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LOCAL INDEX THEOREM IN NONCOMMUTATIVE GEOMETRY

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Abstract

We consider the multiplicative property of index cocycles associated to a spectral triple. We construct a perturbation of the JLO-cocyle and prove that it is multiplicative on the cochain level with respect to the exterior product of theta-summable spectral triples and the product in entire cyclic theory. Furthermore, we show that this cocycle admits an asymptotic expansion if the spectral triple admits the analogue of an Euler vector field. Using Laurent series, we obtain a multiplicative local index cocycle.

Contents

List of Symbols

$C^\infty(M)$ smooth complex valued functions on M

$C^\infty(M,E)$ smooth sections of $E \to M$

D (odd) elliptic operator

$D_1 \times_{\mathrm{alg}} D_2$ direct product of D_1 and D_2, page 90

E complex vector bundle

$HC^\bullet(\mathcal{A})$ cyclic cohomology, page 52

$HH^\bullet(\mathcal{A},\mathcal{A}^*)$ Hochschild cohomology, page 52

$HH_\bullet(\mathcal{A},\mathcal{A})$ Hochschild homology, page 52

$HN_\bullet(\mathcal{A})$ negative cycllic homology, page 52

$HP^\bullet(\mathcal{A})$ periodic cyclic cohomology, page 52

$H^\bullet_{\mathrm{cpt}}$ de Rham cohomology with compact support

K^i Atiyah-Hirzebruch topological K^i-theory

$L^2(M,E)$ L^2-sections of $E \to M$

M smooth manifold

P elliptic operator

P^* adjoint of P, page 98

$P^n(\mathbb{C})$ projective space of $\mathbb{C}^n$, page 4

R_λ resolvent $(\lambda - \Delta)^{-1}$, page 44

S periodicity map, page 53

TM tangent bundle of M

V real vector space

$[-,-]$ commutator, page 7

$(\mathcal{A},\mathfrak{H},D)$ spectral triple, page 8

$(\mathcal{A},\mathfrak{H},F)$ Fredholm module, page 19

$\mathcal{A}$ associative algebra

$\mathcal{A}^+$ minimal unitization of $\mathcal{A}$, page 18

Ψ_{cl} classical pseudodifferential operators, page 34

$\mathbb{C}^{1,1}$ standard two-dimensional graded Hilbert space, page 16

$\mathrm{Cl}(V,g)$ complex Clifford algebra associated to (V,g), page 12

Cl_n Clifford algebra of $\mathbb{R}^n$ with respect to the standard inner product, page 13

$\mathcal{D}$ differential operators, page 34

$\mathcal{D}_{\mathrm{cpt}}$ compactly supported differential operators, page 36

Δ invertible positive self-adjoint operator, page 28

End endomorphisms

Γ Gamma function, page 42

Im imaginary part

$\mathbb{K}$ classifying space for K^0, page 3

$\mathcal{L}$ algebra of bounded operators, page 86

$\mathcal{L}^{(p,\infty)}$ Dixmier p-class, page 95

$\mathcal{L}^p$ Schatten p-class, page 94

$\Omega(\mathcal{A})$ universal enveloping differential graded algebra of $\mathcal{A}$, page 49

$\Omega^\bullet M$ differential forms on M

$\Omega_\bullet^{\mathrm{SO}}$ oriented bordism, page 3

Op^t operators of analytic order at most t, page 32

Ψ filtered algebra, pseudodifferential operators, page 32

Re real part

$\mathcal{S}(\mathbb{R})$ Schwartz functions on $\mathbb{R}$, page 59

Σ^n standard n-simplex, page 68

$\mathrm{Spec}(D)$ Spectrum of D, page 87

Todd Todd class, page 2

Tr	operator trace, page 94
Tr_ω	Dixmier trace associated to ω, page 96
W^s	Sobolev s-space, page 7
$\mathbb{Z}$	integers
$\alpha \times \beta$	shuffle product of α and β, page 68
$\alpha \times' \beta$	cyclic shuffle product of α and β, page 73
$\bar{D}$	closure of D
β	Bott class, page 4
$\mathrm{Ch}[\sigma]$	Chern character of σ, page 2
coker	cokernel, page 97
T^*M	cotangent bundle of M
dim	dimension
dom	domain
$\mathrm{dom}^\infty(\delta)$	infinite domain of δ, page 37
γ	grading operator
im	image, page 97
κ	Karoubi operator, page 51
ker	kernel, page 97
$\mathcal{W}$	Weyl algebra, page 34
Ad	adjoint representation, page 14
Vol	volume form, page 2
$\mathbb{M}_n$	$n \times n$ complex matrices, page 13
$\mu_n(T)$	n-th singular value of T, page 92
$\rtimes$	crossed product
σ_P	principal symbol of P, page 1
σ_i	Pauli matrices, page 13
$\star$	Hodge star operator, page 2
Str	operator supertrace, page 57
$\mathfrak{F}$	Fredholm module

$\mathfrak{S}$	standard one-dimensional 2-multigraded spectral triple, page 16
$\mathfrak{S}_M$	signature class of M, page 3
τ_n	(super)trace of dimension n, page 53
ε_M	signature grading on M, page 2
ε_i	generators of the Clifford algebra corresponding to the standard basis of $\mathbb{R}^n$, page 13
$\wedge$	wedge product of differential forms
ζ_X	zeta function associated to X, page 42
c_M	Clifford symbol of M, page 2
e	idempotent
$f(D)$	functional calculus, page 87
g	symmetric bilinear form, page 12
i	square root of -1
q	projection from T^*M to M, page 2
u	formal parameter of degree 2, page 52
v	formal parameter of degree -2, page 52
$\binom{z}{k}$	binomial coefficient, page 38
B	Connes boundary operator, page 51
$C_\bullet$	Hochschild chain complex, page 51
$C^\bullet$	Hochschild cochain complex, page 51
$CN_\bullet$	negative cyclic chain complex, page 52
$CP^\bullet$	periodic cyclic cochain complex, page 52
$C_c^\infty(M)$	compactly supported smooth functions on M
Index_s	super-index, page 100
$\mathcal{K}$	compact operators, page 92
Σ	dimension spectrum, page 42
Str	super-trace, page 101
b	Hochschild boundary operator, page 51
$\mathcal{B}^\bullet$	(b, B)-cochain complex, page 52
d	differential on Ω, page 50

$\dim_s$ super-dimension, page 100

dom domain of

$\mathfrak{H}$ Hilbert space

$\mathfrak{D}$ spectral triple

$\mathrm{Index}_{\mathrm{a}}$ analytical index, page 2

Index Fredholm index, page 98

$\mathrm{Index}_{\mathrm{t}}$ topological index, page 2

Index_D index map associated to D, page 24

$\mathscr{R}$ analogue of the Euler vector field, page 46

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Chapter 1

Introduction

In this thesis, we study index theory for spectral triples.

A *Fredholm operator* is a linear map between two vector spaces whose kernel and cokernel are both finite-dimensional. The *Fredholm index* of such an operator is defined to be the dimension of the kernel minus the dimension of the cokernel.

Fredholm operators arise naturally in geometry and computing their index using geometric data is an important problem. Following Hirzebruch's proof of the signature theorem [Hir56], Atiyah and Singer solved the index problem for a general linear elliptic partial-differential operator on a closed manifold [AS63, Pal65, CS65a, CS65b]. We give a quick overview of their solution, restricting to the case of a *order-one elliptic* partial-differential operator on an *even* dimensional oriented manifold. This will serve as a motivation for many of the following developments.

The Atiyah-Singer Index Theorem.

We refer to [Pal65, BD82a, BD82b, LM89, Roe98, HR00] for details. Functional analytical notions are referred to the Appendix and references therein. Let M be a closed oriented manifold of even dimension and let E^0 and E^1 be complex vector bundles over M. Let

$$P : C^\infty(M, E^0) \to C^\infty(M, E^1)$$

be a order-one partial-differential operator with *principal symbol*[1]

$$\sigma_P : T^*M \to \mathrm{Hom}(E^0, E^1).$$

Suppose that P is *elliptic*, that is, the principal symbol σ_P is invertible outside the zero-section (which we identify with M) of T^*M. Then P is a Fredholm operator and its Fredholm index is

[1] In a coordinate chart, if $P = \sum a_i(x)\frac{\partial}{\partial x_i}$, then $\sigma_P(x, \xi) = \sum a_i(x)\xi_i$. In a coordinate free language, we can write $[P, f] = \sigma_P(df)$.

called the *analytical index* of P. The following is one of the standard ways to prove the Fredholm property for an elliptic operator using the theory of *Sobolev spaces* (cf. 3.1.6):

(i) Choose a smooth measure on M and Hermitian structures on E^0 and E^1. Let $L^2(M, E^0 \oplus E^1)$denote the corresponding graded Hilbert space. Construct an unbounded odd operator on $L^2(M, E^0 \oplus E^1)$, with domain $C^\infty(M, E^0 \oplus E^1)$, by $D := \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}$.

(ii) Next, show that D is essentially self-adjoint and its closure $\bar{D}$ has compact resolvents, using the basic elliptic estimate and the Rellich Lemma. This implies that $\bar{D}$ is Fredholm.

(iii) Finally, use elliptic regularity to deduce that D, and consequently P, is Fredholm.

Now we describe the geometric side. Since σ_P is invertible outside the zero-section M of T^*M and M is compact, it defines a K^0-element $[\sigma_P] \in K^0(T^*M)$ by the difference construction (cf. [LM89, page 169]). Let $\mathrm{Ch}[\sigma_P] \in H^\bullet_{\mathrm{cpt}}(T^*M)$ denote the Chern character of $[\sigma_P]$ and let $\mathrm{Todd}(TM \otimes \mathbb{C}) \in H^\bullet(M)$ denote the Todd class of the complexified tangent bundle $TM \otimes \mathbb{C} \to M$ of M. Then the Atiyah-Singer index theorem says that

$$\mathrm{Index}(P) = \int_{T^*M} \mathrm{Ch}[\sigma_P] \wedge q^*\mathrm{Todd}(TM \otimes \mathbb{C}),$$

where T^*M is given the orientation coming from its almost complex structure (cf. [LM89, page 254-255]) and $q : T^*M \to M$ is the projection.

The left-hand-side of the index theorem is called the *topological index* of P. The strategy of the proof is to show that both the analytical and the topological indices behave nicely with respect to certain natural operations and then use algebraic topology to reduce the problem to checking on a small class of explicit examples. We describe these steps in a bit more detail.

Reduction to Twisted Signatures.

The correspondence $[\sigma_P] \mapsto \mathrm{Index}(P)$ gives a well-defined additive map $\mathrm{Index}_{\mathrm{a}} : K^0(T^*M) \to \mathbb{Z}$ (cf. [Pal65, Theorem XV.1.1]), again called the analytical index map. On the other hand, there is a natural additive map $\mathrm{Index}_{\mathrm{t}} : K^0(T^*M) \to \mathbb{R}$, called the toplogical index map, given by

$$\mathrm{Index}_{\mathrm{t}}(e) := \int_{T^*M} \mathrm{Ch}(e) \wedge q^*\mathrm{Todd}(TM \otimes \mathbb{C}), \quad e \in K^0(T^*M).$$

We need to show that $\mathrm{Index}_{\mathrm{a}} = \mathrm{Index}_{\mathrm{t}}$ on $K^0(T^*M)$.

Fix a Riemannian structure on M and let Vol denote the associated volume form giving the orientation. Then the *Hodge star* operator $\star : \Omega^\bullet M \to \Omega^\bullet M$, determined by $\beta \wedge \star\alpha = \langle \alpha, \beta \rangle \,\mathrm{Vol}$, satisfies $\star^2 = (-1)^p$ on Ω^p. The *signature grading* on $\Omega^\bullet M \otimes \mathbb{C}$ is the grading given by $\varepsilon_M = i^{k+p(p+1)}\star$ on $\Omega^p M \otimes \mathbb{C}$, where $\dim M = 2k$.

Consider the symbol $c_M : T^*M \to \mathrm{End}(\Lambda^\bullet T^*M \otimes \mathbb{C})$ given by

$$c_M(x, \xi) := \xi \wedge -(\xi\wedge)^*,$$

where $(\xi\wedge)^*$ denote the adjoint of the operator $\xi\wedge$. Then $c_M(x,\xi)^2 = -||\xi||^2 \cdot 1$, and thus c_M is elliptic. Moreover, it is *odd* with respect to the signature grading ε_M, hence determines a class $\mathfrak{S}_M := [c_M, \varepsilon_M]$ in $K^0(T^*M)$, called the *signature class* of M. It follows from Hodge theory (cf. [Pal65, Theorem V.3.5]) that the analytical index of $\mathfrak{S}_M$ is given by the signature of the intersection form

$$H^k(M) \otimes H^k(M) \to \mathbb{R}, \quad (\alpha, \beta) \mapsto \int \alpha \wedge \beta,$$

hence the name. The topological index is given by the Hirzebruch L-class (cf. [Pal65, Proposition V.3.6]), and the Atiyah-Singer index theorem specializes to the Hirzebruch signature theorem.

A *twisted signature class* is a class in $K^0(T^*M)$ of the form $q^*a \cdot \mathfrak{S}_M$, $a \in K^0(M)$. Twisted signatures form a finite-index subgroup in $K^0(T^*M)$ (cf. [Pal65, Proof of Theorem XV.4.3]), hence it remains to show that

$$\mathrm{Index}_{\mathrm{a}}(q^*a \cdot \mathfrak{S}_M) = \mathrm{Index}_{\mathrm{t}}(q^*a \cdot \mathfrak{S}_M), \quad a \in K^0(M).$$

Bordism Invariance.

One of the fundamental properties of the index is bordism invariance. Let $a \in K^0(M)$ be a virtual vector bundle over M. Suppose that $(M, a) = \partial(N, b)$, that is, M is the oriented boundary of a compact oriented manifold N and $a \in K^0(M)$ is the restriction of $b \in K^0(N)$. Then, by [Pal65, Theorem XV.4(c)],

$$\mathrm{Index}_{\mathrm{a}}(q^*a \cdot \mathfrak{S}_M) = 0 = \mathrm{Index}_{\mathrm{t}}(q^*a \cdot \mathfrak{S}_M).$$

Let $\mathbb{K} = \mathbb{Z} \times BU$ denote the classifying space for K^0. Then an element of the *oriented bordism ring* $\Omega^{\mathrm{SO}}_\bullet(\mathbb{K})$ is given by (the bordism class of) a pair (M, a), where M is a closed oriented manifold and $a \in K^0(M)$. We write $[M, a]$ for the corresponding class. Bordism invariance implies that the index maps give maps $\Omega^{\mathrm{SO}}_\bullet(\mathbb{K}) \to \mathbb{R}$ by

$$\mathrm{Index}_{\mathrm{a}}[M, a] := \mathrm{Index}_{\mathrm{a}}(q^*a \cdot \mathfrak{S}_M),$$
$$\mathrm{Index}_{\mathrm{t}}[M, a] := \mathrm{Index}_{\mathrm{t}}(q^*a \cdot \mathfrak{S}_M).$$

We need to show that $\mathrm{Index}_{\mathrm{a}}[M, a] = \mathrm{Index}_{\mathrm{t}}[M, a]$ for any $[M, a] \in \Omega^{\mathrm{SO}}_\bullet(\mathbb{K})$.

Additivity.

The following *additivity* properties of the index maps[2] are clear (cf. [Pal65, Theorem XV.4(a)]):

(1) $\mathrm{Index}[M, a_1 + a_2] = \mathrm{Index}[M, a_1] + \mathrm{Index}[M, a_2]$ and

(2) $\mathrm{Index}[M_1 \sqcup M_2, a_1 \oplus a_2] = \mathrm{Index}[M_1, a_1] + \mathrm{Index}[M_2, a_2]$.

Multiplicativity

Further simplification is obtained from the *multiplicativity* of the index maps (cf. [Pal65, Theorem XV.4(b)]):

[2] If an expression is true for both $\mathrm{Index}_{\mathrm{a}}$ and $\mathrm{Index}_{\mathrm{t}}$, we simply write Index.

(3) $\operatorname{Index}[M_1 \times M_2, a_1 \otimes a_2] = \operatorname{Index}[M_1, a_1] \cdot \operatorname{Index}[M_2, a_2]$.

Using the computation of $\Omega_\bullet^{\mathrm{SO}}(BU) \otimes \mathbb{Q}$ by Thom (cf. [Tho54]), Conner and Floyd showed (cf. [CF64]) that a *ring homomorphism* $I : \Omega_\bullet^{\mathrm{SO}}(\mathbb{K}) \to \mathbb{R}$ satisfying $I[M, a_1 + a_2] = I[M, a_1] + I[M, a_2]$ is determined by its values on $[P^{2k}(\mathbb{C}), 1]$, $k \in \mathbb{Z}_{\geq 1}$ and $[S^2, \beta]$, where $\beta \in K^0(S^2)$ is the Bott class. The proof of the index theorem is completed by checking that both indices satisfy (cf. [Pal65, Theorem XV.4(d)]:

$$\operatorname{Index}[P^{2k}(\mathbb{C}), 1] = 1, \quad k \in \mathbb{Z}_{\geq 1},$$
$$\operatorname{Index}[S^2, \beta] = 1.$$

Vector Bundle Modification and K-homology

Subsequently, Atiyah and Singer produced (cf. [AS68a, AS68b]) a simplified proof which doesn't involve bordism and which deals with all symbols directly, not just twisted signature symbols. The main ingredient is a construction called *vector bundle modification*, a kind of twisted product. It allows one to use *Bott periodicity* [Bot59] in place of Thom's result [Tho54]. Understanding vector bundle modification in the context of index cocycles is an interesting problem, which will be dealt with elsewhere.

Baum and Douglas noticed that bordism theory in conjunction with vector bundle modification can be used to construct a geometric model for K-homology theory [BD82a, BD82b]. Index theory amounts to identifying $K_0(\mathrm{pt})$ with $\mathbb{Z}$.

Consider the additive group $\Omega_\bullet^{\mathrm{Spin}^c}(\mathbb{K})$ of Spin^c *bordism* classes. Its elements are given by pairs (M, a), where M is a Spin^c manifold and $a \in K^0(M)$. We divide it by the relations

1. $[M, a_1] + [M, a_2] = [M, a_1 + a_2]$ and
2. $[M, a] = [\widehat{M}, \widehat{a}]$, where $[\widehat{M}, \widehat{a}]$ is a vector bundle modification of $[M, a]$.

and denote the quotient by $K_0^{\mathrm{Spin}^c}(\mathrm{pt})$. Then Bott periodicity can be used to show that $K_0^{\mathrm{Spin}^c}(\mathrm{pt})$ is a free abelian group generated by $[\mathrm{pt}, 1]$. The twisted signature theorem for Spin^c manifolds is then proved by showing that both the analytical and the topological indices give a map $K_0^{\mathrm{Spin}^c}(\mathrm{pt}) \to \mathbb{R}$ which send the generator $[\mathrm{pt}, 1]$ to 1. See [BHS07] for details.

Alternatively, we can use the *stable* bordism group $\varinjlim \Omega_{\bullet+2k}^{\mathrm{framed}}(\mathbb{K})$ of *framed* manifolds equipped with a virtual vector bundle (cf. [BHS07]) or the *Clifford* bordism group of Guentner (cf. [Gue93, Kes99]).

Spectral Triples

A myriad of generalizations of the index theorem is known to date. Spectral triples provide a natural and convenient framework for the study these generalizations and cover most of the interesting examples. The theory starts from the following observation:

Let M be a closed Riemannian spin manifold of even dimension and let $D : C^\infty(M, E) \to C^\infty(M, E)$ be the *Dirac operator* acting on the spinor bundle E. Then all the geometry of

M can be recovered from the triple $(C^\infty(M), L^2(M,E), D)$. Furthermore, index theory can be developed directly on the triple.

A *spectral triple* is a noncommutative generalization of this example. It consists of an associative algebra $\mathcal{A}$ acting on a graded Hilbert space $\mathfrak{H}$ by even and bounded operators and a self-adjoint odd operator D with locally compact resolvents that almost commutes with $\mathcal{A}$ (cf. Definition 2.1.2). Associated to such a triple is an *index map* (cf. Definition 2.2.18)

$$\mathrm{Index}_D : K_0(\mathcal{A}) \to \mathbb{Z}.$$

At the expense of losing commutativity, we gain greater flexibility: group actions, foliations and quantum groups all give rise to spectral triples. In this generalized situation, the role of the de Rham homology theory is played by the cyclic cohomology theory. Index problems then reduce to the study of *index cocycles* $\varphi_D \in HP^0(\mathcal{A})$ (cf. Definition 4.1.18):

$$\mathrm{Index}_D(e) = \langle \varphi_D, e \rangle\,, \quad e \in K_0(\mathcal{A}).$$

What can we say about the stability properties of index cocycles under various operations on spectral triples? In the classical case, Getzler showed that the Atiyah-Patodi-Singer index theorem can be lifted to entire cyclic theory (cf. [Get93]). In the noncommutative context, while no natural notion of bordism for spectral triples is known, a sensible replacement is given by *bounded perturbation*, see [Roe96, page 30], [Bla98, Definition 17.2.4]. The invariance of index cocycles under bounded perturbation is well-understood and follows from homotopy invariance.

In this thesis, we study the multiplicative property of index cocycles. Our main result is the construction of a *multiplicative local* index cocycle (see Theorem 5.2.7). As we saw earlier, in nice geometric situations, multiplicativity allows one to concentrate on a few generating examples that one can check the index theorem directly by computing both the analytical and the topological indices.

Description of Contents

Now we describe the contents of this thesis.

In Chapter 2, we give the basic definitions. In Section 2.1, we define spectral triples and look at various examples. We also look at standard constructions on spectral triples – sum, product and perturbation – and give a concise account of multigraded spectral triples. Section 2.2 is devoted to the definition and the study of basic properties of the index map. We note that while it is rather straightforward to define the index map for a *unital* spectral triple, the general case requires the introduction of Fredholm modules (see Lemma 2.2.15).

In Chapter 3, we study analytical properties of spectral triples. In Section 3.1, we study the Sobolev spaces associated to an invertible positive self-adjoint operator, generalizing the classical Sobolev spaces associated to a closed manifold. We then develop calculi of differential and pseudodifferential operators along the lines of [CM95] and [Hig04a]. For the convenience of the reader, we reproduce their characterization of regular spectral triples as the ones that admit a (pseudo)differential calculus. This allows one to deduce that the product of regular spectral triples is again regular. In Section 3.2, we reproduce the main result of [Hig04b].

In Chapter 4, we study abstract index formulas. Since the content of this chapter can be considered standard by now, we only give a brief account. Note that our naming convention is *not* standard. This is because we need to distinguish six different index cocycles. In Section 4.1 we review cyclic theory in order to fix the notation. In Section 4.2, we first introduce the Chern character of a Fredholm module. Then, as an application, we get the Connes-Chern character and the Connes-Karoubi character. In Section 4.3, we consider two improper cocycles: the Connes character and the JLO character. These are entire cyclic cocycles, but instead of developing entire cyclic theory, we concentrate on the (proper) cocycles constructed from these improper cocycles, namely, the transgression character and the Connes-Moscovici character. The last one is important because of its local nature. In the case of a Dirac operator on a closed spin manifold, this gives the classical local index theorem.

In Chapter 5, we study multiplicative properties of the JLO character. In Section 5.1, we show that the JLO character is multiplicative with respect to the Hochschild shuffle product and compute the effect of the cyclic shuffle product on the JLO character. This allows us to perturb the JLO character to obtain a character multiplicative on the cochain level. In Section 5.2, with the aid of the meromorphic continuation results from Chapter 3, we localize the perturbed JLO character and obtain a formal Laurent series.

In the last chapter, Chapter 6, we talk about future directions. Our general aim, so far unmet, is to use the freedom gained from considering cocycles with values in formal Laurent series to construct more easily analyzable cocycles in para-Riemannian index theory, and then apply the multiplicativity properties to calculate the index class. In Section 6.1, we review the current state of affairs of para-Riemmannian index theory and construct an asymptotic morphism representing the equivariant signature class. In Section 6.2, we specialize to conformal actions.

In the Appendix, we collect some basic facts from functional analysis. It serves as a convenient place to refer to standard results about self-adjoint operators, compact operators and Fredholm operators. Some of the important results which are not usually covered in a conventional functional analysis course are given with proof.

Chapter 2

Spectral Triples and Index Theory

2.1 Definition and Examples of Spectral Triples

Here we recall basic definitions surrounding the concept of a spectral triple and introduce some examples.

2.1.1 Spectral Triples

In order to motivate the definition, we start with a classic example. See [HR00, Chapter 10] for the details.

Example 2.1.1. Let M be a closed manifold with a fixed smooth measure and let D be an order-one symmetric partial-differential operator acting on the smooth sections $C^\infty(M, E)$ of a graded Hermitian vector bundle E. Then D is admits a unique self-adjoint extension $\bar{D}$ on $L^2(M, E)$ with domain $W^1 = \{\xi \in L^2(M, E) \mid D\xi \in L^2(M, E) \text{ weakly}\}$(cf. [HR00, Corollary 10.2.6]). Furthermore, for any $f \in C^\infty(M)$, we have

(i) $f \cdot W^1 \subseteq W^1$ and

(ii) $[\bar{D}, f] \coloneqq \bar{D}f - f\bar{D}$ extends to a bounded operator on $L^2(M, E)$.

Indeed, if $f \in C^\infty(M)$ and $\xi \in C^\infty(M, E)$ are smooth, then so is $f\xi$. Hence multiplication by $f \in C^\infty(M)$ preserves the domain $\mathrm{dom}(D) = C^\infty(M, E)$ of D and so it makes sense to consider the commutator $[D, f] : C^\infty(M, E) \to C^\infty(M, E)$. Now, property (ii) follows from the identity

$$[\bar{D}, f]\xi = [D, f]\xi = \sigma_D(df)\xi, \quad \xi \in C^\infty(M, E),$$

where $\sigma_D : T^*M \to \mathrm{End}(E)$ is the *principal symbol* of D. Property (i) follows from property (ii): If $\xi \in W^1$ then $D\xi \in L^2(M, E)$ and

$$\langle D\eta, f\xi\rangle = \langle \eta, fD\xi + \sigma_D(df)\xi\rangle\,, \quad \eta \in \mathrm{dom}(D),$$

i.e. $D(f\xi)$ also belongs to $L^2(M,E)$ or equivalently $f\xi \in W^1$.

If in addtion D is *elliptic*, that is, if $\sigma_D(x,\xi)$ is invertible for $\xi \neq 0$, then $\bar{D}$ has *compact resolvents*:

(iii) the operators $(\bar{D} \pm i)^{-1} : L^2(M,E) \to L^2(M,E)$ are compact.

This is proved in [HR00, Proposition 10.4.5].

If M is *not* compact, then there are several difficulties. Firstly, it is no longer true that an order-one symmetric partial-differential operator, with domain the compactly supported smooth sections, is necessarily essentially self-adjoint. A classic example is the differentiation operator $i\frac{d}{dx}$ defined on the open interval $(0,1)$. Secondly, even when it admits a self-adjoint extension, this extension need not satisfy the compact resolvents condition. However, the following weaker condition holds: If D is an order-one essentially self-adjoint elliptic partial-differential operator on M, then

(iii′) $f \cdot (\bar{D} \pm i)^{-1} : L^2(M,E) \to L^2(M,E)$ is compact for any *compactly supported* smooth function $f \in C_c^\infty(M)$. (cf. [HR00, Proposition 10.5.1])

As we will see in Section 2.2, these properties ensure the existence of a well-behaved index theory. The following is an abstraction of the main points.

Definition 2.1.2. A *spectral triple* is a triple $(\mathcal{A}, \mathfrak{H}, D)$, where

(1) $\mathfrak{H} = \mathfrak{H}^0 \oplus \mathfrak{H}^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space and

(2) $\mathcal{A}$ is an associative algebra acting on $\mathfrak{H}$ by *bounded* and *even* operators and

(3) D is a densely defined *self-adjoint odd* operator with domain $\mathrm{dom}(D)$ such that for any element $a \in \mathcal{A}$

  (i) the operator a preserves the domain of D, that is, $a \cdot \mathrm{dom}(D) \subseteq \mathrm{dom}(D)$ and

  (ii) the commutator $[D,a] \coloneqq Da - aD : \mathrm{dom}(D) \to \mathfrak{H}$ extends by continuity to a bounded operator on $\mathfrak{H}$ and

  (iii) the operators $a \cdot (D \pm i)^{-1}$ and $(D \pm i)^{-1}a$ are compact, *i.e.* D has $\mathcal{A}$-*locally compact resolvents.*

We say that $(\mathcal{A}, \mathfrak{H}, D)$ is *unital* if $\mathcal{A}$ is unital and the action is unit preserving and *involutive* if $\mathcal{A}$ is involutive and the action is $*$-preserving.

A few remarks are in order.

Remark 2.1.3. (1) Spectral triples were initially called "K-cycles" in [Con85] and this terminology is also used in [Con94]. However, more recent publications favor the term "spectral triple", so that is what we use.

(2) The condition that D is an *odd* operator becomes crucial when we consider index theory.

(3) If $(\mathcal{A}, \mathfrak{H}, D)$ is either unital or involutive, then it is enough to consider the operators $a(D \pm i)^{-1}$ in condition (3iii). Indeed, if $(\mathcal{A}, \mathfrak{H}, D)$ is unital than taking $a = 1$, we see that D has compact resolvents and therefore $(D \pm i)^{-1} a$ is compact for any $a \in \mathcal{A}$. If $(\mathcal{A}, \mathfrak{H}, D)$ is involutive, then $(D \pm i)^{-1} a = (a^*(D \mp i)^{-1})^*$ are automatically compact.

Definition 2.1.4. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple and let $p \in [1, \infty)$. We say that $(\mathcal{A}, \mathfrak{H}, D)$ is

(a) *p-summable* if the operators $a \cdot (D \pm i)^{-1}$ and $(D \pm i)^{-1} \cdot a$ belong to the Schatten p-ideal $\mathcal{L}^p(\mathfrak{H})$ for all $a \in \mathcal{A}$;

(b) *(p, ∞)-summable* if the operators $a \cdot (D \pm i)^{-1}$ and $(D \pm i)^{-1} \cdot a$ belong to the Dixmier p-ideal $\mathcal{L}^{(p,\infty)}(\mathfrak{H})$ for all $a \in \mathcal{A}$;

(c) *θ-summable* if $e^{-tD^2} \in \mathcal{L}^1(\mathfrak{H})$, for all $t > 0$.

We say that a spectral triple is *finitely summable* if it is p-summable for some p. When the spectral triple is clear from the context, we simply say that the number p is *summable*.

The following lemma enables us to work with essentially self-adjoint operators.

Lemma 2.1.5. *Let $\mathcal{A}$ be an algebra acting on a graded Hilbert space $\mathfrak{H}$ by bounded and even operators. Let D be an* essentially self-adjoint *odd operator on $\mathfrak{H}$ with domain* $\mathrm{dom}(D)$*. Assume that for any $a \in \mathcal{A}$*

(i) $a \cdot \mathrm{dom}(D) \subseteq \mathrm{dom}(D)$ and

(ii) $[D, a]$ extends to a bounded operator and

(iii) $a \cdot (D \pm i)^{-1}$ and $(D \pm i)^{-1} \cdot a$ extend to compact operators.

Let $\bar{D}$ be the (self-adjoint) closure of D. Then $(\mathcal{A}, \mathfrak{H}, \bar{D})$ is a spectral triple.

Proof. The arguments of Example 2.1.1 carry over *ad verbatim.* □

Definition 2.1.6. A *pre-spectral triple* is a triple $(\mathcal{A}, \mathfrak{H}, D)$ satisfying the conditions given in Lemma 2.1.5. The *closure* a pre-spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ is the spectral triple $(\mathcal{A}, \mathfrak{H}, \bar{D})$, where $\bar{D}$ is the closure of D.

Example 2.1.7 (Smooth Manifolds)**.** Let M be a smooth manifold equipped with a smooth measure and let D be an order-one symmetric odd elliptic partial-differential operator acting on the compactly supported smooth sections $C_{\mathrm{c}}^\infty(M, E)$ of a graded Hermitian bundle $E \to M$. Assume that M is *complete* for D, that is, there exists a smooth proper function $g : M \to \mathbb{R}$ such that $[D, g]$ is a bounded operator on $L^2(M, E)$. Then, by [HR00, Proposition 10.2.10], D is essentially self-adjoint and

$$(C_{\mathrm{c}}^\infty(M), L^2(M, E), D)$$

is a pre-spectral triple by Example 2.1.1. Applying Lemma 2.1.5 we get a spectral triple

$$[D] \coloneqq (C_{\mathrm{c}}^\infty(M), L^2(M, E), \bar{D}).$$

It is a theorem of Weyl that $[D]$ is $(\dim M, \infty)$-summable: see [Shu01, Theorem II.15.2] for the compact case, the general case can be deduced from the compact case using the methods in [Roe88, Chapter 9]. An alternative proof is given in Theorem 3.2.11.

Notice that $[D]$ is naturally involutive and it is unital if and only if M is compact.

Riemannian manifolds have natural elliptic operators associated.

Example 2.1.8 (Signature)**.** Let M be a complete Riemannian manifold of even dimension $2k$. Then M has a natural volume form and hence a smooth measure. Equip $\Omega^\bullet M \otimes \mathbb{C}$ with the *signature grading* given by the standard involution coming from the Hodge-star operator:

$$\beta \wedge \varepsilon_M \alpha = i^{-k-p(p-1)} \langle \alpha, \beta \rangle \,\mathrm{Vol}, \qquad \alpha \in \Omega^p M, \beta \in \Omega^{2k-p} M.$$

Let $D = d + d^*$ be the *de Rham operator* acting on $E = \Lambda^\bullet T^*M \otimes \mathbb{C}$. Then D is odd and M is complete for D and thus D is essentially self-adjoint (cf. [HR00, Proposition 10.2.11]). Hence $[D]$ gives a spectral triple by Example 2.1.7.

Moreover, any orientation preserving isometry of M gives rise to an even unitary operator on $L^2(M, E)$ that commutes with D. Hence we get a spectral triple

$$(C_c^\infty(M) \rtimes \Gamma, L^2(M, E), \bar{D})$$

for any discrete group Γ of orientation preserving isometries of M.

Example 2.1.9 (Finite-Dimensional)**.** Let $\mathfrak{H}$ be a finite-dimensional graded Hilbert space. Then any algebra $\mathcal{A}$ and any odd self-adjoint operator D acting on $\mathfrak{H}$ gives rise to a 1-summable spectral triple $(\mathcal{A}, \mathfrak{H}, D)$.

Sum and Product of Spectral Triples

Lemma 2.1.10. *Let $(\mathcal{A}_1, \mathfrak{H}_1, D_1)$ and $(\mathcal{A}_2, \mathfrak{H}_2, D_2)$ be spectral triples. Then*

$$(\mathcal{A}_1 \oplus \mathcal{A}_2,\ \mathfrak{H}_1 \oplus \mathfrak{H}_2,\ D_1 \oplus D_2)$$

is a spectral triple and

$$\big(\mathcal{A}_1 \otimes \mathcal{A}_2,\ \mathfrak{H}_1 \mathbin{\widehat{\otimes}} \mathfrak{H}_2,\ D_1 \times_{\mathrm{alg}} D_2\big)$$

is a pre-spectral triple.

Proof. Leaving the trivial case of the direct sum to the reader, we consider the product. The operator $D_1 \times_{\mathrm{alg}} D_2$ is odd by definition and essentially self-adjoint by Theorem A.1.6.

Let $a_1 \otimes a_2 \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and let $a_1 \mathbin{\widehat{\otimes}} a_2$ denote the corresponding element of $\mathcal{L}(\mathfrak{H}_1) \mathbin{\widehat{\otimes}} \mathcal{L}(\mathfrak{H}_2)$. Then for any $\xi_1 \mathbin{\widehat{\otimes}} \xi_2 \in \mathrm{dom}(D_1) \mathbin{\widehat{\otimes}}_{\mathrm{alg}} \mathrm{dom}(D_2)$, the element $(a_1 \mathbin{\widehat{\otimes}} a_2)(\xi_1 \mathbin{\widehat{\otimes}} \xi_2) = a\xi_1 \mathbin{\widehat{\otimes}} a_2\xi_2$ also belongs to $\mathrm{dom}(D_1) \mathbin{\widehat{\otimes}}_{\mathrm{alg}} \mathrm{dom}(D_2)$. Hence

$$(a_1 \mathbin{\widehat{\otimes}} a_2) \cdot \mathrm{dom}(D_1 \times_{\mathrm{alg}} D_2) \subseteq \mathrm{dom}(D_1 \times_{\mathrm{alg}} D_2).$$

Moreover, the operator

$$[D_1 \times_{\mathrm{alg}} D_2, a_1 \mathbin{\widehat{\otimes}} a_2] = [D_1, a_1] \mathbin{\widehat{\otimes}} a_2 + a_1 \mathbin{\widehat{\otimes}} [D_2, a_2]$$

is bounded and the operator

$$(a_1 \mathbin{\widehat{\otimes}} a_2) \cdot e^{-(D_1 \times D_2)^2} = a_1 e^{-D_1^2} \mathbin{\widehat{\otimes}} a_2 e^{-D_2^2}$$

is compact (cf. Lemma A.1.9). By Lemma A.2.10, $D_1 \times D_2$ has locally compact resolvents. The proof is complete.

□

This allows us to make the following definition.

Definition 2.1.11. Let $\mathfrak{D}_1 = (\mathcal{A}_1, \mathfrak{H}_1, D_1)$ and $\mathfrak{D}_2 = (\mathcal{A}_2, \mathfrak{H}_2, D_2)$ be spectral triples. Then their *sum* and *product* spectral triples are defined, respectively, as

$$\begin{aligned} \mathfrak{D}_1 + \mathfrak{D}_2 &\coloneqq (\mathcal{A}_1 \oplus \mathcal{A}_2, \mathfrak{H}_1 \oplus \mathfrak{H}_2, D_1 \oplus D_2) \\ \mathfrak{D}_1 \times \mathfrak{D}_2 &\coloneqq \left(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathfrak{H}_1 \mathbin{\widehat{\otimes}} \mathfrak{H}_2, D_1 \times D_2\right). \end{aligned}$$

Lemma 2.1.12. *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be spectral triples.*

If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are finitely summable, then so are $\mathfrak{D}_1 + \mathfrak{D}_2$ and $\mathfrak{D}_1 \times \mathfrak{D}_2$.

If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are θ-summable, then so are $\mathfrak{D}_1 + \mathfrak{D}_2$ and $\mathfrak{D}_1 \times \mathfrak{D}_2$.

Proof. We only demonstrate the (less trivial) case of the product. It follows from Lemma A.1.9 that if $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are θ-summable then so is $\mathfrak{D}_1 \times \mathfrak{D}_2$. Now assume that $\mathfrak{D}_1$ is p_1-summable and $\mathfrak{D}_2$ is p_2-summable. Let $a_1 \otimes a_2 \in \mathcal{A}_1 \otimes \mathcal{A}_2$ and let $T = (a_1 \mathbin{\widehat{\otimes}} a_2)(D_1 \times D_2 \pm i)^{-1}$. Then the inequality

$$\begin{aligned} T^*T &= (a_1^* \mathbin{\widehat{\otimes}} a_2^*)(D_1^2 \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} D_2^2 + 1)^{-1}(a_1 \mathbin{\widehat{\otimes}} a_2) \\ &\leq a_1^*(D_1^2 + 1)^{-\frac{1}{2}} a_1 \mathbin{\widehat{\otimes}} a_2^*(D_2^2 + 1)^{-\frac{1}{2}} a_2 \end{aligned}$$

shows that T is of Schatten $\mathcal{L}^{2\cdot\max\{p_1,p_2\}}$-class. Hence $\mathfrak{D}_1 \times \mathfrak{D}_2$ is $2 \cdot \max\{p_1, p_2\}$-summable. □

Perturbation of a Spectral Triple

Finally, we look at perturbations (see Theorem A.1.10).

Lemma 2.1.13. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be an spectral triple and let S be an odd symmetric D-bounded operator with relative bound < 1. Suppose that for any $a \in \mathcal{A}$,* *(i)* $a \cdot \mathrm{dom}(S) \subseteq \mathrm{dom}(S)$ *and*

(ii) $[S, a]$ *extends to a bounded operator.*

Then the triple $(\mathcal{A}, \mathfrak{H}, D{+}S)$ *is a spectral triple. Moreover,* $(\mathcal{A}, \mathfrak{H}, D{+}S)$ *has the same summability as* $(\mathcal{A}, \mathfrak{H}, D)$*.*

Proof. The operator $D{+}S$ with domain $\mathrm{dom}(D{+}S) \coloneqq \mathrm{dom}(D)$ is self-adjoint by the Kato-Rellich theorem A.1.10. Hence the domain of $D + S$ is preserved by elements of $\mathcal{A}$ and

$$[D + S, a] = [D, a] + [S, a]$$

is bounded for any $a \in \mathcal{A}$. Using the notations from the proof of Theorem A.1.10, we have

$$\begin{gathered} a \cdot (\lambda - (D + S))^{-1} = a \cdot (\lambda - D)^{-1}(1 - C)^{-1}, \\ (-\lambda - (D + S))^{-1} \cdot a = (1 - C^*)^{-1}(-\lambda - D)^{-1} \cdot a, \end{gathered}$$

for $\lambda = it$, $t \gg 1$, therefore $D + S$ has locally compact resolvents. The summability statement follows from the identity above. $\square$

Note that a *bounded* operator S is D-bounded with relative bound 0 and satisfies the two conditions (i, ii) above automatically.

Definition 2.1.14. In the situation of Lemma 2.1.13, we say that $(\mathcal{A}, \mathfrak{H}, D{+}S)$ is a *perturbation* of $(\mathcal{A}, \mathfrak{H}, D)$. If S is bounded, then we say that it is a *bounded perturbation* of $(\mathcal{A}, \mathfrak{H}, D)$.

2.1.2 Multigraded Spectral Triples

When we deal with odd dimensional manifolds, we need to consider an odd variant of the notion of a spectral triple. For the sake of completeness, we give a concise introduction to odd and multigraded spectral triples.

Clifford Algebras

Our main technical tool is Clifford algebras. We recall the basic definitions briefly, following the "topologist's convention" for sign. See [ABS64, Roe88, LM89, BGV92, HR00] for details.

Definition 2.1.15. Let V be a real vector space and let $g : V \times V \to \mathbb{R}$ be a *symmetric bilinear* form on V. An $\mathbb{R}$-linear map c from V to a unital algebra over $\mathbb{C}$ is said to be a *Clifford map* if it satisfies

$$c(v)c(w) + c(w)c(v) = -2g(v, w), \quad v, w \in V.$$

The (complex) *Clifford algebra* $\mathrm{Cl}(V, g)$ associated to (V, g) is an universal/initial unital algebra over $\mathbb{C}$ equipped with a Clifford map $c : V \to \mathrm{Cl}(V, g)$.

The existence of such algebras is a simple exercise in abstract algebra (cf. [LM89, page 8]). Universality means that for any unital algebra B over $\mathbb{C}$ and Clifford map $b : V \to B$ there exist a *unique* algebra map $a : \mathrm{Cl}(V, g) \to B$ such that $b = a \circ c$:

$$\begin{tikzcd} & \mathrm{Cl}(V,g) \arrow[d, "a"] \\ V \arrow[ur, "c"] \arrow[r, "b"'] & B \end{tikzcd}$$

In particular, Clifford algebras associated to a fixed (V, g) are all isomorphic, allowing us to talk about "the" Clifford algebra by abuse of language.

Since the map $-c : V \to \mathrm{Cl}(V, g)$ is a Clifford map, it induces an algebra map $\varepsilon : \mathrm{Cl}(V, g) \to \mathrm{Cl}(V, g)$, by universality, and $\varepsilon \circ \varepsilon = 1$. We grade $\mathrm{Cl}(V, g)$ using ε. Using the opposite algebra $\mathrm{Cl}(V, g)^{\mathrm{op}}$ of $\mathrm{Cl}(V, g)$ and the Clifford map $-c : V \to \mathrm{Cl}(V, g)^{\mathrm{op}}$, we also see that there is a conjugate linear involution $* : \mathrm{Cl}(V, g) \to \mathrm{Cl}(V, g)$. By definition, $c(v) \in \mathrm{Cl}(V, g)$ is odd and skew-adjoint, *i.e.* $\varepsilon c(v) = -c(v)\varepsilon$ and $c(v)^* = -c(v)$ for $v \in V$.

Definition 2.1.16. Let $n \in \mathbb{Z}_{\geq 0}$ be a positive integer. We write Cl_n for the Clifford algebra of $\mathbb{R}^n$ with respect to the standard inner product and $\varepsilon_1, \ldots, \varepsilon_n$ for the generators corresponding to the standard basis of $\mathbb{R}^n$.

By definition, we have

$$\varepsilon_i^* = -\varepsilon_i, \quad 1 \leq i \leq n \quad \text{and} \quad \varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_i = -2\delta_{ij}, \quad 1 \leq i, j \leq n.$$

Throughout the paper, we use the following notation.

Definition 2.1.17. The *Pauli matrices* are

$$\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Lemma 2.1.18. *The Pauli matrices satisfy*

$$\sigma_i^* = \sigma_i, \quad 1 \leq i \leq 3 \quad \textit{and} \quad \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}, \quad 1 \leq i, j \leq 3.$$

Proof. Straightforward computation. □

A useful identity is $\sigma_1\sigma_2 = i\sigma_3$ (and its cyclic cousins).

Notation 2.1.19. We write $\mathbb{M}_n$ for the $*$-algebra of $n \times n$ complex matrices.

Lemma 2.1.20 ([LM89] Theorem I.4.3)**.** *As graded $*$-algebras*

(a) $\mathrm{Cl}_0 \cong \mathbb{C}$ *with the trivial grading* 1*;*

(b) $\mathrm{Cl}_1 \cong \mathbb{C}^2$ *with the standard odd grading* σ_1*;*

(c) $\mathrm{Cl}_2 \cong \mathbb{M}_2$ *with the standard even grading* $\mathrm{Ad}_{\sigma_3} : x \mapsto \sigma_3^{-1} x \sigma_3$*;*

(d) $\mathrm{Cl}_n \mathbin{\widehat{\otimes}} \mathrm{Cl}_m \cong \mathrm{Cl}_{n+m}$ *for any* n*,* m*.*

Proof. There is nothing to prove for part (a) and part (d) follows from universality, or more concretely, the generators corresponding to the standard basis of $\mathbb{R}^{n+m}$ are given by

$$\varepsilon_1 \mathbin{\widehat{\otimes}} 1, \ldots, \varepsilon_n \mathbin{\widehat{\otimes}} 1, 1 \mathbin{\widehat{\otimes}} \varepsilon_1, \ldots, 1 \mathbin{\widehat{\otimes}} \varepsilon_m.$$

The isomorphism for (b) is given by $\varepsilon_1 \mapsto \begin{bmatrix} i & \\ & -i \end{bmatrix}$ and the isomorphism for (c) is given by $\varepsilon_1 \mapsto i\sigma_1$ and $\varepsilon_2 \mapsto i\sigma_2$. □

Corollary 2.1.21. *As graded ∗-algebras*

(a) $\mathrm{Cl}_{2n} \cong \mathbb{M}_2 \otimes \mathbb{M}_{2^{n-1}}$ *with grading* $\mathrm{Ad}_{\sigma_3} \otimes 1$*;*

(b) $\mathrm{Cl}_{2n+1} \cong \mathbb{C}^2 \otimes \mathbb{M}_{2^n}$ *with grading* $\sigma_1 \otimes 1$*.*

Proof. In view of Lemma 2.1.20, it is enough to show that

(a) there exists an ∗-automorphism of $\mathbb{M}_2 \otimes \mathbb{M}_2$ that intertwines the two gradings $\mathrm{Ad}_{\sigma_3} \otimes \mathrm{Ad}_{\sigma_3}$ and $\mathrm{Ad}_{\sigma_3} \otimes 1$; and

(b) there exists an ∗-automorphism of $\mathbb{C}^2 \otimes \mathbb{M}_2$ that intertwines the two gradings $\sigma_1 \otimes \mathrm{Ad}_{\sigma_3}$ and $\sigma_1 \otimes 1$.

The identification for (a) is given by Ad_u, where $u \coloneqq 1 \otimes \frac{1+\sigma_3}{2} + \sigma_1 \otimes \frac{1-\sigma_3}{2}$ is a unitary and $(\sigma_3 \otimes \sigma_3) \cdot u = u \cdot (\sigma_3 \otimes 1)$; in matrix form, we have $u = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_1 \end{bmatrix}$ and the intertwining relation is

$$\begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \sigma_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_1 \end{bmatrix} \cdot \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix}.$$

The identification for (b) is given by $\frac{1+\sigma_3}{2} \otimes 1 + \frac{1-\sigma_3}{2} \otimes \mathrm{Ad}_{\sigma_3}$; in matrix form, this can be written as $\begin{bmatrix} 1 & \\ & \mathrm{Ad}_{\sigma_3} \end{bmatrix}$ and then the intertwining relation is

$$\begin{bmatrix} 0 & \mathrm{Ad}_{\sigma_3} \\ \mathrm{Ad}_{\sigma_3} & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{Ad}_{\sigma_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \mathrm{Ad}_{\sigma_3} \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

□

Remark 2.1.22. We note that the two gradings Ad_{σ_3} and 1 on $\mathbb{M}_2$ are *not* equivalent, since σ_3 is not conjugate to 1. It is only after tensoring by $(\mathbb{M}_2, \mathrm{Ad}_{\sigma_3})$ or $(\mathbb{C}^2, \sigma_1)$ they become equivalent.

Multigraded Spectral Triples

Now we introduce multigraded spectral triples.

Definition 2.1.23. Let $n \in \mathbb{Z}_{\geq 0}$ be a positive integer. An *n-multigraded Hilbert space* is a graded Hilbert space equipped with an action of Cl_n, *i.e.* we have *odd* and *skew-adjoint* elements $\varepsilon_i \in \mathcal{L}(\mathfrak{H})$, $1 \leq i \leq n$ satisfying $\varepsilon_i \varepsilon_j + \varepsilon_j \varepsilon_i = -2\delta_{i,j}$ for all $1 \leq i, j \leq n$. An *n-multigraded spectral triple* is a spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ such that $\mathfrak{H}$ is n-multigraded and the following compatibility relations are satisfied:

- $[\varepsilon_i, a] = 0$ for any $a \in \mathcal{A}$ and $1 \leq i \leq n$.
- $[\varepsilon_i, D] \coloneqq \varepsilon_i D + D\varepsilon_i = 0$ for $1 \leq i \leq n$.

Remark 2.1.24. A spectral triple is the same thing as a 0-multigraded spectral triple. As noted by Ezra Getzler in [Get93, Appendix. Clifford modules], a n-multigraded Hilbert space can also be considered as a Hilbert Cl_n-module and these two viewpoints are equivalent.

First we note that multigraded spectral triples behave nicely with respect to the product construction.

Lemma 2.1.25. *The product of a n-multigraded and a m-multigraded Hilbert spaces is naturally $(n+m)$-multigraded. Similarly for spectral triples.*

Proof. In view of Lemma 2.1.20(d), it is enough to note that operators of the form odd $\widehat{\otimes}\, 1$ and $1 \,\widehat{\otimes}$ odd always *anticommute.* $\square$

The following is the quintessential example of a multigraded spectral triple.

Example 2.1.26 (Dirac Operators)**.** Let M be a closed Riemannian manifold of dimension n. Let E be a Clifford bundle and let D be a Dirac-type operator on E. Then right multiplication on E gives $L^2(M, E)$ the structure of a n-multigrading and $(C^\infty(M), L^2(M, E), D)$ is an example of a n-multigraded spectral triple.

Odd Spectral Triples

Another important class of examples multigraded spectral triples come from *odd* spectral triples defined below.

Definition 2.1.27. An *odd* spectral triple is a triple $\mathfrak{D} = (\mathcal{A}, \mathfrak{H}, D)$, where

(1) $\mathfrak{H}$ is a (trivially graded) Hilbert space and

(2) $\mathcal{A}$ is an associative algebra acting on $\mathfrak{H}$ by bounded operators

(3) D is a densely defined self-adjoint operator with domain $\mathrm{dom}(D)$ such that for any $a \in \mathcal{A}$

   (i) $a \cdot \mathrm{dom}(D) \subseteq \mathrm{dom}(D)$ and

(ii) $[D,a] \coloneqq Da - aD$ extends to a bounded operator on $\mathfrak{H}$ and

(iii) $a \cdot (D \pm i)^{-1}$ and $(D \pm i)^{-1} \cdot a$ are compact.

Let σ_i be the Pauli matrices (cf. Definition 2.1.17) and let $\mathbb{C}^{1,1}$ denotes the standard two-dimensional Hilbert space $\mathbb{C}^2$ equipped with the grading σ_3.

Lemma 2.1.28. *There is a natural bijective correspondence between odd spectral triples and* 1*-multigraded spectral triples given by*

$$S : (\mathcal{A}, \mathfrak{H}, D) \mapsto \left(\mathcal{A}, \mathfrak{H} \widehat{\otimes} \mathbb{C}^{1,1}, D \widehat{\otimes} \sigma_2\right),$$

where the 1*-multigrading on* $\mathfrak{H} \widehat{\otimes} \mathbb{C}^{1,1}$ *is given by* $\varepsilon_1 \mapsto 1 \widehat{\otimes} i\sigma_1$.

See [Con85, Lemma I.7.1] and [Con94, Proposition IV.A.13].

Proof. We note that the grading on $\mathfrak{H} \widehat{\otimes} \mathbb{C}^{1,1}$ is given by $1 \widehat{\otimes} \sigma_3$. Then it is clear that

$$\left(\mathcal{A}, \mathfrak{H} \widehat{\otimes} \mathbb{C}^{1,1}, D \widehat{\otimes} \sigma_2\right)$$

is a 1-multigraded spectral triple.

To reverse the construction, let γ be the grading operator on the Hilbert space and consider the 1-eigenspace of $-\gamma\varepsilon_1$. Then, first note that $-\gamma\varepsilon_1 = -(1 \widehat{\otimes} \sigma_3) \cdot (1 \widehat{\otimes} i\sigma_1) = 1 \widehat{\otimes} \sigma_2$. Since the 1-eigenspace of $\sigma_2 : \mathbb{C}^{1,1} \to \mathbb{C}^{1,1}$ is one-dimensional, spanned by $\eta \coloneqq \begin{bmatrix} 1 \\ i \end{bmatrix} \in \mathbb{C}^{1,1}$, the 1-eigenspace of $1 \widehat{\otimes} \sigma_2$ can identified with $\mathfrak{H}$ via $\mathfrak{H} \ni \xi \mapsto \xi \widehat{\otimes} \eta \in \mathfrak{H} \widehat{\otimes} \mathbb{C}^{1,1}$ and under this identification $D \widehat{\otimes} \sigma_2$ corresponds to D. □

Definition 2.1.29. Let $\mathbb{C}$ act on $\mathbb{C}^{1,1}$ by $1 \mapsto \frac{1+\sigma_3}{2} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let Cl_2 act on $\mathbb{C}^{1,1}$ by Lemma 2.1.20(c), *i.e.* $\varepsilon_1 \mapsto i\sigma_1$ and $\varepsilon_2 \mapsto i\sigma_2$. Define

$$\mathfrak{S} \coloneqq (\mathbb{C}, \mathbb{C}^{1,1}, 0).$$

Clearly $\mathfrak{S}$ is a 2-multigraded spectral triple.

Lemma 2.1.30 ([HR00] Appendix A.3.4.)**.** *For $n \in \mathbb{Z}_{\geq 0}$, there is a natural bijective correspondence between n-multigraded spectral triples and $n+2$-multigraded spectral triples given by*

$$S : (\mathcal{A}, \mathfrak{H}, D) \mapsto (\mathcal{A}, \mathfrak{H}, D) \times \mathfrak{S}$$

Proof. By Lemma 2.1.25, $S(\mathcal{A}, \mathfrak{H}, D) = (\mathcal{A}, \mathfrak{H}, D) \times \mathfrak{S}$ is indeed a $n+2$-multigraded spectral triple. To reverse the construction, we consider the 1-eigenspace of $i\varepsilon_{n+1}\varepsilon_{n+2}$. Indeed, under the identifications of Lemma 2.1.20,

$$i\varepsilon_{n+1}\varepsilon_{n+2} = i \cdot (1 \mathbin{\widehat{\otimes}} i\sigma_1) \cdot (1 \mathbin{\widehat{\otimes}} i\sigma_2) = 1 \mathbin{\widehat{\otimes}} \sigma_3$$

and σ_3 is precisely the grading operator for $\mathbb{C}^{1,1}$. $\square$

In view of Lemma 2.1.28 and Lemma 2.1.30, we decree that an *odd* spectral triple is *-1-multigraded.*

Lemma 2.1.31. *Let $\mathfrak{D}_1 = (\mathcal{A}_1, \mathfrak{H}_1, D_1)$ and $\mathfrak{D}_2 = (\mathcal{A}_2, \mathfrak{H}_2, D_2)$ be odd spectral triples. Then their* product *corresponds to the (0-multigraded) spectral triple*

$$\mathfrak{D}_1 \times \mathfrak{D}_2 \coloneqq \left(\mathcal{A}_1 \otimes \mathcal{A}_2, \mathfrak{H}_1 \otimes \mathfrak{H}_2 \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, D_1 \otimes 1 \mathbin{\widehat{\otimes}} \sigma_2 + 1 \otimes D_2 \mathbin{\widehat{\otimes}} \sigma_1\right).$$

In matrix notation, we can write the operator as

$$\begin{bmatrix} 0 & -D_1 \otimes i + 1 \otimes D_2 \\ D_1 \otimes i + 1 \otimes D_2 & 0 \end{bmatrix}.$$

Proof. We claim that $S(\mathfrak{D}_1 \times \mathfrak{D}_2)$ can be naturally identified with $S\mathfrak{D}_1 \times S\mathfrak{D}_2$ as a 2-multigraded spectral triple. Indeed, we have

$$\begin{aligned} S(\mathfrak{D}_1 \times \mathfrak{D}_2) &= \mathfrak{D}_1 \times \mathfrak{D}_2 \times \mathfrak{S} \\ &= (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathfrak{H}_1 \otimes \mathfrak{H}_2 \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, D_1 \otimes 1 \mathbin{\widehat{\otimes}} \sigma_2 \mathbin{\widehat{\otimes}} 1 + 1 \otimes D_2 \mathbin{\widehat{\otimes}} \sigma_1 \mathbin{\widehat{\otimes}} 1), \end{aligned}$$

and the 2-multigrading is given by $\varepsilon_1 \mapsto 1 \otimes 1 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} i\sigma_1$ and $\varepsilon_2 \mapsto 1 \otimes 1 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} i\sigma_2$. On the other hand, using the fact that $\mathfrak{H}_1$ and $\mathfrak{H}_2$ are trivially graded, we have

$$\begin{aligned} S\mathfrak{D}_1 \times S\mathfrak{D}_2 &= (\mathcal{A}_1, \mathfrak{H}_1 \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, D_1 \mathbin{\widehat{\otimes}} \sigma_2) \times (\mathcal{A}_2, \mathfrak{H}_2 \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, D_2 \mathbin{\widehat{\otimes}} \sigma_2) \\ &= (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathfrak{H}_1 \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1} \mathbin{\widehat{\otimes}} \mathfrak{H}_2 \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, D_1 \mathbin{\widehat{\otimes}} \sigma_2 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} D_2 \mathbin{\widehat{\otimes}} \sigma_2) \\ &\cong (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathfrak{H}_1 \otimes \mathfrak{H}_2 \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, D_1 \otimes 1 \mathbin{\widehat{\otimes}} \sigma_2 \mathbin{\widehat{\otimes}} 1 + 1 \otimes D_2 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} \sigma_2) \end{aligned}$$

and the 2-multigrading is given by $\varepsilon_1 \mapsto 1 \otimes 1 \mathbin{\widehat{\otimes}} i\sigma_1 \mathbin{\widehat{\otimes}} 1$ and $\varepsilon_2 \mapsto 1 \otimes 1 \mathbin{\widehat{\otimes}} 1 \mathbin{\widehat{\otimes}} i\sigma_1$. Now note that the two lists

$$\begin{aligned} &\{i\sigma_2 \mathbin{\widehat{\otimes}} 1, i\sigma_1 \mathbin{\widehat{\otimes}} 1, 1 \mathbin{\widehat{\otimes}} i\sigma_1, 1 \mathbin{\widehat{\otimes}} i\sigma_2\} \quad \text{and} \\ &\{i\sigma_2 \mathbin{\widehat{\otimes}} 1, 1 \mathbin{\widehat{\otimes}} i\sigma_2, i\sigma_1 \mathbin{\widehat{\otimes}} 1, 1 \mathbin{\widehat{\otimes}} i\sigma_1\} \end{aligned}$$

both give the standard presentation for $\mathrm{Cl}_4 \cong \mathbb{M}_4$. Hence there exists a unitary u that intertwines the former one to the latter one. Since

$$\begin{aligned} \sigma_2 \mathbin{\widehat{\otimes}} 1 \cdot \sigma_1 \mathbin{\widehat{\otimes}} 1 \cdot 1 \mathbin{\widehat{\otimes}} \sigma_1 \cdot 1 \mathbin{\widehat{\otimes}} \sigma_2 = \sigma_3 \mathbin{\widehat{\otimes}} \sigma_3 \\ \sigma_2 \mathbin{\widehat{\otimes}} 1 \cdot 1 \mathbin{\widehat{\otimes}} \sigma_2 \cdot \sigma_1 \mathbin{\widehat{\otimes}} 1 \cdot 1 \mathbin{\widehat{\otimes}} \sigma_1 = \sigma_3 \mathbin{\widehat{\otimes}} \sigma_3 \end{aligned}$$

we see that u intertwines the gradings as well. Hence $1 \otimes 1 \mathbin{\widehat{\otimes}} u$ gives the desired unitary intertwining the two 2-multigraded spectral triples. $\square$

Remark 2.1.32. Often the product of odd operators is described as

$$D_1 \otimes 1 \mathbin{\widehat{\otimes}} \sigma_1 + 1 \otimes D_2 \mathbin{\widehat{\otimes}} \sigma_2 = \begin{bmatrix} 0 & D_1 \otimes 1 - i \otimes D_2 \\ D_1 \otimes 1 + i \otimes D_2 & 0 \end{bmatrix},$$

for instance in [Con94, page 445]. This corresponds to changing the orientation of $\mathbb{C}^{1,1}$.

2.2 Index Theory

In this section, we review index theory for spectral triples. We assume that the reader is familiar with K-theory.

2.2.1 Fredholm Modules

First we define Fredholm modules which are the bounded analogues of spectral triples. These were first defined by Atiyah as the cycles for K-homology (cf. [Ati68, Ati70]) and later developed into a full fledged theory by Kasparov (cf. [Kas75, Kas88]). A modern account of analytic K-homology theory can be found in [HR00].

We will follow mostly [Con85].

Definition 2.2.1. A *Fredholm module* is a triple $(\mathcal{A}, \mathfrak{H}, F)$, where

(1) $\mathfrak{H} = \mathfrak{H}^0 \oplus \mathfrak{H}^1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space and

(2) $\mathcal{A}$ is an associative algebra acting on $\mathfrak{H}$ by bounded operators and

(3) F is a *supersymmetry*, that is, a bounded *self-adjoint odd* operator on $\mathfrak{H}$ with $F^2 = 1$, satisfying

   (i) for any $a \in \mathcal{A}$ the commutator $[F, a]$ is a compact operator on $\mathfrak{H}$.

Definition 2.2.2. We say that a Fredholm module $(\mathcal{A}, \mathfrak{H}, F)$ is *p-summable* for $p \in [1, \infty)$, if $[F, \mathcal{A}] \subseteq \mathcal{L}^p(\mathfrak{H})$. When the Fredholm module is clear from the context, we simply say that the number p is summable.

As in the case of spectral triples, we say that $(\mathcal{A}, \mathfrak{H}, F)$ is *unital* if $\mathcal{A}$ is unital and the action is unital and *involutive* if $\mathcal{A}$ is involutive and the action is $*$-preserving.

Notation 2.2.3. Let $\mathcal{A}^+$ denote the minimal unitization of $\mathcal{A}$.

As a vector space $\mathcal{A}^+ := \mathbb{C} \oplus \mathcal{A}$ and the product is given by $(\lambda, a) \cdot (\mu, b) := (\lambda\mu, \lambda b + a\mu + ab)$.

Theorem 2.2.4. *Let $(\mathcal{A}, \mathfrak{H}, F)$ be a Fredholm module. We extend the action of $\mathcal{A}$ on $\mathfrak{H}$ to a unital action of $\mathcal{A}^+$. Then for any $e \in \mathcal{A}^+ \otimes \mathbb{M}_n$ idempotent, the operator $F_e = e(F \otimes 1)e : e(\mathfrak{H} \otimes \mathbb{C}^n) \to e(\mathfrak{H} \otimes \mathbb{C}^n)$ is odd and Fredholm with parametrix F_e itself. The pairing given by*

$$\langle F, e\rangle := \mathrm{Index}_s(F_e),$$

where Index_s is the super-index[1]*, determines an additive map $K_0(\mathcal{A}^+) \to \mathbb{Z}$.*

Proof. We refer to [Ati70, Kas75] for the full proof and show only the first statement.

Considering the Fredholm module $(\mathcal{A} \otimes 1, \mathfrak{H} \otimes \mathbb{C}^n, F \otimes 1)$, we may assume that $n = 1$. Then $e - F_e^2 = eF[F, e]e$ is compact since $[F, e]$ is compact and e and F are bounded. $\square$

Since we have a natural split-exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^+ \longrightarrow \mathbb{C} \longrightarrow 0\ ,$$

we have a natural identification

$$K_0(\mathcal{A}) \cong \ker(K_0(\mathcal{A}^+) \to K_0(\mathbb{C})).$$

In fact, for a non-unital algebra $\mathcal{A}$, this is usually taken as the definition of $K_0(\mathcal{A})$.

Definition 2.2.5. Let $(\mathcal{A}, \mathfrak{H}, F)$ be a Fredholm module. The *index map* associated to $(\mathcal{A}, \mathfrak{H}, F)$ is defined by the composition

$$\mathrm{Index}_{(\mathcal{A},\mathfrak{H},F)} : K_0(\mathcal{A}) \to K_0(\mathcal{A}^+) \to \mathbb{Z}.$$

It is not difficult to see that the index map is compatible with unitization:

Lemma 2.2.6. *Let $(\mathcal{A}, \mathfrak{H}, F)$ be a Fredholm module. Then the following diagram is commutative:*

$$\begin{tikzcd}
K_0(\mathcal{A}) \arrow[r] \arrow[dr, "\mathrm{Index}_{(\mathcal{A},\mathfrak{H},F)}"'] & K_0(\mathcal{A}^+) \arrow[d, "\mathrm{Index}_{(\mathcal{A}^+,\mathfrak{H},F)}"] \\
 & Z
\end{tikzcd}$$

$\square$

Remark 2.2.7. The main advantage of using Fredholm modules is the following: in order to define the K_0-group of a non-unital algebra $\mathcal{A}$, we need to consider idempotents in $\mathcal{A}^+ \otimes \mathbb{M}_n$. This is no problem for Fredholm modules because if $(\mathcal{A}, \mathfrak{H}, F)$ is a Fredholm module then so is $(\mathcal{A}^+, \mathfrak{H}, F)$. On the other hand, if $(\mathcal{A}, \mathfrak{H}, D)$ is a spectral triple then so is $(\mathcal{A}^+, \mathfrak{H}, D)$ *only if* D had compact resolvents to begin with, which is a non-trivial condition that is *not* satisfied by many interesting examples including elliptic operators on non-compact manifolds.

[1] The super-index of an odd Fredholm operator $\begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$ is defined to be the Fredholm index of P. See A.3

The index map is homotopy invariant:

Lemma 2.2.8. *Let $\mathcal{A}$ be an algebra and let $\pi_t : \mathcal{A} \to \mathcal{L}(\mathfrak{H})$, $t \in [0,1]$ be a pointwise norm-continuous path of actions of $\mathcal{A}$ on $\mathfrak{H}$. Let F_t be a norm-continuous path of supersymmetries such that the commutator $[F_t, \pi_t(a)]$ is compact for any $a \in \mathcal{A}$. Then the index maps*

$$\mathrm{Index}_{F_t} : K_0(\mathcal{A}) \to \mathbb{Z}$$

do not *depend on t.*

Proof. Follows from the homotopy invariance of the Fredholm index (cf. Lemma A.3.5), since $\pi_t(e)F_t\pi_t(e)F_t\pi_t(e)$ is norm-continuous. □

2.2.2 Spectral Triples

Now we define the index map for spectral triples through Fredholm modules. There are several ways to construct a Fredholm module out of a spectral triple. We start with the simplest one.

Lemma 2.2.9. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple. Assume that D is invertible and has compact resolvents. Let the* phase *of D be given by*

$$F := D|D|^{-1}.$$

Then $(\mathcal{A}, \mathfrak{H}, F)$ is a Fredholm module. If $(\mathcal{A}, \mathfrak{H}, D)$ is p-summable then so is $(\mathcal{A}, \mathfrak{H}, F)$.

Proof. It is clear if, in addition, the commutator $[|D|, a]$ is bounded for any $a \in \mathcal{A}$:

$$\begin{aligned}[F, a] &= [D, a]|D|^{-1} + D[|D|^{-1}, a] \\ &= ([D, a] - F[|D|, a])|D|^{-1}.\end{aligned}$$

See [GBVF01, Lemma 10.18] for the general case. □

Remark 2.2.10. The condition that D is invertible is not a serious restriction, we can define $F = D|D|^{-1}$ as $\mathrm{sgn}(D)$ by functional calculus, where sgn is the Borel function on $\mathbb{R}$ given by $\mathrm{sgn}(x) := \begin{cases} x|x|^{-1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$. Then $(\mathcal{A}, \mathfrak{H}, F)$ is, what's called, a pre-Fredholm module (cf. [Con85, Appendix I.2]). For any idempotent $e \in \mathcal{A}^+ \otimes \mathbb{M}_n$, the operator F_e is again a Fredholm operator and we can define the index pairing $\langle F, e\rangle$ as the superindex of F_e. Moreover, there is a canonical way to construct a Fredholm module from a pre-Fredholm module. However, for simplicity we restrict to Fredholm modules in this thesis. See Example 2.2.14.

Hence for a *unital*[2] spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ with D invertible, we can define an *index map* by

$$\mathrm{Index}_{D|D|^{-1}} : K_0(\mathcal{A}) \to \mathbb{Z}.$$

[2] so that D has compact resolvents

Moreover, for this index can be computed in terms of the spectral triple directly. To show this we consider an alternative way to construct Fredholm modules (cf. [Con85, Proposition I.6.1]).

Lemma 2.2.11. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple. Assume that D is invertible and $D^{-1}[D,a]$ is compact for any $a \in \mathcal{A}$. Let $\mathcal{A}$ act on $\mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}$ by*

$$a \mapsto \begin{bmatrix} a & 0 \\ 0 & D^{-1}aD \end{bmatrix}.$$

Then $(\mathcal{A}, \mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, 1 \mathbin{\widehat{\otimes}} \sigma_1)$ is a Fredholm module. If $(\mathcal{A}, \mathfrak{H}, D)$ is p-summable then so is the Fredholm module $(\mathcal{A}, \mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, 1 \mathbin{\widehat{\otimes}} \sigma_1)$.

Proof. It follows from the properties of Pauli matrices (cf. Lemma 2.1.18) that $F = 1 \mathbin{\widehat{\otimes}} \sigma_1$ is a supersymmetry *i.e.* F is odd and self-adjoint and $F^2 = 1$. To complete the proof it is enough to note that

$$[F, a] = \begin{bmatrix} 0 & D^{-1}aD - a \\ a - D^{-1}aD & 0 \end{bmatrix} = D^{-1}[D,a] \mathbin{\widehat{\otimes}} (-i\sigma_2).$$

□

Let $\mathfrak{F} = (\mathcal{A}, \mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, 1 \mathbin{\widehat{\otimes}} \sigma_1)$ and $D = \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix} = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$. Then

$$D^{-1}aD = \begin{bmatrix} P^{-1}aP & 0 \\ 0 & Q^{-1}aQ \end{bmatrix}$$

and we see that, in fact, $\mathfrak{F}$ is the sum of two Fredholm modules $\mathfrak{F}^0 := (\mathcal{A}, \mathfrak{H}^0 \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, 1 \mathbin{\widehat{\otimes}} \sigma_1)$ and $\mathfrak{F}^1 := (\mathcal{A}, \mathfrak{H}^1 \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, 1 \mathbin{\widehat{\otimes}} \sigma_1)$, where the actions are given by

$$\begin{bmatrix} a & 0 \\ 0 & P^{-1}aP \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a & 0 \\ 0 & Q^{-1}aQ \end{bmatrix}$$

respectively. Let $e \in \mathcal{A}^+$ be an idempotent and let f denote (the closure of) $P^{-1}eP$. Then

$$F_e = \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} 0 & ef \\ fe & 0 \end{bmatrix}$$

and therefore $fe : e\mathfrak{H}^0 \to f\mathfrak{H}^0$ is a Fredholm operator with parametrix $ef : f\mathfrak{H}^0 \to e\mathfrak{H}^0$. Hence,

$$\begin{aligned} \operatorname{Index}_{\mathfrak{F}^0}(e) &= \operatorname{Index}_s(F_e) \\ &= \operatorname{Index}(fe : e\mathfrak{H}^0 \to f\mathfrak{H}^0) \\ &= \operatorname{Index}(ePe : e\mathfrak{H}^0 \to e\mathfrak{H}^1) \\ &= \operatorname{Index}_s(eDe : e\mathfrak{H} \to e\mathfrak{H}). \end{aligned}$$

In the second last equality, we used the fact that P is closed.

Lemma 2.2.12. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple. Assume that D is invertible and has compact resolvents. Let $\mathfrak{F}^0$ be the Fredholm module constructed above. Then for any idempotent $e \in \mathcal{A}^+ \otimes \mathbb{M}_n$,*

$$\mathrm{Index}_{D|D|^{-1}}(e) = \mathrm{Index}_{\mathfrak{F}^0}(e) = \mathrm{Index}_s(D_e),$$

where $D_e : e(\mathfrak{H} \otimes \mathbb{C}^n) \to e(\mathfrak{H} \otimes \mathbb{C}^n)$ is the (odd closed Fredholm) operator $e(D \otimes 1)e$ with domain $e(\mathrm{dom}(D) \otimes \mathbb{C}^n) \subseteq \mathfrak{H} \otimes \mathbb{C}^n$. In particular, if in addition $(\mathcal{A}, \mathfrak{H}, D)$ is involutive and e is a projection, i.e. *self-adjoint idempotent, then*

$$\mathrm{Index}_{D|D|^{-1}}(e) = \dim_s \ker D_e.$$

See [Con85, Corollary I.6.3].

Proof. The path $D|D|^{-t}$, $t \in [0, 1]$, gives a homotopy from D to $F = D|D|^{-1}$. □

Remark 2.2.13. If $(\mathcal{A}, \mathfrak{H}, D)$ is involutive, then so is $(\mathcal{A}, \mathfrak{H}, F)$ and

$$\mathrm{Index}_{\mathfrak{F}^1}(e^*) = \mathrm{Index}(e^* P^* e^* : e^* \mathfrak{H}^1 \to e^* \mathfrak{H}^0) = -\mathrm{Index}_{\mathfrak{F}^0}(e).$$

Hence if e is a projection, then $\mathrm{Index}_{\mathfrak{F}}(e) = \mathrm{Index}_{\mathfrak{F}^0}(e) + \mathrm{Index}_{\mathfrak{F}^1}(e) = 0$.

Now we consider the general case. First off, given a spectral triple $(\mathcal{A}, \mathfrak{H}, D)$, "making" D invertible is easy:

Example 2.2.14. Let $\mathbb{C}$ act on $\mathbb{C}^{1,1}$ by $1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and let $\sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ denote the first Pauli matrix. Then $\left(\mathbb{C}, \mathbb{C}^{1,1}, \sigma_1\right)$ is a spectral triple whose index map is given by the natural identification

$$\mathrm{Index}_{\sigma_1} : K_0(\mathbb{C}) \cong \mathbb{Z}, \quad [1] \mapsto 1.$$

Now consider the spectral triple

$$(\mathcal{A} \otimes \mathbb{C}, \mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, D \times \sigma_1) = (\mathcal{A}, \mathfrak{H}, D) \times (\mathbb{C}, \mathbb{C}^{1,1}, \sigma_1).$$

Then $D \times \sigma_1$ is invertible since $(D \times \sigma_1)^2 = (D^2 + 1) \mathbin{\widehat{\otimes}} 1 \geq 1$. In matrix notation, we have

$$D \times \sigma_1 = D \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} \sigma_1 = D \otimes 1 + \gamma \otimes \sigma_1 = \begin{bmatrix} D & \gamma \\ \gamma & D \end{bmatrix} \quad \text{on } \mathfrak{H} \otimes \mathbb{C}^2,$$

where γ is the grading operator on $\mathfrak{H}$. Moreover, for any idempotent $e \in \mathcal{A}$, the Hilbert space $e(\mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1})$ can be identified with $e\mathfrak{H}$ naturally and under this identification $(D \times \sigma_1)_e = D_e$.

However, if $\mathcal{A}$ is not unital, then D may no longer have compact resolvents and it is not clear how we can make it to have compact resolvents or, equivalently, unitize $(\mathcal{A}, \mathfrak{H}, D)$. Nevertheless, we can still construct a Fredholm module.

Lemma 2.2.15. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple. Let $\mathcal{A}^{(2)}$ denote the subalgebra of $\mathcal{A}^+ \otimes \mathbb{M}_2$ given by*

$$\mathcal{A}^{(2)} := \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in \mathbb{M}_2(\mathcal{A}^+) \mid a - b \in \mathcal{A} \right\}$$

acting on $\mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}$ by restricting the natural amplification action $\mathcal{A}^+ \otimes \mathbb{M}_2 \to \mathcal{L}(\mathfrak{H}) \otimes \mathbb{M}_2$. Let

$$F := \frac{1}{\sqrt{D^2+1}} \begin{bmatrix} D & \gamma \\ \gamma & D \end{bmatrix},$$

where γ is the grading operator on $\mathfrak{H}$. Then $(\mathcal{A}^{(2)}, \mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, F)$ is a Fredholm module.

Proof. It is clear that F is a supersymmetry since it is the phase of

$$D \times \sigma_1 = D \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} \sigma_1 = \begin{bmatrix} D & \gamma \\ \gamma & D \end{bmatrix}.$$

Let $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be an element of $\mathcal{A}^{(2)}$ and let $C = (D^2+1)^{-\frac{1}{2}}$. Then we need to show that the commutator

$$\begin{bmatrix} DC & \gamma C \\ \gamma C & DC \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} - \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} DC & \gamma C \\ \gamma C & DC \end{bmatrix} = \begin{bmatrix} [DC, a] & \gamma(Cb - aC) \\ \gamma(Ca - bC) & [DC, b] \end{bmatrix}$$

is compact. For the diagonal elements, note that $[DC, \mathcal{A}^+] = [DC, \mathcal{A}] = [1 - DC, \mathcal{A}] = [f(D), \mathcal{A}]$, where $f : \mathbb{R} \to \mathbb{R}$ is given by $x \mapsto 1 - x(x^2+1)^{-\frac{1}{2}}$. Since f belongs to $C_0(\mathbb{R})$, we see that $[f(D), \mathcal{A}] \subseteq \mathcal{K}$ by Lemma A.2.10, hence $[DC, a]$ and $[DC, b]$ are compact. Similarly, for the off-diagonal elements, write

$$Cb - aC = [C, b] - (a-b)C \quad \text{and} \quad Ca - bC = [C, a] + (a-b)C.$$

Since $C = (x^2+1)^{-\frac{1}{2}}(D)$ and $(x^2+1)^{-\frac{1}{2}}$ belongs to $C_0(\mathbb{R})$, Lemma A.2.10 finishes off the proof. $\square$

Remark 2.2.16. If D has compact resolvents, then the triple $(\mathcal{A}^{(2)}, \mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, D \times \sigma_1)$ forms a spectral triple and $(\mathcal{A}^{(2)}, \mathfrak{H} \mathbin{\widehat{\otimes}} \mathbb{C}^{1,1}, F)$ is its phase.

Hence we get an index map $K_0(\mathcal{A}^{(2)}) \to \mathbb{Z}$. On the other hand, we have a natural short split-exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{A}^{(2)} \longrightarrow \mathcal{A}^+ \longrightarrow 0,$$

where the nontrivial maps are given by $a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mapsto b$. Therefore

$$K_0(\mathcal{A}) \cong \ker(K_0(\mathcal{A}^{(2)}) \to K_0(\mathcal{A}^+))$$

and there is a natural projection map $K_0(\mathcal{A}^{(2)}) \to K_0(\mathcal{A})$.

Notation 2.2.17. Let e and f be idempotents in $\mathcal{A}^+ \otimes \mathbb{M}_n$ such that $e - f \in \mathcal{A} \otimes \mathbb{M}_n$. We write $e \ominus f$ for the idempotent $\begin{bmatrix} e & 0 \\ 0 & f \end{bmatrix}$ in $\mathcal{A}^{(2)} \otimes \mathbb{M}_n$ and $[e \ominus f] \in K_0(\mathcal{A})$ for the projection of the corresponding K_0-class in $K_0(\mathcal{A}^{(2)})$.

Definition 2.2.18. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple. The *index map* for $(\mathcal{A}, \mathfrak{H}, D)$ is given by the composition

$$\mathrm{Index}_{(\mathcal{A},\mathfrak{H},D)} : K_0(\mathcal{A}) \to K_0(\mathcal{A}^{(2)}) \to \mathbb{Z}.$$

We also write Index or Index_D for the index map if there is no risk of confusion. It is consistent with the previous definition:

Lemma 2.2.19. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple. Assume that D is invertible and has compact resolvents. Then the two definitions of the index map coincide, that is, the following natural diagram is commutative:*

$$\begin{tikzcd} K_0(\mathcal{A}) \arrow[r] \arrow[d] & K_0(\mathcal{A}^{(2)}) \arrow[d] \\ K_0(\mathcal{A}^+) \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Proof. Let $e \in \mathcal{A}^+ \otimes \mathbb{C}^n$ be a projection. To simplify the notation we assume that $n = 1$. Then a short computation shows that we need to demonstrate

$$\mathrm{Index}_s(eD|D|^{-1}e) = \mathrm{Index}_s(eD(D^2+1)^{-\frac{1}{2}}e).$$

This follows from the homotopy invariance of the Fredholm index (cf. Lemma A.3.5), since $eD(D^2+t)^{-\frac{1}{2}}e$, $t \in [0,1]$, gives a homotopy between the two operators. Alternatively, we can use the stability of the Fredholm index under compact perturbations since

$$eD\left(\frac{1}{\sqrt{D^2}} - \frac{1}{\sqrt{D^2+1}}\right)e$$

is compact by Lemma A.2.10. □

In the finite-dimensional case, nothing interesting happens – the index map is independent of the operator D:

Example 2.2.20 (Finite-Dimensional)**.** Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple with $\mathfrak{H}$ finite dimensional. Then for any idemptent $e \in \mathcal{A}^+ \otimes \mathbb{M}_n$, we have

$$\mathrm{Index}_D(e) = \dim e(\mathfrak{H}^0 \otimes \mathbb{C}^n) - \dim e(\mathfrak{H}^1 \otimes \mathbb{C}^n) = \dim_s e(\mathfrak{H} \otimes \mathbb{C}^n).$$

Example 2.2.21 (Closure)**.** Let $(\mathcal{A}, \mathfrak{H}, D)$ be a pre-spectral triple and let $(\mathcal{A}, \mathfrak{H}, \bar{D})$ denote its closure. We claim that if D satisfies a certain "regularity" condition then we can compute the index of $(\mathcal{A}, \mathfrak{H}, \bar{D})$ directly from $(\mathcal{A}, \mathfrak{H}, D)$. The simplest one is to require that

$$\mathrm{dom}(D) = \bigcap_{k=1}^{\infty} \mathrm{dom}(\bar{D}^k).$$

Then $\ker(D) = \ker(\bar{D})$. Indeed, if $\xi \in \ker(\bar{D})$, then ξ belongs to $\mathrm{dom}(\bar{D}^k)$ for any k, hence ξ belongs to $\mathrm{dom}(D)$ *i.e.* $\xi \in \ker(D)$.

The following is the mother of all index theorems, due to Atiyah and Singer [AS68a, AS68b].

Example 2.2.22 (Atiyah-Singer)**.** Let M be a closed manifold and let $(C^\infty(M), L^2(M, E), D)$ be a spectral triple as in Example 2.1.7. Let $\sigma_D \in C^\infty(T^*M, \mathrm{End}E)$ denote the principal symbol of D. Then for any projection $e \in C^\infty(M) \otimes \mathbb{M}_n$,

$$\mathrm{Index}_D(e) = \int_{T^*M} \mathrm{Ch}(e)\mathrm{Ch}(\sigma_D)\mathrm{Todd}(T_{\mathbb{C}}M),$$

where $T_{\mathbb{C}}M \in H^\bullet(M)$ is the Todd class of $T_{\mathbb{C}}M \to M$ and T^*M is given the orientation coming from the natural almost complex structure.

Now we take care of a few formalities.

Remark 2.2.23. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a unital spectral triple. It follows from the definition that for any idempotent $e \in \mathcal{A} \otimes \mathbb{M}_n$,

$$\mathrm{Index}_D([e]_{K_0(\mathcal{A})}) = \mathrm{Index}_{D\otimes 1}([e]_{K_0(\mathcal{A}\otimes\mathbb{M}_n)}).$$

Hence, we can assume that idempotents are in $\mathcal{A}$ instead of matrices over $\mathcal{A}$ for index theoretic purposes.

Lemma 2.2.24. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a unital spectral triple and let*

$$A \coloneqq \{a \in \mathcal{L} \mid [D, a] \in \mathcal{L}\}.$$

Then the following holds.

(1) The space A is an algebra and $(A, \mathfrak{H}, D)$ is an involutive *spectral triple.*

(2) The index maps factor as

$$\begin{tikzcd} K_0(\mathcal{A}) \arrow[r, "\mathrm{Index}_{(\mathcal{A},\mathfrak{H},D)}"] \arrow[d] & \mathbb{Z} \\ K_0(A) \arrow[ur, "\mathrm{Index}_{(A,\mathfrak{H},D)}"'] & \end{tikzcd}$$

where the downward map is induced by the inclusion $\mathcal{A} \to A$.

(3) Every idempotent in A is similar to a self-adjoint idempotent.

Proof. The first two statements follow immediately from the definitions. The last statement follows from Kaplansky's lemma (cf. [Bla98, Proof of Proposition 4.6.2]). $\square$

Hence for most index theoretic problems we can assume that elements of K-theory are represented by self-adjoint idempotents, *i.e.* projections. The advantage of using projections is that if $e \in A$ is a projection, then D_e is *self-adjoint*. In this case, $(eAe, e\mathfrak{H}, D_e)$ is a spectral triple and $\langle e, D\rangle = \langle e, D_e\rangle$.

The following is clear.

Lemma 2.2.25 (Sum)**.** *The index map for the sum is additive in the sense that the following diagram is commutative:*

$$\begin{tikzcd}
K_0(\mathcal{A}_1) \oplus K_0(\mathcal{A}_2) \arrow[r] \arrow[d, "\mathrm{Index}_{D_1} \oplus \mathrm{Index}_{D_2}"] & K_0(\mathcal{A}_1 \oplus \mathcal{A}_2) \arrow[d, "\mathrm{Index}_{D_1 \oplus D_2}"] \\
\mathbb{Z} \oplus \mathbb{Z} \arrow[r, "+"] & \mathbb{Z}
\end{tikzcd}$$

$\square$

Lemma 2.2.26 (Product)**.** *The index map is multiplicative in the sense that the following diagram is commutative:*

$$\begin{tikzcd}
K_0(\mathcal{A}_1) \otimes K_0(\mathcal{A}_2) \arrow[r] \arrow[d, "\mathrm{Index}_{D_1} \otimes \mathrm{Index}_{D_2}"] & K_0(\mathcal{A}_1 \otimes \mathcal{A}_2) \arrow[d, "\mathrm{Index}_{D_1 \times D_2}"] \\
\mathbb{Z} \otimes \mathbb{Z} \arrow[r, equal, "\times"] & \mathbb{Z}
\end{tikzcd}$$

Proof. We may assume that the spectral triples are unital and involutive. Then it is enough to show that $\mathrm{Index}_s(D_1 \times D_2) = \mathrm{Index}_s(D_1)\mathrm{Index}_s(D_2)$. Since $(D_1 \times D_2)^2 = D_1^2 \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} D_2^2$, we have

$$\ker(D_1 \times D_2) = \ker(D_1) \mathbin{\widehat{\otimes}} \ker(D_2)$$

and the proof is complete by Lemma 2.2.12. $\square$

Example 2.2.27 (Perturbation)**.** Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple and let $(\mathcal{A}, \mathfrak{H}, D + S)$ be a perturbation of $(\mathcal{A}, \mathfrak{H}, D)$ (cf. Definition 2.1.14). Then

$$\mathrm{Index}_{D+S} = \mathrm{Index}_D : K_0(\mathcal{A}) \to \mathbb{Z}.$$

Indeed, let $e = e^2 \in \mathcal{A}$ be an idempotent. Then $(D + S)_e - D_e = S_e$ and, by the proof of Theorem A.1.10, choosing $t \gg 1$ large enough, we may assume that

$$||S_e(D + it)_e^{-1}|| < 1 \quad \text{and} \quad ||(D + it)_e^{-1} S_e|| < 1.$$

Since $(D + it)_e^{-1}$ is a parametrix for D_e, we see that by Lemma A.3.5, $\mathrm{Index}_s((D + S)_e) = \mathrm{Index}_s(D_e)$.

Chapter 3

Analytical Properties of Spectral Triples

3.1 Regularity

In this section we develop the necessary functional analytic tools to study spectral triples following [CM95, Hig04a, Hig06]. For simplicity we concentrate on the case where the operator $\Delta = D^2$ is invertible.

3.1.1 Sobolev Spaces

Let Δ be an *invertible positive self-adjoint operator* on a Hilbert space $\mathfrak{H}$. We remind the reader that a self-adjoint operator Δ is said to be invertible if $0 \in \mathbb{C}$ is in the resolvent set of Δ and in this case the 0-resolvent , $(0-\Delta)^{-1} = -\Delta^{-1}$, is a bounded operator on $\mathfrak{H}$. A positive self-adjoint operator is invertible if and only if it is *strictly positive i.e.* bounded from below by a *strictly* positive number: $\langle \Delta\xi, \xi\rangle \geq c\,\langle \xi, \xi\rangle$ for some $c > 0$ (one direction is clear, in the other direction c can be taken as $||\Delta^{-\frac{1}{2}}||^{-2}$). In this case, we write $\Delta \geq c$ and note that $\mathrm{Spec}(\Delta) \subseteq [c, \infty)$.

Now we study the complex powers of Δ. See [Shu01] for a comprehensive treatment of complex powers of unbounded operators, including the non-self-adjoint case.

We use the projection valued measure form of the spectral theorem as given in, for example, [RS80, Theorem VIII.6].

For $\xi \in \mathfrak{H}$, let $d(\xi, P_\lambda\xi)$ denote the measure associated to ξ. Then by *loc.cit.*

$$\mathrm{dom}(\Delta^z) = \left\{\xi \in \mathfrak{H} \mid \int_0^\infty |\lambda^{2z}| d(\xi, P_\lambda\xi) < \infty\right\}$$

and

$$\langle \xi, \Delta^z\xi\rangle = \int_0^\infty \lambda^z d(\xi, P_\lambda\xi)$$

on $\xi \in \mathrm{dom}(\Delta^z)$. This uniquely defines Δ^z on $\mathrm{dom}(\Delta^z)$ by polarization.

Let $s \in \mathbb{R}$. Then Δ^s is self-adjoint. Consider the domain $\mathrm{dom}(\Delta^s) \subseteq \mathfrak{H}$ and equip it with the norm

$$\xi \mapsto ||\Delta^s \xi||_{\mathfrak{H}}.$$

Lemma 3.1.1. *For $s \geq t$, we have a continuous inclusion*

$$\mathrm{dom}(\Delta^s) \subseteq \mathrm{dom}(\Delta^t).$$

Moreover, $\mathrm{dom}(\Delta^s)$ *is complete if $s \geq 0$.*

Proof. Let $\Delta \geq c > 0$. Then

$$\begin{aligned}\mathrm{dom}(\Delta^z) &= \left\{\xi \in \mathfrak{H} \mid \int_c^\infty \lambda^{2s} d(\xi, P_\lambda \xi) < \infty\right\} \\ &= \left\{\xi \in \mathfrak{H} \mid \int_1^\infty \lambda^{2s} d(\xi, P_\lambda \xi) < \infty\right\}\end{aligned}$$

and consequently

$$\mathrm{dom}(\Delta^s) \subseteq \mathrm{dom}(\Delta^t)$$

for $s \geq t$. Moreover, $\Delta^{t-s} = (\Delta^{-1})^{s-t}$ is a bounded operator on $\mathfrak{H}$ and

$$||\Delta^t \xi|| \leq ||\Delta^{t-s}|| \, ||\Delta^s \xi||, \quad \xi \in \mathrm{dom}(\Delta^s).$$

Hence, the inclusions are continuous.

Let $s \geq 0$ and let ξ_n be a Cauchy sequence in $\mathrm{dom}(\Delta^s)$ *i.e.* $\xi_n \in \mathrm{dom}(\Delta^s)$ and $||\Delta^s \xi_n - \Delta^s \xi_m|| \to 0$ as n, $m \to \infty$. Then $\Delta^s \xi_n$ is a Cauchy sequence in $\mathfrak{H}$, thus has a limit $\eta \in \mathfrak{H}$. Moreover, $\xi_n - \xi_m = \Delta^{-s}(\Delta^s \xi_n - \Delta^s \xi_m)$ too is a Cauchy sequence in $\mathfrak{H}$ converging to some ξ. By the closedness of Δ^s, we see that $\xi \in \mathrm{dom}(\Delta^s)$ and $\Delta^s \xi = \eta$. This proves that $\mathrm{dom}(\Delta^s)$ is complete. $\square$

Definition 3.1.2. Let Δ be an invertible positive self-adjoint operator. The Δ-*Sobolev space* of order $s \in \mathbb{R}$, denoted $W^s = W^s(\Delta) = W^s(\mathfrak{H}, \Delta)$, is the Hilbert space completion of $\mathrm{dom}(\Delta^{\frac{s}{2}})$. The inner product is given by

$$\langle \xi, \eta \rangle_{W^s} := \left\langle \Delta^{\frac{s}{2}} \xi, \Delta^{\frac{s}{2}} \eta \right\rangle_{\mathfrak{H}}$$

for ξ, $\eta \in \mathrm{dom}(\Delta^{\frac{s}{2}})$.

Note that the invertibility hypothesis guaranties that $|| \cdot ||_{W^s}$ is indeed a norm. In fact, it is equivalent to the norm given by $\xi \mapsto (||\xi||^2 + ||\Delta^{\frac{s}{2}} \xi||^2)^{\frac{1}{2}}$.

Remark 3.1.3. Lemma 3.1.1 shows that no completion is needed if $s \geq 0$ and that we have continuous inclusions $W^s \subseteq W^t$ for all $s \geq t$, s, $t \in \mathbb{R}$. The "one-half" in $\mathrm{dom}(\Delta^{\frac{s}{2}})$ reflects the fact that we are assuming that Δ has "order 2" in some sense. This will be made precise in Definition 3.1.12.

Example 3.1.4. If Δ is *bounded*, then nothing interesting happens: for any $s \in \mathbb{R}$, $W^s = \mathfrak{H}$ as sets (the inner products are different but equivalent).

Hence, from now on we assume that Δ is *unbounded*.

Example 3.1.5. Suppose that Δ has compact resolvent and let $\lambda_n = \mu_n(\Delta^{-1})^{-1}$. Then

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

and there exists a complete orthonormal basis $\{\xi_n\}$ of $\mathfrak{H}$ consisting of eigenvectors: $\Delta\xi_n = \lambda_n\xi_n$. Hence the Sobolev spaces can be identified as the weighted l^2-space:

$$W^s = \{(a_n)_{n=1}^{\infty} \mid \sum_n \lambda_n^s |a_n|^2 < \infty\}.$$

Example 3.1.6 (Classical Sobolev Spaces)**.** Let M be a closed manifold and let Δ_0 be a *strictly positive* order-two elliptic partial differential operator acting on the smooth functions on M. Let $\Delta = \overline{\Delta_0}$ be its closure. Then Δ is invertible and self-adjoint (cf. [HR00, Proposition 10.2.10]) and has compact resolvent (cf. [HR00, Proposition 10.4.5]).

The following summarizes the properties of the associated Sobolev spaces.

Gårding Inequality: As a Hilbert space W^s is equivalent to the “standard” s-Sobolev space (cf. [Shu01, Proposition I.7.3]).

Rellich Lemma: The continuous inclusion $W^s \subset W^t$ is compact for $s > t$ (cf. [Shu01, Theorem I.7.4]).

Sobolev Embedding Lemma: If $s > \frac{1}{2}\dim M + k$, then $W^s \subset C^k(M)$, the space of k-times continuously differentiable functions (cf. [Shu01, Theorem I.7.6]).

Elliptic Regularity: If D is any order-one elliptic partial differential operator on $C^\infty(M)$ and if $\xi \in \mathfrak{H}$ and $D\xi$ belongs to W^s, $s \geq 0$, weakly, then ξ belongs to W^{s+1} (cf. [Shu01, Corollary I.71]).

In particular, $\bigcap_s W^s = C^\infty(M)$. This motivates the following definition.

Definition 3.1.7. Let Δ be an invertible positive self-adjoint operator. The space of Δ-*smooth vectors* is

$$W^\infty := \bigcap_{s\in\mathbb{R}} W^s = \bigcap_{n=0}^{\infty} W^{2n} = \bigcap_{n=0}^{\infty} \operatorname{dom}(\Delta^n).$$

The space W^∞ contains $1_{[0,M]}(\Delta)\mathfrak{H}$ for any $M > 0$, and thus W^∞ is dense in $\operatorname{dom}(\Delta^z)$ for any $z \in \mathbb{C}$. It follows that W^∞ is dense in W^s for any $s \in \mathbb{R}$.

Lemma 3.1.8. *Let $z \in \mathbb{C}$. For any $s \in \mathbb{R}$, the operator $\Delta^z_{|W^\infty} : W^\infty \to \mathfrak{H}$ extends*[1] *to an* isometry

$$\Delta^z : W^{s+2\operatorname{Re}(z)} \to W^s.$$

[1] Note that if $s > 0$ the range actually shrinks to $W^s \subset \mathfrak{H}$.

In particular, $\Delta^z \cdot W^\infty = W^\infty$.

Proof. First note that $\Delta^{i\,\mathrm{Im}(z)}$ is a unitary operator on $\mathfrak{H}$. Therefore, for any $\xi \in W^\infty$,

$$||\Delta^z \xi||_{W^s} = ||\Delta^{\mathrm{Re}(z)+\frac{s}{2}}\xi||_{\mathfrak{H}} = ||\xi||_{W^{s+2\,\mathrm{Re}(z)}}.$$

□

Lemma 3.1.9. *The operator* Δ^z *is essentially self-adjoint on* $W^\infty \subset \mathfrak{H}$.

Proof. Any vector $\xi \in 1_{[0,M]}(\Delta)\mathfrak{H} \subset W^\infty$ is an analytical vector for Δ^z. Indeed, for any $\xi \in 1_{[0,M]}(\Delta)\mathfrak{H}$,

$$||(\Delta^z)^n \xi|| \leq M^{|\,\mathrm{Re}(z)n|}||\xi||$$

and consequently

$$\sum_{n=0}^{\infty} \frac{||(\Delta^z)^n \xi||}{n!} t^n < \infty$$

for any $t > 0$. Since Δ^z preserves W^∞, all the analytical vectors for Δ^z are also analytical for the restriction $\Delta^z_{|W^\infty}$ and applying Nelson's theorem [RS75, Theorem X.39] to $\Delta^z_{|W^\infty}$, we see that Δ^z is essentially self-adjoint on W^∞. □

Another convenient way to express the complex powers Δ^z is using the Cauchy integral formula: for $\mathrm{Re}(z) < 0$,

$$\Delta^z = \frac{1}{2\pi i}\int \lambda^z(\lambda - \Delta)^{-1} d\lambda,$$

where the integral is a contour integral along a downwards pointing vertical line in $\mathbb{C}$ which separates 0 from $\mathrm{Spec}(\Delta)$. Let $\Delta \geq c > 0$ and let $s \in \mathbb{R}$. Then it follows from the spectral theorem that for any $\lambda \notin \mathrm{Spec}(\Delta)$, the resolvent $(\lambda - \Delta)^{-1}$ is a bounded operator on the Sobolev spaces W^s with norm at most $\big((\mathrm{Re}(\lambda) - c)^2 + \mathrm{Im}(\lambda)^2\big)^{-\frac{1}{2}}$. Hence for any $\mathrm{Re}(z) < 0$, the integral converges to a bounded operator on W^s.

More generally, we have the following.

Lemma 3.1.10. *For* $k \in \mathbb{Z}_{\geq 0}$ *and* $\mathrm{Re}(z) < k$,

$$\binom{z}{k}\Delta^{z-k} = \frac{1}{2\pi i}\int \lambda^z(\lambda - \Delta)^{-k-1} d\lambda \quad in \quad \mathcal{L}(W^s),\ s \in \mathbb{R},$$

where the integral is a contour integral along a downwards pointing vertical line in $\mathbb{C}$ *which separates* 0 *from* $\mathrm{Spec}(\Delta)$. □

3.1.2 Operators of Finite Analytic Order

We consider various class of linear operators on W^∞. The algebra of all linear operators $W^\infty \to W^\infty$ is denoted $\mathrm{End}(W^\infty)$.

Example 3.1.11. If an (unbounded) operator P

(i) has domain $\operatorname{dom}(P) \supseteq W^\infty$ and

(ii) preserves W^∞, *i.e.* $P \cdot W^\infty \subseteq W^\infty$,

then the restriction $P_{|W^\infty}$ gives an element of $\operatorname{End}(W^\infty)$. We often write, simply, P for $P_{|W^\infty}$.

Definition 3.1.12. We say that a linear operator $W^\infty \to W^\infty$ has *analytic order at most* $t \in \mathbb{R}$ if it extends by continuity to a *bounded* linear operator $W^{s+t} \to W^s$ for every $s \in \mathbb{R}$. We write

$$\operatorname{Op}^t = \operatorname{Op}^t(\Delta) = \operatorname{Op}^t(\mathfrak{H}, \Delta)$$

for the class of operators of analytic order at most t and define

$$\operatorname{Op} = \operatorname{Op}^\infty := \bigcup_t \operatorname{Op}^t \quad \text{and} \quad \operatorname{Op}^{-\infty} := \bigcap_t \operatorname{Op}^t.$$

Lemma 3.1.13. *The class* Op *of operators with finite analytic order, filtered by analytic order, is a filtered algebra:*

(a) $\operatorname{Op}^s \subseteq \operatorname{Op}^t$ *for* $s \leq t$ *and*

(b) $\operatorname{Op}^s \cdot \operatorname{Op}^t \subseteq \operatorname{Op}^{s+t}$.

In particular, $\operatorname{Op}^0 \subset \operatorname{Op}$ *is a subalgebra and* $\operatorname{Op}^{-\infty} \subset \operatorname{Op}$ *and* $\operatorname{Op}^t \subset \operatorname{Op}^0$, $t \in [-\infty, 0)$ *are two-sided ideals.*

Proof. (a) follows from Remark 3.1.3 and (b) is clear from the definition of analytic order. □

Notice that operators with analytic order at most 0 extend, in particular, to *bounded* linear operators on $\mathfrak{H} = W^0$ allowing us to identify Op^0 with a subalgebra of $\mathcal{L}$.

Example 3.1.14. We see from Lemma 3.1.8 that the operator Δ^z belongs to $\operatorname{Op}^{2\operatorname{Re}(z)}$ for any $z \in \mathbb{C}$. It follows from the spectral theorem that for z, $w \in \mathbb{C}$

$$\Delta^z \cdot \Delta^w = \Delta^{z+w} \quad \text{in Op}.$$

Lemma 3.1.15. *If* Ψ *is a filtered subalgebra of* Op *such that* $\Delta^{\frac{t}{2}}$ *belongs to* Ψ^t *for any* $t \in \mathbb{R}$, *then*

$$\Psi^t = \Delta^{\frac{t}{2}} \Psi^0 = \Psi^0 \Delta^{\frac{t}{2}}.$$

Conversely, if a unital subalgebra $\Psi^0 \subseteq \operatorname{Op}^0$ *satisfies* $\Delta^{\frac{t}{2}} \Psi^0 = \Psi^0 \Delta^{\frac{t}{2}}$ *for any* $t \in \mathbb{R}$, *then* $\Psi^t := \Delta^{\frac{t}{2}} \Psi^0$ *defines a filtered subalgebra of* Op *such that* $\Delta^{\frac{t}{2}}$ *belongs to* Ψ^t.

Proof. For the first statement:

$$\Psi^t = \Delta^{\frac{t}{2}} \Delta^{-\frac{t}{2}} \Psi^t \subseteq \Delta^{\frac{t}{2}} \Psi^{-t} \Psi^t \subseteq \Delta^{\frac{t}{2}} \Psi^0 \subseteq \Psi^t \Psi^0 \subseteq \Psi^t.$$

Similarly for the other side.

For the second statement:

$$\Psi^t \Psi^s = \Delta^{\frac{t}{2}} \Psi^0 \Delta^{\frac{s}{2}} \Psi^0 = \Delta^{\frac{t}{2}} \Delta^{\frac{s}{2}} \Psi^0 \Psi^0 = \Psi^{t+s}$$

and $\Delta^{\frac{t}{2}} \in \Psi^t$, since Ψ^0 is unital. $\square$

Corollary 3.1.16. *For any $t \in \mathbb{R}$,*

$$\mathrm{Op}^t = \Delta^{\frac{t}{2}} \mathrm{Op}^0 = \mathrm{Op}^0 \Delta^{\frac{t}{2}}.$$

$\square$

Corollary 3.1.17. *Let Δ be an invertible positive self-adjoint operator.*

(a) If $\Delta^{-\frac{1}{2}} \in \mathcal{K}$ then $\mathrm{Op}^{-t} \subseteq \mathcal{K}$ for $t > 0$.

(b) If $\Delta^{-\frac{1}{2}} \in \mathcal{L}^p$, $p \geq 1$ then $\mathrm{Op}^{-t} \subseteq \mathcal{L}^{p/t}$ for $0 < t \leq p$.

(c) If $\Delta^{-\frac{1}{2}} \in \mathcal{L}^{(p,\infty)}$, $p \geq 1$ then $\mathrm{Op}^{-t} \subseteq \mathcal{L}^{(p/t,\infty)}$ for $0 < t \leq p$.

Proof. Using the fact that $\mathrm{Op}^{-t} \subseteq \Delta^{-\frac{t}{2}} \mathcal{L}$, these follow from Lemma A.2.1, Lemma A.2.4(a) and Lemma A.2.7(a), respectively. $\square$

3.1.3 Algebra of Generalized Differential and Pseudodifferential Operators

In this subsection, we define and study algebras of generalized differential and pseudodofferential operators.

We start with the classical case.

Example 3.1.18 (Pseudodifferential Operators)**.** Let M be a closed manifold and let Δ be the closure of a strictly positive order-two elliptic partial differential operator acting on the smooth functions on M, as in Example 3.1.6.

We write $\Psi^t = \Psi^t(M)$ for the class of *pseudodifferential operators* of order at most t and let $\Psi = \Psi^\infty := \bigcup \Psi^t$ and $\Psi^{-\infty} := \bigcap \Psi^t$. Then Ψ is a filtered subalgebra of Op:

(a) $\Psi^s \subseteq \Psi^t$ for $s \leq t$.

(b) $\Psi^s \cdot \Psi^t \subseteq \Psi^{s+t}$ (cf. [Shu01, Theorem I.3.4]) and

(c) $\Psi^t \subseteq \mathrm{Op}^t$ (cf. [Shu01, Theorem I.7.1]).

Moreover, we have following extremely important property:

(d) $[\Psi^s, \Psi^t] \subseteq \Psi^{s+t-1}$ (follows from [Shu01, Theorem I.3.4]).

The subalgebra of *differential operators* is denoted $\mathcal{D} = \mathcal{D}(M)$. The filtration of $\mathcal{D}$ is usually indexed by $\mathbb{Z}_{\geq 0}$, but we can extend it to $\mathbb{R}$ by declaring

$$\mathcal{D}^t := \bigcup_{n \leq t,\, n \in \mathbb{Z}_{\geq 0}} \mathcal{D}^n, \quad t \in \mathbb{R}.$$

The subalgebra of *classical pseudodifferential operators* is denoted $\Psi_{\mathrm{cl}} = \Psi_{\mathrm{cl}}(M)$. Thus we have a chain of *filtered* algebras

$$\mathcal{D} \subset \Psi_{\mathrm{cl}} \subset \Psi \subset \mathrm{Op}.$$

We note that a pseudodifferential operator $P \in \Psi^t$ is classical if for every $l \in \mathbb{R}$, P may be decomposed as

$$P = X\Delta^{\frac{t-m}{2}} + R$$

with $X \in \mathcal{D}^m$ and $R \in \Psi^l$ (but *not* all classical differential operators have this property). In particular, for any $t \in \mathbb{R}$, the operator $\Delta^{\frac{t}{2}}$ belongs to $\Psi^t_{\mathrm{cl}} \subset \Psi^t$ and therefore

$$\begin{aligned} \Psi^t_{\mathrm{cl}} &= \Delta^{\frac{t}{2}}\Psi^0_{\mathrm{cl}} = \Psi^0_{\mathrm{cl}}\Delta^{\frac{t}{2}} \\ \Psi^t &= \Delta^{\frac{t}{2}}\Psi^0 = \Psi^0\Delta^{\frac{t}{2}} \end{aligned}$$

by Lemma 3.1.15 and

$$\begin{aligned} [\Delta^{\frac{1}{2}}, \Psi^t_{\mathrm{cl}}] &\subseteq \Psi^t_{\mathrm{cl}} \quad \text{and} \quad [\Delta, \Psi^t_{\mathrm{cl}}] \subseteq \Psi^{t+1}_{\mathrm{cl}} \\ [\Delta^{\frac{1}{2}}, \Psi^t] &\subseteq \Psi^t \quad \text{and} \quad [\Delta, \Psi^t] \subseteq \Psi^{t+1}. \end{aligned}$$

by property (d) above.

In the non-compact case, we have the following classical example.

Example 3.1.19 (Polynomial Weyl Algebra)**.** Consider the usual Lebesgue measure on $\mathbb{R}^n$ and let $\mathfrak{H} := L^2(\mathbb{R}^n)$. The operator

$$1 + \sum_{i=1}^{n} \left(x_i^2 - \frac{\partial^2}{\partial x_i^2} \right)$$

with domain $C_c^\infty(\mathbb{R}^n)$ of compactly supported smooth functions is called the *harmonic oscillator*. It is essentially self-adjoint (cf. [Shu01, Theorem IV.26.2]) and strictly positive.

Let Δ denote the closure. Then Δ is invertible positive self-adjoint operator with compact resolvent (cf. [Shu01, Theorem IV.26.3]). Let $\mathcal{W} = \mathcal{W}_n$ be the algebra of *polynomial* differential operators on $\mathbb{R}^n$ acting on $L^2(\mathbb{R}^n)$. As an algebra, it is generated by $x_1, \dots, x_n$ and $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. We filter $\mathcal{W}$ by requiring that

$$\operatorname{order} x_i := 1 \quad \text{and} \quad \operatorname{order} \frac{\partial}{\partial x_i} := 1$$

for any $1 \leq i \leq n$. (The nonzero degree of x_i compensates the noncompactness of $\mathbb{R}^n$.) Then $\mathcal{W}$

is a filtered subalgebra of Op (cf. [Shu01, Theorem IV.25.2]). Moreover, it satisfies the following stronger version of property 3.1.18(d):

(d′) $[\mathcal{W}^s, \mathcal{W}^t] \subseteq \mathcal{W}^{s+t-2}$.

In particular, we have

$$[\Delta, \mathcal{W}^t] \subseteq \mathcal{W}^t.$$

For a calculus of pseudodifferential and classical pseudodifferential operators on $\mathbb{R}^n$, see [Shu01, Chapter IV] or [Fed96, Chapter 3].

Generalizing these we make the following definition. Just as in the classical case, these provide a convenient framework to study index theoretic problems. See also [Hig06].

Definition 3.1.20. A *filtered* subalgebra $\mathcal{D} \subseteq \mathrm{Op}$ is called an *algebra of generalized differential operators* if $\mathcal{D}$ is closed under the derivation $[\Delta, -]$ and satisfies

$$[\Delta, \mathcal{D}^t] \subseteq \mathcal{D}^{t+1}, \quad t \in \mathbb{R}.$$

A *filtered* subalgebra $\Psi \subseteq \mathrm{Op}$ is called an *algebra of generalized pseudodifferential operators* if Ψ is closed under the derivation $[\Delta^{\frac{1}{2}}, -]$ and satisfies

$$[\Delta^{\frac{1}{2}}, \Psi^t] \subseteq \Psi^t, \quad t \in \mathbb{R}.$$

Remark 3.1.21. (1) We do *not* assume that Δ belongs to $\mathcal{D}$ or Ψ, see Example 3.1.23 below.

(2) The filtration on $\mathcal{D}$ is often indexed by the non-negative integers $\mathbb{Z}_{\geq 0}$. For all practical purposes, we may assume this is the case. If necessary, we can always extend the filtration to $\mathbb{R}$ by

$$\mathcal{D}^t := \bigcup_{n \leq t,\ n \in \mathbb{Z}_{\geq 0}} \mathcal{D}^n, \quad t \in \mathbb{R}.$$

(3) The filtrations on $\mathcal{D}$ or Ψ does *not* have to be the *subspace* filtrations. For instance the filtration on $\mathcal{D}$ can be defined algebraically (this is one place where it is important to have the filtration to be indexed by $\mathbb{Z}_{\geq 0}$), and the interplay between algebraic and analytic filtration will be crucial for our discussion.

Example 3.1.22. The algebras $\mathcal{D}(M)$, $\Psi_{\mathrm{cl}}(M)$, $\Psi(M)$ from Example 3.1.18 and $\mathcal{W}$ from Example 3.1.19 are examples of algebras of generalized differential operators. The algebras $\Psi_{\mathrm{cl}}(M)$ and $\Psi(M)$ are also algebras of generalized pseudodifferential operators.

Example 3.1.23 (Compactly Supported Operators)**.** Let M be a smooth manifold that is not necessarily compact and let Δ_0 be an order-two strictly positive elliptic partial differential operator acting on the compactly supported smooth functions $C_c^\infty(M)$ as in Example 2.1.7. Assume that M is complete for Δ_0 and let Δ denote the closure of Δ_0.

Let $\mathcal{D}_{\mathrm{cpt}} = \mathcal{D}_{\mathrm{cpt}}(M)$ be the algebra of compactly supported partial differential operators on M filtered by degree. Then $\mathcal{D}_{\mathrm{cpt}}$ is an algebra of generalized differential operators. In this example Δ is *not* an element of $\mathcal{D}_{\mathrm{cpt}}$.

Lemma 3.1.24. *Let $\Psi \subseteq \mathrm{Op}$ be an algebra of generalized pseudodifferential operators. Suppose that $\Delta^{\frac{1}{2}}$ belongs to Ψ^1. Then Ψ is an algebra of generalized differential operators.*

In fact, it is enough to assume that $\Delta^{\frac{1}{2}}\Psi^t \subseteq \Psi^{t+1}$ and/or $\Psi^t\Delta^{\frac{1}{2}} \subseteq \Psi^{t+1}$ for any $t \in \mathbb{R}$.

Proof. It follows from the Leibniz rule:

$$[\Delta, P] = [\Delta^{\frac{1}{2}}\Delta^{\frac{1}{2}}, P] = \Delta^{\frac{1}{2}}[\Delta^{\frac{1}{2}}, P] + [\Delta^{\frac{1}{2}}, P]\Delta^{\frac{1}{2}}.$$

□

In the other direction, we have the following lemma.

Lemma 3.1.25. *Let $\mathcal{D}$ be an algebra of generalized differential operators. Let Ψ^t_{cl} denote the space of all $P \in \mathrm{Op}$ such that for any $l \in \mathbb{R}$, P may be decomposed as*

$$P = X\Delta^{\frac{m}{2}} + Q$$

with $X \in \mathcal{D}$ and $\operatorname{order} X + m \leq t$ and $\operatorname{order} Q \leq l$. Then Ψ_{cl} is an algebra of generalized pseudodifferential operators.

First we prove an auxiliary lemma of independent interest.

Notation 3.1.26. Consider a filtered space. We write

$$P \sim \sum_{k\in\Lambda} P_k$$

if for any $l \in \mathbb{R}$, there exists a finite subset $F \subset \Lambda$ such that $P - \sum_{k\in F} P_k$ has order at most l.

Lemma 3.1.27. *Let $z \in \mathbb{C}$ and let $Y \in \mathrm{Op}$. Let $Y^{(0)} := Y$ and $Y^{(k)} := [\Delta, Y^{(k-1)}]$, $k \geq 1$. Then*

$$\Delta^z Y \sim \sum_{k=0}^{\infty} \binom{z}{k} Y^{(k)}\Delta^{z-k}.$$

Note that $\operatorname{order} Y^{(k)} \leq \operatorname{order} Y + k$, so $\operatorname{order}(Y^{(k)}\Delta^{z-k}) \leq \operatorname{order}(Y) + 2\operatorname{Re}(z) - k$.

Proof. Assume $\operatorname{Re}(z) < 0$. Let $R = (\lambda - \Delta)^{-1} \in \mathrm{Op}^{-2}$. Then

$$RY = YR + RY^{(1)}R.$$

Indeed, $0 = [R\cdot(\lambda - \Delta), Y]R = -R[\Delta, Y]R + [R, Y]$. Hence, for any $n \in \mathbb{Z}_{\geq 0}$,

$$RY = YR + Y^{(1)}R^2 + \ldots Y^{(n)}R^{n+1} + RY^{(n+1)}R^{n+1}.$$

Applying the Cauchy integral formula (Lemma 3.1.10), we see that

$$\Delta^z Y = \sum_{k=0}^{n} \binom{z}{k} Y^{(k)} \Delta^{z-k} + \frac{1}{2\pi i} \int \lambda^z (\lambda - \Delta)^{-1} Y^{(n+1)} (\lambda - \Delta)^{-n-1} d\lambda.$$

But the last integral converges absolutely in $\mathcal{L}(W^{s+l}, W^s)$ for $l = \operatorname{order} Y + n + 1 - 2(n+1)$, hence has order at most $\operatorname{order} Y - n - 1$.

For general $z \in \mathbb{C}$, the identity

$$\Delta^{z+1} Y = \Delta^z Y \Delta + \Delta^z Y^{(1)}$$

allows one to reduce to the case $\operatorname{Re}(z) < 0$. □

Proof of Lemma 3.1.25. It follows from Lemma 3.1.27 that Ψ_{cl} is a filtered subalgebra of Op. Moreover, for $P = X\Delta^{\frac{m}{2}} + Q$ in Ψ^t_{cl}, the identity

$$[\Delta^{\frac{1}{2}}, P] \sim \sum_{k=1}^{\infty} \binom{\frac{1}{2}}{k} X^{(k)} \Delta^{\frac{1+m}{2} - k} + [\Delta^{\frac{1}{2}}, Q]$$

shows that $[\Delta^{\frac{1}{2}}, P] \in \Psi^t_{\mathrm{cl}}$. □

Now we ready to define and study the algebra of generalized differential and pseudodifferential operators associated to a spectral triple.

Definition 3.1.28. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple with D invertible and let $\Delta = D^2$. We say that $(\mathcal{A}, \mathfrak{H}, D)$ is *regular* if there exists an algebra of generalized (pseudo)differential operators containing $\mathcal{A} \cup [D, \mathcal{A}]$ in degree 0.

Consider the unbounded derivation $\delta(b) := [\Delta^{\frac{1}{2}}, b]$ on $\mathcal{L}$. The *domain* $\operatorname{dom}(\delta)$ of δ is the set of $b \in \mathcal{L}$ such that

(i) $b \cdot \operatorname{dom}(\Delta^{\frac{1}{2}}) \subseteq \operatorname{dom}(\Delta^{\frac{1}{2}})$ and

(ii) $[\Delta^{\frac{1}{2}}, b]$ extends to a bounded operator on $\mathfrak{H}$.

For $k \geq 2$, we define $\operatorname{dom}(\delta^k)$ inductively as

$$\operatorname{dom}(\delta^k) := \{b \in \operatorname{dom}(\delta) \mid \delta(b) \in \operatorname{dom}(\delta^{k-1})\}$$

and let $\operatorname{dom}^\infty(\delta) := \bigcap_{k=1}^\infty \operatorname{dom}(\delta^k)$.

Suppose that Ψ is an algebra of generalized pseudodifferential operators in Op. Then Ψ^0 is contained in $\operatorname{dom}^\infty(\delta)$. Indeed, if $b \in \Psi^0 \subseteq \mathrm{Op}^0$, then $b \cdot \operatorname{dom}(\Delta^{\frac{1}{2}}) \subseteq \operatorname{dom}(\Delta^{\frac{1}{2}})$ and, being an operator of order 0, the commutator $[\Delta^{\frac{1}{2}}, b] \in \Psi^0 \subseteq \mathrm{Op}^0$ extends to a bounded operator on $\mathfrak{H}$. In other words, b belongs to $\operatorname{dom}(\delta)$. But Ψ^0 is closed under δ, therefore $\Psi^0 \subseteq \operatorname{dom}^\infty(\delta)$.

Hence, if $(\mathcal{A}, \mathfrak{H}, D)$ is a regular spectral triple then $\mathcal{A} \cup [D, \mathcal{A}] \subseteq \operatorname{dom}^\infty(\delta)$. In fact, often this condition is taken as the definition of regularity. We show that the two definitions are equivalent.

Let

$$\Psi^t := \Delta^{\frac{t}{2}} \mathrm{dom}^\infty(\delta), \quad t \in \mathbb{R}.$$

We claim that Ψ is an algebra of generalized pseudodifferential operators, proving that $(\mathcal{A}, \mathfrak{H}, D)$ is regular if $\mathcal{A} \cup [D, \mathcal{A}] \subseteq \mathrm{dom}^\infty(\delta)$.

First we show that Ψ^0 is a subset of Op^0.

Lemma 3.1.29. *Let* $\mathrm{Re}(z) < 0$*. Then for any* $b \in \Psi^0$ *and* $n \in \mathbb{Z}_{\geq 0}$*,*

$$\Delta^{\frac{z}{2}} b = \sum_{k=0}^{n} \binom{z}{k} \delta^k(b) \Delta^{\frac{z-k}{2}} + \frac{1}{2\pi i} \int \lambda^z (\lambda - \Delta^{\frac{1}{2}})^{-1} \delta^{n+1}(b) (\lambda - \Delta^{\frac{1}{2}})^{-n-1} d\lambda,$$

in $\mathcal{L}(\mathfrak{H})$*, where the integral is a contour integral along a downwards pointing vertical line in* $\mathbb{C}$ *which separates* 0 *from* $\mathrm{Spec}(\Delta^{\frac{1}{2}})$*.*

Proof. The proof of Lemma 3.1.27 goes through ad verbatim, when applied to $\Delta^{\frac{1}{2}}$. □

The last integral gives a bounded operator in $\mathcal{L}(\mathfrak{H}, W^{n+1})$ so taking n large enough we see that $||\Delta^{\frac{s}{2}} b\xi|| \leq C \cdot ||\Delta^{\frac{s}{2}} \xi||$ for some $C > 0$. Hence $b \cdot W^s \subseteq W^s$ for $s < 0$. For $s > 0$, the identity

$$\Delta^{\frac{s+1}{2}} b = \Delta^{\frac{s}{2}} b \Delta^{\frac{1}{2}} + \Delta^{\frac{s}{2}} \delta(b)$$

allows one to reduce to the case $s < 0$. Hence $\Psi^0 \subseteq \mathrm{Op}^0$. Now it is clear that $\Delta^{\frac{t}{2}} \Psi^0 \Delta^{-\frac{t}{2}} \subseteq \Psi^0$, for any $t \in \mathbb{R}$, so by Lemma 3.1.15, Ψ is a filtered subalgebra of Op and $\Delta^{\frac{t}{2}} \in \Psi^t$. Finally, for any $b \in \Psi^0$,

$$[\Delta^{\frac{1}{2}}, \Delta^{\frac{t}{2}} b] = \Delta^{\frac{t}{2}} \delta(b) \in \Psi^t,$$

hence Ψ is an algebra of generalized pseudodifferential operators. We note that Ψ is also an algebra of generalized differential operators by Lemma 3.1.24. Conversely if there exist an algebra of generalized differential operators containing $\mathcal{A} \cup [D, \mathcal{A}]$ in degree 0, then Lemma 3.1.25 gives an algebra of generalized pseudodifferential operators containing $\mathcal{A} \cup [D, \mathcal{A}]$ in degree 0.

We summarize the discussion above into a theorem.

Theorem 3.1.30 ([Hig04a], Theorem 3.25)**.** *Let* $(\mathcal{A}, \mathfrak{H}, D)$ *be a spectral triple with* D *invertible and let* $\Delta := D^2$*. Then the following conditions are equivalent:*

(1) There exists an algebra of generalized pseudodifferential operators containing $\mathcal{A} \cup [D, \mathcal{A}]$ *in degree* 0*.*

(2) There exists an algebra of generalized differential operators containing $\mathcal{A} \cup [D, \mathcal{A}]$ *in degree* 0*.*

(3) The set $\mathcal{A} \cup [D, \mathcal{A}]$ *is contained in* $\mathrm{dom}^\infty(\delta)$*.*

A minimal example of an algebra of generalized differential operators can be obtained as follows. Let $\mathcal{D}^0$ denote the subalgebra of Op^0 generated by $\mathcal{A} \cup [D, \mathcal{A}]$ and let

$$\mathcal{D}^k := \sum_{j=1}^{k-1} \mathcal{D}^j \mathcal{D}^{k-j} + [\Delta, \mathcal{D}^{k-1}] + \mathcal{D}^0[\Delta, \mathcal{D}^{k-1}] \subseteq \mathrm{Op}, \quad k \geq 1.$$

Then by induction on k, we have $\mathcal{D}^k \subseteq \Psi^k$ and $\mathcal{D}$ is an algebra of generalized differential operators. We denote this algebra by $\mathcal{D}(\mathcal{A}, \mathfrak{H}, D)$.

Definition 3.1.31. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a regular spectral triple with D invertible and let $\Delta = D^2$. We call $\mathcal{D}(\mathcal{A}, \mathfrak{H}, D)$ the *algebra of generalized differential operators* associated to $(\mathcal{A}, \mathfrak{H}, D)$. The *algebra of generalized pseudodifferential operators* associated to $(\mathcal{A}, \mathfrak{H}, D)$ is $\Psi_{\mathrm{cl}}(\mathcal{A}, \mathfrak{H}, D)$ constructed from $\mathcal{D}(\mathcal{A}, \mathfrak{H}, D)$ by Lemma 3.1.25.

Proposition 3.1.32. *The sum and product of regular spectral triples are again regular.*

Proof. Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be regular spectral triples and let $\mathcal{D}_1$ and $\mathcal{D}_2$ denote the algebras of generalized differential operators respectively. Then $\mathcal{D}_1 \oplus \mathcal{D}_2$ is the algebra of generalized differential operators for $\mathfrak{D}_1 \oplus \mathfrak{D}_2$ and $\mathcal{D}_1 \otimes \mathcal{D}_2$ is the algebra of generalized differential operators for $\mathfrak{D}_1 \times \mathfrak{D}_2$. ∎

3.2 Meromorphic Continuation

3.2.1 Duhamel Algebra

Here we introduce an algebraic tool designed to simplify certain computations common to index cocycles. We already used some of the ideas in Subsection 3.1.3.

Definition 3.2.1. Let $\mathcal{D}$ be a filtered algebra and let $\partial : \mathcal{D} \to \mathcal{D}$ be a derivation of order one, that is, $\partial(XY) = \partial(X)Y + X\partial(Y)$ for any X, $Y \in \mathcal{D}$ and $\partial(\mathcal{D}^\bullet) \subseteq \mathcal{D}^{\bullet+1}$. Let R be a formal variable of order -2. Let $\mathcal{D} * \mathbb{C}[R]$ denote the coproduct of $\mathcal{D}$ and $\mathbb{C}[R]$ in the category of associative filtered algebras over $\mathbb{C}$. We define the *Duhamel algebra* associated to $(\mathcal{D}, \partial)$ as

$$\mathcal{D}_R := \mathcal{D} * \mathbb{C}[R] / \langle RX - XR - R\partial(X)R \rangle .$$

Notice that the order of $R\partial(X)R$ is at most $\operatorname{order}(X) - 1$.

Notation 3.2.2. For $X \in \mathcal{D}$, we write $X^0 := X$ and $X^{(k)} := \partial(X^{(k-1)})$ for $k \geq 1$.

Definition 3.2.3. We call the relation

$$[R, X] = RX^{(1)}R, \quad X \in \mathcal{D}$$

the *Duhamel relation.*

The Duhamel algebra has the following universal property.

Lemma 3.2.4. *Let Ψ be a filtered algebra and let $\partial : \Psi \to \Psi$ be a derivation of order one. Let $R \in \Psi$ be an element of order -2 satisfying the Duhamel relation with respect to ∂. Then any morphism $\mathcal{D} \to \Psi$ intertwining the derivations extends to an algebra map $\mathcal{D}_R \to \Psi$.* $\square$

Recall that we write

$$P \sim \sum_{k \in \Lambda} P_k$$

if for any $l \in \mathbb{R}$, there exists a finite subset $F \subset \Lambda$ such that $P - \sum_{k \in F} P_k$ has order at most l.

Lemma 3.2.5 (Taylor Expansion)**.** *Let $n \geq 1$. Then for any $X \in \mathcal{D}_R$, we have*

$$R^n X \sim \sum_{k=0}^{\infty} \binom{n+k-1}{k} X^{(k)} R^{n+k}.$$

Notice that the order of $X^{(k)} R^{n+k}$ is $\operatorname{order} X + k - 2(n+k) = \operatorname{order} X - 2n - k$. See [CM95, Theorem B.1].

Proof. Let $n = 1$. Then as we saw in the proof of Lemma 3.1.27,

$$\begin{aligned} RX &= XR + RX^{(1)}R \\ &= XR + X^{(1)}R^2 + RX^{(2)}R^2 \\ &= XR + X^{(1)}R^2 + X^{(2)}R^3 + RX^{(3)}R^3 \\ &\ \ \vdots \\ &\sim \sum_{k=0}^{\infty} X^{(k)} R^{k+1}. \end{aligned}$$

For $n \geq 2$, first notice that the equality

$$\sum_{k_0=0}^{k} \binom{n-1+k_0-1}{k_0} = \binom{n+k-1}{k}$$

can easily be shown by induction on k. Hence, again by induction (on n), we see

$$\begin{aligned} R^n X &= RR^{n-1}X \\ &\sim \sum_{k_0=0} \binom{n-1+k_0-1}{k_0} RX^{(k_0)} R^{n-1+k_0} \\ &\sim \sum_{k_0=0, k_1=0} \binom{n-1+k_0-1}{k_0} X^{(k_0+k_1)} R^{n+k_0+k_1} \\ &\sim \sum_{k=0} \binom{n+k-1}{k} X^{(k)} R^{n+k}. \end{aligned}$$ $\square$

As a corollary, we get the following.

Lemma 3.2.6. *For any $X^0, \dots, X^p$ in $\mathcal{D}$, we have*

$$X^0 R \dots X^p R \sim \sum_{k \geq 0} c(k) X^0 X^{1(k_1)} \dots X^{p(k_p)} R^{|k|+p+1} \quad \textit{in } \mathcal{D}_R,$$

where $|k| := k_1 + \dots + k_p$ and the coefficients $c(k)$ are defined by the formula

$$c(k) := \frac{(k_1 + \dots + k_p + p)!}{k_1! \dots k_p!(k_1+1)(k_1+k_2+2)\dots(k_1+\dots+k_p+p)}.$$

See [Hig06, Proof of Proposition 4.14].

Proof. There is nothing to check if $p = 0$. For $p \geq 1$, writing $k = (k', k_p)$, we see that

$$c(k) = c(k') \binom{|k|+p-1}{k_p} = c(k') \binom{|k'|+p+k_p-1}{k_p}$$

and hence by induction on p and the Taylor Expansion 3.2.5, we get

$$\begin{aligned} X^0 R \dots X^p R &= (X^0 R \dots X^{p-1} R) X^p R \\ &\sim \sum_{k' \geq 0} c(k') X^0 X^{1(k_1)} \dots X^{p-1(k_{p-1})} R^{|k'|+p-1+1} X^p R \\ &\sim \sum_{k \geq 0} c(k') \binom{|k'|+p+k_p-1}{k_p} X^0 X^{1(k_1)} \dots X^{p(k_p)} R^{|k'|+p+k_p+1} \\ &\sim \sum_{k \geq 0} c(k) X^0 X^{1(k_1)} \dots X^{p(k_p)} R^{|k|+p+1}. \end{aligned}$$

□

Now we introduce a notation.

Definition 3.2.7. For $X^0, \dots, X^p \in \mathcal{D}$, let $\langle X^0, \dots, X^p \rangle$ denote the element $X^0 R \dots X^p R$ in $\mathcal{D}_R$.

Then $\langle X^0, \dots, X^p \rangle \langle Y^0, \dots, Y^q \rangle = \langle X^0, \dots, X^p, Y^0, \dots, Y^q \rangle$.

Remark 3.2.8. We can write 3.2.6 as

$$\langle X^0, \dots, X^p \rangle \sim \sum_{k \geq 0} c(k) \langle X^0 X^{1(k_1)} \dots X^{p(k_p)}, \underbrace{1, \dots, 1}_{|k|+p} \rangle.$$

Notice that the infinite sum on the right-hand side can be changed to a finite sum at the expense of adding remainder terms of arbitrary low order.

3.2.2 Meromorphic Continuation

One of the crucial ingredients of the local index theory is the theory of local traces. In the classical case of a manifold, this trace was studied by Manin[Man78], Adler[Adl79], Guillemin[Gui85] and

Wodzicki[Wod84, Wod87]. We dedicate this section to the *spectral zeta function* and the *spectral xi function* associated to an algebra of generalized differential operators.

Definition 3.2.9. Let $\mathcal{D}$ be an algebra of generalized differential operators in $\mathrm{Op}(\Delta)$. We say that $\mathcal{D}$ has *analytic dimension* $\dim(\mathcal{D}) \geq 0$ if for any $\mathrm{order}\,(X) \leq q$, the operator $X\Delta^{-z}$ is of trace-class for $\mathrm{Re}(z) > \frac{q+\dim(\mathcal{D})}{2}$.

We say that $\mathcal{D}$ has the *meromorphic continuation property* if, furthermore, for any $X \in \mathcal{D}$ the *spectral zeta function* $\zeta_X(z) := \mathrm{Tr}(X\Delta^{-\frac{z}{2}})$ extends to a meromorphic function on the full complex plane.

The *dimension spectrum* of $\mathcal{D}$ is the smallest set $\Sigma(\mathcal{D})$ such that the poles of ζ_X, $X \in \mathcal{D}$ are contained in $\Sigma(\mathcal{D}) + \mathrm{order}\,(X)$.

We say that $\mathcal{D}$ has *discrete dimension spectrum* if $\Sigma(\mathcal{D})$ is a discrete subset of $\mathbb{C}$ and *simple dimension spectrum* if all poles of ζ_X, $X \in \mathcal{D}$ are simple.

The classical example is the algebra of differential operators on a closed manifold:

Example 3.2.10 (Closed Manifold)**.** Let M be an even dimensional closed manifold. In the situation of Example 3.1.18, the algebra of differential operators $\mathcal{D}(M)$ has analytic dimension $\dim(M)$. Moreover it has the meromorphic continuation property with simple and discrete dimension spectrum $\{0, \ldots, \dim(M)\}$.

The following criterion for the meromorphic continuation property gives many generalizations.

Theorem 3.2.11 ([Hig04b], Theorem 6.6)**.** *Let Δ be an invertible positive self-adjoint operator and let $\mathcal{D} \subseteq \mathrm{Op}(\Delta)$ be an algebra of generalized differential operators with finite analytic dimension. Let $\mathcal{D}^+ := \mathcal{D} \cup \{\Delta\}$. Suppose that there exists elements $A_1, \ldots, A_N$ and $B_1, \ldots, B_N$ in $\mathcal{D}$ such that*

(i) $\mathrm{order}\,[X, A_i] + \mathrm{order}\,[Y, B_i] \leq \mathrm{order}\, X + \mathrm{order}\, Y - 1$ *for all X, $Y \in \mathcal{D}^+$ and*

(ii) $\sum_{i=1}^N [B_i, A_i] = d$ *for some real number $d \in \mathbb{R}$ and*

(iii) if $X \in \mathcal{D}^+$ then

$$\sum_{i=1}^N [X, A_i]B_i = \mathrm{order}\,(X)X + T$$

for some T of lower order, i.e. $\mathrm{order}\, T \leq \mathrm{order}\, X - 1$.

Then $\mathcal{D}$ has the meromorphic continuation property and has a discrete and simple dimension spectrum: $\Sigma(\mathcal{D}) \subseteq d - \mathbb{Z}_{\geq 0}$. Moreover, if

(iv) the filtration $\mathcal{D}^\bullet$ is positive, that is $\mathcal{D}^{\leq -1} = 0$,

then for $X \in \mathcal{D}^q$, the spectral xi function

$$z \mapsto \Gamma(z)\mathrm{Tr}(X\Delta^{-z})$$

has simple poles in $\frac{1}{2}\Sigma(\mathcal{D}^q)$. In particular, $\Sigma(\mathcal{D}^q) \subseteq \{d + q,\ d + q - 1, \ldots\} \backslash \mathbb{Z}_{\leq 0}$. □

Before we prove the theorem, we look at a couple of examples. The first one is really the model behind the theorem.

Example 3.2.12. Let $\mathcal{W}_n$ denote the polynomial Weyl algebra as in Example 3.1.19. Then

$$A_i = x_i \quad \text{and} \quad B_i = \frac{\partial}{\partial x_i}$$

satisfy the conditions of Theorem 3.2.11. Obviously, the filtration on $\mathcal{W}_n$ is positive. Here the operator

$$\sum_{i=1}^{n} [-, A_i] B_i : \mathcal{W}_n \to \mathcal{W}_n$$

play the role of the *Euler vector field* on a manifold.

Example 3.2.13. Let M be a closed manifold and let Δ be an invertible positive second-order elliptic partial differential operator. Let $\mathcal{D}_{\mathrm{c}}(M)$ denote the algebra of compactly supported differential operators on M as in Example 3.1.22. Let $\{\phi_\alpha\}$ be a smooth partition of unity subordinate to a finite cover $\{U_\alpha\}$ of M by coordinate charts. Choose functions ψ_α supported in U_α for which $\phi_\alpha \psi_\alpha = \phi_\alpha$. Then the collection

$$A_{\alpha,i} \coloneqq \psi_\alpha x_i \quad \text{and} \quad B_{\alpha,i} = \phi_\alpha \frac{\partial}{\partial x_i}$$

satisfies the conditions of Theorem 3.2.11 with $d = \dim M$. If $\dim M$ is even, we see that the poles of the spectral zeta function associated to $X \in \mathcal{D}^q(M)$ are located in $\{0, 1, \ldots, \dim M + q\}$.

We consider the example of Connes-Moscovici, again following Higson [Hig04b].

Example 3.2.14. Let P be a para-Riemannian manifold. See Section 6.1 for more detail. We filter the algebra $\mathcal{D}_{\mathrm{c}}(P)$ of compactly supported linear partial differential operators on P by

(i) functions are of order 0

(ii) vector fields are of order at most 2

(iii) vertical vector fields are of order at most 1.

We say that a coordinate chart $(x_1, \ldots, x_{v+n})$ is a *foliation chart* if $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_v}$ spans the vertical bundle V, where $v = \dim V$ and $n = \dim N$. We assume that P has a finite cover of foliation charts.

Let $\Delta : C_{\mathrm{c}}^\infty(P) \to C_{\mathrm{c}}^\infty(P)$ be a scalar *hypoelliptic* operator of order 2, *i.e.* the principal symbol σ_Δ satisfies

$$|\sigma_\Delta(x,\xi)| \geq \varepsilon_x \cdot \left(|\xi_1|^4 + \cdots + |\xi_v|^4 + |\xi_{v+1}|^2 + \cdots + |\xi_{v+n}|^2 \right)^{\frac{1}{2}}$$

for some $\varepsilon_x > 0$, for any foliation chart. Then $(\mathcal{D}_{\mathrm{c}}(P), \Delta)$ is an algebra of generalized differential operators and using the notation from Example 3.2.13, but in a foliation chart, the collection

$$\begin{gathered}\{A_{\alpha,1},\dots,A_{\alpha,v},A_{\alpha,v+1},A_{\alpha,v+1},\dots,A_{\alpha,v+n},A_{\alpha,v+n}\} \quad \text{and}\\ \{B_{\alpha,1},\dots,B_{\alpha,v},B_{\alpha,v+1},B_{\alpha,v+1},\dots,B_{\alpha,v+n},B_{\alpha,v+n}\}\end{gathered}$$

gives the necessary operators for Theorem 3.2.11. Here, all the transverse operators are listed twice.

As a corollary, the poles of the spectral zeta function associated to $X \in \mathcal{D}^q_c(P)$ are located in $\{0, 1, \dots, d+q\}$, where $d = v + 2n$.

Now we proceed with the proof.

Let $\operatorname{Spec}(D)$ denote the spectrum of D and let

$$R_\lambda \coloneqq (\lambda - \Delta)^{-1}, \quad \lambda \notin \operatorname{Spec}(D)$$

denote the resolvent.

Lemma 3.2.15. *The operators R_λ satisfy the Duhamel relation for $[\Delta, -]$:*

$$[R_\lambda, X] = R_\lambda[\Delta, X]R_\lambda.$$

Proof. Applying the Leibniz rule to $1 = R_\lambda \cdot (\lambda - \Delta)$, we get

$$0 = [R_\lambda(\lambda - \Delta), X]R_\lambda = [R_\lambda, X] - R_\lambda[\Delta, X]R_\lambda. \qquad \square$$

Definition 3.2.16. Let $X^0, \dots, X^p \in \mathcal{D}$ be given. Define[2]

$$\begin{gathered}\langle X^0, \dots, X^p\rangle^{\mathrm{H}}_\lambda \coloneqq X^0R_\lambda X^1R_\lambda \dots R_\lambda X^pR_\lambda \qquad \text{and}\\ \langle X^0, \dots, X^p\rangle^{\mathrm{CM}}_z \coloneqq (-1)^p\frac{\Gamma(z)}{2\pi i}\int \lambda^{-z}\langle X^0, \dots, X^p\rangle^{\mathrm{H}}_\lambda d\lambda,\end{gathered}$$

where the integral is evaluated down a vertical line separating 0 and $\operatorname{Spec}(\Delta)$.

If $q = \sum_{i=0}^p \operatorname{order} X^i$ then we say that the product $\langle X^0, \dots, X^p\rangle^{\mathrm{H}}_\lambda$ has *formal order* $r = q - 2(p+1)$ and the integral $\langle X^0, \dots, X^p\rangle^{\mathrm{CM}}_z$ has *type* $l = q - 2p$.

Remark 3.2.17. By functional calculus, the resolvent $R_\lambda = (\lambda - \Delta)^{-1}$ is a bounded operator on any of the Sobolev spaces W^s, with norm bounded by $|\operatorname{Im}(\lambda)|^{-1}$. Hence if $q = \sum \operatorname{order} X^i$, then the integral absolutely converges in $\operatorname{Op}^q(\Delta)$ whenever $\operatorname{Re}(z) + (p+1) > 1$. Indeed, for λ large, the integrand has norm bounded by

$$C \cdot |\operatorname{Im}(\lambda)|^{-(\operatorname{Re}(z)+p+1)}$$

[2]The superscript H stands for Nigel Higson and CM stands for Connes-Moscovici. Higson first noticed the importance of the expressions in relation to the local index theorem. Quillen also used similar expressions in [Qui88a].

as an operator $W^{s+q} \to W^s$, for some scalar constant $C > 0$, whereas for λ small, we can bound the norm by a constant which depends on $\mathrm{Re}(\lambda)$ and C.

Moreover, the resolvents R_λ can also be seen as an operator of order -2, with norm bounded by a constant which depends only on $\mathrm{Re}(\lambda)$. Hence, for any decomposition $p = p_1 + p_2$, with p_1, $p_2 \in \mathbb{Z}_{\geq 0}$, we have that the integral

$$\int \lambda^{-z} \langle X^0, \ldots, X^p \rangle_\lambda^{\mathrm{H}} d\lambda$$

gives an operator of order $q - 2p_1$ for $\mathrm{Re}(z) + p_2 > 0$.

Remark 3.2.18. The $(-1)^p$ sign and the Gamma function will be useful in Subsection 4.3.4 when we consider cyclic cocycles. Also notice that $\mathrm{Tr}\langle X \rangle_z^{\mathrm{CM}} = \Gamma(z)\mathrm{Tr}(X\Delta^{-z})$ is the xi function.

Example 3.2.19. We have, for $p \in \mathbb{Z}_{\geq 0}$,

$$\langle X, \underbrace{1, \ldots, 1}_{p} \rangle_z^{\mathrm{CM}} = \frac{\Gamma(z+p)}{p!} X\Delta^{-z-p} = \frac{1}{p!} \langle X \rangle_{z+p}^{\mathrm{CM}}.$$

Indeed, it is enough to combine the Cauchy integral formula

$$\frac{1}{2\pi i} \int \lambda^{-z} \frac{d\lambda}{(\lambda - \Delta)^{p+1}} = \binom{-z}{p} \Delta^{-z-p}$$

with the *function equation*

$$(-1)^p \Gamma(z) \binom{-z}{p} = \frac{\Gamma(z+p)}{p!}.$$

This integral has type order $(X) - 2p$.

In general we have the following.

Lemma 3.2.20. *We have*

$$\begin{aligned} \langle X^0, \ldots, X^p \rangle_z^{\mathrm{CM}} &\sim \sum_{k \geq 0} (-1)^{|k|+p-p} c(k) \langle X^0 X^{1(k_1)} \ldots X^{p(k_p)}, \underbrace{1, \ldots, 1}_{|k|+p} \rangle_z^{\mathrm{CM}} \\ &= \sum_{k \geq 0} (-1)^{|k|} \frac{\Gamma(z+p+|k|)}{(p+|k|)!} c(k) X^0 X^{1(k_1)} \ldots X^{p(k_p)} \Delta^{-z-p-|k|}. \end{aligned}$$

where $\sim$ means that large finite sums of the right-hand side is equal to the left-hand side modulo integrals of arbitrary small type.

Proof. Enough to integrate the identity in Corollary 3.2.6. □

Lemma 3.2.21. *An integral $\langle X^0, \ldots, X^p \rangle_z^{\mathrm{CM}}$ of type l has analytic order $\leq l - 2\,\mathrm{Re}(z)$ and the function $z \mapsto \mathrm{Tr}\langle X^0, \ldots, X^p \rangle_z^{\mathrm{CM}}$ extends to a holomorphic function on the right half-plane* $\mathrm{Re}(z) \geq \frac{l + \dim(\mathcal{D})}{2}$.

Proof. The k-th term has analytic order at most $q+|k|-2(\operatorname{Re}(z)+p+|k|) = l-2\operatorname{Re}(z)-|k|$ and is of trace-class for $\operatorname{Re}(z)+p+|k| > \frac{q+|k|+\dim(\mathcal{D})}{2}$. Since the Gamma function has no poles in this region, we are done. □

The rest of this subsection is devoted to the proof of the following generalization of Theorem 3.2.11.

Theorem 3.2.22. *Let $\mathcal{D}$ be an algebra of generalized differential operators satisfying the conditions of Theorem 3.2.11 and let $\langle X^0,\dots,X^p\rangle_z^{\mathrm{CM}}$ be an integral of type l. Then the function $z \mapsto \operatorname{Tr}\langle X^0,\dots,X^p\rangle_z^{\mathrm{CM}}$ admits a meromorphic extension to $\mathbb{C}$ with simple poles in $\frac{1}{2}(d+l-\mathbb{Z}_{\geq 0})$.* □

We start with a simple lemma.

Lemma 3.2.23. *The basic estimate*

$$\operatorname{order}[X,A] + \operatorname{order}[Y,B] \leq \operatorname{order} X + \operatorname{order} Y - 1$$

holds for all X, Y in the algebra generated by $\mathcal{D}\cup\{\Delta, R_\lambda\}$.

Proof. Indeed, for any $C \in \mathcal{D}$

$$\operatorname{order}[R_\lambda, C] = \operatorname{order} R_\lambda C^{(1)} R_\lambda \leq \operatorname{order}[\Delta, C] - 4$$

and thus

$$\begin{aligned}
\operatorname{order}[R_\lambda, A] + \operatorname{order}[Y,B] &\leq \operatorname{order}[\Delta, A] + \operatorname{order}[Y,B] - 4 \leq (-2) + \operatorname{order} Y - 1\\
\operatorname{order}[X,A] + \operatorname{order}[R_\lambda, B] &\leq \operatorname{order}[X,A] + \operatorname{order}[\Delta, B] - 4 \leq \operatorname{order} X + (-2) - 1\\
\operatorname{order}[R_\lambda, A] + \operatorname{order}[R_\lambda, B] &\leq \operatorname{order}[\Delta, A] + \operatorname{order}[\Delta, B] - 8 \leq (-2) + (-2) - 1.
\end{aligned}$$

□

Definition 3.2.24. Let

$$\mathscr{R}(Z) \coloneqq \sum_{i=1}^{N} [Z, A_i] B_i, \quad \text{for } Z \in \mathrm{Op}.$$

Lemma 3.2.25. *The operator $\mathscr{R}$ is a derivation of the algebra generated by $\mathcal{D}\cup\{\Delta, R_\lambda\}$ modulo lower order terms in the following sense:*

$$\mathscr{R}(XY) = \mathscr{R}(X)Y + X\mathscr{R}(Y)$$

modulo terms of order $\operatorname{order} X + \operatorname{order} Y - 1$.

Proof. Suppressing the summation from writing, we have

$$\mathscr{R}(XY) = [XY, A]B = [X,A]YB + X[Y,A]B = \mathscr{R}(X)Y + X\mathscr{R}(Y) + [X,A][Y,B]$$

but order $[X, A]$ + order $[Y, B] \leq$ order X + order $Y - 1$. $\square$

Using the condition (iii) of Theorem 3.2.11 we get the following.

Corollary 3.2.26. *Let $q = \sum_{i=0}^{p} \operatorname{order} X^i$ and let $r = q - 2(p+1)$. Then*

$$\mathscr{R}\langle X^0, \ldots, X^p\rangle_\lambda^{\mathrm{H}} = q\langle X^0, \ldots, X^p\rangle_\lambda^{\mathrm{H}} + 2\sum_{j=0}^{p}\langle X^0, \ldots, X^j, \Delta, X^{j+1}, \ldots, X^p\rangle_\lambda^{\mathrm{H}}$$

modulo terms of order $\leq r - 1$.

Proof. We need to show that $\mathscr{R}(R_\lambda) = 2R_\lambda \Delta R_\lambda$ modulo terms of order $-2-1 = -3$. But

$$[R_\lambda, A]B = R_\lambda A^{(1)} R_\lambda B = R_\lambda A^{(1)} B R_\lambda + R_\lambda A^{(1)} R_\lambda B^{(1)} R_\lambda = 2R_\lambda \Delta R_\lambda$$

modulo terms of order $\max\{-2+1-2, \operatorname{order}[\Delta, A] + \operatorname{order}[\Delta, B] - 6\} = -3$. $\square$

Lemma 3.2.27. *We have*

$$\sum_{j=0}^{p}\langle X^0, \ldots, X^j, 1, X^{j+1}, \ldots, X^p\rangle_z^{\mathrm{CM}} = \langle X^0, \ldots, X^p\rangle_{z+1}^{\mathrm{CM}}$$
$$\sum_{j=0}^{p}\langle X^0, \ldots, X^j, \Delta, X^{j+1}, \ldots, X^p\rangle_z^{\mathrm{CM}} = (z+p)\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}$$

Proof. See [Hig04a] Lemma 5.1. and Lemma 7.7. $\square$

Now we relate the order to analytic order.

Combining Lemma 3.2.27 with Corollary 3.2.26, we get the following.

Lemma 3.2.28. *If $\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}$ is an integral of type l,*

$$\mathscr{R}\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}} = (l - 2z)\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}$$

module integrals of type $l - 1$.

Proof. We have

$$\begin{aligned}\mathscr{R}\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}} &= q\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}} - 2\sum_{j=0}^{p}\langle X^0, \ldots, X^j, \Delta, X^{j+1}, \ldots, X^p\rangle_z^{\mathrm{CM}} \\ &= (q - 2(z+p))\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}\end{aligned}$$

modulo integrals of type $l - 1$. $\square$

Now we are ready to give the proof of Theorem 3.2.11

Proof of Theoerm 3.2.11. We have

$$\begin{aligned}\mathscr{R}(Z) + dZ &= \sum_{i=1}^{N}[Z, A_i]B_i + \sum_{i=1}^{N}[B_i, A_i]Z \\ &= \sum_{i=1}^{N}[Z, A_iB_i] + \sum_{i=1}^{N}[B_i, A_iZ]\end{aligned}$$

for any Z. Hence taking $Z = \langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}$, and using the fact that the commutators have vanishing trace, we see that the function $z \mapsto \mathrm{Tr}\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}$ extends to a meromorphic map to the whole complex plane, with simple poles in

$$\left\{\frac{d+l}{2}, \frac{d+l-1}{2}, \ldots\right\}.$$

□

Now let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple.

Definition 3.2.29. We say that $(\mathcal{A}, \mathfrak{H}, D)$ has the *integral meromorphic continuation property* if it is regular and $\mathcal{D}(\mathcal{A}, \mathfrak{H}, D)$ satisfies the conditions of Theorem 3.2.11 with $d \in \mathbb{Z}$.

Proposition 3.2.30. *The sum and product of spectral triples with the integral meromorphic continuation property again have the integral meromorphic continuation property.*

Proof. Regularity is guaranteed by Proposition 3.1.32. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be the corresponding algebras of generalized differential operators and let $A_{11}, \ldots, A_{1N}$; $B_{11}, \ldots, B_{1N}$ and $A_{21}, \ldots, A_{2M}$; $B_{21}, \ldots, B_{2M}$ be the elements required in Theorem 3.2.11. The the lists

$$\begin{aligned}&A_{11} \oplus 0, \ldots, A_{1N} \oplus 0, 0 \oplus A_{21}, \ldots, 0 \oplus A_{2M};\\ &B_{11} \oplus 0, \ldots, B_{1N} \oplus 0, 0 \oplus B_{21}, \ldots, 0 \oplus B_{2M}\end{aligned}$$

and

$$\begin{aligned}&A_{11} \otimes 1, \ldots, A_{1N} \otimes 1, 1 \otimes A_{21}, \ldots, 1 \otimes A_{2M};\\ &B_{11} \otimes 1, \ldots, B_{1N} \otimes 1, 1 \otimes B_{21}, \ldots, 1 \otimes B_{2M}\end{aligned}$$

give the required elements. □

Chapter 4

Index Formulas

This chapter concerns various instances of index formulas for spectral triples.

In section 1 we review cyclic theory. Connes developed the cyclic cohomology theory [Con85] as an analogue of the de Rham theory for noncommutative algebras. Around the same time Loday-Quillen and Tsygan discovered the cyclic homology theory while studying the Lie algebra homology of matrix algebras [LQ84, Tsy83]. Cyclic theory allows us to give analytic solutions to index problems.

In the next two sections we give examples of these analytic solutions, called index cocycles. In order to distinguish these examples, we give them names, but we note that these names are *not* standard and so are the notations.

4.1 Cyclic Theory

We briefly review cyclic theory à la Cuntz-Quillen [CQ95a, CQ95b]. See [Lod92] and [Con94] for a comprehensive treatment.

4.1.1 Definition of Cyclic Cohomology

We consider a *unital* associative algebra $\mathcal{A}$ over $\mathbb{C}$.

Definition 4.1.1. The *universal enveloping differential graded algebra* $\Omega(\mathcal{A}) = (\Omega^\bullet(\mathcal{A}), d)$ of $\mathcal{A}$ is the universal differential graded algebra generated by $\mathcal{A}$ in degree 0.

Thus we have a map $\imath : \mathcal{A} \to \Omega^0(\mathcal{A}) \subset \Omega(\mathcal{A})$ with the property that for any differential graded algebra Ω and a map of algebras $\mathcal{A} \to \Omega^0$ there exists a unique map of differential graded algebras $\Omega(\mathcal{A}) \to \Omega$ such that the diagram

$$\begin{tikzcd}
 & & \Omega(\mathcal{A}) \arrow[d, dashed] \\
\mathcal{A} \arrow[r] \arrow[urr, "\imath"] & \Omega^0 \arrow[r] & \Omega
\end{tikzcd}$$

commutes. By universality, $\Omega(\mathcal{A})$ is unique up to a canonical isomorphism, allowing us to talk about "the" universal enveloping differential graded algebra by abuse of language.

A particular model can be constructed as follows. Let $\mathcal{A}/\mathbb{C}$ denote the quotient space of $\mathcal{A}$ by the scalar multiples of the unit and let $d : \mathcal{A} \to \mathcal{A}/\mathbb{C}$ denote the quotient map. For $p \geq 0$, let

$$\Omega^p(\mathcal{A}) \coloneqq \mathcal{A} \otimes (\mathcal{A}/\mathbb{C})^{\otimes p}$$

and define $d : \Omega^p(\mathcal{A}) \to \Omega^{p+1}(\mathcal{A})$ by

$$d(a^0 \otimes da^1 \otimes \cdots \otimes da^p) \coloneqq 1 \otimes da^0 \otimes da^1 \otimes \cdots \otimes da^p.$$

The product $\Omega^p(\mathcal{A})\otimes\Omega^q(\mathcal{A}) \to \Omega^{p+q}\mathcal{A}$ is defined by induction on p in such a way that the Leibniz rule holds:

for $p = 0$,

$$a^0 \cdot (b^0 \otimes db^1 \otimes \cdots \otimes db^q) \coloneqq a^0 b^0 \otimes db^1 \otimes \cdots \otimes db^q \quad \text{and}$$

for $p \geq 1$,

$$\begin{aligned}(a^0 \otimes da^1 \otimes \cdots \otimes da^p) \cdot (b^0 \otimes db^1 \otimes \ldots \otimes db^q) \coloneqq{}& a^0 \otimes da^1 \otimes \cdots \otimes d(a^p b^0) \otimes db^1 \otimes \cdots \otimes db^q \\ &- (a^0 \otimes da^1 \otimes \cdots \otimes da^{p-1}) \cdot (a^p \otimes db^0 \otimes \cdots \otimes db^q).\end{aligned}$$

Then

$$\Omega(\mathcal{A}) \coloneqq \left(\bigoplus_{p \geq 0} \Omega^p(\mathcal{A}), d \right)$$

is an example of an universal enveloping differential graded algebra for $\mathcal{A}$ (cf. [CQ95a, Proposition 1.1]). From now on, we simply write

$$a^0 da^1 \ldots da^p \quad \text{or} \quad (a^0, a^1, \ldots, a^p)$$

for the element $a^0 \otimes da^1 \otimes \cdots \otimes da^p$ of $\Omega^p(\mathcal{A})$. Also, when there is no risk for confusion, we write Ω for $\Omega(\mathcal{A})$ for brevity.

A more conceptual construction due to Drinfel'd is given in Example 5.1.2.

Remark 4.1.2. It follows that the map $\imath : \mathcal{A} \to \Omega^0(\mathcal{A})$ is in fact an isomorphism of algebras and we thus can and shall identify $\mathcal{A}$ with $\Omega^0(\mathcal{A})$.

Definition 4.1.3. The *Hochschild boundary* and *Connes boundary* operators on $\Omega^\bullet(\mathcal{A})$ are given by, respectively,

$$\begin{aligned} b(a^0, a^1 \dots, a^p) &\coloneqq (a^0a^1, a^2, \dots, a^p) + \sum_{i=1}^{p-1}(-1)^i(a^0, a^1, \dots, a^ia^{i+1}, \dots, a^p) \\ &\quad + (-1)^p(a^pa^0, a^1, \dots, a^{p-1}) \quad \text{and} \\ B(a^0, a^1 \dots, a^p) &\coloneqq \sum_{i=0}^{p}(-1)^{ip}(1, a^i, \dots, a^p, a^0, \dots, a^{i-1}). \end{aligned}$$

The *Karoubi operator* is defined as

$$\kappa \coloneqq 1 - [b, d] = 1 - (bd + db).$$

More explicitly, $\kappa(a^0) = a^0$ and $\kappa(\omega da) = (-1)^{|\omega|}da \cdot \omega$ or

$$\kappa(a^0, a^1 \dots, a^p) = (-1)^{p-1}(1, a^pa^0, a^1, \dots, a^{p-1}) + (-1)^p(a^p, a^0, \dots, a^{p-1}).$$

Thus κ plays the role of the cyclic permutation operator λ in Connes' theory. Note that we have $B = \sum_{j=0}^{p} \kappa^j d$ on $\Omega^p(\mathcal{A})$.

Lemma 4.1.4. *The operators b, B and κ have degree $-1, 1$ and 0, respectively, and satisfies*

(a) $b^2 = B^2 = Bb + bB = 0$

(b) $[\kappa, d] = [\kappa, b] = [\kappa, B] = 0$

(c) $Bd = dB = 0$ *and* $\kappa B = B\kappa = B$.

Proof. See [CQ95b, Section 3]. $\square$

Dually, by transpose, we get operators on

$$\mathrm{Hom}_{\mathbb{C}}(\Omega^\bullet(\mathcal{A}), \mathbb{C}) \coloneqq \bigoplus_{p\geq 0} \mathrm{Hom}_{\mathbb{C}}(\Omega^p, \mathbb{C}),$$

again denoted by b, B and κ. Note that the *support* of an element $\phi = (\phi_p) \in \mathrm{Hom}_{\mathbb{C}}(\Omega^\bullet, \mathbb{C})$ is the set of p such that $\phi_p \neq 0$.

We define the following (simplicially-normalized) complexes.

Definition 4.1.5. Let $\mathcal{A}$ be a unital algebra and let $\Omega^\bullet = \Omega^\bullet(\mathcal{A})$ denote the underlying chain complex of the universal enveloping differential graded algebra of $\mathcal{A}$. We define

$C_\bullet \coloneqq (\Omega^\bullet, b)$ is the *Hochschild* chain complex.

$C^\bullet \coloneqq (\mathrm{Hom}_{\mathbb{C}}(\Omega^\bullet, \mathbb{C}), b)$ is the *Hochschild* cochain complex.

and

$CN_\bullet := (\Omega^\bullet[[v]], b + vB)$ is the *negative cyclic* chain complex, where v is a formal parameter of degree -2.

$\mathcal{B}^\bullet := (\mathrm{Hom}_{\mathbb{C}}(\Omega^\bullet, \mathbb{C})[u], b + uB)$ is the (b, B)-cochain complex, where u is a formal parameter of order 2.

$CP^\bullet := (\mathrm{Hom}_{\mathbb{C}}(\Omega^\bullet, \mathbb{C})[u^{-1}, u], b + uB)$ is the *periodic cyclic* cochain complex, where u is again a formal parameter of order 2.

The (co)homology of these complexes are denoted, respectively, $HH_\bullet(\mathcal{A}, \mathcal{A})$, $HH^\bullet(\mathcal{A}, \mathcal{A}^*)$, $HN_\bullet(\mathcal{A})$, $HC^\bullet(\mathcal{A})$ and $HP^\bullet(\mathcal{A})$ and called the *Hochschild* homology, *Hochschild* cohomology, *negative cycllic* homology, *cyclic* cohomology and *periodic cyclic* cohomology, respectively.

More explicitly, a negative cyclic chain of degree $n \in \mathbb{Z}$ is given by an *infinite* sum

$$\omega = \sum_{k=0}^{\infty} \omega_{n+2k} v^k, \quad \omega_p \in \Omega^p$$

and a cyclic cochain of degree $n \in \mathbb{Z}$ is given by a *finite* sum

$$\phi = \sum_{0 \le 2k \le n} \phi_{n-2k} u^k, \quad \phi_p \in \mathrm{Hom}_{\mathbb{C}}(\Omega^p, \mathbb{C}).$$

In particular, there are no cyclic cochains with strictly negative degree.

Remark 4.1.6. Thinking of u as the dual of v, *i.e.* $\langle u^k, v^l \rangle = \delta_{kl}$, we may identify as follows

$$\begin{aligned} \mathcal{B}^\bullet &= \mathrm{Hom}_{\mathbb{C}[v]}(CN_\bullet, \mathbb{C}[v]) = \mathrm{Hom}_{\mathbb{C}[v]}(\Omega^\bullet[[v]], \mathbb{C}[v]) \quad \text{and} \\ CP^\bullet &= \mathrm{Hom}_{\mathbb{C}[v^{-1}, v]}(CN_\bullet[v^{-1}], \mathbb{C}[v^{-1}, v]) = \mathrm{Hom}_{\mathbb{C}[v^{-1}, v]}(\Omega^\bullet[v^{-1}, v]], \mathbb{C}[v^{-1}, v]]). \end{aligned}$$

The complexes

$$(\Omega[v], b + vB) \quad \text{and} \quad (\mathrm{Hom}_{\mathbb{C}}(\Omega, \mathbb{C})[[u]], b + uB)$$

are acyclic except in dimension 0 (cf. [CQ95b, Proposition 1.2]). Elements of the latter complex are given by infinite sequences (ϕ_p) with *arbitrary* support and these will be called *improper* cochains. There are ways to construct nontrivial cyclic cohomology theories by imposing some growth condition on improper cochains, but we'll not use them here. Some examples are given at the end of the subsection.

Example 4.1.7. If $\mathcal{A} = \mathbb{C}$, then $\Omega = (\mathbb{C}, d = 0)$ and $b = B = 0$. Thus

$$HH_\bullet(\mathbb{C}, \mathbb{C}) \cong \begin{cases} \mathbb{C}, & \bullet = 0 \\ 0, & \bullet \neq 0 \end{cases},$$

$$HH^\bullet(\mathbb{C}, \mathbb{C}^*) \cong \begin{cases} \mathbb{C}, & \bullet = 0 \\ 0, & \bullet \neq 0 \end{cases},$$

$$\begin{aligned}
HN_\bullet(\mathbb{C}) &\cong \mathbb{C}[[v]], \\
HC^\bullet(\mathbb{C}) &\cong \mathbb{C}[u], \\
HP^\bullet(\mathbb{C}) &\cong \mathbb{C}[u^{-1}, u].
\end{aligned}$$

Definition 4.1.8. Multiplication by u gives cochain maps

$$S : \mathcal{B}^\bullet \to \mathcal{B}^{\bullet+2} \tag{4.1.1}$$

$$S : CP^\bullet \to CP^{\bullet+2} \tag{4.1.2}$$

called the *periodicity* maps.

The periodicity map on the periodic cyclic cochains is a cochain *equivalence*. Therefore we consider only two types of periodic cyclic cochains: *even* and *odd*. An *even* periodic cyclic cochain is given by a *finitely supported* sequence $(\phi_0, \phi_2, \dots)$ and an *odd* periodic cyclic cochain is given by a *finitely supported* sequence $(\phi_1, \phi_3, \dots)$.

The natural inclusion $\mathcal{B}^\bullet \to CP^\bullet$ give rise to maps $HC^\bullet \xrightarrow{S} HC^{\bullet+2} \to \dots \to HP^\bullet$.

Lemma 4.1.9. *Let $\mathcal{A}$ be a unital algebra. Then*

$$HP^\bullet(\mathcal{A}) \cong \varinjlim HC^{\bullet+2k}(\mathcal{A}).$$

Proof. Even at the cochain level $CP^\bullet$ is the direct limit of the system

$$\mathcal{B}^\bullet \xrightarrow{S} \mathcal{B}^{\bullet+2} \to \dots .$$

On the other hand, filtered direct limits of vector spaces are exact (cf. [Wei94, Theorem 2.6.15]). □

We also have a pairing $CP \otimes_{\mathbb{C}[v]} CN \to \mathbb{C}$ given by

$$\langle \phi, \omega \rangle := \sum_k \phi_k(\omega_k).$$

Lemma 4.1.10. *Let $\mathcal{A}$ be a unital algebra. Then the pairing above induces a natural pairing*

$$HP^\bullet(\mathcal{A}) \otimes HN_\bullet(\mathcal{A}) \to \mathbb{C}.$$

□

The simples way to construct (b, B)-cocycles is the following. First we identify linear functionals $\tau_n : \Omega^n(\mathcal{A}) \to \mathbb{C}$ to linear functionals $\mathcal{A}^{\otimes(n+1)} \to \mathbb{C}$, again denoted τ_n, vanishing on $(a^0, \dots, a^p)$ if one of the entries a^j is equal to 1 for some $j \geq 1$. Recall that a linear functional $\tau_n : \Omega^n \to \mathbb{C}$ is said to be

closed if $\tau_n(d\omega) = 0$ for any $\omega \in \Omega^{n-1}$ or equivalently $\tau_n(1, a^1, \dots, a^n) = 0$ if $n \geq 1$ and a *(super)trace* if $\tau_n(\omega_1\omega_2) = (-1)^{p_1p_2}\tau_n(\omega_2\omega_1)$ for any $\omega_i \in \Omega^{p_i}$, $p_1 + p_2 = n$.

Lemma 4.1.11. *Let $\tau_n : \Omega^n \to \mathbb{C}$ be a linear map. Then the following statements are equivalent:*

(1) The map τ_n is a closed trace.

(2) The map τ_n satisfies

(i) $\tau(a^0, \ldots, a^n) = (-1)^n\tau(a^n, a^0, \ldots, a^{n-1})$

(ii) $\sum_{i=0}^n \tau_n(a^0, \ldots, a^ia^{i+1}, \ldots, a^{n+1}) + (-1)^{n+1}\tau_n(a^{n+1}a^0, a^1, \ldots, a^n) = 0$

(3) The Hochschild cochain $\tau_n : \Omega^n \to \mathbb{C}$ is a cocycle and $\kappa\tau_n = \tau_n$.

(4) The (b, B)-cochain $\tau_n : \Omega^n \to \mathbb{C}$ is a cocycle and $\kappa\tau_n = \tau_n$.

Proof. See [Con94, Proposition III.1.α.4]. □

Definition 4.1.12. A *cyclic* n-cocycle is a (b, B)-cocycle supported on $\{n\}$ such that $\kappa\phi_n = \phi_n$. We write Z^n_λ for the space of cyclic n-cocycles and B^n_λ for the space of cyclic n-cocycles that are coboundaries.

Definition 4.1.13. The *character* of a closed n-trace τ_n is defined to be the cyclic n-cocycle $\frac{1}{n!}\tau_n$.

Theorem 4.1.14. *Any periodic cyclic cocycle is cohomologous to a cyclic cocycle.*

Proof. Dualize [Lod92, Theorem 2.1.5]). □

Various variations and extensions of cyclic theory are possible. The following are the main examples.

Nonunital: Let $\mathcal{A}^+$ be the minimal unitization of $\mathcal{A}$. As a vector space $\mathcal{A}^+ = \mathcal{A} \oplus \mathbb{C}$ and the multiplication is given by

$$(a, \lambda) \cdot (b, \mu) \coloneqq (ab + \lambda b + a\mu, \lambda\mu).$$

The projection map $\mathcal{A}^+ \to \mathbb{C}$ is a map of algebras. Cyclic cohomology theories are extended to the nonunital case by

$$H^\bullet(\mathcal{A}) \coloneqq \operatorname{coker}(H^\bullet(\mathbb{C}) \to H^\bullet(\mathcal{A}^+)).$$

Most properties of the unital case extend to the nonunital case in a straightforward manner.

Graded: All the definitions work as expected in the graded case, if we follow the Koszul sign rule, *i.e.* whenever we two cross to elements of order p and q respectively, we get a sign $(-1)^{pq}$. See [NT95a]

Topological: If $\mathcal{A}$ is a locally convex topological algebra, we look at continuous multilinear functionals. See also [Pus96].

Bornological: See [Mey07].

4.1.2 Pairing with K-theory

Lemma 4.1.15. *Let $\mathcal{A}$ be a unital algebra.*

(a) Let $e \in \mathcal{A}$ be an idempotent. Then

$$\mathrm{Ch}(e) := e + \sum_{k=1}^{\infty} (-1)^k \frac{(2k)!}{k!}(e - \frac{1}{2})(de)^{2k} v^k$$

is a negative cyclic 0*-cycle. This defines a functorial map*

$$\mathrm{Ch} : K_0(\mathcal{A}) \to HN_0(\mathcal{A}), \quad e \mapsto \mathrm{Ch}(e).$$

(b) Let $u \in \mathcal{A}$ be an invertible element. Then

$$\mathrm{Ch}(u) := \frac{1}{\Gamma(\frac{1}{2})} \sum_{k=0}^{\infty} (-1)^{k+1} k! u^{-1} du (du^{-1} du)^k v^k$$

is a negative cyclic 1*-cycle. This defines a functorial map*

$$\mathrm{Ch} : K_1^{\mathrm{alg}}(\mathcal{A}) \to HN_1(\mathcal{A}), \quad u \mapsto \mathrm{Ch}(u).$$

Proof. See [Lod92, Lemma-Notation 8.3.3, Proposition 8.3.8] for (a) and [Lod92, Lemma 8.4.8, Proposition 8.4.9] for (b). □

As a corollary, we get the following pairing.

Definition 4.1.16. Let $\mathcal{A}$ be a unital algebra.

(a) Let $\phi = (\phi_{2k})$ be an even periodic cyclic cocycle. Then

$$\langle \phi, e \rangle := \langle \phi, \mathrm{Ch}(e) \rangle$$

defines a pairing $HP^0 \times K_0 \to \mathbb{C}$.

(b) Let $\phi = (\phi_{2k+1})$ be an odd periodic cyclic cocycle. Then

$$\langle \phi, u \rangle := \langle \phi, \mathrm{Ch}(u) \rangle$$

defines a pairing $HP^1 \times K_1^{\mathrm{alg}} \to \mathbb{C}$.

Corollary 4.1.17. *Cyclic cocycles pair with K-theory as follows:*

(a) Let ϕ_n be an even *cyclic cocycle ($n \geq 2$). Then*

$$\langle \phi_n, e \rangle = \frac{(-1)^{\frac{n}{2}} \Gamma(n+1)}{\Gamma(\frac{n}{2}+1)} \phi_n(e - \frac{1}{2}, e, \ldots, e), \quad [e] \in K_0(\mathcal{A}).$$

(b) Let ϕ_n be an odd *cyclic cocycle. Then*

$$\langle \phi_n, u\rangle = \frac{(-1)^{\frac{n+1}{2}}\Gamma(n+1)}{2^n\Gamma(\frac{n}{2}+1)}\phi_n(u^{-1}, u, \dots, u^{-1}, u), \quad [u] \in K_1^{\mathrm{alg}}(\mathcal{A}).$$

□

In the following definition we have the periodic cyclic theory in mind most of the time, but the generality is needed to allow statements about entire cocycles.

Definition 4.1.18. Let $H^\bullet$ be cohomology theory that pairs with K-theory.

(a) An *index cocycle* for a spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ is an even cocycle $\phi \in H^0(\mathcal{A})$ satisfying

$$\langle \phi, e\rangle = \mathrm{Index}_{(\mathcal{A},\mathfrak{H},D)}(e)$$

for all $e \in K_0(\mathcal{A})$ and

(b) an *index cocycle* for an odd spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ is an odd cocycle $\phi \in H^1(\mathcal{A})$ satisfying

$$\langle \phi, u\rangle = \mathrm{Index}_{(\mathcal{A},\mathfrak{H},D)}(u)$$

for all $u \in K_1^{\mathrm{alg}}(\mathcal{A})$.

4.1.3 Operations on Cyclic Cocycles and Homotopy Invariance

In this subsection, we collect some standard facts needed to deal with the homotopy invariance of index cocycles.

Lemma 4.1.19. *Let $a \in \mathcal{A}$ be an element. Define*

$$i_a(a^0, \dots, a^p) := \sum_i^p (-1)^i (a^0, \dots, a^i, a, a^{i+1}, \dots, a^p)$$
$$L_a(a^0, \dots, a^p) := \sum_i^p (a^0, \dots, [a, a^i], \dots, a^p).$$

Then we have

$$L_a = [b + B, i_a].$$

Proof. See [GS89, Lemma 1.2]. □

Lemma 4.1.20. *Let $\partial : \mathcal{A} \to \mathcal{A}$ be a derivation of $\mathcal{A}$. Define*

$$e_\partial(a^0, \dots, a^p) := (-1)^{p+1}(\partial(a^p)a^0, a^1, \dots, a^p)$$
$$E_\partial(a^0, \dots, a^p) := \sum_{1\le i\le j\le p} (-1)^{ip+1}(1, a^i, \dots, a^{j-1}, \partial a^j, a^{j+1}, \dots, a^p, a^0, \dots, a^{i-1}) \quad \textit{and}$$

$$L_\partial(a^0,\dots,a^p) \coloneqq \sum_{i=0}^{p}(a^0,\dots,a^{i-1},\partial a^i,a^{i+1},\dots,a^p).$$

Then we have

$$L_\partial = [e_\partial + E_\partial, b+B].$$

Proof. See [Lod92, Proposition 4.1.8]. □

Notice that $L_a = L_{[a,-]}$. For full generalization of these operations, see [NT95a, NT99].

Definition 4.1.21. Let $\phi^t : \Omega \to \mathbb{C}$ be a smooth family of cocycles. We say that ϕ^t is *homotopy invariant* if we have a smooth family ∂^t of derivations such that $\frac{d}{dt}\phi^t = L_{\partial^t}\phi^t$. A cocycle ϕ_D constructed from a spectral triple $(\mathcal{A},\mathfrak{H},D)$ is said to be *homotopy invariant* if $\phi_{D|D|^{-t}}$ is a homotopy invariant family.

Note that this definition is not universal.

Lemma 4.1.22. *If ϕ_D is a homotopy invariant cocycle, then $\phi_F - \phi_D$ is a coboundary. In particular, ϕ_D is an index cocycle if ϕ_F is one.* □

4.2 Fredholm Modules and Index Cocycles

We define the Chern character of a Fredholm module. A Fredholm module $(\mathcal{A},\mathfrak{H},F)$ is said to be *p-summable* for some $p \in [1,\infty)$ if $[F,\mathcal{A}] \subseteq \mathcal{L}^p(\mathfrak{H})$.

Lemma 4.2.1. *Let $(\mathcal{A},\mathfrak{H},F)$ be a p-summable Fredholm module and let $n \geq p$ be an* even *summable integer. Define*

$$\mathrm{ch}_n(a^0,\dots,a^n) \coloneqq \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n+1)}\cdot\frac{1}{2}\mathrm{Str}(F[F,a^0][F,a^1]\dots[F,a^n]) \quad \textit{and}$$
$$\mathrm{Ch}_{n+1}(a^0,\dots,a^{n+1}) \coloneqq \frac{\Gamma(\frac{n}{2}+2)}{\Gamma(n+3)}\mathrm{Str}(Fa^0[F,a^1]\dots[F,a^{n+1}]).$$

Then

$$\mathrm{ch}_n = (n+1)d\mathrm{Ch}_{n+1} = B\mathrm{Ch}_{n+1} \quad \textit{and} \quad b\mathrm{Ch}_{n+1} = -\mathrm{ch}_{n+2}.$$

□

The proof is a straightforward computation. As a corollary we see that ch_n is a cyclic n-cocycle and $\mathrm{ch}_{n+2} = S\mathrm{ch}_n$.

Definition 4.2.2. The *Chern character* of a finitely-summable Fredholm module $(\mathcal{A},\mathfrak{H},F)$ is the periodic even cyclic cocycle given by the formula, for an *even summable* integer n,

$$\mathrm{ch}_n(a^0,\dots,a^n) \coloneqq \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n+1)}\cdot\frac{1}{2}\mathrm{Str}(F[F,a^0][F,a^1]\dots[F,a^n]).$$

Lemma 4.2.3. *The Chern character of a finitely summable Fredholm modules implements the index map:*

$$\mathrm{Index}_F(e) = \langle \mathrm{ch}_F, e \rangle\,, \quad e \in K_0(\mathcal{A}).$$

Proof. See [Con85, Theorem I.3.1]. □

Definition 4.2.4. Let $(\mathcal{A}, \mathfrak{H}, F_i)$, $i \in \{0,1\}$, be p-summable Fredholm modules with actions $\pi_i : \mathcal{A} \to \mathfrak{H}$, $i \in \{0,1\}$, respectively. A *homotopy* is given by a pointwise norm-continuous path of actions $\pi_t : \mathcal{A} \to \mathcal{L}(\mathfrak{H})$, $t \in [0,1]$ and a norm continuous path F_t, $t \in [0,1]$ of supersymmetries on $\mathfrak{H}$ connecting $(\mathcal{A}, \mathfrak{H}, F_0)$ to $(\mathcal{A}, \mathfrak{H}, F_1)$ through p-summable Fredholm modules $(\mathcal{A}, \mathfrak{H}, F_t)$.

Lemma 4.2.5. *The Chern characters of homotopic Fredholm modules are cohomologous.*

Proof. See [Con85, Corollary I.5.3]. □

4.2.1 The Connes-Karoubi Character

Definition 4.2.6. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a unital finitely-summable spectral triple with *invertible* D. We define the *Connes-Karoubi character* $\mathrm{ch}^{\mathrm{CK}}$ of $(\mathcal{A}, \mathfrak{H}, D)$ as the Chern character of the phase Fredholm module $(\mathcal{A}, \mathfrak{H}, D|D|^{-1})$ (cf. Lemma 2.2.9).

Since the Chern character is an index cocycle for $(\mathcal{A}, \mathfrak{H}, F)$ and the index map for $(\mathcal{A}, \mathfrak{H}, D)$ can be defined as Index_F, we get the following.

Lemma 4.2.7. *The Connes-Karoubi character is an index cocycle.* □

4.2.2 The Connes-Chern Character

Definition 4.2.8. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a unital finitely-summable spectral triple with *invertible* D. The *Connes-Chern character* $\mathrm{ch}^{\mathrm{CCh}}$ of $(\mathcal{A}, \mathfrak{H}, D)$ is defined as

$$\frac{1}{2}\left(\mathrm{ch}_{\mathfrak{F}^0} - \mathrm{ch}_{\mathfrak{F}^1}\right)$$

where the Fredholm modules $\mathfrak{F}^0$ and $\mathfrak{F}^1$ are defined on page 21.

A quick computation shows that the Connes-Chern character is given by the formula, for a summable even integer n,

$$\begin{aligned}\mathrm{ch}^{\mathrm{CCh}}_n(a^0, \ldots, a^n) &= \frac{1}{2} \cdot \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n+1)} \cdot \frac{1}{2}\mathrm{Str}(D^{-1}[D,a^0] \ldots D^{-1}[D,a^n]) \cdot \mathrm{Tr}\left(\sigma_3\sigma_1(-i\sigma_2)^{n+1}\right) \\ &= (-1)^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n+1)} \cdot \frac{1}{2}\mathrm{Str}(D^{-1}[D,a^0] \ldots D^{-1}[D,a^n]).\end{aligned}$$

Lemma 4.2.9. *The Connes-Chern character is an index cocycle.*

Proof. We may assume that $\mathcal{A}$ is involutive and $e \in \mathcal{A}$ is a projection (cf. Lemma 2.2.24). Then it follows from Remark 2.2.13. □

Even more is true.

Lemma 4.2.10. *The Connes-Karoubi and the Connes-Chern characters are cohomologous.*

Proof. The Connes-Chern character is homotopy invariant. Hence we can take $D = F$. Then F anticommutes with $[F, a]$ and hence

$$(-1)^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n+1)} \cdot \frac{1}{2} \mathrm{Str}(F[F,a^0]F[F,a^1] \ldots F[F,a^n]) = \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n+1)} \cdot \frac{1}{2} \mathrm{Str}(F[F,a^0] \ldots [F,a^n])$$

and we get the Connes-Karoubi character. □

Remark 4.2.11. In order to define the Connes-Chern character, all we need is to have

$$D^{-1}[D, \mathcal{A}] \subseteq \mathcal{L}^p.$$

Triples $(\mathcal{A}, \mathfrak{H}, D)$ such that $\mathcal{A}$ is an algebra acting on the graded Hilbert space $\mathfrak{H}$ by even bounded operators and D is an odd self-adjoint operator on $\mathfrak{H}$ with domain $\mathrm{dom}(D)$ such that for any $a \in \mathcal{A}$

(i) $a \cdot \mathrm{dom}(D) \subseteq \mathrm{dom}(D)$ and

(ii) $D^{-1}[D, a]$ is compact

form a nice set of triple that generalizes spectral triples (with invertible D) and Fredholm modules. Index theory can be developed along the same lines as spectral triples or Fredholm modules. Alternatively, we could use the construction from Lemma 2.2.11 to convert the index problem into Fredholm module index problem.

4.3 Entire Cyclic Cohomology and Index Theory

The Connes character and the Jaffe-Lesniewski-Osterwalder (JLO) character are index cocycles in the *entire* cyclic theory. Since we did not introduce entire theory properly, we will only consider them as *improper* cocycles. One justification for this decision is that we still don't know how to compute the entire cyclic cohomology groups for spaces as simple as smooth manifolds. However, in many situations we will be able to extract/construct proper periodic cyclic cocycles out of entire cocycles.

4.3.1 The Connes Character

We consider the convolution algebra $\mathscr{L}$ of operator valued distributions over $\mathbb{R}$ and its super version $\widetilde{\mathscr{L}}$ considered by Connes in [Con88].

Let $\mathfrak{H}$ be a graded Hilbert space. An *operator-valued distribution* is a continuous linear map from the Fréchet space of Schwartz functions $\mathcal{S}(\mathbb{R})$ on $\mathbb{R}$ to bounded operators $\mathcal{L}(\mathfrak{H})$ on $\mathfrak{H}$. We say that $T : \mathcal{S}(\mathbb{R}) \to \mathcal{L}(\mathfrak{H})$ is *even* (respectively *odd*) if $T(f)$ is even (respectively odd) for any $f \in \mathcal{S}(\mathbb{R})$.

Definition 4.3.1. Let $\mathscr{L}$ denote the space of operator-valued distributions $T : \mathcal{S}(\mathbb{R}) \to \mathcal{L}(\mathfrak{H})$ with the properties:

1) The support of T is contained in $\mathbb{R}^+ = [0, \infty)$.

2) There exists $r > 0$ and an analytic operator-valued function $t(z)$ for $z \in \cup_{s>0} sU$, where U is the disk with center at 1 and radius r, such that

   (a) $t(s) = T(s)$ for $s \in (0, \infty)$ and

   (b) the function defined for $p \in (1, \infty)$ by

$$h(p) := \sup_{z \in (1/p)U} ||t(z)||_{\mathcal{L}^p}$$

   is bounded above by a polynomial in p as $p \to \infty$.

In particular, $t(1) \in \mathcal{L}^1$ is a trace-class operator.

The space $\mathscr{L}$ is a graded algebra under the convolution product given by, formally,

$$(T_1 \cdot T_2)(s) := \int_0^s T_1(u) T_2(s-u) du$$

(cf. [Con94, Lemma IV.8.β.7]) with unit δ_0 the Dirac mass at 0 and and the functional

$$\tau_0 : \mathscr{L} \to \mathbb{C}, \quad T \mapsto \mathrm{Str}(T(1))$$

is a supertrace on $\mathscr{L}$ (cf. [Con94, Proof of Lemma IV.8.β.8]).

Let $\lambda = \delta_0'$ denote the derivative of the Dirac mass at 0. Then λ is an even central element in $\mathscr{L}$.

Definition 4.3.2. Let $\widetilde{\mathscr{L}}$ denote the graded algebra with $\widetilde{\mathscr{L}^0} := \mathscr{L}^0 \oplus \mathscr{L}^1$ and $\widetilde{\mathscr{L}^1} := \mathscr{L}^1 \oplus \mathscr{L}^0$ with the product

$$(T_0, T_1) \cdot (S_0, S_1) := (T_0 S_0 + \lambda T_1 S_1, T_0 S_1 + T_1 S_0).$$

Then $\widetilde{\mathscr{L}}$ is an associative graded algebra with unit $(\delta_0, 0)$ and the functional

$$\tau : \widetilde{\mathscr{L}} \to \mathbb{C}, \quad (T_0, T_1) \mapsto \mathrm{Tr}(T_1(1))$$

is an odd (super)trace on $\widetilde{\mathscr{L}}$ (cf. [Con94, Lemma IV.8.β.8]).

Definition 4.3.3. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a θ-summable spectral triple and let $\Delta = D^2$. We define $L^z \in \mathscr{L}$, $z \in \mathbb{C}$ and $F \in \widetilde{\mathscr{L}}$ by

$$\begin{aligned} L^z &:= \Gamma(z)^{-1} s^{z-1} e^{-s\Delta} \quad \text{and} \\ F &:= (DL^{\frac{1}{2}}, \gamma L^{\frac{1}{2}}), \end{aligned}$$

where γ is the grading operator on $\mathfrak{H}$.

Lemma 4.3.4. *We have $L^{z_1}L^{z_2} = L^{z_1+z_2}$ for $\Re(z_1) > 0$ and $\Re(z_2) > 0$. In particular, $\underbrace{L \cdot L \cdots L}_{p} = L^p$ for $p \in \mathbb{Z}_{\geq 1}$.*

Proof. It follows from the formula for the Beta function:

$$\int_0^1 s^{z_1-1}(1-s)^{z_2-1}ds = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)}.$$ □

It follows that the element F is odd and $F^2 = (\delta_0, 0)$.

Lemma 4.3.5. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a θ-summable spectral triple. Define*

$$\begin{aligned}\mathrm{ch}_p^{\mathrm{C}}(a^0, \ldots, a^p) &\coloneqq \frac{\Gamma(\frac{p+1}{2})}{\Gamma(p+1)}\tau(Fa^0[F, a^1] \ldots [F, a^p])\\ \psi_p(a^0, \ldots, a^p) &\coloneqq \frac{\Gamma(\frac{p}{2}+1)}{\Gamma(p+1)}\tau(F[F, a^0] \ldots [F, a^p]).\end{aligned}$$

Then

$$b\mathrm{ch}_{p-1}^{\mathrm{C}} = \psi_p = -B\mathrm{ch}_{p+1}^{\mathrm{C}}$$

Proof. Since

$$\frac{\Gamma(\frac{p+1}{2})}{\Gamma(p+1)} = 2\frac{\Gamma(\frac{p-1}{2}+2)}{\Gamma(p-1+3)},$$

the computation of 4.2.1 applies ad verbatum. □

Definition 4.3.6. Let $(\mathcal{A}, \mathfrak{H}, D)$ be an θ-summable spectral triple. The *Connes character* $\mathrm{ch}_\bullet^{\mathrm{C}}$ is defined by the formula

$$\mathrm{ch}_p^{\mathrm{C}}(a^0, \ldots, a^p) \coloneqq \frac{\Gamma(\frac{p+1}{2})}{\Gamma(p+1)} \cdot \tau(Fa^0[F, a^1] \ldots [F, a^p]).$$

Lemma 4.3.7. *The Connes character is an even entire index cocycle. It is cohomologous to the Connes-Karoubi character if $(\mathcal{A}, \mathfrak{H}, D)$ is finitely-summable.*

Proof. See [Con88] for full detail. The main idea is the following:

Let $e \in \mathcal{A}$ be an idempotent. By considering the algebra generated by operators $\{a \in \mathcal{L}(\mathfrak{H}) \mid [D, a] \text{ is bounded}\}$, we may assume that $e \in \mathcal{L}$ is self-adjoint (cf. Lemma 2.2.24). The Connes character is homotopy invariant, hence using the bounded perturbation homotopy $s \mapsto D + s(2e-1)[D, e]$, we may assume that e commutes with D. Finally, in this case, the pairing can be computed as follows

$$\begin{aligned}\left\langle \mathrm{ch}_D^{\mathrm{C}}, e \right\rangle &= \mathrm{ch}_0^{\mathrm{C}}(e)\\ &= \Gamma\left(\frac{1}{2}\right)\tau_0\left(\gamma L^{\frac{1}{2}} e\right)\\ &= \mathrm{Tr}\left(\gamma e \cdot \exp(-D^2)\right)\end{aligned}$$

$$= \mathrm{Tr}_{e\mathfrak{H}}\left(\gamma \exp(-D_e^2)\right)$$
$$= \mathrm{Index}_s(D_e).$$

The last identity is known as the McKean-Singer formula. □

4.3.2 The JLO Character

While the Connes character is technically very elegant, it is hard to compute. A more user-friendly version was defined by Jaffe-Lesniewski-Osterwalder in [JLO88].

Definition 4.3.8. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a θ-summable spectral triple. The *JLO character* $\mathrm{ch}_\bullet^{\mathrm{JLO}}$ is defined by the formula

$$\mathrm{ch}_p^{\mathrm{JLO}}(a^0, \dots, a^p) := \int_{\Sigma^p} \mathrm{Tr}\left(\varepsilon a^0 e^{-s^0\Delta}[D, a^1]e^{-s^1\Delta} \dots [D, a^p]e^{-s^p\Delta}\right) ds,$$

where $\Sigma^p := \{(s^0, \dots, s^p) \mid s^i \geq 0,\ \sum s^i = 1\}$ is the standard p-simplex and $ds = ds^1 \dots ds^p$ is the standard Lebesgue measure on Σ^p with volume $\frac{1}{p!}$.

The main result of [JLO88] is that the JLO character is indeed a cocycle, that is, $(b + uB)\mathrm{ch}_\bullet^{\mathrm{JLO}} = 0$ and that it is an index cocycle in the entire cyclic theory. See [Con91] for the proof that the Connes and JLO characters are indeed cohomologous.

There are various approaches to the JLO cocycle. First we use the language of Subsection 4.3.1.

Definition 4.3.9. Define
$$L := L^1 = e^{-s\Delta} \in \mathscr{L}.$$

Then for $X^0, \dots, X^p \in \mathcal{L}$, we have

$$\begin{aligned}(X^0 L \dots X^p L)(s) &= \int_{s^i \geq 0,\ \sum s^i = s} X^0 e^{-s^0\Delta} \dots X^p e^{-s^p\Delta} ds^1 \dots ds^p \\ &= s^p \int_{\Sigma^p} X^0 e^{-t^0 s\Delta} \dots X^p e^{-t^p s\Delta} dt^1 \dots dt^p.\end{aligned}$$

Thus, for $a^0, \dots, a^p \in \mathcal{A}$,

$$\mathrm{ch}_p^{\mathrm{JLO}}(a^0, \dots, a^p) = \tau_0(a^0 L[D, a^1]L \dots L[D, a^p]L),$$

where $\tau_0 : \mathscr{L} \to \mathbb{C}$ is the supertrace given by $T \mapsto \mathrm{Str}(t(1))$.

The following lemma is used in Subsection 5.1.3.

Lemma 4.3.10. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a θ-summable spectral triple. Then we have*

$$B\mathrm{ch}_D^{\mathrm{JLO}}(a^0, \dots, a^p) = \tau_0([D, a^0]L \dots L[D, a^p]L).$$

Proof. See [GS89, Lemma 2.2]. □

Alternatively, we use the language of Chapter 3.

Lemma 4.3.11. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a θ-summable spectral triple and let $\Delta = D^2$. For any $X^0, \ldots, X^p \in \mathcal{L}$ let*

$$\langle X^0, \ldots, X^p \rangle_s^{\mathrm{JLO}} \coloneqq s^{-\frac{p}{2}} (X^0 L X^1 \ldots L X^p L)(s) \in \mathcal{L}.$$

Then we have

$$\langle X^0, \ldots, X^p \rangle_s^{\mathrm{JLO}} = s^{-\frac{p}{2}} \frac{(-1)^p}{2\pi i} \int e^{-s\lambda} \langle X^0, \ldots, X^p \rangle_\lambda^{\mathrm{H}} d\lambda.$$

Proof. See [Hig06, Lemma A.2]. □

Lemma 4.3.12. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a finitely-summable regular spectral triple with discrete and simple dimension spectrum. Let d denote the analytic dimension. Then for $X^0, \ldots, X^p$ in the algebra generated by $\mathcal{A} \cup [D, \mathcal{A}]$ and $\mathrm{Re}(z) > \frac{d-p}{2}$, we have*

$$\langle X^0, \ldots, X^p \rangle_{z-\frac{p}{2}}^{\mathrm{CM}} = \int_0^\infty s^{z-1} \langle X^0, \ldots, X^p \rangle_s^{\mathrm{JLO}} ds.$$

Proof. See [Hig06, Proposition A.4] □

4.3.3 The Transgression Character

Let $(\mathcal{A}, \mathfrak{H}, D)$ be a θ-summable spectral triple. Then for any $t > 0$, $(\mathcal{A}, \mathfrak{H}, tD)$ is a θ-summable spectral triple, and the JLO character $\mathrm{ch}^{\mathrm{JLO}}_{(\mathcal{A},\mathfrak{H},tD)}$ is defined. If $(\mathcal{A}, \mathfrak{H}, D)$ is finitely-summable, then the limit of $\mathrm{ch}^{\mathrm{JLO}}_{(\mathcal{A},\mathfrak{H},tD)}$ as $t \to \infty$ exists and we get a periodic cyclic cocycle (cf. [CM86, CM93, Qui88b, Hig06]. The following is the result.

Definition 4.3.13. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a finitely-summable spectral triple with *invertible* D. The *transgression character* is given by the formula, for a summable even integer n,

$$\mathrm{ch}_n^{\mathrm{trans}}(a^0, \ldots, a^n) \coloneqq \frac{1}{2} \sum_{j=0}^{n} (-1)^{j+1} \mathrm{Str} \langle [D, a^0], \ldots, [D, a^j], D, [D, a^{j+1}], \ldots, [D, a^n] \rangle_{-\frac{n}{2}}^{\mathrm{CM}},$$

where $\langle X^0, \ldots, X^p \rangle_z^{\mathrm{CM}}$ is given by the contour integral

$$\langle X^0, \ldots, X^p \rangle_z^{\mathrm{CM}} \coloneqq (-1)^p \frac{\Gamma(z)}{2\pi i} \int \lambda^{-z} X^0 (\lambda - \Delta)^{-1} \ldots X^p (\lambda - \Delta)^{-1} d\lambda$$

as in Definition 3.2.16.

This cycle was first discovered by Connes and Moscovici using Loday-Quillen, Tsygan isomorphism [LQ84, Tsy83].

Lemma 4.3.14. *The transgression character is a cyclic index cocycle, cohomologous to the Connes-Karoubi character.*

Proof. The transgression character is homotopy invariant. Hence we may assume that $D = F$. Then F anticommutes with all $[F, a]$, and hence

$$\begin{aligned}
\frac{1}{2}\sum_{j=0}^{n}(-1)^{j+1}\mathrm{Str}\langle[F,a^0],\dots,[F,a^j],F,[F,a^{j+1}],\dots,[F,a^n]\rangle_z^{\mathrm{CM}} \\
&= \frac{n+1}{2}\frac{(-1)^{n+1}\Gamma(z)}{2\pi i}\mathrm{Str}\left(\int \lambda^{-z}\frac{d\lambda}{(\lambda-I)^{n+2}}F[F,a^0]\dots[F,a^n]\right) \\
&= \frac{n+1}{2}(-1)^{n+1}\Gamma(z)\binom{-z}{n+1}\mathrm{Str}(F[F,a^0],\dots,[F,a^n]) \\
&= \frac{\Gamma(z+n+1)}{\Gamma(n+1)}\frac{1}{2}\mathrm{Str}(F[F,a^0],\dots,[F,a^n])
\end{aligned}$$

using the functional equation of the Gamma function, which gives the Connes-Karoubi character $\mathrm{ch}_n^{\mathrm{CK}}$ when evaluated at $z = -\frac{n}{2}$. $\square$

Remark 4.3.15. We chose the notation from [Hig06], but it is easy to see that the definition coincides with the definition given in [CM86] using Lemma 4.3.12.

4.3.4 The Connes-Moscovici Character

Let $(\mathcal{A}, \mathfrak{H}, D)$ be a finitely-summable spectral triple. Again we consider the family $\mathrm{ch}^{\mathrm{JLO}}_{(\mathcal{A},\mathfrak{H},tD)}$, $t > 0$. If it admits an asymptotic expansion as $t \to 0$, we get a local cocycle by isolating the constant coefficient [CM95] (see also Subsection 5.2.1). Following Higson [Hig04a, Hig06], we work with residues directly.

Definition 4.3.16. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a finitely-summable regular spectral triple with discrete and simple dimension spectrum. The *Connes-Moscovici character* is given by

$$\mathrm{ch}_p^{\mathrm{CM}}(a^0,\dots,a^p) := \sum_{k\geq 0} c_{pk}\mathrm{Res}_{z=0}\mathrm{Str}(a^0[D,a^1]^{(k_1)}\dots[D,a^p]^{(k_p)}\Delta^{-z-\frac{p}{2}-|k|}),$$

where the sum is taken over all multi-indices $k = (k_1, \dots, k_p)$ with non-negative integer entries and the constants c_{pk} are given by the formula

$$c_{pk} := \frac{(-1)^{|k|}}{k!}\frac{\Gamma(k_1+\dots+k_p+\frac{p}{2})}{(k_1+1)(k_1+k_2+2)\dots(k_1+\dots k_p+p)}.$$

Here $|k| := k_1 + \dots + k_p$.

Lemma 4.3.17. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a finitely-summable regular spectral triple with discrete and simple dimension spectrum. The Connes-Moscovici character is an index cocycle, cohomologous to the transgression character. In particular, it is cohomologous to the Connes-Karoubi character.*

Proof. See [Hig04a, Hig06] for the full proof. We sketch the main idea.

It is enough to show that the Connes-Moscovici cocycle is cohomologous to the transgression cocycle. We freely use the results from Chapter 3. Define new families of cochains by

$$\Psi_p(a^0, \ldots, a^p) \coloneqq \operatorname{Str} \langle a^0, [D, a^1], \ldots, [D, a^p] \rangle^{\mathrm{CM}}_{z-\frac{p}{2}}$$
$$\Theta_p(a^0, \ldots, a^p) \coloneqq \frac{1}{2} \sum_{j=0}^{n} (-1)^{j+1} \operatorname{Str} \langle a^0, [D, a^1], \ldots, D, [D, a^{j+1}], \ldots, [D, a^n] \rangle^{\mathrm{CM}}_{z-\frac{p+1}{2}}.$$

Then $\mathrm{ch}^{\mathrm{CM}}_p = \operatorname{Res}_{z=0} \Psi_p$ and $\mathrm{ch}^{\mathrm{trans}}_n = B\Theta_{n+1}|_{z=0} = \operatorname{Res}_{z=0} \frac{1}{z} B\Theta_{n+1}$. Moreover, by direct computation we see that

$$b\Theta_{p-1} + B\Theta_{p+1} = z\Psi_p.$$

Hence, $\mathrm{ch}^{\mathrm{CM}}_\bullet - \mathrm{ch}^{\mathrm{trans}}_n = (b + B)\operatorname{Res}_{z=0} \frac{1}{z} \Theta_{\bullet < n}$ is a coboundary for any summable n bigger than the analytical dimension of $(\mathcal{A}, \mathfrak{H}, D)$. $\square$

Chapter 5

Multiplicative Cyclic Cocycles

In this chapter we construct a new local cocycle that is multiplicative with respect to the external product of spectral triples. See 5.2.7 for the precise statement. The main idea comes from [BG94].

In the proof of Bott periodicity and the Atiyah-Singer index theorem, multiplicativity of the index map plays a crucial role. It allows one to concentrate on a few generating examples, where we can check the index theorem directly by computing both the analytical and the topological indices.

Unfortunately, the Connes-Moscovici cocycle is *not* multiplicative at the cochain level. In a sense, we don't expect it to be, because we are looking at residues and residues are not multiplicative in general. In order to make it clear what we mean we first recall briefly about products in cyclic theory.

5.1 The JLO-character and Products

5.1.1 Cup Product of Cyclic Cocycles

Recall that if $\mathcal{A}$ is a unital algebra, $\Omega(\mathcal{A}) = (\Omega^\bullet(\mathcal{A}), d)$ denote the universal differential graded algebra generated by $\mathcal{A}$ (see Subsection 4.1.1).

Definition 5.1.1. Let Ω_1 and Ω_2 be unital differential graded algebras. The *free product* $\Omega_1 * \Omega_2$ is the *coproduct* of Ω_1 and Ω_2 in the category of unital differential graded algebras. The graded tensor product $\Omega_1 \widehat{\otimes} \Omega_2$ is the quotient by the differential graded ideal generated by the supercommutators $[\Omega_1 * 1, 1 * \Omega_2]$.

We remark that coproducts exist in the category of unital differential graded algebras by abstract algebra.

Example 5.1.2 (due to Drinfel'd)**.** Let $\mathcal{A}$ be a unital algebra. Consider $\mathcal{A}$ as a differential graded algebra concentrated in degree 0 with trivial differential. Let ε be a formal element of degree one with $\varepsilon^2 = 0$ and let $\mathbb{C}[\varepsilon] = \mathbb{C} \oplus \mathbb{C}\varepsilon$ denote the algebra of *dual* numbers. Then $(\mathcal{A}, 0) * \left(\mathbb{C}[\varepsilon], \frac{\partial}{\partial\varepsilon}\right)$ is a model for $\Omega(\mathcal{A})$. An explicit isomorphism to the "tensor" model can be constructed as

$$a^0 \varepsilon a^1 \varepsilon \ldots \varepsilon a^p \mapsto a^0 da^1 \ldots da^p.$$

We often write $(a^0, \ldots, a^p)$ for the element $a^0 da^1 \ldots da^p$ for the sake of legibility.

Now we define the cup product following Connes (with different normalization, see Example 5.1.10). Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital algebras. By universality of $\Omega(\mathcal{A}_1 \otimes \mathcal{A}_2)$, we have a map

$$\mu : \Omega(\mathcal{A}_1 \otimes \mathcal{A}_2) \to \Omega(\mathcal{A}_1) \mathbin{\widehat{\otimes}} \Omega(\mathcal{A}_2).$$

If $\tau^1 : \Omega(\mathcal{A}_1) \to \mathbb{C}$ and $\tau^2 : \Omega(\mathcal{A}_2) \to \mathbb{C}$ are closed traces, then $(\tau^1 \mathbin{\widehat{\otimes}} \tau^2) \circ \mu$ is a closed trace on $\mathcal{A}_1 \otimes \mathcal{A}_2$. Moreover, if either τ^1 or τ^2 is a coboundary, then so is $(\tau^1 \mathbin{\widehat{\otimes}} \tau^2) \circ \mu$ (cf. [Con94, Theorem 3.1.α.12]). This allows us to define a product on cyclic cocycles.

Definition 5.1.3. The *cup* product of cyclic cocycles is given by the product of the corresponding closed traces: if $\phi^1 \in Z^n_\lambda(\mathcal{A}_1)$ and $\phi^2 \in Z^m_\lambda(\mathcal{A}_2)$, then

$$\phi^1 \times \phi^2 := \frac{n!m!}{(n+m)!} (\phi^1 \mathbin{\widehat{\otimes}} \phi^2) \circ \mu \in Z^{n+m}_\lambda(\mathcal{A}_1 \otimes \mathcal{A}_2).$$

Since every periodic cyclic cocycle is cohomologous to a cyclic cocycle we get a product:

$$HP^\bullet(\mathcal{A}_1) \otimes HP^\bullet(\mathcal{A}_2) \to HP^\bullet(\mathcal{A}_1 \otimes \mathcal{A}_2).$$

In the next two subsection we construct the product at the cocycle level and study the JLO cocycle with respect to this product.

5.1.2 Shuffle Product in Hochschild Theory

Definition 5.1.4. Let $p,\ q \in \mathbb{Z}_{\geq 1}$ be natural numbers. A *permutation* χ of $\{1, \ldots, p+q\}$ is called a (p,q)-*shuffle* if

$$\chi(1) < \chi(2) < \cdots < \chi(p) \quad \text{and} \quad \chi(p+1) < \chi(p+2) < \cdots < \chi(p+q).$$

Example 5.1.5. Let us write $[\chi^{-1}(1), \ldots, \chi^{-1}(n)]$ for the permutation

$$\chi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \chi(1) & \chi(2) & \cdots & \chi(n) \end{pmatrix}$$

of $\{1, \ldots, n\}$. There are three $(2,1)$-shuffles:

$$[1,2,3],\ [1,3,2] \quad \text{and} \quad [3,1,2]$$

and six $(2,2)$-shuffles:

$$[1,2,3,4],\ [1,3,2,4],\ [1,3,4,2],\quad \text{and}$$
$$[3,1,2,4],\ [3,1,4,2],\ [3,4,1,2].$$

In general, the number of (p,q)-shuffles is $\binom{p}{p+q}$.

Notation 5.1.6. Let Σ^n denote the standard n-simplex given by

$$\Sigma^n \coloneqq \{t = (t^1, \dots, t^n) \in [0,1]^n \mid 0 \le t^1 \le \dots \le t^n \le 1\}.$$

We write $dt = dt^1 \dots dt^n$ for the standard measure on Σ^n with volume $\frac{1}{n!}$.

We let permutations on $\{1, \dots, n\}$ act on $[0,1]^n$ by

$$\chi(t^1, \dots, t^n) \coloneqq (t^{\chi^{-1}(1)}, \dots, t^{\chi^{-1}(n)}).$$

Then for any element $t \in [0,1]^n$, such that all the entries are distinct, there exists a *unique* permutation χ such that $\chi(t)$ is an element of Σ^n, *i.e.* entries of $\chi(t)$ are in increasing order. Therefore, permutations give a decomposition of $[0,1]^n$ into cells, which are in fact n-simplices. Restricting to the appropriate subspaces (which are simplicial), we get the following lemma (cf. [BG94, Proof of Theorem 3.2(2)]).

Lemma 5.1.7. *Let p, q be natural numbers. For a (p,q)-shuffle χ, put*

$$\Sigma(\chi) \coloneqq \{(s,t) \in \Sigma^p \times \Sigma^q \subset [0,1]^{p+q} \mid \chi(s,t) \in \Sigma^{p+q}\}$$

Then the sets $\Sigma(\chi)$ give a decomposition of $\Sigma^p \times \Sigma^q$ into $p+q$-simplices. □

Example 5.1.8. For instance, for $n = 4$,

$$[1,3,4,2](0,1,1/3,1/2) = (0,1/3,1/2,1).$$

Now let the group of permutations of $\{1, \dots, n\}$ act on $\Omega^n(\mathcal{A})$ by

$$\chi(a^0, a^1, \dots, a^n) \coloneqq (-1)^\chi (a^0, a^{\chi^{-1}(1)}, \dots, a^{\chi^{-1}(n)}).$$

Definition 5.1.9. Let $\alpha = (a^0, a^1, \dots, a^p) \in \Omega^p(\mathcal{A}_1)$ and $\beta = (b^0, b^1, \dots, b^q) \in \Omega^q(\mathcal{A}_2)$. The *shuffle product* $\alpha \times \beta \in \Omega^{p+q}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ is defined as

$$\alpha \times \beta \coloneqq \sum_{(p,q)\text{-shuffles}} \chi(a^0 \otimes b^0, a^1 \otimes 1, \dots, a^p \otimes 1, 1 \otimes b^1, \dots, 1 \otimes b^q).$$

Example 5.1.10. Let $\tau^1 : \Omega^n(\mathcal{A}_1) \to \mathbb{C}$ and $\tau^2 : \Omega^m(\mathcal{A}_2) \to \mathbb{C}$ be cyclic cocycles. Then

$$\left\langle \phi^1 \times \phi^2, \alpha^1 \times \alpha^2 \right\rangle = \left\langle \phi^1 \mathbin{\widehat{\otimes}} \phi^2, \alpha^1 \mathbin{\widehat{\otimes}} \alpha^2 \right\rangle.$$

Indeed, note that for any (n,m)-shuffle χ,

$$\mu(\chi(a^0 \otimes b^0, a^1 \otimes 1, \dots, a^n \otimes 1, 1 \otimes b^1, \dots, 1 \otimes b^m)) = \alpha^1 \mathbin{\widehat{\otimes}} \alpha^2.$$

in $\Omega^n(\mathcal{A}_1) \mathbin{\widehat{\otimes}} \Omega^m(\mathcal{A}_2)$. For instance, for $p = 2$ and $q = 1$,

$$\begin{aligned}\mu([1,3,2](a^0 \otimes b^0, a^1 \otimes 1, a^2 \otimes 1, 1 \otimes b^1)) &= (-1)^{[1,3,2]}\mu(a^0 \otimes b^0, a^1 \otimes 1, 1 \otimes b^1, a^2 \otimes 1) \\ &= -a^0 \mathbin{\widehat{\otimes}} b^0 \cdot da^1 \mathbin{\widehat{\otimes}} 1 \cdot 1 \mathbin{\widehat{\otimes}} db^1 \cdot da^2 \mathbin{\widehat{\otimes}} 1 \\ &= a^0 da^1 da^2 \mathbin{\widehat{\otimes}} b^0 db^1 \\ &= (a^0, a^1, a^2) \mathbin{\widehat{\otimes}} (b^0, b^1).\end{aligned}$$

Since the number of (n,m)-shuffles is $\binom{n}{n+m} = \frac{(n+m)!}{n!m!}$, we get the result.

Lemma 5.1.11. *The Hochschild boundary is a graded derivation for the shuffle product: for $\alpha \in \Omega^p(\mathcal{A}_1)$ and $\beta \in \Omega^q(\mathcal{A}_2)$*

$$b(\alpha \times \beta) = b\alpha \times \beta + (-1)^{|\alpha|}\alpha \times b\beta$$

in $\Omega^{p+q-1}(\mathcal{A}_1 \otimes \mathcal{A}_2)$.

Proof. See [Lod92, Proposition 4.2.2]. □

Next we show that the shuffle product is compatible with the external product of spectral triples and the JLO-cocycle (cf. 4.3.8).

Theorem 5.1.12. *Let $(\mathcal{A}_1, \mathfrak{H}_1, D_1)$ and $(\mathcal{A}_2, \mathfrak{H}_2, D_2)$ be θ-summable spectral triples. Then*

$$\mathrm{ch}^{\mathrm{JLO}}_{D_1 \times D_2}(\alpha \times \beta) = \mathrm{ch}^{\mathrm{JLO}}_{D_1}(\alpha)\mathrm{ch}^{\mathrm{JLO}}_{D_2}(\beta)$$

for $\alpha \in \Omega^p(\mathcal{A}_1)$ and $\beta \in \Omega^q(\mathcal{A}_2)$.

Proof. By Lemma 2.1.12,

$$(\mathcal{A}_1, \mathfrak{H}_1, D_1) \times (\mathcal{A}_2, \mathfrak{H}_2, D_2) = (\mathcal{A}_1 \otimes \mathcal{A}_2, \mathfrak{H}_1 \mathbin{\widehat{\otimes}} \mathfrak{H}_2, D_1 \times D_2)$$

is a θ-summable spectral triple. We remind the reader that $D_1 \times D_2$ denotes (the closure of) the operator $D_1 \mathbin{\widehat{\otimes}} 1 + 1 \mathbin{\widehat{\otimes}} D_2$ on $\mathfrak{H}_1 \mathbin{\widehat{\otimes}} \mathfrak{H}_2$.

We introduce some notations for the sake of legibility. Let $D \coloneqq D_1 \times D_2$ and

$$\Delta_1 \coloneqq D_1^2, \quad \Delta_2 \coloneqq D_2^2, \quad \text{and} \quad \Delta \coloneqq D^2 = \Delta_1 \otimes 1 + 1 \otimes \Delta_2.$$

By Lemma A.1.9, we have

$$e^{-r\Delta} = e^{-r\Delta_1} \mathbin{\widehat{\otimes}} e^{-r\Delta_2}.$$

Let $\alpha = (a^0, a^1, \dots, a^p)$, $\beta = (b^0, b^1, \dots, b^q)$ and $\gamma = (c^0, c^1, \dots, c^{p+q}) := (a^0 \otimes b^0, a^1 \otimes 1, \dots, a^p \otimes 1, 1 \otimes b^1, \dots, 1 \otimes b^q)$. Then by definition

$$\mathrm{ch}^{\mathrm{JLO}}_{D_1 \times D_2}(\alpha \times \beta) = \sum_{\chi} \mathrm{ch}^{\mathrm{JLO}}_{D}(\chi(\gamma)),$$

where the summation is over all (p, q)-shuffles χ. Since

$$e^{-r\Delta}[D, c^j] = \begin{cases} e^{-r\Delta_1}[D_1, a^j] \,\widehat{\otimes}\, e^{-r\Delta_2}, & 1 \le j \le p \\ e^{-r\Delta_1} \,\widehat{\otimes}\, e^{-r\Delta_2}[D_2, b^j], & p+1 \le j \le p+q \end{cases}$$

we see that each summand gives

$$\begin{aligned} \mathrm{ch}^{\mathrm{JLO}}_{D}(\chi(\gamma)) &= \int_{r \in \Sigma^{p+q}} (-1)^{\chi} \mathrm{Tr}\Big(\varepsilon_1 \otimes \varepsilon_2 \cdot c^0 \cdot e^{-r^1\Delta}[D, c^{\chi^{-1}(1)}] e^{-(r^2 - r^1)\Delta} \\ &\qquad \cdots [D, c^{\chi^{-1}(p+q)}] e^{-(1 - r^{p+q})\Delta}\Big)\, dr \\ &= \int_{r \in \Sigma^{p+q}} \mathrm{Tr}\Big(\varepsilon_1 a^0 e^{-r^{\chi(1)}\Delta_1}[D_1, a^1] e^{-(r^{\chi(2)} - r^{\chi(1)})\Delta_1} \cdots [D_1, a^p] e^{-(1 - r^{\chi(p)})\Delta_1}\Big) \\ &\quad \cdot \mathrm{Tr}\Big(\varepsilon_2 b^0 e^{-r^{\chi(p+1)}\Delta_2}[D_2, b^1] e^{-(r^{\chi(p+2)} - r^{\chi(p+1)})\Delta_2} \cdots [D_2, b^q] e^{-(1 - r^{\chi(p+q)})\Delta_2}\Big)\, dr \\ &= \int_{(s,t) \in \Sigma(\chi)} \mathrm{Tr}\Big(\varepsilon_1 a^0 e^{-s^1\Delta_1}[D_1, a^1] e^{-(s^2 - s^1)\Delta_1} \cdots [D_1, a^p] e^{-(1 - s^p)\Delta_1}\Big) \\ &\quad \cdot \mathrm{Tr}\Big(\varepsilon_2 b^0 e^{-t^1\Delta_2}[D_2, b^1] e^{-(t^2 - t^1)\Delta_2} \cdots [D_2, b^q] e^{-(1 - t^q)\Delta_2}\Big)\, ds dt. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathrm{ch}^{\mathrm{JLO}}_{D_1}(\alpha) \mathrm{ch}^{\mathrm{JLO}}_{D_2}(\beta) &= \int_{\Sigma^p \times \Sigma^q} \mathrm{Tr}\Big(\varepsilon_1 a^0 e^{-s^1\Delta_1}[D_1, a^1] e^{-(s^2 - s^1)\Delta_1} \dots [D_1, a^p] e^{-(1 - s^p)\Delta_1}\Big) \\ &\cdot \mathrm{Tr}\Big(\varepsilon_2 b^0 e^{-t^1\Delta_2}[D_2, b^1] e^{-(t^2 - t^1)\Delta_2} \dots [D_2, b^q] e^{-(1 - t^q)\Delta_2}\Big)\, ds dt \end{aligned}$$

by Fubini's theorem. Using the decomposition of the product simplex $\Sigma^p \times \Sigma^q$ into $p+q$-simplices given by the shuffle permutations as in Lemma 5.1.7, we get the result. $\square$

5.1.3 Cyclic Shuffle Product in Cyclic Theory

Unfortunately, the shuffle product is not the right product for *cyclic* theory: it is not a derivation with respect to the Connes differential B. The remedy is given by the *cyclic* shuffle product, which we discuss now.

Definition 5.1.13. We order the set $\{(1,0), \dots, (1,p), (2,0), \dots, (2,q)\}$ lexicographically:

$$(1,0) < (1,1) < \cdots < (1,p) < (2,0) < (2,1) < \dots (2,q).$$

A permutation σ of $\{(1,0), \dots, (1,p), (2,0), \dots, (2,q)\}$ is called a *cyclic* $(p+1, q+1)$-*shuffle* if there exists $k \in \{0, \dots, p\}$ and $l \in \{0, \dots, q\}$ such that

$$\sigma(1,k) < \sigma(1,k+1) < \cdots < \sigma(1,p) < \sigma(1,0) < \dots \sigma(1,k-1)$$
$$\sigma(2,l) < \sigma(2,l+1) < \cdots < \sigma(2,q) < \sigma(2,0) < \dots \sigma(2,l-1)$$

and

$$\sigma(1,0) < \sigma(2,0). \tag{5.1.1}$$

Remark 5.1.14. (1) The reason we defined cyclic $(p+1, q+1)$-shuffles instead of cyclic (p,q)-shuffles is purely for convenience: it makes the notations in the proof of Theorem 5.1.27 simpler.

(2) One "explanation" for the condition (5.1.1) is Lemma 5.1.17.

(3) Our definition is different from [Lod92, Definition 4.3.2]: a permutation σ is a cyclic $(p+1, q+1)$-shuffle by J-L. Loday's definition if and only if σ^{-1} is a cyclic $(p+1, q+1)$-shuffle by our definition. Our convention is consistent with [GJ90, page 17] and [BG94, page 11] and it naturally shows up in the proof of Theorem 5.1.27. It was also noted by A. Bauval [Bau98] that this is for other purposes a better definition.

Example 5.1.15. We identify $\{(1,0), \dots, (1,p); (2,0), \dots, (2,q)\}$ with $\{1, \dots, p+q+2\}$ and write

$$[\sigma^{-1}(1), \dots, \sigma^{-1}(p); \sigma^{-1}(p+1), \dots, \sigma^{-1}(p+q+2)]$$

for the permutation

$$\sigma = \begin{pmatrix} (1,0) & \dots & (2,q) \\ \sigma(1,0) & \dots & \sigma(2,q) \end{pmatrix}$$

of $\{(1,0), \dots, (1,p); (2,0), \dots, (2,q)\}$.

There are three cyclic $(2,1)$-shuffles:

$$[1,2;3],\ [1,3;2], \quad \text{and} \quad [2,1;3]$$

and twelve cyclic $(2,2)$-shuffles:

$$\begin{array}{lllll} [1,2;3,4] & [1,3;2,4] & [1,3;4,2] & & \\ [2,1;3,4] & & & & \\ [1,2;4,3] & [1,4;2,3] & [1,4;3,2] & [4,1;2,3] & [4,1;3,2] \\ [2,1;4,3] & [2,4;1,3] & [4,2;1,3] & & \end{array}$$

First we note that cyclic $(p+1,q+1)$-shuffles act on $[0,1]^{p+1} \times [0,1]^{q+1}$ naturally, by permuting the coordinates:

$$\sigma(r^{1,0}, r^{1,1}, \dots, r^{1,p}; r^{2,0}, r^{2,1}, \dots, r^{2,q}) := (r^{\sigma^{-1}(1,0)}, \dots, r^{\sigma^{-1}(1,p)}; r^{\sigma^{-1}(2,0)}, \dots, r^{\sigma^{-1}(2,q)}).$$

Notation 5.1.16. For $x \in [0,2]$, we write

$$x \mod 1 := \begin{cases} x, & 0 \leq x \leq 1 \\ x-1, & 1 < x \leq 2 \end{cases}.$$

For $(r,s,t) \in [0,1]^2 \times [0,1]^p \times [0,1]^q$, let

$$r+s+t := (r^1, r^1+s^1, \ldots, r^1+s^p; r^2, r^2+t^1, \ldots, r^2+t^q) \mod 1.$$

Lemma 5.1.17. *Let p, q be natural numbers. For a cyclic $(p+1,q+1)$-shuffle σ we put*

$$\Sigma(2;\sigma) := \{(r,s,t) \in \Sigma^2 \times \Sigma^p \times \Sigma^q \subset [0,1]^{2+p+q} \mid \sigma(r+s+t) \in \Sigma^{2+p+q}\}.$$

Then the sets $\Sigma(2;\sigma)$ give a decomposition of $\Sigma^2 \times \Sigma^p \times \Sigma^q$ into $2+p+q$-simplices.

This is contained in the proof of [BG94, Theorem 3.2(3)]. For completeness, we give a proof.

Proof. Assume that all entries of $r+s+t$ are distinct. Then since $0 \leq r^1 < r^2 \leq 1$ we see that $\sigma(1,0) < \sigma(2,0)$. Now let $s^0 = 0$ and $t^0 = 0$. Then we can write $r+s+t$ as

$$(r^1+s^0, r^1+s^1, \ldots, r^1+s^p; r^2+t^0, r^2+t^1, \ldots, r^2+t^q) \mod 1.$$

Let k be the first index such that $r^1+s^k > 1$. Then $(r^1+s^0, r^1+s^1, \ldots, r^1+s^q) \mod 1$ is equal to

$$(r^1+s^0, r^1+s^1, \ldots, r^1+s^{k-1}, r^1+s^k-1, \ldots, r^1+s^p-1)$$

and, since $r^1+s^p-1 < r^1+s^0$, by applying a (unique) cyclic permutation we get them in increasing order:

$$(r^1+s^k-1, \ldots, r^1+s^p-1, r^1+s^0, r^1+s^1, \ldots, r^1+s^{k-1}).$$

If there is no k with $r^1+s^k > 1$, then applying $\mod 1$ has no effect and we don't need to do anything. Similarly for $(r^2+t^0, r^2+t^1, \ldots, r^2+t^q)$, there exist an l such that

$$(r^2+t^l-1, \ldots, r^2+t^q-1, r^2+t^0, \ldots, r^2+t^{l-1})$$

is in $[0,1]^{q+1}$ and in increasing order. Thus combining the two we get an element of $\Sigma^{p+1} \times \Sigma^{q+1}$ and so are in position to apply Lemma 5.1.7 to obtain a $(p+1,q+1)$-shuffle.

In summary, there is a unique cyclic $(p+1,q+1)$-shuffle that puts $r+s+t$ into increasing order, *i.e.* an element of Σ^{2+p+q}. Moreover, all cyclic $(p+1,q+1)$-shuffles are used this way. The proof is complete. $\square$

Example 5.1.18. For instance, with $p = q = 1$,

$$\begin{aligned}[1,3;4,2]((0,1/2)+(1/3)+(3/4)) &= [1,3;4,2](0,1/3;1/2,1/4) \\ &= (0,1/4;1/3,1/2).\end{aligned}$$

Let cyclic $(p+1, q+1)$-shuffles act on Ω^{p+q+2} by:

$$\sigma(c^{0,0}, c^{1,0}, \ldots, c^{1,p}, c^{2,0}, \ldots, c^{2,q}) := (-1)^{\sigma}(c^{0,0}, c^{\sigma^{-1}(1,0)}, \ldots, c^{\sigma^{-1}(1,p)}, c^{\sigma^{-1}(2,0)}, \ldots, c^{\sigma^{-1}(2,q)}).$$

Definition 5.1.19. Let $\alpha = (a^0, \ldots, a^p) \in \Omega^p(\mathcal{A}_1)$ and $\beta = (b^0, \ldots, b^q) \in \Omega^q(\mathcal{A}_2)$. Define the *cyclic shuffle product* $\alpha \times' \beta \in \Omega^{p+q+2}(\mathcal{A}_1 \otimes \mathcal{A}_2)$ by

$$\alpha \times' \beta := \sum_{\text{cyclic } (p+1,\, q+1)\text{-shuffles}} \sigma(1 \otimes 1, a^0 \otimes 1, \ldots, a^p \otimes 1, 1 \otimes b^0, \ldots 1 \otimes b^q).$$

Remark 5.1.20. Note that we always have $d\alpha \times' \beta = 0$ and $\alpha \times' d\beta = 0$, since $d^2 = 0$. In particular, for any α and β,

$$B\alpha \times' \beta = 0 \quad \text{and} \quad \alpha \times' B\beta = 0.$$

Lemma 5.1.21. *The obstruction to B being a derivation for the shuffle product is equal (up to sign) to the obstruction of b being a derivation for the cyclic shuffle product. That is for $\alpha \in \Omega^p(\mathcal{A}_1)$ and $\beta \in \Omega^q(\mathcal{A}_2)$*

$$\begin{aligned}&B(\alpha \times \beta) - B\alpha \times \beta - (-1)^{|\alpha|}\alpha \times B\beta \\ &\qquad + b(\alpha \times' \beta) - b\alpha \times' \beta - (-1)^{|\alpha|}\alpha \times' b\beta = 0\end{aligned}$$

in $\Omega^{p+q+1}(\mathcal{A}_1 \otimes \mathcal{A}_2)$.

Proof. See [Lod92, Proposition 4.3.3]. □

Definition 5.1.22. The *exterior product* of simplicially-normalized negative cycles is given by

$$\times + v\times' : CN_\bullet(\mathcal{A}_1) \widehat{\otimes}_{\mathbb{C}[v]} CN_\bullet(\mathcal{A}_2) \to CN_\bullet(\mathcal{A}_1 \otimes \mathcal{A}_2).$$

It follows from Lemma 5.1.11 and Lemma 5.1.21 that the exterior product is a chain map.

Theorem 5.1.23 (Eilenberg-Zilber Theorem for Negative Cyclic Homology)**.** *The exterior product induces isomorphism on homology* i.e. *it is a quasi-isomorphism.*

Proof. See [Lod92, 4.3.1] and [Lod92, Theorem 4.3.8]. □

Theorem 5.1.24. *The Chern character is compatible with the product structure: the diagram*

$$\begin{array}{ccc} K_0(\mathcal{A}_1)\,\widehat{\otimes}\,K_0(\mathcal{A}_2) & \longrightarrow & K_0(\mathcal{A}_1\otimes\mathcal{A}_2) \\ \downarrow & & \downarrow \\ HN_0(\mathcal{A}_1)\,\widehat{\otimes}\,HN_0(\mathcal{A}_2) & \longrightarrow & HN_0(\mathcal{A}_1\otimes\mathcal{A}_2) \end{array}$$

where the vertical maps are given by Chern character and the horizontal maps are given by exterior product is commutative.

Proof. See [Lod92, Proposition 8.3.8]. □

By the Eilenberg-Zilber theorem, in order to define the product of two periodic cyclic cocycles, it is enough to give the values of the product on cycles of the form $\alpha\times\beta+v\alpha\times'\beta$.

Definition 5.1.25. We say that $\phi\in CP^\bullet(\mathcal{A}_1\otimes\mathcal{A}_2)$ is the product of two cocycles $\phi^1\in CP^\bullet(\mathcal{A}_1)$ and $\phi^2\in CP^\bullet(\mathcal{A}_2)$ if for any $\alpha\in\Omega^\bullet(\mathcal{A}_1)$ and $\beta\in\Omega^\bullet(\mathcal{A}_2)$,

$$\langle\phi,\alpha\times\beta+v\alpha\times'\beta\rangle=\left\langle\phi^1\,\widehat{\otimes}\,\phi^2,\alpha\,\widehat{\otimes}\,\beta\right\rangle.$$

Lemma 5.1.26. *The exterior product of (b,B)-cocycles is compatible with the exterior product of cyclic cocycles.*

Proof. In Example 5.1.10 we saw that if ϕ^1 and ϕ^2 are cyclic cocycles, then

$$\left\langle\phi^1\times\phi^2,\alpha^1\times\alpha^2\right\rangle=\left\langle\phi^1\,\widehat{\otimes}\,\phi^2,\alpha^1\,\widehat{\otimes}\,\alpha^2\right\rangle.$$

On the other hand, since ϕ^1 and ϕ^2 are closed traces, $\left\langle\phi^1\times\phi^2,\alpha\times'\beta\right\rangle=0$. Consequently, $\phi^1\times\phi^2$ is the product of ϕ^1 and ϕ^2. □

Theorem 5.1.27. *Let $(\mathcal{A}_1,\,\mathfrak{H}_1,\,D_1)$ and $(\mathcal{A}_2,\,\mathfrak{H}_2,\,D_2)$ be θ-summable spectral triples. Then*

$$\mathrm{ch}^{\mathrm{JLO}}_{D_1\times D_2}(\alpha\times'\beta)=\frac{1}{2}B\mathrm{ch}^{\mathrm{JLO}}_{D_1}(\alpha)B\mathrm{ch}^{\mathrm{JLO}}_{D_2}(\beta)$$

for $\alpha\in\Omega^p(\mathcal{A}_1)$ and $\beta\in\Omega^q(\mathcal{A}_2)$.

Proof. The proof is similar to the proof of Theorem 5.1.12. It depends on the interpretation of the cyclic shuffles as a simplicial decomposition of $\Sigma^2\times\Sigma^p\times\Sigma^q$ into $2+p+q$-simplices. Again we put $D\coloneqq D_1\times D_2$ and

$$\Delta_1\coloneqq D_1^2,\quad \Delta_2\coloneqq D_2^2,\quad\text{and}\quad \Delta\coloneqq D^2=\Delta_1\otimes 1+1\otimes\Delta_2.$$

Let $\gamma=(c^{0,0},c^{1,0},\ldots,c^{1,p},c^{2,0},\ldots,c^{2,q})\coloneqq(1\otimes 1,a^0\otimes 1,\ldots,a^p\otimes 1,1\otimes b^0,\ldots,1\otimes b^q)\in\Omega^{2+p+q}(\mathcal{A}_1\otimes\mathcal{A}_2)$. Then for any cyclic $(p+1,q+1)$-shuffle σ, with $(\sigma(1,0)<\sigma(2,0)$ and)

$$\sigma(1,k)<\cdots<\sigma(1,p)<\sigma(1,0)<\cdots<\sigma(1,k-1)$$

$$\sigma(2,l) < \cdots < \sigma(2,q) < \sigma(2,0) < \cdots < \sigma(2,l-1),$$

we have

$$\begin{aligned}
\mathrm{ch}_D^{\mathrm{JLO}}(\sigma(\gamma)) &= \int_{r\in\Sigma^{2+p+q}} (-1)^\sigma \mathrm{Tr}(\varepsilon_1\otimes\varepsilon_2 \cdot c^{0,0}\cdot e^{-r^{1,0}\Delta}[D,c^{\sigma^{-1}(1,0)}]e^{-(r^{1,1}-r^{1,0})\Delta} \\
&\quad \cdots [D,c^{\sigma^{-1}(2,0)}]e^{-(r^{1,p}-r^{2,0})\Delta}\cdots [D,c^{\sigma^{-1}(2,q)}]e^{-(1-r^{2,q})\Delta})dr \\
&= \int_{r\in\Sigma^{2+p+q}} \pm\mathrm{Tr}\left(\varepsilon_1 e^{-r^{\sigma(1,k)}\Delta_1}[D_1,a^k]e^{-(r^{\sigma(1,k+1)}-r^{\sigma(1,k)})\Delta_1}\dots[D_1,a^{k-1}]e^{-(1-r^{\sigma(1,k-1)})\Delta_1}\right) \\
&\quad \cdot\mathrm{Tr}\left(\varepsilon_2 e^{-r^{\sigma(2,l)}\Delta_2}[D_2,b^l]e^{-(r^{\sigma(2,l+1)}-r^{\sigma(2,l)})\Delta_2}\dots[D_2,b^{l-1}]e^{-(1-r^{\sigma(2,l-1)})\Delta_2}\right)dr \\
&= \int_{r\in\Sigma^{2+p+q}} \mathrm{Tr}\left(\varepsilon_1 e^{-r^{\sigma(1,0)}\Delta_1}[D_1,a^0]e^{-(r^{\sigma(1,1)}-r^{\sigma(1,0)})\Delta_1}\dots[D_1,a^{p}]e^{-(1-r^{\sigma(1,p)})\Delta_1}\right) \\
&\quad \cdot\mathrm{Tr}\left(\varepsilon_2 e^{-r^{\sigma(2,0)}\Delta_2}[D_2,b^0]e^{-(r^{\sigma(2,1)}-r^{\sigma(2,0)})\Delta_2}\dots[D_2,b^{q}]e^{-(1-r^{\sigma(2,q)})\Delta_2}\right)dr.
\end{aligned}$$

Hence using Lemma 5.1.17, we see that

$$\begin{aligned}
&\mathrm{ch}_D^{\mathrm{JLO}}(\alpha\times'\beta) \\
&\quad = \int_{\Sigma^2\times\Sigma^p\times\Sigma^q} \mathrm{Tr}\left(\varepsilon_1 e^{-r_1\Delta}[D_1,a^0]e^{-s^1\Delta_1}[D_1,a^1]e^{-(s^2-s^1)\Delta_1}\dots[D_1,a^p]e^{-(1-s^p-r_1)\Delta_1}\right) \\
&\qquad \cdot\mathrm{Tr}\left(\varepsilon_2 e^{-r_2\Delta_2}[D_2,b^0]e^{-t^1\Delta_2}[D_2,b^1]e^{-(t^2-t^1)\Delta_2}\dots[D_2,b^q]e^{-(1-t^q-r_2)\Delta_2}\right)drdsdt \\
&\quad = \int_{\Sigma^2\times\Sigma^p\times\Sigma^q} \mathrm{Tr}\left(\varepsilon_1 [D_1,a^0]e^{-s^1\Delta_1}[D_1,a^1]e^{-(s^2-s^1)\Delta_1}\dots[D_1,a^p]e^{-(1-s^p)\Delta_1}\right) \\
&\qquad \cdot\mathrm{Tr}\left(\varepsilon_2 [D_2,b^0]e^{-t^1\Delta_2}[D_2,b^1]e^{-(t^2-t^1)\Delta_2}\dots[D_2,b^q]e^{-(1-t^q)\Delta_2}\right)drdsdt \\
&\quad = \frac{1}{2}\mathrm{Tr}\left(\varepsilon_1\langle[D_1,a^0],\dots,[D_1,a^p]\rangle_{D_1}^{\mathrm{JLO}}\right)\mathrm{Tr}\left(\varepsilon_2\langle[D_2,b^0],\dots,[D_2,b^q]\rangle_{D_2}^{\mathrm{JLO}}\right).
\end{aligned}$$

Now Lemma 4.3.10 completes the proof. □

Remark 5.1.28. If one of p or q is even, then the cocycle $\mathrm{ch}^{\mathrm{JLO}}_{D_1\times D_2}$ vanishes on $\Omega^p(\mathcal{A}_1)\times'\Omega^q(\mathcal{A}_2)$ for parity reasons.

Definition 5.1.29. Let $(\mathcal{A},\mathfrak{H},D)$ be a θ-summable spectral triple. We define the *perturbed JLO cocycle* as

$$\mathrm{ch}_\bullet^{\mathrm{pert}} = \mathrm{ch}_\bullet^{\mathrm{JLO}} + \frac{1}{\sqrt{2}}B\mathrm{ch}_{\bullet-1}^{\mathrm{JLO}}$$

This is indeed a cocycle since,

$$\begin{aligned}
(b+B)\mathrm{ch}_\bullet^{\mathrm{pert}} &= (b+B)(\mathrm{ch}_\bullet^{\mathrm{JLO}} + 2^{-\frac{1}{2}}B\mathrm{ch}_{\bullet-1}^{\mathrm{JLO}}) \\
&= 2^{-\frac{1}{2}}bB\mathrm{ch}_{\bullet-1}^{\mathrm{JLO}} \\
&= -2^{-\frac{1}{2}}Bb\mathrm{ch}_{\bullet-1}^{\mathrm{JLO}} \\
&= 2^{-\frac{1}{2}}B(b+B)\mathrm{ch}_{\bullet-1}^{\mathrm{JLO}} \\
&= 0.
\end{aligned}$$

The following is the main theorem of this section. It shows that the perturbed JLO cochain is multiplicative.

Theorem 5.1.30. *Let $(\mathcal{A}_1, \mathfrak{H}_1, D_1)$ and $(\mathcal{A}_2, \mathfrak{H}_2, D_2)$ be θ-summable spectral triples. Then for $\alpha \in \Omega(\mathcal{A}_1)$ and $\beta \in \Omega(\mathcal{A}_2)$*

$$\mathrm{ch}^{\mathrm{pert}}_{D_1 \times D_2}(\alpha \times \beta + v\alpha \times' \beta) = \mathrm{ch}^{\mathrm{pert}}_{D_1}(\alpha)\mathrm{ch}^{\mathrm{pert}}_{D_2}(\beta).$$

Proof. First we introduce some notation. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a θ-summable spectral triple. Let

$$\delta(a^0, a^1 \dots, a^p) \coloneqq \frac{1}{\sqrt{2}}([D, a^0], a^1, \dots, a^p).$$

Note that $[D, a^0]$ does *not* necessarily belong to $\mathcal{A}$, but we can work in $\mathcal{D}^0$, the algebra generated by $\mathcal{A} \cup [D, \mathcal{A}]$. Then

$$\frac{1}{\sqrt{2}} B\mathrm{ch}^{\mathrm{JLO}}_D(\omega) = \mathrm{ch}^{\mathrm{JLO}}_D(\delta\omega)$$

and

$$\mathrm{ch}^{\mathrm{pert}}_D(\omega) = \mathrm{ch}^{\mathrm{JLO}}_D\left((1 + \delta)(\omega)\right).$$

Now we write δ_1, δ_2 and δ_{12} for the δ corresponding to the spectral triples $(\mathcal{A}_1, \mathfrak{H}_1, D_1)$, $(\mathcal{A}_2, \mathfrak{H}_2, D_2)$ and $(\mathcal{A}_1, \mathfrak{H}_1, D_1) \times (\mathcal{A}_2, \mathfrak{H}_2, D_2)$, respectively. Then, by definition,

$$\begin{aligned}
\delta_{12}(\alpha \times \beta) &= \sum_{\chi} (-1)^{\chi} \chi([D_1 \times D_2, a^0 \otimes b^0], a^1 \otimes 1, \dots, a^p \otimes 1, 1 \otimes b^1, \dots, 1 \otimes b^q) \\
&= \sum_{\chi} (-1)^{\chi} \chi([D_1, a^0] \otimes b^0 + a^0 \otimes [D_2, b^0], a^1 \otimes 1, \dots, a^p \otimes 1, 1 \otimes b^1, \dots, 1 \otimes b^q) \\
&= \delta_1(\alpha) \times \beta + \alpha \times \delta_2(\beta)
\end{aligned}$$

and $\delta_{12}(\alpha \times' \beta) = 0$ because all the summands start with the term $[D_1 \times D_2, 1 \otimes 1] = 0$.

Therefore,

$$\begin{aligned}
\mathrm{ch}^{\mathrm{pert}}_{D_1 \times D_2}(\alpha \times \beta + v\alpha \times' \beta) &= \mathrm{ch}^{\mathrm{JLO}}_{D_1 \times D_2}\left((1 + \delta_{12})(\alpha \times \beta + v\alpha \times' \beta)\right) \\
&= \mathrm{ch}^{\mathrm{JLO}}_{D_1 \times D_2}(\alpha \times \beta + v\alpha \times' \beta + \delta_1(\alpha) \times \beta + \alpha \times \delta_2(\beta)) \\
&= \mathrm{ch}^{\mathrm{JLO}}_{D_1}(\alpha)\mathrm{ch}^{\mathrm{JLO}}_{D_2}(\beta) + \mathrm{ch}^{\mathrm{JLO}}_{D_1}(\delta_1\alpha)\mathrm{ch}^{\mathrm{JLO}}_{D_2}(\delta_2\beta) \\
&\quad + \mathrm{ch}^{\mathrm{JLO}}_{D_1}(\delta_1\alpha)\mathrm{ch}^{\mathrm{JLO}}_{D_2}(\beta) + \mathrm{ch}^{\mathrm{JLO}}_{D_1}(\alpha)\mathrm{ch}^{\mathrm{JLO}}_{D_2}(\delta_2\beta) \\
&= \mathrm{ch}^{\mathrm{JLO}}_{D_1}\left((1 + \delta_1)(\alpha)\right)\mathrm{ch}^{\mathrm{JLO}}_{D_2}\left((1 + \delta_2)(\beta)\right) \\
&= \mathrm{ch}^{\mathrm{pert}}_{D_1}(\alpha)\mathrm{ch}^{\mathrm{pert}}_{D_2}(\beta)
\end{aligned}$$

by Theorem 5.1.12 and Theorem 5.1.27. ∎

Corollary 5.1.31. *The perturbed JLO character implements the diagram in Lemma 2.2.26.* ∎

5.2 The Local Perturbed JLO character

5.2.1 Asymptotic Expansion Property

Now we make the perturbed JLO-cocycle local. We employ the same technique as Connes and Moscovici [CM95].

Definition 5.2.1. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple and let $\mathcal{D}^0$ denote the algebra generated by $\mathcal{A} \cup [D, \mathcal{A}]$. We say that $(\mathcal{A}, \mathfrak{H}, D)$ has the *asymptotic expansion property* if $(\mathcal{A}, \mathfrak{H}, D)$ is θ-summable and there exists an $n \in \mathbb{Z}_{\geq 0}$ such that for any $X^0, \dots, X^p \in \mathcal{D}^0$, the map $\mathrm{Str}\langle X^0, X^1, \dots, X^p\rangle_s^{\mathrm{JLO}}$ has an asymptotic expansion as $s \in \mathbb{R}_+^\times$ decreases to 0 of the form:

$$\mathrm{Str}\langle X^0, X^1, \dots, X^p\rangle_s^{\mathrm{JLO}} \sim \sum_{k=-n}^{\infty} \phi_p^k(X^0, \dots, X^p) s^{\frac{k}{2}}. \tag{5.2.1}$$

Since the JLO character is an improper (b, B)-cocycle, it is clear that $\phi^k = (\phi_p^k)$ gives an improper (b, B)-cocycle on $\mathcal{A}$ by

$$(a^0, \dots, a^p) \mapsto \phi_p^k(a^0, [D, a^1], \dots, [D, a^p]).$$

Example 5.2.2. The Dirac spectral triple associated to a closed spin manifold has the asymptotic expansion property.

If $(\mathcal{A}, \mathfrak{H}, D)$ has the asymptotic expansion property then the perturbed JLO character also has an asymptotic expansion:

$$\mathrm{ch}_D^{\mathrm{pert}}(a^0, \dots, a^p) \sim \sum_{k \geq -n} \phi^k(a^0 + 2^{-\frac{1}{2}}[D, a^0], [D, a^1], \dots, [D, a^p]) s^{\frac{k}{2}}.$$

Definition 5.2.3. Let $(\mathcal{A}, \mathfrak{H}, D)$ be a spectral triple with the asymptotic expansion property. Let the asymptotic expansion be given by (5.2.1). We define the *asymptotic JLO character* as

$$\mathrm{ch}_D^{\mathrm{asym}} := \sum_{k \geq -n} \psi^k t^k \in \prod_{p \geq 0} \mathrm{Hom}(\Omega^p, \mathbb{C})[t^{-1}, t]]$$

where $\psi^k = (\psi_p^k)$ is the improper mixed cocycle given by

$$\begin{aligned} \psi_p^k(a^0, \dots, a^p) &:= \phi^k(a^0 + 2^{-\frac{1}{2}}[D, a^0], [D, a^1], \dots, [D, a^p]) \\ &= \begin{cases} \phi^k(a^0, [D, a^1], \dots, [D, a^p]), & \text{if } p \text{ is even} \\ 2^{-\frac{1}{2}} \phi^k([D, a^0], [D, a^1], \dots, [D, a^p]), & \text{if } p \text{ is odd} \end{cases}. \end{aligned}$$

Since $\mathrm{ch}_D^{\mathrm{pert}}$ is a (b, B)-cocycle, we see that ψ^k is indeed an improper *cocycle* for any k.

In the finitely summable case we have the following.

Theorem 5.2.4. *Let $(\mathcal{A}, \mathfrak{H}, D)$ be a finitely summable spectral triple with the asymptotic expansion property. Then $(\mathcal{A}, \mathfrak{H}, D)$ has the meromorphic continuation property with simple and discrete dimension spectrum contained in the set $n - \mathbb{Z}_{\geq 0}$. Moreover, the coefficients ϕ^k can be computed as*

$$\phi^k(X^0, \ldots, X^p) = \mathrm{Res}_{z=-\frac{k+p}{2}} \mathrm{Str}\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}.$$

In particular, the constant coefficient ϕ^0 gives the Connes-Moscovici character:

$$\mathrm{ch}_p^{\mathrm{CM}}(a^0, \ldots, a^p) = \phi_p^0(a^0, [D, a^1], \ldots, [D, a^p]).$$

First we have the following lemma.

Lemma 5.2.5. *Let $f \in C^\infty(\mathbb{R}_+^\times)$ be a function with asymptotic expansion for small s of the form*

$$f(s) \sim \sum_{k \geq -n} f_k s^{\frac{k}{2}}$$

and which decays exponentially at infinity, that is, for some $\lambda > 0$, and s sufficiently large,

$$f(s) \leq Ce^{-s\lambda}.$$

Then we have the following.

1. *The Mellin transform*
$$\mathrm{M}[f](z) := \int_0^\infty s^{z-1} f(s) ds$$
of f is a meromorphic function with simple poles contained in the set $\frac{1}{2}(n - \mathbb{Z}_{\geq 0})$;

2. *For any $k \geq -n$, the coefficient f_k can be computed by $\mathrm{Res}_{z=-\frac{k}{2}} \mathrm{M}[f](z)$.*

Proof. See [BGV92, Lemma 9.34]. □

Proof of Theorem 5.2.4. Since $(\mathcal{A}, \mathfrak{H}, D)$ is finitely summable, $\mathrm{Str}\langle X^0, \ldots, X^p\rangle_s^{\mathrm{JLO}}$ satisfies the exponential decay property (cf. [GBVF01, (10.47)]), hence the lemma is applicable. Moreover, its Mellin transform is given by Lemma 4.3.12:

$$\mathrm{M}[\mathrm{Str}\langle X^0, \ldots, X^p\rangle_s^{\mathrm{JLO}}] = \mathrm{Str}\langle X^0, \ldots, X^p\rangle_{z-\frac{p}{2}}^{\mathrm{CM}}.$$

Thus, the functions $\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}$ have simple poles contained in the set $\frac{1}{2}(n - p - \mathbb{Z}_{\geq 0})$ and

$$\phi^k(X^0, \ldots, X^p) = \mathrm{Res}_{z=-\frac{k+p}{2}} \langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}.$$

□

5.2.2 Integral Meromorphic Continuation Property

Now we look at the question when we have the asymptotic expansion property.

Theorem 5.2.6. *A finitely-summable spectral triple with the integral meromorphic continuation property (cf. Definition 3.2.29) has the asymptotic expansion property.*

Proof. We know that it has simple and discrete dimension spectrum contained in $\{0, 1, \ldots, d\}$ by Theorem 3.2.11. Moreover, since the meromorphic continuation is obtained by applying the Euler vector field repeatedly, we see that they satisfy the following growth condition: *in each strip* $\alpha < \mathrm{Re}(z) < \beta$ *and* $N > 0$*, one has*

$$|\mathrm{Str}\langle X^0, \ldots, X^p\rangle_z^{\mathrm{CM}}| = O(|z|^{-N})$$

as $|z| \to \infty$*.* Now [Hig06, Theorem A.5] completes the proof. $\square$

Next we consider two θ-summable spectral triples $(\mathcal{A}_1, \mathfrak{H}_1, D_1)$ and $(\mathcal{A}_2, \mathfrak{H}_2, D_2)$ with the asymptotic expansion property. Assume that their product also has the asymptotic expansion property. Then the multiplicative property of the perturbed JLO cocycle (Theorem 5.1.30) implies that for any $\alpha \in \Omega(\mathcal{A}_1)$ and $\beta \in \Omega(\mathcal{A}_2)$,

$$\mathrm{ch}^{\mathrm{asym}}_{D_1 \times D_2}(\alpha \times \beta + v\alpha \times' \beta) = \mathrm{ch}^{\mathrm{asym}}_{D_1}(\alpha)\mathrm{ch}^{\mathrm{asym}}_{D_2}(\beta) \quad \text{in} \quad \mathbb{C}[t^{-1}, t]].$$

As a corollary, we obtain the following.

Theorem 5.2.7. *Let* $(\mathcal{A}_1, \mathfrak{H}_1, D_1)$ *and* $(\mathcal{A}_2, \mathfrak{H}_2, D_2)$ *be spectral triples with the integral meromorphic continuation property. Then for any* $\alpha \in \Omega^\bullet(\mathcal{A}_1)$ *and* $\beta \in \Omega^\bullet(\mathcal{A}_2)$*,*

$$\mathrm{ch}^{\mathrm{asym}}_{D_1 \times D_2}(\alpha \times \beta + v\alpha \times' \beta) = \mathrm{ch}^{\mathrm{asym}}_{D_1}(\alpha)\mathrm{ch}^{\mathrm{asym}}_{D_2}(\beta) \quad \textit{in} \quad \mathbb{C}[t^{-1}, t]].$$

Proof. It is enough to note that the product again has the integral meromorphic continuation property by Proposition 3.2.30. $\square$

Chapter 6

Future Directions

Our general aim, so far unmet, is to use the freedom gained from considering cocycles with values in formal Laurent series to construct more easily analyzable cocycles in para-Riemannian index theory, and then apply the multiplicativity properties to calculate the index class. This similar to the strategy Nest and Tsygan used in the proof of their algebraic index theorem [NT95a, NT95b]. At the topological/homotopical level, the signature asymptotic morphism can be modified to accommodate para-Riemannian transformations. However, constructing a *computable* local cocycle out of it seems to be a much harder problem and there is still much that remains mysterious.

6.1 Para-Riemannian Manifolds

Para-Riemannian manifolds were first considered in [CM95]. Other terminologies such as "triangulation" and "upper-triangular structure" are also used.

Definition 6.1.1. Let P be a smooth manifold and $V \subseteq TP$ be an integrable subbundle and let $N = TP/V$ be the transverse bundle. A choice of Euclidean structures on V and N is called a *para-Riemannian* structure. A diffeomorphism $\psi : P \to P$ is said to be a *para-Riemannian transformation* if $\psi' : TP \to TP$ preserves V, *i.e.* $\psi'(V) \subseteq V$, and $\psi' : V \to V$ and $\psi' : N \to N$ are isometries.

We assume that such a structure is fixed and furthermore V and N are both oriented and even-dimensional. Then P has a natural volume form and the complex (bi)graded vector bundle

$$E \coloneqq \Lambda^\bullet V_{\mathbb{C}}^* \mathbin{\widehat{\otimes}} \Lambda^\bullet N_{\mathbb{C}}^*$$

has a natural Hermitian structure. Hence the Hilbert space $L^2(P, E)$ is defined without any additional choice. Fix an isomorphism $j : V \oplus N \to TP$ such that the restriction to V is the inclusion and the associated connection is torsion-free. Then j gives rise to an isomorphism, again denoted by j,

$$j : E \to \Lambda^\bullet T^*_{\mathbb{C}} P.$$

Following Connes and Moscovici, we define the vertical exterior derivative d_V and the horizontal exterior derivative d_H by

$$d_V\omega := (r+1,s)\text{-component of } j^{-1}d(j\omega)$$
$$d_H\omega := (r,s+1)\text{-component of } j^{-1}d(j\omega)$$

for $\omega \in C^\infty(P, \Lambda^r V^*_{\mathbb{C}} \mathbin{\widehat{\otimes}} \Lambda^s N^*_{\mathbb{C}})$.

Definition 6.1.2. We define the *hypoelliptic signature operator* Q on E as the graded sum

$$(d_V^* d_V - d_V d_V^*)(-1)^{\partial_N} + (d_H + d_H^*),$$

where $(-1)^{\partial_N}$ is the parity operator in the transversal direction.

Notice that Q is of order two in the longitudinal direction and of order one in the transversal direction.

Let D be defined by the equation

$$D|D| = Q$$

or, equivalently, be given by the formula (for Q essentially self-adjoint)

$$D := \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{Q}{Q^2+\mu} \mu^{-\frac{1}{4}} \, d\mu.$$

Then it follows from the calculus of pseudodifferential operators on a Heisenberg manifold (cf. [BG88]) that

$$(C^\infty_{\mathrm{c}}(P) \rtimes \Gamma, L^2(P,E), D)$$

is a $(\dim V + 2\dim N, \infty)$-summable spectral triple for any group Γ of diffeomorphisms preserving the para-Riemannian structure (cf. [CM95, Theorem I.1, Lemma I.3]). Moreover, it satisfies the assumptions of the local index theorem (cf. [CM95, Theorem I.2]). See also Example 3.2.14.

Theorem 6.1.3. *Let (P,V) be a para-Riemannian manifold and let*

$$(C^\infty_{\mathrm{c}}(P) \rtimes \Gamma, L^2(P,E), D)$$

be the associated spectral triple. Then the Connes-Moscovici character $\mathrm{ch}^{\mathrm{CM}}_\bullet$ (definition 4.3.16) is an index cocycle for D. In particular, for any idempotent $e \in C^\infty_{\mathrm{c}}(P) \rtimes \Gamma$,

$$\langle D, e\rangle = \sum_{k\geq 0} c_{pk} \mathrm{Res}_{z=0} \mathrm{Str}(e[D,e]^{(k_1)} \dots [D,e]^{(k_p)} |D|^{-2z-p-2|k|}),$$

where the sum is taken over all multi-indices $k = (k_1, \dots, k_p)$ with non-negative integer entries *and the constants c_{pk} are given by the formula*

$$c_{pk} \coloneqq \frac{(-1)^{|k|}}{k!} \frac{\Gamma(k_1 + \cdots + k_p + \frac{p}{2})}{(k_1+1)(k_1+k_2+2)\dots(k_1+\dots k_p+p)}.$$

The term corresponding to $p=0$ and $k=0$ is $\mathrm{Res}_{z=0}(\Gamma(s)\mathrm{Str}(e|D|^{-2z}))$.

6.1.1 Gelfand-Fuks Cohomology

A particularly interesting instance of the construction above can be obtained as follows.

Let M be a smooth manifold of dimension n and let P be the $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n$-bundle over M of Euclidean metrics on TM. Thus a Riemannian structure on M can be identified with a smooth section of $P \to M$. We fix a $\mathrm{GL}_n(\mathbb{R})$-invariant Riemannian structure on $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n$ once and for all. Then P has a canonical para-Riemannian structure, invariant under the action of the full *diffeomorphism group* of M. Since the dimension of $\mathrm{GL}_n(\mathbb{R})/\mathrm{O}_n$ is $\frac{n(n+1)}{2}$, we replace P by $P \times S^1$ for $n = 1, 2 \mod 4$, so that the vertical component has even dimension.

If M is oriented, we can consider all *orientation-preserving diffeomorphisms* by looking at the $\mathrm{GL}_n^+(\mathbb{R})/\mathrm{SO}_n$-bundle over M.

Connes and Moscovici proved in [CM98, CM00, CM01] the existence of a universal cohomological formula for the associated index problem. We refer to *loc.cit.* for the details and record the end result. Let $H^\bullet(\mathrm{WSO}_n)$ denote the oriented *Gelfand-Fuks* cohomology ring (cf. [GF70]).

Theorem 6.1.4. *Let Γ be a discrete group of orientation preserving diffeomorphisms of M. Then there exists a natural characteristic map $\theta : H^\bullet(\mathrm{WSO}_n) \to HP^\bullet(C_c^\infty(P) \rtimes \Gamma)$ and a universal polynomial $L_n \in H^\bullet(\mathrm{WSO}_n)$ such that the identity component of the Connes-Moscovici character of $(C_c^\infty(P) \rtimes \Gamma, L^2(P,E), D)$ is given by $\theta(L_n)$.* □

Explicit formulas of the polynomials L_n are not known.

One crucial observation for the proof of the theorem above is that we can use the operator Q directly, instead of D, in the Connes-Moscovici character. Now instead of the operator Q, we consider the family of operators

$$D_t \coloneqq t(d_V + d_V^*) + t^2(d_H + d_H^*), \quad t \in \mathbb{R}_+^\times.$$

Then while $(\mathcal{A}, \mathfrak{H}, D_t)$ is no longer a spectral triple for a fixed $t \in \mathbb{R}_+^\times$ because $[D_t, \phi^{-1}]$ is not necessarily bounded, we can still construct a K-homology class. For instance, let $\bar{\mathcal{A}}$ denote the C^*-completion of $\mathcal{A}$ in $\mathcal{L}(\mathfrak{H})$. Then

$$C_0(\mathbb{R}) \otimes \bar{\mathcal{A}} \to \mathcal{K}(\mathfrak{H}), \quad g \otimes a \mapsto g(D_{t^{-1}})a$$

gives an asymptotic morphism, that is, an $E(\bar{\mathcal{A}}, \mathbb{C})$-cycle (cf. [GHT00]). Indeed, let $D_V \coloneqq (d_V + d_V^*)(-1)^{\partial_N}$ and let $D_H \coloneqq d_H + d_H^*$. Then for any $\phi^{-1} \in \mathcal{A}$, we have $[D_V, \phi^{-1}] = 0$ and

$[D_H, \phi^{-1}]$ is a" vertical operator", *i.e.* for any $\xi \in \mathrm{dom}(D_t)$,

$$||[D_H, \phi^{-1}]\xi|| \leq C \cdot (||D_V \xi||^2 + ||\xi||^2)^{\frac{1}{2}})$$

for some $C > 0$. Now, since for any $\lambda > 0$,

$$\begin{aligned}[(\lambda + D_t)^{-1}, \phi^{-1}] &= (\lambda + D_t)^{-1}[\phi^{-1}, D_t](\lambda + D_t)^{-1} \\ &= t^2 \cdot (\lambda + D_t)^{-1}[\phi^{-1}, D_H](\lambda + D_t)^{-1},\end{aligned}$$

it is enough to show that $(\lambda + D_t)^{-1}[\phi^{-1}, D_H](\lambda + D_t)^{-1}$ is bounded with norm $o(t^{-2})$ as t goes to 0. But since D_V has bidegree $(1, 0)$ and D_H has bidegree $(0, 1)$, for any $0 < t < \lambda$, we have

$$\begin{aligned}||[\phi^{-1}, D_H]\xi||^2 &\leq C^2(||D_V \xi||^2 + ||\xi||^2) \\ &\leq t^{-2} \cdot C^2(||\lambda\xi||^2 + ||tD_V \xi||^2 + ||t^2 D_H \xi||^2) \\ &= t^{-2} \cdot C^2||(\lambda + D_t)\xi||^2,\end{aligned}$$

for any $\xi \in \mathrm{dom}(D_t)$. Hence, $||(\lambda + D_t)^{-1}[\phi^{-1}, D_H](\lambda + D_t)^{-1}|| \leq \lambda^{-1}Ct^{-1}$.

Alternatively, we see that $(\bar{\mathcal{A}}, \mathfrak{H}, F_t)$, where $F_t \coloneqq D_t(1+D_t^2)^{-\frac{1}{2}}$, gives an asymptotic Fredholm module or a $KE(\bar{\mathcal{A}}, \mathbb{C})$-cycle (cf. [Dum02]).

The advantage of the family D_t is that it a family of first-order elliptic differential operators, hence, usual elliptic theory applies.

6.2 Conformal Transformations

Now, instead of all diffeomorphisms, we consider the group of conformal transformations. This gives a very explicit and easy to work with example.

Let (M, g) be a Riemannian manifold. Recall that a diffeomorphism $\phi : M \to M$ is said to be a *conformal transformation* if there exists a smooth function $c : M \to \mathbb{R}_+^\times$, called the *conformal factor*, such that

$$g_{\phi(m)}(\phi'\xi, \phi'\eta) = c(m)^2 \cdot g_m(\xi, \eta)$$

for any $m \in M$ and ξ, $\eta \in T_m M$.

Let $P = M \times \mathbb{R}_+^\times$ and identify $TP = TM \oplus T\mathbb{R}_+^\times$. We claim that P is naturally a para-Riemannian manifold and conformal transformations of M act on P by para-Riemannian transformations. Indeed, let $V \to P$ denote the *integrable* subbundle

$$V \coloneqq 0 \oplus T\mathbb{R}_+^\times \subset TM \oplus T\mathbb{R}_+^\times$$

and let $N \coloneqq TP/V$. We identify $T\mathbb{R}_+^\times = \mathbb{R}_+^\times \times \mathbb{R}$ and fix a Euclidean structure $g_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *e.g.* $(s, t) \mapsto st$. Also, for $\xi \in T_m M$, $s \in T_\lambda \mathbb{R}_+^\times = \mathbb{R}$, let us write (ξ, s) for the class of $(\xi \oplus s) + V$

in $N_{(m,\lambda)}$. We give Euclidean structures on V and N by

$$\begin{aligned}\langle 0\oplus s, 0\oplus t\rangle_{(m,\lambda)} &\coloneqq g_{\mathbb{R}}(s,t) \quad \text{and}\\ \langle (\xi,s),(\eta,t)\rangle_{(m,\lambda)} &\coloneqq \lambda^{-2} g_m(\xi,\eta).\end{aligned}$$

Now, let $\phi : M \to M$ be a conformal transformation with conformal factor $c : M \to \mathbb{R}_+^\times$. Let $c' : TM \to \mathbb{R}$ denote the (Darboux) derivative of c. We let ϕ act on P by

$$\phi(m,\lambda) = (\phi(m), c(m)\lambda).$$

Then,

$$\phi'(\xi\oplus s) = (\phi'\xi \oplus c'\xi + s)$$

or equivalently

$$\phi' = \begin{bmatrix}\phi' & 0\\ c' & 1\end{bmatrix} : \begin{bmatrix} TM \\ V\end{bmatrix} \to \begin{bmatrix} TM \\ V\end{bmatrix}$$

in matrix notation, and thus ϕ' preserves V. Moreover, ϕ is isometric on V:

$$\begin{aligned}\langle \phi'(0\oplus s), \phi'(0\oplus t)\rangle_{\phi(m,\lambda)} &= \langle 0\oplus s, 0\oplus t\rangle_{(\phi(m),c(m)\lambda)}\\ &= g_{\mathbb{R}}(s,t)\\ &= \langle 0\oplus s, 0\oplus t\rangle_{(m,\lambda)}\end{aligned}$$

and ϕ is isometric on N:

$$\begin{aligned}\langle \phi'(\xi,s), \phi'(\eta,t)\rangle_{\phi(m,\lambda)} &= \langle (\phi'\xi, c'\xi+s), (\phi'\eta, c'\eta+t)\rangle_{(\phi(m),c(m)\lambda)}\\ &= c(m)^{-2}\lambda^{-2} g_{\phi(m)}(\phi'\xi,\phi'\eta)\\ &= \lambda^{-2} g_m(\xi,\eta)\\ &= \langle (\xi,s),(\eta,t)\rangle_{(m,\lambda)}\,.\end{aligned}$$

In summary, we see that $\phi : P \to P$ is a para-Riemannian transformation. Hence the group of conformal transformations of (M,g) act on P by para-Riemannian transformation.

Example 6.2.1. Let $M = \mathbb{T} = \mathbb{R}/\mathbb{Z}$ be equipped with the standard Riemannian structure. Then $TM = \mathbb{T}\times\mathbb{R} \to \mathbb{T}$ is one-dimensional and any diffeomorphism is a conformal transformation, with conformal factor $c(\theta) = |\phi'(\theta)|$, $\theta \in \mathbb{T}$. Indeed, $\phi'(\theta,t) = (\phi(\theta), \phi'(\theta)t)$ in $\mathbb{T}\times\mathbb{R}$, thus for any ξ, $\eta \in TM$,

$$g_{\phi(\theta)}(\phi'\xi,\phi'\eta) = \phi'(\theta)^2 g_\theta(\xi,\eta).$$

6.2.1 Twisted Spectral Triples

An alternative approach using twisted spectral triples is developed in [CM06]. We recall it briefly. For simplicity, let $M = \mathbb{T}$.

Let Γ be a group of orientation preserving diffeomorphisms of $\mathbb{T}$ and let $\mathcal{A} = C^\infty(\mathbb{T}) \rtimes \Gamma$.

Let $\mathfrak{H} = L^2(\mathbb{T})$ and let $\mathcal{A}$ act on $\mathfrak{H}$ by

$$(f \cdot \xi)(\theta) \coloneqq f(\theta)\xi(\theta), \quad f \in C^\infty(\mathbb{T})$$
$$(\phi^{-1} \cdot \xi)(\theta) \coloneqq \phi'(\theta)^{\frac{1}{2}}\xi(\phi(\theta)), \quad \phi \in \Gamma.$$

Then with $D \coloneqq \frac{1}{i}\frac{d}{d\theta}$, the triple $(\mathcal{A}, \mathfrak{H}, D)$ is *not* a spectral triple: $[D, \phi^{-1}]$ is not necessarily bounded. We define $\sigma : \mathcal{A} \to \mathcal{A}$ by

$$\sigma(f) = f \quad \text{and} \quad \sigma(\phi^{-1}) = \phi' \cdot \phi^{-1}.$$

Then $Da - \sigma(a)D$ is bounded for any $a \in \mathcal{A}$.

Definition 6.2.2. Let σ be an automorphism of $\mathcal{A}$. A σ-spectral triple $(\mathcal{A}, \mathfrak{H}, D)$ is given by

(1) a graded Hilbert space $\mathfrak{H}$ and

(2) an algebra $\mathcal{A}$ acting $\mathfrak{H}$ by even bounded operators and

(3) an odd self-adjoint operator D with compact resolvents such that $Da - \sigma(a)D$ is bounded for all $a \in \mathcal{A}$.

We say that $(\mathcal{A}, \mathfrak{H}, D)$ is Lipschitz regular if in addition

$$|D|a - \sigma(a)|D|$$

is bounded for any $a \in \mathcal{A}$.

Then $(C^\infty(\mathbb{T}) \rtimes \Gamma, L^2(\mathbb{T}), D)$ is a Lipschitz regular spectral triple. Index theory can be developed along the same lines as normal spectral triples. However, satisfactory local index theory is still missing.

Appendix

Functional Analysis

A.1 Self-adjoint Operators

We refer to [RS80, Chapter VIII] or [Ped89, Chapter 5] for details, particularly for notions of adjoint and closure used below.

Let $\mathfrak{H}$ be a Hilbert space. We write $\mathcal{L} = \mathcal{L}(\mathfrak{H})$ for the algebra of bounded linear operators on $\mathfrak{H}$ and identify $1 \in \mathbb{C}$ with the identity operator on $\mathfrak{H}$.

Recall that an operator D on $\mathfrak{H}$, with *domain* $\mathrm{dom}(D) \subseteq \mathfrak{H}$, is said to be a *densely defined* operator if $\mathrm{dom}(D)$ is dense in $\mathfrak{H}$. We shall be exclusively concerned with densely defined operators.

We say that a densely defined operator D is *self-adjoint* if $D = D^*$, *i.e.* D is *symmetric* and $\mathrm{dom}(D) = \mathrm{dom}(D^*)$. By the basic criterion for self-adjointness (cf. [RS80, Theorem VIII.3] or [Ped89, Proposition 5.2.5]) a symmetric operator D is self-adjoint if and only if the operators

$$D \pm i : \mathrm{dom}(D) \to \mathfrak{H}$$

are *surjective*. In this case, since

$$||(D \pm i)\xi||^2 = ||\xi||^2 + ||D\xi||^2, \quad \xi \in \mathrm{dom}(D),$$

the operators $D \pm i$ are actually linear isomorphisms and the operators

$$(D \pm i)^{-1} : \mathfrak{H} \to \mathrm{dom}(D) \subseteq \mathfrak{H}$$

are bounded operators on $\mathfrak{H}$.

The main reason to consider self-adjoint operators is that they have a well-behaved spectral theory, also known as functional calculus, which we recall below briefly.

We define the *resolvent set* of an operator D to be the set of all $\lambda \in \mathbb{C}$ for which there exists a bounded operator R_λ in $\mathcal{L}(\mathfrak{H})$, necessarily unique, such that

$$R_\lambda(\lambda - D) \subseteq (\lambda - D)R_\lambda = I.$$

By the closed graph theorem, this is equivalent to the condition that $\lambda - D$ is a linear isomorphism from $\mathrm{dom}(D)$ onto $\mathfrak{H}$.

The compliment of the resolvent set is called the *spectrum* of D, and is denoted by $\mathrm{Spec}(D)$. It is a closed subset of $\mathbb{C}$. The operators R_λ are called the *resolvents* of D and can collectively be considered as a function defined on $\mathbb{C}\backslash\mathrm{Spec}(D)$ with values in $\mathcal{L}(\mathfrak{H})$.

If X is a locally compact Hausdorff space, we write $\mathscr{B}_b(X)$ for the normed $*$-algebra of complex-valued *bounded Borel* functions on X with the supremum norm

$$||f|| := \sup_{x\in X} |f(x)|.$$

If X is a closed subset of $\mathbb{C}$, then it is locally compact and Hausdorff in the subspace topology and if $\lambda \in \mathbb{C}\backslash X$, then the function

$$X \to \mathbb{C}, \quad x \mapsto (\lambda - x)^{-1}$$

is bounded (since $X \subseteq \mathbb{C}$ is closed) and Borel (in fact continuous) *i.e.* it belongs to $\mathscr{B}_b(X)$.

Theorem A.1.1 (Spectral Theorem)**.** *Let D be a self-adjoint operator on a separable Hilbert space $\mathfrak{H}$. Then the spectrum of D is a nonempty closed subset of $\mathbb{R}$ and there exists a unique $*$-algebra map from $\mathscr{B}_b(\mathrm{Spec}(D))$ to $\mathcal{L}(\mathfrak{H})$, denoted $f \mapsto f(D)$, with the following properties:*

(a) If $\lambda \notin \mathrm{Spec}(D)$, then $(\lambda - x)^{-1} \mapsto R_\lambda$, i.e. *the two senses of the equality $R_\lambda = (\lambda - D)^{-1}$ coincide;*

(b) If $f \in \mathscr{B}_b(\mathrm{Spec}(D))$, then $||f(D)|| \leq ||f||$;

(c) If f_n converges to f pointwise and the sequence $||f_n||$ is bounded, then $f_n(D)$ converges to $f(D)$ strongly.

(d) Let $1_{[-M,M]}$ denote the characteristic function of the interval $[-M, M]$, $M > 0$. Then $1_{[-M,M]}(D)\xi \to \xi$, $\xi \in \mathfrak{H}$, as $M \to \infty$.

Proof. See [RS80, Theorem VIII.2 and Theorem VIII.5] or [Ped89, Proposition 5.2.13 and Theorem 5.3.8]. □

As a corollary we see that a self-adjoint operator is bounded if and only if it has bounded spectrum.

The following norm estimate is fundamental in many of our computations.

Notation A.1.2. Let λ be a complex number. We write $\mathrm{Re}(\lambda)$ for the *real* part of λ and $\mathrm{Im}(\lambda)$ for the *imaginary* part of λ.

Lemma A.1.3. *Let D be a self-adjoint operator. Then any $\lambda \in \mathbb{C}$* non-real *number is in the resolvent set of D and*

$$||(\lambda - D)^{-1}|| \leq |\operatorname{Im}(\lambda)|^{-1}.$$

This follows from the spectral theorem immediately, but we give a direct proof to illustrate the main points.

Proof. Since λ is non-real, $\operatorname{Im}(\lambda) \neq 0$ and $\operatorname{Im}(\lambda)^{-1}(D - \operatorname{Re}(\lambda))$ is a self-adjoint operator. Thus, by the basic criterion, $\operatorname{Im}(\lambda)^{-1}(D - \operatorname{Re}(\lambda)) - i$ is surjective. Therefore, $\lambda - D$ is surjective and λ is in the resolvent set of D.

On the other hand, for any $\xi \in \operatorname{dom}(D)$, we have

$$||(\lambda - D)\xi||^2 = ||(\operatorname{Re}(\lambda) - D)\xi||^2 + |\operatorname{Im}(\lambda)|^2||\xi||^2,$$

hence $||(\lambda - D)^{-1}|| \leq \operatorname{Im}(\lambda)$. □

Finally, recall that a symmetric operator D is said to be *essentially self-adjoint* if its *closure* $\bar{D}$ is self-adjoint. A symmetric operator D is essentially self-adjoint if and only if it admits a unique self-adjoint extension (cf. [RS80, page 256]) or equivalently the operators $D \pm i$ have dense range (cf. [RS80, Corollary to Theorem VIII.3]).

Let D be an operator. Elements of $\bigcap_{n=1}^{\infty} \operatorname{dom}(D^n) \subseteq \mathfrak{H}$ are called the *smooth vectors*[1] for D. We say that $\xi \in \mathfrak{H}$ is an *analytical vector* for D if ξ is smooth and

$$\sum_{n=0}^{\infty} \frac{||D^n\xi||}{n!} t^n < \infty$$

for some $t > 0$.

The domain of a self-adjoint operator always contains a dense set of analytical vectors. Indeed, if D is self-adjoint, then any $\xi \in 1_{[-M,M]}(D)\mathfrak{H}$, $M > 0$ is an analytical vector for D, since $||D^n\xi|| \leq M^n||\xi||$ by the spectral theorem, and $\bigcup_{M>0} 1_{[-M,M]}(D)\mathfrak{H}$ is dense in $\mathfrak{H}$ (cf. [RS75, Section X.6]). In the converse direction, we have the following result.

Theorem A.1.4 (Nelson)**.** *Suppose that the analytical vectors of a densely defined symmetric operator form a dense subset inside the domain. Then the operator is essentially self-adjoint.*

Proof. See [RS75, Theorem X.39]). □

Sum and Product of Operators

Definition A.1.5. Let D_1 and D_2 be operators on Hilbert spaces $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively. We define new operators as follows.

[1] See Example 3.1.6 for motivation for the terminology.

The *direct sum* $D_1 \oplus D_2$ of D_1 and D_2 is the operator on $\mathfrak{H}_1 \oplus \mathfrak{H}_2$ given by

$$(D_1 \oplus D_2)(\xi_1 \oplus \xi_2) \coloneqq D_1\xi_2 \oplus D_2\xi_2,$$

with domain $\mathrm{dom}(D_1) \oplus \mathrm{dom}(D_2)$ (which is dense in $\mathfrak{H}_1 \oplus \mathfrak{H}_2$).

The *tensor product* $D_1 \otimes D_2$ of D_1 and D_2 is the operator on $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ given by

$$(D_1 \otimes D_2)(\xi_1 \otimes \xi_2) \coloneqq D_1 x_1 \otimes D_2\xi_2,$$

with domain $\mathrm{dom}(D_1) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2)$ (which is dense in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$).

If D_1 and D_2 extend to *bounded* operators, then so do $D_1 \oplus D_2$ and $D_1 \otimes D_2$. The extensions are again denoted by $D_1 \oplus D_2$ and $D_1 \otimes D_2$ respectively. Then we have

$$\begin{aligned} \mathcal{L}(\mathfrak{H}_1) \oplus \mathcal{L}(\mathfrak{H}_2) &\subseteq \mathcal{L}(\mathfrak{H}_1 \oplus \mathfrak{H}_2) \\ \mathcal{L}(\mathfrak{H}_1) \otimes \mathcal{L}(\mathfrak{H}_2) &\subseteq \mathcal{L}(\mathfrak{H}_1 \otimes \mathfrak{H}_2). \end{aligned}$$

For unbounded operators, we have the following.

Theorem A.1.6. *Let D_1 and D_2 be self-adjoint operators. Then $D_1 \oplus D_2$ is self-adjoint and $D_1 \otimes D_2$, $D_1 \otimes 1 + 1 \otimes D_2$ are essentially self-adjoint. Moreover*

$$\begin{aligned} \mathrm{Spec}(D_1 \oplus D_2) &= \mathrm{Spec}(D_1) \cup \mathrm{Spec}(D_2) \\ \mathrm{Spec}(\overline{D_1 \otimes D_2}) &= \overline{\mathrm{Spec}(D_1) \cdot \mathrm{Spec}(D_2)} \\ \mathrm{Spec}(\overline{D_1 \otimes 1 + 1 \otimes D_2}) &= \overline{\mathrm{Spec}(D_1) + \mathrm{Spec}(D_2)}. \end{aligned}$$

Proof. The case of $D_1 \oplus D_2$ is straightforward since

$$\lambda - (D_1 \oplus D_2) = (\lambda - D_1) \oplus (\lambda - D_2).$$

The other two cases follow from [RS80, Theorem VIII.33] or [RS75, Example X.6.3]. See also the proof of Lemma A.1.9. □

Now we consider graded operators.

Let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ be graded Hilbert spaces with gradings γ_1 and γ_2 respectively. The *graded tensor product* $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$ of $\mathfrak{H}_1$ and $\mathfrak{H}_2$ is the Hilbert space $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ equipped with the grading $\gamma_1 \otimes \gamma_2$.

Definition A.1.7. Let D_1 and D_2 be homogeneous operators on $\mathfrak{H}_1$ and $\mathfrak{H}_2$, respectively.

The *graded tensor product* $D_1 \widehat{\otimes} D_2$ of D_1 and D_2 is the operator on $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$ given by

$$(D_1 \widehat{\otimes} D_2)(\xi_1 \widehat{\otimes} \xi_2) \coloneqq (-1)^{|D_2||\xi_1|} D_1\xi_1 \widehat{\otimes} D_2\xi_2$$

with domain $\mathrm{dom}(D_1)\,\widehat{\otimes}_{\mathrm{alg}}\mathrm{dom}(D_2)$. Here $|\cdot|$ denotes the degree.

The *direct product* $D_1 \times_{\mathrm{alg}} D_2$ of D_1 and D_2 is the operator on $\mathfrak{H}_1 \,\widehat{\otimes}\, \mathfrak{H}_2$ given by

$$D_1 \times_{\mathrm{alg}} D_2 \coloneqq D_1 \,\widehat{\otimes}\, 1 + 1 \,\widehat{\otimes}\, D_2$$

with domain $\mathrm{dom}(D_1)\,\widehat{\otimes}_{\mathrm{alg}}\mathrm{dom}(D_2)$.

Remark A.1.8. Note that the degree of $D_1 \,\widehat{\otimes}\, D_2$ is $|D_1| + |D_2|$.

Lemma A.1.9. *Let D_1 and D_2 be odd self-adjoint operators. Then $D_1 \times_{\mathrm{alg}} D_2$ is essentially self-adjoint. Let $D_1 \times D_2$ denote the closure. Then*

$$e^{-(D_1\times D_2)^2} = e^{-D_1^2} \,\widehat{\otimes}\, e^{-D_2^2}.$$

Proof. We identify $\mathfrak{H}_1 \,\widehat{\otimes}\, \mathfrak{H}_2$ with $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. Then $D_1 \times_{\mathrm{alg}} D_2 = D_1 \otimes 1 + \gamma_1 \otimes D_2$ is symmetric and

$$(D_1 \times_{\mathrm{alg}} D_2)^n = \sum \pm \gamma_1^{n-k} D_1^k \otimes D_2^{(n-k)}$$

on $\mathrm{dom}(D_1^n) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2^n)$.

If ξ_1 and ξ_2 are analytical vectors for D_1 and D_2, respectively, then $\xi_1 \otimes \xi_2$ is a smooth vector for $D_1 \times_{\mathrm{alg}} D_2$ and

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{||(D_1 \times_{\mathrm{alg}} D_2)^n(\xi_1 \otimes \xi_2)||}{n!} t^n &\leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{k}{n} ||D_1^k \xi_1||\, ||D_2^{(n-k)}\xi_2|| t^n \\ &= \left(\sum_{k=0}^{\infty} \frac{||D_1^k \xi_1||}{k!} t^k\right)\left(\sum_{m=0}^{\infty} \frac{||D_2^m \xi_2||}{m!} t^m\right).\end{aligned}$$

Hence, choosing $t > 0$ small, we see that $\xi_1 \otimes \xi_2$ is analytical for $D_1 \times_{\mathrm{alg}} D_2$. Since the finite linear combination of such elementary tensors is dense in $\mathrm{dom}(D_1) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2)$, Nelson's analytical vector theorem (cf. Theorem A.1.4) proves that $D_1 \times_{\mathrm{alg}} D_2$ is essentially self-adjoint on $\mathrm{dom}(D_1) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2)$.

Now, we can apply the spectral theorem to define $e^{-(D_1\times D_2)^2}$. Since

$$(D_1 \times_{\mathrm{alg}} D_2)^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$$

on $\mathrm{dom}(D_1^2) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2^2)$ and $D_1^2 \otimes 1$ and $1 \otimes D_2^2$ commute on this domain, we have

$$(D_1 \times_{\mathrm{alg}} D_2)^{2n} = \sum_{k=0}^{n} \binom{k}{n} D_1^{2k} \otimes D_2^{2(n-k)}$$

on $\mathrm{dom}(D_1^n) \otimes_{\mathrm{alg}} \mathrm{dom}(D_2^n)$. Consequently, for analytical vectors $\xi_j \in 1_{[-M,M]}(D_j)\mathfrak{H}_j$,

$$e^{-(D_1\times D_2)^2}(\xi_1 \otimes \xi_2) = \sum_{n=0}^{\infty} (-1)^n \frac{(D_1^2 \otimes 1 + 1 \otimes D_2^2)^n}{n!}(\xi_1 \otimes \xi_2)$$

$$
\begin{aligned}
&= \left(\sum_{k=0}^{\infty}(-1)^k \frac{D_1^{2k}\xi_1}{k!}\right) \otimes \left(\sum_{m=0}^{\infty}(-1)^m \frac{D_2^{2m}\xi_2}{m!}\right) \\
&= e^{-D_1^2}\xi_1 \otimes e^{-D_2^2}\xi_2.
\end{aligned}
$$

Here the sums converge in norm in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$. The proof is complete, since $e^{-(D_1 \times D_2)^2}$ and $e^{-D_1^2} \widehat{\otimes}\, e^{-D_2^2} = e^{-D_1^2} \otimes e^{-D_2^2}$ are bounded operators and elementary tensors of the form above span a dense subset in $\mathfrak{H}_1 \widehat{\otimes} \mathfrak{H}_2$. □

Perturbations of Operators

Let D and S be operators on $\mathfrak{H}$. We say that S is D-bounded, with relative bound $s \geq 0$, if

(i) $\operatorname{dom}(S) \supseteq \operatorname{dom}(D)$ and

(ii) for some $b \geq 0$ and all $\xi \in \operatorname{dom}(D)$,

$$||S\xi|| \leq s||D\xi|| + b||\xi||.$$

For example, a bounded operator S is D-bounded for any D, with relative bound $s = 0$.

Theorem A.1.10 (Kato-Rellich)**.** *Let D be a self-adjoint operator and let S be a symmetric operator. If S is D-bounded with relative bound < 1, then $D + S$ is self-adjoint on* $\operatorname{dom}(D)$*.*

This is a standard result, but since we need some of the details in Lemma 2.1.13, we give a proof (cf. [RS75, Theorem X.12]).

Proof. Let $s < 1$ denote the relative bound and let $\lambda = it$, $t \neq 0$. We show that $\lambda - (D + S)$ is surjective for $|t| \gg 0$.

For any $\xi \in \mathfrak{H}$, the vector $(\lambda - D)^{-1}\xi$ belongs to $\operatorname{dom}(D) \subseteq \operatorname{dom}(S)$ and

$$
\begin{aligned}
||S(\lambda - D)^{-1}\xi|| &\leq s||D(\lambda - D)^{-1}\xi|| + b||(\lambda - D)^{-1}\xi|| \\
&\leq (s + b|t|^{-1})||\xi||.
\end{aligned}
$$

Hence for $|t| \gg 0$, $C := S(\lambda - D)^{-1}$ is bounded with norm $||C|| < 1$ and $1 - C$ is invertible. On the other hand, on $\operatorname{dom}(D)$

$$\lambda - (D + S) = (1 - C)(\lambda - D)$$

and $\lambda - D$ is surjective. Thus $\lambda - (D + S)$ too is surjective. □

A.2 Compact Operators

We refer to [RS80, Sections VI.5, VI.6] and [GBVF01, Sections 7.5 and 7.C] and [Con94, Section IV.2.] for details.

Let $\mathfrak{H}$ be a separable Hilbert space and let T be a bounded operator on $\mathfrak{H}$. For $n \in \mathbb{Z}_{\geq 0}$, we define the n-th *singular value* of T as

$$\mu_n(T) \coloneqq \inf_{R \in \mathcal{L}(\mathfrak{H})} \{||T - R|| \mid \mathrm{rank} R = n\}.$$

Then

$$||T|| = \mu_0(T) \geq \mu_1(T) \cdots \geq \mu_n(T) \geq \cdots \geq 0$$

and T is *compact* if and only if $\mu_n(T) \to 0$ as $n \to \infty$. We write $\mathcal{K} = \mathcal{K}(\mathfrak{H})$for the set of compact operators on $\mathfrak{H}$. The following properties are easy to check: for S, $T \in \mathcal{L}$ and n, $m \in \mathbb{Z}_{\geq 0}$,

(a) $|\mu_n(S) - \mu_n(T)| \leq ||S - T||$;

(b) $\mu_{n+m}(S + T) \leq \mu_n(S) + \mu_m(T)$;

(c) $\mu_{n+m}(ST) \leq \mu_n(S)\mu_m(T)$.

It follows that $\mathcal{K}$ is a closed two-sided ideal in $\mathcal{L}$. Recall that if T is compact then all nonzero eigenvalues of T have finite multiplicity and $\mathrm{Spec}(T) = \{\text{eigenvalues of } T\} \cup \{0\}$ (cf. [RS80, Theorem VI.15]). If, in addition, T is self-adjoint then, T diagonalizable *i.e.* there exists a complete orthonormal basis of $\mathfrak{H}$ consisting of eigenvectors of T (cf. [RS80, Theorem VI.16]). Finally, for any T compact, $\mu_n(T)$ are precisely the *eigenvalues* of $|T| = (T^*T)^{\frac{1}{2}}$, repeated according to multiplicity and arranged in decreasing order (cf. [RS80, Proof of Theorem VI.17] and/or [GBVF01, Lemma 7.31]). In particular, for $0 \leq T \in \mathcal{K}$

$$\mathrm{Spec}(T) = \{\mu_n(T) \mid n \in \mathbb{Z}_{\geq 0}\} \cup \{0\}.$$

Lemma A.2.1. *If $0 \leq T \in \mathcal{K}$ and f is a continuous function on $[0, \infty)$ vanishing at 0, then $f(T)$ is compact. If moreover, f is positive and monotone-increasing then $f(T)$ is positive and*

$$\mu_n(f(T)) = f(\mu_n(T)).$$

Proof. The compactness and positivity of $f(T)$ follows from general spectral theory of C^*-algebras (cf. [Ped89, Proposition 4.3.15]). But we give an elementary proof, using simple properties of diagonal operators.

We may assume that T is a diagonal operator with diagonal entries $\mu_n(T)$. Then $f(T)$ is a diagonal operator with diagonal entries $f(\mu_n(T))$. Since $\mu_n(T) \to 0$ as $n \to \infty$ and f is continuous at 0, $f(\mu_n(T)) \to f(0) = 0$ as $n \to 0$. Hence $f(T)$ is compact (cf. [Ped89, Theorem 3.3.8]).

If f is positive then $f(\mu_n(T)) \geq 0$ for all $n \in \mathbb{Z}_{\geq 0}$ and consequently $f(T) \geq 0$. If, in addition, f is monotone-increasing then

$$f(\mu_0(T)) \geq f(\mu_1(T)) \geq \ldots$$

is the list of eigenvalues of $f(T) \geq 0$ in decreasing order, hence $\mu_n(f(T)) = f(\mu_n(T))$. $\square$

By the proof of [GBVF01, Lemma 7.31], for $T \in \mathcal{L}$, the singular values can also be written as

$$\begin{aligned}\mu_n(T) &= \inf\{||T_{|V^\perp} : V^\perp \to \mathfrak{H}|| \mid V \subset \mathfrak{H},\, \dim V = n\}\\ &= \inf\{||TP|| \mid P \text{ orthogonal projection of corank } n\}.\end{aligned}$$

The following lemma is due to Hermann Weyl.

Lemma A.2.2. *If $0 \leq S \leq T$ and T is compact, then S is compact and*

$$\mu_n(S) \leq \mu_n(T).$$

Proof. For any $P \in \mathcal{L}$, we have $0 \leq P^*SP \leq P^*TP$ and so $||S^{\frac{1}{2}}P|| \leq ||T^{\frac{1}{2}}P||$. Taking the infimum over all corank n orthogonal projections P, we see that $\mu_n(S^{\frac{1}{2}}) \leq \mu_n(T^{\frac{1}{2}})$. Since T is compact, so is $T^{\frac{1}{2}}$ by Lemaa A.2.1 and so $\mu_n(T^{\frac{1}{2}}) \to 0$ as $n \to \infty$. Hence, $\mu_n(S^{\frac{1}{2}}) \to 0$ as $n \to \infty$ *i.e.* $S^{\frac{1}{2}}$ is compact and consequently S is compact. Finally,

$$\mu_n(S) = \mu_n(S^{\frac{1}{2}})^2 \leq \mu_n(T^{\frac{1}{2}})^2 = \mu_n(T),$$

again by Lemma A.2.1. $\square$

Corollary A.2.3. *If $S^*S \leq T$ and T is compact, then S is compact.* $\square$

We say that a linear subspace $\mathcal{A} \subseteq \mathcal{L}$ is *hereditary* if $0 \leq S \leq T$ and $T \in \mathcal{A}$ implies that S is in $\mathcal{A}$. It is a general property of C^*-algebras that *closed* two-sided ideals are hereditary (cf. [Ped79, Theorem 1.5.2 and Corollary 1.5.3]).

A *symmetrically normed ideal* is an ideal $\mathcal{J} \subset \mathcal{L}$ equipped with a norm $||\cdot||_{\mathcal{J}}$ such that

(i) $||RTS||_{\mathcal{J}} \leq ||R||\,||T||_{\mathcal{J}}||S||$ for $T \in \mathcal{J}$ and R, $S \in \mathcal{L}$ and

(ii) $\mathcal{J}$ is complete in the norm $||\cdot||_{\mathcal{J}}$.

It follows from Lemma A.2.2 that not only is $\mathcal{K}$ hereditary in $\mathcal{L}$, but any symmetrically normed ideal is hereditary in $\mathcal{L}$ (see also [Sim05, Theorem 1.19]). Referring to *loc.cit.* for generalities on symmetrically normed ideals, we review the two main classes of examples: the Schatten ideals and the Dixmier ideals.

Schatten Ideals

We consider the partial sums

$$\sigma_n(T) \coloneqq \sum_{j=0}^{n-1} \mu_j(T).$$

Then, for any $T \in \mathcal{L}(\mathfrak{H})$,

$$0 \leq \sigma_1(T) \leq \cdots \leq \sigma_n(T) \leq \ldots$$

and T is of *trace-class* if and only if $\sup_n \sigma_n(T) < \infty$. A trace-class operator is necessarily compact.

The *Schatten p-ideal* is defined as

$$\mathcal{L}^p = \mathcal{L}^p(\mathfrak{H}) := \{T \in \mathcal{K}(\mathfrak{H}) \mid \sigma(|T|^p) = O(1)\}, \quad \text{for } p \in [1, \infty).$$

These satisfy the following well-known properties.

Lemma A.2.4. *The Schatten p-ideal $\mathcal{L}^p$ is a hereditary ideal in $\mathcal{L}$.*

(a) If $T \geq 0$ is positive, then T belongs to $\mathcal{L}^p$ if and only if T^p belongs to $\mathcal{L}^1$.

(b) If $1 \leq p < q < \infty$, then $\mathcal{L}^p \subset \mathcal{L}^q$.

(c) If $1 \leq p, q < \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, then $\mathcal{L}^p \cdot \mathcal{L}^q \subseteq \mathcal{L}^r$.

Proof. The only non-trivial statement is (c): see [RS75, Proposition IX.4.5]. □

The map $T \mapsto \sup_n \sigma_n(T)$, defined on the set of positive trace-class operators, is additive (cf. [GBVF01, (7.68)]) and extends uniquely to a positive linear functional

$$\mathrm{Tr} : \mathcal{L}^1(\mathfrak{H}) \to \mathbb{C},$$

called the *operator trace* or simply the *trace*. It is indeed a trace: If $T \in \mathcal{L}^1$ and $S \in \mathcal{L}$, then

$$\mathrm{Tr}(TS) = \mathrm{Tr}(ST).$$

A theorem of Lidskij [Lid59] (for a modern treatment see [Sim05, Chapter 3]) states that the trace of a trace-class operator is given by the sum of all its eigenvalues. This has the following very important consequence as noted by Fedosov in [Fed96].

Lemma A.2.5 ([Fed96] Corollary 3.4.6)**.** *If P and Q are two operators (may be unbounded) such that the operators PQ and QP are of trace-class, then*

$$\mathrm{Tr}(PQ) = \mathrm{Tr}(QP).$$

We include the simple proof (assuming Lidskij's theorem) for the convenience of the reader. The following is an analogy of [Tak02, Proposition I.2.1].

Lemma A.2.6. *Let P and Q be operators such that PQ and QP are bounded operators. Then the nonzero eigenvalues of PQ and QP coincide, including multiplicity.*

Proof. Let $\lambda \neq 0$ be an eigenvalue of PQ with a a generalized eigenvector ξ satisfying

$$(PQ - \lambda)^k \xi = 0.$$

Then $Q\xi \neq 0$ (since $\lambda \neq 0$) and

$$(QP - \lambda)^k Q\xi = Q(PQ - \lambda)^k \xi = 0.$$

Hence $Q\xi$ is a generalized eigenvector for QP associated to λ, with the same multiplicity. $\square$

Proof of Lemma A.2.5. Since, by Lidskij's theorem, only the nonzero eigenvalues contribute to the trace, the proof follows from Lemma A.2.6. $\square$

Dixmier Ideals

The *Dixmier p-ideal* is defined as

$$\mathcal{L}^{(1,\infty)} = \mathcal{L}^{(1,\infty)}(\mathfrak{H}) \coloneqq \{T \in \mathcal{K}(\mathfrak{H}) \mid \sigma_n(T) = O(\log n)\}, \quad \text{for } p = 1 \text{ and}$$
$$\mathcal{L}^{(p,\infty)} = \mathcal{L}^{(p,\infty)}(\mathfrak{H}) \coloneqq \{T \in \mathcal{K}(\mathfrak{H}) \mid \sigma_n(T) = O(n^{1-\frac{1}{p}})\}, \quad \text{for } p \in (1, \infty).$$

The following properties are well-known.

Lemma A.2.7. *The Dixmier p-ideal $\mathcal{L}^{(p,\infty)}$ is a hereditary ideal in $\mathcal{L}$.*

(a) If $T \geq 0$ and T belongs to $\mathcal{L}^{(p,\infty)}$, $p > 1$, then T^p belongs to $\mathcal{L}^{(1,\infty)}$.

(b) If $1 \leq p < q < \infty$, then $\mathcal{L}^p \subset \mathcal{L}^{(p,\infty)} \subset \mathcal{L}^q$. $\square$

The converse of a does *not* hold (cf. [GBVF01, Lemma 7.37]).

Now we construct a trace (in fact, a family of traces) on the Dixmier 1-ideal $\mathcal{L}^{(1,\infty)}$. We remark that of the Schatten ideals and the Dixmier ideals, only $\mathcal{L}^1$ and $\mathcal{L}^{(1,\infty)}$ admit nontrivial traces (cf. [DFWW04, Corollary 5.23]).

Let $T \geq 0$ be given. We extend the sequence $\sigma_n(T)$, $n \in \mathbb{Z}_{\geq 1}$ to a piecewise linear function $\sigma_u(T)$ on $[1, \infty)$ and define the λ-*Cesàro mean* of T as

$$\tau_\lambda(T) \coloneqq \frac{1}{\log \lambda} \int_3^\lambda \frac{\sigma_u(T)}{\log u} \frac{du}{u}, \quad \lambda \geq 3.$$

Then, if $T \in \mathcal{L}^{(1,\infty)}$, then $\frac{\sigma_u(T)}{\log u}$ is a bounded function and hence $\tau(T)$ too is a bounded function, *i.e.* an element of $C_b[3, \infty)$, the algebra of bounded continuous functions on $[3, \infty)$. Let

$$\omega : C_b[3, \infty) \to \mathbb{C}$$

be a positive linear functional of norm 1, vanishing on the ideal $C_0[3, \infty)$ of functions vanishing at infinity, *i.e.* a state on $C_b[3, \infty)/C_0[3, \infty)$. Then the map

$$\mathrm{Tr}_\omega(T) \coloneqq \omega(\tau_\lambda(T)),$$

defined for $T \in \mathcal{L}^{(1,\infty)}$ positive, is additive (cf. [GBVF01, Lemma 7.14]) and extends uniquely to a linear map (cf. [GBVF01, Lemma 7.35])

$$\mathrm{Tr}_\omega : \mathcal{L}^{(1,\infty)} \to \mathbb{C}$$

called the *Dixmier trace*. The Dixmier trace is indeed a trace: if $T \in \mathcal{L}^{(1,\infty)}$ and $S \in \mathcal{L}$ then

$$\mathrm{Tr}_\omega(TS) = \mathrm{Tr}_\omega(ST)$$

(cf. [GBVF01, Exercise 7.17]).

An operator $T \in \mathcal{L}^{(1,\infty)}$ is said to be *measurable* if $\mathrm{Tr}_\omega(T)$ is independent of ω. We write $\mathcal{M} = \mathcal{M}(\mathfrak{H})$ for the class of measurable operators. It is a linear subspace of $\mathcal{L}^{(1,\infty)}$ that is invariant under conjugation by invertible operators on $\mathfrak{H}$.

We conclude with a criterion for measurablity. Let T be a positive operator such that $T \in \mathcal{L}^{1+\epsilon}$ for any $\epsilon > 0$. Then the *zeta function*

$$\zeta(z) \coloneqq \sum_{n=0}^{\infty} \mu_n(T)^z$$

is a holomorphic function defined on the right half-plane $\mathrm{Re}(z) > 1$.

Theorem A.2.8. *Let T be a* positive *operator such that $T \in \mathcal{L}^{1+\epsilon}$ for any $\epsilon > 0$ and let $L \geq 0$ be a real number. Then the following conditions are equivalent:*

(1) T is measurable and $\mathrm{Tr}_\omega(T) = L$

(2) $\lim_{\lambda\to\infty} \tau_\lambda(T) = L$

(3) $\lim_{n\to\infty} \frac{\sigma_n(T)}{\log n} = L$

(4) $\lim_{z\to 1^+}(z-1)\zeta(z) = L.$

In particular, if the zeta function $\zeta(z)$ admits a meromorphic extension to a neighbourhood of $1 \in \mathbb{C}$ with a simple pole at 1, then T is measurable and

$$\mathrm{Tr}_\omega(T) = \mathrm{Res}_{s=1}\zeta(z).$$

Proof. The implications (1) $\Longleftrightarrow$ (2) $\Longleftarrow$ (3) are immediate. The implication (2) $\Longrightarrow$ (3) is shown in [LSS05] and the equivalence (3) $\Longleftrightarrow$ (4) is shown in [GBVF01, Section 7.6]. □

For a positive compact T, if

$$\lim_{n\to\infty} n\mu_n(T) = L,$$

then condition (3) is satisfied and T is measurable. However, the converse does not hold (cf. [GBVF01, Example 7.3]).

A.2.1 Compact Resolvent Operators

We say that a self-adjoint operator D has *compact resolvents* if the bounded operators $(D \pm i)^{-1}$ are compact operators.

The following is well-known and easy to show.

Theorem A.2.9. *Let D be a self-adjoint operator. Then the following are equivalent:*

(1) The operator D has compact resolvents.

(2) The resolvent $R_\lambda = (\lambda - D)^{-1}$ is compact for any $\lambda \notin \mathrm{Spec}(D)$.

(3) The operator D is diagonalizable and the diagonal entries go off to infinity. □

Often, we will consider a weaker condition: the operators

$$a \cdot (D \pm i)^{-1} \quad \text{and} \quad (D \pm i)^{-1} \cdot a$$

are compact for all a in some subalgebra $\mathcal{A} \subseteq \mathcal{L}$. In this case, we say that D has $\mathcal{A}$*-locally compact resolvents*, or if the algebra $\mathcal{A}$ is understood from the context, we say simply, D has *locally compact resolvents*. See Example 2.1.1 for the motivation behind the terminology.

The following is a standard result, but since it is very useful we give a proof.

Lemma A.2.10. *Let D be a self-adjoint operator on $\mathfrak{H}$ and let $a \in \mathcal{L}(\mathfrak{H})$ be given. If the operator $a \cdot f(D) \in \mathcal{L}(\mathfrak{H})$ is compact for some nowhere-vanishing function $f \in C_0(\mathbb{R})$, then $a \cdot f(D)$ is compact for all $f \in C_0(\mathbb{R})$. Similarly for the operators $f(D) \cdot a$.*

Proof. Let $\mathcal{C}$ denote the set of all $f \in C_0(\mathbb{R})$ such that $af(D)$ is compact. Then $\mathcal{C}$ is clearly a closed ideal in $C_0(\mathbb{R})$. But any nowhere-vanishing element of $C_0(\mathbb{R})$ generates a dense ideal in $C_0(\mathbb{R})$. Indeed, if $f \in C_0(\mathbb{R})$ is nowhere-vanishing, then $\{fe^{-x^2}, fxe^{-x^2}\}$ generate a dense subalgebra in $C_0(\mathbb{R})$ by the Stone-Weierstrass theorem. Hence $\mathcal{C} = C_0(\mathbb{R})$. □

A.3 Unbounded Fredholm Operators

Our main reference is [Fed96]. Let $P : \mathfrak{H}^0 \to \mathfrak{H}^1$ be a linear operator with dense domain $\mathrm{dom}(P) \subseteq \mathfrak{H}^0$. We define the *kernel*, *image* and *cokernel* of P by, respectively,

$$\ker(P) \coloneqq \{\xi \in \mathrm{dom}(P) \mid P\xi = 0\} \subseteq \mathfrak{H}^0$$

$$\mathrm{im}(P) = P\mathfrak{H}^0 \coloneqq \{P\xi \mid \xi \in \mathrm{dom}(P)\} \subseteq \mathfrak{H}^1$$

$$\mathrm{coker}(P) \coloneqq \mathfrak{H}_2/\mathrm{im}(P).$$

The kernel of a *closed* operator is always closed, but the image is *not* necessarily closed; we consider the cokernel purely algebraically. As in Subsection A.1, we work with densely defined operators exclusively. Recall that if $P : \mathfrak{H}^0 \to \mathfrak{H}^1$ is a closed operator, the adjoint operator

$P^* : \mathfrak{H}^1 \to \mathfrak{H}^0$ is also densely defined and closed and $P^{**} = P$ (apply [RS80, Theorem VIII.1] to the operator $\begin{bmatrix} 0 & 0 \\ P & 0 \end{bmatrix}$ on $\mathfrak{H}^0 \oplus \mathfrak{H}^1$).

An operator $P : \mathfrak{H}_1 \to \mathfrak{H}_2$ is called a *Fredholm operator* if $\ker(P)$ and $\operatorname{coker}(P)$ are finite dimensional. The integer

$$\operatorname{Index}(P) \coloneqq \dim \ker(P) - \dim \operatorname{coker}(P)$$

is called the *index* of P.

The following is standard.

Lemma A.3.1. *Let $P : \mathfrak{H}^0 \to \mathfrak{H}^1$ be a closed Fredholm operator.*

(a) The operator $P^* : \mathfrak{H}^1 \to \mathfrak{H}^0$ is also Fredholm.

(b) The image of P satisfies $\operatorname{im}(P) = \ker(P^*)^\perp$. In particular, $\operatorname{im}(P)$ is closed and

$$\dim \operatorname{coker}(P) = \dim \ker(P^*).$$

(c) There exists a bounded operator $R_0 : \mathfrak{H}^1 \to \mathfrak{H}^0$ such that $1 - R_0 P$ and $1 - P R_0$ are the orthogonal projections onto the finite dimensional subspaces $\ker(P) \subset \mathfrak{H}^0$ and $\ker(P^*) \subset \mathfrak{H}^1$, respectively.

Proof. See [Fed96, Proposition 4.1.2]. □

In particular, if P is closed and Fredholm, then

$$\operatorname{Index}(P) = \dim \ker(P) - \dim \ker(P^*) = -\operatorname{Index}(P^*).$$

A theorem of Atkinson [Atk51] says that a bounded linear operator is Fredholm if and only if it is invertible modulo compact operators. The following is an straightforward extension to unbounded operators.

Lemma A.3.2. *Let $P : \mathfrak{H}^0 \to \mathfrak{H}^1$ be a closed operator. The the following conditions are equivalent:*

(1) The operator P is Fredholm.

(2) There exists a bounded operator $R : \mathfrak{H}^1 \to \mathfrak{H}^0$ such that $1 - RP$ and $1 - PR$ are finite rank operators.

(3) There exists a bounded operator $R : \mathfrak{H}^1 \to \mathfrak{H}^0$ such that $1 - RP$ and $1 - PR$ are compact operators.

Proof. The implication (2) $\Longrightarrow$ (3) is clear. The implication (1) $\Longrightarrow$ (2) follows from part (c) of Lemma A.3.1.

Finally, the implication (3) $\implies$ (1) follows from Atkinson's theorem. Indeed, since $1 - RP$ is compact, the bounded operator RP is Fredholm and thus $\ker(RP)$ is finite dimensional. But $\ker(P) \subseteq \ker(RP)$, therefore $\ker(P)$ too is finite dimensional. Similarly for the cokernel. $\square$

We say that R is a *parametrix* if $1 - RP$ and $1 - PR$ are compact.

Example A.3.3. A self-adjoint operator D with compact resolvents is a Fredholm operator with parametrix $(D + i)^{-1}$.

For index theoretic purposes, the following variation is more useful.

Lemma A.3.4 (Fedosov [Fed74])**.** *A closed operator $P : \mathfrak{H}^0 \to \mathfrak{H}^1$ is Fredholm if and only if, for any $p \geq 1$, there exists a bounded operator $R : \mathfrak{H}^1 \to \mathfrak{H}^0$ such that $1 - RP$ and $1 - PR$ are of Schatten $\mathcal{L}^p$-class. In this case,*

$$\mathrm{Index}(P) = \mathrm{Tr}\left((1 - RP)^k\right) - \mathrm{Tr}\left((1 - PR)^k\right) \tag{A.3.1}$$

for any $k \geq p$.

See also [Hör79], [Con85, Appendix I Proposition 6] and [Fed96, Proposition 4.1.3].

Proof. Let $p = k = 1$. The first part follows from Lemma A.3.2. The identity (A.3.1) holds for the operator R_0 from part (c) of Lemma A.3.1. For general R, consider the operators $(R - R_0)P$ and $P(R - R_0)$. These are of trace-class, thus have the same trace by Lemma A.2.5. Hence,

$$\mathrm{Tr}(1 - RP) - \mathrm{Tr}(1 - PR) = \mathrm{Tr}(1 - R_0P) - \mathrm{Tr}(1 - PR_0).$$

For general $p \geq 1$ and $k \geq p$, let

$$R' = \sum_{j=0}^{k-1} R(1 - PR)^j = \sum_{j=0}^{k-1} (1 - RP)^j R.$$

Then $1 - R'P = (1 - RP)^k$ and $1 - PR' = (1 - PR)^k$ are of trace-class and we may apply the case of $p = k = 1$ to the pair (P, R'). $\square$

Finally, recall that the Fredholm index has the following stability property.

Lemma A.3.5. *Let P_0 and P_1 be closed operators with the same domain. Assume that P_0 is a Fredholm operator with parametrix R and $||R(P_0 - P_1)|| < 1$ and $||(P_0 - P_1)R|| < 1$. Then P_1 is also Fredholm and*

$$\mathrm{Index}(P_0) = \mathrm{Index}(P_1).$$

In particular, if P_t is a family of closed Fredholm operators with a common domain and if there exist parametrices R_t such that R_tP_t and P_tR_t are norm continuous families, then $\mathrm{Index}(P_t)$ *is independent of t.*

Proof. See [Fed96, Proposition 4.1.4 and Corollary 4.1.5] $\square$

Graded Picture

Let $P : \mathfrak{H}^0 \to \mathfrak{H}^1$ be a closed operator. In order to facilitate the use of functional analysis a là Subsection A.1, we employ the usual trick and consider the operator

$$D = \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}$$

on $\mathfrak{H} := \mathfrak{H}^0 \oplus \mathfrak{H}^1$ with domain $\mathrm{dom}(P) \oplus \mathrm{dom}(P^*)$. We consider $\mathfrak{H}$ as a graded Hilbert space with even part $\mathfrak{H}^0$ and odd part $\mathfrak{H}^1$. Then D is odd and self-adjoint (not merely symmetric) and D is Fredholm if and only if P is.

Indeed, let $D = \begin{bmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{bmatrix}$ be an operator on the graded Hilbert space $\mathfrak{H} := \mathfrak{H}^0 \oplus \mathfrak{H}^1$. Then D is odd if and only if it is of the form $D = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$ and in this case one can easily check that

$$\overline{D} = \begin{bmatrix} 0 & \overline{Q} \\ \overline{P} & 0 \end{bmatrix} \quad \text{and} \quad D^* = \begin{bmatrix} 0 & P^* \\ Q^* & 0 \end{bmatrix}.$$

Hence $D = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$ is

closed if and only if both P and Q are closed;

symmetric if and only if $P \subseteq Q^*$ and $Q \subseteq P^*$;

essentially self-adjoint if and only if $\overline{P} = Q^*$ and $\overline{Q} = P^*$;

self-adjoint if and only if P is closed and $Q = P^*$.

Now assume that an odd closed operator $D = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$ is Fredholm with parametrix $\begin{bmatrix} * & R \\ * & * \end{bmatrix}$. Then R is a parametrix for P, hence P is Fredholm. Conversely, if a closed operator P is Fredholm with parametrix R, then $\begin{bmatrix} 0 & R \\ R^* & 0 \end{bmatrix}$ is a parametrix for $D = \begin{bmatrix} 0 & P^* \\ P & 0 \end{bmatrix}$, hence D is Fredholm.

Let $\mathfrak{H}$ be a graded Hilbert space.

The *super-index* of a closed odd Fredholm operator $D : \mathfrak{H} \to \mathfrak{H}$ is

$$\mathrm{Index}_s(D) := \mathrm{Index}(D_{10}).$$

The *super-dimension* of a finite dimensional graded subspace $V \subseteq \mathfrak{H}$ is

$$\dim_s(V) := \dim V^0 - \dim V^1.$$

The *super-trace* of a trace class operator $T \in \mathcal{L}(\mathfrak{H})$ is

$$\mathrm{Str}(T) \coloneqq \mathrm{Tr}(\gamma T),$$

where $\gamma : \mathfrak{H} \to \mathfrak{H}$ is the grading operator.

It follows from the discussion above that if a Fredholm operator D is *odd* and *self-adjoint*, then the components $D_{10} = D_{01}^*$ are Fredholm and

$$\mathrm{Index}_s(D) = \mathrm{Index}(D_{10}) = -\mathrm{Index}(D_{01}) = \dim_s(\ker D).$$

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Vita

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  Abstract: We define and study a bivariant generalization of the topological K-group $K^{\mathrm{top}}(G)$ for a topological group G. We consider the Baum-Connes conjecture in this context and study its relation to the usual Baum-Connes conjecture. Generalizations of the Universal Coefficient Theorem and the Green-Julg theorem are presented. Available at `http://www.arxiv.org/abs/math.KT/0403511`
- Ph.D. Mathematics, Pennsylvania State University, 2009 (Expected)
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  Thesis: *Local Index Theorem In Noncommutative Geometry*
  Abstract: We consider the multiplicative property of index cocycles associated to a spectral triple. We construct a perturbation of the JLO-cocyle and prove that it is multiplicative on the cochain level with respect to the exterior product of theta-summable spectral triples and the product in entire cyclic theory. Furthermore, we show that this cocycle admits an asymptotic expansion if the spectral triple admits the analogue of an Euler vector field. Using Laurent series, we obtain a multiplicative local index cocycle.