained here in (4.15) and (4.16). Firstly, the vector trace component of the torsion tensor (4.15) vanishes, whereas it does not for (B.28). Secondly, the expressions (B.28) and (B.29) contain a β dependent prefactor of $\frac{1}{1+\beta^2}$ which complicated the theory. The effective action then yields GR coupled to the scalar field $\varphi = \sinh\beta$ rather than β, affecting also the coupling with fermions that becomes quite unnatural [81].

B.4.2 Effective Dynamics with Fermions

Alternatively, one can also take the theory described by (B.27), still without the complete Nieh–Yan term, and non-minimally couple fermions to it using the non-minimal coupling introduced in [55] and [57]. Then one still gets a more complicated theory than the one presented in this work because the term quadratic in torsion in the Nieh–Yan invariant is still neglected. The action for this theory is

$$S(e,\omega,\psi,\bar{\psi},\beta) = -\frac{1}{4\kappa}\int \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd} - \frac{1}{2\kappa}\int \beta e_a \wedge e_b \wedge R^{ab} + \frac{i}{2}\int \star e_a \wedge \left[\overline{\psi}\gamma^a(1-i\beta\gamma^5)D\psi - \overline{D\psi}(1-i\beta\gamma^5)\gamma^a\psi\right] . \quad \text{(B.30)}$$

The structure equation can be calculated by varying with respect to ω, i.e.

$$\left(\frac{1}{2}\epsilon^{ab}{}_{cd} + \beta\delta^{[a}_c\delta^{b]}_d\right) d^{(\omega)}(e^a \wedge e^b) - \left(\frac{1}{2}\epsilon^{ab}{}_{cd} + \beta\delta^{[a}_c\delta^{b]}_d\right) \kappa \star e^c J^d + e^a \wedge e^b \wedge d\beta = 0 . \quad \text{(B.31)}$$

One can then solve (B.31) for the torsion 2-form thus obtaining,

$$T^a = -\frac{1}{4}\epsilon^a{}_{bcd}\left(\kappa J^b - \frac{2}{1+\beta^2}\partial^b\beta\right) e^c \wedge e^d + \frac{1}{2}\frac{\beta}{1+\beta^2} e^a \wedge d\beta , \quad \text{(B.32)}$$

and the corresponding contorsion 1-form

$$K^{ab} = \frac{1}{4}\epsilon^{ab}{}_{cd} e^c \left(\kappa J^d - \frac{2}{1+\beta^2}\partial^d\beta\right) + \frac{\beta}{1+\beta^2} e^{[a}\partial^{b]}\beta . \quad \text{(B.33)}$$

This is similar to the result obtained in the main body of this work, but for the presence of the factor $\frac{1}{1+\beta^2}$ and the fact that the torsion tensor (4.15) has a vanishing trace component. The theory in the main body of this work has no β-dependent multiplicative factors nor do such β-dependent multiplicative factors show up in the effective theory either. Whereas, for the theory (B.30) the effective action is more complicated

$$S_{eff} = -\frac{1}{4\kappa}\int \epsilon_{abcd} e^a \wedge e^b \wedge {}^oR^{cd} + \frac{i}{2}\int \star e_a \wedge \left(\overline{\psi}\gamma^a\, {}^oD\psi - \overline{{}^oD\psi}\gamma^a\psi\right) + \frac{3}{16}\kappa\int dV \eta_{ab} J^a J^b - 2\int dV\, J^a\partial_a\beta + \frac{3}{4\kappa}\int dV \frac{1}{1+\beta^2}\eta^{ab}\partial_a\beta\partial_b\beta \quad \text{(B.34)}$$

Finally, we note that in order to have a standard kinetic term for the scalar field, a change of variable is necessary, specifically we have to introduce the field $\varphi = \sinh\beta$, but this inevitably complicates the interaction with fermionic matter. Summarily, the simplicity of (4.19), obtained from the theory described by action (4.5), suggests that considering the full Nieh–Yan term leads to more appealing and natural results, when the interaction with fermions is taken into account.

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