ULTRA-WIDEBAND PLANAR ANTENNA ARRAYS BASED ON
RECURSIVE-PERTURBATION DESIGN TECHNIQUES

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by
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ABSTRACT

This dissertation addresses the need for robust methodologies that are capable of designing ultra-wideband antenna arrays. In order to achieve this level of performance it is necessary to look toward non-periodic, or aperiodic, element distributions. Unlike the widely studied and rather straightforward design of periodic arrays, the design of aperiodic arrays typically involves complex and often non-intuitive element distributions. Adjusting, or perturbing, element locations based on a periodic lattice is one of the traditional approaches but it lacks the ability appreciably extend the bandwidth of an array. Moreover, it has only been shown to be well suited for linear arrays with a relatively small number of elements, not medium to large-N arrays which would require the adjustment of many element locations. This dissertation introduces a design methodology that looks toward iteratively constructed geometries to mitigate these limitations. Rather than adjusting every element location in an array, it is based on adjusting a small number of elements and then exploiting the recursive properties of these geometries to generate large array distributions. A key aspect of this approach is that it reduces the representation of arbitrarily large-N planar arrays to a very small set of parameters. This is significant because it provides for a tractable design problem. Additionally, with a small set of parameters, these methodologies are easily combined with a global optimizer for a robust optimization procedure. The efficacy of the concept of recursive-perturbation will be demonstrated through the introduction of two design techniques. One of these techniques is based on exploiting space-filling curves to generate highly modular, ultra-wideband planar arrays. This approach is explored through several examples, including a 962-element array that maintains a sidelobe level below –10 dB over more than a 10:1 bandwidth. The other technique is based on a perturbation scheme that is incorporated into the basic aperiodic tiling generation process. A variety of examples will be investigated to demonstrate its versatility, including a design that maintains a sidelobe level below –11 dB over more than a 22:1 bandwidth. In addition to ultra-wide bandwidths, there are a number of other desirable properties
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Chapter 1

Introduction

1.1 Statement of Problem

Phased antenna arrays offer a number of advantages over conventional aperture antennas and for some applications they provide the only viable option. The principal advantage lies in their ability to synthesize a range of radiation patterns based on the adjustment of phase and amplitude excitations of individual antenna elements. This allows for an agile pattern that can be rapidly steered within the field of view of the array. Another important advantage is that they offer a high degree of spatial resolution that might otherwise be impossible to achieve using conventional aperture antennas. Array architectures can be roughly divided into two main categories: linear and planar. Linear arrays yield fan beams, which are broad in the direction perpendicular to the length of the arrays and narrow in the parallel direction. Phasing of a linear array permits beam steering in the direction parallel to its length. Planar phased arrays, on the other hand, possess a pencil beam pattern that that offers beam steering over two linearly independent dimensions. This additional degree of control makes planar arrays an attractive choice for a variety of applications [1]-[3].

As is the case with most facets of antenna engineering, phased antenna arrays have their advantages but they are not without their unattractive characteristics. Chief among these are high cost and the appearance of grating lobes. The high cost comes about due to the hundreds or thousands of elements that can make up an array and their associated components, including such things as feed networks and T/R modules. The other unattractive characteristic, grating lobes, are a major issue in antenna array systems. In
receiving mode they may cause the array to receive signals from several directions and ambiguities result. For transmitting, grating lobes cause power to be wasted in undesired directions. In the context of a radar system, grating lobes increase the probability of false alarm and they make it more susceptible to jamming from intentional and unintentional sources [2]. In addition, it should be noted that these unwanted effects are not to limited to being caused by grating lobes. They will also result from the presence of high sidelobes in the radiation pattern of the array.

Considering a canonical periodic array of isotropic sources, grating lobes appear when it operates at a frequency corresponding to an element spacing of \( \lambda \) and for all higher frequencies (or equivalently, for all larger spacings). For spacings smaller than \( \lambda \) the pattern of the array will be devoid of grating lobes. However, as the spacing decreases, effects due to mutual coupling among closely spaced elements become more pronounced; this can lead to issues such as pattern degradation and scan blindness [2]-[4]. Generally, elements are spaced at least \( \lambda/2 \) apart at the lowest operating frequency to minimize these effects. Based on these factors, the optimal bandwidth (lobe level below \(-10\) dB) for this canonical array is from \( \lambda/2 \) to \( \lambda \), or equivalently from a frequency \( f_o \) to \( 2f_o \). This 2:1 bandwidth is sufficient for many systems that may only require a narrow bandwidth with limited beam steering capabilities. However, it is a serious limiting factor for many array applications, especially for cutting edge applications that require wideband or ultra-wideband capabilities [1], [3]. A variety of design approaches have led to the development of antenna elements with wideband and ultra-wideband capabilities, such as tapered-slots [5], spirals [6], [7], and horns [8]. However, utilizing wideband antenna elements in an inherently narrowband array generally results in narrowband performance. It is necessary to couple the wideband elements with a suitable wideband array. Consequently, much attention has been focused on developing array layouts that can meet this need.

Most bandwidth enhancement techniques seek to use non-periodic, or aperiodic, element distributions to break up the periodicity that accounts for the formation of grating lobes at spacings of \( \lambda \) or greater. The various approaches of using aperiodic configurations to generate arrays with improved performance can be roughly grouped
into two main categories: unequally spaced arrays and iteratively-constructed arrays. The category of unequally spaced arrays includes approaches such as various deterministic spacing schemes, perturbed element spacings based on a periodic lattice, and randomly distributed arrays [9]-[14]. More recently, approaches based on fractal and tiling concepts have spawned the area of iteratively-constructed array design [15]-[24]. Both categories of arrays attempt to utilize the added flexibility provided by variable interelement spacing to achieve desirable radiation characteristics that would otherwise not be possible with conventional periodic arrays.

Some of the more traditional design techniques involve using unequally spaced layouts to generate aperiodic array distributions. In [9], linear arrays were designed based on various spacing schemes, such as logarithmic spacings, arithmetic progression spacings, and prime number spacings. These configurations were shown to generate arrays with relatively low sidelobes over a 2:1 bandwidth. In [10], a design technique was presented for producing linear arrays with unequal spacings and a desired sidelobe level with the average spacing many wavelengths apart. A theoretical approach to probabilistically designing large randomly spaced arrays was introduced in [11]. While random arrays have been shown to possess large bandwidths that are void of grating lobes, they can often possess relatively high sidelobe levels [12]. In [25], this concept was extended to subarrays of random arrays. These subarray architectures have the potential for wideband performance while greatly simplifying the fabrication of random arrays. An array with excellent sidelobe suppression over a 5:1 bandwidth was reported [25].

A more recent approach at this problem involves exploiting iteratively-constructed geometries in the design of array layouts. One example is the utilization of element distributions based on fractal geometries. Fractal geometries possess scalable self-similar features and can be generated in a recursive manner. The application of these structures to antenna arrays leads to designs with very interesting and useful properties. For example, the Cantor set [17], [19], Koch fractal [17], [19], and Sierpinski carpets [18], [19] have been used to generate array configurations that are deterministically thinned and possess multiband capabilities. While they offer some interesting properties, these
deterministic fractal arrays tend to lack design flexibility. Fractal-random arrays, first introduced in [15], provide one alternative these deterministic fractal structures. They utilize random fractals to generate linear array configurations that are somewhere between completely ordered and completely disordered. Arrays generated in this manner are naturally thinned and they possess moderate sidelobe suppression over relatively wide bandwidths. Arrays based on fractal tilings have also been shown to exhibit a number of desirable characteristics including low sidelobes over a relatively wide bandwidth and modular architectures [20], [21]. In particular, the Peano-Gosper fractile array has a sidelobe level of −18 dB over a 2.4:1 bandwidth [20]. Aperiodic tilings represent yet another approach at utilizing iteratively-constructed geometries in array design. Many types of aperiodic tilings can be constructed in a similar fashion as fractal structures by iteratively enlarging the tiles and decomposing them into several tiles of the original size. In [16], the radiation properties of planar arrays based on certain categories of aperiodic tilings were investigated. Unlike conventional periodic arrays, certain aperiodic tiling geometries lead to array configurations that exhibit suppressed grating lobes over wide bandwidths; arrays with bandwidths up to 1.5\(\lambda\) were reported in [16].

Combining some of the aforementioned methodologies with an optimization algorithm has proven to be very effective in designing array configurations for specific performance criteria [14], [22]-[24], [26]. Some of these optimizer-based techniques lead to array designs with remarkable characteristics, which otherwise would have been impossible to generate using an analytical- or manual-based approach. One example is the use of a genetic algorithm to perturb the positions of elements located on a thinned linear periodic lattice to eliminate grating lobes during scanning [14]. While effective, this process has not been used to appreciably extend the bandwidth of an array and it has been limited to the design of relatively small arrays; examples of up to 24 elements were shown in [14]. Recently, the fractal-random array concept was combined with a GA in the design of wideband linear antenna arrays [22]-[24]. Modifications to the basic fractal random array structure reduced the design of these arrays to a tractable problem and it allowed for the optimization of small arrays up to very large arrays with thousands of elements. This technique has been used to generate linear arrays with suppressed
sidelobes and no grating lobes over a 40:1 bandwidth [24]. As of yet, this approach has not been shown to be well suited for planar array design.

Despite these developments, design approaches that are capable of producing ultra-wideband planar array layouts are still lacking. The goal of the research in this dissertation is to develop robust design techniques that are capable of meeting this need. In addition to generating wideband and ultra-wideband layouts, they are also capable of generating arrays with a number of other desirable attributes including highly thinned apertures and modular architectures. The following section discusses the technical approach that was used to develop these design techniques.

### 1.2 Technical Approach

The research contained in this dissertation presents the development of new design methodologies for generating planar antenna arrays with ultra-wideband capabilities. In order to achieve this level of performance it is necessary to look towards non-periodic, or aperiodic, element distributions. Unlike the widely studied and rather straightforward design of periodic arrays, the design of aperiodic arrays typically involves complex and often non-intuitive element distributions. The well known method of perturbing element locations based on a periodic lattice is a very effective approach but it lacks the ability to appreciably extend the bandwidth of an array. Moreover, it has only been shown to be well suited for linear arrays with a relatively small number of elements, not medium to large-N arrays which would require the adjustment of many element locations. The design methodologies that will be introduced look toward unique 2-D geometries to mitigate these limitations. Rather than applying perturbations to a periodic lattice, they adjust element locations based on recursively-constructed structures. The key aspect of this approach is that the recursive geometries can be used to reduce the representation of large-N planar arrays to a very small set of parameters. This is significant because it reduces the design of these arrays to a tractable problem. Additionally, with a small set
of parameters, these methodologies are easily combined with a global optimizer for a robust optimization procedure.

One of the novel design methodologies is based on perturbing the geometry of arrays derived from space-filling curves. Similar to fractal structures, the construction of a space-filling curve follows an iterative procedure whereby higher stages are generated based on a set formula of scaling, rotating, and translating copies of lower stages. The particular curves that will be considered here are the Peano-Gosper curve and the recently discovered class of generalized Gosper curves. Conversion to an antenna array basically involves uniformly distributing elements along the length of the curve. Both Peano-Gosper and generalized Gosper arrays exhibit a number of desirable characteristics but they are inherently limited to a bandwidth of approximately 2.4:1. The recursive-perturbation technique is capable of significantly extending their bandwidth through the efficient incorporation of perturbations to their array layouts. The technique is based on perturbing the locations of elements along a stage-1 curve and then using these locations along with the recursive generation properties of the curve to generate higher stages of arrays. The effect of this process is that adjustments to the element locations along the stage-1 curve lead to significant changes to the overall distributions of higher-stage arrays. This recursive-perturbation approach allows large, wideband planar arrays to be generated via the adjustment of only a small number of element locations along a simple curve. Unlike comparable approaches, this eliminates the necessity to adjust the location of every element in the array. In addition to having a wide bandwidth, these arrays also possess a number of other highly desirable characteristics including sparse apertures, modular architectures, and the ability to use recursive formulations for rapid pattern calculations.

The second design technique presented in this dissertation is based on a perturbation scheme that is incorporated into the basic aperiodic tiling generation process. Many types of aperiodic tilings can be constructed in a similar fashion as fractal structures by iteratively enlarging tiles and decomposing them into several tiles of the original size. This iterative \textit{inflation} process is repeated until a sufficiently large tiling is grown. Converting this to an array involves placing elements at the vertices of the tiling and then
scaling and truncating the array to meet design specifications. Arrays based on aperiodic tilings have some attractive properties, such as grating lobe suppression over a wide bandwidth and relatively sparse apertures, but they lack design flexibility and they tend to have a limited bandwidth of suppressed sidelobes. The recursive-perturbation technique adds a high degree of flexibility to these arrays, which permits them to be designed for a variety of characteristics. It is based on placing additional elements within the tiles in the initial phase of the inflation process and then retaining these positions as the overall tiling is grown. The array that results from this process is comprised of the elements at the vertices of the tiling as well as elements at the element(s) within each tile. Due to their iterative construction, widely varied layouts can be obtained through the adjustment of the initial locations of the perturbation elements. Similar to first methodology that was discussed, this feature turns the design of these arrays into a tractable problem whereby arbitrarily large planar array distributions can be designed using only a very small set of parameters. Combining this technique with an optimization algorithm leads to arrays that exhibit wideband and ultra-wideband capabilities as well as other desirable characteristics.

1.3 Original Contributions

The research conducted in the development of these array design methodologies has led to original contributions in several areas including the

- Development of a robust design technique for the generation of wideband, modular, planar phased antenna arrays. The technique is based on the recursive-perturbation of space-filling curve geometries, specifically the Peano-Gosper curve and generalized Gosper curves.
• Development of a recursive formulation for the rapid analysis of the radiation patterns of layouts based on perturbed Peano-Gosper arrays and perturbed generalized Gosper arrays.

• Introduction of a new category of planar antenna arrays, called *generalized Gosper arrays*, that are based distributing elements uniformly along generalized Gosper space-filling curves. These arrays possess highly modular architectures, low sidelobes over a relatively wide bandwidth, and the ability to utilize recursive pattern formulations.

• Integration of the concept of antenna arrays based on perturbed space-filling curves and the recursive array factor formulation with a hybrid optimizer for an automated design methodology. This combination was successfully used to generate layouts with bandwidths (sidelobe suppression below –10 dB) of up to 10.4:1.

• Introduction of a new class of antenna arrays, called *perturbed aperiodic tiling arrays*. The arrays are based on exploiting the inflation properties of aperiodic tilings to generate large planar array configurations that possess only a small set of design parameters.

• Integration of the recursive-perturbation technique with a genetic algorithm and a Pareto genetic algorithm for a robust design methodology. This combination allowed for the discovery of planar array configurations with a variety of highly desirable characteristics, such as ultra-wideband performance (in terms of suppressed grating lobes) and highly sparse element distributions.

• Design of planar antenna array layouts with peak sidelobe levels less than –10 dB over more than a 20:1 bandwidth.
• Some of the developments of this research have been published in the *IEEE Transactions on Antennas and Propagation* [27] and are featured in a chapter of the newly published fourth edition of the Antenna Engineering Handbook, John L. Volakis, Editor [1].

### 1.4 Overview

This dissertation introduces robust design methodologies that capable of generating ultra-wideband planar array layouts. The concepts behind these methods as well as the details of their implementation will be discussed. The various optimization algorithms that were instrumental in exploiting the full potential of these techniques will be examined. Finally, the efficacy of these approaches will be shown via a variety of design examples. This section provides an overview of the material discussed in the remainder of this dissertation.

Chapter 2 provides background information on some of the key aspects of this dissertation, including the unique 2-D geometries that form the basis of the design methodologies and the optimization tools that are instrumental in fully exploiting their potential. In Section 2.1, the properties of aperiodic sets and their corresponding tilings are examined. The common methods used to generate these tilings will be also discussed, with emphasis placed on the inflation technique. Section 2.2 reviews the properties of space-filling curves, specifically the Peano-Gosper and generalized Gosper curves. In Section 2.3, the global and local optimization algorithms that are used in conjunction with the array design methodologies are examined. This includes genetic algorithms, the multiobjective Nondominated Sorting Genetic Algorithm, the Nelder-Mead Simplex optimizer, and a hybrid combination of global and local optimizers.

Chapter 3 examines some of the important aspects in the design and analysis of planar antenna arrays. Chief among these are the array factor formulation for arbitrary planar
array layouts and an efficient directivity formulation. The visible region of an array, which is of vital importance in the design of wideband arrays, will also be discussed.

Chapter 4 examines the characteristics of array layouts based on the structures presented in Chapter 2 and it introduces the novel array design methodologies that are central to this dissertation. The first half of the chapter begins by analyzing the use of aperiodic tilings in the design of antenna arrays. Aperiodic tiling arrays will be shown to exhibit excellent performance in terms of suppressed grating lobes and sparse apertures but lacking in the area of design flexibility and bandwidth. Following this, the novel recursive-perturbation technique will be introduced as a means to dramatically improve the capabilities of these arrays. The latter half of the chapter is focused on utilizing space-filling curves in array design, specifically the Peano-Gosper and generalized Gosper curves. This begins by presenting the use of space-filling curves in array design along with analysis of their desirable attributes and their limitations. A recursive-perturbation technique is then introduced as a simple and effective way to significantly improve the performance of these arrays.

Chapter 5 presents some selected examples of arrays that were generated through the combination of the aperiodic tiling perturbation technique and optimization algorithms. The first two sections provide examples of Penrose and Danzer perturbed tiling arrays that were designed for wideband and ultra-wideband performance. The last section presents some examples of perturbed Danzer tiling arrays that were designed using a multiobjective optimizer. This includes arrays that were designed for suppressed sidelobes over a wide bandwidth and a specific degree of aperture thinning as well as an array that was optimized at multiple frequencies for added sidelobe suppression over an ultra-wide bandwidth.

Chapter 6 presents some selected examples that were designed using the recursive-perturbation technique applied to arrays based on Peano-Gosper and generalized Gosper space-filling curves. In Section 6.1, examples will be shown for Peano-Gosper arrays that were designed for enhanced beam steering capabilities over a 2:1 bandwidth. Section 6.2 presents examples of generalized Gosper arrays that were designed for suppressed sidelobes over a very wide bandwidth. Included in this chapter is an array
that attained an ultra-wide bandwidth through the use of a hybrid-optimizer design approach.

Finally, Chapter 7 summarizes some of the important aspects of the design methodologies and their associated array design examples. Conclusions will be drawn about these design techniques in light of existing design approaches. Proposed future work in this area of antenna engineering will be discussed.
Chapter 2

Background

2.1 Aperiodic Tilings

A 2-D aperiodic tiling consists of a collection of tiles that covers the plane without any gaps or overlaps and is void of any translational symmetry [28], [29]. This type of tiling is in direct contrast to 2-D periodic tilings, which admit translations in two linearly independent directions. Both types of tilings can be constructed using a basic set of shapes, or prototiles, which ranges in size from a single prototile up to a set consisting of an infinite number of prototiles. Matching conditions are often utilized to distinguish between a set of prototiles that is capable of generating an aperiodic tiling and one that generates some other “uninteresting” tiling. These matching conditions usually come in the form of tile decorations or edge matching conditions that force the set of prototiles to form an aperiodic tiling [28], [29]. In the realm of crystallography, matching conditions are analogous to the forces governing the self-assembly of real materials from their constituent atoms and molecules [29]. Tilings that are generated from these aperiodic sets possess visually appealing patterns that contain varying degrees of local and long-range order. The study of these tilings has found widespread appeal in various research areas including crystallography, solid-state physics, discrete geometry, and more recently antenna engineering [16].

Some of the more well known examples out of the myriad of aperiodic tilings include, Penrose ‘kite and dart’, chair, sphinx, binary, Danzer, and pinwheel tilings [28], [29]. Analyzing representative portions of these tilings using optical diffraction illustrates that there is great diversity in their diffraction patterns. On one hand, there exists a
category of tilings that possess diffraction patterns which have distinct features and sharp Bragg peaks [29], [30]. Tilings that fall into this category include the Penrose ‘kite and dart’, chair, and sphinx tilings [28], [29]. Another category consists of tilings that contain no discrete components in their spectrum; their diffraction patterns are characterized by continuous, diffused features. The binary, Danzer, and pinwheel tilings fall into this category [28], [29]. It is worth noting that the classification of the spectral features for both of these tiling categories is valid in the infinite tiling limit. Truncating the tilings to a finite region leads to an effect, appropriately called the finite size effect, which tends to diffuse some of the spectral features of the former category and sharpen some of the features of the latter [29].

One of the aperiodic tilings that is utilized in this paper is based on the well known Penrose “kite and dart” aperiodic set [28], [29]. The kite and dart set consists of two quadrangles, each having two sides of length 1 and two sides of length $\tau = (1+\sqrt{5})/2$, as shown in Figure 2.1a. More detailed background information about the set and its matching conditions can be found in [28]. The specific aperiodic set that is considered here is based on a simple modification of the kite and dart set [28]. It requires splitting the kite and dart quadrangles along their axis of symmetry into two triangles that are also shown in Figure 1a. The matching conditions for this set of prototiles are similar to the Penrose kite and dart set with an additional edge matching condition. The modified Penrose aperiodic set is shown in Figure 2.1b and an example of a portion of an aperiodic tiling formed by this set is shown in Figure 2.2a.
As it will be seen in Section 4.1.1, from an antenna array point of view, a tiling formed by the Penrose kite and dart set and a corresponding tiling formed by the modified set generate identical arrays. This is due to the fact that the locations of the vertices of both tilings are identical [28]. While these tilings form similar antenna arrays, the differences in their prototiles will be important when the perturbation technique is

![Figure 2.1](image-url)  

**Figure 2.1:** (a) Prototiles of the Penrose kite and dart aperiodic set. The dashed lines illustrate how the prototiles can be divided into the isosceles triangles of a (b) variation of the basic kite and dart set. (θ = π/5 and τ = (1+√5)/2).

![Figure 2.2](image-url)  

**Figure 2.2:** Portions of tilings formed by the (a) modified Penrose kite and dart set and (b) Danzer aperiodic set.

As it will be seen in Section 4.1.1, from an antenna array point of view, a tiling formed by the Penrose kite and dart set and a corresponding tiling formed by the modified set generate identical arrays. This is due to the fact that the locations of the vertices of both tilings are identical [28]. While these tilings form similar antenna arrays, the differences in their prototiles will be important when the perturbation technique is
introduced in Section 4.1.2. From this point on, this dissertation will only be concerned with tilings formed by the modified Penrose aperiodic set of Figure 2.1b. To simplify discussion, the prototiles of this set and a corresponding tiling formed by this set will be referred to as Penrose prototiles and a Penrose tiling, respectively.

The other aperiodic tiling that will be utilized in this paper is based on the Danzer aperiodic set [29], [31]. This aperiodic set is comprised of the three prototiles that are shown in Figure 2.3. The prototiles of this set must follow specific matching rules that are similar to those of the modified Penrose set. Equal edge lengths must be put together and vertices must be matched up. Additionally, edges of adjacent triangles must be orientated in the same direction according to edge marking conditions [29], [31]. An example of a portion of a Danzer tiling is shown in Figure 2.2b. Note that the vertices of the Penrose and Danzer aperiodic tilings match up and that there are no bisected edges [28].

![Figure 2.3: Prototiles of the Danzer aperiodic set. ($\theta = \pi/7$).](image)

There are several techniques that are commonly used to generate aperiodic tilings. Perhaps the most intuitive technique involves starting with a single tile and adding additional tiles to it by following specific matching rules. This can be based on a manual procedure or it can be accomplished via an automated algorithm. While this procedure seems to be straightforward, it often falls short when a tiling is generated to a point where no additional tiles can be added via the matching rules. A much better approach is to use a systematic and robust technique such as the projection method [28] or the inflation process [28], [31]-[33]. The inflation process is excellent for the generation of aperiodic
tilings that possess a hierarchical structure [28], such as Penrose and Danzer tilings. Similar to the construction of fractal structures [34]-[36], the inflation process is based on iteratively subdividing large tiles into a collection of smaller ones. Throughout each iteration, tiles are first enlarged and then decomposed into tiles of the original size. This is continued until a sufficiently large tiling is formed. The inflation process was the primary tiling generating technique that was used in this dissertation and as it will be seen in Chapter 4, it plays an integral role in the recursive-perturbation design methodology. An illustration of the process applied to the Danzer aperiodic set is shown in Figure 2.4. Part (b) shows the first stage of the inflation process applied to one of the Danzer prototiles. Note that the matching conditions from part (a) dictate how the edges of the prototiles are permitted to match up during the process. This process is continued in the second through the fourth stages that are shown in parts (c) through (e). Note that for illustration purposes the tilings in this figure have been scaled to have a comparable overall size.
Figure 2.4: (a) Prototiles of the Danzer aperiodic set shown with their edge matching conditions. (b) – (e) The first through the fourth stage of the inflation process applied to one of the prototiles of the Danzer aperiodic set. Note that for illustration purposes the tilings of the inflation stages have been scaled to be comparable in size.
2.2 Space-Filling Curves

Space-filling curves have proven to be useful in a variety of electromagnetics applications. These curves have the unique ability to provide a continuous mapping from a 1-D interval \([0,1]\) to a normalized 2-D space \([0,1]^2\) while passing through every point in the space in the infinite limit of the curve [37]. One of the primary uses of these curves has been in reducing, or miniaturizing, the footprint of resonating structures. For instance space-filling curves have been shown to be effective at reducing the overall length of antennas in a printed monopole configuration [38]-[41]. As it is with most miniaturization methods, this length reduction tends to come at the expense of an increased quality factor and a reduced radiation efficiency [39]-[41]. With a similar design objective, space-filling curves have been utilized to miniaturize planar inverted-F antennas [42]. Space-filling curves have also found use in the design of high-impedance surfaces for use as artificial magnetic conductors [43], [44]. In this dissertation, the application that is of primary importance is antenna array design. Various space filling curves have shown to be very effective at providing the framework for array layouts with desirable characteristics, including low sidelobes, modular architectures, and the ability to use rapid pattern formulation [20], [21].

The Peano-Gosper curve is a space-filling curve with the special properties that it is self-avoiding and that it intersects a triangular lattice [35], [37]. It consists of a single meandering path that fills a normalized 2-D space in its infinite limit while avoiding intersections. The curve is not closed; it has a starting point and an endpoint. The contour that bounds the PG curve is a variation of the Koch island [35]. Typical of space-filling curves, construction of the PG curve is based on an iterative procedure that involves scaling, translating, and rotating copies of its stage-1 generator curve. The PG generator is comprised of an arrangement of seven segments of equal length. The number of segments corresponds to the order, \(N\), of the PG curve. An illustration of the iterative construction of the PG curve is shown in Figure 2.5. The fundamental rule of construction is defined by the relationship shown in Figure 2.5a, which involves on replacing a line segment (initiator line) with a seven-segment curve (generator). This
replacement process is then iteratively repeated to generate higher stages of the curve. For instance, the stage-2 curve can be generated by replacing the segments of the stage-1 curve with scaled, rotated, and translated copies of itself based on the relationship shown in Figure 2.5a. The second through the fifth stages of the PG curve are shown in Figure 2.5b through Figure 2.5e. Note that the space-filling properties of the PG curve become especially apparent in the fourth and fifth stages.

![Figure 2.5](image)

**Figure 2.5:** (a) Initiator line (dashed line) and stage-1 generator (solid line) of the N=7 PG curve. (b) Stage-2 PG curve (thick line) superimposed on the stage-1 generator (thin line). (c) Stage-3, (d) -4, and (e) -5 of the PG curve.

In [45], an automated search process was carried out to discover additional space-filling curves with properties similar to those of the Peano-Gosper curve. These
properties mainly included that curve must consist of a self-avoiding path that passes through all points in an equilateral triangular lattice and that it has a starting point and an endpoint. The automated search process was based on searching for stage-1 generators that satisfy a set of criteria that would lead to curves with the desired space-filling properties. A number of generator curves were discovered, which ranged in size up to \( N=37 \). Later in [46], generators in the range of \( N=43 \) to \( N=61 \) were discovered. Based on their analysis, it was conjectured in [46] that the allowable order of the Gosper generator curves are \( N = 6n + 1 \), where \( n=1,2,3,... \). It was also shown that simple modifications to existing generators could lead to new generators and space-filling curves. This modification process can be repeated, leading to an infinite number of generator configuration possibilities. These newly discovered generators are the foundation for a new class of space-filling curves called generalized Gosper curves. While they possess properties similar to those of the PG curve they tend to have vastly different space-filling paths and boundary contours. To simplify discussion throughout this dissertation, the phrases generalized Gosper curves and Gosper curves will be used interchangeably. This will still distinguish them from the well known, Peano-Gosper curve.

Some representative examples of generalized Gosper curves are shown in Figure 2.6 and Figure 2.7. In these figures, the stage-1 generating curves are shown on the left side and corresponding stage-2 curves are shown superimposed on the stage-1 curve on the right. By comparing these curves with that of Figure 2.5 it is evident that the generalized generators have a greater number of segments and more diverse paths than the PG generating curve. Diversity is seen even amongst generators of the same order. For instance, Figure 2.7 illustrate two different generating curves of order \( N=31 \) and their corresponding stage-2 curves. Examination of their stage-2 curves demonstrates that two generators of the same order can lead to Gosper curves with dissimilar space-filling paths and boundary contours.
Figure 2.6: Stage-1 and stage-2 (a) N=13 and (b) N=19 Gosper curves. The left side of each pair illustrates the stage-1 generating curve and its initiator line (dashed line). The right side illustrates the stage-2 curve (thick line) superimposed on its stage-1 curve (thin line).
The construction of these curves follows the basic procedure that outlined for the PG curve. It is based on beginning with an initiator line and then iteratively applying segment replacement rules to generate the various stages of the curve. For example, the relationship between the initiator and the generator of the N=13 Gosper curve is shown in Figure 2.6a. The stage-2 of Figure 2.6b is generated by replacing each segment of the generator with scaled, rotated, and translated copies of itself. Therefore, the stage-2 curve is comprised of an arrangement of thirteen generator curves.

Figure 2.7: Stage-1 and stage-2 (a) N=31a and (b) N=31b-1 Gosper curves. The left side of each pair illustrates the stage-1 generating curve and its initiator line (dashed line). The right side illustrates the stage-2 curve (thick line) superimposed on its stage-1 curve (thin line).
2.3 Optimization Algorithms

Most optimization problems in science and engineering are not as straightforward as finding a solution to a function with analytical derivatives. Often they are vastly more complicated. Many design problems involve functions that possess varying degrees of nonlinearities and discontinuities as well as boundary conditions and constraints. Some problems involve functions that can only be formulated using numerical methods. Additionally, these designs can possess challenging multi-dimensional domains as well as various bounding conditions. Traditional optimization approaches, such as steepest descent [47]-[50] and Newton’s method [50], are ill-equipped to effectively handle such tasks. Such problems are often better handled using more robust algorithms that are well suited for global searches. There exist a number of global search algorithms that have been developed, including simulated annealing [51], [52], genetic algorithms [26], [53]-[56], and particle swarm optimization [57], [58]. These nature-based algorithms have proven to be versatile tools for a variety of design problems.

A genetic algorithm was the primary design tool for the examples that are provided in Chapter 5 and 6 of this dissertation. Variations of the standard GA were also utilized in some of the examples. One of these variations is the Nondominated Sorting Genetic Algorithm, which is an extension of the basic GA for multiobjective problems. The other is a hybrid approach that is based on a combination of a GA and a local optimizer. The details of these design tools will be discussed in the remainder of this section.

2.3.1 Genetic Algorithm

A genetic algorithm is a global search algorithm that searches through a solution space based on the concepts of evolution and natural selection. Throughout a GA optimization a set of potential solutions is evolved toward a global optimal solution based on pressure applied by a performance-based selection process, the exchange of information among members, and randomly introduced mutations [55]. Potential solutions are represented
by a chromosome, which is comprised of a set of parameters, called genes. Each gene corresponds to a specific parameter that is to be optimized by the GA. For engineering applications, quite often these genes represent physical attributes of the system under consideration or a variable in a function to be optimized. Figure 2.8 contains a general flowchart of a typical genetic algorithm.

![Flowchart of a typical genetic algorithm optimization.](image)

The GA is initiated by generating a population of randomly generated possible solutions. The relative merit of each solution is determined by evaluating its performance based on a defined cost function. The entire population is then sorted based on cost by following the convention that low cost implies high performance. Only population members with adequate performance are carried over to the next generation. This
selection process is typically carried out through a threshold cost value or simply a specified percentage of the population. The underperforming members are eliminated from the population and the voids left behind are filled by genetic crossovers between remaining members. Various schemes can be used to select mates for a genetic crossover. One of the most popular schemes, called tournament selection, selects two pairs of members from the population and then allows the best performing member from each pair to participate in a crossover. Crossovers simply mix the genetic information of two members to generate two new members. Mixing of genetic information involves switching a random portion of a chromosome with a corresponding portion of another chromosome, resulting in the creation of a new chromosome with information from both chromosomes. The process of mate selection and genetic crossover is repeated until the entire population is replenished. Genetic mutations are introduced to this new population with the aim of further exploring the solution space and consequently, lessening the chance of convergence on a local minimum. They are incorporated by randomly changing the values of a small percentage of genes within the population. This completes the initial phase of the GA optimization. The process that was outlined is then repeated over a number of iterations, or generations, until the population converges on a solution.

2.3.2 Non-Dominated Sorting Genetic Algorithm

Genetic algorithms are well suited for optimization problems involving a relatively large number of design parameters and complex solution spaces but one of their limitations is that they are naturally based on evolving only a single objective function. In many design problems it is desirable to optimize two or more design objectives; for instance, an antenna array optimization that has the simultaneous goals of minimizing sidelobe levels and maximizing directivity. A common approach to this problem is to represent the multiple objectives in the form of a single objective function. Perhaps the simplest formulation involves just a weighted sum of the objective functions. However, there are several challenges that arise when taking this approach. The most apparent of which is
that it can be difficult to select the proper weights that will optimize the multiple objectives in the desired manner. For similar objectives this may be less of an issue but for dissimilar objectives it can prove to be very challenging. Sometimes multiple optimizations must be carried out before the “optimal” weighting is determined. Moreover, single objective optimizations tend to restrict the search to a limited region of the solution space and they provide limited results in the form of design tradeoffs among the multiple objectives.

In multiobjective design, there exists an ideal set of solutions, called the Pareto front, that consists of members that are not dominated by any other solutions. Rather than possessing a single optimal solution, the Pareto front provides a range of Pareto optimal solutions that offer various design tradeoffs among the design objectives. This allows a designer to select the particular solution from the front that best meets the problem at hand. The first approach along these lines at a multiobjective optimization was the Vector Evaluated GA (VEGA) [59]. This algorithm uses a modified selection operator such that at each generation a number of subpopulations are generated by performing proportional selection based on each objective function. A later development in multiobjective optimization was the Nondominated Sorting GA (NSGA) [54], [60]. This algorithm utilizes the concepts of nondominated sorting and cost sharing to rank potential solutions. It is based on assigning a rank of one to all nondominated solutions of a population and then applying sharing (penalizing individuals within densely populated areas) among these solutions. They are removed from the population and then the process repeats with the remaining solutions until the entire population is ranked. Reproductive operations, such as selection and crossovers, are performed based on these rankings. NSGA have been applied to a variety of design problems [61]-[64]. More recently, other approaches such as the Strength Pareto Evolutionary Algorithm (SPEA) [65] have been developed. The SPEA uses an external set of nondominated population members that were previously found through the optimization. At each generation, this external nondominated set is updated with new individuals. The strength of an individual in the external set is proportional to the number of individuals that it dominates in the current population. The relative cost of individuals in the current population is assigned
according to the strengths of all of individuals of the external set that dominate it. A clustering routine is also used to aid in the development of a smooth Pareto front. The SPEA has proven to be useful in a variety of design applications [22]-[24], [26], [65].

The particular implementation of a Pareto GA that is used in this dissertation is based on the Non-dominated Sorting Genetic Algorithm. The NSGA has been shown to be adept in a variety of multiobjective optimization problems and is well suited for the multiobjective antenna array designs that will be considered in Chapter 5. The basic structure of the NSGA is similar to that of the GA with some additional dominated ranking and cost sharing routines. A basic flowchart that illustrates the major routines of the NSGA is shown in Figure 2.9.
Figure 2.9: Flowchart of the main operations of the Non-dominated Sorting Genetic Algorithm.
Sorting the population based on dominance is one of the key aspects of the NSGA. A member of the population, A, is considered to be dominated by another solution, B, provided

\[ \text{Cost}_i(B) < \text{Cost}_i(A) \quad \text{for every value of } i \quad (2.1) \]

This condition assumes that the goal of the optimization is to minimize the cost functions. The first step of the NSGA sorting involves determining all of the nondominated solutions within current population. Each of these nondominated solutions is assigned the rank of one. Cost sharing is then imposed on these solutions to restrict the front from heading toward a single region of the solution space and to promote a diverse distribution of solutions along the front. This process, called niching, tends to penalize the rank of solutions that are within densely populated areas while not affecting the rank of solutions in sparse areas. Niching is carried out through determining the relative spacing among solutions that are within a prescribed radius, or niche radius, from a given solution. It can be applied to the phenotype (objective function values) or genotype (parameter values) of solutions, though phenotypic sharing is by far the popular approach [54], [60], [65] and it is the method utilized by the NSGA. The relative distance between solutions is given by

\[ d(i, j) = \sqrt{\sum_{n=1}^{q} (F_{n,i} - F_{n,j})^2} \quad (2.2) \]

where \( F_{n,i} \) is the value of the \( n^{th} \) objective function for the \( i^{th} \) population member and \( F_{n,j} \) is the value of the \( n^{th} \) objective function for the \( j^{th} \) population member function, and \( q \) is the total number of objective functions. An estimate of an appropriate value for the niche radius of phenotypic sharing was developed in [66]. The estimate can be determined by solving for \( \sigma \) in the following

\[ N = \frac{\prod_{i=1}^{q} (\max(F_i) - \min(F_i) + \sigma) - \prod_{i=1}^{q} (\max(F_i) - \min(F_i))}{\sigma^2} \quad (2.3) \]
where \( \sigma \) is the niche radius, \( N \) is the number of members in the population and \( F_i \) is the value of the \( i \text{th} \) objective function. The rank of a particular solution is penalized by based on

\[
\text{Rank}_i = \text{Rank}_i + \sum_{j \in \text{Pop}, j \neq i} S[d(i, j)]
\]

(2.4)

where

\[
S[d(i, j)] = \begin{cases} 
1 - d(i, j)/\sigma, & d(i, j) \leq \sigma \\
0, & d(i, j) > \sigma 
\end{cases}
\]

(2.5)

As it can be seen from this triangular weighting function [67], the rank of a solution is only penalized by another solution if it is within its niche radius. Additionally, the degree to which its rank is penalized is proportional to the separation distance, i.e. the smaller the separation distance the higher the penalty.

The solutions on the nondominated front are removed from the population after cost sharing has been applied. The nondominated front of the remaining population is then determined. These members are assigned a rank of one greater than the largest rank of the previous front (after cost sharing) and then cost sharing is imposed among these current members. This process repeats until the entire population has been ranked. The mate selection routine uses the assigned rankings, rather than values of the cost functions, to evaluate the relative merit of individuals. Members with lower ranks are considered superior.

### 2.3.3 Nelder-Mead Simplex Optimizer

Local search algorithms are a common approach to heuristically solving optimization problems. Typically, local searches start with a candidate solution and then iteratively move to a neighboring solution based information in the neighborhood of the current one. Unlike global search algorithms, local searches are not adept at traversing through solution spaces that contain numerous peaks and valleys. When applied to these
problems, local searches tend to get trapped within the valleys and consequently converge on a local minimum. Thus, unlike global searches, they are highly dependent on the starting point of a search. What local searches do offer is the ability to navigate within the neighborhood of a solution and to rapidly and efficiently converge on the minimum. There exist a variety of local search algorithms, including line search methods [50], steepest descent [47]-[50], Newton’s method [50], [68], and the Nelder-Mead simplex algorithm [69], [70]. Line search methods offer a straightforward approach to minimization but they have a propensity to fail when applied to a problem with several minima. Other approaches, such as steepest descent and Newton’s method, require knowledge of the derivatives of the solution space in order to navigate within the region of a minimum. They are adept at locally optimizing a function but they cannot be readily applied to problems where knowledge of the behavior of the solution space is lacking. The Nelder-Mead simplex algorithm provides a viable alternative to these methods. It is more robust than the line search methods and rather than requiring knowledge of derivatives of the solution space, it only requires the knowledge of the values of the solution space.

The Nelder-Mead simplex algorithm is based on moving a simplex down a slope to a minimum in a solution space. A simplex is the most basic geometric shape that can be formed in an N-dimensional space. The minimum number of points (or sides) of the simplex corresponds to N+1, such as a triangle in 2-D space and a tetrahedron in 3-D space. For a given optimization problem of N variables, an initial simplex of (N+1) points is specified. The cost function at each vertex point is evaluated and the one with the highest cost is rejected and replaced with a new vertex location. Depending on the solution space and the current simplex, this new vertex is calculated based on the appropriate reflection, expansion, and contraction operations. In any case, the objective of this procedure is to move the simplex toward the minimum of the cost function. The process of simplex adjustment is iteratively repeated until a specified stopping criterion is met.

A basic flowchart of the Nelder-Mead simplex algorithm is shown in Figure 2.10. In this figure the N+1 vertices of the simplex are denoted by the $v_i$, each of which specify
the coordinates of the vertex in N-dimensional space. The value of the evaluated cost function of vertex \( v_i \) is denoted \( f(v_i) \). The variables \( v_{\text{max}} \), \( v_{\text{max}-1} \), and \( v_{\text{min}} \) correspond to the vertices with the highest, second highest, and lowest cost function evaluations, respectively. The variable \( v_{\text{avg}} \) corresponds to the centroid of the vertices of the simplex excluding the vertex with the highest cost, \( v_{\text{max}} \). The variables \( a \), \( g \), and \( b \), corresponding to the operations of reflection, expansion, and contraction, have a direct impact on the speed and accuracy of the convergence of the simplex algorithm. Based on the analysis in [69], the best set of these parameters for speed and accuracy of convergence was determined to be \( a=1 \), \( b=\frac{1}{2} \), and \( g=2 \).
Figure 2.10: Flowchart for the Nelder-Mead simplex algorithm.
2.3.4 Hybrid Genetic Algorithm & Nelder-Mead Simplex Optimizer

Robust global optimization algorithms, such as genetic algorithms, are well suited for searching through complex solution spaces that possess a relatively large number of design variables. While typical GAs are designed for global searches, their ability to search locally by exploiting information within the vicinity of apparently good solutions is generally lacking. Thus, they are effective at quickly maneuvering through complex solution spaces, but they tend to not be efficient at making local refinements upon good solutions. Quite often, a GA optimization is characterized by fairly rapid convergence to a reasonably good solution, followed by relatively slow improvement on that solution. The slow refinement process in the latter stages often accounts for a large majority of the total computation time in an optimization.

In some instances, the total runtime to reach convergence in a GA optimization can be very long, especially when cost function evaluations are computationally intensive. One way to mitigate this issue is by incorporating a local search routine into the optimization process with the aim of exploiting the strengths of both the global and local algorithms. Often this hybrid approach leads to a dramatic improvement in time to convergence of the final solution.

There exist a variety of ways that a local search routine can be combined with a GA for hybrid optimization [26], [71]. For instance, a local search algorithm can be used to refine a random initial population prior to seeding a GA-based optimizer. Along these lines, local searches can also be used to refine the population members throughout each generation of a GA optimization. Alternatively, a local optimizer can be used to provide efficient refinement once the GA has pinpointed the area of a global minimum.
Chapter 3

Array Analysis

3.1 Array Factor Formulation

The layout of an arbitrary antenna distribution of $N$ elements is shown in Figure 3.1.

Figure 3.1: Geometry of an arbitrary antenna array layout.
In this figure \( \vec{r}_n \) is a vector directed from the origin to the location \((x_n,y_n,z_n)\) of the \(n^{th}\) antenna element of the array. The unit vector \( \hat{n} \) defines the direction from the origin to an observation point with far-field coordinates \((x,y,z)\). If the array is modeled as a collection of isotropic radiators, then its array factor can be formulated as [4]

\[
AF(\theta,\varphi) = \sum_{n=1}^{N} I_n \exp(j\beta_n) \exp(jk\hat{n} \cdot \vec{r}_n)
\]  

(3.1)

The variables \( I_n \) and \( \beta_n \) correspond to the amplitude and phase of the current excitation of the \(n^{th}\) radiator, respectively. The wave number, \( k \), is equal to \( 2\pi/\lambda \). In rectangular coordinates, the vectors \( \vec{r}_n \) and \( \hat{n} \) are given by

\[
\vec{r}_n = \hat{x} x_n + \hat{y} y_n + \hat{z} z_n
\]

(3.2)

\[
\hat{n} = \frac{\hat{x} x + \hat{y} y + \hat{z} z}{\left( x^2 + y^2 + z^2 \right)^{1/2}}
\]

(3.3)

where \((x,y,z)\) are the coordinates of the far-field observation point. The rectangular coordinates of (3.3) can be transformed to spherical coordinates using the following transformations

\[
x = r \sin \theta \cos \varphi
\]

(3.4)

\[
y = r \sin \theta \sin \varphi
\]

(3.5)

\[
z = r \cos \theta
\]

(3.6)

Substituting (3.4)-(3.6) along with

\[
r = \left( x^2 + y^2 + z^2 \right)^{1/2}
\]

(3.7)
into (3.3) yields

\[ \hat{n} = \hat{i}_x \sin \theta \cos \varphi + \hat{i}_y \sin \theta \sin \varphi + \hat{i}_z \cos \theta \]  \hspace{1cm} (3.8)

The dot product of (3.8) and (3.2) is

\[ \hat{n} \cdot \vec{r}_n = x_n \sin \theta \cos \varphi + y_n \sin \theta \sin \varphi + z_n \cos \theta \]  \hspace{1cm} (3.9)

Substituting (3.9) into (3.1) gives

\[ AF(\theta, \phi) = \sum_{n=1}^N \exp \left[ jk \left( x_n \sin \theta \cos \varphi + y_n \sin \theta \sin \varphi + z_n \cos \theta \right) \right] \]  \hspace{1cm} (3.10)

Alternatively, the array factor can also be formalized in terms of the spherical coordinates of the element locations. Substituting (3.4)-(3.6) into (3.2) yields

\[ \vec{r}_n = \hat{i}_x r_n \sin \theta_n \cos \varphi_n + \hat{i}_y r_n \sin \theta_n \sin \varphi_n + \hat{i}_z r_n \cos \theta_n \]  \hspace{1cm} (3.11)

The dot product of (3.9) and (3.11) is

\[ \hat{n} \cdot \vec{r}_n = r_n \sin \theta_n \cos \varphi_n \sin \theta \cos \varphi + r_n \sin \theta_n \sin \varphi_n \sin \theta \sin \varphi + r_n \cos \theta \cos \theta \]  \hspace{1cm} (3.12)

\[ \hat{n} \cdot \vec{r}_n = r_n \sin \theta_n \sin \theta (\cos \varphi_n \cos \varphi + \sin \varphi_n \sin \varphi) + r_n \cos \theta_n \cos \theta \]  \hspace{1cm} (3.13)

This can be simplified by applying the trigonometric relation \( \cos(A-B) = \cos A \cos B + \sin A \sin B \), which yields

\[ \hat{n} \cdot \vec{r}_n = r_n \sin \theta_n \sin \theta \cos(\varphi_n - \varphi) + r_n \cos \theta_n \cos \theta \]  \hspace{1cm} (3.14)

Substituting (3.14) into (3.1) gives an expression for the array factor of an arbitrary array layout in terms of the spherical coordinates of element locations.
If the array elements are restricted to only residing on the x,y-plane, then (3.15) reduces to

\[ AF(\theta, \varphi) = \sum_{n=1}^{N} I_n \exp(j\beta_n) \exp\left[jkr_n \left(\sin\theta_n \sin\theta\cos(\varphi_n - \varphi) + \cos\theta_n \cos\theta\right)\right] \]  

(3.15)

Adjustment of the phases, \( \beta_n \), of the current excitations leads to control over the direction of the main beam. The main beam can be steered to a far-field direction \((\theta_0, \varphi_0)\) by incorporating the following phase excitations

\[ \beta_n = -kr_n \sin\theta_n \sin\theta_0 \cos(\varphi_n - \varphi_0) \]  

(3.17)

The array factor formulation in (3.16) is only applicable to arrays comprised of idealized isotropic radiators. The element patterns of individual antennas can be incorporated into the formulation as follows

\[ AF(\theta, \varphi) = \sum_{n=1}^{N} EP_n(\theta, \varphi) I_n \exp(j\beta_n) \exp\left(jkr_n \sin\theta_n \sin\theta\cos(\varphi_n - \varphi)\right) \]  

(3.18)

where \( EP_n(\theta, \varphi) \) is the complex representation of the radiated electric field of the \( n^{th} \) element of the array, which takes into account polarization. Accordingly, the resultant array factor, \( AF(\theta, \varphi) \), is also polarized.

### 3.2 Directivity Formulation

The directivity of an array provides a quantitative measure of the directional properties of its radiation pattern. It is defined as the ratio of the radiation intensity in a certain
direction to the average radiation intensity \[4\]. This definition can be formalized as follows

\[
D(\theta, \varphi) = \frac{1}{4\pi} \left| \mathbf{AF}(\theta, \varphi) \right|^2 \int_0^{2\pi} \int_0^\pi \frac{\sin \theta |\mathbf{AF}(\theta, \varphi)|^2 \sin \theta d\theta d\varphi}{\sin \theta d\theta d\varphi}
\]  \hspace{1cm} (3.19)

Quite often, the directivity value that is of primary importance is that of the maximum directivity, \(D\). In this case, (3.19) reduces to

\[
D = \frac{1}{4\pi} \left| \mathbf{AF}(\theta, \varphi) \right|_{\text{max}}^2 \int_0^{2\pi} \int_0^\pi \frac{\sin \theta |\mathbf{AF}(\theta, \varphi)|^2 \sin \theta d\theta d\varphi}{\sin \theta d\theta d\varphi}
\]  \hspace{1cm} (3.20)

Evaluating (3.20) for an antenna array is often computationally intensive since it requires integrating (or summing) the radiation pattern of the array over a sphere. At low frequencies this may not be an issue but at high frequencies a very high resolution is required to ensure that the fine features of the radiation pattern are not missed. For an accurate calculation, this may require evaluating the array factor with a resolution of a hundredth or thousandth of a degree. In [72], an efficient alternative to (3.20) was developed for calculating the maximum directivity of an array of isotropic sources with its main beam directed to broadside. Unlike the traditional approach, this formulation does not require evaluating the array factor over a spherical volume. The formulation is given below

\[
D = \frac{\left( \sum_{n=1}^{N} I_n \right)^2}{\left( \sum_{n=1}^{N} I_n^2 \right) + 2 \sum_{m=2}^{N} \sum_{n=1}^{m-1} j_0 \left( k |\vec{r}_n - \vec{r}_m| \right) + 2 \sum_{m=2}^{N} \sum_{n=1}^{m-1} j_0 \left( k |\vec{r}_n - \vec{r}_m| \right)}
\]  \hspace{1cm} (3.21)

where \(\vec{r}_n\) and \(\vec{r}_m\) are the positional vectors for the \(n^{th}\) and \(m^{th}\) elements in the array, respectively, and \(j_0(x)\) is the spherical Bessel function of the first kind of order zero. The spherical Bessel functions in (3.21) may be represented as
For the case of uniform amplitude excitations, (3.22) reduces to

\[
D = \frac{N^2}{N + 2 \sum_{m=2}^{N} \sum_{n=1}^{m-1} j_0 \left( k |\bar{r}_n - \bar{r}_m| \right)}
\]  

(3.23)

### 3.3 Element Spacings

An important consideration in the design and analysis of an antenna array is its element spacings. Not only are they critical in mitigating issues with mutual coupling but they also impact other facets of array design, including such things as aperture thinning and the cooling of an array [2]. This section introduces the various metrics that were used to quantify the element distributions in this dissertation.

The distance between the \( n^{th} \) element of an array and its nearest neighbor can be determined from

\[
d_n = \min_{m \neq n} |\bar{r}_n - \bar{r}_m|, \quad m=1,2,\ldots,N
\]  

(3.24)

Using (3.24), the minimum, maximum, and average nearest neighbor spacings for an array may be defined as

\[
d_{\text{min}} = \min(d_n), \quad n=1,2,\ldots,N
\]  

(3.25)

\[
d_{\text{max}} = \max(d_n), \quad n=1,2,\ldots,N
\]  

(3.26)

\[
d_{\text{avg}} = \frac{1}{N} \sum_{n=1}^{N} d_n
\]  

(3.27)
3.4 Visible Region

The visible region of an antenna array defines the region over which its array factor formulation corresponds to physically real angles. Examining the array factor within this region, as well as outside, provides valuable insight into various aspects of the performance of an array, such as its capacity for beam steering and wideband operations. This section discusses the basic facets of the visible region of an array and how it can be exploited in the design of the wideband phased arrays.

A convenient way to visualize the array factor is to plot it against the axes $u = \sin \theta \cos \varphi$ and $v = \sin \theta \sin \varphi$. In this $(u,v)$-space the array factor can be plotted over an infinite extent. However, the only portion of this space that corresponds to physically real angles is the region bounded by $u^2 + v^2 < 1$. The features within this visible region are those that will show up in the radiation pattern of an array; the features outside of this region are not realized. The concept of the visible region will be explored further through the use of a canonical planar square-lattice periodic array. A simplified illustration of its array factor in $(u,v)$-space at $f = f_0$ (with a corresponding wavelength of $\lambda$) is shown in Figure 3.2. At this frequency it is assumed that the array has an element spacing of $\lambda/2$. Note that this figure merely illustrates the relative positions of grating lobes in relation to the main beam; it does not provide any information about the sidelobes. However, the locations of the grating lobes will be sufficient for the analysis that is considered here. In this figure the main beam is located at the origin and it is surrounded by a number of grating lobes that are periodically distributed with a spacing of $\lambda/d_{\text{min}}$. The grating lobe distribution is shown truncated but it actually extends out infinitely in $(u,v)$-space. At $f = f_0$, the spacing between grating lobes is 2. This is sufficient to place them outside of the visible region of the array. Therefore, at this frequency, the radiation pattern of an array only contains a main beam and sidelobes. An illustration of the array factor at $f = 2f_0$ is shown in Figure 3.3. A doubling of the frequency causes the grating lobe spacing to be cut in half. This places four grating lobes at the boundary of the visible region. This is the well known condition that limits the
bandwidth of a periodic array to element spacings less than \( \lambda \). A further increase in the operating frequency brings even more of the \((u,v)\)-space into the visible region of the array. An example of this at \( f = 4f_o \) is shown in Figure 3.4. It should be pointed out that different scales are used in the plots of Figure 3.2 through Figure 3.4.

Figure 3.2: Grating lobe locations of a periodic lattice during broadside operation at \( f = f_o \). At this frequency the array has an element spacing of \( \lambda/2 \).

Figure 3.3: Grating lobe locations of a periodic lattice during broadside operation at \( f = 2f_o \). At this frequency the array has an element spacing of \( \lambda \).
Based on the preceding discussion it should be apparent that the visible region of an array at a particular frequency contains the details of the visible region at all lower frequencies. For instance, the various sidelobes and grating lobes that are present at $f = 2f_0$ (see Figure 3.3) will also be present in the visible region at $f = 4f_0$ (see Figure 3.4). This concept is very useful because it means that the array factor at a particular operating frequency provides an indication of the characteristics of the array factor at lower frequencies. This greatly simplifies the design of the wideband and ultra-wideband array layouts based on the recursive-perturbation techniques. Rather than analyzing the array at a number of frequencies, it allows the design of these arrays to focus solely on the lobe suppression at the upper bound of a targeted bandwidth. This ensures that the suppression at all lower frequencies will be equal to or less than this level.

The concept of the visible region in $(u,v)$-space is also useful for analyzing the performance of an array during beam steering. When the main beam is directed away from broadside, there is a corresponding shift in the array factor in the $(u,v)$ plane; this alters the portion of the array factor that falls within the visible region of the array. The extent and direction of this shift corresponds to the elevation and azimuthal angles of
The array factor shifts in the same direction as the azimuthal steering angle and the extent of the shift is proportional to \( \sin(\theta_o) \), where \( \theta_o \) is the elevation angle of the main beam. An illustration of this for main beam steering to \( \theta = 30^\circ \), \( \varphi = 0^\circ \) is shown in Figure 3.5. For this steering angle the array factor is seen to shift by 0.5 in the direction of \( \varphi = 0^\circ \). Accordingly, for a steering angle of \( \theta = 30^\circ \), \( \varphi = 90^\circ \) the array factor shifts toward \( \varphi = 90^\circ \) in the amount of 0.5 (see Figure 3.6). In both cases, the effect of beam steering is to essentially shift the visible region to cover a different portion of the array factor in \((u,v)\)-space.

Figure 3.5: Grating lobe locations of a periodic lattice with its main beam directed to \( \theta = 30^\circ \), \( \varphi = 0^\circ \) at \( f = 2f_o \). At this frequency the array has an element spacing of \( \lambda \).
Some of the examples that will be presented in this dissertation were designed with the objective of enhancing the beam steering capabilities of an array within a conical volume (up to $\theta = \theta_o$) that is centered at broadside. This is achieved by suppressing the lobes that would appear if the main beam of the array were steered to any angle within the volume. One approach to this problem is to determine the peak lobes that appear while the array is steered to a number of different directions. However, this would require evaluating the array factor at numerous scan angles to ensure that the peak lobe is not missed. A much more efficient approach involves calculating the maximum extent of the array factor in $(u,v)$-space that would be encompassed during beam steering. Based on the discussion above, this extended visible region corresponds to the area bounded by $u^2 + v^2 < 1 + \sin \theta_o$. This extended region provides information about the peak lobe (and other lobes) that would appear if the main beam were steered throughout the conical volume. An example of this for steering within a 30° conical volume is shown in Figure 3.7. In this case, the extended visible region is bounded by $u^2 + v^2 < 1.5$. Note that this region encompasses the portions of the array factor that were covered in the beam steering examples of Figures 3.5 and 3.6.
Figure 3.7: Grating lobe locations of a periodic lattice during broadside operation at $f = 2f_0$. In this illustration, the visible region is expanded to show the region of (u,v)-space that is encompassed during beam steering within a 30° conical volume that is centered at broadside.
Chapter 4

Recursive-Perturbation

4.1 Application to Aperiodic Tiling Arrays

The various spectral features that are associated with aperiodic tilings translate over to antenna array design. Given that optical diffraction is analogous to the array factor of antenna theory, similar spectral features will show up in both the diffraction pattern of a tiling and the radiation pattern of a corresponding array. In [16], the radiation properties of several different categories of aperiodic tiling arrays were investigated and found to possess features similar to their associated optical diffraction patterns. Moreover, arrays generated using tilings that possess continuous, diffused diffraction patterns were shown to exhibit relatively wide bandwidths over which grating lobes were suppressed. In this dissertation, the recursive-perturbation design technique is applied to the Danzer tiling, which falls under this category of aperiodic tiling. Results will also be presented for the application of this technique to arrays based on the Penrose tiling, which possesses a more discrete radiation spectrum. Applying the recursive perturbation technique to these two types of aperiodic tilings with dissimilar radiation spectra will demonstrate the versatility of this array design methodology.

The characteristics of a conventional square-lattice periodic array will be used as a baseline for comparison with the aperiodic tiling arrays. The particular array that will be considered has an aperture with the same size and shape as that of tiling arrays. The periodic array that fits within this aperture is comprised of 1793 elements when its adjacent element spacing is $\lambda/2$. The layout of the 1793-element array is shown in Figure 4.1. It has a sidelobe level of -17.41 dB that extends from $f = f_o$ to approximately $f$
Due to multiple regions of constructive interference in its radiation pattern, the array has grating lobes at \( f = 2f_0 \) and for all higher frequencies.

![Figure 4.1: Geometry of a 1793-element square-lattice periodic antenna array.](image)

### 4.1.1 Planar Arrays Based on Aperiodic Tilings

Converting an aperiodic tiling to an antenna array is a straightforward process. As it was first presented in [16], it involves replacing an aperiodic tiling with a set of points that are located at the vertices of the tiling. The locations of these points correspond to the locations of the elements that comprise the antenna array. The next step is to scale and truncate the tiling to meet a desired minimum element spacing requirement and aperture size. For comparison purposes, all of the aperiodic tiling arrays that are presented in this dissertation follow the same scaling and truncation procedure. Each aperiodic tiling is scaled to have a minimum spacing between elements of at least \( 0.5\lambda \) at a frequency \( f_0 \) (with corresponding wavelength \( \lambda \)). The tiling is then truncated such that it fits within a
circular aperture with a radius of $12\lambda$. Any tile that has at least one vertex outside of the circular aperture is eliminated from the tiling. The vertices of the remaining tiles constitute the locations of the elements of the truncated aperiodic array. All of the arrays are orientated such that they lie in the x-y plane. Additionally, the array elements are assumed to be ideal isotropic sources with equal amplitude excitations.

A representative portion of a Penrose aperiodic tiling and its corresponding array layout are shown in Figure 4.2. A plot of the peak sidelobe level versus frequency for these arrays is shown in Figure 4.3. The Penrose tiling array is seen to have a larger bandwidth than the periodic array, extending up to $f = 2.4f_0$. Radiation pattern cuts at a few selected frequencies are shown in Figures 4.4 through 4.6. At $f = f_0$ both arrays have excellent sidelobe suppression. Increasing the frequency to $2f_0$ yields grating lobes in the pattern of the periodic array but the Penrose array still exhibits excellent sidelobe suppression. A further increase to $f = 3f_0$ yields both arrays with high lobes in their patterns. Some of the relevant geometrical and radiation properties of the arrays are listed Tables 4.1 and 4.2. The 1381-element Penrose tiling array has approximately 23% fewer elements than its periodic counterpart. A tradeoff for the reduced number of elements in the Penrose array is a reduction in directivity by 1.02 dB. Due to their comparable apertures, the half-power beamwidths (HPBW) of the arrays are nearly identical.
Figure 4.2: Example of an antenna array configuration that was obtained via a Penrose tiling. The antenna element positions are shown as black dots located at the vertices of the tiles.
Figure 4.3: Sidelobe level performance of a 1381-element Penrose tiling array and a 1793-element square-lattice periodic array.

Figure 4.4: Radiation pattern cuts at (a) $\varphi = 0^\circ$ and (b) $\varphi = 90^\circ$ for the 811-element Danzer array, 1391-element Penrose array, and 1793-element periodic array at $f = f_o$. 
Figure 4.5: Radiation pattern cuts at (a) $\phi = 0^\circ$ and (b) $\phi = 90^\circ$ for the 811-element Danzer array, 1391-element Penrose array, and 1793-element periodic array at $f = 2f_o$.

Figure 4.6: Radiation pattern cuts at (a) $\phi = 0^\circ$ and (b) $\phi = 90^\circ$ for the 811-element Danzer array, 1391-element Penrose array, and 1793-element periodic array at $f = 3f_o$. 
Table 4.1: Geometrical properties of the tiling antenna array designs. The average minimum nearest neighbor spacing and maximum nearest neighbor spacing correspond to operation at $f_0$ with a minimum element spacing of $0.5\lambda$.

<table>
<thead>
<tr>
<th>Array Configuration</th>
<th>Number of Elements</th>
<th>$d_{\text{avg}} / \lambda$</th>
<th>$d_{\text{max}} / \lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penrose</td>
<td>1381</td>
<td>0.526</td>
<td>0.818</td>
</tr>
<tr>
<td>Danzer</td>
<td>811</td>
<td>0.525</td>
<td>1.123</td>
</tr>
<tr>
<td>Periodic</td>
<td>1793</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table 4.2: Radiation properties of the tiling antenna array designs. The sidelobe level, HPBW, and directivity values correspond to operation at $f_0$ with a minimum element spacing $0.5\lambda$. The bandwidth ratio denotes the ratio of the approximate upper frequency bound of the operating band (SLL < -10 dB) of the array to the lower frequency bound, $f_0$.

<table>
<thead>
<tr>
<th>Array Configuration</th>
<th>Sidelobe Level (dB)</th>
<th>HPBW (degrees)</th>
<th>Directivity (dB)</th>
<th>Bandwidth Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Penrose</td>
<td>-17.49</td>
<td>2.44</td>
<td>33.42</td>
<td>2.4:1</td>
</tr>
<tr>
<td>Danzer</td>
<td>-17.28</td>
<td>2.45</td>
<td>28.56</td>
<td>2:1</td>
</tr>
<tr>
<td>Periodic</td>
<td>-17.41</td>
<td>2.46</td>
<td>34.44</td>
<td>2:1</td>
</tr>
</tbody>
</table>

A portion of a Danzer aperiodic tiling and its corresponding antenna array geometry are shown in Figure 4.7. The 811-element Danzer array possesses significantly fewer elements than the Penrose and periodic arrays; it has approximately 55% fewer elements than the periodic array. A plot of the sidelobe performance of the Danzer tiling array and the periodic array is shown in Figure 4.8. The Danzer array is seen to have the same 2:1 bandwidth as the periodic array, though its sidelobe level within this bandwidth is 4 dB higher. It has an extremely wide bandwidth that is void of grating lobes. Contrast this with the conventional periodic array, where grating lobes first appear at a minimum element spacing of $\lambda$. It is noted, however, that even though the Danzer array is void of grating lobes over a very wide bandwidth, the peak sidelobe levels are still relatively high.
(i.e., greater than -10 dB). Radiation pattern cuts at a few selected frequencies are shown in Figures 4.4 through 4.6. Geometrical and radiation properties of the array are listed in Table 4.1 and Table 4.2, respectively.

Figure 4.7: Example of an antenna array configuration that was obtained via a Danzer tiling. The antenna element positions are shown as black dots located at the vertices of the tiles.
The analysis in this section has looked at two examples of representative portions of tilings and their corresponding arrays. The characteristics of these tiling arrays are indicative of what would be observed in arrays from other comparable portions of the tilings. In other words, various portions of a tiling tend to generate array layouts with comparable geometrical and performance characteristics. For instance, negligible variations would be observed in the sidelobe level performance of the Danzer array if it had been created from a different portion of the tiling. Similar observations regarding this property of aperiodic tiling arrays were made in [16]. It is worth noting that this property has a propensity to only be associated with moderate to large size arrays; significant variations may be seen among various small tiling arrays.

Figure 4.8: Sidelobe level performance of a conventional Danzer array, perturbed Danzer array that was optimized at a frequency $f = 10f_o$ (corresponding to a minimum element spacing of $5\lambda$), and a conventional periodic array.
4.1.2 Recursive-Perturbation

In their native form, aperiodic tiling arrays are not well suited for wideband applications. The primary limiting factors are that they possess relatively high sidelobes and they lack design flexibility. However, they do possess some desirable attributes such as grating lobe suppression at high frequencies and highly sparse element distributions. These attractive features warranted further investigation into how to exploit the properties of tiling arrays in wideband array design. This subsequently led to the development of a robust design technique that is based on an extension of the basic inflation process. It is straightforward to implement and leads to a marked improvement in the broadband performance that can be achieved through the use of aperiodic tiling based arrays. The remainder of this section discusses the methodology of this design technique.

The design technique is based on a modification to the basic inflation process. It starts off by placing a point within the boundary of each prototile of an aperiodic set. Following this, the locations of these points are preserved within each prototile as the overall tiling is generated via the inflation process. The result of this is the formation of an aperiodic tiling that contains an additional point within each of its constituent tiles. Converting this to an antenna array yields elements at the vertices of the tiling along with an element at each additional point location within the tiles. The layout of the array can be scaled to have a desired minimum element spacing and then truncated to fit within a desired aperture.

The application of this perturbation process to the Danzer aperiodic set is illustrated in Figure 4.9. An additional point is randomly placed within the boundary of each of the three Danzer prototiles (Figure 4.9a). An iteration of the inflation process is applied to one of the Danzer prototiles (Figure 4.9b). Note that by observing the matching conditions of the tiles it should be apparent that the locations of the points within the tiles are preserved as the inflation process is applied. This is also the case for higher stages of the inflation process (e.g., Figure 4.9c). An example of a truncated portion of a Danzer tiling and its corresponding array geometry is shown in Figure 4.10. From this figure it is easily seen that the perturbed tiling array is also aperiodic.
Figure 4.9: (a) Prototiles of the Danzer aperiodic set shown with their edge matching conditions. Example perturbation point locations are designated by filled circles. (b) First stage of the inflation process for one of the prototiles. The edge matching conditions are shown along with the perturbation point locations. (c) Second stage of the inflation process for one of the prototiles and the corresponding perturbation point locations.
One of the novelties of this recursive-perturbation scheme lies in its ability to generate arbitrarily large planar arrays based on the specification of only a small set of parameters. Moreover, adjustment of the parameters leads to very diverse array configurations. These capabilities result from an effective use of the iterative inflation process and matching conditions of an aperiodic set in the perturbation technique. The inflation process allows a number of perturbation elements to be efficiently distributed throughout an aperiodic tiling. Also, this iterative construction means that the adjustment of a single perturbation location has an impact on the locations of many elements in the overall array. Accordingly, adjustment of a few perturbation locations leads to a variety of array layouts with a range of radiation characteristics. Since they are based on large aperiodic tilings, the resulting arrays can also take on a number of different aperture shapes and sizes, depending on design specifications. The traditional perturbation approaches do not come close to offering all of these desirable features [14].

The perturbation scheme is not limited to the basic method that has been presented thus far. One extension of the basic scheme involves increasing the number of perturbation elements that are added to each prototile. This simple modification allows
for more variation in the geometry of the perturbed arrays while still allowing them to be designed using a small set of design parameters. An illustration of double-point perturbations applied to the Danzer aperiodic set is shown in Figure 4.11. Two points are randomly added to each Danzer prototile (Figure 4.11a). The locations of these points are preserved within each tile during the first (Figure 4.11b) and second (Figure 4.11c) stages of the inflation process. Figure 4.12 shows a representative portion of a perturbed tiling array that was generated using these double-point perturbation locations.

Figure 4.11: (a) Prototiles of the Danzer aperiodic set shown with their edge matching conditions. Example perturbation point locations are designated by filled circles. (b) First stage of the inflation process for one of the prototiles. The edge matching conditions are shown along with the perturbation point locations. (c) Second stage of the inflation process for one of the prototiles and the corresponding perturbation point locations.
Tilings that are generated via the inflation process possess a hierarchical structure that is akin to the scalable features that are associated with fractals. This type of structure is naturally modular, in that a large tiling can be subdivided into several smaller tilings, which in turn, can be subdivided further. From an antenna array viewpoint, modular structures can often be utilized to provide for simplified construction as well as convenient subarray architectures [20], [21]. By judicious selection of a portion of an aperiodic tiling, the potential exists for a highly modular array design that is comprised of a collection of several identical subarrays. If care is taken, modularity can be preserved in designs that are generated using the tiling perturbation technique introduced in this paper. Along with this modularity comes the potential to implement a rapid recursive beamforming algorithm similar to that reported in [20], [21].

Figure 4.12: Example of an array that was created via two perturbation points per prototile and its corresponding tiling. Note that the array is comprised of elements at the vertices of the tiling and additional elements at the locations of the perturbation points. The element locations correspond to those shown in Figure 4.7.
4.1.3 Design Process

The effects of various perturbation locations on the radiation properties of perturbed tiling arrays are not intuitive. One possible method for designing these arrays might involve setting up positional grids within the prototiles and then manually searching via a trial-and-error based procedure. This may produce satisfactory results but it would prove to be very tedious and time consuming, with no guarantee that the optimal positions will be found. The method that will be used in this dissertation couples the perturbation technique with a global-search optimization algorithm. This approach is robust and it proved to be much more effective than a manual-based procedure. The particular algorithms that were used to generate the designs in Chapter 5 were the GA and the NDSA.

The number of design parameters in a perturbed tiling array corresponds to the number of prototiles in its aperiodic set and the number of perturbation points that are added to each prototile. Each perturbation point has an associated (x,y) coordinate within a prototile. Therefore, the total number of parameters in a single-point perturbation array corresponds to the twice the number of tiles in its aperiodic set. Likewise, the number of parameters in a double-point perturbation array is equal to four times the number of prototiles.

The role of the optimization algorithm in the design process was to determine the set of parameters that minimize a specified design objective. All of the examples that will be presented in Chapter 5 share the common design objective of trying to maximize sidelobe suppression at a specified frequency. This frequency typically corresponded to the upper bound frequency on a targeted bandwidth. The visible region of the array’s radiation pattern at this upper bound contains information about the sidelobe levels at all lower frequencies. Therefore, this ensures that all sidelobes at lower frequencies will be equal to or less than those of the optimized upper bound frequency. The cost function that corresponds to this design objective is simply equal to the sidelobe (or grating lobe) level at the upper bound frequency. Evaluating the cost function requires calculating the
radiation pattern of an array over its entire visible region. The array factor is then normalized and its peak sidelobe level is determined.

4.2 Application to Peano-Gosper and Generalized Gosper Curves

4.2.1 Arrays Based on the Peano-Gosper Curve

Peano-Gosper arrays are part of a relatively new family of fractal arrays, known as fractile arrays, that was introduced in [20], [21]. A fractile array is defined to be any array with a boundary contour that forms fractal tiles, or fractiles, which are capable of tiling the plane. This type of fractal array differs from other traditional fractal arrays; such as those reported in [18], [19], [73], which have regular boundaries with elements distributed in a fractal pattern on the interior of the array. The tiling capability of fractile arrays leads to a convenient modular subarray architecture whereby each subarray could be individually controlled to support simultaneous multibeam and multifrequency operation. The modular architecture also has the potential to being exploited to simplify the design and construction of the feed structure of these arrays. This section focuses on a specific type of fractile array that is based on the Peano-Gosper space-filling curve [20], [28], [36], [74]. Elements of the array are uniformly distributed along curve, which leads to a planar configuration with an equilateral triangular lattice on its interior and an irregular closed Koch fractal curve around its perimeter. Examples of the first three stages of the PG curve and the corresponding array element locations are shown in Figure 4.13. The Koch fractal boundary plus its interior form a Gosper island that can be used to cover the plane. Gosper islands are self-similar and can be divided into seven smaller tiles, each representing a scaled copy of the original. A useful feature of PG arrays (PGFA) is that their hierarchal structure can be exploited to develop an iterative
construction scheme and a recursive formulation for the rapid calculation of far-field radiation patterns [20], [72].

Unlike conventional square-lattice periodic arrays, the radiation pattern of a PG array has excellent sidelobe suppression and is void of grating lobes even when its minimum element spacing ($d_{\text{min}}$) is increased to $\lambda$. The grating lobe suppression at this spacing is due to the equilateral triangle element distribution [75] that makes up its interior. While the PG array is void of grating lobes during broadside operation at $d_{\text{min}} = \lambda$, grating lobes develop when its main beam is steered from broadside. This issue can be clearly illustrated by observing, for example, the radiation pattern of a stage-3 PG array with its main beam directed toward broadside compared to the radiation pattern when its main beam is steered away from broadside. Figure 4.14 contains a contour plot of the array factor for a uniformly excited stage-3 PG array with $d_{\text{min}} = \lambda$ and a broadside directed main beam. The array factor is plotted against the axes $u$ and $v$, where $u = \sin\theta \cos\phi$ and $v = \sin\theta \sin\phi$. The extent of the contour plot is limited to the visible region of the array, which corresponds to the region bounded by the equation $u^2 + v^2 \leq 1$. The contour plot illustrates that there are no grating lobes present in the visible region of the array for the broadside case. However, they appear when the main beam is steered away from broadside. Figure 4.15 illustrates a case where two grating lobes appear when the main
beam of the array is steered to $\varphi = 0^\circ$, $\theta = 30^\circ$. Similarly, grating lobes are present under
the same conditions for all other stages of the PG array. This analysis assumes that the
arrays are comprised of isotropic sources.

Figure 4.14: Contour plot of the normalized radiation pattern for a stage-3 Peano-Gosper
array at a minimum spacing $d_{\text{min}} = \lambda$ with the main beam directed towards broadside.
4.2.2 Arrays Based on Generalized Gosper Curves

Similar to the Peano-Gosper curve, generalized Gosper curves provide an excellent framework for planar antenna array layouts. The construction of arrays based on these curves follows the same procedure that was outlined in Section 4.2.1 for the Peano-Gosper array; it involves uniformly distributing elements along the length of the curve. The entire structure is then scaled to meet a desired minimum element spacing, which corresponds to the distance between adjacent elements along the curve. Placement of elements along the curve in this manner results in an element distribution with an
equilateral triangular lattice on its interior that is bounded by an irregular curve around its perimeter. The particular perimeter curve that bounds the array is dependent on the order and stage of its corresponding curve. The number of elements in a generalized Gosper curve array is dependent on the order, \( N \), of its corresponding generating curve and its stage of growth, \( P \). The number of elements is given by

\[
\text{Number of elements} = N^P + 1
\]  

(4.1)

One of the useful features of these arrays is that their element distributions can be iteratively constructed to any arbitrary stage of growth based on a set formula for scaling, translating, and rotating stage-1 arrays. The iterative construction of Gosper arrays allows for the development of a recursive array factor formulation for rapid pattern analysis. The formulation is based on calculating the radiation pattern of stages of the array through the pattern multiplication of stage-1 arrays. It offers a significant decrease in the computation time required to analyze the radiation properties of these arrays. This computation reduction, and likewise the reduced simulation time, becomes more pronounced as the arrays under consideration increase in order and stage. Specific details regarding the implementation of the recursive formulation can be found in the Appendix at the end of this section. As it was described in [20], [72], the current distribution that results from the recursive formulation consists of unity amplitude excitations at the two end points of the Gosper curve and excitations with a magnitude of two on all interior elements. To simplify discussion, this particular almost uniform current distribution shall be referred to as uniform throughout this dissertation.

Some representative examples of Gosper arrays are shown in Figure 4.16. As it can be seen in this figure, a variety of array layouts are generated by the various Gosper curves. Due to the space-filling property of the curves, each Gosper array has an interior distribution of elements that falls on an equilateral triangular-lattice. While they share the same interior lattice distribution, a variety of boundary contours and array sizes are produced by different orders of the curves.
Figure 4.16: Stage-2 (a) Peano-Gosper, (b) N=13, (c) N=19, (d) N=31a, and (e) N=31b-1 curves and their corresponding array element locations.
Generalized Gosper arrays tend to exhibit excellent performance in terms of bandwidth and sidelobe suppression. Unlike conventional square-lattice periodic arrays, they do not suffer from the appearance of grating lobes within the visible region of their radiation pattern at a minimum element spacing of $\lambda$. This attribute results from the triangular-lattice that makes up the interior of Gosper arrays. Additionally, their irregular boundary contours typically lead to greater sidelobe suppression than comparable triangular-lattice arrays with a rectangular boundary contour. The boundary contours also provide for a convenient modular architecture whereby the overall array can be subdivided into a collection of identical subarrays. This type of architecture has the potential for supporting operations as a single array or as a collection of smaller, identical tiled subarrays. Additional details regarding this subarray structure will be provided in the following section.

Representative pattern cuts of stage-2 $N=13$ and $N=19$ Gosper arrays at two operating frequencies are shown in Figure 4.17 and 4.18, respectively. At $f = f_o$ the arrays have an element spacing of $0.5\lambda$ and at $f = 2f_o$ the spacing is $\lambda$. In these figures, as well as for all of the other radiation patterns in this dissertation, it is assumed that the arrays consist of isotropic sources with uniform amplitude excitations (unless otherwise noted). At these frequencies the arrays are seen to exhibit no grating lobes over both cuts. The $N=19$ array in particular is seen to have excellent sidelobe suppression. Over its visible region, it has a peak sidelobe level of $-16.6$ dB for operation from $f = f_o$ to $f = 2.25f_o$. The radiation properties of the stage-2 $N=19$ Gosper array are comparable to those of the stage-3 Peano-Gosper ($N=7$) array. The particular stages were selected such that there is a comparable number of elements in the arrays: 344 elements for the stage-3 PG and 362 elements in the stage-2 $N=19$. Both arrays have excellent sidelobe suppression up to $f = 2.25f_o$. Up to this frequency, the sidelobe level in the visible region of the $N=19$ array is approximately $0.5$ dB lower than that of the PG array. Pattern cuts of the arrays at two different frequencies are shown in Figure 4.19. A contour plot of the radiation pattern of the stage-2 $N=19$ Gosper array at $f = 2f_o$ is shown in Figure 4.20.
Figure 4.17: Normalized radiation pattern cuts at (a) $\varphi = 0^\circ$ and (b) $\varphi = 90^\circ$ for a stage-2 $N=13$ Gosper array at $f = f_o$ and $f = 2f_o$.

Figure 4.18: Normalized radiation pattern cuts at (a) $\varphi = 0^\circ$ and (b) $\varphi = 90^\circ$ for a stage-2 $N=19$ Gosper array at $f = f_o$ and $f = 2f_o$. 
Figure 4.19: Normalized radiation pattern cuts at $\varphi = 0^\circ$ for stage-2 $N=19$ and stage-3 $N=7$ Gosper arrays at $f = f_0$ and $f = 2f_0$. 
Similar to PG arrays, Gosper arrays possess limited beam steering capabilities at $f = 2f_0$ due to the presence of grating lobes in their radiation pattern. This issue is clearly illustrated through the contour plot shown in Figure 4.21. In this figure, the main beam of the stage-2 N=31 array operating at $f = 2f_0$ is directed to $\phi = 0^\circ$, $\theta = 30^\circ$. At this beam steering angle there are two grating lobes present in the visible region of the array. Similar performance is exhibited by other orders and stages of Gosper arrays.

Figure 4.20: Contour plots of the normalized radiation pattern for the stage-2 N=19 Gosper array at $f = 2f_0$ with its main beam directed towards broadside.
4.2.3 Recursive-Perturbation

It the past two sections, arrays based on Peano-Gosper and generalized Gosper space-filling curves have been examined. They have been shown to exhibit a number of favorable characteristics including low sidelobes, modular architectures, and the ability to employ recursive pattern formulations. Their 2.25:1 bandwidth is larger than that of a conventional square-lattice periodic array and for many applications it is more than adequate. However, this bandwidth is insufficient for others that require wider bandwidths and enhanced beam steering capabilities. The design technique that is

Figure 4.21: Contour plots of the normalized radiation pattern for the stage-2 N=19 Gosper array at $f = 2f_0$ with its main beam directed towards $\phi = 0^\circ$, $\theta = 30^\circ$. 
discussed in this section addresses these issues by significantly extending the bandwidth of Peano-Gosper and generalized Gosper arrays. One of the appealing features of the technique is that very large, wideband planar arrays can be designed using only a small set of design parameters.

The primary limiting factor on the bandwidth and beam steering capabilities of Peano-Gosper and Gosper arrays is the triangular-lattice that makes up their interior. The design technique that will be discussed here is based on altering this interior lattice. This has the effect of disrupting the unwanted constructive interference that accounts for the formation of grating lobes, thus extending the usable bandwidth of the arrays. The design technique accomplishes this by introducing perturbations to the locations of the antenna elements that are uniformly distributed along the curve. For large array layouts this would require the adjustment of many element positions, ranging from dozens up to thousands. Rather than adjusting the location of every element in the array, the design technique simply perturbs the locations of elements along a stage-1 generating curve and then uses these locations to recursively generate higher stages of arrays. The effect of the iterative construction is that the perturbation of an element location along the stage-1 curve considerably alters the layout of higher stages. Consequently, adjustment of several elements along the curve can lead to a wide range of planar array geometries and associated radiation characteristics. Traditional perturbation techniques require adjusting the position of every element in an array based on an initial periodic lattice \[13\], \[14\], \[76\]. For planar arrays this process becomes very challenging due to various issues such minimum spacings between elements and increased design complexity due to a large number of design parameters. The novelty of the recursive-perturbation technique lies in its ability to generate large broadband planar arrays based on the displacement of only a small number of elements along a simple 1-D generator curve. Dealing with a small set of element positions along a curve significantly lessens issues involving design complexity and element spacings. It also provides for a tractable design problem that is easily handled by an optimizer-based approach. Additionally, perturbed arrays retain some of the beneficial properties of Peano-Gosper and Gosper arrays, including
convenient modular architectures and the ability to utilize recursive construction and pattern formulations.

An illustration of the perturbation process applied to a Peano-Gosper array is shown in Figure 4.22. The left side of the figure shows an example of element locations that are slightly shifted along the curve from their original position (see Figure 4.13). In order to simplify the perturbation process, only the locations of the six interior elements along the stage-1 curve are modified, while the two endpoints remain fixed. The element locations long the stage-1 curve are then used to generate higher stages of arrays. Stage-2 and stage-3 arrays that were generated using the perturbation locations are shown in parts b and c of Figure 4.22. Note that the perturbed arrays still contain the same number of elements as their unperturbed counterparts depicted in Figure 4.13.

![Figure 4.22: (a) Example of a stage-1 generating array with perturbed element locations. The dashed line denotes the initiator curve and the polar coordinates of the 7th element referenced to the center of the initiator are denoted by r\(_7\) and \(\varphi_7\). (b) Stage-2 and (c) stage-3 arrays that were generated using the element locations of the perturbed stage-1 generating array.](image)

An illustration of the modularity of a perturbed Peano-Gosper array is shown in Figure 4.23. In this figure, seven perturbed stage-3 arrays are arranged to form a single stage-4 array. The elements of the stage-3 arrays are shown superimposed on the stage-4 PG curve. To highlight the subarray architecture, each stage-3 array is shown with its
associated Gosper island bounding curve [36]. Note that the curve that bounds the resultant stage-4 array is also a Gosper island.

Figure 4.23: Construction of a stage-4 array via a tiling of seven perturbed stage-3 Peano-Gosper fractile subarrays. The element locations are superimposed on the stage-4 Peano-Gosper curve and each stage-3 array is bounded by Gosper islands.

Application of the perturbation technique to generalized Gosper arrays follows the same basic procedure that was outlined for Peano-Gosper arrays; endpoint generator
elements remain fixed while the interior elements are permitted to be adjusted along the curve from their initial positions. Due to their larger generators, generalized Gosper arrays possess a greater number of perturbation locations than Peano-Gosper arrays. The number of perturbation elements corresponds to the size of the generator curve. It is equal to one less than the order of the generator. For instance, there are 18 perturbation locations in the case of the N=19 generator. An illustration of the perturbation process applied to an N=19 Gosper array is shown in Figure 4.24. On the left side of the figure, the interior elements along the N=19 generator are shifted along the curve from their initial positions (see Figure 4.16). The locations of elements are preserved in the iterative construction of the stage-2 array that is shown on the right side of the figure.

![Figure 4.24](image)

Figure 4.24: (a) Example of an N=19 Gosper generating array with perturbed element locations. The dashed line represents the initiator curve and polar coordinates of the 6th element referenced to the center of the initiator are denoted by $r_6$ and $\phi_6$. (b) Stage-2 array that was generated using the element locations of the perturbed generating array.

The design of perturbed Peano-Gosper and Gosper arrays requires the proper selection of perturbation element locations subject to design constraints and objectives. One possible way to represent the perturbations is as a two-coordinate (e.g. \{x,y\} or \{r,\theta\}) translation from their original locations in the plane of the array. However, this
would require two variables to specify each perturbation location. An alternate representation, and the one that will be utilized here, specifies the perturbation locations as translational offsets from their original locations along the length of the generator curve. The convention that was used to represent translational offsets designates a negative shift as a translation in the element location toward the starting point (element #1) of the stage-1 curve and vice versa (see Figure 4.22 and 4.24). Additionally, the offsets were normalized such that a maximum shift along the entire length of a segment of the stage-1 curve corresponds to unity. This representation along with the hierarchical structure of the space-filling curves essentially allows the entire geometry of various stages of perturbed arrays to be completely specified by only a small set of parameters.

An important consideration in antenna array design is the spacing between neighboring elements. For practical considerations, it is assumed that the array examples that are presented in Chapter 6 have a minimum element spacing of $\lambda/2$ at their lowest operational frequency $f_o$ (with corresponding wavelength $\lambda$). Accordingly, at higher frequencies this spacing increases in terms of its electrical length. For instance, at a frequency of $2f_o$ the minimum spacing between elements increases to $\lambda$. In order for the perturbation technique to be applied, it is necessary to begin with an unperturbed array that has an element spacing that is greater than that of the targeted design frequency. This allows the elements to be perturbed from their initial locations while not violating the targeted minimum element spacing constraint. For simplicity, an initial unperturbed element spacing of $2d_{\text{min}}$ was selected for all of the design examples, where $d_{\text{min}}$ corresponds to the minimum element spacing at the targeted design frequency. If the element locations are permitted to be translated along more than half of a segment length, then there is a high probability that that randomly generated arrays will have unacceptable minimum element spacings (i.e. $<1\lambda$). However, in order to allow for diversity in the permissible array configurations, perturbations of up to 75% of the length of a segment are permitted.

The perturbation technique that is presented in this paper has the advantage of naturally generating element distributions that are relatively sparse. This is a direct result of beginning the design process with a uniformly spaced array with $2d_{\text{min}}$ and then
perturbing the elements while maintaining a minimum spacing of at least $d_{\text{min}}$. This leads to designs with apertures that are highly thinned compared to conventional square- and triangular-lattice distributions (with the same minimum element spacing). These sparse distributions have the potential to impact several important aspects of an antenna array design, including cost, mutual coupling, beamwidth, bandwidth, and even the efficiency of heat dissipation [1], [2].

### 4.2.4 Design Process

Even though the recursive perturbation technique reduces the design of these arrays to a manageable set of parameters, their design would still prove to be a challenging task for a manual (i.e., trial-and-error) procedure. A much more robust and efficient approach based on optimization algorithms was utilized to generate the designs that will be presented in Chapter 6. Specifically, the process was carried out using a hybrid optimizer that combined a genetic algorithm (GA) with a Nelder-Mead Simplex optimizer. The principal component of the hybrid optimizer is the GA due to its ability to navigate through complex solution spaces with a large number of design parameters. The function of the GA is to provide the global search in the initial phase of the optimization process. Once the GA appears to locate the area of the global minimum (typically signified by convergence of the cost function) the best parameter set from the GA is used to seed the Nelder-Mead simplex optimizer. The simplex optimizer, which is particularly adept at local optimization problems, further tailors the parameter set in an effort to pinpoint the exact global minimum of the solution space. For the designs considered here, this combination proved to be much more efficient at quickly converging towards the global minimum than by using solely the GA.

The GA represents the perturbed arrays as chromosomes that consist of a string of 12-bit genes. The number of genes in the chromosome corresponds to the number of perturbation elements in the array. Whenever the cost function is evaluated, it is necessary to decode each binary gene of the chromosome into floating point form and
then scale and translate the value to be within the acceptable range of the perturbation offsets. Optimizations typically utilize a population of 30 members that is evolved through approximately 200 generations or until a desired level of convergence is obtained. Fifty percent of the population is carried over to the next generation, while new population members are generated using tournament selection and single-point crossovers. The optimization solution space is explored by adding diversity to the population through bit mutations at a rate of 8%. Since a particular minimum element spacing is targeted during the optimization process, it was necessary to add a routine that takes this into account. Arrays with a spacing less than the desired $d_{\text{min}}$ are deemed unacceptable and their cost function is assigned a very poor value. Arrays that do not fall into this category are evaluated in a normal manner using the cost function. Thus, a cutoff in the minimum element spacing is specified but the optimized designs may have a spacing that is equal to or greater than this value. There is one additional caveat to this routine; if a perturbed element is too close to an endpoint of the stage-1 curve, its location is set to be at the midpoint of its corresponding segment of the stage-1 curve. The seeding of the initial GA population also takes the minimum element spacing of an array into account to ensure that the GA begins with members that satisfy the spacing condition.

In Chapter 6, perturbed Peano-Gosper arrays were designed with the objective of extending their beam steering capabilities at a frequency $2f_o$, corresponding to an element spacing of $\lambda$. The desired coverage area for beam steering was a $30^\circ$ conical volume that is centered at broadside, shown in Figure 4.25. To meet this objective, the cost function was set equal to the maximum sidelobe or grating lobe level (in dB) that appears if the main beam is steered throughout the targeted volume. Hence, the goal of the hybrid optimizer was to evolve the perturbed array geometries towards minimizing this function. Evaluation of this particular cost function can be accomplished in several ways. One method involves determining the sidelobe levels of the array as its main beam is steered through a number of azimuthal angles. However, in order to ensure that the peak sidelobe level is not missed it is necessary to sample the array factor at many closely spaced azimuthal angles, especially for larger arrays. A much more efficient approach
involves determining the sidelobe levels that appear in the visible region of the array during broadside operation as well as the sidelobe levels that appear in a portion of $(u,v)$ space that extends past the visible region [77], [78]. Knowledge of the peak sidelobe level within these regions is sufficient to determine the peak level that would appear if the main beam of the array was steered throughout the targeted scan volume. The 30° conical scan volume has a corresponding expanded visible region that is bounded by $u^2 + v^2 \leq 1.5$ (see Section 3.4). The peak sidelobe level within this expanded region corresponds to the peak level that appears if the main beam of the array is steered throughout the targeted scan volume.

![30° Conical Scan Volume](image)

Figure 4.25: 30° conical scan volume that is targeted in the design of the perturbed Peano-Gosper arrays. Beam steering is permitted anywhere within this scan volume. The dashed line shows a beam steering direction that is at the extent of this volume.

In the case of the perturbed Gosper arrays, the design objective was to determine the extent to which the bandwidth of the designs can be enhanced for broadside operation. Although different than that of the PG arrays, the overall aim in both cases was to maximize the visible region over which the sidelobes of the array are well maintained. In the case of the Gosper arrays this is accomplished by minimizing the sidelobes of the array at a frequency that corresponds to an upper bound on a targeted bandwidth. Optimizing at the upper bound ensures that the sidelobes are maintained at or below the
optimized level for all lower frequencies. A number of optimizations were carried out for
different targeted bandwidths and for different orders and stages of perturbed Gosper
arrays. In each case, the bandwidth and sidelobe level performance of the optimized
designs far exceeded those of the optimized Peano-Gosper arrays. The bandwidth of
these designs can also support beam steering over a very wide bandwidth. Each example
that will be shown in Chapter 6 corresponds to the array (of a particular order and stage)
that had maximum bandwidth after being optimized.

Appendix: Recursive Array Factor Formulation for
Perturbed Peano-Gosper and Gosper Arrays

The hierarchical structure of Peano-Gosper and generalized Gosper arrays allows for the
development of an efficient recursive array factor formulation that is based on using
pattern multiplication of stage-1 generator arrays. In [20], [72], a recursive formulation
was presented for calculating the array factor of PG arrays. Through an effective use of
pattern multiplication [4] offered a significant reduction in the simulation time for
analyzing these arrays. The formulation is limited to standard, unperturbed PG arrays
and had to be generalized in order for it to be applicable to perturbed PG arrays as well as
perturbed Gosper arrays. The details of the generalized formulation are provided below.

The scale factor, $\delta$, is defined as the ratio of the length of the initiator to the length of
a segment in the stage-1 curve. The angle $\alpha$ corresponds to the acute angle formed
between the first segment of the stage-1 curve and the initiator (see Figures 4.22 and
4.24). Both $\delta$ and $\alpha$ are dependent on the order of the Gosper array under consideration.
The values of these variables for the N=7 PG array and the N=13 and N=19 Gosper
arrays are:

$$\alpha_s = \arctan\left(\sqrt{3}/5\right)$$  \hspace{1cm} (4.2)
\[ \delta_i = \frac{\sqrt{3}}{2 \sin \alpha} \quad (4.3) \]

\[ \alpha_{i3} = \arctan \left( 2\sqrt{3} \right) \quad (4.4) \]

\[ \delta_{13} = \frac{2\sqrt{3}}{\sin \alpha} \quad (4.5) \]

\[ \alpha_{19} = \arctan \left( \sqrt{3}/4 \right) \quad (4.6) \]

\[ \delta_{19} = \frac{\sqrt{3}}{\sin \alpha} \quad (4.7) \]

With a perturbed stage-1 generator array as the fundamental building block, stages of the PG and Gosper arrays are simply comprised of a collection of three different orientations of generator arrays (see Figure 4.13). The recursive formulation uses pattern multiplication of these orientations of the generator arrays and the underlying hierarchal architecture its associated space-filling curve. Assuming isotropic sources, the array factor of a stage-\( P \) perturbed array (\( P > 1 \)) at a particular far-field location \((\theta, \varphi)\) is conveniently expressed using matrix multiplication as

\[ AF_p(\theta, \varphi) = A_p B_p C \quad (4.8) \]

\[ A_p = \begin{bmatrix} A_{1,p} & A_{2,p} & A_{3,p} \end{bmatrix} \quad (4.9) \]
\[ A_{i,P} = \sum_{m=1,N+1} a_{m,i,P} + 2 \sum_{m=2}^{N} a_{m,i,P} \]  

(4.10)

\[ a_{m,i,P} = \exp\left[ jkr_m \sin \theta \cos \left( \varphi - \varphi_m - \varphi_i + (P-1)\alpha \right) \right] \]  

(4.11)

\[ \varphi_i = (i-1)\frac{2\pi}{3} \]  

(4.12)

where \( x_m, y_m, r_m \) (in wavelengths) and \( \varphi_m \) are associated with the location of the \( m \)th generator element from the center of the initiator (as shown in Figure 4.22). The amplitude excitations have a value of one for the endpoint elements and a value of two for all others. The component of the array factor associated with the structure of the underlying space-filling curve, \( B_p \), is calculated using

\[ B_p = \prod_{q=1}^{P-1} F_q \]  

(4.13)

\[ F_q = \begin{bmatrix} f_{1,i}^{P,q} \end{bmatrix}_{3\times3} \]  

(4.14)

\[ f_{1,i}^{P,q} = \sum_{n \in \mathcal{H}_{i,j}} \exp\left[ jk\delta^q r_n \sin \theta \cos \left( \varphi - \varphi_n - \varphi_i + (P-q-1)\alpha \right) \right] \]  

(4.15)

\[ r_n = \sqrt{X_n^2 + Y_n^2} \]  

(4.16)
where $X_n$ and $Y_n$ correspond to the coordinates (in wavelengths) of the center of the $n^{th}$ segment of the stage-1 curve with origin of the coordinate system assumed to be at the center of the initiator line as illustrated in Figure 4.22. These coordinates can readily be determined using the geometry of the generating curve with the segment length set equal to the desired minimum element spacing between elements. Elements of the matrix $H$ correspond to the orientation of each segment along the generating curve. The matrices of the $N=7$ PG array and the $N=13$ and $N=19$ Gosper arrays are given by (4.19), (4.20), and (4.21), respectively:

$$
\varphi_n = \begin{cases} 
\arctan\left(\frac{Y_n}{X_n}\right), & X_m > 0 \\
0, & X_m = 0 \\
\arctan\left(\frac{Y_n}{X_n}\right) + \pi, & X_m < 0
\end{cases} \quad (4.17)
$$

$$
C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \quad (4.18)
$$

$$
H = \begin{bmatrix} H_n \end{bmatrix}_{(3\times3)} =
\begin{bmatrix}
\{1,3,5,6\} & \{2\} & \{4,7\} \\
\{4,7\} & \{1,3,5,6\} & \{2\} \\
\{2\} & \{4,7\} & \{1,3,5,6\}
\end{bmatrix} \quad (4.19)
$$

$$
H = \begin{bmatrix} H_n \end{bmatrix}_{(3\times3)} =
\begin{bmatrix}
\{1,3,5,8,10\} & \{2,6,7,9,11,13\} & \{4,12\} \\
\{4,12\} & \{1,3,5,8,10\} & \{2,6,7,9,11,13\} \\
\{2,6,7,9,11,13\} & \{4,12\} & \{1,3,5,8,10\}
\end{bmatrix} \quad (4.20)
$$

$$
H = \begin{bmatrix} H_n \end{bmatrix}_{(3\times3)} =
\begin{bmatrix}
\{1,2,6,7,9,11,15,16,17\} & \{5,8,12,13,18,19\} & \{3,4,10,14\} \\
\{3,4,10,14\} & \{1,2,6,7,9,11,15,16,17\} & \{5,8,12,13,18,19\} \\
\{5,8,12,13,18,19\} & \{3,4,10,14\} & \{1,2,6,7,9,11,15,16,17\}
\end{bmatrix} \quad (4.21)
$$
For a particular generating curve, the appropriate forms of $\delta$, $\alpha$, and $H$ from (4.2) through (4.7) and (4.19) through (4.21) should be used in the recursive formulation of (4.8) through (4.18).

The current distribution that results from the recursive formulation [72] consists of unity amplitude excitations on the elements at the two end points of the PG curve and excitations with a magnitude of two on all interior elements. This particular *almost uniform* current distribution is preserved for the perturbed PG and Gosper arrays considered here and shall be referred to as *uniform* throughout this dissertation. In order to take beam steering into account, phase excitations must be incorporated into the recursive formulation. The phase excitations will allow the main beam to be steered to a specified far-field direction, corresponding to $\theta = \theta_o$ and $\phi = \phi_o$. To accomplish this, (4.11) and (4.15) must be generalized in the following way:

\[
a_{m,i,P} = \exp \left[ jkr_m \sin \theta \cos \left( \varphi - \varphi_m - \varphi_i + (P-1)\alpha \right) + j\beta_{i,m,P} \right] \tag{4.22}
\]

\[
\beta_{i,m,P} = -kr_m \sin \theta_o \cos \left[ \varphi_o - \varphi_m - \varphi_i + (P-1)\alpha \right] \tag{4.23}
\]

\[
f_{l,i}^{P,q} = \sum_{n \in H_{i,j}} \exp \left[ jk\delta^q r_n \sin \theta \cos \left( \varphi - \varphi_n - \varphi_i + (P-q-1)\alpha \right) + j\beta_{i,n,q,P} \right] \tag{4.24}
\]

\[
\beta_{i,n,q,P} = -k\delta^q r_n \sin \theta_o \cos \left[ \varphi_o - \varphi_n - \varphi_i + (P-q-1)\alpha \right] \tag{4.25}
\]

In addition to the recursive formulation, it is possible to calculate the array factor of a perturbed array using the more conventional method of sequentially adding the contribution of each element to the array factor using (3.16) and (3.17). The execution
time associated with these two methods was compared to investigate the relative time savings that is afforded through recursion. The calculation of a single cut through the array factor of a perturbed PG array was used as the test simulation for this comparison. The cut was made at an arbitrary angle of \( \theta \) and the array factor was calculated over a 360° azimuthal range with a resolution of 0.25°. Numerous simulations were performed for stages of the PG array and the ratio of the average execution times of the conventional formulation to that of the recursive formulation are listed in . The recursive formulation shows a marked improvement in execution time over the conventional formulation at each high-order stage of the PG array. As expected, the relative time savings of the recursive formulation becomes more dramatic as the stage of the array is increased. On average, the recursive formulation was found to be 156 times faster than the conventional formulation in calculating the array factor of the stage-5 perturbed PG array. It should be noted that the execution times for evaluating the array factor using the conventional formulation did not take into account the time required to generate the perturbed array element distribution. It is expected that the relative time savings listed in Table 4.3 would be much greater if the execution time required to generate the array element distribution was also taken into account.

\[
AF(\theta, \phi) = \sum_{n=1}^{N} I_n \exp\left[j kr_n \sin\theta \cos(\phi - \phi_n) + \beta_n\right] \tag{4.26}
\]

\[
\beta_n = -kr_n \sin\theta_o \cos(\phi_o - \phi_n) \tag{4.27}
\]

where \( r_n \) and \( \phi_n \) are the polar coordinates of the nth element in the array and \( I_n \) is the magnitude of the corresponding element current excitation. The execution time associated with using (22) - (23) was compared with that of (7) - (20) in order to determine the relative time savings that is afforded by the recursive formulation. The calculation of a single cut through the array factor of a perturbed PG array was used as the test simulation for this comparison. The cut was made at an arbitrary angle of \( \theta \) and the array factor was calculated over a 360° azimuthal range with a resolution of
0.25°. Numerous simulations were performed for stages of the PG array and the ratio of the average execution times of the conventional formulation to that of the recursive formulation are listed in Table 4.3. The recursive formulation shows a marked improvement in execution time over the conventional formulation at each high-order stage of the PG array. As expected, the relative time savings of the recursive formulation becomes more dramatic as the stage of the array is increased. On average, the recursive formulation was found to be 156 times faster than the conventional formulation in calculating the array factor of the stage-5 perturbed PG array. It should be noted that the execution times for evaluating the array factor using the conventional formulation did not take into account the time required to generate the perturbed array element distribution. It is expected that the relative time savings listed in Table 4.3 would be much greater if the execution time required to generate the array element distribution was also taken into account.

Table 4.3: Ratio of the average simulation time of the conventional AF formulation to that of the recursive AF formulation for stages of PG arrays.

<table>
<thead>
<tr>
<th>Stage</th>
<th>Ratio of Average Execution Times</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.1</td>
</tr>
<tr>
<td>3</td>
<td>5.3</td>
</tr>
<tr>
<td>4</td>
<td>27.8</td>
</tr>
<tr>
<td>5</td>
<td>156.0</td>
</tr>
</tbody>
</table>
Chapter 5

Aperiodic Tiling Array Designs

In this chapter, the versatility of recursive-perturbation applied to aperiodic tiling arrays is demonstrated through a variety of design examples. In the first section, a design is presented for a wideband array that was generated using the Penrose aperiodic set. The second section presents ultra-wideband arrays that were designed using perturbations of the Danzer aperiodic set. Finally, in the third section the flexibility of the perturbation design technique is explored through multiobjective optimizations. Examples will be shown for arrays that were designed for wideband performance as well as a specific degree of aperture thinning and an ultra-wideband array that was designed for additional sidelobe suppression over a particular portion of its bandwidth.

All of the examples that are presented in this chapter possess the same aperture shape and size. This allows for straightforward comparisons of important characteristics among various designs, such as directivity, half-power beamwidth, and aperture thinning. Since the layouts are simply based on truncated portions of a large tiling, designs with alternate aperture shapes and sizes could readily be generated to meet particular design specifications. All of the examples were also designed such that they have a minimum element spacing of $\lambda/2$ at the lowest intended operating frequency, $f_0$ (with wavelength $\lambda$). The purpose of this element spacing constraint is to lessen the effects of mutual coupling that often arise in arrays with elements that are closely spaced [1], [2], [4]. The analysis in this chapter assumes that the arrays are comprised of isotropic sources with uniform amplitude excitations. All of the single-objective examples represent those arrays that possessed maximum bandwidth via the optimization process. Some of the designs in chapter were originally presented in [27].
5.1 Penrose Tiling Arrays

The first example involves a perturbed Penrose tiling array that was designed for maximum sidelobe suppression at \( f = 5f_o \), corresponding to a minimum element spacing of \( 2.5\lambda \). A single perturbation element was added to each prototile of the Penrose aperiodic set, which accounts for a total of four parameters (x,y location of each perturbation element) that had to be selected. In this particular example, a GA was used to determine the optimal set of perturbation locations that maximize the sidelobe suppression at the targeted frequency. Throughout the optimization process, the perturbed Penrose tiling arrays were scaled and then truncated to satisfy the minimum element spacing and aperture constraints that were discussed at the beginning of this chapter. As a baseline for comparison, the performance of the optimized design will be compared to that of a standard, unperturbed Penrose tiling array. The standard Penrose tiling array that will be used for this analysis is shown in Figure 4.2 and a plot of its sidelobe performance versus frequency is shown in Figure 5.1. At \( f = 5f_o \) the Penrose tiling array has a sidelobe level of –1.45 dB.

The GA was able to find a set of perturbation locations that suppress the sidelobe levels of the perturbed Penrose tiling array to –10.35 dB. This represents a reduction of 8.9 dB from the sidelobe level of the standard Penrose tiling array. A plot of the sidelobe level performance of the single-point perturbed Penrose array, the standard Penrose tiling array, and a 1793-element periodic array is shown in Figure 5.1. Using -10 dB as the sidelobe criteria for the definition of bandwidth, the optimized array has a bandwidth of approximately 5:1. The upper bound on the bandwidth of the array more than doubled via the perturbation process, i.e. from \( f = 2.4f_o \) to \( f = 5f_o \). In addition, the array exhibits moderate sidelobe suppression (< 6 dB) well past the targeted frequency (see Figure 5.1). A representative pattern cut of the array at \( f = 5f_o \) is shown in Figure 5.2. Relevant geometrical and radiation characteristics are listed in Table 5.1.

The layout of the 551-element perturbed Penrose array is shown in Figure 5.3. Although it was not a design constraint, the perturbation process was able to generate a highly thinned element distribution [2], [10], [26], [79]. In addition to having a
significantly wider bandwidth, the perturbed array has 60% fewer elements than the 1381-element Penrose tiling array and nearly 70% fewer elements than the 1793-element periodic array. This can be observed through a comparison of the array layout of Figure 5.3 with those of Figure 4.1 and Figure 4.2. This significant degree of thinning is valuable in a number of facets of array design, such as reducing cost, reducing weight, and improving heat dissipation. For this design, the reduction of elements comes at the expense of a reduction in directivity (see Table 4.2). Some of the examples that are to follow will examine aperture thinning in more detail.

Figure 5.1: Sidelobe level performance of the 551-element Penrose array with single-point perturbations, the 1381-element standard Penrose array, and a 1793-element square-lattice periodic array.
Figure 5.2: Cut of the radiation pattern at $f = 5f_0$ for the Penrose array that was optimized with a single perturbation point per prototile. The right side of the figure shows the array factor as a function of theta plotted from -90 to +90 degrees and the left side of the figure shows a detailed view of the array factor near the main beam.

Figure 5.3: Antenna array configuration of the perturbed Penrose array that was optimized at $f = 5f_0$, corresponding to a minimum element spacing of $2.5\lambda$. 
Table 5.1: Characteristics of the Penrose tiling array that was designed with single-point perturbations. The last four columns correspond to operation at $f = f_0$.

<table>
<thead>
<tr>
<th>Number of Elements</th>
<th>Bandwidth</th>
<th>Peak sidelobe over bandwidth (dB)</th>
<th>Directivity (dB)</th>
<th>HPBW (degrees)</th>
<th>$d_{av}/\lambda$</th>
<th>$d_{max}/\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>551</td>
<td>5:1</td>
<td>-10.35</td>
<td>28.92</td>
<td>2.42</td>
<td>0.633</td>
<td>1.683</td>
</tr>
</tbody>
</table>

5.2 Danzer Tiling Arrays

5.2.1 Single-Point Perturbation Example

The second example involves a perturbed Danzer tiling array that was designed for maximum sidelobe suppression at $f = 10f_0$, corresponding to a minimum element spacing of $5\lambda$. This example utilizes a single perturbation point within each of the three Danzer prototiles; the three perturbation points account for a total of the six design parameters in this example. A GA was used to determine the optimal set of perturbation locations subject to the design objective. A standard, unperturbed Danzer tiling array and a square-lattice periodic array were used as a baseline for comparison with the perturbed array. The layouts of these arrays are shown in Figure 4.7 and Figure 4.1, respectively. Their sidelobe level performance versus frequency is shown in Figure 5.4. At $f = 10f_0$, the Danzer tiling array has a sidelobe level of $-2.2$ dB and the periodic array has a peak lobe level of 0 dB due to the numerous grating lobes in its radiation pattern.

The GA optimization resulted in a design with excellent sidelobe suppression over an ultra-wide bandwidth. At $f = 10f_0$ the sidelobe level was reduced from $-2.2$ dB to $-10.05$ dB and the upper bound on the bandwidth of the array was extended from $f = 2f_0$ to $f = 10.5f_0$. This corresponds to a bandwidth enhancement from 2:1, in the case of the standard Danzer array, to 10.5:1 for the perturbed design. Figure 5.4 provides a plot of
the sidelobe level performance of the optimized design as well as that for the standard Danzer array and the periodic array. In terms of sidelobe suppression, the perturbed design is seen to outperform the standard Danzer array over nearly the entire domain of the plot. A cut of its radiation pattern at the optimized frequency is shown in Figure 5.5 and its element distribution is shown in Figure 5.6. Some of the geometrical and radiation characteristics of the array are listed in Table 5.2.

Figure 5.4: Sidelobe level performance of the 811-element Danzer array with single-point perturbations, the 811-element standard Danzer array, and a 1793-element square-lattice periodic array.
Figure 5.5: Radiation pattern cut at $f = 10f_0$ for the Danzer array with single-point perturbations. The right side of the figure shows the array factor as a function of theta plotted from -90 to +90 degrees and the left side of the figure shows a detailed view of the array factor near the main beam.

Figure 5.6: Antenna array configuration of the perturbed Danzer array that was optimized at $f = 10f_0$, corresponding to a minimum element spacing of $5\lambda$. 
Table 5.2: Characteristics of the Danzer array that was designed with single-point perturbations. The last four columns correspond to operation at $f = f_0$.

<table>
<thead>
<tr>
<th>Number of Elements</th>
<th>Bandwidth</th>
<th>Peak sidelobe over bandwidth (dB)</th>
<th>Directivity (dB)</th>
<th>HPBW (degrees)</th>
<th>$d_{avg}/\lambda$</th>
<th>$d_{max}/\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>811</td>
<td>10.4:1</td>
<td>-10.05</td>
<td>31.26</td>
<td>2.43</td>
<td>0.574</td>
<td>1.172</td>
</tr>
</tbody>
</table>

5.2.2 Double-Point Perturbation Example

The last example possessed the maximum bandwidth that can be attained through the use of single-point perturbations of the Danzer aperiodic set. While this design possesses an ultra-wide bandwidth (>10:1), it was expected that an even greater bandwidth might be attained through the use of additional perturbation elements. The example in this section makes use of two perturbation elements per prototile; this doubles the number of design parameters but it still allows for a manageable optimization problem. In an attempt to extend the upper bound on the bandwidth of the single-point perturbed Danzer array, this optimization was carried out at a slightly higher frequency of $12f_0$.

The double-point perturbation of a Danzer tiling was able to generate an array with a much greater bandwidth than the single-point example. The evolution of sidelobe suppression throughout the optimization process is shown in Figure 5.7. At the design frequency of $12f_0$, the GA discovered a set of perturbation locations that increase the level of sidelobe suppression to $-11.2$ dB. This represents a reduction of 9.2 dB from that of the standard, unperturbed Danzer array. Figure 5.8 shows a plot of the sidelobe performance of the double-point perturbation example. In this plot the array is seen to have a bandwidth that extends well beyond the targeted design frequency, maintaining a sidelobe level below $-10$ dB up to $f = 22f_0$. This ultra-wide bandwidth has an upper bound that is approximately 10 times higher than that of standard Danzer tiling array. And unlike periodic arrays, which possess grating lobes at a spacing of one-wavelength,
the optimized design operates over spacings of $0.5\lambda$ up to $11\lambda$ with excellent sidelobe suppression and no grating lobes. It is even seen to have moderate suppression past $11\lambda$. The geometry of the 431-element optimized design is shown in Figure 5.9. Some of its relevant geometrical and radiation characteristics are listed in Table 5.3.

![Figure 5.7: Evolution of the sidelobe level throughout the optimization of the double-point perturbed Danzer array.](image)
Figure 5.8: Sidelobe level performance of the 431-element perturbed Danzer array that was optimized with two perturbation points per prototile at $f = 12f_o$ (corresponding to a minimum element spacing of $6\lambda$), the standard 811-element Danzer array, and a 1793-element square-lattice periodic array.
The radiation patterns of the optimized design provide some insight into its ultra-wideband performance. Contour plots of its normalized radiation pattern at $f = f_o$ and $f = 20f_o$ are shown in Figures 5.10 and 5.11, respectively (where the array factor is plotted against the axes $u = \sin \theta \cos \phi$ and $v = \sin \theta \sin \phi$). At $f = 20f_o$ the pattern of the optimized design can be described as having a diffused pattern with no high-sidelobe or grating lobe features. Aside from its main beam, there are no regions in its pattern with appreciable constructive interference. A pattern cut of the normalized radiation pattern at this frequency is shown in Figure 5.12a to provide a more detailed view of its structure. The diffused features of the perturbed Danzer array become especially clear when the pattern of Figure 5.11 is compared with that of the 1793-element periodic array.
(Figure 5.13) at the same frequency. As expected, the pattern of the periodic array contains numerous unwanted grating lobes at this frequency. A detailed cut of this pattern is shown in Figure 5.12b.

Figure 5.10: Normalized radiation pattern at $f = f_0$ of the Danzer array that was optimized with two perturbation points per prototile.
Figure 5.11: Normalized radiation pattern at $f = 20f_0$ of the Danzer array that was optimized with two perturbation points per prototile.
Figure 5.12: Cut of the normalized radiation pattern ($\phi = 0^\circ$) at $f = 20f_0$ for the (a) perturbed Danzer array that was optimized with two perturbation points per prototile and the (b) 1793-element periodic array. The right side of the figure shows the array factor as a function of theta plotted from -90 to +90 degrees and the left side of the figure shows a detailed view of the array factor near the main beam.
As it was discussed in Chapter 3, beam steering has the effect of contracting the bandwidth of an array. This reduction in bandwidth becomes quite significant for designs with relatively narrow bandwidths. Such is the case for periodic arrays. For instance, the 2:1 bandwidth of a periodic array during broadside operation is reduced to 1.33:1 for beam steering applications up to $\theta = 30^\circ$. Due to its ultra-wide bandwidth, this effect is much less pronounced in the case of the optimized Danzer tiling array. For instance, the array would still have a 16:1 bandwidth if it were operated for beam steering within a 30° conical volume that is centered at broadside (see Figure 4.25). Extending the scanning to a 60° conical volume would still provide for a design with a 14:1 bandwidth. Figure 5.14 illustrates the sidelobe performance of these two cases. Although the overall bandwidth is reduced, the array is seen to exhibit excellent sidelobe suppression (less than -9 dB) well past its limit.

Figure 5.13: Normalized radiation pattern at $f = 20f_o$ of a 1793-element square-lattice periodic array. The array elements are located within a circular aperture and the minimum element spacing is $10\lambda$. 
In addition to providing ultra-wideband beam steering capabilities, the bandwidth of the optimized array can also be truncated to ensure that mutual coupling between neighboring elements is minimized. For example, the array has an operable bandwidth from \( f = f_o \) (\( d_{\text{min}} = 0.5\lambda \)) up to \( f = 22f_o \) (\( d_{\text{min}} = 11\lambda \)). The array could be operated from \( f = 2f_o \) (\( d_{\text{min}} = \lambda \)) to \( f = 22f_o \) (\( d_{\text{min}} = 0.5\lambda \)) without an appreciable loss of bandwidth while guaranteeing that the minimum element spacing is never less than \( \lambda \). In such a case, the array elements would not be strongly coupled at the lowest operating frequency. The process of bandwidth truncation is not limited to being applied to the double-point perturbation design; it could be applied to all of the designs that are presented in this chapter. Alternatively, rather than using this approach, a minimum element spacing of \( \lambda \).
(or some other spacing) at the lowest intended operating frequency could simply be incorporated as a constraint in the design process.

As it was discussed earlier, an interesting aspect of this array design approach is the significant degree of aperture thinning that is achieved through the use of aperiodic tiling geometries. Compared to their periodic counterparts, arrays based on aperiodic tilings are naturally thinned (i.e. their mean minimum element spacing is greater than $\lambda/2$). Moreover, a side effect of the recursive-perturbation design technique is the potential to further thin aperiodic tiling array apertures. Although it was not included as a design objective, significant aperture thinning was observed in two of the optimized designs. First, the single-point perturbed Penrose array is comprised of 551 elements. Compared to the baseline Penrose aperiodic array (with 1381 elements) the number of elements in the optimized array is reduced by 60%, whereas compared to the 1793-element periodic array this represents a reduction of 69%. Secondly, the double-point optimized Danzer array contains a total of 431 elements. Compared to its baseline Danzer array and periodic counterpart, the number of elements has been thinned by 47% and 76% respectively. These thinned apertures are achieved at the expense of directivity as seen in Tables 4.2 and 5.3.

All of the examples in this chapter were designed to have a similar circular aperture. This allowed for straightforward comparisons among their various characteristics, such as directivity and half-power beamwidth. One of the central features of the aperiodic tiling perturbation technique is that it allows for the design of arrays with apertures of arbitrary shape and size. This is due to the array distributions being based on truncations of large aperiodic tilings. Rather than truncating them to a circular aperture, the arrays could have been truncated to have a rectangular aperture, elliptical aperture, or some other aperture shape. Also, by adjusting the size of the truncation they can be designed to have a larger or smaller aperture. Alternatively, rather than performing an additional optimization, the perturbation locations from an existing design could be simply be used to generate an array distribution with an alternate aperture, which would still retain excellent performance characteristics. For instance, the optimized double-point perturbation locations were used to generate an array that has a circular aperture with a radius of $16\lambda$.
at \( f = f_0 \). This represents an increase of more than 75\% in aperture area and the number of elements over the optimized design with a radius of \( 12\lambda \). This larger design exhibits a narrower beamwidth and comparable sidelobe suppression over a 22:1 bandwidth.

### 5.3 Multi-Objective Optimization

The utility of the recursive-perturbation design technique has been explored through some design examples in the past two sections. It has been shown to be very effective at designing ultra-wideband planar arrays with highly sparse apertures. In this section the efficacy of the perturbation design technique is further explored by investigating its use in multiobjective design. In particular, two cases will be examined. The first case involves designing array layouts that have suppressed sidelobes over a wide bandwidth as well as a specified number of elements within a given aperture. The second case involves tailoring the sidelobe performance of perturbed tiling arrays through the use of multi-frequency optimizations.

#### 5.3.1 Targeting Aperture Thinning and Sidelobe Suppression

The examples that have been presented thus far have demonstrated the effectiveness of the recursive-perturbation technique in generating ultra-wideband arrays with highly sparse distributions. Since sidelobe suppression was the sole objective in the design of these arrays, the thinned apertures that resulted were merely a side effect of the design process. It would be desirable to have a design tool that is capable of not only generating arrays with wideband performance but also offering control over the number of elements that make up the arrays. The wide range of aperture thinning that was observed in the single-objective optimizations indicated that the number of elements could be effectively
incorporated as a parameter in the design process. Additionally, since very wide bandwidths were readily achieved, sacrificing some bandwidth at the expense of a multiobjective optimization would most likely still result in very wideband arrays. The objectives of the design examples that will be presented are to suppress sidelobes at a specific frequency and to try to achieve a specific number of elements within a given aperture. It should be noted that the multi-criteria approach is not limited to these particular objectives; the flexibility of the perturbation technique could be extended to other objectives, such as designing for a targeted level of directivity.

The multiobjective design was applied to Danzer tiling arrays with single-point perturbations. The goal of the NSGA was to determine the optimal set of perturbation locations subject to the two objective functions

\[ F_1 = SLL \text{ (dB)} \]  \hspace{1cm} (1)
\[ F_2 = |N - N_t|/N_t \]  \hspace{1cm} (2)

where \(SLL\) is the peak sidelobe level of the array at the targeted frequency, \(N\) is the number of elements in an array under consideration, and \(N_t\) is the desired number of array elements. The targeted optimization frequency was selected to be \(6f_0\). Note that this is lower than the frequency that was used in the single-point perturbation example of Section 5.2.1. A lower frequency was chosen because it was expected that some bandwidth might have to be sacrificed in order to provide the ability to optimize for a specific number of elements in the array. Several multiobjective optimizations were carried out, each with a different number of targeted elements: \(N_t = 200, 400, 600, 800, 1000\). In each case, sidelobe suppression below –10 dB and excellent agreement with the targeted number of elements were achieved. The following discussion will provide details of two selected optimization examples.

The targeted number of elements in the first example is 400. The final Pareto front from the NSGA optimization is shown in Figure 5.15. The solutions along the Pareto front offer a range of tradeoffs in design objectives. The solution that came closest to the targeted number of elements while maintaining low sidelobes is shown in Figure 5.16a. The design consists of 398 elements, representing a deviation of only 0.5% from the targeted number.
of elements. At the optimized frequency the array has a sidelobe level of $-10.35\,\text{dB}$. A plot of its sidelobe performance is shown in Figure 5.17. The bandwidth of the array is seen to extend past the targeted frequency, up to $f = 7.35f_o$. At the lowest intended operating frequency ($f_o$) the array has a directivity of $25.9\,\text{dB}$.

The targeted number of elements in the second example is 600. The solution that appeared to offer the best tradeoff between the design objectives is shown in Figure 5.16b. It is comprised of 604 elements, corresponding to a deviation of less than $0.7\%$ from the targeted number. A plot of its sidelobe performance is shown in Figure 5.17. The array has a SLL of $-10.57\,\text{dB}$ at $f = 6f_o$ and its bandwidth extends up to $f = 6.8f_o$. As expected, this design has a greater directivity than the 398-element array. Geometrical and radiation characteristics of the two designs are listed in Table 5.4.

![Figure 5.15: Pareto front of the NSGA multiobjective optimization that was carried out on a perturbed Danzer array. The objectives of the design were to minimize the sidelobe level at $f = 6f_o$ and to attain a design with approximately 400 elements.](image-url)
Figure 5.16: Geometry of the (a) 398-element and (b) 604-element perturbed Danzer arrays that were designed using the NSGA.
Table 5.4: Characteristics of the single-point perturbed Danzer arrays that were optimized for sidelobe suppression and a targeted number of elements. The last four columns correspond to operation at $f = f_0$.

<table>
<thead>
<tr>
<th>Number of Elements</th>
<th>Bandwidth</th>
<th>Peak sidelobe over bandwidth (dB)</th>
<th>Directivity (dB)</th>
<th>HPBW (degrees)</th>
<th>$d_{avg}/\lambda$</th>
<th>$d_{max}/\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>398</td>
<td>7.35:1</td>
<td>-10.35</td>
<td>25.9</td>
<td>2.44</td>
<td>0.8</td>
<td>1.63</td>
</tr>
<tr>
<td>604</td>
<td>6.8:1</td>
<td>-10.57</td>
<td>27.77</td>
<td>2.43</td>
<td>0.64</td>
<td>1.34</td>
</tr>
</tbody>
</table>
5.3.2 Dual-Frequency Sidelobe Suppression

All of the examples presented thus far were designed with the objective of suppressing sidelobes at the upper bound of a targeted bandwidth. This ensures that the sidelobes at all lower frequencies will be equal to or less than that of the upper bound. While this objective is well suited for achieving wideband performance, it does not place any consideration on the specific degree of sidelobe suppression at lower operating frequencies. Consequently, some of the optimized designs have a sidelobe level at lower frequencies that is comparable to that of the upper bound of their bandwidth. For instance, the optimized array in Section 5.2.2 has a relatively steady sidelobe level over a majority of its bandwidth; from \( f = 2.5f_0 \) up to the upper bound of \( f = 22f_0 \) there is less than a decibel of deviation in its sidelobe level (see Figure 5.8). Incorporating an additional sampling frequency into the design process provides the potential for tailored sidelobe suppression over specific regions in the bandwidth of an array. In the case of aperiodic tiling arrays, this dual-frequency approach combined with the flexibility of the perturbation design technique allows for significant improvements in the sidelobe suppression at the lower frequencies of ultra-wideband designs.

To illustrate the effectiveness of this approach, a perturbed aperiodic tiling array was designed for wideband performance by optimizing for sidelobe suppression at an upper bound frequency as well as an intermediate frequency. The upper bound and intermediate frequencies were selected to be \( 5f_0 \) and \( 2.5f_0 \), respectively. It was expected that designing at these frequencies would most likely generate an array with considerable sidelobe suppression up to \( f = 2f_0 \) and then a secondary region of sidelobe suppression from \( f = 2.5f_0 \) up to at least \( 5f_0 \). Additionally, as it was seen in the examples of Section 5.2, a side effect of optimization process is that designs can sometimes have additional sidelobe suppression that extends well past the targeted upper bound frequency. The Danzer aperiodic set with double-point perturbations was selected for this dual-frequency optimization. The design of the perturbed array was carried out using the NSGA that was discussed in Chapter 2. The cost functions that were used to meet the design objectives were
\[ F_1 = \text{SLL}_{f_1} \text{ (dB)} \] (3)

\[ F_2 = \text{SLL}_{f_2} \text{ (dB)} \] (4)

where \( \text{SLL}_{f_1} \) is the sidelobe level at the intermediate frequency and \( \text{SLL}_{f_2} \) is sidelobe level at the upper bound frequency.

The final Pareto front of the multiobjective design is shown in Figure 5.18. As it can be seen in this figure, the front offers a range of solutions with tradeoffs in the level of sidelobe suppression at the targeted frequencies. The Pareto solution with \( F_1 = -16.93 \text{ dB} \) and \( F_2 = -13.23 \text{ dB} \) appeared to offer the best overall performance and was selected for further investigation. The selected design is comprised of 678 elements. Radiation and geometrical properties of the array are listed in Table 5.5. Its element distribution is shown in Figure 5.19 and a plot of its sidelobe level versus frequency is shown in Figure 5.20. In accordance with the design objectives, the array has very good sidelobe suppression up to \( f = 5f_o \), with excellent suppression from \( f = f_o \) to \( f = 2.5f_o \). In particular, the array maintains a sidelobe level of \(-16.93 \text{ dB} \) up to \( 2.5f_o \) and then a sidelobe level of \(-13.23 \text{ dB} \) from \( f = 2.5f_o \) to \( f = 5f_o \). Moreover, the array exhibits strong sidelobe suppression (below \(-11 \text{ dB} \)) well past the upper bound target of \( 5f_o \), up to \( 22.7f_o \). The bandwidth and sidelobe suppression of this design exceed those of the other examples that were presented in this chapter. The bandwidth of this design, as well as some of the presented examples, far exceeds that of other planar antenna array layouts that are reported in literature [1]-[3].
Figure 5.18: Pareto front of the NSGA dual-frequency optimization that was carried out on a perturbed Danzer array.

Figure 5.19: Geometry of the perturbed Danzer array that was designed using a multiobjective optimizer.
Figure 5.20: Sidelobe level performance of the 678-element double-point perturbed Danzer array that was designed using a multiobjective optimizer, 811-element standard Danzer array, and 1793-element square-lattice periodic array.

Table 5.5: Characteristics of the double-point perturbed Danzer array that was designed for sidelobe suppression at two frequencies. The last four columns correspond to operation at $f = f_o$.

<table>
<thead>
<tr>
<th>Number of Elements</th>
<th>Bandwidth</th>
<th>Peak sidelobe over bandwidth (dB)</th>
<th>Directivity (dB)</th>
<th>HPBW (degrees)</th>
<th>$d_{avg}/\lambda$</th>
<th>$d_{max}/\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>678</td>
<td>22.7:1</td>
<td>-11.13</td>
<td>30.6</td>
<td>2.44</td>
<td>0.63</td>
<td>1.28</td>
</tr>
</tbody>
</table>
Chapter 6

Peano-Gosper and Generalized Gosper Arrays

The efficacy of applying recursive-perturbation to space-filling curves will be shown in this chapter via several wideband and ultra-wideband array examples. Chapter 4 has already discussed details regarding the optimizer-based design methodology that was coupled with the perturbation technique as well as the general setup for these examples. Consequently, this chapter will mainly focus on the characteristics of the arrays that were generated using this approach. The first section is devoted to perturbed Peano-Gosper arrays that were designed for enhanced beam steering capabilities at $f = 2f_o$. The second section presents some examples of perturbed Gosper arrays that were designed for sidelobe suppression over a wide bandwidth. Among these is a perturbed stage-2 N=31 Gosper array that has excellent sidelobe suppression over more than a 10:1 bandwidth.

6.1 Peano-Gosper Curve

A few stages of perturbed Peano-Gosper arrays were designed for enhanced beam steering at $f = 2f_o$. The targeted scanning volume was a 30° conical volume that is centered at broadside (see Figure 4.25). The design process was carried out for stage-2, stage-3, and stage-4 arrays. In each case, the perturbation process provided a significant level of sidelobe suppression over the standard PG array. The perturbed stage-4 array, for example, has a peak sidelobe level of $-10.2$ dB for scanning within the targeted scan volume. Table 6.1 lists some of the relevant geometrical and radiation characteristics of
the optimized stages of the perturbed PG arrays. The following discussion will examine some selected aspects of these designs.

Table 6.1: Characteristics of the optimized stages of the perturbed Peano-Gosper arrays.
The peak SLL corresponds to the peak lobe level that appears while beam steering through a 30° conical volume at \( f = 2f_o \). The last three columns pertain to operation at \( f = f_o \). The average element spacing corresponds to the average of the spacings between successive elements along the PG curve.

<table>
<thead>
<tr>
<th>Stage</th>
<th>Number of Elements</th>
<th>Peak SLL during scanning (dB)</th>
<th>Directivity (dB)</th>
<th>HPBW (degrees)</th>
<th>Average Element Spacing (( \lambda ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>50</td>
<td>-7.71</td>
<td>16.94</td>
<td>4.0</td>
<td>0.865</td>
</tr>
<tr>
<td>3</td>
<td>344</td>
<td>-9.71</td>
<td>25.27</td>
<td>1.48</td>
<td>0.805</td>
</tr>
<tr>
<td>4</td>
<td>2402</td>
<td>-10.2</td>
<td>33.4</td>
<td>0.56</td>
<td>0.82</td>
</tr>
</tbody>
</table>

The layouts of the standard and perturbed stage-3 PG arrays are shown in Figure 6.1. It is clear from this figure that the adjustment of the six perturbation locations led to a highly altered element distribution. The best way to illustrate the enhanced scanning performance of the optimized stage-3 array is through the examination of its radiation pattern during beam steering. In Figure 6.2, a contour plot is shown for its normalized radiation pattern when its main beam is directed to \( \phi = 0^\circ, \theta = 30^\circ \) at a minimum element spacing of \( \lambda \). Clearly, there are no grating lobes contained within the visible region of the array at this particular scan angle. This is in direct contrast to the standard stage-3 PG array that has several grating lobes present at the same scan angle, as shown in Figure 4.15. Pattern cuts also provide some useful details regarding sidelobe behavior of these arrays during scanning. In Figures 6.3 and 6.4, pattern cuts are shown for the standard and perturbed stage-3 PG arrays with their main beam directed towards \( \phi = 180^\circ, \theta = 30^\circ \). Figure 6.3 contains cuts of the array factor at \( \theta = 30^\circ \) and Figure 6.4 contains cuts at \( \phi = 92.5^\circ \). Note that in these figures the arrays are based on a stage-3 PG curve with a segment length of \( 2\lambda \). In both cuts the pattern of the standard PG array contain at least one grating lobe while those of the perturbed array maintain a reasonable
sidelobe level without the appearance of a grating lobe. The improved broadband scanning performance of the perturbed stage-3 array comes at the expense of increased sidelobe levels at lower frequencies (i.e., smaller element spacings). For example, at $f = f_0$ ($d_{\text{min}} = 0.5\lambda$) the array has a sidelobe level of -9.47 dB during broadside operation, which is 6.73 dB higher than that of its conventional PGFA counterpart.

Figure 6.1: Element distributions of (a) a standard stage-3 Peano-Gosper array and (b) the perturbed stage-3 Peano-Gosper array.
Figure 6.2: Contour plots of the normalized radiation pattern with the main beam steered to $\phi = 0^\circ$, $\theta = 30^\circ$ for the perturbed stage-3 PG array at its optimized spacing.
The stage-4 optimization produced a design with a sidelobe suppression of -10.2 dB while scanning within the targeted 30° conical volume at $f = 2f_o$. If the array were limited

Figure 6.3: Normalized radiation pattern cut at $\theta = 30^\circ$ when the main beam is steered to $\varphi = 180^\circ$, $\theta = 30^\circ$ for (a) the standard stage-3 PG array and (b) the perturbed stage-3 PG array at its optimized spacing. Both arrays are based on a stage-3 PG curve with segment length of $2\lambda$.

Figure 6.4: Normalized radiation pattern cut at $\varphi = 92.5^\circ$ when the main beam is steered to $\varphi = 180^\circ$, $\theta = 30^\circ$ for (a) the standard stage-3 PG array and (b) the perturbed stage-3 PG array at its optimized spacing. Both arrays are based on a stage-3 PG curve with segment length of $2\lambda$.

The stage-4 optimization produced a design with a sidelobe suppression of -10.2 dB while scanning within the targeted 30° conical volume at $f = 2f_o$. If the array were limited
to only broadside operation, it would have a 3:1 bandwidth (SLL ≤ -10 dB) that extends up to $f = 3f_o$. In addition, it maintains a sidelobe level below −7 dB up to $f = 6.4f_o$. This is in contrast to the standard PG array that has grating lobes in its pattern past $f = 2.25f_o$. A plot of the sidelobe performance of these arrays is shown in Figure 6.5. As it can be seen in this plot, the enhanced scanning capabilities and improved sidelobe suppression at higher frequencies come at the expense of higher sidelobes at lower frequencies. For instance, the perturbed stage-4 array has a sidelobe level of −10.7 dB for broadside operation at $f = f_o$; this level is 5.42 dB higher than that of the standard stage-4 array. A pattern cut of the array with its main been directed to the boundary of the conical volume is shown in Figure 6.6.

Figure 6.5: Sidelobe performance of the perturbed stage-4 PG array, the standard stage-4 PG array, and a 1793-element square lattice periodic array.
Finally, from a subarray point-of-view it is worth examining the performance impact that results from interchanging offset values amongst various stages of the PGFA. It was expected that offset values which are optimal for a stage-2 array, for example, are not necessarily optimal for a stage-4 array. Additionally, it was expected that interchanging offset values between different stages would result in only a minor degradation in performance since the array geometry is based on a hierarchical structure. As an extension of these ideas, a stage-3 perturbed PG array was analyzed (via the cost function described in Section 4.2.4) using the offsets that were obtained via the stage-4 optimization. The peak sidelobe level of the array was only 0.355 dB higher than the results obtained through a direct stage-3 optimization. Similarly, generating a stage-4 perturbed PGFA using the optimized stage-3 offsets resulted in a 0.21 dB increase in sidelobe levels. The largest sidelobe increase that resulted from interchanging offsets amongst optimized stages was 1.15 dB. This came about from evaluating a stage-4 array using the offsets from the optimized stage-2 array. Based on this analysis, one can conclude that acceptable performance (low sidelobes and no grating lobes) is still retained even if the optimized offset values are interchanged between stages. This ability to interchange optimized offset values is important if the arrays are being used to support

Figure 6.6: Normalized radiation pattern cut at $\varphi = 0^\circ$ of the perturbed stage-4 PG array with its main beam directed towards $\varphi = 0^\circ$, $\theta = 30^\circ$. The cut corresponds to operation at $f = 2f_0$. The right side of the figure shows the array factor as a function of theta plotted from $-90^\circ$ to $+90^\circ$ and the left side shows a detailed view of the array factor near the main beam.
modular multibeam or multifrequency operations. For instance, the optimized stage-4 PGFA can be subdivided into seven stage-3 modular subarrays as shown in Figure 4.23. As it was shown above, each stage-3 subarray would still have excellent broadband scanning performance even if the entire array were constructed using the optimized stage-4 offset values.

6.2 Generalized Gosper Curves

6.2.1 13-Segment Generator Example

This section presents a stage-3 N=17 perturbed Gosper array that was designed for sidelobe suppression at $f = 6f_0$. The recursive-perturbation process used 13 element locations to control the overall layout of the 2198-element stage-3 array. The optimal perturbation locations determined through the use of a hybrid optimizer are shown in Figure 6.7a. A stage-2 N=13 array that was generated based on these perturbation locations is shown in Figure 6.7b. The perturbed stage-3 array has a peak sidelobe level of $-10.03$ dB at the design frequency of $6f_0$. This particular sidelobe level extends up to approximately $f = 6.9f_0$, which corresponds to a bandwidth of 6.9:1 for applications involving broadside operation. This upper bound is approximately three times higher than that of unperturbed, standard Gosper arrays. For beam steering applications the array still retains wideband capabilities. For instance, the optimized array would still have roughly a 4.6:1 bandwidth while beam steering within a 30° conical volume that is centered at broadside. A plot of the sidelobe performance of the perturbed and standard stage-3 N=13 PG arrays is shown in Figure 6.8. Some of the relevant characteristics of the perturbed array are listed in Table 6.2.
Figure 6.7: (a) Optimized element locations along the generating curve of the stage-3 N=13 Gosper array that was optimized at $f = 6f_0$. (b) Stage-2 array that was constructed using the optimized element locations in (a).
Table 6.2: Characteristics of the standard and perturbed stage-3 N=13 Gosper arrays. The last three columns pertain to operation at $f = f_0$. The average spacing corresponds to the average of the spacings between successive elements along the N=13 Gosper curve.

<table>
<thead>
<tr>
<th></th>
<th>Number of Elements</th>
<th>Bandwidth</th>
<th>Peak SLL over Bandwidth (dB)</th>
<th>Directivity (dB)</th>
<th>HPBW (degrees)</th>
<th>Average Spacing ($\lambda$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perturbed</td>
<td>2198</td>
<td>6.9:1</td>
<td>-10.03</td>
<td>34.87</td>
<td>1.07</td>
<td>0.83</td>
</tr>
<tr>
<td>Standard</td>
<td>2198</td>
<td>2.25:1</td>
<td>-12.78</td>
<td>34.64</td>
<td>2.15</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Analysis of the radiation pattern of the perturbed N=13 array yields an important observation regarding its performance. The pattern cuts of Figure 6.9 will aid in the discussion to follow. In this figure a pattern cut is shown for the perturbed stage-3 N=13 array at its design frequency of $6f_0$. The figure also provides a pattern cut of the standard stage-3 N=13 array with its corresponding curve set equal in size to that of the perturbed array. The particular cut at $\phi = 18.3^\circ$ was selected to illustrate the numerous grating lobes that are present in the case of the standard Gosper array. Of primary importance in this figure is the lack of a well-defined main beam in the patterns of both arrays. For most array applications it is desired to have a distinct main beam that is surrounded by a sharp first-null. As it can be seen in Figure 6.9b, the standard N=13 PG array does not support this type of pattern due to its unconventional aperture. Similarly, since this aperture is relatively unchanged through the use of the perturbation technique, the perturbed N=13 array also exhibits this undesirable characteristic. As it will be seen in the next two sections, this characteristic is not associated with Gosper arrays that have a more balanced aperture contour.
The second example that will be presented in this section is of a stage-2 N=19 perturbed Gosper array that was designed for sidelobe suppression at $f = 6f_o$. The array is comprised of 362 elements and its generator possesses eighteen perturbation locations.

Figure 6.9: Normalized radiation pattern cut at $\varphi = 18.3^\circ$ for the stage-3 N=13 (a) unperturbed Gosper array and the (b) perturbed Gosper array at $f = 6f_o$. At this frequency the unperturbed array was scaled to match the dimensions of its curve to that of the optimized array. The right side of the figure shows the array factor as a function of theta plotted from -90 to +90 degrees and the left side of the figure shows a detailed view of the array factor near the main beam.

### 6.2.2 19-Segment Generator Example

The second example that will be presented in this section is of a stage-2 N=19 perturbed Gosper array that was designed for sidelobe suppression at $f = 6f_o$. The array is comprised of 362 elements and its generator possesses eighteen perturbation locations.
The optimized perturbation locations and the resultant stage-2 array layout are shown in Figure 4.24. At the targeted frequency the sidelobes of the perturbed array were reduced from 0 dB (in the case of the unperturbed array) to –10.06 dB. This level of suppression is maintained slightly past the targeted frequency, up to $f = 6.2f_o$. The upper bound on its bandwidth is approximately three times higher than that of the standard N=19 array and a square-lattice periodic array. A plot of the sidelobe performance of the standard and perturbed stage-2 N=19 arrays is shown in Figure 6.10. Some of the relevant characteristics of the perturbed array are listed in Table 6.3.

Figure 6.10: Sidelobe performance of the optimized stage-2 N=19 Gosper array, standard stage-2 N=19 Gosper array, and a 1793-element square-lattice periodic array with a circular aperture.
Table 6.3: Characteristics of the standard and perturbed stage-2 N=19 Gosper arrays. The last three columns pertain to operation at $f = f_o$. The average spacing corresponds to the average of the spacings between successive elements along the N=19 Gosper curve.

<table>
<thead>
<tr>
<th></th>
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<th>Bandwidth (dB)</th>
<th>Peak SLL over Bandwidth (dB)</th>
<th>Directivity (dB)</th>
<th>HPBW (degrees)</th>
<th>Average Spacing ($\lambda$)</th>
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<td>Perturbed</td>
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<td>6.2:1</td>
<td>-10.06</td>
<td>26.37</td>
<td>2.92</td>
<td>0.907</td>
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<tr>
<td>Standard</td>
<td>362</td>
<td>2.25:1</td>
<td>-16.6</td>
<td>26.78</td>
<td>5.86</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Similar to the case of the perturbed N=13 array, valuable insight is attained through the examination of the radiation patterns of the perturbed and standard N=19 arrays. Pattern cuts of the arrays are shown in Figure 6.11. This particular cut at $\phi = 43^\circ$ was selected to illustrate the numerous grating lobes that are present in the pattern of the standard Gosper array. In this figure the pattern of the optimized array is shown at $f = 6f_o$, corresponding to an element spacing of $3\lambda$. The standard, unperturbed array was scaled such that the dimensions of its curve are equivalent to those of the curve of the perturbed array. Some important observations can be made when comparing these pattern cuts. First, within the region of the main beam the array factor appears relatively unaffected by the perturbation process. For instance, the overall shape and sidelobe suppression of the patterns of the arrays are nearly identical within the four-degree window shown on the left side of Figure 6.11. Additionally, both arrays exhibit sharp nulls adjacent to the main beam. This is in contrast to the N=13 arrays that were analyzed above.
Additional insight gained from Figure 6.11 is that remnants of the grating lobes that are present in the case of the standard, unperturbed array are still seen in the patterns of perturbed array. This can be observed by noting the locations of the peak lobes of the unperturbed and optimized arrays. The pattern cut of the optimized array (Figure 6.11b) has ten peak lobes that are suppressed well below the main beam. Comparing this with Figure 6.11b reveals that the locations of these peak lobes correspond to the locations of the grating lobes of the unperturbed array. This illustrates that remnants of the

Figure 6.11: Normalized radiation pattern at $\varphi = 43^\circ$ for the stage-2 N=19 (a) unperturbed Gosper array and the (b) perturbed Gosper array at $f = 6f_o$. At this frequency the unperturbed array was scaled to match the dimensions of its curve to that of the optimized array. The right side of the figure shows the array factor as a function of theta plotted from -90 to +90 degrees and the left side of the figure shows a detailed view of the array factor near the main beam.
underlying behavior of the unperturbed Gosper array (with its interior triangular lattice) are still present in the behavior of this perturbed Gosper array. It is assumed that this occurrence is primarily due to the restriction that elements are only perturbed along the Gosper curve; this limits the allowable directions for perturbations to only three principle directions (see Figure 4.24). Even with this restriction, the perturbation process is still able to significantly reduce the constructive interference that accounts for the numerous grating lobes present in the case of the standard Gosper array.

### 6.2.3 31-Segment Generator Example

Following the successful design of a wideband array based on the N=19 generator, the recursive-perturbation process was applied to a Gosper array with an even larger generator. The next step up in size from the 19-segment generator is a 31-segment generator. Multiple N=31 curves were reported in [45] and the one that was selected for this investigation is designated the N=31b-1 curve. To simply discussion, this curve will be referred to as simply the N=31 curve for the remainder of this chapter. The first two stages of this curve are shown in Figure 2.7b and an array based the geometry of second stage of the curve is shown in Figure 4.16e. The N=31 generator possesses thirty perturbation elements. Compared to the N=19 generator, this represents an increase of more than a sixty percent in the number of design parameters. It was expected that the additional perturbation locations would lead to more flexibility in the design process and consequently, the potential for improved performance. This was the case in going from the N=7 PG arrays to the N=19 Gosper array; where perturbation of the N=19 array led to a bandwidth with an upper bound that more than doubled that of the N=7 array.

The increased design complexity for the N=31 array makes for a more much more challenging optimization problem. In addition to dealing with a greater number of parameters comes the issue of interelement spacing violations among the thirty perturbation elements. In the adjustment of thirty element locations there is a high probability that at least one set of elements is too closely spaced. One way to mitigate
this issue is to restrict the distance that elements are permitted to be perturbed from their initial positions. However, a consequence of this is that less diversity is shown in the resulting perturbed array layouts. And as it was observed through optimizations, this has a significant impact on the level of performance that can be attained for a design. For the N=31 generator, an allowable perturbation range of fifty percent (along the length of a segment) was found to offer a good tradeoff between mitigating spacing issues and offering design diversity. Even with this restriction, throughout the optimization process a considerable percentage of the arrays that were generated through genetic crossovers were in violation of the minimum spacing constraint. This helps to illustrate the benefit of the recursive-perturbation approach, since trying to adjust the location of every element in the array would certainly prove to be an intractable design problem.

A perturbed stage-2 N=13 array was designed with the goal of maximizing sidelobe suppression at $f = 10f_o$. The design of the array was carried out using the hybrid optimizer. The evolution of the peak sidelobe level throughout the optimization process is shown in Figure 6.12. In this plot, the sidelobe level is plotted versus the number of cost function evaluations. The GA optimization was carried out until an extended period of stagnation was observed in the evolution of the sidelobe level. The solution from the GA was then used to seed the Nelder-Mead optimizer for further refinement. The layout of the final solution is shown in Figure 6.13 and the optimized perturbation locations are shown in Figure 6.14. The hybrid optimizer was able to come up with a design that has a sidelobe level of $-10.3$ dB at $f = 10f_o$. The bandwidth of the array extends slightly past this targeted frequency, up to $f = 10.35f_o$. This upper bound frequency is nearly five times higher than that of the standard Gosper arrays and square-lattice periodic arrays. A plot of the sidelobe performance of these arrays is shown in Figure 6.15. As it can be seen in this plot, the significant bandwidth enhancement comes at the expense of an increase in the sidelobe level at lower frequencies, i.e. from $f = f_o$ to $f = 2.2f_o$. However, for many wideband applications this tradeoff would be justified. Some of the relevant characteristics of the array are listed in Table 6.4. Note that the perturbation process had a negligible impact on the directivity of the stage-2 N=31 Gosper array. This was also
the case for the other Gosper arrays that were considered in this chapter (see Tables 6.2 and 6.3).

Figure 6.12: Evolution of the peak sidelobe level of the perturbed stage-2 N=31 Gosper array that was designed using the hybrid optimizer.
Figure 6.13: Geometry of the optimized stage-2 N=31 perturbed Gosper array.

Figure 6.14: Generating curve and perturbed element locations that correspond to the optimized stage-2 N=31 perturbed Gosper array.
Figure 6.15: Sidelobe performance of the optimized stage-2 N=31 Gosper array, standard stage-2 N=31 Gosper array, and a 1793-element square-lattice periodic array with a circular aperture.

Table 6.4: Characteristics of the standard and perturbed stage-2 N=31 Gosper arrays. The last three columns pertain to operation at $f = f_o$. The average spacing corresponds to the average of the spacings between successive elements along the N=31 Gosper curve.

<table>
<thead>
<tr>
<th></th>
<th>Number of Elements</th>
<th>Bandwidth</th>
<th>Peak SLL over Bandwidth (dB)</th>
<th>Directivity (dB)</th>
<th>HPBW (degrees)</th>
<th>Average Spacing ($\lambda$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perturbed</td>
<td>962</td>
<td>10.35:1</td>
<td>-10.3</td>
<td>31.13</td>
<td>1.78</td>
<td>0.84</td>
</tr>
<tr>
<td>Standard</td>
<td>962</td>
<td>2.25:1</td>
<td>-16.8</td>
<td>31.12</td>
<td>3.56</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Plots of the normalized radiation pattern of the standard and optimized stage-2 N=31 Gosper array help illustrate the excellent sidelobe and grating lobe suppression that was attained through the recursive-perturbation process. At $f = 10f_o$, the radiation pattern of the standard Gosper array is completely dominated by numerous unwanted grating lobes, as shown in Figure 6.16. In this plot the array is seen to possess sixty grating lobes within its visible region. The plot in Figure 6.17 provides a detailed view of some of the grating lobes through a cut of the radiation pattern. The features of its radiation pattern are in contrast to the diffused features that are seen in the radiation pattern of the perturbed Gosper array. Figure 6.18 provides a contour plot of the normalized radiation pattern of the perturbed array and Figure 6.17b shows a detailed view along a cut of this pattern. In these plots the radiation pattern is seen to possess a fairly diffused features with no areas of appreciable constructive interference (aside from the main beam). Additionally, the pattern of the perturbed array does not appear to share any readily identifiable features with the pattern of the standard, unperturbed array. As shown in Figures 6.9 and Figure 6.11, this is in contrast to patterns of the other perturbed Gosper arrays in this chapter.
Figure 6.16: Normalized radiation pattern at $f = 10 f_0$ of the stage-2 N=31 standard, unperturbed Gosper array.
Figure 6.17: Normalized radiation pattern cut at $\varphi = 42^\circ$ for the stage-2 $N=31$ (a) unperturbed Gosper array and the (b) perturbed Gosper array at $f = 6f_o$. At this frequency the unperturbed array was scaled to match the dimensions of its curve to that of the optimized array. The right side of the figure shows the array factor as a function of theta plotted from -90 to +90 degrees and the left side of the figure shows a detailed view of the array factor near the main beam.
6.3 Full-Wave Analysis of a Perturbed Gosper Array

It was expected that mutual coupling would not play a significant role in the performance of perturbed Gosper arrays due to their relatively large average element spacings. This was investigated through the use of full-wave analysis (Method of Moments [80]) of a 32-element stage-1 N=31b-1 perturbed Gosper array. The element locations of the array

Figure 6.18: Normalized radiation pattern at $f = 10f_o$ of the stage-2 N=31 perturbed Gosper array.
correspond to the optimized perturbation locations of the generator that was discussed in previous section. The array was analyzed at several different minimum element spacings (corresponding to various operating frequencies) to investigate the impact of mutual coupling on perturbed Gosper array layout. Rather than utilizing a wideband antenna element, this analysis was based on exploring the characteristics of the array through the use of a narrowband microstrip patch antenna. The patch element was designed for operation at 1 GHz and it was used to investigate the mutual coupling and radiation characteristics of the array layout through the adjustment of the minimum element spacing at this frequency. This type of analysis is adequate to explore the characteristics of perturbed Gosper arrays and to illustrate the effect of their layout on the mutual coupling of microstrip antenna elements. Of particular interest is the operation of these arrays over the lower portion of their bandwidth where the elements are more closely spaced.

The element selected for this investigation was a probe-fed rectangular microstrip patch antenna. It was designed for operation at 1 GHz and is based on a substrate with a dielectric constant of 4 and a thickness of 8 mm. The length and width of the patch are 86.3 mm and 71.5 mm, respectively. The models for the full-wave analysis of the microstrip patch and microstrip patch array assume that they are comprised of a substrate and a ground plane that are infinite in extent.

The baselines for comparison in this investigation are radiation patterns calculated using pattern multiplication of a full-wave simulation of a single antenna element (in isolation) with the array factor of the stage-1 array. The variations between the full-wave analysis and this “idealized” case will illustrate the impact that the array environment has on the radiation characteristics of the Gosper microstrip patch array. At each frequency under investigation the hemispherical patterns of the two models were compared for broadside operation as well as for a number of beam steering angles within a 30˚ conical volume. In addition to pattern comparisons, the driving-point impedance of each patch element was calculated for the full-wave model and compared with that of the single-patch in isolation.
All of the arrays in this chapter were designed with the constraint that their lowest intended operating frequency corresponds to a frequency at which the array has a minimum element spacing of $0.5\lambda$. This common design constraint is typically used to lessen the degree of coupling between neighboring elements. The first case will be investigated in the full-wave Gosper array analysis is operation at 1 GHz with a minimum spacing of $0.5\lambda$ (15 cm). At this spacing, the impact of mutual coupling on the overall radiation characteristics of the array was found to be minimal. In particular, during broadside operation there is excellent agreement between the patterns of the two array models; a variation of less than 0.2 dB is exhibited in their peak sidelobe levels. E-plane and H-plane cuts of the radiation patterns during broadside operation are shown in Figure 6.19. Similarly, comparable agreement between the two models is seen while beam steering within a 30° conical volume. Some representative pattern cuts for scanning along the E-plane and H-plane are shown in Figures 6.20 and 6.21. The excellent agreement in pattern analysis gives an indication that the radiation characteristics of the patch elements are relatively unaffected by the presence of neighboring elements in the Gosper array when the elements are spaced at least $0.5\lambda$ apart. While the radiation characteristics are minimally affected, coupling among radiating elements is seen to impact the driving-point impedance of the patch elements during beam steering. For instance, in some cases the real part of the driving-point impedance of an element drops to nearly 0 ohm. While it was not investigated, this issue might be mitigated through the proper adjustment of the patch probe feed locations. An alternate approach is to operate the array at a slightly higher frequency. This increases the electrical spacing between elements and in turn tends to reduce the degree of coupling between neighboring elements.
Figure 6.19: Radiation pattern cut along (a) $\varphi = 0^\circ$ (H-plane of the microstrip patch elements) and (b) $\varphi = 90^\circ$ (E-plane of the microstrip patch elements) for the perturbed stage-1 $N=31$ Gosper array with $d_{\text{min}} = 0.5\lambda$ at $f = 1$ GHz and its main beam directed to broadside. The centers of the array elements have a spacing of at least of 15 cm.
Figure 6.20: Radiation pattern cut along $\phi = 0^\circ$ for the perturbed stage-1 $N=31$ Gosper array with $d_{min} = 0.5\lambda$ at $f = 1$ GHz and its main beam directed to $\phi = 0^\circ$, $\theta = 30^\circ$. The centers of the array elements have a spacing of at least of 15 cm.

Figure 6.21: Radiation pattern cut along $\phi = 90^\circ$ for the perturbed stage-1 $N=31$ Gosper array with $d_{min} = 0.5\lambda$ at $f = 1$ GHz and its main beam directed to $\phi = 90^\circ$, $\theta = 30^\circ$. The centers of the array elements have a spacing of at least of 15 cm.
Full-wave analysis was performed on the Gosper array at the same operating frequency of 1 GHz but with a slightly larger minimum element spacing of 0.625λ. In terms of the discussion in this chapter, this spacing corresponds to operation at \( f = 1.25f_o \). Similar to the case of \( d_{\text{min}} = 0.5\lambda \), excellent agreement is exhibited between the radiation patterns of the two array models. However, unlike the previous case the values of driving-point impedance remain relatively stable during beam steering. While steering within a 30° conical volume the real part of the driving-point impedance remains within the range of 18 ohms to 210 ohms for each array element. Restricting operations to spacings greater than 0.625λ might be a limiting factor for conventional periodic arrays but it has less of an impact on Gosper arrays due to their relatively wide bandwidths. For instance, the stage-2 \( N=31 \) Gosper array from Section 6.2.3 has a 10:1 bandwidth that corresponds to operations from spacings of 0.5λ to 5λ. Restricting the operation of the array to spacings greater than 0.625λ (or some alternate higher spacing) does not lead to an appreciable loss of bandwidth.

As expected, the effects of mutual coupling become less pronounced as the minimum element spacing is further increased. For example, the range of values of the driving-point impedances during beam steering at \( d_{\text{min}} = \lambda \ (f = 2f_o) \) is significantly reduced from that of \( d_{\text{min}} = 0.625\lambda \). Also, the radiation pattern of the full-wave model is nearly identical to that of idealized model. Some representative pattern cuts at this spacing are shown in Figure 6.22 through Figure 6.24. As shown Figure 6.25, a further increase in minimum element spacing to 4λ leads to nearly indistinguishable patterns.
Figure 6.22: Radiation pattern cut along (a) $\varphi = 0^\circ$ (H-plane of the microstrip patch elements) and (b) $\varphi = 90^\circ$ (E-plane of the microstrip patch elements) for the perturbed stage-1 $N=31$ Gosper array with $d_{\text{min}} = \lambda$ at $f = 1$ GHz and its main beam directed to broadside. The centers of the array elements have a spacing of at least of 15 cm.
Figure 6.23: Radiation pattern cut along $\phi = 0^\circ$ for the perturbed stage-1 $N=31$ Gosper array with $d_{\text{min}} = \lambda$ at $f = 1$ GHz and its main beam directed to $\phi = 0^\circ, \theta = 30^\circ$. The centers of the array elements have a spacing of at least of 15 cm.

Figure 6.24: Radiation pattern cut along $\phi = 90^\circ$ for the perturbed stage-1 $N=31$ Gosper array with $d_{\text{min}} = \lambda$ at $f = 1$ GHz and its main beam directed to $\phi = 90^\circ, \theta = 30^\circ$. The centers of the array elements have a spacing of at least of 15 cm.
Figure 6.25: Radiation pattern cut along (a) $\varphi = 0^\circ$ (H-plane of the microstrip patch elements) and (b) $\varphi = 90^\circ$ (E-plane of the microstrip patch elements) for the perturbed stage-1 $N=31$ Gosper array with $d_{\text{min}} = 4\lambda$ at $f = 1$ GHz and its main beam directed to broadside. The centers of the array elements have a spacing of at least of 15 cm.
Chapter 7

Conclusions and Ideas For Future Research

This dissertation has addressed the need for robust methodologies capable of designing ultra-wideband antenna arrays by introducing the concept of recursive-perturbation. Recursive-perturbation offers a number of advantages over traditional perturbation techniques that involve adjusting element locations based on a periodic lattice. A key advantage lies in its ability to generate very diverse planar array distributions based on only a small number of parameters. Thus, unlike the traditional approach of adjusting every element in an array, this allows the perturbation process to be applied to large planar arrays in an efficient and reasonable manner. Furthermore, with only a small number of parameters, the design of recursively-perturbed arrays is easily carried out through a robust optimization procedure. In this dissertation the effectiveness of recursive-perturbation was demonstrated through the introduction of array perturbation schemes based on the geometries of space-filling curves and aperiodic tilings. The schemes lead to arrays with not only ultra-wideband performance but also a variety of other desirable characteristics.

The hierarchal structure of space-filling curves provides an excellent framework for array design and the use of recursive-perturbation. A design technique was introduced that involves adjusting element locations along a stage-1 generator and then using these locations along with the recursive properties of the curve to generate successively larger arrays. The novelty of this technique lies in its ability to use a small set of element locations along a simple curve to generate large planar arrays. Dealing with only several elements along a curve significantly lessens issues that would arise in trying to adjust
every element location in a planar configuration. It also reduces the perturbation of these
arrays to a tractable problem with a manageable number of parameters. Coupling this
technique with a hybrid optimizer provides for a very robust array design procedure. The
utility of this combination was explored through its application to arrays based on the
Peano-Gosper curve and generalized Gosper curves. A recursive formulation was
developed for calculating the array factor perturbed PG and Gosper arrays. This
dramatically reduces analysis time and it allows for the optimization of very large arrays
that would otherwise be too time consuming. A variety of design examples were
presented, including one that consists of a uniformly excited 962-element perturbed
Gosper array that has sidelobe suppression greater than –10 dB over more than a 10:1
bandwidth. In addition to exhibiting wideband and ultra-wideband capabilities, perturbed
PG and Gosper arrays possess highly modular architectures whereby large arrays can be
divided into a collection of identical subarrays. This offers the potential for a wideband
array system that can be operated as either a single array or as a collection of identical
subarrays to support multi-beam operations.

In addition to space-filling curves, aperiodic tilings were also shown to provide an
excellent framework for applying the application of recursive perturbations. A scheme
was developed that is based on adding perturbation elements to the iterative tiling
generation process. The effect of this scheme is that the adjustment of a few perturbation
elements leads to a wide range of planar array distributions and associated radiation
characteristics. And since these distributions are based on truncations of large aperiodic
tilings, they can take on a variety of aperture shapes and sizes to meet system design
requirements. The tiling perturbation technique was coupled with optimization
algorithms for an automated and robust design process. The utility of this design
approach was explored through several single-objective and multi-objective examples.
One example, involves a uniformly excited 678-element planar array that has a sidelobe
level less than –11 dB over more than a 22:1 bandwidth. This bandwidth is more than
twenty times greater than that of conventional periodic arrays. In addition to ultra-wide
bandwidths, this design technique is also capable of generating highly sparse element
distributions. An example was shown for an ultra-wideband array that has 75% fewer elements than a comparable periodic array.

The research contained within this dissertation has introduced the concept of recursive-perturbation and examined its use in ultra-wideband array design. There exist a number of ways in which future research could build on this work. One example involves exploring the potential for modular architectures in perturbed aperiodic tiling arrays. Their iterative geometries hint at this ability but they may require some modifications or alternate tilings than the ones considered here. Also, these geometries offer the potential to be exploited in the development of a recursive array factor formulation. As was the case for the space-filling arrays, this would dramatically reduce the time to convergence of the optimization design procedure. Along these lines, it may be worthwhile to determine if the geometries of space-filling curves and aperiodic tilings can aid in the design of feed networks of their associated perturbed arrays. And more generally, the concept of recursive-perturbation could be applied to other iteratively constructed geometries, such as fractals. The framework of these geometries may lead to various benefits that are not associated with the techniques presented here.
Bibliography


VITA

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Selected Publications:


