AN ACOUSTIC SUPERPOSITION METHOD
FOR COMPUTING STRUCTURAL RADIATION
IN SPATIALLY DIGITIZED DOMAINS

A Thesis in
Mechanical Engineering
by
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ABSTRACT

This thesis presents a new method for computing acoustic fields of structural radiators based on acoustic superposition methods using meshless, spatially digitized domains (ASMDD). Here the system matrices are assembled knowing only coordinate points in 3D space that describe the geometry of the radiating structure. In contrast to conventional methods, ASMDD does not require cumbersome numerical, high orders of integration over elemental surfaces to populate system matrices. The system’s Greens functions are computed simply between source and receiver locations at their respective points. A new derivation presented in this work provides an analytical solution for coincident source and receiver points where the Greens function is singular. Because of the simplifications used in the ASMDD method, the matrix formulation routines are highly efficient.

A significant contribution of this thesis is the rendering of ASMDD point-based structures in a digital computational domain. The digital domain is a uniform distribution of points equidistant in the $x$, $y$, and $z$ directions. The centroid of each activated voxel (used only as a means for visualizing the 3D surface) represents a point on the structural surface being modeled. Work in this thesis exploits the inherent uniformity of neighboring points to formulate a locally determined outward-pointing, surface normal needed for acoustic radiation problems. The ability of the calculated surface normals to model the curvature of the continuous radiating surface depends on the density of the meshless grid, i.e., higher curvature requires higher grid densities. The attractiveness of the digital domain approach is its simplicity for morphing of structural shapes. Shape iterations in the digitized space reduces to a simple process of activating or deactivating selected points in a contiguous manner depending on the desired shape during an optimization. As an example of this, the ASMDD formulation is used to compute the radiation from a square piston in a cubic baffle. In the first example, the square piston shape is morphed to a hemi-sphere through five evolutionary changes in shape. The sound power calculations are in good agreement with those computed with conventional BEM codes for shape changes that have low changes in curvature. For a second
example, the ASMDD surface points are shown to blend seamlessly with surface vibration of the plate generated via meshless structural dynamics (Meshless Local Petrov Galerkin method - MLPG). This is achieved by solving the modal radiated acoustic power from the plate where the surface velocity is specified by the modal results determined by the MLPG method. The sound power calculations are again in good agreement with those generated via conventional BEM codes.
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Chapter 1
INTRODUCTION

1.1 Motivation

Noise control requirements for manufactured products are becoming more prevalent in competitive marketing. In particular, sound quality is rapidly becoming a functional requirement for successful products. Assessing product sound quality early in the design phase is more cost effective than retrofitting these products with noise control solutions after manufacturing. Therefore, implementing noise control via design variables must be done efficiently since it has a direct impact on consumer cost. When one considers these balanced against the benefits of more efficient virtual design tools for the same products (which were previously expensive), these consumer products can find their way into a larger spectrum of users.

Being aware of these industrial needs, the Pennsylvania State University’s Center for Acoustics and Vibration has been developing a research program aimed at meeting these industrial sound quality needs for over the last decade. The ultimate aim of the noise-control-by-design program is to develop modeling strategies that can be used as tools for designing quiet products virtually, thereby by-passing the cumbersome and expensive path of designing, fabricating, and testing which often requires several iterations before an acceptable design is discovered. The cost savings benefits of such an approach are obvious. Numerical techniques in structural acoustics are utilized by the research program to help meet these needs through the introduction of means by which to virtually assess a product’s acoustic interaction with its service environment [23]. Furthermore, integrating optimization tools with numerical structural/acoustic modeling offers virtual design control aimed at meeting a variety of noise control requirements [22, 25]. It is optimization that allows design to be efficient with resources (i.e. materials, processing time, etc.) to meet product specific requirements at the same time as meeting the acoustic requirements.
Presently, the noise-control-by-design program combines several numerical programs that include structural modal analysis (either experimental or FEM-based), a boundary element program based on superposition methods (POWER), and Matlab-based optimization programs. Design variables include changes in mass and/or stiffness distributions on the surface of a radiating structure. The objective or cost function to be minimized is sound power. The motivation behind this thesis was to add shape optimization to the design tool box as an additional design variable. This is an unexplored area in acoustic radiation studies. The reason is that, under the constraints imposed by the use of meshed surfaces in conventional numerical modeling, the process of remeshing and updating system matrices for every design iteration is simply too cumbersome. A simpler approach must be developed, and it is hoped that developing this technique can offer yet another noise control solution to virtual product design. The benefit of shape optimization in this context can yield a less costly noise control technique for products having previously been restricted to more costly noise control alternatives. Yet, as will be introduced in this thesis, special tools are needed to ensure an efficient and robust execution of structural/acoustic shape optimization.

The essence of this thesis is to adopt a meshless strategy whereby an acoustically radiating surface is simply defined in terms of points rather than elements. The logical extension of this modeling procedure is to represent a surface in a digital format similar to the graphics community that uses pixels to represent surfaces as a collection of “turned on” points. This meshless technique gives way to simplifying the population of system matrices as tedious elemental numerical integrations for each iteration in shape are no longer necessary. In addition, the need for expensive corrective action for distortion in these models is no longer necessary. It should be noted, that meshless, structural mechanic, numerical programs have been under development for over a decade and show promise for their general acceptance as alternatives to the more conventional FEM programs.

This thesis does not develop a self-contained acoustic shape optimization tool that can be integrated in the CAV noise-control-by-design tool kit. However, it does lay the framework for such a program and provides an exciting new approach to solving acoustic radiation problems in a highly efficient and highly simplified manner.
1.2 Background

Figure 1.1 below illustrates the process flow for conventional structural/acoustic shape optimization. Here, the process begins with a virtual meshed structure created by the finite element method (FEM). Conventional FEM coding would define the nodes and elements giving way to computing the elemental area, centroid location, and outward pointing element surface normal. Next, the modal composition of this computational structure can be determined within a frequency band of interest using FEM. These structural modes are then used to represent the surface velocity of the structure in the boundary element method (BEM) used to compute the acoustic radiation from the structure. For this work, the BEM is based upon the method of wave superposition in which special attention is given to the difference in numerical integration schemes between diagonal and off diagonal matrix terms [23, 24]. Even further, the order of the off diagonal integration is dependent upon source and receiver location [23]. Once the surface pressure and velocity are known, the radiated acoustic power can be determined for the optimization objective function. If the convergence criteria have not been met, the optimization would then iterate the shape of the virtual structure. However, changing the shape of the structure results in element distortions, which degrade the computational accuracy of the elements and can render the results suspect to error. Therefore, morphed elements must be checked for numerical integrity, and if necessary, the virtual structure should be remeshed for use in the next FEM and BEM iteration. This process continues until convergence is reached in the objective function.
1.3 Proposed Work

As indicated in Figure 1.1, structural acoustic modeling is conventionally accomplished using the well established finite and boundary element methods (FEM and BEM). However, these techniques are computationally limited when modeling structural acoustic radiation problems. These limitations are due largely in part to the required matrix system equation assembly time and matrix rank deficiency (from nonunique solutions). Current state-of-the-art computational techniques that overcome the aforementioned limitations are termed element free or “meshless” techniques. The historic development of meshless methods was motivated by similar computational
deficiencies of traditional finite and boundary element techniques for modeling crack propagation in fracture mechanics [5]. As such, these meshless methods would also provide similar numerical advantages in structural/acoustic problems. Figure 1.2 illustrates these advantages in the process flow of structural/acoustic shape optimization using meshless techniques. The advantages are evident when one compares the process flow of Figure 1.2 to the process flow of Figure 1.1. From the comparison, it can immediately be seen that the numerical integrity of the elements need not be checked as conventional elements do not exist. As such, the absence of conventional elements eliminates the need for remeshing of the problem domain using complex meshing algorithms. This closes the optimization loop using fewer procedures than in the conventional element case. In addition, the system matrix assembly is simpler as the diagonal and the off diagonal terms are computed without the tedious numerical integration schemes of the conventional approach.

Optimally changing the shape of a structure so as to render it a weak radiator is a computationally demanding task both structurally and acoustically. Considering the structural aspect, when changing the shape of the radiator, the solution space for the eigenvalue problem can no longer be expanded using the eigenvectors from the original solution space. As such, the use of an efficient eigenvalue solver that can rely on expanding the solution space using the same set of basis vectors (such as the Rayleigh-Ritz method) is not feasible. Therefore, a computational method that can efficiently resolve the eigenvalue problem repeatedly is vital for considering modal participation in the acoustic radiation of the structure. As indicated, meshless techniques are able to respond to this computational challenge, by using techniques such as the Meshless Local Petrov Galerkin (MLPG) method. When considering the acoustic aspect, the absence of elements in meshless techniques removes the need for tedious numerical integration schemes over an element dependent upon that elements position in space (which changes for each iteration in shape), and can lend itself to faster matrix assembly time. Furthermore, without the presence of elements in the computational domain, there exists no potential for degradation in the computational accuracy of either the structural or acoustic solutions related to element distortions. As a consequence, there is no need for complex and time consuming remeshing of the computational domain.
However, as our interest lies in performing structural/acoustic shape optimization a way is needed in which to morph a surface. By defining the structure in a “digital” space, potential shape movements can be performed by “activating” uniformly spaced nodal positions already existing in this digital domain. The digital domain ensures that nodal positions are conveniently found next to each other in a contiguous manner given the resolution of the domain, as seen in Figure 1.2.

\[
\{u\} = \nabla \{G\}\{s\} \\
\{p\} = [G]\{s\}
\]

**Diagonal**

*ALL* co-located source/field points use the same *SELF TERM*

**Off-Diagonal**

Point values only

**Figure 1.2:** Meshless Shape Optimization Logic Flow

It is important to note that neither the evaluation of element computational integrity nor remeshing needs to be performed to close the optimization loop when using meshless techniques. The objective of this proposed work is then to develop an efficient and
robust meshless computational tool for the determination of acoustic radiation for use in shape optimization of arbitrary radiating structures. In an effort to realize the proposed objective, the research is divided into the two constituent categories addressed in this work identified as: one, the meshless acoustic radiation problem, and two, the digitized spatial domain defining the structure to optimize.

In keeping with meshless numerical techniques, the objective of this work is to develop a so-called meshless technique (acoustic superposition method using meshless, spatially digitized domains ASMDD), that provides for an efficient structural/acoustic shape optimization. The conventional technique of computational acoustics is based upon the principle of wave superposition, which couples the acoustic radiation of monopole and dipole sources located near a virtual structure’s surface so as to match the volume velocity of the actual vibrating structural surface [23]. In the spirit of meshless methods, ASMDD will resolve the acoustic radiation characteristics of a monopole and a dipole source at a point (a ASMDD node). The free space Greens function couples the radiation of each point source together to acoustically reconstruct the radiating surface. As mentioned, conventional boundary elements require tedious numerical integration over the surface of each radiating boundary element. However, since there exists no elements within the computational domain, meshless acoustics does not require the integration of acoustic pressures and velocities over the surface of an element. The result is a computational acoustic radiation method that consists solely of matrix inversion and multiplication. Matrix assembly is accomplished by knowing only coordinate locations in space. Such reliance on only matrix inversion and multiplication assembled from coordinates in 3-D space provides a computationally efficient method for computing the radiated sound power radiated by a vibrating structure.

Another major objective of this thesis is to develop the spatial domain of the structure from voxels. A voxel is best interpreted as a three dimensional version of a pixel as used in digital image processing. Each ASMDD node will be located at the center of a voxel where each exposed face of each voxel represents the outward surface of the structure being modeled. For each shape iteration, voxels are either activated or deactivated which is analogous to the use of 1’s and 0’s in various digital technologies in engineering and the sciences. It is important to note that the use of voxels is simply for
visualizing that each node is located on a uniform three dimensional grid. The uniform
distribution of ASMDD nodes are used to help determine locally the outward pointing
normal at each ASMDD node. There is no mathematical surface information provided to
the ASMDD nodes.

![Diagram](image)

**Figure 1.3:** Proposed Technical Items Addressed

In summary, there is interest in developing shape optimization as a design tool for
use in optimal structural/acoustic design. Current element based technologies would
require complex adaptive remeshing algorithms to accomplish this task, and are not
readily available for general research use. As such, this thesis attempts to develop its
own shape optimization design tool. The objective in this work is to introduce, derive,
and demonstrate the feasibility of ASMDD for solving acoustic radiation problems. Another objective is to use ASMDD in a digital space where the structure and the evolution of its shape is defined by nodes equally spaced in all three coordinate directions. Here, shape changes are accomplished by activating and deactivating points in a contiguous manner to generate a new shape. As a caveat, it is hoped that the new element-free/meshless ASMDD technique in a digital space possess significant computational savings over the conventional element based techniques giving further support for its use. However, quantitative proof of any computational savings will not be determined in this work as this thesis simply lays the ground work for developing a meshless boundary value solver for use in computational acoustic radiation problems. The scope of this work is to derive the ASMDD formulation as well as introduce the idea of spatially digitized domains. In addition, this work’s scope will show how to find point surface normal vectors from a local neighborhood of digital points. These formulations will then be use to compute the acoustic radiation from an example structure whose shape is manipulated through prescribed changes in shape. Also, this work’s scope will show how the ASMDD technique can blend with a state-of-the-art meshless eigenvalue solver, called the MLPG method. Figure 1.3 highlights in yellow the technical items addressed in this work as they were introduced in the process flow of Figure 1.2.
Chapter 2
SHAPE OPTIMIZATION

Optimization schemes are divided into three categories identified as size, shape, and topological optimization as indicated by Huff and Bernhard [20]. Here, size optimization refers to the use of element parameters such as thickness, geometry, and material properties as design variables. Shape optimization uses structural geometry as the design variable used to achieve the objective. Huff and Bernhard identify topological optimization as the most general of the three as it can include both shape and size parameters as well as other design variables such as stiffeners, holes, or the like. The focus of this research is to optimize the shape of an arbitrarily radiating structure to achieve prescribed acoustic radiation characteristics.

2.1 Shape Optimization in Structural Mechanics

Shape optimization has received very little attention in the field of acoustics and therefore is best introduced through its efforts in structural mechanics as they are much more numerous. However, despite the difference in objective functions when compared to acoustics and vibration (static versus dynamic) the mechanisms and concerns of performing shape changing are very much the same. It should be noted that the following discussion is based upon the use of elements (FEM and BEM). Such a perspective is helpful in supporting the use of meshless strategies when considering the limitations in elemental formulations.

There are numerous applications of shape optimization of structures. For example, Wassermann applied the technique of shape optimization to arch dams using hyperelements with prescribed shape functions [39]. The element mesh is specified within each hyperelement where each element uses traditional linear, quadratic, or cubic interpolation functions. Belegundu and Chandraputla use shape optimization to determine the optimal shape of a culvert using natural shape functions [3]. Bae et al. optimize the shape of an aircraft wing for both aerodynamic and aeroelastic
considerations [2]. Following aircraft wing morphing, Johnston et al. show that optimally changing the shape of an aircraft's wing can reduce the energy needed to maintain flight control [21]. These examples show that shape optimization is an effective and realizable technique for optimal design.

Given its success, a common theme of shape optimization in structural mechanics is most notably the difficulty in efficiently controlling the shape of a structure during optimization iterations. A survey publication by Ding indicates this concern as possible shape movements can deteriorate the accuracy of finite elements and require adaptive remeshing [15]. Another shape optimization survey, by Haftka and Grandhi, shares the same concern as Ding [19]. They state that shape optimization is typically aimed at reducing stresses at or near the boundary of a problem. Traditionally, the nodes located at the boundary of the problem are used as design variables yielding solutions that are not useful let alone to be trusted because of distortions. Ding as well as Haftka and Grandhi point out alternatives to using nodes as design variables. Here they suggest polynomial or spline representations of the boundaries. The design variables then become the polynomial coefficients or piecewise spline control points of each function used to describe the boundary. Then, a mesh is generated within the domain defined by these shape functions to relate changes in the domain shape to changes in the FEM nodes so as to evaluate the objective function. Such a method is aimed at smoothing the iterative shapes at these boundaries. However, both papers address a possible need for mesh refinement. Although automated refinement is the most desirable, sophisticated algorithms are needed to efficiently mesh and monitor the problem domain and boundary for violations in element accuracy. As such, mesh refinement can burden the computational efficiency of structural shape optimization. In addition, Ding suggests three possible mesh refinement strategies which are adding additional elements, increasing the order of the finite element, or a remeshing of the entire problem domain. However, no recommendations are made as to the preference from the previous list based on the problem type. Work by Bennet and Botkin also express the need for adaptive refinements to ensure solution accuracy [6, 9].

In an effort to address element distortions during shape iterations, Robinson shows the extent to which distortions are possible and how to parametrically describe the
possible distortions encountered by a quadrilateral element [35]. The value here is to note the various kinds of distortions and the necessary computations involved to identify the extent of these distortions and possible corrective action. Figure 2.1 shows the nomenclature that Robinson uses to help determine the extent of quadrilateral element warpage.

![Warped Quadrilateral Element and Nomenclature](image)

**Figure 2.1:** Warped Quadrilateral Element and Nomenclature

Robinson defines vectors pointing towards points \(0, 5, 6, 7,\) and \(8\) and vectors \(V_{ij}\) pointing from the local element origin \(0\) towards points \(A, B, C,\) and \(D\) as

\[
\begin{align*}
V_5 &= \frac{1}{2}(V_A + V_B) \\
V_6 &= \frac{1}{2}(V_B + V_C) \\
V_7 &= \frac{1}{2}(V_C + V_D) \\
V_8 &= \frac{1}{2}(V_D + V_A) \\
V_0 &= \frac{1}{4}(V_A + V_B + V_C + V_D)
\end{align*}
\]

Local elemental coordinates, identified in Figure 2.2, are defined using the vector definitions of Equation 2.1 yielding
Finally, Robinson shows that the distortion parameters $e_i$ and $f_i$ in Equation 2.3 can be defined in terms of Equation 2.2. These distortion parameters are used to identify four types of distortions as seen in Figure 2.3.

$$
e_1 = \frac{1}{4} (x_1 + x_2 + x_3 + x_4), \quad f_1 = \frac{1}{4} (y_1 + y_2 + y_3 + y_4)$$

$$e_2 = \frac{1}{4} (-x_1 + x_2 + x_3 - x_4), \quad f_2 = \frac{1}{4} (-y_1 + y_2 + y_3 - y_4)$$

$$e_3 = \frac{1}{4} (-x_1 - x_2 + x_3 + x_4), \quad f_3 = \frac{1}{4} (-y_1 - y_2 + y_3 + y_4)$$

$$e_4 = \frac{1}{4} (x_1 - x_2 + x_3 - x_4), \quad f_4 = \frac{1}{4} (y_1 - y_2 + y_3 - y_4)$$

(2.3)
Equation 2.4 specifies the distortion calculations based on the parameters identified in Figure 2.3. Although Robinson does not specify reasonable distortion limits for each distortion type, user defined limits can be applied to ensure element accuracy as desired.

\[
\text{Aspect Ratio} = \max\left(\frac{e_2}{f_3}, \frac{f_3}{e_2}\right)
\]

\[
\text{Skew} = \frac{e_3}{f_3} \quad \text{Skew}
\]

\[
\text{Taper in y - direction} = \frac{e_4}{e_2}
\]

\[
\text{Taper in x - direction} = \frac{f_4}{f_3}
\]

In addition, Robinson addresses the calculation of quadrilateral degeneration and nonconvex shapes as well. As such, for shape optimization, all of the aforementioned distortions must be monitored to ensure solution accuracy. Triangular elements, although simpler than quadrilateral element, need not worry about convexity issues when using linear interpolation functions (LST), however, they still have the potential to significantly distort causing error in the final solution [36]. Yet, as the order of the interpolating shape function increases, convexity can become an issue. In this work, Salem et al. suggest the possibility of element repair. El-Hamalawi suggests using interior element angles to monitor distortion in contrast to using element lengths as suggested by Robinson [17].
This method lumps some of distortion parameters based on length into one thereby reducing the number of calculations per element. Such calculations are required for each element regardless.

As suggested before, shape control can be accomplished using a multiplicity of polynomial functions or piecewise spline functions to describe the shape of the boundary [8, 10, 18, 37]. Specifically, the polynomial coefficients or the spline control points are used as design variables. The choice must then be made as to how many shaping functions or patches to use so as to allow for the most flexibility in possible shape iterations. However, Rajan and Belegundu offer an alternative to using these piecewise analytical functions. Here Rajan and Belegundu define natural design variables which utilize static displacement field results from nodes permitted to take part in the shape change [34]. These displacement fields are developed from an auxiliary structure by enforcing a unit displacement normal to each “morphable” nodal location and then solving the resulting FEM static problem. The auxiliary structure is the same FEM structure to be optimized, however it is constrained in such a way that only the degrees-of-freedom (dof) of the nodes permitted to take part in the shape change are displaced when enforcing the unit normal displacement. These resulting displacement fields are developed a priori and stored for use in the optimization. The design variables then become scalar multipliers operating on each displacement field to determine the new shape for each iteration yielding

\[
G_{\text{new}}(\vec{x}) = G_{\text{old}}(\vec{x}) + \sum_{i=1}^{N} \beta_i \vec{d}_i ,
\]

where \( G_{\text{new}}(\vec{x}) \) is the new shape, \( G_{\text{old}}(\vec{x}) \) is the previous iterated shape, and \( \vec{d}_i \) is the resulting displacement field developed from the auxiliary structure when enforcing a unit normal displacement at “morphable” node \( i \). The scalar multiplier \( \beta_i \) is the design variable that determines the magnitude of displacement field \( i \) and hence its influence on the entire “morphable” domain of the problem when summed with all \( \beta_i \vec{d}_i \). For example, if a node is not permitted to move during the optimization, the corresponding dof values in \( \vec{d}_i \) will be zero. However, if a node is permitted to move during the optimization the
corresponding dof values in $\vec{d}_i$ will be nonzero. The displacements in Equation 2.5 can be summed as $\vec{d}_i$ is a FEM solution based on linear mechanics.

In summary, shape optimization using finite or boundary elements requires careful and repeated inspection of element distortion and accuracy. Although parametric values can be computed to identify the extent of distortion and hence solution inaccuracy, the procedure can be exhaustive for a domain with large numbers of elements. The use of polynomial or spline functions offers shape control using polynomial coefficients or key domain control points as design variables [10]. As such, a boundary can be reconstructed using piecewise collections of these functions. However, mesh generation is still necessary to relate changes in shape functions to changes in nodal displacements before evaluating the objective function [18]. The efficiency of such a procedure is directly dependent on the efficiency of mesh generation algorithms. The shape control technique developed by Belegundu and Rajan offers control over the entire "morphable" domain through only one design variable [4]. This technique does not rely on intermittent mesh generation as the mesh is beginning update directly. Furthermore, such a technique conveniently coincides with meshless techniques in structural acoustics as shape iterations are based only on a summation of nodal coordinate locations. For a meshless application, the algorithm would not need to monitor element distortions as elements do not exist. However, this technique does not address the possibility of adaptive refinement. Currently, the method would require the development of new static displacement fields should more nodes be necessary. Also, an algorithm to control the placement of additional nodes to accurately capture the structural curvature would be necessary as well.

2.2 Shape Optimization in Structural Acoustics

Having considered shape optimization in detail for structural mechanics using elements, a foundation well suited to exploring the more limited structural acoustic application is established.

Shape optimization in acoustics saw the earliest published works in the 1980’s. It was about this time that Milner optimized a boundary enclosing an acoustic cavity using
triangular parametric finite elements [29]. Here, the finite element matrices, sensitivities, and element interpolating functions were derived as parametric functions of the nodal positions in space. During shape iterations, the problem domain was not remeshed with more elements all using the same low order interpolation shape function (h-version method). Rather, Milner utilized p-version elements where the order of the interpolating polynomials of the elements who share a morphing node increase to accommodate the increasing complexity in shape. As such, no additional elements were added to the problem domain. Although Milner recognized the potential for the finite element mesh to become distorted or elements to become degenerate and hence introduce error into the final solution, the p-version method was identified as least sensitive to mesh distortion when compared to the h-version. Milner stored pre-computed mass and stiffness matrices corresponding to various orders $p$ of interpolating functions to be called upon during shape iterations. Such an approach can potentially avoid costly remeshing, however it requires enough excess storage for a library of mass and stiffness matrices with sufficient orders $p$ of interpolation functions. Yet, as the discussion of shape optimization in structural mechanics points out, higher orders of interpolating shape functions do not exclude the possibility of distortions and/or remeshing within the problem domain.

Christensen and Olhoff [14] performed a shape optimization of a loudspeaker diaphragm. Their objective was to shape the diaphragm so as to achieve a uniformly shaped directivity pattern. Here, the authors exploited the angular axisymmetric geometry of a loudspeaker diaphragm and modeled it only as a line as seen in Figure 2.4.

![Figure 2.4: B-Spline Representation of Speaker Diaphragm](image)

The B-Spline uses 8 modifiers spanned by 9 points to model the diaphragm. The optimization design variables are the 8 modifiers between the 9 points and dictate the
shape of the diaphragm. The use of a spline function to describe the shape of the loudspeaker diaphragm ensures a smooth and continuous result along the radius of the speaker diaphragm. Furthermore, with a one dimensional representation of the surface, the candidate solution does not suffer from distortions as a conventional surface finite element would. Such assurances give way to more efficient optimization and robustness of solution. However, the conveniences used in this work by Christensen and Olhoff are not readily translated to general radiation problems of higher dimension.

Because of the limited history and number of results for shape optimization in computational acoustics, it is of value to determine if indeed changes in shape offer significant changes in the radiated acoustic power. To help understand what the potential differences in radiated acoustic power might be, an example problem is used. The radiated acoustic power from a uniformly pulsating cube with 1 meter sides and a surface velocity of $1 \frac{m}{s}$ is computed using conventional BEM methods. This radiation spectrum is then compared to a uniformly pulsating sphere with a radius of $\frac{\sqrt{3}}{2}$ meters, also computed using conventional BEM methods. To ensure that the cube and sphere are equivalent radiators, the surface velocity of the sphere is determined such that the cube and sphere have the same overall volume velocity. As such, the surface velocity of the sphere is determined as

$$\frac{6 \times 1 m^2 \times 1 m/}{4\pi \left(\frac{\sqrt{3}}{2}\right)^2 m^2} = 0.637 \frac{m}{s}.$$ 

The result of the comparison of the radiation spectrums from the uniformly pulsating cube and pulsating sphere of equal volume velocity is given in Figure 2.5. The media of propagation is air where $\rho = 1.21 \frac{kg}{m^3}$ and $c = 343 \frac{m}{s}$. It can be seen from the plot that the cube has a higher level of radiation than the sphere by approximately 3 dB.
Therefore, the results of Figure 2.5 show that changes in shape have the potential to affect significant changes in the radiation spectrum from a structure. Such results give support to the development of shape optimization as a design tool for use in the optimal design of quiet structures. It should also be noted that if the structures were made with not equal volume velocity (area \times velocity) but rather equal values of area \times velocity^2, then the spectral difference in the radiated acoustic power in the shapes would change at low frequency when compare to Figure 2.5. Then, these structures would have the same high frequency limit.

To summarize, shape optimization has seen successful use in structural mechanics. However, this success relies on being able to control the shape of the structure to optimize, while correcting element distortions that can occur during a change in structural shape. Spline functions, unit structural displacements, and piecewise polynomials are a few of the shape controlling techniques that are currently used. Regardless of the technique, each iteration in shape requires a through evaluation of the computational domain to make sure elements are within required computational tolerances. As such,
element evaluations use parametric distortion values to access the numerical integrity of
the elements making up the problem domain. Highly complex adaptive remeshing
algorithms and local element repair are then used as the corrective actions during an
optimization. To eliminate the use of these highly complex adaptive meshing algorithms,
this work will set the stage for its own shape changing strategy based on using a digitized
spatial domain. Here, the strategy would be to simply activate voxels in space that render
a contiguous surface. Using voxels would ensure that regardless of the iteration in shape,
each meshless point would be representing a uniform and undistorted portion of the
virtual structure. As such, no checks of numerical integrity would be necessary, and
hence highly complex adaptive corrective actions are also eliminated.
3.1 Introduction to Acoustic Boundary Element Analysis

By using a variety of coordinate spaces such as rectangular, spherical, cylindrical, etc., analytical models can be derived to find the sound power radiated by a vibrating body appropriately defined in these spaces. However, for many applications, vibrating bodies are not convenient shapes with separable coordinates that possess closed form solutions. In an effort to determine the sound power radiated by these arbitrarily shaped radiators, numerical techniques must be implemented. As such, the sound power radiated from a vibrating body can be determined in a discrete/numerical manner by using the lumped method of wave superposition.

To begin, it is of value to understand the benefit of the lumped wave superposition method over traditional Boundary Element Method (BEM) techniques for determining sound power radiation. Traditional BEM techniques rely on integrating the governing system’s partial differential equation over the discrete elements in the problem domain using locally defined linear, quadratic, etc. basis functions (i.e. shape functions). Any amount of error is due mainly to the inability of the approximate solution to reproduce the boundary conditions. The advantage of wave superposition is that it allows the reproduction of the boundary conditions exactly by using a set of basis functions that are exact solutions to the system’s governing partial differential equation (i.e. the wave equation).

However, it was determined that, in trying to reproduce the boundary conditions at every point on the boundary of the problem domain, the strategy was impractical [23]. This conclusion gave way to the use of a lumped model of acoustic sources near the centroid of an element close to the structure’s surface. The influence of these acoustic sources is numerically integrated over each element in the problem domain. The use of Green’s functions as basis functions allows for the reproduction of the exact solution only at a point on the surface. As such, the exact solution is approximated by taking an
average of the elemental surface velocity of the surrounding nodes and multiplying it by the elemental area. Then, appropriate source strengths are computed to match this approximated volume velocity. It is important to note that because the approximated volume velocity is determined by averaging the surrounding nodal velocities, the velocity is assumed to vary linearly over the surface of an element. As such, a requirement of the lumped parameter method of wave superposition is the characteristic length of an element remains small compared with the acoustic wavelength.

3.2 Method of Wave Superposition

The lumped parameter method of wave superposition superimposes the radiated field of a finite number of acoustic sources near the surface of the radiator as seen in Figure 3.1,

![Figure 3.1: Superposition of Simple Sources on Original Radiator](image)

where each source $n$ is located at a point $\vec{r}_n$ defined in three space [23]. Figure 3.1 also identifies both the acoustic and structural domains for the arbitrary radiating structure given by $V$ and $S$ respectively. The resulting pressure on the surface of the radiator at a point $\vec{r}_m$ defined in 3D space is expanded using a set of basis functions that satisfy the wave equation and is defined as

$$ p(\vec{r}_m) = \sum_{n=1}^{N} s_n T_n(\vec{r}_m), $$

(3.1)
which has been shown to be equal to the KHIT (Kirchhoff–Helmholtz Integral Theorem) [24]. The function $T_n(\vec{r}_m)$ is a linear combination of simple monopole and dipole basis functions given as

$$T_n(\vec{r}_m) = \alpha G(\vec{r}_m | \vec{r}_n) + \beta V_n G(\vec{r}_m | \vec{r}_n) \cdot \vec{r}_n,$$  \hspace{1cm} (3.2)

where the expansion coefficients $\alpha$ and $\beta$ activate the monopole and dipole sources respectively. The expansion coefficients $\alpha$ and $\beta$ have some combination of either a zero value (deactivated source) or a value of $I$ and $\frac{i}{k}$ respectively (activated source). The use of two terms in the expansion is to overcome the non-uniqueness problem encountered for the interior problem [11]. The goal here is to allow one source in the pair of Equation 3.2 to ensure a unique solution for the exterior radiation problem when the other provides a non-unique solution. Looking more closely at the cause of the non-uniqueness, the finite distribution of acoustic sources as seen in Figure 3.1 encloses a finite volume. This interior volume posses an infinite set of resonant conditions characterized by eigenvalues and eigenvectors. Here, a non-unique solution is encountered when the frequency of the exterior problem coincides with a resonance frequency for the interior problem for the prescribed boundary conditions. This condition thereby renders the determinant of the exterior problem system matrices to be zero giving way to a singular matrix value. Yet, as this condition occurs for only one source type (monopole), the other (dipole) can compensate by providing a unique solution. However, it is interesting to note that the non-uniqueness is a consequence of the computational formulation only and has no physical significance [11].

The wave superposition method uses a volume velocity matching scheme to determine the appropriate source strength $s_n$ to reproduce the original structural surface velocity $v(\vec{r}_n)$. The volume velocity is determined by integrating the modal structural vibration velocity over the area of each element. The linearized Euler equation shows that the particle velocity is related to the acoustic pressure through the relation

$$\rho \frac{\partial v(\vec{r})}{\partial t} \cdot \vec{r}_m = -\nabla p(\vec{r}_m).$$  \hspace{1cm} (3.3)
Assuming an $e^{i\omega t}$ time convention for outward propagating waves, Equation 3.3 can be written as
\[ i\omega \rho v(\bar{r}_m) \cdot \bar{\Pi}_m = -\nabla p(\bar{r}_m). \tag{3.4} \]
Substituting Equation 3.1 into Equation 3.4, dividing through by the constants, and integrating over the area of element $m$ yields a volume velocity of
\[ u(\bar{r}_m) = \frac{i}{kc\rho} \iint \sum_{n=1}^{N} s_n \nabla_{n} \left[ \alpha G(\bar{r}_m | \bar{r}_n) + \beta \nabla_{n} G(\bar{r}_m | \bar{r}_n) \cdot \bar{n}_n \right] \cdot \bar{n}_m \, dS_m, \tag{3.5} \]
where $u_m$ is the volume velocity of element $m$, $s_n$ is the source strength, $\bar{n}_m$ is the unit outward pointing normal of the elemental radiator, $\omega = kc$ is the circular frequency of oscillation, and $\rho$ is the density of the acoustic medium. A graphical representation of Equation 3.5 is seen in Figure 3.2.

\[ \text{Figure 3.2: Graphical Representation of Equation 3.5} \]

Defining a new variable $U_{mn}$ as
\[ U_{mn} = \frac{i}{kc\rho} \iint \sum_{n=1}^{N} \nabla_{n} \left[ \alpha G(r_m | r_n) + \beta \nabla_{n} G(r_m | r_n) \cdot \bar{n}_n \right] \cdot \bar{n}_m \, dS_m, \tag{3.6} \]
and then writing Equation 3.5 in matrix form gives
\[ \{u_m\} = [U_{mn} \{s_n\}]. \tag{3.7} \]
Solving Equation 3.7 for the source strength vector $\{s_n\}$ and substituting into Equation 3.1 yields the surface pressure. Having both the pressure and the velocity the time averaged intensity can be integrated over the problem domain to yield the radiated power as
\[
\Pi_{AV} = \frac{1}{2} \text{Re} \left\{ \iint_S \{ p_m \}^h (\overline{\mathbf{v}}_m \cdot \overline{\mathbf{n}}_m)^h dS \right\},
\]
where \( H \) represents a Hermitian (complex conjugate transpose). All acoustic power calculations made in the remainder of this thesis are in air where the density \( \rho = 1.21 \, \frac{kg}{m^3} \) and the speed of sound \( c = 343 \, \frac{m}{s} \).

### 3.3 Acoustic Superposition using Meshless Spatially Digitized Domains (ASMDD)

In the same spirit as meshless structural mechanics, as will be introduced in Chapter 5, this research develops meshless acoustic boundary value problems for solving structural/acoustic radiation problems. This technique concentrates the acoustic radiation of monopole and dipole sources at points in space throughout the frequency range of interest. Singularities resulting from the coincidence of the source and receiver points are replaced with nonsingular, analytical expressions for the concentrated pressure and velocity. The singularity here occurs within the 3D Greens Function. Figure 3.3 illustrates the acoustic superposition method using meshless spatially digitized domains (ASMDD) for determining self terms (co-located source/receiver) by concentrically locating a monopole and a dipole source some distance \( z \) above a circular surface. This circular surface represents a portion of the surface area of the original radiating structure and is chosen for its convenient symmetry when integrating. Here, it assumes that the nodal distribution for \( N \) nodes is uniform and that all nodal positions each occupy \( \frac{1}{n^m} \) of the total area of the surface being modeled.
Figure 3.3: Monopole and Dipole Sources Radiating to Circle

From Figure 3.3, \( \vec{n} \) and \( \vec{d} \) represent the receiver normal vector and the dipole moment vector, respectively. The radius of the circular surface is \( a \) and the distance between the source and any receiver location on the circular area is \( R = \sqrt{r^2 + z^2} \). The pressure radiated by the monopole and dipole sources is averaged over the area of the circular surface receiver location. The velocity developed at the receiver location due to this pressure on the circular surface is also averaged over the area of the circle. Upon integrating these values, the resulting equations are functions of \( z \). In the limit as \( z \) tends towards zero, the resulting area averaged pressure and velocity values become the self terms for a monopole and dipole source. As expected, their sum yields the self term for a tripole source. For simplicity, the monopole and dipole self terms are derived separately.

3.3.1 ASMDD Monopole Self Term

The pressure field \( p \) radiated by a monopole source is given by the free space 3D Green’s function (the Green’s function of the first kind) multiplied by the source strength \( s \) as

\[
p = s \frac{e^{ikR}}{R}, \tag{3.9}
\]

where \( k \) is the wavenumber and \( R \) is defined as in Figure 3.3. The pressure developed on the circular surface due to the monopole self term is defined as

\[
p_m = s_m g_m = \frac{s_m}{\pi a^2} \int_0^{2\pi} \int_0^a e^{-ik\sqrt{r^2+z^2}} r dr d\theta, \tag{3.10}
\]
where the term \( g_m \) is the nonsingular monopole pressure self term in the diagonal of the Green’s function matrix, and \( s_m \) is the respective source strength. Performing a change of variables as \( u = \sqrt{r^2 + z^2} \) and integrating out the axis-symmetric \( \theta \) variable, the integral in Equation 3.10 becomes

\[
g_m = \frac{2\pi}{\pi a^2} \int e^{-iku} u^{-1} du ,
\]

where finally the integration over the limits yields

\[
g_m = \frac{2\pi i}{k\pi a^2} \left\{ \frac{e^{-ik\sqrt{a^2 + z^2}}}{z} - e^{-ik|z|} \right\}.
\]

The monopole pressure self term \( g_m \) is determined in the limit of Equation 3.12 as

\[
\lim_{z \to 0} \left[ \frac{2\pi i}{k\pi a^2} \left\{ \frac{e^{-ik\sqrt{a^2 + z^2}}}{z} - e^{-ik|z|} \right\} \right] = \frac{2\pi i}{k\pi a^2} \left\{ e^{-ik|z|} - 1 \right\}.
\]

The normal velocity \( v \) at the receiver location resulting from the pressure radiated by the monopole source is determined using the linearized Euler equation seen as

\[
\rho \frac{\partial v}{\partial t} \cdot \vec{n} = -\nabla p ,
\]

where \( \nabla \) is the gradient operator with respect to the receiver location. As such, the normal velocity self term \( v_m \) developed at the receiver location due to the monopole pressure self term \( p_m \) can be determined by substituting Equation 3.9 into Equation 3.14 and averaging the result over the area of the circular surface giving

\[
v_m = s_m \nabla g_m \cdot \vec{n} = \frac{s_m}{io\rho \pi a^2} \int_0^{2\pi} \int_0^a \frac{\partial}{\partial R} \left( \frac{e^{-ikR}}{R} \right) \nabla R \cdot \vec{n} dR d\theta .
\]

It should be noted that Equation 3.15 assumes time harmonic motion as \( e^{i\omega t} \). Performing the same change of variables as before with \( u = \sqrt{r^2 + z^2} \) and integrating out the axis-symmetric \( \theta \) variable yields

\[
\nabla g_m \cdot \vec{n} = \frac{2\pi}{i\rho c k\pi a^2} \int_{\vec{u}} \frac{\partial}{\partial u} \left( \frac{e^{-iku}}{u} \right) \nabla R \cdot \vec{n} du.
\]

Noting that \( \nabla R \cdot \vec{n} \) is simply represented as \( -\frac{z}{u} \) after applying the change of variables, where Equation 3.16 can now be written as
\[
\n\nabla g_m \cdot \bar{n} = \frac{-2\pi}{i \rho c k \pi a^2} \int_{|u|}^{\sqrt{a^2 + z^2}} \frac{\partial}{\partial u} \left( e^{-iku} u \right) z \text{du}.
\] (3.17)

The integrand of Equation 3.17 is left in a convenient form that cancels the integration and differentiation as they are done with respect to the same variable \(u\). Evaluating Equation 3.17 gives

\[
\nabla g_m \cdot \bar{n} = \frac{2\pi i}{\rho c k \pi a^2} \left( e^{-i\sqrt{a^2 + z^2}} - e^{-i|z|} \right) z.
\] (3.18)

As before, the monopole velocity self term is determined in the limit of Equation 3.18 as

\[
\lim_{z \to 0} \left[ \frac{2\pi i}{\rho c k \pi a^2} \left( e^{-i\sqrt{a^2 + z^2}} - e^{-i|z|} \right) z \right] = \frac{-2\pi i}{\rho c k \pi a^2}.
\] (3.19)

### 3.3.2 ASMDD Dipole Self Term

The same strategy applied to derive the monopole pressure and velocity self terms can be applied to derive the same pressure and velocity self terms as generated by the dipole source. The pressure field radiated by a dipole source is given as

\[
p = s \nabla_d e^{-iR \bar{d} / R},
\] (3.20)

where \(\nabla_d\) is the gradient operator with respect to the dipole source location, \(s\) is the source strength, and \(\bar{d}\) is the dipole orientation vector. Equation 3.20 can then be integrated to average the pressure \(p_d\) over the area of the circular surface developed by the dipole self term giving

\[
p_d = s_d g_d = \frac{s_d}{\pi a^2} \int_0^{2\pi} \int_0^a \frac{\partial}{\partial R} \left( e^{-iR \bar{d} / R} \right) \nabla_d R \cdot \bar{d} \text{drd} \theta,
\] (3.21)

where the term \(g_d\) is the nonsingular dipole pressure self term in the diagonal of the Green’s function matrix, and \(s_d\) is the respective source strength. As before, integrating out the axis-symmetric \(\theta\) variable and performing the same change of variables as before with \(u = \sqrt{r^2 + z^2}\) gives the dipole pressure self term as
\begin{align}
\mathbf{g}_d = \frac{1}{\pi a^2} \int_{|\mathbf{r}|} \frac{\partial}{\partial u} \left( \frac{e^{-i\mathbf{u} \cdot \mathbf{R}}}{u} \right) \nabla_{\mathbf{d}} \mathbf{R} \cdot \mathbf{u} du.
\end{align}

(3.22)

The quantity \( \nabla_{\mathbf{d}} \mathbf{R} \cdot \mathbf{u} \) is simply \( \frac{z}{u} \) after applying the change of variables, and now Equation 3.22 can be written as

\begin{align}
\mathbf{g}_d = \frac{1}{\pi a^2} \int_{|\mathbf{r}|} \frac{\partial}{\partial u} \left( \frac{e^{-i\mathbf{u} \cdot \mathbf{R}}}{u} \right) \frac{z}{u} du.
\end{align}

(3.23)

The convenience of the integrand in Equation 3.23 permits the cancellation of the integration and differentiation as they are done with respect to the same variable \( u \). Evaluating Equation 3.23 gives

\begin{align}
\mathbf{g}_d = \frac{2\pi}{\pi a^2} \left\{ \frac{e^{-ik\sqrt{a^2 + z^2}}}{\sqrt{a^2 + z^2}} - \frac{e^{-ik|\mathbf{r}|}}{|\mathbf{r}|} \right\} z.
\end{align}

(3.24)

The dipole pressure self term is then determined in the limit of Equation 3.24 as

\begin{align}
\lim_{z \to 0} \left[ \frac{2\pi}{\pi a^2} \left\{ \frac{e^{-ik\sqrt{a^2 + z^2}}}{\sqrt{a^2 + z^2}} - \frac{e^{-ik|\mathbf{r}|}}{|\mathbf{r}|} \right\} z \right] = -\frac{2\pi}{\pi a^2}.
\end{align}

(3.25)

The normal velocity at the receiver location resulting from the pressure radiated by a dipole source is determined as before using Equation 3.14. Substituting Equation 3.20 into Equation 3.14 and solving for the velocity \( \mathbf{v}_d \) due to the dipole self term yields

\begin{align}
\mathbf{v}_d = s_{\mathbf{d}} \nabla \mathbf{g}_d \cdot \mathbf{n} = -s_{\mathbf{d}} \frac{2\pi}{i\omega \rho a^2} \int_0^a \int_0^{2\pi} \nabla \left\{ \nabla_{\mathbf{d}} \frac{e^{-ikR}}{R} \right\} \cdot \bar{\mathbf{n}} r dr d\theta.
\end{align}

(3.26)

After distributing the gradient operators in Equation 3.26 and integrating out the axisymmetric \( \theta \) variable, the dipole velocity self term is given as

\begin{align}
\nabla \mathbf{g}_d \cdot \mathbf{n} = -\frac{2\pi}{i\omega \rho a^2} \int_0^a \left\{ \nabla \left[ \frac{\partial}{\partial R} \left( \frac{e^{-ikR}}{R} \right) \right] \left( \nabla_{\mathbf{d}} \frac{e^{-ikR}}{R} \right) \cdot \bar{\mathbf{n}} + \frac{\partial}{\partial R} \left[ \frac{e^{-ikR}}{R} \right] \nabla \left[ \nabla_{\mathbf{d}} \frac{e^{-ikR}}{R} \right] \cdot \bar{\mathbf{n}} \right\} r dr,
\end{align}

(3.27)

which can be further simplified as
\[ \nabla g_d \cdot \mathbf{n} = \frac{-2\pi i}{\rho c k \alpha^2} \int_0^\rho \left\{ \frac{\partial^2}{\partial R^2} \left( \frac{e^{-ikR}}{R} \right) \left( \nabla_d R \cdot \mathbf{d} \right) \left( \nabla R \cdot \mathbf{n} \right) + \frac{\partial}{\partial R} \left( \frac{e^{-ikR}}{R} \right) \nabla \left[ \nabla_d R \cdot \mathbf{d} \right] \cdot \mathbf{n} \right\} \,. \]  

(3.28)

Utilizing the same change of variables as before and the vector relationships of \( \nabla_d R \cdot \mathbf{d} = \frac{z}{R} \), \( \nabla R \cdot \mathbf{n} = -\frac{z}{R} \), and \( \nabla \left[ \nabla_d R \cdot \mathbf{d} \right] \cdot \mathbf{n} = \frac{z^2}{R^3} \), Equation 3.28 can then be written as

\[ \nabla g_d \cdot \mathbf{n} = \frac{2\pi}{i\rho c k \alpha^2} \sqrt{\frac{\alpha^2 + z^2}{z^2}} \int_\alpha \left\{ \frac{\partial^2}{\partial u^2} \left( \frac{e^{-iku}}{u} \right) \frac{z^2}{u^2} - \frac{\partial}{\partial u} \left( \frac{e^{-iku}}{u} \right) \frac{z^2}{u^3} \right\} du \,. \]  

(3.29)

For the sake of ease, the integral of Equation 3.29 is distributed and thus allows us to separate the difference in the integrand as

\[ \nabla g_d \cdot \mathbf{n} = \frac{2\pi}{i\rho c k \alpha^2} \left[ \frac{\sqrt{\alpha^2 + z^2}}{u} \frac{\partial}{\partial u} \left( \frac{e^{-iku}}{u} \right) \frac{z^2}{u^2} du - \int_\alpha \frac{\partial}{\partial u} \left( \frac{e^{-iku}}{u} \right) \frac{z^2}{u^3} du \right] \,. \]  

(3.30)

Solving the first integral in Equation 3.30 through integration by parts yields

\[ \frac{2\pi}{i\rho c k \alpha^2} \left[ \frac{z^2}{u} \frac{\partial}{\partial u} \left( \frac{e^{-iku}}{u} \right) \frac{\sqrt{\alpha^2 + z^2}}{u^3} \right]_{|u|} + \int_\alpha \frac{z^2}{u^2} \frac{\partial}{\partial u} \left( \frac{e^{-iku}}{u} \right) du \,. \]  

(3.31)

Solving the second integral in Equation 3.30 through integration by parts yields

\[ \frac{-2\pi}{i\rho c k \alpha^2} \left[ \frac{z^2 e^{-iku}}{u^2} \frac{\sqrt{\alpha^2 + z^2}}{u} \right]_{|u|} + \int_\alpha \frac{2z^2 e^{-iku}}{u^3} du \,. \]  

(3.32)

Combining the integration results of Equation 3.31 and Equation 3.32 yields the integration solution to Equation 3.30 giving

\[ \frac{2\pi}{i\rho c k \alpha^2} \left[ \frac{z^2}{u} \frac{\partial}{\partial u} \left( \frac{e^{-iku}}{u} \right) \frac{\sqrt{\alpha^2 + z^2}}{u^3} \right]_{|u|} + \int_\alpha \frac{z^2}{u^2} \frac{\partial}{\partial u} \left( \frac{e^{-iku}}{u} \right) - \frac{2z^2 e^{-iku}}{u^3} \right] du \,. \]  

(3.33)
Noting that both the integrand and the terms evaluated over the integration limits in Equation 3.33 are simply the resulting expansion of a first derivative of products with respect to \( u \) allows them to be written more simply as

\[
\frac{2\pi}{\iota \rho c k \pi a^2} \left[ \frac{\partial}{\partial u} \left( \frac{e^{-i k u}}{u} \frac{z^2}{u^2} \right) \right]_{\|} + \int_{\|} \frac{\partial}{\partial u} \left( \frac{e^{-i k u}}{u} \frac{z^2}{u^2} \right) du.
\]

(3.34)

As has been seen before, the convenience of the integrand in Equation 3.34 permits the cancellation of the integration and differentiation as they are done with respect to the same variable \( u \). Noting this allows Equation 3.34 to be written as

\[
\frac{2\pi}{\iota \rho c k \pi a^2} \left[ \frac{\partial}{\partial u} \left( \frac{e^{-i k u}}{u} \frac{z^2}{u^2} \right) \right]_{\|}.
\]

(3.35)

Expanding the derivative of Equation 3.35 and applying the limits of integration yields

\[
\nabla g_d \cdot \vec{n} = \frac{2\pi}{\iota \rho c k \pi a^2} \left[ \frac{e^{-i k \sqrt{a^2 + z^2}}}{a^2 + z^2} \left( 1 + ik \sqrt{a^2 + z^2} \right) \frac{z^2}{\sqrt{a^2 + z^2}} - e^{-i k |\vec{n}|} \left( 1 + ik |\vec{z}| \right) \frac{z^2}{|\vec{z}|} \right].
\]

(3.36)

The dipole velocity self term is then determined in the limit of Equation 3.36 as

\[
\lim_{z \to 0} \left[ \frac{2\pi}{\iota \rho c k \pi a^2} \left( \frac{e^{-i k \sqrt{a^2 + z^2}}}{a^2 + z^2} \left( 1 + ik \sqrt{a^2 + z^2} \right) \frac{z^2}{\sqrt{a^2 + z^2}} - e^{-i k |\vec{n}|} \left( 1 + ik |\vec{z}| \right) \frac{z^2}{|\vec{z}|} \right) \right].
\]

(3.37)

However, it is not immediately obvious as to whether or not Equation 3.37 is bounded. Therefore, all cancellations and bounded limit values (those tending to zero) are applied and the result is given below so as to isolate any unbounded terms that survive the cancellations. Equation 3.38 shows these cancellations and limit values as

\[
\lim_{z \to 0} \left[ \frac{2\pi}{\iota \rho c k \pi a^2} \left( \frac{-2\pi}{\iota \rho c k \pi a^2} \left[ ik + \frac{e^{-i k |\vec{n}|}}{|\vec{n}|} \right] \right) = \infty.
\]

(3.38)

From Equation 3.38, it can be seen that the denominator of the exponential term goes to zero, where the numerator goes to unity, thereby rendering Equation 3.38 unbounded. As such, an alternative approach is needed to solve for the velocity term of the dipole self term.

Because the ASMDD self terms are simple acoustic sources integrated over the surface of circular disks of finite area, their behavior is expected to be much like that of
an unbaffled piston radiator in free space. Dividing the known dipole pressure self term by the impedance (the ratio of pressure to velocity at the same location) of the equivalent unbaffled piston radiator would yield the nonsingular and bounded analytical solution for the dipole velocity self term. In recent work by Mellow, et al., they provide the analytical series solution for a piston in a finite baffle [28]. Figure 3.4 shows the geometry of the piston radiator derived by Mellow, et al. The results published by Mellow et al. give results for various ratios of baffle radius, \( b \), to piston radius, \( a \). The unbaffled piston source modeling the ASMDD self terms is given by a \( b:a \) ratio of unity.

![Figure 3.4: Finite Baffled Piston Radiator](image)

Figure 3.4: Finite Baffled Piston Radiator

Figure 3.5 shows the analytical impedance results for an unbaffled piston (i.e. finite baffled piston where \( b:a = 1 \)). For this work, the unbaffled piston impedance is identified as \( Z_{up} \). The complete derivation for the determination of the complex radiation resistance of the unbaffled piston is given in Appendix A.

![Figure 3.5: Normalized Impedance for an Unbaffled Piston Radiator](image)
To summarize, the catalog of ASMDD self terms is presented in Table 3.1. It is important to note that the self terms are not functions of $z$, however they are functions of the characteristic length dimension $a$ of the element area. As before, a node in space can be assigned an area based on the number of nodes where each node is given $\frac{1}{N^{th}}$ the total surface area being modeled. This of course requires a uniform distribution of points on the radiating surface. However, as will be seen in Chapter 4, such a uniform distribution is a consequence of using a digitally defined space. Adjusting for the difference in area between models of various digital resolutions can be related to an adjustment in $a$ seen as

$$a = \sqrt{\frac{S}{\pi N}}.$$  

(3.39)

The ASMDD dipole self terms in Table 3.1 include the $\beta = \frac{i}{k}$ term, which, as mentioned earlier, is used to activate the distribution of dipoles in the method of superposition.

<table>
<thead>
<tr>
<th>Source Type</th>
<th>Pressure Self Term</th>
<th>Velocity Self Term</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monopole</td>
<td>$\frac{2\pi i}{k \pi a^2} { e^{-ik</td>
<td>\textbf{r}</td>
</tr>
<tr>
<td>Dipole</td>
<td>$\frac{-2\pi i}{k \pi a^2}$</td>
<td>$\frac{-2\pi i}{k \pi a^2 Z_{up}}$</td>
</tr>
</tbody>
</table>

Table 3.1: Catalog of ASMDD Self Terms

### 3.3.3: Selfterm Extraction

To qualify the ASMDD self terms in Table 3.1 for numerically solving radiation problems, the exact numerical self terms needed for a model can be extracted from an appropriate numerical model for comparison. Such a model is a uniformly pulsating sphere as it possesses not only geometric symmetry in all directions in 3D space, but its surface pressure and velocity are uniform over the entire radiating surface. Furthermore, the exact solution is known analytically for the radiated pressure both on and off the
surface. Figure 3.6 shows the meshless uniformly pulsating sphere model with a radius of \( a_s = 1 \) meter and surface velocity of \( V_s = 1 \text{m/s} \).

**Figure 3.6:** Meshless Sphere Model

Due to the symmetry in geometry and acoustic surface pressures, computing the self terms is independent of surface location. To begin, the analytical pressure \( p_f \) off the surface at a vector location \( \vec{r} \) is given as

\[
p_f = \frac{i \rho c k a^2 V}{r(1 + ika)} e^{-ik(r-a)}, \tag{3.40}
\]

where \( |\vec{r}| = r \). ASMDD treats each nodal position \( n \) located at \( \vec{r}_n \) in Figure 3.6 as an acoustic source. Therefore, the acoustic pressure \( p_f \) developed due to all \( N \) sources radiating together at the field point is determined by

\[
p_f = \sum_{n=1}^{N} s_n T_n(\vec{r}), \tag{3.41}
\]

where \( T_n(\vec{r}) \) is the linear combination of basis functions which are solutions to the wave equation as presented in Equation 3.2. It is important to note that no singular terms are encountered in Equation 3.41. Since the pulsating sphere radiates uniformly over its entire surface, the sources each radiate equally yielding a common source strength \( s \) of

\[
s = \frac{p_f}{\sum_{n=1}^{N} T_n(\vec{r})}. \tag{3.42}
\]
It should be noted that the common source strength values determined in Equation 3.42 are valid for any point on or off the surface of the sphere. This is due to the fact that $p_f$ contains a harmonic term that decays with respect to $r$. Therefore, $s$ given in Equation 3.42 is the same as if the field point were located on the surface of the sphere at $r = a$.

Next, as the field point is brought to the surface at $r = a$, the pressure can be expressed numerically as before in Equation 3.41. However, a singular acoustic pressure term is now encountered when $|\vec{r}_n| = a$ (expressed as $g_{i1}$). The singular term can be isolated and the surface pressure $p_s$ can written as

$$p_s = sg_{11} + s \sum_{n=2}^{N} T_n(\vec{r}).$$  \hspace{1cm} (3.43)

Finally, the singular term identifying the meshless acoustic self term associated with pressure can be expressed numerically as

$$g_{11} = \frac{p_s + s \sum_{n=2}^{N} T_n(\vec{r})}{s}. \hspace{1cm} (3.44)$$

Similarly, the self term related to the velocity (expressed as $g'_{i1}$) is determined using the surface impedance of the same uniformly pulsating sphere which is expressed analytically as

$$Z = \rho c \left[ \frac{(ka)^2}{1 + (ka)^2} + i \frac{ka}{1 + (ka)^2} \right], \hspace{1cm} (3.45)$$

where the surface velocity is expressed as

$$v_s = \frac{p_s}{Z}. \hspace{1cm} (3.46)$$

The source strength $s$ determined in Equation 3.42 is also used to solve for the velocity since the acoustic pressure should correspond to the respective particle velocity. Solving for $g'_{i1}$ as done in Equation 3.44 yields the singular term identifying the meshless acoustic self term associated with velocity expressed numerically as

$$g'_{11} = \frac{p_s}{Z} + s \sum_{n=2}^{N} \nabla T_n(\vec{r}) \cdot \vec{n} \frac{s}{s}. \hspace{1cm} (3.47)$$
3.3.3.1: Monopole Selfterm Extraction

Plots of the numerically extracted $g_{11}$ and $g'_{11}$ self terms using $N = 60$ nodes are given in Figure 3.7 and Figure 3.8 respectively. The results of Figure 3.7 and Figure 3.8 are generated using the first basis function in the expansion of $T_n(\bar{r})$ where the expansion coefficient $\alpha = 1$ and $\beta = 0$ indicates only a monopole source distribution, as discussed earlier.

![Figure 3.7: Numerically Extracted $g_{11}$ Self Term](image)

**Figure 3.7:** Numerically Extracted $g_{11}$ Self Term
It is important to note that the results plotted in Figures 3.7 and 3.8 represent the required system matrix diagonal terms to match exactly the analytical acoustic pressure radiated from a uniformly pulsating sphere. As such, the results of Equation 3.44 and Equation 3.47 are substituted into the diagonal index of the pressure and velocity system matrices so as to solve for the acoustic surface pressure and radiated sound power. Figure 3.9 shows the numerical results compared to the exact analytical solution for the radiated sound power from the uniformly pulsating sphere. It can be seen that the agreement is excellent and that no nonuniqueness problems are encountered at frequencies corresponding to the internal eigenvalue problem at $ka = n\pi$ [7]. This evidence shows that nonuniqueness problems can be avoided in the presence of a matrix inversion if the diagonal terms are properly determined. Such a conclusion continues to motivate the development of ASMDD as it is feasible for acoustic radiation to be determined using only matrix math with no nonuniqueness problems.
3.3.3.2: Dipole Selfterm Extraction

Again, plots of the numerically extracted $g_{ii}$ and $g'_{ii}$ self terms using $N = 60$ nodes are given in Figure 3.10 and Figure 3.11 respectively. The results of Figures 3.10 and 3.11 are generated using only the second basis function in the expansion of $T_n(r)$ where the expansion coefficient $\alpha = 0$ and $\beta = \frac{i}{k}$ indicates only a dipole source distribution, as detailed previously. Again, these results are substituted into the diagonal index of the pressure and velocity system matrices so as to solve for the acoustic surface pressure and radiated sound power. Figure 3.12 shows the numerical results compared to the exact analytical solution for the radiated sound power from the uniformly pulsating sphere. As expected, the results are in excellent agreement and the solution is without the presence of nonuniqueness problems.
Figure 3.10: Numerically Extracted $g_{11}$ Self Term

Figure 3.11: Numerically Extracted $g_{11}'$ Self Term
3.3.3.3: Tripole Selfterm Extraction

Also, as was indicated earlier, a linear combination of the results plotted in Figures 3.7 and 3.10 and Figures 3.8 and 3.11 respectively represent another unique solution. Here, both expansion coefficients are nonzero with $\alpha = 1$ and $\beta = \frac{i}{k}$ which indicates a tripole source distribution. The extracted tripole self terms are seen in Figure 3.13 and Figure 3.14. Figure 3.15 shows the numerical results compared to the exact analytical solution for the radiated sound power from the uniformly pulsating sphere.
Figure 3.13: Numerically Extracted $g_{11}$ Self Term

Figure 3.14: Numerically Extracted $g'_{11}$ Self Term
3.3.3.4: Validation of ASMDD Selfterms

The generalized analytical ASMDD self terms of Table 3.1 are compared to the numerically extracted values plotted in Figures 3.7 and 3.10 and Figures 3.8 and 3.11. Figure 3.16 shows the comparison of the $g_{11}$ self terms for the monopole distribution ($\alpha = 1$ and $\beta = 0$). Figure 3.17 shows the same comparison for the $g'_{11}$ self terms for the same monopole distribution. The agreement between the numerical and analytical results for a monopole distribution is quite good, although there do exist some minor discrepancies.

**Figure 3.15:** Radiated Sound Power using Numerically Extracted Tripole Self Terms
**Figure 3.16:** Comparison of Extracted \((D \text{ xxx ooo})\) and ASMDD \((A \text{ --})\) \(g_{11}\) Self Terms

**Figure 3.17:** Comparison of Extracted \((D \text{ xxx ooo})\) and ASMDD \((A \text{ --})\) \(g'_{11}\) Self Terms
Figure 3.18: Pulsating Sphere Sound Power using ASMDD Monopole Self Terms

Figure 3.18 shows the resulting radiated sound power after substituting the ASMDD self terms for the monopole distribution into the diagonal indices of the pressure and velocity system matrices. The numerical results show the presence of nonuniquenesses at frequencies corresponding to the eigenvalues of the interior problem at $ka = n\pi$ [7]. This is most likely due to the fact that the analytical self terms do not exactly reproduce the numerically extracted self terms. This lack of exact correlation contributes to some level of rank deficiency in the system matrices allowing nonunique solutions to become a part of the solution space. Yet, the results do show a promising trend that attempts to reproduce the analytically derive radiated sound power.

Lastly, Figures 3.19 and 3.20 show the same comparison of the $g_{11}$ and $g'_{11}$ self terms, respectively, for a dipole distribution. As before, the $g_{11}$ ASMDD self term shows good agreement with the numerically extracted self term. However, the agreement for the $g'_{11}$ self term is less than satisfactory. As indicated by the results from the monopole distribution, even slight deviations in the ASMDD self terms from the numerically
extracted self terms provides enough matrix rank deficiency to absorb nonunique solutions into the solution space for the radiation problem.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{g11_self_point_dipole.png}
\caption{Comparison of Extracted ($D \ xxx \ 000$) and ASMDD ($A - -$) $g_{11}$ Self Terms}
\end{figure}
Figure 3.20: Comparison of Extracted ($D \xxx \ooo$) and ASMDD ($A \rightarrow D$) $g'_{11}$ Self Terms

Figure 3.21 shows the results for the radiated sound power after substituting the ASMDD self terms for the dipole distribution into the diagonal indices of the pressure and velocity system matrices. As expected the results are less than ideal, however, the numerical solution does show the presence of nonunique solutions at the expected frequencies corresponding to the internal eigenvalue problem at $ka = 4.493$ and $ka = 7.725$ [7].
Finally, Figures 3.22 and 3.23 show the same comparison of the $g_{11}$ and $g'_{11}$ self terms, respectively, for a tripole distribution. As already encountered, the $g_{11}$ ASMDD self term shows good agreement with the numerically extracted self term. However, the agreement for the $g'_{11}$ self term is less than satisfactory. Figure 3.24 shows the results for the radiated sound power after substituting the ASMDD self terms for the tripole distribution into the diagonal indices of the pressure and velocity system matrices.
Figure 3.22: Comparison of Extracted ($D$ xxx $ooo$) and ASMDD ($A$ $-$ $-$) $g_{11}$ Self Terms

Figure 3.23: Comparison of Extracted ($D$ xxx $ooo$) and ASMDD ($A$ $-$ $-$) $g'_{11}$ Self Terms
The computational radiated acoustic power results from the analytical self term derivation for the dipole, and hence the tripole term, are less than satisfactory. Because the analytical dipole term failed to reproduce the exact result for the pulsating sphere, only the monopole term will be used in the ASMDD formulation as its resulting fit is quite good. In addition, the frequency range of interest will be limited up to a $ka = 4$ as the analytical monopole solution maintains a good fit up until this limit.

3.4 Noncontiguous Surface Validation

To review, the circular area for the determination of the ASMDD self terms was chosen for its symmetry and geometric convenience when integrating. As seen in Figure 3.3, the ASMDD self terms are found by integrating over a circular area that represents $\frac{1}{N^{th}}$ of the total surface area of the problem domain. However, when considering the influence of each ASMDD self term in the problem domain, the collection of these circular domains of integration do not produce a contiguous surface representation of the
problem domain as conventional elements would. Consider Figure 3.25 which compares a rectangular surface discretized using quadrilateral elements and its equivalent ASMDD representation. Although ASMDD is a meshless computational technique that does not rely on a mathematical definition of the surface, the analytical ASMDD self term formulations are derived from these circular areas on the surface of the problem domain. As such, they carry with them this collective geometric influence yielding a surface equivalence seen in Figure 3.25.

**Figure 3.25: Surface with Conventional Elements and ASMDD Nodes**

The conventional quadrilateral element representation of the rectangular surface is contiguous. Every node defining an element is shared by a neighboring element rendering the surface closed, and each element defines a domain of unique points on the surface being modeled. Yet the ASMDD nodal representation is not contiguous as the surface is not closed throughout and nor do the circular regions, defined for ASMDD self term integration, define unique points on the surface being modeled as they overlap. However, with the ASMDD nodal representation of a structure, such overlap is expected but has no significant effect on the final solution.

To ensure that the ASMDD self term surface integration geometry has little effect on the determination of the ASMDD self terms, a quadrilateral element (here it is a square) is compared to that of the circle with equal surface area in Figure 3.26.

**Figure 3.26: Quadrilateral and Equal Area Circle**
For simplicity, only a monopole source (as seen in Figure 3.26) radiates acoustic pressure to the surface of both the square quadrilateral and the circular domain of self term integration in this example. Because the area averaged acoustic pressure over the surface of the square quadrilateral cannot be resolved in closed form solution as a function of \( z \), the monopole source is located at a constant distance \( z = 1 \text{ m} \) for both cases. In both cases, the source is located concentric to the centroid of the domain of integration. Furthermore, the numerical integration over the square quadrilateral uses eight Gauss points. Figure 3.27 shows the results of the integration of Equation 3.9 (with unit source strength \( s = 1 \text{ Pa-m} \)) for both cases as a function of dimensionless wavenumber. It is important to note that the ordinate axis of Figure 3.27 is nondimensionalized with respect to the characteristic length of the quadrilateral and the circle geometry cases. The surface considered here is that of a 60 point/ASMDD node 1 meter radius sphere thereby giving each self term an equivalent area of \( \frac{4\pi (1)^2}{60} = 0.20944 \text{ m}^2 \). Therefore, the circle area has the characteristic dimension that is its radius \( a = \sqrt{\frac{0.20944 \text{ m}^2}{\pi}} = 0.2582 \text{ m} \), and for the quadrilateral square the characteristic dimension is half the length of the sides, \( r = \frac{\sqrt{0.20944 \text{ m}^2}}{2} = 0.2288 \text{ m} \). Figure 3.27 identifies the area and integrated pressure of the circle as \( A_c \) and \( P_c \) respectively. Figure 3.27 also identifies the area of the quadrilateral and its integrated pressure as \( A_s \) and \( P_s \) respectively.
Figure 3.27: Integration Results Over the Square and Circle versus \( ka \) for \( c \) (circle) and \( kr \) for \( s \) (square)

The results show that for low frequencies \( (k(a,r) < 0.5) \), the integration results of Equation 3.9 are very similar for both cases. This indicates Equation 3.9 does not vary much over the surface of each integration domain at low frequencies. However, for higher frequencies the results are quite different given there is more significant variation in Equation 3.9 at higher frequencies. Yet, the overall interest lies in determining the resulting radiated acoustic pressure from an entire structure consisting of many ASMDD nodes integrated over circular domains each with \( \frac{1}{N^{th}} \) the total area of the problem domain. Therefore, if the above results are plotted on an ordinate that is nondimensionalized with respect to the characteristic length \( A \) of the complete problem domain, the results given in Figure 3.28 show excellent agreement.
The ordinate of Figure 3.28 is nondimensionalized with respect to the characteristic length $A = 1$ meter, the radius of the 60 point/ASMDD node sphere structure being modeled. Therefore, despite the high frequency variation of Equation 3.9 with respect to the spatial domains of integration, this variation over the length of the entire problem domain (the entire sphere) is negligible at all frequencies. Such a conclusion supports the use of circular domains of integration for ASMDD self term derivations.

To summarize, the development of the meshless boundary value solver ASMDD is formulated based on the conventional BEM technique of acoustic wave superposition. Here the acoustic pressure and velocity radiated from monopole and dipole sources are concentrated at a meshless point. These point pressure and velocity values are represented by a closed form solution and are functions of the size of the computational domain. However, the closed form solutions find success for only the monopole type source and are limited to a $ka = 4$. In addition, the ASMDD meshless point formulation relies on integration over a circular area representing $\frac{1}{N^{th}}$ of the computational surface.
area where \( N \) meshless points are used to represent the computational surface. The assembly of these integrated point values represents a noncontiguous computational surface for the complete structure. Yet, despite the lack of assembled continuity on the computational surface, the final solution for the radiated acoustic power is unaffected.
Chapter 4
SPATIALLY DIGITIZED ACOUSTIC DOMAINS

4.1 Introduction

As indicated in the discussion on shape optimization from the structural mechanics point of view, element distortions must be monitored and corrected as needed. This is a tedious process and requires computational efforts that limit the performance of shape optimization when using conventional elements. However, when using meshless computational techniques, element distortions need not be monitored nor corrected as they do not exist. Yet, even without elements or other surface representations, normals still need to be defined to assign surface velocities as inputs to the acoustic superposition program. To find these, it should be noted that the uniformity or “digitized” representation of the computational domain facilitates normal calculations from only local ASMDD nodal information.

For example, consider a sphere whose problem domain is defined by conventional elements as seen in Figure 4.1. Using the meshless acoustic representation of the same problem, the element free sphere is given in the same figure for comparison.

Figure 4.1: Conventional Element and Meshless Problem Domains
Conventionally, the surface normal information needed for acoustic radiation problems is determined from the vectors that define each node in an element. To review, Figure 4.2 shows the vector relationships needed to determine each element surface normal. For simplicity of demonstration, Figure 4.2 shows linear triangular elements.

**Figure 4.2: Conventional Element Normal Vector Relationships**

Nodes 1, 2, and 3 which define element 1 are located in three space with the vectors $V_1$, $V_2$, and $V_3$ respectively. The nodes are numbered about the element with increasing node number in a counterclockwise manner to ensure outward pointing normals. The vectors that point from node 1 to nodes 2 and 3 are given by

$$
V_{21} = V_2 - V_1 \\
V_{31} = V_3 - V_1.
$$

Equation 4.1

The cross product of the vectors determined in Equation 4.1 yield the element surface normal vector $V_{123}$ seen as

$$
V_{123} = V_{21} \times V_{31}.
$$

Equation 4.2

It is important to note that conventional elements such as those seen in Figure 4.2 are derived from connectivity information which is not available in the meshless representation. Therefore, digitizing the problem domain offers a uniformity that carries with it an inherent local connectivity defined simply by the digital neighborhood.

### 4.2 Digital Acoustic Problem Domains

Consider a computational acoustic domain that completely envelops the problem domain with cubic dimensions (i.e. length $x = width y = height z$). Figure 4.3 shows a simple quadratic surface defined by conventional elements. In addition, Figure 4.3 also shows the same simple quadratic surface which is enveloped by a cube subdivided by
internal cubes. The cube defines the extremities of the digitized computational domain and maintains a resolution of $\Delta x \times \Delta y \times \Delta z$ throughout. The resolution values $\Delta x$, $\Delta y$, and $\Delta z$ are defined as $x/N_x$, $y/N_y$, and $z/N_z$ respectively where $N_x$, $N_y$, and $N_z$ are the number of internal divisions made by the digital domain along the $x$, $y$, and $z$ directions respectively.

For the case illustrated in Figure 4.3, $N_x = 4$, $N_y = 4$, and $N_z = 8$, yielding 128 internal divisions.

**Figure 4.3:** Computational Domain Enveloped by the Digital Domain

For every node from the elemental representation of the quadratic surface that is completely contained within an internal cube or voxel, that voxel is activated. A voxel being activated or not being activated is very much like the conventional 1’s and 0’s used in fields such as digital signal processing. Being activated here means that a particular voxel should be used in the representation of the surface being modeled. Figure 4.4 shows the active voxels used to approximate the elemental quadratic surface in Figure 4.3 using 128 internal divisions. As such, the resulting voxel surface requires only 40 of the 128 total internal divisions.
Figure 4.4: Activated Voxels Approximating the Problem Domain

As expected, increasing the resolution of the digital domain also increases the fit between the digital representation and the conventional elemental representation. Figure 4.5 through Figure 4.7 illustrate an increase in digital resolution and the corresponding fit to the initial elemental representation.

Figure 4.5: Digital Representation using 61 Voxels
Despite the fact that voxels are seen in Figure 4.5 through Figure 4.7 to represent the conventional elemental surface, the problem domain remains meshless. Rather, the problem domain does not incorporate surface information. The voxels seen in these figures are present to help the user visualize the surface in a contiguous manner within the digital domain. Not using voxels for the digitized meshless representation makes it difficult for the user to visualize the shape of the surface. Points in 3D space displayed on a 2 dimensional surface (\textit{i.e.} monitor or hardcopy) cannot help the user determine which points line in front of others. For example, the meshless representation of the sphere in Figure 4.1 could be interpreted as a circle and not a sphere. The meshless representation of the problem domain can be derived from the activated voxels where the
meshless points lie at the geometric center of each voxel. Figure 4.8 illustrates the relationship between a voxel and a meshless point.

![Figure 4.8: Voxel and Meshless Point Relationship](image)

Applying this relationship between the digitized quadratic surface seen in Figure 4.7 yields its meshless representation given in Figure 4.9. Again, one can appreciate the benefit of having the voxel representation of the example quadratic surface to aid in understanding the meshless problem domain in a contiguous manner. Furthermore, it also helps to visually enforce the digital uniformity with which the meshless points are distributed.

![Figure 4.9: Meshless Representation of Voxel Surface](image)

### 4.3 Surface Normal Determination

Again, it is important to note that no surface normal information has been made available to the meshless representation. Here we will rely on the inherent connectivity that can be interpreted from the uniform distribution of the meshless problem domain in a digitized acoustic space. To do so, we will make five planar passes through the entire digitized domain: three through the $xy$ plane, one through the $xz$ plane, and one through
the \( yz \) plane. The resulting normal vectors are assigned to the ASMDD nodes that make up the structure in the digital domain. These normals are also the same vectors used in formulating the Greens function system matrices that contain vector relationships for computing the radiated acoustic power.

The first pass through the \( xy \) plane moves in the positive \( z \) direction from the smallest value of \( z \) to the largest value of \( z \). Then, where ever an ASMDD node coincides with a grid point in the digital domain, a 1 is assigned indicating a structural boundary. If no ASMDD nodes are found to coincide with a grid point in the digital domain, a 0 is assigned indicating no structural boundary. Figure 4.10 illustrates the assignment of 1’s and 0’s for this initial pass. Figure 4.10 shows the scan direction taking slices of the digitized structure and the corresponding normal view of the sliced plane showing the assignment of 1’s and 0’s. For ease of viewing, the boundary 1’s are circled in red.

**Figure 4.10:** First Pass in the \( xy \) Plane to identify the Boundary 1’s

The next pass in the \( xy \) plane moves in the positive \( z \) direction from the smallest value of \( z \) to the largest value of \( z \) as before. With all the boundary 1’s assigned, the internal portion of the structure can be determined. To further distinguish between the 0’s and 1’s, 2’s are used to identify the internal portion of the structure. For each scan plane, each row is searched for all 0’s bounded by boundary 1’s. These are then identified as the internal portion of the structure. Figure 4.11 shows the same plane as illustrated in Figure 4.10, this time with the internal portion of the structure identified.
using 2’s. However, with this method of assigning the internal portion of the structure, the structures are limited to convex shapes only.

**Figure 4.11:** Second pass in the $xy$ Plane to identify the Interior 2’s

Once this second scan is complete, a three dimensional array of 0’s, 1’s, and 2’s is compiled for use in the remaining three scans to determine the normal direction vectors. This three dimensional array allows us to scan all planes ($xy$, $xz$, and $yz$) of the three dimensional array searching for matches to a predefined criteria that defines the edge of the structure and the corresponding normal vector at the edge. The edge criteria for normal vector assignment is a catalog of all possible template values of 0’s, 1’s, and 2’s that could identify the edge of the point-based structure. Figure 4.12 shows the template that is used to scan the structural boundary within the three dimensional array planes. The values for the template, listed clockwise about the point of each boundary location (all the locations represented by 1’s) are checked against the catalog of all possible values defining an edge location in the digital space.

**Figure 4.12:** Boundary Template and Catalog Lookup
This catalog of edge values consists of 58 different boundary edge possibilities resulting from a total of $3^4 = 81$ possible template combinations (3 numeric choices at 4 different locations). The 23 possible template combinations not used for boundary edge detection contain either nonconvex boundary edges or points internal to the boundary. Listed below are the 58 possible choices that identify a boundary edge.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 \\
1 & 0 & 2 & 0 \\
1 & 0 & 0 & 2 \\
1 & 2 & 2 & 0 \\
1 & 2 & 0 & 2 \\
1 & 0 & 2 & 2 \\
0 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 \\
0 & 1 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 2 \\
2 & 1 & 2 & 0 \\
2 & 1 & 0 & 2 \\
0 & 1 & 2 & 2 \\
0 & 0 & 1 & 0 \\
2 & 0 & 1 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 1 & 2 \\
2 & 2 & 1 & 0 \\
2 & 0 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
0 & 2 & 1 & 2 \\
0 & 0 & 0 & 1 \\
2 & 0 & 0 & 1 \\
0 & 2 & 0 & 1 \\
0 & 0 & 2 & 1 \\
2 & 2 & 0 & 1 \\
2 & 0 & 2 & 1 \\
0 & 2 & 2 & 1 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 2 & 0 \\
2 & 1 & 1 & 0 \\
2 & 0 & 1 & 1 \\
1 & 2 & 0 & 1 \\
1 & 2 & 1 & 0 \\
2 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 0 & 2 \\
0 & 1 & 1 & 2 \\
0 & 2 & 1 & 1 \\
1 & 0 & 2 & 1 \\
1 & 0 & 1 & 2 \\
0 & 1 & 2 & 1 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2 \\
2 & 2 & 0 & 0 \\
0 & 2 & 2 & 0 \\
0 & 0 & 2 & 2 \\
2 & 0 & 0 & 2
\end{bmatrix}
\]

Once a match in the catalog is made, the template values are negated using a logical NOT. Next multiplying by the appropriate planar $dx_1$ and $dx_2$ length values in their respective directions yields the normal vector at that location. For example, consider the boundary location under investigation in Figure 4.13. The template values listed in a clockwise manner around the point under investigation are (0 0 1 1).
Finding a match between \((0 \ 0 \ 1 \ 1)\) for the point and \((0 \ 0 \ 1 \ 1)\) in the catalog identifies this boundary point as an edge point. Next, as shown in Figure 4.14, the template values are logically negated and the appropriate planar \(dx_i\) and \(dx_j\) values are multiplied yielding a normal vector in the respective 12 plane (\(i.e.\ xy, \ xz,\) or \(yz\)). Rather, if the search plane is the \(xy\) plane, then the appropriate \(dx_i\) and \(dx_j\) values are \(dx\) and \(dy\) respectively; if the search plane is the \(xz\) plane, then the appropriate \(dx_i\) and \(dx_j\) values are \(dx\) and \(dz\) respectively, etc. The resulting vector is as expected for this corner edge location; one unit \(dx_i\) in the \(\hat{i}\) direction and one unit \(dx_j\) in the \(\hat{j}\) direction, indicated by the blue vector lines in Figure 4.14.

\[
\begin{align*}
\text{NOT}(0 & 0 1 1) = (1 1 0 0) \\
\begin{bmatrix}
dx_j \\
dx_i \\
-dx_j \\
-dx_i
\end{bmatrix} = (dx\hat{i} + dx\hat{j}) \\
\end{align*}
\]

Figure 4.14: Normal Vector

Since this template scanning takes place in a plane, there is no contribution to the out of plane portion of the normal vector. Therefore, if the resulting normal vector determined in Figure 4.14 was accomplished in the \(xy\) plane, the vector is more appropriately written as \((dx\hat{i} + dy\hat{j} + 0\hat{k})\). This procedure continues for all three planes and the results are stored. Table 4.1 shows the planar scanning advancements in the \(xy\), \(xz\), and the \(yz\) planes respectively.
Finally, after all three planes have been scanned, the three dimensional normal vector assignment at each point is a vector sum of the results from all three scans. From the example point above, this would yield the vector (unnormalized) as,

\[
\begin{align*}
x y \text{ scan: } & (dx \hat{i} + dy \hat{j} + 0\hat{k}) \\
x z \text{ scan: } & (dx \hat{i} + 0\hat{j} + dz \hat{k}) \\
y z \text{ scan: } & (0\hat{i} + dy \hat{j} + dz \hat{k}) \\
\text{final } xyz & = 2(dx \hat{i} + dy \hat{j} + dz \hat{k})
\end{align*}
\]

### Table 4.1: Normal Vector Assignments for Each Planar Direction

<table>
<thead>
<tr>
<th>Direction</th>
<th>Normal Vector Assignment</th>
</tr>
</thead>
<tbody>
<tr>
<td>xy scan</td>
<td>((dx \hat{i} + dy \hat{j} + 0\hat{k}))</td>
</tr>
<tr>
<td>xz scan</td>
<td>((dx \hat{i} + 0\hat{j} + dz \hat{k}))</td>
</tr>
<tr>
<td>yz scan</td>
<td>((0\hat{i} + dy \hat{j} + dz \hat{k}))</td>
</tr>
<tr>
<td>final xyz</td>
<td>(2(dx \hat{i} + dy \hat{j} + dz \hat{k}))</td>
</tr>
</tbody>
</table>
The final normal assignment result for the preceding sphere example is given in Figure 4.15. From Figure 4.15 it can be seen that the surface normal vectors expected to point out from either voxel faces, edges, or corners do so.

![Final Voxel Normal Vector Assignment](image)

**Figure 4.15:** Final Voxel Normal Vector Assignment

### 4.4 ASMDD Computation for Changes in Shape

Table 4.2 shows the evolution of shape in an example structure to be modeled. Table 4.2 renders the shape using finite elements, voxels, and ASMDD nodes for comparison. Note that the ASMDD node representation is difficult for users to discern the depth of field thereby leaving out valuable contour information to the eye. For review, if these shape iterations were the result of autonomous computer control, each shape would have been determined by activating and/or deactivating voxels in the computational domain. However for sake of demonstrating ASMDD, each evolution in shape is predetermined. The example structure shows a uniformly pulsating cap (red) on a rigid cubic baffle (green).
Table 4.2: Evolution in Shape of Example Structure

This example uses the ASMDD computation to determine the radiated sound power for all five evolutions in shape as well as for the initial cube shape itself. The results for the radiated acoustic power follow and are compared to conventional BEM solutions in the order they appear in Table 4.2. It should be noted that the radiating area of the BEM model is not always equal to the total radiating area of the voxel/ASMDD model. This is because the total number of points making up the radiating surface times \( \frac{1}{N^{th}} \) the area of the BEM surface (\( N \) total points in the voxel/ASMDD model) is not equal the total radiating area of the BEM surface. As such, each of the following acoustic power results are adjusted for the difference in radiating area.

Figure 4.16: Radiated Acoustic Power from Shape #1 in Example Shape Evolution
Figure 4.17: Radiated Acoustic Power from Shape #2 in Example Shape Evolution

Figure 4.18: Radiated Acoustic Power from Shape #3 in Example Shape Evolution
Figure 4.19: Radiated Acoustic Power from Shape #4 in Example Shape Evolution

Figure 4.20: Radiated Acoustic Power from Shape #5 in Example Shape Evolution
It is important to note that as the shape evolves, the curvature of the BEM surface is becoming larger. As such, the voxel/ASMDD surfaces are required to accommodate more curvature in their representation of the BEM surface. However, the normal vectors on the voxel/ASMDD surface are limited to the face, edge, or corner assignments as dictated by the normal vector formulation in Section 4.3. Thus, for the given density of points, the preceding figures, illustrate that the normal vector assignment for the voxel/ASMDD can only approximate the actual normal vectors of the structure being modeled. As a result, we have increasing deviations in the radiated sound power between the two computational schemes as the curvature increases. Obviously, a finer point density is required in the ASMDD calculations. Consider the following figures in the discussion on properly aligned normal vectors.
4.5 ASMDD Normal Vector Limitations

The following discussion helps to illustrate the limitation in normal vector assignments for ASMDD computations in a digital spatial domain. According to the normal formulation of section 4.3, the normal vectors assigned to ASMDD nodes are either $0^\circ$, $45^\circ$, or $90^\circ$ (when considered in 2D) to the global axis. Consider the following table showing an increasing resolution in points/voxels used to model the same continuous circle. The table shows both the continuous circle with normals and its voxel representation with normals in the left column. In the right column is a more focused look at the circle edge comparing both the continuous and voxel based normals. The continuous circle is plotted in blue, along with the corresponding outward pointing normals. The ASMDD representation, in red, of the same circle is superimposed onto the circle with its outward pointing normals as well.
Table 4.3: Continuous and ASMDD Normal Comparison

From the right column it can be seen that the points/voxels required to model the continuous circle shape result in normals that do not correspond with the continuous outward pointing normals. Some points/voxels receive normals that point normal to the voxel face, which is largely skewed from the continuous normal. This suggests that points/voxel normals can only accommodate a limited amount of curvature when using the method of normal assignment from Section 4.3. This is further supported by noting the first two shapes used in the example ASMDD calculation of radiated sound power with evolving shapes in Section 4.4. For these two shapes, the curvature is quite limited and retains most, if not all as in shape one, of its cubic geometry. As such, the resulting radiated sound power determined from ASMDD is a very close match to the BEM results. However, as the curvature becomes more pronounced, there exist larger excursions between ASMDD and the BEM results.

In summary, conventional elements are no longer being used to represent the surface of a structure, and as a result surface normals can no longer be computed using conventional element vector cross products. Therefore, a new method for determining the outward pointing surface normals for a point representation of the structure is formulated and outlined. The procedure for assigning these point based normals uses local planar neighborhoods of points surrounding the point of interest to determine the normal direction. A vector sum for all three planar neighborhoods about a point yields the final outward pointing normal vector. The results show that the normals do indeed point in the appropriate direction as dictated by the local neighborhood of points. At this point, all necessary computational information is available to model an example acoustic radiation problem. Here, the example is a square plate with a uniform surface velocity.
set in a cubic baffle. The shape of the square plate is then morphed over six iterations to create a hemisphere. The radiated acoustic power is computed for each iteration in shape using both the newly formulated ASMDD technique and the conventional BEM technique. The ASMDD and BEM results agree quite well for the first two iterations in shape. However, as the curvature of the shape evolution increases, the deviation between the ASMDD results and the BEM results increases. Given these results, the present technique for point normal vector assignment is limited to low levels of curvature. This is supported when considering the shape changing example structure evaluated in this chapter where shape changes having the lowest low of curvature (i.e. Shape #1 and Shape #2) had the best ASMDD to BEM agreement.
Chapter 5
CURRENT ELEMENT FREE (“MESHLESS”) TECHNIQUES

5.1 Derivation of Meshless Theory in Structural Dynamics

As indicated earlier, using meshless techniques for solving the structural eigenvalue problem assists in alleviating the computational burden associated with structural acoustic shape optimization. Because only points are distributed within the problem domain, potential element distortions due to shape changing are no longer considered. In addition, meshless methods better lend themselves to adaptive refinements where now adding points can avoid the previous book keeping burden associated with updating element connectivity. Although the present thesis work does not incorporate meshless structural dynamic solving for eigenvalue problems, it does introduce the theory and gives the opportunity to learn from and potentially integrate other state-of-the-art meshless computational techniques into future efforts of this work.

Meshless methods developed within the engineering computational mechanics community in the early 1990’s when Nayroles et al. introduced their diffuse element method (DEM) [30]. The DEM was developed in response to the limited ability of FEM to dynamically remesh a problem domain in an efficient manner and to resolve solutions in a continuous and accurate manner for complex geometries. As such, Nayroles et al. developed a solution technique that relies solely on nodal locations within the problem domain. Here, they incorporated a locally weighted least squares technique to locally approximate the solution to the system’s governing partial differential equation as given in Equation 5.1.

\[
    u_{\Omega_s}(\bar{x}) = \sum_{j=1}^{m} p_j(\bar{x}) a_j(\bar{x})
\]  

(5.1)

The exact solution \( u(\bar{x}) \) at \( \bar{x} \) contained within a domain \( \Omega_s (\Omega_s \in \Omega) \) is approximated by \( u_{\Omega_s}(\bar{x}) \). The approximate solution \( u_{\Omega_s}(\bar{x}) \) is expanded using a set of basis functions \( p_j(\bar{x}) \) of order \( m \) multiplied by coefficients \( a_j(\bar{x}) \). This is similar to the FEM technique
of interpolating the solution within the domain of an element, typically using low order polynomials. The coefficients \(a_j(\vec{x})\) needed to approximate \(u(\vec{x})\) within a domain \(\Omega\), containing \(n\) nodal locations are found using the weighted \(L_2\) norm of Equation 5.1 given as

\[
J = \sum_{j=1}^{n} \sum_{m=1}^{m} w(\vec{x} - \vec{x}_m)[p_j(\vec{x})a_j(\vec{x}) - u(\vec{x})]^2,
\]

(5.2)

where \(w(\vec{x} - \vec{x}_m)\) is a monotonically decaying weight function symmetric about \(\vec{x}\). It is important to note that the weight function should be at least \(C^L\) continuous where \(L\) is the largest order of differentiation required. The weight function serves to enforce a localized domain of influence (DOI) for \(u_{\Omega_i}(\vec{x})\) within the problem domain. This is similar to the delta function property of the FEM shape functions that interpolate within the domain of their respective element. The size of the domain of influence at \(\vec{x}\) is given by all points \(\vec{x}_m\) in the problem domain where \(w(\vec{x} - \vec{x}_m) > 0\). A potential weight function identified by Nayroles et al. is given as

\[
w(\vec{x} - \vec{x}_m) = e^{\log(e)(\frac{d}{d_{max}})^p},
\]

(5.3)

where the parameters \(p\) and \(d_{max}\) control the steepness and radial dimension \((d_{max})\) of the DOI about \(\vec{x}\) and \(d = |\vec{x} - \vec{x}_m|\). However, Nayroles et al. do not offer any conclusions as to the selection of a weight function based on the problem being modeled. Finally, the unknown coefficients are determined by minimizing Equation 5.2 with respect to \(a_j(\vec{x})\) seen as

\[
\frac{\partial J}{\partial a} = 0.
\]

(5.4)

Shortly after the introduction of the DEM, Belytschko et al. built upon the work of Nayroles et al. and introduced their Element-Free Galerkin (EFG) method [5]. The EFG method, as the DEM, does not rely on element connectivity but rather only nodal locations within the domain of the problem. Belytschko et al. also called attention to the benefits in adaptive techniques when using the EFG method as the distribution of points within the problem domain does not affect the accuracy of the results. Rather, the
addition of nodes has only a local effect and does not require “remeshing” of the entire problem domain. Such computational strengths provide for an increase in solution accuracy for large stress gradient and stress concentration problems like those modeling crack tip growth and propagation or irregular geometries. Belytschko \textit{et al.} include a similar derivation for approximating the nodal parameters as Nayroles \textit{et al.}, as well as identify its origin and development under the label \textit{Moving Least Squares} (MLS) interpolation attributed to Lancaster and Salkauskas \cite{27}. In addition, they incorporate the weak form of the Galerkin method and introduce a method for the enforcement of essential boundary conditions.

Belytschko \textit{et al.} solve Equation 5.4 for \( a_j(\vec{x}) \) in matrix form yielding

\[
\{a(\vec{x})\} = \{A(\vec{x})\}^{-1}\{B(\vec{x})\}\{u_{\Omega_i}(\vec{x})\}, \tag{5.5}
\]

where

\[
\{A(\vec{x})\} = \sum_{i=1}^{n} w(\vec{x} - \vec{x}_i)(p(\vec{x}_i))^T \{p(\vec{x}_i)\},
\]

\[
\{B(\vec{x})\} = [w(\vec{x} - \vec{x}_1)(p(\vec{x}_1)), \ldots, w(\vec{x} - \vec{x}_n)(p(\vec{x}_n))],
\]

\[
\{u_{\Omega_i}(\vec{x})\} = \{u_1, u_2, \ldots, u_n\}, \quad \forall u_i \in \Omega_s. \tag{5.6}
\]

Substituting Equation 5.6 into Equation 5.1 gives

\[
u_{\Omega_i}(\vec{x}) = \sum_{i=1}^{n} \phi_i(\vec{x})u_i(\vec{x}), \tag{5.7}
\]

where \( \phi_i(\vec{x}) \) now defines the meshless analogue to the commonly referred to FEM shape function. For the remainder of this work, shape function will refer to the meshless shape function and will be defined in Equation 5.8 as

\[
\phi_i(\vec{x}) = \sum_{j=1}^{m} p_j(\vec{x})[A_i(\vec{x})]^{-1}[B_i(\vec{x})]_j. \tag{5.8}
\]

In structural dynamics, the Galerkin method sums the total work done over the problem domain \( \Omega \) and can be described as

\[
\text{Work} = \int_{\Omega} \text{Displacement} \times \frac{L(\text{trial})}{\text{Force}} \ d\Omega, \tag{5.9}
\]
where the test and the trial functions are from the same space and $L$ is the linear differential operator describing the system to be modeled and operates on the trial function. When considering a dynamic system Equation 5.9 is more precisely written as

$$
-\frac{1}{2} \int_\Omega \varepsilon^T \sigma \, d\Omega + \frac{1}{2} \int_\Omega \hat{\varepsilon}^T \rho \, d\Omega = 0, \quad (5.10)
$$

where $\varepsilon$ is the strain vector, $\sigma$ is the stress tensor, $\rho$ is the volumetric mass density, and $\hat{\varepsilon}$ is the velocity of the surface displacement. Equation 5.10 describes the system response within the global boundary $\Gamma$ of the problem domain $\Omega$, yet it does not incorporate the boundary conditions themselves (i.e. the strong form). Belytschko et al. include the essential (geometric) boundary conditions by incorporating Lagrange multipliers into Equation 5.10 yielding

$$
-\frac{1}{2} \int_\Omega \varepsilon^T \sigma \, d\Omega + \frac{1}{2} \int_\Omega \hat{\varepsilon}^T \rho \, d\Omega + \int_\Gamma \lambda \left( u_{\Gamma} - u_{\Gamma} \right) \, d\Gamma = 0, \quad (5.11)
$$

where $u_{\Gamma} = B(u)$ is the boundary condition defined by $B$, the linear differential boundary operator that operates on the trial function. This identifies Equation 5.11 as the weak form of the Galerkin method. The Lagrange multiplier is given by $\lambda$ and represents a “smart force” that forces the displacement approximation $u_{\Gamma}$ at the local boundary $\Gamma_s = \Omega_s \cup \Gamma$ to satisfy the prescribed boundary condition $u_{\Gamma}$. The total work integrated over the domain and boundary of the problem can still be equated to zero even though it includes the work done on the boundary as seen in Equation 5.11. This is possible if the forces $\lambda$ acting at the boundary displaces $u_{\Gamma}$ to $u_{\Gamma}$. As such, Belytschko et al. show that the Lagrange multipliers become a part of the system of equations when solving for the nodal parameters $u_i$.

Substituting Equation 5.7 into Equation 5.11 (neglecting the argument $\bar{x}$) and taking the first variation gives

$$
\int_\Gamma \lambda \left( u_{\Gamma} - u_{\Gamma} \right) \, d\Gamma + \int_\Gamma \delta \lambda \left( u_{\Gamma} \right) \, d\Gamma = 0,
$$

$$
\int_\Gamma \lambda \left( u_{\Gamma} - u_{\Gamma} \right) \, d\Gamma + \int_\Gamma \delta \lambda \left( u_{\Gamma} \right) \, d\Gamma = 0, \quad (5.12)
$$
where \( c \) is the material properties matrix. Belytschko et al. write Equation 5.12 in a more familiar form by identifying analogous stiffness \([K_{ik}]\) and mass \([M_{ik}]\) matrices. The analogies are
\[
K_{ik} = \int_{\Omega} L(\phi_i)^T c L(\phi_k) d\Omega, \\
M_{ik} = \int_{\Omega} \phi_i^T \rho \phi_k d\Omega.
\]
(5.13)

In addition they also define
\[
G_{ik} = \int_{\Gamma} N_k(s) \phi_i d\Gamma, \\
q_k = \int_{\Gamma} N_k(s) u_i d\Gamma,
\]
(5.14)
where \( N_k(s) \) is the Lagrange interpolation along an arc length \( s \) on the boundary \( \Gamma \); \( \rho \) for the Lagrange multipliers. The integration of Equations 5.13 and 5.14 is accomplished using the Gaussian integration technique defined in \( \Omega \). Substituting Equations 5.13 and 5.14 into Equation 5.12 and factoring the variational terms yields the following equation
\[
\delta u_k \left( K_{ik} u_i - \omega^2 M_{ik} u_i \right) + \delta \lambda_k \left( G_{ik} u_i - q_k \right) = 0,
\]
where the variations \( \delta u_k = \delta \lambda_k = 0 \) are trivial solutions thereby yielding a matrix system of equations seen as
\[
\begin{bmatrix}
[K_{ik}] - \omega^2 [M_{ik}] & [G_{ik}]
\end{bmatrix}
\begin{bmatrix}
\{u_i\}
\end{bmatrix}
= 
\begin{bmatrix}
\{0\}
\end{bmatrix},
\]
(5.16)
to be solved for the vector of nodal parameters \( \{u_i\} \).

It should be noted that the dimension of the system of equations in Equation 5.16 exceeds that of conventional FEM where the dimension is known to be \( ndof \times ndof \) (\( ndof \) – number of degrees of freedom). For the EFG method, the problem dimension is now \((ndof + nbc) \times (ndof + nbc)\), where \( nbc \) = number of degrees of freedom for the boundary conditions. In addition, the domain of integration in the EFG method is performed over the entire domain \( \Omega \), which is larger than the domain of integration over a conventional elemental domain. As the integration takes place over the entire domain of the problem, a grid of cells is overlaid on the problem domain to identify the location of the numerical integration quadrature points. It is important to note that although this grid of cells is independent of the nodal locations (i.e. no connectivity), it does not suggest a truly
meshless computational technique in the literal sense of the word. However, the relative performance between the EFG method and conventional FEM should be strongly considered for adaptive and high stress gradient problems.

Ouatouati and Johnson used the same approach as Belytschko et al., however, they enforced their boundary conditions using model order reduction through Singular Value Decomposition (SVD) [31]. This approach begins by assembling a matrix $[C]$ of the essential boundary conditions defined as

$$B(u_{\Gamma_s}) = 0,$$  \hspace{1cm} (5.17)

where $B$ is defined as the linear differential boundary operator operating on $u_{\Gamma_s}$ as before. Substituting Equation 5.7 into Equation 5.17 gives

$$B\left[\{\phi_i\}_s^T\{u_i\}_s\right] = 0,$$  \hspace{1cm} (5.18)

where $[C] = \{\phi_i\}_s^T|_{\Gamma_s}$. However, only those DOI’s that intersect the boundary have non-zero shape function values, or

$$\{\phi_i(\overline{x})\}_s|_{\Gamma_s} \neq 0 \hspace{0.5cm} \forall \overline{x} \in \Gamma_s,$$  \hspace{1cm} (5.19)

which leads to a matrix $[C]$ that is sparse having rows consisting of either all zeros or shape functions according to Equation 5.19. The all zero rows are linearly dependent with respect to one another and thereby yields a singular matrix. As such, Ouatouati and Johnson implement SVD to circumvent the singularity and reduce the order of their model.

Singular Value Decomposition is used to diagnose the rank deficiency of a matrix [33]. For example, consider a square singular matrix $A$ and a column vector $x$ such that $Ax = b$. The range space of $A$ is defined as the solution space that successfully maps $x$ to $b$ and has dimension $r$. The nullspace is defined as the solution space that maps $x$ to zero and has dimension $n$. The singular matrix $A$ has dimension $N$, which is a sum of $r$ and $n$. As such, the SVD method can is used to expand a singular matrix into a product of three special matrices given as

$$[C] = [U][\Sigma][V]^T.$$  \hspace{1cm} (5.20)
The matrix $\Sigma$ is a diagonal matrix of the singular values of matrix $C$. The column indices of matrix $U$ corresponding to the same numbered indices with a nonzero diagonal value in $\Sigma$ makeup the set of orthonormal basis vectors that span the range space of $C$. Similarly, the column indices of matrix $V$ corresponding to the same numbered diagonal indices of zero value in $\Sigma$ makeup the set of orthonormal basis vectors that span the nullspace of $C$. Ouatouati and Johnson show that the matrices $\Sigma$ and $V$ can be partitioned according to the rank value $r$ as

$$
\begin{bmatrix}
\Sigma
\end{bmatrix} = \\
\begin{bmatrix}
\mu_{rr} & 0 & \cdots & 0 \\
0 & \mu_{rr} & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \mu_{rr}
\end{bmatrix}_{(N-r) \times (N-r)}
$$

(5.21)

$$
\begin{bmatrix}
V
\end{bmatrix} = \\
\begin{bmatrix}
V_{N \times r} & V_{N \times (N-r)}
\end{bmatrix}
$$

Ouatouati and Johnson then show that a null coordinate transformation results from

$$
\begin{bmatrix}
C \\
\end{bmatrix}_{N \times N} \begin{bmatrix}
V
\end{bmatrix}_{N \times (N-r)} = 0.
$$

(5.22)

By defining a new set of transformed coordinates as

$$
\begin{bmatrix}
\{u_i\}
\end{bmatrix}_{N \times (N-r)} = \begin{bmatrix}
V
\end{bmatrix}_{N \times (N-r)} \begin{bmatrix}
\{\hat{u}_i\}
\end{bmatrix}_{N \times (N-r)},
$$

(5.23)

the eigenvalue problem of Equation 5.16 can be written in reduced form as

$$
\begin{bmatrix}
\hat{K}_{ik}
\end{bmatrix}_{(N-r) \times (N-r)} - \omega^2 \begin{bmatrix}
\hat{M}_{ik}
\end{bmatrix}_{(N-r) \times (N-r)} \begin{bmatrix}
\hat{u}_i
\end{bmatrix}_{(N-r) \times (N-r)} = 0,
$$

(5.24)

where the transformed mass and stiffness matrices are given as

$$
\begin{align*}
\begin{bmatrix}
\hat{K}_{ik}
\end{bmatrix}_{(N-r) \times (N-r)} &= \begin{bmatrix}
V
\end{bmatrix}_{N \times (N-r)} \begin{bmatrix}
K_{ik}
\end{bmatrix}_{N \times N} \begin{bmatrix}
V
\end{bmatrix}_{N \times (N-r)} \\
\hat{M}_{ik} &= \begin{bmatrix}
V
\end{bmatrix}_{N \times (N-r)} \begin{bmatrix}
M_{ik}
\end{bmatrix}_{N \times N} \begin{bmatrix}
V
\end{bmatrix}_{N \times (N-r)}
\end{align*}
$$

(5.25)

which shows a reduction in the order of the original system from $N \times N$ to $(N-r) \times (N-r)$. Furthermore, the reduced eigenvalue problem incorporates the boundary conditions in the solution. It is important to note that upon solving Equation 5.25, the coordinates must be transformed back using Equation 5.23. Such a system reduction using SVD shows and increased computational savings over the Lagrange multiplier technique proposed by Belytschko et al. by yielding an eigenvalue system of equations with smaller dimensions.

The dimensional comparison is $(N-r) \times (N-r)$ with SVD and $(ndof + nbc) \times (ndof + nbc)$. 
with Lagrange multipliers. However, it is of value to reiterate the fact that the numerical integration still takes place over the entire domain using an overlaid grid of cells to identify the numerical integration quadrature points.

In addition to introducing and implementing a model reduction from a coordinate transform using SVD, Ouatouati and Johnson also perform a parametric study on their weight function in an effort to generalize its application. Here, the weight function used is that first introduced by Belytschko et al. given as

\[
 w(d) = \begin{cases} 
 e^{-(d/c)^k} - e^{-(d_m/c)^k} & \forall d \leq d_m \\
 1 - e^{-(d_m/c)^k} & \forall d > d_m 
\end{cases}
\]  

(5.26)

where \(d\) and \(d_m\) are defined as \(d\) and \(d_{max}\) in Equation 5.3 respectively. The parameters \(c\), \(k\), and \(d_m\) are defined by the user and determine the steepness and radial dimension of the weight function. The results of the parametric study show convergence for the eigenvalues of various structures. This convergence study is not too unlike that performed for FEM when determining if the element size is appropriate to accurately model the system in question. However, without being able to model all possible structures, it is unsure if the convergent parameters are suitable to achieve accurate solutions for all geometries.

Chen et al. also implement the EFG method for composite plate eigenvalue analysis [13]. Here the essential boundary conditions are enforced using the orthogonal transformation technique of SVD [31]. Chen et al. show convergence studies of the normalized eigenvalue versus the parameters of the weight function. Here, the weight function is a quartic spline function written as

\[
 w(d) = \begin{cases} 
 1 - 6d^2 + 8d^3 - 3d^4 & \forall d \leq 1 \\
 0 & \forall d > 1
\end{cases}
\]  

(5.27)

where \(d\) is defined as before in Equation 5.3. Chen et al. do not generalized the weight function of Equation 5.27, however they offer a guideline for determining an adequate size of the DOI requiring that it contain enough nodes to ensure that the matrix \(A\) of Equation 5.6 is nonsingular. They also confirm the conclusions drawn by Belytschko et al. that the manner in the distribution of nodes (i.e. random or uniform) within the problem domain does not significantly impact the results. In addition, Wang et al. also
use the EFG method to perform a convergence study on the weight function parameters associated with Equation 5.26 [38]. Although their conclusions are not generalizable to all structural eigenvalue analysis problems, they do offer a warning against using parameters that unnecessarily increase the DOI of a node. Such an increase can negatively affect the computational benefits that the EFG method offers. As indicated by Chen et al., Wang et al. also suggest to include enough nodes in the DOI to ensure that the matrix $A$ of Equation 5.6 is nonsingular. At this point, only soft upper and lower bounds are suggested for the weight function and its parameters to promote solution accuracy. However, such values can only be identified through an iterative convergence study.

In an effort to develop a truly “meshless” technique Atluri and Zhu introduced the Meshless Local Petrov-Galerkin (MLPG) approach [1]. Here the domain of integration of the weak form of the Galerkin method is simply over the domain where the test function is defined. This idea borrows from the Petrov-Galerkin approach where the test and trial functions can come from different spaces. This offers the opportunity for the domain of integration $\Omega$ to be defined as $\Omega_s$ when the test function is defined to be the weight function. This alleviates the need for an overlaid grid to determine the Gauss point location within the problem domain $\Omega$ for numerical integration. Now, the Gauss points are located within the domain $\Omega_s$. This defines a truly “meshless” approach as each DOI is integrated independently and then jointly assembled into the corresponding mass and stiffness matrices.

For review, a continuous definite integral $f(x,y)$ can be evaluated numerically as [12]

$$
\int_{\Omega_s} f(x,y) dx dy \approx \int_{-1}^{1} \int_{-1}^{1} f(\xi,\eta) d\xi d\eta = \sum_i \sum_j w_i w_j f(\xi_i,\eta_i) \det[J].
$$

The location of the integration points within $\Omega_s$ are defined as a function of the radius of the DOI in a recently published textbook by Liu [26]. Liu relies on a circular symmetry of the DOI and first develops the Gauss point locations for only one quarter as seen in Figure 5.1.
Liu performs a mapping of the quarter circle into polar coordinates that define a rectangular region, and then a mapping of that rectangular region into the parametric space resulting in Figure 5.2.

\begin{align}
  x &= d_{\text{max}} \cos \theta \\
  y &= d_{\text{max}} \sin \theta. 
\end{align} \tag{5.29}

The parametric space is related to the polar coordinates through

\begin{align}
  d &= \frac{d_{\text{max}}}{2} \xi + \frac{d_{\text{max}}}{2} \eta \\
  \theta &= \frac{\pi}{4} \eta + \frac{\pi}{4}. 
\end{align} \tag{5.30}

Finally, in the original coordinate space $\Omega_s$, the location of the numerical integration points (for four point integration) is given as

\begin{align}
  x &= \frac{d_{\text{max}}}{2} (\xi + 1) \cos \left[ \frac{\pi}{4} (\eta + 1) \right] \\
  y &= \frac{d_{\text{max}}}{2} (\xi + 1) \sin \left[ \frac{\pi}{4} (\eta + 1) \right]. \tag{5.31}
\end{align}
Figure 5.3 shows the location of the Gauss points in the parametric space and their corresponding location in the original rectangular coordinate space.

\[
\begin{bmatrix}
\frac{\pi}{4} (\eta + 2k - 1) \\
\frac{\pi}{4} (\eta + 2k - 1)
\end{bmatrix}
\]

However, Equation 5.31 only locates the Gauss points in the first quadrant of \(\Omega_s\). As such the Gauss points in the remainder of the quadrants \(k\) \((k = 1, 2, 3, 4\) incrementing in a counterclockwise manner from positive \(x\)) can be computed from

\[
\begin{align*}
\begin{bmatrix}
x \\
y
\end{bmatrix} &= \frac{d_{\text{max}}}{2} (\xi + 1) \begin{bmatrix}
\cos(\eta + 2k - 1) \\
\sin(\eta + 2k - 1)
\end{bmatrix} \\
\end{align*}
\]

Equation (5.32)

Figure 5.4 shows all the corresponding Gauss points in \(\Omega_s\).

\[
\begin{bmatrix}
\frac{\pi}{4} (\eta + 2k - 1) \\
\frac{\pi}{4} (\eta + 2k - 1)
\end{bmatrix}
\]

Figure 5.4: Gauss Point Location in \(\Omega_s\)

With the rectangular coordinates \(x\) and \(y\) as functions of the parametric coordinates \(\xi\) and \(\eta\), the Jacobian \(J\) in Equation 5.28 can be evaluated giving

\[
\det[J] = \frac{\pi}{4} \left(\frac{d_{\text{max}}}{2}\right)^2 (\xi + 1),
\]

Equation (5.33)
thereby allowing the numerical integral of Equation 5.28 to be solved. Such integration continues over each domain $\Omega_s$ within the domain $\Omega$ to assemble the mass and stiffness matrices.

In this work by Atluri and Zhu they utilized penalty parameters, a well established FEM technique, to enforce the essential boundary conditions. Here, the weak form of the MLPG method is given as

$$-\frac{1}{2} \int_{\Omega} \epsilon^T \sigma \, d\Omega + \frac{1}{2} \int_{\Omega} \dot{\epsilon}^T \rho \, d\Omega + \alpha \int_{\Gamma} \left( u_{r_s} - u_r \right) d\Gamma = 0.$$ (5.34)

Liu points out that the difference between penalty parameters $\alpha_i$ and Lagrange multipliers $\lambda_i$ for enforcing the boundary conditions is that $\alpha_i$ is a constant stiffness value and $\lambda$ is a variable that can vary [26]. As such, the matrix system of equations for the discrete eigenvalue problem does not increase in dimension ($ndof \times ndof$) when enforcing the boundary conditions using $\alpha$.

### 5.2 Example MLPG Method Eigenvalue Problem

To show the effectiveness of meshless techniques, the MLPG technique will be used to solve a typical eigenvalue problem for a rectangular plate. The MLPG solution is then compared to the FEM solution to show correlation between both eigenvectors and eigenvalues. One dimensional coding developed by Dolbow and Belytschko was helpful in producing the results given below [16]. Figure 5.5 shows both the finite element model and the meshless model with the same boundary conditions and number of nodes.

**Figure 5.5:** FEM and Meshless Plate Models

It is important to note that the FEM model shows definite plate elements and simply supported boundary conditions. However, the meshless model shows only nodal locations corresponding to those present in the FEM model. For simplicity, Figure 5.6
depicts a meshless model with a lower nodal density for the ease of viewing the domain of influence, integration points, etc. However, the eigenvalue results are reported using the same number of nodes. As such, Figure 5.6 shows the domain of influence for node 23 in the meshless model and the corresponding integration points identified by the blue circles.

**Figure 5.6: Gauss Point Distribution around Node 23**

The eigenvalue and eigenvector results for the eigenvalue problem using both conventional finite elements and meshless techniques are presented in Table 5.1 for the first five modes. The meshless results were developed using the MLPG method with SVD model order reduction to enforce the boundary conditions.

<table>
<thead>
<tr>
<th>Mode 1: 103.8 Hz</th>
<th>Mode 1: 138.1 Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mode 2: 165.3 Hz</td>
<td>Mode 2: 220 Hz</td>
</tr>
<tr>
<td>Mode 3: 267.9 Hz</td>
<td>Mode 3: 357.8 Hz</td>
</tr>
</tbody>
</table>
Table 5.1: Comparison of FEM and MLPG results

<table>
<thead>
<tr>
<th>Mode 4:</th>
<th>Mode 5:</th>
</tr>
</thead>
<tbody>
<tr>
<td>353.1 Hz</td>
<td>411.5 Hz</td>
</tr>
<tr>
<td>472.9 Hz</td>
<td>551.3 Hz</td>
</tr>
</tbody>
</table>

The resulting eigenmodes are identical indicating corresponding mode shapes. However, the eigenvalues differ in increasing magnitude with increasing mode number. Changing the radius of the domain of influence is one potential solution to converging the eigenvalues. Such effort to converge meshless computational solutions to conventional computational solutions represents an ongoing study in the meshless community [38].

However, these results do indicate the successful development and implementation of meshless techniques for use in computational modeling. The efforts in studying these methods offer an opportunity to discover how the structural mechanics community has integrated meshless techniques into computational modeling so as to exploit the numerical efficiency. Such efforts parallel those as in this work. When considering future efforts in developing this study by integrating a meshless eigenvalue solver, the previous study sets the stage having completed the background work that is necessary prior to integration of this technique into ASMDD shape optimization.

5.3 Integration of MLPG and ASMDD

This section demonstrates the use of the MLPG technique, as previously described and demonstrated, for creating the prescribed surface velocity profiles for use in ASMDD, which was shown to be based on volume velocity matching. This marriage
helps to show the possible integration of state-of-the-art meshless structural mechanics with the newly developed ASMDD technique for meshless boundary value problems for acoustic radiation. Again it is important to review the motivation for this work being efficient shape optimization that accounts for the change in structural dynamics associated with shape. Therefore, it is of value to show that current techniques in meshless structural dynamics can blend seamlessly with the newly formulated meshless boundary value method ASMDD. To demonstrate this, the MLPG method will be used to determine the first ten modes of a simply supported square plate. This plate would be much like the radiating surface of Shape #1 in Table 4.2 of Section 4.4. Of course, as a square plate has axisymmetric geometry, degenerate modes are expected. Such modal degeneracy brings a very real and special case to the analysis of the problem thereby enhancing its complexity. The MLPG square plate will be discretized according to the digital domain so that each displaced point on the MLGP plate will have a corresponding point in the ASMDD problem domain. The displacement at each point will be used to prescribe the surface velocity (displacement times $2\pi if$ for an $e^{j2\pi ft}$ time convention) at each point in the ASMDD problem domain. This velocity distribution, which is a modal combination of the ten plate modes, will then be used by ASMDD to determine the acoustic power radiated by the plate. To begin, the voxel/ASMDD plate is given in Figure 5.7 and the first ten modes of the MLPG square plate are given in Table 5.2. Again, it is important to note that each ASMDD node has a corresponding displaced node in the MLPG mode shape.

![Figure 5.7: Voxel/ASMDD Square Plate](image-url)
<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3,1)</td>
<td>35.6486 Hz</td>
</tr>
<tr>
<td>(2,2)</td>
<td>28.4574 Hz</td>
</tr>
<tr>
<td>(1,2)</td>
<td>17.8512 Hz</td>
</tr>
<tr>
<td>(2,1)</td>
<td>17.8486 Hz</td>
</tr>
<tr>
<td>(1,1)</td>
<td>7.1417 Hz</td>
</tr>
</tbody>
</table>

![Graphs of different modes](image-url)
Once these eigenvalues and eigenvectors are determined, a spectral plot of the surface velocity can be made. Of course, the results of such a plot are dependent upon force and
measurement locations. The force is applied so that all modes within a 0 Hz to 50 Hz frequency band are excited. Since the largest number of half sine waves for the most complex mode from 0 Hz to 50 Hz is three, the force is located in the center of one third of the square plate. Figure 5.8 indicates the forcing and measurement locations at MLPG points 247 (blue circle) and 1327 (green circle), respectively. Figure 5.8 also shows the nodal lines for the largest number of half sine waves for the most complex mode within the frequency range of interest (0 Hz to 50 Hz). Figure 5.9 shows the spectral composition of all ten modes radiating together as seen by the measurement and forcing locations indicated in Figure 5.8. Figure 5.9 is normalized to the largest velocity peak in the entire radiation spectrum.

![Figure 5.8: Forcing and Measurement Locations](image)
Next, ASMDD is used to compute the radiated acoustic power from the square plate above. For comparison, the BEM technique is also utilized to determine the radiated acoustic power. Here, the velocity distribution determined from the MLPG technique is used as the velocity distribution in the elemental model. The BEM model will have a corresponding plate element node at each meshless point in the MLPG/ASMDD model. Rather, each MLPG/ASMDD point in the meshless models is used as the node for each quadrilateral element in the BEM model. Furthermore, the BEM model uses a monopole distribution only (i.e. $\alpha = 1$ and $\beta = 0$) to model the radiation. Figure 5.10 shows the comparison between the results of the BEM and ASMDD methods for computing the radiated sound power spectrum from 0 Hz to 50 Hz. The spectrum is normalized to the largest sound power radiation peak in the entire radiation spectrum.

**Figure 5.9:** Square Plate Surface Velocity Frequency Response
As indicated in section 3.3.3.4, the meshless monopole self term is limited to a value of $ka = 4$, which corresponds to approximately 218 Hz. As this represents an upper bound on the ASMDD formulation, it is of value to determine the computational performance near this upper bound. Therefore, each of the eigenvalues identified in Table 5.2 are shifted by 150 Hz so as to model the radiation from 150 Hz to 200 Hz. Figure 5.11 shows these shifted eigenvalue results in comparison to the BEM results. As before, the spectrum is normalized to the largest sound power radiation peak in the entire radiation spectrum from 150 Hz to 200 Hz.
The results of Figure 5.10 and Figure 5.11 show near identical matches between the ASMDD method and the BEM method within the operating frequency range of the meshless monopole self term. The sound power spectral peaks occur at frequencies that correspond to the same modal frequencies found in the eigenvalue analysis. In addition, the spectral peaks also match in amplitude between the two modeling methods. In short, the meshless computational results in determining the radiated sound power from the square plate are consistent with those produced by the conventional BEM technique. Given these results, the MLPG and ASMDD techniques blend quite well together and their results agree well with those generated by the conventional BEM formulation, giving promise to their use.

Building upon the previous analysis, this next section models the same plate on a cubic baffle. This problem is the same as Shape #1 of Table 4.2 analyzed in section 4.4, however, the modal distribution of the radiating plate is used as the surface velocity distribution on the radiating portion of the structure. The procedure for matching MLPG nodes with ASMDD nodes as well as MLPG nodes with boundary element nodes follows
as described previously for the plate analysis. However, the addition of the cubic baffle with the same mesh density as the plate, as presented in Figure 5.7, increases the size of both models considerably. As such, these cubic baffled models have a reduced computational size in comparison to the previous plate analysis so as to keep the size of the models modest. As a consequence of a coarser model, there is a slight shifting upward in the eigenvalues of the radiating plate presented in Table 5.2, however, the eigenvectors remain the same. As before, the BEM is used for comparison to the ASMDD results. As the BEM method is modeling a closed volume, it uses a distribution of tripole sources to avoid nonunique solutions as described in section 3.2. For convenience, Figure 5.12 shows the ASMDD and BEM cubic baffle models used in the analysis.

![BEM Model and ASMDD Model](image)

Figure 5.12: BEM and ASMDD Models

The comparative analysis of the baffled models is done from 0 Hz to 80 Hz. Figure 5.13 show the sound power radiation results of the cubic baffled plate for the frequency range 0 Hz to 80 Hz. The spectrum is normalized to the largest sound power radiation peak in the frequency band.
Figure 5.13: MLPG/ASMDD and BEM Cubic Baffle Sound Power Radiation Spectrum

The ASMDD results in Figure 5.13 show good agreement with those generated by BEM for the 0 Hz to 80 Hz frequency band, as the first four major spectral peaks differ by no more than 5 dB. The (2,4) and (4,2) degenerate modal pair (modes 9 and 10) occurring at about 72 Hz do not have good agreement as they are pushing the computational accuracy limit of the necessary six points per wavelength. As such, the model results cannot be trusted to give accurate results and would explain the discrepancy. Overall, the agreement between the ASMDD and BEM methods is quite good for the radiating plate on a cubic baffle.

In summary, the following chapter shows the formulation for the current state-of-the-art technique for using element-free/meshless formulations in structural dynamics, called the MLPG technique. As the overall goal for future versions of this work is to perform structural/acoustic shape optimization that accounts for the modal composition of the structure, it is necessary to have a technique that can blend seamlessly with the meshless boundary value solver ASMDD. The results of this chapter show that indeed meshless structural dynamics can blend seamlessly with ASMDD to determine the
radiated acoustic power based on the modal composition of a square plate. Such results show excellent agreement when compared to the results of the same problem using the conventional BEM technique.
6.1 Dissertation Summary

This dissertation lays the framework for a self-contained acoustic shape optimization tool through the development of a highly efficient and simplified “meshless” strategy for solving computational problems in acoustic radiation and placing them in a digitized computational domain.

In the introduction, the limitations of conventional element based computational formulations for use in shape optimization are identified and discussed. Given that the shape of the structure is a design variable, element computational integrity must be monitored and corrected as necessary as the shape evolves. Computing parametric distortion values and comparing them to acceptable limits must be done for all elements participating in the shape change. Should the element distortions become too large, a remeshing of the surface could be necessary, or new elements might need to be created out of distorted elements, etc. As such, the bookkeeping burden associated with updating element connectivity or the use of sophisticated remeshing algorithms for the structure during every design iteration in shape is simply too cumbersome. Furthermore, updating conventional BEM matrices which are completely full to reflect changes in structural shape is also computationally taxing. Populating these matrices requires various orders of numerical integration dependent upon spatial dimension between source and receiver, which is changing for each iteration in shape. It is these limitations in conventional computational modeling that motivates the development of a “meshless” strategy in a digitized domain.

The dissertation demonstrates that modeling acoustic radiation can be accomplished simply using points rather than elements. The governing system matrices are populated according to nonsingular Greens function values between points. However, for collocated source and receiver points on the structures surface the Greens functions are singular. A major contribution to the ground work for acoustic shape optimization in
this dissertation is the derivation for developing a nonsingular analytical formulation for the collocated source and receiver points, called the self terms. The new point based formulation in conjunction with the analytical formulations for the self terms performed well for monopole source distributions. However, our objective was also to include the dipole source type as well to eliminate the nonuniqueness problem. Yet, the results for the dipole source type distribution were less than ideal.

Next, the dissertation introduces a strategy for morphing the shape of a structure as described through the activation and deactivation of points uniformly distributed in the computational domain. It is the logical extension of this point-based acoustic radiation modeling procedure that gives way to representing a surface in a digital format similar to the graphics community that uses pixels to represent surfaces as a collection of “turned on” points. Here, 3D surfaces can be visualized as a collection of voxels where each point describing the surface of the radiating structure is located at the centroid of the voxel. Morphing a structure then becomes a matter of activating and deactivating points in a contiguous manner. But, as for all acoustic radiation problems, surface normal vectors are necessary to complete the computation. As such this dissertation shows, as a second major contribution to the acoustic shape optimization framework, how to assign surface normal vectors to point-based models. The inherent uniformity of a point-based computational domain lends itself to locally determining normal vectors through a series of planar scans through the structure’s surface. The results showed that the assignment of point-based normal vectors perform as expected and meets the second major objective for the thesis; yielding normals that point out of voxel faces, voxel edges, or voxel corners for use as normal vectors at points, and are determined only by a local neighborhood of points.

A combination of the new point-based formulation with the analytical formulations for the self terms and the method for surface normal vector assignments produced good results for the example structure when compared to computations via the move conventional BEM method. The best comparison occurred at low structural curvatures and wavenumbers, $ka < 4$. It was determined that the limit in model curvature, for ASMDD models, was due in part to the lack of uniformity in neighboring normal vector assignments, although the normal vectors did point as required by the code. This
is because the normal vector assignments are based upon neighboring point locations and do not consider the neighboring normal assignment. As such neighboring point normals can deviate from one another to a larger extent than those on the continuous surface.

As a final example, the dissertation reviews the meshless strategy for modeling the dynamics of structures, called MLPG (Meshless Local Petrov Galerkin method). This meshless structural dynamic work, under development for over a decade now, shows promise for its general acceptance as an alternative to the more conventional FEM programs. Such a computational technique blends well with the meshless point-based strategy detailed in this dissertation for modeling acoustic radiation. Here, the MLPG method provides the surface velocity information needed by the new point-based acoustic radiation model. This is achieved by solving the modal radiated acoustic power from a plate where the surface velocity is specified by the modal results determined by the MLPG method. The radiated acoustic power from the plate was analyzed in a free field and on a cubic baffle. The resulting sound power calculations are again in good agreement with those generated via conventional BEM codes. Such results show that meshless acoustic modeling for shape optimization can eventually incorporate the change in structural dynamics for changes in shape using a technique founded on the same point-based strategy.

6.2 Method Assessment

In summary, ASMDD is capable of modeling the acoustic radiation from a structure defined only by points (not with conventional elements) in 3D space. Virtual monopole type acoustic sources alone are used to replicate the radiated acoustic field of the actual structure, and are activated with a source strength determined by a method based upon the conventional BEM formulation of wave superposition. Also, ASMDD blends seamlessly with the current state-of-the-art meshless structural dynamic formulation called MLPG. Blending the ASMDD method with the MLPG formulation gives ASMDD the ability to model acoustic radiation based upon the modal composition of the structure under study without the need for element based eigenvalue solvers.
ASMDD has a frequency limitation of $ka \leq 4$ and is accurate for models with a limited amount of curvature. However, as ASMDD uses only monopole type acoustic sources, it cannot overcome the nonuniqueness problem associated with the interior resonance condition present when modeling closed volumes. Therefore, ASMDD is limited to accurately modeling the acoustic radiation of any closed volume away from the resonance of the interior mode of the closed volume.

6.3 Future Work

Future efforts should concentrate on resolving the fit between the analytical dipole velocity self term and the numerically extracted values. As discussed in Chapter 3, the dipole source is a necessary complement to the monopole source distribution for modeling acoustic radiation from enclosed volumes so as to avoid nonunique solutions.

The method for determining point-based outward pointing normal vectors as detailed in Section 4.3 performs as expected. Point-based normals point out from any convex surface at voxel faces, edges, or corners as dictated by the surrounding voxel neighborhood. However, the formulation of Section 4.3 allows for normal assignments for large values of curvature that are not consistent with the continuous surface normals. As such, future efforts should consider another set of planar scans as illustrated by Table 4.1. These scans might edit existing normal vector assignments so as to be consistent with surrounding normals by ensuring that no normal is greater than some angular limit from the surrounding normals. In addition, using more surrounding points in the interpretation of the edge of the structure could refine the normal vector assignment to include angles other than $0^\circ$, $45^\circ$, or $90^\circ$. These techniques would help to enforce more uniformity from one point-based normal to the next. Improving the normal assignments could help to ensure a better fit for the acoustic radiation results for structures with larger values of curvature. Also, an improved normal-vector-to-curvature fit could improve higher frequency results as well because the ratio of acoustic wavelength to meshless point area is becoming closer to unity in this frequency range. Rather, the computational solution of any model is becoming increasingly sensitive to the ability of the computational model to accurately model the actual structure. Refining the fit of the
point based normal vectors is one way to help improve this model accuracy. In addition to the improvement of the fit of point-based normals, is the adaptation of a point normal assignment to nonconvex structural volumes. Potential shape iterations are very likely to iterate a shape that is nonconvex and the normal assignment routine needs to be able to accommodate such shapes.

Autonomous computer control of the structural shape in the digitized space through optimization is another future effort. Here a strategy for activating and deactivating voxels in a structurally contiguous manner is necessary. This strategy would represent the design variable in shape optimization, changing the shape with each iteration in the optimization. The method of doing so would have to ensure that the iterated shape is contiguous creating a closed volume structure. For example, one solution might be to create a union between various simple geometric shapes to the boundary of the structure.

Finally, implementing the MLPG method in 3D for determining the eigenvalues and eigenvectors for each iterated shape is necessary. This would allow the objective function of the shape optimization to account for the modal composition of the iterated 3D shape. Currently, the understanding of the formulation of the MLPG work, as presented in this thesis, is limited to 2D. However, the theory does extend to 3D given the proper formulation of the Gauss point locations within the domain of influence for a 3D structure. Specifically, future efforts would need to understand how to formulate a 3D domain of influence. Once this is done, the Gauss points are located in the 3D domain of influence as is done in 2D.
BIBLIOGRAPHY


APPENDIX

Derivation of Unbaffled Piston Radiation Resistance

The derivation of the radiation resistance of an unbaffled piston is based upon the work of T. Mellow et al. [28]. Here Mellow derives the radiation resistance for a piston in a finite baffle seen in Figure A.1.

![Figure A.1: Finite Baffled Piston Radiator](image)

The unbaffled piston case is determined from this same formulation for the case where \( b = a \). Mellow shows the complex radiation impedance of the finite baffled piston to be written as

\[
Z_{up} = \frac{2 \rho c}{S} (R_R + iX_R),
\]

(A.1)

where \( S \) is the area of the piston face \( \pi a^2 \), and \( \rho c \) is the impedance of the media through which the radiated sound propagates. \( R_R \) and \( X_R \) are the normalized radiation resistance and reactance components of the radiation resistance, respectively.

Mellow writes the radiation resistance (real part of the radiation impedance) as

\[
R_R = -kb \frac{b^2}{a^2} \sum_{m=0}^{M} \text{Re}\left(\tau_m\right) \left\{1 - \left(1 - \left(\frac{a}{b}\right)^2\right)^{m+3/2}\right\},
\]

(A.2)

and the radiation reactance (imaginary part of the radiation impedance) as

\[
X_R = -kb \frac{b^2}{a^2} \sum_{m=0}^{M} \text{Im}\left(\tau_m\right) \left\{1 - \left(1 - \left(\frac{a}{b}\right)^2\right)^{m+3/2}\right\},
\]

(A.3)

where, in both Equations A.2 and A.3 \( k \) is the wave number from \( \frac{2\pi f}{c} \), where \( f \) is the frequency of oscillation and \( c \) is the speed of sound in the media of propagation. Next, Mellow identifies a set of \( M \) simultaneous equation that, when solved, yield the coefficients \( \tau_m \) for use in the final solutions above given as
\[ \sum_{m=0}^{M} \left[ B_{mq}(kb) - iS_{mq}(kb) \right] \tau_m = -\Phi_q, \quad (A.4) \]

where \( \Phi_q \) is written as
\[
\Phi_q = \frac{a}{b} \sum_{n=1}^{N} \frac{(-1)^q J_1(j_{0n} a/b) \left( j_{0n} \right)}{(q!)^2 J_1^2(j_{0n})} \left( \frac{J_1(j_{0n})}{2} \right)^{2q-1}. \quad (A.5)
\]

From Equation A.5, \( J_1 \) is a first order ordinary Bessel function of the first kind. Also, Mellow defines \( j_{0n} \) as the \( n^{th} \) zero of the zero order ordinary Bessel function of the first kind \( J_0 \), such that \( J_0(j_{0n}) = 0 \). From Equation A.4, Mellow identifies \( B \) as the Bouwkamp function and \( S \) as the Streng function where they are written as
\[
B_{mq}(kb) = \sqrt{\pi} \sum_{r=0}^{q+m} \frac{(-1)^{q+r} \Gamma(m+5/2) \Gamma(q+r+1) \left( kb \right)^{2(q+r)+3}}{\Gamma(r+m+5/2) \Gamma(q+r+5/2) \left( 2 \right)}
\]
\[
S_{mq}(kb) = \sqrt{\pi} \sum_{r=0}^{q-m} \frac{(-1)^{q+r-m} \Gamma(m+5/2) \Gamma(q+r-m-1/2) \left( kb \right)^{2(q+r-m)+3}}{r! (q!)^2 \Gamma(r-m-1/2) \Gamma(q+r-m+1) \left( 2 \right)}
\]

where \( \Gamma \) is the well known Gamma function. Writing the above in matrix form we have
\[
\left[ \left( B_{mq} - iS_{mq} \right) \right] \{ \tau_m \} = -\{ \Phi_q \}, \quad (A.7)
\]

which can be solved for \( \tau_m \) for substitution into Equations A.2 and A.3.
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Education

Ph. D. in Mechanical Engineering - Center for Acoustics and Vibration
The Pennsylvania State University, January 2003 – December 2006; GPA: 3.96

Masters of Science in Mechanical Engineering - Noise and Vibration Laboratory
Western Michigan University, May 2001 – December 2002; GPA: 3.94

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Work Experience

Torque Control Products Division Eaton Corporation
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Product Engineering CO-OP
January 2000-April 2000
Participated in new locking differential product development. Performed analytical calculations and initiated testing for components of the locking differential prototypes.

Torque Control Products Division Eaton Corporation
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Participated in the drafting, processing, and review of manufacturing documentation. Participated in machinery runoffs to determine statistical capability of manufacturing machines.

Conference Publications


Journal Publications


Awards and Honors

- NSF EAPSI Korea Recipient ($3000) 2005
- Member of Psi Kappa Psi (ΨΚΨ) Academic Honor Society 2003
- Kalamazoo Antique Auto Restorers Club (KAARC) Scholarship ($750) 2002
- WMU Graduate Travel Grant ($600); WMU Graduate Fellowship ($12,362) 2001
- Member of Tau Beta Pi (ΤΒΠ) National Engineering Honor Society 2000
- Member of Pi Tau Sigma (ΠΤΣ); Member of Golden Key National Honor Society 1999
- Member of Lee Honors College 1996 - 2001
- Eagle Scout in Boy Scouts of America 1996