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EXISTENCE OF MONETARY STEADY STATES IN A MATCHING MODEL OF MONEY

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Abstract

Existence of a monetary steady state is established for a random matching model with divisible goods, general individual money holdings, and take-it-or-leave-it offers by consumers. For indivisible money, the only assumption is a lower bound on the marginal utility of consumption at zero. For divisible money, there are two additional assumptions: the marginal utility of consumption at zero is bounded above and there is a finite bound on individual money holdings. In each case, the monetary steady state shown to exist has nice properties: the value function, defined on money holdings, is increasing and strictly concave, and the measure over money holdings has full support.

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Chapter 1

Introduction

I establish existence of a monetary steady state in a random matching model with divisible goods, general individual money holdings, and take-it-or-leave-it offers by consumers. Chapter 2 deals with indivisible money and chapter 3 with divisible money. The common background environment is that in papers by Shi (1995) and by Trejos and Wright (1995). The monetary steady state shown to exist has nice properties: the value function, defined on money holdings, is increasing and strictly concave, and the measure over money holdings has full support.

I start with indivisible money. I extend a result of Taber and Wallace (1995), who study indivisible commodity money, money with a direct utility payoff, with a finite bound on individual holdings. They establish existence of a steady state with a concave and strictly increasing value function. I extend their result in two respects. I allow individual money holdings to be unbounded (their finite bound is a simple special case) and I consider fiat money. To deal with fiat money, I show that there exists a steady state for the corresponding commodity money version in which the value of money is bounded away from zero as the direct utility payoff approaches zero.

In chapter 3, I consider divisible money. In contrast to indivisible money, I need to assume bounds on both the marginal utility of consumption at zero and on individual money holdings. My approach is to define a grid on the interval $[0, B]$ with mesh Δ, where

B is the bound on individual money holdings and Δ is the smallest unit of indivisible money. I first embed the steady state for mesh Δ into the space of measures and value functions defined on $[0, B]$. I then show that a steady state for divisible money is the limit of embedded functions as Δ goes to 0. To do this, a suitable topology for the two function spaces has to be selected. I select the weak(*) topology for the space of distribution functions, and the sup norm for the space of value functions. Finiteness of the marginal utility of consumption at zero and finiteness of the bound on individual holdings are sufficient to guarantee that the slope of the value function for indivisible money is bounded as Δ goes to 0. This implies compactness of the space of value functions, and, hence, existence of the limit.

Chapter 2

Indivisible Money

2.1 Introduction

Shi (1995) and Trejos and Wright (1995) introduce a matching model of money with divisible goods. While the model builds on the indivisible goods model of Kiyotaki and Wright (1989), the introduction of divisible goods permits output and prices to be determined as part of an equilibrium. Trejos and Wright show that equilibrium under a bargaining rule is easily formulated for general individual money holdings. However, existence of a monetary equilibrium has been established only for special versions. Here, I give a general existence proof for indivisible money. In particular, under the bargaining rule that potential consumers make take-it-or-leave-it offers, I prove that there exists a steady state with a value function defined on money holdings that is increasing and strictly concave and with a measure over money holdings that has full support. The only assumptions are lower bounds on (a) the marginal utility of consumption at zero and (b) the ratio of the average stock of money to the size of the smallest unit of money.

Proving existence is difficult because the general model has endogenous heterogeneity of money holdings. Most researchers simplify or avoid the endogeneity of the distribution of money holdings by making special assumptions.[1] One exception is Molico

[1] Green and Zhou (1998) and Zhou (1999) assume indivisible goods and divisible money. Green and Zhou (in print) assume divisible goods and divisible money, but make preference assumptions that effectively make goods indivisible. Camera and Corbae (1999) consider the

(1997). He studies the model numerically and claims to find monetary steady states for divisible money and unbounded individual holdings. My results—and those in a companion paper on divisible money—provide a basis for interpreting his numerical results. Another exception is Taber and Wallace (1999), who study indivisible commodity money, money with a direct utility payoff, with a general finite bound on individual holdings. They establish existence of a steady state with a concave and strictly increasing value function. I extend their result in two respects. I allow individual money holdings to be unbounded and I consider fiat money. To deal with fiat money, I show that there exists a steady state for the corresponding commodity money version in which the value of money is bounded away from zero as the direct utility payoff approaches zero.[2]

The properties of the steady state shown to exist—monotonicity and strict concavity of the value function and full support of the measure—are important. One implication is a non-neutrality result. Two economies that have different ratios of average holdings of money to the smallest unit of money have different sets of steady states in terms of allocations. In fact, if the larger ratio is an integer multiple of the smaller ratio, then the set of steady states for the economy with more money is a strict superset of that for the economy with less money. As shown below, this is an immediate implication of the full-support property.

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same model as I study with a finite bound on individual money holdings. For a special region of the parameter space, they construct a steady state in which one unit of money is offered in every trade. Cavalcanti (2000) assumes a unit bound of money holdings and a large number of kinds of money. Shi (1997) and Lagos and Wright (2000) make special assumptions that produce a degenerate distribution of money holdings.

[2]The approach used here is likely to be applicable to models in which the source of heterogeneity in money holdings is preference shocks rather than random meetings.

2.2 The Model

As noted above, the model is essentially that in Shi (1995) and Trejos and Wright (1995).

2.2.1 Environment

Time is discrete, dated as $t \geq 0$. There is a $[0,1]$ continuum of each of $N \geq 3$ types of infinitely lived agents, and there are N distinct produced and perishable types of divisible goods at each date. A type n agent, $n \in \{1, 2, ..., N\}$, produces only type n good and consumes only type $n+1$ good (modulo N). Each agent maximizes expected discounted utility with discount factor $\beta \in (0,1)$. For a type n agent, utility in a period is $u(q_{n+1}) - q_n$, where $q_{n+1} \in \mathbb{R}_+$ is consumption of type $n+1$ good and $q_n \in \mathbb{R}_+$ is production of type n good. The utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is increasing, strictly concave, continuously differentiable, and satisfies $u(0) = 0$ and $u'(\infty) < 1$. In addition, there is a lower bound on $u'(0)$ which is specified later.

There exists a fixed stock of money which is perfectly durable. Money is symmetrically distributed among the N specialization types. Let the average money holding be denoted by $\overline{m}$ and let the (smallest) unit of money be denoted by $\Delta (> 0)$. I assume that Δ is small relative to $\overline{m}$ with a lower bound on $\overline{m}/\Delta$ that is specified later. Also, let the exogenous upper bound of individual money holdings be denoted by B. Although the focus of the paper is unbounded individual holdings ($B = \infty$), I also include the bounded case (B finite). If B is finite, then it is assumed to be large relative to $\overline{m}$ with a lower bound on $B/\overline{m}$ that is specified later. ($B > \overline{m}$ is necessary for trade to occur.)

Also, if B is finite, then B/Δ is assumed to be an integer. Let $B_\Delta = \{0, \Delta, ..., B\}$ denote the set of possible individual holdings of money.

In each period, agents are randomly matched in pairs. A meeting between a type n agent and a type $n+1$ agent is called a single-coincidence meeting. Other meetings are not relevant. In meetings, the agents' types and money holdings are observable, but any other information about an agent's trading history is private.

2.2.2 Definition of Equilibrium

In single-coincidence meetings, the potential consumer makes a take-it-or-leave-it offer, (p, q), where p is the amount of money offered and q is the amount of production demanded. Let $w_t(x)$ be the expected discounted value of holding x amount of money at the start of period t, prior to date t matching, where $w_t : B_\Delta \to \mathbb{R}_+$ is nondecreasing. Consider a date t single-coincidence meeting between a consumer with x amount of money and a producer with m amount of money. Let

$$\Gamma(x, m) = \{p \in B_\Delta : p \leq \min\{x, B - m\}\}, \tag{2.1}$$

the set of feasible offers of money. (As a convention, $\infty - m = \infty$.) Assuming, as is standard, that the producer accepts all offers which leave him no worse off, an optimal offer satisfies $p \in \Gamma(x, m)$ and $q = \beta w_{t+1}(m+p) - \beta w_{t+1}(m)$, where the equality for q says that the lower bound on the producer's gain-from-trade, zero, is attained. Therefore, the consumer's problem reduces to $\max_{p\in\Gamma(x,m)}\{u[\beta w_{t+1}(m+p) - \beta w_{t+1}(m)] + \beta w_{t+1}(x - p)\}$. To express this objective function more succinctly, it is convenient to introduce a symbol for an increment in a function: for any function $g : \mathbb{R} \to \mathbb{R}$, let $g(x, y) \equiv g(x) - g(x - y)$. Using this shorthand and dropping the time subscript on the value function, for a nondecreasing $w : B_\Delta \to \mathbb{R}_+$ and $(x, m) \in B_\Delta^2$, let

$$f(x, m, w) = \max_{p \in \Gamma(x,m)} \{u[\beta w(m + p, p)] + \beta w(x - p)\}, \tag{2.2}$$

and

$$p(x, m, w) = \operatorname*{arg\,max}_{p \in \Gamma(x,m)} \{u[\beta w(m + p, p)] + \beta w(x - p)\}. \tag{2.3}$$

That is, when w is the value of money at the start of the next period, f is the payoff for a consumer with x (pre-trade) who meets a producer with m (pre-trade) while p is the set of optimal offers of money.

Because $p(x, m, w)$ is discrete, and, may, therefore, be multi-valued, it is important for existence to allow all possible randomizations over the elements of $p(x, m, w)$. In order to describe the law of motion for the distribution of money holdings, it is convenient to express randomizations over the post-trade money holdings of consumers. Therefore, I define the set of randomizations, a set of measures on B_Δ, as

$$\Lambda(y, m, w) = \{\lambda(.; y, m, w) : \lambda(x; y, m, w) = 0 \text{ if } x \notin y - p(y, m, w)\}, \tag{2.4}$$

where $\lambda(x; y, m, w)$ is the fraction of consumers with y (pre-trade) in meetings with producers with m (pre-trade) who end up with x.

Let $\pi_t(x)$ denote the fraction of agents holding x amount of money at the start of period t, so that π_t is a measure on B_Δ. The law of motion for π_{t+1} can be expressed as

$$\begin{aligned}\pi_{t+1}(x) &= \frac{N-2}{N}\pi_t(x) + \frac{1}{N}\textstyle\sum_{y,m}\pi_t(y)\pi_t(m)[\lambda(x;y,m,w_{t+1}) \\ &\quad +\lambda(m+y-x;m,y,w_{t+1})]\end{aligned} \tag{2.5}$$

for some

$$\lambda(.;y,m,w_{t+1}) \in \Lambda(y,m,w_{t+1}). \tag{2.6}$$

Note that $\lambda(m+y-x;m,y,w_{t+1})$ is the fraction of producers with y (pre-trade) in meetings with consumers with m (pre-trade) who end up with x. The value function, $w_t(x)$, satisfies

$$w_t(x) = \frac{N-1}{N}\beta w_{t+1}(x) + \frac{1}{N}\textstyle\sum_m \pi_t(m) f(x,m,w_{t+1}). \tag{2.7}$$

This follows from the fact that the payoff to being a producer with x is $\beta w_{t+1}(x)$.

I can now state the relevant definitions.

Definition 1. *Given π_0, a sequence $\{w_t, \pi_{t+1}\}_{t=0}^{\infty}$ is an equilibrium if it satisfies* (2.1)–(2.7). *A monetary equilibrium is an equilibrium with positive consumption and production. A pair (w, π) is a steady state if $\{w_t, \pi_{t+1}\}_{t=0}^{\infty}$ with $w_t = w$ and $\pi_{t+1} = \pi$ for all t is an equilibrium for $\pi_0 = \pi$.*

2.3 Existence of a Monetary Steady State

To establish the existence of a monetary steady state, the following assumptions are maintained from now on.

$(A1) u'(0) > [2/(R\beta)]^2$, where $R \equiv [N - (N-1)\beta]^{-1}$.[3]

$(A2) B \geq 4\overline{m}$.

$(A3) \Delta \leq \overline{m}D/(\beta\overline{W})$, where D is the unique solution of $u'(D) = [2/(R\beta)]^2$ and $\overline{W}$ is the unique positive solution of $N(1-\beta)\overline{W} = u(\beta\overline{W}) + N$.[4]

In this model, existence always requires a lower bound on $u'(0)$ because a producer has to see a future reward from producing. Assumptions $(A2)$ and $(A3)$ say that the set of individual holdings is large enough (relative to the average holding). In what follows, except to the discussion of neutrality and non-neutrality at the end of the paper, it is convenient to normalize the exogenous nominal variables $\overline{m}$, Δ, and B by letting $\overline{m} = 1$.

I start by defining the main correspondences used. These are essentially implied by (2.1) – (2.7). Let $\mathbf{W}$ be the set of concave and nondecreasing functions from B_Δ to $[0, \overline{W}]$. Let $\mathbf{\Pi}$ be the subset of measures on B_Δ satisfying the unit mean condition. Let both $\mathbf{W}$ and $\mathbf{\Pi}$ be equipped with the topology of pointwise convergence.

Let the single-valued map Φ_w on $\mathbf{W} \times \mathbf{\Pi}$ be defined by

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[3] If (w, π) is a steady state, then (2.7) can be written as $w(x) = R\sum_m \pi(m) f(x, m, w)$. This expression for $w(x)$ is used repeatedly below. Also, note that $R < 1$.

[4] As will be shown in Lemma 2.3, $\overline{W} - 1$ can be taken to be an upper bound on steady state value functions. Also, note that $D/(\beta\overline{W}) < 1$ because $\beta[2/(R\beta)]^2 > N(1-\beta)$.

$$\Phi_w(w,\pi)(x) = \frac{N-1}{N}\beta w(x) + \frac{1}{N}\sum\nolimits_m \pi(m) f(x,m,w). \quad (2.8)$$

Let the correspondence Φ_π on $\mathbf{W} \times \mathbf{\Pi}$ be defined by

$$\begin{aligned} \Phi_\pi(w,\pi) \quad &= \quad \{\nu : \nu(x) = \frac{N-2}{N}\pi(x) + \frac{1}{N}\sum\nolimits_{y,m} \pi(y)\pi(m)[\lambda(x;y,m,w) \quad (2.9) \\ &\quad +\lambda(m+y-x;m,y,w)] \text{ for some } \lambda(.;y,m,w) \in \Lambda(y,m,w)\}. \end{aligned}$$

Finally, let $\Phi = (\Phi_w, \Phi_\pi)$.

In what follows, I deal directly with the unbounded case ($B = \infty$). The finite bound situation is a special case. The next lemma establishes important properties of $\mathbf{W} \times \mathbf{\Pi}$ and Φ.

Lemma 2.1. *$(i)\mathbf{W} \times \mathbf{\Pi}$ is compact and metrizable. $(ii)\Phi(w,\pi) \subset \mathbf{W} \times \mathbf{\Pi}$ with $\Phi_w(w,\pi)$ bounded above by $\overline{W} - 1$. $(iii)\Phi$ is convex-valued. $(iv)\Phi$ is upper hemicontinuous.*

Proof. By the Tychonoff Product Theorem (see Aliprantis and Border (1994, page 53)), both $\mathbf{W}$ and $\mathbf{\Pi}$ are compact. By 3.30 Theorem of Aliprantis and Border (1994, page 89), both $\mathbf{W}$ and $\mathbf{\Pi}$ are metrizable. By the definition of Φ_π in (2.9), there is no disposal of money. Hence $\Phi_\pi(w,\pi) \subset \mathbf{\Pi}$. Taber and Wallace (1999) show that $\Phi_w(w,\pi)$ preserves concavity and monotonicity. As regards the bound,

$$\begin{aligned} N\Phi_w(w,\pi)(x) \quad &\leq \quad (N-1)\beta w(x) + u[\beta w(x)] + \beta w(x) \\ &\leq \quad N\beta\overline{W} + u(\beta\overline{W}) = N\overline{W} - N, \end{aligned}$$

where the first inequality follows from (2.8) and the equality from the definition of $\overline{W}$. Because $\Lambda(y,m,w)$ is convex, it follows that Φ_π is convex-valued.

Now we consider part (iv). We begin with three claims.

Claim 1 : *$f(.,.,.)$ is continuous on $B_\Delta^2 \times \mathbf{W}$ and $p(.,.,.)$ is upper hemicontinuous on $B_\Delta^2 \times \mathbf{W}$.* Let $A = \{(x,m,w,p) : (x,m,w) \in B_\Delta^2 \times \mathbf{W}$ and $p \in \Gamma(x,m)\}$. Let $g : A \to \mathbb{R}_+$ be defined by $g(x,m,w,p) = u[\beta w(m+p,p)] + \beta w(x-p)$. Because the value of $g(x,m,w,p)$ only depends on $x, m, w(0), w(\Delta), ..., w(x+m)$, and p, it follows that g is continuous on A. Then claim 1 follows from Berge's Maximum Theorem (see Aliprantis and Border (1994, page 473)).

Claim 2 : *Let $w_n, w \in \mathbf{W}$ with $w_n \to w$. For all x, $f(x,m,w_n) \to f(x,m,w)$ uniformly in m.* Fix x and fix $\varepsilon > 0$. Let m^* be such that $u(\beta x\overline{W}/m^*) < \varepsilon$ and let n be such that $\beta\,|w_n(y) - w(y)| < \varepsilon$ for all $y \le x$. By Claim 1, for sufficiently large n, $|f(x,m,w) - f(x,m,w_n)| < \varepsilon$ for all $m \le m^*$. So we only need to consider $m > m^*$. Let $m > m^*$. Because w is concave and bounded above by $\overline{W}$ and $w(0) \ge 0$, it follows that $\overline{W}/m^* > w(m+x,x)/x$ or $w(m+x,x) < x\overline{W}/m^*$. Let $p \in p(x,m,w)$. Because $g(x,m,w_n,p) \le f(x,m,w_n)$ (for g, see Claim 1), it follows that

$$
\begin{aligned}
& f(x,m,w) - f(x,m,w_n) \\
\le\ & u[\beta w(m+p,p)] - u[\beta w_n(m+p,p)] + \beta[w(x-p) - w_n(x-p)] \\
<\ & u[\beta w(m+p,p)] + \varepsilon \le u[\beta w(m+x,x)] + \varepsilon < u(\beta x\overline{W}/m^*) + \varepsilon < 2\varepsilon.
\end{aligned}
$$

By reversing the roles of $f(x,m,w)$ and $f(x,m,w_n)$, we have $f(x,m,w_n) - f(x,m,w) < 2\varepsilon$. This establishes Claim 2.

Claim 3 : Let the correspondence J on $\mathbf{W} \times \mathbf{\Pi}$ *be defined as*

$$J(w,\pi) = \{\mu : \mu(x) = \textstyle\sum_{y,m} \pi(y)\pi(m)[\lambda(x;y,m,w) + \lambda(m+y-x;m,y,w)] \textit{for some } \lambda(.;y,m,w) \in \Lambda(y,m,w)\}.$$

Let $(w_n, \pi_n), (w,\pi) \in \mathbf{W} \times \mathbf{\Pi}$ *with* $(w_n,\pi_n) \to (w,\pi)$. *There exists a subsequence of* n, *denoted by* j, *such that there exist* $\mu_j \in J(w_j, \pi_j)$ *and* $\mu \in J(w,\pi)$ *with* $\mu_j \to \mu$.

Fix $\varepsilon > 0$. Let $x^* > 1/\varepsilon$ and let n be such that $|\pi_n(x) - \pi(x)| < \varepsilon$ for all $x \leq x^*$. By Claim 1, $p(y,m,.)$ is upper hemicontinuous on $\mathbf{W}$ for all (y,m). Hence there exists a subsequence of n, denoted by j, such that for large j, an element of $p(y,m,w_j)$ coincides with an element of $p \in p(y,m,w)$ for all $y,m \leq x^*$. It follows that there exist $\lambda(.;y,m,w_j) \in \Lambda(y,m,w_j)$ and $\lambda(.;y,m,w) \in \Lambda(y,m,w)$ such that for large j, $\lambda(.;y,m,w_j) = \lambda(.;y,m,w)$ for all $y,m \leq x^*$. Let $\lambda(x;y,m,w_j) + \lambda(m+y-x;m,y,w_j)$ be denoted by $c^{x,j}_{y,m}$ and let $\lambda(x;y,m,w) + \lambda(m+y-x;m,y,w)$ be denoted by $c^{x}_{y,m}$. Now let $\mu_j(x) = \sum_{y,m} \pi_j(y)\pi_j(m)c^{x,j}_{y,m}$ and $\mu(x) = \sum_{y,m} \pi(y)\pi(m)c^{x}_{y,m}$. Because $x^* > 1/\varepsilon$ and the average holding is unity, it follows that both $\mu_j(x)$ and $\mu(x)$ are bounded above by ε for all $x > x^*$. So we only need to consider $x \leq x^*$. For $x \leq x^*$, we have

$$\begin{aligned}
\mu_j(x) - \mu(x) &= \textstyle\sum_{y,m \leq x^*} [\pi_j(y)\pi_j(m) - \pi(y)\pi(m)]c^{x}_{y,m} \\
&\quad + \textstyle\sum_{\text{either } y > x^* \text{ or } m > x^*} [\pi_j(y)\pi_j(m)c^{x,j}_{y,m} - \pi(y)\pi(m)c^{x}_{y,m}] \\
&< 2\textstyle\sum_{y,m \leq x^*} \{[\pi(y)+\varepsilon][\pi(m)+\varepsilon] - \pi(y)\pi(m)\} \\
&\quad + 2\textstyle\sum_{y > x^*} \pi_j(y)\pi_j(m) + 2\sum_{m > x^*} \pi_j(y)\pi_j(m) \\
&< 8\varepsilon + 2\varepsilon^2,
\end{aligned}$$

where the last inequality follows from $\pi_j(m) < \varepsilon$ for all $m > x^*$. By reversing the roles of $\mu_j(x)$ and $\mu(x)$, we have $\mu(x) - \mu_j(x) < 8\varepsilon + 2\varepsilon^2$. This establishes Claim 3.

By 12.6 Corollary of Aliprantis and Border (1994, page 417), Claim 2 implies that for all x, $(w, \pi) \mapsto \sum_m \pi(m) f(x, m, w)$ is continuous. It follows that Φ_w is continuous. By Claim 3, J is upper hemicontinuous. It follows that Φ_π is upper hemicontinuous. ■

Next, I introduce a perturbation of the mapping Φ, which can be interpreted as assigning some direct utility to money. Let the real function h on B_Δ be defined by

$$h(x) = x/4 \text{ if } x \leq 4, \quad h(x) = 1 \text{ if } x > 4.$$

Let $\mathbf{K}$ be the set of concave and nondecreasing functions from B_Δ to $[0, \overline{W} - 1]$. (Note that $\mathbf{K} \subset \mathbf{W}$.) For a positive integer n, let the correspondence $\Phi_n = (\Phi_{w,n}, \Phi_{\pi,n})$ on $\mathbf{K} \times \mathbf{\Pi}$ be defined by

$$\Phi_n = \Phi(w + h/n, \pi). \tag{2.10}$$

Lemma 2.2. *Φ_n has a fixed point.*

Proof. Because $w \in \mathbf{K}$ implies $w + h/n \in \mathbf{W}$ and because $(w, \pi) \mapsto (w + h/n, \pi)$ is continuous, by Lemma 2.1 (*iii*) and (*iv*), Φ_n is convex-valued and upper hemicontinuous. By Lemma 2.1 (*ii*), $\Phi_n(w, \pi) \subset \mathbf{K} \times \mathbf{\Pi}$. Because $\mathbf{K} \subset \mathbf{W}$ is closed, it follows that $\mathbf{K}$ is compact, and, hence, that $\mathbf{K} \times \mathbf{\Pi}$ is compact. Then by Kakutani's fixed point theorem (see Zeidler (1985, page 452)), Φ_n has a fixed point. ■

The next lemma, the main ingredient in the existence proof, establishes a uniform (with respect to n in (2.10)) lower bound on the value functions of the fixed points of Φ_n.

Lemma 2.3. *If* (w, π) *is a fixed point of* Φ_n, *then* $w(4) \geq D/\beta - 1/n$.

Proof. Assume by contradiction that $w(4) < D/\beta - 1/n$. Let $w + h/n$ be denoted by φ. The proof is split into two steps. In the first step, we calculate a lower bound on $f(4, m, \varphi) - f(4 - \Delta, m, \varphi)$ for $m \leq 2$. In the second step, we draw contradictions based on this bound. In this and subsequent proofs, we suppress the dependence of f and p on φ or w. Also, for a measure μ on B_Δ and an interval I, we denote $\mu(I \cap B_\Delta)$ by μI.

Step 1. To get the lower bound, consider two possibilities for $p(4-\Delta, m)$ for each $m \leq 2$, according to whether an element of $p(4-\Delta, m)$ does or does not exceed 2. First, assume $p(4 - \Delta, m) \ni p \geq 2$. Because the consumer with money holding 4 can make the same offer as the consumer with $4 - \Delta$ does, and, hence, get the same amount of the consumption good, it follows that

$$f(4, m) - f(4 - \Delta, m) \geq \beta\varphi(4 - p, \Delta) \geq \beta\varphi(2, \Delta) > \beta w(2, \Delta),$$

where the second inequality follows from concavity of φ. Next, assume $p(4 - \Delta, m) \ni p < 2$. Because $m \leq 2$, we have $p + \Delta + m \leq 4$. Hence the consumer with 4 can make the offer $p + \Delta$ to the producer with m and end up with the same money holding as the consumer with $4-\Delta$. It follows that

$$\begin{aligned}
& f(4,m)-f(4-\Delta,m) \\
\geq\ & u[\beta\varphi(m+p+\Delta,p+\Delta)]-u[\beta\varphi(m+p,p)] \\
>\ & u'[\beta\varphi(m+p+\Delta,p+\Delta)]\beta\varphi(m+p+\Delta,\Delta) \\
>\ & u'(D)\beta\varphi(m+p+\Delta,\Delta)\geq u'(D)\beta\varphi(4,\Delta)>u'(D)\beta w(4,\Delta), \qquad (2.11)
\end{aligned}$$

where the second inequality follows from the mean value theorem and strict concavity of u, the third from $\beta\varphi(m+p+\Delta,p+\Delta)<\beta\varphi(4)=\beta[w(4)+1/n]<D$ and strict concavity of u, and the fourth from concavity of φ. Let $l=\min\{\beta w(2,\Delta),\ u'(D)\beta w(4,\Delta)\}$. Then for $m\leq 2$, $f(4,m)-f(4-\Delta,m)>l$.

Step 2. Because (w,π) is a fixed point of Φ^n, by (2.10) and (2.8), we have

$$w(x,\Delta)=R(N-1)\beta h(x,\Delta)/n+R\textstyle\sum_m\pi(m)[f(x,m)-f(x-\Delta,m)].$$

Because $f(x,m)\geq f(x-\Delta,m)$ for all m, it follows that for $m^*<\infty$,

$$w(x,\Delta)\geq R\textstyle\sum_{m=0}^{m^*}\pi(m)[f(x,m)-f(x-\Delta,m)]. \qquad (2.12)$$

Because the average holding is 1, $\pi[0,2]\geq 1/2$. Then by (2.12) and Step 1,

$$w(4,\Delta)>Rl/2. \qquad (2.13)$$

Now consider the two possible values of l. If $l = u'(D)\beta w(4,\Delta)$, then by (2.13),

$$w(4,\Delta) > Rl/2 = (R\beta/2)u'(D)w(4,\Delta) = [2/(R\beta)]w(4,\Delta) > w(4,\Delta),$$

a contradiction. So it must be that $l = \beta w(2,\Delta)$. Then by (2.13),

$$w(4,\Delta) > Rl/2 = (R\beta/2)w(2,\Delta). \tag{2.14}$$

To rule this out, we calculate a lower bound on $f(2,m) - f(2-\Delta,m)$ for $m \leq 2$. Let $p \in p(2-\Delta,m)$. Because $p \leq 2-\Delta$ and $m \leq 2$, we have $p+\Delta+m \leq 4$. Hence the consumer with 2 can offer $p+\Delta$ to the producer with m. It follows that

$$\begin{aligned} & f(2,m) - f(2-\Delta,m) \\ \geq\ & u[\beta\varphi(m+p+\Delta,p+\Delta)] - u[\beta\varphi(m+p,p)] \\ >\ & u'(D)\beta w(4,\Delta) > u'(D)\beta(R\beta/2)w(2,\Delta), \end{aligned}$$

where the second inequality follows exactly the logic used in (2.11) and the last from (2.14). Let $l' = u'(D)\beta(R\beta/2)w(2,\Delta)$. Then by (2.12) and $\pi[0,2] \geq 1/2$, we have

$$w(2,\Delta) > Rl'/2 = (R\beta/2)^2 u'(D)w(2,\Delta) = w(2,\Delta),$$

a contradiction.[5] ■

[5] In this proof, Δ is simply required to be no greater than unity, the average holding. If the average holding $\overline{m}$ is not unity, then we require $\Delta \leq \overline{m}$ and we redefine h as $h(x) = x/(4\overline{m})$ for $x \leq 4\overline{m}$ and $h(x) = 1$ for $x > 4\overline{m}$. It follows that $w_n(4\overline{m})$ is bounded below by $D/\beta - 1/n$.

In some respects, the ingredients in the proof of Lemma 2.3 have analogues in the simple case with $B = \Delta = 1$. In my proof, I require that there be a set of "poor" agents with positive measure. This set plays the role of the agents with 0 when $B = \Delta = 1$. In the proof, the "poor" set is $[0, 2] \cap B_\Delta$ because there is an obvious lower bound on the measure of this set — namely, 1/2. However, other sets would also work. The agents with 4 are like those with holdings of 1 when $B = \Delta = 1$. Of course, when $B = \Delta = 1$, the monetary steady state can be computed directly because the distribution of money holdings and the offers in trades are fixed. The argument here is complicated because very little is known either about the distribution or the offers that agents make.

Now I show that there is a monetary steady state by taking a limit as the direct utility payoff of money approaches zero.

LEMMA 2.4. *Let $\{(w_n, \pi_n)\}$ be a sequence such that (w_n, π_n) is a fixed point of Φ_n. (i)$\{(w_n, \pi_n)\}$ has at least one limit (accumulation) point, denoted (w, π). (ii)(w, π) is a fixed point of Φ. (iii)$w(0) = 0$ and $w(4) \geq D/\beta$.*

Proof. Because $\mathbf{W} \times \mathbf{\Pi}$ is compact, there is a subsequence of $\{(w_n, \pi_n)\}$ that converges to some $(w, \pi) \in \mathbf{W} \times \mathbf{\Pi}$. To simplify the notation, let $\{(w_n, \pi_n)\}$ represent the subsequence whose limit is (w, π). Because (w_n, π_n) is a fixed point of Φ, it follows from (2.10) that $(w_n, \pi_n) \in \Phi(w_n + h/n, \pi_n)$. Because $(w_n, \pi_n) \to (w, \pi)$, it follows that $(w_n + h/n, \pi_n) \to (w, \pi)$. Because Φ is upper hemicontinuous, $(w_n + h/n, \pi_n) \to (w, \pi)$ and $(w_n, \pi_n) \in \Phi(w_n + h/n, \pi_n)$ imply that there is a subsequence of $\{(w_n, \pi_n)\}$

converging to an element of $\Phi(w, \pi)$. Because $\{(w_n, \pi_n)\}$ itself converges to (w, π), it follows that $(w, \pi) \in \Phi(w, \pi)$. Part (iii) is obvious. ■

Any Lemma 2.4 limit point (w, π) is a monetary steady state according to Definition 1. The next lemma establishes some of the properties of (w, π).

LEMMA 2.5. *Let* (w, π) *be a Lemma* 2.4 *limit point.* $(i) w$ *is concave and strictly increasing.* $(ii) \pi(0) > 0$.

Proof. (i)Concavity is obvious. Assume by contradiction that w is not strictly increasing. By concavity of w, there exists $a > 0$ such that $w(x) = w(a)$ if $x \geq a$ and $w(x) < w(a)$ if $x < a$. (That is, by concavity, the flat portion of w must occur over a set of the form $\{a, a + \Delta, ...\}$.) It follows that $w(a) > 0$, and, hence, that there must be a positive probability that the consumer with a makes an offer $p \geq \Delta$ to some producers. The consumer with $a + \Delta$ has the same probability of meeting those producers and can also make the offer p. If so, he ends up with $a + \Delta - p$ and the consumer with a ends up with $a - p$. But then $a - p < a$ implies $w(a + \Delta - p) > w(a - p)$. This, in turn, implies $w(a + \Delta) > w(a)$, a contradiction.

(ii)Assume by contradiction that $\pi(0) = 0$ and let $a = \min\{x : \pi(x) > 0\}$. It follows that $w(a) > 0$, and, hence, that there must be a positive probability that the consumer with a makes an offer $p \geq \Delta$ to some producers. That is, for some m with $\pi(m) > 0$, $p \in p(a, m)$ with $p \geq \Delta$ occurs with positive possibility. But then $\pi(a)\pi(m) > 0$ implies $\pi(x - p) > 0$, a contradiction. ■

Now I turn to establishing that the steady state measure has full support. In what follows, let (w, π) be a Lemma 2.4 limit point and let *supp* π denote the support of π. The full support result relies on some facts about the optimal offers of money, $p(x, m, w)$, and their dependence on x and m.

LEMMA 2.6. *(i)If* $p_1 \in p(x, m, w)$ *and* $p_2 \in p(x + \Delta, m, w)$*, then* $p_2 - p_1 \in \{0, \Delta\}$. *(The consumer's marginal propensity to spend on a given producer is between* 0 *and* 1.*)*

(ii)If $x_1 < x_2$*, then* $x_1 + \max p(m, x_1, w) \leq x_2 + \min p(m, x_2, w)$. *(For a given consumer, the producer's post-trade money holding is weakly increasing in his pre-trade holding.)*

(iii)If $x_2 \geq x_1$ *and* $m_2 < m_1$*, then* $\max p(x_2, m_2, w) \leq \max\{x_2 - x_1, m_1 - m_2\} + \min p(x_1, m_1, w)$. *(If the consumer is richer and the producer is poorer, then the change in spending is bounded above by the maximum of differences in the consumer's and the producer's holdings.)*

(iv)Assume $\min p(x_1, m_1, w) = 0$ *and* $m_2 \geq m_1$. *If* $x_2 > x_1$*, then* $\max p(x_2, m_2, w) \leq x_2 - x_1$. *If* $x_2 = x_1$*, then* $\min p(x_2, m_2, w) = 0$. *(If a consumer and a producer do not trade, then a richer consumer who meets a richer producer offers at most the consumer's increment.)*

(v)If $x > m$*, then* $0 \notin p(x, m, w)$. *(If the consumer is richer than the producer, then there is trade.)*

Proof. See the Appendix. ■

The next lemma shows that there is no endogenous bound.

LEMMA 2.7. *There is no $x \in B_\Delta$ such that $\pi(m) = 0$ for $m > x$.*

Proof. Assume by contradiction that $\exists\, x = \max\{m : \pi(m) > 0\} < B$. Because w is concave and bounded above by $\overline{W}$ and $w(0) = 0$, it follows that $w(x + \Delta, \Delta) < \Delta\overline{W}/x$. Because the average holding is 1 and $\pi(0) > 0$, it follows that $x > 1$. Then by assumption $(A3)$, we have $w(x+\Delta, \Delta) < \Delta\overline{W} \leq D/\beta$. By the definition of x , $0 \in p(x, x)$. It follows that

$$\beta w(x, \Delta) \geq u[\beta w(x + \Delta, \Delta)] > u'(D)\beta w(x + \Delta, \Delta). \tag{2.15}$$

Also, because $0 \in p(x, x)$ and $\Delta \in \Gamma(x + \Delta, x)$, it follows that

$$f(x + \Delta, x) - f(x, x) \geq u[\beta w(x + \Delta, \Delta)] > u'(D)\beta w(x + \Delta, \Delta), \tag{2.16}$$

where the last inequality follows from comparing the second and last terms in (2.15). Now, either $\pi[0, x) \geq 1/2$ or $\pi(x) \geq 1/2$. If the latter, then

$$\begin{aligned} w(x + \Delta, \Delta) &= R\textstyle\sum_m \pi(m)[f(x + \Delta, m) - f(x, m)] \\ &> R\pi(x)u'(D)\beta w(x + \Delta, \Delta) > w(x + \Delta, \Delta), \end{aligned} \tag{2.17}$$

a contradiction. (Here, the first inequality follows from (2.16) and $f(x + \Delta, m) > f(x, m)$ for all m. For the equality, see footnote 3.) So $\pi[0, x) \geq 1/2$. Fix $m < x$. By Lemma 2.6 (v), $\min p(x, m) > 0$. Because $p(x, m) \subset \Gamma(x + \Delta, m)$, it follows that $f(x + \Delta, m) - f(x, m) \geq \beta w(x, \Delta)$. Then by the logic used in (2.17), we have $w(x + \Delta, \Delta) > R\pi[0, x)\beta w(x, \Delta) > (R/2)u'(D)\beta w(x + \Delta, \Delta) > w(x + \Delta, \Delta)$, a contradiction. (Here, the second inequality follows from (2.15).) ■

The next lemma shows that *supp* π is periodic.

LEMMA 2.8. *supp* $\pi = \{0, b\Delta, 2b\Delta, ...\}$, *where* b *is an integer.*

Proof. Let $a = \min\{x > 0 : \pi(x) > 0\}$. The following proof is written as if $a > \Delta$. It also applies if $a = \Delta$, which is a simple special case. In this proof, we let $i, j \in \mathbb{Z}_+$. Let $n = \max\{i : \min p(a, ia) \geq \Delta\}$.

Claim 1 : $n \geq 1$. Assume by contradiction that $n = 0$. By Lemma 2.6 (*iv*), this implies $0 \in p(a, m)$ for $m > a$. Hence, letting $\rho = \pi(0)$, we have

$$w(a) = R\rho u[\beta w(a)] + R(1-\rho)\beta w(a). \tag{2.18}$$

We also have $w(2a) \geq R\rho\{u[\beta w(a)] + \beta w(a)\} + R(1-\rho)\beta w(2a)$. Comparing this with (2.18), we have

$$w(2a, a) \geq R\rho\beta w(a) + R(1-\rho)\beta w(2a, a). \tag{2.19}$$

Now let $c = [1 - R(1-\rho)\beta]/(R\rho)$. (Note that $c > 1$.) By (2.18) and (2.19), we have $cw(a) = u[\beta w(a)]$ and $cw(2a, a) \geq \beta w(a)$. Let g satisfy $cg = \beta w(a)$. (Note that $w(2a, a) \geq g$.) By $g < w(a)$ and $cw(a) = u[\beta w(a)]$, we have $cg < u(\beta g)$. Then $u[\beta w(2a, a)] \geq u(\beta g) > cg = \beta w(a)$. But $n = 0$ implies $\beta w(a) \geq u[\beta w(2a, a)]$, a contradiction.

Claim 2 : $a \in p(a, ja)$ *for* $j = 1, ..., n$ *and* $\pi(ja) > 0$ *for* $j = 1, ..., n+1$. We proceed by induction: for $j = 1, ..., n$, $\pi(ja) > 0$ implies $a \in p(a, ja)$ and $\pi(ja + a) > 0$. By the definition of a, we only need to establish the induction step. By Lemma 2.6 (*iv*) and the definition of n, $\min p(a, ja) \geq \Delta$. If $p \in p(a, ja)$ with $p \in (0, a)$ occurs with positive probability, then $\pi(a)\pi(ja) > 0$ implies $\pi(a - p) > 0$, which contradicts the definition of a. It follows that $a \in p(a, ja)$ occurs with probability 1, and, hence, that $\pi(ja + a) > 0$.

Claim 3 : $\pi(x) = 0$ *for* $x \neq ia$ *if* $x \leq na + a$. Suppose otherwise. We first establish the following induction argument: for $j = 2, ..., n$, $x \in (ja - a, ja)$ with $\pi(x) > 0$ implies $\pi(x + a) > 0$. To see this, assume that x and j satisfy the conditions. By Claim 2, $a \in p(a, ja - a)$. By Lemma 2.6 (ii), this implies $0 \notin p(a, x)$. It follows that $a \in p(a, x)$ occurs with probability 1, and, hence, that $\pi(x+a) > 0$. By the contradicting assumption and the induction argument, $\exists\ x \in (na, na + a)$ with $\pi(x) > 0$. Because $a \in p(a, na)$, by Lemma 2.6 (ii), min $p(a, x) \geq na + a - x$. Because $0 \in p(a, na + a)$, by Lemma 2.6 (ii), $\max p(a, x) \leq na + a - x$. Hence $p(a, x) = \{na + a - x\}$. But then $\pi(x - na) > 0$, a contradiction.

Claim 4 : $\pi(x) = 0$ *for* $x \neq ia$ *if* $x > na + a$. We proceed by induction: for $j \geq 1$, $\pi(x) = 0$ for $x \neq ia$ if $x \leq na+ja$ implies $\pi(x) = 0$ for $x \neq ia$ if $x \leq na+ja+a$. By Claim 3, the hypothesis holds for $j = 1$. So it suffices to establish the induction step. Assume by contradiction that $\pi(x) = 0$ for some $x \in (na + ja, na + ja + a)$. By Lemma 2.6 (v), $\min p(x, 0) > 0$. By Lemma 2.6 (iii), $\max p(x, 0) \leq na+ja+\min p(a, na+ja) = na+ja$. Because x is not a multiple of a, any feasible value of $p(x, 0)$ makes $\pi(y) > 0$ for some $y \leq na + ja$ where $y \neq ia$, a contradiction.

Claim 5 : $\pi(na + ja) > 0$ *for* $j > 1$. We proceed by induction: for $j \geq 1$, $\pi(na + ja) > 0$ implies $\pi(na + ja + a) > 0$. By Claim 2, the hypothesis holds for $j = 1$. So it suffices to establish the induction step. Let $k = \min\{i : \min p(ia, na + ja) \geq \Delta\}$.

First assume $k > n + j$. Now assume by contradiction that $\pi(na + ja + a) = 0$ and let $l = \min\{i : \pi(ia) > 0, i \geq n+j+2\}$. Note that $p(la, na+ja-a)$ only contains multiples of a. By Lemma 2.6 (iii), $\max p(la, na+ja-a) \leq (l-n-j)a + \min p(na+ja, na+ja)$ $= (l-n-j)a$, where the equality comes from the definition of k and $k > n+j$. By Lemma 2.6 (v), $\min p(la, na+ja-a) > 0$. But then any feasible value of $p(la, na+ja-a)$ makes $\pi(ia) > 0$ for some $n + j < i < l$, a contradiction. So $\pi(na + ja + a) > 0$. Next assume $k \leq n+j$. Note that $p(ka, na+ja)$ only contains positive multiples of a. By the definition of k, $0 \in p(ka - a, na + ja)$. By Lemma 2.6 (iv), this implies $p(ka, na+ja) = \{a\}$. Then by the induction assumption, $\pi(na + ja + a) > 0$.[6] ■

Now I can prove that π has full support. The proof is by contradiction. If b (see Lemma 2.8) exceeds unity, then I can construct a mapping that is concave and strictly increasing and has more than one positive fixed point. However, this mapping can have at most one positive fixed point.[7]

LEMMA 2.9. *supp* $\pi = B_\Delta$.

Proof. By Lemma 2.8, it suffices to prove that $b = 1$. So assume by contradiction that $b \geq 2$. In this proof, we let $i \in \mathbb{N}$ and $j \in \mathbb{Z}_+$.

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[6] For finite B, we first prove Claims 1, 2, and 3. It is clear that B must be at least $na + a$. If $B = na + a$, then the proof is complete. Otherwise we continue to Claims 4 and 5. It is clear that B must be equal to $na + ja + a$ for some $j > 0$.

[7] The proof that the mapping has at most one positive fixed point resembles the proof of Corollary 7.45 of Zeidler (1985, page 309).

First, we introduce some notation. Let $\pi(jb\Delta)$ be denoted by π_j and $w(jb\Delta)$ by w_j. Also, let

$$k_i = w_i - w_{i-1} \text{ and } h_i = w(ib\Delta - \Delta) - w_{i-1}.$$

(Note that if $b = 1$, then $h_i = 0$.) Let $k = (k_1, k_2, ...)$ and $h = (h_1, h_2, ...)$. Let $f(ib\Delta, jb\Delta)$ be denoted by $f_{i,j}$. Now consider $p(ib\Delta, jb\Delta)$. By Lemma 2.6 (i), if $p_1, p_2 \in p(x, m)$, then $|p_2 - p_1| \in \{0, \Delta\}$. Because $b \geq 2$, this implies that there is at most one element of $p(ib\Delta, jb\Delta)$ that is equal to $nb\Delta$ for some $n \in \mathbb{Z}_+$. By Lemma 2.8, any element of $p(ib\Delta, jb\Delta)$ that occurs with positive probability is equal to $nb\Delta$ for some $n \in \mathbb{Z}_+$. Hence, there exists a unique element of $p(ib\Delta, jb\Delta)$ that is equal to $nb\Delta$ for some $n \in \mathbb{Z}_+$ and occurs with probability 1. Let this element be denoted by $\overline{p}(i,j)b\Delta$. Finally, let

$$A_{i0} = \{j : \overline{p}(i,j) = \overline{p}(i-1,j)\} \text{ and } A_{i1} = \{j : \overline{p}(i,j) = \overline{p}(i-1,j) + 1\}.$$

By Lemma 2.6 (i), $A_{i0} \cup A_{i1} = \mathbb{Z}_+$. (Also note that $A_{i0} \cap A_{i1}$ is empty.)

Next, for each pair of (i,j), we define mappings $\phi_{i,j}$ and $\sigma_{i,j}$ according to whether $j \in A_{i0}$ or $j \in A_{i1}$. If $j \in A_{i0}$, then let the mapping $\phi_{i,j} : \mathbb{R}_+^\infty \to \mathbb{R}_+$ be defined by

$$\phi_{i,j}(x) = \beta(x_{i-\overline{p}(i,j)} + w_{i-\overline{p}(i,j)-1}) + u[\beta(w_{j+\overline{p}(i,j)} - w_j)]. \tag{2.20}$$

Note that

$$\phi_{i,j}(k) = f_{i,j}. \tag{2.21}$$

By Lemma 2.6 (i), $j \in A_{i0}$ implies $\overline{p}(i,j)b\Delta \in p(ib\Delta - \Delta, jb\Delta)$. Hence,

$$\phi_{i,j}(h) = f(ib\Delta - \Delta, jb\Delta). \tag{2.22}$$

If $j \in A_{i1}$, then let the mapping $\sigma_{i,j} : \mathbb{R}_+^\infty \rightarrow \mathbb{R}_+$ be defined by

$$\sigma_{i,j}(x) = \beta w_{i-\overline{p}(i,j)} + u[\beta(x_{j+\overline{p}(i,j)} + w_{j+\overline{p}(i,j)-1} - w_j)].$$

Note that

$$\sigma_{i,j}(k) = f_{i,j}. \tag{2.23}$$

By Lemma 2.6 (i), $j \in A_{i1}$ implies $\overline{p}(i,j)b\Delta - \Delta \in p(ib\Delta - \Delta, jb\Delta)$. Hence,

$$\sigma_{i,j}(h) = f(ib\Delta - \Delta, jb\Delta). \tag{2.24}$$

Next, for each i, let the mapping $\theta_i : \mathbb{R}_+^\infty \rightarrow \mathbb{R}_+$ be defined by

$$\begin{aligned} \theta_i(x) \quad = \quad & \frac{N-1}{N}\beta(x_i + w_{i-1}) + \frac{1}{N}[\textstyle\sum_{j\in A_{i0}} \pi_j \phi_{i,j}(x) + \sum_{j \in A_{i1}} \pi_j \sigma_{i,j}(x)] \\ & -w_{i-1}. \end{aligned} \tag{2.25}$$

Let $\theta = (\theta_1, \theta_2, ...)$. By (2.21) and (2.23), we have

$$\theta_i(k) = w_i - w_{i-1} = k_i, \tag{2.26}$$

By (2.22) and (2.24), we have

$$\theta_i(h) = w(ib\Delta - \Delta) - w_{i-1} = h_i. \tag{2.27}$$

(Hence the mapping θ has multiple positive fixed points.) By substituting (2.26) and (2.27) into (2.25), we have

$$k_i = R[\textstyle\sum_{j\in A_{i0}} \pi_j \phi_{i,j}(k) + \sum_{j\in A_{i1}} \pi_j \sigma_{i,j}(k)] - w_{i-1}, \tag{2.28}$$

$$h_i = R[\textstyle\sum_{j\in A_{i0}} \pi_j \phi_{i,j}(h) + \sum_{j\in A_{i1}} \pi_j \sigma_{i,j}(h)] - w_{i-1}. \tag{2.29}$$

Next we make some claims.

Claim 1 : *There exists* $s \in (0,1)$ *such that* $h \geq sk$ *with* $h_i = sk_i$ *for some* i. By Cone Lemma 8.31 (i) of Zeidler (1985, page 292), there exists $s > 0$ such that $h \geq sk$ with $h_i = sk_i$ for some i. By monotonicity of w, $s < 1$.

Claim 2 : $\theta_i(h) \geq \theta_i(sk)$. Because $h \geq sk$, this is obvious.

Claim 3 : $\phi_{i,j}(sk) = s\phi_{i,j}(k)$ *and* $\sigma_{i,j}(sk) > s\sigma_{i,j}(k)$. The equality is obvious. Now let $c \geq 0$. Because $s \in (0,1)$, by strict concavity of u, $u(\beta sk + c) > su(\beta k + c) + (1-s)u(c) \geq su(\beta k + c)$. Then the inequality follows.

Claim 4 : $\theta_i(sk) \geq s\theta_i(k)$, *and, strictly if* A_{i1} *is nonempty.* This follows from Claim 3.

Claim 5 : *If* $h_i = sk_i$, *then* A_{i1} *is empty.* Assume that A_{i1} is nonempty. But then $h_i = \theta_i(h) \geq \theta_i(sk) > s\theta_i(k) = sk_i$, a contradiction. (Here, the first equality follows from (2.27) and the second from (2.26). The first inequality follows from Claim 2 and the second from Claim 4 and the contradicting assumption.)

Claim 6 : $h_1 > sk_1$. Because (w, π) is a steady state, A_{11} is nonempty. Then the result follows from Claim 5.

Now let $n = \min\{i : h_i = sk_i\}$. By Claim 6, $n > 1$. By Claim 5, A_{n1} is empty. Let $Q = R \sum_j \pi_j \{\beta w_{n-\overline{p}(n,j)-1} + u[\beta(w_{j+\overline{p}(n,j)} - w_j)]\}$. Then by (2.29) and (2.28), we have

$$\begin{aligned} h_n - sk_n &= R \sum_j \pi_j [\phi_{n,j}(h) - s\phi_{n,j}(k)] - (1-s)w_{n-1} \\ &= R \sum_j \pi_j \beta(h_{n-\overline{p}(n,j)} - sk_{n-\overline{p}(n,j)}) + (1-s)(Q - w_{n-1}). \end{aligned}$$

Because $j \in A_{n0}$ for all j, we have

$$Q = R \sum_j \pi_j \{\beta w_{n-\overline{p}(n-1,j)-1} + u[\beta(w_{j+\overline{p}(n-1,j)} - w_j)]\} = w_{n-1}.$$

Hence, we have

$$h_n - sk_n = R \sum_j \pi_j \beta(h_{n-\overline{p}(n,j)} - sk_{n-\overline{p}(n,j)}) \geq R\pi_0 \beta(h_{n-\overline{p}(n,0)} - sk_{n-\overline{p}(n,0)}),$$

where the inequality follows from $h \geq sk$. By Lemma 2.6 (v), $\overline{p}(n,0) > 0$. By $0 \in A_{n0}$, $\overline{p}(n,0) < n$. But then $h_i > sk_i$ for $1 \leq i < n$ implies $h_n > sk_n$, a contradiction. ■

Full support allows us to establish strict concavity of the value function.

LEMMA 2.10. *w is strictly concave.*

Proof. See the Appendix. ■

Therefore, I have proved the following proposition.

PROPOSITION 2.1. *Under assumptions $(A1)-(A3)$, which allow for unbounded individual holdings of money, there exists a steady state (w, π) where w is increasing and strictly concave and π has full support.*

One implication of this proposition is related to non-neutrality. To state the result, I first define a notion of equivalence between steady states.

DEFINITION 2. *Let (w, π) and (w', π') be steady states. We say that (w', π') is equivalent to (w, π) if there exists a bijection γ from supp π to supp π' such that if $x \in$ supp π, then $w(x) = w'(\gamma(x))$ and $\pi(x) = \pi'(\gamma(x))$. Let $e \equiv (\overline{m}, \Delta, B)$, the vector of exogenous nominal objects, and let $S(e)$ denote the set of **all** steady states associated with e. We say that $S(e) \subset S(e')$ if $(w, \pi) \in S(e)$ implies that there exists $(w', \pi') \in S(e')$ with (w', π') equivalent to (w, π). We say that $S(e)$ and $S(e')$ are equivalent if $S(e) \subset S(e')$ and $S(e') \subset S(e)$.*

It follows from this definition that equivalence between steady states is symmetric and transitive. Using this definition, I can state the obvious neutrality result. If two economies differ only in their vectors of exogenous nominal objects, e and e', and $e = \theta e'$

for some $\theta \in \mathbb{R}_+$ (as a convention, $\infty = \theta\infty$), then $S(e)$ and $S(e')$ are equivalent.[8] However, equivalence does not hold if e and e' differ but not proportional.

Corollary 2.1. *If $e \neq \theta e'$ for any $\theta \in \mathbb{R}_+$, then $S(e)$ and $S(e')$ are not equivalent. Moreover, if $e = (k\overline{m}, \Delta, kB)$ and $e' = (\overline{m}, \Delta, B)$ where $k \geq 2$ and is an integer, then $S(e') \subset S(e)$.*

Proof. We begin with the first assertion. Without loss of generality, let $e = (\overline{m}, \Delta, B)$ and $e' = (\overline{m}', \Delta, B')$. By the hypothesis, either $\overline{m} \neq \overline{m}'$ or $B \neq B'$. First consider $\overline{m} \neq \overline{m}'$ and assume by contradiction that $S(e)$ and $S(e')$ are equivalent. Without loss of generality, assume that $\overline{m} > \overline{m}'$. Let $(w, \pi) \in S(e)$ be a Proposition 1 steady state and let $(w', \pi') \in S(e')$ be equivalent to (w, π). Let *supp* $\pi' = \{a_0, a_1, ...\}$ with $a_i < a_{i+1}$ for all i. (Note that $a_i \geq i\Delta$.) Because w is strictly increasing and w' is nondecreasing, the bijection γ in Definition 2 from *supp* π to *supp* π' is strictly increasing. That is, $\gamma(i\Delta) < \gamma(j\Delta)$ if $i < j$. Because *supp* $\pi = B_\Delta$, it follows that $\gamma(i\Delta) = a_i$. Hence $\pi(i\Delta) = \pi'(a_i)$. But because $i\Delta \leq a_i$, this implies $\overline{m} \leq \overline{m}'$, a contradiction. Next consider $B \neq B'$. Without loss of generality, assume that B' is finite and $B' < B$. Let $(w, \pi) \in S(e)$ be a Proposition 1 steady state. But because *supp* $\pi = B_\Delta$ and because B'_Δ is a strict subset of B_Δ, no $(w', \pi') \in S(e')$ is equivalent to (w, π).

For the second assertion, let $(w', \pi') \in S(e')$. The following construction of $(w, \pi) \in S(e)$, which is similar to that used to prove neutrality, is well known. For $n \geq 0$

[8] Let $e' = (\overline{m}, \Delta, B)$ and $e = \theta e'$. Let $(w', \pi') \in S(e')$. Under the vector e, let (w, π) be defined as follows. For $n \geq 0$, let $w(n\theta\Delta) = w'(n\Delta)$ and $\pi(n\theta\Delta) = \pi'(n\Delta)$. Notice from (2.3) that $p \in p(x, m, w')$ implies $\theta p \in p(\theta x, \theta m, w)$. Hence, $(w, \pi) \in S(e)$. It is clear that (w, π) is equivalent to (w', π').

and $0 \leq j \leq k-1$, let $w(nk\Delta+j\Delta) = w'(n\Delta)$, $\pi(nk\Delta) = \pi'(n\Delta)$, and $\pi(nk\Delta+j\Delta) = 0$. It is clear that (w, π) is equivalent to (w', π'). ■

A surmise is that if $\overline{m} > \overline{m}'$, then some $s \in S(\overline{m}, \Delta, \infty)$ has more trade and higher average welfare than any $s' \in S(\overline{m}', \Delta, \infty)$, but that remains to be established.

Appendix

Proof of Lemma 2.6

Proof. (i)See Taber and Wallace (1999, page 967).

(ii)It suffices to prove that $x+\max p(m,x) \leq x+\Delta+\min p(m,x+\Delta)$. Assume by contradiction that $p \in p(m,x)$, $p' \in p(m,x+\Delta)$, and $x+p'+\Delta < x+p$. Then $p'+\Delta < p$. Let $a_1 = \beta w(x+p'+\Delta, p')$, $a_2 = \beta w(x+p'+2\Delta, p'+\Delta)$, $b_1 = \beta w(x+p-\Delta, p-\Delta)$, and $b_2 = \beta w(x+p, p)$.

Because $a_2 - a_1 = \beta w(x+p'+2\Delta, \Delta) > 0$ and $b_2 - b_1 = \beta w(x+p, \Delta) > 0$, by the definitions of p and p', we have

$$\frac{u(a_2)-u(a_1)}{a_2-a_1} w(x+p'+2\Delta, \Delta) \quad \leq \quad w(m-p', \Delta), \tag{2.30}$$

$$\frac{u(b_2)-u(b_1)}{b_2-b_1} w(x+p, \Delta) \quad \geq \quad w(m-p+\Delta, \Delta). \tag{2.31}$$

By the definitions of a_i and b_i, we have

$$\begin{aligned} b_1 - a_1 &= \beta[w(x + p - \Delta) - w(x + p' + \Delta) + w(x + \Delta, \Delta)] \\ b_2 - a_2 &= \beta[w(x + p) - w(x + p' + 2\Delta) + w(x + \Delta, \Delta)]. \end{aligned}$$

But, $p' + \Delta < p$ implies $p' + \Delta \leq p - \Delta$ and $p' + 2\Delta \leq p$. Hence, $b_1 > a_1$ and $b_2 > a_2$. Then strict concavity of u implies $\frac{u(a_2)-u(a_1)}{a_2-a_1} > \frac{u(b_2)-u(b_1)}{b_2-b_1}$. This inequality, $p' + 2\Delta \leq p$, and concavity of w imply that the left side of (2.30) exceeds the left side of (2.31). Then by (2.30) and (2.31), $w(m-p', \Delta) > w(m-p+\Delta, \Delta)$. But this contradicts $p' < p - \Delta$ and concavity of w.

*(iii)*Let $a = \max\{x_2 - x_1, m_1 - m_2\}$ and let $p_1 = \min p(x_1, m_1)$. Let $p_2 = a + p_1$. We assume that $x_2 > p_2$ and $m_2 + p_2 < B$; otherwise the result follows immediately. By $m_2 < m_1$ and $p_2 > p_1$, we have $w(m_2+p_2, p_2) > w(m_1+p_1, p_1)$. By $m_2+p_2 \geq m_1+p_1$, we have $w(m_2 + p_2 + \Delta, \Delta) \leq w(m_1 + p_1 + \Delta, \Delta)$. Then we have

$$\begin{aligned} \beta w(x_2 - p_2, \Delta) &\geq \beta w(x_1 - p_1, \Delta) \\ &\geq u[\beta w(m_1 + p_1 + \Delta, p_1 + \Delta)] - u[\beta w(m_1 + p_1, p_1)] \\ &> u[\beta w(m_2 + p_2 + \Delta, p_2 + \Delta)] - u[\beta w(m_2 + p_2, p_2)], \end{aligned} \tag{2.32}$$

where the second inequality follows from $p_1 \in p(x_1, m_1)$ and the third from strict concavity of u. Note that $u[\beta w(m+p, p)] + \beta w(x-p)$, viewed as a function of p, is concave, and, hence, strictly increasing on $[0, \min p(x, m)]$ and strictly decreasing on $[\max p(x, m), \min\{x, B - m\}]$. Then by (2.32), $\max p(x_2, m_2) \leq p_2$.

(iv)First consider $x_2 > x_1$. Let $p = x_2 - x_1$. We assume that $x_1 > 0$ and $m_2 + p < B$; otherwise the result follows immediately. We have $\beta w(x_2 - p, \Delta) = \beta w(x_1, \Delta) \geq u[\beta w(m_1 + \Delta, \Delta)] - u(0) > u[\beta w(m_2 + p + \Delta, p + \Delta)] - u[\beta w(m_2 + p, p)]$, where the first inequality follows from $0 \in p(x_1, m_1)$ and $u(0) = 0$ and the second from strict concavity of u. By the logic used in the proof of part (iii), $\max p(x_2, m_2) \leq p$. Next consider $x_2 = x_1$. We assume that $x_1 > 0$ and $m_2 < B$; otherwise the result follows immediately. We have $\beta w(x_2, \Delta) = \beta w(x_1, \Delta) \geq u[\beta w(m_1 + \Delta, \Delta)] \geq u[\beta w(m_2 + \Delta, \Delta)]$, where the first inequality follows from $0 \in p(x_1, m_1)$. By the logic used in the proof of part (iii), $\min p(x_2, m_2) = 0$.

(v)We have $u[\beta w(\Delta)] > \beta w(\Delta)$; otherwise, by concavity of w, it follows that $u[\beta w(m + \Delta, \Delta)] \leq \beta w(\Delta)$ for all $m \geq \Delta$, and, hence, that $w(\Delta) = 0$. By concavity of u and w, this implies that for $x > m$, $u[\beta w(m + \Delta, \Delta)] > \beta w(m + \Delta, \Delta) \geq \beta w(x, \Delta)$. So $0 \notin p(x, m)$. ■

Proof of Lemma 2.10

Proof. We first prove the following.

Claim : If for each $x > 0$, there exists m such that $p(x, m)$ is a positive singleton, then w is strictly concave.

The proof of the claim is by induction on the set satisfying strict concavity. That is, we show that w is strict concave on $\{0, \Delta, 2\Delta\}$ and then show strict concavity on $\{0, \Delta, ..., x\}$ implies strict concavity on $\{0, \Delta, ..., x, x + \Delta\}$. First, we prove that $2w(\Delta) > w(0) + w(2\Delta) = w(2\Delta)$. Taber and Wallace (1999) show that $2f(x, m) \geq f(x - \Delta, m) + f(x + \Delta, m)$. Because $\pi(0) > 0$ and $f(0, 0) = 0$, it suffices to show that $2f(\Delta, 0) >$

$f(2\Delta, 0)$. By Lemma 2.6 (v), $f(\Delta, 0) = u[\beta w(\Delta)] > \beta w(\Delta)$. There are two possibilities for $p(2\Delta, 0)$. (i)If $\Delta \in p(2\Delta, 0)$, then $f(2\Delta, 0) = u[\beta w(\Delta)] + \beta w(\Delta) < 2u[\beta w(\Delta)] = 2f(\Delta, 0)$. (ii)If $2\Delta \in p(2\Delta, 0)$, then $f(2\Delta, 0) = u[\beta w(2\Delta)] \leq u[2\beta w(\Delta)] < 2u[\beta w(\Delta)] = 2f(\Delta, 0)$, where the first inequality follows from concavity of w and the second from strict concavity of u. Next for the induction step. Let m be such that $p(x, m)$ is a positive singleton. As above, because $\pi(m) > 0$, it suffices to show that $2f(x, m) > f(x-\Delta, m) + f(x+\Delta, m)$. Let $\min p(x-\Delta, m) = p$. By Lemma 2.6 (i), there are three possibilities for $\min p(x + \Delta, m)$. $(i)\min p(x + \Delta, m) = p + \Delta$. Because $p, p + \Delta \in \Gamma(x, m)$ and because $p(x, m)$ is a singleton, it follows that $2f(x, m) > u[\beta w(m + p + \Delta, p + \Delta)] + \beta w(x - p - \Delta) + u[\beta w(m+p, p)] + \beta w(x-p) = f(x - \Delta, m) + f(x+\Delta, m)$. $(ii)\min p(x+\Delta, m) = p$. By Lemma 2.6 (i), $\min p(x + \Delta, m) \geq \max p(x, m) \geq \min p(x - \Delta, m)$. It follows that $p(x, m) = \{p\}$ and $p \geq \Delta$. Therefore, $2f(x, m) - f(x - \Delta, m) - f(x + \Delta, m) = 2\beta w(x - p) - \beta w(x - \Delta - p) - \beta w(x + \Delta - p) > 0$, where the last inequality follows from $p \geq \Delta$ and the induction assumption. $(iii)\min p(x+\Delta, m) = p+2\Delta$. Because $p+\Delta \in \Gamma(x, m)$, it follows that $2f(x, m) - f(x - \Delta, m) - f(x + \Delta, m) \geq 2u[\beta w(m + p + \Delta, p + \Delta)] - u[\beta w(m + p, p)] - u[\beta w(m + p + 2\Delta, p + 2\Delta)] > 0$, where the last inequality follows from strict concavity of u and concavity of w.

Now we can finish the proof of this lemma. By the claim, it suffices to prove that $\forall\ x > 0$, $\exists\ m^*$ such that $p(x, m^*)$ is a positive singleton. Note that $\beta w(m + \Delta, \Delta) < u^{-1}[\beta w(x, \Delta)]$ implies $p(x, m) = \{0\}$. Also note that concavity of w implies $w(m + \Delta, \Delta)/\Delta < \overline{W}/m$. Hence $m > \beta\overline{W}\Delta/u^{-1}[\beta w(x, \Delta)]$ implies $p(x, m) = \{0\}$. By

Lemma 2.6 (v), $0 \notin p(x,0)$. Then $\exists\ y = \max\{m : 0 \notin p(x,m)\}$. By Lemma 2.6 (ii), $p(x,y) = \{\Delta\}$. Then $m^* = y$.[9] ■

[9] For finite B, we can find m^* as follows. If $p(x, B-\Delta) = \{\Delta\}$, then $m^* = B - \Delta$ is as required. Hence, assume $0 \in p(x, B-\Delta)$. Then $\exists\ y = \max\{m : 0 \notin p(x,m)\}$ and $m^* = y$.

Chapter 3

Divisible Money

3.1 Introduction

In Chapter 2, I proved that there exists a monetary steady state in a matching model with a general bound, finite or infinite, on individual holdings of indivisible money. The steady state has nice properties: the value function, defined on money holdings, is increasing and strictly concave, and the distribution function of money holdings has full support. In this chapter, I use that result to establish existence for divisible money.

The transition from indivisible money (a countable support) to divisible money (an uncountable support) is not trivial. My approach is as follows. I embed the nice steady states for indivisible money in the spaces of value functions and measures for divisible money. I then let the unit of indivisible money go to zero and show that the limit is a monetary steady state for divisible money. To carry through this approach, I need continuity of a mapping whose fixed point is a divisible money steady state, and I need compactness of the space on which the mapping is defined. For continuity, the space of value functions is equipped with the sup norm topology. But, then, compactness of the space is challenging. To achieve compactness, I rely on two additional assumptions: a finite marginal utility of consumption at zero and a finite bound on individual money holdings. I show, through a lengthy argument, that these assumptions imply a bound on the slope of the value functions for indivisible-money steady states. That, in turn, implies equicontinuity of the set of the embedded value functions, and, therefore, compactness.

3.2 The Model

The model is that in Chapter 2, except that money is now divisible.

3.2.1 Environment

Time is discrete, dated as $t \geq 0$. There is a $[0, 1]$ continuum of each of $N \geq 3$ types of infinitely lived agents, and there are N distinct produced and perishable types of divisible goods at each date. A type n agent, $n \in \{1, 2, ..., N\}$, produces only type n good and consumes only type $n + 1$ good (modulo N). Each agent maximizes expected discounted utility with discount factor $\beta \in (0, 1)$. For a type n agent, utility in a period is $u(q_{n+1}) - q_n$, where $q_{n+1} \in \mathbb{R}_+$ is consumption of type $n + 1$ good and $q_n \in \mathbb{R}_+$ is production of type n good. The utility function $u : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, strictly concave and continuously differentiable and satisfies $u(0) = 0$ and $u'(\infty) < 1$. In addition, $u'(0)$ is bounded, with bounds that are specified later.

There exists a fixed stock of money which is perfectly durable. Money is symmetrically distributed among the N specialization types. The average money holdings per specialization type is fixed at unity and each individual's money holding is restricted to be in the set $[0, B]$, where B is sufficiently large.

In each period, agents are randomly matched in pairs. In meetings, agents' types and money holdings are observable, but any other information about an agent's trading history is private. In a meeting, the (potential) consumer makes a take-it-or-leave-it offer to the producer.

3.2.2 Definition of Equilibrium

The definition of equilibrium for divisible money is analogous to that for indivisible money.

Let $v_t(x)$ denote the expected discounted value of holding x amount of money at the start of period t, prior to date t matching, where $v_t : [0, B] \to \mathbb{R}_+$ is nondecreasing. Let μ_t denote the Borel measure of money holdings on $[0, B]$ at the start of period t, so that $\mu_t([0, x])$ is the fraction of agents holding money less than or equal to x prior to date t matching.

Let $\widetilde{\Gamma}(x, m)$ denote $\{p \in [0, B] : p \leq \min\{x, B - m\}\}$. It is convenient to denote $g(a_2) - g(a_2 - a_1)$ by $g(a_2, a_1)$ for any function $g : \mathbb{R} \to \mathbb{R}$. Now let

$$\widetilde{f}(x, m, v_t) = \max_{p \in \widetilde{\Gamma}(x,m)} u[\beta v_t(m + p, p)] + \beta v_t(x - p), \tag{3.1}$$

$$\widetilde{p}(x, m, v_t) = \arg \max_{\widetilde{\Gamma}(x,m)} u[\beta v_t(m + p, p)] + \beta v_t(x - p). \tag{3.2}$$

It is important to remember that the first argument of $\widetilde{f}$ and of $\widetilde{p}$ is pretrade money holding of a consumer and the second is that of a producer and that $\widetilde{f}$ is the payoff for a consumer with x who meets a producer with m and $\widetilde{p}$ is the optimal offer of money in the meeting. The value function, $v_t(x)$, satisfies

$$v_t(x) = \frac{N-1}{N}\beta v_{t+1}(x) + \frac{1}{N}\int \widetilde{f}(x, m, v_{t+1})d\mu_t(m). \tag{3.3}$$

To express the law of motion, I need some additional notation. Let

$$\widetilde{g}(x, m, v_t) = x - \widetilde{p}(x, m, v_t), \tag{3.4}$$

$$\widetilde{h}(m, x, v_t) = x + \widetilde{p}(m, x, v_t), \tag{3.5}$$

and let

$$\widetilde{g}^{-1}(x, v_t) = \{(y, m) \in [0, B]^2, \widetilde{g}(y, m, v_t) \le x\}, \tag{3.6}$$

$$\widetilde{h}^{-1}(x, v_t) = \{(m, y) \in [0, B]^2, \widetilde{h}(m, y, v_t) \le x\}. \tag{3.7}$$

Let μ_t^2 denote the product Borel measure $\mu_t \times \mu_t$ on $[0, B]^2$. Then the law of motion for μ_{t+1} can be expressed as

$$\mu_{t+1}([0, x]) = \frac{N-2}{N}\mu_t([0, x]) + \frac{1}{N}\mu_t^2(\widetilde{g}^{-1}(x, v_{t+1})) + \frac{1}{N}\mu_t^2(\widetilde{h}^{-1}(x, v_{t+1})). \tag{3.8}$$

Definition 3. *Given μ_0, a sequence $\{v_t, \mu_{t+1}\}_{t=0}^{\infty}$ is an equilibrium if it satisfies* (3.1)–(3.8). *A monetary equilibrium is an equilibrium with positive consumption and production. A pair (v, μ) is a steady state if $\{v_t, \mu_{t+1}\}_{t=0}^{\infty}$ with $v_t = v$ and $\mu_{t+1} = \mu$ is an equilibrium for $\mu_0 = \mu$.*

3.3 Steady States for Indivisible Money

Because my approach is to approximate a steady state for divisible money using steady states for indivisible money, I need some notation and results for indivisible money.

Let the unit of indivisible money be denoted by Δ, the set $\{0, \Delta, 2\Delta, ..., B\}$ by B_Δ, and the set $\{p \in B_\Delta, p \leq \min\{x, B-m\}\}$ by $\Gamma_\Delta(x, m)$. Let (w_Δ, π_Δ) be a steady state for indivisible money with Δ as the unit of money, where w_Δ is the value function of money holdings and π_Δ is the measure of money holdings. Let

$$f(x, m, w_\Delta) = \max_{p \in \Gamma_\Delta(x,m)} u[\beta w_\Delta(m+p, p)] + \beta w_\Delta(x-p), \tag{3.9}$$

$$p(x, m, w_\Delta) = \arg \max_{p \in \Gamma_\Delta(x,m)} u[\beta w_\Delta(m+p, p)] + \beta w_\Delta(x-p). \tag{3.10}$$

Note that f is the indivisible money counterpart of $\widetilde{f}$ in (3.1) and p is that of $\widetilde{p}$ in (3.2).

The next two lemmas were proved as Proposition 2.1 and Lemma 2.6 respectively. The first lemma gives existence of monetary steady states for indivisible money and the second describes the dependence of the optimal offer in (3.10) on the money holdings of the consumer and the producer.

LEMMA 3.1. *Let* $R \equiv [(N-(N-1)\beta]^{-1}$. *If* $B \geq 8$ *and* $u'(0) > [4/(R\beta)]^2$, *then for* $\Delta > 0$, *there exists a monetary steady state* (w_Δ, π_Δ) *with* w_Δ *increasing and strictly concave and with* $w_\Delta(0) = 0$ *and* $D/\beta \leq w_\Delta(B) < \overline{W}$, *where* D *is the unique solution of* $u'(D) = [4/(R\beta)]^2$ *and* $\overline{W}$ *is the unique positive solution of* $N(1-\beta)\overline{W} = u(\beta\overline{W})$.

LEMMA 3.2. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state.*

(i) If $x > y$, *then* $0 \notin p(x, y, w_\Delta)$.

(ii) If $p_1 \in p(x,m,w_\Delta)$ *and* $p_2 \in p(x+\Delta,m,w_\Delta)$*, then* $p_2 - p_1 \in \{0,\Delta\}$.

(iii)If $m_1 < m_2$, $p_1 \in p(x,m_1,w_\Delta)$*, and* $p_2 \in p(x,m_2,w_\Delta)$*, then* $m_1 + p_1 \leq m_2 + p_2$.

(iv)If $0 \in p(x',m',w_\Delta)$, $x > x'$*, and* $m \geq m' - (x - x')$*, then* $\max p(x,m,w_\Delta) \leq x - x'$.

The results in the next lemma are essentially steps in the proof of Lemma 2.3. From now on, for an interval I, let $\pi_\Delta I$ denote the measure of $I \cap B_\Delta$ implied by π_Δ.

LEMMA 3.3. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state.*

(i)If $\varkappa_0, \varkappa_1, \varkappa_2 \in B_\Delta$ *is such that* $\varkappa_0 < \varkappa_1 \leq \varkappa_2$ *and* $\varkappa_0 + \varkappa_2 \leq 2\varkappa_1$*, then* $w_\Delta(\varkappa_2, \Delta) > R\pi_\Delta[0,\varkappa_0]\beta u'(\overline{W})\ w_\Delta(\varkappa_1, \Delta)$.

(ii) If $\varkappa_1, \varkappa_2 \in B_\Delta$ *is such that* $\varkappa_1 \leq \varkappa_2 \leq B/2$ *and* $\pi_\Delta[\varkappa_1, \varkappa_2] \geq 1/4$*, then* $w_\Delta(2\varkappa_2) - w_\Delta(\varkappa_1) \geq D/\beta$.

Proof. See the Appendix. ■

The main goal of the next section is to establish boundedness of $w_\Delta(\Delta)/\Delta$ as $\Delta \to 0$. To this end, boundedness of $u'(0)$ is assumed. The following assumptions are maintained from now on.

(A)$\infty > B \geq 8$ and $\infty > u'(0) > [4/(R\beta)]^2$, where $R \equiv [(N-(N-1)\beta]^{-1}$.

3.4 Boundedness of the Slope of the Value Function for Indivisible Money

The proof of boundedness is built on the following idea. If $u'(0)$ is finite and $w_\Delta(\Delta)/\Delta$ is sufficiently large, then almost all agents must have money holdings near zero, which is impossible. The argument leading to this contradiction proceeds as follows. Let x_1 be an amount of money at which the marginal value of money is sufficiently large. It follows that x_1 is small. Because $u'(0)$ is finite, consumers with holdings no greater than x_1 only trade with producers with holdings no greater than some x_2, where x_2 is also such that the marginal value of money is sufficiently large. It follows that x_2 is small. However, producers with holdings no greater than x_1 trade with all consumers with holdings greater than x_1. Because the outflow from and inflow into $\{0, \Delta, ..., x_1\}$ are equal, the measure of the set $\{x_1, x_1 + \Delta, ..., x_2\}$ or a close approximation to it is bounded from below. By induction, I can show that this implies that almost all agents have money holdings near zero, the contradiction. The proof relies on several lemmas. To state these lemmas, I need some notation. The notation is grouped in a definition to facilitate subsequent reference to it.

Definition 4. *(i)Let (w_Δ, π_Δ) be a Lemma 3.1 steady state. For $x_0 \in B_\Delta \backslash \{0\}$, let $\{x_n\}_{n \geq 1}$ be defined by*

$$x_n = \max\{x : p(x_{n-1}, x, w_\Delta) \neq \{0\}\} + \Delta. \tag{3.11}$$

Also, let $\{z_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ be defined by

$$z_n = \min\{x : \min p(x, 0, w_\Delta) \geq x_n\} \text{ and } y_n = \max\{x_n + x_{n+1}, z_n\}. \tag{3.12}$$

(ii)Let $\rho \equiv \frac{N(1-\beta)}{[u'(0)-1]\beta}$, $L \equiv \min\{n \in \mathbb{N} : 2^n \geq 2B - 1\}$, *and* $K \equiv \sum_{j=1}^{L} 2^{j-1}$. *For* $(s_1, s_2) \in [\rho, 1] \times [\frac{\rho^2}{K}, 1]$, *consider the quadratic equation* $x^2 + s_1 x - s_2 = 0$. *Because* $(s_1, s_2) > (0, 0)$, *this equation has a unique positive root. Denote the positive root by* $g(s_1, s_2)$ *and let* $g^* \equiv \min g(s_1, s_2)$. *(Note that* $g^* > 0$.*) Also, let int* $: \mathbb{R} \to \mathbb{Z}$ *be defined by* $int(x) = \min\{n \in \mathbb{Z} : n \geq x\}$ *and let* $\sigma : \mathbb{N} \to \mathbb{N}$ *be defined by* $\sigma(n) = int(2^{-n}/g^*)$. *Finally, let* $J \equiv \sigma(1)$.

Note that x_n in (3.11) is the money holding of the poorest producers with whom consumers with x_{n-1} do not trade, while z_n in (3.12) is that of the poorest consumers who offer at least x_n to producers with 0. Existence of x_n is given in Lemma 3.4, while a sufficient condition for existence of z_n and y_n is given in Lemma 3.7.

LEMMA 3.4. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state. Let* $x_0 \in B_\Delta \backslash \{0\}$ *and let* $\{(x_n, y_n, z_n)\}$ *and* ρ *satisfy Definition* 4.

(i)$\{x_n\}$ *exists and is nondecreasing.*

(ii)If $\{z_n\}$ *exists, then it is nondecreasing.*

(iii)If $\{y_n\}$ *exists, then it is nondecreasing.*

(iv)$\pi_\Delta[0, x_n) > \rho$ *for* $n \geq 1$.

Proof. In this and subsequent proofs, the subscript Δ is deleted from (w_Δ, π_Δ) when it is not needed. Part (i) follows from Lemma 3.2 (i). Part (ii) follows from part (i) and Lemma 3.2 (ii). Part (iii) follows from parts (i) and (ii). Now consider part (iv). Let $a = \max\{x : p(\Delta, x) \neq \{0\}\} + \Delta$. So a is the implied x_1 if $x_0 = \Delta$. By part (i),

$x_1 \leq x_n$. By Lemma 3.2 (*ii*), $a \leq x_1$. Hence it suffices to show $\pi[0,a) > \rho$. We have

$$\begin{aligned} w(\Delta) &= R\{\textstyle\sum_{m=0}^{a-\Delta} \pi(m)u[\beta w(m+\Delta,\Delta)] + \sum_{m=a}^{B} \pi(m)\beta w(\Delta)\} \\ &< R\{\textstyle\sum_{m=0}^{a-\Delta} \pi(m)u'(0)\beta w(m+\Delta,\Delta) + \pi[a,B]\beta w(\Delta)\} \\ &< R\pi[0,a)u'(0)\beta w(\Delta) + R\{1-\pi[0,a)\}\beta w(\Delta), \end{aligned}$$

where the equality follows from the definition of x_1, the first inequality from strict concavity of u and $u(0) = 0$, and the second inequality from concavity of w. The desired result follows immediately. ■

The following lemma shows that $\{(x_n, y_n, z_n)\}$ is a "measure exhausting" sequence.

Lemma 3.5. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state. Let* $x_0 \in B_\Delta \backslash \{0\}$ *and let* $\{(x_n, y_n, z_n)\}$ *and* ρ, K, *and* g^* *satisfy Definition* 4. *Assume* y_n *exists.*

(*i*) *If* $\pi_\Delta[y_n, B] - \pi_\Delta[x_{n+1}, y_n) < \frac{\rho}{K}$, *then* $\pi_\Delta[y_n, B] < \frac{1}{2}\{1 - \pi[0, x_n) + \frac{\rho}{K}\}$.

(*ii*) *If* $\pi_\Delta[y_n, B] - \pi_\Delta[x_{n+1}, y_n) \geq \frac{\rho}{K}$, *then* $x_{n+1} > x_n$ *and* $\pi_\Delta[x_n, x_{n+1}) \geq g^*$.

Proof. In this and subsequent proofs, w_Δ is suppressed from the list of arguments of f and p (see (3.9) and (3.10)) when it is not needed.

(*i*) We have

$$1 \geq \pi[0, x_n) + \pi[x_{n+1}, y_n) + \pi[y_n, B] > \pi[0, x_n) + 2\pi[y_n, B] - \frac{\rho}{K},$$

where the last inequality follows from the hypothesis. The inequality between the first and third terms gives the conclusion.

(ii)First consider the outflow from $[0, x_n)$. Note that producers with $x \geq x_n$ do not contribute to the outflow. Consider producers with $x < x_n$ and consumers with $m \geq y_n$. We have

$$x_n \leq \min p(z_n, 0) \leq \min p(y_n, 0) \leq x + \min p(y_n, x) \leq x + \min p(m, x),$$

where the first inequality follows from the definition of z_n, the second from Lemma 3.2 (ii), the third from Lemma 3.2 (iii), and the fourth from Lemma 3.2 (ii). So a lower bound on the outflow is $\pi[0, x_n)\pi[y_n, B]$.

Next consider the inflow into $[0, x_n)$. Note that consumers with $x < x_n$ do not contribute to the inflow and that by the definition of $\{x_n\}$, $p(x_n, x_{n+1}) = \{0\}$. We start with consumers with $x \geq y_n$ and producers with $m \geq 0$ and apply Lemma 3.2 (iv) with $(x', m') = (x_n, x_{n+1})$. It follows from $y_n \geq x_n + x_{n+1}$ that $m \geq 0 \geq m' - (x - x')$, and, hence, that $\max p(x, m) \leq x - x' = x - x_n$ or $x - \max p(x, m) \geq x_n$. Therefore, consumers with $x \geq y_n$ do not contribute to the inflow. Now we consider consumers with $x \geq x_n$ and producers with $m \geq x_{n+1}$ and apply Lemma 3.2 (iv) with $(x', m') = (x_n, x_{n+1})$. It follows that $m \geq x_{n+1} \geq m' - (x - x')$, and, hence, that $\max p(x, m) \leq x - x' = x - x_n$ or $x - \max p(x, m) \geq x_n$. Therefore, consumers with $x \geq x_n$ do not contribute to the inflow if they meet producers with $m \geq x_{n+1}$. So an upper bound on the inflow is $\pi[x_n, y_n)\pi[0, x_{n+1})$.

Because (w, π) is a steady state, the outflow from and inflow into $[0, x_n)$ are equal. Therefore,

$$\pi[0, x_n)\pi[y_n, B] \leq \pi[x_n, y_n)\pi[0, x_{n+1}). \tag{3.13}$$

By the hypothesis, $\pi[y_n, B] > \pi[x_{n+1}, y_n)$. So (3.13) implies $x_{n+1} > x_n$. Now write $\pi[x_n, y_n)$ as $\pi[x_n, x_{n+1}) + \pi[x_{n+1}, y_n)$ and $\pi[0, x_{n+1})$ as $\pi[0, x_n) + \pi[x_n, x_{n+1})$ and insert these into (3.13). Then, letting $x \equiv \pi[x_n, x_{n+1})$, (3.13) is equivalent to $0 \leq x^2 + s_1 x - s_2$, where $s_1 = \pi[0, x_n) + \pi[x_{n+1}, y_n) > \rho$ (by Lemma 3.4 (iv)) and $s_2 = \pi[0, x_n)\{\pi[y_n, B] - \pi[x_{n+1}, y_n)\} > \frac{\rho^2}{K}$ (by Lemma 3.4 (iv) and the hypothesis). Hence, $(s_1, s_2) \in [\rho, 1] \times [\frac{\rho^2}{K}, 1]$. This implies $\pi[x_n, x_{n+1}) \geq g^*$. ■

The next lemma, an application of Lemma 3.5, provides the ingredients for an induction argument.

LEMMA 3.6. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state and let* ρ, J, K, *and* σ *satisfy Definition* 4.

(i)*Let* $x_0 \in B_\Delta \backslash \{0\}$ *and let* $\{(x_n, y_n, z_n)\}$ *satisfy Definition* 4. *If* $y_J = y_{\sigma(1)}$ *exists, then* $\pi_\Delta[y_J, B] < \frac{1}{2}[1 - \frac{K-1}{K}\rho]$.

(ii)*Assume* $\omega_{i-1} \in B_\Delta \backslash \{0\}$ *and* $\pi_\Delta[\omega_{i-1}, B] < \frac{1}{2^{i-1}} - \frac{1}{2^{i-1}} \frac{K - \sum_{j=1}^{i-1} 2^{j-1}}{K}\rho$. *Let* $x_0 = \omega_{i-1}$ *and let* $\{(x_n, y_n, z_n)\}$ *satisfy Definition* 4. *Let* $y_{\sigma(i)}$ *be denoted by* ω_i. *If* ω_i *exists, then* $\pi_\Delta[\omega_i, B] < \frac{1}{2^i} - \frac{1}{2^i} \frac{K - \sum_{j=1}^{i} 2^{j-1}}{K}\rho$.

Proof. (i)If $\pi[x_{n+1}, y_n) - \pi[y_n, B] < \frac{\rho}{K}$ for some $1 \leq n \leq J$, then

$$\pi[y_J, B] \leq \pi[y_n, B] < \frac{1}{2}\{1 - \pi[0, x_n) + \frac{\rho}{K}\} < \frac{1}{2}[1 - \frac{K-1}{K}\rho],$$

where the first inequality follows from Lemma 3.4 (iii), the second from Lemma 3.5 (i), and the last from Lemma 3.4 (iv). If $\pi[x_{n+1}, y_n) - \pi[y_n, B] \geq \frac{\rho}{K}$ for all $1 \leq n \leq J$, then Lemma 3.5 (ii) implies $x_{n+1} > x_n$ and $\pi[x_n, x_{n+1}) \geq g^*$ for all $1 \leq n \leq J$. Hence,

$$\pi[0, y_J) \geq \pi[0, x_1) + \textstyle\sum_{n=1}^{J} \pi[x_n, x_{n+1}) > \rho + Jg^* \geq \frac{1}{2} + \rho,$$

where the second inequality follows from Lemma 3.4 (iv) and the last from the definition of J. This implies $\pi[y_J, B] = 1 - \pi[0, y_J) < \frac{1}{2} - \rho$.

(ii)If $\pi[x_{n+1}, y_n) - \pi[y_n, B] < \frac{\rho}{K}$ for some $1 \leq n \leq \sigma(i)$, then

$$\begin{aligned}\pi[\omega_i, B] &\leq \pi[y_n, B] < \frac{1}{2}\{1 - \pi[0, x_n) + \frac{\rho}{K}\} \\ &\leq \frac{1}{2}\{1 - 1 + \pi[x_0, B] + \frac{\rho}{K}\} < \frac{1}{2^i} - \frac{1}{2^i}\frac{K - \sum_{j=1}^{i} 2^{j-1}}{K}\rho,\end{aligned}$$

where the first inequality follows from Lemma 3.4 (iii), the second from Lemma 3.5 (i), the third from $\pi[0, x_n) \geq 1 - \pi[x_0, B]$, and the last from the hypothesis. If $\pi[x_{n+1}, y_n) - \pi[y_n, B] \geq \frac{\rho}{K}$ for all $1 \leq n \leq \sigma(i)$, then Lemma 3.5 (ii) implies $x_{n+1} > x_n$ and $\pi[x_n, x_{n+1}) \geq g^*$ for all $1 \leq n \leq \sigma(i)$. Hence,

$$\begin{aligned}\pi[0, \omega_i) &\geq \pi[0, x_1) + \textstyle\sum_{n=1}^{\sigma(i)} \pi[x_n, x_{n+1}) \geq 1 - \pi[x_0, B] + \sigma(i) g^* \\ &> 1 - \frac{1}{2^{i-1}} + \frac{1}{2^{i-1}}\frac{K - \sum_{j=1}^{i-1} 2^{j-1}}{K}\rho + \frac{1}{2^i} \\ &= 1 - \frac{1}{2^i} + \frac{1}{2^{i-1}}\frac{K - \sum_{j=1}^{i-1} 2^{j-1}}{K}\rho > 1 - \frac{1}{2^i} + \frac{1}{2^i}\frac{K - \sum_{j=1}^{i} 2^{j-1}}{K}\rho,\end{aligned}$$

where the third inequality follows from the hypothesis and the definition of $\sigma(i)$. The conclusion follows from $\pi[\omega_i, B] = 1 - \pi[0, \omega_i)$. ■

Consider the sequence $\{\omega_i\}_{i=1}^{i^*}$ defined by Lemma 3.6 (ii). If it exists for sufficiently large i^* and if ω_{i^*} is sufficiently small, then we get a contradiction to the assumption that the mean of money holdings is unity. The rest of this section shows that if $w_\Delta(\Delta)/\Delta$ is unbounded as $\Delta \to 0$, then there exists $\{\omega_i\}_{i=1}^{i^*}$ satisfying those conditions. As the first step, the next lemma gives a sufficient condition for existence of z_n and y_n in Definition 4.

LEMMA 3.7. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state. Let* $x_0 \in B_\Delta \backslash \{0\}$ *and let* $\{(x_n, y_n, z_n)\}$ *satisfy Definition* 4.

$(i) w_\Delta(x_n, \Delta) > [u'(0)]^{-n}\ w_\Delta(x_0, \Delta).$

$(ii) x_n < \overline{W}[u'(0)]^n\ [w_\Delta(x_0, \Delta)/\Delta]^{-1}.$

(iii)*If* $w_\Delta(x_0, \Delta)/\Delta$ *is sufficiently large, then* z_n *(and, hence,* y_n*) exists.*

(iv)*If* z_n *exists, then* $w_\Delta(z_n - x_n, \Delta) > u'(\overline{W}) w_\Delta(x_n, \Delta)$.

Proof. First consider parts (i) and (ii). We have

$$\begin{aligned} w(x_{n-1}, \Delta)/\Delta &\le (1/\beta) u[\beta w(x_n, \Delta)]/\Delta < u'(0) w(x_n, \Delta)/\Delta \\ &\le u'(0) w(x_n)/x_n < u'(0)\overline{W}/x_n, \end{aligned}$$

where the first inequality follows from the definition of x_n, the second from strict concavity of u and $u(0) = 0$, the third from concavity of w and $w(0) = 0$, and the last from the definition of $\overline{W}$. A comparison of the first term with the third term gives $w(x_n, \Delta) > [u'(0)]^{-1} w(x_{n-1}, \Delta)$, while a comparison of the first and last terms gives $x_n < \overline{W}[u'(0)][w(x_{n-1}, \Delta)/\Delta]^{-1}$. The desired results follow in an obvious way.

Next consider part (*iii*). By part (*i*) of this lemma, if $w(x_0, \Delta)/\Delta$ is sufficiently large, then $x_n < 1/2$ and $w(x_n, \Delta)/\Delta > \overline{W}/[u'(\overline{W})(B-1/2)]$. By concavity of w, $w(B - x_n + \Delta, \Delta)/\Delta < \overline{W}/(B - x_n)$. It follows that $u'(\overline{W})w(x_n, \Delta)/\Delta > w(B - x_n + \Delta, \Delta)/\Delta$. If z_n does not exist, then $\{x : \min p(x, 0) \geq x_n\}$ is empty. But this implies

$$\begin{aligned} w(B - x_n + \Delta, \Delta) &> u[\beta w(x_n)]//\beta - u[\beta w(x_n - \Delta)]/\beta \\ &> u'[\beta w(x_n)]w(x_n, \Delta) > u'(\overline{W})w(x_n, \Delta), \end{aligned}$$

a contradiction. Hence sufficiently large $w(x_0, \Delta)/\Delta$ implies that z_n exists. Because sufficiently large $w(x_0, \Delta)/\Delta$ implies small x_n and small x_{n+1}, it follows that y_n exists.

Finally consider part (*iv*). By definition, $\min p(z_n - \Delta, 0) \leq x_n - \Delta$ and $\min p(z_n, 0) \geq x_n$. Then by Lemma 3.2 (*ii*), $x_n - \Delta \in p(z_n - \Delta, 0)$. Hence,

$$\begin{aligned} & w(z_n - x_n, \Delta) = w[z_n - \Delta - (x_n - \Delta), \Delta] \\ \geq \; & u[\beta w(x_n)]/\beta - u[\beta w(x_n - \Delta)]/\beta > u'[\beta w(x_n)]w(x_n, \Delta) > u'(\overline{W})w(x_n, \Delta). \end{aligned}$$

■

The next two lemmas are important for establishing existence of $\{\omega_i\}_{i=1}^{i^*}$.

Lemma 3.8. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state. Let* $x_0 = \Delta$ *and let* $\{(x_n, y_n, z_n)\}$ *and* J *satisfy Definition* 4. *Assume* y_J *exists. Then there exists* $C_0 > 0$*, not dependant on* Δ*, such that* $w_\Delta(2x_1, \Delta) > C_0 w_\Delta(\Delta)$.

Proof. See the Appendix. ■

Lemma 3.9. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state. Let* $x_0 \in B_\Delta \backslash \{0\}$ *and let* $\{(x_n, y_n, z_n)\}$ *satisfy Definition* 4. *Assume* y_n *exists. Then there exist* $c_n > 0$ *and* $C > 0$*, not dependant on* Δ*, such that* $w_\Delta(y_n, \Delta) > \min\{c_n w_\Delta(x_0, \Delta), C w_\Delta(\Delta)\}$.

Proof. See the Appendix. ■

Now I can give a sufficient condition for existence of $\{\omega_i\}_{i=1}^{i^*}$.

Lemma 3.10. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state and let* J *and* L *satisfy Definition* 4.

(i)*Let* $x_0 = \Delta$ *and let* $\{(x_n, y_n, z_n)\}$ *satisfy Definition* 4. *If* $w_\Delta(\Delta)/\Delta$ *is sufficiently large, then* y_J *exists and* $w_\Delta(y_J, \Delta) > \xi_1 w_\Delta(\Delta)$*, where* $\xi_1 > 0$ *does not depend on* Δ.

(ii)*Let* $\{\omega_i\}_{i=1}^{L}$ *be the sequence defined in Lemma* 3.6 (ii) *with* $\omega_1 \equiv y_J$ *as defined in part* (i). *If* $w_\Delta(\Delta)/\Delta$ *is sufficiently large, then* $\{\omega_i\}_{i=1}^{L}$ *exists and* $w_\Delta(\omega_i, \Delta) > \xi_i w_\Delta(\Delta)$*, where* $\xi_i > 0$ *does not depend on* Δ.

Proof. (i)By Lemma 3.7 (iii), sufficiently large $w(\Delta)/\Delta$ implies that y_J exists. By Lemma 3.9, $w(y_J, \Delta) > \min\{c_J w_\Delta(\Delta, \Delta), C w_\Delta(\Delta)\} = \min\{c_J, C\} w_\Delta(\Delta)$. Hence we can put $\xi_1 = \min\{c_J, C\} = \min\{c_{\sigma(1)}, C\}$.

(ii)Given part (i), it suffices to prove that if $w(\omega_{i-1},\Delta) > \xi_{i-1}w(\Delta)$, where $\xi_{i-1} > 0$ does not depend on Δ, then $w(\omega_i,\Delta) > \xi_i w(\Delta)$, where $\xi_i > 0$ does not depend on Δ. By definition, $\omega_i = y_{\sigma(i)}$, where $\{(x_n, y_n, z_n)\}$ satisfies Definition 4 (i) with $x_0 = \omega_{i-1}$. Because $w(\omega_{i-1},\Delta) > \xi_{i-1}w(\Delta)$, by Lemma 3.7 (iii), sufficiently large $w(\Delta)/\Delta$ implies that ω_i exists. By Lemma 3.9, $w(\omega_i,\Delta) > \min\{c_{\sigma(i)}w(\omega_{i-1},\Delta), Cw(\Delta)\} > \min\{c_{\sigma(i)}\xi_{i-1}, C\}w(\Delta)$. Hence we can put $\xi_i = \min\{c_{\sigma(i)}\xi_{i-1}, C\}$. ■

I can now prove boundedness of $w_\Delta(\Delta)/\Delta$ as $\Delta \to 0$.

PROPOSITION 3.1. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state. Under assumption* (A), *there exists* $\overline{M}$, *not dependent on* Δ, *such that* $w_\Delta(\Delta)/\Delta < \overline{M}$.

Proof. Assume by contradiction that $w_\Delta(\Delta)/\Delta$ is unbounded as $\Delta \to 0$.. Then by Lemma 3.10, for (w_Δ, π_Δ) with sufficiently large $w_\Delta(\Delta)/\Delta$, the sequence $\{\omega_i\}_{i=1}^{L}$ defined in that lemma 3.10 (ii) exists. By Lemma 3.6 (i), $\pi_\Delta[\omega_1, B] < \frac{1}{2}[1 - \frac{K-1}{K}\rho]$. By Lemma 3.6 (ii) and induction,

$$\pi_\Delta[\omega_L, B] < \frac{1}{2^L} - \frac{1}{2^L}\frac{K - \sum_{j=1}^{L} 2^{j-1}}{K}\rho = \frac{1}{2^L} \leq \frac{1}{2B-1}, \tag{3.14}$$

where the equality and the last inequality follow from the definitions of L and K. By Lemma 3.10 (ii), $w_\Delta(\omega_L,\Delta) > \xi_L w_\Delta(\Delta)$. Therefore

$$\overline{W} > w_\Delta(\omega_L) \geq \omega_L w_\Delta(\omega_L,\Delta)/\Delta > \xi_L \omega_L w_\Delta(\Delta)/\Delta.$$

where the second inequality follows from concavity of w_Δ and $w_\Delta(0) = 0$. It follows that $\omega_L < \overline{W}[\xi_L w_\Delta(\Delta)/\Delta]^{-1}$, and, hence, that $\omega_L < 1/2$ for sufficiently large $w_\Delta(\Delta)/\Delta$. Let $\pi = \pi_\Delta[\omega_L, B]$. Because the mean of money holdings is 1, it follows that $1 < \pi B + (1-\pi)\omega_L < \pi B + (1-\pi)/2$, and, hence, that $\pi = \pi_\Delta[\omega_L, B] > \frac{1}{2B-1}$. But this contradicts (3.14). ■

The next lemma, a corollary of Proposition 3.1, is useful for a later proof.

LEMMA 3.11. *Let* (w_Δ, π_Δ) *be a Lemma* 3.1 *steady state. Let* q^* *be such that* $u'(q^*) = 1$. *If* Δ *is sufficiently small and if* $x > m$, *then* $\min p(x, m, w_\Delta) \geq \min\{q^*/(2\beta\overline{M}), (x - m)/2\}$.

Proof. Let $p \in B_\Delta$ with $p < \min\{q^*/(2\beta\overline{M}), (x-m)/2\}$. By Proposition 3.1, $w(\Delta)/\Delta < \overline{M}$. By concavity of w, this implies for sufficiently small Δ, $\beta w(m + p + \Delta, p + \Delta) < q^*$, and, hence, $u'[\beta w(m + p + \Delta, p + \Delta)] > 1$. Because $p < (x-m)/2$, concavity of w and sufficiently small Δ imply $w(x - p, \Delta) \leq w(m + p + \Delta, \Delta)$. It follows that

$$
\begin{aligned}
& u[\beta w(m+p+\Delta, p+\Delta)] - u[\beta w(m+p, p)] \\
> \; & u'[\beta w(m+p+\Delta, p+\Delta)]\beta w(m+p+\Delta, \Delta) > \beta w(x-p, \Delta).
\end{aligned}
$$

and, hence, that $\min p(x, m) > p$. ■

3.5 Existence of a Monetary Steady State

In this section, I first define the function spaces used. The spaces are compact. By compactness, I obtain a candidate for a monetary steady state. The value function of the candidate is shown to be strictly increasing with a nonzero left derivative at B. Using this fact, I define the mapping implied by (3.1) – (3.8) on a compact domain containing the candidate, a domain whose value functions are strictly increasing. The properties of the domain simplify the argument establishing continuity of the mapping. By continuity, it follows that the candidate is indeed a steady state.

Let $\mathbf{V}_1$ be the set of nondecreasing and concave functions from $[0, B]$ to $[0, \overline{W}]$ with the right derivative at 0 bounded above by $[\frac{1}{N}u'(0)+1]\beta\overline{M}$.[1] (Note that $[\frac{1}{N}u'(0)+1]\beta > 1$.) $\mathbf{V}_1$ is equipped with the sup norm topology. Let the left and right derivatives of a concave function v be denoted by v'_- and v'_+ respectively. Let $\mathbf{V} = \{v \in V_1 : v'_+(0) \leq \overline{M}\}$.

Let $\mathbf{\Lambda}$ be the set of Borel measures on $[0, B]$ with unit mean. $\mathbf{\Lambda}$ is equipped with the weak*-topology.

Now I embed a Lemma 3.1 steady state (w_Δ, π_Δ) in $\mathbf{V} \times \mathbf{\Lambda}$ as follows. Let $b \in B_\Delta \backslash \{B\}$ and $x \in [b, b + \Delta]$. Then let

$$v_\Delta(x) \equiv w_\Delta(b) + (x - b)w_\Delta(b + \Delta, \Delta)/\Delta, \tag{3.15}$$

$$\mu_\Delta(b) \equiv \pi_\Delta(b) \text{ and } \mu_\Delta(x) \equiv 0 \ \forall \ x \notin \{b, b + \Delta\}. \tag{3.16}$$

[1] For a concave function defined on an interval in $\mathbb{R}$, existence of left and right derivatives and other results used in the proof of Lemma 3.12 are well known (see Aliprantis and Border (1994), page 133, 4.26 Theorem and 4.27 Theorem).

That is, v_Δ is the linear interpolation of w_Δ and μ_Δ is the extension of π_Δ to $[0, B]$. Let $\{(v_\Delta, \mu_\Delta)\}_\Delta$ denote the sequence by letting $\Delta = 10^{-n}$, $n = 1, 2, ...$ The next lemma gives a candidate for a monetary steady state.

LEMMA 3.12. *(i)If $v \in \mathbf{V}_1$, then v is continuous. (ii)$\mathbf{V}_1 \times \mathbf{\Lambda}$ is compact and metrizable. (iii)$\mathbf{V} \times \mathbf{\Lambda}$ is compact and metrizable. (iv)The sequence $\{(v_\Delta, \mu_\Delta)\}_\Delta$ has at least one limit (accumulation) point in $\mathbf{V} \times \mathbf{\Lambda}$, denoted (v^*, μ^*). (v)v^* is nondecreasing and concave with $v^*(B) \geq D/\beta$.*

Proof. Let $v \in \mathbf{V}_1$. By concavity, v is continuous on $(0, B)$. By concavity and monotonicity, v is continuous at B. By boundedness of $v'_+(0)$, v is continuous at 0. Then part (i) follows immediately. Because v is continuous and concave with bounded $v'_+(0)$, it follows that $\mathbf{V}_1$ is equicontinuous. Because $\mathbf{V}_1$ is bounded and closed, it follows that $\mathbf{V}_1$ is compact. Because $\mathbf{V} \subset \mathbf{V}_1$ is closed, it follows that $\mathbf{V}$ is compact. Because $[0, B]$ is compact and metrizable, by 12.10 Theorem of Aliprantis and Border (1994) (see page 419), $\{\mu : \mu$ is a Borel measure on $[0, B]\}$ is compact and metrizable. Because $\mu \to \int x d\mu$ defines a continuous functional, $\mathbf{\Lambda}$ is closed, and, hence, compact. Then parts (ii) and (iii) follow immediately. By Proposition 3.1, $(v_\Delta, \mu_\Delta) \in \mathbf{V} \times \mathbf{\Lambda}$ all Δ. Then parts (iv) and (v) follow immediately. ■

The next lemma shows that the value function of a Lemma 3.12 limit point has nonzero left derivative at B.

LEMMA 3.13. *Let $\{(v_\Delta, \mu_\Delta)\}_\Delta$ be a Lemma 3.12 sequence with limit (v, μ). (i)There exists $\underline{M} > 0$, not dependant on Δ, such that $v_\Delta(B, \Delta)/\Delta \geq \underline{M}$. (ii)$v'_-(B) \geq \underline{M}$.*

Proof. We first prove that v is strictly increasing.

(Note that we have not yet established that v is a monetary steady state. Hence the proof of the claim is not a simple generalization of its indivisible money counterpart (see Lemma 2.5).) Assume that v is not strictly increasing. Because v is concave and nondecreasing, there exists a unique $b_2 \in (0, B)$ with $v(x) = v(B)$ for $x \geq b_2$ and $v(x) < v(b_2)$ for $x < b_2$. Let b_0 be the unique solution for $v(b_2) - v(b_0) = D/2$. Recall by assumption, $\Delta = 10^{-n}$ for some positive integer n. By taking large n and taking some arbitrarily close approximation, we can assume that $b_0 10^n$, $b_2 10^n \in \mathbb{N}$ (and, hence, $b_0, b_2 \in B_\Delta$): the approximation, if taken, can satisfy $v(b_2) = v(B)$, $v(b_2) - v(b_0) < D$, and $v(b_2) - v(b_1) > 0$ (b_1 is defined below), which are the essential requirements for b_0 and b_2.

First assume $b_2 > 4$. Then let

$$b_1 = b_2 - \min\{b_2 - 2, B - b_2\}/10, \;\; b_3 = (b_2 - b_1) + b_2.$$

(Note that $b_0, b_2 \in B_\Delta \Rightarrow b_1, b_3 \in B_\Delta$.) Let $\varepsilon > 0$ satisfy $[v(b_2) - v(b_1) - \varepsilon]/\varepsilon > [R\beta u'(\overline{W})/4]^{-1}$. For sufficiently small Δ, we have

$$\begin{aligned} w_\Delta(b_1, \Delta)/w_\Delta(b_3, \Delta) &\geq \frac{[w_\Delta(b_2) - w_\Delta(b_1)]/(b_2 - b_1)}{[w_\Delta(b_3) - w_\Delta(b_2)]/(b_3 - b_2)} = \frac{w_\Delta(b_2) - w_\Delta(b_1)}{w_\Delta(b_3) - w_\Delta(b_2)} \\ &= \frac{v_\Delta(b_2) - v_\Delta(b_1)}{v_\Delta(b_3) - v_\Delta(b_2)} \geq \frac{v(b_2) - v(b_1) - \varepsilon}{v(b_3) - v(b_2) + \varepsilon} \\ &= [v(b_2) - v(b_1) - \varepsilon]/\varepsilon > [R\beta u'(\overline{W})/4]^{-1}, \end{aligned} \tag{3.17}$$

where the first inequality follows from concavity of w_Δ, the first equality from $b_2 - b_1 = b_3 - b_2$, the second inequality from $\lim v_\Delta = v$, and the last equality from $v(b_3) = v(b_2)$. Consider (w_Δ, π_Δ) that satisfies (3.17) and apply Lemma 3.3 (i) with $\varkappa_0 = 2$, $\varkappa_1 = b_1$, and $\varkappa_2 = b_3$. (Note that $2 + b_3 - b_1 = 2 + 2(b_2 - b_1) \leq 2 + 2 \times \frac{b_2 - 2}{10} < b_2 - \frac{b_2 - 2}{10} \leq b_1$.) It follows that $w_\Delta(b_3, \Delta) > R\pi_\Delta[0,2]\beta u'(\overline{W})w_\Delta(b_1, \Delta) \geq [R\beta u'(\overline{W})/2]w_\Delta(b_1, \Delta) > [R\beta u'(\overline{W})/4]w_\Delta(b_1, \Delta)$, which contradicts (3.17).

Next assume $b_2 \leq 4$. Then let

$$b_1 = b_2 - \min\{b_2 - b_0, B - b_2\}/10, \;\; b_3 = (b_2 - b_1) + b_2$$

(Note that $b_0, b_2 \in B_\Delta \Rightarrow b_1, b_3 \in B_\Delta$.) As in (3.17), for sufficiently small Δ, we again have

$$w_\Delta(b_1, \Delta)/w_\Delta(b_3, \Delta) > [R\beta u'(\overline{W})/4]^{-1}. \tag{3.18}$$

Because $v(B) - v(b_0) = v(b_2) - v(b_0) < D$ and $\lim v_\Delta = v$, for sufficiently small Δ,

$$v_\Delta(B) - v_\Delta(b_0) < D. \tag{3.19}$$

Consider (w_Δ, π_Δ) that satisfies (3.18) and (3.19). Either $\pi_\Delta[b_0, B] \leq 3/4$ or $\pi_\Delta[b_0, B] > 3/4$. If the latter, then from $\pi_\Delta(4, B] < 1/4$, it follows that $\pi_\Delta[b_0, 4] > 1/2$. Because $B \geq 8$, by Lemma 3.3 (ii), this implies

$$v_\Delta(B) - v_\Delta(b_0) = w_\Delta(B) - w_\Delta(b_0) \geq D/\beta, \tag{3.20}$$

which contradicts (3.19). So $\pi_\Delta[b_0, B] \leq 3/4$. It follows that $\pi_\Delta[0, b_0] \geq 1/4$. Now we apply Lemma 3.3 (i) with $\varkappa_0 = b_0$, $\varkappa_1 = b_1$, and $\varkappa_2 = b_3$. (Note that $b_0 + b_3 - b_1 = b_0 + 2(b_2 - b_1) \leq b_0 + 2 \times \frac{b_2 - b_0}{10} < b_2 - \frac{b_2 - b_0}{10} \leq b_1$.) It follows that $w_\Delta(b_3, \Delta) > R\pi_\Delta[0, b_0]\beta u'(\overline{W})w_\Delta(b_1, \Delta) \geq [R\beta u'(\overline{W})/4]w_\Delta(b_1, \Delta)$, which contradicts to (3.18). Therefore, v is strictly increasing.

Now we can prove part (i). Let $p = \min\{q^*/(2\beta\overline{M}), (B-3)/2\}$ (for q^*, see Lemma 3.11). By letting Δ be sufficiently small and taking some arbitrarily close approximation, we can assume $p/\Delta \in B_\Delta$. Fix $m \in [0, 2]$. By Lemma 3.11, $\min p(B - \Delta, m, w_\Delta) \geq p$. It follows that $f(B, m, w_\Delta) - f(B - \Delta, m, w_\Delta) \geq \beta w_\Delta(B - p, \Delta)$, and, hence, that $w_\Delta(B, \Delta) \geq (R\beta/2)w_\Delta(B - p, \Delta)$. On the other hand, for sufficiently small Δ, we have

$$\frac{w_\Delta(B - p, \Delta)}{\Delta} = \frac{v_\Delta(B - p, \Delta)}{\Delta} \geq \frac{v_\Delta(B, p)}{p} > \frac{v(B, p)}{2p} > 0,$$

where the first inequality follows from concavity of v_Δ, the second from $\lim v_\Delta = v$, and the last from the claim. Hence, for sufficiently small Δ, it follows that

$$\frac{v_\Delta(B, \Delta)}{\Delta} = \frac{w_\Delta(B, \Delta)}{\Delta} \geq \frac{R\beta}{2}\frac{w_\Delta(B - p, \Delta)}{\Delta} > \frac{R\beta}{2}\frac{v(B, p)}{2p} > 0.$$

By construction, for each v_Δ, the left derivative at B is positive. Then part (i) follows immediately. Consequently, part (ii) follows. ■

Let $\{(v_\Delta, \mu_\Delta)\}_\Delta$ be a Lemma 3.12 sequence with limit (v^*, μ^*). By Lemma 3.13, the left derivative of v^* at B is bounded below by some $\underline{M} > 0$. Let $\mathbf{V}_0 = \{v \in \mathbf{V}$:

$v'_{-}(B) \geq \underline{M}\}$. Note that $\mathbf{V}_0 \subset \mathbf{V} \subset \mathbf{V}_1$ and that by Lemma 3.13, $v_\Delta \in \mathbf{V}_0$ for all Δ. Also note that $\mathbf{V}_0$ is closed, and, hence, compact. Let the mapping $T = (T_v, T_\mu)$ on $\mathbf{V}_0 \times \mathbf{\Lambda}$ be defined by

$$T_v(v,\mu)(x) = \frac{N-1}{N}\beta v\,(x) + \frac{1}{N}\int \widetilde{f}(x,m,v)d\mu(m), \tag{3.22}$$

and

$$T_\mu(v,\mu)([0,x]) = \frac{N-2}{N}\mu([0,x]) + \frac{1}{N}\mu^2(\widetilde{g}^{-1}(x,v)) + \frac{1}{N}\mu^2(\widetilde{h}^{-1}(x,v)), \tag{3.23}$$

where $\mu^2 \equiv \mu\times\mu$. Note that T is essentially given by (3.1) – (3.8) and that by Definition 3, a fixed point of T is a monetary steady state. The next two lemmas give important properties of T.

Lemma 3.14. *If $(v,\mu) \in \mathbf{V}_0 \times \mathbf{\Lambda}$, then $T(v,\mu)$ is single-valued and $T(v,\mu) \in \mathbf{V}_1 \times \mathbf{\Lambda}$.*

Proof. Because v is concave and strictly increasing, $u[\beta v(m+p,p)]+\beta v(x-p)$, viewed as a function in p, is strictly concave. It follows that $\widetilde{p}(x,m,v)$ (see (3.2)) is a singleton, and, hence, that $T(v,\mu)$ is single-valued. Let $T_v(v,\mu)$ be denoted by Tv and $T_\mu(v,\mu)$ by $T\mu$. It is obvious that $T\mu \in \mathbf{\Lambda}$ and that Tv is strictly increasing, $Tv(0) \geq 0$ and $Tv(B) < \overline{W}$. Now let $0 \leq x_1 < x_2 \leq B$, $0 < \lambda < 1$, and $x = \lambda x_1 + (1-\lambda)x_2$. It is easy to see that $\lambda\widetilde{p}(x_1,m,v) + (1-\lambda)\widetilde{p}(x_2,m,v) \in [0, \min\{x, B-m\}]$. This implies $\widetilde{f}(x,m,v) \geq \lambda\widetilde{f}(x_1,m,v) + (1-\lambda)\widetilde{f}(x_2,m,v)$, and, hence, concavity of Tv. Finally, for

$x > 0$, it follows from $Tv(0) \geq 0$ and $v'_+(0) \leq \overline{M}$ that

$$\begin{aligned} Tv(x) - Tv(0) &\leq Tv(x) = \frac{N-1}{N}\beta v\,(x) + \frac{1}{N}\int \widetilde{f}(x,m,v)d\mu(m) \\ &< \frac{N-1}{N}\beta\overline{M}x + \frac{1}{N}[u'(0)\beta\overline{M}x + \beta\overline{M}x] \\ &= [\frac{1}{N}u'(0) + 1]\beta\overline{M}x, \end{aligned}$$

and, hence, that $Tv'_+(0) < [\frac{1}{N}u'(0) + 1]\beta\overline{M}$. Therefore, $Tv \in \mathbf{V}_1$. ■

LEMMA 3.15. *T is continuous on $\mathbf{V}_0 \times \mathbf{\Lambda}$.*

Proof. Fix $(v, \mu) \in \mathbf{V}_0 \times \mathbf{\Lambda}$ and let $\{(v_n, \mu_n)\}$ be a sequence in $\mathbf{V}_0 \times \mathbf{\Lambda}$ converging to (v, μ). First, we have the following,

Claim : $\widetilde{p}(x, m, v_n) \to \widetilde{p}(x, m, v)$ uniformly in (x, m).

Because $\widetilde{p}(x, m, v)$ is a singleton and $(x, m) \longmapsto \min\{x, B - m\}$ is continuous, the theorem of maximum implies that $\widetilde{p}(.,.,.) : [0, B]^2 \times \mathbf{V}_0 \to [0, B]$ is continuous. Because $[0, B]^2 \times \mathbf{V}_0$ is compact, $\widetilde{p}(.,.,.)$ is uniformly continuous. Therefore, $\widetilde{p}(x, m, v_n) \to \widetilde{p}(x, m, v)$ uniformly in (x, m).

Next, let $\phi(x) = \int \widetilde{f}(x, m, v)d\mu(m)$ and $\phi_n(x) = \int \widetilde{f}(x, m, v_n)d\mu_n(m)$. By the claim, for any x, $\widetilde{p}(x, m, v_n) \to \widetilde{p}(x, m, v)$ uniformly in m, so $\widetilde{f}(x, m, v_n) \to \widetilde{f}(x, m, v)$ uniformly in m. Hence, by 12.6 Corollary of Aliprantis and Border (1994, page 417), $\phi_n(x) \to \phi(x)$ pointwise. Because ϕ_n and ϕ are continuous and increasing, it follows that $\phi_n(x) \to \phi(x)$ uniformly, and, hence, that $T_v(v_n, \mu_n)(x) \to T_v(v, \mu)(x)$ uniformly. Therefore, $T_v(v_n, \mu_n) \to T_v(v, \mu)$.

Next, note that $\mu_n \to \mu$ implies $\mu_n^2 \to \mu^2$. By the claim and continuity of $g(.,.,v_n)$ and $g(.,.,v)$, for any (x,m) and $(x_n,m_n) \to (x,m)$, $g(x_n,m_n,v_n) \to g(x,m,v)$. Then by Theorem 5.5 of Billingsley (1968, page 34), $\mu_n^2(\widetilde{g}^{-1}(.,v_n)) \to \mu^2(\widetilde{g}^{-1}(.,v))$. Similarly, we have $\mu_n^2(\widetilde{g}^{-1}(.,v_n)) \to \mu^2(\widetilde{g}^{-1}(.,v))$. Therefore, $T_\mu(v_n,\mu_n) \to T_\mu(v,\mu)$. ■

The next lemma shows that any Lemma 3.12 limit point in $\mathbf{V}_0 \times \mathbf{\Lambda}$ is a fixed point of T.

LEMMA 3.16. *Let* $(v,\mu) \in \mathbf{V}_0 \times \mathbf{\Lambda}$ *be a Lemma* 3.12 *limit point. Then* $T(v,\mu) = (v,\mu)$.

Proof. Let $\{(v_\Delta,\mu_\Delta)\}_\Delta$ be a Lemma 3.12 sequence whose limit is (v,μ). Let *dist* be a metric of $\mathbf{V}_1 \times \mathbf{\Lambda}$. Let $T_v(v_\Delta,\mu_\Delta)$ be denoted by Tv_Δ and $T_\mu(v_\Delta,\mu_\Delta)$ by $T\mu_\Delta$. Because $(Tv_\Delta, T\mu_\Delta) = T(v_\Delta,\mu_\Delta) \to T(v,\mu)$ (by Lemma 3.15) and because

$$dist[T(v,\mu),(v,\mu)] \le dist[T(v,\mu),(Tv_\Delta,T\mu_\Delta)] + dist[(Tv_\Delta,T\mu_\Delta),(v,\mu)],$$

it suffices to prove $Tv_\Delta \to v$ and $T\mu_\Delta \to \mu$.

We start by proving $Tv_\Delta \to v$. Fix $\varepsilon > 0$ and let Δ be such that $\|v_\Delta - v\| < \varepsilon$ and $v(\Delta) < \varepsilon$. First consider $x \in B_\Delta$. By the definition of Tv_Δ,

$$Tv_\Delta(x) = \frac{N-1}{N}\beta v_\Delta(x) + \frac{1}{N}\textstyle\sum_{m \in B_\Delta} \pi_\Delta(m)\widetilde{f}(x,m,v_\Delta). \tag{3.24}$$

Recall that v_Δ is extended from w_Δ, where w_Δ is an indivisible money steady state value function. Hence

$$v_\Delta(x) = \frac{N-1}{N}\beta v_\Delta(x) + \frac{1}{N}\sum_{m\in B_\Delta} \pi_\Delta(m)\overline{f}(x,m,v_\Delta),$$

where

$$\overline{f}(x,m,v_\Delta) = \max_{p\in\Gamma_\Delta(x,m)} u[\beta v_\Delta(m+p,p)] + \beta v_\Delta(x-p). \tag{3.25}$$

and $\Gamma_\Delta(x,m) = \{p \in B_\Delta, p \leq \min\{x, B-m\}\}$. Clearly $\widetilde{f}(x,m,v_\Delta) \geq \overline{f}(x,m,v_\Delta)$ all m. Now fix m and let $p = \widetilde{p}(x,m,v_\Delta)$. A lower bound on $\overline{f}(x,m,v_\Delta)$ can be obtained by taking $p' \in [p, p+\Delta] \cap B_\Delta$ in (3.25). Hence

$$\begin{aligned} & \widetilde{f}(x,m,v_\Delta) - \overline{f}(x,m,v_\Delta) \\ \leq\ & \beta[v_\Delta(x-p) - v_\Delta(x-p')] < v_\Delta(x-p) - v_\Delta(x-p') \\ <\ & v(x-p) - v(x-p') + 2\varepsilon \leq v(\Delta) + 2\varepsilon < 3\varepsilon, \end{aligned}$$

where the third inequality follows from $\|v_\Delta - v\| < \varepsilon$, the fourth from $p' - p \in [0,\Delta]$, and the last from $v(\Delta) < \varepsilon$. Hence $0 \leq Tv_\Delta(x) - v_\Delta(x) < (3/N)\varepsilon$. So for $x \in B_\Delta$,

$$|Tv_\Delta(x) - v(x)| \leq |Tv_\Delta(x) - v_\Delta(x)| + |v_\Delta(x) - v(x)| < (1 + 3/N)\varepsilon. \tag{3.26}$$

Next, consider $x \notin B_\Delta$. Let $a_1 = int(x/\Delta)\Delta$ and $a_0 = a_1 - \Delta$. Note that $|v(a_0) - v(x)| \leq v(\Delta)$ and $|v(a_1) - v(x)| \leq v(\Delta)$. We have

$$
\begin{aligned}
& |Tv_\Delta(x) - v(x)| < \max\{|Tv_\Delta(a_0) - v(x)|\, , |Tv_\Delta(a_1) - v(x)|\} \\
\leq \; & \max\{|Tv_\Delta(a_0) - v(a_0)| + |v(a_0) - v(x)|\, , \\
& |Tv_\Delta(a_1) - v(a_1)| + |v(a_1) - v(x)|\} \\
\leq \; & (2 + 3/N)\varepsilon,
\end{aligned}
$$

where the first inequality follows from monotonicity of Tv_Δ and the last from (3.26) and $v(\Delta) < \varepsilon$. Then $Tv_\Delta \to v$ follows immediately.

Now we prove $T\mu_\Delta \to \mu$. First, we introduce some notation. For each $(y, m) \in [0, B]^2$, let

$$\overline{p}(y, m, v_\Delta) = \arg \max_{p \in \Gamma_\Delta(y,m)} u[\beta v_\Delta(m + p, p)] + \beta v_\Delta(y - p).$$

Note that $\overline{p}(y, m, v_\Delta)$ may be multivalued and that for any $p \in \overline{p}(y, m, v_\Delta)$, $p - \widetilde{p}(y, m, v_\Delta) \in [-\Delta, \Delta]$. Let $\overline{g}(y, m, v_\Delta) = y - \overline{p}(y, m, v_\Delta)$ and $\overline{h}(m, y, v_\Delta) = y + \overline{p}(m, y, v_\Delta)$. Recall that μ_Δ is extended from π_Δ, where π_Δ is an indivisible money steady state measure. Hence

$$
\begin{aligned}
\mu_\Delta([0, x]) &= \frac{N-2}{N}\mu_\Delta([0, x]) + \frac{1}{N}\overline{G}_\Delta([0, x]) + \frac{1}{N}\overline{H}_\Delta([0, x]), \\
\overline{G}_\Delta([0, x]) &= \textstyle\sum_{(y,m) \in B_\Delta^2} \overline{\omega}_{1\Delta}([0, x]; y, m)\mu_\Delta(y)\mu_\Delta(m), \\
\overline{H}_\Delta([0, x]) &= \textstyle\sum_{(m,y) \in B_\Delta^2} \overline{\omega}_{2\Delta}([0, x]; m, y)\mu_\Delta(m)\mu_\Delta(y),
\end{aligned}
$$

where $\overline{\omega}_1(.;y,m)$ and $\overline{\omega}_2(.;m,y)$ are measures defined on $[0,B]$ satisfying

$$\overline{\omega}_{1_\Delta}(\overline{g}(y,m,v_\Delta);y,m) = \overline{\omega}_{2\Delta}(\overline{h}(m,y,v_\Delta);m,y) = 1.$$

By the definition of $T\mu_\Delta$,

$$\begin{aligned} T\mu_\Delta([0,x]) &= \frac{N-2}{N}\mu_\Delta([0,x]) + \frac{1}{N}\widetilde{G}_\Delta([0,x]) + \frac{1}{N}\widetilde{H}_\Delta([0,x]), \\ \widetilde{G}_\Delta([0,x]) &= \textstyle\sum_{(y,m)\in B_\Delta^2} \widetilde{\omega}_{1\Delta}([0,x];y,m)\mu_\Delta(y)\mu_\Delta(m), \\ \widetilde{H}_\Delta([0,x]) &= \textstyle\sum_{(m,y)\in B_\Delta^2} \widetilde{\omega}_{2\Delta}([0,x];m,y)\mu_\Delta(m)\mu_\Delta(y), \end{aligned}$$

where $\widetilde{\omega}_{1\Delta}(.;y,m)$ and $\widetilde{\omega}_{2\Delta}(.;m,y)$ are measures defined on $[0,B]$ satisfying

$$\widetilde{\omega}_{1\Delta}(\widetilde{g}(y,m,v_\Delta);y,m) = \widetilde{\omega}_{2\Delta}(\widetilde{h}(m,y,v_\Delta);m,y) = 1.$$

Next, we introduce a metric to metricize the weak* topology of $\mathbf{\Lambda}$. Let $\mu_1, \mu_2 \in \mathbf{\Lambda}$ and let F_1, F_2 be the corresponding distribution of μ_1, μ_2 respectively. The metric d_L, called Lévy distance, is defined by (see Huber (1981, page 25))

$$d_L(\mu_1,\mu_2) = \inf\{\varepsilon : \forall x, F_1(x-\varepsilon) - \varepsilon \le F_2(x) \le F_1(x+\varepsilon) + \varepsilon\}.$$

Given this metric, it suffices to show that $d_L(\widetilde{G}_\Delta, \overline{G}_\Delta) \to 0$ and $d_L(\widetilde{H}_\Delta, \overline{H}_\Delta) \to 0$. But because for any (y,m) and any $p \in \overline{p}(y,m,v_\Delta)$, $p - \widetilde{p}(y,m,v_\Delta) \in [-\Delta,\Delta]$, it follows that for any x,

$$\overline{G}_{\Delta}([0,\min\{0,x-\Delta]\}) \leq \widetilde{G}_{\Delta}([0,x]) \leq \overline{G}_{\Delta}([0,\max\{B,x+\Delta]\}),$$

and, hence, that $d_L(\widetilde{G}_{\Delta},\overline{G}_{\Delta}) \leq \Delta$. Similarly, we have $d_L(\widetilde{H}_{\Delta},\overline{H}_{\Delta}) \leq \Delta$. ■

By Lemma 3.16, any Lemma 3.12 limit point in $\mathbf{V}_0 \times \mathbf{\Lambda}$ is a monetary steady state. The next lemma establishes strict concavity of the steady state value function.

LEMMA 3.17. *Let $(v,\mu) \in \mathbf{V}_0 \times \mathbf{\Lambda}$ be a Lemma* 3.12 *limit point. Then v is strictly concave.*

Proof. Let $0 \leq x_1 < x_2 \leq B$, $0 < \lambda < 1$, and $x = \lambda x_1 + (1-\lambda)x_2$. Because $T(v,\mu) = (v,\mu)$, $v(x_2) > 0$ implies $\exists\ A \subset [0,B]$ with $\mu(A) > 0$ and $\widetilde{p}(x_2,m,v) > 0$ $\forall\ m \in A$. But then $\lambda\widetilde{p}(x_1,m,v) + (1-\lambda)\widetilde{p}(x_2,m,v) \in [0,\min\{x,B-m\}]$ implies $\widetilde{f}(x,m,v) > \lambda\widetilde{f}(x_1,m,v) + (1-\lambda)\widetilde{f}(x_2,m,v)$. This gives strict concavity of v. ■

The next proposition summarizes the results in this section.

PROPOSITION 3.2. *Under assumption (A), there exists a monetary steady state (v,μ) with v increasing and strictly concave.*

3.6 The Support of the Steady State

Now I turn to establishing that the steady state distribution function has full support. In what follows, let (v,μ) be a Proposition 3.2 steady state and let *supp*

μ denote the support of μ. Part of the next lemma describes the dependence of the optimal offer in (3.2) on the money holdings of the consumer and the producer.

LEMMA 3.18. *(i)If $\widetilde{p}(x', m', v) = 0$, $x \geq x'$, and $m \geq m' - (x - x')$, then $\widetilde{p}(x, m, v) \leq x - x'$.*

(ii)If $m_1 < m_2$, then $m_1 + \widetilde{p}(x, m_1, v) \leq m_2 + \widetilde{p}(x, m_2, v)$.

(iii)If $y > x$, then $\widetilde{p}(y, m, v) - \widetilde{p}(x, m, v) \in [0, y - x]$.

(iv)If $x > m$, then $\widetilde{p}(x, m, v) \geq \min\{q^*/(\beta\overline{M}), (x - m)/2\}$ (for q^*, see Lemma 3.11).

(v)Let $x, m \in$ supp μ and let Q be the intersection of $[0, B]$ and an open interval. If either $x - \widetilde{p}(x, m, v) \in Q$ or $x + \widetilde{p}(m, x, v) \in Q$, then $\mu(Q) > 0$.

Proof. The proof of parts $(i) - (iii)$ is essentially the same as the proof of the corresponding parts of Lemma 2.6. The proof of part (iv) is essentially the same as the proof of Lemma 3.11. Now consider part (v). If $x - \widetilde{p}(x, m) \in Q$, then by continuity of $\widetilde{p}$, $\exists\ \varepsilon > 0$ with $x' - \widetilde{p}(x', m') \in Q$, $\forall\ (x', m') \in (\min\{0, x - \varepsilon\}, \max\{B, x + \varepsilon\}) \times (\min\{0, m - \varepsilon\}, \max\{B, m + \varepsilon\})$. But then $x, m \in$ *supp* μ implies $\mu(Q) > 0$. The other part of part (v) follows in the same way. ■

The next lemma shows that the bound on money holdings is binding.

LEMMA 3.19. *If $x < B$, then $\mu([0, x]) < 1$.*

Proof. Assume by contradiction that $\min\{x : \mu([0, x]) = 1\} = a < B$. Note that $a \in$ *supp* μ. Either $\widetilde{p}(a, a) = 0$ or $\widetilde{p}(a, a) > 0$. If the latter, then $\widetilde{p}(a, a) + a > a$. Because $a \in$

supp μ, by Lemma 3.18 (v), this implies $\mu((a,B]) > 0$, a contradiction. So $\widetilde{p}(a,a) = 0$, which implies $v'_-(a) \geq u'(0)v'_+(a)$. Now either $\mu(a) < 1/4$ or $\mu(a) \geq 1/4$. Assume the latter. Note that $\widetilde{p}(a,a) = 0$ implies $\widetilde{f}(a+\varepsilon,y) - \widetilde{f}(a,a) \geq u[\beta v(a+\varepsilon,\varepsilon)]\ \forall\, \varepsilon \in (0, B-a)$, and, hence,

$$\beta v(a+\varepsilon,\varepsilon) > \beta R\mu(a)u[\beta v(a+\varepsilon,\varepsilon)] \geq (R\beta/4)u[\beta v(a+\varepsilon,\varepsilon)]. \tag{3.27}$$

Because $(R\beta/4)u'(0) > 1$, the equation $\varkappa = (R\beta/4)u(\varkappa)$ has a unique positive solution for $\varkappa$. A comparison of the first and last terms in (3.27) implies that $\beta v(a+\varepsilon,\varepsilon)$ is bounded below by that positive solution. But then $\beta v'_+(a)$ is infinite, a contradiction. So $\mu(a) < 1/4$. Hence $\exists\, z < a$ with $\mu([0,z]) \geq 3/4$. Now let $p = \min\{q^*/(\beta\overline{M}), (a-z)/2\}$ and let $\varepsilon \in (0,p)$. Fix $m \in [0,z]$. By Lemma 3.18 (iv), $\widetilde{p}(a,m) \geq p$. This implies

$$\widetilde{f}(a+\varepsilon,m) - \widetilde{f}(a,m) \geq \beta v(a-p+\varepsilon,\varepsilon) > \beta v'_-(a)\varepsilon,$$

and, hence,

$$v(a+\varepsilon,\varepsilon) > R\mu([0,z])\beta v'_-(a)\varepsilon \geq (R\beta 3/4)u'(0)v'_+(a)\varepsilon > v'_+(a)\varepsilon.$$

where the second inequality follows from $\mu([0,z]) \geq 3/4$ and $v'_-(a) \geq u'(0)v'_+(a)$. But because v is strictly concave, this gives a contradiction. ■

The next lemma gives a sufficient condition for full support.

Lemma 3.20. *If* $\inf\{x : \mu((0,x)) > 0\} = 0$, *then supp* $\mu = [0,B]$.

Proof. Assume by contradiction that $\exists\, z_1 < y_1$ with $\mu((z_1, y_1)) = 0$. Without loss of generality, assume that $z_1 = \inf\{x : \mu((x, y_1)) = 0\}$ and $y_1 = \sup\{x : \mu((z_1, x)) = 0\}$. By the hypothesis, $\mu((0, x)) > 0\ \forall\ x > 0$, and, hence, $z_1 > 0$. By Lemma 3.19, either $y_1 < B$ or $\mu(y_1) > 0$. Note that $z_1, y_1 \in supp\ \mu$. Let $c = y_1 - z_1$. Fix $\varepsilon \in (0, \frac{c}{2B})$. We begin with the following claim.

Claim : $\exists\ 0 < z_2 < y_2 < z_1$ *with* $\mu((z_2, y_2)) = 0$ *and* $y_2 - z_2 \geq (1 - 2\varepsilon)c$.

First assume $\widetilde{p}(z_1, z_1) = 0$. Let $A_{z_1} = (\min\{0, z_1 - c + c\varepsilon\}, \min\{1, z_1 - c\varepsilon\})$. Assume $\mu(A_{z_1}) > 0$ and let $z \in A_{z_1} \cap supp\ \mu$. By Lemma 3.18 (i), $\widetilde{p}(z_1, z_1) = 0$ implies $\widetilde{p}(y_1, z) \leq y_1 - z_1 = c$. Either $\widetilde{p}(y_1, z) = c$ or $\widetilde{p}(y_1, z) < c$. If the latter, then $y_1 - \widetilde{p}(y_1, z) > z_1$. By Lemma 3.18 (iv), $y_1 > z$ implies $\widetilde{p}(y_1, z) > 0$ or $y_1 - \widetilde{p}(y_1, z) < y_1$. It follows that $y_1 - \widetilde{p}(y_1, z) \in (z_1, y_1)$. Because $y_1, z \in supp\ \mu$, by Lemma 3.18 (v), this implies $\mu((z_1, y_1)) > 0$, a contradiction. So $\widetilde{p}(y_1, z) = c$. It follows that $z + \widetilde{p}(y_1, z) \in (z_1, y_1)$. Because $z, y_1 \in supp\ \mu$, by Lemma 3.18 (v), this implies $\mu((z_1, y_1)) > 0$, a contradiction. Therefore, $\mu(A_{z_1}) = 0$. By the hypothesis, $z_1 - c + c\varepsilon > 0$; otherwise, $\mu((0, \min\{1, z_1 - c\varepsilon\})) = 0$. Hence we can put $z_2 = z_1 - c + c\varepsilon$ and $y_2 = \min\{1, z_1 - c\varepsilon\}$.

Next assume $\widetilde{p}(z_1, z_1) > 0$. Let $\underline{z} = \max\{x : \widetilde{p}(x, z_1) = 0\}$. Either $z_1 - \underline{z} \geq c$ or $z_1 - \underline{z} < c$. If the latter, then by Lemma 3.18 (i), $\widetilde{p}(z_1, z_1) < c$, and, hence, $z_1 + \widetilde{p}(z_1, z_1) < y_1$. Because $\widetilde{p}(z_1, z_1) > 0$, it follows that $z_1 + \widetilde{p}(z_1, z_1) \in (z_1, y_1)$. Because $z_1 \in supp\ \mu$, by Lemma 3.18 (v), this implies $\mu((z_1, y_1)) > 0$, a contradiction. So $z_1 - \underline{z} \geq c$. Let $A_{\underline{z}} = (\underline{z} + c\varepsilon, \underline{z} + c - c\varepsilon)$. Assume $\mu(A_{\underline{z}}) > 0$ and let $z \in A_{\underline{z}} \cap supp\ \mu$. By Lemma 3.18 (i), $\widetilde{p}(\underline{z}, z_1) = 0$ implies $\widetilde{p}(z, z_1) \leq z - \underline{z} \leq c - c\varepsilon$, and, hence, $z_1 + \widetilde{p}(z, z_1) < y_1$. By the definition of $\underline{z}$, $z > \underline{z}$ implies $\widetilde{p}(z, z_1) > 0$ or $z_1 + \widetilde{p}(z, z_1) > z_1$. It follows that $z_1 + \widetilde{p}(z, z_1) \in (z_1, y_1)$. Because $z_1, z \in supp\ \mu$, by Lemma 3.18 (v), this implies $\mu((z_1, y_1)) > 0$, a contradiction. Therefore, $\mu(A_{\underline{z}}) = 0$. Because $z_1 - \underline{z} \geq c$, we can put $z_2 = \underline{z} + c\varepsilon$ and $y_2 = \underline{z} + c - c\varepsilon$, which establishes the claim.

Now we can apply the claim to z_2 and y_2 and find $0 < z_3 < y_3 < z_2$ with $\mu((z_3, y_3)) = 0$ and $y_3 - z_3 \geq (1 - 2\varepsilon)^2 c$. Continuing in this way, at the n^{th} step, we have $0 < z_n < y_n < z_{n-1}$ with $\mu((z_n, y_n)) = 0$ and $y_n - z_n \geq (1 - 2\varepsilon)^{n-1} c$. But because $\varepsilon < \frac{c}{2B}$, $c \sum_{i=1}^{n} (1 - 2\varepsilon)^{i-1} > B$ for sufficiently large n, a contradiction. ■

Now I can prove that a Proposition 3.2 steady state measure has full support.

PROPOSITION 3.3. *Let (v, μ) be a Proposition 3.2 steady state. Then μ has full support.*

Proof. By Lemma 3.20, it suffices to prove that $\inf\{x : \mu((0, x)) > 0\} = a = 0$. Assume by contradiction that $a > 0$. Note that $a \in supp\ \mu$.

There are three exhaustive cases. The first two have indivisible money counterparts, while the last one does not.

Case 1 : $\mu(0) = 0$. So $\mu([a, B]) = 1$. Either $\widetilde{p}(a, a) > 0$ or $\widetilde{p}(a, a) = 0$. If the latter, then by Lemma 3.18 (i), $\widetilde{p}(a, m) = 0$ for $m \geq a$. But because $\mu([a, B]) = 1$, this implies $v(a) = 0$, a contradiction. So $\widetilde{p}(a, a) > 0$ or $a - \widetilde{p}(a, a) < a$. Because $a \in supp\ \mu$, by Lemma 3.18 (v), this implies $\mu([0, a)) > 0$, a contradiction.

Case 2 : $\mu(0) > 0$ and $\mu(a) > 0$. So both 0 and a are mass points and there is no mass between them. By the argument in the proof of Lemma 2.8, this implies $B/a \in \mathbb{N}$ and $supp\ \mu = \{0, a, 2a, ..., B - a, B\}$. Let $I = B/a$. As we show in the proof of Lemma 2.9, this implies the existence of a mapping $\theta : \mathbb{R}^I_+ \to \mathbb{R}^I_+$ with the following properties.

θ is concave, strictly increasing, $\theta(0) \geq 0$, and θ has multiple positive fixed points, in this case, a continuum of them. That, however, is impossible.

Case $3 : \mu(0) > 0$ and $\mu(a) = 0$. So 0 is a mass point but a is not and there is no mass between them.

By the definition and Lemma 3.18 (v), $\widetilde{p}(a, a) \in \{0, a\}$. But by the argument in Claim 1 in the proof of Lemma 2.8, $\widetilde{p}(a, a) = 0$ implies a contradiction. Hence $\widetilde{p}(a, a) = a$. Now let $A = supp\ \mu \cap \{m > a : \widetilde{p}(a, m) > 0\}$. By the definition and the hypothesis of a and continuity of $\widetilde{p}$, $\exists\ \varepsilon > 0$ such that $(a, a + \varepsilon) \subset A$. Hence A is nonempty. Let $b = \sup A$. Note that $b \in supp\ \mu$.

Again by the definition of a and Lemma 3.18 (v), $\widetilde{p}(a, m) \in \{0, a\}$ for all $m \in supp\ \mu$. It follows that $\widetilde{p}(a, m) = a$ for $m \in A$. By continuity of $\widetilde{p}$, it follows that $\widetilde{p}(a, b) = a$. Now by Lemma 3.18 (iii), $x > a$ implies $p(x, b) \geq a$. By the definition and the hypothesis of a, $\exists\ \varepsilon_0 \in (0, a)$ with $a + \varepsilon_0 \in supp\ \mu$. It follows that $\widetilde{p}(a + \varepsilon_0, b) \geq a$. Because $\varepsilon_0 < a$, by Lemma 3.18 (v), this implies $\widetilde{p}(a + \varepsilon_0, b) = a + \varepsilon_0$. Again by the definition and the hypothesis of a, $\exists\ \delta \in (0, \varepsilon_0/2)$ such that $(a, a + \delta) \cap supp\ \mu$ is nonempty. By Lemma 3.18 (iii), $\widetilde{p}(a + \varepsilon_0, b) = a + \varepsilon_0$ implies $\widetilde{p}(x, b) = x$ for $x \in (a, a + \delta)$. By Lemma 3.18 (v), this implies that $[b + a, b + a + \delta] \cap supp\ \mu$ is nonempty.

We next show that $\widetilde{p}(a, m) > 0$ for $m \in [b + a, b + a + \delta]$. Consider $\widetilde{p}(\varepsilon_0/2, b + a + \varepsilon_0/2)$. If this is zero, then by Lemma 3.18 (i), $\widetilde{p}(a + \varepsilon_0, b) \leq a + \varepsilon_0/2$, a contradiction. So $\widetilde{p}(\varepsilon_0/2, b + a + \varepsilon_0/2) > 0$. Then by Lemma 3.18 (iii), $\widetilde{p}(a, b + a + \varepsilon_0/2) > 0$. But because $\delta < \varepsilon_0/2$, again by Lemma 3.18 (i), this implies $\widetilde{p}(a, m) > 0$ for $m \in [b + a, b + a + \delta]$.

It follows that $[b + a, b + a + \delta] \subset A$, a contradiction. $\blacksquare$

3.7 Discussion

In contrast to what was assumed for indivisible money, boundedness of $u'(0)$ plays a role for divisible money. It is used to get compactness, or, more exactly, equicontinuity of the set of the embedded value functions of indivisible money steady states. If there were another way to get compactness of the set of embedded value functions, then that assumption would not be needed.

Appendix

Proof of Lemma 3.3

Proof. (i)Fix $m \in [0, \varkappa_0]$. First assume $p(\varkappa_2 - \Delta, m) \ni p \geq \varkappa_2 - \varkappa_1$. Because consumers with $\varkappa_2$ can also offer p, it follows that $f(\varkappa_2, m) - f(\varkappa_2 - \Delta, m) \geq \beta w(\varkappa_2 - p, \Delta) \geq \beta w(\varkappa_1, \Delta)$. Next assume $p(\varkappa_2 - \Delta, m) \ni p < \varkappa_2 - \varkappa_1$. Because consumers with $\varkappa_2$ can offer $p + \Delta$, it follows that

$$\begin{aligned} & f(\varkappa_2, m) - f(\varkappa_2 - \Delta, m) \\ \geq \;& u[\beta w(m + p + \Delta, p + \Delta)] - u[\beta w(m + p, p)] \\ > \;& u'[\beta w(m + p + \Delta, p + \Delta)]\beta w(m + p + \Delta, \Delta) > \beta u'[\beta w(B)]w(\varkappa_1, \Delta). \end{aligned}$$

By the definition of $\overline{W}$, it follows that $f(\varkappa_2, m) - f(\varkappa_2 - \Delta, m) > \beta u'(\overline{W})w(\varkappa_1, \Delta)$ for $m \in [0, \varkappa_0]$. Because (w, π) is a steady state, it follows that

$$\begin{aligned} w(\varkappa_2, \Delta) &> R\sum_{m=0}^{\varkappa_0} \pi(m)[f(\varkappa_2, m) - f(\varkappa_2 - \Delta, m)] \\ &> R\pi[0, \varkappa_0]\beta u'(\overline{W})w(\varkappa_1, \Delta). \end{aligned}$$

(ii)Assume $w(2\varkappa_2) - w(\varkappa_1) < D/\beta$. It follows that $w(\varkappa_1 + \varkappa_2, \varkappa_2) = w(\varkappa_1 + \varkappa_2) - w(\varkappa_1) < D/\beta$. Fix $m \in [\varkappa_1, \varkappa_2]$. The logic here is similar to that in the proof of part (i). If $p(2\varkappa_2 - \Delta, m) \ni p \geq \varkappa_2$, then $f(2\varkappa_2, m) - f(2\varkappa_2 - \Delta, m) \geq \beta w(\varkappa_2, \Delta)$. If $p(\varkappa_2 - \Delta, m) \ni p < \varkappa_2$, then

$$\begin{aligned} & f(2\varkappa_2, m) - f(2\varkappa_2 - \Delta, m) \qquad (3.28) \\ > \;& u'[\beta w(m + p + \Delta, p + \Delta)]\beta w(m + p + \Delta, \Delta) \\ \geq \;& \beta u'[\beta w(m + \varkappa_2, \varkappa_2)]w(m + \varkappa_2, \Delta) \\ \geq \;& \beta u'[\beta w(\varkappa_1 + \varkappa_2, \varkappa_2)]w(2\varkappa_2, \Delta) > \beta u'(D)w(2\varkappa_2, \Delta). \end{aligned}$$

Therefore, if $w(\varkappa_2, \Delta) \geq u'(D)w(2\varkappa_2, \Delta)$, then

$$\begin{aligned} w(2\varkappa_2, \Delta) &> R\pi[\varkappa_1, \varkappa_2]\beta u'(D)w(2\varkappa_2, \Delta) \\ &\geq u'(D)(R\beta/4)w(2\varkappa_2, \Delta) > w(2\varkappa_2, \Delta), \end{aligned}$$

a contradiction. Hence, $w(\varkappa_2, \Delta) < u'(D)w(2\varkappa_2, \Delta)$. It follows that

$$w(2\varkappa_2, \Delta) > R\pi[\varkappa_1, \varkappa_2]\beta w(\varkappa_2, \Delta) \geq (R\beta/4)w(\varkappa_2, \Delta).$$

Again, fix $m \in [\varkappa_1, \varkappa_2]$ and note that consumers with $\varkappa_2$ can offer one more unit of money to producers with m than consumers with $\varkappa_2 - \Delta$. By the logic similar to that used in (3.28), we get $f(\varkappa_2, m) - f(\varkappa_2 - \Delta, m) > \beta u'(D)w(2\varkappa_2, \Delta)$. Hence,

$$\begin{aligned} w(\varkappa_2, \Delta) &> R\pi[\varkappa_1, \varkappa_2]\beta u'(D)w(2\varkappa_2, \Delta) \\ &> u'(D)(R\beta/4)^2 w(\varkappa_2, \Delta) > w(\varkappa_2, \Delta), \end{aligned}$$

a contradiction. ■

Proof of Lemma 3.8

We first prove three claims.

Claim 0 *For* $n \geq 2$, $w_\Delta(n\Delta, \Delta) > (R\rho\beta)^{n-1} w_\Delta(\Delta)$.

Proof. Fix $m \in [0, x_1)$. By the definition of x_1, $\max p(\Delta, m) \geq \Delta$. By Lemma 3.2 (*ii*), this implies $p(n\Delta - \Delta, m) \ni p_n \geq \Delta \ \forall \ n \geq 2$. Because consumers with $n\Delta$ can also offer p_n to producers with m, it follows that $f(n\Delta, m) - f(n\Delta - \Delta, m) \geq \beta w(n\Delta - p_n, \Delta) \geq \beta w(n\Delta - \Delta, \Delta)$. Because $\pi[0, x_1) > \rho$ (see Lemma 3.4 (*iv*)), it follows that $w(n\Delta, \Delta) > R\rho\beta w(n\Delta - \Delta, \Delta)$. The desired result follows in an obvious way by noting that $w(\Delta, \Delta) = w(\Delta)$. ■

Claim 1 *If* $x_{J+1} < 2x_1$ and $z_J \geq 4x_1$, *then* $w_\Delta(2x_1, \Delta) > u'(\overline{W})[u'(0)]^{-J} w_\Delta(\Delta)$.

Proof. By Lemma 3.4 (i), $x_J \leq x_{J+1}$. Hence $x_J < 2x_1$. It follows that

$$\begin{aligned} w(2x_1, \Delta) &> w(z_J - x_J, \Delta) > u'(\overline{W}) w(x_J, \Delta) > u'(\overline{W})[u'(0)]^{-J} w(\Delta, \Delta) \\ &> u'(\overline{W})[u'(0)]^{-J} w(\Delta), \end{aligned}$$

where the first inequality follows from strict concavity of w, the second from Lemma 3.7 (iv), and the third from Lemma 3.7 (i). ■

Claim 2 *If* $x_{J+1} < 2x_1$ and $z_J < 4x_1$, *then there exists* $\alpha > 0$, *not dependent on* Δ, *such that* $w_\Delta(2x_1, \Delta) > \alpha w_\Delta(\Delta)$.

Proof. By Lemma 3.4 (i), $x_J \leq x_{J+1}$. Hence $y_J < 4x_1$. By Lemma 3.6 (i), this implies

$$\pi[0, 4x_1] > 1/2. \tag{3.29}$$

Let $d = D/\beta$. There are six mutually exclusive and exhaustive cases.

Case 1 : $x_1 \leq \frac{6\overline{W}}{d}\Delta$. Hence $2x_1 \leq int(12\overline{W}/d)\Delta$. By concavity of w, $w(2x_1, \Delta) \geq w[int(12\overline{W}/d)\Delta, \Delta]$. By Claim 0, this implies

$$w(2x_1, \Delta) > (R\rho\beta)^{int(12\overline{W}/d)-1} w(\Delta). \tag{3.30}$$

Case 2 : $x_1 > \frac{6\overline{W}}{d}\Delta$ and $w(8x_1) - w(4x_1) \geq d/3$. It follows that

$$w(2x_1, \Delta) > \frac{w(8x_1) - w(4x_1)}{4x_1}\Delta > \frac{d}{12x_1}\Delta > \frac{d}{12\overline{W}u'(0)} w(\Delta), \tag{3.31}$$

where the first inequality follows from strict concavity of w and the last from Lemma 3.7 (*ii*).

Case 3 : $x_1 > \frac{6\overline{W}}{d}\Delta$, $w(8x_1) - w(4x_1) < d/3$, and $w(4x_1) \leq 2d/3$. Hence $w(8x_1) < d$. This case is impossible: by Lemma 3.3 (*ii*), (3.29) implies $w(8x_1) \geq D/\beta = d$.

The remaining 3 cases involve assumptions either about $x_1/2$ or $(x_1 - \Delta)/2$, whichever is in B_Δ. The arguments are written under the assumption that $x_1/2 \in B_\Delta$. If not, then $(x_1 - \Delta)/2 \in B_\Delta$ would appear in place of $x_1/2$.

Case 4 : $x_1 > \frac{6\overline{W}}{d}\Delta$, $w(8x_1) - w(4x_1) < d/3$, $w(4x_1) > 2d/3$, $w(4x_1) - w(x_1/2) \leq 2d/3$, and $\pi[x_1/2, 4x_1] \geq 1/4$. Hence $w(8x_1) - w(x_1/2) < d$. This case is impossible: by Lemma 3.3 (*ii*), $\pi[x_1/2, 4x_1] \geq 1/4$ implies $w(8x_1) - w(x_1/2) \geq D/\beta = d$.

Case 5 : $x_1 > \frac{6\overline{W}}{d}\Delta$, $w(8x_1) - w(4x_1) < d/3$, $w(4x_1) > 2d/3$, $w(4x_1) - w(x_1/2) \leq 2d/3$, and $\pi[x_1/2, 4x_1] < 1/4$. By (3.29) and the hypothesis, $\pi[0, x_1/2] > 1/4$. Now we apply Lemma 3.3 (*i*) with $\varkappa_0 = x_1/2$, $\varkappa_1 = x_1 + kx_1/2$, and $\varkappa_2 = x_1 + (k+1)x_1/2$ for $k = 0, 1$.[2] Because $\pi[0, x_1/2] > 1/4$, it follows that

$$w[x_1 + (k+1)x_1/2, \Delta] > [R\beta u'(\overline{W})/4] w(x_1 + kx_1/2, \Delta). \quad (3.32)$$

Therefore,

$$w(2x_1, \Delta) > [R\beta u'(\overline{W})/4]^2 w(x_1, \Delta) > [R\beta u'(\overline{W})/4]^2 [u'(0)]^{-1} w(\Delta), \quad (3.33)$$

[2] If $x_1/2 \notin B_\Delta$, then we can take $\varkappa_0 = (x_1 - \Delta)/2$, $\varkappa_1 = x_1 + k(x_1 + \Delta)/2$, and $\varkappa_2 = x_1 + (k+1)(x_1 + \Delta)/2$ for $k = 0, 1$.

where the first inequality follows from (3.32) and the last from Lemma 3.7 (i).

Case 6 : $x_1 > \frac{6\overline{W}}{d}\Delta$, $w(8x_1) - w(4x_1) < d/3$, $w(4x_1) > 2d/3$, and $w(4x_1) - w(x_1/2) > 2d/3$. Let $a_0 = \min\{x : d/3 > w(x) - w(x_1/2) \geq d/6\}$ and $a_2 = \max\{x : d/3 > w(4x_1) - w(x) \geq d/6\}$. Because $w(4x_1) - w(x_1/2) > 2d/3$, by concavity of w, existence of a_0 implies existence of a_2. We claim that a_0 exists: otherwise, concavity of w implies $w(x_1/2 + \Delta, \Delta) \geq d/3$, and, hence,

$$w(x_1/2) > \frac{x_1}{2}\frac{w(x_1/2 + \Delta, \Delta)}{\Delta} \geq \frac{x_1}{2\Delta}\frac{d}{3} > \frac{6\overline{W}}{d}\frac{d}{6} = \overline{W},$$

a contradiction. Note that $4x_1 > a_2 > a_0 > x_1/2$ and that

$$w(a_2, \Delta) > \frac{w(4x_1) - w(a_2)}{4x_1 - a_2}\Delta > \frac{d}{6}\frac{1}{3.5x_1}\Delta > \frac{d}{21\overline{W}u'(0)}w(\Delta), \tag{3.34}$$

where the first inequality follows from strict concavity of w, the second from the definition of a_2, and the last from Lemma 3.7 (ii).

Let $a_1 = a_2 - (a_0 - x_1/2)$. By definition, $w(4x_1) - w(a_2) < d/3$ and $w(a_0) - w(x_1/2) < d/3$, while the latter, by concavity of w, implies $w(a_2) - w(a_1) < d/3$. Hence

$$w(4x_1) - w(a_1) = w(4x_1) - w(a_2) + w(a_2) - w(a_1) < 2d/3.$$

By the hypothesis, this implies $w(8x_1) - w(a_1) < d$. Now either $\pi[a_1, 4x_1] < 1/4$ or $\pi[a_1, 4x_1] \geq 1/4$. If the latter, then by Lemma 3.3 (ii), $w(8x_1) - w(a_1) \geq D/\beta = d$, a contradiction. So $\pi[a_1, 4x_1] < 1/4$. By (3.29), this implies $\pi[0, a_1] > 1/4$.

By definition, $w(a_0) - w(x_1/2) \geq d/6$. It follows that

$$\overline{W} > w(x_1/2) > \frac{x_1}{2}\frac{w(a_0) - w(x_1/2)}{a_0 - x_1/2} \geq \frac{dx_1}{12}\frac{1}{a_0 - x_1/2},$$

and, hence, that

$$a_2 - a_1 = a_0 - x_1/2 > dx_1/(12\overline{W}).$$

Let

$$i_0 = int[\frac{3x_1/2}{dx_1/(12\overline{W})}] = int(18\overline{W}/d).$$

Because $a_2 > x_1/2$, we have $a_2 + i_0(a_2 - a_1) > x_1/2 + 3x_1/2 = 2x_1$. Now we apply Lemma 3.3 (i) with $\varkappa_0 = a_1$, $\varkappa_1 = a_2 + k(a_2 - a_1)$, and $\varkappa_2 = a_2 + (k+1)(a_2 - a_1)$ for $k = 0, 1, ..., i_0 - 1$. Because $\pi[0, a_1] > 1/4$, it follows that

$$w[a_2 + (k+1)(a_2 - a_1), \Delta)] > [R\beta u'(\overline{W})/4]w[a_2 + k(a_2 - a_1), \Delta]. \tag{3.35}$$

Therefore,

$$\begin{aligned} w(2x_1, \Delta) &> w[a_2 + i_0(a_2 - a_1)] > [R\beta u'(\overline{W})/4]^{i_0} w(a_2, \Delta) \\ &> \frac{[R\beta u'(\overline{W})/4]^{i_0} d}{21\overline{W}u'(0)} w(\Delta), \end{aligned} \tag{3.36}$$

where the first inequality follows from strict concavity of w, the second from (3.35), and the last from (3.34).

Because cases 3 and 4 are impossible, by (3.30), (3.31), (3.33), and (3.36), the desired α exists. ■

Now I can finish the proof of Lemma 3.8.

Proof. If $x_{J+1} \geq 2x_1$, then

$$w(2x_1, \Delta) \geq w(x_{J+1}, \Delta) > [u'(0)]^{-(J+1)} w(\Delta),$$

where the last inequality follows from Lemma 3.7 (i). If $x_{J+1} < 2x_1$, then by Claim 1 and Claim 2, $w(2x_1, \Delta) > \min\{u'(\overline{W})[u'(0)]^{-J}, \alpha\} w(\Delta)$. Hence the desired C_0 exists. ■

Proof of Lemma 3.9

Let $a = \max\{x : p(\Delta, x) \neq \{0\}\}$. Hence a is the implied x_1 if $x_0 = \Delta$. By Lemma 3.2 (ii), $a \leq x_1$. Then by Lemma 3.4 (i), $a \leq x_n$ for $n \geq 1$. The proof is split into two claims.

Claim 1 *If $x_{n+1} \geq 2a$, then there exists $c_n > 0$, not dependant on Δ, such that $w_\Delta(y_n, \Delta) > c_n w_\Delta(x_0, \Delta)$.*

Proof. We first derive a useful inequality. We apply Lemma 3.3 (i) with $\varkappa_0 = a$, $\varkappa_1 = x_{n+1} + kx_n/2$, and $\varkappa_2 = x_{n+1} + (k+1)x_n/2$ for $k = 0, 1$. (By the hypothesis, $x_{n+1} \geq 2a \Rightarrow a + x_n/2 \leq x_{n+1}/2 + x_n/2 \leq x_{n+1}$.) [3] Because $\pi[0, a) > \rho$ (Lemma 3.4

[3] If $x_n/2 \notin B_\Delta$, then we can take $\varkappa_0 = a$, $\varkappa_1 = x_{n+1} + k(x_n + \Delta)/2$, and $\varkappa_2 = x_{n+1} + (k + 1)(x_n + \Delta)/2$ for $k = 0, 1$. Note that $x_{n+1} \geq 2a$ and $x_n/2 \notin B_\Delta \Rightarrow$ either $x_{n+1} \geq 2a + \Delta$ or $x_{n+1} > x_n \Rightarrow a + (x_n + \Delta)/2 \leq x_{n+1}$.

(iv)), it follows that

$$w[x_{n+1} + (k+1)x_n/2, \Delta] > R\rho\beta u'(\overline{W})w(x_{n+1} + kx_n/2, \Delta),$$

and, hence, that

$$w(x_{n+1} + x_n, \Delta) > [R\rho\beta u'(\overline{W})]^2 w(x_{n+1}, \Delta). \tag{3.37}$$

Next we discuss three mutually exclusive and exhaustive cases.

Case $1: z_n < x_{n+1} + x_n$. Hence, $y_n = x_{n+1} + x_n$. By (3.37),

$$w(y_n, \Delta) > [R\rho\beta u'(\overline{W})]^2 w(x_{n+1}, \Delta). \tag{3.38}$$

Case $2: z_n \geq x_{n+1} + x_n$ and $z_n \leq 3x_n$. Now, $y_n = z_n$. We apply Lemma 3.3 (i) with $\varkappa_0 = a$, $\varkappa_1 = x_{n+1} + x_n$, and $\varkappa_2 = 3x_n$. Because $\pi[0, a) > \rho$, it follows that

$$w(3x_n, \Delta) > R\rho\beta u'(\overline{W})w(x_{n+1} + x_n, \Delta) > [R\rho\beta u'(\overline{W})]^3 w(x_{n+1}, \Delta),$$

where the last inequality follows from (3.37). Therefore,

$$w(y_n, \Delta) > [R\rho\beta u'(\overline{W})]^3 w(x_{n+1}, \Delta). \tag{3.39}$$

Case 3 : $z_n \geq x_{n+1} + x_n$ and $z_n > 3x_n$. Again, $y_n = z_n$. We apply Lemma 3.3 (*i*) with $\varkappa_0 = a$, $\varkappa_1 = z_n - x_n$, and $\varkappa_2 = z_n$. Because $\pi[0, a) > \rho$, it follows that

$$\begin{aligned} w(y_n, \Delta) &= w(z_n, \Delta) > R\rho\beta u'(\overline{W}) w(z_n - x_n, \Delta) \\ &> R\rho\beta [u'(\overline{W})]^2 w(x_n, \Delta) \geq R\rho\beta [u'(\overline{W})]^2 w(x_{n+1}, \Delta), \end{aligned} \tag{3.40}$$

where the second inequality follows from Lemma 3.7 (*iv*).

By Lemma 3.7 (*i*), $w(x_{n+1}, \Delta) > [u'(0)]^{-(n+1)} w(x_0, \Delta)$. Hence, by (3.38), (3.39), and (3.40), the desired c_n exists. ■

Claim 2 *If $x_{n+1} < 2a$, then there exists $C > 0$, not dependant on Δ, such that $w_\Delta(y_n, \Delta) > C w_\Delta(\Delta)$.*

Proof. The proof of this claim is analogous to the proof of Claim 1. We first derive a useful inequality. We apply Lemma 3.3 (*i*) with $\varkappa_0 = a$, $\varkappa_1 = 2a + kx_n/2$, and $\varkappa_2 = 2a + (k+1)x_n/2$ for $k = 0, 1$. (By the hypothesis, $x_{n+1} < 2a \Rightarrow x_n/2 < a$.)[4] Because $\pi[0, a) > \rho$, it follows that

$$w[2a + (k+1)x_n/2, \Delta] > [R\rho\beta u'(\overline{W})] w(2a + kx_n/2, \Delta),$$

and, hence, that

$$w(2a + x_n, \Delta) > [R\rho\beta u'(\overline{W})]^2 w(2a, \Delta). \tag{3.42}$$

[4]If $x_n/2 \notin B_\Delta$, then we can take $\varkappa_0 = a$, $\varkappa_1 = 2a + k(x_n + \Delta)/2$, and $\varkappa_2 = 2a + (k+1)(x_n + \Delta)/2$ for $k = 0, 1$. Note that $x_{n+1} < 2a \Rightarrow x_{n+1} + \Delta \leq 2a \Rightarrow (x_n + \Delta)/2 \leq a$.

Next we discuss three mutually exclusive and exhaustive cases.

Case 1 : $z_n < x_{n+1} + x_n$. Hence, $y_n = x_{n+1} + x_n$. By (3.42) and $x_{n+1} < 2a$,

$$w(y_n, \Delta) > [R\rho\beta u'(\overline{W})]^2 w(2a, \Delta). \tag{3.43}$$

Case 2 : $z_n \geq x_{n+1} + x_n$ and $z_n \leq 3x_n$. Now, $y_n = z_n$. We apply Lemma 3.3 (i) with $\varkappa_0 = a$, $\varkappa_1 = 2a + x_n$, and $\varkappa_2 = 3x_n$. Because $\pi[0, a) > \rho$, it follows that

$$w(3x_n, \Delta) > R\rho\beta u'(\overline{W}) w(2a + x_n, \Delta) > [R\rho\beta u'(\overline{W})]^3 w(2a, \Delta),$$

where the last inequality follows from (3.42). Therefore,

$$w(y_n, \Delta) > [R\rho\beta u'(\overline{W})]^3 w(2a, \Delta). \tag{3.45}$$

Case 3 : $z_n \geq x_{n+1} + x_n$ and $z_n > 3x_n$. Again, $y_n = z_n$. We apply Lemma 3.3 (i) with $\varkappa_0 = a$, $\varkappa_1 = z_n - x_n$, and $\varkappa_2 = z_n$. As in (3.40), we have $w(y_n, \Delta) > R\rho\beta[u'(\overline{W})]^2 w(x_{n+1}, \Delta)$. Then by $x_{n+1} < 2a$,

$$w(y_n, \Delta) > R\rho\beta[u'(\overline{W})]^2 w(2a, \Delta). \tag{3.46}$$

Recall that a is the implied x_1 if $x_0 = \Delta$. Then by Lemma 3.8, $w(2a, \Delta) > C_0 w(\Delta)$. Hence by (3.43), (3.45), and (3.46), the desired C exists. ■

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